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Abstract

We analyze the valuation of American options under the forward performance criterion introduced by Musiela and Zariphopoulou (2008). In this framework, the performance criterion evolves forward in time without reference to a specific future time horizon. Moreover, risk preferences change with stochastic market conditions, which is natural as investors are clearly more risk averse in economic downtimes. We examine two applications: the valuation of American options with stochastic volatility and the modeling of early exercises of American-style employee stock options. We also study the marginal forward performance price, which is related to the classical marginal utility price introduced by Davis (1997). Among others, we find that, for arbitrary preferences, the marginal forward indifference price is always independent of the investor's wealth and is represented as pricing the claim under the minimal martingale measure.

1 Introduction

Utility maximization theory has been central to quantifying rational investment decisions and risk-averse valuations of assets at least since the work of von Neumann and Morgenstern in the 1940s. In the Merton (1969) problem of continuous-time portfolio optimization, utility is defined at some time horizon in the future, when investment decisions are assessed in terms of expected utility of wealth. For portfolios involving derivatives, and associated utility indifference pricing problems, derivatives payoffs or random endowments may be realized at random times, which requires the specification of utility at other times, not just at a single terminal time. This consideration is particularly important for investment and pricing problems involving defaultable securities or American options.

One way to address this issue is to consider the definition of utility at the time of a random cash flow as analogous to specifying what the investor does with the endowment thereafter. Any answer to the latter question necessarily involves details of the market in which he might invest, and utilities and markets are inextricably linked. Some examples include Oberman and Zariphopoulou (2003), Leung and Sircar (2009a) for utility indifference pricing of American options, and Leung et al. (2008) and Jaimungal and Sigloch (2010) for defaultable securities. This approach allows for comparing utilities of wealth at different times. However, as is common in classical utility indifference pricing, the investor's risk

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preferences at intermediate times and the optimal investment decisions still directly depend on an *a priori* chosen investment horizon.

This horizon dependence issue has been addressed by one of the authors and Musiela through the construction of the forward performance criterion (see, for example, Musiela and Zariphopoulou (2008)). In this approach, the investor's utility is specified at an *initial* time, and his risk preferences at subsequent times evolve *forward* without reference to any specific ultimate time horizon. This results in a stochastic utility process, called the *forward performance process*, which satisfies certain properties so that it evolves consistently with the random market conditions. Hence, this approach necessarily connects risk preferences with market models. The risk profile of a given investor is no longer considered separately from his investment opportunities and the market. This is entirely natural: the current economic crisis has clearly shown increased risk aversion in investors as the market has fallen.

In this paper, we develop an indifference valuation methodology based on the forward performance criterion. Specifically, we study the valuation of a long position in an American option in an incomplete diffusion market model. This leads us to a combined stochastic control and optimal stopping problem. In Section 2, our main objective is to analyze the optimal trading and exercising strategies that maximize the option holder's forward performance from the dynamic portfolio together with the option payoff upon exercise. We also study the holder's *forward indifference price* for the American option, which is defined by comparing his optimal expected forward performance with and without the derivative (see Definition 2).

In Section 3, we discuss the *exponential* forward indifference valuation of an American option in a stochastic volatility model. Using the analytical properties of the exponential forward performance, we show that the forward indifference price is wealth-independent, and derive its associated variational inequality as well as a dual representation. We also provide a comparative analysis between the forward and classical exponential indifference prices. For instance, we show that the forward indifference price representation involves a relative entropy minimization (up to a stopping time) with respect to the *minimal martingale measure* (MMM), as opposed to the *minimal entropy martingale measure* (MEMM) that arises in the classical exponential utility indifference price (see, among others, Rouge and El Karoui (2000) and Delbaen et al. (2002) for European claims, and Leung and Sircar (2009b) for American claims). This contrasting difference is also reflected in the asymptotics results of indifference prices discussed here.

Another application is the modeling of early exercises of employee stock options (ESOs), which are American-style call options written on the firm's stock granted to the employee as a form of compensation. In Section 4, we assume a forward performance criterion for the employee and investigate the impacts of various factors, such as wealth and risk tolerance function, on the employee's exercise timing. In particular, we find that the employee tends to exercise the ESO earlier when his wealth approaches zero.

Lastly, in Section 5, we introduce an alternative valuation mechanism for American options based on the *marginal* forward performance. In the classical utility framework, as introduced by Davis (1997), the *marginal utility price* represents the per-unit price that a risk-averse investor is willing to pay for an infinitesimal position in a contingent claim. In general, the marginal utility price is closely linked to the investor's utility function and the market setup, and it only becomes wealth independent under very special circumstances (see Kramkov and Sirbu (2006) for details). We adapt the classical definition to our forward performance framework and give a definition of the *marginal forward indifference price*. In contrast to the classical marginal utility price, the marginal forward indifference price turns out to be independent of the holder's wealth and forward performance criterion, and is equivalent to pricing under the *minimal martingale measure*. Section 6 concludes the paper and discusses extensions for future investigation.

2 Forward Investment Performance Measurement and Indifference Valuation

We fix a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with a filtration $(\mathcal{F}_t)_{t\geq 0}$ that satisfies the usual conditions of right continuity and completeness. In addition, all stochastic processes considered in this paper are continuous-path processes. The financial market consists of two liquidly traded assets, namely, a riskless money market account and a stock. The money market account has the price process B that satisfies

$$dB_t = r_t B_t \, dt, \tag{1}$$

with $B_0 = 1$,where $(r_t)_{t\geq 0}$ is a non-negative $\mathcal{F}_t$-adapted interest rate process. We shall work with discounted cash flows throughout.

The discounted stock price S is modeled as a continuous Itô process satisfying

$$dS_t = S_t \sigma_t \left(\lambda_t \, dt + dW_t \right), \tag{2}$$

with $S_0 > 0$, where $(W_t)_{t\geq 0}$ is an $\mathcal{F}_t$-adapted standard Brownian motion. The Sharpe ratio $(\lambda_t)_{t\geq 0}$ is a bounded $\mathcal{F}_t$-adapted process, and the volatility coefficient $(\sigma_t)_{t\geq 0}$ is strictly positive bounded $\mathcal{F}_t$-adapted process. Moreover, we assume that a strong solution exists for the SDE (2).

Starting with initial endowment $x \in \mathbb{R}$, the investor dynamically rebalances his portfolio allocations between the stock S and the money market account B. Under the self-financing trading condition, the discounted wealth satisfies

$$dX_t^\pi = \pi_t \sigma_t (\lambda_t \, dt + dW_t), \tag{3}$$

where $(\pi_t)_{t\geq 0}$ represents the discounted cash amount invested in S. As is a common choice for Itô markets (see Section 6C of Duffie (2001)), the set of admissible strategies $\mathcal{Z}$ consists of all self-financing $\mathcal{F}_t$-adapted processes $(\pi_t)_{t\geq 0}$ such that $I\!E\{\int_0^s \sigma_t^2 \pi_t^2 \, dt\} < \infty$ for each $s > 0$. For $0 \leq t \leq s$, we denote by $\mathcal{Z}_{t,s}$ the set of admissible strategies over the period $[t, s]$.

In the standard Merton portfolio optimization problem, risk aversion is modeled by a deterministic utility function $\hat{U}(x)$ defined at some fixed terminal time T and the investor's risk preferences (or indirect utility) at intermediate times are inferred backwards. Starting at time $t \leq T$ with $\mathcal{F}_t$-measurable wealth X_t, the Merton value function is given by

$$M_t(X_t) = \operatorname*{ess\,sup}_{\pi \in \mathcal{Z}_{t,T}} I\!E \left\{ \hat{U}(X_T^\pi) | \, \mathcal{F}_t \right\}. \tag{4}$$

When the *dynamic programming principle* holds, the Merton problem can be written as

$$M_t(X_t) = \operatorname*{ess\,sup}_{\pi \in \mathcal{Z}_{t,s}} I\!E \left\{ M_s(X_s^\pi) | \, \mathcal{F}_t \right\}, \qquad 0 \leq t \leq s \leq T. \tag{5}$$

Some well-known examples when (5) holds include i) markets with Markovian dynamics where the optimal portfolio allocation can be found by solving a Bellman PDE; ii) when the utility is of exponential type, in which case (5) holds under quite general semimartingale models (see Mania and Schweizer (2005); Leung and Sircar (2009b)); and iii) when expected utility is replaced by a dynamic time-consistent concave utility functional, defined, for instance, from a BSDE in Itô markets (see Klöppel and Schweizer (2006); Cheridito and Kupper (2009)). The dynamic programming principle (5) is taken as the *defining characteristic* of the forward performance criterion.

Nevertheless, in the forward performance framework, the investor's utility function $u_0(x)$ is defined at time 0, and his performance criterion evolves forward in time. We adapt the definition of the forward performance process given by Musiela and Zariphopoulou (2008):

Definition 1 *An $\mathcal{F}_t$-adapted process $(U_t(x))_{t\geq 0}$ is a forward performance process if:*

1. *it satisfies the initial datum $U_0(x) = u_0(x)$, $x \in \mathbb{R}$, where $u_0 : \mathbb{R} \mapsto \mathbb{R}$ is an increasing and strictly concave function of x,*
2. *for each $t \geq 0$, the mapping $x \mapsto U_t(x)$ is increasing and strictly concave in $x \in \mathbb{R}$, and*
3. *for $0 \leq t \leq s < \infty$, we have*

$$U_t(X_t) = \operatorname*{ess\,sup}_{\pi\in\mathcal{Z}_{t,s}} I\!E\{U_s(X_s^\pi)|\,\mathcal{F}_t\}, \tag{6}$$

with any $\mathcal{F}_t$-measurable starting wealth X_t.

As condition 3 indicates, the forward performance process $(U_t(X_t^\pi))_{t\geq 0}$ is a $(\mathbb{P}, \mathcal{F}_t)$ supermartingale for any strategy π, and a martingale if there exists an optimal admissible strategy π^* for (6). In related studies, this is also referred to as the horizon-unbiased condition in Henderson and Hobson (2007) and the self-generating condition in Žitković (2009).

The above definition does not explicitly require the existence of π^*. As is common in the classical utility maximization, the existence and characterization of the optimal strategy are challenging questions and depend on the market structure and utility function used. Related research for forward performance processes includes Musiela and Zariphopoulou (2009), El Karoui and M'Rad (2010), and Žitković (2009) (for exponential preferences). In this paper, however, our analysis will focus on a class of *explicit* forward performance processes (see Theorem 3), whose optimal strategies have been completely characterized in the recent papers Berrier et al. (2009) and Musiela and Zariphopoulou (2010). Our objective is to apply forward performance to the indifference pricing of American options.

2.1 Forward Indifference Price

We introduce the forward indifference valuation from the perspective of the holder of an American option. The option payoff is characterized by an $\mathcal{F}_t$-adapted bounded process $(g_t)_{0\leq t\leq T}$ with a finite expiration date T. The collection of admissible exercise times is the set of stopping times with respect to $\mathcal{F}_{0,T} = (\mathcal{F}_t)_{0\leq t\leq T}$ taking values in $[0, T]$. For $0 \leq t \leq s \leq T$, we denote by $\mathcal{T}_{t,s}$ the set of stopping times bounded by $[t, s]$.

The option holder chooses his dynamic trading strategy π and exercise time τ so as to maximize his expected forward performance from investing in S and the money market account and from receiving the option payoff. This leads to a combined stochastic control and optimal stopping problem. We define

$$V_t(X_t) = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{t,T}} \operatorname*{ess\,sup}_{\pi\in\mathcal{Z}_{t,\tau}} I\!E\left\{U_\tau(X_\tau^\pi + g_\tau)\,|\,\mathcal{F}_t\right\}, \qquad t \in [0, T], \tag{7}$$

which is the holder's value process starting at time t with wealth X_t.

In the classical case with a terminal utility function $\hat{U}$, the holder's optimal investment problem is to solve

$$\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{t,T}} \operatorname*{ess\,sup}_{\pi\in\mathcal{Z}_{t,\tau}} I\!E\left\{M_\tau(X_\tau^\pi + g_\tau) \,|\, \mathcal{F}_t\right\},$$

where M is the solution to the Merton problem defined in (4). In this formulation, M plays the role of intermediate utility at times $\tau \leq T$, and corresponds to specifying that option proceeds received at exercise time τ are re-invested in the Merton optimal strategy, up till time T. By contrast, the forward performance process U specifies utilities at all times, without reference to any specific horizon.

The holder's forward indifference price p_t for the American option g is defined as the cash amount such that the option holder is indifferent between two positions: optimal investment with the American option, and optimal investment without the American option but with extra wealth p_t.

Definition 2 *The holder's forward indifference price process $(p_t)_{0\leq t\leq T}$ for the American option is defined by the equation*

$$V_t(X_t) = U_t(X_t + p_t), \qquad t \in [0, T], \tag{8}$$

where V_t and U_t are given in (7) and Definition 1 respectively.

The forward indifference price is useful for characterizing the option holder's optimal exercise time τ^*. Under appropriate integrability conditions (Karatzas and Shreve, 1998, Appendix D), the optimal stopping time is the first time the value process reaches the reward process. From (7) and (8), we have

$$\begin{aligned}
\tau_t^* &= \inf\{t \leq s \leq T \,:\, V_s(X_s) = U_s(X_s + g_s)\} \\
&= \inf\{t \leq s \leq T \,:\, U_s(X_s + p_s) = U_s(X_s + g_s)\} \\
&= \inf\{t \leq s \leq T \,:\, p_s = g_s\}.
\end{aligned} \tag{9}$$

The representation (9) implies that the option holder will exercise the American option as soon as the forward indifference price reaches (from above) the option payoff. It allows us to analyze the holder's optimal exercise policy through his forward indifference price.

In Sections 3 and 4, we will focus our study on two specific financial applications: i) the valuation of an American option written on a stock S with stochastic volatility under forward performance criterion of *exponential* type (to be defined in (24)), and ii) modeling early exercises of employee stock options (ESOs), where we go beyond the exponential forward performance criterion.

2.2 Forward Performance of Generalized CARA/CRRA Type

Henceforth, we will use a special class of forward performance processes introduced by Musiela and Zariphopoulou (2008), namely, the time-monotone ones. They are represented by the compilation of a deterministic function $u(x,t)$ which models the investor's dynamic risk preference, and an increasing process $(A_t)_{t\geq 0}$ that solely depends on the market. Recently, Berrier et al. (2009) and Musiela and Zariphopoulou (2010) studied various properties of this family of forward performance and gave a dual representation.

Theorem 3 *(Proposition 3 of Musiela and Zariphopoulou (2008)) Define the stochastic process*

$$A_t = \int_0^t \lambda_s^2\, ds, \qquad t \geq 0. \tag{10}$$

Suppose $u : \mathbb{R} \times \mathbb{R}_+ \mapsto \mathbb{R}$ is $\mathcal{C}^{3,1}$, strictly concave, increasing in the first argument, and satisfies the partial differential equation

$$u_t = \frac{1}{2} \frac{u_x^2}{u_{xx}}, \tag{11}$$

with initial condition $u(x, 0) = u_0(x)$, where $u_0 \in \mathcal{C}^3(\mathbb{R})$. Then, the process $U_t(x)$ defined by

$$U_t(x) = u(x, A_t), \qquad t \geq 0, \tag{12}$$

is a forward performance process. Moreover, the strategy π^* defined by

$$\pi_t^* = -\frac{\lambda_t}{\sigma_t} \frac{u_x(X_t^*, A_t)}{u_{xx}(X_t^*, A_t)}, \qquad t \geq 0, \tag{13}$$

where $X^* = X^{\pi^*}$ is the associated wealth process following (3), is the candidate optimal strategy in (6).

A quantity that plays a crucial role in the description of the wealth and portfolio processes (X^*, π^*) is the so-called *local risk tolerance* function $R : \mathbb{R} \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ defined by

$$R(x, t) = -\frac{u_x(x, t)}{u_{xx}(x, t)}. \tag{14}$$

With this notation, the dynamics of X^* is

$$dX_t^* = R(X_t^*, A_t)\lambda_t \left(\lambda_t \, dt + dW_t\right). \tag{15}$$

Furthermore, if $u \in \mathcal{C}^4$, then, by differentiation, R is solution of the fast diffusion equation, namely

$$R_t + \frac{1}{2} R^2 R_{xx} = 0. \tag{16}$$

One way to construct forward performance criteria is to look for solutions of this PDE. In Zariphopoulou and Zhou (2008), the following two-parameter family of risk tolerance functions was introduced:

$$R(x, t; \alpha, \beta) = \sqrt{\alpha x^2 + \beta e^{-\alpha t}}, \quad \alpha, \beta > 0. \tag{17}$$

We illustrate an example of this risk tolerance in Figure 1.

There are several reasons to work with this family. First, it yields, in the limit, risk tolerance functions that resemble those related to the three most popular cases, specifically, the *exponential*, *power* and *logarithmic*. Indeed,

$$\begin{aligned} &\lim_{\alpha \to 0} R(x, t; \alpha, \beta) = \sqrt{\beta} && \text{(exponential)}, \\ &\lim_{\beta \to 0} R(x, t; \alpha, \beta) = \alpha x, \quad x \geq 0 && \text{(power)}, \\ &\lim_{\beta \to 0} R(x, t; 1, \beta) = x, \quad x > 0 && \text{(logarithmic)}. \end{aligned} \tag{18}$$

We remark that $R(x, t; \alpha, \beta)$ is well-defined for wealth $x \in (-\infty, \infty)$, except in the limit case $\beta \downarrow 0$. In Figure 2, we illustrate the limit in (18) where the risk tolerance function converges to the constant $\sqrt{\beta}$ as $\alpha \downarrow 0$. As we will see in Section 3, the constant risk tolerance corresponds to the forward performance measure of exponential type.

Zariphopoulou and Zhou (2008) compute the corresponding dynamic risk preference function $u(x, t)$:

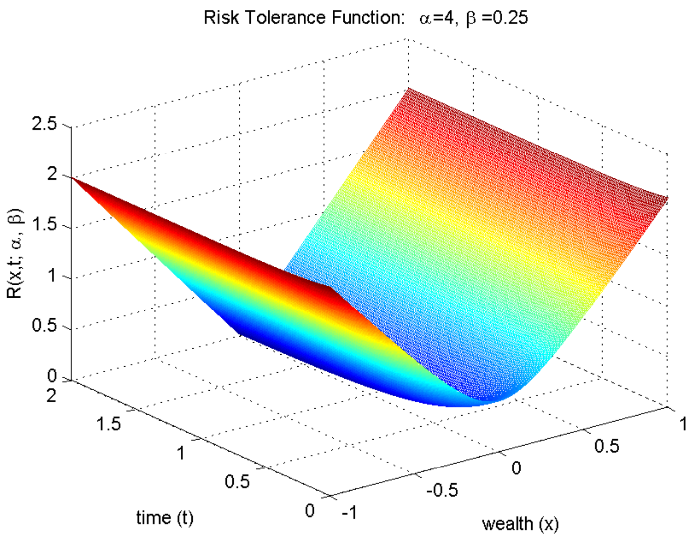

Figure 1: *The risk tolerance function $R(x,t;\alpha,\beta)$ with $\alpha=4$, and $\beta=0.25$. For any fixed wealth x, the risk tolerance decreases over time. At any fixed time, the risk tolerance increases as wealth decreases or increases away from zero.*

Proposition 4 *(Proposition 3.2 of Zariphopoulou and Zhou (2008)) The dynamic risk preference function $u(x,t;\alpha,\beta)$ associated with $R(x,t;\alpha,\beta)$ in (17) is given by*

$$u(x,t;\alpha,\beta) \;=\; m\frac{\kappa^{1+\frac{1}{\kappa}}}{\alpha-1}e^{\frac{1-\kappa}{2}t}\frac{\frac{\beta}{\kappa}e^{-\alpha t}+(1+\kappa)x(\kappa x+\sqrt{\alpha x^2+\beta e^{-\alpha t}})}{(\kappa x+\sqrt{\alpha x^2+\beta e^{-\alpha t}})^{1+\frac{1}{\kappa}}}+n, \quad \alpha\neq 1, \tag{19}$$

$$u(x,t;1,\beta) \;=\; \frac{m}{2}\left(\log(x+\sqrt{x^2+\beta e^{-t}})-\frac{e^{-t}}{\beta}x\left(x-\sqrt{x^2+\beta e^{-t}}\right)-\frac{t}{2}\right)+n, \quad \alpha=1,$$

where $\kappa=\sqrt{\alpha}$, and $m>0, n\in\mathbb{R}$ are constants of integration.

As mentioned earlier, in the context of the domain of the local risk tolerance, the function $u(x,t;\alpha,\beta)$ is also well-defined for all $x\in\mathbb{R}$, except in the limit case $\beta\downarrow 0$. This property is particularly useful in indifference valuation for it eliminates the nonnegativity constraints on the investor's wealth (with and without the claim in hand).

3 American Options under Stochastic Volatility

In this section, we study the forward indifference valuation of an American option in a stochastic volatility model. We work with the exponential forward performance, which, as mentioned in the previous section, corresponds to the parameter choice $\alpha=0$. A comparative analysis with the classical exponential utility indifference pricing is provided in Section 3.4.

The underlying stock price S, discounted by the money market account, is modeled as a diffusion process satisfying

$$dS_t = S_t\sigma(Y_t)(\lambda(Y_t)\,dt + dW_t). \tag{20}$$

The Sharpe ratio $\lambda(Y_t)$ and volatility coefficient $\sigma(Y_t)$ are driven by a *non-traded* stochastic factor Y which evolves according to

$$dY_t = b(Y_t)\,dt + c(Y_t)\,(\rho dW_t + \sqrt{1-\rho^2}d\hat{W}_t), \tag{21}$$

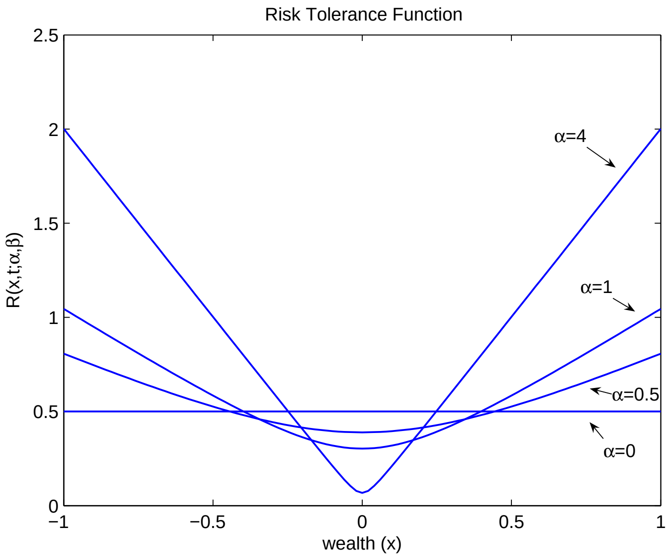

Figure 2: *As α decreases from 4 to 0, with $\beta = 0.25$ and $t = 1$, the risk tolerance function $R(x, t; \alpha, \beta)$ converges to the constant level $\sqrt{\beta} = 0.5$, as predicted by the limit in (18).*

with correlation coefficient $\rho \in (-1, 1)$. The processes W and $\hat{W}$ are two independent standard Brownian motions defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, where $\mathcal{F}_t$ is taken to be the augmented σ-algebra generated by $((W_u, \hat{W}_u); 0 \leq u \leq t)$. The volatility function $\sigma(\cdot)$ and the diffusion coefficient $c(\cdot)$ are smooth, positive and bounded. The Sharpe ratio $\lambda(\cdot)$ is bounded continuous, and $b(\cdot)$ is Lipschitz continuous on $\mathbb{R}$. The American option yields payoff $g(S_\tau, Y_\tau, \tau)$ at exercise time $\tau \in [0, T]$, where $g(\cdot, \cdot, \cdot)$ is a smooth and bounded function.

The holder of the American option g dynamically trades in S and the money market. His discounted trading wealth follows

$$dX_t^\pi = \pi_t \sigma(Y_t)(\lambda(Y_t)\, dt + dW_t), \tag{22}$$

where $(\pi_t)_{t\geq 0}$ is the discounted cash amount invested in S.

3.1 Exponential Forward Indifference Price

We model the American option holder's risk preferences by the exponential forward performance process. By setting the parameter $\alpha = 0$ in (17), the risk tolerance becomes a constant $\sqrt{\beta}$ (see (18) and Figure 2). In turn, (11) and (14) yield the corresponding exponential risk preference function $u(x, t)$:

$$u(x, t) = -e^{-\gamma x + \frac{t}{2}}, \tag{23}$$

with $\gamma = 1/\sqrt{\beta}$. The parameter γ can be considered as the investor's *local risk aversion*. Applying Theorem 3, we obtain the exponential forward performance process

$$U_t^e(x) = -e^{-\gamma x + \frac{1}{2}\int_0^t \lambda(Y_s)^2 ds}, \qquad t \geq 0. \tag{24}$$

As defined in (7), the holder's forward performance value process is given by

$$\begin{aligned} V_t^e(X_t) &= \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \operatorname*{ess\,sup}_{\pi \in \mathcal{Z}_{t,\tau}} \mathbb{E}\left\{ -e^{-\gamma(X_\tau^\pi + g(S_\tau, Y_\tau, \tau))} e^{\frac{1}{2}\int_0^\tau \lambda(Y_s)^2 ds} \mid \mathcal{F}_t \right\} \\ &= e^{\frac{1}{2}\int_0^t \lambda(Y_s)^2 ds} \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \operatorname*{ess\,sup}_{\pi \in \mathcal{Z}_{t,\tau}} \mathbb{E}\left\{ -e^{-\gamma(X_\tau^\pi + g(S_\tau, Y_\tau, \tau))} e^{\frac{1}{2}\int_t^\tau \lambda(Y_s)^2 ds} \mid \mathcal{F}_t \right\}. \end{aligned} \tag{25}$$

We observe that the second term in (25) is the value of a combined stochastic control and optimal stopping problem. Working under the Markovian stochastic volatility market (20)-(21), we look for a candidate optimal $\mathcal{F}_t$-adapted Markovian strategy by studying the associated HJB variational inequality.

To facilitate notation, we introduce the following differential operators and Hamiltonian:

$$\mathcal{L}_{SY}v = \frac{1}{2}\sigma(y)^2 s^2 v_{ss} + \rho c(y)\sigma(y)sv_{sy} + \frac{1}{2}c(y)^2 v_{yy} + \lambda(y)\sigma(y)sv_s + b(y)v_y,$$
$$\mathcal{L}^0_{SY}v = \frac{1}{2}\sigma(y)^2 s^2 v_{ss} + \rho c(y)\sigma(y)sv_{sy} + \frac{1}{2}c(y)^2 v_{yy} + (b(y) - \rho c(y)\lambda(y))v_y,$$

and

$$\mathcal{H}(v_{xx}, v_{xy}, v_{xs}, v_x) = \max_{\pi}\left(\frac{\pi^2\sigma(y)^2}{2}v_{xx} + \pi\left(\rho\sigma(y)c(y)v_{xy} + \sigma(y)^2 sv_{xs} + \lambda(y)\sigma(y)v_x\right)\right).$$

Note that $\mathcal{L}^{SY}$ and $\mathcal{L}^0_{SY}$ are, respectively, the infinitesimal generators of the Markov process $(S_t, Y_t)_{t\geq 0}$ under the historical measure $\mathbb{P}$ and the minimal martingale measure Q^0 (to be defined in (32)).

Next, we consider the HJB variational inequality

$$\begin{cases} V_t + \mathcal{L}_{SY}V + \mathcal{H}(V_{xx}, V_{xy}, V_{xs}, V_x) + \dfrac{\lambda(y)^2}{2}V \leq 0, \\ V(x,s,y,t) \geq -e^{-\gamma(x+g(s,y,t))}, \\ \big(V_t + \mathcal{L}_{SY}V + \mathcal{H}(V_{xx}, V_{xy}, V_{xs}, V_x) + \dfrac{\lambda(y)^2}{2}V\big)\cdot\big(-e^{-\gamma(x+g(s,y,t))} - V(x,s,y,t)\big) = 0, \\ V(x,s,y,T) = -e^{-\gamma(x+g(s,y,T))}, \end{cases} \tag{26}$$

for $(x,s,y,t) \in \mathbb{R}\times\mathbb{R}_+\times\mathbb{R}\times[0,T]$. Given a solution $V(x,s,y,t)$ to (26) that is $\mathcal{C}^{2,2,2,1}$ except across a lower-dimensional optimal exercise boundary, one can show by standard verification arguments (see, for example, Theorem 4.2 of Oksendal and Sulem (2005)) that V is the value function for the combined optimal control/stopping problem in (25). Therefore, we can write

$$V^e_t(X_t) = e^{\frac{1}{2}\int_0^t \lambda(Y_s)^2 ds}\, V(X_t, S_t, Y_t, t). \tag{27}$$

Remark 5 *As is usual in classical indifference pricing problems, the existence of a solution (in the appropriate regularity class) to the HJB equation or variational inequality is a non-trivial issue, and is typically verified in specific cases. For instance, in the classical exponential utility indifference pricing for American options, Oberman and Zariphopoulou (2003) show the viscosity (weak) solution of the HJB equation for the value function. For our analysis in this section, we assume the existence of a unique solution to the variational inequality (26) with the regularity needed for the verification arguments.*

Applying (24) and (27) to Definition 2, the option holder's exponential forward indifference price function $p(x,s,y,t)$ is given by

$$p(x,s,y,t) = -\frac{1}{\gamma}\log\big(-V(x,s,y,t)\big) - x. \tag{28}$$

Then, we substitute (28) into the variational inequality (26) to derive the variational equality for $p(x,s,y,t)$. As it turns out, the indifference price is independent of the wealth argument

x and it solves the free boundary problem

$$\begin{cases} p_t + \mathcal{L}^0_{SY} p - \frac{1}{2}\gamma(1-\rho^2)c(y)^2 p_y^2 \leq 0, \\ p(s,y,t) \geq g(s,y,t), \\ \left(p_t + \mathcal{L}^0_{SY} p - \frac{1}{2}\gamma(1-\rho^2)c(y)^2 p_y^2\right) \cdot (g(s,y,t) - p(s,y,t)) = 0, \\ p(s,y,T) = g(s,y,T), \end{cases} \tag{29}$$

for $(s,y,t) \in \mathbb{R}_+ \times \mathbb{R} \times [0,T]$.

By the first-order condition in (26) and the formula (28), the optimal hedging strategy $(\tilde{\pi}^*_t)_{0\leq t\leq T}$ can be expressed in terms of the indifference price, namely,

$$\tilde{\pi}^*_t = \frac{\lambda(Y_t)}{\gamma\sigma(Y_t)} + \frac{S_t}{\gamma} p_s(S_t, Y_t, t) + \frac{\rho c(Y_t)}{\gamma\sigma(Y_t)} p_y(S_t, Y_t, t).$$

The first term in this expression is the optimal strategy in (13) when there is no claim, while the second and third terms correspond to the extra demand in stock S due to changes in S and the stochastic factor Y, respectively.

The optimal exercise time is the first time that the indifference price reaches the option payoff:

$$\tau^*_t = \inf\{t \leq u \leq T \,:\, p(S_u, Y_u, u) = g(S_u, Y_u, u)\}\,. \tag{30}$$

In practice, one can numerically solve the variational inequality (29) to obtain the optimal exercise boundary which represents the critical levels of S and Y at which the option should be exercised. We remark that the indifference price, the optimal hedging and exercising strategies are all wealth independent. The same phenomenon occurs in the classical indifference valuation with exponential utility.

3.2 Dual Representation

The option holder's forward performance maximization in (27) can be considered as the primal optimization problem, and it yields the first expression for the forward indifference price in (28). Our objective is to derive a dual representation for the forward indifference price, which turns out to be related to pricing the American option with entropic penalty. This result will allow us to express the price in a way analogous to the classical exponential indifference price. We carry out this comparison in Section 3.4.

First, we denote by $\mathcal{M}(\mathbb{P})$ the set of equivalent local martingale measures with respect to $\mathbb{P}$ on $\mathcal{F}_T$. As is well-known (see, for example, Frey (1997) and Romano and Touzi (1997)), these measures are characterized its density process with respect to $\mathbb{P}$, which is given by the stochastic exponential

$$Z^\phi_t = \frac{dQ^\phi}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \exp\left(-\frac{1}{2}\int_0^t \left(\lambda(Y_s)^2 + \phi_s^2\right)\,ds - \int_0^t \lambda(Y_s)\,dW_s - \int_0^t \phi_s\,d\hat{W}_s\right), \tag{31}$$

where $(\phi_t)_{0\leq t\leq T}$ is a progressively measurable process satisfying (i) $\int_0^T \phi_s^2\,ds < \infty$ $\mathbb{P}$-a.s., and (ii) $\mathbb{E}\{Z^\phi_T\} = 1$. By Girsanov's Theorem, it follows that $W^\phi_t = W_t + \int_0^t \lambda(Y_s)ds$ and $\hat{W}^\phi_t = \hat{W}_t + \int_0^t \phi_s\,ds$ are independent Q^ϕ-Brownian motions. The process ϕ is commonly referred to as the *volatility risk premium* for the second Brownian motion $\hat{W}$.

When $\phi = 0$, we obtain the minimal martingale measure (MMM) Q^0, given by

$$\frac{dQ^0}{d\mathbb{P}} = \exp\left(-\frac{1}{2}\int_0^T \lambda(Y_s)^2\,ds - \int_0^T \lambda(Y_s)\,dW_s\right). \tag{32}$$

In fact, we can express Q^ϕ in terms of Q^0, namely,

$$\frac{dQ^\phi}{dQ^0} = \frac{dQ^\phi}{d\mathbb{P}} \Big/ \frac{dQ^0}{d\mathbb{P}} = \exp\left(-\frac{1}{2}\int_0^T \phi_s^2\,ds - \int_0^T \phi_s\,d\hat{W}_s^0\right). \tag{33}$$

We also denote the density process of Q^ϕ with respect to Q^0 by $Z_t^{\phi,0} = I\!E^{Q^0}\{\frac{dQ^\phi}{dQ^0}|\mathcal{F}_t\}$.

Treating Q^0 as the prior risk-neutral measure, we define the conditional relative entropy $H_t^\tau(Q^\phi|Q^0)$ of Q^ϕ with respect to Q^0 over the interval $[t,\tau]$ as

$$H_t^\tau(Q^\phi|Q^0) = I\!E^{Q^\phi}\left\{\log\frac{Z_\tau^{\phi,0}}{Z_t^{\phi,0}}|\mathcal{F}_t\right\}.$$

Direct computation from (33) shows that this relative entropy is, in fact, a quadratic penalization on the risk premium ϕ. In other words,

$$H_t^\tau(Q^\phi|Q^0) = \frac{1}{2}I\!E^{Q^\phi}\left\{\int_t^\tau \phi_s^2\,ds|\mathcal{F}_t\right\}. \tag{34}$$

Let us denote the set of equivalent local martingale measures with finite relative entropy (with respect to Q^0) as

$$\mathcal{M}_f := \left\{Q^\phi \in \mathcal{M}(\mathbb{P}) \,:\, I\!E^{Q^\phi}\left\{\int_0^T \phi_t^2\,dt\right\} < \infty\right\}.$$

Next, we give a duality result for the exponential forward indifference price.

Proposition 6 *Let $p(s,y,t)$ be the solution to (29). Define Q^{ϕ^*} by $dQ^{\phi^*}/d\mathbb{P} = Z_T^{\phi^*}$ (as in (31)) with*

$$\phi_t^* = -\gamma c(Y_t)\sqrt{1-\rho^2}\,p_y(S_t,Y_t,t), \qquad 0 \le t \le T, \tag{35}$$

and assume that $Q^{\phi^*} \in \mathcal{M}_f$. Then, $p(s,y,t)$ is the solution of the following combined stochastic control and optimal stopping problem:

$$p(S_t,Y_t,t) = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{t,T}}\ \operatorname*{ess\,inf}_{Q^\phi\in\mathcal{M}_f}\left(I\!E^{Q^\phi}\{g(S_\tau,Y_\tau,\tau)|\mathcal{F}_t\} + \frac{1}{\gamma}H_t^\tau(Q^\phi|Q^0)\right), \tag{36}$$

and Q^{ϕ^*} is the associated optimal measure.

Proof. It is straightforward to check that the HJB variational inequality for the stochastic control/stopping problem in (36) is identical to (29). The associated optimal control ϕ^* is given in (35), and Q^{ϕ^*} is the corresponding optimal probability measure. ■

The dual representation (36) provides an alternative interpretation of the forward indifference price. In essence, the holder tries to value the American option over a set of equivalent local martingale measures, and his selection criterion for the optimal pricing measure is based on relative entropic penalization. Indeed, the second term in (36) is the relative entropy of a candidate measure Q^ϕ with respect to the MMM Q^0 up to the exercise time. This leads the holder to assign the corresponding optimal risk premium ϕ^* in (35). We will compare this result to its classical analogue in Section 3.4.

3.3 Risk Aversion and Volume Asymptotics

Proposition 6 provides a convenient representation for analyzing the exponential forward indifference price's sensitivity with respect to risk aversion and the number of options held. First, let us consider a risk-averse investor with local risk aversion γ who holds $a > 0$ units of American options, and suppose that all a units are constrained to be exercised simultaneously. In this case, the holder's indifference price $p(s, y, t; \gamma, a)$ is again given by (36) but with the payoff $g(S_\tau, Y_\tau, \tau)$ replaced by $ag(S_\tau, Y_\tau, \tau)$. The optimal exercise time $\tau^*(a, \gamma)$ is the first time that the forward indifference price reaches the payoff from exercising all a units:

$$\tau^*(a, \gamma) = \inf\{t \leq u \leq T \,:\, p(S_u, Y_u, u; \gamma, a) = ag(S_u, Y_u, u)\}. \tag{37}$$

Proposition 7 *Fix $a > 0$ and $t \in [0, T]$. If $\gamma_2 \geq \gamma_1 > 0$, then*

$$p(s, y, t; \gamma_2, a) \leq p(s, y, t; \gamma_1, a)$$

and

$$\tau^*(a, \gamma_2) \leq \tau^*(a, \gamma_1), \quad \textit{almost surely.}$$

Proof. For $\gamma_2 \geq \gamma_1 > 0$, it follows from (36) that $p(s, y, t; \gamma_2, a) \leq p(s, y, t; \gamma_1, a)$. Therefore, as γ increases, $p(s, y, t; \gamma, a)$ decreases, while the payoff $ag(s, y, t)$ does not depend on γ. By (37), this leads to a shorter exercise time (almost surely). ■

Furthermore, we observe that, as γ increases to infinity, the penalty term in the indifference price representation (36) vanishes. Consequently, we deduce the following limit:

$$\lim_{\gamma\to\infty} p(s, y, t; \gamma, a) = a \cdot \sup_{\tau\in\mathcal{T}_{t,T}} \inf_{Q^\phi\in\mathcal{M}_f} I\!\!E^{Q^\phi} \left\{ g(S_\tau, Y_\tau, \tau) | \, S_t = s, Y_t = y \right\}. \tag{38}$$

This limiting price is commonly referred to as the sub-hedging price of the American options (see, for example, Karatzas and Kou (1998)). On the other hand, as the holder's risk aversion γ decreases to zero, one can deduce from (36) that it is optimal not to deviate from the prior measure Q^0 (i.e. $\phi = 0$), yielding zero entropic penalty. This leads to valuing the American options under the MMM Q^0, namely,

$$\lim_{\gamma\to 0} p(s, y, t; \gamma, a) = a \cdot \sup_{\tau\in\mathcal{T}_{t,T}} I\!\!E^{Q^0} \left\{ g(S_\tau, Y_\tau, \tau) | \, S_t = s, Y_t = y \right\}. \tag{39}$$

We have provided the formal arguments for these risk-aversion limits. For more technical details, we refer the reader to Leung and Sircar (2009b) who have shown these asymptotic results for the traditional exponential indifference price of American options in a general semimartingale framework, and their proofs can be easily adapted here.

Finally, we observe from (36) the volume-scaling property:

$$p(s, y, t; \gamma, a)/a = p(s, y, t; a\gamma, 1).$$

Hence, as the number of options held increases, the holder's average indifference price $p(s, y, t; \gamma, a)/a$ decreases, and by (37) the options will be exercised earlier. Moreover, the indifference price limits in (38)-(39) lead to the following limits:

$$\lim_{a\to\infty} \frac{p(s, y, t; \gamma, a)}{a} = \sup_{\tau\in\mathcal{T}_{t,T}} \inf_{Q^\phi\in\mathcal{M}_f} I\!\!E^{Q^\phi} \left\{ g(S_\tau, Y_\tau, \tau) | \, S_t = s, Y_t = y \right\}$$

and

$$\lim_{a\to 0} \frac{p(s, y, t; \gamma, a)}{a} = \sup_{\tau\in\mathcal{T}_{t,T}} I\!\!E^{Q^0} \left\{ g(S_\tau, Y_\tau, \tau) | \, S_t = s, Y_t = y \right\}.$$

3.4 Comparison with the Classical Exponential Utility Indifference Price

In this section, we provide a comparative analysis between the classical and forward indifference valuation. We start with a brief review of the classical indifference pricing with exponential utility under stochastic volatility models. We refer the reader to, for example, Sircar and Zariphopoulou (2005) and Benth and Karlsen (2005) for European options, and Oberman and Zariphopoulou (2003) for American options.

In the traditional setting, the investor's risk preferences at time T are modeled by the exponential utility function $-e^{-\gamma x}$, $\gamma > 0$. In the stochastic volatility model described in (20)-(21), the value function of the Merton problem is

$$M(x,y,t) = \sup_{\pi \in \mathcal{Z}_{t,T}} \mathbb{E}\left\{-e^{-\gamma X_T^\pi} | \, X_t = x, Y_t = y\right\}, \tag{40}$$

with $(X_t^\pi)_{t\geq 0}$ given by (22). As is well known, the function M admits a separation of variables due to the exponential utility.

Proposition 8 *The value function $M(x,y,t)$ is given by*

$$M(x,y,t) = -e^{-\gamma x} f(y,t)^{\frac{1}{1-\rho^2}}, \tag{41}$$

where ρ is the correlation coefficient in (21), and f solves

$$f_t + \mathcal{L}_Y^0 f = \frac{1}{2}(1-\rho^2)\lambda(y)^2 f, \tag{42}$$

for $(x,t) \in \mathbb{R} \times [0,T)$, with $f(y,T) = 1$, for $y \in \mathbb{R}$. The operator $\mathcal{L}_Y^0$ is the infinitesimal generator of Y under the MMM Q^0, and is given by

$$\mathcal{L}_Y^0 f = (b(y) - \rho c(y)\lambda(y))\, f_y + \frac{1}{2} c(y)^2 f_{yy}.$$

Details can be found, for example, in Theorem 2.2 of Sircar and Zariphopoulou (2005).

If the American option g is held, then the investor seeks the optimal trading strategy and exercise time to maximize the expected utility from trading wealth plus the option's payoff. Upon exercise of the option, the investor will reinvest the contract proceeds, if any, to his trading portfolio, and continue to trade up to time T. As a consequence, the holder faces the optimization problem

$$\hat{V}(x,s,y,t) = \sup_{\tau \in \mathcal{T}_{t,T}} \sup_{\pi \in \mathcal{Z}_{t,\tau}} \mathbb{E}\left\{\, M\left(X_\tau^\pi + g(S_\tau, Y_\tau, \tau), Y_\tau, \tau\right) \mid X_t = x, S_t = s, Y_t = y \,\right\}, \tag{43}$$

where M is defined in (40).

The indifference price of the American option $\hat{p}$ is found from the equation

$$M(x,y,t) = \hat{V}(x - \hat{p}(x,s,y,t), s, y, t). \tag{44}$$

Using (41) and (44), we obtain the formula

$$\hat{V}(x,s,y,t) = -e^{-\gamma(x+\hat{p}(x,s,y,t))} f(y,t)^{\frac{1}{1-\rho^2}}. \tag{45}$$

To derive the variational inequality for the indifference price, one can use the variational inequality for V and then apply the transformation (45). Again, the choice of exponential utility yields wealth-independent indifference prices, i.e. $\hat{p}(x,s,y,t)=\hat{p}(s,y,t)$. We obtain

$$\begin{cases}\hat{p}_t+\mathcal{L}^E_{SY}\hat{p}-\dfrac{1}{2}\gamma(1-\rho^2)c(y)^2\hat{p}_y^2\le 0,\\ \hat{p}(s,y,t)\ge g(s,y,t),\\ \left(\hat{p}_t+\mathcal{L}^E_{SY}\hat{p}-\dfrac{1}{2}\gamma(1-\rho^2)c(y)^2\hat{p}_y^2\right)\cdot\left(g(s,y,t)-\hat{p}(s,y,t)\right)=0,\\ \hat{p}(s,y,T)=g(s,y,T),\end{cases}\tag{46}$$

for $(s,y,t)\in\mathbb{R}_+\times\mathbb{R}_+\times[0,T]$. Here,

$$\mathcal{L}^E_{SY}w=\mathcal{L}^0_{SY}w+l(y,t)c(y)\sqrt{1-\rho^2}w_y,$$

where

$$l(y,t)=\frac{1}{\sqrt{1-\rho^2}}c(y)\frac{f_y(y,t)}{f(y,t)}.$$

As shown in Section 2 of Sircar and Zariphopoulou (2005), $l(y,t)$ is smooth and bounded, and is the risk premium corresponding to the minimal entropy martingale measure (MEMM) Q^E, namely,

$$\frac{dQ^E}{d\mathbb{P}}=\exp\left(-\frac{1}{2}\int_0^T(\lambda(Y_s)^2+l(Y_s,s)^2)\,ds-\int_0^T\lambda(Y_s)\,dW_s+\int_0^T l(Y_s,s)\,d\hat{W}_s\right).\tag{47}$$

Therefore, the operator $\mathcal{L}^E_{SY}$ is the infinitesimal generator of (S,Y) under Q^E.

It is important to notice that the fundamental difference between variational inequalities (29) and (46) lies in the operators $\mathcal{L}^0_{SY}$ and $\mathcal{L}^E_{SY}$. Indeed, these variational inequalities reflect the special roles of the MMM in the forward indifference setting and the MEMM in the classical model. The MEMM also arises in the dual representation of $\hat{p}$. Namely, one can show that (see Proposition 2.8 of Leung and Sircar (2009b))

$$\hat{p}(S_t,Y_t,t)=\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{t,T}}\operatorname*{ess\,inf}_{Q^\phi\in\mathcal{M}_f}\left(\mathbb{E}^{Q^\phi}\left\{g(S_\tau,Y_\tau,\tau)|\mathcal{F}_t\right\}+\frac{1}{\gamma}H_t^\tau(Q^\phi|Q^E)\right).\tag{48}$$

This duality result bears a striking resemblance to (36), except here the relative entropy term is computed with respect to Q^E rather than with respect to Q^0.

In the classical setting, the computation of the indifference price involves two steps, namely, first solving PDE (42) followed by variational inequality (46). However, in the forward indifference valuation, the indifference price can be obtained by solving only one variational inequality (29). Hence, with exponential performance, the forward indifference formulation allows for more efficient computation than in the classical framework.

Remark 9 *If the claim is written on Y only, say with payoff function $g(y,t)$, then the indifference price does not depend on S. Applying a logarithmic transformation to the variational inequality (29), the nonlinear variational inequality can be linearized. As a result, in addition to (36), the forward indifference price admits the representation:*

$$p(y,t)=-\frac{1}{\gamma(1-\rho^2)}\log\inf_{\tau\in\mathcal{T}_{t,T}}\mathbb{E}^{Q^0}\left\{e^{-\gamma(1-\rho^2)g(Y_\tau,\tau)}\,|Y_t=y\right\}.$$

In contrast, the classical exponential utility indifference price of an American option with the same payoff function $g(y,t)$ can be found in Oberman and Zariphopoulou (2003), and it is given by

$$\hat{p}(y,t) = -\frac{1}{\gamma(1-\rho^2)} \log \inf_{\tau \in \mathcal{T}_{t,T}} I\!E^{Q^E} \left\{ e^{-\gamma(1-\rho^2) g(Y_\tau,\tau)} | Y_t = y \right\},$$

with Q^E given in (47). Again, we see that Q^0 in the forward performance framework plays a similar role as Q^E in the classical setting.

In summary, we have studied the exponential forward indifference price through its associated variational inequality and dual representation, which have very desirable structures and interpretations due to the nice analytic properties of the exponential forward performance. We have also seen that higher risk aversion leads to earlier exercise times.

4 Modeling Early Exercises of Employee Stock Options

Now, we consider the problem of exercising employee stock options (ESOs) under a forward performance criterion. These options are American calls granted by a company to its employees as a form of compensation. A typical ESO contract prohibits the employee from selling the option and from hedging by short selling the firm's stock. The sale and hedging restrictions may induce the employee to exercise the ESO early and invest the option proceeds elsewhere. In fact, empirical studies (for example, Bettis et al. (2005)) show that employees tend to exercise their ESOs very early. Recent studies, for example, Henderson (2005) and Leung and Sircar (2009a), apply classical indifference pricing to ESO valuation. In those papers, the employee was assumed to have a classical exponential utility specified at the expiration date T of the options. Here, we assume a forward performance criterion for the employee, which is not anchored to a specific future time, and then compute the optimal exercise strategy. Modeling the employee's exercise timing is crucial to the accurate valuation of ESOs.

We assume that the employee dynamically trades in a liquid correlated market index and a riskless money market account in order to partially hedge against his ESO position. Alternative hedging strategies for ESOs have also been proposed. For instance, Leung and Sircar (2009b) considered combining static hedges with market-traded European or American puts with the dynamic investment in the market index.

We focus our study on the case of a single ESO. Typically, ESOs have a vesting period during which they cannot be exercised early. The incorporation of a vesting period amounts to lifting the employee's pre-vesting exercise boundary to infinity to prevent exercise, but leaving the post-vesting policy unchanged. The case with multiple ESOs can be studied as a straightforward extension to our model though the numerical computations will be more complex and time-consuming; see Grasselli and Henderson (2009) for the case of multiple perpetual ESOs with exponential utility. Our main objective is to examine the non-trivial effects of forward investment performance criterion on the employee's optimal exercise timing.

4.1 The Employee's Optimal Forward Performance with an ESO

We assume that the money market account yields a constant interest rate $r \geq 0$. The *discounted* prices of the market index and the firm's stock are modeled as correlated lognormal processes, namely,

$$dS_t = S_t \sigma \left(\lambda \, dt + dW_t \right), \qquad \text{(traded)} \qquad (49)$$

$$dY_t = bY_t \, dt + cY_t \left(\rho dW_t + \sqrt{1-\rho^2} \, d\hat{W}_t \right), \qquad \text{(non-traded)} \qquad (50)$$

where λ, σ, b, c are constant parameters. The discounted ESO payoff, with expiration date T, is given by

$$g(Y_\tau, \tau) = (Y_\tau - Ke^{-r\tau})^+, \quad \text{for } \tau \in \mathcal{T}_{0,T}.$$

Note that this market setup is nested in the Itô diffusion market described in Section 2. Here, the Sharpe ratio λ of S is now a constant, and the option payoff is independent of S. The employee dynamically trades in the index Y and the money market account, so his discounted wealth process satisfies

$$dX_t^\pi = \pi_t \sigma \left(\lambda \, dt + dW_t \right). \tag{51}$$

We proceed with the employee's forward performance criterion $U_t(x)$. First, we adopt the risk tolerance function in (17), namely, $R(x,t) = \sqrt{\alpha x^2 + \beta e^{-\alpha t}}$, and the corresponding dynamic risk preference function $u(x,t)$ given in Proposition 4. Then, we apply Theorem 3 to obtain the employee's forward performance $U_t(x) = u(x, \lambda^2 t)$. In turn, the employee's maximal forward performance in the presence of the ESO is given by

$$V(x,y,t) = \sup_{\tau \in \mathcal{T}_{t,T}} \sup_{\pi \in \mathcal{Z}_{t,\tau}} \mathbb{E} \left\{ u(X_\tau^\pi + g(Y_\tau, \tau), \lambda^2 \tau) \,|\, X_t = x, Y_t = y \right\}. \tag{52}$$

In contrast to the stochastic volatility problem in Section 3, the state variable S is no longer needed, but we work with a more general forward performance criterion than the exponential one.

To solve for the employee's value function, we look for a solution to the following HJB variational inequality:

$$\begin{cases} V_t + \mathcal{L}_Y V - \dfrac{(\rho c y V_{xy} + \lambda V_x)^2}{2V_{xx}} \leq 0, \\ V(x,y,t) \geq u(x + g(y,t), \lambda^2 t), \\ \big(V_t + \mathcal{L}_Y V - \dfrac{(\rho c y V_{xy} + \lambda V_x)^2}{2V_{xx}}\big) \cdot \big(u(x + g(y,t), \lambda^2 t) - V(x,y,t)\big) = 0, \\ V(x,y,T) = u(x + g(y,T), \lambda^2 T), \end{cases} \tag{53}$$

for $(x,y,t) \in \mathbb{R} \times \mathbb{R}_+ \times [0,T]$. We remark that the variational inequality (53) is highly nonlinear, and it can be simplified only for very special local utility functions (e.g. the exponential utility). Here, we do not attempt to address the related existence, uniqueness, and regularity questions.

4.2 Numerical Solutions

We apply a *fully explicit* finite-difference scheme to numerically solve (53) for the employee's optimal exercising strategy. First, we restrict the domain $\mathbb{R} \times \mathbb{R}_+ \times [0,T]$ to a finite domain $\mathcal{D} = \{(x,y,t) : -L_1 \leq x \leq L_2,\, 0 \leq y \leq L_3,\, 0 \leq t \leq T\}$, where L_k, $k = 1,2,3$, are chosen to be sufficiently large to preserve the accuracy of the numerical solutions. The numerical scheme also requires a number of boundary conditions. For $y = 0$, we have $Y_t = 0$, $t \geq 0$, and thus the ESO becomes worthless. Therefore, we set

$$V(x,0,t) = u(x, \lambda^2 t).$$

When Y hits the high level L_3, we assume that the employee will exercise the ESO there, implying the condition

$$V(x, L_3, t) = u(x + g(L_3, t), \lambda^2 t).$$

We also need to set appropriate boundary conditions along $x = -L_1$ and $x = L_2$. To this end, we adopt the Dirichlet boundary conditions

$$V(-L_1, y, t) = u(-L_1 + g(y,t), \lambda^2 t), \qquad V(L_2, y, t) = u(L_2 + g(y,t), \lambda^2 t),$$

which imply that the employee will exercise the ESO at these boundaries.

Then, a uniform grid is applied on $\mathcal{D}$ with nodes $\{(x_i, y_j, t_n) \, : \, i = 0, ..., I; \, j = 0, ..., J; \, n = 0, ..., N\}$, with $\Delta x = (L_1 + L_2)/I$, $\Delta y = L_3/J$, and $\Delta t = T/N$ being the grid spacings. Next, we use the discrete approximations $V_{i,j}^n \approx V(x_i, y_j, t_n)$ where $x_i = i\Delta x$, $y_j = j\Delta y$, and $t_n = n\Delta t$. Also, we discretize the partial differential inequality in (53) by approximating the x and y derivatives with central differences:

$$\frac{\partial V}{\partial x}(x_i, y_j, t_n) \approx \frac{V_{i+1,j}^n - V_{i-1,j}^n}{2\Delta x}, \qquad \frac{\partial^2 V}{\partial x^2}(x_i, y_j, t_n) \approx \frac{V_{i+1,j}^n - 2V_{i,j}^n + V_{i-1,j}^n}{\Delta x^2}, \tag{54}$$

$$\frac{\partial V}{\partial y}(x_i, y_j, t_n) \approx \frac{V_{i,j+1}^n - V_{i,j-1}^n}{2\Delta y}, \qquad \frac{\partial^2 V}{\partial y^2}(x_i, y_j, t_n) \approx \frac{V_{i,j+1}^n - 2V_{i,j}^n + V_{i,j-1}^n}{\Delta y^2}, \tag{55}$$

and

$$\frac{\partial^2 V}{\partial x \partial y}(x_i, y_j, t_n) \approx \frac{V_{i+1,j+1}^n - V_{i+1,j-1}^n + V_{i-1,j-1}^n - V_{i-1,j+1}^n}{4\Delta x \Delta y}.$$

The t-derivative is approximated by the backward difference

$$\frac{\partial V}{\partial t}(x_i, y_j, t_n) \approx \frac{V_{i,j}^{n+1} - V_{i,j}^n}{\Delta t}.$$

This results in an explicit finite-difference scheme which solves for all $V_{i,j}^n$ iteratively backward in time starting at $t_N = T$.

During each time step, the inequality constraint $V(x, y, t) \geq u(x + g(y,t), \lambda^2 t)$ is enforced. By comparing the value function and the obstacle term, we identify the *continuation region* $\mathcal{C}$ where the ESO is not exercised, and the *exercise region* $\mathcal{E}$ where the ESO is exercised, namely

$$\mathcal{C} = \{(x, y, t) \in \mathbb{R} \times \mathbb{R}_+ \times [0, T] \, : \, V(x, y, t) > u(x + g(y,t), \lambda^2 t)\}, \tag{56}$$

$$\mathcal{E} = \{(x, y, t) \in \mathbb{R} \times \mathbb{R}_+ \times [0, T] \, : \, V(x, y, t) = u(x + g(y,t), \lambda^2 t)\}. \tag{57}$$

From the numerical example in Figure 3, we observe that the value function dominates the obstacle term. At any time t and wealth x, we locate the optimal stock price level $y^*(x, t)$ that separates the two regions $\mathcal{C}$ and $\mathcal{E}$. As a result, the employee will exercise the ESO as soon as Y_t hits the threshold $y^*(X_t, t)$:

$$\tau^* = \inf\{0 \leq t \leq T \, : \, Y_t = y^*(X_t, t)\}. \tag{58}$$

In the case of call options, the boundary lies above the strike K. Figure 4 shows an example of the optimal exercise boundary for the ESO.

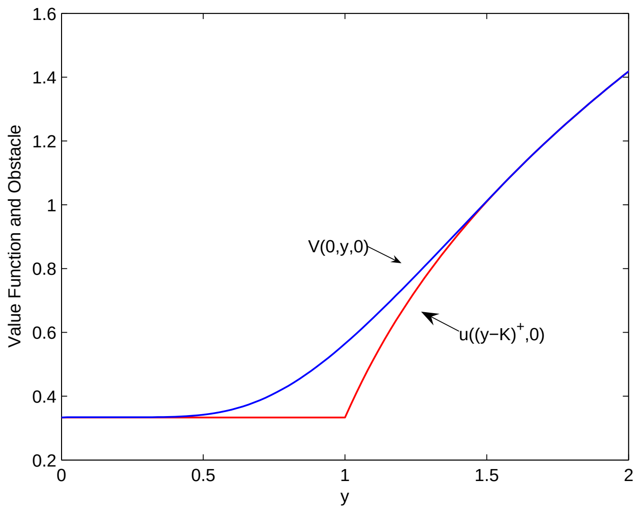

Figure 3: *The value function $V(x,y,t)$ dominates the obstacle term $u(x+g(y,t),\lambda^2 t)$. The parameters are $\lambda = 33\%$, $\sigma = 35\%$, $b = 6\%$, $c = 40\%$, $\rho = 50\%$, $r = 1\%$, $K = 1$, $T = 1$, $\alpha = 4$, $\beta = 0.25$. At $t = 0$ and $x = 0$, the critical stock price $y^*(0,0) = 1.58$ is the point at which the value function touches the obstacle term (above the strike).*

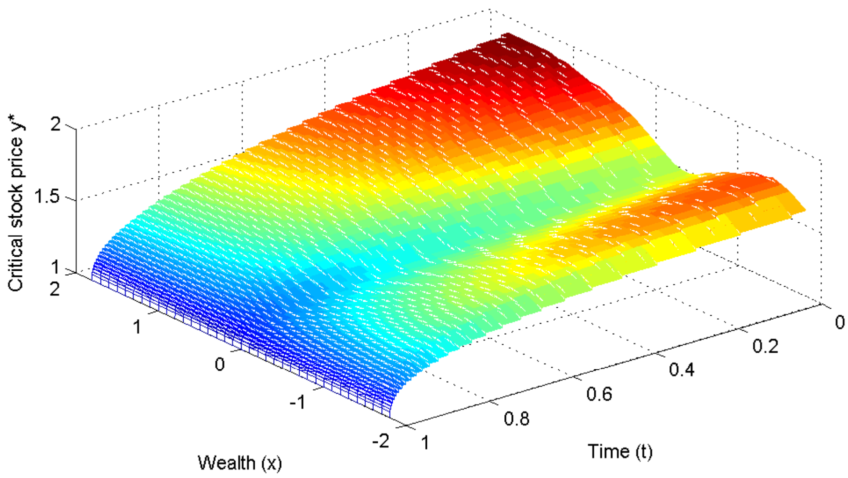

Figure 4: *The optimal exercise policy is characterized by the critical stock price $y^*(x,t)$ as a function of wealth x and time t. It decreases as time approaches maturity. In addition, it tends to shift lower as wealth is near zero.*

4.3 Behavior of the Optimal Exercise Policy

We illustrate the employee's optimal exercise boundary in Figure 4. Not surprisingly, the exercise boundary $y^*(x,t)$ decreases with respect to time, which implies that the employee is willing to exercise the ESO at a lower stock price as it gets closer to expiry.

From Figure 5, we observe that the exercise boundary is wealth dependent. The employee tends to delay exercising the ESO when his wealth deviates away from zero. We can gain some intuition from our choice of risk tolerance function $R(x,t;\alpha,\beta)$. As wealth approaches zero, the employee's risk tolerance decreases (recall Figure 1), or equivalently, risk aversion increases. Higher risk aversion influences the employee to exercise earlier to secure small gains rather than waiting for future uncertain payoffs.

Finally, we show in Figure 6 that the exercise boundary tends to shift upward for higher values of α and β, given the initial wealth $x=0$. The effect of β is intuitive because the risk tolerance function is increasing with respect to β. Therefore, the option holder with a higher β is effectively less risk averse and may be willing to hold on to the ESO longer.

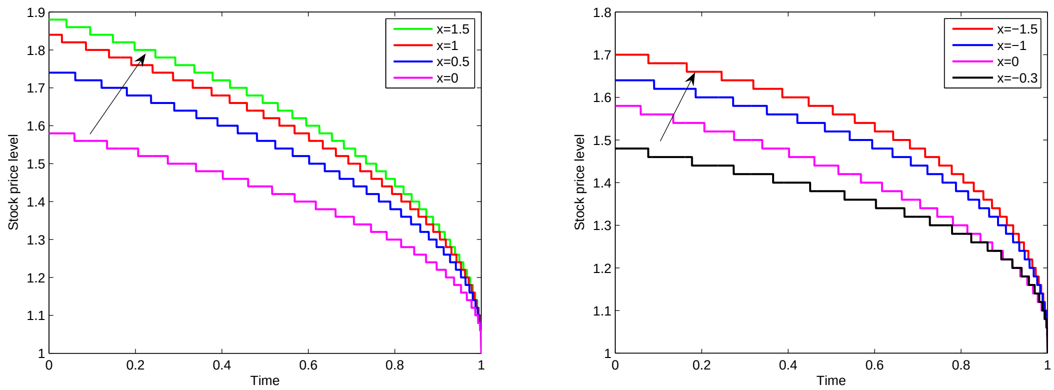

Figure 5: *(Left): The exercise boundary shifts upward as wealth increases from zero. (Right): The exercise boundary is the lowest when wealth* $x=-0.3$*. As wealth further decreases from* -0.3 *to* -1.5*, the exercise boundary rises again. The parameters here are the same as in Figure 3.*

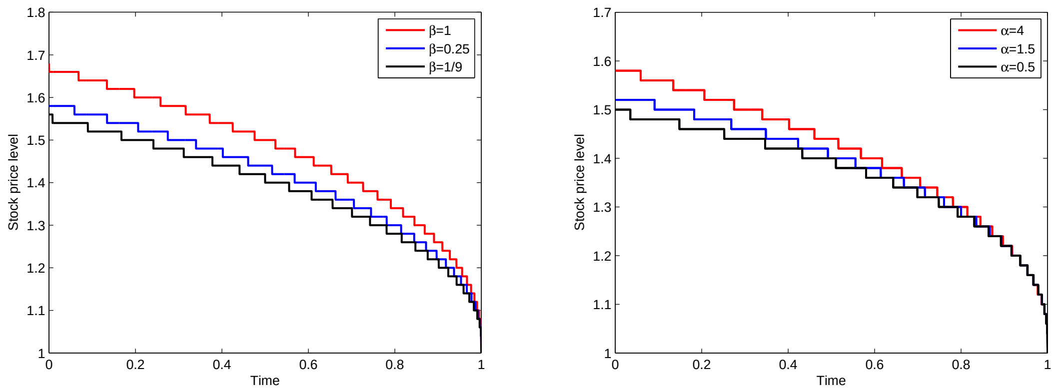

Figure 6: *(Left): A higher value of* β *leads to a higher exercise boundary (at initial wealth* $x=0$*). (Right): A higher value of* α *shifts the exercise boundary upward (initial wealth* $x=0$*). The parameters here are taken to be same as those in Figure 3, except for* α *and* β *specified in the figures above.*

5 Marginal Forward Indifference Price of American Options

In this section, we introduce the marginal forward indifference price of American options. A related concept in the classical utility framework is the *marginal utility price* introduced by Davis (1997), which is useful as an approximation for pricing a small number of claims. For completeness and the upcoming comparison with the forward analogue, we provide a brief review of the marginal utility price in the diffusion market.

5.1 The Classical Marginal Utility Price

In traditional utility maximization, risk aversion is modeled by a deterministic utility function $\hat{U}(x)$ defined at time T. In the Itô diffusion market introduced in Section 2, the investor dynamically trades between the money market and stock S, and his wealth process follows (3). The corresponding Merton portfolio optimization problem is given in (4).

Next, suppose that the investor decides to buy δ units of a European claim, each offering payoff $C_T \in \mathcal{F}_T$. The marginal utility price is the per-unit price that the investor is willing to pay for an infinitesimal position ($\delta \approx 0$) in the claim. This problem has been studied in detail by Davis (1997, 2001), who shows by a formal small δ expansion that the investor's marginal utility price at time t is given by

$$\hat{h}_t = \frac{\mathbb{E}\left\{\hat{U}'(\hat{X}_T^*)\, C_T \,|\, \mathcal{F}_t\right\}}{M_t'(X_t)}, \qquad t \in [0,T], \tag{59}$$

where $\hat{X}_T^*$ is the optimal terminal wealth for the Merton problem in (4), and $\hat{U}'$ and M_t' are the derivatives with respect to the wealth-argument. Kramkov and Sirbu (2006) adopt (59) as the definition of the marginal utility price, which we also adapt to the case of American options.

Definition 10 *The marginal utility price process $(h_t)_{0\le t\le T}$ for an American option with payoff process $(g_t)_{0\le t\le T}$ is defined as*

$$h_t = \frac{\operatorname{ess\,sup}_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}\left\{M_\tau'(\hat{X}_\tau^*)\, g_\tau \,|\, \mathcal{F}_t\right\}}{M_t'(X_t)}, \tag{60}$$

where $M_t(X_t)$ is given in (4).

Among others, one important question is under what conditions does the marginal utility price become independent of the investor's wealth. In the classical setting for options without early exercise, wealth-independence of marginal utility prices is very rare. In fact, Kramkov and Sirbu (2006) show that only exponential and power utilities yield wealth-independent marginal utility prices for any payoff and in any financial market.

5.2 The Marginal Forward Indifference Price Formula

Following the classical formulation, we introduce the marginal forward indifference price for our model. Henceforth, we will give the definitions and results based on the Itô diffusion market settings described in Section 2, where the discounted stock price S follows (2) and the option holder's trading wealth X_t follows (3).

Definition 11 *Let $U_t(x) = u(x, A_t)$ (see Theorem 3) be the investor's forward performance process. The marginal forward indifference price process $(\tilde{p}_t)_{0 \le t \le T}$ for an American option with an $\mathcal{F}_t$-adapted bounded payoff process $(g_t)_{0 \le t \le T}$ is defined as*

$$\tilde{p}_t = \frac{\operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} I\!E \left\{ u_x \left(X^*_\tau, A_\tau \right) g_\tau \middle| \mathcal{F}_t \right\}}{u_x(X_t, A_t)}, \tag{61}$$

where $A_t = \int_0^t \lambda_s^2 ds$ and the optimal wealth process X^* follows (15).

At first glance, the marginal forward indifference price in (61) might depend on the holder's risk preferences and wealth. Nevertheless, as the next proposition shows, the marginal forward indifference price is *independent* of both of these elements, and is simply given as the expected discounted payoff under the MMM, regardless of the investor's forward performance criterion.

Proposition 12 *The marginal forward indifference price of an American option with payoff process $(g_t)_{0 \le t \le T}$ is given by*

$$\tilde{p}_t = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} I\!E^{Q^0} \{ g_\tau \,|\, \mathcal{F}_t \}, \tag{62}$$

where Q^0 is the MMM. Consequently, $\tilde{p}_t$ is independent of both the holder's wealth and his forward performance.

Proof. Comparing (61) and (62), we observe that it is sufficient to show that

$$\frac{u_x \left(X^*_\tau, A_\tau \right)}{u_x \left(X^*_t, A_t \right)} = \exp \left(-\frac{1}{2} \int_t^\tau \lambda_s^2 \, ds - \int_t^\tau \lambda_s \, dW_s \right), \quad \tau \in \mathcal{T}_{t,T}. \tag{63}$$

Since λ is bounded, this leads to the desired measure change from the historical measure $\mathbb{P}$ to the MMM Q^0.

Applying Itô's formula to $u_x(X^*_t, A_t)$ and using the SDE (15) for X^* gives

$$\begin{aligned} du_x \left(X^*_t, A_t \right) = {} & \lambda_t^2 \left(u_{xt}(X^*_t, A_t) + R(X^*_t, A_t) u_{xx}(X^*_t, A_t) + \frac{R(X^*_t, A_t)^2}{2} u_{xxx}(X^*_t, A_t) \right) dt \\ & + \lambda_t R(X^*_t, A_t) u_{xx}(X^*_t, A_t) \, dW_t. \end{aligned} \tag{64}$$

Next, we show that the drift vanishes. First, it follows from differentiating $u(x, t)$ in (11) that

$$u_{xt} = u_x - \frac{u_x^2 u_{xxx}}{2 u_{xx}^2}.$$

Using this and the fact that $R(x, t) = -u_x(x, t)/u_{xx}(x, t)$ to (64), we see that the drift in (64) becomes zero. As a result, the SDE (64) simplifies to

$$\begin{aligned} du_x \left(X^*_t, A_t \right) &= \lambda_t R(X^*_t, A_t) u_{xx}(X^*_t, A_t) \, dW_t \\ &= \lambda_t u_x(X^*_t, A_t) \, dW_t. \end{aligned}$$

This implies that the process $\left(u_x \left(X^*_t, A_t \right) \right)_{t \ge 0}$ is given by the stochastic exponential representation in (63). Hence, by a change of measure, formula (62) follows. ∎

Proposition 12 illustrates a crucial feature of the forward indifference pricing mechanism. If we consider that, in a general Itô diffusion market, different investors adopt different forward performances according to Theorem 3, then their marginal forward indifference prices for an American claim will necessarily be the same, regardless of their wealth and choices of forward performance. In particular, this is true for the stochastic volatility model in (20)-(21) and the basis risk model in (49)-(50). In contrast, the classical marginal utility price for a general utility function is typically wealth and utility dependent (Kramkov and Sirbu, 2006, Theorem 7). In the basis risk model as a special case, Kramkov and Sirbu (2006) show that the marginal utility price is also found from pricing under the MMM, coinciding the forward counterpart, even though they are derived from different mechanisms.

6 Conclusions and Extensions

In summary, we have discussed the forward indifference valuation for American options in an incomplete model with a stochastic factor. We have applied it to value American options with an exponential forward performance criterion and to model the early exercises of ESOs. The option holder's optimal hedging and exercising strategies are found from solving the underlying variational inequalities.

The forward indifference valuation mechanism is profoundly different from the one in the classical approach. This is best illustrated in Section 3, in which the exponential forward indifference price is expressed in terms of relative entropy minimization with respect to the MMM, rather than the MEMM as is the case in the traditional setting. The MMM also plays a crucial role as the pricing measure for the marginal forward indifference price. In contrast to the classical marginal utility price, the marginal forward indifference price is independent of both the investor's wealth and risk preferences.

Several major challenges and interesting problems remain for future investigation. These include the existence and regularity results for the variational inequalities associated with the optimal forward performance and the forward indifference price. The nonlinearity of the variational inequalities also requires the development of efficient numerical schemes. Moreover, even though we have focused on the valuation of American options, it is important to examine its impact in the host of other applications where traditional utility valuation has been used, for example, credit derivatives (Leung et al., 2008; Jaimungal and Sigloch, 2010), volatility derivatives (Grasselli and Hurd, 2008), insurance products (Bayraktar and Ludkovski, 2009), and order book modeling (Avellaneda and Stoikov, 2008). In all of these, exponential utility is chosen for its convenient analytic properties. Forward performance provides a convenient tool to i) move away from exponential utility, and ii) remove the horizon dependence.

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