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# MARKOWITZ'S MEAN-VARIANCE PORTFOLIO SELECTION WITH REGIME SWITCHING: A CONTINUOUS-TIME MODEL*

XUN YU ZHOU† AND G. YIN‡

**Abstract.** A continuous-time version of the Markowitz mean-variance portfolio selection model is proposed and analyzed for a market consisting of one bank account and multiple stocks. The market parameters, including the bank interest rate and the appreciation and volatility rates of the stocks, depend on the market mode that switches among a finite number of states. The random regime switching is assumed to be independent of the underlying Brownian motion. This essentially renders the underlying market *incomplete*. A Markov chain modulated diffusion formulation is employed to model the problem. Using techniques of stochastic linear-quadratic control, mean-variance efficient portfolios and efficient frontiers are derived explicitly in *closed forms*, based on solutions of two systems of linear ordinary differential equations. Related issues such as a minimum-variance portfolio and a mutual fund theorem are also addressed. All the results are markedly different from those for the case when there is no regime switching. An interesting observation is, however, that if the interest rate is deterministic, then the results exhibit (rather unexpected) similarity to their no-regime-switching counterparts, even if the stock appreciation and volatility rates are Markov-modulated.

**Key words.** continuous time, regime switching, Markov chain, mean-variance, portfolio selection, efficient frontier, linear-quadratic control

**AMS subject classifications.** Primary, 90A09; Secondary, 93E20

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## 1. Introduction.

Recently there has been an increasing interest in financial market models whose key parameters, such as the bank interest rate, stocks appreciation rates, and volatility rates, are modulated by some Markov processes. This is motivated by the need of more realistic models that better reflect random market environment. A factor that dominates the movement of a stock is the trend of the market. To reflect the market trend, it is necessary to allow the key parameters to respond to the general market movements. One such formulation is the regime switching model, where the market parameters depend on the market mode that switches among a finite number of states. The market mode could reflect the state of the underlying economy, the general mood of investors in the market, and other economic factors. For example, the market can be roughly divided as "bullish" and "bearish," while the market parameters can be quite different in the two modes. One could certainly introduce more intermediate states between the two extremes. A regime switching model can be formulated mathematically as a stochastic differential equation (SDE) whose coefficients are modulated by a continuous-time Markov chain. Such models have been mainly employed in the literature to deal with options; see Barone-Adesi and Whaley [1], Di Masi, Kabanov, and Runggaldier [6], Guo [10], Buffington and

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†Department of Systems Engineering and Engineering Management, The Chinese University of Hong Kong, Shatin, Hong Kong (xyzhou@se.cuhk.edu.hk). The research of this author was supported in part by the RGC Earmarked Grants CUHK 4435/99E and CUHK 4175/00E.

‡Department of Mathematics, Wayne State University, Detroit, MI 48202 (gyin@math.wayne.edu). The research of this author was supported in part by the National Science Foundation under grants DMS-9877090 and DMS-0304928.

Elliott [3], and Yao, Zhang, and Zhou [26]. In addition, recently Zhang [30] studied an optimal stock selling rule for a Markov-modulated Black–Scholes model (see also [27] for a stochastic optimization approach).

In this paper, we develop a continuous-time version of the Nobel prize winning mean-variance portfolio selection model with regime switching and attempt to derive closed-form solutions for efficient portfolios and efficient frontier. The mean-variance model was originally proposed by Markowitz [20, 21] for portfolio construction in a single period. One of the salient features of his model is as follows: It enables an investor to seek the highest return after specifying his/her acceptable risk level that is quantified by the variance of the return. The mean-variance approach has become the foundation of modern finance theory and has inspired numerous extensions and applications. One natural extension is to investigate *dynamic* mean-variance models. Along this line, multiperiod mean-variance portfolio selection was studied in, for example, Samuelson [23], Hakansson [11], and Pliska [22] among others. On the other hand, continuous-time mean-variance hedging problems were attacked by Duffie and Richardson [7] and Schweizer [24], where optimal dynamic strategies were derived, based on the projection theorem, to hedge contingent claims in incomplete markets. In [7], the result was derived under the assumption that all the coefficients (interest rate, volatility rate, etc.) are deterministic, time-invariant constants. The model considered in [24] is mathematically general; however, the solution is based on an abstract martingale measure and is thus not easily decipherable.

It should be noted that the research works on dynamic portfolio selections have been dominated by those of maximizing expected utility functions of the terminal wealth. In the utility model, besides the difficulty in eliciting utility functions from the investors, tradeoff between the risk and return is implicit, making an investment decision much less intuitive. In this sense, Markowitz's mean-variance approach has not been fully utilized in the utility approach.

Using the recently developed stochastic linear-quadratic (LQ) control framework [4, 5, 29], Zhou and Li [31] studied the mean-variance problem for a continuous-time model from another angel. By embedding the original (not readily solvable) problem into a tractable auxiliary problem, following a similar embedding technique introduced in Li and Ng [19] for the multiperiod model, it was shown that this auxiliary problem in fact is a stochastic optimal LQ problem and can be solved explicitly by LQ theory. Such an approach establishes a natural connection of the portfolio selection problems and standard stochastic control models. The theory of stochastic control is rich, and many mathematical machineries are available; see Fleming and Soner [9] and Yong and Zhou [29], which provides an opportunity for treating more complicated situations. For example, a portfolio selection problem with random coefficients was solved in [16] using LQ theory and backward SDEs, a problem with short sell prohibition was studied in [15] via LQ and viscosity solution theories, and a mean-variance hedging problem was treated in [12] within the LQ framework.

In this work, we focus on a continuous-time mean-variance model modulated by a Markov chain representing the regime switching. The random switching of the market modes is assumed to be independent of the Brownian motion in defining the stock prices. Therefore, the underlying market is essentially incomplete, as the regime switching constitutes an additional dimension of uncertainty that cannot be perfectly hedged by any combination of the stocks and the bank account. We formulate the problem as a Markov-modulated stochastic LQ control model with a terminal constraint representing the expected payoff of the investor. The feasibility due to the constraint is first addressed under a very mild condition. Then, using Lagrange multiplier techniques, the problem is converted to an unconstrained problem. We proceed with the solution of the unconstrained problem based on two systems of ordinary differential equations (ODEs). This leads to the analytic expressions of the efficient portfolios in a feedback form as well as the efficient frontier. In addition, the minimum variance is explicitly derived. In fact, one needs only to solve two systems of linear ODEs in order to completely determine the efficient portfolios/frontier of the underlying mean-variance problem. It is interesting, though rather expected, that the efficient frontier is no longer a straight line in the mean-standard deviation diagram. However, if the interest rate is independent of the Markov chain, then the efficient frontier becomes a straight line again, and the one-fund theorem is preserved, even if the appreciation and volatility rates of the stocks are random (i.e., Markov-modulated).

It should be noted that in our model the wealth process is allowed to take negative values, representing the bankruptcy situation. This is due to our definition of admissible portfolios. (A portfolio is defined to be a vector consisting of the dollar values of different stocks.) In most of the literature, a portfolio contains the *fractions* of wealth in stocks, which *automatically* ensures the positivity of the wealth process (see Remark 1). In our model, requiring a nonnegative wealth process imposes an *additional state constraint*, which is a very difficult problem from the stochastic control point of view. We are not able to treat such a case in this paper and will defer it to later consideration. We remark that portfolio selection problems with constraints on wealth have been studied by many researchers, mostly in the realm of utility optimization. In particular, Korn and Trautmann considered in [14], for the first time, a mean-variance problem with nonnegative terminal constraint and without regime switching; see also Korn [13, Chapter 4]. The basic idea presented in these references is to reduce the problem to one finding an optimal attainable terminal wealth, the latter being a quadratic optimization problem. Then the efficient portfolio is the one that duplicates the optimal attainable terminal wealth.

The rest of the paper is arranged as follows. Section 2 begins with the precise problem formulation. Section 3 is concerned with the feasibility issue of the underlying model. Section 4 proceeds with the solution of the unconstrained optimization problem. The efficient frontier is obtained in section 5. Section 6 specializes in the case when the interest rate is independent of the modulating Markov chain. Finally, concluding remarks are made in the last section.

## 2. Problem formulation.

Throughout the paper, let $(\Omega, \mathcal{F}, P)$ be a fixed complete probability space on which are defined a standard $d$-dimensional Brownian motion $W(t) \equiv (W_1(t), \dots, W_d(t))'$ and a continuous-time stationary Markov chain $\alpha(t)$ taking value in a finite state space $\mathcal{M} = \{1, 2, \dots, l\}$ such that $W(t)$ and $\alpha(t)$ are independent of each other. The Markov chain has a generator $Q = (q_{ij})_{l\times l}$ and stationary transition probabilities

$$p_{ij}(t) = P(\alpha(t) = j|\alpha(0) = i), \quad t \geq 0, \quad i, j = 1, 2, \dots, l. \tag{2.1}$$

Define $\mathcal{F}_t = \sigma\{W(s), \alpha(s) : 0 \leq s \leq t\}$. We denote by $L^2_{\mathcal{F}}(0, T; \mathbb{R}^m)$ the set of all $\mathbb{R}^m$-valued, measurable stochastic processes $f(t)$ adapted to $\{\mathcal{F}_t\}_{t\geq 0}$, such that $E\int_0^T |f(t)|^2 dt < +\infty$. We will also use the following notation.

*Notation.*

| | |
|---|---|
| $M'$: | the transpose of any vector or matrix $M$; |
| $m_j$: | the $j$th component of any vector $M$; |
| $\|M\|$: | $=\sqrt{\sum_{i,j} m_{ij}^2}$ for any matrix vector $M = (m_{ij})$ |
| | or $=\sqrt{\sum_j m_j^2}$ for any vector $M = (m_j)$; |
| $\mathrm{tr}(M)$: | the trace of a square matrix $M$; |
| $C([0,T];X)$: | the Banach space of $X$-valued continuous functions on $[0,T]$ endowed with the maximum norm $\|\cdot\|$ for a given Hilbert space $X$; |
| $C^2([0,T]\times\mathbb{R}^n)$: | the space of all twice continuously differentiable functions on $[0,T]\times\mathbb{R}^n$; |
| $L^2(0,T;X)$: | the Hilbert space of $X$-valued integrable functions on $[0,T]$ endowed with the norm $(\int_0^T \| f(t) \|_X^2\, dt)^{\frac{1}{2}}$ for a given Hilbert space $X$. |

Consider a market in which $d+1$ assets are traded continuously. One of the assets is a bank account whose price $P_0(t)$ is subject to the stochastic ODE

$$\begin{cases} dP_0(t) = r(t,\alpha(t))P_0(t)dt, & t \in [0,T], \\ P_0(0) = p_0 > 0, \end{cases} \tag{2.2}$$

where $r(t,i) \geq 0,\ i = 1,2,\dots,l$, are given as the interest rate processes corresponding to different market modes. The other $d$ assets are stocks whose price processes $P_m(t)$, $m = 1,2,\dots,d$, satisfy the system of SDEs

$$\begin{cases} dP_m(t) = P_m(t)\left\{b_m(t,\alpha(t))dt + \displaystyle\sum_{n=1}^{d} \sigma_{mn}(t,\alpha(t))dW_n(t)\right\}, & t \in [0,T], \\ P_m(0) = p_m > 0, \end{cases} \tag{2.3}$$

where for each $i = 1,2,\dots,l$, $b_m(t,i)$ is the appreciation rate process and $\sigma_m(t,i) := (\sigma_{m1}(t,i),\dots,\sigma_{md}(t,i))$ is the volatility or the dispersion rate process of the $m$th stock, corresponding to $\alpha(t) = i$.

Define the volatility matrix

$$\sigma(t,i) := \begin{pmatrix} \sigma_1(t,i) \\ \vdots \\ \sigma_d(t,i) \end{pmatrix} \equiv (\sigma_{mn}(t,i))_{d\times d} \quad \text{for each } i = 1,\dots,l. \tag{2.4}$$

We assume throughout this paper that the nondegeneracy condition

$$\sigma(t,i)\sigma(t,i)' \geq \delta I \quad \forall t \in [0,T] \ \text{ and } \ i = 1,2,\dots,l \tag{2.5}$$

is satisfied for some $\delta > 0$. We also assume that all the functions $r(t,i), b_m(t,i), \sigma_{mn}(t,i)$ are measurable and uniformly bounded in $t$.

Suppose that the initial market mode $\alpha(0) = i_0$. Consider an agent with an initial wealth $x_0 > 0$. These initial conditions are fixed throughout the paper. Denote by $x(t)$ the total wealth of the agent at time $t \geq 0$. Assuming that the trading of shares takes place continuously and that transaction cost and consumptions are not considered, then one has (see, e.g., [29, p. 57])

$$
(2.6)\quad \begin{cases} dx(t) &= \left\{ r(t,\alpha(t))x(t) + \sum_{m=1}^{d} \left[ b_m(t,\alpha(t)) - r(t,\alpha(t)) \right] u_m(t) \right\} dt \\ &\qquad + \sum_{n=1}^{d} \sum_{m=1}^{d} \sigma_{mn}(t,\alpha(t)) u_m(t) dW_n(t), \\ x(0) &= x_0 > 0, \ \ \alpha(0) = i_0, \end{cases}
$$

where $u_m(t)$ is the total market value of the agent's wealth in the $m$th asset, $m = 0, 1, \ldots, d$, at time $t$. We call $u(\cdot) = (u_1(\cdot), \ldots, u_d(\cdot))'$ a *portfolio* of the agent. Note that once $u(\cdot)$ is determined, $u_0(\cdot)$, the asset in the bank account is completely specified since $u_0(t) = x(t) - \sum_{i=1}^{d} u_i(t)$. Thus, in our analysis to follow, only $u(\cdot)$ is considered.

Setting

$$
(2.7)\quad B(t,i) := (b_1(t,i) - r(t,i), \ldots, b_d(t,i) - r(t,i)), \ i = 1, 2, \ldots, l,
$$

we can rewrite the wealth equation (2.6) as

$$
(2.8)\quad \begin{cases} dx(t) = [r(t,\alpha(t))x(t) + B(t,\alpha(t))u(t)]dt + u(t)'\sigma(t,\alpha(t))dW(t), \\ x(0) = x_0, \ \ \alpha(0) = i_0. \end{cases}
$$

DEFINITION 2.1. *A portfolio* $u(\cdot)$ *is said to be admissible if* $u(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^d)$ *and the SDE* (2.8) *has a unique solution* $x(\cdot)$ *corresponding to* $u(\cdot)$*. In this case, we refer to* $(x(\cdot), u(\cdot))$ *as an* admissible (wealth, portfolio) pair*.*

*Remark* 1. Most works in the literature define a portfolio, say, $\pi(\cdot)$, as the fractions of wealth allocated to different stocks. That is,

$$
(2.9)\quad \pi(t) = \frac{u(t)}{x(t)}, \ \ t \in [0, T].
$$

With this definition, (2.8) can be rewritten as

$$
(2.10)\quad \begin{cases} dx(t) = x(t)[r(t,\alpha(t)) + B(t,\alpha(t))\pi(t)]dt + x(t)\pi(t)'\sigma(t,\alpha(t))dW(t), \\ x(0) = x_0, \ \ \alpha(0) = i_0. \end{cases}
$$

It is well known that this equation has a solution that can be expressed explicitly as an exponential of certain process, which therefore must be *automatically* positive if the initial wealth $x_0$ is positive. The reason for this guaranteed positivity of wealth is because the very definition of the portfolio, (2.9), has implicitly assumed that $x(t) \neq 0$; hence $x = 0$ becomes a natural barrier of the wealth process. It is our view, however, that a wealth process with possible zero or negative values is theoretically and practically sensible at least for some circumstances. Hence the nonnegativity of the wealth is better imposed as an *additional constraint*, rather than as a built-in feature, of the model. In our formulation, a portfolio is well defined even if the wealth is zero or negative, and the nonnegativity of the wealth, if so required, would be a constraint.

The agent's objective is to find an admissible portfolio $u(\cdot)$, among all the admissible portfolios whose expected terminal wealth is $Ex(T) = z$ for some given $z \in \mathbb{R}^1$, so that the risk measured by the variance of the terminal wealth

$$
(2.11)\quad \operatorname{Var} x(T) \equiv E[x(T) - Ex(T)]^2 = E[x(T) - z]^2
$$

is minimized. Finding such a portfolio $u(\cdot)$ is referred to as the *mean-variance portfolio selection problem*. Specifically, we have the following formulation.

DEFINITION 2.2. *The mean-variance portfolio selection is a constrained stochastic optimization problem, parameterized by* $z \in \mathbb{R}^1$*:*

$$\left\{ \begin{array}{ll} \textit{minimize} & J_{\mathrm{MV}}(x_0, i_0, u(\cdot)) := E[x(T) - z]^2, \\ \textit{subject to} & \left\{ \begin{array}{l} Ex(T) = z, \\ (x(\cdot), u(\cdot)) \ \ \textit{admissible}. \end{array} \right. \end{array} \right. \tag{2.12}$$

*Moreover, the problem is called* feasible *if there is at least one portfolio satisfying all the constraints. The problem is called* finite *if it is feasible and the infimum of* $J_{\mathrm{MV}}(x_0, i_0, u(\cdot))$ *is finite. Finally, an optimal portfolio to the above problem, if it ever exists, is called an* efficient portfolio *corresponding to* $z$*, and the corresponding* $(\mathrm{Var}\ x(T), z) \in \mathbb{R}^2$ *and* $(\sigma_{x(T)}, z) \in \mathbb{R}^2$ *are interchangeably called an* efficient point*, where* $\sigma_{x(T)}$ *denotes the standard deviation of* $x(T)$*. The set of all the efficient points is called the* efficient frontier.

*Remark* 2. While in the above definition, an efficient portfolio is broadly defined for any given $z \in \mathbb{R}^1$; in the subsequent context we will see that it is practically sensible only for $z$ greater than or equal to certain value. Also, the shape of the efficient frontier depends on whether it is plotted in the mean-variance plane or mean-standard deviation plane. In what follows, we will specify which one we are referring to only when ambiguity might arise.

*Remark* 3. The mean-variance portfolio selection problem may be defined in some different, albeit equivalent, ways. For example, in [31] the problem is formulated as a multiobjective optimization problem. It should be noted that the model in this paper is a faithful replication in form of the original Markowitz single-period model.

## 3. Feasibility.

Since the problem (2.12) involves a terminal constraint $Ex(T) = z$, in this section, we derive conditions under which the problem is at least feasible. First, the following generalized Itô lemma [2] for Markov-modulated processes is useful.

LEMMA 3.1. *Given an* $n$*-dimensional process* $x(\cdot)$ *satisfying*

$$dx(t) = b(t, x(t), \alpha(t))dt + \sigma(t, x(t), \alpha(t))dW(t)$$

*and a number of functions* $\varphi(\cdot, \cdot, i) \in C^2([0, T] \times \mathbb{R}^n)$*,* $i = 1, 2, \ldots, l$*, we have*

$$d\varphi(t, x(t), \alpha(t)) = \Gamma\varphi(t, x(t), \alpha(t))dt + \varphi_x(t, x(t), \alpha(t))'\sigma(t, x(t), \alpha(t))dW(t),$$

*where*

$$\begin{aligned} \Gamma\varphi(t, x, i) := \quad & \varphi_t(t, x, i) + \varphi_x(t, x, i)'b(t, x, i) \\ & + \frac{1}{2}\mathrm{tr}[\sigma(t, x, i)'\varphi_{xx}(t, x, i)\sigma(t, x, i)] + \sum_{j=1}^{l} q_{ij}\varphi(t, x, j). \end{aligned}$$

Consider a portfolio $u^0(t) \equiv 0$, corresponding to the one that puts all the money in the bank account. The associated wealth process $x^0(\cdot)$ satisfies

$$\left\{ \begin{array}{l} dx^0(t) = r(t, \alpha(t))x^0(t)dt, \\ x^0(0) = x_0, \quad \alpha(0) = i_0, \end{array} \right. \tag{3.1}$$

with its expected terminal wealth

$$z^0 := Ex^0(T) = Ee^{\int_0^T r(s, \alpha(s))ds}x_0. \tag{3.2}$$

LEMMA 3.2. *Let $\psi(\cdot,i)$, $i=1,2,\ldots,l$, be the solutions to the following system of linear ODEs:*

$$\begin{cases} \dot\psi(t,i) = -r(t,i)\psi(t,i) - \displaystyle\sum_{j=1}^{l} q_{ij}\psi(t,j), \\ \psi(T,i)=1, \qquad i=1,2,\ldots,l. \end{cases} \tag{3.3}$$

*Then the mean-variance problem* (2.12) *is feasible for every $z\in\mathbb{R}^1$ if and only if*

$$\gamma := E\int_0^T |\psi(t,\alpha(t))B(t,\alpha(t))|^2 dt > 0. \tag{3.4}$$

*Proof.* To prove the "if" part, construct a family of admissible portfolios $u^\beta(\cdot)=\beta u(\cdot)$ for $\beta\in\mathbb{R}^1$, where

$$u(t) = B(t,\alpha(t))'\psi(t,\alpha(t)). \tag{3.5}$$

Let $x^\beta(\cdot)$ be the wealth process corresponding to $u^\beta(\cdot)$. By linearity of the wealth equation, we have $x^\beta(t)=x^0(t)+\beta y(t)$, where $x^0(\cdot)$ satisfies (3.1) and $y(\cdot)$ is the solution to the following equation:

$$\begin{cases} dy(t) = [r(t,\alpha(t))y(t) + B(t,\alpha(t))u(t)]dt + u(t)'\sigma(t,\alpha(t))dW(t), \\ y(0)=0, \quad \alpha(0)=i_0. \end{cases} \tag{3.6}$$

Therefore, problem (2.12) is feasible for every $z\in\mathbb{R}^1$ if there exists $\beta\in\mathbb{R}^1$ such that $z = Ex^\beta(T) \equiv Ex^0(T)+\beta Ey(T)$. Equivalently, (2.12) is feasible for every $z\in\mathbb{R}^1$ if $Ey(T)\neq 0$. However, applying the generalized Itô formula (Lemma 3.1) to $\varphi(t,x,i)=\psi(t,i)x$, we have

$$\begin{aligned} &d[\psi(t,\alpha(t))y(t)] \\ &\quad= \Bigg\{\psi(t,\alpha(t))[r(t,\alpha(t))y(t)+B(t,\alpha(t))u(t)]dt - r(t,\alpha(t))\psi(t,\alpha(t))y(t) \\ &\qquad - \sum_{j=1}^{l} q_{\alpha(t)j}\psi(t,j)y(t)\Bigg\}dt + \sum_{j=1}^{l} q_{\alpha(t)j}\psi(t,j)y(t)dt + \{\cdots\}dW(t) \\ &\quad= \psi(t,\alpha(t))B(t,\alpha(t))u(t)dt + \{\cdots\}dW(t). \end{aligned}$$

Integrating from 0 to $T$, taking expectation, and using (3.5), we obtain

$$Ey(T) = E\int_0^T \psi(t,\alpha(t))B(t,\alpha(t))u(t)dt = E\int_0^T |\psi(t,\alpha(t))B(t,\alpha(t))|^2 dt. \tag{3.7}$$

Consequently, $Ey(T)\neq 0$ if (3.4) holds.

Conversely, suppose that problem (2.12) is feasible for every $z\in\mathbb{R}^1$. Then for each $z\in\mathbb{R}^1$, there is an admissible portfolio $u(\cdot)$ so that $Ex(T)=z$. However, we can always decompose $x(t)=x^0(t)+y(t)$, where $y(\cdot)$ satisfies (3.6). This leads to $Ex^0(T)+Ey(T)=z$. However, $Ex^0(T)\equiv z_0$ is independent of $u(\cdot)$; thus it is necessary that there is a $u(\cdot)$ with $Ey(T)\neq 0$. It follows then from (3.7) that (3.4) is valid. $\square$

THEOREM 3.3. *The mean-variance problem* (2.12) *is feasible for every* $z \in \mathbb{R}^1$ *if and only if*

$$E \int_0^T |B(t, \alpha(t))|^2 dt > 0. \tag{3.8}$$

*Proof.* By virtue of Lemma 3.2, it suffices to prove that $\psi(t,i) > 0$ for all $t \in [0,T]$, $i = 1, 2, \ldots, l$. To this end, note that (3.3) can be rewritten as

$$\begin{cases} \dot{\psi}(t,i) = [-r(t,i) - q_{ii}]\, \psi(t,i) - \displaystyle\sum_{j \neq i}^{l} q_{ij} \psi(t,j), \\ \psi(T,i) = 1, \quad i = 1, 2, \ldots, l. \end{cases} \tag{3.9}$$

Treating this as a system of terminal-valued ODEs, a variation-of-constant formula yields

$$\psi(t,i) = e^{-\int_t^T [-r(s,i) - q_{ii}] ds} + \int_t^T e^{-\int_t^s [-r(\tau,i) - q_{ii}] d\tau} \sum_{j \neq i}^{l} q_{ij} \psi(s,j) ds, \; i = 1, 2, \ldots, l. \tag{3.10}$$

Construct a sequence $\{\psi^{(k)}(\cdot, i)\}$ (known as the Picard sequence) as follows:

$$\begin{aligned} &\psi^{(0)}(t,i) = 1, \; t \in [0,T], \; i = 1, 2, \ldots, l, \\ &\psi^{(k+1)}(t,i) = e^{-\int_t^T [-r(s,i) - q_{ii}] ds} + \int_t^T e^{-\int_t^s [-r(\tau,i) - q_{ii}] d\tau} \sum_{j \neq i}^{l} q_{ij} \psi^{(k)}(s,j) ds, \; t \in [0,T], \\ &\qquad\qquad i = 1, 2, \ldots, l, \; k = 0, 1, \ldots. \end{aligned}$$

Noting that $q_{ij} \geq 0$ for all $j \neq i$, we have

$$\psi^{(k)}(t,i) \geq e^{-\int_t^T [-r(s,i) - q_{ii}] ds} > 0, \; k = 0, 1, \ldots.$$

On the other hand, it is well known that $\psi(t,i)$ is the limit of the Picard sequence $\{\psi^{(k)}(t,i)\}$ as $k \to \infty$. Thus $\psi(t,i) > 0$. This proves the desired result. ∎

COROLLARY 3.4. *If* (3.8) *holds, then for any* $z \in \mathbb{R}^1$, *an admissible portfolio that satisfies* $Ex(T) = z$ *is given by*

$$u(t) = \frac{z - z^0}{\gamma} B(t, \alpha(t))' \psi(t, \alpha(t)), \tag{3.11}$$

*where* $z^0$ *and* $\gamma$ *are given by* (3.2) *and* (3.4), *respectively.*

*Proof.* This is immediate from the proof of the "if" part of Lemma 3.2. ∎

COROLLARY 3.5. *If* $E \int_0^T |B(t, \alpha(t))|^2 dt = 0$, *then any admissible portfolio* $u(\cdot)$ *results in* $Ex(T) = z^0$.

*Proof.* This is seen from the proof of the "only if" part of Lemma 3.2. ∎

*Remark* 4. The condition (3.8) is very mild. For example, (3.8) holds as long as there is one stock whose appreciation-rate process is different from the interest-rate process at any market mode, which is obviously a practically reasonable assumption. On the other hand, if (3.8) fails, then Corollary 3.5 implies that the mean-variance problem (2.12) is feasible only if $z = z^0$. This is a pathological and trivial case that does not warrant further consideration. Therefore, from this point on we shall assume that (3.8) holds or, equivalently, that the mean-variance problem (2.12) is feasible for any $z$.

Having addressed the issue of feasibility, we proceed with the study of optimality. The mean-variance problem (2.12) under consideration is a dynamic optimization problem with a constraint $Ex(T) = z$. To handle this constraint, we apply the Lagrange multiplier technique. Define

$$\begin{aligned} J(x_0, i_0, u(\cdot), \lambda) :=& \quad E\big\{|x(T) - z|^2 + 2\lambda[x(T) - z]\big\} \\ =& \quad E[x(T) + \lambda - z]^2 - \lambda^2, \qquad \lambda \in \mathbb{R}^1. \end{aligned} \tag{3.12}$$

Our first goal is to solve the following unconstrained problem parameterized by the Lagrange multiplier $\lambda$:

$$\begin{cases} \text{minimize} & J(x_0, i_0, u(\cdot), \lambda) = E\big[x(T) + \lambda - z]^2 - \lambda^2 \\ \text{subject to} & (x(\cdot), u(\cdot)) \ \text{ admissible.} \end{cases} \tag{3.13}$$

This turns out to be a Markov-modulated stochastic LQ optimal control problem, which will be solved in the next section.

**4. Solution to the unconstrained problem.** In this section we solve the unconstrained problem (3.13). First define

$$\rho(t,i) := B(t,i)[\sigma(t,i)\sigma(t,i)']^{-1}B(t,i)', \ i = 1, 2, \ldots, l. \tag{4.1}$$

Consider the following two systems of ODEs:

$$\begin{cases} \dot{P}(t,i) = [\rho(t,i) - 2r(t,i)]\, P(t,i) - \displaystyle\sum_{j=1}^{l} q_{ij} P(t,j), \\ P(T,i) = 1, \quad i = 1, 2, \ldots, l, \end{cases} \tag{4.2}$$

and

$$\begin{cases} \dot{H}(t,i) = r(t,i)H(t,i) - \dfrac{1}{P(t,i)} \displaystyle\sum_{j=1}^{l} q_{ij} P(t,j)[H(t,j) - H(t,i)], \\ H(T,i) = 1, \quad i = 1, 2, \ldots, l. \end{cases} \tag{4.3}$$

The existence and uniqueness of solutions to the above two systems of equations are evident as both are linear with uniformly bounded coefficients.

Proposition 4.1. *The solutions of* (4.2) *and* (4.3) *must satisfy* $P(t,i) > 0$ *and* $0 < H(t,i) \le 1$ *for all* $t \in [0,T]$, $i = 1, 2, \ldots, l$. *Moreover, if for a fixed* $i$, $r(t,i) > 0$ *a.e.* $t \in [0,T]$, *then* $H(t,i) < 1$ *for all* $t \in [0,T)$.

*Proof.* The assertion $P(t,i) > 0$ can be proved in exactly the same way as that of $\psi(t,i) > 0$; see the proof of Theorem 3.3. Having proved the positivity of $P(t,i)$, one can then show $H(t,i) > 0$ using the same argument because now $P(t,j)/P(t,i) > 0$.

To prove that $H(t,i) \le 1$, first note that the system of ODEs

$$\begin{cases} \dfrac{d}{dt}\widetilde{H}(t,i) = -\dfrac{1}{P(t,i)} \displaystyle\sum_{j=1}^{l} q_{ij} P(t,j)[\widetilde{H}(t,j) - \widetilde{H}(t,i)], \\ \widetilde{H}(T,i) = 1, \quad i = 1, 2, \ldots, l, \end{cases} \tag{4.4}$$

has the only solutions $\widetilde{H}(t,i) \equiv 1$, $i = 1, 2, \dots, l$, due to the uniqueness of solutions. Set

$$\widehat{H}(t,i) := \widetilde{H}(t,i) - H(t,i) \equiv 1 - H(t,i),$$

which solves the following equations:

$$\tag{4.5} \begin{cases} \dfrac{d}{dt}\widehat{H}(t,i) &= r(t,i)\widehat{H}(t,i) - r(t,i) - \dfrac{1}{P(t,i)}\displaystyle\sum_{j=1}^{l} q_{ij}P(t,j)[\widehat{H}(t,j) - \widehat{H}(t,i)] \\ &= \left[ r(t,i) + \displaystyle\sum_{j\neq i} q_{ij} \right] \widehat{H}(t,i) - r(t,i) - \dfrac{1}{P(t,i)} \displaystyle\sum_{j\neq i} q_{ij} P(t,j)\widehat{H}(t,j), \\ \widehat{H}(T,i) &= 0, \quad i = 1, 2, \dots, l. \end{cases}$$

A variation-of-constant formula leads to

$$\tag{4.6} \widehat{H}(t,i) = \int_t^T e^{-\int_t^s [r(\tau,i) + \sum_{j\neq i} q_{ij}]d\tau} \left[ r(s,i) + \frac{1}{P(s,i)} \sum_{j\neq i} q_{ij} P(s,j) \widehat{H}(s,j) \right].$$

A similar trick using the construction of Picard's sequence yields that $\widehat{H}(t,i) \geq 0$. In addition, $\widehat{H}(t,i) > 0$ for all $t \in [0,T)$ if $r(t,i) > 0$ a.e. $t \in [0,T]$. The desired result then follows from the fact that $\widehat{H}(t,i) = 1 - H(t,i)$. □

*Remark* 5. Equation (4.2) is a Riccati-type equation that arises naturally in studying the stochastic LQ control problem (3.13), whereas (4.3) is used to handle the nonhomogeneous terms involved in (3.13); see the proof of Theorem 4.2 below. On the other hand, $H(t,i)$ has a financial interpretation: for fixed $(t,i)$, $H(t,i)$ is a *deterministic* quantity representing the *risk-adjusted discount factor* at time $t$ when the market mode is $i$ (note that the interest rate itself is random); see also Remark 11 in what follows.

THEOREM 4.2. *Problem* (3.13) *has an optimal feedback control*

$$\tag{4.7} u^*(t,x,i) = -[\sigma(t,i)\sigma(t,i)']^{-1}B(t,i)'[x + (\lambda - z)H(t,i)].$$

*Moreover, the corresponding optimal value is*

$$\tag{4.8} \begin{aligned} &\inf\nolimits_{u(\cdot)\text{admissible}} J(x_0, i_0, u(\cdot), \lambda) \\ &\quad = [P(0,i_0)H(0,i_0)^2 + \theta - 1](\lambda - z)^2 \\ &\qquad + 2[P(0,i_0)H(0,i_0)x_0 - z](\lambda - z) + P(0,i_0)x_0^2 - z^2, \end{aligned}$$

*where*

$$\tag{4.9} \begin{aligned} \theta &:= E\int_0^T \sum_{j=1}^{l} q_{\alpha(t)j} P(t,j)[H(t,j) - H(t,\alpha(t))]^2 dt \\ &= \sum_{i=1}^{l}\sum_{j=1}^{l} \int_0^T P(t,j)p_{i_0 i}(t) q_{ij}[H(t,j) - H(t,i)]^2 dt \geq 0, \end{aligned}$$

*with the transition probabilities* $p_{i_0 i}(t)$ *given by* (2.1).

*Proof.* Let $u(\cdot)$ be any admissible control and $x(\cdot)$ be the corresponding state trajectory of (2.8). Applying the generalized Itô formula (Lemma 3.1) to

$$\varphi(t,x,i) = P(t,i)[x+(\lambda-z)H(t,i)]^2,$$

we obtain

(4.10)

$$\begin{aligned}
&d\Big\{P(t,\alpha(t))[x(t)+(\lambda-z)H(t,\alpha(t))]^2\Big\}\\
&\ = P(t,\alpha(t))\Big\{u(t)'[\sigma(t,\alpha(t))\sigma(t,\alpha(t))']u(t)+2u(t)'B(t,\alpha(t))'[x(t)+(\lambda-z)H(t,\alpha(t))]\\
&\quad +2r(t,\alpha(t))[x(t)+(\lambda-z)H(t,\alpha(t))]^2\Big\}dt\\
&\quad -2(\lambda-z)[x(t)+(\lambda-z)H(t,\alpha(t))]\sum_{j=1}^{l} q_{\alpha(t)j}P(t,j)[H(t,j)-H(t,\alpha(t))]dt\\
&\quad +[\rho(t,\alpha(t))-2r(t,\alpha(t))]P(t,\alpha(t))[x(t)+(\lambda-z)H(t,\alpha(t))]^2dt\\
&\quad -\sum_{j=1}^{l} q_{\alpha(t)j}P(t,j)[x(t)+(\lambda-z)H(t,\alpha(t))]^2dt\\
&\quad +\sum_{j=1}^{l} q_{\alpha(t)j}P(t,j)[x(t)+(\lambda-z)H(t,j)]^2dt+\{\cdots\}dW(t)\\
&\ = P(t,\alpha(t))\Big\{u(t)'[\sigma(t,\alpha(t))\sigma(t,\alpha(t))']u(t)+2u(t)'B(t,\alpha(t))'[x(t)+(\lambda-z)H(t,\alpha(t))]\\
&\quad +\rho(t,\alpha(t))[x(t)+(\lambda-z)H(t,\alpha(t))]^2\Big\}dt\\
&\quad +(\lambda-z)^2\sum_{j=1}^{l} q_{\alpha(t)j}P(t,j)[H(t,j)-H(t,\alpha(t))]^2dt+\{\cdots\}dW(t)\\
&\ = P(t,\alpha(t))[u(t)-u^*(t,x(t),\alpha(t))]'[\sigma(t,\alpha(t))\sigma(t,\alpha(t))'][u(t)-u^*(t,x(t),\alpha(t))]dt\\
&\quad +(\lambda-z)^2\sum_{j=1}^{l} q_{\alpha(t)j}P(t,j)[H(t,j)-H(t,\alpha(t))]^2dt+\{\cdots\}dW(t),
\end{aligned}$$

where $u^*(t,x,i)$ is defined as the right-hand side of (4.7). Integrating the above from 0 to $T$ and taking expectations, we obtain

$$\begin{aligned}
&E[x(T)+\lambda-z]^2\\
&\quad = P(0,i_0)[x_0+(\lambda-z)H(0,i_0)]^2+\theta(\lambda-z)^2\\
&\qquad +E\int_0^T P(t,\alpha(t))[u(t)-u^*(t,x(t),\alpha(t))]'[\sigma(t,\alpha(t))\sigma(t,\alpha(t))']\\
&\qquad\qquad \times[u(t)-u^*(t,x(t),\alpha(t))]dt.
\end{aligned}\tag{4.11}$$

Consequently,

$$\begin{aligned}
&J(x_0,i_0,u(\cdot),\lambda)\\
&\quad = E[x(T)+\lambda-z]^2-\lambda^2\\
&\quad = [P(0,i_0)H(0,i_0)^2+\theta-1](\lambda-z)^2+2[P(0,i_0)H(0,i_0)x_0-z](\lambda-z)\\
&\qquad + P(0,i_0)x_0^2-z^2\\
&\qquad +E\int_0^T P(t,\alpha(t))[u(t)-u^*(t,x(t),\alpha(t))]'[\sigma(t,\alpha(t))\sigma(t,\alpha(t))']\\
&\qquad\qquad \cdot[u(t)-u^*(t,x(t),\alpha(t))]dt.
\end{aligned}\tag{4.12}$$

Since $P(t,\alpha(t)) > 0$ by Proposition 3.1, it follows immediately that the optimal feedback control is given by (4.7) and the optimal value is given by (4.8), provided that the corresponding equation (2.8) under the feedback control (4.7) has a solution. However, under (4.7), the system (2.8) is a nonhomogeneous linear SDE with coefficients modulated by $\alpha(t)$. Since all the coefficients of this linear equation are uniformly bounded and $\alpha(t)$ is independent of $W(t)$, the existence and uniqueness of the solution to the equation are straightforward based on a standard successive approximation scheme.

Finally, since

$$\theta = \sum_{i \neq j} \int_0^T P(t,j) p_{i_0 i}(t) q_{ij} [H(t,j) - H(t,i)]^2 dt$$

and $q_{ij} \geq 0$ for all $i \neq j$, we must have $\theta \geq 0$. This completes the proof. $\square$

**5. Efficient frontier.** In this section we proceed to derive the efficient frontier for the original mean-variance problem (2.12).

THEOREM 5.1 (efficient portfolios and efficient frontier). *Assume that* (3.8) *holds. Then we have*

$$P(0,i_0)H(0,i_0)^2 + \theta - 1 < 0. \tag{5.1}$$

*Moreover, the efficient portfolio corresponding to $z$, as a function of the time $t$, the wealth level $x$, and the market mode $i$, is*

$$u^*(t,x,i) = -[\sigma(t,i)\sigma(t,i)']^{-1} B(t,i)'[x + (\lambda^* - z)H(t,i)], \tag{5.2}$$

*where*

$$\lambda^* - z = \frac{z - P(0,i_0)H(0,i_0)x_0}{P(0,i_0)H(0,i_0)^2 + \theta - 1}. \tag{5.3}$$

*Furthermore, the optimal value of* $\operatorname{Var} x(T)$, *among all the wealth processes $x(\cdot)$ satisfying $Ex(T) = z$, is*

$$\begin{aligned} &\operatorname{Var} x^*(T) \\ &\quad = \frac{P(0,i_0)H(0,i_0)^2 + \theta}{1 - \theta - P(0,i_0)H(0,i_0)^2} \left[ z - \frac{P(0,i_0)H(0,i_0)}{P(0,i_0)H(0,i_0)^2 + \theta} x_0 \right]^2 \\ &\qquad + \frac{P(0,i_0)\theta}{P(0,i_0)H(0,i_0)^2 + \theta} x_0^2. \end{aligned} \tag{5.4}$$

*Proof.* By assumption (3.8) and Theorem 3.3, the mean-variance problem (2.12) is feasible for any $z \in \mathbb{R}^1$. Moreover, using exactly the same approach as in the proof of Theorem 4.2, one can show that problem (2.12) *without* the constraint $Ex(T) = z$ must have a finite optimal value; hence so does the problem (2.12). Therefore, (2.12) is finite for any $z \in \mathbb{R}^1$. Since $J_{\mathrm{MV}}(x_0, i_0, \pi(\cdot))$ is strictly convex in $u(\cdot)$ and the constraint function $Ex(T) - z$ is affine in $u(\cdot)$, we can apply the well-known duality theorem (see, e.g., [17, p. 224, Theorem 1][1]) to conclude that for any $z \in \mathbb{R}^1$, the

[1]To be precise, one should apply [17, p. 236, Problem 7] together with the proof of [17, p. 224, Theorem 1] in our case, as there is an *equality* constraint, $Ex(T) = z$, in (2.12). To be able to use the result there, one needs to check a condition posed in [17, p. 236, Problem 7]; namely, 0 is an interior point of the set $\mathcal{T} := \{Ex(T) - z | x(\cdot)$ is the wealth process of an admissible portfolio $u(\cdot)\}$. In the present case this condition is implied by Theorem 3.3, which essentially yields that $\mathcal{T} = \mathbb{R}^1$.

optimal value of (2.12) is

$$J^*_{\rm MV}(x_0,i_0)=\sup_{\lambda\in\mathbb{R}^1}\ \inf_{u(\cdot)\text{ admissible}} J(x_0,i_0,u(\cdot),\lambda)>-\infty. \tag{5.5}$$

By Theorem 4.2, $\inf_{u(\cdot)\text{ admissible}} J(x_0,i_0,u(\cdot),\lambda)$ is a quadratic function (4.8) in $\lambda-z$. It follows from the finiteness of the supremum value of this quadratic function (see (5.5)) that

$$P(0,i_0)H(0,i_0)^2+\theta-1\le 0.$$

Now, if

$$P(0,i_0)H(0,i_0)^2+\theta-1=0,$$

then again by Theorem 4.2 and (5.5) we must have

$$P(0,i_0)H(0,i_0)x_0-z=0$$

for every $z\in\mathbb{R}^1$, which is a contradiction. This proves (5.1). On the other hand, in view of (5.5), we maximize the quadratic function (4.8) over $\lambda-z$ and conclude that the maximizer is given by (5.3), whereas the maximum value is given by the right-hand side of (5.4). Finally, the optimal control (5.2) is obtained by (4.7) with $\lambda=\lambda^*$. □

The efficient frontier (5.4) reveals explicitly the tradeoff between the mean (return) and variance (risk) at the terminal. Quite contrary to the case without Markovian jumps [31], the efficient frontier in the present case is no longer a perfect square (or, equivalently, the efficient frontier in the mean-standard deviation diagram is no longer a straight line). As a consequence, one is not able to achieve a risk-free investment. This, certainly, is expected since now the interest rate process is modulated by the Markov chain, and the interest rate risk cannot be perfectly hedged by any portfolio consisting of the bank account and stocks (as with the case studied in [16]) because the Markov chain is independent of the Brownian motion.

Nevertheless, the expression (5.4) does disclose the *minimum variance*, namely, the minimum possible terminal variance achievable by an admissible portfolio, along with the portfolio that attains this minimum variance.

THEOREM 5.2 (minimum variance). *The minimum terminal variance is*

$$\text{Var } x^*_{\min}(T)=\frac{P(0,i_0)\theta}{P(0,i_0)H(0,i_0)^2+\theta}x_0^2\ge 0 \tag{5.6}$$

*with the corresponding expected terminal wealth*

$$z_{\min}:=\frac{P(0,i_0)H(0,i_0)}{P(0,i_0)H(0,i_0)^2+\theta}x_0 \tag{5.7}$$

*and the corresponding Lagrange multiplier $\lambda^*_{\min}=0$. Moreover, the portfolio that achieves the above minimum variance, as a function of the time $t$, the wealth level $x$, and the market mode $i$, is*

$$u^*_{\min}(t,x,i)=-[\sigma(t,i)\sigma(t,i)']^{-1}B(t,i)'[x-z_{\min}H(t,i)]. \tag{5.8}$$

*Proof.* The conclusions regarding (5.6) and (5.7) are evident in view of the efficient frontier (5.4). The assertion $\lambda^*_{\min}=0$ can be verified via (5.3) and (5.7). Finally, (5.8) follows from (5.2). □

*Remark* 6. As a consequence of the above theorem, the parameter $z$ can be restricted to $z \geq z_{\min}$ when one defines the efficient frontier for the mean-variance problem (2.12).

THEOREM 5.3 (mutual fund theorem). *Suppose an efficient portfolio $u_1^*(\cdot)$ is given by* (5.2) *corresponding to $z = z_1 > z_{\min}$. Then a portfolio $u^*(\cdot)$ is efficient if and only if there is a $\mu \geq 0$ such that*

$$u^*(t) = (1-\mu)u_{\min}^*(t) + \mu u_1^*(t), \quad t \in [0,T], \tag{5.9}$$

*where $u_{\min}^*(\cdot)$ is the minimum variance portfolio defined in Theorem* 5.2.

*Proof.* We first prove the "if" part. Since both $u_{\min}^*(\cdot)$ and $u_1^*(\cdot)$ are efficient, by the explicit expression of any efficient portfolio given by (5.2), $u^*(t) = (1-\mu)u_0^*(\cdot) + \mu u_1^*(t)$ must be in the form of (5.2) corresponding to $z = (1-\mu)z_{\min} + \mu z_1$ (also noting that $x^*(\cdot)$ is linear in $u^*(\cdot)$). Hence $u^*(\cdot)$ must be efficient.

Conversely, suppose $u^*(\cdot)$ is efficient corresponding to a certain $z \geq z_{\min}$. Write $z = (1-\mu)z_{\min} + \mu z_1$ with some $\mu \geq 0$. Multiplying

$$u_{\min}^*(t) = -[\sigma(t,\alpha(t))\sigma(t,\alpha(t))']^{-1}B(t,\alpha(t))'[x_{\min}^*(t) - z_{\min}H(t,\alpha(t))]$$

by $(1-\mu)$, multiplying

$$u_1^*(t) = -[\sigma(t,\alpha(t))\sigma(t,\alpha(t))']^{-1}B(t,\alpha(t))'[x_1^*(t) + (\lambda_1^* - z_1)H(t,\alpha(t))]$$

by $\mu$, and summing them up, we obtain that $(1-\mu)u_{\min}^*(t) + \mu u_1^*(t)$ is represented by (5.2) with $x^*(t) = (1-\mu)x_{\min}^*(t) + \mu x_1^*(t)$ and $z = (1-\mu)z_{\min} + \mu z_1$. This leads to (5.9). ∎

*Remark* 7. The above mutual fund theorem implies that any investor needs only to invest in the minimum variance portfolio and another prespecified efficient portfolio in order to achieve the efficiency. Note that in the case where all the market parameters are deterministic [31], the corresponding mutual fund theorem becomes the *one-fund theorem*, which yields that any efficient portfolio is a combination of the bank account and a given efficient risky portfolio (known as the *tangent fund*). This is equivalent to the fact that the fractions of wealth among the stocks are the same among all efficient portfolios. However, in the present Markov-modulated case, this feature is no longer available.

**6. A special case: Interest rate unaffected by the Markov chain.** In this section we consider a special case where the interest-rate process does not respond to the change in the market mode, namely, $r(t,i) = r(t)$ for any $i = 1,2,\ldots,l$, whereas the appreciation-rate and volatility-rate processes still do. This stems from the situations where substantial changes in the interest-rate process are much less frequent than those in the other processes. For example, the interest rate may typically change on a bimonthly, or even less often, basis, whereas the stock market mode may switch on a weekly, or more frequent, basis. It turns out that the results obtained in the previous sections can be substantially simplified in this case.

The key to the simplification is that when $r(t,i) = r(t)$, the only solutions to (4.3) are

$$H(t,i) = e^{-\int_t^T r(s)ds} \quad \forall i = 1,2,\ldots,l \tag{6.1}$$

due to the uniqueness of solutions to (4.3). It follows then from (4.9) that

$$\theta = 0. \tag{6.2}$$

As a result, Theorem 5.1 reduces to the following result.

THEOREM 6.1. *Assume that* (3.8) *holds and that* $r(t,i) = r(t)$ *for all* $i = 1, 2, \ldots, l$. *Then we must have*

$$P(0, i_0) < e^{2\int_0^T r(s)ds}. \tag{6.3}$$

*Moreover, the efficient portfolio corresponding to* $z$*, as a function of the time* $t$*, the wealth level* $x$*, and the market mode* $i$*, is*

$$u^*(t,x,i) = -[\sigma(t,i)\sigma(t,i)']^{-1}B(t,i)'[x + (\lambda^* - z)e^{-\int_t^T r(s)ds}], \tag{6.4}$$

*where*

$$\lambda^* - z = \frac{z - P(0,i_0)e^{-\int_0^T r(s)ds}x_0}{P(0,i_0)e^{-2\int_0^T r(s)ds} - 1}. \tag{6.5}$$

*Furthermore, the optimal value of* Var $x(T)$*, among all the wealth processes* $x(\cdot)$ *satisfying* $Ex(T) = z$*, is*

$$\text{Var}\, x^*(T) = \frac{P(0,i_0)e^{-2\int_0^T r(s)ds}}{1 - P(0,i_0)e^{-2\int_0^T r(s)ds}}\left[z - e^{\int_0^T r(s)ds}x_0\right]^2. \tag{6.6}$$

*Proof.* This is straightforward by Theorem 5.1, together with (6.1) and (6.2). □

*Remark* 8. Note that in this case the efficient frontier involves a perfect square, even if the market parameters of the stocks are all random. The *capital market line* (see, e.g., [18]) in the mean-standard deviation diagram is

$$Ex^*(T) = e^{\int_0^T r(t)dt}x_0 + \sqrt{\frac{1 - P(0,i_0)e^{-2\int_0^T r(s)ds}}{P(0,i_0)e^{-2\int_0^T r(s)ds}}}\,\sigma_{x^*(T)}. \tag{6.7}$$

Therefore, the price of risk is given by

$$p = \sqrt{\frac{1 - P(0,i_0)e^{-2\int_0^T r(s)ds}}{P(0,i_0)e^{-2\int_0^T r(s)ds}}},$$

which depends only on the initial market mode $i_0$.

*Remark* 9. Clearly the minimum terminal variance in this case is zero, corresponding to putting all the money in the bank account. Moreover, $z_{\min} = e^{\int_0^T r(t)dt}x_0$. Consequently, the mutual fund theorem (Theorem 5.3) specifies that any efficient portfolio is a combination of the bank account and a given efficient portfolio. In other words, the one-fund theorem is valid in this case. In particular, the proportions of the stocks in all the efficient portfolios are the same under a particular market mode, irrespective of the wealth level and risk preference of the investors. This, in turn, will lead to the so-called market portfolio and *capital asset pricing model* (CAPM); see [18].

*Remark* 10. If we further assume that all the appreciation-rate and volatility-rate processes are independent of the market mode $i$, then $P(t,i) = e^{-\int_0^T [\rho(s) - 2r(s)]ds}$ for each $i = 1, 2, \ldots, l$ are the only solutions to (4.2). In this case, all the results reduce to those of [31].

*Remark* 11. We see from (6.1) that the functions $H(t,i)$, which are keys in our main results Theorems 5.1–5.3, are nothing else than a generalization of the discount factor between the present time to the terminal time under different market modes. Note that Proposition 4.1 stipulates that if the interest rate $r(t,i) > 0$ for a mode $i$, then the corresponding $H(t,i) < 1$, representing a genuine discount.

**7. Concluding remarks.** We have developed mean-variance optimal portfolio selection for a market with regime switching. The formulation allows the market to have random switching among a finite number of possible configurations that are modulated by a continuous-time Markov chain. Such a setup takes into consideration the discrete changes in a regime across which the behavior of the corresponding market could be markedly different. Our main effort has been devoted to obtaining efficient portfolios and an efficient frontier. It is interesting to note that for the Markov-modulated model, the efficient frontier is no longer a perfect square, except in the case when the interest rate is independent of the Markov chain.

There are several interesting problems that deserve further investigation. One is a model with nonnegativity constraints on the terminal wealth. As discussed earlier, this would render a stochastic LQ control problem with a sample-wise state constraint, which is a very challenging problem. Another problem is one with transaction costs. Although with the rapidly growing use of on-line trading, transaction costs nowadays represent a very small, if not at all negligible, portion of the total transacted values, the problem with transaction costs is theoretically interesting as it leads to a singular stochastic control problem whose solution would normally exhibit very different behavior than its no-transaction counterpart. In particular, with transaction costs optimal strategies would no longer be continuously trading strategies as opposed to the no-transaction case. In some sense, one motivation of introducing the transaction costs is to limit the changes in the optimal strategy. Indeed, Soner and Touzi [25] considered a market, in the absence of transaction costs, with the so-called gamma constraints in order to restrict the unbounded variation of the portfolios under consideration. Yet another problem is to remove the assumption that the Markov chain is independent of the underlying Brownian motion. Note that the mean-variance portfolio selection with the Brownian motion adapted random market coefficients has been completely solved in [16]. The model of this paper represents another "extreme" where the random coefficients are entirely independent of the Brownian motion. A more general model where the randomness in the coefficients is neither adapted to nor independent of the Brownian motion may be tackled by decomposing the problem into the two extremes that have been solved. On the other hand, the Markov chain describing the regime switching is assumed to be *completely* observable in this paper. A more realistic and theoretically interesting model is that the Markov chain is "hidden" and only partially observable through the stock prices. In this case, one needs to first perform filtering in order to estimate the state of the current regime before making efficient investment strategies. In [8], Elliott, Malcolm, and Tsoi developed schemes to estimate the appreciation rate, the volatility, and the generator of the underlying Markov chain. Estimates of the generator were also obtained via the stochastic approximation method in [28]. These estimation techniques may be used in conjunction with the portfolio selection approach presented in this work. Finally, a corresponding discrete-time model will be useful in the actual computing. In addition, to take into consideration that the Markov chain may have a large state space, an interesting problem is to reduce complexity via a singular perturbation approach.

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