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The Master equation in mean field theory

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ABSTRACT

In his lectures at *College de France*, P.L. Lions introduced the concept of Master equation, see [8] for Mean Field Games. It is introduced in a heuristic fashion, from the prospective as a system of partial differential equations, that the equation is associated to a Nash equilibrium for a large, but finite, number of players. The method, also explained in [3], composed of a formalism of derivations. The interest of this equation is that it contains interesting particular cases, which can be studied directly, in particular the system of HJB–FP (Hamilton–Jacobi–Bellman, Fokker–Planck) equations obtained as the limit of the finite Nash equilibrium game, when the trajectories are independent, see [6]. Usually, in mean field theory, one can bypass the large Nash equilibrium, by introducing the concept of representative agent, whose action is influenced by a distribution of similar agents, and obtains directly the system of HJB–FP equations of interest, see for instance [1]. Apparently, there is no such approach for the Master equation. We show here that it is possible. We first do it for the Mean Field type control problem, for which we interpret completely the Master equation. For the Mean Field Games itself, we solve a related problem, and obtain again the Master equation.

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RÉSUMÉ

Dans son cours au Collège de France, P.L. Lions a inroduit le concept d'équation maîtresse, pour les jeux à champ moyen [8]. Ceci a été fait d'une manière heuristique, à partir du système d'équations aux dérivées partielles associé à un équilibre de Nash, pour un nombre fini, mais grand, de joueurs. La méthode repose sur un formalisme de dérivations. L'intérêt de cette équation maîtresse est qu'elle contient des cas particuliers intéressants, qui peuvent être étudiés dicretement, en particulier le système des équations HJB–FP, Hamilton–Jacobi–Bellman & Fokker Planck, obtenu comme la limite d'un équilibre de Nash, lorsque les trajectoires sont indépendantes [6]. De manière générale, dans la théorie des jeux à champ moyen,

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on peut ne pas passer par l'équilibre de Nash pour un grand nombre de joueurs, en introduisant le concept d'agent représentatif, dont le comportement dépend d'une distribution d'agents similaires à l'agent représentatif [1]. Une telle possibilité n'avait pas été mise en avant, pour l'équation maîtresse. On montre que c'est possible. On le montre d'abord pour les problèmes de contrôle de type champ moyen, et on caractérise complètement l'équation maîtresse. Pour les jeux champ moyen, on résout un problème relatif à ce cas, et on obtient à nouveau l'équation maîtresse.
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1. Introduction

Since we do not intend to give complete proofs, we proceed formally, by assuming relevant smoothness structure whenever it eases the argument. We consider functions $f(x,m,v)$, $g(x,m,v)$, $h(x,m)$ and $\sigma(x)$ where $x \in \mathbb{R}^n$; m is a probability measure on $\mathbb{R}^n$, but we shall retain ourselves mostly in the regular case, in which m represents the probability density, assumed to be in $L^2(\mathbb{R}^n)$ and v is a control in $\mathbb{R}^d$. The functions f and h are scalar, g is a vector in $\mathbb{R}^n$ and $\sigma(x)$ is a $n \times n$ matrix. All these functions are differentiable in all arguments. In the case of the differentiability with respect to m, we use the concept of Gâteaux differentiability. Indeed, $F : L^2(\mathbb{R}^n) \to \mathbb{R}$ is said to be Gâteaux differentiable if there uniquely exists $\frac{\partial F}{\partial m}(m) \in L^2(\mathbb{R}^n)$, such that

$$\frac{d}{d\theta} F(m+\theta\tilde{m})|_{\theta=0} = \int_{\mathbb{R}^n} \frac{\partial F}{\partial m}(m)(\xi)\tilde{m}(\xi)d\xi.$$

The second order derivative is a linear map from $L^2(\mathbb{R}^n)$ into itself, defined by

$$\frac{d}{d\theta}\frac{\partial F}{\partial m}(m+\theta\tilde{m})(\xi)|_{\theta=0} = \int_{\mathbb{R}^n} \frac{\partial^2 F}{\partial m^2}(m)(\xi,\eta)\tilde{m}(\eta)d\eta.$$

We can state the second order Taylor's formula as

$$F(m+\tilde{m}) = F(m) + \int_{\mathbb{R}^n} \frac{\partial F}{\partial m}(m)(\xi)\tilde{m}(\xi)d\xi + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\partial^2 F}{\partial m^2}(m)(\xi,\eta)\tilde{m}(\xi)\tilde{m}(\eta)d\xi d\eta.$$

Consider a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ on which various Wiener processes are defined. We define first a standard Wiener process $w(t)$ in $\mathbb{R}^n$. To avoid cumbersome of notation, if there is no ambiguity, we shall suppress the arguments of functions in the rest of this paper.

We first introduce the classical mean field type control problem in which the state dynamic is represented by the stochastic differential equation of McKean–Vlasov type:

$$\begin{cases} dx = g\big(x, m_{v(\cdot)}, v(x)\big)dt + \sigma(x)dw, \\ x(0) = x_0 \end{cases} \tag{1.1}$$

in which $v(x)$ is a feedback and $m_{v(\cdot)}(x,t)$ is the probability density of the state $x(t)$. The initial value x_0 is a random variable independent of the Wiener process $w(\cdot)$. This density is well-defined provided that there is invertibility of $a(x) = \sigma(x)\sigma^*(x)$. We define the second order differential operator

$$\mathcal{A}\varphi(x) = -\frac{1}{2}\sum_{i,j} a_{ij}(x)\frac{\partial^2 \varphi(x)}{\partial x_i \partial x_j}$$

and its adjoint

$$\mathcal{A}^*\varphi(x) = -\frac{1}{2}\sum_{i,j}\frac{\partial^2(a_{ij}(x)\varphi(x))}{\partial x_i \partial x_j}.$$

The mean field type control problem is to minimize the cost functional

$$J\big(v(\cdot)\big) = \mathbb{E}\left[\int_0^T f\big(x(t), m_{v(\cdot)}(t), v\big(x(t)\big)\big)dt + h\big(x(T), m_{v(\cdot)}(T)\big)\right]. \tag{1.2}$$

In the classical mean field games problem, one should first fix $m(\cdot) \in C([0,T]; L^2(\mathbb{R}^n))$ as a given parameter in the state equation

$$\begin{cases} dx = g\big(x, m, v(x)\big)dt + \sigma(x)dw, \\ x(0) = x_0, \end{cases} \tag{1.3}$$

and the objective functional

$$J\big(v(\cdot), m(\cdot)\big) = \mathbb{E}\left[\int_0^T f\big(x(t), m(t), v\big(x(t)\big)\big)dt + h\big(x(T), m(T)\big)\right]. \tag{1.4}$$

The mean field games problem looks for an equilibrium $\hat{v}(\cdot), m(\cdot)$ such that

$$\begin{cases} J\big(\hat{v}(\cdot), m(\cdot)\big) \le J\big(v(\cdot), m(\cdot)\big), & \forall v(\cdot), \\ m(t) \text{ is the probability density of } \hat{x}(t), & \forall t \in [0,T], \end{cases} \tag{1.5}$$

where $\hat{x}(\cdot)$ is the solution of (1.3) corresponding to the equilibrium pair $\hat{v}(\cdot), m(\cdot)$.

2. Master equation for the classical case

We refer to problems (1.1), (1.2) and (1.3), (1.4), (1.5) as the classical case. We define the Hamiltonian $H(x, m, q) : \mathbb{R}^n \times L^2(\mathbb{R}^n) \times \mathbb{R}^n \to \mathbb{R}$ as

$$H(x, m, q) = \inf_v \big(f(x, m, v) + q \cdot g(x, m, v)\big)$$

and the optimal value of v is denoted by $\hat{v}(x, m, q)$. We then set

$$G(x, m, q) = g\big(x, m, \hat{v}(x, m, q)\big).$$

2.1. Mean field type control problem

The mean field type control problem is easily transformed into a control problem in which the state is a probability density process $m_{v(\cdot)}$, which satisfies the solution of the Fokker–Planck equation

$$\begin{cases} \dfrac{\partial m_{v(\cdot)}}{\partial t} + \mathcal{A}^* m_{v(\cdot)} + \operatorname{div}\big(g\big(x, m_{v(\cdot)}, v(x)\big)m_{v(\cdot)}(x)\big) = 0, \\ m_{v(\cdot)}(x, 0) = m_0(x). \end{cases} \tag{2.1}$$

Here $m_0(x)$ is the density of the initial value x_0. The objective functional $J(v(\cdot))$ can be written as

$$J\big(v(\cdot)\big) = \int_0^T \int_{\mathbb{R}^n} f\big(x, m_{v(\cdot)}(t), v(x)\big) m_{v(\cdot)}(x,t)\,dx\,dt + \int_{\mathbb{R}^n} h\big(x, m_{v(\cdot)}(T)\big) m_{v(\cdot)}(x,T)\,dx. \tag{2.2}$$

We next use the traditional invariant embedding approach. Define a family of control problems indexed by initial conditions (m, t):

$$\begin{cases} \dfrac{\partial m_{v(\cdot)}}{\partial s} + \mathcal{A}^* m_{v(\cdot)} + \operatorname{div}\big(g\big(x, m_{v(\cdot)}, v(x)\big) m_{v(\cdot)}(x)\big) = 0, \\ m_v(x,t) = m(x), \end{cases} \tag{2.3}$$

$$J_{m,t}\big(v(\cdot)\big) = \int_t^T \int_{\mathbb{R}^n} f\big(x, m_{v(\cdot)}(s), v(x)\big) m_{v(\cdot)}(x,s)\,dx\,ds + \int_{\mathbb{R}^n} h\big(x, m_{v(\cdot)}(T)\big) m_{v(\cdot)}(x,T)\,dx, \tag{2.4}$$

and we set

$$V(m,t) = \inf_{v(\cdot)} J_{m,t}\big(v(\cdot)\big). \tag{2.5}$$

We can then write the Dynamic Programming equation satisfied by $V(m,t)$. By standard arguments, one obtains[3]

$$\begin{cases} \dfrac{\partial V}{\partial t} - \displaystyle\int_{\mathbb{R}^n} \dfrac{\partial V(m)}{\partial m}(\xi) \mathcal{A}^* m(\xi)\,d\xi \\ \qquad + \inf_v \left(\displaystyle\int_{\mathbb{R}^n} f\big(\xi, m, v(\xi)\big) m(\xi)\,d\xi - \displaystyle\int_{\mathbb{R}^n} \dfrac{\partial V(m)}{\partial m}(\xi) \operatorname{div}\big(g\big(\xi, m, v(\xi)\big) m(\xi)\big)\,d\xi \right) = 0, \\ V(m,T) = \displaystyle\int_{\mathbb{R}^n} h(x,m) m(x)\,dx. \end{cases} \tag{2.6}$$

By setting

$$U(x,m,t) = \frac{\partial V(m,t)}{\partial m}(x),$$

we can rewrite (2.6) as

$$\frac{\partial V}{\partial t} - \int_{\mathbb{R}^n} \mathcal{A} U(\xi, m, t) m(\xi)\,d\xi + \int_{\mathbb{R}^n} H(\xi, m, DU) m(\xi)\,d\xi = 0 \tag{2.7}$$

since the optimization in v can be done inside the integral. We next differentiate (2.7) with respect to m which gives

$$\begin{aligned} &\frac{\partial}{\partial m}\left[\int_{\mathbb{R}^n} H\big(\xi, m, DU(\xi, m, t)\big) m(\xi)\,d\xi \right](x) \\ &\quad = H\big(x, m, DU(x)\big) + \int_{\mathbb{R}^n} \frac{\partial}{\partial m} H\big(\xi, m, DU(\xi)\big)(x) m(\xi)\,d\xi + \int_{\mathbb{R}^n} G\big(\xi, m, DU(\xi)\big) m(\xi) D_\xi \frac{\partial}{\partial m} U(\xi, m, t)(x)\,d\xi. \end{aligned} \tag{2.8}$$

[3] This equation has also been obtained, independently, by M. Laurière and O. Pironneau [7].

Hence under sufficient continuous differentiability, using

$$\frac{\partial}{\partial m}U(\xi,m,t)(x)=\frac{\partial}{\partial m}U(x,m,t)(\xi)=\frac{\partial^2 V(m,t)}{\partial m^2}(x,\xi), \tag{2.9}$$

we obtain the master equation

$$\begin{cases} -\dfrac{\partial U}{\partial t}+\mathcal{A}U+\displaystyle\int_{\mathbb{R}^n}\frac{\partial}{\partial m}U(x,m,t)(\xi)(\mathcal{A}^*m(\xi)+\operatorname{div}\big(G\big(\xi,m,DU(\xi)\big)m(\xi)\big)d\xi \\ \quad =H\big(x,m,DU(x)\big)+\displaystyle\int_{\mathbb{R}^n}\frac{\partial}{\partial m}H\big(\xi,m,DU(\xi)\big)(x)m(\xi)d\xi, \\ U(x,m,T)=h(x,m)+\displaystyle\int_{\mathbb{R}^n}\frac{\partial}{\partial m}h(\xi,m)(x)m(\xi)d\xi. \end{cases} \tag{2.10}$$

The probability density, corresponding to the optimal feedback control, is given by

$$\begin{cases} \dfrac{\partial m}{\partial t}+\mathcal{A}^*m+\operatorname{div}\big(G(x,m,DU)m(x)\big)=0, \\ m(x,0)=m_0(x). \end{cases}$$

Define $u(x,t)=U(x,m(t),t)$, then clearly, from (2.10) we obtain

$$\begin{cases} -\dfrac{\partial u}{\partial t}+\mathcal{A}u=H\big(x,m,Du(x)\big)+\displaystyle\int_{\mathbb{R}^n}\frac{\partial}{\partial m}H\big(\xi,m,Du(\xi)\big)(x)m(\xi)d\xi, \\ u(x,T)=h(x,m)+\displaystyle\int_{\mathbb{R}^n}\frac{\partial}{\partial m}h(\xi,m)(x)m(\xi)d\xi \end{cases} \tag{2.11}$$

which together with the FP equation

$$\begin{cases} \dfrac{\partial m}{\partial t}+\mathcal{A}^*m+\operatorname{div}\big(G(x,m,Du)m(x)\big)=0, \\ m(x,0)=m_0(x) \end{cases} \tag{2.12}$$

form the system of coupled HJB–FP equations of the classical mean field type control problem, see [1].

2.2. Mean Field Games

In Mean Field Games, we cannot have a Bellman equation, similar to (2.6), (3.11), since the problem is not simply a control problem. However, if we first fixed parameter $m(\cdot)$ in (1.3), (1.4) we have a standard control problem. We introduce the state dynamics and the cost functional accordingly

$$\begin{cases} dx(s)=g\big(x(s),m,v\big(x(s)\big)\big)ds+\sigma\big(x(s)\big)dw, \\ x(t)=x, \end{cases} \tag{2.13}$$

$$J_{x,t}\big(v(\cdot),m(\cdot)\big)=\mathbb{E}\left[\int_t^T f\big(x(s),m(s),v\big(x(s)\big)\big)ds+h\big(x(T),m(T)\big)\right]. \tag{2.14}$$

If we set

$$u(x,t) = \inf_{v(\cdot)} J_{x,t}\big(v(\cdot), m(\cdot)\big)$$

in which we omit to write explicitly the dependence of u in m. Then $u(x,t)$ satisfies Bellman equation

$$\begin{cases} -\dfrac{\partial u}{\partial t} + \mathcal{A}u = H\big(x, m, Du(x)\big), \\ u(x,T) = h(x,m). \end{cases} \tag{2.15}$$

For mean field games, we next require that m must be the probability density of the optimal state, hence

$$\begin{cases} \dfrac{\partial m}{\partial t} + \mathcal{A}^* m + \operatorname{div}\big(G(x,m,Du)m(x)\big) = 0, \\ m(x,0) = m_0(x) \end{cases} \tag{2.16}$$

and this is the system of HJB–FP equations, corresponding to the classical Mean Field Games problem. We can check that, if one considers the Master equation

$$\begin{cases} -\dfrac{\partial U}{\partial t} + \mathcal{A}U + \displaystyle\int_{\mathbb{R}^n} \dfrac{\partial}{\partial m} U(x,m,t)(\xi)(\mathcal{A}^* m(\xi) + \operatorname{div}\big(G\big(\xi, m, DU(\xi)\big)m(\xi)\big)d\xi \\ \quad = H\big(x, m, DU(x)\big), \\ U(x,m,T) = h(x,m), \end{cases} \tag{2.17}$$

then

$$u(x,t) = U\big(x, m(t), t\big). \tag{2.18}$$

Combining (2.17) and (2.16), we obtain easily (2.15), and hence (2.18) if the problem is well-posed.

3. Stochastic mean field type control

3.1. Preliminaries

If we look at the formulation (2.1), (2.2) of the mean field type control problem, it is a deterministic problem, although at the origin it was a stochastic one, see (1.1), (1.2). We now consider a stochastic version of (2.1), (2.2) or a doubly stochastic version of (1.1), (1.2). Let us begin with this one. Assume that there is a second standard Wiener process $b(t)$ with values in $\mathbb{R}^n$; $b(t)$ and $w(t)$ are independent which are also independent of x_0. We set $\mathcal{B}^t = \sigma(b(s)\colon s \le t)$ and $\mathcal{F}^t = \sigma(x_0, b(s), w(s)\colon s \le t)$. The control $v(x,t)$ at time t is a feedback, but not deterministic, i.e. the functional form of v is $\mathcal{B}^t$ adapted. We consider the stochastic McKean–Vlasov equation

$$\begin{cases} dx = g\big(x, m_{v(\cdot)}(t), v(x,t)\big)dt + \sigma(x)dw + \beta db(t), \\ x(0) = x_0 \end{cases} \tag{3.1}$$

in which $m_{v(\cdot)}(t)$ represents the conditional probability density of $x(t)$, given the σ-algebra $\mathcal{B}^t$. The stochastic mean field type control problem aims at minimizing the objective functional

$$J\big(v(\cdot)\big) = \mathbb{E}\left[\int_0^T f\big(x(t), m_{v(\cdot)}(t), v\big(x(t),t\big)\big)dt + h\big(x(T), m_{v(\cdot)}(T)\big)\right]. \tag{3.2}$$

3.2. Conditional probability

Let $y(t) = x(t) - \beta b(t)$, then the process $y(t)$ satisfies the equation

$$\begin{cases} dy = g\big(y(t) + \beta b(t), m_{v(\cdot)}(t), v\big(y(t) + \beta b(t), t\big)\big)dt + \sigma\big(y(t) + \beta b(t)\big)dw, \\ y(0) = x_0. \end{cases} \tag{3.3}$$

If we fix $b(s)$, $s \le t$, then the conditional probability of $y(t)$ is simply the probability density arising from the Wiener process $w(t)$, in view of the independence of $w(t)$ and $b(t)$. It is the function $p(y,t)$ solution of

$$\begin{cases} \dfrac{\partial p}{\partial t} - \dfrac{1}{2}\displaystyle\sum_{ij}\left(a_{i,j}\big(y + \beta b(t)\big)\frac{\partial^2 p}{\partial y_i \partial y_j}\right) + \operatorname{div}\big(g\big(y + \beta b(t), m_{v(\cdot)}(t), v\big(y + \beta b(t), t\big)\big)p\big) = 0, \\ p(y,0) = m_0(y). \end{cases} \tag{3.4}$$

The conditional probability density of $x(t)$ given $\mathcal{B}^t$ is $m_{v(\cdot)}(x,t) = p(x - \beta b(t), t)$, and hence

$$\partial_t m_{v(\cdot)} = \left(\frac{\partial p}{\partial t} + \frac{1}{2}\beta^2 \Delta p\right)\big(x - \beta b(t), t\big)dt - \beta D p\big(x - \beta b(t), t\big)db(t).$$

We thus have

$$\begin{cases} \partial_t m_{v(\cdot)} + \left(\mathcal{A}^* m_{v(\cdot)} - \dfrac{1}{2}\beta^2 \Delta m_{v(\cdot)} + \operatorname{div}\big(g\big(x, m_{v(\cdot)}(t), v(x,t)\big)m_{v(\cdot)}\big)\right)dt + \beta D m_{v(\cdot)} db(t) = 0, \\ m_{v(\cdot)}(x,0) = m_0(x), \end{cases} \tag{3.5}$$

and the objective functional (3.2) can be written as

$$J\big(v(\cdot)\big) = \mathbb{E}\left[\int_0^T \int_{\mathbb{R}^n} f\big(x, m_{v(\cdot)}(t), v(x,t)\big)m_{v(\cdot)}(x,t)dxdt + \int_{\mathbb{R}^n} h\big(x, m_{v(\cdot)}(T)\big)m_{v(\cdot)}(x,T)dx\right]. \tag{3.6}$$

The problem becomes a stochastic control problem for a distribution-valued parameter system. Using the invariant embedding again, we consider the family of problems indexed by m, t

$$\begin{cases} \partial_s m_{v(\cdot)} + \left(\mathcal{A}^* m_{v(\cdot)} - \dfrac{1}{2}\beta^2 \Delta m_{v(\cdot)} + \operatorname{div}\big(g\big(x, m_{v(\cdot)}(s), v(x,s)\big)m_{v(\cdot)}\big)\right)ds + \beta D m_{v(\cdot)} db(s) = 0, \\ m_{v(\cdot)}(x,t) = m(x), \end{cases} \tag{3.7}$$

and

$$J_{m,t}\big(v(\cdot)\big) = \mathbb{E}\left[\int_t^T \int_{\mathbb{R}^n} f\big(x, m_{v(\cdot)}(s), v(x,s)\big)m_{v(\cdot)}(x,s)dxdt + \int_{\mathbb{R}^n} h\big(x, m_{v(\cdot)}(T)\big)m_{v(\cdot)}(x,T)dx\right]. \tag{3.8}$$

Set

$$V(m,t) = \inf_{v(\cdot)} J_{m,t}\big(v(\cdot)\big),$$

then $V(m,t)$ satisfies the Dynamic Programming equation

$$\begin{cases} \dfrac{\partial V}{\partial t} - \displaystyle\int_{\mathbb{R}^n} \dfrac{\partial V(m,t)}{\partial m}(\xi)\Big(\mathcal{A}^* m(\xi) - \dfrac{1}{2}\beta^2 \Delta m(\xi)\Big) d\xi \\ \quad + \dfrac{1}{2}\beta^2 \displaystyle\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \dfrac{\partial^2 V(m,t)}{\partial m^2}(\xi,\eta) Dm(\xi) Dm(\eta) d\xi d\eta \\ \quad + \inf_v \Big(\displaystyle\int_{\mathbb{R}^n} f\big(\xi,m,v(\xi)\big) m(\xi) d\xi - \int_{\mathbb{R}^n} \dfrac{\partial V(m,t)}{\partial m}(\xi) \operatorname{div}\big(g\big(\xi,m,v(\xi)\big)m(\xi)\big) d\xi \Big) = 0, \\ V(m,T) = \displaystyle\int_{\mathbb{R}^n} h(x,m) m(x) dx. \end{cases} \tag{3.9}$$

3.3. Master equation

To obtain the master equation, we define again $U(x,m,t) = \frac{\partial V(m,t)}{\partial m}(x)$ and we have the terminal condition

$$U(x,m,T) = h(x,m) + \int_{\mathbb{R}^n} \frac{\partial h(\xi,m)}{\partial m}(x) m(\xi) d\xi, \tag{3.10}$$

and from (3.9) we obtain

$$\begin{aligned} &\frac{\partial V}{\partial t} - \int_{\mathbb{R}^n} \Big(\mathcal{A}U - \frac{1}{2}\beta^2 \Delta U\Big)(\xi,m,t) m(\xi) d\xi \\ &\quad + \frac{1}{2}\beta^2 \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\partial U(\xi,m,t)}{\partial m}(\eta) Dm(\xi) Dm(\eta) d\xi d\eta + \int_{\mathbb{R}^n} H(\xi,m,DU) m(\xi) d\xi = 0. \end{aligned} \tag{3.11}$$

We then differentiate (3.11) with respect to m to obtain the master equation. We note that

$$\begin{aligned} &\frac{\partial}{\partial m}\Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\partial U(\xi,m,t)}{\partial m}(\eta) Dm(\xi) Dm(\eta) d\xi d\eta \Big)(x) \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\partial^2 U(x,m,t)}{\partial m^2}(\xi,\eta) Dm(\xi) Dm(\eta) d\xi d\eta - 2 \operatorname{div}\Big(\int_{\mathbb{R}^n} \frac{\partial U(x,m,t)}{\partial m}(\eta) Dm(\eta) d\eta \Big), \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial m}\Big(\int_{\mathbb{R}^n} H(\xi,m,DU) m(\xi) dx \Big)(x) &= H\big(x,m,DU(x)\big) + \int_{\mathbb{R}^n} \frac{\partial}{\partial m} H\big(\xi,m,DU(\xi)\big)(x) m(\xi) d\xi \\ &\quad + \int_{\mathbb{R}^n} \frac{\partial}{\partial m}\big(DU(\xi,m,t)\big)(x) G\big(\xi,m,DU(\xi)\big) m(\xi) d\xi. \end{aligned}$$

Next, using (2.9) in terms of those in the present setting, we have

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{\partial}{\partial m}\big(DU(\xi,m,t)\big)(x) G\big(\xi,m,DU(\xi)\big) m(\xi) d\xi &= \int_{\mathbb{R}^n} D_\xi \Big(\frac{\partial}{\partial m} U(\xi,m,t)(x) \Big) G\big(\xi,m,DU(\xi)\big) m(\xi) d\xi \\ &= -\int_{\mathbb{R}^n} \frac{\partial}{\partial m} U(\xi,m,t)(x) \operatorname{div}\big(G\big(\xi,m,DU(\xi)\big) m(\xi)\big) d\xi \end{aligned}$$

$$= -\int\limits_{\mathbb{R}^n} \frac{\partial}{\partial m} U(x,m,t)(\xi)\,\mathrm{div}\big(G\big(\xi,m,DU(\xi)\big)m(\xi)\big)d\xi.$$

Collecting results, we obtain the Master equation

$$\begin{cases} -\dfrac{\partial U}{\partial t} + \mathcal{A}U - \dfrac{1}{2}\beta^2 \Delta U \\ \quad + \displaystyle\int\limits_{\mathbb{R}^n} \frac{\partial}{\partial m} U(x,m,t)(\xi)\Big(\mathcal{A}^* m(\xi) - \frac{1}{2}\beta^2 \Delta m(\xi) + \mathrm{div}\big(G\big(\xi,m,DU(\xi)\big)m(\xi)\big)\Big)d\xi \\ \quad - \dfrac{1}{2}\beta^2 \displaystyle\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \frac{\partial^2 U(x,m,t)}{\partial m^2}(\xi,\eta)Dm(\xi)Dm(\eta)d\xi d\eta + \beta^2\,\mathrm{div}\Big(\int\limits_{\mathbb{R}^n} \frac{\partial U(x,m,t)}{\partial m}(\xi)Dm(\xi)d\xi\Big) \\ \quad = H\big(x,m,DU(x)\big) + \displaystyle\int\limits_{\mathbb{R}^n} \frac{\partial}{\partial m} H\big(\xi,m,DU(\xi)\big)(x)m(\xi)d\xi; \\ U(x,m,T) = h(x,m) + \displaystyle\int\limits_{\mathbb{R}^n} \frac{\partial h(\xi,m)}{\partial m}(x)m(\xi)d\xi. \end{cases} \tag{3.12}$$

We note that this equation reduces to (2.10) when $\beta = 0$.

3.4. System of HJB–FP equations

We first check that we can derive from the Master equation a system of coupled stochastic HJB–FP equations. Consider the conditional probability density process corresponding to the optimal feedback $\hat{v}(x,m,DU(x,m,t))$, which is the solution of

$$\begin{cases} \partial_t m + \Big(\mathcal{A}^* m - \dfrac{1}{2}\beta^2 \Delta m + \mathrm{div}\big(G(x,m,DU)m\big)\Big)dt + \beta Dm.db(t) = 0, \\ m(x,0) = m_0(x). \end{cases} \tag{3.13}$$

Set $u(x,t) = U(x,m(t),t)$, we obtain

$$\begin{cases} -\partial_t u + \Big(\mathcal{A}u - \dfrac{1}{2}\beta^2 \Delta u\Big)dt + \beta^2\,\mathrm{div}\Big(\displaystyle\int\limits_{\mathbb{R}^n} \frac{\partial U(x,m,t)}{\partial m}(\xi)Dm(\xi)d\xi\Big)dt \\ \quad = \Big(H\big(x,m,Du(x)\big) + \displaystyle\int\limits_{\mathbb{R}^n} \frac{\partial}{\partial m} H\big(\xi,m,Du(\xi)\big)(x)m(\xi)d\xi\Big)dt + \beta \int\limits_{\mathbb{R}^n} \frac{\partial U(x,m,t)}{\partial m}(\xi)Dm(\xi)d\xi db(t), \\ u(x,T) = h(x,m) + \displaystyle\int\limits_{\mathbb{R}^n} \frac{\partial h(\xi,m)}{\partial m}(x)m(\xi)d\xi. \end{cases} \tag{3.14}$$

Let

$$B(x,t) = \int\limits_{\mathbb{R}^n} \frac{\partial U(x,m,t)}{\partial m}(\xi)Dm(\xi)d\xi, \tag{3.15}$$

we can rewrite (3.13), (3.14) as follows, by noting the Itô's correction term of u that involves the second derivative of U with respect to m,

$$\begin{cases} -\partial_t u + \left(\mathcal{A}u - \frac{1}{2}\beta^2\Delta u\right)dt + \beta^2 \operatorname{div} B(x,t)dt \\ \quad = \left(H\big(x,m,Du(x)\big) + \int\limits_{\mathbb{R}^n} \frac{\partial}{\partial m} H\big(\xi,m,Du(\xi)\big)(x)m(\xi)d\xi\right)dt + \beta B(x,t)db(t), \\ u(x,T) = h(x,m) + \int\limits_{\mathbb{R}^n} \frac{\partial h(\xi,m)}{\partial m}(x)m(\xi)d\xi. \end{cases}$$

$$\begin{cases} \partial_t m + \left(\mathcal{A}^* m - \frac{1}{2}\beta^2\Delta m + \operatorname{div}\big(G(x,m,Du)m\big)\right)dt + \beta Dmdb(t) = 0, \\ m(x,0) = m_0(x). \end{cases} \tag{3.16}$$

Since the equation for u is a backward stochastic partial differential equation (BSPDE), the solution is expressed by the pair $(u(x,t), B(x,t))$ which is adapted to the filtration $\mathcal{B}^t$.

3.5. Obtaining the system of stochastic HJB–FP equations by calculus of variations

In this section, we are going to check that the system (3.16) can be also obtained by calculus of variations techniques, without referring to the Master equation. This is similar to approach in the deterministic case, see [1]. We go back to the formulation (3.5), (3.6). Let $\hat{v}(x,t)$ be an optimal feedback (it is a random field adapted to $\mathcal{B}^t$). We denote $m(x,t) = m_{\hat{v}(\cdot)}(x,t)$, which is therefore the solution of

$$\begin{cases} \partial_t m + \left(\mathcal{A}^* m - \frac{1}{2}\beta^2\Delta m + \operatorname{div}\big(g\big(x,m(t),\hat{v}(x,t)\big)m\big)\right)dt + \beta Dmdb(t) = 0, \\ m(x,0) = m_0(x). \end{cases} \tag{3.17}$$

We can compute its Gâteaux differential

$$\tilde{m}(x,t) = \frac{d}{d\theta} m_{\hat{v}(\cdot)+\theta v(\cdot)}(x,t)|_{\theta=0},$$

which satisfies

$$\begin{cases} \partial_t \tilde{m} + \left(\mathcal{A}^* \tilde{m} - \frac{1}{2}\beta^2\Delta \tilde{m} + \operatorname{div}\big(g\big(x,m(t),\hat{v}(x,t)\big)\tilde{m}\big)\right)dt + \beta D\tilde{m}db(t) \\ \quad + \operatorname{div}\left(\left[\int\limits_{\mathbb{R}^n} \frac{\partial g(x,m,\hat{v}(x))}{\partial m}(\xi)\tilde{m}(\xi)d\xi + \frac{\partial g}{\partial v}\big(x,m,\hat{v}(x)\big)v(x)\right]m(x)\right)dt = 0, \\ \tilde{m}(x,0) = 0. \end{cases} \tag{3.18}$$

We next compute the Gâteaux differential of the cost functional

$$\begin{aligned} &\frac{d}{d\theta} J\big(\hat{v}(\cdot) + \theta v(\cdot)\big)\big|_{\theta=0} \\ &\quad = \mathbb{E}\Bigg[\int\limits_0^T \int\limits_{\mathbb{R}^n} f\big(x,m(t),\hat{v}(x,t)\big)\tilde{m}(x,t)dxdt + \int\limits_0^T \int\limits_{\mathbb{R}^n} \frac{\partial f}{\partial v}\big(x,m(t),\hat{v}(x,t)\big)v(x)m(x)dxdt \\ &\qquad + \int\limits_0^T \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \frac{\partial f}{\partial m}\big(x,m(t),\hat{v}(x,t)\big)(\xi)\tilde{m}(\xi,t)m(x,t)dxd\xi dt \\ &\qquad + \int\limits_{\mathbb{R}^n} h\big(x,m(T)\big)\tilde{m}(x,T)dx + \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \frac{\partial}{\partial m} h\big(x,m(T)\big)(\xi)\tilde{m}(\xi,T)m(x,T)dx\Bigg]. \end{aligned} \tag{3.19}$$

As an adjoint equation, consider the BSPDE:

$$\begin{cases} -\partial_t u = \Big(f\big(x,m,\hat{v}(x)\big) + Dug\big(x,m,\hat{v}(x)\big) - \mathcal{A}u + \dfrac{1}{2}\beta^2 \Delta u - \beta^2 \operatorname{div} B(x,t) \\ \qquad + \displaystyle\int_{\mathbb{R}^n} \frac{\partial}{\partial m} f\big(\xi,m,\hat{v}(\xi)\big)(x)m(\xi)d\xi + \int_{\mathbb{R}^n} Du(\xi)\frac{\partial}{\partial m} g\big(\xi,m,\hat{v}(\xi)\big)(x)m(\xi)d\xi \Big) dt \\ \qquad + \beta B(x,t)db(t), \\ u(x,T) = h(x,m) + \displaystyle\int_{\mathbb{R}^n} \frac{\partial}{\partial m} h(\xi,m)(x)m(\xi)d\xi. \end{cases} \tag{3.20}$$

We can then write

$$\begin{aligned} \frac{d}{d\theta} J\big(\hat{v}(\cdot)+\theta v(\cdot)\big)\big|_{\theta=0} = \mathbb{E}\int_0^T \Big[\int_{\mathbb{R}^n} &\frac{\partial f}{\partial v}\big(x,m(t),\hat{v}(x,t)\big)v(x)m(x)dx \\ &+ \int_{\mathbb{R}^n} \tilde{m}(x,t)\Big[-\partial_t u + \Big(\mathcal{A}u - \frac{1}{2}\beta^2\Delta u + \beta^2 \operatorname{div} B \Big) - Du.g\big(x,m,\hat{v}(x)\big) \\ &- \int_{\mathbb{R}^n} Du(\xi)\frac{\partial}{\partial m} g\big(\xi,m,\hat{v}(\xi)\big)(x)m(\xi)d\xi \Big] dx \Big] dt \\ &+ \mathbb{E}\int_{\mathbb{R}^n} u(x,T)\tilde{m}(x,T)dx. \end{aligned}$$

But

$$\partial_t \int_{\mathbb{R}^n} u(x,t)\tilde{m}(x,t)dx = \int_{\mathbb{R}^n} \partial_t u(x,t)\tilde{m}(x,t)dx + \int_{\mathbb{R}^n} \partial_t \tilde{m}(x,t)u(x,t)dx + \beta^2 \int_{\mathbb{R}^n} D\tilde{m}(x,t)B(x,t)dx,$$

therefore

$$\partial_t \int_{\mathbb{R}^n} u(x,t)\tilde{m}(x,t)dx - \int_{\mathbb{R}^n} \partial_t u(x,t)\tilde{m}(x,t)dx + \beta^2 \int_{\mathbb{R}^n} \operatorname{div} B(x,t)\, \tilde{m}(x,t)dx = \int_{\mathbb{R}^n} \partial_t \tilde{m}(x,t)u(x,t)dx$$

which implies

$$\begin{aligned} \frac{d}{d\theta} J\big(\hat{v}(\cdot)+\theta v(\cdot)\big)\big|_{\theta=0} = \mathbb{E}\int_0^T \int_{\mathbb{R}^n} &\frac{\partial f}{\partial v}\big(x,m(t),\hat{v}(x,t)\big)v(x)m(x)dxdt \\ &+ \mathbb{E}\int_0^T \int_{\mathbb{R}^n} u(x,t)\Big[\partial_t \tilde{m}(x,t) + \Big(\mathcal{A}^*\tilde{m} - \frac{1}{2}\beta^2\Delta\tilde{m} + \operatorname{div}\big(g\big(x,m(t),\hat{v}(x,t)\big)\tilde{m}\big) \Big) dt \\ &+ \operatorname{div}\Big(m(x)\int_{\mathbb{R}^n} \frac{\partial g(x,m,\hat{v}(x))}{\partial m}(\xi)\tilde{m}(\xi)d\xi \Big)\, dt \Big] dx. \end{aligned}$$

From (3.18) we obtain

$$\begin{aligned}\frac{d}{d\theta}J\big(\hat{v}(\cdot)+\theta v(\cdot)\big)\big|_{\theta=0} &= \mathbb{E}\int_0^T\int_{\mathbb{R}^n}\frac{\partial f}{\partial v}\big(x,m(t),\hat{v}(x,t)\big)v(x)m(x)dxdt \\ &\quad -\mathbb{E}\int_0^T\int_{\mathbb{R}^n}u(x,t)\operatorname{div}\bigg(\frac{\partial g}{\partial v}\big(x,m,\hat{v}(x)\big)v(x)m(x)\bigg)dxdt \\ &= \mathbb{E}\int_0^T\int_{\mathbb{R}^n}\bigg(\frac{\partial f}{\partial v}\big(x,m(t),\hat{v}(x,t)\big)v(x)+Du\frac{\partial g}{\partial v}\big(x,m,\hat{v}(x,t)\big)v(x)\bigg)m(x)dxdt,\end{aligned}$$

and since v is arbitrary, for if the support of m is $\mathbb{R}^n$, $\hat{v}(x,t)$ satisfies (because of optimality)

$$\frac{\partial f}{\partial v}\big(x,m(t),\hat{v}(x,t)\big)+Du\frac{\partial g}{\partial v}\big(x,m,\hat{v}(x,t)\big)=0,$$

and therefore $\hat{v}(x,t)=\hat{v}(x,m(t),Du(x,t))$.

One obtains immediately that the pair $(u(x,t),m(x,t))$ is a solution of the system of stochastic HJB–FP equations (3.16).

4. Stochastic mean field games

4.1. General comments

How to obtain a system of coupled HJB–FP equations in the case of stochastic mean field games? In the deterministic case, see Section 2.2, the idea was to consider $m(t)$ as a parameter, and to solve a standard stochastic control problem. For this problem, one can use standard Dynamic Programming to obtain the HJB equation, depending on the parameter $m(t)$. One expresses next the fixed point property, namely that $m(t)$ is the probability density of the optimal state. This leads to the FP equation.

By analogy with what was done in the framework of mean field type control, the HJB equation becomes stochastic, and $m(t)$ becomes the conditional probability density. This motivates the model we develop in this section.

4.2. The model

Adopt the notation of Section 3.1. We recall that the state equation is the solution of

$$\begin{cases} dx=g\big(x,m(t),v(x,t)\big)dt+\sigma(x)dw+\beta db(t),\\ x(0)=x_0,\end{cases}\tag{4.1}$$

and $\mathcal{B}^t=\sigma(b(s),s\le t)$, $\mathcal{F}^t=\sigma(x_0,b(s),w(s),s\le t)$. This time $m(t)$ is a given process adapted to $\mathcal{B}^t$ with values in $L^2(\mathbb{R}^n)$. We again consider feedback controls which are field processes adapted to the filtration $\mathcal{B}^t$.

We follow the theory developed by Shi-ge Peng [9]. We define the function

$$u(x,t)=\inf_{v(\cdot)}\mathbb{E}^{\mathcal{B}^t}\Bigg[\int_t^T f\big(x(s),m(s),v\big(x(s),s\big)\big)ds+h\big(x(T),m(T)\big)\Bigg],\tag{4.2}$$

and show that it satisfies a stochastic Hamilton–Jacobi–Bellman equation. Although, it is possible to proceed directly and formally from the definition of $u(x,t)$, to find out the stochastic HJB equation, the best is to postulate the equation and to use a verification argument. The equation is

$$\begin{cases} -\partial_t u + \left(\mathcal{A}u - \frac{1}{2}\beta^2 \Delta u\right) dt + \beta^2 \operatorname{div} B(x,t) dt = H\left(x, m(t), Du(x)\right) dt + \beta B(x,t) db(t), \\ u(x,T) = h\left(x, m(T)\right). \end{cases} \tag{4.3}$$

If we can solve (4.3) for the pair $(u(x,t), B(x,t))$, $\mathcal{B}^t$-adapted field processes, then the verification argument is as follows: for any $\mathcal{B}^t$-adapted field process $v(x,t)$, we write

$$H\left(x, m(t), Du(x,t)\right) \le f\left(x, m(t), v(x,t)\right) + Du(x,t) g\left(x, m(t), v(x,t)\right).$$

Consider the process $x(t)$, the solution of the state equation (4.1), we compute the Ito differential $d\,u(x(t),t)$ by using a generalized Itô's formula due to Kunita, which gives

$$\begin{aligned} du\left(x(t),t\right) = {} & \left(Dug\left(x(t), m(t), v\left(x(t),t\right)\right) + \frac{1}{2}\operatorname{tr}\left(a\left(x(t)\right) D^2 u\right) + \frac{\beta^2}{2}\Delta u\right) dt \\ & + Du\left(\sigma\left(x(t)\right) dw + \beta db(t)\right) + \partial_t u\left(x(t),t\right) - \beta^2 \operatorname{div}\left(B\left(x(t),t\right)\right) dt. \end{aligned}$$

Hence we have the inequality, by using (4.3),

$$\begin{aligned} & h\left(x(T), m(T)\right) + \int_t^T f\left(x(s), m(s), v\left(x(s),s\right)\right) ds \\ & \quad \ge u(x,t) + \int_t^T Du\left(x(s)\right)\left(\sigma\left(x(s)\right) dw(s) + \beta db(s)\right) - \beta \int_t^T B\left(x(s)\right) db(s), \end{aligned}$$

from which we get

$$u(x,t) \le \mathbb{E}^{\mathcal{B}^t}\left[\int_t^T f\left(x(s), m(s), v\left(x(s),s\right)\right) ds + h\left(x(T), m(T)\right)\right],$$

and a similar development used for the optimal feedback yields the equality, hence the property (4.2) follows.

Next, consider the optimal feedback $\hat{v}(x, m(t), Du(x,t))$ and impose the fixed point property that $m(t)$ is conditional probability density of the optimal state, we get the stochastic FP equation

$$\begin{cases} \partial_t m + \left(\mathcal{A}^* m - \frac{1}{2}\beta^2 \Delta m + \operatorname{div}\left(G(x, m, Du) m\right)\right) dt + \beta D m db(t) = 0, \\ m(x,0) = m_0(x). \end{cases} \tag{4.4}$$

We thus have obtained the pair of HJB–FP equations for the stochastic mean field game problem, (4.3) and (4.4).

4.3. The Master equation

We shall derive the Master equation by writing $u(x,t) = U(x, m(t), t)$. We must have, by using (4.4),

$$\begin{aligned} \partial_t u = \Bigg[& \frac{\partial U}{\partial t} - \int_{\mathbb{R}^n} \frac{\partial U(x, m(t), t)}{\partial m}(\xi)\left(\mathcal{A}^* m(\xi) - \frac{1}{2}\beta^2 \Delta m(\xi) + \operatorname{div}\left(G\left(\xi, m(t), DU(\xi)\right) m(\xi)\right)\right) d\xi \\ & + \frac{1}{2}\beta^2 \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\partial^2 U(x, m(t), t)}{\partial m^2}(\xi, \eta) Dm(\xi) Dm(\eta) d\xi d\eta \Bigg] dt \end{aligned}$$

$$- \beta \int_{\mathbb{R}^n} \frac{\partial U(x,m(t),t)}{\partial m}(\xi) Dm(\xi) d\xi db(t).$$

Comparing terms with (4.3), and letting

$$B(x,t) = \int_{\mathbb{R}^n} \frac{\partial U(x,m(t),t)}{\partial m}(\xi) Dm(\xi) d\xi, \tag{4.5}$$

we have the master equation

$$\begin{cases} -\dfrac{\partial U}{\partial t} + \mathcal{A}U - \dfrac{1}{2}\beta^2 \Delta U + \displaystyle\int_{\mathbb{R}^n} \dfrac{\partial U(x,m,t)}{\partial m}(\xi)\Big(\mathcal{A}^* m(\xi) - \dfrac{1}{2}\beta^2 \Delta m(\xi) + \operatorname{div}\big(G\big(\xi,m,DU(\xi)\big)m(\xi)\big)\Big) d\xi \\ \qquad + \beta^2 \operatorname{div}\Big(\displaystyle\int_{\mathbb{R}^n} \dfrac{\partial U(x,m,t)}{\partial m}(\xi) Dm(\xi) d\xi\Big) - \dfrac{1}{2}\beta^2 \displaystyle\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \dfrac{\partial^2 U(x,m,t)}{\partial m^2}(\xi,\eta) Dm(\xi) Dm(\eta) d\xi d\eta \\ \quad = H\big(x,m,DU(x)\big), \\ U(x,m,T) = h(x,m). \end{cases} \tag{4.6}$$

Remark 1. It is very important to notice that the property

$$\frac{\partial U(x,m,t)}{\partial m}(\xi) = \frac{\partial U(\xi,m,t)}{\partial m}(x) \tag{4.7}$$

is true for the Master equation of the Mean Field type control problem (3.12) and not true for the Master equation of the Mean Field games problem, (4.6). We shall notice this discrepancy in the case of linear quadratic (LQ) problems, in which explicit formulas can be obtained.

4.4. Remarks on the approach of Carmona–Delarue [4]

Independently of our work, Carmona and Delarue [4] have put recently on Arxiv, an article in which they try to interpret the Master equation differently. To synthesize our approach, we state that there is a Bellman equation for the Mean field type control problem, in which the space variable is in an infinite dimensional space. The Master equation is obtained by taking the gradient in this space variable. This differentiation is taken in the sense of Frechet. It is also possible to derive from the Master equation a coupled system of stochastic HJB–FP equations, in which the solution of the HJB equation appears not as a value function, but as an adjoint equation for an infinite dimensional stochastic control problem. This system can be also obtained directly, and the Master equation is then a way to decouple the coupled equations. In the case of mean field games, there is no Bellman equation, so we introduce the system of HJB–FP equations directly, with a fixed point approach. The Master equation is then obtained by trying to decouple the HJB equation from the Fokker–Planck equation. Carmona and Delarue try to derive the Master equation from a common optimality principle of Dynamic Programming, with constraints. The differences between the Mean field games and Mean field type control cases stem from the way the minimization is performed. It results that for the Mean field type control, our Master equation is different from that of Carmona–Delarue. To some extent, our approach is more analytic (sufficient conditions) whereas the approach of Carmona–Delarue is more probabilistic (necessary conditions). In particular, Carmona–Delarue rely on the lift-up approach introduced by P.L. Lions in his lectures. This method connects functions of random variables to functions of probability measures. Both works are largely formal, and a full comparison is a daunting work, which is left for future work. It may be interesting to also combine these ideas.

5. Linear quadratic problems

5.1. Assumptions and general comments

For linear quadratic problems, we know that explicit formulas can be obtained. We shall see that we can solve explicitly the Master equation in both cases, and recover all results obtained in the LQ case. Symmetric and nonsymmetric Riccati equations naturally come in the present framework. It is important to keep in mind that in the Master equation, the argument m is a general element of $L^2(\mathbb{R}^n)$ and not necessarily a probability. When we use the value arising from the FP equation, we get of course a probability density, provided that the initial condition is a probability density. To synthesize formulas, we shall use the following notation: If $m \in L^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ then we set

$$m_1 = \int\limits_{\mathbb{R}^n} m(\xi)d\xi, \qquad y = \int\limits_{\mathbb{R}^n} \xi m(\xi)d\xi. \tag{5.1}$$

We then take

$$\begin{aligned} f(x,m,v) &= \frac{1}{2}\left[x^*Qx + v^*Rv + (x-Sy)^*\bar{Q}(x-Sy)\right], \\ g(x,m,v) &= Ax + \bar{A}y + Bv, \\ h(x,m) &= \frac{1}{2}\left[x^*Q_Tx + (x-S_Ty)^*\bar{Q}_T(x-S_Ty)\right], \\ \sigma(x) &= \sigma, \end{aligned} \tag{5.2}$$

and hence $a(x) = a = \sigma\sigma^*$. We deduce easily

$$H(x,m,q) = \frac{1}{2}x^*(Q+\bar{Q})\,x - x^*\bar{Q}Sy + \frac{1}{2}y^*S^*\bar{Q}Sy - \frac{1}{2}q^*BR^{-1}B^*q + q^*(Ax+\bar{A}y) \tag{5.3}$$

and

$$G(x,m,q) = Ax + \bar{A}y - BR^{-1}B^*q. \tag{5.4}$$

5.2. Mean field type control Master equation

We begin with Bellman equation (3.11)

$$\begin{cases} \dfrac{\partial V}{\partial t} - \displaystyle\int\limits_{\mathbb{R}^n}\left(\mathcal{A}U - \frac{1}{2}\beta^2\Delta U\right)(\xi,m,t)m(\xi)d\xi \\ \qquad + \dfrac{1}{2}\beta^2 \displaystyle\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \frac{\partial U(\xi,m,t)}{\partial m}(\eta)Dm(\xi)Dm(\eta)d\xi d\eta + \int\limits_{\mathbb{R}^n} H(\xi,m,DU)m(\xi)d\xi = 0, \\ V(m,T) = \displaystyle\int\limits_{\mathbb{R}^n} h(\xi,m)m(\xi)d\xi, \end{cases} \tag{5.5}$$

in which

$$U(\xi,m,t) = \frac{\partial V(m,t)}{\partial m}(\xi). \tag{5.6}$$

We can rewrite (5.5) under the present linear quadratic setting

$$\begin{cases} \dfrac{\partial V}{\partial t} + \displaystyle\int_{\mathbb{R}^n} \left(\frac{1}{2} \operatorname{tr} a D_\xi^2 U(\xi, m, t) + \frac{1}{2} \beta^2 \Delta_\xi U(\xi, m, t) \right) m(\xi) d\xi \\ \quad + \dfrac{1}{2} \beta^2 \displaystyle\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sum_{i=1}^n \frac{\partial}{\partial \xi_i} \frac{\partial}{\partial \eta_i} \left(\frac{\partial U(\xi, m, t)}{\partial m}(\eta) \right) m(\xi) m(\eta) d\xi d\eta \\ \quad + \displaystyle\int_{\mathbb{R}^n} \bigg[\frac{1}{2} \xi^* (Q + \bar{Q}) \xi - \xi^* \bar{Q} S y + \frac{1}{2} y^* S^* \bar{Q} S y \\ \quad - \dfrac{1}{2} \big(DU(\xi, m, t) \big)^* B R^{-1} B^* DU(\xi, m, t) + \big(DU(\xi, m, t) \big)^* (A\xi + \bar{A} y) \bigg] m(\xi) d\xi = 0, \\ V(m, T) = \displaystyle\int_{\mathbb{R}^n} \frac{1}{2} \xi^* (Q_T + \bar{Q}_T) \xi m(\xi) d\xi - \frac{1}{2} y^* \big(S_T^* \bar{Q}_T + \bar{Q}_T S_T \big) y + \frac{1}{2} y^* S_T^* \bar{Q}_T S_T y \, m_1. \end{cases} \tag{5.7}$$

We look for a solution V in (5.7) of the form in which $P(t)$ and $\Sigma(t, m_1)$ are symmetric matrices

$$V(m, t) = \frac{1}{2} \int_{\mathbb{R}^n} \xi^* P(t) \xi m(\xi) d\xi + \frac{1}{2} y^* \Sigma(t, m_1) y + \lambda(t, m_1). \tag{5.8}$$

Clearly, the terminal condition of (5.7) yields that

$$P(T) = Q_T + \bar{Q}_T, \qquad \Sigma(T, m_1) = S_T^* \bar{Q}_T S_T m_1 - \big(S_T^* \bar{Q}_T + \bar{Q}_T S_T \big), \qquad \lambda(T, m_1) = 0. \tag{5.9}$$

Next

$$U(\xi, m, t) = \frac{1}{2} \xi^* P(t) \xi + y^* \Sigma(t, m_1) \xi + \frac{1}{2} y^* \frac{\partial \Sigma(t, m_1)}{\partial m_1} y + \frac{\partial \lambda(t, m_1)}{\partial m_1}, \tag{5.10}$$

and hence

$$D_\xi U(\xi, m, t) = P(t) \xi + \Sigma(t, m_1) y,$$
$$D_\xi^2 U(\xi, m, t) = P(t),$$
$$\frac{\partial U(\xi, m, t)}{\partial m}(\eta) = \eta^* \Sigma(t, m_1) \xi + y^* \frac{\partial \Sigma(t, m_1)}{\partial m_1} (\xi + \eta) + \frac{1}{2} y^* \frac{\partial^2 \Sigma(t, m_1)}{\partial m_1^2} y + \frac{\partial^2 \lambda(t, m_1)}{\partial m_1^2}.$$

We can see that the property (4.7) is satisfied, thanks to the symmetry of the matrix $\Sigma(t, m_1)$. We need

$$\sum_{i=1}^n \frac{\partial}{\partial \xi_i} \frac{\partial}{\partial \eta_i} \frac{\partial U(\xi, m, t)}{\partial m}(\eta) = \operatorname{tr} \Sigma(t, m_1).$$

With these calculations, we can proceed on Eq. (5.7) and obtain

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{R}^n} \xi^* \frac{d}{dt} P(t) \xi m(\xi) d\xi + \frac{1}{2} y^* \frac{\partial}{\partial t} \Sigma(t, m_1) y + \frac{\partial \lambda(t, m_1)}{\partial t} \\ & \quad + \left(\frac{1}{2} \operatorname{tr} a P(t) + \frac{\beta^2}{2} \operatorname{tr} P(t) \right) m_1 + \frac{1}{2} \beta^2 \operatorname{tr} \Sigma(t, m_1) (m_1)^2 + \int_{\mathbb{R}^n} \bigg[\frac{1}{2} \xi^* (Q + \bar{Q}) \xi - \xi^* \bar{Q} S y + \frac{1}{2} y^* S^* \bar{Q} S y \\ & \quad - \frac{1}{2} \big(P(t) \xi + \Sigma(t, m_1) y \big)^* B R^{-1} B^* \big(P(t) \xi + \Sigma(t, m_1) y \big) + \big(P(t) \xi + \Sigma(t, m_1) y \big)^* (A\xi + \bar{A} y) \bigg] m(\xi) d\xi = 0. \end{aligned}$$

We can identify terms and obtain

$$\frac{\partial \lambda(t,m_1)}{\partial t} + \left(\frac{1}{2}\operatorname{tr} aP(t) + \frac{\beta^2}{2}\operatorname{tr} P(t)\right)m_1 + \frac{1}{2}\beta^2 \operatorname{tr} \Sigma(t,m_1)(m_1)^2 = 0.$$

Therefore, from the final condition $\lambda(T,m_1) = 0$, it follows that

$$\lambda(t,m_1) = \int_t^T \left(\frac{1}{2}\operatorname{tr} aP(s) + \frac{\beta^2}{2}\operatorname{tr} P(s)\right)ds\, m_1 + \frac{1}{2}\beta^2 \int_t^T \operatorname{tr} \Sigma(s,m_1)ds\,(m_1)^2. \tag{5.11}$$

Recalling that $y = \int_{\mathbb{R}^n} \xi m(\xi)d\xi$ and identifying quadratic terms in ξ (within the integral) and in y respectively, it follows easily that

$$\begin{cases} \dfrac{dP}{dt} + PA + A^*P - PBR^{-1}B^*P + Q + \bar{Q} = 0, \\ P(T) = Q_T + \bar{Q}_T, \end{cases} \tag{5.12}$$

and

$$\begin{cases} \dfrac{d\Sigma}{dt} + \Sigma\big(A + \bar{A}m_1 - BR^{-1}B^*P\big) + \big(A + \bar{A}m_1 - BR^{-1}B^*P\big)^*\Sigma \\ \qquad - \Sigma BR^{-1}B^*\Sigma m_1 + S^*\bar{Q}Sm_1 - \bar{Q}S - S^*\bar{Q} + P\bar{A} + \bar{A}^*P = 0, \\ \Sigma(T,m_1) = S_T^*\bar{Q}_T S_T\, m_1 - \big(S_T^*\bar{Q}_T + \bar{Q}_T S_T\big). \end{cases} \tag{5.13}$$

We obtain formula (5.8) with the values of $P(t)$, $\Sigma(t,m_1)$, $\lambda(t,m_1)$ given by Eqs. (5.12), (5.13), (5.11). We next turn to the Master equation. The function $U(x,m,t)$ is given by (5.10). Let us set $\Gamma(t,m_1) = \frac{\partial \Sigma(t,m_1)}{\partial m_1}$. From (5.13) we obtain easily

$$\begin{cases} \dfrac{d\Gamma}{dt} + \Gamma\big(A + \bar{A}m_1 - BR^{-1}B^*(P + \Sigma m_1)\big) + \big(A + \bar{A}m_1 - BR^{-1}B^*(P + \Sigma m_1)\big)^*\Gamma \\ \qquad + S^*\bar{Q}S - \Sigma BR^{-1}B^*\Sigma + \Sigma\bar{A} + \bar{A}^*\Sigma = 0, \\ \Gamma(T,m_1) = S_T^*\bar{Q}_T S_T. \end{cases} \tag{5.14}$$

So we can write

$$\begin{aligned} U(x,m,t) &= \frac{1}{2}x^*P(t)x + y^*\Sigma(t,m_1)x + \frac{1}{2}y^*\Gamma(t,m_1)y \\ &\quad + \int_t^T \left(\frac{1}{2}\operatorname{tr} aP(s) + \frac{\beta^2}{2}\operatorname{tr} P(s)\right)ds + \frac{1}{2}\beta^2 \int_t^T \operatorname{tr} \Gamma(s,m_1)ds\,(m_1)^2 + \beta^2 \int_t^T \operatorname{tr} \Sigma(s,m_1)ds m_1. \end{aligned} \tag{5.15}$$

We want to check that this functional is the solution of the Master equation (3.12). We rewrite the Master equation as follows under the linear quadratic setting

$$\begin{cases} -\dfrac{\partial U}{\partial t} - \dfrac{1}{2}\operatorname{tr} aD^2U - \dfrac{1}{2}\beta^2\Delta U \\ \quad - \displaystyle\int_{\mathbb{R}^n} \left(\frac{1}{2}\operatorname{tr} aD_\xi^2 \frac{\partial}{\partial m}U(x,m,t)(\xi) + \frac{1}{2}\beta^2\Delta_\xi \frac{\partial}{\partial m}U(x,m,t)(\xi)\right) m(\xi)d\xi \\ \quad - \displaystyle\int_{\mathbb{R}^n} D_\xi \frac{\partial}{\partial m}U(x,m,t)(\xi).G\big(\xi,m,DU(\xi,m,t)\big)m(\xi)d\xi \\ \quad - \dfrac{1}{2}\beta^2 \displaystyle\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \sum_{i=1}^n \frac{\partial}{\partial \xi_i}\frac{\partial}{\partial \eta_i}\left(\frac{\partial^2 U(x,m,t)}{\partial m^2}(\xi,\eta)\right) m(\xi)m(\eta)d\xi d\eta \\ \quad - \beta^2 \displaystyle\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\int_{\mathbb{R}^n} \frac{\partial}{\partial \xi_i}\frac{\partial U(x,m,t)}{\partial m}(\xi)m(\xi)d\xi\right) \\ \quad = \dfrac{1}{2}x^*(Q+\bar{Q})x - x^*\big(\bar{Q}S + S^*\bar{Q}\big)y + \dfrac{1}{2}y^*S^*\bar{Q}Sy + y^*S^*\bar{Q}S\,xm_1 \\ \quad - \dfrac{1}{2}\big(DU(x,m,t)\big)^* BR^{-1}B^*DU(x,m,t) + \big(DU(x,m,t)\big)^*(Ax+\bar{A}y) \\ \quad + \displaystyle\int_{\mathbb{R}^n} \big(DU(\xi,m,t)\big)^* m(\xi)d\xi\,\bar{A}x, \\ U(x,m,T) = \dfrac{1}{2}x^*(Q_T+\bar{Q}_T)x - x^*\big(\bar{Q}_TS_T + S_T^*\bar{Q}_T\big)y + \dfrac{1}{2}y^*S_T^*\bar{Q}_TS_Ty + y^*S_T^*\bar{Q}_TS_T\,xm_1. \end{cases} \tag{5.16}$$

We look for a solution of the form

$$U(x,m,t) = \frac{1}{2}x^*P(t)x + y^*\Sigma(t,m_1)x + \frac{1}{2}y^*\Gamma(t,m_1)y + \mu(t,m_1) \tag{5.17}$$

with $\Sigma(t,m_1)$, $\Gamma(t,m_1)$ symmetric. We can calculate that

$$\begin{aligned} DU(x,m,t) &= P(t)x + \Sigma(t,m_1)y, \\ D^2U(x,m,t) &= P(t), \\ \frac{\partial}{\partial m}U(x,m,t)(\xi) &= \big(x^*\Sigma(t,m_1) + y^*\Gamma(t,m_1)\big)\xi + y^*\frac{\partial}{\partial m_1}\Sigma(t,m_1)x \\ &\quad + \frac{1}{2}y^*\frac{\partial}{\partial m_1}\Gamma(t,m_1)y + \frac{\partial}{\partial m_1}\mu(t,m_1), \\ D_\xi\frac{\partial}{\partial m}U(x,m,t)(\xi) &= \Sigma(t,m_1)x + \Gamma(t,m_1)y, \\ D_\xi^2\frac{\partial}{\partial m}U(x,m,t)(\xi) &= 0, \\ \frac{\partial^2}{\partial m^2}U(x,m,t)(\xi,\eta) &= \eta^*\Gamma(t,m_1)\xi + x^*\frac{\partial}{\partial m_1}\Sigma(t,m_1)(\xi+\eta) + y^*\frac{\partial}{\partial m_1}\Gamma(t,m_1)(\xi+\eta) \\ &\quad + y^*\frac{\partial^2}{\partial m_1^2}\Sigma(t,m_1)x + \frac{1}{2}y^*\frac{\partial^2}{\partial m_1^2}\Gamma(t,m_1)y + \frac{\partial^2}{\partial m_1^2}\mu(t,m_1), \\ \sum_{i=1}^n \frac{\partial}{\partial \xi_i}\frac{\partial}{\partial \eta_i}\left(\frac{\partial^2 U(x,m,t)}{\partial m^2}(\xi,\eta)\right) &= \operatorname{tr}\,\Gamma(t,m_1), \\ \sum_{i=1}^n \frac{\partial}{\partial x_i}\frac{\partial}{\partial \xi_i}\left(\frac{\partial}{\partial m}U(x,m,t)(\xi)\right) &= \operatorname{tr}\,\Sigma(t,m_1). \end{aligned}$$

Substituting all these results in the Master equation (5.16) yields

$$
\begin{aligned}
&-\left(\frac{1}{2}x^*\frac{d}{dt}P(t)x + y^*\frac{d}{d}\Sigma(t,m_1)x + \frac{1}{2}y^*\frac{d}{dt}\Gamma(t,m_1)y + \frac{d}{dt}\mu(t,m_1)\right) - \left(\frac{1}{2}\operatorname{tr} aP(t) + \frac{\beta^2}{2}\operatorname{tr} P(t)\right)\\
&\quad - \big(\Sigma(t,m_1)x + \Gamma(t,m_1)y\big)^*\big(Ay + \bar{A}ym_1 - BR^{-1}B^*(Py + \Sigma ym_1)\big)\\
&\quad - \frac{\beta^2}{2}\operatorname{tr} \Gamma(t,m_1)m_1^2 - \beta^2\operatorname{tr} \Sigma(t,m_1)m_1\\
&= \frac{1}{2}x^*(Q+\bar{Q})x - x^*\big(\bar{Q}S + S^*\bar{Q}\big)y + \frac{1}{2}y^*S^*\bar{Q}Sy + y^*S^*\bar{Q}S\,xm_1\\
&\quad - \frac{1}{2}\big(P(t)x + \Sigma(t,m_1)y\big)^*BR^{-1}B^*\big(P(t)x + \Sigma(t,m_1)y\big)\\
&\quad + \big(P(t)x + \Sigma(t,m_1)y\big)^*(Ax + \bar{A}y) + \big(P(t)y + \Sigma(t,m_1)ym_1\big)^*\bar{A}x.
\end{aligned}
$$

Comparing coefficients, one checks easily that $P(t)$, $\Sigma(t,m_1)$, $\Gamma(t,m_1)$ satisfy Eqs. (5.12), (5.13), (5.14). Hence $\Sigma(t,m_1)$ is symmetric and $\Gamma(t,m_1) = \frac{\partial \Sigma(t,m_1)}{\partial m_1}$. Also

$$
\frac{d}{dt}\mu(t,m_1) + \frac{1}{2}\operatorname{tr} aP(t) + \frac{\beta^2}{2}\operatorname{tr} P(t) + \frac{\beta^2}{2}\operatorname{tr} \Gamma(t,m_1)m_1^2 + \beta^2\operatorname{tr} \Sigma(t,m_1)m_1 = 0.
$$

Therefore $\mu(t,m_1) = \frac{\partial \lambda(t,m_1)}{\partial m_1}$ and we recover the formula $U(x,m,t) = \frac{\partial V(m,t)}{\partial m}(x)$. We can state the following proposition:

Proposition 2. *If we assume (5.2), then the solution of the Bellman equation (5.5) is given by formula (5.8) with $P(t)$, $\Sigma(t,m_1)$, $\lambda(t,m_1)$ given respectively by Eqs. (5.12), (5.13), (5.11). The solution of the Master equation (3.12) is given by formula (5.15), in which $\Gamma(t,m_1)$ is the solution of (5.14).*

5.3. System of stochastic HJB–FP equations for Mean Field type control

We now consider the solution of the stochastic Fokker–Planck equation

$$
\begin{cases}
\partial_t m + \left(\mathcal{A}^* m - \frac{1}{2}\beta^2\Delta m + \operatorname{div}\big(G(x,m,DU)m\big)\right)dt + \beta Dm\,db(t) = 0,\\
m(x,0) = m_0(x)
\end{cases}
\tag{5.18}
$$

for the linear quadratic Mean Field type control problem, assuming the initial condition m_0 is Gaussian with mean $\bar{x}_0$ and covariance matrix Π_0. If we call $m(t) = m(x,t)$ the solution, then

$$
m_1(t) = \int\limits_{\mathbb{R}^n} m(\xi,t)d\xi = 1, \qquad y(t) = \int\limits_{\mathbb{R}^n} \xi m(\xi,t)d\xi.
$$

Next

$$
\begin{gathered}
DU\big(x,m(t),t\big) = P(t)x + \Sigma(t)y(t),\ \Sigma(t) = \Sigma(t,1),\\
G\big(x,m,DU\big(x,m(t),t\big)\big) = \big(A - BR^{-1}B^*P(t)\big)x + \big(\bar{A} - BR^{-1}B^*\Sigma(t)\big)y(t).
\end{gathered}
$$

From (5.18) by using the test function x, and then integrating in x, we get easily that

$$
\begin{cases}
dy = \big(A + \bar{A} - BR^{-1}B^*P(t)\big)y(t)dt - BR^{-1}B^*\Sigma(t)y(t)dt + \beta db(t),\\
y(0) = \bar{x}_0.
\end{cases}
\tag{5.19}
$$

We define

$$u(x,t) = U\big(x, m(t), t\big) = \frac{1}{2}x^*P(t)x + x^*r(t) + s(t)$$

with

$$r(t) = \Sigma(t)y(t), \qquad s(t) = \frac{1}{2}y(t)^*\Gamma(t)y(t) + \mu(t) \tag{5.20}$$

in which $\Gamma(t) = \Gamma(t,1)$ and $\mu(t) = \mu(t,1)$. An easy calculation, by taking account of (5.13) and (5.19) yields

$$\begin{cases} -dr = \big(A^* + \bar{A}^* - PBR^{-1}B^*\big)rdt + \big(S^*\bar{Q}S - \bar{Q}S - S^*\bar{Q} + P\bar{A} + \bar{A}^*P\big)ydt - \beta\Sigma db(t), \\ r(T) = \big(S_T^*\bar{Q}_TS_T\,\mathbf{1} - S_T^*\bar{Q}_T - \bar{Q}_TS_T\big)y(T), \end{cases} \tag{5.21}$$

and we can rewrite (5.19) as

$$\begin{cases} dy = \big(A + \bar{A} - BR^{-1}B^*P(t)\big)y(t)dt - BR^{-1}B^*r(t)dt + \beta db(t), \\ y(0) = \bar{x}_0, \end{cases} \tag{5.22}$$

and the pair $(y(t), r(t))$ becomes the solution of a system of forward–backward SDE. Considering the fundamental matrix $\Phi_P(t,s)$ associated to the matrix $A + \bar{A} - BR^{-1}B^*P(t)$:

$$\begin{cases} \dfrac{\partial\Phi_P(t,s)}{\partial t} = \big(A + \bar{A} - BR^{-1}B^*P(t)\big)\Phi_P(t,s), \quad \forall t > s, \\ \Phi_P(s,s) = I, \end{cases}$$

then we can write

$$\begin{aligned} r(t) = {} & \Phi_P^*(T,t)\big(S_T^*\bar{Q}_TS_T\,\mathbf{1} - S_T^*\bar{Q}_T - \bar{Q}_TS_T\big)y(T) \\ & + \int_t^T \Phi_P^*(s,t)\big(S^*\bar{Q}S - \bar{Q}S - S^*\bar{Q} + P(s)\bar{A} + \bar{A}^*P(s)\big)y(s)ds - \beta\int_t^T \Phi_P^*(s,t)\Sigma(s)db(s). \end{aligned}$$

This relation implies

$$\begin{aligned} r(t) = {} & \Phi_P^*(T,t)\big(S_T^*\bar{Q}_TS_T\mathbf{1} - S_T^*\bar{Q}_T - \bar{Q}_TS_T\big)\mathbb{E}^{\mathcal{B}^t}y(T) \\ & + \int_t^T \Phi_P^*(s,t)\big(S^*\bar{Q}S - \bar{Q}S - S^*\bar{Q} + P(s)\bar{A} + \bar{A}^*P(s)\big)\mathbb{E}^{\mathcal{B}^t}y(s)ds. \end{aligned} \tag{5.23}$$

In the system (5.22), (5.23) the external function $\Sigma(s)$ does not appear anymore, but $r(t)$ is the solution of an integral equation, instead of a backward SDE. Finally, from (5.20), by taking differentiation and then integrating from t to T, we have

$$\begin{aligned} s(t) = {} & \frac{1}{2}y(T)^*S_T^*\bar{Q}_TS_Ty(T) + \int_t^T \left(\frac{1}{2}\operatorname{tr} aP(s) + \frac{\beta^2}{2}\operatorname{tr} P(s) + \beta^2\operatorname{tr}\,\Sigma(s)\right)ds \\ & + \int_t^T \left(\frac{1}{2}y(s)^*S^*QSy(s) - \frac{1}{2}r(s)^*BR^{-1}B^*r(s) + r(s)^*\bar{A}y(s)\right)ds - \beta\int_t^T y(s)^*\Gamma(s)db(s). \end{aligned} \tag{5.24}$$

Since $s(t)$ is adapted to $\mathcal{B}^t$, we can write

$$s(t) = \int_t^T \left(\frac{1}{2}\operatorname{tr} aP(s) + \frac{\beta^2}{2}\operatorname{tr} P(s) + \beta^2 \operatorname{tr} \Sigma(s)\right)ds + \frac{1}{2}\mathbb{E}^{\mathcal{B}^t} y(T)^* S_T^* \bar{Q}_T S_T y(T)$$
$$+ \mathbb{E}^{\mathcal{B}^t} \int_t^T \left(\frac{1}{2} y(s)^* S^* Q S y(s) - \frac{1}{2} r(s)^* B R^{-1} B^* r(s) + r(s)^* \bar{A} y(s)\right) ds. \tag{5.25}$$

The results obtained contains that in Bensoussan et al. [2] as a special case when $\beta = 0$.

5.4. Mean Field Games Master equation

The Master equation (4.6) under the LQ setting is

$$\begin{cases} -\dfrac{\partial U}{\partial t} - \dfrac{1}{2}\operatorname{tr} aD^2U - \dfrac{1}{2}\beta^2 \Delta U \\ \quad - \displaystyle\int_{\mathbb{R}^n} \left(\frac{1}{2}\operatorname{tr} aD_\xi^2 \frac{\partial}{\partial m} U(x,m,t)(\xi) + \frac{1}{2}\beta^2 \Delta_\xi \frac{\partial}{\partial m} U(x,m,t)(\xi)\right) m(\xi) d\xi \\ \quad - \displaystyle\int_{\mathbb{R}^n} D_\xi \frac{\partial}{\partial m} U(x,m,t)(\xi). G\big(\xi, m, DU(\xi,m,t)\big) m(\xi) d\xi \\ \quad - \dfrac{1}{2}\beta^2 \displaystyle\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \sum_{i=1}^n \frac{\partial}{\partial \xi_i}\frac{\partial}{\partial \eta_i}\left(\frac{\partial^2 U(x,m,t)}{\partial m^2}(\xi,\eta)\right) m(\xi) m(\eta) d\xi d\eta \\ \quad - \beta^2 \displaystyle\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\int_{\mathbb{R}^n} \frac{\partial}{\partial \xi_i}\frac{\partial U(x,m,t)}{\partial m}(\xi) m(\xi) d\xi\right) \\ \quad = \dfrac{1}{2} x^*(Q+\bar{Q})x - x^*\bar{Q}Sy + \dfrac{1}{2} y^* S^* \bar{Q} S y \\ \quad - \dfrac{1}{2}\big(DU(x,m,t)\big)^* BR^{-1}B^* DU(x,m,t) + \big(DU(x,m,t)\big)^*(Ax + \bar{A}y), \\ U(x,m,T) = \dfrac{1}{2} x^*(Q_T + \bar{Q}_T)x - x^* \bar{Q}_T S_T y + \dfrac{1}{2} y^* S_T^* \bar{Q}_T S_T y. \end{cases} \tag{5.26}$$

We look for a solution of the form

$$U(x,m,t) = \frac{1}{2} x^* P(t) x + x^* \Sigma(t,m_1) y + \frac{1}{2} y^* \Gamma(t,m_1) y + \mu(t,m_1). \tag{5.27}$$

Using the terminal condition, we cannot have a symmetric $\Sigma(t,m_1)$ this time.
We can also compute the derivatives

$$\begin{aligned} DU(x,m,t) &= P(t)x + \Sigma(t,m_1)y, \\ D^2U(x,m,t) &= P(t), \\ \frac{\partial}{\partial m} U(x,m,t)(\xi) &= x^*\Sigma(t,m_1)\xi + y^*\Gamma(t,m_1)\xi + x^* \frac{\partial}{\partial m_1}\Sigma(t,m_1) y \\ &\quad + \frac{1}{2} y^* \frac{\partial}{\partial m_1}\Gamma(t,m_1) y + \frac{\partial}{\partial m_1}\mu(t,m_1), \\ D_\xi \frac{\partial}{\partial m} U(x,m,t)(\xi) &= \Sigma^*(t,m_1)x + \Gamma(t,m_1)y, \\ D_\xi^2 \frac{\partial}{\partial m} U(x,m,t)(\xi) &= 0, \end{aligned}$$

$$\frac{\partial^2}{\partial m^2}U(x,m,t)(\xi,\eta) = \left(x^*\frac{\partial}{\partial m_1}\Sigma(t,m_1) + y^*\frac{\partial}{\partial m_1}\Gamma(t,m_1)\right)(\xi+\eta) + \eta^*\Gamma(t,m_1)\xi + x^*\frac{\partial^2}{\partial m_1^2}\Sigma(t,m_1)y + \frac{1}{2}y^*\frac{\partial^2}{\partial m_1^2}\Gamma(t,m_1)y + \frac{\partial^2}{\partial m_1^2}\mu(t,m_1),$$

$$\sum_{i=1}^n \frac{\partial}{\partial \xi_i}\frac{\partial}{\partial \eta_i}\left(\frac{\partial^2 U(x,m,t)}{\partial m^2}(\xi,\eta)\right) = \operatorname{tr}\Gamma(t,m_1),$$

$$\sum_{i=1}^n \frac{\partial}{\partial x_i}\frac{\partial}{\partial \xi_i}\frac{\partial U(x,m,t)}{\partial m}(\xi) = \operatorname{tr}\Sigma(t,m_1).$$

We apply these formulas in the Master equation (5.26) to obtain

$$\begin{aligned}
&-\left(\frac{1}{2}x^*\frac{d}{dt}P(t)x + x^*\frac{d}{dt}\Sigma(t,m_1)y + \frac{1}{2}y^*\frac{d}{dt}\Gamma(t,m_1)y + \frac{d}{dt}\mu(t,m_1)\right) - \left(\frac{1}{2}\operatorname{tr} aP(t) + \frac{1}{2}\beta^2\operatorname{tr}P(t)\right) \\
&\quad - \big(\Sigma^*(t,m_1)x + \Gamma(t,m_1)y\big)^*\big(Ay + \bar{A}ym_1 - BR^{-1}B^*(Py + \Sigma ym_1)\big) \\
&\quad - \frac{\beta^2}{2}\operatorname{tr}\Gamma(t,m_1)m_1^2 - \beta^2\operatorname{tr}\Sigma(t,m_1)m_1 \\
&= \frac{1}{2}x^*(Q+\bar{Q})x - x^*\bar{Q}Sy + \frac{1}{2}y^*S^*\bar{Q}Sy \\
&\quad - \frac{1}{2}\big(P(t)x + \Sigma(t,m_1)y\big)^*BR^{-1}B^*\big(P(t)x + \Sigma(t,m_1)y\big) \\
&\quad + \big(P(t)x + \Sigma(t,m_1)y\big)^*(Ax + \bar{A}y).
\end{aligned}$$

Comparing coefficients, we obtain

$$\begin{cases} \dfrac{d}{dt}P(t) + PA + A^*P - PBR^{-1}B^*P + Q + \bar{Q} = 0, \\ P(T) = Q_T + \bar{Q}_T. \end{cases} \tag{5.28}$$

$$\begin{cases} \dfrac{d\Sigma}{dt} + \Sigma\big(A + \bar{A}m_1 - BR^{-1}B^*P\big) + \big(A^* - PBR^{-1}B^*\big)\Sigma - \Sigma BR^{-1}B^*\Sigma m_1 - \bar{Q}S + P\bar{A} = 0, \\ \Sigma(T,m_1) = -\bar{Q}_TS_T. \end{cases} \tag{5.29}$$

$$\begin{cases} \dfrac{d\Gamma}{dt} + \Gamma\big(A + \bar{A}m_1 - BR^{-1}B^*(P + \Sigma m_1)\big) + \big(A + \bar{A}m_1 - BR^{-1}B^*(P + \Sigma m_1)\big)^*\Gamma \\ \qquad + S^*\bar{Q}S - \Sigma BR^{-1}B^*\Sigma + \Sigma\bar{A} + \bar{A}^*\Sigma = 0, \\ \Gamma(T,m_1) = S_T^*\bar{Q}_TS_T. \end{cases} \tag{5.30}$$

$$\begin{cases} \dfrac{d}{dt}\mu(t,m_1) + \dfrac{1}{2}\operatorname{tr} aP(t) + \dfrac{\beta^2}{2}\operatorname{tr}P(t) + \dfrac{\beta^2}{2}\operatorname{tr}\Gamma(t,m_1)m_1^2 + \beta^2\operatorname{tr}\Sigma(t,m_1)m_1 = 0, \\ \mu(T,m_1) = 0. \end{cases} \tag{5.31}$$

Remark 3. The function $\Gamma(t,m_1)$ is no more the derivative of $\Sigma(t,m_1)$ with respect to m_1.

Proposition 4. *If we assume (5.2) then the solution of the Master equation (5.26) is given by formula (5.27), with $P(t)$ solution of (5.28), $\Sigma(t,m_1)$ solution of (5.29), $\Gamma(t,m_1)$ solution of (5.30) and $\mu(t,m_1)$ solution of (5.31).*

5.5. System of stochastic HJB–FP equations for mean field games

We consider in the LQ case the system of stochastic HJB–FP equations for Mean Field games, namely (4.3) and (4.4). We first consider the FP equation, which is the same as for the Mean Field type control, namely (5.18). With the notation of Section 5.3, we have

$$\begin{cases} dy = \big(A + \bar{A} - BR^{-1}B^*P(t)\big)y(t)dt - BR^{-1}B^*r(t)dt + \beta db(t), \\ y(0) = \bar{x}_0 \end{cases}$$

with $r(t) = \Sigma(t)y(t)$ and $\Sigma(t) = \Sigma(t,1)$. We obtain that $r(t)$ satisfies

$$\begin{cases} -dr = \big(A^* - PBR^{-1}B^*\big)rdt + (\bar{Q}S - P\bar{A})ydt - \beta\Sigma db(t), \\ r(T) = -\bar{Q}_T S_T y(T) \end{cases}$$

and

$$u(x,t) = U\big(x, m(t), t\big) = \frac{1}{2}x^*P(t)x + x^*r(t) + s(t).$$

We have

$$s(t) = \frac{1}{2}y^*\Gamma(t)y + \mu(t).$$

Since $\Gamma(t), y(t), \mu(t)$ are solutions of equations identical to the Mean Field type control, we again obtain

$$\begin{aligned} s(t) = {} & \frac{1}{2}y(T)^*S_T^*\bar{Q}_T S_T y(T) + \int_t^T \left(\frac{1}{2}\operatorname{tr} aP(s) + \frac{\beta^2}{2}\operatorname{tr} P(s) + \beta^2 \operatorname{tr} \Sigma(s)\right)ds \\ & + \int_t^T \left(\frac{1}{2}y(s)^*S^*QSy(s) - \frac{1}{2}r(s)^*BR^{-1}B^*r(s) + r(s)^*\bar{A}y(s)\right)ds - \beta\int_t^T y(s)^*\Gamma(s)db(s). \end{aligned}$$

The result coincides with that in Bensoussan [2] when $\beta = 0$.

6. NASH equilibrium for a finite number of players

6.1. The problem for a finite number of players

We consider here N players. Each of them has a state $x^i(t) \in \mathbb{R}^n$. We define the vector of states $x(t) \in \mathbb{R}^{nN}$ as

$$x(t) = \big(x^1(t), \cdots, x^N(t)\big).$$

We denote $x^i_l(t)$, $l = 1, \cdots, n$, the components of the state $x^i(t)$. The states evolve according to the following dynamics:

$$\begin{cases} dx^i(s) = g\big(x, v^i(x)\big)ds + \sigma\big(x^i\big)dw^i(s) + \beta db(s), \quad s > t, \\ x^i(t) = x^i \end{cases} \tag{6.1}$$

in which $v^i(x) \in \mathbb{R}^d$ is the control of player i. The processes $w^i(s)$ and $b(s)$ are independent standard Wiener processes in $\mathbb{R}^n$. We set, as usual, $a(x^i) = \sigma(x^i)\sigma(x^i)^*$. It is convenient to denote $v(x) = (v^1(x), \cdots, v^N(x))$. We introduce the cost functional of each player

$$J^i\big(x,t;v(\cdot)\big) = \mathbb{E}\left[\int_t^T f\big(x(s), v^i\big(x(s)\big)ds + h\big(x(T)\big)\right]. \tag{6.2}$$

We notice that the trajectories and cost functionals are linked only through the states and not through the controls. Consider the Hamiltonian

$$H(x,q) = \inf_v \big[f(x,v) + q.g(x,v)\big]. \tag{6.3}$$

Denote $\hat{v}(x,q)$ the minimizer in the Hamiltonian, then we set $G(x,q) = g(x, \hat{v}(x,q))$. We next consider the system of PDEs, $i = 1, \cdots, N$,

$$\begin{cases} -\dfrac{\partial u^i}{\partial t} - \dfrac{1}{2}\displaystyle\sum_{j=1}^N \operatorname{tr}\big(a(x^j)D^2_{x^j}u^i\big) - \dfrac{\beta^2}{2}\operatorname{tr}\displaystyle\sum_{j,k=1}^N D^2_{x^jx^k}u^i - \sum_{j\neq i} D_{x^j}u^i.G\big(x, D_{x^j}u^j\big) = H\big(x, D_{x^i}u^i\big), \\ u^i(x,T) = h(x), \end{cases} \tag{6.4}$$

and define the feedbacks

$$\hat{v}^i(x) = \hat{v}\big(x, D_{x^i}u^i(x)\big), \tag{6.5}$$

which form a Nash equilibrium for the differential game (6.1), (6.2). This means first that

$$u^i(x,t) = J^i\big(x,t;\hat{v}(\cdot)\big), \tag{6.6}$$

and if we use the notation

$$v(x) = \big(v^i(x), \bar{v}^i(x)\big),$$

in which $\bar{v}^i(x)$ represents all the components of $v(x)$, except $v^i(x)$, to emphasize the control of player i, then

$$u^i(x,t) \leq J^i\big(x,t;v^i(x), \bar{\hat{v}}^i(x)\big), \quad \forall v^i(x). \tag{6.7}$$

We next apply this framework in the following case. Consider $f(x,m,v)$, $g(x,m,v), h(x,m)$ as in the previous sections (with $x \in \mathbb{R}^n$), and assume this time that the argument m is no more an element of $L^2(\mathbb{R}^n)$ but a probability measure in $\mathbb{R}^n$. We define

$$f\big(x, v^i(x)\big) = f\left(x^i, \frac{1}{N-1}\sum_{j\neq i}\delta_{x^j}, v^i(x)\right), \qquad g\big(x, v^i(x)\big) = g\left(x^i, \frac{1}{N-1}\sum_{j\neq i}\delta_{x^j}, v^i(x)\right),$$
$$h(x) = h\left(x^i, \frac{1}{N-1}\sum_{j\neq i}\delta_{x^j}\right)$$

in which $x = (x^1, \cdots, x^N)$. The Hamiltonian becomes

$$H\left(x^i, \frac{1}{N-1}\sum_{j\neq i}\delta_{x^j}, q\right) = \inf_v\left(f\left(x^i, \frac{1}{N-1}\sum_{j\neq i}\delta_{x^j}, v\right) + q.g\left(x^i, \frac{1}{N-1}\sum_{j\neq i}\delta_{x^j}, v\right)\right).$$

Consider the optimal feedback $\hat{v}(x^i, \frac{1}{N-1}\sum_{j\neq i}\delta_{x^j}, q)$, we have

$$G\left(x^i, \frac{1}{N-1}\sum_{j\neq i}\delta_{x^j}, q\right) = g\left(x^i, \frac{1}{N-1}\sum_{j\neq i}\delta_{x^j}, \hat{v}\left(x^i, \frac{1}{N-1}\sum_{j\neq i}\delta_{x^j}, q\right)\right).$$

Hence the system of PDEs as in (6.4) becomes, for $i = 1, \cdots, N$,

$$\begin{cases} -\dfrac{\partial u^i}{\partial t} - \dfrac{1}{2}\displaystyle\sum_{j=1}^{N} \operatorname{tr}\big(a(x^j)D^2_{x^j}u^i\big) - \dfrac{\beta^2}{2}\operatorname{tr}\displaystyle\sum_{j,k=1}^{N} D^2_{x^j x^k}u^i - \sum_{j\neq i} D_{x^j}u^i.G\left(x^i, \frac{1}{N-1}\sum_{k\neq j}\delta_{x^k}, D_{x^j}u^j\right) \\ \quad = H\left(x^i, \dfrac{1}{N-1}\displaystyle\sum_{j\neq i}\delta_{x^j}, D_{x^i}u^i\right), \\ u^i(x,T) = h\left(x, \dfrac{1}{N-1}\displaystyle\sum_{j\neq i}\delta_{x^j}\right). \end{cases} \tag{6.8}$$

We want to interpret (6.8) as a Master equation, which is the way P.L. Lions has introduced the concept of Master equation.

6.2. Discussion on the derivative with respect to a probability measure

For a functional $F(m)$ where m is in $L^2(\mathbb{R}^n)$, we have defined the concept of derivative by simply using the concept of Gâteaux differentiability. When m represents the density of a probability measure, the functional becomes a functional of a probability measure, but of a special kind. Suppose that F extends to a general probability measure, then the concept of differentiability does not extend. For instance if x^j is a point in $\mathbb{R}^n$, neither can the concept of differentiability be extended to $F(\delta_{x^j})$, nor more generally to $F(\frac{\sum_{j=1}^K \delta_{x^j}}{K})$.

Nevertheless, we may have the following situation: the functional $F(m)$ is differentiable in m, and the function

$$\Phi(x) = \Phi\big(x^1, \cdots, x^N\big) = F\left(\frac{\sum_{j=1}^K \delta_{x^j}}{K}\right) \tag{6.9}$$

is differentiable in x. Note that differentiability refers to two different set up, one with respect to arguments in $L^2(\mathbb{R}^n)$ and the other one, with respect to arguments in $\mathbb{R}^{nN}$. We want to study the link between these two concepts.

Proposition 5. *Assume that $F(m)$ and $\Phi(x)$ are sufficiently differentiable in their own sense, and that expressions below are well-defined. Then we have the following relations:*

$$\int_{\mathbb{R}^n} D_\xi \frac{\partial F(m)}{\partial m}(\xi).B(\xi)m(\xi)d\xi = \sum_{j=1}^{K} D_{x^j}\Phi(x).B\big(x^j\big), \tag{6.10}$$

$$\int_{\mathbb{R}^n} \Delta_\xi \frac{\partial F(m)}{\partial m}(\xi)m(\xi)d\xi + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \operatorname{tr} D_\xi D_\eta\left(\frac{\partial^2 F(m)}{\partial m^2}(\xi,\eta)\right)m(\xi)m(\eta)d\xi d\eta = \sum_{j,k=1}^{K} \operatorname{tr}\big(D_{x^j}D_{x^k}\Phi(x)\big), \tag{6.11}$$

$$\int_{\mathbb{R}^n} \operatorname{tr}\left(a(\xi)D^2_\xi\left(\frac{\partial F(m)}{\partial m}(\xi)\right)\right)m(\xi)d\xi \approx \sum_{j=1}^{K} \operatorname{tr}\big(a\big(x^j\big)D^2_{x^j}\Phi(x)\big), \quad \textit{for large } K, \tag{6.12}$$

where $m = \frac{1}{K}\sum_{j=1}^K \delta_{x^j}$.

Proof. We first comment how to understand these relations. The left hand side makes sense for sufficiently smooth functionals $F(m)$. The results are functionals of m, defined on $L^2(\mathbb{R}^n)$. Suppose that we can interpret them in the case m is replaced by $\frac{\sum_{j=1}^K \delta_{x^j}}{K}$. We obtained functions of $x = (x^1, \cdots, x^N) \in \mathbb{R}^{nN}$. The statement tells that these functions are identical to those on the right hand side, in which $\Phi(x)$ is defined by (6.9).

For (6.12), it is only an approximation valid for large K. To illustrate, consider the particular case:

$$F(m) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Gamma(u,v)m(u)m(v)dudv \tag{6.13}$$

in which $\Gamma(u,v) = \Gamma(v,u)$ is twice continuously differentiable. We have

$$\begin{aligned}
\frac{\partial F(m)}{\partial m}(\xi) &= 2\int_{\mathbb{R}^n} \Gamma(\xi,v)m(v)dv,\\
D_\xi \frac{\partial F(m)}{\partial m}(\xi) &= 2\int_{\mathbb{R}^n} D_\xi\Gamma(\xi,v)m(v)dv,\\
D_\xi^2\left(\frac{\partial F(m)}{\partial m}(\xi)\right) &= 2\int_{\mathbb{R}^n} D_\xi^2\Gamma(\xi,v)m(v)dv,\\
\frac{\partial^2 F(m)}{\partial m^2}(\xi,\eta) &= 2\Gamma(\xi,\eta),\\
D_\xi D_\eta \frac{\partial^2 F(m)}{\partial m^2}(\xi,\eta) &= 2D_\xi D_\eta\Gamma(\xi,\eta).
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n} D_\xi \frac{\partial F(m)}{\partial m}(\xi).B(\xi)m(\xi)d\xi = 2\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} D_u\Gamma(u,v).B(u)m(u)m(v)dudv,\\
&\int_{\mathbb{R}^n} \Delta_\xi \frac{\partial F(m)}{\partial m}(\xi)m(\xi)d\xi + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \operatorname{tr} D_\xi D_\eta\left(\frac{\partial^2 F(m)}{\partial m^2}(\xi,\eta)\right)m(\xi)m(\eta)d\xi d\eta\\
&\quad = 2\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Delta_u\Gamma(u,v)m(u)m(v)dudv + 2\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \operatorname{tr} D_u D_v\Gamma(u,v)m(u)m(v)dudv,\\
&\int_{\mathbb{R}^n} \operatorname{tr}(a(\xi)D_\xi^2\left(\frac{\partial F(m)}{\partial m}(\xi)\right)m(\xi)d\xi = 2\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \operatorname{tr}\big(a(u)D_u^2\Gamma(u,v)\big)m(u)m(v)dudv.
\end{aligned}$$

We now apply these formulas with $m(x) = \frac{\sum_{j=1}^K \delta_{x^j}(x)}{K}$ which yields

$$\begin{aligned}
&\int_{\mathbb{R}^n} D_\xi \frac{\partial F(m)}{\partial m}(\xi).B(\xi)m(\xi)d\xi = \frac{2}{K^2}\sum_{j,k=1}^K D_{x^j}\Gamma\big(x^j,x^k\big).B\big(x^j\big),\\
&\int_{\mathbb{R}^n} \Delta_\xi \frac{\partial F(m)}{\partial m}(\xi)m(\xi)d\xi + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \operatorname{tr} D_\xi D_\eta\left(\frac{\partial^2 F(m)}{\partial m^2}(\xi,\eta)\right)m(\xi)m(\eta)d\xi d\eta\\
&\quad = \frac{2}{K^2}\sum_{j,k=1}^K \Delta_{x^j}\Gamma\big(x^j,x^k\big) + \frac{2}{K^2}\sum_{j\neq k=1}^K \operatorname{tr}\big(D_{x^j}D_{x^k}\Gamma\big(x^j,x^k\big)\big),
\end{aligned}$$

$$\int_{\mathbb{R}^n} \operatorname{tr}\big(a(\xi)D_\xi^2\Big(\frac{\partial F(m)}{\partial m}(\xi)\Big)m(\xi)d\xi = \frac{2}{K^2}\sum_{j,k=1}^K \operatorname{tr}\big(a\big(x^j\big)D_{x^j}^2\Gamma\big(x^j,x^k\big)\big).$$

We have to be careful in interpreting these formulas: $D_{x^j}\Gamma(x^j,x^k)$ represents the gradient with respect to the first argument, even when $k=j$. We have the similar convention for $\Delta_{x^j}\Gamma(x^j,x^k)$ or $D_{x^j}^2\Gamma(x^j,x^k)$. Observe that under this particular case, $\Phi(x)=\frac{1}{K^2}\sum_{j,k=1}^K\Gamma(x^j,x^k)$. Hence, we have

$$D_{x^j}\Phi(x)=\frac{2}{K^2}\sum_{k=1}^K D_{x^j}\Gamma\big(x^j,x^k\big),$$

and therefore

$$\sum_{j=1}^K D_{x^j}\Phi(x).B\big(x^j\big)=\frac{2}{K^2}\sum_{j=1}^K\sum_{k=1}^K D_{x^j}\Gamma\big(x^j,x^k\big).B\big(x^j\big)$$

which proves (6.10).

With $\Phi(x)=\frac{1}{K^2}\sum_{j\neq k=1}^K\Gamma(x^j,x^k)+\frac{1}{K^2}\sum_{j=1}^K\Gamma(x^j,x^j)$, we consider next

$$D_{x^j}D_{x^k}\Phi(x)=\frac{2}{K^2}D_{x^j}D_{x^k}\Gamma\big(x^j,x^k\big),\quad \text{if } j\neq k,$$

and

$$\Delta_{x^j}\Phi(x)=\frac{2}{K^2}\sum_{k\neq j=1}^K\Delta_{x^j}\Gamma\big(x^j,x^k\big)+\frac{1}{K^2}\Delta_{x^j}\Gamma\big(x^j,x^j\big)=\frac{2}{K^2}\sum_{k=1}^K\Delta_{x^j}\Gamma\big(x^j,x^k\big).$$

Hence,

$$\begin{aligned}\sum_{j,k=1}^K \operatorname{tr}\big(D_{x^j}D_{x^k}\Phi(x)\big) &= \sum_{j\neq k=1}^K \operatorname{tr}\big(D_{x^j}D_{x^k}\Phi(x)\big)+\sum_{j=1}^K\Delta_{x^j}\Phi(x)\\ &= \frac{2}{K^2}\sum_{j\neq k=1}^K \operatorname{tr}\big(D_{x^j}D_{x^k}\Gamma\big(x^j,x^k\big)\big)+\frac{2}{K^2}\sum_{j,k=1}^K\Delta_{x^j}\Gamma\big(x^j,x^k\big),\end{aligned}$$

and thus (6.11) is obtained.

Finally, observe that

$$D_{x^j}^2\Phi(x)=\frac{2}{K^2}\sum_{k=1}^K D_{x^j}^2\Gamma\big(x^j,x^k\big),$$

and hence

$$\sum_{j=1}^K \operatorname{tr}\big(a\big(x^j\big)D_{x^j}^2\Phi(x)\big)=\frac{2}{K^2}\sum_{j,k=1}^K \operatorname{tr}\big(a\big(x^j\big)D_{x^j}^2\Gamma\big(x^j,x^k\big)\big)$$

which also implies (6.12). We note that (6.12) is exact under this particular case.

We begin by proving (6.10) in the general case. Consider the probability measure defined by

$$m(x,s)=\frac{1}{K}\sum_{j=1}^{K}\delta_{x^j(s)}(x),$$

where $x^j(s)$ satisfies

$$\begin{cases}\dfrac{dx^j(s)}{ds}=B\big(x^j(s)\big),\\ x^j(0)=x^j.\end{cases}$$

We have

$$m(x,0)=m(x)=\frac{\sum_{j=1}^{K}\delta_{x^j}(x)}{K}.$$

The probability measure satisfies the degenerate Fokker–Planck equation in the weak sense

$$\frac{\partial m}{\partial s}+\operatorname{div}(mB)=0,$$

and one checks easily that, from the differentiability of $F(m)$

$$\frac{d}{ds}F\big(m(s)\big)\big|_{s=0}=\int\limits_{\mathbb{R}^n}D_\xi\frac{\partial F(m)}{\partial m}(\xi)B(\xi)m(\xi)d\xi.$$

On the other hand, by the definition of $\Phi(x)$, one has

$$F\big(m(s)\big)=\Phi\big(x^1(s),\cdots,x^K(s)\big)$$

and thus

$$\frac{d}{ds}F\big(m(s)\big)\big|_{s=0}=\sum_{j=1}^{K}D_{x^j}\Phi(x).B\big(x^j\big)$$

and (6.10) is obtained.

We now turn to (6.11). We again consider the probability distribution $m(x,s)=\frac{1}{K}\sum_{j=1}^{K}\delta_{x^j(s)}(x)$, with this time

$$x^j(s)=x^j+\beta b(s).$$

We can check that the probability distribution $m(x,s)$ satisfies the stochastic partial differential equation

$$\begin{cases}\partial_s m(x,s)-\dfrac{\beta^2}{2}\Delta m(x,s)ds+\beta Dm(x,s).db(s)=0,\\ m(x,0)=\dfrac{1}{K}\displaystyle\sum_{j=1}^{K}\delta_{x^j}(x).\end{cases}\tag{6.14}$$

Indeed, this is obtained by taking a test function and writing

$$\int\varphi(\xi)m(\xi,s)d\xi=\frac{1}{K}\sum_{j=1}^{K}\varphi\big(x^j(s)\big).$$

We then expand the right hand side, by using Itô's formula, and obtain

$$d\int \varphi(\xi)m(\xi,s)d\xi = \beta\frac{1}{K}\sum_{j=1}^{K} D\varphi\big(x^j(s)\big)db(s) + \frac{\beta^2}{2}\frac{1}{K}\sum_{j=1}^{K}\Delta\varphi\big(x^j(s)\big)ds.$$

Hence we have

$$\begin{aligned} d\int \varphi(\xi)m(\xi,s)d\xi &= \beta\int m(\xi,s)D\varphi(\xi).db(s) + \frac{\beta^2}{2}\int m(\xi,s)\Delta\varphi(\xi)d\xi ds \\ &= -\beta\int \varphi(\xi)Dm(\xi,s).db(s)d\xi + \frac{\beta^2}{2}\int \varphi(\xi)\Delta m(\xi,s)d\xi ds, \end{aligned}$$

and (6.14) follows immediately. We next write

$$\begin{aligned} dF\big(m(s)\big) &= \int\limits_{\mathbb{R}^n} \frac{\partial F(m(s))}{\partial m}(\xi)\left[-\beta Dm(\xi,s).db(s)d\xi + \frac{\beta^2}{2}\Delta m(\xi,s)d\xi ds\right] \\ &\quad + \frac{\beta^2}{2}\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \frac{\partial^2 F(m)}{\partial m^2}(\xi,\eta)Dm(\xi,s)Dm(\eta,s)d\xi d\eta ds. \end{aligned}$$

It follows that

$$\begin{aligned} &dF\big(m(s)\big) \\ &\quad = \beta\int\limits_{\mathbb{R}^n} D_\xi\frac{\partial F(m(s))}{\partial m}(\xi)m(\xi,s).db(s)d\xi \\ &\qquad + \frac{\beta^2}{2}\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} D_\xi D_\eta\frac{\partial^2 F(m)}{\partial m^2}(\xi,\eta)m(\xi,s)m(\eta,s)d\xi d\eta ds + \frac{\beta^2}{2}\int\limits_{\mathbb{R}^n}\Delta_\xi\frac{\partial F(m(s))}{\partial m}(\xi)m(\xi,s)d\xi ds. \end{aligned}$$

Hence

$$\frac{d}{ds}\mathbb{E}F\big(m(s)\big)|_{s=0} = \frac{\beta^2}{2}\left(\int\limits_{\mathbb{R}^n}\Delta_\xi\frac{\partial F(m)}{\partial m}(\xi)m(\xi)d\xi + \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} D_\xi D_\eta\frac{\partial^2 F(m)}{\partial m^2}(\xi,\eta)m(\xi)m(\eta)d\xi d\eta\right). \quad (6.15)$$

On the other hand

$$F\big(m(s)\big) = \Phi\big(x^1(s),\cdots,x^K(s)\big)$$

and by Itô's formula again

$$dF\big(m(s)\big) = \beta\sum_{j=1}^{K} D_{x^j}\Phi\big(x(s)\big).db(s) + \frac{\beta^2}{2}\sum_{j,k}\operatorname{tr} D_{x^j}D_{x^k}\Phi\big(x(s)\big)ds,$$

which implies

$$\frac{d}{ds}\mathbb{E}\big[F\big(m(s)\big)\big]\big|_{s=0} = \frac{\beta^2}{2}\sum_{j,k}\operatorname{tr} D_{x^j}D_{x^k}\Phi(x).$$

Comparing with (6.15), we obtain the formula (6.11).

We finally prove (6.12). We have $m(x,s) = \frac{1}{K}\sum_{j=1}^{K} \delta_{x^j(s)}(x)$, with

$$x^j(s) = x^j + \sigma\big(x^j(s)\big)dw^j(s)$$

in which the $w^j(s)$ are independent standard Wiener processes in $\mathbb{R}^n$. We can write

$$d\int \varphi(\xi)m(\xi,s)d\xi = \frac{1}{K}\sum_{j=1}^{K} D\varphi\big(x^j(s)\big).\sigma\big(x^j(s)\big)dw^j(s) + \frac{1}{2}\int m(\xi,s)\,\mathrm{tr}\big(a(\xi)D^2\varphi(\xi)\big)d\xi ds. \tag{6.16}$$

It is not possible, this time, to write a closed form equation for $m(x,s)$. However, for K large, since the processes $x^j(s)$ are independent, $m(x,s)$ is close to its average, which satisfies the Fokker–Planck equation in a weak sense:

$$\frac{\partial m}{\partial s} = \frac{1}{2}\sum_{\alpha,\beta=1}^{n} \frac{\partial^2(a_{\alpha\beta}(\xi)m)}{\partial\xi_\alpha\partial\xi_\beta}.$$

Next, consider then $F(m(s))$. We check easily that

$$\frac{d}{ds}F\big(m(s)\big)\big|_{s=0} = \frac{1}{2}\int\limits_{\mathbb{R}^n} \mathrm{tr}(a(\xi)D_\xi^2\left(\frac{\partial F(m)}{\partial m}(\xi)\right)m(\xi)d\xi. \tag{6.17}$$

Since $F(m(s)) = \Phi(x^1(s),\cdots,x^K(s))$, we can write

$$F\big(m(s)\big) \approx \mathbb{E}\Phi\big(x^1(s),\cdots,x^K(s)\big),$$

and from Itô's formula we get easily

$$\frac{d}{ds}F\big(m(s)\big)\big|_{s=0} \approx \frac{1}{2}\sum_{j=1}^{K}\mathrm{tr}\big(a\big(x^j\big)D_{x^j}^2\Phi(x)\big),$$

and the result (6.12) follows. □

6.3. Interpretation of the Master equation for Mean Field Games

Consider Eq. (4.6) which we write as follows

$$\begin{cases} -\dfrac{\partial U}{\partial t} - \dfrac{1}{2}\mathrm{tr}\big(a(x)D^2U\big) - \dfrac{1}{2}\beta^2\Delta U - \displaystyle\int\limits_{\mathbb{R}^n}\left(\frac{1}{2}\mathrm{tr}\,a(\xi)D_\xi^2\frac{\partial U(x,m,t)}{\partial m}(\xi) + \frac{1}{2}\beta^2\Delta_\xi\frac{\partial U(x,m,t)}{\partial m}(\xi)\right)m(\xi)d\xi \\ \qquad - \displaystyle\int\limits_{\mathbb{R}^n} D_\xi\frac{\partial U(x,m,t)}{\partial m}(\xi).G\big(\xi,m,DU(\xi)\big)\,m(\xi)d\xi - \beta^2\int\limits_{\mathbb{R}^n}\mathrm{tr}\,D_xD_\xi\left(\frac{\partial U(x,m,t)}{\partial m}(\xi)\right)m(\xi)d\xi \\ \qquad - \dfrac{1}{2}\beta^2\displaystyle\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}\mathrm{tr}\,D_\xi D_\eta\left(\frac{\partial^2 U(x,m,t)}{\partial m^2}(\xi,\eta)\right)m(\xi)m(\eta)d\xi d\eta \\ \quad = H\big(x,m,DU(x)\big), \\ U(x,m,T) = h(x,m). \end{cases} \tag{6.18}$$

We have the following proposition:

Proposition 6. *If the solution of (6.18) $U(x,m,t)$ is sufficiently smooth, then the functions*

$$u_i(x,t) = u_i\big(x^1,\cdots,x^N,t\big) = U\bigg(x^i, \frac{\sum_{j\neq i}\delta_{x^j}(\cdot)}{N-1}, t\bigg), \quad \textit{for } i=1,\cdots,N, \tag{6.19}$$

satisfy approximately the system of relations (6.8).

Proof. We take $x = x^i$ and $m = \frac{\sum_{j\neq i}\delta_{x^j}(\cdot)}{N-1}$, and we make use of Proposition 5. We have the equivalences

$$\frac{\partial U(x^i, \frac{1}{N-1}\sum_{j\neq i}\delta_{x^j}, t)}{\partial t} = \frac{\partial u^i(x,t)}{\partial t},$$

$$\frac{1}{2}\operatorname{tr}\bigg(a(x^i)D^2_{x^i}U\bigg(x^i, \frac{1}{N-1}\sum_{j\neq i}\delta_{x^j}, t\bigg)\bigg) = \frac{1}{2}\operatorname{tr}\big(a(x^i)D^2_{x^i}u^i(x,t)\big),$$

$$\frac{1}{2}\beta^2\Delta_{x^i}U\bigg(x^i, \frac{1}{N-1}\sum_{j\neq i}\delta_{x^j}, t\bigg) = \frac{1}{2}\beta^2\Delta_{x^i}u^i(x,t),$$

$$\int_{\mathbb{R}^n}\frac{1}{2}\operatorname{tr} a(\xi)D^2_\xi\frac{\partial U(x^i,m,t)}{\partial m}(\xi)\, m(\xi)d\xi \sim \sum_{j\neq i}\frac{1}{2}\operatorname{tr}\big(a(x^j)D^2_{x^j}u^i(x,t)\big),$$

$$\int_{\mathbb{R}^n} D_\xi\frac{\partial U(x^i,m,t)}{\partial m}(\xi).G\big(\xi,m,DU(\xi)\big)\, m(\xi)d\xi = \sum_{j\neq i}D_{x^j}u^i(x,t).G\bigg(x^i, \frac{1}{N-1}\sum_{k\neq j}\delta_{x^k}, D_{x^j}u^j(x,t)\bigg),$$

$$\frac{1}{2}\beta^2\bigg(\int_{\mathbb{R}^n}\Delta_\xi\frac{\partial U(x^i,m,t)}{\partial m}(\xi)m(\xi)d\xi + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\operatorname{tr} D_\xi D_\eta\bigg(\frac{\partial^2 U(x^i,m,t)}{\partial m^2}(\xi,\eta)\bigg)m(\xi)m(\eta)d\xi d\eta\bigg)$$
$$= \frac{1}{2}\beta^2\sum_{j,k\neq i}\operatorname{tr}\big(D^2_{x^jx^k}u^j(x,t)\big),$$

$$\beta^2\int_{\mathbb{R}^n}\operatorname{tr} D_{x^i}D_\xi\bigg(\frac{\partial U(x^i,m,t)}{\partial m}(\xi)\bigg)m(\xi)d\xi = \beta^2\sum_{j\neq i}\operatorname{tr}\big(D^2_{x^jx^i}u^j(x,t)\big),$$

$$H\big(x^i,m,D_{x^i}U(x^i,m,t)\big) = H\bigg(x^i, \frac{1}{N-1}\sum_{k\neq j}\delta_{x^k}, D_{x^i}u^i(x^i,t)\bigg).$$

Using these equivalences, we check easily that the Master equation (6.18) with $x = x^i$ and $m = \frac{1}{N-1}\sum_{j\neq i}\delta_{x^j}$ reduces to the system (6.8) with functions $u^i(x,t)$ given by (6.19). It is an approximation, only because of (6.12). The reduction is exact, when $\sigma(x) = 0$. □

7. Application to systemic risk

7.1. The problem

We discuss here an application of *Mean Field Games and Systemic Risk* introduced by R. Carmona, J.P. Fouque, and L.H. Sun [5]. There are N players, and the state equations are given as follows:

$$\begin{cases} dx^i(s) = \bigg[\alpha\bigg(\frac{1}{N}\sum_{j=1}^N x^j(s) - x^i(s)\bigg) + v^i(s)\bigg]ds + \sigma dw^i(s) + \beta db(s), \quad s > t, \\ x^i(t) = x^i. \end{cases} \tag{7.1}$$

Each player has the payoff

$$J^i\big(x,t;v(\cdot)\big) = \mathbb{E}\Bigg[\int_t^T \Bigg\{\frac{1}{2}\big(v^i\big(x(s)\big)\big)^2 - \lambda v^i\big(x(s)\big)\Bigg(\frac{1}{N}\sum_{j=1}^N x^j(s) - x^i(s)\Bigg) + \frac{\mu}{2}\Bigg(\frac{1}{N}\sum_{j=1}^N x^j(s) - x^i(s)\Bigg)^2\Bigg\}ds + \frac{c}{2}\Bigg(\frac{1}{N}\sum_{j=1}^N x^j(T) - x^i(T)\Bigg)^2\Bigg]. \tag{7.2}$$

In [5], the players are banks lending and borrowing from each other. The state x^i represents the log-monetary reserve of bank i. The term

$$\alpha\Bigg(\frac{1}{N}\sum_{j=1}^N x^j(s) - x^i(s)\Bigg) = \frac{\alpha}{N}\sum_{j=1}^N \big(x^j(s) - x^i(s)\big)$$

represents the sum of rates at which bank i borrows or lends to bank j, with $\alpha > 0$. The control v^i represents the rate of lending or borrowing to the central bank. The pay-off contains a penalty term on the difference between each reserve level and the average. The parameter λ is positive, and the corresponding term models an incentive to borrow ($v^i > 0$) when the reserve of bank i is smaller than the average, or to lend in the opposite case. The players look for a Nash equilibrium.

7.2. Mean Field Game approximation

The preceding problem is a particular case of the problem (6.1), (6.2). So we can consider the mean field analogue. Using our notations

$$m_1 = \int_{\mathbb{R}^n} m(\xi)d\xi = 1, \qquad y = \int_{\mathbb{R}^n} \xi m(\xi)d\xi, \tag{7.3}$$

we introduce the functions:

$$\begin{aligned} f(x,m,v) &= \frac{1}{2}v^2 - \lambda v(y-v) + \frac{\mu}{2}(y-x)^2, \\ g(x,m,v) &= \alpha(y-x) + v, \\ h(x,m) &= \frac{c}{2}(y-x)^2. \end{aligned} \tag{7.4}$$

It is then easy to check that

$$H(x,m,q) = \frac{\mu-\lambda^2}{2}(y-x)^2 + q(\alpha+\lambda)(y-x) - \frac{1}{2}q^2,$$

and

$$G(x,m,q) = (\alpha+\lambda)(y-x) - q.$$

To preserve convexity, [5] assume that $\mu - \lambda^2 > 0$. The Master equation (4.6) under this setting becomes

$$
\begin{cases}
-\dfrac{\partial U}{\partial t} - \dfrac{1}{2}(\sigma^2+\beta^2)\dfrac{\partial^2 U}{\partial x^2} - \dfrac{1}{2}(\sigma^2+\beta^2)\displaystyle\int_R \dfrac{\partial^2}{\partial \xi^2}\dfrac{\partial U(x,m,t)}{\partial m}(\xi)m(\xi)d\xi \\
\quad -\displaystyle\int_R \dfrac{\partial}{\partial \xi}\dfrac{\partial U(x,m,t)}{\partial m}(\xi)G\Big(\xi,m,\dfrac{\partial}{\partial \xi}U(\xi,m,t)\Big)m(\xi)d\xi - \beta^2\displaystyle\int_R \dfrac{\partial}{\partial x}\dfrac{\partial}{\partial \xi}\dfrac{\partial U(x,m,t)}{\partial m}(\xi)m(\xi)d\xi \\
\quad -\dfrac{1}{2}\beta^2\displaystyle\int_R\int_R \dfrac{\partial}{\partial \xi}\dfrac{\partial}{\partial \eta}\dfrac{\partial^2 U(x,m,t)}{\partial m^2}(\xi,\eta)m(\xi)m(\eta)d\xi d\eta \\
\quad = \dfrac{\mu-\lambda^2}{2}(y-x)^2 + (\alpha+\lambda)(y-x)\dfrac{\partial U(x,m,t)}{\partial x} - \dfrac{1}{2}\Big(\dfrac{\partial U(x,m,t)}{\partial x}\Big)^2, \\
U(x,m,T) = \dfrac{c}{2}(y-x)^2.
\end{cases}
\tag{7.5}
$$

7.3. Solution

To get an explicit solution of (7.5), we postulate that

$$
U(x,m,t) = \frac{1}{2}(x-y)^2 P(t) + R(m_1,t). \tag{7.6}
$$

Clearly, we have

$$
P(T) = c, \qquad R(m_1,T) = 0 \tag{7.7}
$$

from the terminal condition. An easy calculation, by using (5.28) to (5.31), leads to

$$
\frac{dP}{dt} - 2(\alpha+\lambda)P - P^2 + \mu - \lambda^2 = 0
$$

and

$$
R(m_1,t) = \frac{1}{2}\big(\sigma^2 + \beta^2(m_1-1)^2\big)\int_t^T P(s)ds.
$$

The corresponding FP equation becomes

$$
\begin{cases}
\partial_t m + \Big(-\dfrac{1}{2}(\sigma^2+\beta^2)\dfrac{\partial^2 m}{\partial x^2} + (\alpha+\lambda+P(t))\dfrac{\partial}{\partial x}\big((y-x)m\big)\Big)dt + \beta\dfrac{\partial m}{\partial x}db(t) = 0, \\
m(x,0) = m_0(x).
\end{cases}
$$

From the definition of y in (7.3), we have

$$
y(t) = y_0 + \beta b(t). \tag{7.8}
$$

The solution of the stochastic HJB equation (4.3) is then

$$
u(x,t) = u\big(x,m(t),t\big) = \frac{1}{2}\big(x-y_0-\beta b(t)\big)^2 P(t) + R(t), \tag{7.9}
$$

with $R(t) = R(1,t)$. We have $B(x,t) = (x-y(t))$. We here recover the results in [5].

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