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Graphon Mean Field Games and the GMFG Equations

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Abstract— **Networks are ubiquitous in modern society and the need to analyse, design and control them is evident. However many technical and social networks apparently grow unboundedly over time. This has the undesirable consequence that, inevitably, any method founded upon techniques whose effectiveness decreases with the size of the network will eventually be overwhelmed. This paper presents a framework called Graphon Mean Field Game (GMFG) theory for the analysis and control of non-cooperative dynamical game systems distributed over networks of unbounded size. This work is based upon the recently developed and profoundly influential graphon theory of large networks and their infinite limits. A theory for the centralized control of asymptotically infinite networks has already been formulated within the framework of dynamical systems on graphons [Gao and Caines, CDC 2017]. The current work greatly extends that analysis to populations of competing dynamical agents for which the game theoretic equilibria are expressed in terms of the newly defined Graphon Mean Field (GMFG) equations, these being a significant generalization of the classical MFG PDEs. Furthermore, in this paper, existence and uniqueness theorems for GMFG equations are given together with a sketch of the corresponding epsilon-Nash theory for GMFG systems.**

I. Introduction

One overall strategy which may be adopted to confront the situation of huge system complexity, and of unlimited recursive growth, as occurs with various populations and networks, is to perform an end-run around the phenomenon by passing directly to some sort of infinite limit. This approach has a distinguished history, since it is the conceptual principle which underlays the creation of statistical mechanics in the 19th Century which gave a rigorous mathematical formulation for the prediction of the behaviour of gases of various molecular weights subject to temperature and pressure variations. Central to this achievement was the celebrated Boltzmann Equation [1]. In fact examples of continuum methods for the study of phenomena which at sufficiently fine level have a discrete dynamical description abound in mathematical physics and the life sciences. One may cite, for instance, the PDEs of fluid mechanics [2], and the Fokker-Plank-Kolmogorov (FPK) equations for the flow of probabilities [3], [4], which of course appear in a host of applications including, for instance, epidemiology and ecology. A recent example of this fundamental approach in systems and control science is indeed that of Mean Field Game theory which originated in the early years of this century, and a second, roughly contemporaneous example, is that of the infinite limits of graphs and networks which is called graphon theory.

There have been some mathematically rigorous studies of MFG systems with state values in finite graphs (see e.g. [5]), and of MFG systems where the agent subsystems are defined at the nodes (vertices) of finite random Erdös-Rényi graphs [6]. In that work the issue of system behaviour subject to the unbounded growth of the network has not been tackled directly, perhaps for want of an appropriate theory. However, a beautiful unifying theory of graph limits has been developed in recent years, and this work up to 2012 is summarized in the important monograph [7]. This theory gives a rigorous formulation of the notion of limits for infinite sequences of networks of increasing size (as measured by the number of nodes) and is developed in [8], [9], [10], [11], [7] among other works. The theory is still undergoing vigorous growth, but it has already revolutionized certain areas of combinatorics and discrete mathematics by connecting them with topology, functions and measures within the domain of mathematical analysis.

The first application of graphon theory in systems and control theory appeared in 2017 in [12], [13], [14], while initial work on static game theoretic equilibria for infinite populations on graphons has been reported in [15]. However the work in [14], and its further development in [16], treats the centralized and distributed control of arbitrarily large networks of dynamical control systems for which a direct solution would be completely intractable due to the cardinality of the network. Approximate control is achieved by solving control problems on the infinite limit graphon and then applying control laws derived from those solutions on the finite network of interest. The analogy with the strategies for finding feedback laws resulting in ε-Nash equilibria in the MFG framework is obvious.

A natural framework for the formulation of large scale decentralized control on networks and hence for game theoretic problems involving agents distributed over large networks is given by Mean Field Game theory defined on graphons. The resulting basic idea and the associated fundamental equations for what we term graphon Mean Field Game (GMFG) systems are the subject of the current paper. The GMFG equations are of great generality since they permit the study, in the limit, of both dense and sparse, infinite networks of non-cooperative dynamical agents. Moreover the classical MFG equations are retrieved when the communication over the infinite network (modelled as a graphon) involves uniform weightings of a direct influence of all agents on the network on every other agent on the network. In relation to non-uniform weightings in mean field games, an early

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analysis of linear quadratic models can be found in [17].

II. The Concept of a Graphon

The basic idea of the theory of graphons is that the edge structure of each finite cardinality network is represented by a step function density on the unit square in $\mathbb{R}^2$ and a metric (called the cut-metric and based upon a graph theoretic notion) is introduced. The set of finite graphs endowed with the cut metric then gives rise to a metric space, and the completion of this space is the space of *graphons*. Graphons are represented by bounded symmetric Lebesgue measurable functions $W : [0,1]^2 \to [0,1]$ which can be interpreted as weighted graphs on the vertex set $[0,1]$.

To be specific, unless otherwise stated, the term "graphon" here is used to refer to measurable functions $W_1 : [0,1]^2 \to [-1,1]$ and $\mathbf{G_1^{sp}}$ denotes the space of graphons. Let $\mathbf{G^{sp}}$ denote the space of all symmetric measurable functions $W : [0,1]^2 \to \mathbb{R}$.

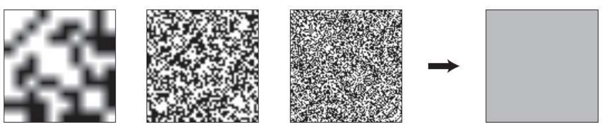

Fig. 1: E-R Graph Sequence Converging to Limit [7]

The cut norm of a graphon then has the expression:

$$\|W\|_\square = \sup_{M,T\subset[0,1]} \left| \int_{M\times T} W(x,y)dxdy \right| \quad (1)$$

with the supremum taking over all measurable subsets M and T of $[0,1]$. Denote the set of measure preserving bijections $[0,1] \to [0,1]$ by $S_{[0,1]}$. The *cut metric* between two graphons V and W is then given by

$$d_\square(W,V) = \inf_{\phi\in S_{[0,1]}} \|W^\phi - V\|_\square, \quad (2)$$

where $W^\phi(x,y) := W(\phi(x),\phi(y))$.

The space $(\mathbf{G_1^{sp}}, d_\square)$ is compact and this still holds if $\mathbf{G_1^{sp}}$ is replaced by any uniformly bounded subset of $\mathbf{G^{sp}}$ closed in the $d_\square$ distance [7]. Sets in $\mathbf{G_1^{sp}}$ compact with respect to the L^2 metric are compact with respect to the cut metric. It follows that if a graphon sequence is Cauchy in the L^2 metric then it is also a Cauchy sequence in the cut metric and the limits are identical in $\mathbf{G_1^{sp}}$.

III. Graphon Mean Field Game Systems and the MFG Equations

A. The Standard MFG Model

In the diffusion based models of large population games the state evolution of a collection of N agents $\mathscr{A}_i, 1 \le i \le N < \infty$, is specified by a set of N controlled stochastic differential equations (SDEs). A simplified form of the general case is given by the following set of controlled SDEs which for each agent $\mathscr{A}_i$ includes state coupling with *all* other agents.

$$dx_i(t) = \frac{1}{N}\sum_{j=1}^{N} f(t,x_i(t),u_i(t),x_j(t))dt + \sigma dw_i(t), \quad (3)$$

where for the ith agent $\mathscr{A}_i$, $x_i \in \mathbb{R}^n$ is the state, $u_i \in \mathbb{R}^m$ the control input, and $w_i \in \mathbb{R}^r$ a standard Wiener process, and where $\{w_i, 1 \le i \le N\}$ are independent processes. For simplicity, all collections of system initial conditions are taken to be independent and have finite second moment.

The dynamics of a generic agent $\mathscr{A}_i$ in the infinite population limit of this system is then described by the controlled McKean-Vlasov (MV) equation

$$dx_i = f[x_i,u_i,\mu_t]dt + \sigma dw_i, \quad 0 \le t \le T, \quad (4)$$

where $\mu_t(\cdot)$ denotes the distribution of the state of the generic agent in the population at $t \in [0,T], f[x,u,\mu_t] := \int_{\mathbb{R}} f(x,u,y)\mu_t(dy)$ and where the initial condition measure μ_0 is specified.

The finite population mean field cost function with coupling with *all* other agents is given by

$$J_i^N(u_i,u_{-i}) := E\int_0^T (1/N)\sum_{j=1}^{N} L(x_i(t),u_i(t),x_j(t))dt, \quad (5)$$

where $1 \le i \le N$ and $L(\cdot)$ is the pairwise cost rate function, and u_{-i} denotes the controls of all agents other than $\mathscr{A}_i$. Setting the infinite population running cost to be $L[x,u,\mu_t] = \int_{\mathbb{R}} L(x,u,y)\mu_t(dy)$ the corresponding infinite population expected cost for a generic agent $\mathscr{A}_i$ takes the form

$$J_i(u_i,\mu) \ := \ E\int_0^T L[x_i(t),u_i(t),\mu_t]dt. \quad (6)$$

B. The Graphon MFG Model and Its Equations

In this section the standard MFG equations (see e.g. [18], [19]) will be generalized so that they subsume the standard (implicitly uniform totally connected) dense network case and cover the fully general graphon limit network case. Specifically, the agent index i will be identified with the ith node in an N node network and in the infinite population graphon limit this will be taken to map to $\alpha \in [0,1]$. It is important to note here that *although the network is assumed dense it is not assumed to be uniformly totally connected*; indeed, the connection structure of the infinite network is represented precisely by its graphon $\mathbf{G} = \{g(\alpha,\beta), 0 \le \alpha,\beta \le 1\}$.

The generalized Graphon-MFG scheme below on $[0,T]$ is given by the linked equations for (i) the value function V_α for a generic agent's stochastic control problem when all other agents' control laws are fixed and generating the given local mean field μ_α and the graphon local mean field μ_β, (ii) the FPK for the MV-SDE for the local mean field of the generic agent, and (iii) the specification of the best response feedback law.

The key feature of the generalized graphon MFG construction beyond the standard MFG scheme is that at any agent in a dense network the averaged dynamics (3) and cost function (5) decompose into averages of neighbouring subpopulations distributed on the network edges incident upon that agent's node plus a standard local differential dynamics. In the limit, the summed subpopulation averages are given by an integral over the local mean field measures of the neighbouring agents. For notational simplicity, we present the graphon MFG framework with scalar individual states and controls, i.e., $n=m=r=1$. Its extension to the vector case is evident. Specifically, (suppressing the time index on the measures for simplicity of notation) we have the *Graphon Mean Field Game (GMFG) equations*:

$$\text{[HJB]}(\alpha) \quad -\frac{\partial V_\alpha(t,x)}{\partial t} = \inf_{u\in U}\left\{\widetilde{f}[x,u,\mu_G;g_\alpha]\frac{\partial V_\alpha(t,x)}{\partial x} + \widetilde{l}[x,u,\mu_G;g_\alpha]\right\} + \frac{\sigma^2}{2}\frac{\partial^2 V_\alpha(t,x)}{\partial x^2}, \quad (7)$$

$$V_\alpha(T,x)=0, \quad (t,x)\in[0,T]\times\mathbb{R}, \quad \alpha\in[0,1],$$

$$\text{[FPK]}(\alpha) \quad \frac{\partial p_\alpha(t,x)}{\partial t} = -\frac{\partial\{\widetilde{f}[x,u^0(x,\mu_G;g_\alpha)p_\alpha(t,x)\}}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 p_\alpha(t,x)}{\partial x^2}, \quad (8)$$

$$\text{[BR]}(\alpha) \quad u^0(x_\alpha,\mu_G;g_\alpha) =: \varphi(t,x_\alpha|\mu_G;g_\alpha)$$

where

$$f[x_\alpha,u_\alpha,\mu_G;g_\alpha] := \int_{[0,1]}\int_R f(x_\alpha,u_\alpha,x_\beta)g(\alpha,\beta)\mu_\beta(dx_\beta)d\beta, \quad (9)$$

which gives the complete local graphon dynamics via the sum

$$\widetilde{f}[x_\alpha,u_\alpha,\mu_G;g_\alpha] := f_0(x_\alpha,u_\alpha)+f[x_\alpha,u_\alpha,\mu_G;g_\alpha], \quad (10)$$

and where the individual agent α has the performance function given by

$$J_\alpha(u_\alpha,\mu_G) = E\int_0^T \widetilde{l}[x_\alpha(t),u_\alpha(t),\mu_G;g_\alpha]dt, \quad (11)$$

where the individual cost function is specified by

$$\tilde{l}[x_\alpha,u_\alpha,\mu_G;g_\alpha] = \int_0^1\int_{\mathbb{R}} l(x_\alpha,u_\alpha,x_\beta,g(\alpha,\beta)\mu_\beta(dx_\beta)d\beta.$$

Here the densities $p_\alpha(t,x)$ of the measures $\mu_\alpha\equiv\mu_\alpha(t)$ are assumed to exist, and where, to complete the specification of the GMFG equations, we define the following terms: the *graphon local mean field* μ_α and the corresponding set or ensemble of all the *local mean fields* $\mu_G=\{\mu_\beta, 0\le\beta\le 1\}$, the *local dynamics* $f_0(x_\alpha,u_\alpha)$, the *network local dynamics* $f(x_\alpha,u_\alpha,x_\beta)$, and the *graphon averaged local dynamics* $f[x_\alpha,u_\alpha,\mu_G;g_\alpha]$ of the α indexed system and the graphon function $g_\alpha=\{g(\alpha,\beta);0\le\beta\le 1\}$.

To have some intuition behind (10), we provide a probabilistic interpretation below. Let $[0,1]$ be assigned a grid of N points α_k, $1\le k\le N$, to model the locations of N agents. At given time t, let the states of the agents be independent random variables X_{α_k} with distribution μ_{α_k}. Now given $X_{\alpha_i}=x_{\alpha_i}$ and deterministic u_{α_i}, consider the network average term

$$S_i = \frac{1}{N}\sum_{k\neq i}^N f(x_{\alpha_i},u_{\alpha_i},X_{\alpha_k})a_{\alpha_i\alpha_k}.$$

The exclusion of the self term of $k=i$ has no effect on the approximation below but will ease our specification. Suppose $a_{\alpha_i\alpha_k}$, $k\neq i$, are $N-1$ independent random variables taking values in $\{0,1\}$ implying independently generated links, which are also independent of X_{α_k}, $k\neq i$. Further assume $Ea_{\alpha_i\alpha_k}=g(\alpha_i,\alpha_k)$ and that there exists a fixed C such that

$$E|f(x,u,X_{\alpha_k})|^2\le C_f.$$

Then it can be shown that $Var(S_i)\le C_f/N$. This suggests we introduce ES_i as a good approximation of S_i, where

$$ES_i = \frac{1}{N}\sum_{k\neq i}^N\int_{\mathbb{R}} f(x_{\alpha_i},u_{\alpha_i},y)\mu_{\alpha_k}(dy)g(\alpha_i,\alpha_k).$$

Under suitable conditions, the summation is further approximated by $f[x_{\alpha_i},u_{\alpha_i},\mu_G;g_{\alpha_i}]$.

In a similar manner, $\widetilde{l}$ is defined based on two functions $l_0(x_\alpha,u_\alpha)$, $l(x_\alpha,u_\alpha,x_\beta)$ and g_α. Note that in (7) and (8), μ_G depends on time t and may be written as $\mu_G(t)$.

This completes the GMFG specification. Finally, parallel to the standard MFG case, the stochastic differential equation

$$\text{[MV-SDE]}(\alpha) \quad dx_\alpha = \widetilde{f}[x_\alpha,u_\alpha,\mu_G;g_\alpha]dt+\sigma dw_t, \quad 0\le t\le T, \quad \alpha\in[0,1], \quad (12)$$

corresponds to the PDE [FPK](α), where the initial condition is $p_\alpha(0,\cdot)$. We observe that to solve the SDE (12) the set of SDEs for all $\alpha\in[0,1]$ must be simultaneously solved.

We notice that we retrieve the simplest standard MFG framework, where the agents' dynamics and costs are uniform, and where the network is totally connected with uniform link weights, by setting $\{g(\alpha,\beta)=1, 0\le\alpha,\beta\le 1\}$. Then at each node (by uniformity) the local MKV state distributions may be taken to be equal to that at a nominal node α, in other words $\mu_\beta(dx_\beta)=\mu_\alpha(dx_\beta)$ for all β. In that case, at the instant t, by (10), $\widetilde{f}[x,u,\mu_G(t);g_\alpha]$ takes the standard MKV dynamics form $f[x,u,\mu_\alpha(t)]$, and the local and graphon local measures are equal: $\mu_\alpha(t)=:\mu(t)$, for all α. Evidently this system wide mean field state distribution is given by the standard form of equation (8) (see [18], [19]).

It is to be noted that the GMFG best response control of each agent generalizes the fundamental property of a solution to the MFG equations in the standard case where for a generic agent *the feedback control (BR) generates a Nash equilibrium* with the best response strategy depending only on the agent's state, x_α, and the standard mean field, namely the generic agent's state distribution $\mu(t)$.

Finally we note that the standard case of controlled diffusion processes is simply obtained by setting $\{g(\alpha,\beta)=0; 0\le\alpha,\beta\le 1\}$, which totally disconnects the network and results simply in $\widetilde{f}[x,u,\mu_G(t);g_\alpha]=f_0(x,u)$, and $\widetilde{l}[x,u,\mu_G(t);g_\alpha]=l_0(x,u)$.

In order to analyze the solvability of the GMFG equations, we need to restrict $\mu_G(\cdot)$ from a certain class. We say $\{\mu_G(t), 0\le t\le T\}$ is from the admissible set $\mathscr{M}_{[0,T]}$ if

M1) for each fixed t, $\int_B \mu_\beta(t,dy)$ is a Lebesgue measurable function of β;

M2) there exists $\eta \in (0,1]$ such that for any bounded and Lipschitz continuous function ϕ on $\mathbb{R}$,

$$\sup_{\beta\in[0,1]} \left|\int_{\mathbb{R}} \phi(y)\mu_\beta(t_1,dy) - \int_{\mathbb{R}} \phi(y)\mu_\beta(t_2,dy)\right| \le C_h|t_1-t_2|^\eta$$

where C_h may be selected to depend only on the Lipschitz constant $\mathrm{Lip}(\phi)$ for ϕ.

Remark 1: Condition M1) ensures the integration with respect to $d\beta$ in (10) is well defined. By condition M2), the drift term in the HJB equation has a certain time continuity, which facilitates the existence analysis of the best response.

A collection of measures on some measurable space which are indexed by the vertex set $[0,1]$ is called a measure ensemble. Thus, for each fixed t, $\mu_G(t)$ is a measure ensemble.

C. Assumptions for the Existence Analysis

We introduce the following assumptions:

(H1) U is a compact set.

(H2) $f(x,u,y)$ and $l(x,u,y)$ ($f_0(x,u)$ and $l_0(x,u)$, resp.) are continuous and bounded functions on $\mathbb{R}\times U\times\mathbb{R}$ ($\mathbb{R}\times U$, resp.), and are Lipschitz continuous in (x,y) (in x, resp.) uniformly with respect to u.

(H3) For f_0, f and l_0, l, their first and second derivatives with respect to x are all uniformly continuous and bounded in $\mathbb{R}\times U\times\mathbb{R}$ (or $\mathbb{R}\times U$).

(H4) $f(x,u,y)$ ($f_0(x,u)$, resp.) is Lipschitz continuous in u, uniformly with respect to (x,y) (to x, resp.).

(H5) For any $q\in\mathbb{R}$, $\alpha\in[0,1]$ and any probability measure ensemble μ_G satisfying M1), the set

$$\begin{aligned} S(x,q) &= \arg\min_u [q(\tilde{f}[x,u,\mu_G;g_\alpha]) + \widetilde{l}[x,u,\mu_G;g_\alpha]] \\ &= \arg\min_u [q(f_0(x,u) + f[x,u,\mu_G;g_\alpha]) \\ &\qquad + \widetilde{l}[x,u,\mu_G;g_\alpha]] \end{aligned}$$

is a singleton, and the resulting u as a function of (x,q), is Lipschitz continuous in (x,q), uniformly with respect to μ_G and g_α.

D. Stochastic Analysis Tools

Although the GMFG equation system only involves $\{\mu_G(t), 0\le t\le T\}$, which may be viewed as a collection of marginals at different vertices, it is necessary to develop the existence analysis in the underlying probability spaces (see related discussions in [20, p.240]).

We begin by introducing some analytic preliminaries. For the space $C_T = C([0,T],\mathbb{R})$, we specify a σ-algebra $\mathscr{F}_T$ induced by all cylindrical sets of the form $\{x(\cdot)\in C_T : x(t_i)\in B_i, 1\le i\le l \text{ for some } l\}$, where B_i is a Borel set. Let M_T denote the space of all probability measures on $(C_T,\mathscr{F}_T)$. The canonical process X is defined by $X_t(\omega)=\omega_t$ for $\omega\in C_T$.

On C_T, we introduce the metric $\rho(x,y) = \sup_t |x(t) - y(t)| \wedge 1$. Then (C_T,ρ) is a complete metric space. Based on ρ, we introduce the Wasserstein metric on M_T. For $m_1, m_2 \in M_T$, denote

$$\begin{aligned} &D_T(m_1,m_2) = \\ &\inf_m \int_{C_T\times C_T} (\sup_{s\le T}|X_s(\omega_1)-X_s(\omega_2)|\wedge 1)\,dm(\omega_1,\omega_2), \end{aligned}$$

where m is called a coupling as a probability measure on $(C_T,\mathscr{F}_T)\times(C_T,\mathscr{F}_T)$ with the first and second marginals as m_1 and m_2. Then (M_T,D_T) is a complete metric space [21].

We introduce the following product of probability measure spaces $\prod_{\alpha\in[0,1]}(C_T,\mathscr{F}_T,m_\alpha)$, where each individual space is interpreted as the path space of the agent at vertex α with a corresponding probability measure m_α. Denote the product of spaces of probability measures

$$M_{T,G} = \prod_{\alpha\in[0,1]} M_T.$$

An element in $M_{T,G}$ is a measure ensemble. Given a measure ensemble m_G, the projection operator Proj_α picks up its component associated with $\alpha\in[0,1]$.

For two measure ensembles $m_G := (m_\alpha)_{\alpha\in[0,1]}$ and $\bar{m}_G := (\bar{m}_\alpha)_{\alpha\in[0,1]}$ in $M_{T,G}$, define

$$d(m_G,\bar{m}_G) = \sup_{\alpha\in[0,1]} D_T(m_\alpha,\bar{m}_\alpha).$$

Lemma 1: $(M_{T,G}, d)$ is a *complete metric space.*

Proof. If $\{m_G(k), k\ge 1\}$ is a fundamental sequence in $M_{T,G}$, then each sequence $\{\mathrm{Proj}_\alpha(m_G(k)), k\ge 1\}$ (of probability measures) is a fundamental sequence in the complete metric space M_T and so has a limit within M_T. ■

Given the probability measure m_α, we determine the t-marginal as follows: $\mu_\alpha(t,B) = m_\alpha(\{x(\cdot)\in C_T : x(t)\in B\})$ for a Borel set $B\subset\mathbb{R}$, which will simply be denoted by $\mu_\alpha(t)$. Consider $m_G\in M_{T,G}$ and denote the time t marginal on a measure ensemble by the following rule

$$\mu_G(t) := (\mu_\alpha(t))_{\alpha\in[0,1]} = \mathrm{Marj}_t(m_G). \tag{13}$$

For a given t, this can be interpreted as a measure valued function defined on the vertex set $[0,1]$.

E. Existence Theorem

Step 1 – The best response map. First we let

$$\{\mu_G(t), 0\le t\le T\} \in \mathscr{M}_{[0,T]} \tag{14}$$

be fixed. We determine the best response for the α-agent as a feedback

$$u_\alpha = \phi_\alpha(t,x|\mu_G(\cdot)), \quad \alpha\in[0,1].$$

By (H1)-(H5), the function ϕ_α is bounded and continuous on $[0,T]\times\mathbb{R}$, and Lipschitz continuous in x. Applying the control law ϕ_α for all the agents, we obtain by (12) the closed-loop state process which, further, determines the measure m_α on $(C_T,\mathscr{F}_T)$. Next we introduce the mappings

$$(\phi_\alpha)_{\alpha\in[0,1]} = \Gamma(\mu_G(\cdot))$$

and

$$(m_\alpha)_{\alpha\in[0,1]} = \widehat{\Gamma}\circ\Gamma(\mu_G(\cdot)).$$

Thus $\widehat{\Gamma}$ maps the set of best responses to a measure ensemble. Now the existence analysis may be formulated as the fixed point problem to find (m_α) such that

$$(m_\alpha)_{\alpha\in[0,1]} = \widehat{\Gamma}\circ\Gamma\circ\{\mathrm{Marj}_t(m_\alpha)_{\alpha\in[0,1]}, 0\le t\le T\}.$$

Step 2 – *Sensitivity condition*. In the search for the fixed point, we consider

$$m_G = (m_\alpha)_{\alpha\in[0,1]} \in \widehat{\Gamma}\circ\Gamma(\mathscr{M}_{[0,T]}), \tag{15}$$

from which we determine μ_G and the next ϕ_α. By considering m_G of the form (15), we can ensure sufficient regularity of a subsequently generated $\mu_G(\cdot)$ by employing M1) and M2) which is further used to solve the HJB equation.

When $\bar{m}_G \in \widehat{\Gamma}\circ\Gamma(\mathscr{M}_{[0,T]})$ is used, we let the corresponding solution be denoted by $\bar{\mu}_G$ and $\bar{\phi}_\alpha(t,x|\bar{\mu}_G(\cdot))$, and we introduce the regularity assumption: For some $c_1>0$,

$$\sup_{t,x,\alpha}|\phi_\alpha(t,x)|\mu_G) - \bar{\phi}_\alpha(t,x|\bar{\mu}_G)| \le c_1 D_T(m_G,\bar{m}_G). \tag{16}$$

This is a generalization from the finite class model in [20] where an illustration via a linear model is presented.

Step 3 – *Regeneration of the state processes*. Once the α-agent has applied the strategy ϕ_α as given in Step 2, let the resulting distribution of the state process be denoted by m_α^{new}, which further determines m_G^{new}. This is done in parallel for $\bar{m}_G$ to generate $\bar{m}_\alpha^{\mathrm{new}}$ and next $\bar{m}_G^{\mathrm{new}}$. As in [20, Lemma 9], we can show

$$D_T(m_\alpha^{\mathrm{new}},\bar{m}_\alpha^{\mathrm{new}}) \le c_2 \sup_{t,x}|\phi_\alpha(t,x|\mu_G(\cdot)) - \bar{\phi}_\alpha(t,x|\bar{\mu}_G(\cdot))|$$

for some constant c_2 not depending on α.

Step 4. By the estimates in Steps 2 and 3, we obtain

$$D_T(m_\alpha^{\mathrm{new}},\bar{m}_\alpha^{\mathrm{new}}) \le c_1c_2 D_T(m_G,\bar{m}_G).$$

Since α is arbitrary, it follows that

$$D_T(m_G^{\mathrm{new}},\bar{m}_G^{\mathrm{new}}) \le c_1c_2 D_T(m_G,\bar{m}_G).$$

Based on the above steps, we state the main result on existence and uniqueness of solutions to the GMFG equation system.

Theorem 2: If the gain condition $c_1c_2<1$ holds, there exists a unique solution to the GMFG equations, which (i) gives the feedback control best response (BR) strategy $\varphi(t,x_\alpha|\mu_G;g_\alpha)$ depending only upon the agent's state and the graphon local mean fields (i.e. $(x_\alpha,\mu_G;g_\alpha)$), and (ii) generates a Nash equilibrium. ■

Remark 2: By SDE estimates, one can obtain refined bound information on c_2. When the network coupling effect is weak, a small value for c_2 can be obtained.

Remark 3: For linear models, a verification of (16) can be done under certain model parameters, as in [20]. This part is itself an interesting though challenging topic. It is related to perturbation analysis of quasi-linear PDEs.

IV. THE ANALYSIS OF LIPSCHITZ FEEDBACK

The main analysis in Section III is based on Lipschitz feedback. This section provides a concrete model to check such a condition. We consider the following model where the dynamics is affine in the control u, i.e.,

$$f_0(x,u) = f_0(x)u, \quad f(x,u,y) = f(x,y)u,$$

and the cost is determined from

$$l_0(x,u) = l_0(x)u^2, \quad l(x,u,y) = l(x,y)u^2.$$

Given $\mu_G(t)$, $t\in[0,T]$, we check the minimizer of

$$p(f_0(x)+f[x,\mu_G(t);g_\alpha])u + (l_0(x)+l[x,\mu_G(t);g_\alpha])u^2$$

in the HJB equation

$$V_t(t,x) = \min_u[V_x(f_0+f)u + (l_0+l)u^2] + \frac{\sigma^2}{2}V_{xx}. \tag{17}$$

For the analysis of the control law, we introduce:

C1) The control set $U=[a,b]$ is a compact interval.

C2) f_0, f, l_0, l satisfy H1)-H4), and $l+l_0\ge c_0$ for some constant $c_0>0$.

Proposition 3: Suppose $\mu_G(\cdot)\in\mathscr{M}_{[0,T]}$. Then both $f[x,\mu_G(t);g_\alpha]$ and $l[x,\mu_G(t);g_\alpha]$ are Holder continuous in t uniformly with respect to x.

Proof. We estimate $|f[x,\mu_G(t_1);g_\alpha] - f[x,\mu_G(t_2);g_\alpha]|$ by using the Lipschitz condition of f and condition M2) on $\mathscr{M}_{[0,T]}$. This proves the Hölder continuity for $f[x,\mu_G(t);g_\alpha]$. The other function can be similarly checked. ■

Proposition 4: $V_x(t,x)$ is continuous and bounded on $[0,T]\times\mathbb{R}$ and Lipschitz continuous in x, uniformly with respect to t.

Proof. First of all, for the given μ_G, the HJB equation (17) is quasi-linear parabolic PDE with Hölder continuity in t. It has a classical solution V on $Q_0=(0,T)\times\mathbb{R}$. By the method in [22, p.209], we can show that V_x is bounded.

Moreover, by the approximation argument in [22, p.210], we can show V_x satisfies a Hölder continuity condition on each fixed set $(0,T)\times(|x|<C_0)$. Now the term $\widetilde{f}V_x+\widetilde{l}$ is Hölder continuous function of (t,x) on compact sets. By [23, sec. 4.10], we further obtain the bound for V_{xx} on compact sets. Due to C2), the bound estimate for V_{xx} does not depend on the particular choice of the compact set of a given diameter. This shows boundedness of V_{xx}. Hence V_x is Lipschitz in x. ■

Proposition 5: The optimal control law $\phi(t,x)$ in (17) is a bounded Lipschitz continuous function of x.

Proof. The boundedness is obvious from compactness of U. It is clear that the optimal control law is the minimizer $u^2+\frac{V_x(f_0+f)}{l_0+l}u$ subject to $u\in[a,b]$.

Now consider the following auxiliary problem to minimize u^2+su subject to $u\in[a,b]$, where s is a parameter. Then we can explicitly determine a function $F_{min}(z)=a$ for $z<a$, $=z$ for $z\in[a,b]$, $=b$ for $z>b$, such that the minimizer is given by $\hat{u}=F_{min}(s)$. Then F_{min} is Lipschitz continuous. Finally, by using Lipschitz continuity and boundedness of the function F_{min}, V_x, f_0,f,l_0+l, we can further show that the optimal control law $\phi(t,x)$ is a Lipschitz continuous function of x. ■

V. The performance issue

In the MFG case it is shown [20], [19] that the joint strategy $\{u_i^o(t) = \varphi_i(t, x_i(t)|\mu_t), 1 \le i \le N\}$ yields an ε-Nash equilibrium for all ε, i.e. for all $\varepsilon > 0$, there exists $N(\varepsilon)$ such that for all $N \ge N(\varepsilon)$

$$J_i^N(u_i^\circ, u_{-i}^\circ) - \varepsilon \le \inf_{u_i \in \mathscr{U}_i} J_i^N(u_i, u_{-i}^\circ) \le J_i^N(u_i^\circ, u_{-i}^\circ). \tag{18}$$

This form of approximate Nash equilibrium is in fact a principal result of all the MFG work in the contributions in the sequence [20], [19] to [24] and many other works in the literature.

That is to say, the infinite population solution is related to the sufficiently large finite population behaviour by an ε-Nash equilibrium property which states that the value of the *performance function of any agent in a finite population can be reduced (i.e. improved) by at most* ε *if it changes unilaterally from the infinite MFG population feedback law* while all other agents remain with the infinite population based control strategies.

This basic ε-Nash equilibrium result in MFG theory and its expected form in GMFG theory are not only theoretically significant but are vital for the application the application of MFG derived control laws since the solution of the MFG and GMFG equations is necessarily simpler than the effectively intractable task of finding the solution to the game problems for the large finite population systems. Indeed, this was one of the original motives for the creation of MFG theory since it provides a feedback control design methodology for systems of intractable complexity. Furthermore it is a basic feature of the graphon systems control theory [12].

For the graphon MFG case analyzed in this paper there is a double limit to be considered: first, the limit as the population of agents goes to infinity on a given finite graph, and, second, the limit where the graph converges to its graphon limit with the associated family of stochastic systems converging to a limit form. An unavoidable feature of this analysis will be that, just as in the graphon-systems control case, the application of the infinite network control to a finite network will require some form of sampling of the infinite BR control law for its application over a finite set of nodes.

VI. Conclusion

For future work, it is of great interest to rigorously establish ε-Nash equilibrium results in analogy with those in MFG theory and to develop computational techniques for GMFG problems.

There also potentially exists a direction of extension of MFG theory to Sparse Networks. Consider a (maximum) node degree d infinite network graph $G^\infty = (N^\infty, E^\infty)$, where the ith agent $\mathscr{A}_i$ with state x_i is located at the node n_i which lies in the neighbourhood N_i with boundary ∂N_i. By the definition of G^∞, the boundary ∂N_i contains at most d nodes and correspondingly at most d agents located at those nodes. The main change in passing from the graphon case to the sparse, or graphing, case is that the bounded measurable functions of the graphons are replaced with measures on the unit square which for any α are concentrated on a finite set of weighted unit measures indexed by β. So the first construction of an MFG theory for sparse networks would replace the measures appearing in the GMFG equations by such graphing measures.

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