

EXPONENTIAL HEDGING WITH OPTIMAL STOPPING AND APPLICATION TO EMPLOYEE STOCK OPTION VALUATION*

TIM LEUNG[†] AND RONNIE SIRCAR[‡]

Abstract. We study the problem of hedging early exercise (American) options with respect to exponential utility within a general incomplete market model. This leads us to construct a duality formula involving relative entropy minimization and optimal stopping. We further consider claims with multiple exercises, and static-dynamic hedges of American claims with other European and American options. The problem is important for accurate valuation of employee stock options (ESOs), and we demonstrate this in a standard diffusion model. We find that incorporating static hedges with market-traded options induces the holder to delay exercises and increases the ESO cost to the firm.

Key words. financial mathematics, utility indifference pricing, optimal stopping, American options, employee stock options

AMS subject classifications. 60G40, 91B28, 91B70, 93E20

DOI. 10.1137/080718930

1. Introduction. Many applications of financial mathematics involve the optimization of expected utility of wealth, combined with one or many optimal stopping decisions, over a finite time horizon. Typical of these are hedging, indifference valuation, or asset management of portfolios containing American (early exercise) derivative securities. At the same time, the exponential utility function has become popular because of its basis for the entropic convex risk measure, which has convenient dynamic and analytic properties that make it amenable for computations. A very general duality theory with problems of relative entropy minimization has been developed in [1, 7, 11, 20] and [32], among others, for exponential hedging of claims with no early exercise feature. Our goal here is to develop the analogous duality formula for exponential hedging of American claims, under minimal assumptions on the underlying price process. This is then applied to the problem of employee stock option (ESO) valuation, illustrated within a standard diffusion-based financial model.

A key feature of the problems we are interested in is the finite time horizon that corresponds naturally to the expiration date of the American claim. Allowing for cash flows at (optimally chosen) prior times requires specification of how the payoff of the exercised American option is invested thereafter. Here the time-consistency of exponential utility (or the self-generating property of its associated Merton function) is crucial for tractability. We refer to [30] for a discussion and alternative specifications. There is, of course, an enormous literature on utility maximization problems of mixed optimal stopping/control type, over an infinite horizon, with utility functions defined on \mathbb{R}^+ , where stopping represents the decision to get out or retire from investing.

*Received by the editors March 19, 2008; accepted for publication (in revised form) January 16, 2009; published electronically April 15, 2009.

<http://www.siam.org/journals/sicon/48-3/71893.html>

[†]Department of Applied Mathematics and Statistics, Johns Hopkins University, Baltimore, MD 21218 (timleung@jhu.edu). This author's work was supported by NSF grant DMS-0456195 and a Charlotte Elizabeth Procter Fellowship.

[‡]Operations Research and Financial Engineering (ORFE) Department, Princeton University, Sherrerd Hall, Princeton, NJ 08544 (sircar@princeton.edu). This author's work was supported by NSF grants DMS-0456195 and DMS-0739195.

We refer to [22] for details and references. Superhedging of American claims under portfolio constraints was studied in [21].

Valuation of ESOs has become important since the Financial Accounting Standards Board (FASB) required their inclusion in firms' accounting statements since 2005, where previously they had been exempted. The method of valuation is a controversial topic, with a trade-off between simplicity and realism, coupled with a political lobby to push for whatever proposal results in booking a lower value. The major concern in ESO valuation that moves the problem away from standard *no-arbitrage* pricing methods is the hedging restriction: employees cannot take short positions in their firm's stock. However, there are other possibilities that can be taken into account.

In the approach we analyze here, the employee is able to dynamically invest in the market index and take static positions in market-traded vanilla options written on the firm's stock. This is a natural advance from earlier utility-based approaches using just certainty equivalent with no market trading (for example, [17]), then incorporating dynamic hedging with a correlated index (in [3, 13, 15, 26]), and now incorporating market options data for more accurate calibration. However, high transaction costs discourage frequent option trades, so recent work (for example, [4, 19]) has focused on static hedging with options, which involves purchasing a portfolio of standard options at initiation and no trades afterwards. The combination of a dynamic trading strategy and static positions, which is referred to as *static-dynamic* hedging, leads us to study how market prices of traded put options affect the employee's optimal exercising strategy. As shown in sections 4.1 and 4.2, the optimal static position is found from the *Fenchel–Legendre transform* of the employee's indifference price as a function of the number of puts, evaluated at the market price. In Propositions 5.2 and 5.3, we find that static hedges with put options induce the employee to delay exercises, which in general leads to a higher ESO cost.

The paper is structured as follows. In section 2, we investigate the case of a single American claim in a semimartingale framework. This allows us, in section 3, to extend our results to the case of American claims with multiple exercising rights. In section 4, we incorporate European and American put options into the investor's hedging strategy. We solve for the optimal exercising strategy along with the optimal static hedge. In section 5, we apply our methodology to ESO valuation in a diffusion framework. We study early exercises through the examination of the employee's optimal exercise boundaries, and illustrate their impact on the ESO cost.

2. Dynamic hedging of American claims with single exercise. We begin with the problem of exponential hedging of an American option using the underlying asset, within a general incomplete market model. We derive the dual problem in Proposition 2.4. To characterize the optimal exercise time, we introduce the indifference price in section 2.3. Finally, we give results on large and small risk aversion limits as well as quantity asymptotics in section 2.5.

2.1. Notation and assumptions. In the background, we fix an investment horizon with a finite terminal time T , which is chosen to coincide with the expiration date of all securities in our model. We work on a probability space (Ω, \mathcal{F}, P) with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$, which satisfies the usual conditions of right continuity and completeness. Hence, all processes are assumed to have right continuous paths with left limits. We adopt the shorthand for taking conditional expectations: $\mathbb{E}_t\{\cdot\} = \mathbb{E}\{\cdot | \mathcal{F}_t\}$.

The basic trading assets consist of a riskless asset (e.g., the bank account), that pays interest at constant rate $r \geq 0$, and a risky asset (e.g., the market index), whose discounted price process is a nonnegative \mathbb{F} -locally bounded semimartingale $(S_t)_{0 \leq t \leq T}$. We denote by $(X_t)_{0 \leq t \leq T}$ the discounted trading wealth process with a dynamic trading strategy θ_t which represents the number of shares held at time t . The set of admissible strategies is defined in (2.1) below. With initial capital X_t at time $t \in [0, T]$, the discounted wealth at a later time $u \in [t, T]$ is given by

$$X_u = X_t + G_{t,u}(\theta), \quad \text{with} \quad G_{t,u}(\theta) := \int_t^u \theta_s dS_s.$$

The stochastic integral $G_{t,u}(\theta)$ is the discounted capital gains or losses from trading with strategy θ from time t to u . We write $G(\theta)$ for the process $(G_{0,t}(\theta))_{0 \leq t \leq T}$.

We denote by \mathcal{T} the set of all stopping times with respect to \mathbb{F} taking values in $[0, T]$. This will be the collection of all admissible exercise times for American claims considered in this paper. For any stopping times $s, u \in \mathcal{T}$ with $s \leq u$, we set $\mathcal{T}_{s,u} := \{\tau \in \mathcal{T} : s \leq \tau \leq u\}$.

The sets of absolutely continuous (equivalent) local martingale measures for S with respect to P are denoted by

$$\begin{aligned} \mathbb{P}_a(P) &:= \{Q \ll P \mid S \text{ is a local } (Q, \mathbb{F})\text{-martingale}\}, \\ \mathbb{P}_e(P) &:= \{Q \sim P \mid S \text{ is a local } (Q, \mathbb{F})\text{-martingale}\}. \end{aligned}$$

For any measure Q , the relative entropy of Q with respect to P is given by

$$H(Q|P) := \begin{cases} \mathbb{E}^Q \left\{ \log \frac{dQ}{dP} \right\}, & Q \ll P, \\ +\infty, & \text{otherwise.} \end{cases}$$

We introduce $\mathbb{P}_f(P)$, the set of measures in $\mathbb{P}_a(P)$ with finite relative entropy with respect to P , and define the set of admissible strategies as

$$(2.1) \quad \Theta(P) := \{\theta \in L(S) \mid G(\theta) \text{ is a } (Q, \mathbb{F}) \text{ martingale for all } Q \in \mathbb{P}_f(P)\},$$

where $L(S)$ is the set of \mathbb{F} -predictable S -integrable \mathbb{R} -valued processes. For notational simplicity, we write, respectively, $\mathbb{P}_a, \mathbb{P}_e, \mathbb{P}_f, \Theta$ for $\mathbb{P}_a(P), \mathbb{P}_e(P), \mathbb{P}_f(P), \Theta(P)$ when no ambiguity arises. When we specify the trading horizon $[s, u]$, we write $\Theta_{s,u}$ to denote the set of admissible strategies over the period $[s, u]$.

Throughout, we assume that there exists some equivalent local martingale measure with finite relative entropy with respect to P .

ASSUMPTION 2.1. $\mathbb{P}_f \cap \mathbb{P}_e \neq \emptyset$.

By Theorems 2.1 and 2.2 of [11], this assumption ensures the existence of a unique measure $Q^E \in \mathbb{P}_f \cap \mathbb{P}_e$ that minimizes the relative entropy with respect to P over all measures in \mathbb{P}_f , that is,

$$Q^E = \arg \min_{Q \in \mathbb{P}_f} H(Q|P).$$

This measure is called the minimal entropy martingale measure (MEMM). By Theorem 2.3 of [11] and Theorem 2.1 of [20], it has a density of the form

$$(2.2) \quad \frac{dQ^E}{dP} = c_E \exp(G_{0,T}(\theta^E))$$

for some $\theta^E \in \Theta$ and $\log c_E = H(Q^E|P) < \infty$. We can derive from (2.2) the density process of Q^E with respect to P :

$$(2.3) \quad Z_t^E := \mathbb{E}_t \left\{ \frac{dQ^E}{dP} \right\} = c_E \mathbb{E}_t \left\{ e^{G_{0,T}(\theta^E)} \right\}.$$

In general, for any two measures Q^a, Q^b such that $Q^a \ll Q^b$, we write the density process of Q^a with respect to Q^b as

$$Z_t^{Q^a, Q^b} := \mathbb{E}_t^{Q^b} \left\{ \frac{dQ^a}{dQ^b} \right\}, \quad t \in [0, T].$$

It is well known that exponential utility optimization is closely linked to the minimization of relative entropy. The dynamic version of the problem involves minimizing the *conditional relative entropy* with respect to P , so we state here its definition in our notation.

DEFINITION 2.2. For any $t \in [0, T]$ and $Q \in \mathbb{P}_f(P)$, the conditional relative entropy of Q with respect to P at time t is

$$(2.4) \quad H_t^T(Q|P) := \mathbb{E}_t^Q \left\{ \log \frac{Z_T^{Q,P}}{Z_t^{Q,P}} \right\}.$$

Remark 2.3. For any $t \in [0, T]$ and $Q \in \mathbb{P}_f(P)$, the random variable $\log Z_t^{Q,P}$ is Q -integrable (see Lemma 3.3 of [7]), so the conditional relative entropy is well defined. As is well known, an application of Jensen's inequality yields $H_t^T(Q|P) \geq 0$.

By Proposition 4.1 of [20], the MEMM Q^E also minimizes the conditional relative entropy $H_\tau^T(Q|P)$ at any $\tau \in \mathcal{T}$. That is,

$$(2.5) \quad \operatorname{ess\,inf}_{Q \in \mathbb{P}_f(P)} H_\tau^T(Q|P) = H_\tau^T(Q^E|P), \quad \tau \in \mathcal{T}.$$

Throughout, we consider a risk-averse investor whose risk preferences are described by the exponential utility function $U : \mathbb{R} \mapsto \mathbb{R}_-$ defined by

$$U(x) = -e^{-\gamma x}, \quad x \in \mathbb{R},$$

where $\gamma > 0$ is the coefficient of absolute risk aversion. Precisely, $U(x)$ is the investor's utility for having (discounted) wealth x at time T . As discussed in the introduction, exponential utility is a convenient convex (but not coherent) risk measure (up to a log and change in sign), as studied in [10], with good dynamic (time-consistency) properties [24], and it is the only (nonlinear) utility function of this kind [5].

2.2. Exponential hedging of an American option. In order to formulate the problem of hedging an American claim, we first need to consider an investment problem in which the risk-averse investor without any claims dynamically trades in the riskless and risky assets throughout the horizon $[0, T]$. This is a well-studied problem first introduced by Merton [28]. For an investor with starting wealth X_t at time $t \in [0, T]$, the maximal expected utility from terminal wealth is

$$(2.6) \quad M(t, X_t) := \operatorname{ess\,sup}_{\theta \in \Theta_{t,T}} \mathbb{E}_t \{ U(X_T) \}.$$

We consider an American claim, denoted by A , with a payoff process $(A_t)_{0 \leq t \leq T}$ which we assume to be bounded and adapted to \mathbb{F} . We will use the boundedness

of A in the proof of Proposition 2.4. More relaxed assumptions on the payoff for European-style contingent claims (with no early exercise) can be found in [1] and [7], but these still exclude European call options with geometric Brownian motion. We do not attempt to relax the assumptions on the American claim payoff A in this paper.

The holder of the claim maximizes expected utility by choosing the optimal exercise policy and dynamic trading strategy. Upon exercise, the holder immediately reinvests the contract proceeds, if any, into the trading portfolio, and continues to trade up to time T . This means that the investor will face the Merton problem after exercising the claim. At time $t \in [0, T]$, the holder's value function is given by

$$(2.7) \quad V(t, X_t; A) := \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \operatorname{ess\,sup}_{\theta \in \Theta_{t,\tau}} \mathbb{E}_t \{M(\tau, X_\tau + A_\tau)\}.$$

The value function in (2.7) is the primal problem, for which we will derive the dual in Proposition 2.4. For exponential utility, Delbaen et al. [7] derive a duality result for investment including a European-style contingent claim (with no early exercise), which involves optimizing the expected payoff over a set of martingale measures but penalized by an entropy distance from the historical measure. Since our optimal hedging problem includes an American claim (with the possibility of early exercise), the dual value function also involves finding the holder's optimal exercise time in addition to the optimal measure. The next proposition is the main result in this section.

PROPOSITION 2.4. *The dual of the value function is given by*

$$(2.8) \quad \begin{aligned} V(t, X_t; A) \\ = U(X_t) \cdot \exp \left(- \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \operatorname{ess\,inf}_{Q \in \mathbb{P}_f(P)} \left(\mathbb{E}_t^Q \{ \gamma A_\tau \} + H_t^\tau(Q|P) + \mathbb{E}_t^Q \{ H_\tau^T(Q^E|P) \} \right) \right). \end{aligned}$$

In developing this duality result, we shall make use of some useful properties of the Merton problem.

2.2.1. Properties of the Merton function. First, we observe a separation of variables and a duality formula for the Merton function.

PROPOSITION 2.5. *For an investor with starting wealth X_τ at $\tau \in \mathcal{T}$, the Merton function admits a separation of variables*

$$(2.9) \quad M(\tau, X_\tau) = U(X_\tau) \cdot \mathbb{E}_\tau \left\{ e^{G_{\tau,T}(\theta^E)} \right\},$$

and its dual is given by

$$(2.10) \quad M(\tau, X_\tau) = U(X_\tau) \cdot \exp \left(- \operatorname{ess\,inf}_{Q \in \mathbb{P}_f(P)} H_\tau^T(Q|P) \right).$$

Moreover, the optimal trading strategy is $\theta^* := -\frac{\theta^E}{\gamma}$, and the optimal measure in (2.10) is Q^E .

Proof. Recall that $X_T = X_\tau + G_{\tau,T}(\theta)$, and so by (2.6),

$$(2.11) \quad M(\tau, X_\tau) = \operatorname{ess\,sup}_{\theta \in \Theta_{\tau,T}} \mathbb{E}_\tau \{U(X_T)\} = U(X_\tau) \cdot \operatorname{ess\,inf}_{\theta \in \Theta_{\tau,T}} \mathbb{E}_\tau \left\{ e^{-\gamma G_{\tau,T}(\theta)} \right\}.$$

For any $\theta \in \Theta_{\tau,T}$, we can apply a change of measure from P to Q^E using the density in (2.3) to obtain

$$\mathbb{E}_{\tau} \left\{ e^{-\gamma G_{\tau,T}(\theta)} \right\} = \mathbb{E}_{\tau} \left\{ e^{G_{\tau,T}(\theta^E)} \right\} \cdot \mathbb{E}_{\tau}^{Q^E} \left\{ e^{-G_{\tau,T}(\gamma\theta + \theta^E)} \right\} \geq \mathbb{E}_{\tau} \left\{ e^{G_{\tau,T}(\theta^E)} \right\},$$

where the last inequality follows from Jensen's inequality and that $G(\gamma\theta + \theta^E)$ is a Q^E -martingale. The inequality becomes an equality when $\theta = -\frac{\theta^E}{\gamma}$, so the infimum in (2.11) is attained, and (2.9) follows.

On the other hand, we observe from (2.5) that the right-hand side of (2.10) is $U(X_{\tau}) \cdot \exp(-H_{\tau}^T(Q^E|P))$. By the definitions of Z^E in (2.3) and conditional relative entropy in (2.4), we can write

$$(2.12) \quad H_{\tau}^T(Q^E|P) = \mathbb{E}_{\tau}^{Q^E} \left\{ \log \frac{e^{G_{0,T}(\theta^E)}}{\mathbb{E}_{\tau} \{ e^{G_{0,T}(\theta^E)} \}} \right\} = -\log \mathbb{E}_{\tau} \{ e^{G_{\tau,T}(\theta^E)} \},$$

where we have used the fact that $G(\theta^E)$ is a Q^E -martingale. Hence, by exponentiating (2.12) and comparing it with (2.9), we obtain (2.10). \square

Next, we show that the Merton function satisfies the following dynamic programming property. It is called the self-generating condition in [30], and horizon-unbiased condition in [16].

PROPOSITION 2.6. *With starting wealth X_t at $t \in [0, T]$, the Merton function satisfies*

$$M(t, X_t) = \operatorname{ess\,sup}_{\theta \in \Theta_{t,\tau}} \mathbb{E}_t \{ M(\tau, X_{\tau}) \}, \quad \tau \in \mathcal{T}_{t,T}.$$

Proof. Recall that $X_{\tau} = X_t + G_{t,\tau}(\theta)$. By the separation of variables formula (2.9) and a change of measure from P to Q^E , we have

$$\begin{aligned} \operatorname{ess\,sup}_{\theta \in \Theta_{t,\tau}} \mathbb{E}_t \{ M(\tau, X_{\tau}) \} &= \operatorname{ess\,inf}_{\theta \in \Theta_{t,\tau}} \mathbb{E}_t \left\{ U(X_t + G_{t,\tau}(\theta)) \cdot \mathbb{E}_{\tau} \{ e^{G_{\tau,T}(\theta^E)} \} \right\} \\ &= U(X_t) \cdot \mathbb{E}_t \left\{ e^{G_{t,T}(\theta^E)} \right\} \cdot \operatorname{ess\,inf}_{\theta \in \Theta_{t,\tau}} \mathbb{E}_t^{Q^E} \left\{ e^{-\gamma G_{t,\tau}(\theta)} e^{-G_{t,\tau}(\theta^E)} \right\} \\ (2.13) \quad &= M(t, X_t) \cdot \operatorname{ess\,inf}_{\theta \in \Theta_{t,\tau}} \mathbb{E}_t^{Q^E} \left\{ e^{-G_{t,\tau}(\gamma\theta + \theta^E)} \right\}. \end{aligned}$$

Next, applying Jensen's inequality and the fact that $G(\gamma\theta + \theta^E)$ is a Q^E -martingale, we obtain for every $\theta \in \Theta_{t,\tau}$ that

$$\mathbb{E}_t^{Q^E} \left\{ e^{-G_{t,\tau}(\gamma\theta + \theta^E)} \right\} \geq 1,$$

and equality is attained at $\theta^* = -\frac{\theta^E}{\gamma}$. This implies that θ^* attains the infimum in (2.13) and the infimum is 1. This completes the proof. \square

2.2.2. Proof of Proposition 2.4. To prove Proposition 2.4, we shall also need the following lemma regarding entropy minimization.

LEMMA 2.7. *For any $t \in [0, T]$ and $\tau \in \mathcal{T}_{t,T}$, we have*

$$(2.14) \quad \operatorname{ess\,inf}_{Q \in \mathbb{P}_f(P)} H_t^T(Q|P) = \operatorname{ess\,inf}_{Q \in \mathbb{P}_f(P)} \left(H_t^T(Q|P) + \mathbb{E}_t^Q \left\{ \operatorname{ess\,inf}_{Q \in \mathbb{P}_f(P)} H_{\tau}^T(Q|P) \right\} \right).$$

Proof. The definition of conditional relative entropy gives the simple equality

$$H_t^T(Q|P) = H_t^\tau(Q|P) + \mathbb{E}_t^Q \{ H_\tau^T(Q|P) \}.$$

Taking infimum on both sides, we easily deduce the inequality

$$(2.15) \quad \operatorname{ess\,inf}_{Q \in \mathbb{P}_f(P)} H_t^T(Q|P) \geq \operatorname{ess\,inf}_{Q \in \mathbb{P}_f(P)} \left(H_t^\tau(Q|P) + \mathbb{E}_t^Q \left\{ \operatorname{ess\,inf}_{Q \in \mathbb{P}_f(P)} H_\tau^T(Q|P) \right\} \right).$$

Next, on the right-hand side of (2.14), we observe from (2.5) that Q^E solves the inner minimization. The outer minimization depends on the density process $Z^{Q,P}$ only over the stochastic interval $[t, \tau]$, so it is unchanged if we minimize over the set

$$\mathcal{P}_\tau := \{Q \in \mathbb{P}_f : Z_u^{Q,P} = Z_u^E \text{ for } u \geq \tau\}.$$

Hence, we write the right-hand side of (2.14) as

$$\operatorname{ess\,inf}_{Q \in \mathcal{P}_\tau} \left(H_t^\tau(Q|P) + \mathbb{E}_t^Q \{ H_\tau^T(Q^E|P) \} \right) = \operatorname{ess\,inf}_{Q \in \mathcal{P}_\tau} H_t^T(Q|P) \geq \operatorname{ess\,inf}_{Q \in \mathbb{P}_f(P)} H_t^T(Q|P).$$

This gives the reverse inequality to (2.15) and thus completes the proof. \square

To complete the proof of Proposition 2.4, for any $t \in [0, T]$ and $\tau \in \mathcal{T}_{t,\tau}$ we define the measure P^A by

$$(2.16) \quad \frac{dP^A}{dP} := c_A \cdot e^{-\gamma A_\tau}, \quad \text{with } c_A^{-1} = \mathbb{E}\{e^{-\gamma A_\tau}\}.$$

Given that A is bounded, we have $c_A \in (0, \infty)$ and $P^A \sim P$. Next, we want to show that

$$(2.17) \quad \mathbb{P}_f(P) = \mathbb{P}_f(P^A).$$

Indeed, for a measure $Q \ll P$, the relation

$$E^Q \left\{ \log \frac{dQ}{dP} \right\} = E^Q \left\{ \log \frac{dQ}{dP^A} \right\} + E^Q \left\{ \log \frac{dP^A}{dP} \right\}$$

and (2.16) imply

$$H(Q|P) = H(Q|P^A) + \log c_A - \gamma \mathbb{E}^Q \{ A_\tau \}.$$

For $Q \in \mathbb{P}_f(P)$, $H(Q|P)$ is finite. Since the last two terms on the right-hand side are also finite (due to the boundedness of A), we conclude that $H(Q|P^A)$ is also finite, and therefore, $\mathbb{P}_f(P) \subseteq \mathbb{P}_f(P^A)$. The reverse inclusion can be shown using similar arguments.

Furthermore, using

$$(2.18) \quad \log \frac{Z_\nu^{Q,P^A}}{Z_s^{Q,P^A}} = \log \frac{Z_\nu^{Q,P}}{Z_s^{Q,P}} - \log \frac{Z_\nu^{P^A,P}}{Z_s^{P^A,P}}, \quad s, \nu \in \mathcal{T}, \quad s \leq \nu,$$

and that $Z_u^{P^A,P} = Z_\tau^{P^A,P}$ for $u \in [\tau, T]$, we conclude that

$$(2.19) \quad H_\tau^T(Q|P) = H_\tau^T(Q|P^A).$$

We apply the duality formula (2.10) (with starting wealth $X_\tau + A_\tau$) along with a change of measure from P to P^A , and then use (2.17) and (2.19) to write the value function

$$\begin{aligned} V(t, X_t; A) &= \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \operatorname{ess\,sup}_{\Theta_{t,\tau}(P)} \mathbb{E}_t \left\{ U(X_\tau) \cdot e^{-\gamma A_\tau} \cdot \exp \left(- \operatorname{ess\,inf}_{Q \in \mathbb{P}_f(P)} H_\tau^T(Q|P) \right) \right\} \\ &= \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} c_A^{-1} Z_t^{P^A, P} \operatorname{ess\,sup}_{\Theta_{t,\tau}(P^A)} \mathbb{E}_t^{P^A} \left\{ U(X_\tau) \cdot \exp \left(- \operatorname{ess\,inf}_{Q \in \mathbb{P}_f(P^A)} H_\tau^T(Q|P^A) \right) \right\}. \end{aligned}$$

Next, applying Proposition 2.6 and Lemma 2.7 with the prior measure being P^A , we have

$$\begin{aligned} V(t, X_t; A) &= \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} c_A^{-1} Z_t^{P^A, P} U(X_t) \cdot \exp \left(- \operatorname{ess\,inf}_{Q \in \mathbb{P}_f(P^A)} H_t^T(Q|P^A) \right) \\ &= \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} c_A^{-1} Z_t^{P^A, P} U(X_t) \cdot \exp \left(- \operatorname{ess\,inf}_{Q \in \mathbb{P}_f(P^A)} \left(H_t^\tau(Q|P^A) \right. \right. \\ &\quad \left. \left. + \mathbb{E}_t^Q \left\{ \operatorname{ess\,inf}_{Q \in \mathbb{P}_f(P^A)} H_\tau^T(Q|P^A) \right\} \right) \right) \\ &= \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} c_A^{-1} Z_t^{P^A, P} U(X_t) \cdot \exp \left(- \operatorname{ess\,inf}_{Q \in \mathbb{P}_f(P^A)} \left(H_t^\tau(Q|P^A) \right. \right. \\ &\quad \left. \left. + \mathbb{E}_t^Q \{ H_\tau^T(Q^E|P) \} \right) \right), \end{aligned} \tag{2.20}$$

where the last equality follows from (2.17) and (2.19) and the fact that Q^E is the entropy-minimizing measure. Lastly, we use definition (2.16) and the equality (2.18) to write the conditional relative entropy of Q with respect to P^A in terms of its entropy with respect to P :

$$H_t^\tau(Q|P^A) = \mathbb{E}_t^Q \left\{ \log \frac{Z_\tau^{Q, P}}{Z_\tau^{Q^E, P}} + \gamma A_\tau \right\} - \log c_A + \log Z_t^{P^A, P}.$$

Substituting this into (2.20), we immediately obtain (2.8).

2.3. The indifference price. It is often better for intuitive purposes to characterize optimal exercising strategies in terms of indifference prices, which we introduce next. A holder's indifference price of an American claim is defined as the reduction in wealth such that the holder's value function V is the same as the Merton function M from investment without the claim. For time $t \in [0, T]$, denote $p_t \equiv p_t(A)$ as the indifference price of claim A . It is defined by the equation

$$(2.21) \quad V(t, X_t - p_t; A) = M(t, X_t).$$

Next, we give some general expressions for the indifference price.

PROPOSITION 2.8. *The indifference price can be written as*

$$(2.22) \quad p_t = -\frac{1}{\gamma} \log \left(- \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \operatorname{ess\,sup}_{\theta \in \Theta_{t,\tau}} \mathbb{E}_t^{Q^E} \left\{ -e^{-\gamma(G_{t,\tau}(\theta) + A_\tau)} \right\} \right)$$

and

$$(2.23) \quad p_t = \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \operatorname{ess\,inf}_{Q \in \mathbb{P}_f(P)} \left(\mathbb{E}_t^Q \{ A_\tau \} + \frac{1}{\gamma} H_t^\tau(Q|P) + \frac{1}{\gamma} \mathbb{E}_t^Q \{ H_\tau^T(Q^E|P) \} \right) - \frac{1}{\gamma} H_t^T(Q^E|P).$$

If $\mathbb{P}_f(P) = \mathbb{P}_f(Q^E)$, then the last representation can be simplified as

$$(2.24) \quad p_t = \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \operatorname{ess\,inf}_{Q \in \mathbb{P}_f(Q^E)} \left(\mathbb{E}_t^Q \{A_\tau\} + \frac{1}{\gamma} H_t^\tau(Q|Q^E) \right).$$

Proof. Applying the separation of variables formula (2.9) and a change of measure from P to Q^E using the density in (2.3), we write

$$\begin{aligned} V(t, X_t; A) &= \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \operatorname{ess\,inf}_{\theta \in \Theta_{t,\tau}} \mathbb{E}_t \left\{ U(X_t + G_{t,\tau}(\theta) + \gamma A_\tau) \cdot \mathbb{E}_\tau \{e^{G_{\tau,T}(\theta^E)}\} \right\} \\ &= -U(X_t) \cdot \mathbb{E}_t \{e^{G_{t,T}(\theta^E)}\} \cdot \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \operatorname{ess\,sup}_{\theta \in \Theta_{t,\tau}} \mathbb{E}_t^{Q^E} \left\{ -e^{-\gamma A_\tau} e^{-\gamma G_{t,\tau}(\theta)} e^{-G_{t,\tau}(\theta^E)} \right\} \\ &= -M(t, X_t) \cdot \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \operatorname{ess\,sup}_{\theta \in \Theta_{t,\tau}} \mathbb{E}_t^{Q^E} \left\{ -e^{-\gamma A_\tau} e^{-\gamma G_{t,\tau}(\theta + \frac{\theta^E}{\gamma})} \right\} \\ (2.25) \quad &= -M(t, X_t) \cdot \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \operatorname{ess\,sup}_{\theta \in \Theta_{t,\tau}} \mathbb{E}_t^{Q^E} \{U(A_\tau + G_{t,\tau}(\theta))\}. \end{aligned}$$

Since it follows from (2.21) that

$$(2.26) \quad \frac{V(t, X_t; A)}{-M(t, X_t)} = U(p_t),$$

substitution of (2.25) into (2.26) yields (2.22).

The second expression (2.23) follows from the dual (2.8) and the definition (2.21). To show (2.24), we use the simple equality

$$(2.27) \quad H_t^\tau(Q|P) = H_t^\tau(Q|Q^E) + \mathbb{E}_t^Q \left\{ \log \frac{Z_\tau^E}{Z_t^E} \right\}, \quad Q \in \mathbb{P}_f(P),$$

where, by (2.2), the last term can be written as

$$(2.28) \quad \mathbb{E}_t^Q \left\{ \log \frac{Z_\tau^E}{Z_t^E} \right\} = -\mathbb{E}_t^Q \{H_\tau^T(Q^E|P)\} + H_t^T(Q^E|P) + \mathbb{E}_t^Q \{G_{t,\tau}(\theta^E)\}.$$

The assumption $\mathbb{P}_f(P) = \mathbb{P}_f(Q^E)$ implies that the last term is zero. To complete the proof, we substitute (2.27) and (2.28) into (2.23). \square

Remark 2.9. A sufficient condition for $\mathbb{P}_f(P) = \mathbb{P}_f(Q^E)$ is $\frac{dQ^E}{dP} \in L^2(P)$, as shown in Lemma 2 of [19]. In general, we have $\mathbb{P}_f(P) \subseteq \mathbb{P}_f(Q^E)$.

2.4. The optimal exercise time. In this section, we provide a characterization for the holder's optimal exercise time by analyzing the indifference price, which is expressed in (2.22) and (2.23) as two joint stochastic control and optimal stopping problems. First, we rewrite (2.22) as

$$(2.29) \quad U(p_t) = \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \operatorname{ess\,sup}_{\theta \in \Theta_{t,\tau}} \mathbb{E}_t^{Q^E} \{U(G_{t,\tau}(\theta) + A_\tau)\},$$

which can be regarded as a special example of a *cooperative* stochastic game. The second expression yields a *noncooperative* stochastic game:

$$(2.30) \quad p_t = \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \operatorname{ess\,inf}_{Q \in \mathbb{P}_f(P)} \mathbb{E}_t^Q \{A_\tau + l_{t,\tau}^{Q,\gamma}\},$$

where the penalty term is

$$(2.31) \quad l_{t,\tau}^{Q,\gamma} := \frac{1}{\gamma} \left[H_{\tau}^T(Q^E|P) + \log \frac{Z_{\tau}^{Q,P}}{Z_t^{Q,P}} - H_t^T(Q^E|P) \right].$$

The structures of expressions (2.29) and (2.30) are very similar to the stochastic games studied in [23], and we will apply some of the results from that paper here. In the theory of optimal stopping, it is common to require quasi left-continuity for the associated processes (see, among others, [9, 34, 35]). This assumption is quite general, and it allows for the processes commonly used in finance, including diffusion processes and Lévy processes. In fact, all standard Markov processes are quasi-left-continuous. Guasoni [14] studies optimal investment with quasi-left-continuous asset prices subject to transaction costs.

DEFINITION 2.10. A process $(Y_t)_{0 \leq t \leq T}$ is P -quasi-left-continuous if for any increasing sequence of stopping times $(\tau_n)_{n \in \mathbb{N}} \subseteq \mathcal{T}$ and with $\tau := \lim_{n \rightarrow \infty} \tau_n \in \mathcal{T}$ we have $\lim_{n \rightarrow \infty} Y_{\tau_n} = Y_{\tau}$, P -a.s.

Remark 2.11. If Y is P -quasi-left-continuous, then it is also quasi-left-continuous with respect to any $Q \in \mathbb{P}_f(P)$. This also means that Y is $\mathbb{P}_f(P)$ -quasi-left-continuous in the sense of Definition 2.9 of [23].

ASSUMPTION 2.12. The processes $(A_t)_{0 \leq t \leq T}$, $(S_t)_{0 \leq t \leq T}$, and $(l_{0,t}^{Q,\gamma})_{0 \leq t \leq T}$ are quasi-left-continuous with respect to every $Q \in \mathbb{P}_f(P)$.

PROPOSITION 2.13. For any $t \in [0, T]$, the optimal exercise time for (2.22) or (2.29) is given by

$$(2.32) \quad \tau_t^* = \inf\{t \leq u \leq T : p_u = A_u\}$$

so that

$$p_t = -\frac{1}{\gamma} \log \left(-\operatorname{ess\,sup}_{\theta \in \Theta_{t,\tau_t^*}} \mathbb{E}_t^{Q^E} \left\{ -e^{-\gamma(G_{t,\tau_t^*}(\theta) + A_{\tau_t^*})} \right\} \right).$$

Proof. Since S is P -quasi-left-continuous, so is the process $G(\theta)$ for any feasible $\theta \in \Theta$. By Remark 2.11, the process $(U(G_t(\theta) + A_t))_{0 \leq t \leq T}$ is Q^E -quasi-left-continuous. It is also bounded above by zero. Then, by Theorem 2.10 of [23], we have for any $t \in [0, T]$ that

$$U(p_t) = \operatorname{ess\,sup}_{\theta \in \Theta_{t,\tau_t^*}} \mathbb{E}_t^{Q^E} \left\{ -e^{-\gamma(G_{t,\tau_t^*}(\theta) + A_{\tau_t^*})} \right\},$$

with τ_t^* given by (2.32). \square

The optimal exercise time (2.32) is the first time when the indifference price equals the payoff from immediate exercise. This is highly intuitive because the indifference price is the minimum amount of money the holder demands in order to forgo the claim. At the optimal exercise time, the claim payoff is sufficient to induce the holder to exercise.

Turning our attention to expression (2.30), we want to show that the order of choosing the optimal exercise time (via essential supremum) and optimal measure (via essential infimum) does not alter the problem.

PROPOSITION 2.14. For any $t \in [0, T]$, define the upper value process by

$$\bar{p}_t := \operatorname{ess\,inf}_{Q \in \mathbb{P}_f(P)} \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_t^Q \left\{ A_{\tau} + l_{t,\tau}^{Q,\gamma} \right\},$$

and the stopping time $\bar{\tau}_t := \inf\{t \leq u \leq T : \bar{p}_u = A_u\}$. Then, for any $Q \in \mathbb{P}_f(P)$ and $\tau \in [t, \bar{\tau}_t]$, we have

$$(2.33) \quad \bar{p}_t \leq \mathbb{E}_t^Q \{\bar{p}_\tau + l_{t,\tau}^{Q,\gamma}\}.$$

In particular, when $\tau = \bar{\tau}_t$, we have the equality

$$(2.34) \quad \bar{p}_t = \operatorname{ess\,inf}_{Q \in \mathbb{P}_f(P)} \mathbb{E}_t^Q \{A_{\bar{\tau}_t} + l_{t,\bar{\tau}_t}^{Q,\gamma}\}.$$

The proof is given in Appendix A.1. The last equality indicates that $\bar{\tau}_t$ is optimal for the upper value process \bar{p}_t . The next proposition, which follows easily from Proposition 2.14, shows that the upper value process is the same as the indifference price process, and $\bar{\tau}_t$ is in fact equal to τ_t^* .

PROPOSITION 2.15. *For any $t \in [0, T]$, the indifference price and upper value processes are the same; i.e., $\bar{p}_t = p_t$. Consequently, the corresponding optimal stopping times are identical; i.e., $\bar{\tau}_t = \tau_t^*$.*

Proof. We always have $\bar{p}_t \geq p_t$. The preceding proposition gives the other direction:

$$\bar{p}_t = \operatorname{ess\,inf}_{Q \in \mathbb{P}_f(P)} \mathbb{E}_t^Q \{A_{\bar{\tau}_t} + l_{t,\bar{\tau}_t}^{Q,\gamma}\} \leq \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \operatorname{ess\,inf}_{Q \in \mathbb{P}_f(P)} \mathbb{E}_t^Q \{A_\tau + l_{t,\tau}^{Q,\gamma}\} = p_t. \quad \square$$

2.5. Risk aversion and volume asymptotics. We now analyze the effects of risk aversion and holding volume on indifference prices and the corresponding optimal exercise times. To this end, let us consider an investor with risk aversion parameter γ who holds $\alpha > 0$ units of an American claim A , and suppose that all α units have to be exercised simultaneously. By definition (2.7), the holder's value function is given by $V(t, X_t; \alpha A)$. The corresponding indifference price, denoted by $p_t(\alpha, \gamma)$, is defined by the equation

$$V(t, X_t; \alpha A) = M(t, X_t + p_t(\alpha, \gamma)).$$

The optimal exercise time is the first time that the indifference price reaches the payoff from exercising all claims:

$$(2.35) \quad \tau_t^*(\alpha, \gamma) = \inf\{t \leq u \leq T : p_u(\alpha, \gamma) = \alpha A_u\}.$$

The following result establishes monotonicity in the risk aversion coefficient.

PROPOSITION 2.16. *Let $\gamma_2 \geq \gamma_1 > 0$. Then, for any $t \in [0, T]$ and $\alpha \geq 0$, we have $p_t(\gamma_2, \alpha) \leq p_t(\gamma_1, \alpha)$ and $\tau_t^*(\alpha, \gamma_2) \leq \tau_t^*(\alpha, \gamma_1)$. That is, a higher risk aversion implies an earlier optimal exercise time of the American claims.*

Proof. By definition (2.31) of the penalty term, we have

$$\mathbb{E}_t^Q \{l_{t,\tau}^{Q,\gamma}\} = \gamma^{-1} \left(H_t^\tau(Q|P) + \mathbb{E}_t^Q \{H_\tau^T(Q^E|P)\} - H_t^T(Q^E|P) \right).$$

It easily follows from Lemma 2.7 that $\mathbb{E}_t^Q \{l_{t,\tau}^{Q,\gamma}\} \geq 0$. For $\gamma_2 \geq \gamma_1 > 0$, we have $\mathbb{E}_t^Q \{l_{t,\tau}^{Q,\gamma_1}\} \geq \mathbb{E}_t^Q \{l_{t,\tau}^{Q,\gamma_2}\}$, and therefore $p_t(\gamma_1, \alpha) \geq p_t(\gamma_2, \alpha)$. As the risk aversion parameter γ increases, the indifference price decreases, but the payoff αA does not depend on γ . This leads to a shorter optimal exercise time according to (2.35). \square

Next, to analyze the large risk aversion limit, we recall that the *subhedging price* of the American claim A is defined as

$$c_t := \operatorname{ess\,inf}_{Q \in \mathbb{P}_e(P)} \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_t^Q \{A_\tau\}, \quad t \in [0, T].$$

See, for example, [21]. As an investor becomes more risk-averse, the price he is willing to pay for A tends to its subhedging price. The idea is that the penalty term vanishes as risk aversion increases to infinity.

PROPOSITION 2.17. *For any $t \in [0, T]$, we have*

$$\lim_{\gamma \rightarrow \infty} p_t(\alpha, \gamma) = \alpha c_t.$$

The proof is given in Appendix A.2. A consequence of this result is that, at the large risk aversion limit, the pricing rule will become linear in the quantity of claims, and the investor will exercise all the American claims A at a time independent of α .

As the investor's risk aversion diminishes to zero, he tends to price the American claim under the MEMM, Q^E . This limiting price is the American analogue to Davis-price for European-style contingent claims (see [6]).

PROPOSITION 2.18. *For any $t \in [0, T]$, we have*

$$\lim_{\gamma \rightarrow 0} p_t(\alpha, \gamma) = \alpha \cdot \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_t^{Q^E} \{A_\tau\}.$$

Proof. The proof is a slight extension of Proposition 1.3.4 of [1] and is omitted. \square

We observe from (2.23) the *volume-scaling* property that for $\alpha > 0$, $\alpha^{-1} p_t(\alpha, \gamma) = p_t(1, \alpha\gamma)$. The simultaneous exercise assumption is essential for this property to hold. As the number of claims held increases, it follows from Proposition 2.16 that the *average* indifference price for holding α units of A decreases. By (2.35), the optimal exercise time $\tau_t^{\alpha*}$ is the first time the average indifference price hits the claim payoff, so the holder tends to exercise the claims earlier when the holding volume increases.

The risk aversion limits in Propositions 2.17 and 2.18 can be rewritten as volume limits:

$$\lim_{\alpha \rightarrow \infty} \frac{p_t(\alpha, \gamma)}{\alpha} = c_t, \quad \lim_{\alpha \rightarrow 0} \frac{p_t(\alpha, \gamma)}{\alpha} = \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_t^{Q^E} \{A_\tau\}.$$

Finally, we point out that the indifference price for the claim A lies within the no-arbitrage price interval. That is,

$$c_t \leq p_t(1, \gamma) \leq \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_t^{Q^E} \{A_\tau\} \leq \operatorname{ess\,sup}_{Q \in \mathbb{P}_e(P)} \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_t^Q \{A_\tau\},$$

where the leftmost term is the subhedging price and the rightmost term is the superhedging price.

The indifference price possibly possesses other properties of interest. For instance, Becherer [1] and Mania and Schweizer [27] point out that the indifference price for holding β units of European-style claims is concave as a function of β , and strict concavity and differentiability with respect to β is established in [18]. However, in our case with American claims, these problems become more complicated, and we do not address them here.

3. Dynamic hedging of American claims with multiple exercises. For American claims with multiple exercise rights, the holder can exercise separately and has to choose the optimal multiple exercise times for all claims held. Suppose that an investor is dynamically hedging a long position in $n \geq 2$ integer units of claim A . For any $t \in [0, T]$ and $i \leq n$, we denote by $\tau_t^{(i)*}$ the optimal exercise time of the next

claim when i units remain unexercised at time t . After exercising one claim at time $\tau_t^{(i)*}$, the investor has $(i - 1)$ units left. If the holder exercises multiple units at the same time, then some exercise times may coincide.

Upon exercising any American claim(s), the investor immediately reinvests the contract proceeds into his trading portfolio until time T . The investor's value function for holding $i \geq 2$ units of A at time t is given recursively by

$$(3.1) \quad V^{(i)}(t, X_t) = \operatorname{ess\,sup}_{\tau_i \in \mathcal{T}_{t,T}} \operatorname{ess\,sup}_{\theta \in \Theta_{t,\tau_i}} \mathbb{E}_t \left\{ V^{(i-1)}(\tau_i, X_{\tau_i} + A_{\tau_i}) \right\},$$

with $V^{(1)}(t, X_t) = V(t, X_t; A)$. The optimal exercise times can be characterized via indifference prices for holding multiple claims, which we define next.

DEFINITION 3.1. *For any $t \in [0, T]$, the holder's indifference price for holding $i \leq n$ claims with multiple exercises is defined as the random variable $p_t^{(i)}$ such that*

$$(3.2) \quad V^{(i)}(t, X_t - p_t^{(i)}) = M(t, X_t).$$

Substituting (3.2) into (3.1), we observe that the value function

$$\begin{aligned} V^{(i)}(t, X_t) &= \operatorname{ess\,sup}_{\tau_i \in \mathcal{T}_{t,T}} \operatorname{ess\,sup}_{\theta \in \Theta_{t,\tau_i}} \mathbb{E}_t \left\{ V^{(i-1)}(\tau_i, X_{\tau_i} + A_{\tau_i}) \right\} \\ &= \operatorname{ess\,sup}_{\tau_i \in \mathcal{T}_{t,T}} \operatorname{ess\,sup}_{\theta \in \Theta_{t,\tau_i}} \mathbb{E}_t \left\{ M \left(\tau_i, X_{\tau_i} + A_{\tau_i} + p_{\tau_i}^{(i-1)} \right) \right\} \\ (3.3) \quad &= V(t, X_t; A + p^{(i-1)}). \end{aligned}$$

The last equality in (3.3) provides a crucial connection between dynamic hedging for claims with single exercise and for claims with multiple exercises. We can derive the dual for $V^{(i)}(t, X_t)$ and the indifference price $p_t^{(i)}$ by just replacing A by $A + p^{(i-1)}$ in Propositions 2.4 and 2.8.

Next, we give the following general expressions for the indifference price.

PROPOSITION 3.2. *The indifference price $p_t^{(i)}$ can be written recursively by*

$$(3.4) \quad p_t^{(i)} = -\frac{1}{\gamma} \log \left(-\operatorname{ess\,sup}_{\tau_i \in \mathcal{T}_{t,T}} \operatorname{ess\,sup}_{\theta \in \Theta_{t,\tau_i}} \mathbb{E}_t^{Q^E} \left\{ -e^{-\gamma(G_{t,\tau_i}(\theta) + A_{\tau_i} + p_{\tau_i}^{(i-1)})} \right\} \right).$$

An alternative expression for the indifference price is

$$(3.5) \quad p_t^{(i)} = \operatorname{ess\,sup}_{\tau_i \in \mathcal{T}_{t,T}} \operatorname{ess\,inf}_{Q \in \mathbb{P}_f(P)} \mathbb{E}_t^Q \{ A_{\tau_i} + p_{\tau_i}^{(i-1)} + l_{t,\tau_i}^{Q,\gamma} \},$$

where $l_{t,\tau_i}^{Q,\gamma}$ is defined in (2.31).

In Proposition 2.16, we showed that higher risk aversion reduces the indifference price $p_t(\alpha, \gamma)$ for American claims with simultaneous exercise. The same also holds in the case with multiple exercises. We denote the indifference price by $p_t^{(i)}(\gamma)$ to indicate the dependence on γ .

PROPOSITION 3.3. *For any time $t \in [0, T]$ and integer $i \geq 1$, the indifference price $p_t^{(i)}(\gamma)$ is decreasing with respect to γ .*

Proof. As shown in Proposition 2.16, both $p_t^{(1)}(\gamma)$ and $\mathbb{E}_t^Q \{ l_{t,\tau}^{Q,\gamma} \}$ (for any $\tau \in \mathcal{T}_{t,T}$) decrease with γ . Suppose that the same is true for $p_t^{(i-1)}(\gamma)$. Then, we have that

$\mathbb{E}^Q\{p_t^{(i-1)}(\gamma) + l_{t,\tau_i}^{Q,\gamma}\}$ decreases with γ , and therefore $p_t^{(i)}(\gamma)$ is also decreasing with γ . As a result, the proposition holds by induction. \square

To understand the impact of holding volume on the exercise times, we consider two investors holding i and $(i+1)$ units of claims, respectively, and compare the optimal exercise times $\tau_t^{(i)*}$ and $\tau_t^{(i+1)*}$. Applying the techniques in section 3.3, we have

$$(3.6) \quad \tau_t^{(i)*} = \inf \left\{ t \leq u \leq T : p_u^{(i)} - p_u^{(i-1)} = A_u \right\},$$

and we can address the question by showing that the exercise times are ordered if and only if the indifference price is a concave function of i : $p_t^{(i+1)} - p_t^{(i)} \leq p_t^{(i)} - p_t^{(i-1)}$.

PROPOSITION 3.4. *For $t \in [0, T]$ and integer $i \geq 1$, the optimal exercise times are ordered in the sense $\tau_t^{(i+1)*} \leq \tau_t^{(i)*}$ if and only if the indifference price $p_t^{(i)}$ is a concave function of i .*

Proof. First, we define the Snell envelope

$$\hat{p}_t^{(i)}(Q) := \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_t^Q \left\{ A_\tau + p_\tau^{(i-1)} + l_{t,\tau}^{Q,\gamma} \right\} \quad \text{for } Q \in \mathbb{P}_f(P).$$

Note that, $p_t^{(i)} = \operatorname{ess\,inf}_{Q \in \mathbb{P}_f} \hat{p}_t^{(i)}(Q)$. It is well known that the process $(\hat{p}_t^{(i)}(Q) + l_{0,t}^{Q,\gamma})_{0 \leq t \leq T}$ is a Q -supermartingale. Consequently,

$$\hat{p}_t^{(i)}(Q) \geq \mathbb{E}_t^Q \left\{ \hat{p}_\tau^{(i)}(Q) + l_{t,\tau}^{Q,\gamma} \right\} \geq \mathbb{E}_t^Q \left\{ p_\tau^{(i)} + l_{t,\tau}^{Q,\gamma} \right\}.$$

Now, we suppose $\tau_t^{(i+1)*} \leq \tau_t^{(i)*} \leq \tau_t^{(i-1)*}$. Applying (2.33) of Proposition 2.14 with A replaced by $A + p^{(i-1)}$, it follows that

$$p_t^{(i-1)} \leq \mathbb{E}_t^Q \left\{ p_\tau^{(i-1)} + l_{t,\tau}^{Q,\gamma} \right\} \quad \text{for } \tau \in [t, \tau_t^{(i)*}], \quad Q \in \mathbb{P}_f(P).$$

Hence, we have the following inequalities:

$$\begin{aligned} p_t^{(i+1)} + p_t^{(i-1)} &= \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,\tau_t^{(i)*}}} \operatorname{ess\,inf}_{Q \in \mathbb{P}_f(P)} \mathbb{E}_t^Q \left\{ A_\tau + p_\tau^{(i)} + l_{t,\tau}^{Q,\gamma} \right\} + p_t^{(i-1)} \\ &\leq \operatorname{ess\,inf}_{Q \in \mathbb{P}_f(P)} \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,\tau_t^{(i)*}}} \left(\mathbb{E}_t^Q \left\{ A_\tau + p_t^{(i-1)} \right\} + \mathbb{E}_t^Q \left\{ p_\tau^{(i)} + l_{t,\tau}^{Q,\gamma} \right\} \right) \\ &\leq \operatorname{ess\,inf}_{Q \in \mathbb{P}_f(P)} \left(\operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,\tau_t^{(i)*}}} \mathbb{E}_t^Q \left\{ A_\tau + p_\tau^{(i-1)} + l_{t,\tau}^{Q,\gamma} \right\} + \hat{p}_t^{(i)}(Q) \right) \\ &\leq \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_t^{Q^{(i)*}} \left\{ A_\tau + p_\tau^{(i-1)} + l_{t,\tau}^{Q^{(i)*},\gamma} \right\} + \hat{p}_t^{(i)}(Q^{(i)*}) \\ &= 2p_t^{(i)}, \end{aligned}$$

where

$$Q^{(i)*} = \arg \min_{Q \in \mathbb{P}_f(P)} \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_t^Q \left\{ A_\tau + p_\tau^{(i-1)} + l_{t,\tau}^{Q,\gamma} \right\}.$$

Hence, we have $p_t^{(i+1)} + p_t^{(i-1)} \leq 2p_t^{(i)}$, and therefore the indifference prices are concave in i .

On the other hand, if the indifference price is a concave function of i , then from the inequality $p_t^{(i+1)} - p_t^{(i)} \leq p_t^{(i)} - p_t^{(i-1)}$ and the definition of the optimal exercise in (3.6), we can easily conclude that $\tau_t^{(i+1)*} \leq \tau_t^{(i)*}$. \square

4. Static-dynamic hedges for American claims. In addition to investing in the riskless and risky assets, the holder of claim A can also reduce risk exposure by purchasing some market-traded claims. For instance, the holder of an American call option can remove some downside risk by buying and holding some European or American puts written on the same underlying asset. As transaction costs on derivative securities are less negligible than on stocks, the static hedging strategies considered here involve purchasing a portfolio of market-traded claims at initiation and no trades afterwards. To avoid arbitrage, the prices of the market-traded claims are assumed to lie between the subhedging and superhedging prices. In this section, we analyze the combination of the static and dynamic hedging strategies, called static-dynamic hedges, for a long position in an American claim. An application to ESO valuation is analyzed in section 5.

4.1. Hedging with European options. While holding one unit of American claim A , the investor purchases β European options each with bounded payoff, $B \in \mathcal{F}_T$, expiring at the same date T for simplicity. If the investor exercises claim A at time $\tau \in \mathcal{T}$, then immediately his trading wealth increases by A_τ , and only European claims remain. At that point, the value function is

$$\operatorname{ess\,sup}_{\theta \in \Theta_{\tau,T}} \mathbb{E}_\tau \{ U(X_\tau + A_\tau + G_{\tau,T}(\theta) + \beta B) \} = M(\tau, X_\tau + A_\tau + h_\tau(\beta B)),$$

where $h_\tau(\beta B)$ is the indifference price for holding β units of B at time τ . From this, we define the investor's value function as

$$(4.1) \quad \tilde{V}(t, X_t; \beta B) = \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \operatorname{ess\,sup}_{\theta \in \Theta_{t,\tau}} \mathbb{E}_t \{ M(\tau, X_\tau + A_\tau + h_\tau(\beta B)) \}.$$

PROPOSITION 4.1. *For any $t \in [0, T]$ and $\beta \in \mathbb{R}$, the value function can be written as*

$$(4.2) \quad \tilde{V}(t, X_t; \beta B) = V(t, X_t; A + h(\beta B)).$$

Therefore, the indifference price for holding claim A and β units of B is $p_t(A + h(\beta B))$, and the optimal exercise time is given by

$$(4.3) \quad \tilde{\tau}_t^*(\beta) = \inf \{ t \leq u \leq T : p_u(A + h(\beta B)) = A_u + h_u(\beta B) \}.$$

Proof. We observe from (4.1) that the value function, $\tilde{V}(t, X_t; \beta B)$, is equivalent to that with dynamic hedge but for a different claim $A + h(\beta B)$ instead of A . Hence, we have (4.2). This equality also allows us to derive the dual of \tilde{V} and then the corresponding indifference price, denoted by $p_t(A + h(\beta B))$, by replacing A_τ with $A_\tau + h_\tau(\beta B)$ in Propositions 2.4 and 2.8, respectively. Then, it easily follows from Proposition 2.13 that the optimal exercise time is given by (4.3). \square

If the European claims B cost π each, then the investor's wealth is reduced by the amount $\beta\pi$. Therefore, the value function is given by $\tilde{V}(t, X_t - \beta\pi; \beta B)$, and the indifference price is given by $p_t(A + h(\beta B)) - \beta\pi$. However, for any fixed β , the cost does not affect the optimal exercise time.

The investor chooses the optimal static hedge, β^* , to maximize his value function, which turns out to be equivalent to maximizing the indifference price:

$$(4.4) \quad \beta^* = \arg \max_{\beta} \tilde{V}(t, X_t - \beta\pi; \beta B) = \arg \max_{\beta} p_t(A + h(\beta B)) - \beta\pi.$$

Hence, the optimal quantity (when it exists) of European options to purchase is found from the Fenchel–Legendre transform of the indifference price $p_t(A + h(\beta B))$ as a function of β , evaluated at the market price π . The market price of European puts controls the optimal static hedge in (4.4), which indirectly affects the holder's optimal exercise time.

4.2. Hedging with American options. The holder of claim A may be restricted to purchasing American options to form a partial static hedge. In fact, most nonindex options on stocks are American. We consider an American claim D with an adapted bounded discounted payoff process $(D_t)_{0 \leq t \leq T}$. For instance, D can be the payoff of an American put written on the same underlying asset as A . However, as in section 2.5, we shall make the simplification that all the American options used for the static hedge have to be exercised simultaneously.

Fix a time $t \leq T$ at which the investor, while holding claim A , purchases α units of claim D from the market for the price $\$ \pi'$ each. The investor needs to decide which claim(s) to exercise first and when. To this end, we recall that $V(t, X_t; A)$ and $V(t, X_t; \alpha D)$ represent the investor's value functions for holding, respectively, only A and only α units of D .

We denote by (p^A, τ^A) and $(p^{\alpha D}, \tau^{\alpha D})$ the corresponding pairs of indifference prices and optimal exercise times. The investor's value function is given by

$$\hat{V}(t, X_t; \alpha D) := \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \operatorname{ess\,sup}_{\theta \in \Theta_{t,\tau}} \mathbb{E}_t \{ \max(V(\tau, X_\tau + \alpha D_\tau; A), V(\tau, X_\tau + A_\tau; \alpha D)) \}.$$

Next, we simplify the problem and derive the corresponding optimal exercise times.

PROPOSITION 4.2. *For any $t \in [0, T]$ and $\alpha \in \mathbb{R}$, the value function can be written as*

$$(4.5) \quad \hat{V}(t, X_t; \alpha D) = V(t, X_t; R^\alpha),$$

with $R^\alpha := \max(\alpha D + p^A, A + p^{\alpha D})$. Therefore, the investor's indifference price for holding A and α units of D is $p_t(R^\alpha)$, and the optimal exercise time is given by

$$(4.6) \quad \hat{\tau}_t^*(\alpha) = \min(\tau_t^{AD}(\alpha), \tau_t^{DA}(\alpha)),$$

where

$$(4.7) \quad \tau_t^{AD}(\alpha) = \inf \{ t \leq u \leq T : p_u(R^\alpha) = A_u + p_u^{\alpha D} \},$$

$$(4.8) \quad \tau_t^{DA}(\alpha) = \inf \{ t \leq u \leq T : p_u(R^\alpha) = \alpha D_u + p_u^A \}.$$

Proof. By the definitions of indifference prices $p^{\alpha D}$ and p^A , we have

$$\begin{aligned} \hat{V}(t, X_t; \alpha D) &= \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \operatorname{ess\,sup}_{\theta \in \Theta_{t,\tau}} \mathbb{E}_t \{ \max(M(\tau, X_\tau + \alpha D_\tau + p_\tau^A), M(\tau, X_\tau + A_\tau + p_\tau^{\alpha D})) \} \\ &= \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \operatorname{ess\,sup}_{\theta \in \Theta_{t,\tau}} \mathbb{E}_t \{ M(\tau, X_\tau + \max(\alpha D_\tau + p_\tau^A, A_\tau + p_\tau^{\alpha D})) \} \\ &= V(t, X_t; R^\alpha). \end{aligned}$$

The last equality means that the investor's value function is reduced to the dynamic hedging case for a single yet complex claim R^α , paying R_τ^α at any exercise time

$\tau \in \mathcal{T}_{t,T}$. This observation allows us to apply the results in section 2 to derive the indifference price $p_t(R^\alpha)$ by replacing A with R^α in (2.22) and (2.23). For any fixed α , the investor's optimal time to exercise the first claim, which could be either A or D , is given by

$$\begin{aligned}\hat{\tau}_t^*(\alpha) &= \inf\{t \leq u \leq T : p_u(R^\alpha) = \max(\alpha D_u + p_u^A, A_u + p_u^{\alpha D})\} \\ &= \min(\tau_t^{AD}(\alpha), \tau_t^{DA}(\alpha)),\end{aligned}$$

where $\tau_t^{AD}(\alpha)$ and $\tau_t^{DA}(\alpha)$ are given by (4.7) and (4.8), respectively. \square

When the market price of the static hedge is incorporated, the investor's wealth is reduced by the amount $\alpha\pi'$, and the value function becomes $\hat{V}(t, X_t - \alpha\pi'; \alpha D)$. The investor's indifference price for holding A and purchasing α units of D at price π' each is given by $p_t(R^\alpha) - \alpha\pi'$. In view of (4.4), the investor's optimal static hedge, denoted by α^* if it exists, is given by

$$\alpha^* = \arg \max_{\alpha} p_t(R^\alpha) - \alpha\pi'.$$

In this section, we have formulated the basic framework for static-dynamic hedging of American claims with simultaneous exercise, in which the static hedging instruments are all exercised at once. The problem of static-dynamic hedging American claims with multiple exercising rights using other multiple exercising American claims can be formulated similarly using the principle of dynamic programming. To write down the value function, one has to first consider the maximal expected utilities from all possible orders of exercises. As the number of multiple exercising claims increases, the value function becomes very tedious, though straightforward, to write down. Moreover, the resulting optimal exercising strategies will also be too complex to describe. As an approximation, one can limit the number of exercise opportunities to a small finite number, or just adopt the case with simultaneous exercise.

5. ESO valuation. We will apply the mechanism of dynamic hedging and static-dynamic hedging to ESO valuation. ESOs are American call options written on the firm's stock granted to the employee as a form of compensation. In most cases, the ESOs are not exercisable until a prespecified vesting period has elapsed. Typically, the terms of an ESO contract stipulate that the employee is not allowed to sell or transfer the option, short sell the firm's stock, or take short positions in call options written on the firm's stock. These restrictions prevent the employee from perfectly hedging his ESO. Hence, the employee faces a constrained investment problem in which he has to decide how to optimally hedge and exercise his ESO.

Henderson [15] studies a valuation model for a European ESO that captures the employee's risk aversion and dynamic investment in the market index. Grasselli and Henderson [12] study the case of multiple American ESOs with infinite maturity. For ESOs of American type with multiple exercise rights and finite maturity, Leung and Sircar [26] study the combined effect of risk aversion, dynamic hedging, vesting, and job termination risk on the optimal exercise policy and the corresponding ESO cost.

Here we will consider only the hedging and valuation of a single ESO. We summarize the results on dynamic hedging of an ESO in section 5.1. In this case, the employee trades in a market index and the bank account, but not the firm's stock. Then, in sections 5.2.1 and 5.2.2, we will augment the employee's trading strategy by incorporating, respectively, static hedges with European and American puts on the firm's stock. Our goal is to analyze the nontrivial effects of static-dynamic hedges

on the employee's optimal exercising strategies (Figures 5.2 and 5.3), and study the impact of these strategies on the ESO cost to the firm (section 5.3).

5.1. Dynamic hedge for one ESO. The market index S and company stock price Y are described by the following SDEs:

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t^1, \\ dY_t &= (\nu - q)Y_t dt + \eta Y_t (\rho dW_t^1 + \rho' dW_t^2), \end{aligned}$$

with constant parameters $\mu, \sigma, \nu, q, \eta > 0$, correlation coefficient $\rho \in (-1, 1)$, and $\rho' = \sqrt{1 - \rho^2}$. The two independent Brownian motions, W^1 and W^2 , are defined on the given probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, and $(\mathcal{F}_t)_{0 \leq t \leq T}$ is the augmented filtration generated by these two processes. Since the processes are continuous, Assumption 2.12 of quasi left-continuity is satisfied.

Now suppose, at time $t \leq T$, that the employee holds an ESO with maturity T and a vesting period $t_v \leq T$. We will assume no vesting in this section, and address the effect of vesting in section 5.1.1. At any exercise time τ , the employee receives the payoff $A(Y_\tau) := (Y_\tau \wedge N - K)^+$, where N is a very large constant such that this payoff equals that of a call option except for unrealistically high stock prices. Throughout the period $[t, T]$, the employee can hedge to trade dynamically in the market index and the bank account that pays interest at constant rate $r \geq 0$. A trading strategy $\{\theta_u; t \leq u \leq T\}$ is the cash amount invested in the market index S , and it is deemed admissible, denoted by $\theta \in \Theta_{t,T}$, if it is \mathcal{F}_u -progressively measurable and satisfies the integrability condition $\mathbb{E}\{\int_t^T \theta_u^2 du\} < \infty$. The employee's trading wealth evolves according to

$$dX_t = [\theta_t(\mu - r) + rX] dt + \theta_t \sigma dW_t^1.$$

As in section 2.2, the employee faces the Merton investment problem after exercise. At any time $t \leq T$ with wealth $\$x$, the employee's maximal expected utility is given by

$$M(t, x) = \sup_{\Theta_{t,T}} \mathbb{E} \{ -e^{-\gamma X_T} \mid X_t = x \} = -e^{-\gamma x e^{r(T-t)}} e^{-\frac{(\mu-r)^2}{2\sigma^2}(T-t)}.$$

The employee's value function at time $t \in [0, T]$, given that his wealth $X_t = x$ and the company stock price $Y_t = y$, is

$$(5.1) \quad V(t, x, y) = \sup_{\tau \in T_{t,T}} \sup_{\Theta_{t,\tau}} \mathbb{E} \{ M(\tau, X_\tau + A(Y_\tau)) \mid X_t = x, Y_t = y \}.$$

To facilitate the presentation, we use the following shorthand for conditional expectations,

$$\mathbb{E}_{t,x,y} \{ \cdot \} = \mathbb{E} \{ \cdot \mid X_t = x, Y_t = y \}, \quad \mathbb{E}_{t,y} \{ \cdot \} = \mathbb{E} \{ \cdot \mid Y_t = y \},$$

and introduce the differential operators

$$\begin{aligned} \mathcal{L}^E u &= \frac{\eta^2 y^2}{2} \frac{\partial^2 u}{\partial y^2} + \left(\nu - q - \rho \frac{\mu - r}{\sigma} \eta \right) y \frac{\partial u}{\partial y}, \\ \mathcal{A}^q u &= \frac{\partial u}{\partial t} + \mathcal{L}^E u - ru - \frac{1}{2} \gamma (1 - \rho^2) \eta^2 y^2 e^{r(T-t)} \left(\frac{\partial u}{\partial y} \right)^2. \end{aligned}$$

The operator \mathcal{L}^E is the infinitesimal generator of Y under the minimal entropy martingale measure, Q^E , and the second operator \mathcal{A}^{ql} is *quasi-linear*.

Due to the exponential utility function, the value function has a separation of variables (see [31]):

$$(5.2) \quad V(t, x, y) = M(t, x) \cdot H(t, y)^{\frac{1}{(1-\rho^2)}}.$$

The function H solves a linear free boundary problem

$$(5.3) \quad H_t + \mathcal{L}^E H \geq 0, \quad H \leq \kappa,$$

$$(5.4) \quad (H_t + \mathcal{L}^E H) \cdot (\kappa - H) = 0,$$

for $(t, y) \in [0, T) \times (0, +\infty)$, where $\kappa(t, y) = e^{-\gamma(1-\rho^2)A(y)e^{r(T-t)}}$. The terminal condition is

$$(5.5) \quad H(T, y) = e^{-\gamma(1-\rho^2)A(y)}.$$

The employee's indifference price for holding the ESO, denoted by $p(t, y)$, is defined via $V(t, x, y) = M(t, x + p(t, y))$. It satisfies the quasi-linear variational inequality

$$(5.6) \quad \mathcal{A}^{ql} p \leq 0, \quad p \geq A(y),$$

$$(5.7) \quad \mathcal{A}^{ql} p \cdot (A(y) - p) = 0,$$

for $(t, y) \in [0, T) \times (0, +\infty)$, with $p(T, y) = A(y)$. From Proposition 2.13, the employee's optimal exercise time is given by

$$(5.8) \quad \tau_t^* = \inf \{ t \leq u \leq T : p(u, Y_u) = A(Y_u) \}.$$

In practice, we numerically solve the free boundary problem for H (in (5.3)–(5.5)) to obtain the employee's exercise boundary, which is the critical stock price at time t . It is given by

$$y^*(t) = \inf \{ y \geq 0 : H(t, y) = \kappa(t, y) \} \quad \text{for } t \in [0, T].$$

A numerical example of this optimal exercise boundary is shown in Figure 5.2 below. Details of the numerical scheme, verification, and existence results can be found in our previous analysis of the dynamic hedging case [26].

Remark 5.1. The variational inequality for p in (5.6)–(5.7) is connected with a reflected BSDE (backward stochastic differential equation), in which the driver has quadratic growth. On that front, Kobylanski et al. [25] study the link between a quadratic reflected BSDE with a bounded obstacle and the corresponding variational inequality, and provide an example of pricing an American option with exponential utility. For a study on pricing European claims with exponential utility using BSDE, we refer the reader to [32].

5.1.1. Effects of vesting. When a vesting period of t_v years is imposed, the employee cannot exercise the ESO during $[0, t_v)$, but the postvesting exercising strategy will be unaffected. The employee's value function at time $t \in [0, T]$ is given by

$$(5.9) \quad F(t, x, y) = \sup_{\tau \in \mathcal{T}_{t \vee t_v, T}} \sup_{\Theta_{t, \tau}} \mathbb{E}_{t, x, y} \{ M(\tau, X_\tau + A(Y_\tau)) \}.$$

Observe that $F(t, x, y) \leq V(t, x, y)$ due to exercise restriction before t_v , but we have $F(t, x, y) = V(t, x, y)$ for $t \geq t_v$. The optimal exercise time associated with $F(t, x, y)$, denoted by τ_t^F , is simply $\tau_t^* \vee t_v = \tau_{t_v}^*$, and the employee's postvesting exercising strategy is unaffected by the vesting provision. Therefore, it is sufficient to solve the employee's hedging problem for the no-vesting case and raise the prevesting part of the exercise boundary to infinity.

5.1.2. ESO cost to the firm. In general, the firm is able to hedge or transfer the ESO liability, so we assume that the firm is risk-neutral, which is in compliance with financial regulations.¹ By no-arbitrage arguments, the firm stock price follows the following SDE under the risk-neutral measure \mathbb{Q} :

$$(5.10) \quad dY_u = (r - q) Y_u du + \eta Y_u dW_u^{\mathbb{Q}},$$

where $W^{\mathbb{Q}}$ is a \mathbb{Q} -Brownian motion.

The firm's granting cost is given by the no-arbitrage price of a barrier-type call option written on stock Y , with strike K and maturity T . The barrier for this option is the employee's optimal exercise boundary y^* , and the option pays as soon as the firm's stock reaches the boundary y^* . Due to vesting, the employee does not exercise before t_v , nor in the region $\mathcal{C} = \{(t, y) : t_v \leq t < T, 0 \leq y < y^*(t)\}$. The cost of a vested ESO at time $t \geq t_v$, given that the stock price Y_t is y and the ESO is still alive, is given by

$$(5.11) \quad C(t, y) = \mathbb{E}_{t,y}^{\mathbb{Q}} \left\{ e^{-r(\tau_t^* - t)} A(Y_{\tau_t^*}) \right\}.$$

Then, the cost of an unvested ESO at time $t \leq t_v$, given that the stock price $Y_t = y$, is given by

$$(5.12) \quad \tilde{C}(t, y) = \mathbb{E}_{t,y}^{\mathbb{Q}} \left\{ e^{-r(t_v - t)} C(t_v, Y_{t_v}) \right\}.$$

Associated with $C(t, y)$ and $\tilde{C}(t, y)$ are two PDE problems, which we numerically solve using an implicit finite-difference method. Next, we incorporate static-dynamic hedges into our valuation methodology.

5.2. Static-dynamic hedging with put options. In addition to dynamic hedging, the employee can reduce risk exposure by taking static positions in other derivative securities. For examples, they can use synthetic instruments such as a collar contract (which involves simultaneous purchase of a put and sale of a call), as discussed in [2], or basket options written on correlated underlying assets [33]. In this section, we incorporate static hedging with two simple derivatives—European and American put options, which are easily available in the market to employees for most publicly traded companies. Our goal is to examine the impact of incorporating static hedges with these options on the employee's ESO exercise policy and the corresponding ESO cost.

5.2.1. Hedging with European puts. For simplicity, we assume that the employee purchases $\beta \geq 0$ units of European puts with the same maturity T and strike K' , even though a wide array of European puts with various strikes and expiration dates could be available. We take the market price of each European put, denoted by

¹In paragraph A13 of *Statement of Financial Accounting Standards No. 123 (revised)*, the FASB approves the use of risk-neutral models.

π , as the Black–Scholes price, since we model the company stock price as following a geometric Brownian motion.

Following our formulation in section 4.1, we first write down the indifference price for holding β European puts. By Theorem 3 in [29], the indifference price can be written as

$$h(t, y; \beta) = -\frac{1}{\gamma(1-\rho^2)e^{r(T-t)}} \log \mathbb{E}^{Q^E} \left\{ e^{-\gamma(1-\rho^2)\beta(K'-Y_T)^+} \mid Y_t = y \right\},$$

where \mathbb{E}^{Q^E} indicates that the expectation is taken under the minimal entropy martingale measure. The indifference price can be found from solving the quasi-linear PDE

$$\mathcal{A}^q h = 0,$$

for $(t, y) \in [0, T) \times (0, +\infty)$, with $h(T, y) = \beta(K' - y)^+$. Then, by (4.1), the value function for holding an ESO and β European puts is given by

$$\tilde{V}(t, x, y; \beta) = \sup_{\tau \in \mathcal{T}_{t,T}} \sup_{\Theta_{t,\tau}} \mathbb{E}_{t,x,y} \{ M(\tau, X_\tau + A(Y_\tau) + h(\tau, Y_\tau; \beta)) \}.$$

Incorporating the cost of European puts, the employee's value function becomes $\tilde{V}(t, x - \beta\pi, y; \beta)$. We apply the same transformation (5.2) to \tilde{V} . That is, we let

$$\tilde{V}(t, x - \beta\pi, y; \beta) = M(t, x - \beta\pi) \cdot \tilde{H}(t, y; \beta)^{\frac{1}{1-\rho^2}}.$$

Then, \tilde{H} solves the linear variational inequality

$$(5.13) \quad \begin{aligned} \tilde{H}_t + \mathcal{L}^E \tilde{H} &\geq 0, & \tilde{H} &\leq \tilde{\kappa}, \\ (\tilde{H}_t + \mathcal{L}^E \tilde{H}) \cdot (\tilde{\kappa} - \tilde{H}) &= 0, \end{aligned}$$

for $(t, y) \in [0, T) \times (0, +\infty)$, where $\tilde{\kappa}(t, y; \beta) = e^{-\gamma(1-\rho^2)(A(y)+h(t,y;\beta))e^{r(T-t)}}$. The terminal condition is

$$(5.14) \quad \tilde{H}(T, y; \beta) = e^{-\gamma(1-\rho^2)(A(y)+\beta(K'-y)^+)}.$$

In practice, we apply standard finite-difference methods to numerically solve (5.13)–(5.14) for the optimal exercise boundary, and we compute the indifference price for holding an ESO along with β units of European puts using the formula

$$(5.15) \quad \tilde{p}(t, y; \beta) = -\frac{1}{\gamma(1-\rho^2)e^{r(T-t)}} \log \tilde{H}(t, y; \beta).$$

For ESO hedging, the employee considers only long positions in the European puts ($\beta \geq 0$). According to (4.4), the optimal quantity of European puts to purchase, β^* , is found from the Fenchel–Legendre transform of the indifference price $\tilde{p}(t, y; \beta)$ as a function of β , evaluated at the market price π . That is,

$$\beta^* = \arg \max_{0 \leq \beta < \infty} \tilde{p}(t, y; \beta) - \beta\pi.$$

We illustrate how to determine β^* through a numerical example in Figure 5.1. Having determined β^* , we use the corresponding exercise boundary to compute the cost of the ESO to the firm, following the steps in section 5.1.

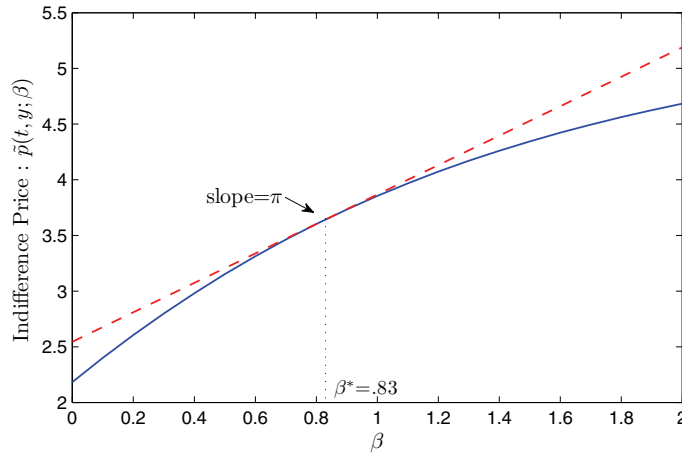


FIG. 5.1. The optimal static hedge β^* is the point at which the indifference price (solid curve) has slope equal to the market price π . The parameters are $K = K' = 10$, $T = 10$, $r = 5\%$, $q = 0\%$, $\nu = 8\%$, $\eta = 30\%$, $(\mu - r)/\sigma = 20\%$, $\rho = 30\%$, $\gamma = 0.3$. The Black-Scholes put price is $\pi = 1.322$.

From Proposition 4.1, the employee's optimal exercise time, for any fixed $\beta \geq 0$, is given by

$$(5.16) \quad \tilde{\tau}_t^*(\beta) = \inf \{ t \leq u \leq T : \tilde{p}(u, Y_u; \beta) = A(Y_u) + h(u, Y_u; \beta) \}.$$

The combination of risk aversion and static hedge has a profound impact on the employee's optimal exercising strategy. In the presence of hedging restrictions, it is well known that a risk-averse employee may find it optimal to exercise an American option early even if the underlying stock pays no dividend (see, for example, [8]). We see a similar effect in our model, but we also identify opposite effects of risk aversion and static hedges with puts on the employee's optimal exercise policy. In the next proposition, we compare the optimal exercise times τ^* and $\tilde{\tau}^*(\beta)$ and show that long positions in European puts will delay the employee's ESO exercise. In essence, the put options offer protection from the stock's downward movement, which effectively makes the employee less conservative in exercising an ESO. This effect can be seen in the numerical example in Figure 5.2, where the employee's optimal exercise boundary shifts upward when European puts are used.

PROPOSITION 5.2. *For every $\beta \geq 0$, we have $\tilde{p}(t, y; \beta) \geq p(t, y) + h(t, y; \beta)$, for $t \in [0, T]$, $y \in \mathbb{R}_+$. From this, it follows that $\tilde{\tau}_t^*(\beta) \geq \tau_t^*$.*

Proof. Fix a $\beta \geq 0$. We first observe that $\tilde{p}(T, y; \beta) = p(T, y) + h(T, y; \beta)$, for $y \geq 0$. From (5.15), we can derive the variational inequality for $\tilde{p}(t, y; \beta)$ and express it in the following form:

$$\mathcal{M}(\tilde{p}(t, y; \beta)) := \min\{-\mathcal{A}^{ql}\tilde{p}(t, y; \beta), \tilde{p}(t, y; \beta) - h(t, y; \beta) - A(y)\} = 0.$$

We want to show that $\mathcal{M}(p(t, y) + h(t, y; \beta)) \leq 0$ since this will imply $\tilde{p}(t, y; \beta) \geq p(t, y) + h(t, y; \beta)$ by the comparison principle (see [31]). To this end, we consider

$$-\mathcal{A}^{ql}(p(t, y) + h(t, y; \beta)) = -\mathcal{A}^{ql}p - \mathcal{A}^{ql}h + \frac{1}{2}\gamma(1 - \rho^2)\eta^2 y^2 e^{r(T-t)} p_y h_y.$$

Notice that the second term is zero, and the last term is nonpositive because p is nondecreasing with y (it is the indifference price of an American call option), but h is

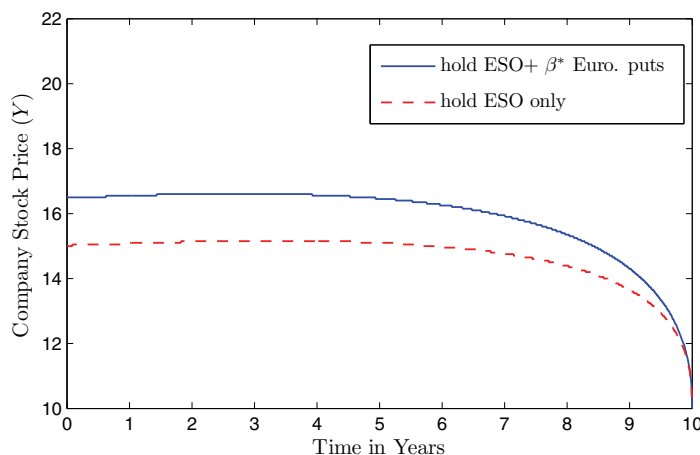


FIG. 5.2. The employee who hedges the ESO dynamically will exercise the option as soon as the firm's stock hits the lower dashed boundary. If static-dynamic hedges with European puts are used, the employee will exercise the ESO later at the upper solid boundary. The parameters are the same as those in Figure 5.1.

nonincreasing with y (it is the indifference price of β European put options). Hence, we have

$$\begin{aligned} \mathcal{M}(p(t, y) + h(t, y; \beta)) &= \min \left\{ -\mathcal{A}^{ql}p + \frac{1}{2}\gamma(1 - \rho^2)\eta^2 y^2 e^{r(T-t)} p_y h_y, p(t, y) - A(y) \right\} \\ &\leq \min \{ -\mathcal{A}^{ql}p, p(t, y) - A(y) \} = 0, \end{aligned}$$

because $p(t, y)$ satisfies (5.6). The inequality for the exercise times follows from (5.8) and (5.16). \square

This result illustrates a major difference between utility-based optimal exercising policies and risk-neutral (or no-arbitrage) exercise times. In the standard no-arbitrage framework, the pricing rule is linear in the quantity of securities, and the optimal exercising strategy for a derivative is unaffected by whether the holder also holds other derivatives, whereas utility-based stopping rules are strongly affected by the holder's hedging strategy. Because the dynamic-static hedge with European puts induces the employee to delay the exercise and capture more time-value of the option, the ESO cost in general will be higher than in the case with only dynamic hedge (see section 5.3).

5.2.2. Hedging with American puts. As an alternative to European puts, the employee can use American puts as a partial static hedge. American puts provide more flexibility in the timing of exercise, and lead to more complicated and interesting exercising strategies. The actual choice of hedging derivatives depends on their availability and current market prices. The employee purchases α units of identical American puts written on the firm's stock with maturity T and strike K' at the cost of $\$ \pi'$ each. We assume that all the puts will be exercised simultaneously.

The employee needs to decide which option (ESO or American puts) to exercise first. As in section 4.2, we first consider the expected utilities from holding only the ESO and only the American puts. For the ESO, we have the value function $V(t, x, y)$

given by (5.1). We define the value function for holding α American puts as

$$v_D(t, x, y; \alpha) := \sup_{\tau \in \mathcal{T}_{t,T}} \sup_{\Theta_{t,\tau}} \mathbb{E}_{t,x,y} \{ M(\tau, X_\tau + \alpha(K' - Y_\tau)^+) \}.$$

Also, let $\{p^D(t, y; \alpha), \tau_t^D(\alpha), y^D\}$ be the corresponding indifference price, optimal exercise time, and optimal exercise boundary. As shown in Proposition 4.2, the investor's value function is given by $\hat{V}(t, x - \alpha\pi', y; \alpha)$, where

$$\hat{V}(t, x, y; \alpha) = \sup_{\tau \in \mathcal{T}_{t,T}} \sup_{\Theta_{t,\tau}} \mathbb{E}_{t,x,y} \{ M(\tau, X_\tau + R(\tau, Y_\tau; \alpha)) \},$$

with $R(t, y; \alpha) := \max \{ A(y) + p^D(t, y; \alpha), \alpha(K' - y)^+ + p(t, y) \}$. The value function admits a separation of variables:

$$\hat{V}(t, x - \alpha\pi', y; \alpha) = M(t, x - \alpha\pi') \cdot \hat{H}(t, y; \alpha)^{\frac{1}{(1-\rho^2)}}.$$

The resulting free boundary problem for \hat{H} is

$$(5.17) \quad \begin{aligned} \hat{H}_t + \mathcal{L}^E \hat{H} &\geq 0, & \hat{H} &\leq g, \\ (\hat{H}_t + \mathcal{L}^E \hat{H}) \cdot (g - \hat{H}) &= 0, \end{aligned}$$

for $(t, y) \in [0, T] \times (0, +\infty)$, with terminal condition

$$(5.18) \quad \hat{H}(T, y) = e^{-\gamma(1-\rho^2)(A(y)+\alpha(K'-y)^+)}.$$

The obstacle term is given by

$$g(t, y; \alpha) = \min \left\{ e^{-\gamma(1-\rho^2)A(y)e^{r(T-t)}} H^D(t, y; \alpha), e^{-\gamma(1-\rho^2)\alpha(K'-y)^+ e^{r(T-t)}} H(t, y) \right\},$$

where the function $H^D(t, y; \alpha)$ satisfies the same free boundary problem as that for $H(t, y)$ (in (5.3)–(5.5)) but with $A(y)$ replaced by $\alpha(K' - y)^+$.

The employee's indifference price for holding an ESO and buying α American puts at cost π' each is $\hat{p}(t, y; \alpha) - \alpha\pi'$, where

$$(5.19) \quad \hat{p}(t, y; \alpha) = -\frac{1}{\gamma(1-\rho^2)e^{r(T-t)}} \log \hat{H}(t, y; \alpha).$$

For each fixed $\alpha \geq 0$, we numerically solve the variational inequality (5.17)–(5.18) for the optimal exercise boundaries and compute the indifference price using (5.19). Then, we determine the optimal static hedge, α^* , by maximizing $\hat{p}(t, y; \alpha) - \alpha\pi'$ over different values of $\alpha \in [0, \infty)$.

From Proposition 4.2, for any fixed $\alpha \geq 0$, the optimal time to exercise the first option(s) (ESO or puts) is given by

$$(5.20) \quad \begin{aligned} \hat{\tau}_t^*(\alpha) &= \inf \{ t \leq u \leq T : \hat{p}(u, Y_u; \alpha) = R(u, Y_u; \alpha) \} \\ &= \min \{ \tau_t^{AD}(\alpha), \tau_t^{DA}(\alpha) \}, \end{aligned}$$

where

$$(5.21) \quad \tau_t^{AD}(\alpha) := \inf \{ t \leq u \leq T : \hat{p}(u, Y_u; \alpha) = A(Y_u) + p^D(u, Y_u; \alpha) \},$$

$$(5.22) \quad \tau_t^{DA}(\alpha) := \inf \{ t \leq u \leq T : \hat{p}(u, Y_u; \alpha) = \alpha(K' - Y_u)^+ + p(u, Y_u) \}.$$

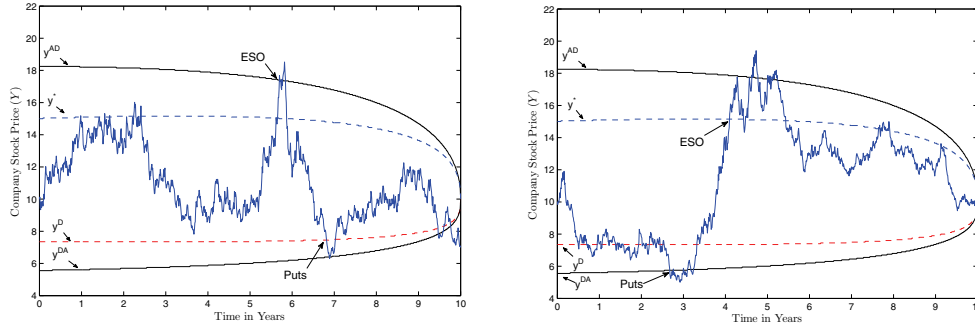


FIG. 5.3. The order of exercises is determined by whether the stock price hits the upper or lower solid boundary first. (Left) If the stock price touches the upper solid boundary (y^{AD}) first, then the employee exercises the ESO here, and then the puts later when the price crosses the lower dashed boundary (y^D). (Right) If the stock price touches the lower solid boundary (y^{DA}) first, then the employee exercises all the puts here, and then the ESO later when the price crosses the upper dashed boundary (y^*). The parameters are the same as those in Figure 5.1, with $\alpha^* = 1.32$.

The exercise times τ_t^{AD} and τ_t^{DA} represent, respectively, the times to exercise the ESO first and the American puts first. They are characterized as the first time that the firm's stock hits the respective exercise boundaries, $y^{AD}, y^{DA} : [0, T] \mapsto \mathbb{R}_+$. As illustrated in Figure 5.3 (left), if the stock reaches the boundary y^{AD} first, which means $\hat{\tau}_t^* = \tau_t^{AD} \leq \tau_t^{DA}$, then the employee will exercise the ESO first at the boundary y^{AD} . After this exercise, the employee will exercise the remaining American puts when the stock reaches the exercise boundary y^D . In the other scenario, illustrated in Figure 5.3 (right), the stock hits the boundary y^{DA} before hitting y^{AD} ; that is, $\hat{\tau}_t^* = \tau_t^{DA} \leq \tau_t^{AD}$.

The positions in the ESO and American puts exhibit interactive effects on the risk-averse employee's optimal exercise times. In the next proposition, we will show that $\tau_t^{AD} \geq \tau_t^*$, which implies that static-dynamic hedges with American puts delay the employee's ESO exercise compared to the case with a dynamic hedge only. This effect can also be seen in Figure 5.3 where the boundary y^{AD} dominates y^* ; the employee's optimal exercise boundary for the ESO is lifted upward when American puts are used. Also, the long ESO position induces the employee to delay his American put exercise; that is, $\tau_t^{DA} \geq \tau_t^D$.

PROPOSITION 5.3. *For every $\alpha \geq 0$, we have $\hat{p}(t, y; \alpha) \geq p(t, y) + p^D(t, y; \alpha)$, for $t \in [0, T], y \in \mathbb{R}_+$. Consequently, it follows that $\tau_t^{AD}(\alpha) \geq \tau_t^*$, $\tau_t^{DA}(\alpha) \geq \tau_t^D(\alpha)$, and $\hat{\tau}_t^*(\alpha) \geq \min\{\tau_t^*, \tau_t^D(\alpha)\}$.*

Proof. Fix any $\alpha \geq 0$. From (5.19), $\hat{p}(t, y; \alpha)$ solves the following variational inequality:

$$\hat{\mathcal{M}}\hat{p}(t, y; \alpha) := \min \{ -\mathcal{A}^q \hat{p}(t, y; \alpha), \hat{p}(t, y; \alpha) - R(t, y; \alpha) \} = 0,$$

with $\hat{p}(T, y; \alpha) = A(y) + \alpha(K' - y)^+$.

We want to show

$$(5.23) \quad \hat{\mathcal{M}}(p(t, y) + p^D(t, y; \alpha)) \leq 0,$$

which will imply $\hat{p}(t, y; \alpha) \geq p(t, y) + p^D(t, y; \alpha)$ by the comparison principle. First,

we observe that $\hat{p}(T, y; \alpha) \geq p(T, y) + p^D(T, y; \alpha)$. Next, we consider

$$(5.24) \quad -\mathcal{A}^{ql}(p + p^D) = -\mathcal{A}^{ql}p - \mathcal{A}^{ql}p^D + \frac{1}{2}\gamma(1 - \rho^2)\eta^2 y^2 e^{r(T-t)} p_y p_y^D,$$

where $-\mathcal{A}^{ql}p, -\mathcal{A}^{ql}p^D \geq 0$ from (5.6). If $-\mathcal{A}^{ql}p > 0$, then $p(t, y) = A(y)$ must hold by (5.7). In this case, we have

$$p(t, y) + p^D(t, y; \alpha) = A(y) + p^D(t, y; \alpha) \leq R(t, y; \alpha),$$

which implies (5.23). The same is true if $-\mathcal{A}^{ql}p^D > 0$. The only remaining possibility is that $\mathcal{A}^{ql}p = \mathcal{A}^{ql}p^D = 0$. However, since the last term in (5.24) is nonpositive because p is nondecreasing with y while p^D is nonincreasing with y , this also yields (5.23). Hence, we have shown the first part of the proposition. Then, we deduce from (5.21) and (5.22) that $\tau_t^{AD}(\alpha) \geq \tau_t^*$ and $\tau_t^{DA}(\alpha) \geq \tau_t^D(\alpha)$, and the last claim follows immediately. \square

5.2.3. The ESO cost. The ESO cost can be computed using the exercise boundaries corresponding to optimal static hedge with α^* American put options. For notational simplicity, we will suppress the dependence of various quantities on α . For example, we write $\hat{\tau}_t^*$ for $\hat{\tau}_t^*(\alpha^*)$. Under the risk-neutral measure \mathbb{Q} , the company stock evolves according to (5.10). At time $t \in [0, T]$, given that the stock price Y_t is y and the ESO is still alive, the cost of the ESO is given by

$$c(t, y) = \mathbb{E}_{t,y}^{\mathbb{Q}} \left\{ e^{-r(\tau_t^{AD}-t)} A(Y_{\tau_t^{AD}}) 1_{\{\tau_t^{AD} \leq \tau_t^{DA}\}} + e^{-r(\tilde{\tau}_t-t)} A(Y_{\tilde{\tau}_t}) 1_{\{\tau_t^{AD} > \tau_t^{DA}\}} \right\},$$

where $\tilde{\tau} = \tau_{\tau_t^{DA}}^*$. By repeated conditioning at τ_t^{DA} , the cost becomes

$$c(t, y) = \mathbb{E}_{t,y}^{\mathbb{Q}} \left\{ e^{-r(\tau_t^{AD}-t)} A(Y_{\tau_t^{AD}}) 1_{\{\tau_t^{AD} \leq \tau_t^{DA}\}} + e^{-r(\tau_t^{DA}-t)} C(\tau_t^{DA}, Y_{\tau_t^{DA}}) 1_{\{\tau_t^{AD} > \tau_t^{DA}\}} \right\},$$

with $C(t, y)$ defined in (5.11). The PDE problem for $c(t, y)$ is

$$(5.25) \quad c_t + \frac{\eta^2}{2} y^2 c_{yy} + (r - q) y c_y - r c = 0,$$

for $(t, y) \in \mathcal{C}$, the continuation region defined in section 5.1.2, and the boundary conditions

$$(5.26) \quad \begin{aligned} c(t, y^{AD}(t)) &= A(y^{AD}(t)), & 0 \leq t < T, \\ c(t, y^{DA}(t)) &= C(t, y^{DA}(t)), & 0 \leq t < T, \\ c(T, y) &= A(y), & y^{DA}(T) \leq y \leq y^{AD}(T). \end{aligned}$$

ESOs usually have an initial *vesting period* during which they cannot be exercised. In the cases with dynamic hedging and static-dynamic hedging with European puts, there are no option exercises during the vesting period. In contrast, when hedging with American puts, the employee may exercise the puts anytime. This leads us to consider the prevesting exercising strategy for the American puts. Suppose a vesting period of t_v years. If the employee exercises the puts prior to time t_v , then he will receive the option payoff and continue to hold the ESO until at least t_v . Incorporating vesting is important, straightforward, but tedious, and we omit the details of the implementation.

TABLE 5.1
See text for discussion.

	Dynamic hedge	Static-dynamic hedge with European puts	Static-dynamic hedge with American puts	Black–Scholes (perfect hedge)
$t_v = 0$	3.0330	3.4572 ($\beta^* = 0.83$)	3.6931 ($\alpha^* = 1.32$)	5.2567
$t_v = 2$	3.4376	3.7136 ($\beta^* = 0.86$)	3.8396 ($\alpha^* = 1.25$)	5.2567
$t_v = 4$	3.9932	4.1516 ($\beta^* = 0.87$)	4.1941 ($\alpha^* = 1.07$)	5.2567

5.3. The impact of static-dynamic hedges. We present a numerical example to illustrate the impacts of various hedging strategies on the ESO cost to the firm. Table 5.1 compares the ESO costs for the cases with and without vesting under the parameter values given in Figures 5.2 and 5.3. The entries in the last column are the Black–Scholes prices of a ten-year at-the-money American option on the firm’s stock. They are the same under different vesting provisions because the optimal time to exercise is at expiry. The entries in the second column are the costs when the employee hedges only dynamically as in section 5.1. The next two columns are from the cases of static-dynamic hedges with European puts and American puts, respectively.

The Black–Scholes model gives the highest cost across different vesting periods because it assumes that the ESO holder can perfectly hedge. When the employee can hedge only dynamically with the market index, the ESO cost is only 58% of the Black–Scholes value in the case with no vesting period. From left to right, we notice that the ESO cost increases as the employee adopts more effective hedging strategies. For instance, in the first row, static hedges with European puts and American puts increase the costs by 14% and 22%, respectively, compared with the cost from a dynamic hedge, but they remain significantly lower than the Black–Scholes price. The situation is similar when vesting is imposed, but vesting drives the ESO cost closer to the Black–Scholes value by preventing the employee from exercising early. In the limit of $t_v = T$, the ESO becomes a European option and can be exercised only at expiry. The cost under various hedging strategies will coincide with the Black–Scholes value.

Appendix.

A.1. Proof of Proposition 2.14. The proposition is basically an application of Propositions 3.1 and 3.2 of [23]. We adapt their proofs here in our notation and settings. First, we need Assumption 2.12 on the quasi left-continuity of $(A_t)_{0 \leq t \leq T}$ and $(l_{0,t}^{Q,\gamma})_{0 \leq t \leq T}$. Also, we need to check that

$$(A.1) \quad \sup_{0 \leq t \leq T} |l_{0,t}^{Q,\gamma}| \in L^1(Q) \quad \text{for } Q \in \mathbb{P}_f.$$

To this end, we write

$$\begin{aligned} |l_{0,t}^{Q,\gamma}| &= |H_t^T(Q^E|P) + \log Z_t^{Q,P} - H(Q^E|P)| \\ &\leq |E_t^{Q^E}\{\log Z_T^E\}| + |\log Z_t^E| + |\log Z_t^{Q,P}| + H(Q^E|P). \end{aligned}$$

The last term is a finite constant. By Lemma 3.3 of [7] and Lemma 4.2 of [20], we have

$$\sup_{0 \leq t \leq T} |E_t^{Q^E}\{\log Z_T^E\}|, \quad \sup_{0 \leq t \leq T} |\log Z_t^E|, \quad \sup_{0 \leq t \leq T} |\log Z_t^{Q,P}| \in L^1(Q) \quad \text{for } Q \in \mathbb{P}_f(P).$$

Hence, condition (A.1) is satisfied. Next, we define the Snell envelope

$$(A.2) \quad \hat{p}_t^Q := \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_t^Q \left\{ A_\tau + l_{t,\tau}^{Q,\gamma} \right\} \quad \text{for } Q \in \mathbb{P}_f(P).$$

Note that $p_t = \operatorname{ess\,inf}_{Q \in \mathbb{P}_f(P)} \hat{p}_t^Q$. Under Assumption 2.12 and integrability condition (A.1), the optimal stopping time is given by

$$\hat{\tau}_t^Q = \inf\{t \leq u \leq T : \hat{p}_u^Q = A_u\}.$$

It is well known that the process $(\hat{p}_t^Q + l_{0,t}^{Q,\gamma})_{0 \leq t \leq T}$ is a Q -supermartingale and has the following property:

$$(A.3) \quad \mathbb{E}_t^Q \left\{ \hat{p}_\tau^Q + l_{t,\tau}^{Q,\gamma} \right\} = \hat{p}_t^Q = \mathbb{E}_t^Q \left\{ A_{\hat{\tau}_t^Q} + l_{t,\hat{\tau}_t^Q}^{Q,\gamma} \right\}, \quad \tau \in [t, \hat{\tau}_t^Q].$$

To show (2.33), we observe that $\bar{\tau}_t \leq \hat{\tau}_t^Q$ since $\bar{p}_t \leq \hat{p}_t^Q$. Therefore, $\tau \in [t, \bar{\tau}_t]$ implies $\tau \in [t, \hat{\tau}_t^Q]$, which justifies the use of (A.3) later in this proof. As in Appendix A of [23], one can show that the collection $\{\hat{p}_t^Q : Q \in \mathbb{P}_f(P)\}$ is closed under pairwise optimization, so there exists a sequence $(Q^k)_{k \in \mathbb{N}}$ such that $\bar{p}_t = \lim_{k \rightarrow \infty} \downarrow \hat{p}_t^{Q^k}$. For all k , the restriction of process $Z^{Q^k,P}$ over $[0, \tau]$ does not affect \hat{p}_τ^Q , so this portion of $Z^{Q^k,P}$ can be chosen to equal $Z^{Q,P}$. Therefore, for $\tau \in [t, \bar{\tau}_t]$, we have by the monotone convergence theorem and a measure change

$$\begin{aligned} \mathbb{E}_t^Q \left\{ \bar{p}_\tau + l_{t,\tau}^{Q,\gamma} \right\} &= \lim_{k \rightarrow \infty} \downarrow \mathbb{E}_t^Q \left\{ \hat{p}_\tau^{Q^k} + l_{t,\tau}^{Q,\gamma} \right\} \\ &= \lim_{k \rightarrow \infty} \downarrow \mathbb{E}_t \left\{ \frac{Z_\tau^{Q^k,P}}{Z_t^{Q^k,P}} \left(\hat{p}_\tau^{Q^k} + l_{t,\tau}^{Q^k,\gamma} \right) \right\} = \lim_{k \rightarrow \infty} \downarrow \mathbb{E}_t^{Q^k} \left\{ \hat{p}_\tau^{Q^k} + l_{t,\tau}^{Q^k,\gamma} \right\} \\ (A.4) \quad &\geq \operatorname{ess\,inf}_{Q \in \mathbb{P}_f(P)} \mathbb{E}_t^Q \left\{ \hat{p}_\tau^Q + l_{t,\tau}^{Q,\gamma} \right\} = \operatorname{ess\,inf}_{Q \in \mathbb{P}_f(P)} \hat{p}_t^Q = \bar{p}_t. \end{aligned}$$

Using (A.4) and that $\bar{p}_{\bar{\tau}_t} = A_{\bar{\tau}_t}$ at $\bar{\tau}_t$, we have $\bar{p}_t \leq \operatorname{ess\,inf}_{Q \in \mathbb{P}_f(P)} \mathbb{E}_t^Q \{ A_{\bar{\tau}_t} + l_{t,\bar{\tau}_t}^{Q,\gamma} \}$. The reverse inequality follows easily from the definition of \bar{p}_t . Hence, (2.34) follows, and $\bar{\tau}_t$ is optimal for \bar{p}_t .

A.2. Proof of Proposition 2.17. We first consider the simple equality

$$\begin{aligned} \lim_{\gamma \rightarrow \infty} p_t(\alpha, \gamma) &= \operatorname{ess\,inf}_{\gamma > 0} \operatorname{ess\,inf}_{Q \in \mathbb{P}_f \cap \mathbb{P}_e} \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_t^Q \left\{ \alpha A_\tau + l_{t,\tau}^{Q,\gamma} \right\} \\ (A.5) \quad &= \operatorname{ess\,inf}_{Q \in \mathbb{P}_f \cap \mathbb{P}_e} \operatorname{ess\,inf}_{\gamma > 0} \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_t^Q \left\{ \alpha A_\tau + l_{t,\tau}^{Q,\gamma} \right\}. \end{aligned}$$

Following the same line of thought as in the proof of Proposition 2.14, we can switch the infimum and supremum in the last equality to obtain

$$(A.6) \quad \operatorname{ess\,inf}_{\gamma > 0} \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_t^Q \left\{ \alpha A_\tau + l_{t,\tau}^{Q,\gamma} \right\} = \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \operatorname{ess\,inf}_{\gamma > 0} \mathbb{E}_t^Q \left\{ \alpha A_\tau + l_{t,\tau}^{Q,\gamma} \right\} = \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_t^Q \{ \alpha A_\tau \}.$$

Substituting (A.6) into (A.5), we get

$$\lim_{\gamma \rightarrow \infty} p_t(\alpha, \gamma) = \operatorname{ess\,inf}_{Q \in \mathbb{P}_f \cap \mathbb{P}_e} \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_t^Q \{ \alpha A_\tau \} = \alpha c_t.$$

The last equality is due to Lemma 3.4 of [7] stating that the set $\{\frac{dQ}{dP}|Q \in \mathbb{P}_f(P) \cap \mathbb{P}_e(P)\}$ is $L^1(P)$ -dense in $\{\frac{dQ}{dP}|Q \in \mathbb{P}_e(P)\}$. See also Proposition 5.1 and Corollary 5.1 of [7] for a similar proof for European claims.

REFERENCES

- [1] D. BECHERER, *Rational Hedging and Valuation with Utility-Based Preferences*, Doctoral dissertation, School of Mathematics and Natural Sciences, Technical University of Berlin, Berlin, 2001.
- [2] J. C. BETTIS, J. M. BIZJAK, AND M. L. LEMMON, *Managerial ownership, incentive contracting, and the use of zero-cost collars and equity swaps by corporate insiders*, J. Financial Quant. Anal., 36 (2001), pp. 345–370.
- [3] J. CARPENTER, R. STANTON, AND N. WALLACE, *Optimal Exercise of Executive Stock Options and Implications for Firm Cost*, working paper, Department of Finance, New York University, New York, 2008.
- [4] P. CARR, K. ELLIS, AND V. GUPTA, *Static hedging of exotic options*, J. Finance, 53 (1998), pp. 1165–1190.
- [5] P. CHERIDITO AND M. KUPPER, *Recursivity of indifference prices and translation-invariant preferences*, preprint, Department of Operations Research and Financial Engineering, Princeton University, Princeton, NJ, 2008.
- [6] M. DAVIS, *Option pricing in incomplete markets*, in Mathematics of Derivatives Securities, M. Dempster and S. Pliska, eds., Cambridge University Press, Cambridge, UK, 1997, pp. 227–254.
- [7] F. DELBAEN, P. GRANDITS, T. RHEINLÄNDER, D. SAMPERI, M. SCHWEIZER, AND C. STRICKER, *Exponential hedging and entropic penalties*, Math. Finance, 12 (2002), pp. 99–123.
- [8] J. DETEMPLE AND S. SUNDARESAN, *Nontraded asset valuation with portfolio constraints: A binomial approach*, Rev. Financial Stud., 12 (1999), pp. 835–872.
- [9] N. EL KAROUI, *Les aspects probabilistes du contrôle stochastique*, in Lecture Notes in Math. 876, Springer-Verlag, Berlin, 1981, pp. 73–238.
- [10] H. FÖLLMER AND A. SCHIED, *Stochastic Finance: An Introduction in Discrete Time*, 2nd ed., de Gruyter Stud. Math., Walter de Gruyter, Berlin, 2004.
- [11] M. FRITELLI, *The minimal entropy martingale measure and the valuation problem in incomplete markets*, Math. Finance, 10 (2000), pp. 39–52.
- [12] M. GRASSELLI AND V. HENDERSON, *Risk aversion and block exercise of executive stock options*, J. Econom. Dynam. Control, 33 (2009), pp. 109–127.
- [13] M. R. GRASSELLI, *Nonlinearity, Correlation and the Valuation of Employee Stock Options*, preprint, McMaster University, Hamilton, ON, 2005.
- [14] P. GUASONI, *Optimal investment with transaction costs and without semimartingales*, Ann. Appl. Probab., 12 (2002), pp. 1227–1246.
- [15] V. HENDERSON, *The impact of the market portfolio on the valuation, incentives and optimality of executive stock options*, Quant. Finance, 5 (2005), pp. 1–13.
- [16] V. HENDERSON AND D. HOBSON, *Horizon-unbiased utility functions*, Stochastic Process. Appl., 117 (2007), pp. 1621–1641.
- [17] S. HUDDART, *Employee stock options*, J. Accounting Econom., 18 (1994), pp. 207–231.
- [18] A. İLHAN, M. JONSSON, AND R. SIRCAR, *Optimal investment with derivative securities*, Finance Stoch., 9 (2005), pp. 585–595.
- [19] A. İLHAN AND R. SIRCAR, *Optimal static-dynamic hedges for barrier options*, Math. Finance, 16 (2005), pp. 359–385.
- [20] Y. KABANOV AND C. STRICKER, *On the optimal portfolio for the exponential utility maximization: Remarks to the six-author paper*, Math. Finance, 12 (2002), pp. 125–134.
- [21] I. KARATZAS AND S. KOU, *Hedging American contingent claims with constrained portfolios*, Finance Stoch., 2 (1998), pp. 215–258.
- [22] I. KARATZAS AND H. WANG, *Utility maximization with discretionary stopping*, SIAM J. Control Optim., 39 (2000), pp. 306–329.
- [23] I. KARATZAS AND I. ZAMFIRESCU, *Game approach to the optimal stopping problem*, Stochastics, 77 (2005), pp. 401–435.
- [24] S. KLÖPPEL AND M. SCHWEIZER, *Dynamic indifference valuation via convex risk measures*, Math. Finance, 17 (2006), pp. 599–627.
- [25] M. KOBYLANSKI, J. P. LEPELTIER, M. C. QUENEZ, AND S. TORRES, *Reflected BSDE with superlinear quadratic coefficient*, Probab. Math. Statist., 22 (2002), pp. 51–83.

- [26] T. LEUNG AND R. SIRCAR, *Accounting for risk aversion, vesting, job termination risk and multiple exercises in valuation of employee stock options*, Math. Finance, 19 (2009), pp. 99–128.
- [27] M. MANIA AND M. SCHWEIZER, *Dynamic exponential utility indifference valuation*, Ann. Appl. Probab., 15 (2005), pp. 2113–2143.
- [28] R. MERTON, *Lifetime portfolio selection under uncertainty: The continuous time model*, Rev. Econom. Stud., 51 (1969), pp. 247–257.
- [29] M. MUSIELA AND T. ZARIPHOPULOU, *An example of indifference pricing under exponential preferences*, Finance Stoch., 8 (2004), pp. 229–239.
- [30] M. MUSIELA AND T. ZARIPHOPULOU, *Investment and valuation under backward and forward dynamic exponential utilities in a stochastic factor model*, in Advances in Mathematical Finance, R. Elliott, M. Fu, R. Jarrow, and J.-Y. Yen, eds., Birkhäuser Boston, Cambridge, MA, 2007, pp. 303–334.
- [31] A. OBERMAN AND T. ZARIPHOPULOU, *Pricing early exercise contracts in incomplete markets*, Comput. Management Sci., 1 (2003), pp. 75–107.
- [32] R. ROUGE AND N. EL KAROUI, *Pricing via utility maximization and entropy*, Math. Finance, 10 (2000), pp. 259–276.
- [33] D. SCHIZER, *Executive Stock Options and Derivatives: Tax as Corporate Governance Ally*, working paper, Columbia Law School, Columbia University, New York, 1999.
- [34] A. SHIRYAEV, *Optimal Stopping Rules*, Springer-Verlag, Berlin, 1978.
- [35] M. THOMPSON, *Continuous parameter optimal stopping problems*, Probab. Theory Related Fields, 19 (1971), pp. 302–318.