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A CERTAIN PROPERTY OF SOLUTIONS OF PARABOLIC EQUATIONS WITH MEASURABLE COEFFICIENTS

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N. V. KRYLOV AND M. V. SAFONOV

ABSTRACT. In this paper Harnack's inequality is proved and the Hölder exponent is estimated for solutions of parabolic equations in nondivergence form with measurable coefficients. No assumptions are imposed on the smallness of scatter of the eigenvalues of the coefficient matrix for the second derivatives.

Bibliography: 9 titles.

Theorems concerning Harnack's inequality and estimates of Hölder norms of solutions of second order elliptic and parabolic equations play an important role in the theory of linear and nonlinear equations. The first general results of De Giorgi and Nash concerning Hölder continuity of solutions, obtained about twenty years ago, gave strong impetus to the development of the theory of elliptic and parabolic equations of *divergence* form, and were extended by many authors. Without listing the huge number of works bearing on the above theme, we merely point to the monographs [1]–[4], in which the reader will find a history of the matter (see especially [4]), the results in their most general form, and an extensive bibliography.

In the present article we shall take up the proof of Harnack's inequality and Hölder continuity of solutions for linear parabolic equations of *nondivergence* form with *measurable* coefficients. The theory of such equations, as well as of the corresponding elliptic equations in E_d for $d \geqslant 3$, is essentially in the initial stage of development, although a priori estimates for Hölder norms of solutions of elliptic equations were obtained by Cordes [5] back in 1956, i.e. even earlier than the corresponding result of De Giorgi for equations in divergence form. Such a difference in the role of similar results for the theory of equations in various forms is caused, above all, by the specifics of the divergence equations.

Certain restrictions on the distribution of eigenvalues of the matrix of coefficients of the highest order derivatives participate in Cordes' considerations. Under restrictions on the distribution of eigenvalues which are different from Cordes' assumptions, Harnack's inequality and Hölder continuity of solutions of elliptic and parabolic equations have been proved by Landis and his students (see [3]). In our work, similar restrictions are not imposed. As regards our methods, they are different from Cordes' methods, and are very close to Landis' methods. It should be emphasized that the basic source for us, not only of the methods but also of the ideas, is the exceptionally simply and clearly written book of Landis [3] and the probabilistic interpretation of the arguments given there.

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Notwithstanding that theorems concerning Harnack's inequality and Hölder continuity of solutions do not play such a large role for the theory of nondivergence equations as they do for the theory of divergence equations, it should nevertheless be noted that from our results, using results and methods which are well known in the theory of differential equations, quite significant information concerning linear and nonlinear equations can be obtained. The reader may familiarize himself with how this can be done in [1]–[4]. Here, let us just point out that the results of this paper automatically carry over to elliptic equations, by considering functions depending only on the space variables x as functions of (t, x) which are constant with respect to t, and formally adding a t-derivative to the elliptic operator.

This work has been discussed in seminars directed by E. M. Landis and V. A. Kondrat'ev, and by O. A. Ladyženskaja and N. N. Ural'ceva. The authors are sincerely grateful to them and to the students in those seminars for extremely useful discussions.

This article consists of four sections. In §1 the proof of Harnack's inequality is reduced to the investigation of the auxiliary function $\gamma(\beta)$. §2 is devoted to the preparation for the proof of the basic theorem concerning the properties of this function, which is proved in §3. In §4, Hölder continuity of solutions of parabolic equations is studied.

Throughout this article, ∂D is the boundary of the set D, $a \wedge b = \min(a, b)$, $a \vee b = \max(a, b)$, $a_+ = a \vee 0$, $a_- = (-a) \vee 0$, and $N = N(\cdots)$ means that the constant N depends only on the arguments contained in the parentheses.

§1. Derivation of Harnack's inequality from properties of the function $\gamma(\beta)$

Let E_d be a d-dimensional euclidean space with a fixed orthonormal basis, $E_{d+1} = \{(t, x): t \in (-\infty, \infty), x = (x^1, \ldots, x^d) \in E_d\}$, and let μ be a fixed number, $\mu \in (0, 1]$. Consider the parabolic operator

$$L = a^{ij}(t, x)\frac{\partial^2}{\partial x^i \partial x^j} + b^i(t, x)\frac{\partial}{\partial x^i} - c(t, x) - \frac{\partial}{\partial t}, ^{(1)} \tag{1.1}$$

concerning whose coefficients, besides measurability, we shall assume that for all $(t, x) \in E_{d+1}$ and $\lambda \in E_d$

$$\mu \|\lambda\|^2 \leqslant a^{ij}(t, x)\lambda^i\lambda^j \leqslant \mu^{-1}\|\lambda\|^2,$$

$$\|b(t, x)\| \leqslant \mu^{-1}, \quad 0 \leqslant c(t, x) \leqslant \mu^{-1}, \tag{1.2}$$

where

$$\|\lambda\|^2 = \sum_{i=1}^{d} (\lambda^i)^2, \quad b(t, x) = (b^1(t, x), \ldots, b^d(t, x)).$$

Denote

$$|x| = \max_{i=1,\ldots,d} |x^i|, \qquad Q(\theta, R) = (0, \theta R^2) \times \{x \in E_d : |x| < R\}.$$

We shall prove Harnack's inequality in the following form.

(1)Here, and throughout this article, repeated indices presume summation from 1 to d.

THEOREM 1.1. *Let $\theta > 1$, $R \leqslant 2$, $u \in W^{1,2}_{d+1}(Q(\theta, R))$, $u \geqslant 0$ in $Q(\theta, R)$, and $Lu =$ (a.e. $Q(\theta, R)$). Then there exists a constant N, depending only on θ, μ and d, such th $u(R^2, 0) \leqslant Nu(\theta R^2, x)$(2) for $|x| \leqslant \frac{1}{2}R$. Moreover, when $(1-\theta)^{-1}$ and μ^{-1} vary withi finite bounds, N also varies within finite bounds.*

The proof of this theorem is based on the use of properties of the function $\gamma(\beta)$, whic is introduced in the following way. For a region $Q \subset E_{d+1}$ we denote by $\mathfrak{A}(Q)$ the set c all functions $u \in W^{1,2}_{d+1}(Q)$ for each of which one can find an operator L of the forr (1.1) with coefficients satisfying (1.2), such that $Lu \leqslant 0$ (a.e. Q). Let $\mathfrak{A}^+(Q)$ be th subset of $\mathfrak{A}(Q)$ consisting of all functions u which are nonnegative on Q. Let $Q_1 =$ $Q(1, 1)$.

For a Borel set $\Gamma \subset E_{d+1}$, denote by $|\Gamma|$ the $d+1$-dimensional Lebesgue measure c Γ. For $\beta \in [0, 1]$, set

$$\mathfrak{A}_\beta = \mathfrak{A}^+(Q_1) \cap \{u : |Q_1 \cap \{(t, x) : u(t, x) \geqslant 1\}| \geqslant \beta |Q_1|\},$$

$$\gamma(\beta) = \inf\left\{u(1, x) : |x| \leqslant \frac{1}{2},\ u \in \mathfrak{A}_\beta\right\}.$$

Let us discuss some properties of the function $\gamma(\beta)$. It is understood that $0 \leqslant \gamma(\beta) \leqslant$ 1 and $\gamma(\beta)$ is nondecreasing with β. Further, if $u \in \mathfrak{A}^+(Q(1, R))$ for $R \leqslant 1$, then, as i easy to check, $u(tR^2, xR) \in \mathfrak{A}^+(Q_1)$. Hence, using the definition of $\gamma(\beta)$, as well as th parallel displacement transformation, it is easy to derive the following statement.

LEMMA 1.1. *Let $(t_0, x_0) \in E_{d+1}$, $R \in (0, 1]$, $Q = (t_0, x_0) + Q(1, R)$, $\beta \in [0, 1]$, $u \in$ $\mathfrak{A}^+(Q)$, $\varepsilon > 0$, and*

$$|Q \cap \{(t, x) : u(t, x) \geqslant \varepsilon\}| \geqslant \beta |Q|.$$

Then $u(t_0 + R^2, x) \geqslant \varepsilon\gamma(\beta)$ for $|x - x_0| \leqslant R/2$.

In terms of the function $\gamma(\beta)$, we prove a variant of the so-called growth lemmas c [3].

LEMMA 1.2. *Let $(t_0, x_0) \in E_{d+1}$, $R \in (0, 1]$, $Q = (t_0, x_0) + Q(1, R)$, $\beta \in [0, 1]$, $\nu >$ ($-u \in \mathfrak{A}(Q)$, $u(t_0 + R^2, x_0) \geqslant \nu$, and*

$$\left|Q \cap \left\{(t, x) : u(t, x) \leqslant \frac{\nu}{2}\right\}\right| \geqslant \beta |Q|.$$

Then

$$\sup_Q u(t, x) \geqslant \frac{\nu}{2}\left(1 + \frac{1}{1 - \gamma(\beta)}\right).$$

PROOF. Assume the contrary and set $w = (2u - \nu)/\nu$. Then, for some $\varepsilon > 0$, in Q w have

$$w \leqslant \frac{1}{1 - \gamma(\beta) + \varepsilon} \equiv a, \quad z \equiv 1 - \frac{w}{a} \in \mathfrak{A}^+(Q).$$

(2)The space $W^{1,2}_{d+1}(Q(\theta, R))$ is embedded in $C(\bar{Q}(\theta, R))$, i.e. for each function $u \in W^{1,2}_{d+1}(Q(\theta, R))$ the exists a continuous function on $\bar{Q}(\theta, R)$ coinciding with u (a.e.). We shall always be considering this functio

Moreover, wherever $u \leqslant v/2$, obviously $z \geqslant 1$. By Lemma 1.1 we have $z(t_0 + R^2, 0) \geqslant \gamma(\beta)$. The latter inequality gives $1 - 1/a \geqslant \gamma(\beta)$, since $w(t_0 + R^2, 0) \geqslant 1$. Hence, right away we get a contradiction to the definition of a. The lemma is proved.

We shall prove the following theorem concerning the function γ in §3.

THEOREM 1.2. *$\gamma(\beta) > 0$ for $\beta > 0$, and $\gamma(\beta)\uparrow 1$ as $\beta \uparrow 1$.*

For the proof of Theorem 1.1, we also need

LEMMA 1.3. *Let $R \in (0, 2]$, $\kappa \in (0, 1]$, $\varepsilon \in (0, 1]$, $\kappa R^2 \leqslant \tau \leqslant \kappa^{-1}R^2$, $x_1^i < x_2^i$, $2\kappa R \leqslant x_2^i - x_1^i \leqslant \kappa^{-1}R$, $x_1^i + \kappa R \leqslant x_0^i \leqslant x_2^i - \kappa R$, $i = 1, \dots, d$, $Q = \{(t, x): 0 < t < \tau,\ x_1^i < x^i < x_2^i,\ i = 1, \dots, d\}$, $u \in \mathfrak{A}^+(Q)$, $u(0, x) \geqslant 1$ for $(0, x) \in \overline{Q}$, and $|x - x_0| \leqslant \varepsilon R$. Then there exist constants $\delta = \delta(\mu, d, \kappa) > 0$ and $m = m(\mu, d, \kappa)$ such that $u(t, x) \geqslant \delta\varepsilon^m$ for $(t, x) \in \overline{Q}$, $\kappa R^2 \leqslant t \leqslant \tau$, $x_1^i + \kappa R \leqslant x^i \leqslant x_2^i - \kappa R$, $i = 1, \dots, d$.*

PROOF. First of all, using the transformation

$$(t, x) \to \left(\frac{t}{R^2}, \frac{x - x_0}{R}\right),$$

we easily reduce the whole question to the case $R = 1$, $x_0 = 0$.

Fix a point $(t_*, x_*) \in \overline{Q}$ such that $\kappa \leqslant t_* \leqslant \tau$ and $x_1^i + \kappa \leqslant x_*^i \leqslant x_2^i - \kappa$, $i = 1, \dots, d$. We must prove that $u(t_*, x_*) \geqslant \delta\varepsilon^m$. In this regard, it suffices to consider the case $t_* = \tau$ (instead of Q one can take $Q \cap \{0 < t < t_*\}$). Moreover, without loss of generality, we shall assume that $2\varepsilon^2 < \kappa^2$.

Set $\xi = \frac{1}{2}\kappa^3$ and $y = \tau^{-1}x_*$. Obviously, $|y| = \tau^{-1}|x_*| \leqslant \kappa^{-2}$. Let us introduce the set

$$D = \{(t, x): 0 < t < \tau,\ \|x - ty\|^2 < \xi t + \varepsilon^2\}.$$

It is entirely contained in the oblique cylinder $\{(t, x): 0 < t < \tau,\ \|x - ty\|^2 < \xi\tau + \varepsilon^2\}$. Since $\xi\tau + \varepsilon^2 \leqslant \kappa^2$, and $\tau y = x_*$, the bases of this cylinder belong to $\overline{Q}$; consequently the cylinder itself, and along with it D, also, belongs to Q.

So, $D \subset Q$. The parabolic boundary of D is the union of the sets $\partial_1 D$ and $\partial_2 D$, where $\partial_1 D = \{(t, x): 0 \leqslant t \leqslant \tau,\ \|x - ty\|^2 = \xi t + \varepsilon^2\}$ and $\partial_2 D = \{(t, x): t = 0,\ \|x\| < \varepsilon\}$. For $(t, x) \in \overline{D}$, set

$$z = z(t, x) = \frac{x - ty}{\sqrt{\xi t + \varepsilon^2}},$$

$$r = \|z\|^2 = \frac{\|x - ty\|^2}{\xi t + \varepsilon^2}, \quad \psi = \psi(t, x) = \frac{(1 - r)^2}{(\xi t + \varepsilon^2)^n},$$

where we shall define the constant n below. It is clear that $0 \leqslant r < 1$ on D and $r = 1$ on $\partial_1 D$.

Further, let the operator L of the form (1.1), satisfying (1.2), be such that $Lu \leqslant 0$ (a.e. Q). A simple computation shows that

$$(\xi t + \varepsilon^2)^{n+1} L\psi = A_1 + A_2 + A_3,$$

where

$$A_1 = (1 - r)^2[n\xi - (\xi t + \varepsilon^2)c],$$

$$A_2 = 8a^{ij}z^iz^j,$$

$$A_3 = 2(r - 1)[2\delta^{ij}a^{ij} + 2(x^i - ty^i)b^i + 2(x^i - ty^i)y^i + \xi r].$$

We have $A_2 \geqslant 8\mu\|z\|^2 = 8\mu r$. Moreover, all the terms in square brackets in the expression for A_3 are bounded in modulus by constants depending only on μ, d and κ. Upon choosing $r_0 = r_0(\mu, d, \kappa) < 1$ sufficiently close to 1, we get $A_2 + A_3 \geqslant 0$ for $r_0 \leqslant r \leqslant 1$. For $0 \leqslant r \leqslant r_0$ we have $(1 - r)^2 \geqslant (1 - r_0)^2 > 0$; therefore, by choosing a sufficiently large $n = n(\mu, d, \kappa)$, one can arrange for the inequality $A_1 + A_2 + A_3 \geqslant 0$ to hold on $\overline{D}$.

Thus, $L\psi \geqslant 0$ on $\overline{D}$ for an appropriate choice of n. We have $L(u - \varepsilon^{2n}\psi) \leqslant 0$ (a.e. D), $u - \varepsilon^{2n}\psi = u \geqslant 0$ on $\partial_1 D$ and $u - \varepsilon^{2n}\psi \geqslant 1 - \varepsilon^{2n}\psi = 1 - (1 - r)^2 \geqslant 0$ on $\partial_2 D$. By the maximum principle (see [7] or [8]), $u \geqslant \varepsilon^{2n}\psi$ on $\overline{D}$. In particular,

$$u(t_*, x_*) = u(\tau, x_*) \geqslant \varepsilon^{2n}\psi(\tau, x_*) = \frac{\varepsilon^{2n}}{(\xi\tau + \varepsilon^2)^n} \geqslant \frac{\varepsilon^{2n}}{\kappa^{2n}}.$$

The lemma is proved.

PROOF OF THEOREM 1.1. Using the transformation $(t, x) \to (4R^{-2}t, 2R^{-1}x)$, we can easily see that it suffices to prove the theorem only for $R = 2$. If, once again, a suitable transformation of the form $t \to \alpha(t)$ is used, then the general case can be reduced to the case $\theta = 2$. For $\theta = R = 2$, obviously, the optimal desired constant N is a decreasing function of μ. Therefore the last statement of the theorem is a consequence of the first and the above arguments.

In the proof of the first statement, we shall assume as before that $\theta = R = 2$. We find, by Lemma 1.3, constants δ and m for $\kappa = \frac{1}{2}$, and by Theorem 1.2 we determine $\beta > 0$ so that

$$\frac{1}{2}\left(1 + \frac{1}{1 - \gamma(1 - \beta)}\right) \geqslant 2^m.$$

Set

$$v(r) = u(4, 0)(1 - r)^{-m} \quad (r \in [0, 1)),$$

$$Q(r) = \{(t, x): |x| \leqslant r,\ 0 \leqslant 4 - t \leqslant r^2\}, \quad n(r) = \max_{Q(r)} u(t, x).$$

Define r_0 as the largest root of the equation $n(r) = v(r)$. Since $n(0) = v(0)$, $v(r) \to \infty$ as $r \uparrow 1$, and u is continuous and bounded on $Q(2, 2)$, it follows that r_0 is well defined and $r_0 < 1$. Let $(t_1, x_1) \in Q(r_0)$ and $v(r_0) = u(t_1, x_1)$.

Denote

$$Q = \left\{(t, x): 0 < t_1 - t < \frac{(1 - r_0)^2}{4},\ |x - x_1| < \frac{1 - r_0}{2}\right\}.$$

It is easy to check that $\overline{Q} \subset Q((1 + r_0)/2)$, and so, by the definition of r_0, on $\overline{Q}$ we have

$$u < u(4, 0)\left(1 - \frac{1 + r_0}{2}\right)^{-m} = 2^m v(r_0) \leqslant \frac{v(r_0)}{2}\left(1 + \frac{1}{1 - \gamma(1 - \beta)}\right).$$

Hence, by Lemma 1.2, we conclude that

$$\left|Q \cap \left\{u \leqslant \frac{v(r_0)}{2}\right\}\right| < (1 - \beta)|Q|, \quad \left|Q \cap \left\{u > \frac{v(r_0)}{2}\right\}\right| > \beta|Q|.$$

By Lemma 1.1, for $|x - x_1| \leqslant (1 - r_0)/4$ we get $u(t_1, x) \geqslant \frac{1}{2}\nu(r_0)\gamma(\beta)$. Finally, by Lemma 1.3 ($\varepsilon = (1 - r_0)/8$ and $\tau = 8 - t_1$), for $|x| \leqslant 1$ we have

$$u(8, x) \geqslant \delta\left(\frac{1-r_0}{8}\right)^m \frac{\nu(r_0)}{2}\gamma(\beta) = \delta 2^{-3m-1}\gamma(\beta)\, u(4, 0).$$

The theorem is proved.

§2. Some facts from measure theory

In this section, we shall prove some auxiliary results from measure theory which are needed for the proof of Theorem 1.2. Statements of the type of Lemma 2.1, which are close in spirit as well as proof to the Hertz-Stein theorem (see Theorem 5.1 in Chapter II of [6]), E. M. Landis calls (aptly in our opinion) lemmas on the "crawling of ink spots". In the same style, Lemma 2.3 can be called a lemma on the "crawling of ink spots in the wind", and Lemma 2.4 a lemma on the "crawling of ink spots in a hot wind".

Consider a measurable set $\Gamma \subset Q_1 = Q(1, 1)$ and numbers $\xi, \eta, \zeta \in (0, 1)$. We shall construct, given Γ, ξ, η and ζ, the open sets D^1, D^2 and D^3. For this, let us denote by $\mathfrak{B}$ the system of all sets Q of the form $(t_0, x_0) + Q(1, R)$ such that $Q \subset Q_1$, $|Q \cap \Gamma| > \xi|Q|$. If $Q = (t_0, x_0) + Q(1, R) \in \mathfrak{B}$, then we set

$$Q^1 = \left\{(t_0 - 3R^2, x_0) + Q\left(\frac{7}{9}, 3R\right)\right\} \cap Q_1,$$

$$Q^2 = \left\{(t, x) : t_0 + R^2 < t < t_0 + R^2 + \frac{4R^2}{\eta},\ |x_0 - x| < 3R,\ |x| < 1\right\},$$

$$Q^3 = \left\{\left(t_0 + R^2 + \frac{4R^2}{\eta},\ x_0\right) + (t, x) : \left(t_0 + R^2 + \frac{4R^2}{\eta}, x_0\right) + (\zeta^{-2}t, \zeta^{-1}x) \in Q^2\right\}.$$

We observe that the box Q^1 is obtained by intersecting Q_1 with a box that is constructed from boxes congruent to Q (to within a boundary) as from bricks: three layers of bricks on each side of Q in the t-direction, each layer consisting of a central brick congruent to Q and adjoined on all sides and corners in the x-direction by similar bricks in a single layer. The box Q^2 is set against Q on the right in the t-direction; the dimensions in the x- and y-directions are the same as for Q^1 but $4/\eta$ times longer in the t-direction. It is quite possible that Q^2 will extend beyond the boundary of Q_1. We note for what follows that if $A(Q)$ is the length of $Q^1 \cup Q^2$ in the t-direction and $B(Q)$ is the length of Q^2 in the t-direction ($B(Q) = 4R^2\eta^{-1}$), then

$$A(Q) \leqslant (1+\eta)B(Q). \tag{2.1}$$

The box Q^3 is obtained from Q^2 by a contraction with center lying on the right side, whose space coordinate is x_0. In the t-direction this contraction is by a factor ζ^{-2}, and in the x-direction by a factor ζ^{-1}.

Finally, we write

$$D^i = \bigcup_{Q \in \mathfrak{B}} Q^i, \quad i = 1, 2, 3.$$

Since Q^i is open, D^i is also open.

LEMMA 2.1. *$|\Gamma \setminus D^1| = 0$, and if $|\Gamma| \leqslant \xi|Q_1|$, then $|\Gamma| \leqslant \xi|D^1|$.*

PROOF. The first assertion follows from the fact that almost every point of Γ is a point of density and $\xi < 1$. Therefore almost every point of Γ belongs to one of the sets $Q \in \mathfrak{B}$ for Q of sufficiently small dimensions.

Next, let us represent the open set D^1, up to a set of measure zero, as the union of "binary" boxes of the form $(t_0, x_0) + Q(1, R)$. We do this in the following way. Partition the box Q_1 with hyperplanes $t = \frac{1}{4}$, $t = \frac{1}{2}$, $t = \frac{3}{4}$ and $x^i = 0$, $i = 1, \ldots, d$, into 2^{d+2} congruent boxes, to which we shall not associate their boundaries. Then Q_1 is represented, up to a set of measure zero, as the union of the open boxes $Q(i_1)$, $i_1 = 1, \ldots, 2^{d+2}$, each of which is congruent to $\{(t, x)\colon 0 < t < \frac{1}{4},\ 0 < x^i < 1,\ i = 1, \ldots, d\}$.

We define by induction the set of boxes $\mathfrak{C}_n$. In $\mathfrak{C}_1$ we include those and only those $Q(i_1)$ which lie in D^1. If $Q(i_1) \not\subset D^1$, then, analogously to Q_1, we partition $Q(i_1)$ into 2^{d+2} boxes $Q(i_1, i_2)$, $i_2 = 1, \ldots, 2^{d+2}$, each of which is congruent to $\{(t, x)\colon 0 < t < \frac{1}{4^2},\ 0 < x^i < \frac{1}{2},\ i = 1, \ldots, d\}$. In $\mathfrak{C}_2$ we include those and only those $Q(i_1, i_2)$ for which $Q(i_1) \not\subset D^1$ but $Q(i_1, i_2) \subset D^1$. If $Q(i_1) \in \mathfrak{C}_1$ for all i_1, then we put $\mathfrak{C}_2$ equal to the empty set. If $Q(i_1, \ldots, i_n)$ and $\mathfrak{C}_n$ are already constructed, we define $Q(i_1, \ldots, i_{n+1})$ by partitioning the boxes $Q(i_1, \ldots, i_n)$ not belonging to D^1 into 2^{d+2} boxes, each of which is congruent to $\{(t, x)\colon 0 < t < 1/4^{n+1},\ 0 < x^i < 1/2^n,\ i = 1, \ldots, d\}$. Afterwards, we include in $\mathfrak{C}_{n+1}$ those and only those $Q(i_1, \ldots, i_{n+1})$ for which $Q(i_1, \ldots, i_n) \not\subset D^1$ but $Q(i_1, \ldots, i_{n+1}) \subset D^1$.

It is easy to see that distinct elements of $\mathfrak{C}_n$ are disjoint, as are elements of $\mathfrak{C}_m$ and $\mathfrak{C}_n$, for $m \neq n$. Moreover,

$$|D^1 \setminus (\bigcup_{n \geqslant 1} \bigcup_{Q \in \mathfrak{C}_n} Q)| = 0, \quad |D^1| = \sum_{n=1}^{\infty} \sum_{Q \in \mathfrak{C}_n} |Q|,$$

$$|\Gamma| = |\Gamma \cap D^1| = \sum_{n=1}^{\infty} \sum_{Q \in \mathfrak{C}_n} |\Gamma \cap Q|. \tag{2.2}$$

Note that if $Q(i) \subset D^1$ for all i, then $|D^1| = |Q_1|$ and there is nothing to prove. Therefore it can be assumed that $Q(i) \not\subset D^1$ for some i. Under this hypothesis, let us prove that for all $n \geqslant 1$ and $Q \in \mathfrak{C}_n$

$$|\Gamma \cap Q| < \xi|Q|. \tag{2.3}$$

By (2.2) this suffices for the proof of the lemma. Assume the contrary. Let, for example, $|\Gamma \cap Q(i_0)| \geqslant \xi|Q(i_0)|$. Then $Q(i_0) \in \mathfrak{B}$. Set $Q = Q(i_0)$ and construct, given Q, the set Q^1. Obviously $Q^1 = Q_1$; but then $D^1 = Q_1$ and $Q(i) \subset D^1$ for all i, in contradiction to the hypothesis that $Q(i) \not\subset D^1$ for some i. Let $Q = Q(i_1, \ldots, i_{n+1}) \in \mathfrak{C}_{n+1}$ for $n \geqslant 1$ and $|\Gamma \cap Q| \geqslant \xi|Q|$. Then $Q \in \mathfrak{B}$, and obviously $Q(i_1, \ldots, i_n) \subset Q^1 \subset D^1$. This, however, is impossible, since $\mathfrak{C}_{n+1}$ contains portions of partitions only of those $Q(i_1, \ldots, i_n)$ which do not lie entirely in D^1. This contradiction proves (2.3), and with it the lemma.

LEMMA 2.2. *Let $\kappa \geqslant 1$ and $-\infty \leqslant t_1 < t_2 \leqslant \infty$, let A be the set of all subintervals of (t_1, t_2), let $B \subset A$, and let the function $g\colon A \to A$ be defined so that $|g(I)| \leqslant \kappa|I|$ for $I \in A$, where $|g(I)|$ and $|I|$ are the (one-dimensional) Lebesgue measures of $g(I)$ and I. Further, let $g(I_1) \subset g(I_2)$ if $I_1 \subset I_2$. Then*

$$|\bigcup_{I \in B} g(I)| \leqslant \kappa |\bigcup_{I \in B} I|.$$

PROOF. The set $\bigcup\{I: I \in B\}$ is open and can be represented as the union of certain nonintersecting intervals I_n. Therefore

$$\Big|\bigcup_{I\in B} g(I)\Big| \leqslant \Big|\bigcup_n \bigcup_{I\subset I_n} g(I)\Big| \leqslant \Big|\bigcup_n g(I_n)\Big| \leqslant \sum_n \varkappa |I_n| = \varkappa \Big|\bigcup_{I\in B} I\Big|,$$

which was to be shown.

LEMMA 2.3. $|D^1| \leqslant (1+\eta)|D^2|$.

PROOF. Obviously it suffices to prove that $|D^1 \cup D^2| \leqslant (1+\eta)|D^2|$, and for that, by Fubini's theorem, it suffices to prove that for any x the one-dimensional Lebesgue measure of the set $\{t: (t,x) \in D^1 \cup D^2\}$ does not exceed the product of $1+\eta$ and the measure of $\{t: (t,x)\in D^2\}$. For the proof of the latter, in Lemma 2.2 take $\kappa = 1+\eta$, $t_1 = -\infty$, $t_2 = \infty$ and $B = \{\{t: (t,x) \in Q^2\}: Q \in \mathfrak{B}\}$, and define $g(I)$ as the result of expanding I by a factor of κ, with center at the right endpoint of I. After this, the assertion we require will follow from (2.1). The lemma is proved.

The set Q^3 can be obtained from Q^2 as the result of the $(d+1)$st successive shrinking, of which the first proceeds along the t-axis with coefficient ζ^{-2}, and all the rest take place along each coordinate axis separately with coefficient ζ^{-1}. The first shrinking leads to a decrease in the volume of D^2 by no more than a factor of ζ^{-2}. This is shown as in the preceding lemma. Each of the shrinkings along the x-coordinate axes leads to a decrease in volume by no more than a factor of ζ^{-1}. Let us prove this using Lemma 2.2. By Fubini's theorem, it suffices to prove that if B_1 is some set of intervals whose centers lie in $(-1, 1)$, and B is the set of all intervals each of which is obtained as the result of shrinking the intersection of some interval $I \in B_1$ with $(-1, 1)$ towards the center of I by a factor of ζ^{-1}, then

$$\Big|(-1, 1) \cap \bigcup_{I_1\in B_1} I_1\Big| \leqslant \zeta^{-1} \Big|\bigcup_{I\in B} I\Big|. \tag{2.4}$$

We take $t_1 = -t_2 = -1$ and $\kappa = \zeta^{-1}$ in Lemma 2.2, and define g. For $I \in A$, let $f(I)$ be the interval which is the expansion of I by a factor of ζ^{-1} from the center of I. If $f(I) \subset (-1, 1)$, set $g(I) = f(I)$. If $1 \in f(I)$ or $-1 \in f(I)$, let $g(I)$ be the intersection with $(-1, 1)$ of the interval with right (respectively, left) endpoint at the point 1 (-1) and of length equal to that of $f(I)$. It is easy to see that if the interval $I \in B$ comes from $I_1 \in B_1$, then $g(I) = I_1 \cap (-1, 1)$. Therefore (2.4) follows from Lemma 2.2. These arguments show the validity of

LEMMA 2.4. $|D^2| \leqslant \zeta^{-(d+2)}|D^3|$.

Let us sum up the results of this section.

LEMMA 2.5. *If* $|\Gamma| \leqslant \xi|Q_1|$, *then* $|\Gamma| \leqslant \xi(1+\eta)\zeta^{-(d+2)}|D^3|$.

§3. Proof of Theorem 1.2

For the proof of Theorem 1.2, we need three lemmas, in the latter two of which are proved, separately, both assertions of Theorem 1.2.

LEMMA 3.1. *There exists a constant* $\varepsilon = \varepsilon(\mu, d) > 0$ *such that for* $\lambda \geqslant 1$, *for any operator* L *of the form* (1.1) *with coefficients satisfying condition* (1.2), *and for* $\varphi(t,x) = \cosh[\varepsilon\sqrt{\lambda}\,(\|x\|^2 - t)]$, *on* Q_1 *we have* $L\varphi - \lambda\varphi \leqslant 0$.

PROOF. Using a simple computation, we see that

$$\lambda^{-1}\varphi^{-1}(L\varphi - \lambda\varphi) = 4a^{ij}x^ix^j\varepsilon^2 - 1 - \frac{c}{\lambda} + (2\delta^{ij}a^{ij} + 2x^ib^i + 1)\,\varepsilon\lambda^{-\frac{1}{2}} \tanh\,[\varepsilon\sqrt{\lambda}(\|x\|^2 - t)].$$

Hence, from the boundedness of the function $\tanh r$ and the boundedness of the region Q_1, the statement follows in an almost obvious fashion. The lemma is proved.

LEMMA 3.2. $\gamma(\beta)\uparrow 1$ *as* $\beta\uparrow 1$.

PROOF. Let $u \in \mathfrak{A}_\beta$, and let L be an operator of the form (1.1) satisfying conditions (1.2) and such that $Lu \leqslant 0$ (a.e. Q_1). Let $|x_0| \leqslant \frac{1}{2}$, and fix $\gamma \in (0, 1)$. Set

$$Q = \left\{(t, x): \frac{3}{4} < t < 1,\ \|x - x_0\|^2 \leqslant \frac{1}{4}\right\},$$

$$\psi(t,x) = \varphi(t-1, x-x_0)\left(\cosh\frac{\varepsilon\sqrt{\lambda}}{4}\right)^{-1}, \qquad v = u + \psi - 1 + \frac{1}{\lambda\mu},$$

where φ and ε are defined in Lemma 3.1, and choose the constant $\lambda \geqslant 1$ so that

$$1 - \frac{1}{\lambda\mu} - \left(\cosh\frac{\varepsilon\sqrt{\lambda}}{4}\right)^{-1} \geqslant \gamma.$$

It is understood that $Q \subset Q_1$, and for $\frac{3}{4} \leqslant t \leqslant 1$ and $\|x - x_0\|^2 = \frac{1}{4}$ or $t = \frac{3}{4}$ and $\|x - x_0\|^2 \leqslant \frac{1}{4}$ we have $\psi \geqslant 1$, $u \geqslant 0$ and $v \geqslant 1/\lambda\mu \geqslant 0$. Moreover, on Q

$$Lv - \lambda v = Lu + L\psi - \lambda\psi + \lambda(1-u) + (c - \mu^{-1}) - \frac{c}{\lambda\mu} \leqslant \lambda(1-u)_+.$$

Since $\lambda \geqslant 1$, we have $\|b\| \leqslant \mu^{-1}(c + \lambda)$, and by Theorem 3 of [7] or Theorem 3.2 of [8] we get that there exists a constant N, depending only on μ and d, such that

$$v(1, x_0) \geqslant -N\|\lambda(1-u)_+\|_{\mathcal{L}_{d+1}(Q)}. \tag{3.1}$$

From (3.1) we conclude that

$$u(1, x_0) \geqslant 1 - \psi(1, x_0) - \frac{1}{\lambda\mu} - N\lambda\,|Q \cap \{u < 1\}|^{\frac{1}{d+1}}$$

$$\geqslant \gamma - N\lambda(|Q_1| - |Q_1 \cap \{u \geqslant 1\}|)^{\frac{1}{d+1}} \geqslant \gamma - N\lambda|Q_1|^{\frac{1}{d+1}}(1-\beta)^{\frac{1}{d+1}}.$$

Consequently,

$$\varliminf_{\beta\uparrow 1} \gamma(\beta) \geqslant \gamma, \tag{3.2}$$

and since γ can be taken to be an arbitrary number in (0, 1), the lower limit in (3.2) is not less than 1. On the other hand, as was already mentioned, $\gamma(\beta) \leqslant 1$ and $\gamma(\beta)$ is nondecreasing with β. The lemma is proved.

The following lemma completes the proof of Theorem 1.2.

LEMMA 3.3. $\gamma(\beta) > 0$ *for* $\beta > 0$.

PROOF. Let β^* be the upper bound of those β for which $\gamma(\beta) = 0$. Obviously, $\gamma(\beta) = 0$ for $\beta \in [0, \beta^*)$, if $\beta^* > 0$. We need to prove that $\beta^* = 0$. Assume the contrary: $\beta^* > 0$. By Lemma 3.2, we find $\xi \in (0, 1)$ so that $\gamma(\xi^2) \geqslant \frac{1}{2}$. It is understood that $\beta^* \leqslant \xi^2$. We further determine numbers $\beta_1 < \beta^* < \beta_2$ and $\eta \in (0, 1)$ so that

$$\beta_2 \leqslant 1, \quad \beta^* < \frac{\beta_1}{(1+\eta)\sqrt{\xi}}, \quad \beta_2 = \frac{1}{2}\left(\beta^* + \frac{\beta_1}{(1+\eta)\sqrt{\xi}}\right).$$

Since $\beta_1 < \beta^* < \beta_2$, we have $\gamma(\beta_2) > 0$ and $\gamma(\beta_1) = 0$. Now let $u \in \mathfrak{A}_{\beta_1}$ and $\Gamma = Q_1 \cap \{u \geqslant 1\}$. If $|\Gamma| \geqslant \xi|Q_1|$, then

$$u(1, x) \geqslant \gamma(\xi) \geqslant \frac{1}{2}, \quad |x| \leqslant \frac{1}{2}. \tag{3.3}$$

Let us see how we might estimate $u(1, x)$ for $|\Gamma| < \xi|Q_1|$: For that, take $\zeta = \xi^{1/2(d+2)}$, and for selected Γ, ξ, η and ζ, as in §2, construct the sets D^2 and D^3. We consider two cases:

1) $|D^3 \setminus Q_1| \leqslant (\beta_2 - \beta^*)|Q_1|$,
2) $|D^3 \setminus Q_1| > (\beta_2 - \beta^*)|Q_1|$.

In the first case, by Lemma 2.5,

$$\beta_1|Q_1| \leqslant |\Gamma| \leqslant \xi(1+\eta)\xi^{-\frac{1}{2}}|D^3|, \quad \frac{\beta_1}{(1+\eta)\sqrt{\xi}}|Q_1| \leqslant |D^3|,$$

$$\beta_2|Q_1| = \left[\frac{\beta_1}{(1+\eta)\sqrt{\xi}} - (\beta_2 - \beta^*)\right]|Q_1| \leqslant |D^3 \cap Q_1|. \tag{3.4}$$

Further, for $Q = (t_0, x_0) + Q(1, R) \in \mathfrak{B}$ we have $u(t_0 + R^2, x) \geqslant \gamma(\xi) \geqslant \frac{1}{2}$, if $|x - x_0| < \frac{1}{2}R$ (Lemma 1.1). Taking $t_0 + R^2$ as the coordinate origin along the t-axis, from Lemma 1.3 ($\kappa = \eta/4 \wedge (1 - \zeta)$ and $\varepsilon = \frac{1}{2}$) we easily find that there exists a constant $\delta = \delta(\mu, d, \eta) > 0$ such that $u(t, x) \geqslant \delta$ in $Q^3 \cap Q_1$. Since Q is an arbitrary element of $\mathfrak{B}$, $u \geqslant \delta$ on $D^3 \cap Q_1$. Consequently, by (3.4) and the definition of $\gamma(\beta_2)$,

$$u(1, x) \geqslant \delta\gamma(\beta_2), \quad |x| \leqslant \frac{1}{2}. \tag{3.5}$$

In the second case, $|D^2 \setminus Q_1| > (\beta_2 - \beta^*)|Q_1|$, and hence D^2 has nonempty intersection with the hyperplane $t = 1 + (\beta_2 - \beta^*)$. Hence a box $Q = (t_0, x_0) + Q(1, R) \in \mathfrak{B}$ can be found for which Q^2 intersects this hyperplane. Then

$$|x_0| \leqslant 1 - R, \quad 1 + (\beta_2 - \beta^*) < t_0 + R^2 + \frac{4R^2}{\eta} \leqslant 1 + \frac{4R^2}{\eta},$$
$$R^2 > \frac{1}{4}\eta(\beta_2 - \beta^*) \equiv R_0^2.$$

Denote

$$\alpha = (1+\xi)^{-\frac{1}{d+2}}, \quad t_1 = 1 - (1 - \alpha^2)R_0^2, \quad Q' = (t_0, x_0) + Q(1, \alpha R).$$

Since $t_0 + R^2 \leqslant 1$, we have $t_0 + \alpha^2R^2 \leqslant t_1 < 1$. Moreover,

$$|\Gamma \cap Q'| \geqslant |\Gamma \cap Q| - |Q \setminus Q'| \geqslant \xi|Q| - |Q|(1 - \alpha^{d+2}) = \xi^2|Q'|.$$

By the choice of ξ we have $\gamma(\xi^2) \geqslant \frac{1}{2}$; consequently, $u(t_0 + \alpha^2R^2, x) \geqslant \frac{1}{2}$ for $|x - x_0| < \frac{1}{2}\alpha R$; in particular, for $|x - x_0| \leqslant \frac{1}{2}\alpha R_0$. By the inequality $t_0 + \alpha^2R^2 \leqslant t_1 < 1$, there exists by Lemma 1.3 a constant $\delta_1 > 0$, depending only on μ, d, t_1 and αR_0, such that

$$u(1, x) \geqslant \delta_1, \quad |x| \leqslant \frac{1}{2}. \tag{3.6}$$

Finally, in all cases, from (3.3), (3.5) and (3.6) we have

$$u(1, x) \geqslant \min\left(\delta_1, \delta\gamma(\beta_2), \frac{1}{2}\right) \equiv \delta_2, \quad |x| \leqslant \frac{1}{2},$$

where $\delta_2 > 0$ and does not depend on $u \in \mathfrak{A}_{\beta_1}$ or $x \in \{y: |y| \leqslant \frac{1}{2}\}$. Consequently, $\gamma(\beta_1) \geqslant \delta_2 > 0$, which contradicts the equality $\gamma(\beta_1) = 0$ obtained previously. The lemma is proved.

§4. Hölder continuity of solutions of parabolic equations

As in §1, we shall consider parabolic operators (1.1) with coefficients satisfying conditions (1.2). Moreover, we define

$$C_r(t_0, x_0) = \{(t, x) : t_0 - r^2 < t < t_0, \|x - x_0\| < r\},$$

and for $z_1 = (t_1, x_1)$ and $z_2 = (t_2, x_2)$ we let

$$|z_1 - z_2| = \|x_1 - x_2\| + |t_1 - t_2|^{\frac{1}{2}}.$$

LEMMA 4.1. *Let $u \in W^{1,2}_{d+1}(C_{2r}(t_0, x_0))$, $r \leqslant 1$, $Lu = 0$ (a.e. $C_{2r}(t_0, x_0)$) and $c \equiv 0$. Then there exist constants $N = N(\mu, d)$ and $\alpha = \alpha(\mu, d) > 0$ such that for any $z_1, z_2 \in \overline{C}_r(t_0, x_0)$*

$$|u(z_1) - u(z_2)| \leqslant Nr^{-\alpha}|z_1 - z_2|^{\alpha} \sup_{C_{2r}(t_0, x_0)} |u(t, x)|. \tag{4.1}$$

If, moreover, $b \equiv 0$, then (4.1) is true for $r \geqslant 1$ as well.

PROOF. Using the transformation $(t, x) \to (r^{-2}t, r^{-1}x)$, we can easily see that it suffices to prove (4.1) just for $r = 1$. We assume that $r = 1$ and set $D = C_2(t_0, x_0)$. As is well known (see, for example, [1] or [3]), for the proof of (4.1) it suffices to prove the existence of a constant $\xi = \xi(\mu, d) > 1$ such that as soon as $R \in (0, 1]$ and $Q = (t_1, x_1) + Q(1, R) \subset D$, then for $Q' = (t_0 + \frac{3}{4}R^2, x_0) + Q(1, \frac{1}{2}R)$

$$\operatorname{osc}\{u; Q\} \geqslant \xi \operatorname{osc}\{u; Q'\}, \tag{4.2}$$

where $\operatorname{osc}\{u; \Gamma\}$ is understood to be the difference between the upper and lower bounds of u on Γ. Again we use standard arguments from [1] and [3], which, using coordinate shift and scaling transformations, as well as consideration of $N_1u + N_2$ in place of u(3)

for suitable constants N_1 and N_2, reduce the proof of (4.2) to the proof of the inequality

$$\sup_{Q_1} v(t, x) \geqslant \eta(\mu, d) > 1 \tag{4.3}$$

under the hypothesis that the function v given on $\overline{Q}_1$ satisfies the equation $Lv = 0$ (a.e. Q_1), its maximum on the closure of $Q' = (\frac{3}{4}, 0) + Q(1, \frac{1}{2})$ is equal to $+1$, its minimum on $\overline{Q}'$ is equal to -1, and

$$|Q_1 \cap \{v \leqslant 0\}| \geqslant \frac{1}{2}|Q_1|. \tag{4.4}$$

Further, if $(t_1, x_1) \in \overline{Q}'$ and $v(t_1, x_1) = 1$, then using the transformation $t \to t_1^{-1}t$, one can arrange for the function $\tilde{v}(t, x) = v(t_1 t, x)$ to equal $+1$ at the point $(1, x_1)$ and satisfy the equation $\tilde{L}\tilde{v} = 0$ (a.e. Q_1) for some operator $\tilde{L}$ whose coefficients satisfy (1.2) with constant $\tilde{\mu} = \tilde{\mu}(\mu) > 0$ not depending on t_1 (note that $\frac{3}{4} \leqslant t_1 \leqslant 1$). In this case (4.4) guarantees that

$$|Q_1 \cap \{\tilde{v} \leqslant 0\}| \geqslant \frac{1}{4}|Q_1|. \tag{4.5}$$

Let us show that if $\gamma(\beta)$ is calculated according to $\tilde{\mu}$ instead of μ, then

$$\sup_{Q_1} \tilde{v}(t, x) \geqslant \frac{1}{1 - \gamma\left(\frac{1}{4}\right)}. \tag{4.6}$$

Assume the contrary. Then, for some $\varepsilon > 0$, on Q_1 we have $w \equiv 1 - (1 - \gamma(\frac{1}{4}) + \varepsilon)\tilde{v} \geqslant 0$. Moreover, wherever $\tilde{v} \leqslant 0$, clearly $w \geqslant 1$. By (4.5) we conclude that $w \in \mathfrak{A}_{1/4}$ (in the definition of $\mathfrak{A}_\beta$ we take $\tilde{\mu}$ instead of μ). By the definition of $\gamma(\frac{1}{4})$ we get $w(1, x) \geqslant \gamma(\frac{1}{4})$ for $|x| \leqslant \frac{1}{2}$. Since $|x_1| \leqslant \frac{1}{2}$, we have $\gamma(\frac{1}{4}) \leqslant w(1, x_1) = \gamma(\frac{1}{4}) - \varepsilon$, which is impossible. Inequality (4.6) is proved. From it (4.3) obviously follows. The lemma is proved.

COROLLARY 4.1 (Two-sided Liouville's Theorem). *If $c \equiv 0$, $b \equiv 0$, $u \in W^{1,2}_{d+1}(C_r(0, 0))$ for all r, $Lu = 0$ almost everywhere in $(-\infty, 0) \times E_d$ and u is bounded on this set, then* $u = \text{const}$.

Indeed, from (4.1) as $r \to \infty$ it follows that $u = \text{const}$.

LEMMA 4.2. *Let $u \in W^{1,2}_{d+1}(C_{2r}(t_0, x_0))$, $r \leqslant 1$, and $u = 0$ on the left face and the lateral boundary of the cylinder $C_{2r}(t_0, x_0)$. Then there exists a constant $N = N(\mu, d)$ such that on $\overline{C}_{2r}(t_0, x_0)$*

$$|u| \leqslant N r^{\frac{d}{d+1}} \| Lu \|_{\mathscr{L}_{d+1}(C_{2r}(t_0, x_0))}. \tag{4.7}$$

PROOF. As in the previous proof, it suffices just to consider $r = 1$. By Theorem 3 of [7], there exists a constant $N_1 = N_1(\mu, d)$ such that

$$|u| \leqslant N_1 \| Lu - \mu^{-1}u \|_{\mathscr{L}_{d+1}(C_2(t_0, x_0))}.$$

Here let us substitute $u \exp(-\mu^{-1}t)$ in place of u; then right away we get (4.7) with

LEMMA 4.3. *Let $u \in W^{1,2}_{d+1}(C_{2r}(t_0, x_0))$, $r \leqslant 1$ and $c \equiv 0$. Then there exists a constant $N = N(\mu, d)$ such that for $z_1, z_2 \in \overline{C}_r(t_0, x_0)$*

$$|u(z_1) - u(z_2)| \leqslant N\Big(\sup_{C_{2r}(t_0, x_0)} |u|\, r^{-\alpha} |z_1 - z_2|^\alpha + r^{\frac{d}{d+1}} \| Lu \|_{\mathscr{L}_{d+1}(C_{2r}(t_0, x_0))}\Big). \tag{4.8}$$

If, moreover, $b \equiv 0$, then (4.8) is valid for $r \geqslant 1$ as well.

PROOF. In (4.8) it is easy to pass to the limit from smooth u and operators L with smooth coefficients to the arbitrary u and L under consideration. Therefore it can be assumed that u and the coefficients of L are smooth functions. In this case, for the proof of (4.8) let us represent u in the form $u_1 + u_2$, where $Lu_1 = 0$ and $Lu_2 = Lu$; $u_1 = u$ and $u_2 = 0$ on the left face and lateral surface of $C_{2r}(t_0, x_0)$. This kind of representation with smooth functions u_1 and u_2 is well known to be possible (see, for example, [4]). Afterwards, it suffices to use Lemmas 4.1 and 4.2, and the fact that

$$\sup_{C_{2r}(t_0, x_0)} |u_1| \leqslant \sup_{C_{2r}(t_0, x_0)} |u|$$

by the maximum principle. The lemma is proved.

Now we can prove the basic results of this section. In the theorems to follow below, α is the constant whose existence was proved in Lemma 4.1, $\sigma = \alpha d/(d + \alpha(d + 1))$, and D is a region in E_{d+1}; for $z = (t, x) \in \overline{D}$ we set

$$\rho(z) = \inf(|z - z'| : z' = (t', x') \in \partial D,\ t' < t)\ \ (^4)$$

and for $z_1, z_2 \in \overline{D}$ let $\rho(z_1, z_2) = \rho(z_1) \wedge \rho(z_2)$ and $\tilde{\rho}(z_1, z_2) = 1 \wedge \rho(z_1, z_2)$. We shall formulate the theorem on Hölder continuity of solutions of parabolic equations, in a special case, in the spirit of Theorem 3 of [4], Chapter IV, §6.

THEOREM 4.1. *Let $u \in W^{1,2}_{d+1}(D)$, $z_0, z_1 \in D$, $c \equiv 0$, $b \equiv 0$ and $4|z_0 - z_1| \leqslant \rho(z_0, z_1)$. There exists a constant $N = N(\mu, d)$ such that for any finite $\rho \in [4|z_0 - z_1|, \rho(z_0, z_1)]$*

$$|u(z_0) - u(z_1)| \leqslant N\rho^{-\sigma}\big(\sup_D |u| + \rho^{\frac{d}{d+1}} \| Lu \|_{\mathscr{L}_{d+1}(D)}\big)|z_0 - z_1|^\sigma.$$

PROOF. Let $z_0 = (t_0, x_0)$ and $z_1 = (t_1, x_1)$. Without loss of generality, it can be assumed that $t_1 \leqslant t_0$. Denote

$$\xi = \frac{\alpha(d+1)}{d + \alpha(d+1)}, \quad r = 4^{\xi - 1}\rho^{1-\xi}|z_0 - z_1|^\xi. \tag{4.9}$$

As is easy to check, by the conditions on ρ we have $2r \leqslant \frac{1}{2}\rho(z_0, z_1)$ and $|z_0 - z_1| \leqslant r$. Hence $z_1 \in \overline{C}_r(z_0)$ and $C_{2r}(z_0) \subset D$. Consequently, Lemma 4.3 applies. It remains to note that the right-hand side of (4.8) for the chosen r is equal to

$$N\left(\frac{4}{\rho}\right)^\sigma \left[\sup_{C_{2r}(z_0)} |u| + \left(\frac{\rho}{4}\right)^{\frac{d}{d+1}} \| Lu \|_{\mathscr{L}_{d+1}(C_{2r}(z_0))}\right] |z_0 - z_1|^\sigma. \tag{4.10}$$

The theorem is proved.

THEOREM 4.2. *Let $u \in W^{1,2}_{d+1}(D)$, $z_0, z_1 \in D$ and $4|z_0 - z_1| \leqslant \tilde{\rho}(z_0, z_1)$. Then there exists a constant $N = N(\mu, d)$ such that*

$$|u(z_0) - u(z_1)| \leqslant N\,[\tilde{\rho}(z_0, z_1)]^{-\sigma}\left(\sup_D |u| + \|Lu\|_{\mathscr{L}_{d+1}(D)}\right)|z_0 - z_1|^{\sigma}. \quad (4.11)$$

PROOF. We set $\rho = \tilde{\rho}(z_0, z_1)$ and carry out the same arguments as in the previous proof, noting that this time $\rho \leqslant 1$ and $r \leqslant 1$ ($r \leqslant \frac{1}{4}$). Then we estimate the left-hand side of (4.11) by (4.10), with the expression Lu in (4.10) replaced by $Lu + cu$. It remains to use once more the facts that $\rho \leqslant 1$ and $r \leqslant 1$, and that

$$\|Lu + cu\|_{\mathscr{L}_{d+1}(C_{2r}(z_0))} \leqslant \|Lu\|_{\mathscr{L}_{d+1}(C_{2r}(z_0))} + N \sup_D |u|,$$

where $N = N(\mu, d)$. The theorem is proved.

An immediate consequence of Theorem 4.2 is

THEOREM 4.3. *Let the region $D' \subset D$, and let $u \in W^{1,2}_{d+1}(D)$. Denote*

$$\rho = \inf\{|(t', x') - (t, x)| : (t', x') \in D',\ (t, x) \in \partial D,\ t < t'\}$$

and assume that $\rho > 0$. Then for some constant $N = N(\mu, d, \rho)$, for all $z_0, z_1 \in D'$,

$$|u(z_0) - u(z_1)| \leqslant N\left(\sup_D |u| + \|Lu\|_{\mathscr{L}_{d+1}(D)}\right)|z_0 - z_1|^{\sigma}. \quad (4.12)$$

Indeed, if $4|z_0 - z_1| \leqslant \rho \wedge 1$, then in (4.12) one can take

$$N = N_1(\mu, d, \rho) = N(\mu, d)\,(\rho \wedge 1)^{-\sigma},$$

where $N(\mu, d)$ is the constant from (4.11). If, however, $4|z_0 - z_1| \geqslant \rho \wedge 1$, then one can take

$$N = N_2(\mu, d, \rho) = 2 \cdot 4^{\sigma} \cdot (\rho \wedge 1)^{-\sigma}.$$

In any case, $N = N_1 \vee N_2$ will be suitable in (4.12).

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ON DIFFERENTIAL IRREDUCIBILITY OF A CLASS OF DIFFERENTIAL EQUATIONS

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V. H. SALIHOV

ABSTRACT. In this paper a method is developed for proving algebraic independence of the solutions of linear differential equations. The result is applied to prove algebraic independence of the values at an algebraic point of a class of E-functions.

Bibliography: 9 titles.

§1. Introduction and statement of results

This article is devoted to the development of a method for proving differential irreducibility over $\mathbf{C}(z)$ of a class of linear differential equations with coefficients in $\mathbf{C}(z)$. This result, together with a general theorem concerning algebraic independence of values of E-functions that was proved by A. B. Šidlovskiĭ [1], enables us to establish transcendence and algebraic independence of the values at algebraic points of a class of hypergeometric E-functions. The statements and outlines of the proofs of the theorems given below were published in [8].

We consider the differential equation

$$y^{(m)} + q_{m-1}y^{(m-1)} + \dots + q_1 y' + q_0 y = q, \quad (1)$$

where $q, q_0, \dots, q_{m-1} \in \mathbf{C}(z)$.

Let $\mathfrak{L}$ be an arbitrary field of analytic functions which contains $\mathbf{C}(z)$.

DEFINITION 1 (see [4], §2). Equation (1) is said to be *differentially reducible over* $\mathfrak{L}$ if there exists a nontrivial solution y of (1) such that the functions $y, y', \dots, y^{(m-1)}$ are algebraically dependent over $\mathfrak{L}$.

As V. A. Oleĭnikov has shown [4], differential irreducibility of equation (1) over $\mathbf{C}(z)$ is equivalent to unsolvability of a certain set of partial differential equations in algebraic functions of several variables. To solve this problem, Oleĭnikov, developing ideas of Siegel [3], made use of the expansions of algebraic functions of several variables in series of the form

$$f(z, z_0, \dots, z_s) = \sum_{n=N}^{\infty} h_n(z_0, \dots, z_s)\, z^{-\frac{n}{e}},$$

where $N \in \mathbf{Z}$ and $e \in \mathbf{N}$.

Oleĭnikov's method, despite its general nature, could not be successfully applied to equations (1) of order greater than three.

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