On Uniqueness and Existence of Viscosity Solutions of Fully Nonlinear Second-Order Elliptic PDE's

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Abstract

We prove several comparison and existence theorems for viscosity solutions of fully nonlinear degenerate elliptic equations. One of them extends some recent uniqueness results by Jensen. Some establish the uniqueness of solutions for second-order Isaacs' equations and hence include the uniqueness results for Bellman equations by P.-L. Lions. Our comparison results apply even for discontinuous solutions and so Perron's method readily yields the existence of continuous solutions.

1. Introduction

In this paper we are concerned with fully nonlinear second-order degenerate elliptic partial differential equations (PDE in short) of the form

$$F(x, u, Du, D^2u) = 0 \tag{1.1}$$

in an open subset Ω of $\mathbb{R}^n$. F is a real-valued function on $\Gamma = \Omega \times \mathbb{R} \times \mathbb{R}^n \times S^n$, where S^n denotes the space of $n \times n$ real symmetric matrices equipped with the usual ordering, u stands for the real-valued unknown function on Ω, $Du = (\partial u/\partial x_i)$ and $D^2u = (\partial^2 u/\partial x_i \partial x_j)$ denote, respectively, the gradient and Hessian of u. The operator (or function) F and equation (1.1) are called *degenerate elliptic* if, whenever $(x, r, p, X) \in \Gamma$, $Y \in S^n$ and $Y \geqq O$,

$$F(x, r, p, X + Y) \leqq F(x, r, p, X). \tag{1.2}$$

We remark that this "ellipticity" is opposite to the usual definition of ellipticity. For example, the operator $-\Delta$, the minus Laplacian, is degenerate elliptic in our notation. We also remark that any first-order PDE is degenerate elliptic. Because of this strong degeneracy we cannot expect PDE's (1.1) to have C^2 solutions in general. This requires us to adapt an appropriate notion of weak solutions.

The notion of viscosity solution was introduced for nonlinear first-order PDE's by M. G. Crandall and P.-L. Lions [7]. Its basic idea is to put the derivatives on a test function via the maximum principle and originates in earlier papers by L. C. Evans [9], [10]. This notion has been quite successful in the existence and uniqueness theory of solutions of Hamilton–Jacobi equations. For this we refer to G. Barles [2], M. G. Crandall, L. C. Evans, and P.-L. Lions [5],

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M. G. Crandall, H. Ishii and P.-L. Lions [6], M. G. Crandall and P.-L. Lions [7], [8], H. Ishii [14]–[17], P. E. Souganidis [28] and references therein. Shortly afterwards P.-L. Lions [24] extended the definition of viscosity solution to second-order PDE's. This is the notion of weak solutions we shall use in this paper. Moreover, in [24] he established fairly general uniqueness results for Bellman equations, i.e., second-order degenerate elliptic PDE's of the form

$$\sup L^{\alpha}(x, u, Du, D^2u) = 0, \tag{1.3}$$

where the supremum is taken over a given set $\mathscr{A}$ with respect to α and L^{α} is a linear degenerate elliptic operator for $\alpha \in \mathscr{A}$. We refer to W. H. Fleming and R. W. Rishel [12], N. V. Krylov [20] and P.-L. Lions [23], [24] for the connection of Bellman equations with optimal control.

The proof of uniqueness results in [24] is based on the theory of stochastic optimal control and its main idea is to identify the viscosity solution of (1.3) with the value function of the associated optimal control problem. It has been a challenging open problem, since [24] appeared, to establish an extension of the uniqueness results covering PDE's (1.1) with F which does not have any convexity as that in (1.3) and thus applies to Isaacs' equations (see equation (1.4) below). It has also been an interesting problem to prove the uniqueness results without using stochastic methods. A few attempts in this direction were made. We refer to S. Aizawa and Y. Tomita [1] and N. Yamada [29]. The proof of uniqueness results in [1] and [29] relies on the existence of C^2 approximate solutions. Thus it is unlike the proof in the case of first-order PDE's and these uniqueness results can only be applied to PDE's of special types. It was quite recent that R. Jensen [19] came up with some important ideas to solve the open problem and improved the situation. In fact, he proved that if F is continuous, independent of x, and degenerate elliptic, and the function $r \to F(x, r, p, X) - \gamma r$ is nondecreasing on $\mathbb{R}$ for some $\gamma > 0$ and all (x, p, X), then the Lipschitz continuous viscosity solution of PDE's (1.1) attaining prescribed Dirichlet data on $\partial\Omega$ is unique. Although this result has obvious deficiencies, he made two important observations. One is that any Lipschitz continuous viscosity subsolution of (1.1) can be approximated by a semi-convex function which is "almost" a viscosity subsolution of (1.1). The other is that if Ω is bounded, $u \in C(\overline{\Omega})$ is semi-convex in Ω and $\max_{\partial\Omega} u < \max_{\Omega} u = u(y)$ for some $y \in \Omega$, then for any $\varepsilon > 0$ there are points $p \in \mathbb{R}^n$ and $z \in \Omega$ satisfying $|p| \leqq \varepsilon$ such that the function $x \to u(x) + \langle p, x \rangle$ on $\overline{\Omega}$ attains its maximum at z and has the second differential at z.

Our purpose here is to refine Jensen's method and to establish an extension of the uniqueness results [24] due to P.-L. Lions. It turns out that the Lipschitz continuity (and even the continuity) requirement of viscosity solutions is unnecessary in the above uniqueness result due to R. Jensen. Also, our results cover completely the uniqueness results in P.-L. Lions [24] for the Dirichlet problem for

Bellman equations and, moreover, apply to second-order Isaacs' equations, i.e., second-order degenerate elliptic PDE's of the form

$$\inf_{\beta \in \mathscr{B}} \sup_{\alpha \in \mathscr{A}} L^{\alpha\beta}(x, u, Du, D^2u) = 0, \tag{1.4}$$

where $\mathscr{A}$ and $\mathscr{B}$ are given sets and $L^{\alpha\beta}$ is a linear degenerate elliptic operator for $\alpha \in \mathscr{A}$ and $\beta \in \mathscr{B}$. The reader should consult the role of Isaacs' equations in differential games, e.g. M. Nisio [26], E. N. Barron, L. C. Evans, R. Jensen [3], L. C. Evans, P. Souganidis [11] and references therein. Moreover, our proof is purely analytical and does not rely on techniques from stochastic optimal control and differential games.

The key idea we shall use here is to introduce a new degenerate elliptic PDE on $\Omega \times \Omega$ the operator of which is "completely degenerate" for functions of the form $(x, y) \to \phi(x - y)$ and which the function $(x, y) \to u(x) - v(y)$ satisfies for any solutions u and v of the original PDE. To illustrate this idea, let us take the uniformly elliptic operator $-\Delta$. Let f be a real-valued function on Ω. If u and v are C^2 solutions of $-\Delta u = f(x)$ in Ω, then the function $w(x, y) = u(x) - v(y)$ satisfies

$$-\Delta_x w - 2\Delta_{x,y} w - \Delta_y w = f(x) - f(y) \quad \text{in} \quad \Omega \times \Omega,$$

where

$$\Delta_x = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}, \quad \Delta_{x,y} = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i \partial y_i} \quad \text{and} \quad \Delta_y = \sum_{i=1}^{n} \frac{\partial^2}{\partial y_i^2}.$$

If we set $\phi(x, y) = \psi(x - y)$ for $\psi \in C^2(\mathbb{R}^n)$, then

$$\Delta_x \phi + 2\Delta_{x,y} \phi + \Delta_y \phi = 0 \quad \text{on} \quad \mathbb{R}^{2n}.$$

Moreover, the operator $-\Delta_x - 2\Delta_{x,y} - \Delta_y$ is degenerate elliptic on $\Omega \times \Omega$. A similar idea is found in A. Brandt [4]. Of course, the situation is more complicated when we deal with viscosity solutions and elliptic operators which depend on the space variable x. This idea is precisely formulated in Proposition 5.1. With this proposition in hand our approach to the existence and uniqueness problem for (1.1) parallels that for first-order PDE's (see e.g. [6], [17]).

The plan of this paper is as follows: In Section 2, we recall the definition of viscosity solutions. We present our main results on bounded domains Ω in Section 3. Section 4 is devoted to the study of ε-envelopes of functions which are one of the key ideas due to R. Jensen. In Section 5, we give a proposition which enables us to prove our main results similarly to the case of first-order PDE's. In Section 6, we give the proofs of the main results on bounded domains. In Section

7, we formulate and prove the extensions of the results in Section 3 to unbounded domains Ω.

It is not hard to formulate, in the spirit of this paper, the existence and uniqueness results for the Cauchy problem

$$\frac{\partial u}{\partial t} + F(t, x, u, Du, D^2u) = 0 \quad \text{in} \quad (0,T) \times \Omega,$$

$$u(0, x) = u_0(x) \qquad \text{for} \quad x \in \Omega.$$

Here $T > 0$, u_0: $\Omega \to \mathbb{R}$ and F: $(0,T) \times \Omega \times \mathbb{R} \times \mathbb{R}^n \times S^n \to \mathbb{R}$ are given and u: $[0,T] \times \Omega \to \mathbb{R}$ is the unknown. However, we do not treat this subject here and will come back to it in a forthcoming paper.

We use the following notation: For $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean norm of x. For x, $y \in \mathbb{R}^n$, $\langle x, y\rangle$ denotes the Euclidean inner product of x and y.

For $x \in \mathbb{R}^n$ and $r \geqq 0$, $B(x; r)$ stands for the closed ball of radius r and center x. $M(m,n)$ denotes the space of real $m \times n$ matrices. For $A \in M(m,n)$, ${}^TA$ and $\|A\|$ stand, respectively, for the transpose of A and the norm of A, i.e., $\|A\| = (\Sigma_{i=1}^m \Sigma_{j=1}^n a_{ij}^2)^{1/2}$ if $A = (a_{ij})$. For $A \in M(n,n)$, $\operatorname{Tr} A$ denotes the trace of A. Finally the unit and zero matrices will be denoted by I and O, respectively.

After having completed this work, the author learned that R. Jensen, P.-L. Lions, and P. E. Souganidis [30] had obtained a comparison result similar to Theorem 7.1 for bounded, uniformly continuous, viscosity sub- and supersolutions. We should also mention the work by P.-L. Lions and P. E. Souganidis [31] which reviews recent developments closely related to the subject of this paper and clarifies the role of the convexity of F in the proof of the uniqueness results in [24].

2. Definition of a Viscosity Solution

We recall the definition of a viscosity solution and some of its basic properties in this section.

Let Ω be an open subset of $\mathbb{R}^n$. For any function u: $\Omega \to \mathbb{R}$ we define u^*: $\overline{\Omega} \to \mathbb{R} \cup \{-\infty, \infty\}$ by

$$\text{(2.1)} \qquad u^*(x) = \lim_{r \downarrow 0} \sup\{u(y) \colon y \in B(x; r) \cap \Omega\} \quad \text{for} \quad x \in \overline{\Omega},$$

and $u_* : \overline{\Omega} \to \mathbb{R} \cup \{-\infty, \infty\}$ by

$$\text{(2.2)} \qquad u_*(x) = \lim_{r \downarrow 0} \inf\{u(y) \colon y \in B(x; r) \cap \Omega\} \quad \text{for} \quad x \in \overline{\Omega}.$$

Obviously $u^* \geqq u \geqq u_*$ on Ω, u^* is upper semi-continuous (u.s.c. in short) on $\overline{\Omega}$ and u_* is lower semi-continuous (l.s.c. in short) on $\overline{\Omega}$. Note also that $u_* =$

$(-u)^*$ and that if u is u.s.c. at $x \in \Omega$, then $u^*(x) = u(x)$. Functions u^* and u_* are called, respectively, the *u.s.c. and l.s.c. envelopes* of u.

Let $F\colon \Gamma = \Omega \times \mathbb{R} \times \mathbb{R}^n \times S^n \to \mathbb{R}$ be continuous, and consider the nonlinear PDE

$$F(x, u, Du, D^2u) = 0 \quad \text{in} \quad \Omega. \tag{2.3}$$

Any function $u\colon \Omega \to \mathbb{R}$ is called a *viscosity subsolution* of (2.3) if $u^*(x) < \infty$ for $x \in \overline{\Omega}$ and if, whenever $\phi \in C^2(\Omega)$, $y \in \Omega$ and $(u^* - \phi)(y) = \max_\Omega(u^* - \phi)$,

$$F(y, u^*(y), D\phi(y), D^2\phi(y)) \leqq 0.$$

Similarly any function $u\colon \Omega \to \mathbb{R}$ is called a *viscosity supersolution* of (2.3) if $u_*(x) > -\infty$ for $x \in \overline{\Omega}$ and if, whenever $\phi \in C^2(\Omega)$, $y \in \Omega$ and $(u_* - \phi)(y) = \min_\Omega(u_* - \phi)$,

$$F(y, u_*(y), D\phi(y), D^2\phi(y)) \geqq 0.$$

We call a function $u\colon \Omega \to \mathbb{R}$ a *viscosity solution* of (2.3) if it is both a viscosity sub- and supersolution of (2.3). The requirement for any subsolution (respectively supersolution) u to satisfy $u^*(x) < \infty$ (respectively $u_*(x) > -\infty$) for $x \in \overline{\Omega}$ may be a little too stringent. This requirement is, however, convenient for us to state comparison results.

We note that $u\colon \Omega \to \mathbb{R}$ is a viscosity subsolution of (2.3) if and only if $v = -u$ is a viscosity supersolution of

$$-F(x, -v, -Dv, -D^2v) = 0 \quad \text{in} \quad \Omega.$$

It is easy to see that a C^2 function is a classical subsolution of (2.3) if and only if it is a viscosity subsolution of (2.3). Moreover, if a viscosity subsolution u of (2.3) has the first and second superdifferentials at a point $y \in \Omega$, i.e., if

$$u^*(x) \leqq u^*(y) + \langle p, x - y\rangle + \tfrac{1}{2}\langle X(x - y), x - y\rangle + o(|x - y|^2)$$

holds for some $p \in \mathbb{R}^n$ and $X \in S^n$ as $x \to y$, then we have

$$F(y, u^*(y), p, X) \leqq 0.$$

For a proof of this assertion we refer to the proof of Lemma 2.2 in L. C. Evans [9].

Before closing this section we give three basic properties of viscosity solutions.

PROPOSITION 2.1 (stability). *For $k \in \mathbf{N}$ let F_k be a continuous function on Γ and u_k a continuous function on $\overline{\Omega}$. Assume that u_k is a viscosity subsolution of*

$F_k(x, u, Du, D^2u) = 0$ in Ω for $k \in \mathbf{N}$ and that $F_k(x, r, p, X) \to F(x, r, p, X)$ uniformly on compact subsets of Γ and $u_k(x) \to u(x)$ uniformly on $\bar{\Omega}$ for some functions F and u as $k \to \infty$. Then u is a viscosity subsolution of (2.3) *with this F.*

For a proof of this proposition we refer to P.-L. Lions [24]. See also [9] where the above result is implicit.

PROPOSITION 2.2. *Let F be a continuous function on Γ. Let S be a nonempty family of viscosity subsolutions of* (2.3). *Define a function u on Ω by*

$$u(x) = \sup\{v\{(x): v \in S\} \quad \textit{for} \quad x \in \Omega\}.$$

Assume $u^*(x) < \infty$ for $x \in \bar{\Omega}$. Then u is a viscosity subsolution of (2.3).

PROPOSITION 2.3 (existence of solution). *Let F be a continuous function on Γ. Suppose that there is a viscosity subsolution f and a viscosity supersolution g of* (2.3) *satisfying $f \leqq g$ on Ω and $f, g \in C(\bar{\Omega})$. Then there is a viscosity solution u of* (2.3) *which satisfies $f \leqq u \leqq g$ on Ω.*

The requirement that $f, g \in C(\bar{\Omega})$ is stronger than what is actually needed. We refer to [17], [18] for this and the above two propositions.

3. Main Results on Bounded Domains

In this section we state our main theorems which will be proved and extended in Sections 6 and 7, respectively.

We call a continuous function m: $[0, \infty) \to [0, \infty)$ a *modulus* if $m(0) = 0$ and if it is nondecreasing and concave on $[0, \infty)$.

We first consider the case where F is independent of the space variable x.

THEOREM 3.1. *Let Ω be a bounded open subset of $\mathbb{R}^n$ and let F: $\mathbb{R} \times \mathbb{R}^n \times S^n \to \mathbb{R}$ be continuous and degenerate elliptic. Assume the function $r \to F(r, p, X)$ is nondecreasing on $\mathbb{R}$ for $(p, X) \in \mathbb{R}^n \times S^n$. Let u and v be, respectively, viscosity sub- and supersolutions of*

$$u + F(u, Du, D^2u) = 0 \quad \textit{in} \quad \Omega. \tag{3.1}$$

Assume $u^*(x) \leqq v_*(x)$ for $x \in \partial\Omega$. Then $u^* \leqq v_*$ on Ω.

Remarks. (a) This theorem obviously yields a uniqueness result for solutions of the Dirichlet problem to (3.1). The above theorem thus generalizes [19], Theorem 3.1, (α). (b) Under the assumptions of this theorem, there is a modulus m_0 which satisfies $u^*(x) - v_*(y) \leqq m_0(|x - y|)$ for $(x, y) \in \partial(\Omega \times \Omega)$. From the conclusion of the theorem we find that there is a modulus m for which $u^*(x) - v_*(y) \leqq m(|x - y|)$ for $(x, y) \in \Omega \times \Omega$. As the proof in Section 6 shows, one can take $m = m_0$.

We have the following existence result.

THEOREM 3.2. *Assume the hypotheses of Theorem* 3.1 *concerning* Ω *and* F. *Suppose there is a viscosity subsolution* f *and a viscosity supersolution* g *of* (3.1). *Suppose moreover that* $f, g \in C(\overline{\Omega})$, $f \leqq g$ *in* Ω *and* $f = g$ *on* $\partial\Omega$. *Then there is a viscosity solution* u *of* (3.1) *satisfying* $u \in C(\overline{\Omega})$ *and* $f \leqq u \leqq g$ *on* $\overline{\Omega}$.

Next we deal with a second-order Isaacs' equation. Let $\mathscr{A}$ and $\mathscr{B}$ be given nonempty sets. Let Ω be an open subset of $\mathbb{R}^n$. We write $\Lambda = \Omega \times \mathscr{A} \times \mathscr{B}$. Let $\Sigma = (\sigma_{ij})$: $\Lambda \to M(m, n)$, $b = (b_i)$: $\Lambda \to \mathbb{R}^n$, c: $\Lambda \to \mathbb{R}$ and d: $\Lambda \to \mathbb{R}$ be given. We define $A = (a_{ij})$: $\Lambda \to S^n$ by

$$A(x, \alpha, \beta) = {}^T\Sigma(x, \alpha, \beta)\Sigma(x, \alpha, \beta) \quad \text{for} \quad (x, \alpha, \beta) \in \Lambda$$

and F: $\Gamma \to \mathbb{R}$ by

$$\begin{aligned} F(x, r, p, X) = &\inf_{\beta \in \mathscr{B}} \sup_{\alpha \in \mathscr{A}} \{ -\operatorname{Tr} A(x, \alpha, \beta) X + \langle b(x, \alpha, \beta), p \rangle \\ &+ c(x, \alpha, \beta) r + d(x, \alpha, \beta) \} \quad \text{for} \quad (x, r, p, X) \in \Gamma. \end{aligned} \tag{3.2}$$

We use the following assumptions:

(A1) Functions A, b, c and d are bounded on Λ.

(A2) Functions Σ and b are Lipschitz continuous in the space variable x; more precisely,

$$\sup \frac{\|\Sigma(x, \alpha, \beta) - \Sigma(y, \alpha, \beta)\|}{|x - y|} < \infty,$$

and

$$\sup \frac{|b(x, \alpha, \beta) - b(y, \alpha, \beta)|}{|x - y|} < \infty,$$

where the supremum is taken for all $(x, \alpha, \beta), (y, \alpha, \beta) \in \Lambda$ with $x \neq y$.

(A3) Functions c and d are continuous in the variable x; more precisely,

$$\lim_{r \downarrow 0} \sup\{ |f(x, \alpha \beta) - f(y, \alpha, \beta)| : (x, \alpha, \beta), (y, \alpha, \beta) \in \Lambda, |x - y| \leqq r \} = 0$$

holds for $f = c, d$.

(A4) $\inf\{ c(x, \alpha, \beta) : (x, \alpha, \beta) \in \Lambda \} > 0$.

We note that F is well defined, continuous and degenerate elliptic under these assumptions.

THEOREM 3.3. *Assume* Ω *is bounded and that* (A1)–(A4) *hold. Let* u *and* v *be, respectively, viscosity sub- and supersolutions of*

$$F(x, y, Du, D^2u) = 0 \quad \textit{in} \quad \Omega. \tag{3.3}$$

Assume $u^*(x) \leqq v_*(x)$ *for* $x \in \partial\Omega$. *Then* $u^* \leqq v_*$ *on* Ω.

Remarks. (i) Under the hypotheses of this theorem, there is a constant $M > 0$ for which $u(x) \leqq M$ and $v(x) \geqq -M$ for $x \in \Omega$ as well as a modulus m_0 for which $u^*(x) - v_*(y) \leqq m_0(|x-y|)$ for $(x, y) \in \partial(\Omega \times \Omega)$. Also, the conclusion implies that there is a modulus m such that $u^*(x) - v_*(y) \leqq m(|x-y|)$ for $x, y \in \Omega$. Our proof in Section 6 shows that the modulus m can be chosen to depend on u and v only through M and m_0. (ii) The following example shows that the Lipschitz continuity assumption on Σ in (A2) is in a sense optimal in order to claim a comparison assertion for (3.3). The example is a linear elliptic PDE

$$u - \operatorname{Tr} A(x) D^2 u = 0 \quad \text{in} \quad \mathbb{R}^n, \tag{3.4}$$

with

$$A(x) = \frac{|x|^{2-\alpha}}{(n-1)^{\alpha}}\left(I - \frac{1}{|x|^2} x \otimes x\right)$$

for some $\alpha \in (0,2)$, where $x \otimes x$ denotes the matrix $(x_i x_j)_{1 \leqq i, j \leqq n}$. If we define

$$\Sigma(x) = \frac{|x|^{1-\alpha/2}}{((n-1)\alpha)^{1/2}}\left(I - \frac{1}{|x|^2} x \otimes x\right) \quad \text{for} \quad x \in \mathbb{R}^n,$$

then $\Sigma(x) \in S^n$ and $\Sigma^2(x) = A(x)$ for $x \in \mathbb{R}^n$, and $\Sigma \in C^{0,1-\alpha/2}(\mathbb{R}^2)$. Define $u, v \colon \mathbb{R}^n \to \mathbb{R}$ by

$$u(x) = \exp|x|^{\alpha} \quad \text{and} \quad v(x) = \begin{cases} \exp|x|^{\alpha} & \text{if} \quad x \neq 0, \\ 0 & \text{if} \quad x = 0. \end{cases}$$

A simple calculation shows that

$$u(x) - \operatorname{Tr} A(x) D^2 u(x) = v(x) - \operatorname{Tr} A(x) D^2 v(x) = 0$$

for $x \neq 0$. Noting that $u \in C(\mathbb{R}^n)$, v is l.s.c. in $\mathbb{R}^n$, $D^+u(0) = \varnothing$ and $A(0) = 0$, we conclude that u and v are both viscosity solutions of (3.4). Finally, notice that if $\Omega \subset \mathbb{R}^n$ is an open subset and $0 \in \Omega$, then $u = v$ on $\partial\Omega$ and $u \nleqq v$ in Ω. Thus we see that if we replace the Lipschitz continuity on Σ in (A2) by the Hölder continuity with exponent less than one, then Theorem 3.3 does not hold any more.

As an existence result we have:

THEOREM 3.4. *Assume the hypotheses of Theorem* 3.3. *Suppose that there is a viscosity subsolution f and a viscosity supersolution g of* (3.3). *Assume in addition that $f \leqq g$ in Ω and $f = g$ on $\partial\Omega$ and that f and g are continuous on $\overline{\Omega}$. Then there is a viscosity solution u of* (3.3) *satisfying*

$$u \in C(\overline{\Omega}) \quad \textit{and} \quad f \leqq u \leqq g \quad \textit{on} \quad \overline{\Omega}.$$

4. ε-Envelopes

We shall define the upper and lower ε-envelopes of functions and show some of their properties. In proving the uniqueness result, R. Jensen [19] used ε-envelopes and showed their importance in the study of viscosity solutions. He restricted himself, however, to the study of ε-envelopes for Lipschitz continuous functions.

Let Ω be an open subset of $\mathbb{R}^n$ and u a real-valued u.s.c. function on $\bar{\Omega}$. For $\varepsilon > 0$ we define a function $u^\varepsilon\colon \bar{\Omega} \to \mathbb{R}$ which we call the *upper ε-envelope* of u by

$$(4.1)\qquad u^\varepsilon(x) = \max\left\{u(y) + \left(\varepsilon^2 - |x-y|^2\right)^{1/2}\colon y \in B(x;\varepsilon) \cap \bar{\Omega}\right\}.$$

Observe that $u(x) + \varepsilon \leqq u^\varepsilon(x) \leqq \max\{u(y)\colon y \in B(x;\varepsilon) \cap \bar{\Omega}\} + \varepsilon$ for $x \in \bar{\Omega}$. Moreover, the upper ε-envelopes have the following properties whose proof we leave to the reader.

PROPOSITION 4.1. *If u is a real-valued u.s.c. function on $\bar{\Omega}$, then the upper ε-envelope u^ε of u is u.s.c. on $\bar{\Omega}$ for $\varepsilon > 0$ and $u^\varepsilon(x) \downarrow u(x)$ for $x \in \bar{\Omega}$ as $\varepsilon \downarrow 0$.*

The lower ε-envelope u_ε of a real-valued l.s.c. function u on $\bar{\Omega}$ is defined similarly on $\bar{\Omega}$ by

$$(4.2)\qquad u_\varepsilon(x) = \min\left\{u(y) - \left(\varepsilon^2 - |x-y|^2\right)^{1/2}\colon y \in B(x;\varepsilon) \cap \bar{\Omega}\right\}.$$

We remark that $u_\varepsilon = -(-u)^\varepsilon$; this formula converts a property of upper ε-envelopes into the corresponding property of lower ε-envelopes.

The above definition of ε-envelopes differs from that of corresponding ones in [19] in appearance but actually they are equivalent; the proof of this we leave to the reader.

The values $u^\varepsilon(x)$ and $u_\varepsilon(x)$ depend on u only through its values on $B(x;\varepsilon) \cap \bar{\Omega}$. Therefore, if u is u.s.c. (respectively l.s.c.) on Ω, then the upper ε-envelope u^ε (respectively the lower ε-envelope u_ε) is defined on

$$\Omega_\varepsilon = \{x \in \Omega\colon \operatorname{dis}(x, \partial\Omega) > \varepsilon\}.$$

The next proposition was first proved by R. Jensen [19] in the case when subsolutions are Lipschitz continuous.

PROPOSITION 4.2. *Let F be a real-valued continuous function on Γ. Let u be an u.s.c. viscosity subsolution of*

$$(4.3)\qquad F(x, u, Du, D^2u) = 0 \quad \textit{in} \quad \Omega.$$

Let $\varepsilon > 0$, and define a continuous function F_ε on $\Gamma_\varepsilon = \Omega_\varepsilon \times \mathbb{R} \times \mathbb{R}^n \times S^n$ by

$$F_\varepsilon(x, r, p, X) = \min\{F(y, s, p, X)\colon (y, s) \in B((x, r); \varepsilon)\}$$

for $(x, r, p, X) \in \Gamma_\varepsilon$. Then the upper ε-envelope u^ε of u is a viscosity subsolution of

$$F_\varepsilon(x, u, Du, D^2u) = 0 \quad in \quad \Omega_\varepsilon.$$

Proof: Let $\phi \in C^2(\Omega_\varepsilon)$ and $x_0 \in \Omega_\varepsilon$ satisfy

$$(u^\varepsilon - \phi)(x_0) = \max(u^\varepsilon - \phi).$$

By the definition of the upper ε-envelope, we see that

$$u^\varepsilon(x_0) = u(y_0) + \left(\varepsilon^2 - |x_0 - y_0|^2\right)^{1/2} \quad \text{for some} \quad y_0 \in B(x_0; \varepsilon) \subset \Omega,$$

and hence

$$u(y) + \left(\varepsilon^2 - |x - y|^2\right)^{1/2} - \phi(x) \leqq u(y_0) + \left(\varepsilon^2 - |x_0 - y_0|^2\right)^{1/2} - \phi(x_0)$$

for all $x \in \Omega_\varepsilon$ and $y \in B(x; \varepsilon)$. We choose $x = y - y_0 + x_0$ in this last inequality to find that the function $y \to u(y) - \phi(y - y_0 + x_0)$ attains a maximum at $y = y_0$. Therefore, by the definition of a viscosity solution, we have

$$F\left(y_0, u(y_0), D\phi(x_0), D^2\phi(x_0)\right) \leqq 0.$$

Since $|y_0 - x_0|^2 + (u(y_0) - u^\varepsilon(x_0))^2 = \varepsilon^2$, this concludes the proof.

PROPOSITION 4.3. *Let $\varepsilon > 0$ and $\phi \in C^1(\Omega)$. Let u be an u.s.c. function on Ω and u^ε the upper ε-envelope of u. Assume $u^\varepsilon - \phi$ attains its maximum at a point $x_0 \in \Omega_\varepsilon$. Then u^ε is semi-convex in a neighborhood of x_0. More precisely, there is a constant C, depending only on ε and $|D\phi(x_0)|$, and a neighborhood U of x_0 such that the function $x \to u^\varepsilon(x) + C|x|^2$ is convex on U.*

Proof: We choose $y_0 \in B(x_0, \varepsilon)$ so that

$$u^\varepsilon(x_0) = u(y_0) + \left(\varepsilon^2 - |x_0 - y_0|^2\right)^{1/2}, \tag{4.4}$$

and observe that (x_0, y_0) is a maximum point of the function $(x, y) \to u(y) + (\varepsilon^2 - |x - y|^2)^{1/2} - \phi(x)$.

First of all we show that $|x_0 - y_0| < \varepsilon$. To do this we suppose $|x_0 - y_0| = \varepsilon$ and get a contradiction. The function ψ on $[0, 1]$ defined by

$$\psi(t) = u(y_0) + \left(\varepsilon^2 - (1 - t)^2|y_0 - x_0|^2\right)^{1/2} - \phi\left(x_0 + t(y_0 - x_0)\right) \tag{4.5}$$

has its maximum at $t = 0$. Therefore,

$$\frac{\psi(t) - \psi(0)}{t} \leqq 0 \quad \text{for} \quad 0 < t \leqq 1,$$

and hence, sending $t \downarrow 0$, we obtain a contradiction.

The inequality

$$u^\varepsilon(x) \geqq u(y_0) + \left(\varepsilon^2 - |x - y_0|^2\right)^{1/2} \quad \text{for} \quad x \in B(x_0; r),$$

where $r = \varepsilon - |x_0 - y_0|$, yields

$$\liminf_{x \to x_0} u^\varepsilon(x) \geqq u^\varepsilon(x_0).$$

This together with the upper semicontinuity of u^ε proves that u^ε is continuous at x_0.

Now the function ψ can be defined near $t = 0$ by (4.5) and attains a maximum at $t = 0$. Hence $\psi'(0) = 0$ and so, by a simple computation, we see that

$$y_0 = x_0 - \varepsilon\left(|D\phi(x_0)|^2 + 1\right)^{-1/2} D\phi(x_0).$$

This formula uniquely determines the point y_0 for which (4.4) is satisfied uniquely. In other words,

$$\text{(4.6)} \quad u(y) + \left(\varepsilon^2 - |x_0 - y|^2\right)^{1/2} < u^\varepsilon(x_0) \quad \text{for} \quad y \in B(x_0; \varepsilon) \setminus \{y_0\}.$$

Fix any δ such that $|x_0 - y_0| < \delta < \varepsilon$. The above inequality implies that there is a $\gamma > 0$ such that

$$\text{(4.7)} \quad \max\left\{u(y) + \left(\varepsilon^2 - |x - y|^2\right)^{1/2} : \delta \leqq |x - y| \leqq \varepsilon\right\} < u^\varepsilon(x) \qquad \text{for} \quad x \in B(x_0; \gamma).$$

Indeed, if this were false, then there would be sequences $\{x_k\}$ and $\{y_k\}$ of points of $\mathbb{R}^n$ such that

$$x_k \to x_0 \quad \text{as} \quad k \to \infty,$$

and

$$u^\varepsilon(x_k) = u(y_k) + \left(\varepsilon^2 - |x_k - y_k|^2\right)^{1/2}, \qquad \delta \leqq |x_k - y_k| \leqq \varepsilon.$$

Passing to the limit, we find that

$$u^\varepsilon(x_0) \leqq u(\bar{y}) + \left(\varepsilon^2 - |x_0 - \bar{y}|^2\right)^{1/2} \quad \text{for some} \quad \bar{y} \in \overline{B(x_0; \varepsilon) \setminus B(x_0; \delta)},$$

which contradicts (4.6) and hence proves (4.7) for some $\gamma > 0$. Fix $0 < \gamma < \frac{1}{3}(\varepsilon - \delta)$ so that (4.7) holds. Inequality (4.7) guarantees that, for $x \in B(x_0; \gamma)$,

$$u^\varepsilon(x) = \max\left\{ u(y) + \left(\varepsilon^2 - |x-y|^2\right)^{1/2} : y \in B(x;\delta) \right\},$$

and hence

$$u^\varepsilon(x) = \max\left\{ u(y) + \left(\varepsilon^2 - |x-y|^2\right)^{1/2} : y \in B(x_0;\delta+\gamma) \right\}.$$

Observe that if we define a function f_y on $B(x_0;\gamma)$ for $y \in B(x_0;\delta+\gamma)$ by

$$f_y(x) = u(y) + \left(\varepsilon^2 - |x-y|^2\right)^{1/2},$$

then

$$D^2 f_y(x) \geqq -\varepsilon^2\left\{\varepsilon^2 - (\delta+2\gamma)^2\right\}^{-3/2} I.$$

Thus we conclude that $x \to u^\varepsilon(x) + C|x|^2$, with $C = \frac{1}{2}\varepsilon^2\{\varepsilon^2 - (\delta+2\gamma)^2\}^{-3/2}$, is convex on $B(x_0;\gamma)$. Finally we remark that the choice of δ depends only on ε and $|D\phi(x_0)|$ while the positive number γ can be chosen as small as desired.

5. A Key Proposition

Let $F, G\colon \Gamma \to \mathbb{R}$ be continuous. For $\varepsilon > 0$ we set

$$\Omega_\varepsilon = \{ x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon \}$$

as in Section 4 and define continuous functions $F_\varepsilon, G^\varepsilon\colon \Omega_\varepsilon \times \mathbb{R} \times \mathbb{R}^n \times S^n \to \mathbb{R}$ by

$$F_\varepsilon(x,r,p,X) = \min\{ F(y,s,p,X) : (y,s) \in B((x,r);\varepsilon)\}$$

and

$$G^\varepsilon(x,r,p,X) = \max\{ G(y,s,p,X) : (y,s) \in B((x,r);\varepsilon)\}.$$

The following proposition will play an important role later on.

PROPOSITION 5.1. *Let u be an u.s.c. viscosity subsolution of*

$$F(x,u,Du,D^2u) = 0 \quad in \quad \Omega,$$

and v a l.s.c. viscosity supersolution of

$$G(x,v,Dv,D^2v) = 0 \quad in \quad \Omega.$$

Let $\varepsilon > 0$ and $\phi \in C^2(\Omega)$. Set $w(x,y) = u^\varepsilon(x) - v_\varepsilon(y)$ for $(x,y) \in \Omega_\varepsilon \times \Omega_\varepsilon$. Suppose $w - \phi$ attains a maximum at some $(\bar{x},\bar{y}) \in \Omega_\varepsilon \times \Omega_\varepsilon$. Then there is a *constant $C > 0$, depending only on ε and $|D\phi(\bar{x}, \bar{y})|$, and matrices $X, Y \in S^n$ such that*

$$-CI \leqq \begin{pmatrix} X & O \\ O & Y \end{pmatrix} \leqq D^2\phi(\bar{x}, \bar{y}),$$

$$F_\varepsilon(\bar{x}, u^\varepsilon(\bar{x}), D_x\phi(\bar{x}, \bar{y}), X) \leqq 0,$$

and

$$G^\varepsilon(\bar{y}, v_\varepsilon(\bar{y}), -D_y\phi(\bar{x}, \bar{y}), -Y) \geqq 0.$$

Moreover, the constant C remains bounded as ε and $|D\phi(\bar{x}, \bar{y})|$ range over compact subsets of $(0, \infty)$ and $\mathbb{R}$, respectively.

To prove this proposition, we need the next two lemmas.

LEMMA 5.2. *Let U be a bounded open subset of $\mathbb{R}^m$ and w a Lipschitz continuous function on $\bar{U}$. Assume $w(y) = \max_U w > \max_{\partial U} w$ for some $y \in U$ and that w is semi-convex on U. Then for any $\varepsilon > 0$ there are points $p \in \mathbb{R}^m$ and $z \in U$ satisfying $|p| \leqq \varepsilon$ such that the function $x \to w(x) + \langle p, x \rangle$ on U attains a maximum at z and has the second differential at z.*

This lemma is due to R. Jensen as already remarked in Section 1. For a proof of this lemma we refer to [19], Lemmas 3.10, 3.15.

LEMMA 5.3. *For any $C > 0$ the subset $K = \{X \in S^n\colon -CI \leqq X \leqq CI\}$ of S^n is compact.*

Proof: By the definition of ordering in S^n we have

$$-C|\xi|^2 \leqq \langle X\xi, \xi \rangle \leqq C|\xi|^2 \quad \text{for} \quad \xi \in \mathbb{R}^n. \tag{5.1}$$

Let $X \in K$ and let x_{ij} denote the (i, j) component of X for $i, j = 1, \cdots, n$. Let e_i denote the unit vector of $\mathbb{R}^n$ with unity as its i-th component. Take $\xi = e_i$ in (5.1), to find

$$|x_{ii}| \leqq C \quad \text{for} \quad i = 1, \cdots, n.$$

Next take $\xi = e_i + e_j$ with $i \neq j$ in (5.1), to find

$$2|x_{ij}| \leqq |x_{ii}| + |x_{jj}| + 2C \leqq 4C.$$

This guarantees

$$|x_{ij}| \leqq 2C \quad \text{for} \quad i, j = 1, \cdots, n,$$

and hence the desired compactness.

Proof of Proposition 5.1: Replacing ϕ by $(x, y) \to \phi(x, y) + |x - \bar{x}|^4 + |y - \bar{y}|^4$ if necessary, we may assume that $w - \phi$ attains a strict maximum over $\Omega_\varepsilon \times \Omega_\varepsilon$ at $(\bar{x}, \bar{y})$. By Proposition 4.3 we see that there is an open neighborhood U of $(\bar{x}, \bar{y})$ and a constant C_0 for which the function $(x, y) \to w(x, y) + \frac{1}{2}C_0(|x|^2 + |y|^2)$ is convex on U. Lemma 5.2 now guarantees that there are sequences $\{(x_k, y_k)\} \subset U$ and $\{p_k\}, \{q_k\} \subset \mathbb{R}^n$ having the properties (i)–(iii): (i) $(x_k, y_k) \to (\bar{x}, \bar{y})$ and $p_k, q_k \to 0$ as $k \to \infty$. (ii) w has the second differential at (x_k, y_k) for $k \in \mathbf{N}$. (iii) The function $(x, y) \to w(x, y) - \phi(x, y) - \langle p_k, x\rangle + \langle q_k, y\rangle$ attains its maximum over U at (x_k, y_k) for $k \in \mathbf{N}$. These yield: $D(w - \phi)(x_k, y_k) = (p_k, -q_k)$, $D^2(w - \phi)(x_k, y_k) \leqq O$,

$$F_\varepsilon\big(x_k, u^\varepsilon(x_k)\, Du^\varepsilon(x_k), D^2u^\varepsilon(x_k)\big) \leqq 0,$$

and

$$G^\varepsilon\big(y_k, v_\varepsilon(y_k), Dv_\varepsilon(y_k), D^2v_\varepsilon(y_k)\big) \geqq 0$$

for $k \in \mathbf{N}$. That is, setting

$$X_k = D^2u^\varepsilon(x_k) \quad \text{and} \quad Y_k = -D^2v_\varepsilon(y_k) \quad \text{for} \quad k \in \mathbf{N},$$

we have

$$C_0 I \leqq \begin{pmatrix} X_k & O \\ O & Y_k \end{pmatrix} \leqq D^2\phi(x_k, y_k),$$

$$F_\varepsilon\big(x_k, u^\varepsilon(x_k), D_x\phi(x_k, y_k) + p_k, X_k\big) \leqq 0,$$

and

$$G^\varepsilon\big(y_k, v_\varepsilon(y_k), -D_y\phi(x_k, y_k) + q_k, -Y_k\big) \geqq 0$$

for $k \in \mathbf{N}$.

Since $-CI \leqq X_k$, $Y_k \leqq CI$ for some $C > 0$ and all $k \in \mathbf{N}$, we see from Lemma 5.3 that there is an increasing sequence $\{k_j\} \subset \mathbf{N}$ and matrices X, $Y \in S^n$ such that $X_{k_j} \to X$, $Y_{k_j} \to Y$ in S^n as $j \to \infty$. Moreover, the semi-convexity of w implies the continuity of w on U. Thus, sending $k = k_j \to \infty$, we obtain

$$C_0 I \leqq \begin{pmatrix} X & O \\ O & Y \end{pmatrix} \leqq D^2\phi(\bar{x}, \bar{y}),$$

$$F_\varepsilon\big(\bar{x}, u^\varepsilon(\bar{x}), D_x\phi(\bar{x}, \bar{y}), X\big) \leqq 0,$$

and

$$G^\varepsilon\big(\bar{y}, v_\varepsilon(\bar{y}), -D_y\phi(\bar{x}, \bar{y}), -Y\big) \geqq 0.$$

Finally, we remark that the constant C_0 given in the proof of Proposition 4.3 has the desired dependence on ε and $|D\phi(\bar{x}, \bar{y})|$.

6. Proofs of Main Results on Bounded Domains

Proof of Theorem 3.1: Let Ω, F, u and v be as in Theorem 3.1. Since $u^*(x) < \infty$ and $v_*(x) > -\infty$ for $x \in \overline{\Omega}$ and since $\overline{\Omega}$ is compact, we find that $u^*(x) \leqq M$ and $v_*(x) \geqq -M$ for some constant $M > 0$ and all $x \in \overline{\Omega}$. Since the function $x \to F(x, 0, 0, O)$ is bounded on Ω, choosing M large enough, we may assume that the functions $w(x) = -M, M$ are, respectively, viscosity sub- and supersolutions of (3.1). Replacing u and v, respectively, by $(\max\{u, -M\})^*$ and $(\min\{v, M\})_*$, we may assume that u and v are defined as real-valued functions on $\overline{\Omega}$ and, respectively, u.s.c. and l.s.c. on $\overline{\Omega}$. Select a modulus m so that $u(x) - v(y) \leqq m(|x - y|)$ for $(x, y) \in \partial(\Omega \times \Omega)$. Set

$$w(x, y) = u(x) - v(y) \quad \text{for} \quad (x, y) \in \overline{\Omega} \times \overline{\Omega}.$$

We choose families $\{a_\lambda\}_{\lambda \in A}$ and $\{b_\lambda\}_{\lambda \in A}$ of positive numbers, with A a suitable index set, so that

$$m(r) = \inf\{a_\lambda r + b_\lambda \colon \lambda \in A\} \quad \text{for} \quad r \geqq 0.$$

Fix $\lambda \in A$ and $\delta > 0$, and define $\phi \in C^2(\mathbb{R}^{2n})$ by

$$\phi(x, y) = a_\lambda\big(|x - y|^2 + \delta\big)^{1/2} + b_\lambda \quad \text{for} \quad x, y \in \mathbb{R}^n.$$

We want to prove that $w \leqq \phi$ on $\Omega \times \Omega$, which obviously ensures our assertion. To do this, we suppose that $\sup_{\Omega \times \Omega}(w - \phi) > 0$, and then get a contradiction. We set

$$w^\varepsilon(x, y) = u^\varepsilon(x) - v_\varepsilon(y) \quad \text{for} \quad (x, y) \in \overline{\Omega} \times \overline{\Omega} \quad \text{and} \quad \varepsilon > 0.$$

We know by Proposition 4.1 that $w^\varepsilon(z) \downarrow w(z)$ for $z \in \overline{\Omega} \times \overline{\Omega}$ as $\varepsilon \downarrow 0$ and that w^ε is u.s.c. on $\overline{\Omega} \times \overline{\Omega}$. Hence we see by Dini's lemma that

$$\Big\{(w^\varepsilon - \phi)(z) - \max_{\partial(\Omega \times \Omega)}(w - \phi)\Big\}^+ \downarrow 0 \quad \text{uniformly for} \quad z \in \partial(\Omega \times \Omega)$$

as $\varepsilon \downarrow 0$. Note also that $\max_{\partial(\Omega \times \Omega)}(w - \phi) < 0 < \sup_{\Omega \times \Omega}(w - \phi)$. Thus we find that $w^\varepsilon - \phi$ attains a maximum over $\Omega \times \Omega$ at a point of $\Omega_\varepsilon \times \Omega_\varepsilon$ if $\varepsilon > 0$ is sufficiently small.

We choose an $\varepsilon > 0$ so that $w^\varepsilon - \phi$ attains its maximum over $\Omega \times \Omega$ at a point $(\bar{x}, \bar{y}) \in \Omega_\varepsilon \times \Omega_\varepsilon$. We may assume that $w^\varepsilon(\bar{x}, \bar{y}) \geqq 3\varepsilon$. By Proposition 5.1, there are matrices $X, Y \in S^n$ such that

$$\begin{pmatrix} X & O \\ O & Y \end{pmatrix} \leqq D^2\phi(\bar{x}, \bar{y}), \tag{6.1}$$

$$u^\varepsilon(\bar{x}) - \varepsilon + F\big(u^\varepsilon(\bar{x}) - \varepsilon, D_x\phi(\bar{x}, \bar{y}), X\big) \leqq 0, \tag{6.2}$$

and

$$v_\varepsilon(\bar{y}) + \varepsilon + F\big(v_\varepsilon(\bar{y}) + \varepsilon, -D_y\phi(\bar{x}, \bar{y}), -Y\big) \geqq 0. \tag{6.3}$$

Inequality (6.1) is equivalent to saying that

$$\langle Xp, p\rangle + \langle Yq, q\rangle \leqq \langle D_x^2\phi(\bar{x}, \bar{y})p, p\rangle + \langle D_xD_y\phi(\bar{x}, \bar{y})q, p\rangle$$

$$+\langle {}^TD_xD_y\phi(\bar{x}, \bar{y})p, q\rangle + \langle D_y^2\phi(\bar{x}, \bar{y})q, q\rangle \quad \text{for} \quad p, q \in \mathbb{R}^n,$$

where D_x^2, D_xD_y and D_y^2 denote, respectively, operators $(\partial^2/\partial x_i\partial x_j)$, $(\partial^2/\partial x_i\partial y_j)$ and $(\partial^2/\partial y_i\partial y_j)$. Taking $q = p$ in the above inequality and noting that

$$D_x^2\phi(\bar{x}, \bar{y}) + D_xD_y\phi(\bar{x}, \bar{y}) + {}^TD_xD_y\phi(\bar{x}, \bar{y}) + D_y^2\phi(\bar{x}, \bar{y}) = O,$$

we see that $X \leqq -Y$. Hence

$$u^\varepsilon(\bar{x}) - \varepsilon + F(u^\varepsilon(\bar{x}) - \varepsilon, D_x\phi(\bar{x}, \bar{y}), -Y) \leqq 0,$$

by (6.2) and the degenerate ellipticity of F. Taking into account that $D_x\phi = -D_y\phi$ and the monotonicity of $F(r, p, X)$ in r we have

$$u^\varepsilon(\bar{x}) - \varepsilon \leqq v_\varepsilon(\bar{y}) + \varepsilon;$$

a contradiction. We have thus completed the proof.

Proof of Theorem 3.2: Let the assumptions of Theorem 3.2 be satisfied. By Proposition 2.3 there is a viscosity solution u of

$$u + F(u, Du, D^2u) = 0 \quad \text{in} \quad \Omega \tag{6.4}$$

satisfying $f \leqq u \leqq g$ on Ω. Then $f \leqq u_* \leqq u^* \leqq g$ on $\bar{\Omega}$ since $f, g \in C(\bar{\Omega})$, and hence $u^*(x) = u_*(x)$ for $x \in \partial\Omega$. Moreover, u^* and u_* are, respectively, viscosity sub- and supersolutions of (6.4). We thus conclude from Theorem 3.1 that $u^* \leqq u_*$ on Ω. Therefore, $u^* = u_*$ on $\bar{\Omega}$ and $u^* = u$ on Ω. Thus u^* has all the required properties.

Before going into the proof of Theorem 3.3 we prepare two lemmas.

LEMMA 6.1. *Let B and C be $m \times n$ matrices. Then the $2n \times 2n$ matrix*

$$\begin{pmatrix} {}^TBB & {}^TBC \\ {}^TCB & {}^TCC \end{pmatrix}$$

is symmetric and non-negative definite.

Proof: We have only to note that the above $2n \times 2n$ matrix is a product of two matrices ${}^T(BC) \in M(2n, m)$ and $(BC) \in M(m, 2n)$.

LEMMA 6.2. *Let $L > 0, l > 0, \varepsilon > 0$, and let m be a modulus. Then there is a nondecreasing concave C^2 function ψ on $[0, \infty)$ such that*

$$\psi(0) < \varepsilon, \quad m(r) \leqq \psi(r) \quad \textit{for} \quad 0 \leqq r \leqq l,$$

and

$$\psi(r) - L\left(r^2|\psi''(r)| + r\psi'(r)\right) - m(r) \geqq 0 \quad \textit{for} \quad 0 \leqq r \leqq l,$$

where the prime stands for the differentiation with respect to r.

Proof: Let $L > 0, l > 0, \varepsilon > 0$, and let m be a modulus. Note that, for $\delta > 0, 0 < \gamma \leqq 1$ and $0 \leqq r \leqq l + 1$, we have

$$m(r) \leqq m(\delta) + \frac{m(\delta)}{\delta} r \leqq m(\delta) + \frac{m(\delta)}{\delta}(l + 1)^{1-\gamma} r^{\gamma}.$$

Choose a $\delta > 0$ so that $m(\delta) < \frac{1}{2}\varepsilon$, and set $\gamma = \min\{1, 1/4L\}$. We define g: $[0, \infty) \to [0, \infty)$ by

$$g(r) = m(\delta) + \frac{2m(\delta)}{\delta}(l + 1)^{1-\gamma} r^{\gamma}.$$

We calculate that

$$\begin{aligned} &g(r) - L\left(r^2|g''(r)| + rg'(r)\right) - m(r) \\ &\geqq \frac{m(\delta)}{\delta}(l + 1)^{1-\gamma} r^{\gamma} - 2L\gamma(2 - \gamma)\frac{m(\delta)}{\delta}(l + 1)^{1-\gamma} r^{\gamma} \geqq 0 \end{aligned}$$

for $0 \leqq r \leqq l + 1$. Finally, we choose a $\sigma \in (0, 1)$ so small that $g(\sigma) < \varepsilon$. Then the function ψ defined by $\psi(r) = g(r + \sigma)$ for $r \geqq 0$ has all the desired properties.

Proof of Theorem 3.3: Let Ω, F, u and v be as in Theorem 3.3. Let m_0 and $M > 0$ be as in the remark after Theorem 3.3. Assume (A1)–(A4) hold. Let

$$\lambda = \inf\{c(x, \alpha, \beta) : (x, \alpha, \beta) \in \Lambda\}.$$

By (A4) we have $\lambda > 0$. Set $l = \operatorname{diam} \Omega$. Replacing m_0 by a new modulus, we may assume by virtue of (A1) and (A3) that

$$M|c(x, \alpha, \beta) - c(y, \alpha, \beta)| \leqq \tfrac{1}{2}\lambda m_0(|x - y|)$$

and

$$|d(x, \alpha, \beta) - d(y, \alpha, \beta)| \leqq \tfrac{1}{2}\lambda m_0(|x - y|)$$

for $(x, \alpha, \beta), (y, \alpha, \beta) \in \Lambda$. By virtue of (A2) there is a constant $L_0 > 0$ such that

$$\|\Sigma(x, \alpha, \beta) - \Sigma(y, \alpha, \beta)\|^2 \leqq \lambda L_0 |x - y|^2$$

and

$$|b(x, \alpha, \beta) - b(y, \alpha, \beta)| \leqq \lambda L_0 |x - y|$$

for $(x, \alpha, \beta), (y, \alpha, \beta) \in \Lambda$.

Fix any $\delta > 0$, and select a nondecreasing concave C^2 function ψ_δ on $[0, \infty)$ such that

$$\psi_\delta(0) < \delta, \qquad m_0(r) \leqq \psi_\delta(r) \quad \text{for} \quad 0 \leqq r \leqq l + 1,$$

and

$$\psi_\delta(r) - (2n + 1) L_0\big(r^2 |\psi_\delta''(r)| + r\psi_\delta'(r)\big) - m_0(r) \geqq 0$$

for $0 \leqq r \leqq l + 1$. Fix any $\eta \in (0, 1)$. We set

$$\phi(x, y) = \psi_\delta\Big(\big(|x - y|^2 + \eta^2\big)^{1/2}\Big) + \eta \quad \text{for} \quad x, y \in \mathbb{R}^n.$$

Clearly, we have

$$\phi \in C^2(\mathbb{R}^{2n}), \qquad m_0(|x - y|) \leqq \phi(x, y) \quad \text{for} \quad x, y \in \bar{\Omega},$$

$$D_x\phi(x, y) = -D_y\phi(x, y) \quad \text{for} \quad x, y \in \mathbb{R}^n,$$

and

$$D_x^2\phi(x, y) = D_y^2\phi(x, y) = -D_x D_y \phi(x, y) = -{}^T D_x D_y \phi(x, y) \quad \text{for} \quad x, y \in \mathbb{R}^n.$$

Moreover, we have

$$\phi(x, y) - L_0\big(|x - y|^2 \|D_x^2\phi(x, y)\| + |x - y|\,|D_x\phi(x, y)|\big) - m_0(|x - y|) \geqq \eta \quad \text{for} \quad x, y \in \Omega, \tag{6.5}$$

since

$$|D_x\phi(x, y)| \leqq \psi_\delta'(\tilde{r})$$

and

$$\|D_x^2\phi(x, y)\| \leqq n\big\{|\psi_\delta''(\tilde{r})| + 2\psi_\delta'(\tilde{r})|x - y|^{-1}\big\} \quad \text{for} \quad x, y \in \mathbb{R}^n,$$

where $\tilde{r} = (|x - y|^2 + \eta^2)^{1/2}$.

As in the proof of Theorem 3.1 we may assume that u and v are defined as real-valued functions on $\overline{\Omega}$ and, respectively, u.s.c. and l.s.c. on $\overline{\Omega}$. We set

$$w(x, y) = u(x) - v(y) \quad \text{for} \quad (x, y) \in \overline{\Omega} \times \overline{\Omega}$$

and

$$w^\varepsilon(x, y) = u^\varepsilon(x) - v_\varepsilon(y) \quad \text{for} \quad \varepsilon > 0 \quad \text{and} \quad x, y \in \overline{\Omega}.$$

We intend to show $w \leqq \varphi$ on $\overline{\Omega} \times \overline{\Omega}$. To this end, we suppose the contrary and obtain a contradiction. Then we deduce as in the proof of Theorem 3.1 that there is an $\varepsilon_0 > 0$ such that for each $0 < \varepsilon \leqq \varepsilon_0$ the function $w^\varepsilon - \phi$ attains a maximum over $\Omega \times \Omega$ at a point of $\Omega_\varepsilon \times \Omega_\varepsilon$. Let $0 < \varepsilon \leqq \varepsilon_0$ and assume that $(x, y) \in \Omega_\varepsilon \times \Omega_\varepsilon$ is a point where $w^\varepsilon - \phi$ attains its maximum. By our tentative assumption we have $w^\varepsilon > \phi$ at (x, y). By Proposition 5.1, there are matrices X, Y of S^n such that

$$\begin{pmatrix} X & O \\ O & Y \end{pmatrix} \leqq D^2\phi(x, y), \tag{6.6}$$

$$F_\varepsilon(x, u^\varepsilon(x), D_x\phi(x, y), X) \leqq 0, \tag{6.7}$$

and

$$F^\varepsilon(y, v_\varepsilon(y), -D_y\phi(x, y), -Y) \geqq 0. \tag{6.8}$$

Relations (6.7) and (6.8) imply that there are points $\xi \in B(x; \varepsilon)$, $\eta \in B(y; \varepsilon)$ and real numbers $r \in B(u^\varepsilon(x); \varepsilon)$, $s \in B(v_\varepsilon(y); \varepsilon)$ such that

$$F(\xi, r, D_x\phi(x, y), X) \leqq 0 \tag{6.9}$$

and

$$F(\eta, s, -D_y\phi(x, y), -Y) \geqq 0 \tag{6.10}$$

hold.

We fix $(\alpha, \beta) \in \mathscr{A} \times \mathscr{B}$. For the time being we put

$$B = \Sigma(\xi, \alpha, \beta) \quad \text{and} \quad C = \Sigma(\eta, \alpha, \beta).$$

From Lemma 6.1, the $2n \times 2n$ matrix

$$\begin{pmatrix} {}^TBB & {}^TBC \\ {}^TCB & {}^TCC \end{pmatrix}$$

is symmetric and non-negative definite. This and (6.6) yield

$$\operatorname{Tr}\begin{pmatrix} {}^TBB & {}^TBC \\ {}^TCB & {}^TCC \end{pmatrix}\left(\begin{pmatrix} X & O \\ O & Y \end{pmatrix} - D^2\phi(x, y)\right) \leqq 0,$$

since the trace of a product of two non-negative definite matrices of S^n is non-negative. Using this observation, we calculate that

$$\begin{aligned}\operatorname{Tr}({}^TBBX + {}^TCCY) &\leqq \operatorname{Tr}\Big({}^TBBD_x^2\phi(x, y) + {}^TBC^TD_xD_y\phi(x, y) \\ &\qquad + {}^TCBD_xD_y\phi(x, y) + {}^TCCD_y^2\phi(x, y)\Big) \\ &= \operatorname{Tr}\big\{({}^TBB - {}^TBC - {}^TCB + {}^TCC)D_x^2\phi(x, y)\big\} \\ &= \operatorname{Tr}{}^T(B - C)(B - C)D_x^2\phi(x, y) \\ &\leqq \|B - C\|^2\,\|D_x^2\phi(x, y)\|.\end{aligned}$$

Thus

$$\operatorname{Tr}(A(\xi, \alpha, \beta)X + A(\eta, \alpha, \beta)Y) \leqq \lambda L_0|\xi - \eta|^2\|D_x^2\phi(x, y)\|.$$

This and (6.10) yield

$$\begin{aligned}&\inf_{\beta\in\mathscr{B}}\ \sup_{\alpha\in\mathscr{A}}\{-\operatorname{Tr}A(\xi, \alpha, \beta)X + \langle b(\eta, \alpha, \beta), D_x\phi(x, y)\rangle \\ &\qquad + c(\eta, \alpha, \beta)s + d(\eta, \alpha, \beta)\} + \lambda L_0|\xi - \eta|^2\|D_x^2\varphi(x, y)\| \geqq 0.\end{aligned}$$

Subtracting this from (6.9) yields

$$\begin{aligned}&\inf_{(\alpha,\beta)\in\mathscr{A}\times\mathscr{B}}\{c(\xi, \alpha, \beta)r - c(\eta, \alpha, \beta)s\} - \tfrac{1}{2}\lambda m_0(|\xi - \eta|) \\ &\qquad -\lambda L_0|\xi - \eta|\,|D_x\phi(x, y)| - \lambda L_0|\xi - \eta|^2\|D_x^2\phi(x, y)\| \leqq 0.\end{aligned}$$

Since

$$\inf_{(\alpha,\beta)\in\mathscr{A}\times\mathscr{B}}\{c(\xi, \alpha, \beta)r - c(\eta, \alpha, \beta)s\} \geqq \lambda(r - s) - \tfrac{1}{2}\lambda m_0(|\xi - \eta|)$$

if $r \geqq s$, we thus have

$$\begin{aligned}&\lambda(r - s) - \lambda m_0(|\xi - \eta|) - \lambda L_0|\xi - \eta|\,|D_x\phi(x, y)| \\ &\qquad -\lambda L_0|\xi - \eta|^2\|D_x^2\phi(x, y)\| \leqq 0.\end{aligned}$$

Dividing this by $\lambda > 0$ and choosing $\varepsilon > 0$ small enough, we have

$$w^\varepsilon(x, y) - L_0\big(|x - y|^2\|D_x^2\phi(x, y)\| + |x - y|\,|D_x\phi(x, y)|\big) - m_0(|x - y|) < \eta.$$

This contradicts (6.5), which proves that $w \leqq \phi$ on $\Omega \times \Omega$. Thus we see that $w(x,y) \leqq \psi_\delta(|x-y|)$ for $x, y \in \Omega$.

Finally, we define m: $[0, \infty) \to [0, \infty)$ by

$$m(r) = \inf\{\psi_\delta(r) : \delta > 0\}.$$

It is easy to see that $m(0) = 0$ and that m is nondecreasing, u.s.c. and concave on $[0, \infty)$. One easily deduces that m is continuous on $[0, \infty)$ and hence m is a modulus. To complete the proof, we have just to note that $w(x, y) \leqq m(|x-y|)$ for $x, y \in \Omega$.

Once Theorem 3.3 is proved, the proof of Theorem 3.4 is quite similar to that of Theorem 3.2. Thus we leave it to the reader.

7. Results on Unbounded Domains

We do not assume in this section that Ω is bounded. We make use of the following assumptions for a real-valued function F on Γ.

(A5) For each $(x, p, X) \in \Omega \times \mathbb{R}^n \times S^n$ the function $r \to F(x, r, p, X)$ is nondecreasing on $\mathbb{R}$.

(A6) There is a modulus m_F such that

$$|F(x, r, p, X) - F(y, r, p, X)| \leqq m_F(|x-y|(|p|+1))$$

for $x, y \in \Omega$ and $(r, p, X) \in \mathbb{R} \times \mathbb{R}^n \times S^n$.

(A7) For each $R > 0$ there is a modulus σ_R such that

$$|F(x, r, p, X) - F(x, r, q, Y)| \leqq \sigma_R(|p-q| + \|X-Y\|)$$

for $(x, r) \in \Omega \times \mathbb{R}$, $p, q \in B(0; R)$ and $X, Y \in S^n$ with $\|X\|, \|Y\| \leqq R$.

THEOREM 7.1. *Assume that F is continuous and degenerate elliptic. Assume* (A5)–(A7) *hold. Let u and v be, respectively, viscosity sub- and supersolutions of*

$$u + F(x, u, Du, D^2u) = 0 \quad \text{in} \quad \Omega. \tag{7.1}$$

Suppose that

$$u(x) \leqq C(|x|+1) \quad \text{and} \quad v(x) \geqq -C(|x|+1) \tag{7.2}$$

for some constant $C > 0$ and all $x \in \Omega$. Suppose, moreover, that $u^*(x) \leqq v_*(x)$ for $x \in \partial\Omega$ and that

$$u(x) \leqq u^*(y) + m_0(|x-y|) \quad \text{and} \quad v(x) \geqq v_*(y) - m_0(|x-y|) \tag{7.3}$$

for some modulus m_0 and all $x \in \Omega$ and $y \in \partial\Omega$. Then there is a modulus m, *depending on u and v only through C and* m_0, *such that* $u^*(x) - v_*(y) \leqq m(|x-y|)$ *for* $x,\ y \in \Omega$.

Remarks. (a) This generalizes Theorem 3.1 and [1], Theorem 2. Moreover, this includes some of the recent uniqueness results for Hamilton–Jacobi equations. For this see [6], [8], [15], [16]. (b) The above assumptions are somewhat superfluous. That is, if $\partial\Omega \neq \varnothing$, then (7.3) implies that $u(x) \leqq C(|x|+1)$ and $v(x) \geqq -C(|x|+1)$ for some constant $C > 0$ and all $x, y \in \Omega$. Conversely, if $\Omega = \mathbb{R}^n$, then (7.3) is void.

LEMMA 7.1. *Let* $\varepsilon > 0$, $l > 0$ *and let* m *be a modulus. Then there is a nondecreasing concave* C^2 *function* ψ *on* $[0,\infty)$ *such that*

$$\psi(0) < \varepsilon, \quad m(r) \leqq \psi(r) \quad \text{for} \quad 0 \leqq r \leqq l,$$

and

$$\psi(r) - m(r\psi'(r) + r) \geqq 0 \quad \text{for} \quad 0 \leqq r \leqq l.$$

Proof: Let $\varepsilon > 0$, $l > 0$ and let m be a modulus. Noting that $m(r) \leqq m(\delta) + m(\delta)r/\delta$ for $r \geqq 0$ and $\delta > 0$, we can choose an $L > 0$ so that $m(r) \leqq \frac{1}{2}\varepsilon + Lr$ for $r \geqq 0$. By Lemma 6.2, there is a nondecreasing concave C^2 function ϕ on $[0,\infty)$ such that

$$\phi(0) < \tfrac{1}{2}\varepsilon, \quad m(r) \leqq \phi(r) \quad \text{for} \quad 0 \leqq r \leqq l,$$

and

$$\phi(r) - L\big(r^2|\phi''(r)| + r\phi'(r)\big) - m(r) \geqq 0 \quad \text{for} \quad 0 \leqq r \leqq l.$$

Taking into account that $m(r+s) \leqq m(r) + m(s)$ for $r,\ s \geqq 0$, we see that if we set $\psi(r) = \phi(r) + \frac{1}{2}\varepsilon$ for $r \geqq 0$, then ψ has the desired properties.

Proof of Theorem 7.1: Let F, u, v, C and m_0 be as in Theorem 7.1. Let m_F and σ_R, with $R > 0$, be moduli from (A6) and (A7). For simplicity we write $\sigma(r, R) = \sigma_R(r)$ for $r \geqq 0$ and $R > 0$. We may assume that $m_0(r) \leqq C(r+1)$ and $m_F(r) \leqq C(r+1)$ for $r \geqq 0$. We may also assume that u and v are defined as real-valued functions on $\overline{\Omega}$ and, respectively, u.s.c. and l.s.c. on $\overline{\Omega}$.

For $\varepsilon > 0$ we define a function w^ε on $\overline{\Omega} \times \overline{\Omega}$ by

$$w^\varepsilon(x, y) = u^\varepsilon(x) - v_\varepsilon(y) \quad \text{for} \quad x, y \in \overline{\Omega}.$$

We see from (7.2) that

$$w^\varepsilon(x, y) \leqq C\big(|x| + |y| + 2 + 2\varepsilon\big) + 2\varepsilon$$

for $\varepsilon > 0$ and $x, y \in \overline{\Omega}$. We now examine the behavior of w^ε near $\partial(\Omega \times \Omega)$.

From assumption (7.3) we find that

$$u^\varepsilon(x) \leqq u(y) + m_0(|x-y|) + m_0(\varepsilon) + \varepsilon$$

and

$$v_\varepsilon(x) \geqq v(y) - m_0(|x-y|) - m_0(\varepsilon) - \varepsilon$$

for $\varepsilon > 0$, $x \in \overline{\Omega}$ and $y \in \partial\Omega$. Fix $\varepsilon > 0$ and $y \in \overline{\Omega} \setminus \Omega_\varepsilon$. Then there is a point $\eta \in \partial\Omega$ such that $|y - \eta| \leqq \varepsilon$. Since $u(\eta) \leqq v(\eta)$ by assumption, we have

$$\begin{aligned} u^\varepsilon(x) &\leqq v(\eta) + m_0(|x-\eta|) + m_0(\varepsilon) + \varepsilon \\ &\leqq v_\varepsilon(y) + m_0(|y-\eta|) + m_0(|x-\eta|) + 2m_0(\varepsilon) + 2\varepsilon \\ &\leqq v_\varepsilon(y) + m_0(|x-y|) + 4m_0(\varepsilon) + 2\varepsilon \end{aligned}$$

for $x \in \overline{\Omega}$. Similarly we have

$$u^\varepsilon(x) \leqq v_\varepsilon(y) + m_0(|x-y|) + 4m_0(\varepsilon) + 2\varepsilon$$

for $\varepsilon > 0$, $x \in \overline{\Omega} \setminus \Omega_\varepsilon$ and $y \in \overline{\Omega}$. Thus,

$$w^\varepsilon(x, y) \leqq m_0(|x-y|) + 4m_0(\varepsilon) + 2\varepsilon \tag{7.4}$$

for $\varepsilon > 0$ and $(x, y) \in (\overline{\Omega} \times \overline{\Omega}) \setminus (\Omega_\varepsilon \times \Omega_\varepsilon)$.

Now we consider the case $\varepsilon = 1$ and prove that there is a constant $M > 0$ depending only on C such that

$$w^1(x, y) \leqq (C+1)|x-y| + M \quad \text{for} \quad x, y \in \Omega. \tag{7.5}$$

We choose a family $\{g_R\}_{R>0}$ of C^2 functions on $\mathbb{R}^n$ with the following properties: (i) $g_R(x) = 0$ for $R > 0$ and $x \in B(0; R)$; (ii) $g_R(x)/|x| \to 1$ as $|x| \to \infty$, and (iii)

$$L = \sup\left\{|Dg_R(x)| + \|D^2 g_R(x)\|\colon x \in \mathbb{R}^n,\ R > 0\right\} \quad \text{is finite.}$$

Let $M > 0$ be a large constant to be chosen later on. We set $C_1 = C + 1$ and $\psi(x, y) = C_1(|x-y|^2 + 1)^{1/2} + M$ for $x, y \in \mathbb{R}^n$. To show (7.5), we suppose that $\sup_{\Omega\times\Omega}(w^1 - \psi) > 0$, and will obtain a contradiction with M sufficiently large. For $R > 0$ we define

$$\phi(x, y) = \psi(x, y) + C_1 g_R(x) \quad \text{for} \quad x, y \in \mathbb{R}^n.$$

Choosing $R > 0$ large enough, we have $\sup_{\Omega\times\Omega}(w^1 - \phi) > 0$. We see from the definition of ϕ that $\sup_{\Omega\times\Omega}(w^1 - \phi) < \infty$ and that if $R_1 > 0$ is large enough, $z \in \Omega \times \Omega$ and $|z| \geqq R_1$, then $(w^1 - \phi)(z) < 0$. Henceforth we assume that

$M > 4m_0(1) + 2$. By (7.5) we have

$$(w^1 - \phi)(z) < 0 \quad \text{for} \quad z \in (\bar{\Omega} \times \bar{\Omega}) \setminus (\Omega_1 \times \Omega_1).$$

Thus there is a point $(x, y) \in \Omega_1 \times \Omega_1$, where $w^1 - \phi$ attains its maximum over $\Omega \times \Omega$. By Proposition 5.1 there is a constant $C_2 > 0$, depending only on C_1 and L, and matrices $X, Y \in S^n$ such that

$$(7.6) \qquad -C_2 I \leqq \begin{pmatrix} X & O \\ O & Y \end{pmatrix} \leqq D^2\phi(x, y),$$

$$(7.7) \qquad u^1(x) - 1 + F_1\big(x, u^1(x), D_x\phi(x, y), X\big) \leqq 0,$$

and

$$(7.8) \qquad v_1(y) + 1 + F^1\big(y, v_1(y), -D_y\phi(x, y), -Y\big) \geqq 0.$$

Using (A5) we have, from inequalities (7.7) and (7.8),

$$u^1(x) - 1 + F\big(\xi, u^1(x) - 1, D_x\phi(x, y), X\big) \leqq 0$$

and

$$v_1(y) + 1 + F\big(\eta, v_1(y) + 1, -D_y\phi(x, y), -Y\big) \geqq 0$$

for some $\xi \in B(x; 1)$ and $\eta \in B(y; 1)$. Inequality (7.6) implies that $\|X\|$ and $\|Y\|$ are bounded by a constant, say C_3, depending only on C_1 and L as R varies.

Note also that $|D_y\phi(x, y)| \leqq C_1$, $|D_x\phi(x, y)| \leqq C_1(1 + L)$ and $w^1(x, y) > \phi(x, y) > 2$. Therefore, putting $r = u^1(x) - 1$ and $C_4 = C_1(1 + L) + C_3$ and using (A5)–(A7), we have

$$\begin{aligned} 2 &\geqq w^1(x, y) + F(\xi, r, 0, O) - F(\eta, r, 0, O) - 2\sigma(C_4, C_4) \\ &\geqq w^1(x, y) - C(|\xi - \eta| + 1) - 2\sigma(C_4, C_4) \\ &\geqq w^1(x, y) - C(|x - y| + 3) - 2\sigma(C_4, C_4) \\ &\geqq M - 3C - 2\sigma(C_4, C_4). \end{aligned}$$

Choosing $M > \max\{4m_0(1) + 2, 3C + 2 + 2\sigma(C_4, C_4)\}$, we arrive at a contradiction. Thus (7.5) is proved for some $M > 0$.

To proceed, we fix $M > 0$ so that (7.5) holds. Let $\delta > 0$. By Lemma 7.1, there is a nondecreasing concave C^2 function ψ_δ on $[0, \infty)$ satisfying

$$\psi_\delta(0) < \delta, \quad \psi_\delta(1) \geq C + 1 + M, \quad m_0(r) \leqq \psi_\delta(r) \quad \text{for} \quad 0 \leqq r \leqq 2,$$

and

$$\psi_\delta(r) - m_F\big(r\psi_\delta'(r) + r\big) \geqq 0 \quad \text{for} \quad 0 \leqq r \leqq 2.$$

(Use Lemma 7.1 with m defined by $m(r) = m_0(r) + m_F(r) + (C + 1 + M)r$.)
We want to show that

$$(7.9) \qquad u(x) - v(y) \leqq \psi_\delta(|x - y|)$$

for $(x, y) \in \Delta = \{(\xi, \eta) \in \Omega \times \Omega \colon |\xi - \eta| < 1\}$. To this end, we prove that, for each $0 < \gamma < 1$, there is an $\varepsilon > 0$ such that

$$(7.10) \qquad w^\varepsilon(x, y) \leqq \psi_\delta\Big(\big(|x - y|^2 + \gamma\big)^{1/2}\Big) + \gamma \quad \text{for} \quad (x, y) \in \Delta.$$

We set $C_5 = \sup\{|\psi_\delta'(r)| \colon 0 \leqq r \leqq 2\} + 1$. We fix any $\gamma \in (0,1)$, and choose an $\varepsilon \in (0,1)$ so that $m_F(2\varepsilon C_5) + 4m_0(\varepsilon) + 2\varepsilon < \gamma$. We set

$$\psi(x, y) = \psi_\delta\Big(\big(|x - y|^2 + \gamma\big)^{1/2}\Big) + \gamma \quad \text{for} \quad (x, y) \in \mathbb{R}^{2n}.$$

In order to verify (7.10) for these γ and ε, we shall suppose that $\sup_\Delta(w^\varepsilon - \psi) > 0$ and obtain a contradiction. Let $\beta \in (0,1)$, and define

$$\phi(x, y) = \psi(x, y) + \beta g_R(x) \quad \text{for} \quad (x, y) \in \mathbb{R}^{2n}.$$

If $R > 0$ is sufficiently large, then $\sup_\Delta(w^\varepsilon - \phi) > 0$. We fix such an $R > 0$. We see from (7.5) that $w^\varepsilon - \phi$ attains its maximum at a point $(x, y) \in \overline{\Delta}$. From (7.4) and (7.5) we find that $(x, y) \in (\Omega_\varepsilon \times \Omega_\varepsilon) \cap \Delta$. Again, by Proposition 5.1 there is a constant $C_6 > 0$, depending only on ε, C_5 and L, and matrices $X, Y \in S^n$ such that

$$(7.11) \qquad -C_6 I \leqq \begin{pmatrix} X & O \\ O & Y \end{pmatrix} \leqq D^2\phi(x, y),$$

$$(7.12) \qquad u^\varepsilon(x) - \varepsilon + F_\varepsilon\big(x, u^\varepsilon(x), D_x\phi(x, y), X\big) \leqq 0,$$

and

$$(7.13) \qquad v_\varepsilon(y) + \varepsilon + F^\varepsilon\big(y, v_\varepsilon(y), -D_y\phi(x, y), -Y\big) \geqq 0.$$

From (7.12) and (7.13) we have

$$w^\varepsilon(x, y) + F(\xi, u^\varepsilon(x) - \varepsilon, D_x\phi(x, y), X)$$

$$-F\big(\eta, v_\varepsilon(y) + \varepsilon, -D_y\phi(x, y), -Y\big) \leqq 2\varepsilon$$

for some $\xi \in B(x; \varepsilon)$ and $\eta \in B(y; \varepsilon)$. As in the proof of Theorem 3.1 we have $X + Y \leqq \beta D^2 g_R(x)$. By (7.11) there is a constant $C_7 > 0$ independent of β for which $\|Y\| \leqq C_7$ as β varies. We put $C_8 = C_7 + L + C_5$. Then $|D_y\phi(x, y)| + \|Y\| \leqq C_8$ and $|D_x\phi(x, y)| + \beta\|D^2 g_R(x)\| + \|Y\| \leqq C_8$. Thus, writing

$$r = u^\varepsilon(x) - \varepsilon \quad \text{and} \quad s = \left(|x - y|^2 + \gamma\right)^{1/2}$$

and using (A5)–(A7) and the degenerate ellipticity of F, we have

$$\begin{aligned} 2\varepsilon &\geqq w^\varepsilon(x, y) + F\left(\xi, r, D_x\phi(x, y), -Y + \beta D^2 g_R(x)\right) \\ &\quad -F\left(\eta, r, -D_y\phi(x, y), -Y\right) \\ &\geqq w^\varepsilon(x, y) + F\left(\xi, r, D_x\psi(x, y), -Y\right) \\ &\quad -F\left(\eta, r, -D_y\psi(x, y), -Y\right) - \sigma(\beta L, C_8) \\ &\geqq \psi(s) + \gamma - m_F\left(s|\psi_\delta'(s)| + s\right) - m_F(2\varepsilon C_5) - \sigma(\beta L, C_8). \end{aligned}$$

Sending $\beta \downarrow 0$, we have $\gamma \leqq m_F(2\varepsilon C_5) + 2\varepsilon$. This contradiction proves (7.9).

From (7.5) and (7.9) we see that

$$u(x) - v(y) \leqq \psi_\delta(|x - y|) + (C + 1 + M)|x - y|$$

for $(x, y) \in \Omega \times \Omega$. Now define

$$m(r) = \inf\{\psi_\delta(r) + (C + 1 + M)r \colon \delta > 0\} \quad \text{for} \quad r \geqq 0.$$

Then m is a modulus having the required property.

Remark. In Theorem 7.1 assumption (7.3) can be replaced by the following: There is a modulus m_0 for which

$$u^*(x) - v_*(y) \leqq m_0(|x - y|) \quad \text{for} \quad (x, y) \in \partial(\Omega \times \Omega).$$

To see this, we have only to repeat the argument of the above proof with the function $(x, y) \to u(x) - v_\varepsilon(y)$ on $\bar{\Omega} \times \bar{\Omega}$ in place of w^ε. (See also the remark after Theorem 7.2 which is useful when $u^*(x) = -\infty$ or $v_*(x) = \infty$ for some $x \in \partial\Omega$ and replace u and v by new functions.)

Corresponding to Theorem 7.1 we have an existence result:

THEOREM 7.2. *Let F be as in Theorem 7.1. Suppose there is a viscosity sub-solution f and a viscosity supersolution g of (7.1). Assume that $f, g \in C(\bar{\Omega})$, $f \leqq g$ on Ω and $f = g$ on $\partial\Omega$. Assume moreover that*

$$|f(x) - f(y)| \leqq m_0(|x - y|) \quad \textit{and} \quad |g(x) - g(y)| \leqq m_0(|x - y|)$$

for some modulus m_0 and all $x, y \in \bar{\Omega}$. Then there is a viscosity solution u of (7.1) *which satisfies $f \leqq u \leqq g$ on $\bar{\Omega}$ and $|u(x) - u(y)| \leqq m(|x - y|)$ for some modulus m and all $x, y \in \bar{\Omega}$.*

Remark. In the case $\Omega = \mathbb{R}^n$ the existence of those f and g which satisfy the conditions in the above theorem follows from assumptions (A5)–(A7). Indeed, if $\Omega = \mathbb{R}^n$, then $\partial\Omega = \varnothing$. And if F satisfies (A5)–(A7), $C_1 > 0$ is sufficiently large and $C_2 > 0$ is sufficiently large compared with C_1, then the functions

$$f(x) = -C_1(|x|^2 + 1)^{1/2} - C_2 \quad \text{and} \quad g(x) = C_1(|x|^2 + 1)^{1/2} + C_2 \quad \text{on} \quad \Omega$$

are, respectively, viscosity sub- and supersolutions of (7.1).

Proof: By Proposition 2.3 there is a viscosity solution u of (7.1) satisfying $f \leqq u \leqq g$ on Ω. Clearly, we have $f \leqq u_* \leqq u^* \leqq g$ on $\bar{\Omega}$, and so

$$u^*(x) \leqq u_*(y) + g(x) - g(y) \leqq u_*(y) + m_0(|x - y|)$$

and

$$u_*(x) \geqq u^*(y) + f(x) - f(y) \geqq u^*(y) - m_0(|x - y|)$$

for $y \in \partial\Omega$ and $x \in \bar{\Omega}$. Also we have

$$u^*(x) \leqq g(x) \leqq C(|x| + 1) \quad \text{and} \quad u_*(x) \geqq f(x) \geqq -C(|x| + 1)$$

for some constant C and all $x \in \Omega$. Thus we see by Theorem 7.1 that there is a modulus m for which $u^*(x) - u_*(y) \leqq m(|x - y|)$ for $x, y \in \bar{\Omega}$. In particular, $u^* = u_*$ on $\bar{\Omega}$. Thus u^* is the viscosity solution of (7.1) satisfying the desired properties.

Next we give two comparison results on solutions of Isaacs' equations. Let $\mathscr{A}$, $\mathscr{B}$, Σ, A, b, c and d be as in Section 3. Let F be the function defined by (3.2). We use the assumptions (A1)$'$ and (A3)$'$ instead of (A1) and (A3).

(A1)$'$ For each bounded subset B of Ω functions A, b, c and d are bounded on $B \times \mathscr{A} \times \mathscr{B}$.

(A3)$'$ Functions $f = c, d$ satisfy

$$\lim_{r \downarrow 0} \sup\{|f(x, \alpha, \beta) - f(y, \alpha, \beta)|: x, y \in B,$$
$$(\alpha, \beta) \in \mathscr{A} \times \mathscr{B}, |x - y| \leqq r\} = 0$$

for bounded subsets B of Ω.

THEOREM 7.3. *Assume that* (A1)$'$, (A2), (A3)$'$ *and* (A4) *hold. Let u and v be, respectively, viscosity sub- and supersolutions of*

$$F(x, u, Du, D^2u) = 0 \quad in \quad \Omega. \tag{7.14}$$

If Ω is unbounded, then assume that

$$\lim_{x\in\Omega,\, |x|\to\infty} \frac{u^+(x)}{\log|x|} = 0 \quad and \quad \lim_{x\in\Omega,\, |x|\to\infty} \frac{v^-(x)}{\log|x|} = 0.$$

Suppose that $u^*(x) \leqq v_*(x)$ for $x \in \partial\Omega$. Then $u^* \leqq v_*$ on Ω.

Remarks. (a) This extends Theorem 3.3 and [24], Corollaries II.1, II.2.

(b) Under the hypotheses of this theorem, for each $R > 0$ there is a modulus μ_R such that $u^*(x) - v_*(y) \leqq \mu_R(|x-y|)$ for $(x, y) \in \partial(\Omega \times \Omega) \cap B(0; R)$ and also there is a continuous function ω on $[0, \infty)$ satisfying $\lim_{r\to\infty}\omega(r) = 0$ for which

$$u(x) \leqq \omega(|x|)\log(|x| + 2) \quad \text{and} \quad v(x) \geqq -\omega(|x|)\log(|x| + 2)$$

for $x, y \in \Omega$. The conclusion implies that for each $R > 0$ there is a modulus m_R such that $u(x) - v(y) \leqq m_R(|x - y|)$ for $x, y \in \Omega \cap B(0; R)$. As the proof below shows, this family $\{m_R\}_{R>0}$ can be selected to depend on u and v only through ω and $\{\mu_R\}$.

Proof: Let F, u and v be as in Theorem 7.3. Define g: $\mathbb{R}^n \to \mathbb{R}$ by $g(x) = \log(|x|^2 + 1)$. Let $\varepsilon > 0$ and $\delta > 0$, and put

$$\tilde{u}(x) = u(x) - \varepsilon g(x) - \delta \quad \text{and} \quad \tilde{v}(x) = v(x) + \varepsilon g(x) + \delta \quad \text{for} \quad x \in \Omega.$$

By assumptions (A1)$'$ and (A2) there is a constant $C > 0$ such that

$$\|A(x, \alpha, \beta)\| \leqq C(|x|^2 + 1) \quad \text{and} \quad |b(x, \alpha, \beta)| \leqq C(|x| + 1)$$

for $(x, \alpha, \beta) \in \Lambda$. We may assume that

$$|Dg(x)| \leqq C(|x| + 1)^{-1} \quad \text{and} \quad |D^2g(x)| \leqq C(|x|^2 + 1)^{-1} \quad \text{for} \quad x \in \mathbb{R}^n.$$

From these and (A4) we see easily that $\tilde{u}$ and $\tilde{v}$ are viscosity sub- and supersolutions, respectively, of

$$F(x, u, Du, D^2u) = -\lambda\delta + 2\varepsilon C^2 \quad \text{in} \quad \Omega$$

and

$$F(x, u, Du, D^2u) = \lambda\delta - 2\varepsilon C^2 \quad \text{in} \quad \Omega,$$

where $\lambda = \inf\{c(x, \alpha, \beta)\colon (x, \alpha, \beta) \in \Lambda\}$. Hereafter we let $\delta = 2\varepsilon C^2/\lambda$. This guarantees that $\tilde{u}$ and $\tilde{v}$ are, respectively, viscosity sub- and supersolutions of (7.14).

By assumption, there is an $L > 0$ such that $\tilde{u}(x) < 0$ and $\tilde{v}(x) > 0$ if $x \in \Omega$ and $|x| \geqq L$. Applying Theorem 3.3 with the set $\{x \in \Omega\colon |x| < L\}$ in place of Ω, we find that there is a modulus m_ε for which $\tilde{u}(x) - \tilde{v}(y) \leqq m_\varepsilon(|x - y|)$ holds for $x, y \in \Omega \cap B(0; L)$. For this m_ε we clearly have

$$u(x) - v(y) \leqq m_\varepsilon(|x - y|) + \varepsilon(g(x) + g(y)) + 2\varepsilon C^2/\lambda \quad \text{for} \quad x, y \in \Omega.$$

If we set

$$m_R(r) = \inf\{\psi_\varepsilon(r) + 2\varepsilon \log(R^2 + 1) + 2\varepsilon C^2/\lambda\colon \varepsilon > 0\}$$

for $R > 0$ and $r \geqq 0$, then m_R is a modulus having the required property.

We make the following assumption on Σ and b.

(A8) For each $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ for which

$$\|\Sigma(x, \alpha, \beta)\| \leqq C_\varepsilon + \varepsilon|x|^2 \quad \text{and} \quad |b(x, \alpha, \beta)| \leqq C_\varepsilon + \varepsilon|x|$$

hold for $(x, \alpha, \beta) \in \Lambda$.

THEOREM 7.4. *Assume that* (A1), (A2), (A3)′, (A4) *and* (A8) *hold. Let u and v be, respectively, viscosity sub- and supersolutions of* (7.14). *Assume that there is an integer k such that*

$$\sup_{x\in\Omega} \frac{u(x)}{(|x| + 1)^k} < \infty \quad \text{and} \quad \inf_{x\in\Omega} \frac{v(x)}{(|x| + 1)^k} > -\infty.$$

Suppose that $u^*(x) \leqq v_*(x)$ for $x \in \partial\Omega$. Then $u^* \leqq v_*$ on Ω.

Remarks. (a) This theorem generalizes [24], Theorem II.1, ii.

(b) The same remark as remark (b) after Theorem 7.3 is valid.

(c) A more precise formulation like Theorem 1.5 of [15] is possible.

Outline of proof: Follow the proof of Theorem 7.3 using the function $x \to (|x|^2 + 1)^{k+1}$ in place of g.

Our final result is this:

THEOREM 7.5. *Let Ω be unbounded. Assume there is a viscosity subsolution f and a viscosity supersolution g of* (7.14). *Suppose that $f, g \in C(\overline{\Omega})$, $f \leqq g$ in Ω and $f = g$ on $\partial\Omega$.* (i) *If* (A1)′, (A2), (A3)′ *and* (A4) *are satisfied and*

$$\lim_{x\in\Omega,\, |x|\to\infty} \frac{f^-(x)}{\log|x|} = 0 \quad \text{and} \quad \lim_{x\in\Omega,\, |x|\to\infty} \frac{g^+(x)}{\log|x|} = 0,$$

then there is a viscosity solution u of (7.14) *satisfying $u \in C(\overline{\Omega})$ and $f \leqq u \leqq g$ on $\overline{\Omega}$.* (ii) *If* (A1)′, (A2)′, (A3)′, (A4) *and* (A8) *are satisfied and*

$$\inf_{x\in\Omega} \frac{f(x)}{(|x|+1)^k} > -\infty \quad \text{and} \quad \sup_{x\in\Omega} \frac{g(x)}{(|x|+1)^k} < \infty$$

for some integer k, then there is a viscosity solution u of (7.14) *having the properties $u \in C(\overline{\Omega})$ and $f \leqq u \leqq g$ on $\overline{\Omega}$.*

The proof of this assertion is similar to that of Theorem 3.2, and we omit giving it here.

Remark. Let $\Omega = \mathbb{R}^n$. Assume in addition to (A1)′, (A2) and (A4) that

$$\lim_{r\to\infty} \sup\left\{\frac{|d(x,\alpha,\beta)|}{\log|x|} : (x,\alpha,\beta) \in \Lambda, |x| \geqq r\right\} = 0.$$

Then we can construct those functions f, g which satisfy the conditions in assertion (i) of the above theorem. Indeed such a function g is given as the infimum of functions

$$g_k(x) = \frac{1}{k}\log(|x|^2+1) + C_k \quad \text{on} \quad \mathbb{R}^n,$$

with $k \in \mathbb{R}$, where the C_k are sufficiently large, while f is given by $f = -g$. Similarly, if we assume in addition to (A4) and (A8) that

$$\sup\left\{\frac{|d(x,\alpha,\beta)|}{(|x|+1)^k} : (x,\alpha,\beta) \in \Lambda\right\} < \infty \quad \text{for some integer} \quad k,$$

then we can easily find those functions f, g which satisfy the conditions in assertion (ii) of the above theorem.

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