Finance Stoch (2013) 17:839–870
DOI 10.1007/s00780-013-0213-8

Outperformance portfolio optimization via the equivalence of pure and randomized hypothesis testing

Tim Leung · Qingshuo Song · Jie Yang

Received: 17 January 2012 / Accepted: 24 March 2013 / Published online: 27 August 2013

Abstract We study the portfolio optimization problem of maximizing the outperformance probability over a random benchmark through dynamic trading with a fixed initial capital. Under a general incomplete market framework, this stochastic control problem can be formulated as a composite pure hypothesis testing problem. We analyze the connection between this *pure* testing problem and its *randomized* counterpart, and from the latter we derive a dual representation for the maximal outperformance probability. Moreover, in a complete market setting, we provide a closed-form solution to the problem of beating a leveraged exchange traded fund. For a general benchmark under an incomplete stochastic factor model, we provide the Hamilton–Jacobi–Bellman PDE characterization for the maximal outperformance probability.

Keywords Portfolio optimization · Quantile hedging · Neyman–Pearson lemma · Stochastic benchmark · Hypothesis testing

JEL Classification G10 · G12 · G13 · D81

Mathematics Subject Classification (2010) 60H30 · 91G10

T. Leung
Department of Industrial Engineering and Operations Research, Columbia University, New York, NY 10027, USA
e-mail: leung@ieor.columbia.edu

Q. Song (✉)
Department of Mathematics, City University of Hong Kong, Hong Kong, Hong Kong
e-mail: song.qingshuo@cityu.edu.hk

J. Yang
Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, Chicago, IL 60607, USA
e-mail: jyang06@math.uic.edu

1 Introduction

Portfolio optimization problems with an objective to exceed a given benchmark arise very commonly in portfolio management among both institutional and individual investors. For many hedge funds, mutual funds and other investment portfolios, their performance is evaluated relative to the market indices, e.g., the S&P 500 index or the Russell 1000 index. In this paper, we consider the problem of maximizing the outperformance probability over a random benchmark through dynamic trading with a fixed initial capital. Specifically, given an initial capital $x > 0$ and a random benchmark F, how can one construct a dynamic trading strategy $(\pi_t)_{0\le t\le T}$ in order to maximize the probability of the "success event" where the terminal trading wealth $X_T^{x,\pi}$ exceeds F, i.e., $\mathbb{P}[X_T^{x,\pi} \ge F]$?

In the existing literature, outperformance portfolio optimization has been studied by [3, 5, 30] among others. It has also been studied in the context of quantile hedging by Föllmer and Leukert [11]. In particular, Föllmer and Leukert show that the quantile hedging problem can be formulated as a *pure* hypothesis testing problem. In statistical terminology, this approach seeks to determine a *test*, taking values 0 or 1, that minimizes the probability of type-II-error, while limiting the probability of type-I-error by a prespecified acceptable significance level. The maximal success probability can be interpreted as the *power* of the test. The Föllmer–Leukert approach permits the use of an important result from statistics, namely the Neyman–Pearson lemma (see, for example, [18, Theorem 3.2.1]), to characterize the optimal success event and determine its probability.

On the other hand, outperformance portfolio optimization can also be viewed as a special case of shortfall risk minimization, that is, to minimize the quantity $\rho(-(F - X_T^{x,\pi})^+)$ for some specific risk measure $\rho(\cdot)$. As is well known (see [6, 12, 24, 26]), shortfall risk minimization with a convex risk measure can be solved via an equivalent *randomized* hypothesis testing problem. In fact, the problem to maximize the success probability $\mathbb{P}[X_T^{x,\pi} \ge F]$ is equivalent to minimizing the shortfall risk $\mathbb{P}[X_T^{x,\pi} < F] = \rho(-(F - X_T^{x,\pi})^+)$ with respect to the risk measure defined by $\rho(Y) := \mathbb{P}[Y < 0]$ for any random variable Y. However, this risk measure $\rho(\cdot)$ does not satisfy either convexity or continuity. Hence, a natural question is:

(Q) Is outperformance optimization equivalent to randomized hypothesis testing?

In Sect. 3.1, we show that outperformance portfolio optimization in a general incomplete market is equivalent to a pure hypothesis testing. Moreover, we illustrate that the outperformance probability, or equivalently the associated pure hypothesis testing value, can be strictly smaller than the value of the corresponding randomized hypothesis testing (see Examples 2.4 and 3.4). Therefore, the answer to (Q) is negative in general. This also motivates us to analyze sufficient conditions for the equivalence of pure and randomized hypothesis testing problems (see Theorem 2.10). In turn, our result is applied to give sufficient conditions for the equivalence of outperformance portfolio optimization and the corresponding randomized hypothesis testing problem (see Theorem 3.5).

The main benefit of such an equivalence is that it allows us to utilize the representation of the randomized testing value to compute the optimal outperformance probability. Moreover, the sufficient conditions established herein are amenable for verification and are applicable to many typical finance markets. We provide detailed illustrative examples in Sect. 3.2 for a complete market and in Sect. 3.3 for a stochastic volatility model.

Among other results, we provide an explicit solution to the problem of outperforming a leveraged fund in a complete market. In a stochastic volatility market, we show that for a constant or stock benchmark, the investor may optimally assign a zero volatility risk premium, which corresponds to the minimal martingale measure (MMM). This in turn allows an explicit solution for the success probability in a range of cases in this incomplete market. With the general form of benchmark, the value function can be characterized by an HJB equation in the framework of stochastic control theory.

The paper is structured as follows. In Sect. 2, we analyze the generalized composite pure and randomized hypothesis testing problems, and study their equivalence. Then we apply the results to solve the related outperformance portfolio optimization in Sect. 3, with examples in both complete and incomplete diffusion markets. Section 4 concludes the paper and discusses a number of extensions. Finally, we include a number of examples and proofs in the Appendix.

2 Generalized composite hypothesis testing

In the background, we fix a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We denote by $\mathbb{E}[\,\cdot\,]$ the expectation under $\mathbb{P}$ and by L^0_+ the space of all nonnegative $\mathcal{F}$-measurable random variables, equipped with the topology of convergence in probability. The randomized tests and pure tests are represented by the two collections of random variables taking values in $[0,1]$ and $\{0,1\}$, respectively, and are denoted by

$$\mathcal{X} = \{X : \Omega/\mathcal{F} \to [0,1]/\mathcal{B}([0,1])\} \quad \text{and} \quad \mathcal{I} = \{X : \Omega/\mathcal{F} \to \{0,1\}/2^{\{0,1\}}\}.$$

In addition, $\mathcal{G}$ and $\mathcal{H}$ are two given collections of nonnegative $\mathcal{F}$-measurable random variables.

2.1 Randomized composite hypothesis testing

First, we consider a randomized composite hypothesis testing problem. For $x > 0$, define

$$V(x) := \sup_{X \in \mathcal{X}} \inf_{G \in \mathcal{G}} \mathbb{E}[GX] \tag{2.1}$$

$$\text{subject to} \quad \sup_{H \in \mathcal{H}} \mathbb{E}[HX] \le x. \tag{2.2}$$

From the statistical viewpoint, $\mathcal{G}$ and $\mathcal{H}$ correspond to the collections of alternative hypotheses and null hypotheses, respectively. The solution X can be viewed as the most powerful test, and $V(x)$ is the power of X, where x is the significance level or size of the test.

For any set of random variables $\tilde{\mathcal{H}} \subset L^0_+$, we define a collection of randomized tests by

$$\mathcal{X}_x^{\tilde{\mathcal{H}}} := \{X \in \mathcal{X} : \mathbb{E}[HX] \leq x,\ \forall H \in \tilde{\mathcal{H}}\}.$$

Then the problem in (2.1), (2.2) can be equivalently expressed as

$$V(x) = \sup_{X \in \mathcal{X}_x^{\mathcal{H}}} \inf_{G \in \mathcal{G}} \mathbb{E}[GX]. \tag{2.3}$$

When no ambiguity arises, we write $\mathcal{X}_x = \mathcal{X}_x^{\mathcal{H}}$ for simplicity.

For the upcoming results, we denote the convex hull of $\mathcal{H}$ by $\mathrm{co}(\mathcal{H})$ and the closure (with respect to the topology of convergence in probability) of $\mathrm{co}(\mathcal{H})$ by $\overline{\mathrm{co}(\mathcal{H})}$. Also, we define the set

$$\mathcal{H}_x := \left\{H \in L^0_+ : \mathbb{E}[HX] \leq x\ \forall X \in \mathcal{X}_x^{\mathcal{H}}\right\}.$$

From the definitions together with Fatou's lemma, it is straightforward to check that $\mathcal{H}_x$ is convex and closed, containing $\mathcal{H}$. Furthermore, we observe that $\mathcal{X}_x^{\mathcal{H}} = \mathcal{X}_x^{\tilde{\mathcal{H}}}$ for an arbitrary $\tilde{\mathcal{H}}$ satisfying $\mathcal{H} \subset \tilde{\mathcal{H}} \subset \mathcal{H}_x$. Hence, the randomized testing problem in (2.1), (2.2), and therefore $V(x)$ in (2.3), will stay invariant if $\mathcal{H}$ is replaced by $\tilde{\mathcal{H}}$ as above. More precisely, we have the following:

Lemma 2.1 *Let $\tilde{\mathcal{H}} \subset L^0_+$ be an arbitrary set satisfying $\mathcal{H} \subset \tilde{\mathcal{H}} \subset \mathcal{H}_x$. Then $V(x)$ in (2.3) is equivalent to*

$$V(x) = \sup_{X \in \mathcal{X}_x^{\tilde{\mathcal{H}}}} \inf_{G \in \mathcal{G}} \mathbb{E}[GX]. \tag{2.4}$$

In particular, one can take $\tilde{\mathcal{H}} = \mathrm{co}(\mathcal{H})$ or $\tilde{\mathcal{H}} = \mathcal{H}_x$.

This randomized hypothesis testing problem is similar to that studied by Cvitanić and Karatzas [7], except that G and H in (2.1)–(2.3) are not necessarily the Radon–Nikodým derivatives for probability measures. In this slight generalization, H can vary among $\mathcal{H}$, which allows statistical hypothesis testing with different significance levels depending on H. To see this, one can divide (2.2) by $\mathbb{E}[H]$ for each $H \in \mathcal{H}$, resulting in a confidence level of $x/\mathbb{E}[H]$ (see also Remark 5.2 in [25]). Similarly to [7] and [19], **we make the following standing assumption throughout Sect. 2**:

Assumption 2.2 *Assume that $\mathcal{G}$ and $\mathcal{H}$ are subsets of L^0_+ with $\sup_{X \in \mathcal{G} \cup \mathcal{H}} \mathbb{E}[X] < \infty$, and $\mathcal{G}$ is convex and closed.*

The following theorem gives a characterization of the solution for (2.3).

Theorem 2.3 *Under Assumption 2.2, there exists*

$$(\hat{G}, \hat{H}, \hat{a}, \hat{X}) \in \mathcal{G} \times \overline{\mathrm{co}(\mathcal{H})} \times [0, \infty) \times \mathcal{X}_x$$

satisfying

$$\hat{X} = I_{\{\hat{G}>\hat{a}\hat{H}\}} + B I_{\{\hat{G}=\hat{a}\hat{H}\}} \quad \textit{for some } B : \Omega/\mathcal{F} \to [0, 1]/\mathcal{B}([0, 1]), \tag{2.5}$$

$$\mathbb{E}[H\hat{X}] \le \mathbb{E}[\hat{H}\hat{X}] = x \quad \forall H \in \mathcal{H}, \tag{2.6}$$

$$\mathbb{E}[\hat{G}\hat{X}] \le \mathbb{E}[G\hat{X}] \quad \forall G \in \mathcal{G}. \tag{2.7}$$

In particular, $\hat{X}$ and B satisfying (2.5)–(2.7) can be chosen to be measurable with respect to $\sigma(\mathcal{G} \cup \mathcal{H})$, the smallest σ-algebra generated by the random variables in $\mathcal{G} \cup \mathcal{H}$. Moreover, $V(x)$ of (2.3) is given by

$$V(x) = \mathbb{E}[\hat{G}\hat{X}] = \inf_{a\ge 0} \Big\{ xa + \inf_{\mathcal{G}\times\mathrm{co}(\mathcal{H})} \mathbb{E}[(G - aH)^+] \Big\}, \tag{2.8}$$

which is continuous, concave, and nondecreasing in $x \in [0, \infty)$. Furthermore, $(\hat{G}, \hat{H})$ and $(\hat{G}, \hat{H}, \hat{a})$ respectively attain the infimum of

$$(G, H) \mapsto \mathbb{E}[(G - \hat{a}H)^+] \quad \textit{and} \quad (G, H, a) \mapsto xa + \mathbb{E}[(G - aH)^+]. \tag{2.9}$$

Proof First, we apply the equivalence between (2.3) and (2.4) from Lemma 2.1 and the fact that $\mathcal{X}_x^{\mathcal{H}} = \mathcal{X}_x^{\mathrm{co}(\mathcal{H})}$. Also, $\overline{\mathrm{co}(\mathcal{H})}$ is convex and closed. If there is a sequence $(H_n) \subset \overline{\mathrm{co}(\mathcal{H})}$ such that $H_n \to H$ almost surely in $\mathbb{P}$, then $H_n \to H$ in probability and $H \in \overline{\mathrm{co}(\mathcal{H})}$. Therefore, we apply the procedures in [7, Proposition 3.2, Theorem 4.1] to obtain the existence of $(\hat{G}, \hat{H}, \hat{a}, \hat{X}) \in \mathcal{G} \times \overline{\mathrm{co}(\mathcal{H})} \times [0, \infty) \times \mathcal{X}_x$ satisfying (2.5)–(2.7), the optimality of (2.9), and the representation

$$V(x) = \mathbb{E}[\hat{G}\hat{X}] = \inf_{a\ge 0} \Big\{ xa + \inf_{\mathcal{G}\times\overline{\mathrm{co}(\mathcal{H})}} \mathbb{E}[(G - aH)^+] \Big\}. \tag{2.10}$$

Specifically, we replace the two probability density sets in [7] by the L^1-bounded sets $\mathcal{G}$ and $\mathcal{H}$ for our problem and their $\mathcal{H}_x$ by $\overline{\mathrm{co}(\mathcal{H})}$. At the infimum, $V(x)$ in (2.10) becomes (see [7, Proposition 3.2(i)])

$$V(x) = x\hat{a} + \mathbb{E}[(\hat{G} - \hat{a}\hat{H})^+]\}.$$

Note that $\hat{H}$ belongs to $\overline{\mathrm{co}(\mathcal{H})}$ but not necessarily to $\mathrm{co}(\mathcal{H})$. Nevertheless, there exists a sequence $(H_n) \subset \mathrm{co}(\mathcal{H})$ satisfying $H_n \to \hat{H}$ in probability. By the fact that any subsequence then contains an almost surely convergent subsequence, together with the dominated convergence theorem, it follows that $\mathbb{E}[(\hat{G} - \hat{a}H_n)^+] \to \mathbb{E}[(\hat{G} - \hat{a}\hat{H})^+]$, and hence the representation (2.8) follows.

Next, for arbitrary $x_1, x_2 \geq 0$, the inequality

$$\begin{aligned}
&\frac{1}{2}\big(V(x_1)+V(x_2)\big)\\
&=\frac{1}{2}\left(\inf_{\substack{a\geq 0\\(G,H)\in\mathcal{G}\times\mathrm{co}(\mathcal{H})}}\mathbb{E}[x_1a+(G-aH)^+]+\inf_{\substack{a\geq 0\\(G,H)\in\mathcal{G}\times\mathrm{co}(\mathcal{H})}}\mathbb{E}[x_2a+(G-aH)^+]\right)\\
&\leq \inf_{\substack{a\geq 0\\(G,H)\in\mathcal{G}\times\mathrm{co}(\mathcal{H})}}\mathbb{E}\left[\frac{1}{2}(x_1+x_2)a+(G-aH)^+\right]\\
&=V\left(\frac{x_1+x_2}{2}\right)
\end{aligned}$$

implies the concavity of $V(x)$. This, together with boundedness, yields the continuity.

Finally, we observe that if $(\hat{G},\hat{H},\hat{a},\hat{X})\in\mathcal{G}\times\overline{\mathrm{co}(\mathcal{H})}\times[0,\infty)\times\mathcal{X}_x$ satisfies (2.5)–(2.7), then $(\hat{G},\hat{H},\hat{a},\widetilde{X})\in\mathcal{G}\times\overline{\mathrm{co}(\mathcal{H})}\times[0,\infty)\times\mathcal{X}_x$ with

$$\widetilde{X}:=I_{\{\hat{G}>\hat{a}\hat{H}\}}+\widetilde{B}I_{\{\hat{G}=\hat{a}\hat{H}\}},\quad \text{where } \widetilde{B}:=\mathbb{E}[B|\sigma(\mathcal{G}\cup\mathcal{H})],$$

also satisfies (2.5)–(2.7). Hence, $\hat{X}$ and B can be chosen $\sigma(\mathcal{G}\cup\mathcal{H})$-measurable. □

Comparing to the similar result by Cvitanić and Karatzas [7], we have improved the representation of $V(x)$ in (2.8), where the minimization in H is conducted over the smaller set $\mathrm{co}(\mathcal{H})$ instead of $\mathcal{H}_x$. This will be useful for our application to outperformance portfolio optimization (see Sect. 3) since it is easier to identify and work with the set $\mathrm{co}(\mathcal{H})$ in a financial market. Moreover, the minimizer $\hat{a}$ in Theorem 2.3 above belongs to $[0,\infty)$, rather than to $(0,\infty)$ according to Proposition 3.1 and Lemma 4.3 in [7]. In Appendix A.2, we provide an example where $\hat{a}=0$ as well as a sufficient condition for $\hat{a}>0$.

We recall from Lemma 2.1 that $V(x)$ of (2.3) is invariant to replacing $\mathcal{H}$ with any larger set $\widetilde{\mathcal{H}}$ such that $\mathcal{H}\subset\widetilde{\mathcal{H}}\subset\mathcal{H}_x$. In Theorem 2.3, we observe that (2.8) also stays valid even if $\mathrm{co}(\mathcal{H})$ is replaced by any larger set $\widetilde{\mathcal{H}}$ such that $\mathrm{co}(\mathcal{H})\subset\widetilde{\mathcal{H}}\subset\mathcal{H}_x$. However, the same does not hold if $\mathrm{co}(\mathcal{H})$ is replaced by the original *smaller* set $\mathcal{H}$. We illustrate this technical point in Example A.1 of Appendix A.1.

It is also interesting to note that one can take $\widetilde{\mathcal{H}}$ as the bipolar of $\mathcal{H}$ without changing the objective value. Recall that by the bipolar theorem (see Theorem 1.3 of [4]), the bipolar $\mathcal{H}^{oo}$ of $\mathcal{H}$ is the smallest convex, closed, and solid set containing $\mathcal{H}$. To see that we can indeed take $\widetilde{\mathcal{H}}=\mathcal{H}^{oo}$, denote the polar of $\mathcal{A}\subset L^0_+$ by $\mathcal{A}^o:=\{X\in L^0_+:\mathbb{E}[AX]\leq 1,\forall A\in\mathcal{A}\}$ and write $x\mathcal{A}=\{xA:A\in\mathcal{A}\}$. Then

$$\mathcal{X}_x^{\mathcal{H}}=(x\mathcal{H}^o)\cap\mathcal{X}\subset x\mathcal{H}^o\quad\text{and}\quad\mathcal{H}_x=x(\mathcal{X}_x^{\mathcal{H}})^o\supset x(x\mathcal{H}^o)^o=\mathcal{H}^{oo}\supset\overline{\mathrm{co}(\mathcal{H})}.$$

Precisely, the last inclusion $\mathcal{H}^{oo}\supset\overline{\mathrm{co}(\mathcal{H})}$ above is due to the bipolar theorem. Moreover, $\overline{\mathrm{co}(\mathcal{H})}$ could be not solid and strictly smaller than the bipolar $\mathcal{H}^{oo}$; see Example 2.4.

2.2 On the equivalence of randomized and pure hypothesis testing

According to Theorem 2.3, if the random variable B in (2.5) can be chosen as an indicator function satisfying (2.5)–(2.7), then the associated solution $\hat{X}$ of (2.5) will also be an indicator and therefore a *pure test*! This leads to an interesting question: When does a pure test solve the randomized composite hypothesis testing problem?

Motivated by this, we define the pure composite hypothesis testing problem

$$V_1(x) := \sup_{X \in \mathcal{I}} \inf_{G \in \mathcal{G}} \mathbb{E}[GX]$$

$$\text{subject to} \quad \sup_{H \in \mathcal{H}} \mathbb{E}[HX] \le x$$

for $x > 0$. This is equivalent to solving

$$V_1(x) = \sup_{X \in \mathcal{I}_x} \inf_{G \in \mathcal{G}} \mathbb{E}[GX], \tag{2.11}$$

where $\mathcal{I}_x := \{X \in \mathcal{I} : \mathbb{E}[HX] \le x \ \forall H \in \mathcal{H}\}$ consists of all candidate pure tests.

From their definitions we see that $V(x) \ge V_1(x)$. However, one cannot expect $V_1(x) = V(x)$ in general, as seen in the next simple example from [19].

Example 2.4 Fix $\Omega = \{0, 1\}$ and $\mathcal{F} = 2^{\Omega}$ with $\mathbb{P}[0] = \mathbb{P}[1] = 1/2$. Define the collections $\mathcal{G} = \{G : G(0) = G(1) = 1\}$ and $\mathcal{H} = \{H : H(0) = 1/2, H(1) = 3/2\}$. In this simple setup, direct computations yield:

1. For the randomized hypothesis testing, $V(x)$ is given by

$$V(x) = \begin{cases} \mathbb{E}[4xI_{\{0\}}] = 2x & \text{if } 0 \le x < 1/4; \\ \mathbb{E}[I_{\{0\}} + \frac{4x-1}{3} I_{\{1\}}] = \frac{2x+1}{3} & \text{if } 1/4 \le x < 1; \\ \mathbb{E}[1] = 1 & \text{if } x \ge 1. \end{cases}$$

2. For the pure hypothesis testing, $V_1(x)$ is given by

$$V_1(x) = \begin{cases} \mathbb{E}[0] = 0 & \text{if } 0 \le x < 1/4; \\ \mathbb{E}[I_{\{0\}}] = \frac{1}{2} & \text{if } 1/4 \le x < 1; \\ \mathbb{E}[1] = 1 & \text{if } x \ge 1. \end{cases}$$

In the above, the inequality $V_1(x) < V(x)$ holds almost everywhere in $[0, 1]$. In fact, $V_1(x)$ is not concave and continuous, while $V(x)$ is.

Remark 2.5 In Example 2.4, $V(x)$ turns out to be the smallest concave majorant of $V_1(x)$. However, this is not always true. We provide a counterexample in Appendix A.3.

If there is a pure test that solves both the pure and randomized composite hypothesis testing problems, then the equality $V_1(x) = V(x)$ must follow. An important question is: When does this phenomenon of equality occur?

Corollary 2.6 *Let* $(\hat{G}, \hat{H}, \hat{a}, \hat{X}) \in \mathcal{G} \times \overline{\mathrm{co}(\mathcal{H})} \times [0, \infty) \times \mathcal{X}_x$ *be given by Theorem* 2.3. *Then* B *in* (2.5) *must satisfy*

(i) *If* $\mathbb{E}[\hat{H} I_{\{\hat{G} > \hat{a}\hat{H}\}}] = x$, *then* $B = 0$.
(ii) *If* $\mathbb{E}[\hat{H} I_{\{\hat{G} \geq \hat{a}\hat{H}\}}] = x > \mathbb{E}[\hat{H} I_{\{\hat{G} > \hat{a}\hat{H}\}}]$, *then* $B = 1$.

Proof In view of the existence of $\hat{X}$ in Theorem 2.3 and its form in (2.5), B as specified in each case above is the unique choice that satisfies $\mathbb{E}[\hat{H}\hat{X}] = x$; see (2.6). □

Corollary 2.6 presents two examples where the optimal test $\hat{X}$ is indeed a pure test. In the remaining case where $\mathbb{E}[\hat{H} I_{\{\hat{G} \geq \hat{a}\hat{H}\}}] > x > \mathbb{E}[\hat{H} I_{\{\hat{G} > \hat{a}\hat{H}\}}]$, B is a random variable taking values in $[0, 1]$. When $\mathcal{G}$ and $\mathcal{H}$ are singletons, we have the following:

Corollary 2.7 *Assume that* $\mathcal{G} = \{\hat{G}\}$ *and* $\mathcal{H} = \{\hat{H}\}$ *are singletons, and*

$$\mathbb{E}[\hat{H} I_{\{\hat{G} \geq \hat{a}\hat{H}\}}] > x > \mathbb{E}[\hat{H} I_{\{\hat{G} > \hat{a}\hat{H}\}}].$$

Then B *in* (2.5) *can be taken as the constant*

$$B_0 := \frac{x - \mathbb{E}[\hat{H} I_{\{\hat{G} > \hat{a}\hat{H}\}}]}{\mathbb{E}[\hat{H} I_{\{\hat{G} = \hat{a}\hat{H}\}}]} > 0.$$

Proof This follows from direct computation to verify (2.5)–(2.7) in Theorem 2.3. □

In Corollary 2.7, we see that when $\mathbb{E}[\hat{H} I_{\{\hat{G} \geq \hat{a}\hat{H}\}}] > x > \mathbb{E}[\hat{H} I_{\{\hat{G} > \hat{a}\hat{H}\}}]$, the choice of $B = B_0 \in (0, 1)$ yields a nonpure test $\hat{X}$; see (2.5). Nevertheless, our next lemma shows that under an additional condition, one can alternatively choose an indicator in place of B and obtain a pure test.

Lemma 2.8 *Assume that* $\mathcal{G} = \{\hat{G}\}$ *and* $\mathcal{H} = \{\hat{H}\}$ *are singletons, and there exists an* $\mathcal{F}$*-measurable random variable* Y *such that the function*

$$g(y) = \mathbb{E}[\hat{H} I_{\{Y < y\}}], \quad y \in \mathbb{R}, \tag{2.12}$$

is continuous. Then there exists a pure test $\hat{X}$ *that solves both problems* (2.3) *and* (2.11).

Proof If $(\hat{G}, \hat{H}, \hat{a})$ satisfies either (i) or (ii) of Corollary 2.6, then Corollary 2.6 implies that $\hat{X}$ must be an indicator. Next, we discuss the other case where $(\hat{G}, \hat{H}, \hat{a})$ satisfies $\mathbb{E}[\hat{H} I_{\{\hat{G} \geq \hat{a}\hat{H}\}}] > x > \mathbb{E}[\hat{H} I_{\{\hat{G} > \hat{a}\hat{H}\}}]$. Define a function $g_1(\cdot)$ by

$$g_1(y) = \mathbb{E}[\hat{H} I_{\{\hat{G} = \hat{a}\hat{H}\} \cap \{Y < y\}}].$$

Note that $g_1(\cdot)$ is right-continuous since for any $y \in \mathbb{R}$,

$$\begin{aligned}|g_1(y+\varepsilon) - g_1(y)| &= \mathbb{E}[\hat{H} I_{\{\hat{G}=\hat{a}\hat{H}\}\cap\{y\le Y<y+\varepsilon\}}] \\ &\le \mathbb{E}[\hat{H} I_{\{y\le Y<y+\varepsilon\}}] \\ &= g(y+\varepsilon) - g(y) \longrightarrow 0 \quad \text{as } \varepsilon \searrow 0\end{aligned}$$

by the continuity of $g(\cdot)$. Similar arguments show that $g_1(\cdot)$ is also left-continuous. Also, observe that

$$\lim_{y\to-\infty} g_1(y) = 0, \quad \text{and} \quad \lim_{y\to\infty} g_1(y) = \mathbb{E}[\hat{H} I_{\{\hat{G}=\hat{a}\hat{H}\}}] > x - \mathbb{E}[\hat{H} I_{\{\hat{G}>\hat{a}\hat{H}\}}].$$

Therefore, there exists $\hat{y} \in \mathbb{R}$ satisfying

$$g_1(\hat{y}) = x - \mathbb{E}[\hat{H} I_{\{\hat{G}>\hat{a}\hat{H}\}}].$$

Now we can simply set

$$\bar{X} = I_{(\{\hat{G}=\hat{a}\hat{H}\}\cap\{Y<\hat{y}\})\cup\{\hat{G}>\hat{a}\hat{H}\}} = I_{\{\hat{G}>\hat{a}\hat{H}\}} + I_{\{Y<\hat{y}\}} I_{\{\hat{G}=\hat{a}\hat{H}\}}. \tag{2.13}$$

One can directly verify that the above $\bar{X}$ belongs to $\mathcal{X}_x$ and satisfies (2.5)–(2.7) with the choice of $B = I_{\{Y<\hat{y}\}}$. □

In Lemma 2.8, if the cumulative distribution function (c.d.f.) $F_Y(y) = \mathbb{P}[Y \le y]$ of the random variable Y is continuous, then $y \mapsto \mathbb{P}[Y < y]$ is also continuous. As a result, $y \mapsto g(y)$ of (2.12) must be continuous due to $\mathbb{E}[\hat{H}] < \infty$ of Assumption 2.2, and Lemma 2.8 is still valid if one replaces the continuity assumption on g by the continuity of $F_Y(\cdot)$. Note that Y need not be independent of $\mathcal{G}$ and $\mathcal{H}$. Next, we establish a similar result for the case where $\mathcal{G}$ and $\mathcal{H}$ are not singletons.

Lemma 2.9 *Assume that there exists an $\mathcal{F}$-measurable random variable Y independent of $\sigma(\mathcal{G} \cup \mathcal{H})$ with continuous cumulative distribution function. Then there exists a pure test $\bar{X}$ that solves both problems* (2.3) *and* (2.11).

Proof First, we define $U = F_Y(Y)$, which is uniformly distributed by the continuity of $F_Y(\cdot)$ and independent of $\sigma(\mathcal{G} \cup \mathcal{H})$. Let $(\hat{G}, \hat{H}, \hat{a}, \hat{X}) \in \mathcal{G} \times \overline{\mathrm{co}(\mathcal{H})} \times [0,\infty) \times \mathcal{X}_x$ be chosen as in Theorem 2.3, where $\hat{X}$ is measurable with respect to $\sigma(\mathcal{G} \cup \mathcal{H})$. Then we show that the indicator

$$\bar{X} := I_{\{U<\hat{X}\}}$$

also solves the problem (2.11) by checking (2.5)–(2.7). Indeed, $\bar{X}$ satisfies (2.5) since it admits the form

$$\bar{X} = I_{\{\hat{G}>\hat{a}\hat{H}\}} + I_{\{U<B\}} I_{\{\hat{G}=\hat{a}\hat{H}\}}$$

with the same B in (2.5). Next, for any random variable $M \in \mathcal{G} \cup \mathcal{H}$, we use the tower property to obtain

$$\begin{aligned}\mathbb{E}[M\bar{X}] &= \mathbb{E}[MI_{\{U<\hat{X}\}}]\\ &= \mathbb{E}\big[\mathbb{E}[MI_{\{U<\hat{X}\}}|\sigma(\mathcal{G}\cup\mathcal{H})]\big]\\ &= \mathbb{E}\big[M\mathbb{E}[I_{\{U<\hat{X}\}}|\sigma(\mathcal{G}\cup\mathcal{H})]\big]\\ &= \mathbb{E}[M\hat{X}].\end{aligned}$$

In the last equality, we have used the fact that $\mathbb{E}[I_{\{U<c\}}] = \mathbb{P}\{U < c\} = c$ for $c \in [0, 1]$ together with the measurability of $\hat{X}$ with respect to $\sigma(\mathcal{G}\cup\mathcal{H})$, which yields that $\hat{X} = \mathbb{E}[I_{\{U<\hat{X}\}}|\sigma(\mathcal{G}\cup\mathcal{H})]$ almost surely in $\mathbb{P}$.

Hence, we have $\mathbb{E}[H\hat{X}] = \mathbb{E}[H\bar{X}]$ and $\mathbb{E}[G\hat{X}] = \mathbb{E}[G\bar{X}]$ for all $(G, H) \in \mathcal{G}\times\mathcal{H}$, and this implies that $\bar{X}$ satisfies both (2.6) and (2.7). As a consequence, the indicator $\bar{X}$ indeed solves both the pure and randomized test problems by the definition. □

The fact that an independent random variable appears in the equivalence between pure and randomized testing problems is quite natural. Indeed, in hypothesis testing, statisticians may interpret the randomized test by a pure test combined with an independent random variable drawn from a uniform distribution. In Lemma 2.9, we have introduced the uniform random variable $F_Y(Y)$ to the same effect.

Next, we summarize a number of sufficient conditions that are amenable for verification.

Theorem 2.10 *Suppose that one of the following conditions is satisfied*:

(C1) *$\mathcal{G}$ and $\mathcal{H}$ are singletons, and there exists an $\mathcal{F}$-measurable random variable with a continuous c.d.f. with respect to $\mathbb{P}$.*
(C2) *There exists a continuous $\mathcal{F}$-measurable random variable independent of $\sigma(\mathcal{G}\cup\mathcal{H})$.*
(C3) *For all x satisfying $0 < x < \sup_{H\in\mathcal{H}}\mathbb{E}[H]$, the associated optimal triplet $(\hat{G}, \hat{H}, \hat{a}) \in \mathcal{G}\times\overline{\mathrm{co}(\mathcal{H})}\times[0,\infty)$ given by Theorem 2.3 satisfies $\mathbb{P}[\hat{G} = \hat{a}\hat{H}] = 0$.*

Then $V_1(x) = V(x)$, and there exists an indicator function $\hat{X}$ that solves problems (2.3) and (2.11) simultaneously. Furthermore, $x \mapsto V_1(x)$ is continuous, concave, and nondecreasing.

Proof In view of Lemma 2.8 and Corollary 2.6, we get $V_1(x) = V(x)$ under either (C1) or (C3). On the other hand, (C2) also implies $V_1(x) = V(x)$ by Lemma 2.9.

Since $V_1(x) = V(x)$ for all $x \geq 0$, by repeating the same arguments as above, $V_1(\cdot)$ inherits from $V(\cdot)$ in Theorem 2.3 to be continuous, concave, and nondecreasing. □

Note that condition (C1) in Theorem 2.10 is slightly stronger than (2.12). However, condition (C1) is convenient to be used to solve quantile hedging problems in a financial market. Comparing conditions (C1) and (C2) in Theorem 2.10, (C2) works for cases where $\mathcal{G}$ and $\mathcal{H}$ are not singletons, but it requires that the continuous random variable be independent of $\sigma(\mathcal{G}\cup\mathcal{H})$. In contrast, (C1) does not require such an independence.

Remark 2.11 As it turns out, one cannot remove the independence requirement on the continuous random variable in (C2) of Theorem 2.10. For the purpose of illustration, we provide a counterexample in Appendix A.4.

Remark 2.12 In this section, our analysis is conducted in $L^0_+(\Omega, \mathcal{F}, \mathbb{P})$ with the topology given by convergence in probability. This differs from our short proceedings paper [19], which summarized a small number of similar results in $L^1_+(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{P}$-a.s. convergence. Moreover, the current paper has revised the main results, especially Theorems 2.3 and 2.10, and provides new lemmas and detailed proofs.

3 Outperformance portfolio optimization

We now discuss a portfolio optimization problem whose objective is to maximize the probability of outperforming a random benchmark. Applying our preceding analysis and the generalized Neyman–Pearson lemma, we examine the problem in both complete and incomplete markets.

3.1 Characterization via pure hypothesis testing

We fix $T > 0$ as the investment horizon and let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$ be a filtered complete probability space satisfying the usual conditions. The market consists of a liquidly traded risky asset and a riskless money market account. For notational simplicity, we assume a zero risk-free interest rate, which amounts to working with cash flows discounted by the risk-free rate. We model the risky asset price by an $(\mathcal{F}_t)$-adapted locally bounded nonnegative semimartingale process $(S_t)_{0\le t\le T}$.

The class of equivalent local martingale measures, denoted by $\mathcal{Q}$, consists of all probability measures $\mathbb{Q} \sim \mathbb{P}$ on $\mathcal{F}_T$ such that the stock price S is a $\mathbb{Q}$-local martingale. We assume no-arbitrage in the sense of no free lunch with vanishing risk (NFLVR). According to [8] (or Chap. 8 of [9]), this is a necessary and sufficient condition to have a nonempty set $\mathcal{Q}$ for the locally bounded semimartingale process. We denote the associated set of Radon–Nikodým densities by

$$\mathcal{Z} := \left\{ \frac{d\mathbb{Q}}{d\mathbb{P}} : \mathbb{Q} \in \mathcal{Q} \right\}.$$

Given an initial capital x and a self-financing trading strategy $(\pi_u)_{0\le u\le T}$ representing the number of shares in S, the investor's wealth process satisfies

$$X_t^{x,\pi} = x + \int_0^t \pi_u \, dS_u.$$

Each admissible trading strategy π is an $(\mathcal{F}_t)$-predictable process with the property that the stochastic integral $\int_0^t \pi_u \, dS_u$ is well defined and $X_t^{x,\pi} \ge 0 \; \forall t \in [0, T]$, $\mathbb{P}$-a.s. See Definition 8.1.1 of [9]. We denote the set of all admissible strategies by $\mathcal{A}(x)$.

The benchmark is modeled by a nonnegative F_T-measurable random variable F. We denote by F_0 the superhedging price, which is the smallest capital needed to achieve $\mathbb{P}[X_T^{x,\pi} \geq F] = 1$ for some strategy $\pi \in \mathcal{A}(x)$. Then F_0 satisfies (see, e.g., [10])

$$F_0 := \sup_{Z \in \mathcal{Z}} \mathbb{E}[ZF], \tag{3.1}$$

which we assume to be finite throughout Sect. 3. Note that with a smaller initial capital $x < F_0$, the success probability $\mathbb{P}[X_T^{x,\pi} \geq F]$ is < 1 for any $\pi \in \mathcal{A}(x)$.

Our objective is to maximize over all admissible trading strategies the success probability with $x < F_0$. Specifically, we solve the optimization problem

$$\widetilde{V}(x) := \sup_{x_1 \leq x} \sup_{\pi \in \mathcal{A}(x_1)} \mathbb{P}[X_T^{x_1,\pi} \geq F] \tag{3.2}$$

$$= \sup_{\pi \in \mathcal{A}(x)} \mathbb{P}[X_T^{x,\pi} \geq F], \quad x \geq 0. \tag{3.3}$$

The second equality (3.3) is a consequence of the monotonicity of the mapping $x \mapsto \sup_{\pi \in \mathcal{A}(x)} \mathbb{P}[X_T^{x,\pi} \geq F]$. Clearly, $\widetilde{V}(x)$ is increasing in x. Moreover, if $F > 0$ $\mathbb{P}$-a.s., then $\widetilde{V}(0) = 0$ due to the nonnegative wealth constraint.

Scaling property If the benchmark is scaled by a factor $\beta \geq 0$, then what is its effect on the success probability, given any fixed initial capital? To address this, we first define

$$\widetilde{V}(x; \beta) := \sup_{\pi \in \mathcal{A}(x)} \mathbb{P}[X_T^{x,\pi} \geq \beta F].$$

Proposition 3.1 *For any fixed $x > 0$, the success probability has the following properties*:

(i) *The mapping $\beta \mapsto \widetilde{V}(x; \beta)$ is nonincreasing for $\beta \geq 0$.*
(ii) *$\widetilde{V}(\beta x; \beta) = \widetilde{V}(x; 1)$ for $\beta \geq 0$.*
(iii) *If $\tilde{V}(\cdot)$ is right-continuous at x in the first argument, then*

$$\lim_{\beta \to \infty} \widetilde{V}(x; \beta) = \mathbb{P}[F = 0]. \tag{3.4}$$

(iv) *$\widetilde{V}(x; \beta) = 1$ for $0 \leq \beta \leq \frac{x}{F_0}$.*

Proof First, we observe that $\widetilde{V}(x; \beta) = \sup_{\pi \in \mathcal{A}(x/\beta)} \mathbb{P}[X_T^{x/\beta,\pi} \geq F]$. Therefore, increasing β means reducing the initial capital for beating the same benchmark F, so (i) holds. Substituting x with βx, we obtain (ii). To show (iii), we write

$$\widetilde{V}(x; \beta) = \sup_{\pi \in \mathcal{A}(x)} \left(\mathbb{P}[X_T^{x,\pi} \geq \beta F, F = 0] + \mathbb{P}[X_T^{x,\pi} \geq \beta F, F > 0]\right)$$

$$= \mathbb{P}[F = 0] + \sup_{\pi \in \mathcal{A}(x)} \mathbb{P}[X_T^{x,\pi} \geq \beta F, F > 0]. \tag{3.5}$$

Focusing on the second term of (3.5), it suffices to consider an arbitrary strictly positive benchmark $F_+ > 0$. We deduce from (i) and $\widetilde{V}(0) = 0$ that

$$\lim_{\beta\to\infty} \sup_{\pi\in\mathcal{A}(x/\beta)} \mathbb{P}[X_T^{x/\beta,\pi} \geq F_+] = \lim_{x\to 0} \sup_{\pi\in\mathcal{A}(x)} \mathbb{P}[X_T^{x,\pi} \geq F_+] = 0.$$

This, together with (3.5), implies the limit (3.4).

Lastly, when the initial capital exceeds the superhedging price of β units of F, i.e., $x \geq \beta F_0$, the success probability $\widetilde{V}(x;\beta)$ equals 1, and hence (iv) holds. □

Proposition 3.1 points out that for any initial capital x, the success probability $\widetilde{V}(\beta x;\beta)$ stays constant whenever the initial capital and benchmark are simultaneously scaled by $\beta > 0$, and hence, there is *no economy of scale*.

Remark 3.2 For any fixed $x > 0$, the success probability $\widetilde{V}(x;\beta)$ is neither convex nor concave in β. This can be easily inferred from the properties of $\widetilde{V}$ shown in Proposition 3.1 and is illustrated in Fig. 1 below.

Next, we show that the portfolio optimization problem (3.2) admits a dual representation as a pure hypothesis testing problem. Such a connection was first pointed out by Föllmer and Leukert [11] in the context of quantile hedging.

Proposition 3.3 *The value function $\widetilde{V}(x)$ of (3.2) is equal to the solution of a pure hypothesis testing problem, that is, $\widetilde{V}(x) = V_1(x)$ where*

$$\begin{aligned} V_1(x) = &\sup_{A\in\mathcal{F}_T} \mathbb{P}[A] \\ \textit{subject to} \quad &\sup\nolimits_{Z\in\mathcal{Z}} \mathbb{E}[ZFI_A] \leq x. \end{aligned} \tag{3.6}$$

Furthermore, if there exists $\hat{A} \in \mathcal{F}_T$ that solves (3.6), then $\widetilde{V}(x) = \mathbb{P}[\hat{A}]$, and the associated optimal strategy π^* is a superhedging strategy with $X_T^{x,\pi^*} \geq FI_{\hat{A}}$ $\mathbb{P}$-a.s.

Proof If we set $\mathcal{H} = \{ZF : Z \in \mathcal{Z}\}$ and $\mathcal{G} = \{1\}$, then the right-hand side of (3.6) resembles the pure hypothesis testing problem in (2.11).

(1) First, we prove that $V_1(x) \geq \widetilde{V}(x)$. For an arbitrary $\pi \in \mathcal{A}(x)$, define the success event $A^{x,\pi} := \{X_T^{x,\pi} \geq F\}$. Then $\sup_{Z\in\mathcal{Z}} \mathbb{E}[ZFI_{A^{x,\pi}}]$ is the smallest amount needed to superhedge $FI_{A^{x,\pi}}$. By the definition of $A^{x,\pi}$, we have that $X_T^{x,\pi} \geq FI_{A^{x,\pi}}$, i.e., the initial capital x is sufficient to superhedge $FI_{A^{x,\pi}}$. This implies that $A^{x,\pi}$ is a candidate solution to V_1 since the constraint $x \geq \sup_{Z\in\mathcal{Z}} \mathbb{E}[ZFI_{A^{x,\pi}}]$ is satisfied. Consequently, for any $\pi \in \mathcal{A}(x)$, we have $V_1(x) \geq \mathbb{P}[A^{x,\pi}]$. Since $\widetilde{V}(x) = \sup_{\pi\in\mathcal{A}(x)} \mathbb{P}[A^{x,\pi}]$ by (3.2), we conclude.

(2) Now we show the reverse inequality $V_1(x) \leq \widetilde{V}(x)$. Let $A \in \mathcal{F}_T$ be an arbitrary set satisfying the constraint $\sup_{Z\in\mathcal{Z}} \mathbb{E}[ZFI_A] \leq x$. This implies a super-replication by some $\pi \in \mathcal{A}(x)$ such that $\mathbb{P}[X_T^{x,\pi} \geq FI_A] = 1$. In turn, this yields $\mathbb{P}[X_T^{x,\pi} \geq F] \geq \mathbb{P}[A]$. Therefore, $\widetilde{V}(x) \geq \mathbb{P}[A]$ by (3.2). Thanks to the arbitrariness of A, $\widetilde{V}(x) \geq V_1(x)$ holds.

In conclusion, $\widetilde{V}(x) = V_1(x)$. Moreover, if a set $\hat{A}$ satisfies $\widetilde{V}(x) = \mathbb{P}[\hat{A}]$, then the corresponding strategy π that superhedges FI_A is the solution of (3.2). □

Applying our analysis from Sect. 2.2, we seek to connect the outperformance portfolio optimization problem, via its pure hypothesis testing representation, to a randomized hypothesis testing problem. We first state an explicit example (see [19]) where outperformance portfolio optimization is equivalent to pure hypothesis testing by Proposition 3.3, but not to the randomized counterpart.

Example 3.4 Consider $\Omega = \{0, 1\}$, $\mathcal{F} = 2^{\{0,1\}}$, and the real probability given by $\mathbb{P}[0] = \mathbb{P}[1] = 1/2$. Suppose that the stock price follows a one-period binomial tree,

$$S_0(0) = S_0(1) = 2; \qquad S_T(0) = 5, \qquad S_T(1) = 1.$$

We take as benchmark $F = 1$ at T. We determine by direct computation the maximum success probability given an initial capital $x \geq 0$. To this end, we notice that any possible strategy with initial capital x is c shares of stock plus $x - 2c$ dollars of cash at $t = 0$. Then the terminal wealth X_T is

$$X_T = \begin{cases} 5c + (x - 2c) = x + 3c, & \omega = 0, \\ c + (x - 2c) = x - c, & \omega = 1. \end{cases}$$

Due to the nonnegative wealth constraint $X_T \geq 0$ a.s., we require that $-\frac{x}{3} \leq c \leq x$. Now, we can write $\widetilde{V}(x)$ as

$$\widetilde{V}(x) = \max_{-\frac{x}{3} \leq c \leq x} \mathbb{P}[X_T \geq 1] = \frac{1}{2} \max_{-\frac{x}{3} \leq c \leq x} \big(I_{\{x+3c \geq 1\}} + I_{\{x-c \geq 1\}}\big). \tag{3.7}$$

As a result, for different values of initial capital x, we have:

1. If $x < 1/4$, then

   $$x + 3c \leq x + 3x = 4x < 1$$

   and

   $$x - c \leq x + \frac{x}{3} = \frac{4x}{3} < 1/3,$$

   which implies that both indicators are zero, i.e., $\widetilde{V}(x) = 0$.
2. If $1/4 \leq x < 1$, then we can take $c = 1/4$, which leads to $x + 3c \geq 1$, i.e., $\widetilde{V}(x) \geq 1/2$. On the other hand, $\widetilde{V}(x) < 1$. From this and from (3.7) we conclude that $\widetilde{V}(x) = 1/2$.
3. If $x \geq 1$, then we can take $c = 0$, and $\widetilde{V}(x) = 1$.

With reference to the value functions $V(x)$ (randomized hypothesis testing) and $V_1(x)$ (pure hypothesis testing) from Example 2.4, we conclude that $\widetilde{V}(x) = V_1(x) \neq V(x)$.

As in Theorem 2.10, we now provide sufficient conditions for the equivalence between outperformance portfolio optimization and randomized hypothesis testing.

Theorem 3.5 *Suppose that one of the two conditions below is satisfied*:

1. $\mathcal{Z}$ *is a singleton, and there exists an* $\mathcal{F}_T$*-measurable random variable with continuous cumulative distribution function under* $\mathbb{P}$.
2. *For all* $a \in (0, \infty)$, *the minimizer* $\hat{Z}_a := \arg\min \mathbb{E}[xa + (1 - aZF)^+]$ *satisfies* $\mathbb{P}[a\hat{Z}_a F = 1] = 0$.

Then:

(i) *The value function* $\widetilde{V}(x)$ *of* (3.2) *admits the representation*

$$\widetilde{V}(x) = \inf_{a \geq 0, Z \in \mathcal{Z}} \mathbb{E}[xa + (1 - aZF)^+]. \tag{3.8}$$

(ii) $\widetilde{V}(x)$ *is continuous, concave, and nondecreasing in* $x \in [0, \infty)$, *taking values from the minimum* $\widetilde{V}(0) = \mathbb{P}[F = 0]$ *to the maximum* $\widetilde{V}(x) = 1$ *for* $x \geq F_0$.

Proof Proposition 3.3 implies that $\widetilde{V}(x)$ is equal to the value $V_1(x)$ of the pure testing problem with $\mathcal{H} := \{FZ : Z \in \mathcal{Z}\}$ and $\mathcal{G} := \{1\}$. Since conditions 1 and 2 imply (C1) and (C3) of Theorem 2.3, respectively, this also implies that $V_1(x)$ of pure testing is equal to $V(x)$ of randomized testing for all $x \geq 0$. Note that $F_0 < \infty$ implies that $\mathcal{H}$ is L^1-bounded. Hence, Assumption 2.2 is satisfied along with the convexity of the set $\mathcal{H}$. Thus, the representation (3.8) follows directly from (2.8) of Theorem 2.3.

It remains to observe from (3.8) that $\widetilde{V}(x) \leq 1$ by taking $a = 0$. When $x = 0$, the success event coincides with $\{F = 0\}$, so the lower bound is $\widetilde{V}(0) = \mathbb{P}[F = 0]$. □

Remark 3.6 Condition 1 of Theorem 3.5, together with (2.5), recovers Proposition 2.1 by Spivak and Cvitanić [30] with zero maintenance margin, (i.e., $A = 0$ in Eq. (2.30) of [30]). Furthermore, our pure test in (2.13) also reveals the structure of their set E.

In Theorem 3.5, condition 2 is typical in the quantile hedging literature (see, e.g., [11, 17]), but it can be violated even in the simple Black–Scholes model; see Sect. 3.2.1 (case 1). In such cases, one may alternatively check condition 1 in order to apply Theorem 3.5.

In the following sections, we discuss the applications of this result in both complete and incomplete diffusion market models.

3.2 A complete market model

Let W be a standard Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$. The financial market consists of a liquid risky stock and a riskless money market account. For notational simplicity, we assume a zero interest rate, which amounts to expressing cash flows in the money market account numeraire. Under the historical measure, the stock price evolves according to

$$dS_t = S_t \sigma(S_t)\big(\theta(S_t)\,dt + dW_t\big),$$

where $\theta(\cdot)$ is the Sharpe ratio function, and $\sigma(\cdot)$ is the volatility function. We assume that both $\theta(\cdot)$ and $\sigma(\cdot)$ satisfy Lipschitz-continuity and uniform boundedness on the domain $\mathbb{R}$ throughout Sect. 3.2. For any admissible strategy $\pi \in \mathcal{A}(x)$, the investor's wealth process associated with strategy π and initial capital x is given by

$$dX_t^{x,\pi} = \pi_t S_t \sigma(S_t)\big(\theta(S_t)\,dt + dW_t\big).$$

The investor's objective is to maximize the probability of beating the benchmark $F = f(S_T)$ for some measurable function f of at most linear growth. Since a perfect replication is possible by trading S and the money market account, the market is complete, and there exists a unique EMM $\mathbb{Q}$ defined by

$$Z_t := \frac{d\mathbb{Q}}{d\mathbb{P}}\bigg|_{\mathcal{F}_t} = \exp\bigg(-\frac{1}{2}\int_0^t \theta^2(S_u)\,du - \int_0^t \theta(S_u)\,dW_u\bigg).$$

Moreover, the superhedging price is simply the risk-neutral value $F_0 = \mathbb{E}^{\mathbb{Q}}[f(S_T)]$, which is a special case of (3.1). Given an initial capital $x < F_0$, the investor faces the optimization problem

$$\widetilde{V}(x) = \sup_{\pi \in \mathcal{A}(x)} \mathbb{P}[X_T^{x,\pi} \geq f(S_T)]. \tag{3.9}$$

Proposition 3.7 *$\widetilde{V}(x)$ is a continuous, nondecreasing, and concave function in x. It admits the dual representation*

$$\widetilde{V}(x) = \inf_{a \geq 0} \big\{xa + \mathbb{E}\big[\big(1 - aZ_T f(S_T)\big)^+\big]\big\}. \tag{3.10}$$

Proof First, Proposition 3.3 implies $\widetilde{V}(x) = V_1(x)$ (from pure hypothesis testing). Also, since $\mathcal{Z} = \{Z\}$ is a singleton and W_T has a continuous c.d.f. with respect to $\mathbb{P}$, the first condition of Theorem 3.5 yields the equivalence of pure and randomized hypothesis testing, i.e., $\widetilde{V}(x) = V_1(x) = V(x)$. □

For computing the value of $\widetilde{V}(x)$ in this complete market model, Proposition 3.7 turns the original stochastic control problem (3.9) into a static optimization (over $a \geq 0$) in (3.10). In the dual representation, the expectation can be interpreted as pricing a claim under the measure $\mathbb{Q}$, namely,

$$q(a) := \mathbb{E}^{\mathbb{Q}}\big[\big(Z_T^{-1} - af(S_T)\big)^+\big].$$

Hence, $\widetilde{V}(x)$ is the Legendre transform, evaluated at x, of the price function $q(a)$.

3.2.1 Benchmark based on the traded asset

In this section, we assume that θ and σ are constant, so S is a geometric Brownian motion (GBM). We consider a class of benchmarks of the form $f(S_T) = \beta S_T^p$ for $\beta > 0$, $p \in \mathbb{R}$. This includes the constant benchmark ($p = 0$) and those based on multiples of the traded asset S ($p = 1$) and its power.

One interpretation of power-type benchmarks is in terms of leveraged exchange traded funds (ETFs). ETFs are investment funds liquidly traded on stock exchanges. They provide leverage, access, and liquidity to investors for various asset classes and typically involve strategies with a constant leverage (e.g., double-long/short). They also serve as benchmarks for fund managers. Since its introduction in the mid-1990s, the ETF market has grown to over 1000 funds with aggregate value exceeding \$1 trillion.

Specifically, a long-leveraged ETF $(L_t)_{t\geq 0}$ based on the underlying asset S with a constant leverage factor $p\geq 0$ is constructed by investing p times the fund value pL_t in S and borrowing $(p-1)L_t$ from the bank. The resulting fund price L satisfies the SDE (see [1] and [15])

$$dL_t = pL_t\frac{dS_t}{S_t} = L_t(p\theta\sigma\, dt + p\sigma\, dW_t). \tag{3.11}$$

For a short-leveraged fund $p\leq 0$, the manager shorts the amount $-pL_t$ of S and keeps $(-p+1)L_t$ in the bank. The fund price L again satisfies the SDE (3.11) with $p\leq 0$. Hence, L is again a GBM and can be expressed in terms of S as

$$\frac{L_t}{L_0} = \left(\frac{S_t}{S_0}\right)^p \exp\left(\frac{p(1-p)\sigma^2}{2}t\right). \tag{3.12}$$

As a result, the objective to outperform a p-leveraged ETF L_T leads to a special example of the power benchmark $\hat{\beta}S_T^p$ with $\hat{\beta} = L_0 S_0^{-p}\exp(\frac{p(1-p)\sigma^2}{2}T)$. In practice, typical leverage factors are $p = 1, 2, 3$ (long) and $-1, -2, -3$ (short).

In general, for any (β, p), the risk-neutral price of the benchmark $f(S_T) = \beta S_T^p$ is

$$F_0 = \beta S_0^p \exp\left(\frac{\sigma^2}{2}p(p-1)T\right). \tag{3.13}$$

Clearly, if $x\geq F_0$, the success probability is 1, so the challenge is to achieve an outperformance using less initial capital. Then a direct computation using (3.10) and (3.13) yields that

$$\widetilde{V}(x) = \inf_{a\geq 0}\left\{xa + \mathbb{E}\left[\left(1 - aF_0\exp\left(-\frac{1}{2}(p\sigma-\theta)^2T + (p\sigma-\theta)W_T\right)\right)^+\right]\right\}. \tag{3.14}$$

To solve for $\widetilde{V}(x)$, we divide the problem into two cases:

1. If $p\sigma = \theta$, then $ZF = F_0$ a.s., so condition 2 in Theorem 3.5 is violated, but condition 1 holds and is used. Consequently, (3.14) simplifies to

$$\widetilde{V}(x) = \inf_{a\geq 0}\{xa + (1-aF_0)^+\} = \begin{cases} 1 & \text{if } x\geq F_0,\\ x/F_0 & \text{if } x < F_0,\end{cases} \tag{3.15}$$

   and the corresponding minimizers are $\hat{a} = 0$ and $\hat{a} = F_0^{-1}$, respectively.

2. If $p\sigma \neq \theta$, then $\widetilde{V}(x) = 1$ if $x \geq F_0$; otherwise, direct computations yield that

$$\widetilde{V}(x) = \inf_{a\geq 0} \left\{ xa + \Phi\big(d_2(a; p\sigma - \theta)\big) - aF_0\Phi\big(d_1(a; p\sigma - \theta)\big)\right\} \tag{3.16}$$

$$= x\hat{a} + \Phi\big(d_2(\hat{a}; p\sigma - \theta)\big) - \hat{a}F_0\Phi\big(d_1(\hat{a}; p\sigma - \theta)\big), \tag{3.17}$$

where d_i are

$$d_1(a; z) = \frac{-\ln(aF_0) - 0.5Tz^2}{|z|\sqrt{T}}, \qquad d_2(a; z) = \frac{-\ln(aF_0) + 0.5Tz^2}{|z|\sqrt{T}}. \tag{3.18}$$

Note that the infimum is reached at $\hat{a}$ that solves

$$\mathbb{E}\left[F_0 \hat{H} I_{\{\hat{a}F_0\hat{H}<1\}}\right] = x, \tag{3.19}$$

where $\hat{H} = \exp(-\frac{1}{2}(p\sigma - \theta)^2 T + (p\sigma - \theta)W_T)$. Let $d\tilde{\mathbb{Q}} = \hat{H}\, d\mathbb{P}$; then (3.19) implies that

$$\tilde{\mathbb{Q}}\left[\hat{H} < \frac{1}{\hat{a}F_0}\right] = \frac{x}{F_0},$$

which is equivalent to

$$\tilde{\mathbb{Q}}\left[(p\sigma - \theta)\big(W_T + (\theta - p\sigma)T\big) < -\ln(\hat{a}F_0) - \frac{1}{2}(p\sigma - \theta)^2 T\right] = \frac{x}{F_0}.$$

Since $W_T + (\theta - p\sigma)T \sim \mathcal{N}(0, T)$ under $\tilde{\mathbb{Q}}$, the optimal $\hat{a}$ is given by

$$\hat{a} = h\big(\Phi^{-1}(x/F_0)\big), \tag{3.20}$$

where

$$h(y) = \exp\big(-y|p\sigma - \theta|\sqrt{T} - 0.5(p\sigma - \theta)^2 T - \ln F_0\big).$$

In the above example, one can also compute the initial capital needed to achieve a prespecified success probability simply by inverting $\tilde{V}(x)$ in (3.16) and (3.15); see Fig. 1(a). Also, note that $\tilde{V}(x)$ depends on β via F_0 in (3.13). In Fig. 1(b) we see that $\widetilde{V}(x;\beta)$ decreases from 1 to 0 as β increases to infinity, which is consistent with the limit (3.4).

While the superhedging price F_0 is computed from $\mathbb{Q}$, the maximal success probability $\widetilde{V}(x)$ is based on the historical measure $\mathbb{P}$. In other words, as we vary the Sharpe ratio θ, the required initial capital x to achieve a given success probability will change, but F_0, the cost to guarantee outperformance, remains unaffected; see Fig. 1(a).

In Fig. 2, we look at the probability to outperform an ETF under different leverages. From (3.12) we note that $F_0 = \mathbb{E}^{\mathbb{Q}}[L_T] = L_0$. Then we apply formula (3.17) to obtain the success probability $\widetilde{V}(x)$ for different values of capital x and leverage p. As shown, for every fixed x, moving the leverage p further away from zero increases the success probability. In other words, for any fixed success probability, highly (long/short) leveraged ETFs require lower initial capital for the outperformance portfolio. The comparison between long and short ETFs with the same mag-

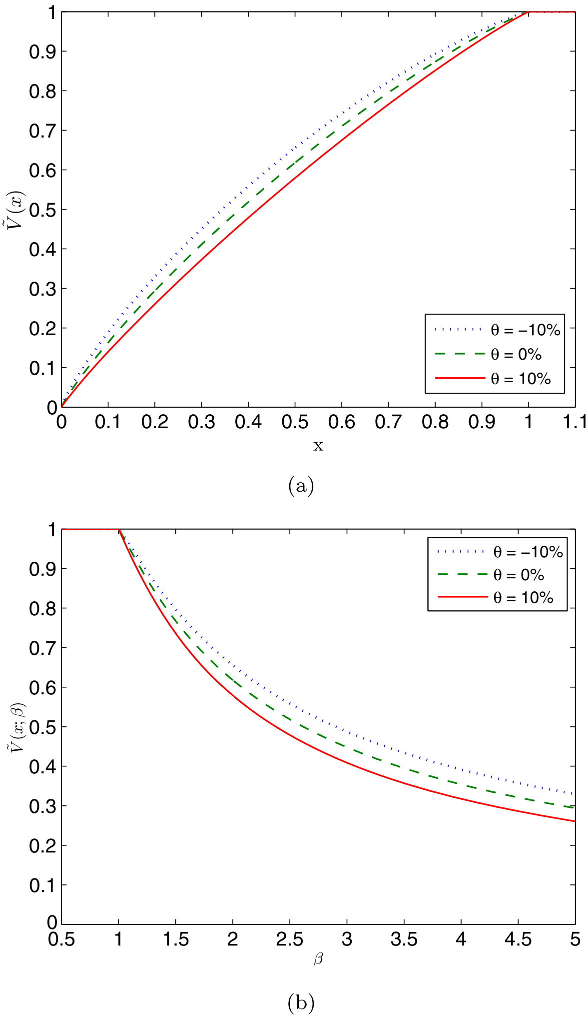

Fig. 1 The benchmark is $F(S_T) = \beta S_T$, and the default parameters are $S_0 = 1$, $\sigma = 30\ \%$, and $T = 1$. (*Top*) With $\beta = 1$, the maximum success probability $\widetilde{V}(x)$ increases with initial capital x, and plateaus at 1 when $x > S_0$. For any fixed success probability, a lower Sharpe ratio θ requires a lower initial capital x. (*Bottom*) With initial capital $x = 1$, $\widetilde{V}(x;\beta)$ takes value 1 and then decreases to 0 as β increases to infinity. Observe that $\widetilde{V}(x;\beta)$ is not simply convex or concave even over the range $[0.5, 5]$ of β and converges to 0 as $\beta \to \infty$ according to (3.4)

nitude of leverage $|p|$ depends on the sign of θ. In particular, we observe from (3.17) and (3.20) that when $\theta = 0$, the success probability $\widetilde{V}(x)$ is the same for $\pm p$, and the surface $\widetilde{V}(x)$ is symmetric around $p = 0$.

Remark 3.8 In a related study, Föllmer and Leukert [11, Sect. 3] considered quantile hedging for a call option in the Black–Scholes market. Their solution method involves first conjecturing the form of the success events under two scenarios. Alternatively, one can also study the quantile hedging problem via randomized hypothesis testing. From (3.10) we can compute the maximal success probability from $\widetilde{V}(x) = \inf_{a\geq 0}\{xa + \mathbb{E}[(1 - aZ_T(S_T - K)^+)^+]\}$, which will yield exactly the same closed-form result in [11, Eqs. (3.15) and (3.27)]. This approach eliminates the need to conjecture a priori the success events.

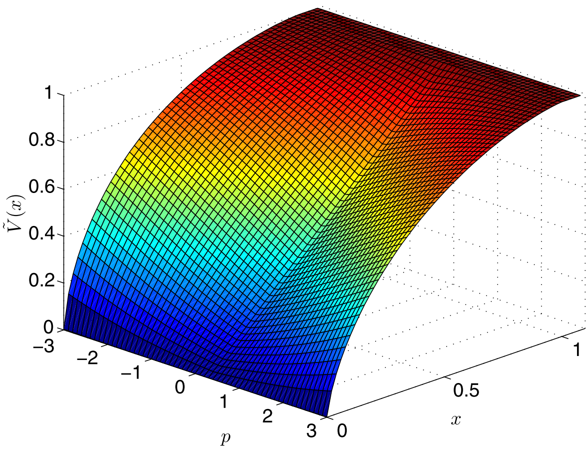

Fig. 2 Outperformance probability surface over leverage p and initial capital x. For any fixed x, the probability $\widetilde{V}(x)$ increases as leverage p increases/decreases from zero. This means that highly leveraged ETFs are easier benchmarks to beat

3.3 A stochastic factor model

Let $(W, \hat{W})$ be a two-dimensional standard Brownian motion on a given filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$. We consider a liquid stock whose price follows the SDE

$$dS_t = S_t\sigma(Y_t)\big(\theta(Y_t)\,dt + dW_t\big), \tag{3.21}$$

where θ is the Sharpe ratio function, and the stochastic factor Y follows

$$dY_t = b(Y_t)\,dt + c(Y_t)\big(\rho\, dW_t + \sqrt{1-\rho^2}\,d\hat{W}_t\big). \tag{3.22}$$

This is a standard stochastic factor/volatility model that can be found in [23, 29], among others. The parameter $\rho \in (-1, 1)$ accounts for the correlation between S and Y. In addition, we assume that the functions $\sigma(\cdot)$, $\theta(\cdot)$, $b(\cdot)$, and $c(\cdot)$ satisfy the Lipschitz-continuity and uniform boundedness on the domain $\mathbb{R}$.

With initial capital x and strategy $\pi \in \mathcal{A}(x)$, the wealth process satisfies

$$dX_t^{x,\pi} = \pi_t S_t\sigma(Y_t)\big(\theta(Y_t)\,dt + dW_t\big).$$

Let Λ^a denote the collection of all $(\mathcal{F}_t)$-progressively measurable processes $\lambda : (0, T) \times \Omega \to \mathbb{R}$ satisfying $\int_0^T \lambda_t^2 dt < \infty$ almost surely in $\mathbb{P}$, and

$$\Lambda^b = \Big\{\lambda \in \Lambda^a : \operatorname*{ess\,sup}_{0\le t\le T} |\lambda_t| < \infty \text{ in } \mathbb{P}\Big\}$$

be the subset of Λ^a consisting of almost surely bounded processes. Define, for any $\lambda \in \Lambda^a$,

$$\tilde{Z}_T^{\lambda} = \exp\bigg(-\frac{1}{2}\int_0^T \theta^2(Y_t)\,dt - \int_0^T \theta(Y_t)\,dW_t - \frac{1}{2}\int_0^T \lambda_t^2\,dt - \int_0^T \lambda_t\,d\hat{W}_t\bigg). \tag{3.23}$$

Recall that $\mathcal{Z}$ is the collection of Radon–Nikodým densities between equivalent local martingale measures and the historical probability measure $\mathbb{P}$. We also define a set

Λ by $\Lambda = \{\lambda \in \Lambda^a : \tilde{Z}_T^\lambda \in \mathcal{Z}\}$. For any $\lambda \in \Lambda^b$, it is easy to check $\mathbb{E}[\tilde{Z}_T^\lambda] = 1$ by the Novikov condition, and this implies $\Lambda^b \subset \Lambda \subset \Lambda^a$. The process $\lambda \in \Lambda$ is commonly referred to as the risk premium for the nontraded Brownian motion $\hat{W}$. In particular, the choice of $\lambda = 0$ results in the minimal martingale measure (MMM) $\mathbb{Q}^0$ (see [13]).

3.3.1 The role of the minimal martingale measure

Let us consider a benchmark of the form $F = \beta S_T^\delta$, where $\delta \in \{0, 1\}$. This includes the constant and stock benchmarks. Following (3.2), we consider the optimization problem

$$\widetilde{V}(x) = \sup_{\pi \in \mathcal{A}(x)} \mathbb{P}[X_T^{x,\pi} \geq \beta S_T^\delta]. \tag{3.24}$$

Proposition 3.9 *Suppose that $c_1 < |\theta(y) - \delta\sigma(y)| < c_2$ for all $(y, \delta) \in \mathbb{R} \times \{0, 1\}$ and some positive constants c_1 and c_2. Then the value function $\widetilde{V}(x)$ in (3.24) is a nondecreasing, continuous, and concave function satisfying*

$$\widetilde{V}(x) = \inf_{a \geq 0} \big\{xa + \mathbb{E}\big[(1 - a\beta S_0^\delta \tilde{Z}_T^0)^+\big]\big\}. \tag{3.25}$$

To show this, we use the following result, which is a variation of [16, Exercise 2.3.2.3], and the proofs of (5.3) and (5.6) in [7].

Lemma 3.10 *Let B be a standard Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$, and $a, b \in \Lambda^b$ such that*

$$\int_0^T a_t^2\, dt \geq \int_0^T b_t^2\, dt \quad \mathbb{P}\text{-a.s.}$$

Define, for $0 \leq t \leq T$, the two processes

$$Z_t^a := \exp\bigg(-\frac{1}{2}\int_0^t a_u^2\, du - \int_t^t a_u\, dB_u\bigg),$$

$$Z_t^b := \exp\bigg(-\frac{1}{2}\int_0^t b_u^2\, du - \int_t^t b_u\, dB_u\bigg).$$

For any convex function $\psi : \mathbb{R} \to \mathbb{R}$, we have

$$\mathbb{E}[\psi(Z_T^a)] \geq \mathbb{E}[\psi(Z_T^b)].$$

Proof Define

$$\tau^a(s) := \inf\bigg\{t \geq 0 : \int_0^t a_u^2\, du > s\bigg\}, \qquad \tau^b(s) := \inf\bigg\{t \geq 0 : \int_0^t b_u^2\, du > s\bigg\}.$$

Then, since the processes $\int a_u\, dB_u$ and $\int b_u\, dB_u$ are local martingales, the time-changed processes

$$B_t^a := \int_0^{\tau^a(t)} a_u\, dB_u, \qquad B_t^b := \int_0^{\tau^b(t)} b_u\, dB_u$$

are standard Brownian motions adapted to the time-changed filtrations $\{\mathcal{F}_{\tau^a(t)} : t > 0\}$ and $\{\mathcal{F}_{\tau^b(t)} : t > 0\}$ on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$, respectively. Define

$$T^a := \int_0^T a_u^2\,du, \qquad T^b := \int_0^T b_u^2\,du.$$

Then it follows that $\tau^a(T^a) = \tau^b(T^b) = T$, and

$$\mathbb{E}[\psi(Z_T^a)] = \mathbb{E}\bigg[\psi\bigg(\exp\bigg(-\frac{1}{2}T^a - B^a_{T^a}\bigg)\bigg)\bigg],$$
$$\mathbb{E}[\psi(Z_T^b)] = \mathbb{E}\bigg[\psi\bigg(\exp\bigg(-\frac{1}{2}T^b - B^b_{T^b}\bigg)\bigg)\bigg].$$

With the martingale $\exp(-\frac{1}{2}t - B_t^a)$ and the convex function ψ, Jensen's inequality implies that $\psi(\exp(-\frac{1}{2}t - B_t^a))$ is a submartingale. Moreover, $C \ge T^a \ge T^b$ for some constant C almost surely with respect to $\mathbb{P}$. Therefore, $\mathbb{E}[\psi(Z_T^a)] \ge \mathbb{E}[\psi(Z_T^b)]$. □

Proof of Proposition 3.9 Applying Theorem 3.5, the associated randomized hypothesis testing is given by

$$\widetilde{V}(x) = \inf_{a\ge 0,\lambda\in\Lambda} \{xa + \mathbb{E}[(1 - a\beta \tilde{Z}_T^\lambda S_T^\delta)^+]\},$$

where, according to (3.21) and (3.23),

$$\begin{aligned}\tilde{Z}_T^\lambda S_T^\delta = S_0^\delta \exp\bigg(\delta(\delta-1)\int_0^T \frac{\sigma^2(Y_t)}{2}\,dt\bigg)\exp\bigg(-\int_0^T \frac{\lambda_t^2}{2}\,dt - \int_0^T \lambda_t\,d\hat{W}_t\bigg)\\ \times \exp\bigg(-\int_0^T \frac{(\delta\sigma(Y_t) - \theta(Y_t))^2}{2}\,dt - \int_0^T \big(\theta(Y_t) - \delta\sigma(Y_t)\big)\,dW_t\bigg).\end{aligned}$$

Note that for $\delta \in \{0, 1\}$, $\tilde{Z}^\lambda S^\delta$ can be rewritten as

$$\tilde{Z}_T^\lambda S_T^\delta = S_0^\delta \exp\bigg(-\int_0^T \frac{\alpha_t^2 + \lambda_t^2}{2}\,dt - \int_0^T \sqrt{\alpha_t^2 + \lambda_t^2}\,dB_t\bigg),$$

where $\alpha_t := \theta(Y_t) - \delta\sigma(Y_t)$, and B is the standard Brownian motion defined by

$$dB_t = \frac{-\alpha_t\,dW_t - \lambda_t\,d\hat{W}_t}{\sqrt{\alpha_t^2 + \lambda_t^2}}.$$

Hence, for each $\lambda \in \Lambda$, the process $\tilde{Z}^\lambda S^\delta$ is in fact a $\mathbb{P}$-martingale for $\delta \in \{0, 1\}$ due to the boundedness of the volatility σ. On the other hand, the representation of $\widetilde{V}(x)$ at the beginning of this proof also applies if one changes the domain of the infimum from Λ to the smaller set Λ^b, i.e.,

$$\widetilde{V}(x) = \inf_{a\ge 0,\lambda\in\Lambda^b} \{xa + \mathbb{E}[(1 - a\beta \tilde{Z}_T^\lambda S_T^\delta)^+]\}.$$

Indeed, the above representation of $\tilde{V}(x)$ is a direct consequence of the following statement: For each $\lambda \in \Lambda$, there exists a family $\{\lambda^\beta \in \Lambda^b : \beta > 0\}$ satisfying

$$\lim_{\beta\to\infty} \mathbb{E}[(1 - a\beta \tilde{Z}_T^{\lambda^\beta} S_T^\delta)^+] = \mathbb{E}[(1 - a\beta \tilde{Z}_T^{\lambda} S_T^\delta)^+].$$

To see this, one can take $\lambda^\beta \in \Lambda^b$ as a stopped process of the form $\lambda_t^\beta = \lambda_{t\wedge\tau^{\lambda,\beta}}$ for $\tau^{\lambda,\beta} = \inf\{t > 0 : |\lambda_t| \geq \beta\}$ and then apply the dominated convergence theorem. In view of Lemma 3.10, the minimizer of $\inf_{\lambda\in\Lambda^b} \mathbb{E}[(1 - a\beta \tilde{Z}_T^{\lambda} S_T^\delta)^+]$ is $\hat{\lambda} \equiv 0$ for any fixed $a \geq 0$, so we conclude (3.25). On the other hand, since α^2 is a positive process bounded away from zero, applying Proposition A.5 and Girsanov's theorem, we have $\mathbb{P}[-\int_0^T \frac{1}{2}\alpha_t^2\, dt - \int_0^T \alpha_t\, dB_t = c] = 0$, and hence $\mathbb{P}[\tilde{Z}^{\hat{\lambda}_T} S_T^\delta = c] = 0$ for any constant c and $\delta \in \{0, 1\}$. So we have verified the second condition of Theorem 3.5 and conclude that $\widetilde{V}(x) = V_1(x) = V(x)$ together with Proposition 3.3. □

Proposition 3.9 shows that among all candidate EMMs, the MMM $\mathbb{Q}^0$ is optimal for $\widetilde{V}(x)$. In other words, when the benchmark is a constant or the final stock price S_T, the objective to maximize the outperformance probability induces the investor to assign a zero risk premium ($\lambda = 0$) for the second Brownian motion $\hat{W}$ under the stochastic factor model (3.21), (3.22). Interestingly, this is true for all choices of θ, σ, b, c, and ρ for (S, Y). Furthermore, if $\alpha_t = \theta(Y_t) - \delta\sigma(Y_t)$ is constant, then the expectation in (3.25) and hence the success probability $\widetilde{V}(x)$ can be computed explicitly.

Corollary 3.11 *Suppose that $\theta(Y_t) - \delta\sigma(Y_t) = \alpha$ for some constant $\alpha \in \mathbb{R} \setminus \{0\}$. Then $\widetilde{V}(x)$ is given by*

$$\widetilde{V}(x) = \begin{cases} 1 & \text{if } x \geq \beta S_0^\delta, \\ x\hat{a} + \Phi(d_2(\hat{a}; -\alpha)) - \hat{a} S_0^\delta \Phi(d_1(\hat{a}; -\alpha)) & \text{if } x < \beta S_0^\delta, \end{cases}$$

where d_1 and d_2 are given in (3.18), *and $\hat{a}$ in* (3.20).

3.3.2 General benchmark and the HJB characterization

More generally, consider a stochastic benchmark in the form $F = f(S_T, Y_T)$ for some measurable function f. The outperformance portfolio optimization value is

$$\widetilde{V}(t, s, x, y) = \sup_{\pi\in\mathcal{A}(x)} \mathbb{P}^{t,s,x,y}[X_T^{x,\pi} \geq f(S_T, Y_T)]$$

with the notation $\mathbb{P}^{t,s,x,y}[\cdot] = \mathbb{P}[\cdot\, |S_t = s, X_t = x, Y_t = y]$. We define

$$U(t, s, y, z) := \inf_{\lambda\in\Lambda_t} \mathbb{E}^{t,s,y}\big[\big(1 - Z_T^{z,\lambda} f(S_T, Y_T)\big)^+\big], \tag{3.26}$$

where $\Lambda_t = \{(\lambda_s)_{t\leq s\leq T} | \lambda \in \Lambda\}$, $\mathbb{E}^{t,s,y}[\,\cdot\,] = \mathbb{E}[\,\cdot\, |S_t = s, Y_t = y]$, and Z is given by

$$Z_u^{z,\lambda} = z + \int_t^u Z_v^{z,\lambda}\big(-\theta(Y_v)\, dW_v - \lambda_v\, d\hat{W}_v\big). \tag{3.27}$$

In view of Theorem 3.5, if $\mathbb{P}[Z_T^{a,\lambda} f(S_T, Y_T) = 1] = 0$ for all a, then we have

$$\widetilde{V}(t,s,x,y) = \inf_{a\geq 0}\left\{xa + \inf_{\lambda\in\Lambda_t} \mathbb{E}^{t,s,y}\left[\left(1 - aZ_T^{1,\lambda} f(S_T, Y_T)\right)^+\right]\right\}$$

$$= \inf_{a\geq 0}\left\{xa + \inf_{\lambda\in\Lambda_t} \mathbb{E}^{t,s,y}\left[\left(1 - Z_T^{a,\lambda} f(S_T, Y_T)\right)^+\right]\right\} \tag{3.28}$$

$$= \inf_{a\geq 0}\{xa + U(t,s,y,a)\}. \tag{3.29}$$

We specify the associated HJB PDE for U. To this end, we define for any $\lambda \in \mathbb{R}$ the differential operator

$$\begin{aligned}\mathcal{L}^\lambda w &= s\theta(y)\sigma(y)w_s + \frac{1}{2}s^2\sigma^2(y)w_{ss} + b(y)w_y + \frac{1}{2}c^2(y)w_{yy}\\ &\quad + \frac{1}{2}\left(\theta^2(y) + \lambda^2\right)z^2 w_{zz} + s\sigma(y)c(y)\rho w_{sy}\\ &\quad - sz\sigma(y)\theta(y)w_{sz} + zc(y)\left(-\theta(y)\rho - \lambda\sqrt{1-\rho^2}\right)w_{yz}.\end{aligned}$$

Define the domains $\mathcal{O} = (0,\infty)\times(-\infty,\infty)\times(0,\infty)$, $\mathcal{O}_T = (0,T)\times\mathcal{O}$. Also, denote by $C^{1,2}(\mathcal{O}_T)$ the collection of all functions on $\mathcal{O}_T$ that are continuously differentiable in t and twice continuously differentiable in (s,y,z).

First, we have the standard verification theorem, which presumes the existence of a classical solution.

Theorem 3.12 *If there exists $w \in C^{1,2}(\mathcal{O}_T)\cap C(\overline{\mathcal{O}}_T)$ satisfying the PDE*

$$w_t + \inf_{\lambda\in\mathbb{R}} \mathcal{L}^\lambda w = 0 \tag{3.30}$$

with $w(T,s,y,z) = (1 - zf(s,y))^+$, then $w \leq U$ on $\mathcal{O}_T$. Let $\hat{\lambda} : \mathbb{R}^4 \to \mathbb{R}$ be a function satisfying

$$\mathcal{L}^{\hat{\lambda}(t,s,y,z)} w(t,s,y,z) = \inf_{\lambda\in\mathbb{R}} \mathcal{L}^\lambda w(t,s,y,z) = 0. \tag{3.31}$$

If there exists a unique solution $\hat{Z}$ for Eq. (3.27) with λ_ν of (3.27) being replaced by $\hat{\lambda}(\nu, S_\nu, Y_\nu, \hat{Z}_\nu)$, then $w = U$ on $\mathcal{O}_T$.

Furthermore, if $\mathbb{P}[Z_T^{a,\lambda} f(S_T, Y_T) = 1] = 0$ for all a, then there exists some $\hat{a} = \hat{a}(t,s,x,y)$ which solves

$$\mathbb{E}^{t,s,y}\left[Z_T^{\hat{a},\hat{\lambda}} f(S_T, Y_T) I_{\{Z_T^{\hat{a},\hat{\lambda}} f(S_T,Y_T)<1\}}\right] = \hat{a}x, \tag{3.32}$$

and

$$\widetilde{V}(t,s,x,y) = \mathbb{P}^{t,s,x,y}[Z_T^{\hat{a},\hat{\lambda}} f(S_T, Y_T) < 1]. \tag{3.33}$$

Proof We follow the standard arguments for verification theorems (Theorem 5.5.1 of [31]). First, for any (S, Y, Z^λ) with initial value (s,y,z) at time t, we have

$$\begin{aligned}&w(t,s,y,z)+\mathbb{E}^{t,s,y,z}\left[\int_t^T \mathcal{L}^\lambda w(\nu,S_\nu,Y_\nu,Z_\nu)\,d\nu\right]\\&=\mathbb{E}^{t,s,y,z}[w(T,S_T,Y_T,Z_T)]\\&=\mathbb{E}^{t,s,y}\big[\big(1-Z_T^{z,\lambda}f(S_T,Y_T)\big)^+\big].\end{aligned}$$

The last equality above holds by the terminal condition of the PDE. Also observe that $\mathbb{E}^{t,s,y,z}[\int_t^T \mathcal{L}^\lambda w(\nu,S_\nu,Y_\nu,Z_\nu)\,d\nu]$ is always nonnegative, and so we have

$$w(t,s,y,z)\le \mathbb{E}^{t,s,y}\big[\big(1-Z_T^{z,\lambda}f(S_T,Y_T)\big)^+\big].$$

So we conclude $w\le U$ by arbitrariness of λ. On the other hand, if we take $\hat\lambda$ of (3.31) in the above, then it yields instead of inequality the equality

$$w(t,s,y,z)=\mathbb{E}^{t,s,y}\big[\big(1-Z_T^{z,\lambda}f(S_T,Y_T)\big)^+\big].$$

By the definition (3.26), the right-hand side is always greater than or equal to U, and this implies $w\ge U$.

Applying (3.28) and (3.29), the optimizer $\hat a$ for $V(t,s,x,y)$ is derived from (2.6) of Theorem 2.3 with $\hat H=Z_T^{\hat a,\hat\lambda}f(S_T,Y_T)$ and $\hat X=I_{\{Z_T^{\hat a,\hat\lambda}f(S_T,Y_T)<1\}}$. In turn, this yields (3.32) and (3.33) via (2.8). □

Recall our assumption on the functions σ,θ,b,c. In general, the HJB equation need not have a classical solution. However, one can show that U of (3.26) is the unique solution of the HJB equation (3.30) in the viscosity sense.

Proposition 3.13 *The dual function U in (3.26) is the unique bounded continuous viscosity solution of (3.30) with final condition $w(T,s,y,z)=(1-zf(s,y))^+$ for all $(s,y,z)\in\mathcal{O}$.*

Proof First, it can be shown that U is a viscosity subsolution (resp. supersolution) by using the Feynman–Kac formula on its super- (resp. sub-) test functions. For details, we refer to the similar proof in [2, Appendix].

For proving the uniqueness, we transform the domain from $\mathcal{O}$ to $\mathbb{R}$ by defining $x=(x_1,x_2,x_3):=(e^s,y,e^z)$ and setting $v(t,x):=w(t,s,y,z)$. Then (3.30) is equivalent to

$$\inf_{\lambda\in\mathbb{R}}(v_t+\widetilde{L}^\lambda v)(t,x)=0,\qquad (t,x)\in(0,T)\times\mathbb{R}^3, \tag{3.34}$$

where

$$\begin{aligned}\widetilde{L}^\lambda v=&\frac12\sigma^2(x_2)v_{x_1x_1}+\frac12 c^2(x_2)v_{x_2x_2}+\frac12\big(\theta^2(x_2)+\lambda^2\big)v_{x_3x_3}\\&+\sigma(x_2)c(x_2)\rho v_{x_1x_2}-\sigma(x_2)\theta(x_2)v_{x_1x_3}\\&+c(x_2)\big(-\rho\theta(x_2)-\sqrt{1-\rho^2}\lambda\big)v_{x_2x_3}\\&+\left(\theta(x_2)-\frac12\sigma(x_2)\right)\sigma(x_2)v_{x_1}+b(x_2)v_{x_2}-\frac12\big(\theta^2(x_2)+\lambda^2\big)v_{x_3}.\end{aligned}$$

Now that it is in the standard form (3.34), so the uniqueness of the solution v, and thus of w, follows from the comparison result in [14, Theorem 4.1]. □

4 Conclusions and extensions

We have studied the outperformance portfolio optimization problem in complete and incomplete markets. The mathematical model is related to generalized composite pure and randomized hypothesis testing problems. We have established a connection between these two testing problems and then have used it to address our portfolio optimization problem. The maximal success probability exhibits special properties with respect to benchmark scaling, while the outperformance portfolio optimization does not enjoy economy of scale. In various cases, we have obtained explicit solutions to the outperformance portfolio optimization problem. In a stochastic volatility model, we have shown the special role played by the minimal martingale measure. With a general benchmark, an HJB characterization is available for the outperformance probability. An alternative approach is a characterization via a BSDE solution for its dual representation (see [20] and [21]).

There are a number of avenues for future research. Most naturally, one can consider quantile hedging in other incomplete markets, with specific market frictions and trading constraints. Another extension involves claims with cash flows over different (random) times, such as American options and insurance products, rather than a payoff at a fixed terminal time.

On the other hand, the result on the composite hypothesis testing can be also applied to problems with model uncertainty. To illustrate this point, consider a trader who receives x from selling a contingent claim with terminal random payoff $F \in [0, K]$ at time T. The objective is to minimize the risk of the terminal liability $-F$ in terms of *average value at risk*,

$$\begin{aligned} AVaR(-F) &:= \max_{\mathbb{Q}\in\mathcal{Q}_\lambda} \mathbb{E}^{\mathbb{Q}}[F] \\ \text{subject to} \quad & \inf_{Z\in\mathcal{Z}} \mathbb{E}[ZF] \geq x, \end{aligned}$$

with the set of measures $\mathcal{Q}_\lambda := \{\mathbb{Q} \ll \mathbb{P} \,|\, \frac{d\mathbb{Q}}{d\mathbb{P}} \leq \frac{1}{\lambda}\ \mathbb{P}\text{-a.s.}\}$ for $\lambda \in (0, 1]$. In fact, we can convert this problem into a randomized composite hypothesis testing problem as in (2.3). To this end, we define $X := (K - F)/K$ and then write $AVaR(-F) = K - K V_\lambda(x)$, where $V_\lambda(x)$ solves

$$\begin{aligned} V_\lambda(x) &= \sup_{X\in\mathcal{X}} \inf_{\mathbb{Q}\in\mathcal{Q}_\lambda} \mathbb{E}^{\mathbb{Q}}[X] \\ \text{subject to} \quad & \sup_{Z\in\mathcal{Z}} \mathbb{E}[ZX] \leq \frac{K - x}{K}. \end{aligned}$$

Following the analysis in this paper, one can obtain the properties of the value function $V_\lambda(x)$ as well as the structure of the solution.

Finally, the outperformance portfolio optimization problem in Sect. 3 is formulated with respect to a fixed reference measure $\mathbb{P}$. This corresponds to applying the theoretical results of Sect. 2 with the set $\mathcal{G} = \{1\}$; cf. the proofs of Proposition 3.3 and Theorem 3.5. It is also possible to incorporate model uncertainty by replacing the reference measure $\mathbb{P}$ by a class of probability measures $\mathcal{M}$. In this setup, the portfolio optimization problem becomes

$$V_{\mathcal{M}}(x) := \sup_{\pi \in \mathcal{A}(x)} \inf_{\mathbb{M} \in \mathcal{M}} \mathbb{M}[X_T^{x,\pi} \geq F], \quad x \geq 0.$$

This is a special case of the hypothesis testing problems discussed in Sect. 2, where the original set $\mathcal{G}$ can be interpreted as the set containing the Radon–Nikodým densities $d\mathbb{M}/d\mathbb{P}$ with $\mathbb{M} \in \mathcal{M}$. For related studies on the robust quantile hedging problem, we refer to [27] and [28].

Acknowledgements The authors would like to thank the Editor and two anonymous referees for their insightful remarks, as well as Jun Sekine, Birgit Rudloff, and James Martin for their helpful discussions.

Tim Leung's work is partially supported by NSF grant DMS-0908295. Qingshuo Song's work is partially supported by SRG grant 7002818 and GRF grant CityU 103310 of Hong Kong.

Appendix

A.1 The role of co($\mathcal{H}$) in $V(x)$

In this example, we show that the representation of $V(x)$ in (2.8) does not hold if $\mathrm{co}(\mathcal{H})$ is replaced by the smaller set $\mathcal{H}$.

Example A.1 Let $\Omega = [0, 1]$, and let $\mathbb{P}$ be Lebesgue measure, i.e., $\mathbb{P}[(a, b)] = b - a$ for $a \leq b$. Let $\mathcal{G} = \{G \equiv 1\}$ and $\mathcal{H} = \{H_1, H_2\}$ with

$$H_1(\omega) = I_{\{1/2 \leq \omega \leq 1\}} + 1, \qquad H_2(\omega) = I_{\{0 \leq \omega \leq 1/2\}} + 1, \quad \omega \in \Omega.$$

For the randomized hypothesis testing problem (2.3) with $x = 1$, it is easy to see, e.g., from (2.8), that

$$V(1) = \inf_{a \geq 0} \left\{ xa + \inf_{\mathcal{G} \times \mathrm{co}(\mathcal{H})} \mathbb{E}[(G - aH)^+] \right\} \Big|_{x=1} = \frac{2}{3},$$

along with the optimizers

$$\hat{G} = 1, \qquad \hat{H} = \frac{1}{2}(H_1 + H_2), \qquad \hat{a} = 2/3.$$

In this simple example, the uniqueness follows immediately.

Now, if one switches from $\mathrm{co}(\mathcal{H})$ to $\mathcal{H}$ in (2.8), then a strictly larger value will result; in fact,

$$\inf_{a \geq 0} \left\{ xa + \inf_{\mathcal{G} \times \mathcal{H}} \mathbb{E}[(G - aH)^+] \right\} \Big|_{x=1} = \frac{3}{4} > \frac{2}{3} = V(1).$$

A.2 On the positivity of $\hat{a}$

Theorem 2.3 shows that the minimizer $\hat{a}$ takes values in $[0, \infty)$ rather than $(0, \infty)$ as claimed in Proposition 3.1 and Lemma 4.3 in Cvitanić and Karatzas [7]. To illustrate this issue, we first give an example where $\hat{a}$ takes the value zero. Then we provide a sufficient condition for $\hat{a} > 0$.

Example A.2 Let $N_g := \bigcap_{G\in\mathcal{G}}\{G = 0\}$ and $x > 0$.

(i) If $\mathbb{P}[N_g] = 1$, then $\mathbb{E}[(G - aH)^+] = 0$ for all G, H, a. Thus, $\hat{a} = 0$ is the unique minimizer of $\{xa + \inf_{\mathcal{G}\times\mathcal{H}} \mathbb{E}[(G - aH)^+]\}$.
(ii) If $0 < \mathbb{P}[N_g] < 1$ and $x > \sup_{H\in\mathcal{H}} \mathbb{E}[(HI_{N_g^c})]$, then there also exists a counterexample such that $\hat{a} = 0$ minimizes $\{xa + \inf_{\mathcal{G}\times\mathcal{H}} \mathbb{E}[(G - aH)^+]\}$. Let us consider the following scenario. The sample space is $\Omega = [0, 1]$, $\mathbb{P}$ is Lebesgue measure on $[0, 1]$, $\mathcal{G} = \{G\}$ with $G = 2I_{[1/2,1]}$, and $\mathcal{H} = \{H\}$ with $H \equiv 1$. Then one can check that $N_g = \{G = 0\} = [0, 1/2)$, which in turn implies that $G = I_{N_g^c}/\mathbb{P}(N_g^c)$, and

$$xa + \inf_{\mathcal{G}\times\mathcal{H}} \mathbb{E}[(G - aH)^+] = xa + \mathbb{E}[(G - zH)^+]$$
$$= \begin{cases} xa & \text{if } a \geq \frac{1}{P[N_g^c]}, \\ 1 + a(x - \mathbb{P}[N_g^c]) & \text{if } 0 \leq a < \frac{1}{\mathbb{P}[N_g^c]}. \end{cases} \tag{A.1}$$

Since $x > \sup_{H\in\mathcal{H}} \mathbb{E}[HI_{N_g^c}] = \mathbb{P}[N_g^c]$, $\hat{a} = 0$ is the unique minimizer of (A.1).

Proposition A.3 *If*

$$0 < x < \sup_{\mathcal{H}} \mathbb{E}\Big[HI_{\bigcap\limits_{G\in\mathcal{G}}\{G>0\}}\Big], \tag{A.2}$$

then there exists $(\hat{G}, \hat{H}, \hat{a}, \hat{X}) \in \mathcal{G} \times \overline{\mathrm{co}(\mathcal{H})} \times (0, \infty) \times \mathcal{X}_x$ *satisfying* (2.5)–(2.7). *In particular,*

$$\hat{a} = \arg\min_{a\geq 0} \Big\{xa + \inf_{\mathcal{G}\times\overline{\mathrm{co}(\mathcal{H})}} \mathbb{E}[(G - aH)^+]\Big\} > 0.$$

Proof Define the function $f_x(a) := xa + \inf_{\mathcal{G}\times\overline{\mathrm{co}(\mathcal{H})}} \mathbb{E}[(G - aH)^+]$, which is Lipschitz-continuous (see Lemma 4.1 of [7]). Since $f_x(0) = \inf_{\mathcal{G}} \mathbb{E}[G]$ is in $[0, \infty)$ and $\lim_{a\to\infty} f_x(a) = \infty$, there exists a finite $\hat{a} \geq 0$ that minimizes $f_x(a)$.

Now suppose that $\hat{a} = 0$ is a minimizer of $f_x(a)$. Then it follows that $f_x(a) \geq f_x(0)$, $\forall a > 0$, which leads to

$$\begin{aligned} xa &\geq \inf_{\mathcal{G}} \mathbb{E}[G] - \inf_{\mathcal{G}\times\overline{\mathrm{co}(\mathcal{H})}} \mathbb{E}[(G - aH)^+] \\ &\geq \mathbb{E}[\tilde{G}] - \inf_{\overline{\mathrm{co}(\mathcal{H})}} \mathbb{E}[(\tilde{G} - aH)^+] \\ &\geq a \sup_{\overline{\mathrm{co}(\mathcal{H})}} \mathbb{E}\big[HI_{\{\tilde{G}\geq aH\}}\big] \geq a \sup_{\mathcal{H}} \mathbb{E}\big[HI_{\{\tilde{G}\geq aH\}}\big]. \end{aligned} \tag{A.3}$$

In (A.3), $\tilde{G}$ minimizes $\mathbb{E}[G]$ over $\mathcal{G}$, and its existence follows from convexity and closedness of $\mathcal{G}$. Taking the limit $a \searrow 0$ yields a contradiction to (A.2) because

$$x \geq \sup_{\mathcal{H}} \mathbb{E}\big[H I_{\{\tilde{G}>0\}}\big] \geq \sup_{\mathcal{H}} \mathbb{E}\big[H I_{\cap_{\mathcal{G}}\{G>0\}}\big].$$

Hence, we conclude that $\hat{a} > 0$. □

A.3 Counterexample for Remark 2.5

Let $\Omega = \{\omega_1, \omega_2\}$, $P[\{\omega_1\}] = P[\{\omega_2\}] = 1/2$. Then any random variable in $\mathcal{G}, \mathcal{H}$ or in $\mathcal{X}_x$, $\mathcal{I}_x$ can be represented as a point in $\mathbb{R}^2$. Let $\mathcal{H}$ be the line segment connecting $(2,4)$ and $(6,2)$, and $\mathcal{G} = \{(2,2)\}$. Given $x \geq 0$, $\mathcal{X}_x$ is the convex quadrangle with four vertices $(0,0)$, $(x/3, 0)$, $(x/5, 2x/5)$, $(0, x/2)$ intersected with $\{(x_1, x_2) \mid 0 \leq x_1, x_2 \leq 1\}$. For all $H = (h_1, h_2) \in \mathcal{H}$ and $X = (x_1, x_2)$, the constraint $\mathbb{E}[HX] \leq x$ implies that $\frac{h_1}{2}x_1 + \frac{h_2}{2}x_2 \leq x$. The set $\{(x_1, x_2) : \frac{h_1}{2}x_1 + \frac{h_2}{2}x_2 \leq x\}$ is the lower half-plane bounded by the line $h_1x_1 + h_2x_2 = 2x$, which passes through $(x/5, 2x/5)$ since $h_1 + 2h_2 = 5$. Hence, we have

$$V(x) = \sup_{(x_1,x_2)\in\mathcal{X}_x} (x_1 + x_2), \quad \text{and} \quad V_1(x) = \sup_{(x_1,x_2)\in\mathcal{I}_x} (x_1 + x_2),$$

where $\mathcal{I}_x = \mathcal{X}_x \cap \{(0,0), (0,1), (1,0), (1,1)\}$. In summary, the values are given by the following table:

x	$V(x)$	$V_1(x)$
$0 \leq x < 2$	$\frac{3}{5}x$	0
$2 \leq x < \frac{5}{2}$	$\frac{3}{5}x$	1
$\frac{5}{2} \leq x < 4$	$\frac{x}{3} + \frac{2}{3}$	1
$x \geq 4$	2	2

By inspecting the values of $V_1(x)$ we see that its smallest concave majorant must take the value $\frac{x}{2}$ in $[0, 4]$. Therefore, $V(x)$ is not the smallest concave majorant of $V_1(x)$.

A.4 Counterexample for Remark 2.11

With reference to Theorem 2.3, we show via an example that one cannot remove the independence requirement in (C2) of Theorem 2.10 when $\mathcal{G}$ and $\mathcal{H}$ are not both singletons.

Example A.4 Let $\Omega = \{0,1\} \times [0,1]$, $\mathcal{F}_T = \mathcal{B}(\Omega)$. Let μ be Lebesgue measure on $[0,1]$. Define $\mathbb{P}$ by

$$\mathbb{P}[\{0\} \times A] = \mathbb{P}[\{1\} \times A] = \frac{1}{2}\mu(A) \quad \forall A \in \mathcal{B}([0,1]).$$

Let $H_0 : \{0,1\} \to \mathbb{R}$ be given by $H_0(0) = 1/2$ and $H_0(1) = 3/2$, and $f : [0,1] \to \mathbb{R}$ be an arbitrarily fixed probability density function. Define the set

$$\mathcal{H} = \{H : \Omega \to \mathbb{R} : H(\alpha, a) = H_0(\alpha) f(a),\ (\alpha, a) \in \Omega\}$$

and the singleton $\mathcal{G} = \{G \equiv 1\}$. Let U be a uniform random variable on $(\Omega, \mathcal{F}_T, \mathbb{P})$, so that $\mathbb{P}[U \le a] = a$ for $a \in [0, 1]$.

The pure hypothesis testing problem is

$$V_1 = \sup_{A \in \mathcal{F}_T} \mathbb{E}[I_A] \quad \text{subject to} \quad \sup_{H \in \mathcal{H}} \mathbb{E}[H I_A] \le 1/2.$$

Direct computation gives the success set $\hat{A} = \{0\}$ and the value of the pure hypothesis test $V_1 = 1/2$. On the other hand, the randomized hypothesis testing problem is

$$V = \sup_{X \in \mathcal{X}} \mathbb{E}[X] \quad \text{subject to} \quad \sup_{H \in \mathcal{H}} \mathbb{E}[H X] \le 1/2.$$

We find that $\hat{H}(\alpha, a) = H_0(\alpha)$ and $\hat{X} = I_{\{\alpha=0\}} + 1/3 I_{\{\alpha=1\}}$ solve this randomized hypothesis test with the optimal value $V = 2/3$.

This shows that the values of the pure and randomized hypothesis tests are different. If one were to construct an indicator version of the randomized test as in (2.13), namely

$$\bar{X} := I_{\{\alpha=0\}} + I_{\{\alpha=1\}} I_{\{U<1/3\}},$$

then although this test $\bar{X}$ still satisfies $\mathbb{E}[\hat{H}\bar{X}] = 1/2$, it does not solve either the pure or the randomized hypothesis test. Indeed, for $\tilde{H}(\alpha, a) = 3 I_{a<1/3} H_0(\alpha) \in \mathcal{H}$, we observe the violation $\mathbb{E}[\tilde{H}\bar{X}] = 1 > 1/2$.

A.5 A property of nondegenerate martingales

On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $(\mathcal{F}_t)_{0 \le t \le 1}$, we denote by W a standard Brownian motion. Let Y be a $(\mathbb{P}, \mathcal{F}_t)$-martingale defined by

$$Y_t = \int_0^t \sigma_r \, dW_r, \qquad t \in [0, 1],$$

where (σ_t) is a bounded $(\mathcal{F}_t)$-adapted process.

Proposition A.5 *Assume that $c < \sigma_t < C$ for some positive constants c and C. Then*

$$\mathbb{P}[Y_1 = b] = 0$$

for all constants b.

To prove this proposition, we use the following two facts. We first define the function $f : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to [0, 1]$ by

$$f(x, y, u) = \mathbb{P}[W_t = u \text{ for some } t \in (x, y)].$$

1. By direct computation we obtain

$$\sup_{u \in \mathbb{R}} f(x, y, u) = f(x, y, 0) < 1.$$

2. By a scaling argument we have

$$f(\lambda x, \lambda y, u) = f\left(x, y, \frac{u}{\sqrt{\lambda}}\right) \quad \forall \lambda > 0.$$

Now we are ready to present the

Proof of Proposition A.5 Since Y is a continuous process,

$$\{Y_1 = b\} \in \sigma(\{\mathcal{F}_t : t < 1\}) =: \mathcal{F}_{1-}.$$

By Lévy's zero–one law we have

$$I_{\{Y_1=b\}} = \lim_{t\uparrow 1} \mathbb{P}[Y_1 = b|\mathcal{F}_t\} \quad \text{a.s.}$$

Therefore, it is enough to show that there exists $a \in (0, 1)$ such that

$$\mathbb{P}[Y_1 = b|\mathcal{F}_t] < a < 1 \quad \forall t \in (0, 1).$$

Note that the martingale $(Y_s|Y_t = u : s > t)$ has the same distribution as a time-changed Brownian motion starting from state u. Combining this with the estimate $c^2(1-t) \le \int_t^1 \sigma_r^2 dr \le C^2(1-t)$, we have for some standard Brownian motion B that

$$\begin{aligned}\mathbb{P}[Y_1 = b|Y_t = u] &= \mathbb{P}\big[B_r = b - u \text{ for some } r \in \big(c^2(1-t), C^2(1-t)\big)\big] \\ &= f\left(c^2, C^2, \frac{b-u}{\sqrt{1-t}}\right) \le f(c^2, C^2, 0).\end{aligned}$$

Since $f(c^2, C^2, 0)$ is independent of t and strictly less than 1, we can simply take $a = f(c^2, C^2, 0)$. □

One may wonder whether the condition on σ in Proposition A.5 can be relaxed to $\sigma_t > 0$ a.s. for all t. The answer is negative, as shown by the counterexample in [22].

References

1. Avellaneda, M., Zhang, S.: Path-dependence of leveraged ETF returns. SIAM J. Financ. Math. **1**, 586–603 (2010)
2. Bayraktar, E., Song, Q., Yang, J.: On the continuity of stochastic control problems on bounded domains. Stoch. Anal. Appl. **29**, 48–60 (2011)
3. Bayraktar, E., Huang, Y.-J., Song, Q.: Outperforming the market portfolio with a given probability. Ann. Appl. Probab. **22**, 1465–1494 (2012)
4. Brannath, W., Schachermayer, W.: A bipolar theorem for $L^0_+(\Omega, \mathcal{F}, \mathbf{P})$. In: Séminaire de Probabilités, XXXIII. Lecture Notes in Math., vol. 1709, pp. 349–354. Springer, Berlin (1999)
5. Browne, S.: Reaching goals by a deadline: digital options and continuous time active portfolio management. Adv. Appl. Probab. **31**, 551–577 (1999)
6. Cvitanić, J.: Minimizing expected loss of hedging in incomplete and constrained markets. SIAM J. Control Optim. **38**, 1050–1066 (2000)

7. Cvitanić, J., Karatzas, I.: Generalized Neyman–Pearson lemma via convex duality. Bernoulli **7**, 79–97 (2001)
8. Delbaen, F., Schachermayer, W.: A general version of the fundamental theorem of asset pricing. Math. Ann. **300**, 463–520 (1994)
9. Delbaen, F., Schachermayer, W.: The Mathematics of Arbitrage. Springer, Berlin (2006)
10. El Karoui, N.E., Quenez, M.-C.: Dynamic programming and pricing of contingent claims in an incomplete market. SIAM J. Control Optim. **33**, 29–66 (1995)
11. Föllmer, H., Leukert, P.: Quantile hedging. Finance Stoch. **3**, 251–273 (1999)
12. Föllmer, H., Schied, A.: Convex measures of risk and trading constraints. Finance Stoch. **6**, 429–447 (2002)
13. Föllmer, H., Schweizer, M.: Hedging of contingent claims under incomplete information. In: Davis, M.H.A., Elliott, R.J. (eds.) Applied Stochastic Analysis. Stochastics Monographs, vol. 5, pp. 389–414. Gordon and Breach, London (1990)
14. Giga, Y., Goto, S., Ishii, H., Sato, M.-H.: Comparison principle and convexity preserving properties for singular degenerate parabolic equations on unbounded domains. Indiana Univ. Math. J. **40**, 443–470 (1991)
15. Jarrow, R.: Understanding the risk of leveraged ETFs. Finance Res. Lett. **7**, 135–139 (2010)
16. Jeanblanc, M., Yor, M., Chesney, M.: Mathematical Methods for Financial Markets. Springer, London (2009)
17. Krutchenko, R.N., Melnikov, A.V.: Quantile hedging for a jump-diffusion financial market model. In: Kohlmann, M., Tang, S. (eds.) Trends in Mathematics: Workshop of the Mathematical Finance Research Project, Konstanz, Germany, 5–7 October 2000, pp. 215–229. Birkhäuser, Basel (2001)
18. Lehmann, E.L., Romano, J.P.: Testing Statistical Hypotheses, 3rd edn. Springer, New York (2005)
19. Leung, T., Song, Q., Yang, J.: Generalized hypothesis testing and maximizing the success probability in financial markets. In: Proceedings of the International Conference on Business Intelligence and Financial Engineering (ICBIFE) (2011). Available at http://ssrn.com/abstract=2292771
20. Ma, J., Yong, J.: Forward-Backward Stochastic Differential Equations and Their Applications. Lecture Notes in Mathematics, vol. 1702. Springer, Berlin (1999)
21. Ma, J., Zhang, J.: Representation theorems for backward stochastic differential equations. Ann. Appl. Probab. **12**, 1390–1418 (2002)
22. MathOverflow, A non-degenerate martingale. Website (version: 2011–12–24). http://mathoverflow.net/questions/84216
23. Romano, M., Touzi, N.: Contingent claims and market completeness in a stochastic volatility model. Math. Finance **7**, 399–410 (1997)
24. Rudloff, B.: Convex hedging in incomplete markets. Appl. Math. Finance **14**, 437–452 (2007)
25. Rudloff, B., Karatzas, I.: Testing composite hypotheses via convex duality. Bernoulli **16**, 1224–1239 (2010)
26. Schied, A.: On the Neyman–Pearson problem for law-invariant risk measures and robust utility functionals. Ann. Appl. Probab. **14**, 1398–1423 (2004)
27. Schied, A.: Optimal investments for robust utility functionals in complete market models. Math. Oper. Res. **30**, 750–764 (2005)
28. Sekine, J.: On a robustness of quantile-hedging: complete market's case. Asia-Pac. Financ. Mark. **6**, 195–201 (1999)
29. Sircar, R., Zariphopoulou, T.: Bounds and asymptotic approximations for utility prices when volatility is random. SIAM J. Control Optim. **43**, 1328–1353 (2005)
30. Spivak, G., Cvitanić, J.: Maximizing the probability of a perfect hedge. Ann. Appl. Probab. **9**, 1303–1328 (1999)
31. Yong, J., Zhou, X.-Y.: Stochastic Controls: Hamiltonian Systems and HJB Equations. Springer, New York (1999)