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ACCOUNTING FOR RISK AVERSION, VESTING, JOB TERMINATION RISK AND MULTIPLE EXERCISES IN VALUATION OF EMPLOYEE STOCK OPTIONS

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We present a valuation framework that captures the main characteristics of employee stock options (ESOs), which financial regulations now require to be expensed in firms' accounting statements. The value of these options is much less than Black–Scholes prices for corresponding market-traded options due to the suboptimal exercising strategies of the holders, which arise from risk aversion, trading and hedging constraints, and job termination risk. We analyze the combined effect of all of these factors along with the standard ESO features of multiple exercising rights, and vesting periods. This leads to the study of a chain of nonlinear free-boundary problems of reaction-diffusion type. We find that job termination risk, vesting, finite maturity and non-zero interest rates are significant contributors to the ESO cost. However, we find that in the presence of vesting, the impact of allowing multiple exercise rights on ESO cost is negligible.

Key Words: employee stock options, American options, risk aversion, reaction-diffusion equations.

1. INTRODUCTION

Employee Stock options (ESOs) are call options granted by a firm to its employees as a form of benefit in addition to salary. They provide both compensation and incentive to the employees. Since the mid 1980s, stock options have become an important component of compensation in the United State. According to Hall and Murphy (2002), 94% of S&P 500 companies granted options to their top executives, and the total value accounted for 47% of total pay for the CEOs.

Due to the extensive use of ESOs, the Financial Accounting Standards Board (FASB) has become concerned about the cost of these options to shareholders. In the past decade, the reporting of the granting cost of such options has changed from optional to mandatory. In 2004, under *Statement of Financial Accounting Standards No. 123 (revised)*, FASB required firms to estimate and report "the grant-date fair value " of the ESOs issued. This gives rise to the need to create a reasonable valuation method for these options.

In order to determine the cost of ESOs to the firm, it is important to understand the characteristics of ESOs, and distinguish them from market-traded options. Typically, ESOs are American call options (i.e., they can be exercised at any time during the exercise

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window), with long maturity ranging from 5 to 15 years. In most cases, the ESOs are not immediately exercisable. The firm usually wants to maintain the incentive effect by prohibiting the employee holders from exercising during a certain period from the grant date. This period is called the *vesting* period. During the vesting period, the holder's departure from the firm, voluntarily or forced, will lead to forfeiture of his option (i.e., it becomes worthless).

Once endowed with an ESO, the employee cannot sell it, or hedge against his position by short selling the company stock,[1] but he can partially hedge his position by trading other securities, for example, the S&P 500 index. The sale and hedging restrictions may induce the employee to exercise the ESO early and invest the option proceeds elsewhere. The employee's risk preference and his available investment opportunities directly affect his exercise behavior.

Another vital feature for ESO valuation is the possibility that the employee suddenly leaves the firm before the ESO matures. At any time, an employee could be fired by the employer, or leave the firm voluntarily. If the departure happens during the vesting period, then the option is forfeited, and the ESO costs the firm nothing in this case. If the ESO holder leaves after the vesting period, then, at the time of departure, the holder may exercise the option, and the firm pays the proceeds, if any.

All these features—vesting, sale and hedging restrictions, the employee's exercise behavior and the risk of sudden job termination—have significant bearing on the fair value of ESOs. Hence, FASB requires valuation models to capture the unique characteristics of ESOs (see *Appendix A: Implementation Guideline* in the FASB statement 123R). Our primary objective in this paper is to provide a model that can accommodate all these characteristics and determine the cost of ESOs to the firm. Moreover, we want to address the challenging question: how do these characteristics influence the employee's exercise policy and the firm's granting cost? Our model will be useful not only in improving the precision of ESOs expensing, but will also shed light on executives' exercising behavior.

As empirical studies on ESOs suggest, the majority of ESOs holders tend to exercise early, often right after the vesting period. For instance, Huddart and Lang (1996), Marquardt (2002), and Bettis et al. (2005) point out that, for ESOs with 10 years to maturity, the average exercise time is between 4 and 5 years. This deviates from the prediction made by no-arbitrage pricing theory. For instance, in the case of an American call written on a non-dividend paying underlying stock, no-arbitrage pricing models conclude that the holder should never exercise early. This early exercise phenomenon indicates that no-arbitrage theory is inadequate for determining the exercise policy for ESOs.

To account for the employee's early exercise, FASB proposes an expensing approach by adjusting the Black–Scholes (B–S) model. In particular, it recommends substituting the option expiration date with the expected time to exercise. Although this expensing method is very simple and convenient, it is far from accurate. Jennergren and Naslund (1993), Hemmer et al. (1994), and Huddart and Lang (1996) conclude that this adjusted B–S model fails to capture the employee's exercise behavior and overstates the cost of the ESOs to the firm.

In this paper, we propose an ESO valuation model that explains this phenomenon in several ways. First, we illustrate that early exercises are optimal for a risk-averse ESO

[1] According to Section 16(c) of the U.S. Securities Exchange Act, executives are precluded from short-selling the shares of their employer. The FASB statement 123R (see paragraph B80) indicates that "many public entities have established share trading policies that effectively extend that prohibition to other employees." This short sales restriction has been adopted in the literature on ESOs; e.g., Huddart (1994) and Carpenter (1998).

holder with sale and hedging restrictions. Secondly, we show that the possibility of job termination induces the employee to adopt a more conservative exercising strategy, indirectly leading to early exercises. Moreover, job termination prior to maturity directly leads to early exercises by forcing the employee to exercise the ESO.

Our valuation model, in its simplest form, consists of two steps. First, we consider a risk-averse ESO holder who is subject to employment termination and constrained from selling the option or shorting the company stock, but is allowed to trade a partially correlated asset, such as the market index. The holder tries to decide when to exercise the ESO so that his expected utility of wealth is maximized. As a result, the holder obtains an exercise boundary. In technical terms, the ESO holder faces a stochastic control problem with optimal stopping. This problem is then formulated as a free boundary problem, from which we obtain the holder's exercise boundary. Next, the firm will use this boundary to find the cost of issuing this ESO. From the firm's perspective, the cost of this ESO is given by the no-arbitrage price of a barrier-type call option subject to early exercise due to job termination, where the barrier is the employee's optimal exercise boundary.

In our model, the employee's optimal exercise boundary differs from that in no-arbitrage pricing theory because the latter assumes the availability of a perfect hedge and the risk-neutrality of the holder. By no-arbitrage pricing theory, the holder's optimal exercise boundary is the one that maximizes the expected discounted payoff of the option. For this reason, we call it the *price-maximizing* boundary. To the contrary, the risk-averse employee in our model, who is constrained from selling the option and shorting the company stock, has no perfect hedge. The employee's exercise boundary is the one that maximizes the expected utility of holding the ESO, so we also call it the *utility-maximizing* boundary.

By incorporating job termination risk, we obtain a nonlinear free boundary problem of reaction-diffusion type for the employee's investment problem. Reaction-diffusion equations arise in utility problems in incomplete markets, for example, in portfolio choice with recursive utility (Tiu 2004), and indifference pricing with interacting Itô and point processes (Becherer 2004; Becherer and Schweizer 2005), and indifference pricing in credit risk (Sircar and Zariphopoulou 2006). In this paper, we study the existence of solution and the properties of the free boundary for this problem.

We also include the case in which the employee is granted multiple ESOs and partial exercises are allowed. In the traditional no-arbitrage pricing theory for American options, the holder's exercise boundary for one American call is identical to that for multiple American calls. In other words, the holder always exercises all the options at the same time. However, it is well documented that ESO holders tend to gradually exercise fractions of their options through maturity. See for instance, Huddart and Lang (1996).

In the context of no-arbitrage pricing for swing options, Carmona and Touzi (2008) study the holder's multiple exercise policy under the assumption that successive exercises are separated by small time intervals. Several authors have used the utility-based framework to explain the optimality of partial exercises of American options in incomplete markets. For perpetual American options with zero interest rate, Henderson (2006) provides an analytic formula for the employee's exercise thresholds. For American-style ESOs with finite maturity, Jain and Subramanian (2004), Grasselli (2005), and Rogers and Scheinkman (2007) numerically determine the employee's optimal exercise policy, but in the absence of sudden job termination risk. All these authors show that partial exercises could be optimal for the option holder under certain constraints. In this paper, we incorporate vesting and job termination risk, and provide a characterization for the optimal exercise time and a numerical scheme for the employee's optimal exercise boundaries. In particular, when a vesting period is imposed, the cost of ESOs with multiple exercise rights and the cost with simultaneous exercise constraint are almost the same (see Figure 6.2).

We have a parametric model for ESO valuation which, given reasonable data, can be calibrated. This is a straightforward test of validity that may be used to select between various models. In addition to evaluating the behavioral assumptions described by the utility formulation, one can also design empirical tests to address questions relevant to ESO valuation. For example, do exercise patterns of ESOs with and without simultaneous exercise constraint conform to Figure 6.1? With other parameters carefully controlled, do similar employees with different job termination rates exercise their ESOs according to Figure 4.1? For this purpose, empirical data of ESO exercises that is well-segmented based on employees' attributes, including age, position, and the time and cause of job termination, is highly desirable.

The rest of the paper is organized as follows. Section 2 provides an overview of related studies of stock options valuation. In Section 3, we formulate our valuation model for a single ESO. In Sections 4 and 5, we examine the employee's exercise policy and the ESO cost, respectively. In Section 6, we extend our model to the case with multiple exercises. The impact of multiple exercises on the employee's exercising strategy and the ESO cost are studied in Section 6.3.

2. RELATED STUDIES

The wide use of employee stock options has led to a growing literature on their valuation. One approach is to risk-neutrally price the options, with the employee's optimal exercise boundary exogenously specified. Hull and White (2004), and Cvitanic et al. (2008) are examples of this approach. Hull and White propose that the employee's exercise boundary be flat. Cvitanic et al. (2008) propose an exponentially decaying barrier. These *ad hoc* exercise boundaries are independent of the model parameters like the employee's exit rate, and the company stock's drift and volatility. As we will see later, the employee's exercise boundary changes considerably with all these parameters (Figures 4.1–4.3).

Alternatively, other researchers model early exercises as the first arrival of an exogenous counting process. For example, Jennergren and Naslund (1993) use the first jump time of some exogenous Poisson process. The Poisson process in their model serves as a proxy for all the factors that cause early exercises, including voluntary and involuntary job termination, and the holder's desire to voluntarily exercise early. On the other hand, Carr and Linetsky (2000) propose an intensity-based framework for European-style stock option valuation, in which the intensities for voluntary early exercises and job termination depend on the company stock price and time. These two models neglect the impact of the risk of job termination on the employee's own exercising strategy. In Section 4, we will show that job termination risk induces the ESO holder to adopt a more conservative exercising strategy (Proposition 4.3 and Figure 4.1 (left-hand side)).

Another approach is to investigate the effects of non-tradability and hedging restrictions. Huddart (1994), Kulatilaka and Marcus (1994) and Chance and Yang (2005) develop binomial tree models that compute the certainty equivalent price of the stock option. Huddart (1994) and Detemple and Sundaresan (1999) show in a binomial model that, in the presence of hedging restrictions, a risk-averse employee may find it optimal to exercise his American-style ESO early even if the underlying stock pays no dividend. They use this result to rationalize the well-known phenomenon that employees tend to exercise their ESOs long before the maturity. We also obtain similar results for our model (see Section 4). Hall and Murphy (2002) use a certainty-equivalence framework to analyze the divergence between the firm's cost of issuing ESOs and the value to employees. Although these models incorporate the effect of holder's risk aversion, they do not consider the fact that the option holder can dynamically trade a partially correlated asset. This is remediated by Henderson (2005), where the methodology of *indifference pricing* in valuing ESOs is introduced. She provides a valuation model for a European ESO that captures the employee's risk aversion.

Oberman and Zariphopoulou (2003) consider indifference pricing for American call options with finite maturity. The holder's indifference price is the solution to a quasilinear variational inequality, which they numerically solve to obtain the optimal exercise boundary. In our case, the ESO holder's investment problem also incorporates vesting, job termination risk, and multiple exercises. In particular, the job termination risk leads to a nonlinear free boundary problem of reaction-diffusion type. Existence of the generalized solution to the problem is provided in the Appendix. Henderson (2006) considers multiple perpetual American options, where the option holder cannot trade the underlying asset, but can invest in a partially spanning asset. A closed-form formula for the holder's exercising barriers is given, but under the assumption that the interest rate is zero. Our Section 5.2 demonstrates the effects of finite maturity and non-zero interest rate on ESO value are not minor. Some other ESO characteristics such as reload and reset provisions are analyzed under the perpetual assumption in the no-arbitrage framework by Sircar and Xiong (2007), but we do not consider those here.

3. THE ESO VALUATION MODEL

In this section, we present our valuation model for a single employee stock option. We will extend it to multiple issues in Section 6. To start our formulation, we consider a market with a riskless bank account that pays interest at constant rate r, and two risky assets, namely, the company stock, and a market index. The employee can only trade the bank account and the market index, but not the company stock. The latter is modeled as a diffusion process that satisfies

$$dY_u = (\nu - q)Y_u\,du + \eta Y_u\,dW_u\,, \quad u \geq t\,, \tag{3.1}$$

with $Y_t = y > 0$. The coefficients ν, q and η are constant. Here, ν and η are the stock's expected return and volatility respectively. We also assume that the stock pays a constant proportional dividend q continuously over time.

The market index is another lognormal process that is partially correlated with the company stock

$$dS_u = \mu S_u\,du + \sigma S_u\,dB_u\,, \quad u \geq t\,, \tag{3.2}$$

with $S_t = S > 0$. The constant parameters μ and σ are, respectively, the market index's expected return and volatility. The two Brownian motions B and W are defined on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_u), \mathbb{P})$, where $\mathcal{F}_u$ is the augmented σ-algebra generated by $\{W_s, B_s; 0 \leq s \leq u\}$, and their instantaneous correlation is $\rho \in (-1, 1)$. The employee can use S to partially hedge away some of the risk in their portfolio, with some remaining idiosyncratic risk. In reality, the employee can trade more than one asset. If so, the aggregate of the traded assets is proxied by the index S.

The employee stock option in this paper is an American call option on the company stock with maturity T (typically 10 years, see Marquardt 2002), with strike K and a vesting period $t_v \leq T$ (typically 2–4 years). At the exercise time, the firm sells a new stock issue to the employee at the price K. Following the arguments in Hull and White (2004) and FASB statement 123R, we work under the assumption that the possible dilution effect is anticipated by the market and already reflected in the stock price immediately after the ESO grant.

Due to vesting, the employee cannot exercise the option before t_v. If the employee leaves the firm during the vesting period, then the option becomes worthless. If the employee's departure happens after the vesting period, then he must exercise the ESO if it is in-the-money. As the vesting period increases to maturity, the ESO becomes a European call—the holder can exercise the ESO only at maturity.

The modeling of job termination is a delicate and important issue that has a crucial impact on ESO valuation, as we demonstrate in Figure 5.2. The fact that the horizon of the valuation problem is typically much shorter than the contractual term of the ESO has even been recognized in the FASB proposal, in which it recommends that the ESO maturity be shortened according to the job termination risk. On the one hand, it would be nice to develop and estimate a detailed model to account for the causes of job termination that separate voluntary and involuntary exits, and the classification of employees, for example, by age. In particular, external opportunities that tempt the employee to depart and exercise the ESO early might be considered. On the other hand, data are scarce and likely not well-segmented according to the identity of employee, or even the cause of job termination. Therefore, the literature has adopted reduced-form modeling that bypasses direct modeling of an individual employee's personal employment choices and potential inducement from external offers. Models that involve more complex information, including the fortellability of the employee's voluntary exit, are topics for future development as more comprehensive empirical data becomes available.

In our model, the employee's (voluntary or involuntary) employment termination time, denoted by τ^λ, is represented by an exponential random variable with parameter λ that is independent of the Brownian motions W and B. In Remark 5.1, we address how to adapt our formulation to more complex τ^λ. The rate of job termination λ can be estimated from the firm's historical data. For instance, one can take the inverse of the average time to job termination. We illustrate the payoff structure of the ESO in Figure 3.1.

3.1. The Employee's Investment Problem

Since the employee cannot sell the ESO, or form a perfect hedge, it is important to consider his risk aversion. To this end, we represent his risk preference with the exponential utility function $U(x) = -e^{-\gamma x}$, with a positive constant absolute risk aversion γ.

To solve the employee's investment problem, it is sufficient to consider the case with zero vesting. When vesting increases from zero, it effectively lifts the employee's *pre-vesting* exercise boundary to infinity, but leaves his *post-vesting* exercise policy unaffected. Now suppose, at time $t \in [0, T]$, the employee is endowed with an ESO and some positive wealth. The employee's investment problem is to decide *when* to exercise the option. We define $\mathcal{T}_{t,T}$ as the set of stopping times (with respect to the filtration $(\mathcal{F}_u)$) taking values in $[t, T]$. Throughout the *entire period* $[t, T]$, the employee is assumed to trade dynamically in the bank account and the market index. A trading strategy $\{\theta_u; t \leq u \leq T\}$ is the cash amount invested in the market index S, and it is deemed admissible if it is

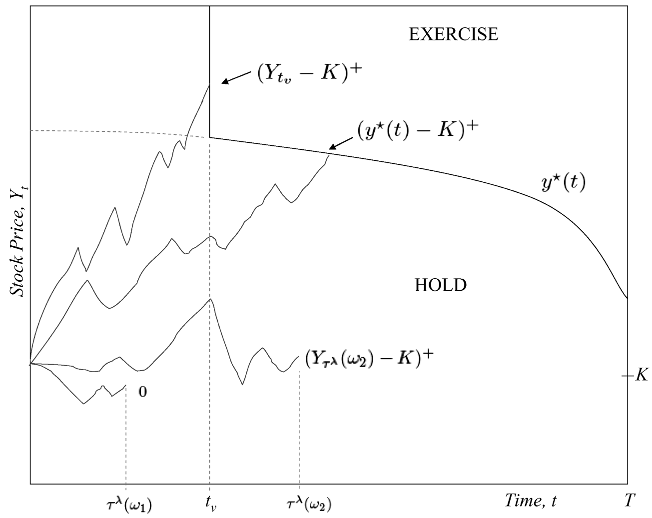

FIGURE 3.1. ESO payoff structure. The bottom path represents the scenario where the employee leaves the firm during the vesting period, resulting in forfeiture of the ESO. In the next path above, the employee is forced to exercise the ESO early due to job termination. The second from the top path hits the optimal exercise boundary $y^\star(t)$ after vesting, so the employee exercises the ESO there. The top one represents that the employee exercises the ESO immediately at the end of vesting.

$\mathcal{F}_u$-progressively measurable and satisfies the integrability condition $\mathbb{E}\{\int_t^T \theta_u^2\, du\} < \infty$. The set of admissible strategies over the period $[t, T]$ is denoted by $\mathcal{Z}_{t,T}$. For $u \geq t$, the employee's trading wealth evolves according to

$$dX_u^\theta = [\theta_u(\mu - r) + r X_u]\, du + \theta_u \sigma\, dB_u\,, \qquad X_t = x\,. \tag{3.3}$$

Upon the exercise of the option, either voluntarily or forced due to job termination, the employee will add the contract proceeds to his portfolio, and continue to optimally invest in the bank account and market index up to the maturity date T. Therefore, from the exercise time till the expiration date, the employee, who no longer holds an ESO, faces the classical Merton problem of optimal investment. According to Merton (1969), if an investor has x dollars at time $t \leq T$ and invests dynamically in the bank account and the market index until time T, then his maximal expected utility is given by

$$\begin{aligned} M(t, x) &= \sup_{\mathcal{Z}_{t,T}} \mathbb{E}\left\{ -e^{-\gamma X_T} \mid X_t = x \right\} \\ &= -e^{-\gamma x e^{r(T-t)}} e^{-\frac{(\mu-r)^2}{2\sigma^2}(T-t)}. \end{aligned} \tag{3.4}$$

To interpret this, we can think of the first part $-e^{-\gamma x e^{r(T-t)}}$ as the utility from merely saving the proceeds in the bank account. The factor $e^{-\frac{(\mu-r)^2}{2\sigma^2}(T-t)}$ increases the utility (which is negative) due to the fact that the employee can invest in the market index, in addition to the bank account. Observe that, for any fixed x, M is decreasing with t.

We formulate the ESO holder's investment problem as a stochastic utility maximization with optimal stopping. We shall use the following shorthands for conditional expectations:

$$\mathbb{E}_{t,y}\{\cdot\} = \mathbb{E}\{\cdot \mid Y_t = y\}, \qquad \mathbb{E}_{t,x,y}\{\cdot\} = \mathbb{E}\{\cdot \mid X_t = x,\, Y_t = y\}.$$

The employee's value function at time $t \in [0, T]$, given that he has not departed the firm and that his wealth $X_t = x$ and company stock price $Y_t = y$, is

$$\begin{aligned} V(t,x,y) &= \sup_{\tau\in\mathcal{T}_{t,T}} \sup_{\mathcal{Z}_{t,\tau}} \mathbb{E}_{t,x,y}\left\{M(\hat{\tau}, X_{\hat{\tau}} + (Y_{\hat{\tau}} - K)^+)\right\} \\ &= \sup_{\tau\in\mathcal{T}_{t,T}} \sup_{\mathcal{Z}_{t,\tau}} \mathbb{E}_{t,x,y}\left\{-e^{-\gamma(X_{\hat{\tau}}+(Y_{\hat{\tau}}-K)^+)e^{r(T-\hat{\tau})}}e^{-\frac{(\mu-r)^2}{2\sigma^2}(T-\hat{\tau})}\right\}, \end{aligned} \tag{3.5}$$

where $\hat{\tau} = \tau \wedge \tau^{\lambda}$. Observe that we are explicitly optimizing the expected utility over all stopping times, and over all trading strategies θ before τ. The *post-exercise* trading is implicitly optimized by the solution to the Merton problem M. Both of the expectations in (3.4) and (3.5) are taken under the historical measure, $\mathbb{P}$. By standard arguments from the theory of optimal stopping, the employee's optimal exercise time is given by

$$\tau^{\star} := \inf\left\{t \le u \le T : V(u, X_u, Y_u) = M(u,\, X_u + (Y_u - K)^+)\right\}. \tag{3.6}$$

3.2. ESO Cost to the Firm

It turns out that the employee's optimal exercise time and the corresponding exercise boundary can be obtained by solving a free boundary problem. This will be discussed in the next section. Meanwhile, let us explain how to use the employee's exercise boundary to determine the ESO cost to the firm. In accordance with the FASB rules,[2] we assume that the company stock evolves according to the following diffusion process under the risk-neutral measure $\mathbb{Q}$:

$$dY_u = (r-q)Y_u\,du + \eta Y_u\,dW_u^{\mathbb{Q}}, \quad u \ge t; \qquad Y_t = y,$$

where $W^{\mathbb{Q}}$ is a $\mathbb{Q}$-Brownian motion, which also independent of the job termination time τ^{λ}. As in Carr and Linetsky (2000), we assume that job termination rate is identical under both measures $\mathbb{P}$ and $\mathbb{Q}$; that is, the job termination risk is unpriced. By no-arbitrage arguments, the firm's granting cost is given by the no-arbitrage price of a barrier-type call option subject to early exercise due to job termination. The barrier is the employee's optimal exercise boundary. It is possible that the employee will leave the firm before the vesting period ends, or job termination arrives before the stock reaches the optimal boundary. In the first case, the ESO is forfeited. In the latter case, the employee is forced to exercise the option immediately. We must consider both cases in order to accurately determine the ESO value to the firm.

We first consider the cost of an *vested* ESO. Suppose the vesting period is t_v years. At time $t \ge t_v$, given that the stock price Y_t is y and the ESO is still alive, the cost of the ESO is given by

$$\begin{aligned} C(t,y) &= \mathbb{E}^{\mathbb{Q}}_{t,y}\left\{e^{-r(\tau^{\star}\wedge\tau^{\lambda}-t)}(Y_{\tau^{\star}\wedge\tau^{\lambda}} - K)^+\right\} \\ &= \mathbb{E}^{\mathbb{Q}}_{t,y}\left\{e^{-(r+\lambda)(\tau^{\star}-t)}(Y_{\tau^{\star}} - K)^+ + \int_t^{\tau^{\star}} e^{-(r+\lambda)(u-t)}\lambda(Y_u - K)^+du\right\}. \end{aligned} \tag{3.7}$$

[2] In paragraph A13 of FASB 123R, it specifically requires the use of "techniques that are used to establish trade prices for derivative instruments," and approves the use of risk-neutral models. Even if a firm does not hedge its ESOs, it should calculate and report the cost generated from such models.

Next, we consider the unvested ESO. Let $\tilde{C}(t, y)$ be the cost of the unvested ESO at time $t \le t_v$ given that it is still alive and the stock price $Y_t = y$. It is given by

$$\tilde{C}(t, y) = \mathbb{E}^{\mathbb{Q}}_{t,y} \left\{ e^{-r(t_v - t)} C(t_v, Y_{t_v}) \mathbf{1}_{\{\tau^\lambda > t_v\}} \right\}. \tag{3.8}$$

In Section 5, we will present the PDE problems for $C(t, y)$ and $\tilde{C}(t, y)$.

4. THE EMPLOYEE'S EXERCISE POLICY

We proceed to determine the employee's post-vesting optimal exercise boundary, and provide a characterization for it. Afterward, we will investigate how various parameters influence the employee's exercising strategy.

4.1. The Free Boundary Problem of Reaction-Diffusion Type

The employee's optimal exercise boundary is not known *ex ante*; it has to be inferred from the solution to the free boundary problem associated with the value function V. Let us introduce the following differential operators

$$\mathcal{L} = \frac{\eta^2 y^2}{2} \frac{\partial^2}{\partial y^2} + \rho\theta\sigma\eta y \frac{\partial^2}{\partial x \partial y} + \frac{\theta^2\sigma^2}{2} \frac{\partial^2}{\partial x^2} + (\nu - q) y \frac{\partial}{\partial y} + [\theta(\mu - r) + r x] \frac{\partial}{\partial x},$$

which is the infinitesimal generator of (X, Y), and

$$\tilde{\mathcal{L}} = \frac{\eta^2 y^2}{2} \frac{\partial^2}{\partial y^2} + \left(\nu - q - \rho \frac{\mu - r}{\sigma} \eta \right) y \frac{\partial}{\partial y},$$

which is the infinitesimal generator of Y under the minimal entropy martingale measure (defined later in (4.9)). Also, we define the utility rewarded for immediate exercise

$$\begin{aligned} \Lambda(t, x, y) &= M(t, x + (y - K)^+) \\ &= -e^{-\gamma (x + (y-K)^+) e^{r(T-t)}} e^{-\frac{(\mu - r)^2}{2\sigma^2}(T-t)}. \end{aligned}$$

By dynamic programming principle, the value function V is conjectured to solve the following complementarity problem

$$\begin{gathered} \lambda(\Lambda - V) + V_t + \sup_\theta \mathcal{L} V \le 0, \\ V \ge \Lambda, \\ \left(\lambda(\Lambda - V) + V_t + \sup_\theta \mathcal{L} V \right) \cdot (\Lambda - V) = 0, \end{gathered} \tag{4.1}$$

for $(t, x, y) \in [0, T) \times \mathbb{R} \times (0, +\infty)$. The boundary conditions are

$$\begin{aligned} V(T, x, y) &= -e^{-\gamma(x + (y-K)^+)}, \\ V(t, x, 0) &= -e^{-\gamma x e^{r(T-t)}} e^{-\frac{(\mu - r)^2}{2\sigma^2}(T-t)}. \end{aligned} \tag{4.2}$$

This free boundary problem can be simplified by a separation of variables and power transformation

$$V(t, x, y) = M(t, x) \cdot H(t, y)^{\frac{1}{(1-\rho^2)}}. \tag{4.3}$$

This is possible due to the exponential utility function see (see Oberman and Zariphopoulou (2003), for a similar transformation).

Then, the free boundary problem for H is of *reaction-diffusion* type.

$$H_t + \tilde{\mathcal{L}}H - (1-\rho^2)\lambda H + (1-\rho^2)\lambda b(t,y)H^{-\hat{\rho}} \geq 0, \tag{4.4}$$

$$H(t,y) \leq \kappa(t,y),$$

$$\left(H_t + \tilde{\mathcal{L}}H - (1-\rho^2)\lambda H + (1-\rho^2)\lambda b(t,y)H^{-\hat{\rho}}\right)\cdot(\kappa(t,y) - H(t,y)) = 0,$$

for $(t,y) \in [0,T] \times (0,+\infty)$, where

$$\hat{\rho} = \frac{\rho^2}{1-\rho^2}, \quad b(t,y) = e^{-\gamma(y-K)^+ e^{r(T-t)}}, \quad \text{and} \quad \kappa(t,y) = e^{-\gamma(1-\rho^2)(y-K)^+ e^{r(T-t)}}.$$

The boundary conditions are

$$\begin{aligned} H(T,y) &= e^{-\gamma(1-\rho^2)(y-K)^+}, \\ H(t,0) &= 1. \end{aligned} \tag{4.5}$$

Observe that if $\lambda = 0$, the reaction-diffusion term will disappear, and the problem will become linear. This problem for H implies that the employee's optimal exercise time is independent of X and S. Therefore, we define the employee's optimal exercise boundary as the function $y^\star : [0,T] \mapsto \mathbb{R}_+$, where $y^\star(t)$ is the critical stock price at time t. That is,

$$y^\star(t) = \inf\{y \geq 0 \,:\, H(t,y) = \kappa(t,y)\}. \tag{4.6}$$

In practice, we numerically solve this free boundary problem to obtain the employee's exercise boundary $y^\star$. Then, the employee's optimal exercise time is the first time that the company stock reaches $y^\star$. That is,

$$\tau^\star = \inf\left\{t \leq u \leq T \,:\, Y_u = y^\star(u)\right\}. \tag{4.7}$$

The function H has the following probabilistic representation

$$\begin{aligned} H(t,y) = \inf_{\tau\in\mathcal{T}_{t,T}} \tilde{\mathbb{E}}_{t,y} \Bigg\{ & e^{-(1-\rho^2)\lambda(\tau-t)}\kappa(\tau, Y_\tau) \\ & + \int_t^\tau e^{-(1-\rho^2)\lambda(u-t)}(1-\rho^2)\lambda b(u,Y_u)H(u,Y_u)^{-\hat{\rho}}\,du \Bigg\}. \end{aligned} \tag{4.8}$$

The expectation is taken under the measure $\tilde{\mathbb{P}}$ defined by

$$\tilde{\mathbb{P}}(A) = \mathbb{E}\left\{\exp\left(-\frac{\mu-r}{\sigma}B_T - \frac{1}{2}\frac{(\mu-r)^2}{\sigma^2}T\right)\mathbf{1}_A\right\}, \quad A \in \mathcal{F}_T. \tag{4.9}$$

The measure $\tilde{\mathbb{P}}$ is a martingale measure that has the minimal entropy relative to $\mathbb{P}$ (see Fritelli 2000). This measure arises frequently in indifference pricing theory. For instance, Musiela and Zariphopoulou (2004) use it to express the writer's value function for an European call option. We will use this probabilistic representation to prove the existence of a unique solution to the free boundary problem for H in the Appendix.

4.2. Characterization of the Employee's Exercise Boundary

The function H, defined in (4.3), turns out to be related to the employee's indifference price for the ESO, which will allow us to characterize the employee's optimal exercise time. We are primarily interested in the *cost* of an ESO to the firm, not the employee's indifference price. Nevertheless, the indifference price is a useful concept in analyzing the employee's exercise behavior.

DEFINITION 4.1. The ESO holder's indifference price of an ESO (without vesting) is defined as the function $p \equiv p(t, x, y)$ such that

$$M(t, x) = V(t, x - p, y). \tag{4.10}$$

As we shall see, due to the exponential utility function, the indifference price is in fact a function of only t and y. By Definition (4.10) and the transformation (4.3), one can deduce the following fact.

PROPOSITION 4.1. *The employee's indifference price for the ESO, denoted by p, satisfies*

$$p(t, y) = -\frac{1}{\gamma(1-\rho^2)e^{r(T-t)}} \log H(t, y)\,, \tag{4.11}$$

or equivalently,

$$V(t, x, y) = M(t, x) \cdot e^{-\gamma p(t,y) e^{r(T-t)}}. \tag{4.12}$$

With this, we can write the original free boundary problem (4.1)–(4.2) in terms of p:

$$\begin{aligned} & p_t + \tilde{\mathcal{L}}p - rp - \frac{1}{2}\gamma(1-\rho^2)\eta^2 y^2 e^{r(T-t)} p_y^2 + \frac{\lambda}{\gamma}\left(1 - b(t, y)e^{\gamma p e^{r(T-t)}}\right) \le 0\,, \\ & p \ge (y-K)^+, \\ & \left(p_t + \tilde{\mathcal{L}}p - rp - \frac{1}{2}\gamma(1-\rho^2)\eta^2 y^2 e^{r(T-t)} p_y^2 + \frac{\lambda}{\gamma}\left(1 - b(t, y)e^{\gamma p e^{r(T-t)}}\right)\right) \\ & \cdot((y-K)^+ - p) = 0, \end{aligned} \tag{4.13}$$

for $(t, y) \in [0, T] \times (0, +\infty)$. The boundary conditions are

$$\begin{aligned} p(T, y) &= (y-K)^+\,, \\ p(t, 0) &= 0\,. \end{aligned} \tag{4.14}$$

Finally, we use equation (4.10) or (4.12) to express the employee's optimal exercise time $\tau^\star$ in terms of p:

$$\begin{aligned} \tau^\star &:= \inf\{t \le u \le T : V(u, X_u, Y_u) = \Lambda(u, X_u, Y_u)\} \\ &= \inf\left\{t \le u \le T : M(u, X_u + p(u, Y_u)) = M(u, X_u + (Y_u - K)^+)\right\} \\ &= \inf\left\{t \le u \le T : p(u, Y_u) = (Y_u - K)^+\right\}. \end{aligned} \tag{4.15}$$

This provides a nice interpretation for the ESO holder's optimal exercising strategy: the holder will exercise the ESO as soon as his indifference price reaches (from above) the ESO payoff. For other utility functions, this interpretation still holds although the indifference price and the optimal exercise time may depend on wealth.

According to the standard no-arbitrage pricing theory, the price-maximizing boundary for an American call on a dividend-paying stock is monotonically decreasing with time. To understand this, we recall that the boundary represents the stock price where the value of an American call equals the payoff from immediate exercise. Note that the value of an American call, for a fixed stock level, is decreasing over time, and the payoff from immediate exercise is time-independent. Therefore, the critical stock price decreases

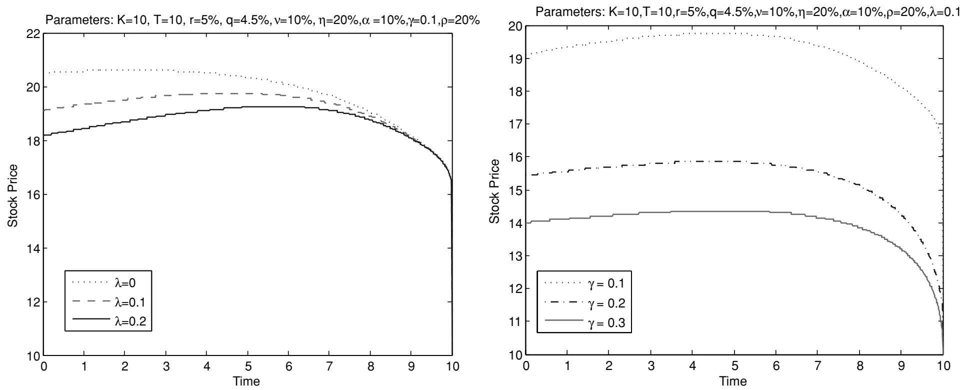

FIGURE 4.1. Effects of job termination risk and risk aversion: (Left-hand side) Higher job termination risk lowers the exercise boundary. (Right-hand side) As risk aversion γ increases, the employee's exercise boundary moves downward.

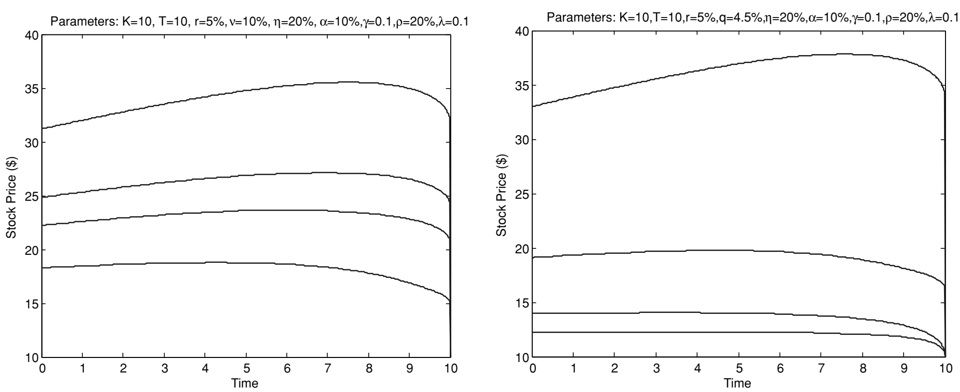

FIGURE 4.2. Effects of dividend rate and drift: (Left-hand side) The employee's exercise boundary exists even when the dividend rate is zero. It shifts downward as dividend rate, q, increases (from top to bottom, $q = 0\%, 2\%, 3\%, 5\%$). (Right-hand side) The employee's exercise boundary is also monotone with respect to the stock's drift (from top to bottom, $\nu = 15\%, 10\%, 5\%, 0\%$).

over time. However, in our model, the utility-maximizing boundary is not always monotonically decreasing with time. The reason is that the utility rewarded for exercising the ESO, instead of being time-independent, is decreasing over time. Since both the value function and the reward from immediate exercise decreases over time, it is possible that the critical stock price is increasing for a certain period of time (see Figures 4.1–4.3). In the special case of zero interest rate and no job termination risk, we can prove that the exercise boundary is non-increasing with time.

PROPOSITION 4.2. *Assume $\lambda = r = 0$. The utility-maximizing boundary is non-increasing with time.*

Proof. First, observe that H is non-decreasing with time. Indeed, setting $\lambda = r = 0$ in equation (4.8), we get

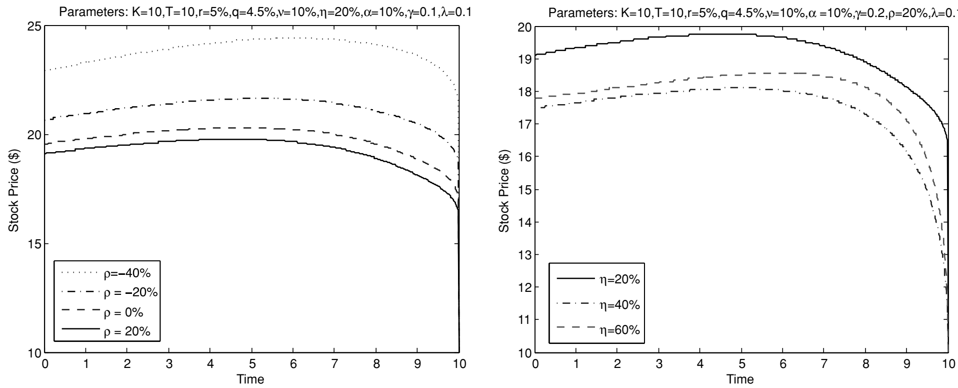

FIGURE 4.3. Effects of correlations and volatilities: (Left-hand side) When the market index (partial spanning asset) has a positive Sharpe ratio of 10%, the ESO holder's exercise boundary moves upward as the correlation becomes more negative. (Right-hand side) The employee's exercise boundary moves downward when stock volatility increases from 20% to 40%. When the volatility is raised to 60%, it moves upwards again. This shows that the employee's exercise boundary is not monotone with respect to the stock volatility.

$$\begin{aligned} H(t,y) &= \inf_{\tau\in\mathcal{T}_{t,T}} \tilde{\mathbb{E}}_{t,y}\left\{e^{-\gamma(1-\rho^2)(Y_\tau-K)^+}\right\} \\ &= \inf_{\tau\in\mathcal{T}_{0,T-t}} \tilde{\mathbb{E}}_{0,y}\left\{e^{-\gamma(1-\rho^2)(Y_\tau-K)^+}\right\}, \end{aligned} \tag{4.16}$$

where we have use the time-homogeneity of Y for the second equality. For any $s \le t$, we have $\mathcal{T}_{0,T-t} \subseteq \mathcal{T}_{0,T-s}$, so $H(s,y) \le H(t,y)$. Next, fix any $y > 0$ and let $s \le t$. If the employee should exercise at (s,y), that is, $H(s,y) = e^{-\gamma(1-\rho^2)(y-K)^+}$, then we want to show the same is true at (t,y). But this is clear from the chain of inequalities

$$e^{-\gamma(1-\rho^2)(y-K)^+} = H(s,y) \le H(t,y) \le e^{-\gamma(1-\rho^2)(y-K)^+}.$$

Hence, the employee should also exercise at (t,y). □

4.3. Effects of Parameters on the Employee's Exercise Policy

Let us first study the effect of job termination risk. Figure 4.1 (left-hand side) shows that higher job termination risk leads to a lower exercise boundary. In other words, the risk of job termination induces the employee to adopt a more conservative exercising strategy.

PROPOSITION 4.3. *Let λ_1, λ_2 be the job termination rates such that $\lambda_2 \ge \lambda_1$. Then, the utility-maximizing boundary associated with λ_1 dominates that with λ_2.*

Proof. First, the indifference price satisfies the variational inequality:

$$\begin{aligned} \min\Big\{ & -p_t - \tilde{\mathcal{L}}p + rp + \tfrac{1}{2}\gamma(1-\rho^2)\eta^2 y^2 e^{r(T-t)} p_y^2 \\ & + \frac{\lambda}{\gamma}\left(b(t,y)e^{\gamma p e^{r(T-t)}} - 1\right), p - (y-K)^+ \Big\} = 0. \end{aligned} \tag{4.17}$$

Let $p_1(t, y)$ and $p_2(t, y)$ be the indifference prices associated with λ_1 and λ_2, respectively. Since the coefficient of λ is non-negative, the left-hand side is non-decreasing with λ. Then, substituting $p_2(t, y)$ into the variational inequality for $p_1(t, y)$ will render the left-hand side less than or equal to zero. Therefore, $p_2(t, y)$ is a subsolution to the variational inequality for $p_1(t, y)$, so $p_2(t, y) \leq p_1(t, y)$. We conclude from (4.15) that the optimal exercise time corresponding to λ_1 is longer than or equal to that corresponding to λ_2, which implies that the utility-maximizing boundary corresponding to λ_1 dominates that corresponding to λ_2. □

Empirical studies on ESOs by Hemmer et al. (1996), Huddart and Lang (1996), and Marquardt (2002) show that most ESO holders exercise well before the options expire. Our model allows us to rationalize this phenomenon. First, Figure 4.1 illustrates that early exercise is optimal for a risk-averse employee. Also, by Proposition 4.3, the risk of job termination induces the employee to lower his exercise boundary, leading to even earlier exercise. In other words, even if the job termination does not happen before maturity, the employee still lowers his exercise boundary due to the *risk* of job termination. Lastly, when job termination actually happens prior to maturity, then the employee is forced to give up or exercise the ESO. All these contribute to the early exercise phenomenon.

In our numerical example depicted in Figure 4.1 (right-hand side), the employee's exercise boundary shifts downward as risk aversion increases. Heuristically, a higher risk aversion implies a greater tendency to lock in sure profit now, rather than waiting for a higher but uncertain return in the future. This means that a more risk-averse holder would exercise the option at a lower critical price. Therefore, we have

PROPOSITION 4.4. *The indifference price is non-increasing with risk aversion. The utility-maximizing boundary of a less risk-averse ESO holder dominates that of a more risk-averse ESO holder.*

Proof. We consider the variational inequality in the previous proposition. The p_y^2 term is non-decreasing with γ. Differentiating the nonlinear term with respect to γ, we get

$$\frac{\lambda}{\gamma^2}\left\{1 + \phi(t, y)e^{\phi(t,y)} - e^{\phi(t,y)}\right\} \geq 0\,,$$

with $\phi(t, y) := \gamma(p(t, y) - (y - K)^+)e^{r(T-t)} \geq 0$. Hence, the nonlinear term is also non-decreasing with γ. By comparison principle, this implies the indifference price p is non-increasing with γ. The second assertion follows from the characterization of the optimal exercise time (see equation (4.15)). □

In the ESO valuation models proposed by Hull and White (2004), and Cvitanic et al. (2008), the employee's exercise boundary is exogenously specified and does not change with the dividend rate, drift, and volatility of the company stock. Empirical studies have shown that these parameters influence the employee's exercise behavior. For example, Bettis et al. (2005) point out that ESOs are exercised earlier in firms with higher dividend yields. This is reasonable because a higher dividend rate entices the employee to own the company stock share and receive the dividend. In our model, the employee's exercise policy is consistent with this empirical result. We summarize our results in the following propositions, and illustrate them in Figures 4.2–4.3.

PROPOSITION 4.5. *The ESO holder's utility-maximizing boundary shifts upward as the dividend rate q decreases, or as the firm's average growth rate ν increases.*

Proof. Again, we consider the variational inequality (4.17), and notice that q and ν only appears in the term $-(\nu - q - \rho\frac{\mu-r}{\sigma}\eta)yp_y$. One can deduce from (3.5) that the value function V is non-decreasing with y, so by equation (4.12) p is also non-decreasing with y, so $p_y \geq 0$. Therefore, the term $-(\nu - q - \rho\frac{\mu-r}{\sigma}\eta)yp_y$ is non-decreasing with q and non-increasing with ν. By comparison principle, we conclude that the indifference price is non-increasing with q and non-decreasing with ν. Then, the proposition follows from (4.15). □

REMARK 4.1. Standard option pricing theory shows that an American call value increases with respect to volatility. By examining the variational inequality (4.17), we notice that the indifference price p is not monotonically increasing with respect to η. Therefore, we expect non-monotonicity of the utility-maximizing boundary with respect to η. We illustrate this in Figure 4.3 (right-hand side). As volatility rises, the exercise boundary tends to fall first and then rise slightly. This is also observed by Henderson (2006) and Carpenter (2005).

Now suppose the employee can choose between two hedging instruments with correlations being, respectively, ρ and $-\rho$, and both have the same positive Sharpe ratio. Then, which one should the employee use to hedge? Heuristically, if the employee hedges with the ESO with a positively (resp. negatively) correlated asset with a positive Sharpe ratio, then he needs to short (resp. long) the asset. But a short position is less favorable than a long position to a risk-averse investor, so the negatively correlated asset should be preferred.

PROPOSITION 4.6. *Assume* $\alpha := \frac{\mu-r}{\sigma} > 0$. *Fix any number* $\rho \in (0, 1)$. *Denote by* p_+ *and* p_- *the indifference prices corresponding to* ρ *and* $-\rho$ *respectively. Then, we have* $p_- \geq p_+$. *Moreover, the utility-maximizing boundary corresponding to* $-\rho$ *dominates that corresponding to* ρ.

Proof. We consider the variational inequality (4.17). Since $\alpha > 0$ and $p_y \geq 0$, the p_y term is non-decreasing in ρ. Therefore, p_+ is a subsolution to the variational inequality for p_-, so $p_+ \leq p_-$. The last statement in the proposition follows from (4.15) and that $p_- \geq p_+$. □

REMARK 4.2. Following from the preceding proof, if $\alpha < 0$, then the opposite happens. In the case of zero Sharpe ratio ($\alpha = 0$), we have $p_+ = p_-$, and the two exercise boundaries coincide. If $\alpha = \rho = 0$, then the employee does not trade in the market index. In this special case, the employee will exercise early even if the firm's stock pays no dividends (see Huddart 1994; Villeneuve 1999).

When the hedging instrument has a positive Sharpe ratio, the employee would prefer a negative correlation than a positive one. As the correlation becomes even more negative, the employee can hedge more risk away. Consequently, the employee's indifference price increases and he tends to wait longer before exercise. As a result, the utility-maximizing boundary should move upward. This is illustrated in Figure 4.3 (left-hand side), and proved in the following proposition.

PROPOSITION 4.7. *Assume* $\alpha := \frac{\mu-r}{\sigma} > 0$. *Then, the indifference price is non-increasing with respect to* ρ, *for* $\rho \leq 0$. *Moreover, the utility-maximizing boundary moves upward with as* ρ *decreases from* 0 *to* -1.

Proof. From variational inequality (4.17), we collect the terms with ρ and define $g(\rho, y, p_y) := \alpha\rho\eta y p_y - \frac{1}{2}\gamma\eta^2 y^2 e^{r(T-t)} p_y^2 \rho^2$. The function g is quadratic in ρ and is non-decreasing for $\rho \leq \frac{\alpha}{\gamma\eta y p_y e^{r(T-t)}}$. Since $\frac{\alpha}{\gamma\eta y p_y e^{r(T-t)}}$ is positive, when $\rho \leq 0$, the left-side of the above variational inequality is non-decreasing with ρ. Then, by comparison principle, more negative correlation leads to higher indifference price. The last assertion follows from (4.15). ☐

4.4. Numerical Solution

To obtain the employee's exercise boundaries, we numerically solve (4.4)–(4.5). Our numerical method utilizes the backward Euler finite-difference stencil on a uniform grid. The constraint $H \leq \kappa$ is enforced by the projected successive-over-relaxation (PSOR) algorithm, which iteratively solves the implicit time-stepping equations, while preserving the constraint between iterations. Similar numerical schemes can be found in Wilmott et al. (1995).

For computational implementation, we restrict the domain $[0, T] \times \mathbb{R}_+$ to a finite domain $\mathcal{D} = \{(t, y) : 0 \leq t \leq T,\ 0 \leq y \leq R\}$, where R is sufficiently large to preserve the accuracy of the numerical solutions. Then, we introduce a uniform grid on $\mathcal{D}$ with nodes $\{(t_k, y_j) : k = 0, 1, \ldots, N; j = 0, 1, \ldots, M\}$, with $\Delta t = T/N$, and $\Delta y = R/M$ being the grid spacings. Next, we apply discrete approximations $H_j^k \approx H(t_k, y_j)$ where $t_k = k\Delta t$, and $y_j = j\Delta y$.

We discretize the PDI (4.4). We approximate the y-derivatives by central differences

$$\frac{\partial H}{\partial y}(t_k, y_j) \approx \frac{H_{j+1}^k - H_{j-1}^k}{2\Delta y}, \qquad \frac{\partial^2 H}{\partial y^2}(t_k, y_j) \approx \frac{H_{j+1}^k - 2H_j^k + H_{j-1}^k}{\Delta y^2}, \tag{4.18}$$

and the t-derivative by the backward Euler scheme

$$\frac{\partial H}{\partial t}(t_k, y_j) \approx \frac{H_j^{k+1} - H_j^k}{\Delta t}. \tag{4.19}$$

Furthermore, we use the explicit approximation for the reaction-diffusion term

$$-\lambda(1-\rho^2)H + \lambda(1-\rho^2)b(t, y)H^{-\hat{\rho}} \approx -\lambda(1-\rho^2)H_j^{k+1} + f_j^{k+1}\big(H_j^{k+1}\big)^{-\hat{\rho}},$$

where $f_j^{k+1} = \lambda(1-\rho^2)b(t_{k+1}, y_j)$. We refer interested readers to Glowinski (1984) for a detailed account on numerical methods for nonlinear variational inequalities. With these approximations, we solve the discretized version of (4.4)–(4.5) backward in time using PSOR algorithm, and locate the free boundary at t_k, $y^\star(t_k)$, by comparing the values of H_j^k and $\kappa(t_k, y_j)$. The numerically-estimated free boundaries are shown in Figures 4.1–4.3.

5. ANALYSIS OF THE ESO COST

With reference to the definitions (3.7) and (3.8), we now present the PDE formulations for the costs of a vested and an unvested ESO. Suppose the firm imposes a vesting period of t_v years, and denote $y^\star(t)$ as the employee's exercise boundary. The holder does not exercise in the regions $\mathcal{C} = \{(t, y) : t_v \leq t < T,\ 0 \leq y < y^\star(t)\}$ and $\mathcal{V} = \{(t, y) : 0 \leq t < t_v,\ 0 \leq y\}$. The cost of an unvested ESO, $C(t, y)$, satisfies the *inhomogeneous* PDE

$$C_t + \frac{\eta^2}{2}y^2 C_{yy} + (r-q)yC_y - (r+\lambda)C + \lambda(y-K)^+ = 0, \tag{5.1}$$

for $(t, y) \in \mathcal{C}$, and the boundary conditions

$$
\begin{aligned}
&C(t, 0) = 0, \quad t_v \le t \le T, \\
&C(t, y^\star(t)) = (y^\star(t) - K)^+, \quad t_v \le t < T, \\
&C(T, y) = (y - K)^+, \quad 0 \le y \le y^\star(T).
\end{aligned} \tag{5.2}
$$

The inhomogeneous term, $\lambda(y - K)^+$ captures the effect that the ESO may be exercised due to job termination with a probability $\lambda\, dt$ over an infinitesimal period dt.[3] Next, the cost of an unvested ESO, $\tilde{C}(t, y)$, satisfies the *homogeneous* PDE

$$
\tilde{C}_t + \frac{\eta^2}{2} y^2 \tilde{C}_{yy} + (r - q) y \tilde{C}_y - (r + \lambda)\tilde{C} = 0\,, \tag{5.3}
$$

for $(t, y) \in \mathcal{V}$, and the boundary conditions

$$
\begin{aligned}
&\tilde{C}(t, 0) \;= 0\,, \quad 0 \le t \le t_v\,, \\
&\tilde{C}(t_v, y) = C(t_v, y)\,, \quad y \ge 0\,.
\end{aligned} \tag{5.4}
$$

Given the boundary curve $y^\star(t)$, we solve these two PDE problems numerically using the implicit finite-difference methods computed in Section 4.4.

In the following subsections, we study the effects of risk aversion, vesting, and job termination risk on the cost of an ESO, and compare our results with other models.

5.1. Effects of Vesting, Risk Aversion, and Job Termination Risk

We first analyze the effects of vesting and risk aversion in the absence of job termination risk. Recall that the risk-averse holder's optimal exercise boundary in general does not maximize the expected discounted payoff of the ESO. From the firm's perspective, the employee's risk-averse attitude means that the ESO costs less than the no-arbitrage price of the corresponding American call. If the risk aversion is small, then the ESO cost would be between the values of an American call and a European call on the company stock with the same strike and maturity. However, as the employee becomes more risk averse, his utility-maximizing boundary shifts downward, getting further away from the price-maximizing boundary. Consequently, the cost of the ESO decreases as risk aversion increases. If the ESO holder is sufficiently risk-averse, the cost of the ESO to the firm could be even lower than a European call on the company stock with the same strike and maturity.

If the firm imposes vesting on the ESO, then any exercise before the end of the vesting period is prevented. Effectively, the pre-vesting part of the employee's utility-maximizing boundary is lifted to infinity. Since vesting imposes discipline on the employee which restrains the employee's risk-averse behavior, it could increase the expected discounted payoff, implying a higher cost to the firm (see Figure 5.1). We can prove this for the case of no dividend and no job termination risk.

PROPOSITION 5.1. *If $\lambda = q = 0$, then the ESO cost is non-decreasing with respect to the length of the vesting period. Moreover, this cost is dominated by the Black–Scholes price of the European call option written on company stock with the same strike and maturity.*

[3] See, for example, Carr and Linetsky (2000), for a similar application.

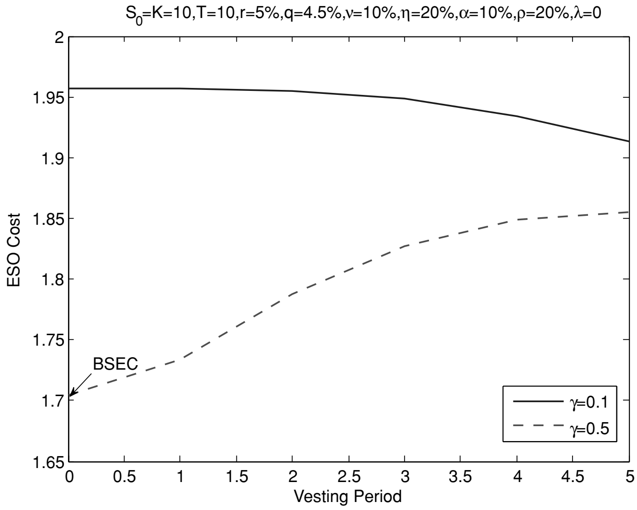

FIGURE 5.1. Effect of vesting: The marker "BSEC " represents the no-arbitrage price of a European call with the same strike and maturity as the ESO. In the absence of job termination risk, the cost of an ESO held by a very risk-averse employee increases (from close to BSEC) with respect to vesting. In the low risk aversion case, the cost decreases with vesting but stays above BSEC.

Proof. Let $0 < a < b < T$. Denote by $\tau_a^\star$ and $\tau_b^\star$ the employee's exercise time when the vesting periods are a and b years respectively. Then, we have $\tau_a^\star \leq \tau_b^\star \leq T$. Since the discounted payoff process $\{e^{-rs}(Y_s - K)^+\}_{s\geq 0}$ is a $\mathbb{Q}$-submartingale (see Karatzas and Shreve 1998), it follows from Optional Stopping Theorem that

$$\mathbb{E}_{t,y}^{\mathbb{Q}}\left\{e^{-r(\tau_a^\star-t)}\left(Y_{\tau_a^\star}-K\right)^+\right\} \leq \mathbb{E}_{t,y}^{\mathbb{Q}}\left\{e^{-r(\tau_b^\star-t)}\left(Y_{\tau_b^\star}-K\right)^+\right\} \leq \mathbb{E}_{t,y}^{\mathbb{Q}}\left\{e^{-r(T-t)}(Y_T-K)^+\right\}.$$

From this, we can conclude that the granting cost is non-decreasing with vesting, and is dominated by the price of the corresponding European call. □

The consideration of the employee's risk aversion gives us an important insight to the cost structure of ESOs to the firm—vesting may involve additional cost. While the firm may be able to maintain the incentive effect of the ESOs and impose discipline on ESO exercises, they may also have to pay for these benefits. This is not reflected by ESO valuation models that assume risk-neutrality of the employee. If the employee were to hedge perfectly and thus were risk-neutral, then the ESO cost would certainly decrease with vesting.

When the risk of job termination is present, the employee adopts a more conservative exercising strategy. Moreover, it potentially shortens the life of an ESO, resulting in either forfeiture of the option or early suboptimal exercise. Therefore, in general, higher job termination risk should reduce the ESO cost (see Figure 5.2). The next proposition proves this for the case of zero dividend rate. On the other hand, as the vesting period lengthens, the employee becomes more likely to depart before vesting ends. As illustrated in Figure 5.2, vesting significantly reduces the ESO cost to the firm.

PROPOSITION 5.2. *Assume $q = 0$. A higher job termination risk decreases the cost of both vested and unvested ESOs.*

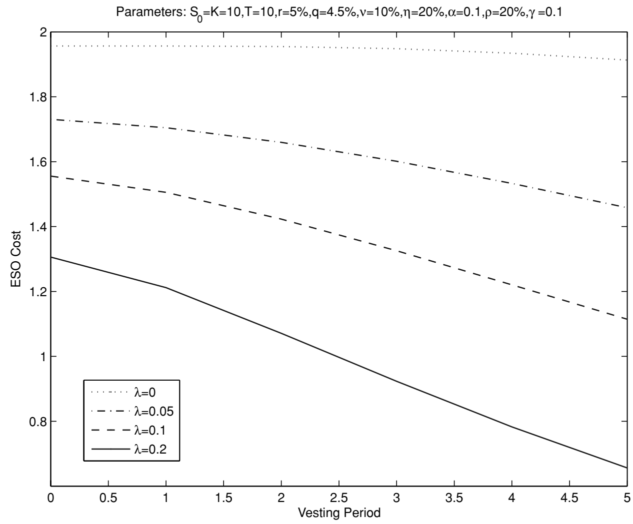

FIGURE 5.2. Effect of job termination risk: The ESO cost decreases significantly as the job termination risk rises. Furthermore, the cost decreases as vesting period lengthens due to the increasing likelihood of forfeiture.

Proof. We first consider the value of a vested ESO. Define the operator $\mathcal{L}_1$ such that

$$\mathcal{L}_1 C(t, y) = C_t + (r - q)yC_y + \frac{\eta^2}{2} y^2 C_{yy} - (r + \lambda)C + \lambda(y - K)^+ . \tag{5.5}$$

Let λ_1, λ_2 be the job termination rates such that $\lambda_2 \geq \lambda_1 \geq 0$. Let $C_i(t, y)$ and $\tau_i^\star$ be the cost of a vested ESO and optimal exercise time corresponding to λ_i, for $i = 1, 2$. By PDE (5.1), we have $\mathcal{L}_1 C_1 = 0$. Due to the $\mathbb{Q}$-submartingale property of the process $\{e^{-rs}(Y_s - K)^+\}_{s\geq 0}$, we have $C_i(t, y) \geq (y - K)^+$. Consequently, direct substitution shows that $\mathcal{L}_1 C_2 \geq 0$.

Next, we apply Itô's formula to the function

$$V(t, Y_t) = e^{(r+\lambda_1)t}\, C_2(t, Y_t) + \int_0^t e^{-(r+\lambda_1)s} \lambda_1 (Y_s - K)^+ \, ds. \tag{5.6}$$

Then, due to $\mathcal{L}_1 C_2 \geq 0$ and the Optional Sampling Theorem, the following holds for any $\tau \geq t$:

$$\mathbb{E}^{\mathbb{Q}}_{t,y}\{V(\tau, Y_\tau)\} \geq V(t, y).$$

In particular, we take $\tau = \tau_2^\star \leq \tau_1^\star$, then we get

$$\begin{aligned} C_2(t, y) &\leq \mathbb{E}^{\mathbb{Q}}_{t,y}\left\{e^{-(r+\lambda_1)(\tau_2^\star - t)} C_2\left(\tau_2^\star, Y_{\tau_2^\star}\right) + \int_t^{\tau_2^\star} e^{-(r+\lambda_1)(s-t)} \lambda_1 (Y_s - K)^+ \, ds\right\} \\ &= \mathbb{E}^{\mathbb{Q}}_{t,y}\left\{e^{-r(\tau_2^\star \wedge \tau^{\lambda_1} - t)} \left(Y_{\tau_2^\star \wedge \tau^{\lambda_1}} - K\right)^+\right\} \\ &\leq \mathbb{E}^{\mathbb{Q}}_{t,y}\left\{e^{-r(\tau_1^\star \wedge \tau^{\lambda_1} - t)} \left(Y_{\tau_1^\star \wedge \tau^{\lambda_1}} - K\right)^+\right\} = C_1(t, y) . \end{aligned} \tag{5.7}$$

Hence, the job termination risk reduces the cost of a vested ESO. As for unvested ESOs, we notice that the job termination risk reduces the terminal values of an unvested ESO

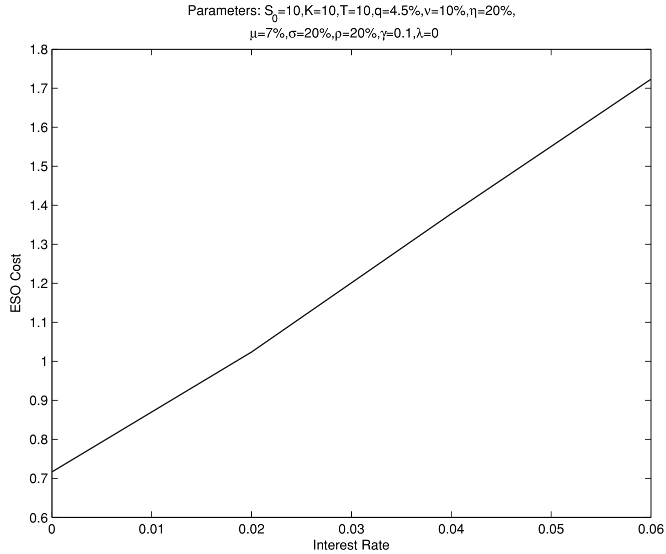

FIGURE 5.3. Effect of interest rate: As interest rate increases from 0% to 6%, the cost of the ESO doubles.

and increases the probability of forfeiture during the vesting period. Therefore, the job termination risk reduces the value of an unvested ESO. □

5.2. Comparison with Other Models

We compare our model with the ones proposed by Henderson (2006) and Grasselli (2005). In Henderson (2006), the interest rate is assumed to be zero. As shown in Figure 5.3, interest rate has a significant bearing on the ESO cost to the firm, so the assumption of zero interest rate is not benign for valuation of ESOs which are long-dated, as is usual. Moreover, the Henderson (2006) model also assumes that the ESO is perpetual. Consequently, the employee's exercise boundary is flat and tends to be very high (see Figure 5.4). In fact, her model concludes that the ESO holder will never exercise in the case of $\frac{\nu-q}{\eta} \geq \frac{\mu}{\sigma}\rho + \frac{\eta}{2}$ (which is equivalent to the drift of $\log Y_t$, with $r = 0$, being non-negative under the minimal entropy martingale measure).

Taking into account positive interest rate and finite maturity, but not job termination risk, Grasselli (2005) obtains a lower cost. Our model incorporates the risk of job termination and vesting, which further reduce the ESO cost. The following table shows the different ESO costs under the parameter values given in Figure 5.4. The first entry is the Black–Scholes price of a ten-year European option, which in this case of no dividend is equal to the American price. The next two entries are Henderson (2006) and Grasselli (2005) models specialized to just one option. The last two entries add job termination risk and then vesting.

Black–Scholes	Henderson	Grasselli	$+\lambda = 0.1$	+3-yr vesting
4.878	4.510	3.412	2.597	2.491

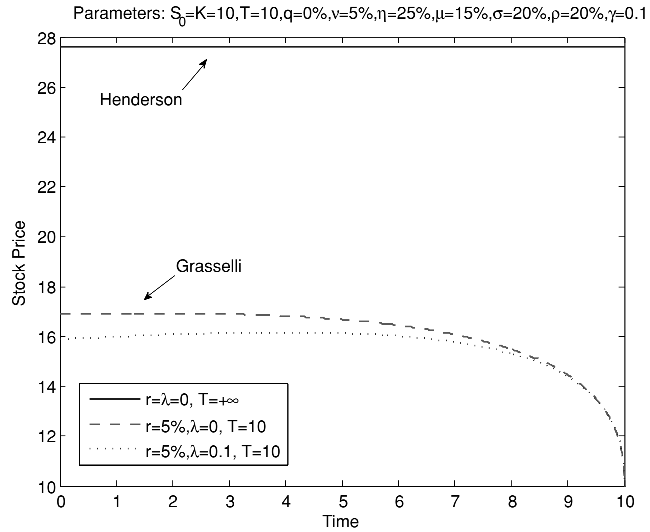

FIGURE 5.4. Comparing exercise boundaries: The model by Henderson (2006) gives a high flat exercise boundary. Grasselli (2005) corresponds to the middle boundary. The bottom boundary from our model accounts for the presence of job termination risk. The parameter values are chosen so that the exercise boundary by Henderson (2006) is finite.

We observe that risk-aversion lowers the cost by about 8% in the perpetual approximation, or by about 30% when we retain finite maturity, but then job termination risk reduces the cost by a further 17% of the Black–Scholes value, and vesting by yet another 2% in this example.

REMARK 5.1. Our formulation can easily be adapted to the case where the job termination time τ^λ is a non-predictable stopping time with a stochastic intensity process. We can define the intensity process $\{\lambda_t\}_{t\geq 0}$ by $\lambda_t = \lambda(Y_t)$ where $\lambda(\cdot)$ is a bounded continuous non-negative function of the firm's stock price Y. In that case, we replace λ with $\lambda(y)$ in our variational inequalities, PDEs, and numerical scheme. However, the estimation of the function $\lambda(\cdot)$ is significantly more difficult than that of a constant parameter. We have implemented the numerical solution with a variety of intensity functions, including that $\lambda(y)$ is decreasing with y. It seems that such generalization does not bring much additional insight to the exercise policy and other features discussed in this section.

6. VALUATION MODEL WITH MULTIPLE EXERCISES

We extend the model to the case in which the employee is granted multiple ESOs which may be exercised separately. In particular, we are interested in characterizing the employee's optimal exercise strategy. As before, it is sufficient to consider the employee's investment problem with no vesting, and then re-introduce the vesting period when we calculate the cost to the firm.

6.1. Formulation

The Employee's Investment Problem. We follow our formulation in Section 3. The only difference is that, at time $t \in [0, T]$, the employee is granted n American-styled ESOs with the same strike and maturity. The employee is risk-averse, subject to employment termination risk, and constrained as in Section 3. He needs to decide the exercise policies for his options. Let us denote by τ_i the exercise time when i options remain unexercised. Then, τ_n is the first exercise time, and τ_1 is the last one. If we keep track of the number of options exercised, then τ_{n-i} is the exercise time of the $(i+1)$th option. We require that $\tau_i \in \mathcal{T}_{t,T}$, and clearly, we have $\tau_n \leq \cdots \leq \tau_1$. If the employee exercises multiple options at the same time, then some exercise times may coincide.

Throughout the period $[t, T]$, the employee dynamically invests his wealth, using admissible strategies $\theta \in \mathcal{Z}_{t,T}$, in the bank account and the market index. Hence, his trading wealth follows (3.3). At every discretionary exercise time, τ_i, the employee invests the option payoff into his trading portfolio. However, at the job termination time τ^λ, he must exercise *all remaining options*. After exiting from the firm, the employee is assumed to invest the contract proceeds into his trading portfolio and continue trading till time T. Given that the employee has not departed the firm and holds $i \geq 2$ unexercised options at time t, his value function is given by

$$V^{(i)}(t,x,y) = \sup_{t\leq\tau_i\leq T} \sup_{\mathcal{Z}_{t,\tau_i}} \mathbb{E}_{t,x,y} \Big\{ V^{(i-1)}\left(\tau_i,\, X_{\tau_i} + (Y_{\tau_i} - K)^+,\, Y_{\tau_i}\right) \cdot \mathbf{1}_{\{\tau_i < \tau^\lambda\}} + M\left(\tau^\lambda,\, X_{\tau^\lambda} + i\,(Y_{\tau^\lambda} - K)^+\right) \cdot \mathbf{1}_{\{\tau_i \geq \tau^\lambda\}} \Big\},$$

with $V^{(1)} = V$ in (3.5) and $V^{(0)} = M$ in (3.4). The second term inside the expectation means that, if the job termination arrives before his next exercise time, the employee must exercise or forgo all i options and invest the proceeds, if any, into the market index and the bank account. Otherwise, as the first term reveals, the employee will exercise his next option at the optimal exercise time, and faces the investment problem again with $i-1$ unexercised options.

The Free Boundary Problem. The value function $V^{(i)}$ solves the variational inequality

$$\lambda\left(M\left(t, x+i(y-K)^+\right) - V^{(i)}\right) + V_t^{(i)} + \sup_\theta \mathcal{L}V^{(i)} \leq 0, \tag{6.1}$$

$$V^{(i)}(t,x,y) \geq V^{(i-1)}(t, x+(y-K)^+, y),$$

$$\left(\lambda(M(t, x+i(y-K)^+) - V^{(i)}) + V_t^{(i)} + \sup\nolimits_\theta \mathcal{L}V^{(i)}\right) \cdot \left(V^{(i-1)}\left(t, x+(y-K)^+, y\right) - V^{(i)}(t,x,y)\right) = 0, \tag{6.2}$$

$(t,x,y) \in [0,T) \times \mathbb{R} \times (0,+\infty)$, with boundary conditions

$$\begin{aligned} V^{(i)}(T,x,y) &= -e^{-\gamma(x+i(y-K)^+)}, \\ V^{(i)}(t,x,0) &= -e^{-\gamma x e^{r(T-t)}} e^{-\frac{(\mu-r)^2}{2\sigma^2}(T-t)}. \end{aligned} \tag{6.3}$$

Next, we simplify the above variational inequalities by applying the transformation

$$V^{(i)}(t,x,y) = M(t,x) \cdot H^{(i)}(t,y)^{\frac{1}{(1-\rho^2)}} \tag{6.4}$$

so that $H^{(i)}$ satisfies

$$H_t^{(i)} + \tilde{\mathcal{L}}H^{(i)} - (1-\rho^2)\lambda H^{(i)} + (1-\rho^2)\lambda b(t,y)^i (H^{(i)})^{-\hat{\rho}} \geq 0, \tag{6.5}$$

$$H^{(i)} \leq \kappa(t,y) H^{(i-1)},$$

$$\left(H_t^{(i)} + \tilde{\mathcal{L}}H^{(i)} - (1-\rho^2)\lambda H^{(i)} + (1-\rho^2)\lambda b(t,y)^i (H^{(i)})^{-\hat{\rho}}\right) \cdot \left(\kappa(t,y)H^{(i-1)} - H^{(i)}\right) = 0,$$

for $(t, y) \in [0, T] \times [0, +\infty)$. The boundary conditions are

$$
\begin{aligned}
H^{(i)}(T, y) &= e^{-\gamma(1-\rho^2)i(y-K)^+}, \\
H^{(i)}(t, 0) &= 1.
\end{aligned}
\tag{6.6}
$$

Associated with each $H^{(i)}$, there is free boundary, denoted by $y_i^\star : [0, T] \mapsto \mathbb{R}_+$, such that

$$
y_i^\star(t) := \inf\left\{y \geq 0 \,:\, H^{(i)}(t, y) = \kappa(t, y) H^{(i-1)}(t, y)\right\}, \qquad \text{for } t \in [0, T].
$$

The boundary $y_i^\star$ represents the employee's optimal exercise boundary for the next option when i options remain unexercised. In Section 4, we have solved the problem and obtained $H^{(i)}$ for the case $i = 1$. Given we know $H^{(1)}$, we use our numerical method discussed in Section 4.4 to solve the free boundary problem (6.5)–(6.6) for $H^{(2)}$. We continue this procedure to solve for all $H^{(i)}$, and obtain the associated free boundaries, $y_i^\star$, for $i = 1, \ldots, n$.

The Cost of Multiple Issues. Given the boundaries $\{y_i^\star, i = 1, 2, \ldots, n\}$, we can calculate the cost of the ESOs to the firm. Following our assumptions in Section 3.2, we first consider the value of a vested ESO. For a vesting period of t_v years, the employee's optimal exercise time when there are i options remaining is

$$
\tau_i^\star = \inf\left\{t \leq u \leq T \,:\, Y_u = y_i^\star(u)\right\}, \qquad t > t_v\,, \qquad i = 1, 2, \ldots, n.
\tag{6.7}
$$

Define $C^{(i)}(t, y)$ as the cost of i vested ESOs at time $t \geq t_v$ when the stock price is y dollars, assuming it is still alive. It satisfies the following recursive relationship

$$
\begin{aligned}
C^{(i)}(t, y) = \mathbb{E}^{\mathbb{Q}}_{t,y} \Big\{ & e^{-r(\tau^\lambda - t)}\, i\, (Y_{\tau^\lambda} - K)^+ \mathbf{1}_{\{\tau^\lambda \leq \tau_i^\star\}} \\
& + e^{-r(\tau_i^\star - t)} \Big[\big(Y_{\tau_i^\star} - K\big)^+ + C^{(i-1)}\big(\tau_i^\star, Y_{\tau_i^\star}\big) \Big] \mathbf{1}_{\{\tau^\lambda > \tau_i^\star\}} \Big\}.
\end{aligned}
\tag{6.8}
$$

The function $C^{(i)}(t, y)$ satisfies the following *inhomogeneous* PDE

$$
C_t^{(i)} + \frac{\eta^2}{2} y^2 C_{yy}^{(i)} + (r - q) y C_y^{(i)} - (r + \lambda) C^{(i)} + \lambda i (y - K)^+ = 0\,,
\tag{6.9}
$$

in the region $\{(t, y) : t_v \leq t \leq T, 0 \leq y \leq y_i^\star(t)\}$, and satisfies the boundary conditions

$$
\begin{aligned}
& C^{(i)}(t, 0) = 0, && t_v \leq t \leq T, \\
& C^{(i)}\big(t, y_i^\star(t)\big) = \big(y_i^\star(t) - K\big)^+ + C^{(i-1)}\big(t, y_i^\star(t)\big), && t_v \leq t < T, \\
& C^{(i)}(T, y) = i(y - K)^+, && 0 \leq y \leq y_i^\star(T).
\end{aligned}
\tag{6.10}
$$

To solve this system of PDEs, we apply the implicit finite-difference approximation discussed in Section 4.4. Since we have already calculated $C^{(1)}(t, y)$ in Section 3, we can use it to solve for $C^{(2)}(t, y), \ldots, C^{(n)}(t, y)$.

Next, we consider the unvested ESOs. Let $\tilde{C}^{(i)}(t, y)$ be the cost of i unvested ESOs at time t when the stock price is y dollars, assuming it is still alive.

$$
\tilde{C}^{(i)}(t, y) = \mathbb{E}^{\mathbb{Q}}_{t,y} \left\{ e^{-r(t_v - t)} C^{(i)}(t_v, Y_{t_v}) \mathbf{1}_{\{\tau^\lambda > t_v\}} \right\}.
$$

We have the following *homogeneous* PDE for $\tilde{C}^{(i)}$.

$$
\tilde{C}_t^{(i)} + \frac{\eta^2}{2} y^2 \tilde{C}_{yy}^{(i)} + (r - q) y \tilde{C}_y^{(i)} - (r + \lambda) \tilde{C}^{(i)} = 0
\tag{6.11}
$$

in the region $\{(t, y) : 0 \le t \le t_v, y \ge 0\}$, with the boundary conditions

$$\begin{aligned} \tilde{C}^{(i)}(t, 0) &= 0\,, && 0 \le t \le t_v\,, \\ \tilde{C}^{(i)}(t_v, y) &= C^{(i)}(t_v, y)\,, && 0 \le y \le y_i^\star(t_v)\,. \end{aligned} \tag{6.12}$$

Since $C^{(i)}(t_v, y)$ is the terminal condition for the PDE formulation for $\tilde{C}^{(i)}$, we must solve the PDE problem for $C^{(i)}$ before $\tilde{C}^{(i)}$. Again, we use an implicit finite-difference method for both PDE problems.

6.2. Characterization of the Employee's Exercise Boundaries

DEFINITION 6.1. The employee's indifference price for holding $i \le n$ ESOs with multiple exercises is defined as the function $p^{(i)} \equiv p^{(i)}(t, x, y)$ such that

$$M(t, x) = V^{(i)}(t, x - p^{(i)}, y)\,.$$

From this, we have $p^{(0)} = 0$ because $V^{(0)}(t, x, y) = M(t, x)$. Also, $p^{(1)}$ is the same as the indifference price in Definition 4.1. Again, due to the exponential utility function, the indifference price is a function of only t and y, and it is related to $H^{(i)}$ and $V^{(i)}$ in the following way.

PROPOSITION 6.1. *The indifference price $p^{(i)}$ satisfies*

$$p^{(i)}(t, y) = -\frac{1}{\gamma(1-\rho^2)e^{r(T-t)}} \log H^{(i)}(t, y), \tag{6.13}$$

and

$$V^{(i)}(t, x, y) = M(t, x) \cdot e^{-\gamma p^{(i)}(t,y) e^{r(T-t)}}\,. \tag{6.14}$$

Due to this proposition, we obtain a variational inequality that is equivalent to (6.1)–(6.3):

$$p_t^{(i)} + \tilde{\mathcal{L}} p^{(i)} - r p^{(i)} - \frac{1}{2}\gamma(1-\rho^2)\eta^2 y^2 e^{r(T-t)} \big(p_y^{(i)}\big)^2 + \frac{\lambda}{\gamma}\left(1 - b(t, y)^i e^{\gamma p^{(i)} e^{r(T-t)}}\right) \le 0\,,$$

$$p^{(i)} \ge p^{(i-1)} + (y - K)^+\,, \tag{6.15}$$

$$\begin{aligned} &\Big(p_t^{(i)} + \tilde{\mathcal{L}} p^{(i)} - r p^{(i)} - \frac{1}{2}\gamma(1-\rho^2)\eta^2 y^2 e^{r(T-t)} \big(p_y^{(i)}\big)^2 + \frac{\lambda}{\gamma}\Big(1 - b(t, y)^i e^{\gamma p^{(i)} e^{r(T-t)}}\Big)\Big) \\ &\quad \cdot \big((y-K)^+ + p^{(i-1)} - p^{(i)}\big) = 0, \end{aligned}$$

for $(t, y) \in [0, T] \times [0, +\infty)$. The boundary conditions are

$$\begin{aligned} p^{(i)}(T, y) &= i(y - K)^+, \\ p^{(i)}(t, 0) &= 0. \end{aligned}$$

The employee's optimal exercise time for the next option when there are i unexercised ESOs can be expressed in terms of indifference prices.

$$\begin{aligned} \tau_i^\star &= \inf\big\{t \le u \le T \,:\, V^{(i)}(u, X_u, Y_u) = V^{(i-1)}(u, X_u + (Y_u - K)^+, Y_u)\big\} \\ &= \inf\big\{t \le u \le T \,:\, p^{(i)}(u, Y_u) - p^{(i-1)}(u, Y_u) = (Y_u - K)^+\big\}\,. \end{aligned} \tag{6.16}$$

To understand the meaning of this, let us define the following.

DEFINITION 6.2. For a holder with i ESOs, we define $w^{(i+1)} \equiv w^{(i+1)}(t, x, y)$ as the **premium** that this holder is willing to pay in order to receive one extra ESO (i.e., the $(i+1)$th option). In other words, $w^{(i+1)}$ satisfies

$$V^{(i+1)}(t, x - w^{(i+1)}, y) = V^{(i)}(t, x, y). \tag{6.17}$$

From this definition, $w^{(1)}$ equals the employee's indifference price for holding one ESO. The premium can be written as the difference of two indifference prices.

PROPOSITION 6.2. *For $t \leq T$ and $y \in \mathbb{R}_+$, we have $w^{(i)}(t, y) = p^{(i)}(t, y) - p^{(i-1)}(t, y)$.*

Proof. We use Definition (6.17) to obtain the equality

$$M(t, x) = V^{(i)}(t, x - p^{(i)}, y) = V^{(i-1)}(t, x - p^{(i-1)}, y).$$

From this we have

$$V^{(i)}(t, x, y) = V^{(i-1)}(t, x + p^{(i)} - p^{(i-1)}, y).$$

Therefore, $w^{(i)} := p^{(i)} - p^{(i-1)}$ satisfies (6.17). □

Proposition 6.2 implies that we can rewrite the optimal exercise time as

$$\tau_i^\star = \inf\left\{t \leq u \leq T \,:\, w^{(i)}(u, Y_u) = (Y_u - K)^+\right\},$$

which means that the employee holding i ESOs should exercise the next option as soon as the payoff from immediate exercise is higher than the amount he is willing to pay for it. Under the assumption of exponential utility, the indifference prices $p^{(i)}$ are wealth-independent, which implies that the premiums $w^{(i)}$ and the optimal exercise time are also wealth-independent. For a general utility function, the indifference prices, premiums, and the optimal exercise time may all depend on wealth.

6.3. The Impact of Multiple Exercises

We first study the effect of multiple exercises on the employee's exercise policy. In the traditional no-arbitrage pricing theory for American options, the option holder always exercises all the options at the same time. In our model, the risk-averse employee exercises his ESOs at different critical price levels (see Figure 6.1). This is because the employee's premium for an additional option diminishes with respect to the number of options he already owns. As a result, the employee tends to exercise the first option very early, and the last one much later. Similar exercise behaviors can be found in Grasselli (2005), Rogers and Scheinkman (2007), and Henderson (2006).

In order to study the impact of multiple exercises, we compare it with the case with simultaneous exercise constraint. This constraint allows the employee to choose only one exercise time for all his ESOs. This is equivalent to the single issue case with the ESO payoff multiplied by the number of options. As Figure 6.1 illustrates, his boundary lies somewhere in the middle.

Finally, we examine the effect of multiple issues on the firm's granting cost. As the number of ESOs increases, the employee tends to adopt a more conservative exercising strategy for every additional option, which in turns results in a diminishing marginal cost. This is depicted in Figure 6.2. When there is no vesting, the cost of ESOs with simultaneous exercise constraint dominates that with multiple exercise rights (see Figure 6.2 (left-hand side)). This is because the simultaneous exercise constraint prevents the employee from exercising too early, leading to a higher expected discounted payoff.

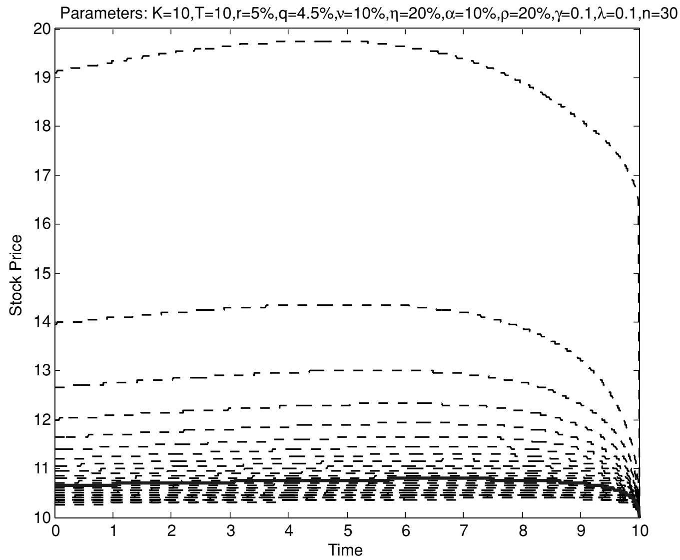

FIGURE 6.1. Multiple exercise boundaries: The dashed curves represent the exercise boundaries for an employee with 30 ESOs with multiple exercise rights. The bottom one corresponds to the first option exercised, and the top one corresponds to the last option exercised. When the employee is granted 30 ESOs with simultaneous exercise constraint, his exercise boundary (solid line) lies somewhere in the middle.

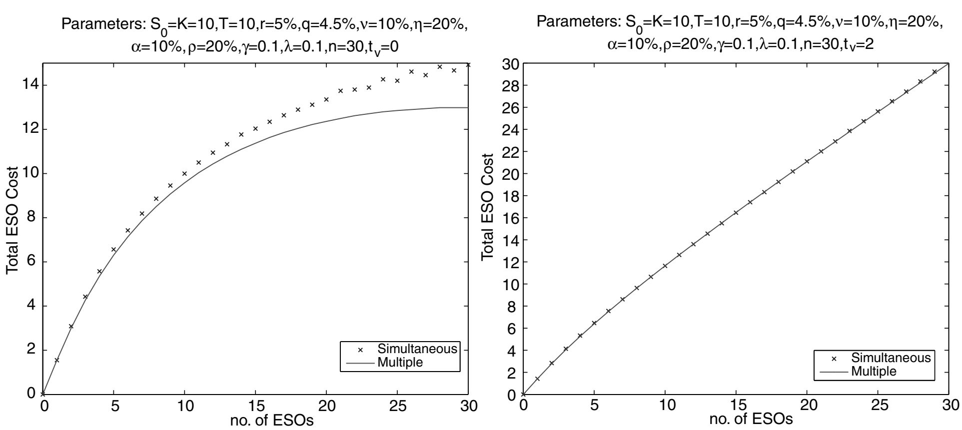

FIGURE 6.2. Effect of multiple exercises: (Left-hand side) When there is no vesting, the cost of ESOs with simultaneous exercise constraint dominates that with multiple exercise rights. (Right-hand side) However, when vesting is imposed, the difference in costs almost disappears.

However, when a two-year vesting period is imposed, the costs are almost the same (see Figure 6.2 (right-hand side)). In the case of multiple exercises, the majority of the employee's exercise boundaries are very low, so it is very likely that the company stock will be above most of them, leading to exercises at the end of the vesting period. Similarly, in the constrained case, the low boundary implies that the employee will probably exercise all his ESOs at the end of the vesting period. This result is consistent with the well-known early exercise phenomenon in ESO empirical studies. As a result, in the presence of vesting, the right of multiple exercises has negligible influence on the firm's granting cost.

APPENDIX: EXISTENCE OF A GENERALIZED SOLUTION

In this Appendix, we investigate the existence and uniqueness of solution to the nonlinear free boundary problem (4.4)–(4.5). The problem has a singularity as y goes to infinity, that is, when the obstacle term κ approaches zero. To circumvent this difficulty, we alter the obstacle term slightly

$$\hat{\kappa}(t, y) = e^{-\gamma(1-\rho^2)(y\wedge L-K)^+ e^{r(T-t)}},$$

with $L < \infty$. Notice $\hat{\kappa} \geq e^{-\gamma(1-\rho^2)Le^{rT}} =: \epsilon > 0$. This change imposes a positive lower bound on the obstacle, and we can choose L sufficiently large so that the error is negligible. For practical purposes, this free boundary is numerically solved on a bounded domain, which renders the obstacle term bounded away from zero. So far, we have seen that the employee's exercise boundary exists and is bounded above. In such cases, the free boundary problems corresponding to the original obstacle κ and the altered obstacle $\hat{\kappa}$ give the same solution if L is chosen sufficiently large.

For any bounded function $w : [0, T] \times [0, +\infty) \mapsto [\epsilon_L, 1]$ (with constant ϵ_L to be specified later), and stopping time $\tau \in \mathcal{T}_{t,T}$, we define

$$g(t, y; \tau, w) := \tilde{\mathbb{E}}_{t,y}\left\{e^{-(1-\rho^2)\lambda(\tau-t)}\hat{\kappa}(\tau, Y_\tau) + \int_t^\tau e^{-(1-\rho^2)\lambda(u-t)}\tilde{b}(u, Y_u)w(u, Y_u)^{-\hat{\rho}}\, du\right\},$$

where $\tilde{b}(t, y) := \lambda(1 - \rho^2)b(t, y)$ is bounded:

$$0 \leq \tilde{b}(t, y) \leq \lambda(1 - \rho^2) =: M < \infty.$$

Also, we define an operator Γ by

$$\Gamma w(t, y) = \inf_{\tau\in\mathcal{T}_{t,T}} g(t, y; \tau, w)\,. \tag{A.1}$$

Then, Γw is also bounded in $[\epsilon_L, 1]$. In view of (4.8), the generalized solution to (4.4)–(4.5) is the function that satisfies

$$G(t, y) = \Gamma G(t, y)\,. \tag{A.2}$$

Note that the solution G appears on both sides of the equation. Moreover, we observe that G is bounded:

$$e^{-(1-\rho^2)\lambda(T-t)}e^{-\gamma(1-\rho^2)(L-K)} \leq G(t, y) \leq \hat{\kappa}(t, y) \leq 1\,,$$

so we set $\epsilon_L = e^{-(1-\rho^2)(\lambda T+\gamma(L-K))}$. Next, we show that Γ is a contraction mapping and thus has a fixed point.

PROPOSITION A.1. *The operator Γ is a contraction mapping on the space of functions bounded in $[\epsilon_L, 1]$ with respect to the norm*

$$||v||_\beta := \sup_{(t,y)\in[0,T]\times[0,L]} e^{-\beta(T-t)}\,|v(t, y)|$$

for $0 < \beta < \infty$ sufficiently large. Also, Γ has a unique fixed point.

Notice the norm $||\cdot||_\beta$ is equivalent to the supremum-norm $||\cdot||_\infty$. We prepare to prove this proposition with two useful inequalities.

LEMMA A.1. *The following inequality holds:*

$$|\Gamma w_1 - \Gamma w_2| \leq \sup_{\tau\in\mathcal{T}_{t,T}} |g(t, y; \tau, w_1) - g(t, y; \tau, w_2)|.$$

Proof. By definition (A.1),we have

$$\begin{aligned}|\Gamma w_1 - \Gamma w_2| &= \Big| \inf_{\tau\in\mathcal{T}_{t,T}} g(t, y; \tau, w_1) - \inf_{\tau\in\mathcal{T}_{t,T}} g(t, y; \tau, w_2)\Big| \\ &\leq \sup_{\tau\in\mathcal{T}_{t,T}} |g(t, y; \tau, w_1) - g(t, y; \tau, w_2)|\,.\end{aligned}$$

Another useful inequality is that, for $a, c > 1$, $\hat{\rho} \in [1, \infty)$, we have $|a^{-\hat{\rho}} - c^{-\hat{\rho}}| \leq |a - c|$. Now, we can prove the proposition. □

Proof. Let w_1 and w_2 be two functions bounded in $[\epsilon_L, 1]$. We obtain

$$\begin{aligned}&e^{-\beta(T-t)}|\Gamma w_1 - \Gamma w_2| \\ &\leq e^{-\beta(T-t)} \sup_{\tau\in\mathcal{T}_{t,T}} \left| \mathbb{E}_{t,y}\left\{ \int_t^\tau \tilde{b}(s, Y_s) \left(w_1(s, Y_s)^{-\hat{\rho}} - w_2(s, Y_s)^{-\hat{\rho}}\right) ds\right\} \right| \\ &\leq e^{-\beta(T-t)} \sup_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}_{t,y}\left\{ \int_t^\tau \tilde{b}(s, Y_s)\epsilon_L^{-\hat{\rho}} \left| \left(\frac{w_1(s, Y_s)}{\epsilon_L}\right)^{-\hat{\rho}} - \left(\frac{w_2(s, Y_s)}{\epsilon_L}\right)^{-\hat{\rho}} \right| ds\right\} \\ &\leq e^{-\beta(T-t)} \sup_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}_{t,y}\left\{ \int_t^\tau \tilde{b}(s, Y_s)\epsilon_L^{-\hat{\rho}} \left| \frac{w_1(s, Y_s)}{\epsilon_L} - \frac{w_2(s, Y_s)}{\epsilon_L} \right| ds\right\} \\ &\leq e^{-\beta(T-t)} M\epsilon_L^{-\hat{\rho}-1} ||w_1 - w_2||_\beta \sup_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}_{t,y}\left\{ \int_t^\tau e^{\beta(T-s)}\, ds\right\} \\ &\leq \frac{M\epsilon_L^{-\frac{1}{1-\rho^2}}}{\beta} ||w_1 - w_2||_\beta\,.\end{aligned}$$

If we choose $\beta > M\epsilon_L^{-\frac{1}{1-\rho^2}}$, then Γ is a contraction mapping with respect to the norm $||\cdot||_\beta$. Consequently, Γ has a unique fixed point. □

In conclusion, the problem (4.4)–(4.5) has a unique generalized solution. In particular, the generalized solution can be approximated by the sequence $\{G_n\}_{n\geq 0}$, starting with

$$G_0(t, y) := \inf_{\tau\in\mathcal{T}_{t,T}} \tilde{\mathbb{E}}_{t,y} \left\{e^{-(1-\rho^2)\lambda(\tau-t)}\hat{\kappa}(\tau, Y_\tau)\right\},$$

and the rest defined via $G_{n+1}(t, y) = \Gamma G_n(t, y)$, for $n \geq 0$. Then, every member of the sequence is bounded below by ϵ_L, and the contraction mapping property of Γ will ensure convergence to the solution.

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