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Approximating functionals of local martingales under lack of uniqueness of the Black–Scholes PDE solution

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When the underlying stock price is a strict local martingale process under an equivalent local martingale measure, the Black–Scholes PDE associated with a European option may have multiple solutions. In this paper, we study an approximation for the smallest hedging price of such an European option. Our results show that a class of rebate barrier options can be used for this approximation. Among them, a specific rebate option is also provided with a continuous rebate function, which corresponds to the unique classical solution of the associated parabolic PDE. Such a construction makes existing numerical PDE techniques applicable for its computation. An asymptotic convergence rate is also studied when the knock-out barrier moves to infinity under suitable conditions.

Keywords: Black–Scholes PDE; Non-uniqueness; Financial bubbles; Local martingales; Euler's approximation; Convergence rate

JEL Classifications: C2, C63

1. Introduction

In a financial market equipped with a unique equivalent local martingale measure (ELMM) $\mathbb{P}$, the smallest hedging price of an European option is the conditional expectation of the payoff with respect to the probability $\mathbb{P}$, see Fernholz and Karatzas (2010), Jeanblanc *et al.* (2009). In contrast to the probabilistic representation, the option price can be also characterized as the unique solution of its associated Black–Scholes PDE, provided that PDE has a unique classical solution.

The necessary and sufficient condition for the unique solvability of the Black–Scholes PDE is that the underlying stock price is a true martingale process, see Bayraktar and Xing (2010). In other words, if the stock price is a strict local martingale, then there exist multiple solutions for the Black–Scholes PDE.

Moreover, the option price may be one of the many solutions of the Black–Scholes PDE. For this reason, the option value may very likely be mispriced by calculating a solution of the associated Black–Scholes PDE with the presence of multiple solutions. In economic terms, a bubble is a deviation between the trading price of an asset and its underlying arbitrage-free value. Therefore, one of the interpretations of financial bubbles is given by the differences between multiple solutions of the Black–Scholes PDE. Recently, the discussions on bubbles within this framework have drawn much attention from both practitioners and theoretical researchers, since financial crashes were partially caused by a systematic over-pricing (under-pricing) followed by market corrections. We refer the reader for more details in this regard to Cox and Hobson (2005), Ekström and Tysk (2009) and the references therein. In this work, we will focus on the following related problem proposed by Fernholz and Karatzas (2010):

(Q) *How can one find a feasible numerical solution convergent to the option price under the lack of uniqueness of the solutions for the associated Black–Scholes PDE?*

We first examine the existing numerical schemes on the CEV model of Example 2.1, where the option price can be explicitly identified. There are typically two kinds of numerical schemes in this vein (Wilmott 2007). One is a Monte Carlo method by discretizing the probability representation, the other is a PDE numerical method by discretizing the truncated version of PDE.

Unfortunately, Examples 2.2 and 2.3 show that the classical Euler–Maruyama approximation (for the Monte Carlo method) and the finite difference method (for the PDE numerical method) lead to a strictly larger value than the desired option price. Motivated by these two examples, question (Q) boils down to the following two problems:

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(Q1) Find a feasible approximation for a Monte Carlo method, and its convergence rate;
(Q2) Find a feasible approximation for a PDE numerical method, and its convergence rate.

In short, this work intends to find a feasible approximation to the smallest superhedging price. It turns out that the value function can be obtained by a limit of a series of appropriate rebate option prices, which can be estimated by some usual Monte Carlo method, see Corollary 3.2. For the demonstration purpose, Example 3.3 presents a correction of Euler's scheme given by Example 2.2.

However, Corollary 3.2 may not be utilized for the approximation by PDE numerical method, since it may cause a discontinuity at the corner of the terminal-boundary datum. Therefore, a specific rebate option based on an appropriately modified terminal payoff is proposed with its price continuous up to the boundary, so that its price corresponds to the unique classical solution of its associated parabolic PDE. Such a construction makes existing numerical PDE techniques applicable for practical computations, see Theorem 3.4.

The rest of the paper is outlined as follows. In the next section, we give the precise formulation of the underlying problem. Section 3 presents main results, and related proofs are relegated to Section 4 for the reader's convenience. The last section summarizes the work.

2. Problem formulation

Throughout this paper, we use K as a generic constant, and $\mathbb{R}^+ = (0, \infty)$, $\bar{\mathbb{R}}^+ = \mathbb{R}^+ \cup \{0\}$. If A is a sub-set of $\mathbb{R} \times [0, T]$, then $C(A)$ denotes the set of all continuous real functions on A, $C^{2,1}(A)$ denotes a collection of all functions $\varphi : A \mapsto \mathbb{R}$ such that φ_{xx} and φ_t belong to $C(A)$. $D_\gamma(A)$ denotes the set of all measurable functions $\varphi : A \to \bar{\mathbb{R}}^+$ satisfying growth condition

$$\varphi(x, t) \le K(1 + |x|^\gamma), \quad \forall (x, t) \in A. \tag{2.1}$$

$C_\gamma(A) := C(A) \cap D_\gamma(A)$ denotes the set of all continuous functions satisfying γ-growth. Also, the parabolic domain under consideration is given by $Q := \mathbb{R}^+ \times (0, T)$. In addition, we also consider the truncated domains of $Q_\beta := (0, \beta) \times (0, T)$ and $Q^\alpha_\beta := (\alpha, \beta) \times (0, T)$ for $0 < \alpha < \beta$.

We consider a single stock in the presence of the unique equivalent local martingale measure (ELMM) $\mathbb{P}$, under which the deflated price process follows

$$dX(s) = \sigma(X(s))dW(s), \qquad X(t) = x \ge 0, \tag{2.2}$$

where W is a standard Brownian motion with respect to a given filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = \{\mathcal{F}_s : s \ge t\})$ satisfying usual conditions. We impose the following two conditions on functions f and σ:

(A1) σ is locally Hölder continuous with exponent $\frac{1}{2}$ satisfying $\sigma(x) > 0$ for all $x \in \mathbb{R}^+$, and $\sigma(0) = 0$.
(A2) $f : \bar{\mathbb{R}}^+ \to \bar{\mathbb{R}}^+$ is a $C_\gamma(\bar{\mathbb{R}}^+)$ payoff function for some $\gamma \in [0, 1]$.

By Proposition 2.13 together with Example 2.14 of Karatzas and Shreve 1991), the assumption (A1) on σ ensures there exists a unique strong solution of (2.2) with absorbing state at zero.

For a given contingent claim $f(X(T))$ with a fixed maturity $T > 0$, the smallest hedging price has the form of

$$V(x, t) = \mathbb{E}_{x,t}[f(X(T))] := \mathbb{E}[f(X^{x,t}(T))|\mathcal{F}_t]. \tag{2.3}$$

In the above, we suppress the superscripts (x, t) in $X^{x,t}$, and write $\mathbb{E}_{x,t}[\ \cdot\ ]$ to indicate the conditional expectation with respect to $\mathbb{P}$ computed under these initial conditions. We emphasize here that $f(X^{x,t}(T))$ is integrable and hence the value V of (2.3) is well defined, since super-martingale property of $X(T)$ and Jensen's inequality together with the growth condition (A2) implies that

$$0 \le \mathbb{E}_{x,t}[f(X(T))] \le K(1 + \mathbb{E}_{x,t}[X^\gamma(T)]) \le K(1 + x^\gamma). \tag{2.4}$$

Recently, Ekström and Tysk (2009) shows that the value function V of (2.3) is the $C^{2,1}(Q) \cap C(\overline{Q})$ solution of $BS(Q, f)$, where $BS(Q, f)$ refers to Black–Scholes equation

$$BS(Q, f) \begin{cases} (E)\ u_t + \frac{1}{2}\sigma^2(x)u_{xx} = 0 & \text{on } Q = \mathbb{R}^+ \times (0, T) \\ (TD)\ u(x, T) = f(x) & \text{on } \forall x \in (0, \infty) \\ (BD)\ u(0, t) = f(0) & \text{on } \forall t \in (0, T]. \end{cases} \tag{2.5}$$

However, the next example taken from Cox and Hobson (2005) shows that the value function V may not be the unique solution of $BS(Q, f)$ when the deflated price process X is a strict local martingale.

Example 2.1 [(CEV model)] Recall that the CEV model describes a process which evolves according to the following stochastic differential equation:

$$dX(s) = X^\lambda(s)dW(s).$$

The parameter λ controls the relationship between the volatility and the price, and is the key feature of the model. When $\lambda < 1$, we see the so-called leverage effect, commonly observed in equity markets, where the volatility of a stock increases as its price falls. Conversely, in commodity markets, we often observe $\lambda > 1$, the so-called inverse leverage effect, whereby the volatility of the price of a commodity tends to increase as its price increases, See the related discussion Geman and Shih (2009) and the references therein.

Suppose the stock price follows a strict local martingale process $dX(s) = X^2 dW(s)$, with the initial $X(t) = x > 0$. Consider $V(x, t) = \mathbb{E}_{x,t}[X(T)]$. Then, V can be computed explicitly as

$$V(x, t) = x\Big(1 - 2\Phi\Big(-\frac{1}{x\sqrt{T - t}}\Big)\Big). \tag{2.6}$$

One can verify V satisfies $BS(Q, f)$. Yet, another trivial solution is $u(x, t) = x$. □

Now, V is one of the possibly multiple solutions of $BS(Q, f)$. With the existence of multiple solutions to PDE $BS(Q, f)$, our question is

- If we use any of existing PDE numerical methods on $BS(Q, f)$, or any of existing Monte Carlo methods on (2.3), does it converge to the desired value function among multiple solutions of PDE?

Unfortunately, the answer is NO in general. In fact, the next trivial example shows that the classical Monte Carlo method by Euler–Maruyama approximation does *not* lead to the desired value $V(x,t)$ of (2.6) of Example 2.1.

Example 2.2 Consider the strong Euler–Maruyama (EM) approximation to Example 2.1 with step size Δ,

$$X_{n+1}^{\Delta} = X_n^{\Delta} + \sigma(X_n^{\Delta})(W(n\Delta+\Delta) - W(n\Delta)), \quad X_0^{\Delta} = x.$$

Let $X^{\Delta}(\cdot)$ be the piecewise constant interpolation of $\{X_n^{\Delta} : n \geq 0\}$, i.e.

$$X^{\Delta}(s) = X^{\Delta}_{[s/\Delta]}, \quad \forall s > 0. \tag{2.7}$$

Since $\{X_n^{\Delta} : n \geq 0\}$ is a martingale, the approximated value function simply leads to a wrong value

$$V_{\Delta}(x,t) := \mathbb{E}_{x,t}[X^{\Delta}(T)] = \mathbb{E}_{x,t}[X^{\Delta}_{(T-t)/\Delta}] = x > V(x,t). \quad \square$$

Similar to Monte Carlo method, one can also prove that the finite difference method on PDE $BS(Q,f)$ also leads to a wrong value.

Example 2.3 Black–Scholes PDE associated to the CEV model Example 2.1 is $BS(Q,f)$ of (2.5) with $\sigma(x) = x^2$ and $f(x) = x$. To use the finite difference method (FDM) in the above PDE, we shall truncate the domain and put artificial boundary conditions on the upper barrier $\{(x,t) : 0 \leq t \leq T\}$ for large enough $\beta > 0$. For this purpose, we impose boundary conditions, which *asymptotes* the option price as suggested by Wilmott (2007), i.e.

$$u(\beta,t) = \beta, \quad \forall 0 \leq t \leq T.$$

With step size Δ^2 in space variable x and Δ in time variable t, one can easily check upward finite difference scheme backward in time yields trivial numerical solution $u^{\Delta}(x,t) = x$ for any small $\Delta > 0$. $\square$

To this end, our work is to resolve the following questions: How can one find a feasible approximation to fix the failure of the convergence to the option price V of (2.3) approximated by Monte Carlo method and FDM? If possible, what is the convergence rate?

3. Main result

In this sub-section, we present the main results, and the proofs will be relegated to the next section.

3.1. Approximation by Monte Carlo method

We consider the following up-rebate option price: suppose the up barrier is given by a positive constant $\beta > x > 0$ and stopping time τ^{β} (suppressing the initial condition (x,t)) is the first hitting time of the stock price $X(s)$ to the barrier β before the maturity T, i.e.

$$\tau^{\beta} := \tau^{x,t,\beta} = \inf\{s > t : X^{x,t}(s) \geq \beta\} \wedge T. \tag{3.1}$$

For some non-negative function g, the payoff of the rebate option consists of

(i) rebate payoff $g(\beta)$, if $\tau^{\beta} < T$;
(ii) otherwise, terminal payoff $f(X(T))$.

Then, the rebate option price V^{β} is of the form

$$V^{\beta}(x,t) = \mathbb{E}_{x,t}\left[g(\beta)\mathbf{1}_{\{\tau^{\beta}<T\}} + f(X(T))\mathbf{1}_{\{\tau^{\beta}=T\}}\right], \tag{3.2}$$

Since V^{β} is a functional of f and g, and we may write $V^{\beta,g,f}$ instead of V^{β} whenever it needs an explicit emphasis on its dependence of f and g. It turns out that the the option price of (2.3) can be obtained by a limit of a series of appropriate rebate option prices.

THEOREM 3.1 *Assume* (A1–A2). *Suppose the rebate payoff g satisfies one of the following two conditions:*

(i) $g(x)$ *is of sub-linear growth, i.e.* $\lim_{x\to\infty} \frac{g(x)}{x} = 0$;
(ii) $g(x)$ *is of linear growth, i.e.* $\limsup_{x\to\infty} \frac{g(x)}{x} < \infty$, *and* $X^{x,t}$ *is a martingale.*

Then, we have the convergence for V^{β} *of* (3.2),

$$\lim_{\beta\to\infty} V^{\beta}(x,t) = V(x,t).$$

In addition, if $g \in D_{\eta}(\mathbb{R}^+)$ *with* $\gamma \vee \eta < 1$, *then the convergence rate is the order of* $1-(\gamma \vee \eta)$ *as* $\beta \to \infty$, *i.e.*

$$|(V - V^{\beta})(x,t)| \leq K\beta^{-(1-(\gamma\vee\eta))}, \quad \forall x < \beta. \tag{3.3}$$

The proof of Theorem 3.1 is given in Section 4, and it shows that Monte Carlo method on the expression V^{β} of (3.2) actually leads to the correct estimation of the option price V, provided that the rebate function g is appropriately chosen. Among the many choices of g, the simplest one shall be taking $g \equiv 0$, which is summarized in this below as Corollary 3.2. Nevertheless, Theorem 3.1 is useful in the sense that it gives more flexibility in choosing the function g. In particular, it is potentially useful to design PDE numerics with continuous terminal-boundary datum, see Section 3.2.

COROLLARY 3.2
Let

$$V^{\beta,0,f} = \mathbb{E}_{x,t}[f(X(T))\mathbf{1}_{\{\tau^{\beta}=T\}}].$$

Then, $\lim_{\beta\to\infty} V^{\beta,0,f}(x,t) = V(x,t)$ *pointwisely in* (x,t). *Furthermore, if* $f(x) = O(x^{\gamma})$ *as* $x \to \infty$ *with some constant* $\gamma < 1$, *then its convergence rate is*

$$|(V - V^{\beta,0,f})(x,t)| = O(\beta^{-1+\gamma}), \quad \text{as } \beta \to \infty.$$

Next, we will fix Monte Carlo method in Example 2.2 based on convergence result in Theorem 3.1.

Example 3.3 Let's extend $f : \bar{\mathbb{R}}^+ \mapsto \mathbb{R}$ to $f : \mathbb{R} \mapsto \mathbb{R}$ by $f(x) = f(0)$ for $x < 0$. Recall that X follows a strict local martingale process $dX(s) = X^2 dW(s)$, with the initial $X(t) = x > 0$. Corollary 3.2 implies that the rebate option price is convergent to the smallest hedging price, i.e.

$$V^{\beta}(x,t) := \mathbb{E}_{x,t}[f(X(T))\mathbf{1}_{\{\tau^{\beta}\geq T\}}] \to V(x,t) \quad \text{as } \beta \to \infty.$$

Now, rather than using Euler's approximation $X^{\Delta}(\cdot)$ of (2.7), we propose Euler's polygonal approximation $\bar{X}^{\Delta}$ as suggested by Gyongy and Krylov (1996) and Gyongy (1998), that is, for $m(s) := t + \Delta \cdot [\frac{s-t}{\Delta}]$

$$\bar{X}^{\Delta}(s) = \bar{X}^{\Delta}(m(s)) + (\bar{X}^{\Delta}(m(s)))^2(W(s) - W(m(s))).$$

Theorem 2.4 and Remark 2.5 of Gyongy (1998) imply that $\bar{X}^{\Delta}$ converges to X in distribution, denoted by $\bar{X}^{\Delta} \Rightarrow X$ as $\Delta \to 0$. Let τ^{β}_{Δ} be the first hitting time of $\bar{X}^{\Delta}$ to the barrier β, and consider the modified Monte Carlo scheme to approximate $V(x,t)$ of Example 2.2 given by

$$V^{\beta}_{\Delta}(x,t) := \mathbb{E}_{x,t}\Big[f(\bar{X}^{\Delta}(T))\mathbf{1}_{\{\tau^{\beta}_{\Delta}\geq T\}}\Big].$$

Note that, one can rewrite $V^{\beta}_{\Delta}(x,t) = \mathbb{E}_{x,t}[F(\bar{X}^{\Delta})]$ and $V^{\beta}(x,t) = \mathbb{E}_{x,t}[F(\bar{X})]$, respectively, where $F : C[0,T] \mapsto \mathbb{R}$ is a functional given by

$$F(\alpha) := f(\alpha(T)) \cdot \mathbf{1}_{\{\tau^{\beta}(\alpha)\geq T\}}, \quad \text{and}$$
$$\tau^{\beta}(\alpha) = \inf\{s > t : \alpha(s) \geq \beta\}, \quad \forall \alpha \in C[0,T].$$

Moreover, F is is continuous on $C[0,T]$ with respect to Skorohod topology almost surely in $\mathbb{P}X^{-1}$, since its decomposition $F(\alpha) = F_1(\alpha)\cdot(F_2 \circ F_3(\alpha))$ satisfies

- $F_1(\alpha) = f(\alpha(T)) : C[0,T] \mapsto \mathbb{R}$ is continuous with respect to Skorohod topology,
- $F_2(\alpha) = \mathbf{1}_{\{a\geq T\}} : \mathbb{R} \mapsto [0,1]$ is discontinuous only at one point $\{T\}$,
- $F_3(\alpha) = \tau^{\beta}(\alpha) : C[0,T] \mapsto \mathbb{R}$ is only discontinuous at most null set $\mathcal{N}$ (with respect to $\mathbb{P}X^{-1}$, i.e. $\mathbb{P}X^{-1}(\mathcal{N}) = 0$) in the space $C[0,T]$ by VI.2.11 of Jacod and Shiryaev (2003).

Thus, $F(\bar{X}^{\Delta}) \Rightarrow F(X)$ in distribution by continuous mapping theorem, see Theorem 2.7 of Billingsley (1999). Together with the boundedness of F by $f(\beta)$, it yields that $\mathbb{E}_{x,t}[F(\bar{X}^{\Delta})] \to \mathbb{E}_{x,t}[F(\bar{X})]$, or equivalently

$$V^{\beta}_{\Delta}(x,t) \to V^{\beta}(x,t), \quad \text{as } \Delta \to 0. \tag{3.4}$$

As a result, the option price $V(x,t)$ of (2.3) can be approximated by using Monte Carlo method on the above rebate options, i.e.

$$\lim_{\beta\to\infty}\lim_{\Delta\to 0} V^{\beta}_{\Delta}(x,t) = V(x,t).$$

□

Regarding the estimation by Monte Carlo method, one may take the simplest choice $g(\beta) \equiv 0$ for the rebate payoff as of Corollary 3.2, see also Ekström *et al.* (2008). However, Corollary 3.2 cannot be utilized for the approximation by PDE numerical method, since it may cause a discontinuity at the corner (β, T) of the terminal-boundary datum when $f(\beta) \neq 0$.

3.2. Approximation by PDE numerical method

For the above Monte Carlo method on (3.2), yet another to be mentioned is a drawback in the computation by PDE numerical methods due to the possible discontinuity of the boundary-terminal data.

To illustrate this issue, we write Black–Scholes PDE associated to the rebate option price $V^{\beta}(x,t)$ of (3.2),

$$\begin{cases} u_t + \dfrac{1}{2}\sigma^2(x)u_{xx} = 0 \quad \text{on } Q_{\beta} := (0,\beta)\times(0,T); \\ u(x,T) = f(x), \quad \forall x \in [0,\beta]; \\ u(0,t) = f(0), \quad u(\beta,t) = g(\beta),\ \forall t \in (0,T). \end{cases} \tag{3.5}$$

Note that, PDE (3.5) has a discontinuous corner at the point (β, T) if $g(\beta) \neq f(\beta)$. Also, recall that the choice of $g(\beta) = f(\beta)$ may not be possible, like in CEV model of Example 2.1.

It is well known that, if $g(\beta) \neq f(\beta)$ and the boundary-terminal data is discontinuous, then one cannot expect the unique solution of (3.5) continuous up to the boundary. Furthermore, the discontinuity and the singularity at the corner propagate the numerical errors quickly throughout its entire domain for the numerical PDE methods, such as finite element method (FEM) or finite difference method (FDM), see more discussions in Song *et al.* (2007) and the references therein. Therefore, the unique solvability and the regularity of the solution are crucial to make use of the existing PDE numerical methods.

To avoid this error propagation due to the discontinuity of the boundary-terminal data, we provide an alternative choice to (3.2) by revising the terminal payoff: consider a rebate option of barrier β with

(i) zero rebate payoff, i.e. $g(\beta) \equiv 0$;
(ii) and a revised terminal payoff

$$f^{\beta}(x) = f(x)\mathbf{1}_{\{x\leq\beta/2\}} + \frac{2f(x)(\beta - x)}{\beta}\mathbf{1}_{\{\beta/2<x\leq\beta\}}. \tag{3.6}$$

In this case, the rebate option price

$$\widetilde{V}^{\beta}(x,t) = \mathbb{E}_{x,t}[f^{\beta}(X(T))\mathbf{1}_{\{\tau^{\beta}=T\}}] \tag{3.7}$$

is associated to PDE

$$\begin{cases} (E)_{\beta} \quad u_t + \dfrac{1}{2}\sigma^2(x)u_{xx} = 0 \quad \text{on } Q_{\beta} := (0,\beta)\times(0,T); \\ (BD) \quad u(0,t) = f(0), \quad u(\beta,t) = 0,\ \forall t \in (0,T); \\ (TD)_{\beta}\ u(x,T) = f^{\beta}(x), \quad \forall x \in [0,\beta]. \end{cases} \tag{3.8}$$

Observe that, the revised terminal data f^{β} not only makes the terminal-boundary data continuous at the corner (β, T), but also preserves Hölder regularity of the original terminal data f regardless how large the value β is.

Although PDE (3.8) is degenerate at $x = 0$, one can still have unique classical solution by utilizing Shauder's interior estimate. Also, its solution is indeed equal to the revised rebate option price $\widetilde{V}^{\beta}$ of (3.7), see Lemma 4.7. Moreover, by using comparison principle twice on two different truncated domains, one can show its unique solution $\widetilde{V}^{\beta}$ must be convergent to the desired value V of (2.3).

THEOREM 3.4 *Assume* (A1–A2). *Then,* $\widetilde{V}^{\beta}$ *of* (3.7) *is the unique* $C^{2,1}(Q_{\beta}) \cap C(\overline{Q}_{\beta})$ *solution of PDE* (3.8)*, and*

$$\lim_{\beta\to\infty}\widetilde{V}^{\beta}(x,t) = V(x,t),\ \forall (x,t) \in Q.$$

In addition, if $\gamma < 1$ *in* (A2)*, then the convergence rate is*

$$|\widetilde{V}^{\beta} - V|(x,t) \leq K\beta^{-1+\gamma}.$$

Thanks to the Theorem 3.4, one can use either well established FDM or FEM on PDE (3.8) for a large β to estimate the smallest superhedging price.

4. Proof of main results

In this section, we will first characterize the value function V. Based on the properties of V, we can estimate $|V - V^{\beta}|$ to prove Theorem 3.1, and $|V - \widetilde{V}^{\beta}|$ to prove Theorem 3.4, respectively.

4.1. Characterization of the option price V

We have seen that the option price V of (2.3) is one of the solutions of $BS(Q, f)$. To proceed, we need to identify which solution corresponds to the option price V among many. This enables us to establish the connection between parabolic partial differential equation $BS(Q, f)$ and probability representation (2.3).

PROPOSITION 4.1 *Assume* (A1–A2). *Then, value function V of* (2.3) *is*

(1) *the smallest lower bounded $C^{2,1}(Q) \cap C_\gamma(\overline{Q})$ solution of $BS(Q, f)$.*
(2) *the unique $C^{2,1}(Q) \cap C(\overline{Q})$ solution of $BS(Q, f)$ in the class of functions with at most linear growth if and only if σ satisfies*

$$\int_1^\infty \frac{x}{\sigma^2(x)} dx = \infty. \tag{4.1}$$

Proof Theorem 3.2 of Ekström and Tysk (2009) shows that V is a $C^{2,1}(Q) \cap C(\overline{Q})$ solution of $BS(Q, f)$. The estimation (2.4) also implies that $V \in C_\gamma(\overline{Q})$. For the necessary and sufficient condition on uniqueness, we refer the proof to Bayraktar and Xing (2010). It remains to show V is the smallest lower bounded solution. Sometimes, we use X to denote $X^{x,0}$ without ambiguity in this proof. Note that, by pathwise uniqueness of the solution to (2.2)

$$\begin{aligned} Y(t) \triangleq V(X^{x,0}(t), t) &= \mathbb{E}[f(X^{X^{x,0}(t),t}(T))|\mathcal{F}_t] \\ &= \mathbb{E}[f(X^{x,0}(T))|\mathcal{F}_t] \end{aligned}$$

is a martingale process. Suppose $\hat{V} \in C^{2,1}(Q) \cap C(\overline{Q})$ is an arbitrary lower bounded solution of $BS(Q, f)$, then Ito's formula applying to $\hat{Y}(t) \triangleq \hat{V}(X(t), t)$ leads to

$$\hat{Y}(t) = \hat{V}(X(0), 0) + \int_0^t \hat{V}_x(X(s), s)\sigma(X(s))dW(s),$$

and $\hat{Y}(t)$ is a lower bounded local martingale, hence is a supermartingale. Therefore, we have

$$\hat{Y}(0) \geq \mathbb{E}[\hat{Y}(T)] = \mathbb{E}[f(X(T))] = Y(0)$$

and this implies

$$\hat{V}(x, 0) \geq V(x, 0).$$

We can similarly prove for $\hat{V}(x, t) \geq V(x, t)$ for all t. □

PROPOSITION 4.2 *Assuming* (A1–A2), *$BS(Q, f)$ only admits non-negative solution in the space of lower bounded $C^{2,1}(Q) \cap C(\overline{Q})$ functions.*

Proof Proposition 4.1 shows that V is the smallest lower bounded solution of PDE. Since $V \geq 0$ by definition of (2.3), it implies any lower bounded solution u satisfies $u \geq V \geq 0$. □

In Example 2.1, we have seen that $BS(Q, f)$ of CEV model has multiple solutions. We continue this model to demonstrate Proposition 4.2, a solution smaller than V must be unbounded from below.

Example 4.3 By Proposition 4.1, the explicit solution $V \geq 0$ of (2.6) in CEV model smallest lower bounded solution of $BS(Q, f)$. In fact one can find,

$$v(x, t) = x\Big(1 - \lambda\Phi\Big(-\frac{1}{x\sqrt{T-t}}\Big)\Big), \lambda > 2$$

is a smaller solution, i.e. $v \leq V$ in Q. However, v is not lower-bounded, i.e. $v(x, t) \to -\infty$ as $x \to \infty$.

4.2. Proof of Theorem 3.1

Recall that the domain of the value function V is given on the domain $\overline{Q}$ of (2.5), and its related truncated domain Q_β is given by (3.5). Let $\varphi : \overline{Q} \to \bar{\mathbb{R}}^+$ be a measurable function. We introduce the truncated value function $V^{\beta,\varphi}$ for convenience,

$$V^{\beta,\varphi}(x, t) = \begin{cases} \mathbb{E}_{x,t}[\varphi(X(\tau^\beta), \tau^\beta)], & \forall (x, t) \in \overline{Q}_\beta, \\ \varphi(x, t) & \text{otherwise.} \end{cases} \tag{4.2}$$

where the stopping time τ^β of (3.1) is the first hitting time to the barrier β before the terminal time T. By the above definition,

$$V^{\beta,\varphi_1}(x, t) = V^\beta(x, t)$$

for the V^β of (3.2), if we set

$$\varphi_1(x, t) = g(x)\mathbf{1}_{\{t<T\}} + f(x)\mathbf{1}_{\{t=T\}}. \tag{4.3}$$

With the above set-up, to prove Theorem 3.1, our goal is to estimate $|V^{\beta,\varphi_1} - V|$ as $\beta \to \infty$ with φ_1 of (4.3) and the constraint on g given in Theorem 3.1. We emphasize here, φ_1 may not be continuous up to the boundary, i.e. $\varphi_1 \notin C(\overline{Q})$ when $g(x) < f(x)$ for some $x > 0$.

LEMMA 4.4 *Assume* (A1–A2). *Then,*

(i) *$V(x, t) = V^{\beta,V}(x, t)$ for all $0 < x < \beta$.*
(ii) *If $\varphi, \psi : \overline{Q} \to \bar{\mathbb{R}}^+$ are two measurable functions satisfying $\varphi \geq \psi$ on $\overline{Q} \setminus Q_\beta$, then*

$$V^{\beta,\varphi} \geq V^{\beta,\psi}, \ \forall \beta > 0.$$

Proof $X^{x,t}$ is the unique strong solution of (2.2) due to (A1). Therefore, the conclusion follows from the following simple derivation using tower property and strong Markov property:

$$\begin{aligned} V(x, t) &= \mathbb{E}[f(X^{x,t}(T))|\mathcal{F}_t] \\ &= \mathbb{E}[f(X^{x,t}(T))|\mathcal{F}_{\tau^\beta}]|\mathcal{F}_t] \\ &= \mathbb{E}[\mathbb{E}[f(X^{X(\tau^\beta),\tau^\beta}(T))|\mathcal{F}_{\tau^\beta}]|\mathcal{F}_t] \\ &= \mathbb{E}[V(X(\tau^\beta), \tau^\beta)|\mathcal{F}_t] \\ &= V^{\beta,V}(x, t). \end{aligned}$$

Monotonicity of $V^{\beta,\varphi}$ in φ follows directly from the definition of $V^{\beta,\varphi}$ of (4.2). □

It is noted that, two results of Lemma 4.4 correspond to uniqueness and comparison principle of its associated PDE. However, we provide the probabilistic proof Lemma 4.4 , since we want to cover potentially discontinuous function φ_1 of (4.3), in which uniqueness may not remain true.

LEMMA 4.5 *Assume* (A1–A2) *and $g \geq 0$. Then, V^{β,φ_1} defined by* (4.2) *and* (4.3) *satisfies*

$$\lim_{\beta\to\infty} V^{\beta,\varphi_1}(x, t) \geq V(x, t).$$

In addition, equality holds in the above if and only if

$$\lim_{\beta\to\infty} \mathbb{E}_{x,t}\Big[g(\beta)\mathbf{1}_{\{\tau^\beta<T\}}\Big] = 0 \tag{4.4}$$

Proof We start with the following observation: the solution $X := X^{t,x}$ of (2.2) does not explode almost surely by (Karatzas and Shreve, 1991, 5.5.3), i.e.

$$\lim_{\beta\to\infty} \tau^\beta = T, \quad \text{a.s.-}\mathbb{P} \tag{4.5}$$

Due to this fact together with Monotone Convergence Theorem, we obtain following identities:

$$\begin{aligned} &\lim_{\beta\to\infty} \mathbb{E}_{x,t}\Big[f(X(T))\mathbf{1}_{\{\tau^\beta=T\}}\Big] \\ &= \mathbb{E}_{x,t}\Big[\lim_{\beta\to\infty} f(X(T))\mathbf{1}_{\{\tau^\beta=T\}}\Big] \\ &= \mathbb{E}_{x,t}\Big[f(X(T))\Big] = V(x,t). \end{aligned} \tag{4.6}$$

By the definition of φ_1 of (4.3), this results in

$$\begin{aligned} &\lim_{\beta\to\infty} V^{\beta,\varphi_1}(x,t) \\ &= \lim_{\beta\to\infty} \mathbb{E}_{x,t}[\varphi_1(X(\tau^\beta),\tau^\beta)\mathbf{1}_{\{\tau^\beta<T\}}] \\ &\quad + \lim_{\beta\to\infty} \mathbb{E}_{x,t}[\varphi_1(X(\tau^\beta),\tau^\beta)\mathbf{1}_{\{\tau^\beta=T\}}] \\ &= \lim_{\beta\to\infty} \mathbb{E}_{x,t}[\varphi_1(\beta,\tau^\beta)\mathbf{1}_{\{\tau^\beta<T\}}] \\ &\quad + \lim_{\beta\to\infty} \mathbb{E}_{x,t}[f(X(T))\mathbf{1}_{\{\tau^\beta=T\}}] \\ &= \lim_{\beta\to\infty} \mathbb{E}_{x,t}[g(\beta)\mathbf{1}_{\{\tau^\beta<T\}}] + V(x,t). \end{aligned}$$

Rearranging the above identity, we have

$$V(x,t) = \lim_{\beta\to\infty} V^{\beta,\varphi_1}(x,t) - \lim_{\beta\to\infty} \mathbb{E}_{x,t}[g(\beta)\mathbf{1}_{\{\tau^\beta<T\}}]. \tag{4.7}$$

Note that three terms in (4.7) are all non-negative. Hence, $\lim_{\beta\to\infty} V^{\beta,\varphi_1}(x,t) \geq V(x,t)$ and equality holds if and only if (4.4) holds. □

As mentioned in (4.5), the solution $X^{x,t}$ of (2.2) does not explode almost surely, and this can be rewritten as

$$\mathbb{P}(\tau^{x,t,\beta} < T) \to 0 \quad \text{as } \beta \to \infty.$$

An interesting question about this is that, how fast does the above probability converge to zero? The answer to this question is indeed useful to obtain the convergence rate of the truncated approximation.

Proposition 4.6 *Fix $(x,t) \in Q$ and assume* (A1–A2). *As $\beta \to \infty$, stopping time $\tau^{x,t,\beta}$ of* (3.1) *satisfies*

(1) $\mathbb{P}\{\tau^{x,t,\beta} < T\} = O(1/\beta)$.
(2) *Moreover,* $\mathbb{P}\{\tau^{x,t,\beta} < T\} = o(1/\beta)$ *if and only if* $\{X^{t,x}(s) : t \leq s \leq T\}$ *is a martingale .*

Proof By taking $g(x) = f(x) = x$ in (4.7),

$$\lim_{\beta\to\infty} \mathbb{E}_{x,t}[X(\tau^\beta)] = \lim_{\beta\to\infty} \beta\mathbb{P}\{\tau^{x,t,\beta} < T\} + \mathbb{E}_{x,t}[X(T)].$$

For all $\beta > x$, since $\{X^{x,t}(\tau^\beta \wedge s) : s > t\}$ is a bounded local martingale, hence it is a martingale. So, $\mathbb{E}_{x,t}[X(\tau^\beta)] = x$ for all $\beta > x$. Rearranging the above identity, we have

$$\lim_{\beta\to\infty} \beta\mathbb{P}\{\tau^{x,t,\beta} < T\} = x - \mathbb{E}_{x,t}[X(T)]. \tag{4.8}$$

(4.8) implies

(1) Since $\mathbb{E}_{x,t}[X(T)] \geq 0$, $\lim_{\beta\to\infty} \beta\mathbb{P}\{\tau^{x,t,\beta} < T\} \leq x < \infty$, which shows $\mathbb{P}\{\tau^{x,t,\beta} < T\} = O(1/\beta)$.
(2) $\{X^{t,x}(s) : t \leq s \leq T\}$ is a martingale if and only if $x = \mathbb{E}_{x,t}[X(T)]$, if and only if $\mathbb{P}\{\tau^{x,t,\beta} < T\} = o(1/\beta)$.

□

Finally, we are now ready for the proof of Theorem 3.1.

Proof [**Proof of Theorem 3.1**] We first show its convergence, then obtain convergence rate.

(a) Regarding its convergence, it is enough to verify (4.4) by Lemma 4.5. Note that

$$\begin{aligned} &\lim_{\beta\to\infty} \mathbb{E}_{x,t}\Big[g(\beta)\mathbf{1}_{\{\tau^\beta<T\}}\Big] \\ &\leq \lim_{\beta\to\infty} \frac{g(\beta)}{\beta} \lim_{\beta\to\infty} \beta\mathbb{P}\{\tau^{x,t,\beta} < T\}. \end{aligned}$$

(i) If g is of sub-linear growth, then $\lim_{\beta\to\infty} \frac{g(\beta)}{\beta} = 0$. Hence, (4.4) holds due to the first result of Proposition 4.6;
(ii) On the other hand, if $X^{t,x}$ is a martingale, then we have $\lim_{\beta\to\infty} \beta\mathbb{P}\{\tau^{x,t,\beta} < T\} = 0$ from the second result of Proposition 4.6, and (4.4) remains true provided that g is of linear growth.

(b) Since $V(x,t) = V^{\beta,V}(x,t)$ for all $\beta > x$ by Lemma 4.4, we have the following identity:

$$\begin{aligned} (V - V^{\beta,\varphi_1})(x,t) &= (V^{\beta,V} - V^{\beta,\varphi_1})(x,t) \\ &= \mathbb{E}[(V - \varphi_1)(X(\tau^\beta),\tau^\beta)\mathbf{1}_{\{\tau^\beta<T\}}]. \end{aligned}$$

Setting $\overline{V} := \sup_{t\in[0,T)} V(x,t)$, we can rewrite

$$|(V-V^{\beta,\varphi_1})(x,t)| \leq (|\overline{V}|+|g|)(\beta)\mathbb{E}[\mathbf{1}_{\{\tau^\beta<T\}}]. \tag{4.9}$$

Since $\overline{V} \in C_\gamma(\mathbb{R}^+)$ by Proposition 4.1 and $g \in D_\eta(\mathbb{R}^+)$, we have $|\overline{V} + g| \in D_{\gamma\vee\eta}(\mathbb{R}^+)$. Hence, write (4.9) by Proposition 4.6

$$\begin{aligned} |(V - V^{\beta,\varphi_1})(x,t)| &\leq (|\overline{V}| + |g|)(\beta)O(1/\beta) \\ &\leq K\beta^{(\gamma\vee\eta)-1}, \end{aligned}$$

which finally results in (3.3). □

4.3. *Proof of Theorem 3.4*

Lemma 4.7 *Assume* (A1-A2). *Then,* $\widetilde{V}^\beta$ *of* (3.7) *is the unique solution of* (3.8) *in the space of* $C^{2,1}(Q_\beta) \cap C(\overline{Q}_\beta)$.

Proof Fix $(x,t) \in Q_\beta$. Take $\alpha \in (0, x/2)$. Recall $Q^\alpha_\beta = Q_\beta \cap (\overline{Q}_\alpha)^c$ be an open set. Also, define

$$\tau^{\alpha,\beta} = \inf\{s > t : (X^{x,t}(s), s) \notin Q^\alpha_\beta\}.$$

Due to the uniform ellipticity,

$$V^{\alpha,\beta}(x,t) := \mathbb{E}_{x,t}[f^\beta(X(\tau^{\alpha,\beta}))] \tag{4.10}$$

is the unique classical solution of

$$\begin{cases} u_t + \frac{1}{2}\sigma^2(x)u_{xx} = 0, & \text{on } Q^\alpha_\beta = (\alpha,\beta)\times(0,T) \\ u(\beta,t) = 0,\ u(\alpha,t) = f^\beta(\alpha), & \forall t \in (0,T) \\ u(x,T) = f^\beta(x), & \forall x \in [\alpha,\beta]. \end{cases} \tag{4.11}$$

If we restrict $V^{\alpha,\beta}$ on the subdomain $Q^{x/2}_\beta$, it solves following PDE uniquely,

$$\begin{cases} u_t + \frac{1}{2}\sigma^2(x)u_{xx} = 0, & \text{on } Q^{x/2}_\beta = (x/2,\beta)\times(0,T) \\ u(\beta,t) = 0,\ u(x/2,t) = V^{\alpha,\beta}(x/2,t), & \forall t \in (0,T) \\ u(x,T) = f^\beta(x), & \forall x \in [x/2,\beta]. \end{cases} \tag{4.12}$$

Furthermore, by using Shauder estimate Theorem 4.9 together with Theorem 5.9 of Lieberman (1996), one can have an estimate on the weighted Hölder norm, i.e.

$$|V^{\alpha,\beta}|^*_{2.5,Q^{x/2}_\beta} \leq K|V^{\alpha,\beta}|_{0,Q^{x/2}_\beta}$$

for some constant K independent to α. On the other hand, by definition (4.10), we have

$$\begin{aligned}|V^{\alpha,\beta}|_{0,Q^{x/2}_\beta} &= \sup_{Q^{x/2}_\beta} V^{\alpha,\beta} \leq \sup_{x\in[0,\beta]} |f^\beta(x)| \\ &\leq \sup_{x\in[0,\beta]} |f^\beta(x)| \leq K\end{aligned}$$

for some K independent to α. Let $d = \frac{1}{2}\min\{x, \beta-x, t, T-t\}$, which must be less than the minimum distance of x to any point in the parabolic boundary $\partial^* Q^\alpha_\beta$. Consider a neighbourhood of x given by $N_x(r) := (x-r, x+r)\times(t-r, t+r)$. By definition of the weighted norm (Page 47 of Lieberman (1996)), we finally have the following α-uniform estimate on $N_x(d)$,

$$|V^{\alpha,\beta}|_{2.5,N_x(d)} \leq |V^{\alpha,\beta}|^*_{2.5,Q^{x/2}_\beta} \leq K$$

Therefore, Arzela–Ascoli Theorem implies that there exists a subsequence of $\{V^{\alpha,\beta} : \alpha \in (0, x/2)\}$, which is uniformly convergent to a function u on $N_x(d)$, i.e.

$$V^{\alpha,\beta} \to u \text{ as } \alpha \to 0, \quad \text{uniformly on } N_x(d).$$

The uniform convergence implies that the limit function is $u \in C^{2,1}(N_x(d))$. Using the facts of almost sure convergence $\tau^{\alpha,\beta} \to \tau^\beta$, together with dominated convergence theorem, one can check that

$$\begin{aligned}\lim_{\alpha\to 0} V^{\alpha,\beta}(x,t) &= \mathbb{E}_{x,t}[\lim_{\alpha\to 0} f^\beta(X(\tau^{\alpha,\beta}))] \\ &= \widetilde{V}^\beta(x,t), \text{ pointwisely.}\end{aligned}$$

Hence, $\widetilde{V}^\beta = u$ solves $(E)_\beta$ of (3.8) in the classical sense. By bounded convergence theorem, one can also show $\widetilde{V}^\beta(x,t) \in C(\overline{Q}_\beta)$ from the facts

$$\lim_{x\to\beta} \widetilde{V}^\beta(x,t) = 0, \lim_{x\to 0} \widetilde{V}^\beta(x,t) = f(0), \lim_{t\to T} \widetilde{V}^\beta(x,t) = f(x), \tag{4.13}$$

Thus, we conclude $\widetilde{V}^\beta$ is the classical solution of (3.8). Moreover, strong solution satisfies maximum principle, and hence the uniqueness follows from Corollary 2.4 of Lieberman (1996). □

Now, we are ready to prove Theorem 3.4.

Proof [**Proof of Theorem 3.4**] $\widetilde{V}^\beta$ is the unique solution of (3.8) by Lemma 4.7. Fix $(x_0, t_0) \in Q$ and $\beta > 2x_0$. We will use comparison principle of Lemma 4.4 twice to obtain the desired results. Define $\varphi_2 : \overline{Q} \mapsto \mathbb{R}$ by

$$\varphi_2(x,t) = f(x)\mathbf{1}_{\{t=T\}}.$$

Since $\varphi_2(x,t) \leq \widetilde{V}^\beta$ on $\partial^* Q_{\beta/2}$, we can apply Lemma 4.4 on $\overline{Q}_{\beta/2}$ to obtain $V^{\beta/2,\varphi_2}(x_0,t_0) \leq \widetilde{V}^\beta(x_0,t_0)$. Similarly, since $\widetilde{V}^\beta \leq \varphi_2(x,t)$ on $\partial^* Q_\beta$ by its definition, we apply Lemma 4.4 on $\overline{Q}_\beta$ to obtain $V^{\beta,\varphi_2}(x_0,t_0) \geq \widetilde{V}^\beta(x_0,t_0)$. Thus, we have inequality

$$V^{\beta/2,\varphi_2}(x_0,t_0) \leq \widetilde{V}^\beta(x_0,t_0) \leq V^{\beta,\varphi_2}(x_0,t_0). \tag{4.14}$$

Taking $\lim_{\beta\to\infty}$ in the above inequality and using Theorem 3.1, all three terms shall converge to the same value $V(x_0,t_0)$. The rate of the convergence is the combined result of (4.14) and (3.3). □

5. Further remarks

This paper studies an approximation to the smallest hedging price of European options using rebate options. From the mathematical point of view, this work concerns an approximation of the value function V of (2.3) by truncating the domain Q and imposing suitable Cauchy-Dirichlet data g.

The main result on convergence, Theorem 3.1, provides that, if the function g is chosen to satisfy sub-linear growth in x uniformly in $t \in [0, T)$, then the truncated value $V^{\beta,g}$ converges to V. This enables practitioners to adopt EM methods on a big enough truncated domain Q_β to get a value close to V, as demonstrated in Example 3.3.

On the other hand, to adopt numerical PDE techniques, continuous Cauchy-Dirichlet data are desired to get a good approximation. However, if the payoff f is given with linear growth, g is taken to be of sub-linear growth in x for the purpose of convergence by Theorem 3.1, then it is not possible to have a continuous solution of the Black–Scholes PDE. Alternatively, we provide continuous Cauchy-Dirichlet data by modifying the terminal payoff appropriately.

Finally, based on the convergence result, it will be also very interesting to compare different kinds of Monte Carlo method, such as the Euler method, Markov chain approximation and its counterpart the finite difference method under the lack of the uniqueness, see Kushner (1990), Jin *et al.* (2012), Jin *et al.* (2011), Jiang and Dai (2004), Yin *et al.* (2009a), Yin *et al.* (2009b) and others.

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