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MEAN FIELD GAME THEORY WITH A PARTIALLY OBSERVED MAJOR AGENT*

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Abstract. Mean field game (MFG) theory where there is a major agent and many minor agents (MM-MFG) has been formulated for both the linear quadratic Gaussian (LQG) case and for the case of nonlinear state dynamics and nonlinear cost functions. In this framework, even asymptotically (as the population size N approaches infinity), and in contrast to the situation without major agents, the mean field term becomes stochastic due to the stochastic evolution of the state of the major agent; furthermore, the best response control actions of the minor agents depend on the state of the major agent as well as the stochastic mean field. In a decentralized environment, one is led to consider the situation where the agents are provided only with partial information on the major agent's state; in this work such a scenario is considered for systems with nonlinear dynamics and cost functions, and an ϵ-Nash MFG theory is developed for this MM-MFG setup. The approach to the problem of partially observed MM-MFG systems adopted in this work is to follow the procedure of constructing the associated completely observed system via the application of nonlinear filtering theory; consequently, as a first step, nonlinear filtering equations are obtained for partially observed stochastic dynamical systems whose state equations contain a measure term corresponding to the distribution of the solution of a state process. Stochastic control theory for systems with random parameters is next generalized to the partially observed case by lifting the analysis to the infinite-dimensional domain. To achieve this, the Itô–Kunita lemma is first generalized to processes taking values in a subset of L^1 consisting of the space of solutions of the conditional density process generated by the filtering equations. The existence and uniqueness of solutions to the MFG system of equations is next established by a fixed point argument in the Wasserstein space of random probability measures in which the robustness property of nonlinear filtering theory is used. Finally, the ϵ-Nash property of such a solution is analyzed in this setting where the state consists of finite- and infinite-dimensional (density) valued stochastic processes.

Key words. mean field games, partially observed stochastic control, stochastic Hamilton–Jacobi–Bellman equation, Nash equilibria, nonlinear filtering, backward stochastic differential equations, infinite-dimensional stochastic differential equations

AMS subject classifications. 35Q83, 35R60, 60G35, 60H15, 91A10, 91A23, 93A14, 93E20

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1. Introduction.

1.1. Fundamentals. The design and analysis of stochastic dynamical games becomes highly complex, both analytically and computationally, as the number of agents increases. Among such systems, large population noncooperative dynamic games with mean field couplings have been introduced and explicit solutions obtained. More specifically, for such a class of games, Nash Certainty Equivalence was developed in a series of papers (see [23], [21], [24], and [22], among others) by Huang, Caines, and Malhamé. Independently, a closely related approach for such problems was developed by Lasry and Lions in [33], [32], and [34], among others, where the term Mean Field Game (MFG) was introduced. In summary, MFG theory considers games with a

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large number of stochastic dynamical agents such that each agent interacts with a mass effect (i.e., the average) of other agents via couplings in their individual cost functions and individual dynamics where each agent has a negligible influence on the overall system asymptotically in population size, but the mass affect on any agent is significant (see [7] for an overview of the subject and [11], [10] for a detailed analysis within a simplified framework). A key result of the theory is that if the agents in a finite population system apply the infinite population equilibrium strategies, then the resulting strategies yield an approximate Nash equilibrium. Such a theory is referred to as ϵ-Nash mean field game theory.

A recent series of works consider MFG theory where there is a major agent in addition to the minor agents (which is denoted by the abbreviation MM-MFG); we refer the reader to [19] for the linear quadratic Gaussian (LQG) case and [35] for the case of nonlinear state dynamics and nonlinear cost functions. In this framework, the major agent has a significant influence, i.e., asymptotically nonvanishing, on any minor agents, and a fundamental feature for this setup has been established (see [19] and [35]) when there is only one major agent. In contrast to the situation without major agents, the mean field term becomes stochastic due to the stochastic evolution of the state of the major agent. As a result, the MM-MFG stochastic systems consist of a set of equations for the major agent and the generic minor agent which consists of a stochastic Hamilton–Jacobi–Bellman (SHJB) equation, this yields a best response control law, and stochastic McKean–Vlasov (SMV) type state dynamics in which the measure term is stochastic. This class of stochastic differential equations (SDE) has the property that the drift and the diffusion coefficients of the state equation depend on the law of the state random variable; a rigorous treatment of the existence and uniqueness results for these equations is available in the literature; see, e.g., [41]. In addition to the above works, MM-MFG setup with major-minor agents has also been considered in [4] and [14], where in the former the authors generalized the MM-MFG setup to the case where the mean field is determined by the control policy of the major agent (hence, using the term dominating agent), and in the latter a probabilistic approach is taken for the MM-MFG problem in which the state of the major agent exists in both the state dynamics and the cost functions. Finally, we refer the reader to [12] and [13] for the study of the master equation and common noise in MFG, respectively.

1.2. Partially observed MFG and nonlinear filtering. An important implication of MFG theory is that when there is no major agent, the agents do not need to know the other agents' states since their effect is completely represented in the deterministic and precomputable mean field. However, when there is a major agent, MM-MFG theory demonstrates that the best response control actions generating ϵ-Nash equilibria depend on both the state of the major agent and the stochastic measure flow, where the latter corresponds to the mean field, which depends in turn on the Brownian noise of the major agent; consequently, in order to realize the best response control, this information is required by each of the minor agents. In a decentralized environment one is led to consider the situation where the agents are only provided with partial information on the major agent's state and the mean field term. Therefore, the control of partially observed MM-MFG systems entails in principle the estimation of the major agent's state and the stochastic measure of the generic minor agent. This scenario was first considered in [8] and [9] for LQG MM-MFG systems, where the situation for LQG MFG systems (i.e., without major agent) is considered in [20], while in the current work we consider partially observed MM-MFG systems with nonlinear dynamics and cost functions. Initial investigations in this direction were reported in [37] and [38].

We recall that the nonlinear dynamics of the major agent in the MM-MFG system in the infinite population limit belong to a class of SMV (without consistency requirement) SDEs, and hence, in order to analyze partially observed MM-MFG system, it is first required to develop estimation theory for such systems. Consequently, we first obtain the nonlinear filtering equations for a process whose dynamics are of such SMV type. We remark that nonlinear filtering theory for MV systems is discussed in [39], where the filtering equations in the normalized and unnormalized forms for state and measure estimations are obtained. Notice though that for the partially observed stochastic optimal control problems (SOCPs), the filtering equations in the form of conditional densities play an important role, and hence we develop the filtering equations for conditional densities. Among others, we refer the reader to [2] and [3] for fundamental results on partially observed SOCPs.

1.3. Main results and strategy of analysis. Following the derivation of the nonlinear filtering equations, we next consider the MM-MFG problem with partial observations on the major agent state provided to the minor agents individually. By adopting the procedure of constructing the associated completely observed system via the application of nonlinear filtering theory, we analyzed the existence and uniqueness of Nash equilibria in this setting as well as the ϵ-Nash equilibrium property. It should, however, be noted that since in the associated fully observed model the state process is L^1-valued, the analysis of the associated fully observed SOCP consists of developing a theory for SOCPs with random coefficients and with infinite-dimensional state.

Considering the diverse contributions in MFG theory, one could follow different approaches to analyzing such a partially observed model, as in the case of completely observed models of both MM-MFG and MFG; see [33], [34], [4], [14], [21], and [35], where, among these, we follow the approach where the limiting MFG system is represented by a couple of SMV-type equations, a couple of Hamilton–Jacobi–Bellman (HJB) equations with stochastic coefficients (SHJB), and their corresponding best response processes. Notice, however, that in our analysis, we need to work on infinite-dimensional objects as the state process, such as the conditional density for the nonlinear filter, and necessary extensions must be developed for such systems.

The main results, based on the above discussion, are achieved via the following contributions:

(i) The derivation of an SHJB equation for the completely observed SOCP which consists of an L^1-valued state process and has random coefficients in the both state dynamics and cost functions.

(ii) The extension of the Itô–Kunita formula [29, Theorem 8.1] to an infinite-dimensional Banach space setting in order to derive such an SHJB equation.

(iii) The existence and uniqueness of a backward stochastic differential equation (BSDE) which takes values in a function space defined on a space of functions and which the value function satisfies.

(iv) The existence and uniqueness analysis of the consistent solution to a controlled SMV SDE where the control functions depend on the conditional density.

(v) The robustness analysis of the nonlinear filter associated to the conditional density.

(vi) The fixed point analysis in the space of random probability measures and the existence and uniqueness of solutions of the stochastic MFG system which consists of the following:

(a) An SOCP with random coefficient processes and with finite-dimensional state component, and another SOCP with random coefficient processes but with infinite-dimensional state component. These SOCPs yield forward adapted stochastic best response control processes determined from the solution of (backward-in-time) SHJB equations in their associated domains.

(b) Two stochastic coefficient McKean–Vlasov (SMV) equations which characterize the state of the major agent and the measure determining the mean field behavior of the minor agents.

(vii) Finally, ϵ-Nash equilibria property of the best response control processes of the partially observed MM-MFG model obtained in (vi).

1.4. Applications. Finally, it is worth mentioning that partially observed MM-MFG theory has many potential applications in economic and social models, including power markets with large consumers and large utilities together with many domestic consumers and generators using smart meters. In addition, there are potential applications in the area of financial markets such as optimal execution problems where a major agent liquidates a large portion of shares, and a large number of minor agents (high-frequency traders) detect and trade along with the liquidator [25].

1.5. Organization of the paper and notation. The organization of the paper is as follows. In section 2 we define the MM-MFG problem with nonlinear dynamics and nonlinear cost functions. In section 3 we provide an SMV approximation to the partially observed MFG problem considered. We develop nonlinear filtering equations for MFG systems in the conditional density form in section 4. In sections 5 and 6 we discuss partially observed MM-MFG systems and present the main results. We conclude the paper with section 7.

Throughout the paper we use the following notation. For a matrix A, A^T and $\mathbf{tr}(A)$ denote the transpose and the trace, respectively. D_x and D^2_{xx} denote the gradient and Hessian operators with respect to the variable x. $\langle \cdot, \cdot \rangle$ denotes the inner product in Euclidean spaces. $T \in (0, \infty)$ is the terminal time. Let $\mathbb{S}$ be a metric space. Then $\mathcal{B}(\mathbb{S})$ denotes its Borel σ-algebra, $\mathcal{P}(\mathbb{S})$ denotes the space of probability measures on $\mathbb{S}$, and $C_b(\mathbb{S})$ denotes the space of continuous and bounded functions on $\mathbb{S}$. $\mathbb{E}$ denotes expectation, and the conditional expectation with respect to a σ-field $\mathcal{F}$ is denoted by $\mathbb{E}_{\mathcal{F}}$. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ be a filtered probability space satisfying the usual conditions. For a given Banach space H we denote by $L^2_{\mathcal{F}}\big([0,T]; H\big)$ the space of all $\{\mathcal{F}_t\}_{t\geq 0}$-adapted H-valued processes $f(t,\omega)$ such that $\mathbb{E}\int_0^T \|f(t,\omega)\|^2_H dt < \infty$. We use the notation $\mathbb{E}_\omega h(z) := \int h(z,\omega) P_\omega(d\omega)$ for any function $h(z,\omega)$ and sample point $\omega \in \Omega$. All SDEs in the paper are of Itô type.

2. ϵ-Nash game theory for nonlinear MM-MFG systems. Following the setup in [35], consider a dynamic game with a major agent $\mathcal{A}_0$ and N minor agents, $\{\mathcal{A}_i, 1 \leq i \leq N\}$, where N is very large. The dynamics of the agents are given by the following controlled SDEs on $\big(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P\big)$:

(1)
$$dz_0^N(t) = \frac{1}{N}\sum_{j=1}^N f_0\left(t, z_0^N(t), u_0^N(t), z_j^N(t)\right) dt + \frac{1}{N}\sum_{j=1}^N \sigma_0\left(t, z_0^N(t), z_j^N(t)\right) dw_0(t),$$

(2)
$$dz_i^N(t) = \frac{1}{N}\sum_{j=1}^{N} f\left(t, z_i^N(t), u_i^N(t), z_j^N(t)\right) dt + \frac{1}{N}\sum_{j=1}^{N} \sigma\left(t, z_i^N(t), z_j^N(t)\right) dw_i(t),$$

with initial conditions $z_i^N(t) = z_i(0)$, where (i) $z_0^N(t) : [0,T] \to \mathbb{R}^n$ is the state of the major agent $\mathcal{A}_0$ and $z_i^N(t) : [0,T] \to \mathbb{R}^n$ is the state of the minor agent $\mathcal{A}_i$; (ii) $u_0^N(t) : [0,T] \to U_0$ and $u_i^N(t) : [0,T] \to U$ are the control inputs of the major and the minor agent $\mathcal{A}_i$, respectively; (iii) $f_0 : [0,T] \times \mathbb{R}^n \times U_0 \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma_0 : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$, $f : [0,T] \times \mathbb{R}^n \times U \times \mathbb{R}^n \to \mathbb{R}^n$, and $\sigma : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$; and (iv) $\{(w_0(t), w_i(t))_{t\geq 0},\ 1 \leq i \leq N\}$ are independent standard Brownian motions in $\mathbb{R}^m$. The initial states of the agents $\{z_0(0), z_i(0), 1 \leq i \leq N\}$ are independent of each other and of $\{(w_0(0), w_i(0)),\ 1 \leq i \leq N\}$. Let $\mathcal{F}_t = \sigma\{z_i(0), w_i(s),\ 0 \leq i \leq N,\ 0 \leq s \leq t\}$ and $\mathcal{F}_t^{w_0} = \sigma\{z_0(0), w_0(s), 0 \leq s \leq t\}$ be the two filtrations which are augmented by all the P-null sets in $\mathcal{F}$. For $0 \leq j \leq N$ denote $u_{-j}^N := \{u_0^N, \ldots, u_{j-1}^N, u_{j+1}^N, \ldots, u_N^N\}$. The objective of each agent is to minimize its finite time horizon nonlinear cost function given by

$$J_0^N(u_0^N, u_{-0}^N) := \mathbb{E}\int_0^T \frac{1}{N}\sum_{j=1}^{N} L_0\left(t, z_0^N(t), u_0^N(t), z_j^N(t)\right) dt, \tag{3}$$

$$J_i^N(u_i^N, u_{-i}^N) := \mathbb{E}\int_0^T \frac{1}{N}\sum_{j=1}^{N} L\left(t, z_i^N(t), u_i^N(t), z_0^N(t), z_j^N(t)\right) dt, \tag{4}$$

where $L_0 : [0,T] \times \mathbb{R}^n \times U_0 \times \mathbb{R}^n \to \mathbb{R}_+$ and $L : [0,T] \times \mathbb{R}^n \times U \times \mathbb{R}^n \to \mathbb{R}_+$ are the nonlinear cost-coupling functions.

Notice that in the setup described by (1)–(4), the major agent $\mathcal{A}_0$ has nonnegligible influence on minor agents through the cost function. We should, however, note that the results presented in the rest of the paper can be generalized to the case where the state of the major agent exists in both the dynamics and the cost functions of the minor agents. In this regard, recall that in the fixed point analysis of the completely observed MM-MFG, the major agent's state exists in both the dynamics and the cost functions (see [35, sections 5 and 6]); however, the dynamics are assumed not to depend on $z_0^N(\cdot)$ in the ϵ-Nash argument (see [35, section 7]). We now state the system assumptions.

2.1. Assumptions. Let $F_N(x) := (1/N)\sum_{i=1}^N \mathbf{1}_{\{\mathbb{E}z_i(0)\leq x\}}$ denote the empirical distribution of N minor agents where $\mathbf{1}_{\{\mathbb{E}z_i(0)<x\}} = 1$ if $\mathbb{E}z_i(0) < x$, and $\mathbf{1}_{\{\mathbb{E}z_i(0)<x\}} = 0$ otherwise. Let us now state the system assumptions.

- (A1) The initial states $\{z_j(0), 0 \leq j \leq N\}$ are $\mathcal{F}_0$-adopted, mutually independent, independent of all Brownian motions, and satisfy $\sup_{j\in\{0,1,\ldots,N\}} \mathbb{E}|z_j(0)|^2 \leq k < \infty$, where k is independent of N.
- (A2) For any $\phi \in C_b(\mathbb{R}^n)$, $\lim_{N\to\infty} \int \phi(x) dF_N(x) = \int \phi(x) dF(x)$, i.e., $\{F_N : N \geq 1\}$ converges to a distribution F weakly.
- (A3) U_0 and U are compact metric spaces.
- (A4) $f_0(t,x,u,y)$, $\sigma_0(t,x,y)$, $f(t,x,u,y)$, and $\sigma(t,x,y)$ are continuous and bounded in all their parameters and Lipschitz continuous in (x,y). In addition, their first order derivatives with respect to x are all uniformly continuous and bounded with respect to all their parameters and Lipschitz continuous in y.
- (A5) $f_0(t,x,u,y)$ and $f(t,x,u,y)$ are Lipschitz continuous in u.

(A6) $L_0(t,x,u,y)$ and $L(t,x,u,y,z)$ are continuous and bounded with respect to all their parameters and Lipschitz continuous in (x,y,z). In addition, their first order derivatives with respect to x are all uniformly continuous and bounded with respect to all their parameters and Lipschitz continuous in (y,z).

(A7) (Nondegeneracy assumption.) There exists a positive constant α such that

$$\sigma_0(t,x,y)\sigma_0^T(t,x,y) \geq \alpha I, \quad \sigma(t,x,y)\sigma^T(t,x,y) \geq \alpha I \quad \forall (t,x,y). \tag{5}$$

We now give a summary of the main results derived in [35]. Following the approach for the analysis of MFG without major agents that introduced in [21] (which is motivated by [41, section I.1]), an approximation to the MFG system given by (1)–(4) is obtained via SMV-type SDEs [35, Theorem 3.1]. Next, the overall asymptotic ($N \to \infty$) MFG problem is decomposed into two SOCPs with random coefficient processes which yield forward-adapted stochastic best response control processes determined from the solution of SHJB equations. More explicitly, we have the following SMV-type SDEs:

$$dz_0(t) = f_0[t, z_0, u_0(t,z_0), \mu_t(\omega)]dt + \sigma_0[t, z_0, \mu_t(\omega)]dw_0(t,\omega), \tag{6}$$

$$dz(t) = f[t, z, u(t,z), \mu_t(\omega)]dt + \sigma[t, z(t), \mu_t(\omega)]dw(t), \tag{7}$$

with given initial conditions $z_0(0) = z_0(0)$ and $z(0) = z(0)$, with $\big(\mu_t(\omega)\big)_{0\leq t\leq T}$ satisfying $P\left(z(t) \leq \alpha | \mathcal{F}_t^{w_0}\right) = \int_{-\infty}^{\alpha} \mu_t(\omega, dx)$ for all $\alpha \in \mathbb{R}^n$, and with $z(0)$ having the measure $\mu_0(dx) = dF(x)$, where F is defined in (A1). Notice that in order to emphasize the particular dependence on the measure term, we use the notation $[\cdot]$ to indicate that $f_0\left[t, z_0, u_0, \mu_t\right] = \int_{\mathbb{R}^n} f_0\left(t, z_0, u_0, x\right) \mu_t(dx)$ for $\mu_t \in \mathcal{P}\left(\mathbb{R}^n\right)$. The dynamics in (6) and (7) represent the closed loop state dynamics of the major and generic minor agents, respectively, in the infinite population limit. Consequently, the two nonstandard SOCPs for the major and generic minor agents can be formulated as follows. Let $\mathcal{U}_0 := \{u(\cdot) \in U_0 : u$ is adapted to $\mathcal{F}_t^{w_0}$ and $\mathbb{E}\int_0^T |u(t)|^2 dt < \infty\}$ and $\mathcal{U} := \{u(\cdot) \in U : u$ is adapted to $\mathcal{F}_t^{w_0,w}$ and $\mathbb{E}\int_0^T |u(t)|^2 dt < \infty\}$. Then the following hold.

Major agent's SOCP at infinite population:

$$\begin{aligned} dz_0(t) &= f_0[t, z_0(t), u_0(t), \mu_t(\omega)]dt + \sigma_0[t, z_0(t), \mu_t(\omega)]dw_0(t), \\ J_0(u_0) &:= \mathbb{E}\int_0^T L[t, z_0(t), u_0(t), \mu_t(\omega)]dt. \end{aligned}$$

Generic minor agent's SOCP at infinite population:

$$\begin{aligned} dz_i(t) &= f[t, z_i(t), u(t), \mu_t(\omega)]dt + \sigma[t, z_i(t), \mu_t(\omega)]dw_i(t), \\ J_i(u) &:= \mathbb{E}\int_0^T L[t, z_i(t), u(t), z_0(t), \mu_t(\omega)]dt, \\ \mu_t(\omega) &:= \mathcal{L}\big(z_i(t)|\mathcal{F}_t^{w_0}\big). \end{aligned}$$

Here $J_0(u_0)$ is to be minimized over $\mathcal{U}_0$, $J_i(u_i)$ is to be minimized over $\mathcal{U}$, $\mathcal{L}$ denotes the conditional law, and $u_0(t)$ and $u(t)$ are $\mathcal{F}_t^{w_0}$-adapted best response control processes (i.e., the unique solutions satisfying [35, (5.14)–(5.19)]. The fundamental contribution of the nonlinear MM-MFG theory can then be summarized by the fact that the set of control laws given by $u_0^N = u_0$, $u_i^N = u, i = 1, \ldots, N$, when applied to (1) and (2) generates an ϵ-Nash equilibrium for the cost functions defined in (3) and (4) [35, Theorem 7.2]. One essential point in this result is that the state of the major agent and the stochastic measure induced by the generic minor agent, $(z_0(t), \mu_t(\omega))_{0\leq t\leq T}$, are completely observed by the minor agents.

3. McKean–Vlasov approximation with partial observation. Motivated by the analysis in [21] and [35], we now proceed with the game in the infinite population limit and introduce a set of SMV-type SDEs that approximate and decouple the closed loop partially observed MM-MFG system. We first remark that in such a setup the control policies depend on the recursively generated local estimates of the major agent's state process. Hence, let for a process $\beta_i(t)$, $0 \le t \le T$, $\mathcal{F}_t^{\beta_i}$ be given filtrations, and let $\zeta_i(t) \in L^2_{\mathcal{F}^{\beta_i}}\big([0,T];G\big)$ for $1 \le i \le N$, where G is a normed vector space. Notice that $\beta_i(\cdot)$ can be replaced by the nonlinear filter that we will obtain in section 4. Let $\eta_0(t,\omega,x) : [0,T] \times \Omega \times \mathbb{R}^n \to U_0$ be an arbitrary $\mathcal{F}_t^{w_0}$-adapted stochastic process, and let $\eta(t,\omega_i,x,y) : [0,T] \times \Omega \times \mathbb{R}^n \times G \to U$ such that the following holds:

(M1) $\eta_0(t,\omega,x)$ and $\eta(t,\omega_i,x,y)$ are Lipschitz continuous in x and y, $\eta_0(t,\omega,0) \in L^2_{\mathcal{F}_t^{w_0}}([0,T];U_0)$, and $\eta(t,\omega,0,0) \in L^2_{\mathcal{F}_t^{\beta_i}}([0,T];U)$.

We assume that $\eta_0(t,x) := \eta_0(t,\omega,x)$ and $\eta(t,x,y) := \eta(t,\omega_i,x,y)$ are used by the major and minor agents, respectively, as their control laws in (1) and (2) such that $u_0 = \eta_0$ and $u_i = \eta$ for $1 \le i \le N$. We then obtain the following closed loop equations for $1 \le i \le N$, $0 \le t \le N$, $\hat{z}_0^N(0) = z_0(0)$, and $\hat{z}_i^N(0) = z_i(0)$:

$$d\hat{z}_0^N(t) = \frac{1}{N}\sum_{j=1}^N f_0\left(t, \hat{z}_0^N(t), \eta_0(t,\hat{z}_0^N), \hat{z}_j^N(t)\right)dt + \frac{1}{N}\sum_{j=1}^N \sigma_0\left(t, \hat{z}_0^N(t), \hat{z}_j^N(t)\right)dw_0(t),$$

$$d\hat{z}_i^N(t) = \frac{1}{N}\sum_{j=1}^N f\left(t, \hat{z}_i^N(t), \eta(t,\hat{z}_i^N,\zeta_i), \hat{z}_j^N(t)\right)dt + \frac{1}{N}\sum_{j=1}^N \sigma\left(t, \hat{z}_i^N(t), \hat{z}_j^N(t)\right)dw_i(t).$$

Notice that since $\zeta_i(\cdot)$ is exogenous for $\hat{z}_i^N(\cdot)$, one can show that under (A4), (A5), and (M1) the above system has a unique solution $\left(\hat{z}_0^N, \dots, \hat{z}_N^N\right)$ by generalizing [43, Theorem 6.16, p. 49]. We now introduce the McKean–Vlasov system for the major and generic minor agents where the generic minor agent's McKean–Vlasov system shall contain the estimation of the major agent's state via nonlinear filtering equations:

$$d\bar{z}_0(t) = f_0\left[t, \bar{z}_0(t), \eta_0(t,\bar{z}_0(t)), \mu_t(\omega)\right]dt + \sigma_0\left[t, \bar{z}_0(t), \mu_t(\omega)\right]dw_0(t,\omega), \tag{8}$$

$$d\bar{z}(t) = f\left[t, \bar{z}(t), \eta(t,\bar{z}(t),\zeta(t)), \mu_t(\omega)\right]dt + \sigma\left[t, \bar{z}(t), \mu_t(\omega)\right]dw(t), \tag{9}$$

$$dy(t) = h\left(t, \bar{z}_0(t)\right)dt + d\nu(t), \quad 0 \le t \le T,\ \bar{z}_0(0) = z_0(0),\ \bar{z}(0) = z(0),\ y(0) = 0, \tag{10}$$

where $\big(w_0(\cdot), w(\cdot), \nu(\cdot)\big)$ denote the independent standard Brownian motion in $\mathbb{R}^m$, $\mathbb{R}^m$, and $\mathbb{R}$, respectively, which are also independent of $(z_0(0), z(0))$, and $h : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ is a bounded function with Lipschitz parameters. Furthermore, we define the conditional measure term by $\mu_t(\omega) = \mathcal{L}\left(\bar{z}(t)|\mathcal{F}_t^{w_0}\right)$. Finally, $\zeta(t)$ is the $\mathcal{F}_t^y$-adapted filtering equation for the conditional density. For the system given by (8) and (9), the triplet $\big(\bar{z}_0(\cdot), \bar{z}(\cdot), \mu_{(\cdot)}(\omega)\big)$ is called a consistent solution if $\left(\bar{z}_0(\cdot), \bar{z}(\cdot)\right)$ are solutions to (8) and (9), respectively, and at the same time if $\mu_t = \mathcal{L}(\bar{z}(t)|\mathcal{F}_t^{w_0})$ for $0 \le t \le T$. We remark that under (A3), (A4), and (M1) it can be shown that a unique consistent solution to the above McKean–Vlasov system exists [41, Theorem 1.1] (see also section 5). Let us also introduce

$$d\bar{z}_0(t) = f_0\left[t, \bar{z}_0(t), \eta_0(t,\bar{z}_0(t)), \mu_t(\omega)\right]dt + \sigma_0\left[t, \bar{z}_0(t), \mu_t(\omega)\right]dw_0(t,\omega), \tag{11}$$

$$d\bar{z}_i(t) = f\left[t, \bar{z}_i(t), \eta(t,\bar{z}_i(t),\zeta_i(t)), \mu_t(\omega)\right]dt + \sigma\left[t, \bar{z}_i(t), \mu_t(\omega)\right]dw_i(t), \tag{12}$$

$$dy_i(t) = h\left(t, \bar{z}_0(t)\right)dt + d\nu_i(t,\omega_i),\ 0 \le t \le T,\ 1 \le i \le N, \tag{13}$$

with ($\bar{z}_i(0) = z_i(0)$) and $y_i(0) = 0$, $0 \leq i \leq N$, where $(w_i(t), \nu_i(t))_{0\leq t\leq T}$ are independent Brownian motions in $\mathbb{R}^m$ and $\mathbb{R}$, respectively, and $\mu_t = \mathcal{L}(\bar{z}(t)|\mathcal{F}_t^{w_0})$. These equations can be considered as N independent copies of (8)–(10). We now state a decoupling result such that each $\hat{z}_i^N(t)$, $1 \leq i \leq N$, has the limit $\bar{z}_i(t)$ in the infinite population limit.

Theorem 1. *Assume* (A1), (A3)–(A5)*, and* (M1) *hold. Then we have*

$$\sup_{0\leq i\leq N}\sup_{0\leq t\leq T}\mathbb{E}|\hat{z}_i^N(t) - \bar{z}_i(t)| = O\left(1/\sqrt{N}\right), \tag{14}$$

where $O(\frac{1}{\sqrt{N}})$ depends on the terminal time T.

The proof of the theorem is in Appendix A.

Before concluding this section, we make the following remark. For $0 \leq t \leq T$, let $F_t^{\bar{z}_0} := \sigma\{\bar{z}_0(s) : 0 \leq s \leq t\}$. Then note that Theorem 1 still holds when $\mu_t(\omega) = \mathcal{L}(\bar{z}(t)|\mathcal{F}_t^{\bar{z}_0})$. This follows by observing that (141) holds under such a measure. This fact will be further exploited in later sections.

An equivalent method for characterizing the SMV of the above type is to express (9) in the form of a stochastic coefficient Fokker–Planck–Kolmogorov equation:

$$\begin{aligned} dp(t,\omega,x) = -\Bigg(&\Big\langle \nabla_x, f[t,x,\eta(t,x,\zeta),\mu_t(\omega)]p(t,\omega,x)\Big\rangle \\ &+\frac{1}{2}\mathbf{tr}\Big\langle \nabla_{xx}^2, a[t,\omega,x]p(t,\omega,x)\Big\rangle\Bigg)dt,\ p(0,\omega,x) = p(0,x) \end{aligned} \tag{15}$$

in $[0,T]\times\mathbb{R}^n$, where $p(t,\omega,x)dx = \mu_t(\omega,dx)$ (a.s.), i.e., $p(t,\omega,x)$ is the conditional density of $\bar{z}$ conditioned on $\mathcal{F}_t^{\bar{z}_0}$(or $F_t^{w_0}$) and $a[t,\omega,x] := \sigma[t,x,\mu(\omega)]\sigma[t,x,\mu(\omega)]$.

4. Nonlinear filtering equations for partially observed MM-MFG. In this section we allow the minor agents to partially observe the state of the major agent in a distributed manner as shown in (17). Therefore, we first develop nonlinear filtering theory for stochastic processes whose dynamics are of SMV type. Let $\nu_i(t)$ be a standard Brownian motion in $\mathbb{R}^n$ independent of $\{(w_0(t), w_i(t))_{t\geq 0},\ 1 \leq i \leq N\}$, of the initial conditions $\{z_0(0), z_i(0)\}$, $1 \leq i \leq N$, and of the other noise processes $\big\{(\nu_{-i}^N(t)_{t\geq 0}\big\}$. The observation dynamics of the minor agent i is given by

$$dz_0(t) = f[t,z_0(t),u_0(t,z_0(t)),\mu_t(\omega)]dt + \sigma[t,z_0(t),\mu_t(\omega)]dw_0(t), \tag{16}$$

$$dy_i(t,\tilde{\omega}) = h\left(t,z_0(t)\right)dt + d\nu_i(t,\tilde{\omega}). \tag{17}$$

Notice that the state process given in (16) is the SMV-type approximation of the major agent's state dynamics at the infinite population limit. We now determine a recursive expression for $\mathbb{E}_{\mathcal{F}_t^{y_i}}\left[\ell\left(z_0\left(t\right)\right)\right]$ for $\ell \in C_b^2\left(\mathbb{R}^n\right)$, where $\mathcal{F}_t^{y_i} := \sigma\{y_i(s) : s \leq t\}$. Consider the major agent's state $z_0(t)$. For a fixed measure flow, we have

$$\begin{aligned} &f_0[t,z_0,u_0(t,z_0),\mu_t(\omega)]dt + \sigma_0[t,z_0,\mu_t(\omega)]dw_0(t) \\ &\quad= \int_{\mathbb{R}^n} f_0\left(t,z_0,u_0(t,z_0),x\right)\mu_t(dx,\omega)dt + \int_{\mathbb{R}^n}\sigma_0(t,z_0,x)\mu_t(dx,\omega)dw_0(t) \\ &:= f_0\left(t,z_0,u_0(t,z_0),\omega\right)dt + \sigma_0(t,z_0,\omega)dw_0(t), \end{aligned} \tag{18}$$

where $f_0 : [0,T]\times\mathbb{R}^n\times U_0\times\Omega \to \mathbb{R}^n$ and $\sigma_0 : [0,T]\times\mathbb{R}^n\times\Omega \to \mathbb{R}^{n\times m}$ are $\mathcal{F}_t^{w_0}$-adapted functions. Hence, the filtering problem is now given by the SDEs

$$dz_0(t) = f_0\left(t,z_0,u_0(t,z_0),\omega\right)dt + \sigma_0(t,z_0,\omega)dw_0(t), \tag{19}$$

$$dy_i\left(t,\tilde{\omega}\right)=h\left(t,z_0(t)\right)dt+d\nu_i\left(t,\tilde{\omega}\right), \tag{20}$$

where the problem reduces to the estimation of the state process whose dynamics are driven by random functions. We need to assume the following:

(A8) $h(t,x):[0,T]\times\mathbb{R}^n\to\mathbb{R}^d$ is bounded and Lipschitz continuous in x uniformly in t.

For $0\le t\le T$, let $\pi_0(t,\cdot):=P\left(z_0(t)\in\cdot|\mathcal{F}_t^{y_i}\right)$. Nonlinear filtering theory for the problem described by (19)–(20) is considered in [39], where the following stochastic partial differential equations (SPDEs) for the unnormalized and normalized conditional distributions (see [39, Theorems 3.3] and [39, Theorems 3.4], respectively) are obtained: For $\ell\in C_b^2\left(\mathbb{R}^n\right)$,

$$\begin{aligned}\hat{\mathbb{E}}_{\mathcal{F}_t^{y_i}}\left[M(t)\ell(z_0(t))\right]&=\hat{\mathbb{E}}_{\mathcal{F}_0^{y_i}}\left[\ell(z_0(0))\right]+\int_0^t\hat{\mathbb{E}}_{\mathcal{F}_s^{y_i}}\left[M(s)\mathcal{L}(s,\omega)\ell(z_0(s))\right]ds\\&\quad+\int_0^t\hat{\mathbb{E}}_{\mathcal{F}_s^{y_i}}\left[M(s)\ell(z_0(s))h^T\left(s,z_0(s)\right)\right]dy_i(s),\end{aligned} \tag{21}$$

$$\begin{aligned}\mathbb{E}_{\mathcal{F}_t^{y_i}}\left[\ell(z_0(t))\right]&=\mathbb{E}_{\mathcal{F}_0^{y_i}}\left[\ell(z_0(0,\omega))\right]+\int_0^t\mathbb{E}_{\mathcal{F}_s^{y_i}}\left[\mathcal{L}(s,\omega)\ell(z_0(s))\right]ds\\&\quad+\int_0^t\left[\mathbb{E}_{\mathcal{F}_s^{y_i}}\left[\ell(z_0)h^T\left(s,z_0(s)\right)\right]-\mathbb{E}_{\mathcal{F}_s^{y_i}}\left[\ell(z_0(s))\right]\mathbb{E}_{\mathcal{F}_s^{y_i}}\left[h^T\left(s,z_0(s)\right)\right]\right]dI_i(s)\end{aligned} \tag{22}$$

with initial conditional distribution $\pi_i(0)\in\mathcal{P}(\mathbb{R}^n)$. Here

$$M(t)^{-1}:=\exp\left(-\int_0^t h\left(s,z_0(s)\right)d\nu_i(s)-\frac{1}{2}\int_0^t|h\left(s,z_0(s)\right)|^2ds\right) \tag{23}$$

is an exponential $\mathcal{F}_t:=\sigma\{w_0(\tau),\nu_i(\tau);0\le\tau\le t\}$-martingale, and $\hat{\mathbb{E}}$ denotes expectation under the probability $\hat{P}$ which is absolutely continuous with respect to P and defined as the Radon–Nikodym derivative $\frac{d\hat{P}}{dP}|_{\mathcal{F}_T}=M(T)^{-1}$. Furthermore,

$$\mathcal{L}(t,\omega)\ell:=\frac{1}{2}\sum_{i,j=1}^n a_{0_{ij}}\partial^2_{x_ix_j}\ell+\sum_{i=1}^n f_{0_i}\partial_{x_i}\ell \tag{24}$$

with $a_0[t,\omega,x]:=\sigma_0\left(t,x,\omega\right)\sigma_0^T\left(t,x,\omega\right)$ and the innovation process defined as follows: $I_i(t)=y_i(t)-\int_0^t\mathbb{E}_{\mathcal{F}_s^{y_i}}\left[h(s,z_0(s))\right]ds$, which is an $\mathcal{F}_t^{y_i}$-Brownian motion under the measure P.

The proofs of the filtering equations in (21) and (22) are based on the extension of the nonlinear filtering theory to the systems whose dynamics are driven by random functions which can be found in [39]. Nonetheless, we provide an argument to show why the randomness in the system dynamics does not affect the analysis. Consider the random functions $f_0\left(t,z_0,u_0(t,z_0),\omega\right)$ and $\sigma_0(t,z_0,\omega)$ and note that they are both $\mathcal{F}_t^{w_0}$-adapted. Hence, there exist $\eta:[0,T]\times\mathbb{R}^n\times U\times C\left([0,T];\mathbb{R}\right)\to\mathbb{R}$ and $\varsigma:[0,T]\times\mathbb{R}^n\times C\left([0,T];\mathbb{R}\right)\to\mathbb{R}$ such that

$$\begin{aligned}f_0\left(t,z_0(t),u_0(t,z_0),\omega\right)&=\eta\big(t,z_0(t),u_0\left(t,z_0(t)\right),w_0(\cdot\wedge t,\omega)\big),\\\sigma_0(t,z_0(t),\omega)&=\varsigma\big(t,z_0(t),w_0(\cdot\wedge t,\omega)\big)\quad P\text{-a.s. }\omega\in\Omega\qquad\forall t\in[0,T]\end{aligned} \tag{25}$$

where we use the notation $a\wedge b:=\min(a,b)$. Define the $n+m$-dimensional state process $\hat{z}_0(t)=\big(z_0(t),z(t)\big)$ by

$$\begin{aligned}dz_0(t)&=\eta\big(t,z_0(t),\varphi(t,z_0(t)),z(\cdot\wedge t,\omega)\big)dt+\varsigma\big(t,z_0(t),z(\cdot\wedge t,\omega)\big)dw(t,\omega),\\dz(t,\omega)&=dw_0(t,\omega).\end{aligned} \tag{26}$$

It is now clear that the dynamics in (26) are no more general than those considered in the literature; see, e.g., [16, Part I, Chapter 3].

4.1. Optimal filtering for the conditional density for MM-MFG systems. The above results provide a recursive expression for the conditional expectation and, hence, for the conditional distribution. However, in order to convert the partially observed MM-MFG into a fully observed one, we need an expression for the conditional density. Among different paths, we follow [26, Theorem 11.2.1] to obtain the optimal filtering equations for the conditional density that each minor agent generates. Let us first introduce the following adjoint operator on $C^2(\mathbb{R}^n)$: For $x \in \mathbb{R}^n$, $t \in [0,T]$ and assuming that all the derivatives exist, set

$$\mathcal{L}^*(t,\omega)\theta(x) = \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_i} a_{0_{ij}}[t,\omega,x]\theta(x) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f_{0_i}\theta(x). \tag{27}$$

THEOREM 2. *In addition to the conditions defined in assumptions* (A1), (A3), (A4), (A5), *and* (A8), *we make the following assumptions:*

(D1) *Assume that the conditional distribution $\pi_i = P(z_0(t)|\mathcal{F}_t^{y_i})$ has a probability density φ_i with respect to Lebesgue measure, i.e., for $A \in \mathcal{B}(\mathbb{R}^n)$, $\pi_i(t,A,\tilde{\omega}) = \int_A \varphi_i(t,x,\tilde{\omega})dx$, where $\varphi_i(\cdot)$ is $(t,x,\tilde{\omega})$-measurable and $\mathcal{F}_t^{y_i}$-adapted for each $t \in [0,T]$.*

(D2) *Assume that for each t,*

$$\int_{\mathbb{R}^n} |\mathcal{L}^*\varphi_i(t,x,\tilde{\omega})|\, dx < \infty, \quad \int_0^T \int_{\mathbb{R}^n} |\mathcal{L}^*\varphi_i(t,x,\tilde{\omega})|\, dxdt < \infty \ \ a.s. \tag{28}$$

(D3) *Let $h^\delta(t,x) := h^T(t,x) - \mathbb{E}_{\mathcal{F}_t^{y_i}}[h^T(t,x)]$. Then assume that for each t,*

$$\int_{\mathbb{R}^n} \left|\varphi_i(t,x,\tilde{\omega})h^\delta(t,x)\right| dx < \infty \ \ a.s. \tag{29}$$

and

$$\mathbb{E}\int_0^T \left[\int_{\mathbb{R}^n} \left|\varphi_i(t,x,\tilde{\omega})h^\delta(t,x)\right| dx\right]^2 dt < \infty. \tag{30}$$

Then $\varphi_i(t,x,\tilde{\omega})$ satisfies the following stochastic integral equation: For every t,

$$\begin{aligned}\varphi_i(t,x,\tilde{\omega}) &= \varphi_i(0,r) + \int_0^t \mathcal{L}^*\varphi_i(s,x,\tilde{\omega})ds \\ &\quad + \int_0^t \varphi_i(s,x,\tilde{\omega})\left\{h^T(s,x) - \int_{\mathbb{R}^n} h^T(s,x')\varphi_i(s,x',\tilde{\omega})dx'\right\}dI_i(s)\end{aligned} \tag{31}$$

for a.e. x with probability 1 *and where $\varphi_i(0,x)$ is the initial conditional density and $I_i(t)$ is the innovations process, which is a Brownian motion, defined above.*

Proof. The main steps of the proof are based on interchanging the order of integrations in (22). Observe first that for the ith minor agent, the left-hand side of (22) is simply $\int_{\mathbb{R}^n} \ell(x)\varphi_i(t,x,\tilde{\omega})dx$. Furthermore, assume that density of $P(z_0(0) \in A)$ exists and denote it by $p(0,x)$. Hence, the first term on the right-hand side becomes $\int_{\mathbb{R}^n} \ell(x)p(0,x)dx$. Now by assumption, $\varphi_i(0,x)$ is the density of $P(z_0(0) \in A|\mathcal{F}_t^{y_i})$, and therefore we have the initial condition $\varphi_i(0,x) = p(0,x)$ for a.e. x, with probability 1. Let us now consider the term $\int_0^t \mathbb{E}_{\mathcal{F}_s^{y_i}}\left[\mathcal{L}(s,\omega)\ell(z_0(s))\right] ds$. Observe that

$$\begin{aligned}\int_0^t \mathbb{E}_{\mathcal{F}_s^{y_i}}\left[\mathcal{L}(s,\omega)\ell(z_0(s))\right] ds &= \int_0^t \int_{\mathbb{R}^n} \mathcal{L}(s,\omega)\ell(x)\varphi_i(s,x,\tilde{\omega})dxds \\ &= \int_0^t \int_{\mathbb{R}^n} \ell(x)\mathcal{L}^*(s,\omega)\varphi_i(s,x,\tilde{\omega})dxds \qquad (32)\\ &= \int_{\mathbb{R}^n} \ell(x)\int_0^t \mathcal{L}^*(s,\omega)\varphi_i(s,x,\tilde{\omega})dsdx \qquad \text{(a.s.)}, \qquad (33)\end{aligned}$$

where (33) follows from (D2) and Fubini's theorem. We shall now consider the last term in (22). Observe first that

$$\begin{aligned}&\int_0^t \left[\mathbb{E}_{\mathcal{F}_s^{y_i}}\left[\ell(z_0(s))h^T(s,z_0(s))\right] - \mathbb{E}_{\mathcal{F}_s^{y_i}}\left[\ell(z_0(s))\right]\mathbb{E}_{\mathcal{F}_s^{y_i}}\left[h^T(s,z_0(s))\right]\right] dI_i(s) \\ &= \int_0^t \left[\int_{\mathbb{R}^n} \ell(x)h^T(s,x)\varphi_i(s,x)dx - \int_{\mathbb{R}^n}\ell(x)\varphi_i(s,x)dx\int_{\mathbb{R}^n} h^T(s,x)\varphi_i(s,x)dx\right] dI_i(s) \\ &= \int_0^t \left(\int_{\mathbb{R}^n} \ell(x)\left[\varphi_i(s,x)\left\{h^T(s,x) - \int_{\mathbb{R}^n} h^T(s,x')\varphi_i(s,x')dx'\right\}\right]dx\right) dI_i(s). \qquad (34)\end{aligned}$$

The goal is to show that (34) is identical to

$$\int_{\mathbb{R}^n} \ell(x)\left(\int_0^t \left[\varphi_i(s,x)\left\{h^T(s,x) - \int_{\mathbb{R}^n} h^T(s,x')\varphi_i(s,x')dx'\right\}\right] dI_i(s)\right) dx \ \text{ a.s.} \qquad (35)$$

for every $t \in [0,T]$. Let $H(t,x) := \varphi_i(t,x,\tilde{\omega})\big\{h^T(t,x) - \int_{\mathbb{R}^n} h^T(t,x')\varphi_i(t,x',\tilde{\omega})dx'\big\}$, and we will show that

$$\int_0^t \left[\int_{\mathbb{R}^n} \ell(x)H(s,x)dx\right] dI_i(s) = \int_{\mathbb{R}^n} \ell(x)\left[\int_0^t H(s,x)dI_i(s)\right] dx. \qquad (36)$$

Let $L^1(t,\tilde{\omega})$ and $L^2(t,\tilde{\omega})$ denote the left- and right-hand sides of (36), respectively. It suffices to show that $\mathbb{E}(L^1(t,\tilde{\omega})M(t,\tilde{\omega})) = \mathbb{E}(L^2(t,\tilde{\omega})M(t,\tilde{\omega}))$ for every $M(t,\tilde{\omega})$ of the form $\mathbb{E}(M(t,\tilde{\omega})) + \int_0^t \Psi(s,\tilde{\omega})dI_i(s)$, where $\Psi(s,\tilde{\omega})$ is $(s,\tilde{\omega})$-measurable and $\mathcal{F}_s^{y_i}$-adapted, and $\mathbb{E}\int_0^t \Psi^2(s,\tilde{\omega})ds < \infty$. Observe also that $L^k(t,\tilde{\omega})$, $k = 1,2$, are $(t,\tilde{\omega})$-measurable and $\mathcal{F}_t^{y_i}$-adapted, and $\mathbb{E}L^k(t,\tilde{\omega}) = 0$ and $\mathbb{E}(L^k(t,\tilde{\omega}))^2 < \infty$. Furthermore,

$$\mathbb{E}(L^1 M) = \int_0^t \mathbb{E}\left[\int_{\mathbb{R}^n} \ell(x)H(s,x)dx\Psi(s,\tilde{\omega})\right] ds \ \text{ a.s.}, \qquad (37)$$

where, due to (30), we have

$$\begin{aligned}\int_0^t \mathbb{E}\int_{\mathbb{R}^n} |\ell(x)\Psi(s)H(s,x)|\, dxds &\le \int_0^t \left(\mathbb{E}\Psi(s)^2\right)^{\frac{1}{2}} \left\{\mathbb{E}\left(\int_{\mathbb{R}^n} |\ell(x)H(s,x)|\, dx\right)^2\right\}^{\frac{1}{2}} ds \\ &\le \left[\int_0^t \mathbb{E}\Psi(s)^2 ds \int_0^t \mathbb{E}\left(\int_{\mathbb{R}^n} |\ell(x)H(s,x)|\, dx\right)^2 ds\right]^{\frac{1}{2}} < \infty.\end{aligned}$$

Hence, by Fubini's theorem we obtain

$$\begin{aligned}\int_0^t \mathbb{E}\left[\int_{\mathbb{R}^n} \ell(x)H(s,x)dx\Psi(s,\tilde{\omega})\right] ds &= \int_{\mathbb{R}^n} \ell(x)\mathbb{E}\left[\int_0^t H(s,x)\Psi(s,\tilde{\omega})ds\right] dx \\ &= \mathbb{E}(L^2 M), \qquad (38)\end{aligned}$$

which proves (36) and hence (35). As a result, the filtering equation in (22) reduces to the following: For $t \in [0,T]$

$$\int_{\mathbb{R}^n} \ell(x)\left[\varphi_i(t,x,\tilde{\omega}) - p(0,x) - \int_0^t \mathcal{L}^*(s)\varphi_i(s,x,\tilde{\omega})ds - \int_0^t H(s,x)dI_i(s)\right]dx = 0 \tag{39}$$

a.s. for every $\ell \in C^2(\mathbb{R}^n)$. To complete the proof, let us further restrict the class of test functions to the $C_c^2(\mathbb{R}^n)$ class of twice continuously differentiable functions on $\mathbb{R}^n$ such that ℓ, ℓ', and ℓ'' have compact support. Let us denote the expression inside the brackets in (39) by $\Xi_t(x,\tilde{\omega})$. Notice that due to (D2) and (D3) we have that for each t, $\int_{\mathbb{R}^n} |\Xi_t(x,\tilde{\omega})|dx < \infty$ a.s. Recall now that $L^1(\mathbb{R}^n)$ is separable and, furthermore, $C_c^2(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$. Hence, (39) can be shown to imply that $P\big[\tilde{\omega} : \Xi_t(x,\tilde{\omega}) = 0 \text{ for a.e. } x\big] = 1$, which proves the theorem. ∎

5. Partially observed MM-MFG systems. In this section we formulate a partially observed MM-MFG problem in the infinite population limit. This entails defining two SOCPs for the major and the generic minor agent where the major agent has complete observation of its own state, and minor agents have only partial information on the state of the major agent. Since the minor agents have partial observations, we need to convert the system into an equivalent fully observed model via application of nonlinear filtering theory that we developed in the previous section.

5.1. SOCP of the major agent. Since the major agent is assumed to have complete observation on its own state, the analysis in [35, section 5.1] holds. Consider the following SOCP:

$$\begin{aligned} dz_0(t) &= f_0\left[t, z_0(t), u_0(t), \mu_t(\omega)\right]dt + \sigma_0\left[t, z_0(t), \mu_t(\omega)\right]dw_0(t), \ z_0(0) = z_0(0), \\ J_0(u_0) &:= \mathbb{E}\int_0^T L_0[t, z_0(t), u_0(t), \mu_t(\omega)]dt, \end{aligned} \tag{40}$$

where the goal is to minimize $J_0(u_0)$ over $\mathcal{U}_0 := \big\{u_0(\cdot) \in U_0 : u_0(t)$ is $\mathcal{F}_t^{w_0}$-adapted and $\mathbb{E}\int_0^T |u_0(t)|^2 dt < \infty\big\}$. The solution of this SOCP is given by the following stochastic MFG (SMFG) system:

[M-SHJB]

$$\begin{aligned} -d\phi_0(t,\omega,x) = \Big[&H_0\left(t,\omega,x,D_x\phi_0(t,\omega,x)\right) + \langle \sigma_0\left[t,x,\mu_t(\omega)\right], D_x\psi_0(t,\omega,x)\rangle \\ &- \psi_0^T(t,\omega,x)dw_0(t) + \frac{1}{2}\mathbf{tr}\big(a_0[t,\omega,x]D_{xx}^2\phi_0(t,\omega,x)\big)\Big]dt, \ \phi_0(T,x) = 0, \end{aligned} \tag{41}$$

[M-best response]

$$\begin{aligned} u_0^o(t,\omega,x) &\equiv u_0^o\big(t,x|(\mu_s(\omega))_{0\le s\le T}\big) \\ &:= \arg\inf_{u_0\in U_0} H_0^{u_0}\left(t,\omega,x,u_0,D_x\phi_0(t,\omega,x)\right) \\ &\equiv \arg\inf_{u_0\in U_0}\big\{f_0\left[t,x,u_0,\mu_t(\omega)\right], D_x\phi_0(t,\omega,x)\rangle + L_0[t,x,u_0,\mu_t(\omega)]\big\}, \end{aligned} \tag{42}$$

[M-SMV]

$$dz_0^o(t) = f_0\left[t, z_0^o, u_0^o(t,z_0^o), \mu_t(\omega)\right]dt + \sigma_0\left[t, z_0^o, \mu_t(\omega)\right]dw_0(t), \ \ z_0^o(0) = z_0(0), \tag{43}$$

where $\phi_0(t,\omega,x)$ is the value function of the major agent's SOCP which is $\mathcal{F}_t^{w_0}$-adapted, $x(t) \in \mathbb{R}^n$ is the initial condition for $z_0(\cdot)$, $a_0[t,\omega,x] := \sigma_0[t,x,\mu_t(\omega)]$

$\sigma_0\left[t,x,\mu_t(\omega)\right]^T$, and the stochastic Hamiltonian is given by

$$H_0(t,\omega,x,q) := \inf_{u_0\in U_0}\big\{\langle f_0\left[t,x,u_0,\mu_t(\omega)\right],q\rangle + L_0\left[t,x,u_0,\mu_t(\omega)\right]\big\}.$$

Observe that the solution to the backward-in-time SHJB given by (41) is given by the forward-in-time $\mathcal{F}_t^{w_0}$-adapted pair $\big(\phi_0(t,\omega,x),\psi_0(t,\omega,x)\big)$.

5.2. SOCP of the generic minor agent. Let $z_0^o(\cdot)$ be the solution of the major agent's SMV given in (43), and let $\mu_t(\omega)$, $0\le t\le T$, be an exogenous nominal minor agent stochastic measure process such that $\mu_0(dx)=dF(x)$, where F is defined in (A1). Note that in section 5.3 $\mu_t(\omega)$ will be characterized via the MFG consistency condition as the random measure of minor agents' mean field behavior approximating the empirical distribution produced by all minor agents in the infinite population limit. The state and observation dynamics for the generic minor agent are given by

$$dz_i(t) = f\left[t,z_i(t),u_i(t),\mu_t(\omega)\right]dt + \sigma\left[t,z_i(t),\mu_t(\omega)\right]dw_i(t), \tag{44}$$

$$dy_i(t) = h\left(t,z_0^o(t)\right)dt + d\nu_i(t), \tag{45}$$

where $\mu_t(\omega) = \mathcal{L}\left(z(t)|\mathcal{F}_t^{w_0}\right)$. Let $(I_i(t))_{0\le t\le T}$ denote the innovations process associated to $y_i(t)$, and set $\mathcal{F}_t^{(z_i,y_i)} := \sigma\{y_i(s),z_i(s);0\le s\le t\}$. Note that since $(\nu_i(t))_{0\le t\le T}$ is independent of $(w_0(t))_{0\le t\le T}$ and $h(\cdot)$ is bounded, the innovations conjecture holds and that $F_t^{y_i}=F_t^{I_i}$, $0\le t\le T$. For the ith generic minor agent consider the cost function

$$J_i(u_i) = \mathbb{E}\int_0^T L\big[t,z_i(t),u_i(t),z_0^o(t),\mu_t(\omega)\big]dt, \tag{46}$$

where $J(u_i)$ is to be minimized over $\mathcal{U} := \big\{u_i(\cdot)\in U : u(t)\text{ is }\mathcal{F}_t^{(z_i,y_i)}\text{-adapted and }\mathbb{E}\int_0^T|u_i(t)|^2dt<\infty\big\}$. Consider now the filtering SPDE given by (31). Following [2], define

$$G_k := \left\{p\in L^1(\mathbb{R}^n) : \|p\|_k = \int_{\mathbb{R}^n}\big(1+|x|^k\big)|p(x)|dx<\infty\right\}$$

for some $k\ge 1$, which depends on the density of the major agent's initial state $z_0(0)$, and note that on the filtered probability space $\big(\Omega,\mathcal{F},(\mathcal{F}_t^{y_i})_{t\ge 0},P\big)$, a solution, denoted by $\varphi(t)$ (we skip $\tilde\omega$ dependence), is an $\mathcal{F}_t^{y_i}$-adapted, G_k-valued process (see also [43, p. 91] for a Sobolev space approach to the solution of filtering equation). Consider now the cost function for a generic minor agent, and let $\mathbb{X} := \mathbb{R}^n\times U\times\mathbb{R}^n\times\mathcal{P}(\mathbb{R}^n)$. We have

$$\begin{aligned}
&\mathbb{E}\int_0^T L\big[s,z_i(s),u_i(s),z_0^o(s),\mu_s(\omega)\big]ds = \mathbb{E}\mathbb{E}_{\mathcal{F}_T^{(z_i,y_i)}}\int_0^T L\big[s,z_i(s),u_i(s),z_0^o(s),\mu_s(\omega)\big]ds\\
&\overset{\text{(i)}}{=} \mathbb{E}\int_0^T \mathbb{E}_{\mathcal{F}_s^{(z_i,y_i)}}L\big[s,z_i(s),u_i(s),z_0^o(s),\mu_s(\omega)\big]ds \quad\text{a.s.}\\
&= \mathbb{E}\int_0^T\int_{\mathbb{X}} L\big[s,z_i(s),u_i(s),z_0^o(s),\mu_s(\omega)\big]dP\big(z_i(s),u_i(s),z_0^o(s),\mu_s(\omega)|\mathcal{F}_s^{(z_i,y_i)}\big)ds \quad\text{a.s.}\\
&\overset{\text{(ii)}}{=} \mathbb{E}\int_0^T\int_{\mathbb{X}} L\big[s,z_i(s),u_i(s),z_0^o(s),\mu_s(\omega)\big]\\
&\qquad\qquad dP\left(z_i(s),u_i(s),\mu_s(\omega)\middle|\mathcal{F}_s^{(z_i,y_i)}\right)dP\left(z_0^o(s)\middle|\mathcal{F}_s^{(z_i,y_i)}\right)ds \quad\text{a.s.}
\end{aligned}$$

$$\overset{\text{(iii)}}{=} \mathbb{E}\int_0^T \bigg(\int_{\mathbb{X}} L\big[s, z_i(s), u_i(s), z_0^o(s), \mu_s(\omega)\big]$$
$$dP\left(z_i(s), u_i(s), \mu_s(\omega)\big|\mathcal{F}_s^{(z_i,y_i)}\right) d\left(\int \varphi_i(s,x)dx\right)\bigg)ds \quad \text{a.s.}$$

$$= \mathbb{E}\int_0^T \big(L\left[t, z_i(s), u_i(s), \cdot, \mu_s(\omega)\right], \varphi_i(s,\cdot)\big)ds \tag{47}$$

$$:= \mathbb{E}\int_0^T \mathbf{L}\left[s, z_i(s), \varphi_i(s), u_i(s), \mu_s(\omega)\right] ds, \tag{48}$$

where (i) is due to (A4) and $u_i \in \mathcal{U}$, and consequently [42, Lemma 5.4] holds; (ii) is valid since $u_i \in \mathcal{U}$; (iii) is due to Theorem 2; and, finally, in (47) the notation $(\cdot,\cdot)$ stands for $(\alpha,\beta) = \int_{\mathbb{R}^n} \alpha(x)\beta(x)dx$. Let $\mathsf{z}_i(t) := \big(z_i(t), \varphi_i(t)\big)$, $\mathsf{w}_i(t) := \big(w_i(t), I_i(t)\big)$ and

$$\mathsf{F}(\mathsf{z}_i) = \begin{bmatrix} f\big[t, z_i(t), u_i(t), \mu_t(\omega)\big] \\ \mathcal{L}^*(t,\omega)\varphi_i(t) \end{bmatrix},$$
$$\mathsf{G}(\mathsf{z}_i) = \begin{bmatrix} \sigma\big[t, z_i(t), \mu_t(\omega)\big] & \mathbf{0}_d^T \\ \mathbf{0}_m & \varphi_i(t)\big(h^T(t) - h^T * \varphi_i(t)\big) \end{bmatrix},$$

where $h^T * \varphi_i(t) := \int_{\mathbb{R}^n} h^T(t,x)\varphi_i(t,x,\tilde{\omega})dx$. Based on the cost function obtained in (48) and by recalling that $\varphi_i(t)$ is $\mathcal{F}_t^y$-adapted, we formulate the following completely observed SOCP for the generic minor agent:

$$d\mathsf{z}_i(t) = \mathsf{F}\big(\mathsf{z}_i(t)\big)dt + \mathsf{G}\big(\mathsf{z}_i(t)\big)d\mathsf{w}_i(t), \tag{49}$$

$$\inf_{u_i\in\mathcal{U}} J(u) = \inf_{u\in\mathcal{U}} \mathbb{E}\bigg[\int_0^T \mathbf{L}\big[t, z_i(t), \varphi_i(t), u_i(t), \mu_s(\omega)\big]dt\bigg], \tag{50}$$

where we explicitly indicate the dependence of the generic minor agent's random measure on the sample point $\omega \in \Omega$. The rest of the analysis for the generic minor agent's SOCP is based on obtaining the SHJB equation for which we need to extend the Itô–Kunita formula for systems with infinite-dimensional state space. Note that for a function $f(t,x) : [0,T] \times \mathsf{X} \to \mathbb{R}$ defined on Banach space X, we denote its first and second order Fréchet derivative at $x \in \mathsf{X}$ by $\mathsf{D}_x f(t,x)$ and $\mathsf{D}_{xx}^2 f(t,x)$, respectively. Furthermore, the notations $\cdot$ and $[\cdot,\cdot]$ are used for operators on Banach spaces.

Theorem 3 (Itô–Kunita lemma for density-valued processes). *Let $(\Omega, \mathcal{F}, P)$ be a complete probability space equipped with a right continuous increasing family $\mathcal{F}_t$, $t \geq 0$, of sub-σ-fields of $\mathcal{F}$. Let $\Delta(t,x,p)$, $t \geq 0$, $x \in \mathbb{R}^d$, $p \in G_k$, be a stochastic process continuous in (t,x,p) a.s. satisfying the following:*

(i) *For each $t \geq 0$, $\Delta(\cdot,x,p)$ is a C^2-map from $\mathbb{R}^d \times G_k$ to $\mathbb{R}$ a.s. where the differentiation is defined in the Fréchet sense and their Riesz representations are given by*

$$\mathsf{D}_p\Delta(s,x,p)\cdot\eta = \int_{\mathbb{R}^n} D_p\Delta(s,x,p)(z)\eta(z)dz, \tag{51}$$
$$\mathsf{D}_{pp}^2\Delta(s,x,p)\cdot[\eta,\eta'] = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} D_{pp}^2\Delta(s,x,p)(z,z')\eta(z)\eta'(z')dzdz'$$

with $\eta(\cdot)$, $\eta'(\cdot) \in G_{k-1}$, and the Kernels satisfy

$$|D_p\Delta(s,x,p)(z)| \leq \gamma_p(s,x,\|p\|_l)(1+|z|^l), \tag{52}$$

$$|D_x D_p \Delta(s,x,p)(z)| \le \gamma_{xp}(s,x,\|p\|_l)(1+|z|^l), \tag{53}$$

$$\left|D^2_{pp}\Delta(s,x,p)(z,z')\right| \le \gamma_{pp}(s,x,\|p\|_l)(1+|z|^l)(1+|z'|^l) \tag{54}$$

for some $l \le k-1$, γ_p, γ_{xp}, and γ_{pp} which are continuous on $[0,T]\times\mathbb{R}^d\times\mathbb{R}_+$.

(ii) *For each (x,p), $\Delta(t,\cdot)$ is a continuous semimartingale represented by*

$$\Delta(t,x,p) = \Delta(0,x,p) - \int_0^t Q(s,x,p)ds + \sum_{k=1}^m \int_0^t \Sigma_k(s,x,p)dW_k(s), \tag{55}$$

where $W_1(\cdot),\dots,W_m(\cdot)$ are standard Brownian motions, $\Sigma_k(s,x,p)$, $s\ge 0$, $(x,p)\in\mathbb{R}^d\times G_k$ are random fields that are continuous in (x,p) such that

(a) *for each $s\ge 0$, $\Sigma_k(s,\cdot)$ are adapted processes;*

(b) *for each (x,p), $\Sigma_k(\cdot,x,p)$ are C^2-map from $\mathbb{R}^d\times G_k$ to $\mathbb{R}$ a.s.*

Now let $\alpha(\cdot) = (\alpha_1(\cdot),\dots,\alpha_d(\cdot))$ be a continuous semimartingale of the form

$$d\alpha_i(t) = f_i(t)dt + \sum_{k=1}^m \kappa_{ik}(t)dW_k(t) + \sum_{k=1}^m \zeta_{ik}(t)dB_k(t),\ 1\le i\le d, \tag{56}$$

where f_i and $\kappa_i = (\kappa_{i1},\dots,\kappa_{im})$, $\zeta_i = (\zeta_{i1},\dots,\zeta_{im})$, $1\le i\le d$, are adapted processes, $B_1(\cdot),\dots,B_m(\cdot)$ are standard Brownian motions independent of $W_1(\cdot),\dots,W_m(\cdot)$, f_i is integrable a.s., and κ_i and ζ_i are square integrable a.s. Also let the G_k-valued process $\beta(t,z)$ satisfy

$$d\beta(t,z) = g(t,z)dt + \varsigma(t,z)dY(t), \tag{57}$$

where $Y(\cdot)$ is a Brownian motion independent of $\{B_k(\cdot),W_k(\cdot),k=1,\dots,m\}$, g and ς satisfy that $g\in\mathcal{M}_{l,1}[\mathcal{F}_t]\cap\mathcal{M}_{l,2}[\mathcal{F}_t]$ and $\varsigma\in\mathcal{M}_{l,2}[\mathcal{F}_t]\cap\mathcal{M}_{l,4}[\mathcal{F}_t]$, where for $0\le t\le T$, $\mathcal{M}_{l,j}[\mathcal{F}_t] := \big\{\xi(t,z) : \xi(t,z)$ is $\mathcal{F}_t$-adapted, and $\mathbb{E}\int_0^T\left(\int_{\mathbb{R}^n}\left(1+|z|^l\right)|\xi(t,z)|dz\right)^j dt < \infty\big\}$; see [2]. Then

$$\begin{aligned}
&d\Delta\big(t,\alpha(t),\beta(t,z)\big) \\
&= -Q\big(t,\alpha(t),\beta(t,z)\big)dt + \sum_{k=1}^m \Sigma_k(t,\alpha,\gamma)dW_k(t) + \sum_{i=1}^d \mathsf{D}_{x_i}\Delta\big(t,\alpha(t),\beta(t,z)\big)f_i(t)dt \\
&+\mathsf{D}_p\Delta\big(t,\alpha(t),\beta(t,z)\big)\cdot g(t,z)dt + \sum_{i=1}^d\sum_{k=1}^m \mathsf{D}_{x_i}\Delta\big(t,\alpha(t),\beta(t,z)\big)\kappa_{ik}(t)dW_k(t) \\
&+\sum_{i=1}^d\sum_{k=1}^m \mathsf{D}_{x_i}\Delta\big(t,\alpha(t),\beta(t,z)\big)\zeta_{ik}(t)dB_k(t) + \sum_{i=1}^d\sum_{k=1}^m \mathsf{D}_{x_i}\Sigma_k\big(t,\alpha(t),\beta(t,z)\big)\kappa_{ik}(t)dt \\
&+\mathsf{D}_p\Delta\big(t,\alpha(t),\beta(t,z)\big)\cdot\varsigma(t)dY(t) + \frac{1}{2}\sum_{i,j=1}^d\sum_{k=1}^m \mathsf{D}^2_{x_ix_j}\Delta\big(t,\alpha(t),\beta(t,z)\big)\kappa_{ik}(t)\kappa_{jk}(t)dt \\
&+\frac{1}{2}\sum_{i,j=1}^d\sum_{k=1}^m \mathsf{D}^2_{x_ix_j}\Delta\big(t,\alpha(t),\beta(t,z)\big)\zeta_{ik}(t)\zeta_{jk}(t)dt + \frac{1}{2}\mathsf{D}^2_{pp}\Delta\big(t,\alpha(t),\beta(t,z)\big)\cdot[\varsigma(t),\varsigma(t)]dt,
\end{aligned} \tag{58}$$

where $\mathsf{D}_\cdot$ denote the partial Fréchet derivative in the indicated direction, $\cdot$ notation is used for the operator action, and $[\cdot,\cdot]$ notation denotes the second-level map operation.

The proof of the above theorem is in Appendix B, and in the rest of the paper we will refer to the assumptions (i) and (ii) as *Itô–Kunita regularity conditions*.

We now derive an HJB equation for the SOCP described by (44)–(46). Such an equation can be considered HJB for SOCPs with infinite-dimensional state space and random parameters for which, to the best of our knowledge, such a theory has not been discussed in the literature. Hence, let $0 \le t \le T$, $\big(z_i(t), \varphi_i(t)\big) = \big(x(t), p(t)\big) := (x, p)$. Denote the value function by $\Phi_i\big(t, x(t), p(t)\big)$. We have

$$\Phi_i\,(t, x, p) = \inf_{u_i \in \mathcal{U}} \mathbb{E}_{\mathcal{F}_t^{I_i}} \int_t^T \big(L\,[s, z_i(s), u_i(s), \cdot, \mu_s(\omega)]\,, \varphi_i(s, \cdot)\big) ds. \tag{59}$$

Remark that for $0 \le t \le T$, the right-hand side of (59) is $F_t^{I_i}$-adapted. Consequently, by the principle of optimality and the martingale representation theorem (see [35] and [36]), the value function satisfies

$$\Phi_i\big(t, x, p\big) = \int_t^T \Gamma_i\,(s, x, p)\, ds - \int_t^T \Psi_i^T\big(s, x, p\big) dI_i(s), \;\; (x, p) \in \mathbb{R}^n \times G_k, \tag{60}$$

where $\Gamma_i\big(s, x, p\big)$ and $\Psi_i^T\big(s, x, p\big)$ are $\mathcal{F}_s^{I_i}$-adapted, $\mathbb{R}_+$-valued processes. Following [36], we shall now provide a backward stochastic partial differential equation (BSPDE) that $\big(\Phi_i\big(t, x, p\big), \Psi_i\big(t, x, p\big)\big)$ satisfy. Before presenting the SHJB equation, it is worth emphasizing the following.

Remark 5.1. The usual approach in the literature for the partially observed SOCP is to use the unnormalized conditional density filtering equations (see (21) for the unnormalized distribution equations) as the state variable in the completely observed formulation [2], [3]. Using the unnormalized form has two important benefits: the SPDE for the unnormalized conditional density is linear in the initial density, and the cost function, which needs to be modified with respect to the unnormalized measure, also becomes linear in the initial density. Linearity then plays an essential role in proving that Fréchet derivatives for the cost function defined in terms of the unnormalized density exist with respect to the infinite-dimensional component. Note that the cost function has linear fractional dependence on the initial density under the normalized density. In this work, rather than using the unnormalized measure and, consequently, the established existence of Fréchet derivatives [2, Theorem 5.1], we proceed with the normalized measure and assume the existence of Fréchet derivatives. The reason for selecting such a path is due to the fact that the state dynamics and the cost functions of the SOCP have random coefficients, which causes complications in the construction of weak solutions. We note that such a path will be employed in our future works and in the complete generalization of [36] to entirely partially observed models.

Let $\mu_t(\omega^o) := \mathbb{E}_{\mathcal{F}_t^{I_i}} \mu_t(\omega)$ and, hence, $f\,[t, x, u, \mu_t(\omega^o)] := \int f\,(t, x, u, y)\, \mu_t(\omega^o, dy)$ and $\mathbf{L}\,[t, x, p, u, \mu_t(\omega^o)] := \int \mathbf{L}\,(t, x, p, u, y)\, \mu_t(\omega^o, dy)$, where $(t, x, u, p) \in [0, T] \times \mathbb{R}^n \times U \times G_k$.

THEOREM 4. *Define the stochastic Hamiltonian* $H : [0, T] \times \Omega \times \mathbb{R}^n \times G_k \times \mathbb{R}^n \to \mathbb{R}$*:*

$$H\,(t, \omega^o, x, p, q) := \inf_{u \in U} \left[\langle f\left[t, x, u, \mu_t(\omega^o)\right], q\rangle + \big(L\left[t, x, u, \cdot, \mu_t(\omega^o)\right], p(\cdot)\big)\right]. \tag{61}$$

Let $\hat{p}(t) := p(t)\big(h^T(t) - h^T * p(t)\big)$ *and* $a[t, \omega^o, x] := \sigma[t, x, \mu_t(\omega^o)]\sigma^T[t, x, \mu_t(\omega^o)]$.

Consider the following BSPDE:

$$-d\Phi_i\left(t,\omega^o,x,p\right)=\Big[H\big(t,\omega^o,x,p,D_x\Phi_i\left(t,\omega^o,x,p\right)\big)+\mathsf{D}_p\Phi_i\left(t,\omega^o,x,p\right)\cdot\mathcal{L}^*(t)p(t)$$
$$+\frac{1}{2}\mathbf{tr}\left(a[t,\omega^o,x]D^2_{xx}\Phi\left(t,\omega^o,x,p\right)\right)+\frac{1}{2}\mathsf{D}^2_{pp}\Phi_i\left(t,\omega^o,x,p\right)\cdot\big[\hat{p}(t),\hat{p}(t)\big]\Big]dt$$
$$-\Psi_i^T\big(t,\omega^o,x,p\big)dI_i(t),\ \ \Phi_i\left(T,x,p\right)=0,\ \ (t,x,p)\in[0,T]\times\mathbb{R}^n\times G_k, \tag{62}$$

where $I_i(t)$ is the innovations process, which is a Brownian motion under P. Then we have the following:

(i) (HJB equation.) *The pair $\big(\Phi_i\big(t,x,p\big),\Psi_i\big(t,x,p\big)\big)$ as defined in* (60) *satisfies* (62)*.*

(ii) (Verification theorem.) *Conversely, assume that a solution to* (62)*, denoted $\big(\phi_i(t,x,p),\psi_i(t,x,p)\big)$, exists, together with infimizing control $u_i^o(t,x,p):[0,T]\times\mathbb{R}^n\times G_k\to U$, which moreover satisfies the Itô–Kunita regularity conditions of Theorem* 3*. Assume further that $H\big(t,\omega^o,x,p,\partial_x\phi_i(t,x,p)\big)$ is smooth with respect to (t,x,p) and $u_i^o(t,x,p)$ is regular enough such that the feedback control system for* (44) *is well-posed. Then $\big(\phi_i(t,x,p),\psi_i(t,x,p)\big)$ coincides with the value function pair $\big(\Phi_i(t,x,p),\Psi_i(t,x,p)\big)$.*

For the proof, we remark that since $(z_i(t),\varphi_i(t))\in\mathbb{R}^n\times G_k$-valued and $\Phi_i(\cdot)$ is $\mathcal{F}_t^{I_i}$-adapted, one needs to extend the results in [36] to the case where the state process has both finite-dimensional and density-valued components. The main step of such an analysis consists of extending further the extended Itô–Kunita formula [29, Theorem 8.1] for $\mathbb{R}^n\times G_k$-valued stochastic processes that we have obtained in Theorem 3. We remark that an extension of Itô's lemma and, consequently, of the HJB equations for unnormalized density-valued processes is given in [2] for cases in which the cost function is not random.

Proof. We first prove (i), which depends on the application of the extended Itô–Kunita lemma developed in Theorem 3. In the first step, we show that the pair $\big(\Phi_i(t,x,p),\Psi_i(t,x,p)\big)$ as in (60) satisfies (62). Notice first that

$$\Phi_i\big(t,x,p\big)=\inf_{u_i\in\mathcal{U}}\mathbb{E}_{\mathcal{F}_t^{I_i}}\int_t^T\left(L\left[t,z_i(s),u_i(s),\cdot,\mu_s(\omega)\right],\varphi_i(s,\cdot)\right)ds \tag{63}$$
$$=\inf_{u_i\in\mathcal{U}}\mathbb{E}_{\mathcal{F}_t^{I_i}}\int_t^{t+\delta}\left(L\left[t,z_i(s),u_i(s),\cdot,\mu_s(\omega)\right],\varphi_i(s,\cdot)\right)ds+\Phi_i\big(t+\delta,z_i(t+\delta),\varphi_i(t+\delta)\big).$$

Applying Theorem 3 to the process $\Phi_i\big(t,z_i(t),\varphi_i(t)\big)$ defined in (60) yields (taking $n=1$ for simplicity)

$$\Phi_i\big(t,z_i(t),\varphi_i(t)\big)$$
$$=\int_t^T\Gamma_i\big(s,z_i(s),\varphi_i(s)\big)ds+\int_t^T\partial_x\phi_i\big(s,z_i(s),\varphi_i(s)\big)f\left[s,z_i(s),u_i(s),\mu_s(\omega)\right]ds$$
$$+\int_t^T\mathsf{D}_p\Phi\big(s,z_i(s),\varphi_i(s)\big)\cdot\mathcal{L}^*(s)\varphi_i(s)ds+\int_t^T\partial_x\Phi_i\big(s,z_i(s),\varphi_i(s)\big)\sigma\left[t,z_i(s),\mu_s(\omega)\right]dw_i(s)$$
$$+\int_t^T\mathsf{D}_p\Phi_i\big(s,z_i(s),\varphi_i(s)\big)\cdot\left(\varphi_i(s)\Big\{h^T(s)-\int_{\mathbb{R}^n}h^T(s,x')\varphi_i(s,x',\tilde{\omega})dx'\Big\}\right)dI_i(s)$$
$$+\frac{1}{2}\int_t^T\partial^2_{xx}\Phi_i\big(s,z_i(s),\varphi_i(s)\big)\sigma\left[s,z_i(s),\mu_s(\omega)\right]\sigma\left[s,z_i(s),\mu_s(\omega)\right]^T ds$$

$$+\frac{1}{2}\int_t^T \mathsf{D}^2_{pp}\Phi_i\big(s, z_i(s), \varphi_i(s)\big)\cdot\Big[\varphi(s)\Big\{h^T(s) - \int_{\mathbb{R}^n} h^T(s,x')\varphi_i(s,x',\tilde{\omega})dx'\Big\}$$

$$\varphi_i(s)\Big\{h^T(s) - \int_{\mathbb{R}^n} h^T(s,x')\varphi_i(s,x',\tilde{\omega})dx'\Big\}\Big]\,ds. \tag{64}$$

Therefore, setting $\hat{\varphi}_i(t) = \varphi_i(t)\big(h^T(t) - h^T * \varphi_i(t)\big)$, we have that

$$\begin{aligned}
&\Phi_i\big(t+\delta, z_i(t+\delta), \varphi_i(t+\delta)\big) - \Phi_i\big(t,x,p\big)\\
&= \mathbb{E}_{\mathcal{F}_t^{I_i}}\Big[\int_t^{t+\delta}\Gamma_i\big(s, z_i(s), \varphi_i(s)\big)ds + \int_t^{t+\delta}\partial_x\Phi_i\big(s, z_i(s), \varphi_i(s)\big) f\left[s, z_i(s), u_i(s), \mu_s\right] ds\\
&+ \int_t^{t+\delta}\mathsf{D}_p\Phi_i\big(s, z_i(s), \varphi_i(s)\big)\cdot\mathcal{L}^*(s)\varphi_i(s)ds + \frac{1}{2}\int_t^{t+\delta}\partial^2_{xx}\Phi_i\big(s, z_i(s), \varphi_i(s)\big)a[s,\omega,z_i(s)]ds\\
&\qquad + \frac{1}{2}\int_t^{t+\delta}\mathsf{D}^2_{pp}\Phi_i\big(s, z_i(s), \varphi_i(s)\big)\cdot\left[\hat{\varphi}_i(s), \hat{\varphi}_i(s)\right] ds\Big],
\end{aligned} \tag{65}$$

where (65) follows since the right-hand side of (64) is $\mathcal{F}_t^{I_i}$-measurable. Combining (65) with (63), we obtain

$$\begin{aligned}
\delta^{-1}\inf_{u_i\in\mathcal{U}}\mathbb{E}_{\mathcal{F}_t^{I_i}}\int_t^{t+\delta}\Big[\;&\big(L\big[s, z_i(s), u_i(s), \cdot, \mu_s(\omega)\big], \varphi_i(s,\cdot)\big) + \Gamma_i\big(s, z_i(s), \varphi_i(s)\big)\\
&+ \partial_x\Phi_i\big(s, z_i(s), \varphi_i(s)\big) f\left[s, z_i(s), u_i(s), \mu_s(\omega)\right]\\
&+ \mathsf{D}_p\Phi_i\big(s, z_i(s), \varphi_i(s)\big)\cdot\mathcal{L}^*(s)\varphi_i(s)\\
&+ \frac{1}{2}\partial^2_{xx}\Phi_i\big(s, z_i(s), \varphi_i(s)\big)a[s,\omega,z_i(s)]ds\\
&+ \frac{1}{2}\mathsf{D}^2_{pp}\Phi_i\big(s, z_i(s), \varphi_i(s)\big)\cdot\left[\hat{\varphi}_i(s), \hat{\varphi}_i(s)\right]\Big]ds = 0.
\end{aligned} \tag{66}$$

Taking $\delta \to 0$, we have

$$\begin{aligned}
\Gamma_i(t,x,p) = -\Big[&H\big(t, \omega^o, x, p, \partial_x\Phi_i(t,x,p)\big) + \mathsf{D}_p\Phi_i\left(t,x,p\right)\cdot\mathcal{L}^*(t)p\\
&+\frac{1}{2}\mathbf{tr}\left(a[t,\omega^o,x]D^2_{xx}\Phi\left(t,x,p\right)\right) + \frac{1}{2}\mathsf{D}^2_{pp}\Phi\left(t,x,p\right)\cdot\big[\hat{p},\hat{p}\big]\Big],
\end{aligned} \tag{67}$$

where in (67) we note the smoothing operation which follows from

$$\begin{aligned}
\mathbb{E}_{\mathcal{F}_t^{I_i}}\partial_x\Phi_i\big(t,x,p\big) f\left[t,x,u(t),\mu_t(\omega)\right] &= \partial_x\Phi_i\big(t,x,p\big)\mathbb{E}_{\mathcal{F}_t^{I_i}} f\left[t,x,u_i(t),\mu_t(\omega)\right]\\
&= \partial_x\Phi_i\big(t,x,p\big)\int\mathbb{E}_{\mathcal{F}_t^{I_i}} f\left(t,x,u_i(t),y\right)\mu_t(\omega,dy)\\
&= \partial_x\Phi_i\big(t,x,p\big)\int f\left(t,x,u_i(t),y\right)\mathbb{E}_{\mathcal{F}_t^{I_i}}\mu_t(\omega,dy)\\
&= \partial_x\Phi_i\big(t,x,p\big) f\left[t,x,u_i(t),\mu_t(\omega^o)\right],
\end{aligned} \tag{68}$$

where (68) follows from $\mu_t(\omega^o) := \mathbb{E}_{\mathcal{F}_t^{I_i}}\mu_t(\omega)$. Substituting (67) into (60), we get (62).

We now prove the converse to (i), namely (ii), the verification theorem, where it is required to show that a sufficiently smooth solution to (62) with $\Gamma_i(\cdot)$ as given in

(60) coincides with the value function pair $\big(\Phi_i(t,x,p),\Psi_i(t,x,p)\big)$. We first introduce the functional form of Hamiltonian. Let $H^u : [0,T]\times\Omega\times\mathbb{R}^n\times G_k\times U\times\mathbb{R}^n\to\mathbb{R}$

$$\begin{aligned}&H^u\left(t,\omega^o,x(t),p(t),u(t),q(t)\right)\\ &\qquad := \big\langle f\big[t,x(t),u(t),\mu_t(\omega^o)\big],q(t)\big\rangle + \big(L\left[t,x(t),u(t),\cdot,\mu_t(\omega^o)\right],p(t,\cdot)\big).\end{aligned} \tag{69}$$

Let the pair $\big(\phi_i(t,x,p),\psi_i(t,x,p)\big)$ be a solution to (62) satisfying the given smoothness conditions. Let $u_i^o(t,x,p)$ be a $\big(\mathcal{B}\left([0,T]\right)\times\mathcal{B}\left(\mathbb{R}\right)\times\mathcal{B}\left(G_k\right);\mathcal{B}\left(U\right)\big)$-measurable function such that

$$\begin{aligned}&H\big(t,\omega^o,x,p,\partial_x\phi_i(t,x,p)\big)\\ &\qquad = \big\langle f\big[t,x,u_i^o,\mu_t(\omega^o)\big],\partial_x\phi_i(t,x(t),p(t))\big\rangle + \big(L\big[t,x,u_i^o,\cdot,\mu_t(\omega^o)\big],p(\cdot)\big).\end{aligned}$$

Recall also that by the assumption on $u_i^o(t,x,p)$, the feedback control system

$$\begin{aligned}dz_i(s) = f\left[s,z_i(s),u_i^o(s),\mu_s(\omega)\right]ds + \sigma\left[s,z_i(s),\mu_s(\omega)\right]dw_i(s),\\ (z_i(t),\varphi_i(t)) = (x,p),\ \ t\le s\le T,\end{aligned}$$

is well-posed. We now show that $\big(\phi_i(t,x,p),\psi_i(t,x,p)\big)$ coincides with $\big(\Phi_i(t,x,p),\Psi_i(t,x,p)\big)$ and that $u_i^o\big(s,z_i(s),\varphi_i(s)\big)$ minimizes the cost function (50).

Let $\tilde{u}_i(s)\in\mathcal{U}$ be another admissible control, and let $\tilde{z}_i(s)$ be the solution of

$$d\tilde{z}_i(s)=f\left[s,\tilde{z}_i(s),\tilde{u}_i(s),\mu_s(\omega)\right]ds+\sigma\left[t,\tilde{z}_i(s),\mu_s(\omega)\right]dw_i(s),\ \ (\tilde{z}_i(t),\varphi_i(t))=(x,p) \tag{70}$$

for $t\le s\le T$. Consider $\phi_i(t,x,p)$. From the extended It$\hat{\text{-}}$o-Kunita lemma, Theorem 3, we have

$$\begin{aligned}d\phi_i\big(t,\tilde{z}_i(t),\varphi_i(t)\big) &= \partial_t\phi_i\big(t,\tilde{z}_i(t),\varphi_i(t)\big)dt + \partial_x\phi_i\big(t,\tilde{z}_i(t),\varphi_i(t)\big)f\left[t,\tilde{z}_i(t),\tilde{u}_i(t),\mu_t(\omega)\right]\\ &+\mathsf{D}_p\phi_i\big(t,\tilde{z}_i(t),\varphi_i(t)\big)\cdot\mathcal{L}^*(t)\varphi_i(t)dt + \partial_x\phi_i\big(t,\tilde{z}_i(t),\varphi_i(t)\big)\sigma\left[t,\tilde{z}_i(t),\mu_t(\omega)\right]dw_i(t)\\ &+\frac{1}{2}\partial^2_{xx}\phi_i\big(t,\tilde{z}_i(t),\varphi_i(t)\big)\sigma\left[t,\tilde{z}_i(t),\mu_t(\omega)\right]\sigma\left[t,\tilde{z}_i(t),\mu_t(\omega)\right]^T dt\\ &+\mathsf{D}_p\phi_i\big(t,\tilde{z}_i(t),\varphi_i(t)\big)\cdot\hat{\varphi}_i(t)dI(t) + \frac{1}{2}\mathsf{D}^2_{pp}\phi_i\big(t,\tilde{z}_i(t),\varphi_i(t)\big)\cdot\left[\hat{\varphi}_i(t),\hat{\varphi}_i(t)\right]dt.\end{aligned} \tag{71}$$

Consequently, observing that

$$\begin{aligned}&\mathbb{E}_{\mathcal{F}_t^{I_i}}\int_t^T \mathsf{D}_p\phi_i\big(s,\tilde{z}_i(s),\varphi_i(s)\big)\cdot\hat{\varphi}(s)dI_i(s)\\ &\qquad = \mathbb{E}_{\mathcal{F}_t^{I_i}}\int_t^T \partial_x\phi_i\big(t,\tilde{z}_i(s),\varphi_i(s)\big)\sigma\left[s,\tilde{z}_i(s),\mu_s(\omega)\right]dw_i(s) = 0,\end{aligned} \tag{72}$$

we obtain

$$\begin{aligned}&\mathbb{E}_{\mathcal{F}_t^{I_i}}\phi_i\big(T,\tilde{z}_i(T),\varphi_i(T)\big) = \phi_i\big(t,x,p\big) + \mathbb{E}_{\mathcal{F}_t^{I_i}}\Bigg[\int_t^T\Bigg(\mathsf{D}_p\phi_i\big(s,\tilde{z}_i(s),\varphi_i(s)\big)\cdot\mathcal{L}^*(s)\varphi_i(s)\\ &+\partial_x\phi_i\big(s,\tilde{z}_i(s),\varphi_i(s)\big)f\left[s,\tilde{z}_i(s),\tilde{u}_i(s),\mu_s(\omega)\right] + \frac{1}{2}\partial^2_{xx}\phi_i\big(s,\tilde{z}_i(s),\varphi_i(s)\big)a[s,\omega,\tilde{z}_i(s)]\\ &\qquad+\frac{1}{2}\mathsf{D}^2_{pp}\phi_i\big(s,\tilde{z}_i(s),\varphi_i(s)\big)\cdot\left[\hat{\varphi}_i(s),\hat{\varphi}_i(s)\right]\\ &-\Big[H\big(s,\omega^o,\tilde{z}_i(s),\varphi_i(s),\partial_x\phi_i(s,\tilde{z}_i(s),\varphi_i(s))\big) + \mathsf{D}_p\phi_i\left(s,\tilde{z}_i(s),\varphi_i(s)\right)\cdot\mathcal{L}^*(s)\varphi_i(s)\end{aligned}$$

$$+\frac{1}{2}\left(\partial^2_{xx}\phi_i\left(s,\tilde{z}_i(s),\varphi_i(s)\right)a[s,\omega,\tilde{z}_i(s)]\right)+\frac{1}{2}\mathsf{D}^2_{pp}\phi_i\left(s,\tilde{z}_i(s),\varphi_i(s)\right)\cdot\left[\hat{\varphi}_i(s),\hat{\varphi}_i(s)\right]\Big]\Big)ds\Big]$$

$$=\phi_i\big(t,x,p\big)+\mathbb{E}_{\mathcal{F}_t^{I_i}}\left[\int_t^T\Big(H^u\big(t,\omega^o,\tilde{z}_i(s),\varphi_i(s),\tilde{u}_i(s),\partial_x\phi_i(t,\tilde{z}_i(s),\varphi_i(s))\big)\right. \tag{73}$$

$$\left.-\big(L\big[t,\tilde{z}_i(s),\cdot,\tilde{u}_i(s),\mu_s(\omega)\big],\varphi_i(s,\cdot)\big)-H\big(t,\omega^o,\tilde{z}_i(s),\varphi_i(s),\partial_x\phi_i(t,\tilde{z}_i(s),\varphi_i(s))\big)\Big)ds\right],$$

which by the infimization in the definition of H yields

$$\phi_i(t,x,p)\leq\mathbb{E}_{\mathcal{F}_t^{I_i}}\left[\int_t^T\big(L\big[s,\tilde{z}_i(s),\tilde{u}_i(s),\cdot,\mu_s(\omega)\big],\varphi_i(s,\cdot)\big)\right]=J\big(t,x,p,\tilde{u}_i(\cdot)\big), \tag{74}$$

where we explicitly indicate dependence on the initial conditions in $J(\cdot)$. On the other hand, if we take $u_i^o\big(t,z_i(s),\varphi_i(s)\big)$ as a control, applying Theorem 3 once again, then, following similar steps, we obtain

$$\mathbb{E}_{\mathcal{F}_t^{I_i}}\phi_i\big(T,z_i(T),\varphi_i(T)\big)$$

$$=\phi_i\big(t,x,p\big)+\mathbb{E}_{\mathcal{F}_t^{I_i}}\left[\int_t^T\Big(H^u\big(t,\omega^o,z_i(s),\varphi_i(s),u_i^o(s),\partial_x\phi_i(t,z_i(s),\varphi_i(s))\big)\right. \tag{75}$$

$$\left.-\big(L\big[t,z_i(s),\cdot,u_i^o(s),\mu_s(\omega)\big],\varphi_i(s,\cdot)\big)-H\big(t,\omega^o,z_i(s),\varphi_i(s),\partial_x\phi_i(t,z_i(s),\varphi_i(s))\big)\Big)ds\right].$$

It then follows from the definition of $u_i^o(t)$ that

$$\phi_i\big(t,x,p\big)=\mathbb{E}_{\mathcal{F}_t^{I_i}}\left[\int_t^T\big(L\big[s,z_i(s),u_i^o(s),\cdot,\mu_s(\omega)\big],\varphi_i(s,\cdot)\big)\right]=J\big(t,x,p,u_i^o(\cdot)\big). \tag{76}$$

Finally, (74) and (76) together with (59) imply that

$$\phi_i\big(t,x,p\big)=\inf_{u_i\in\mathcal{U}}J\big(t,x,p,u_i(\cdot)\big)=\Phi_i(t,x,p) \tag{77}$$

and, consequently, $\psi_i(t,x,p)=\Psi_i(t,x,p)$. This completes the proof for $n=1$. □

Remark 5.2. The properties of the solution to the nonlinear filtering equations have been well studied in the literature. Depending on the initial density, solutions to such SPDEs have been shown to belong in Sobolev spaces, UMD Banach spaces (i.e., spaces in which martingale differences are unconditional), and others; see, e.g., the discussion in [6]. Furthermore, for SDEs taking values in those infinite-dimensional spaces the theory of stochastic calculus has already been developed, and one can alternatively use such spaces in the analysis.

Remark 5.3. The analysis of the existence and uniqueness of solutions to the BSPDE (62) is nontrivial. Nevertheless, the existence assumptions for the solutions to such equations is standard in the partially observed SOCP literature; see the verification step [2, Theorem 6.2] and note the assumption on the existence of solutions to the Mortensen equation [2, equation (6.10)], which is a PDE with domain G_k.

Recall that by BSDE theory, a solution to the SHJB equation (62) is forward in time $\mathcal{F}_t^{I_i}$-adapted. Consequently, the stochastic best response process of the generic minor agent's completely observed SOCP defined by (49) and (50) is given by

$$\begin{aligned}u_i^o\big(t,\omega^o,x,p\big)&\equiv u_i^o\big(t,\omega^o,x,p|\mu_s(\omega^o)_{0\leq s\leq t}\big)\\&:=\arg\inf_{u\in U}H^u\left[t,\omega^o,x,p,u,D_x\phi_i(t,\omega^o,x,p)\right]\\&\equiv\arg\inf_{u\in U}\big\{\big\langle f\big[t,x,u,\mu_t(\omega^o)\big],D_x\phi_i(t,\omega^o,x,p)\big\rangle+\mathbf{L}\big[t,x,p,u,\mu_t(\omega^o)\big]\big\},\end{aligned} \tag{78}$$

where the infimum exists a.s. since the functions in H^u are continuous and U is compact. Notice that u_i^o is a forward-in-time $\mathcal{F}_t^{I_i}$-adapted process. By substituting u_i^o into the minor agent's state dynamics, we get the following closed loop SMV equation:

$$dz_i^o(t,\omega,\omega',\omega^o) = f\left[t, z_i^o, u_i^o(t,\omega^o, z^o,\varphi), \mu_t(\omega)\right]dt + \sigma\left[t, z_i^o, \mu_t(\omega)\right]dw(t,\omega') \tag{79}$$

with $z^o(0) = z(0)$.

We also note that the conditional mean field for minor agent i, $\mathbb{E}_{F_t^{I_i}}\mu_t(\omega)$, can be expressed (79) in the form of a stochastic coefficient Fokker–Planck–Kolmogorov equation:

$$\begin{aligned} dp(t,\omega^o,x) = -\mathbb{E}_{F_t^{I_i}}\Big(&\big\langle \nabla_x, f[t,x,u^o(t,x,\varphi),\mu_t(\omega)]p(t,\omega,x)\big\rangle \\ &+\frac{1}{2}\mathrm{tr}\big\langle \nabla_{xx}^2, a[t,\omega,x]p(t,\omega,x)\big\rangle\Big)dt,\ p(0,\omega,x) = p(0,x) \end{aligned} \tag{80}$$

in $[0,T]\times\mathbb{R}^n$, where $p(t,\omega,x)dx = \mu_t(\omega,dx)$ (a.s.), i.e., $p(t,\omega,x)$ is the conditional density of z_i^o conditioned on $\mathcal{F}_t^{w_0}$ and $p(t,\omega^o,x)dx = \mathbb{E}_{F_t^{I_i}}\mu_t(\omega,dx)$ (a.s.).

It is worthwhile noting that the conditional mean field $\mathbb{E}_{F_t^{I_i}}\mu_t(\omega,dx)$ will be a function of $(\varphi_i(t))_{0\le t\le T}$ when the objective mean field $\mu_t(\omega)$ is defined as the law of $z_i^o(t)$ conditioned on $\mathcal{F}_t^{z_0^o}$. It can further be simplified when $\mu_t(\omega)$ is defined as $\mu_t(\omega) = P\left(z^o(t)|z_0^o(t)\right)$. In that case the conditional mean field of a generic agent can be computed by

$$\mathbb{E}_{F_t^{I_i}}\mu_t(\omega,dx) = \int P\left(z_i^o(t)|z_0^o(t)\right)P\left(z_0^o(t)|F_t^{I_i}\right)dz_0^o, \tag{81}$$

where $P\big(z_0^o(t)\in A|F_t^{I_i}\big) = \int_A \varphi_i(t,x)dx$ for $A\in\mathcal{B}(\mathbb{R}^n)$. Hence, the conditional mean at time t becomes a function of $\varphi_i(t)$. A related path can also be employed by generalizing the nonlinear filters of minor agents via including $\big(w_0(t)\big)_{0\le t\le T}$ in their partial observation, and a similar conclusion can be reached. To continue, according to the McKean–Vlasov approximation result in section 3, the generic minor agent's statistical properties approximate the empirical distribution produced by all minor agents. Hence, let $\hat{\mu}_t(\omega) := P\left(z^o(t)\le\alpha|\mathcal{F}_t^{w_0}\right) = \int_\infty^\alpha \hat{\mu}_t(\omega,dx)$ a.s. for all $\alpha\in\mathbb{R}^n$ and $0\le t\le T$, where $z^o(t)$ is the state process of the generic minor agent and $\hat{\mu}_0(dx) = \mu_0(dx) = dF(x)$, with F is defined in (A1).

5.3. Partially observed mean field game consistency condition. Following [21], [33], and [35], we shall close the measure and control mapping loop by setting $\hat{\mu}_t(\omega) = \mu_t(\omega)$ a.s., $0\le t\le T$. MFG consistency is demonstrated via the major agent's SMFG system consisting of (41), (42), and (43) as well as the generic minor agent's SMFG system, which consists of (61), (62), (78), and (79). Let us denote the generic minor agent state by $z^o(t,\omega,\omega',\omega^o)$ and its best response by $u^o(t,x,p)$. The solution of the partial MM-SMFG system consists of the 9-tuple system

$$\begin{aligned} \Big(&\phi_0(t,\omega,x),\psi_0(t,\omega,x),u_0^o(t,\omega,x),z_0^o(t,\omega),\varphi(t),\phi(t,\omega^o,x,p),\psi(t,\omega^o,x,p), \\ &u^o(t,\omega^o,x,p),z^o(t)\Big),\ \mu_t(\omega) := P\left(z^o(t)|\mathcal{F}_t^{w_0}\right), \end{aligned} \tag{82}$$

where $z^o(t)$ generates the conditional random law.

$$
\begin{tikzcd}
\mu_{(\cdot)}(\omega) \arrow[r, "\text{M-SHJB}"] & \big(\phi_0(\cdot,\omega,x),\psi_0(\cdot,\omega,x)\big) \arrow[rr, "\text{M-SBR}"] & & u_0^o(\cdot,\omega,x) \arrow[d, "\text{M-SMV}"] \\
u^o(\cdot,\omega^o,x,p) \arrow[u, "\text{m-SMV}"] & \big(\phi(\cdot,\omega^o,x,p),\psi(\cdot,\omega^o,x,p)\big) \arrow[l, "\text{m-SBR}"'] & \varphi(\cdot,x) \arrow[l, "\text{m-SHJB}"'] & z_0^o(\cdot,\omega) \arrow[l, "\text{NLF}"']
\end{tikzcd}
$$

FIG. 1. *Partially observed MM-MFG system.*

5.4. Existence and uniqueness of solutions to the partially observed major and minor mean field game system. In order to complete the partially observed MM-SMFG consistency solution, we need to establish the existence and uniqueness of the above 9-tuple system. Following [21] and [35], we provide sufficient conditions for a map that goes from the random conditional measure of minor agents, i.e., $\mu_{(\cdot)}(\omega)$, back to itself via (41)–(43) and (61), (62), (78), and (79) such that the map is a contraction operator on the product space of random probability measures (see Figure 1). We note that in contrast to the completely observed case, there exist two conditional measures in the partially observed MM-SMFG system.

For such an analysis, we first introduce the Wasserstein metric on the space of probability measures. On the Banach space $C([0,T];\mathbb{R}^n)$ define the metric $\rho_T(x,y) = \sup_{0\le t\le T}|x(t)-y(t)|^2\wedge 1$, where $\wedge$ denotes the minimum. Let $C_\rho := (C([0,T];\mathbb{R}^n);\rho_T)$ and note that C_ρ is a Polish space. Let $\mathcal{M}(C_\rho)$ be the space of all Borel probability measures μ on $C([0,T];\mathbb{R}^n)$ such that $\int |x|^2 d\mu(x) < \infty$. Hence as in [21] and [35], one can define a stochastic process as a generic random process with the sample space $C([0,T];\mathbb{R}^n)$, i.e., $x(t,\omega) = \omega(t)$ for $\omega \in C([0,T];\mathbb{R}^n)$. Using ρ_T, we now define the Wasserstein metric on $\mathcal{M}(C_\rho)$:

$$
D_T^\rho(\mu,\nu) = \inf_{\gamma\in\Pi(\mu,\nu)}\left[\int_{C_\rho\times C_\rho}\rho_T\left(x(\omega_1),x(\omega_2)\right)d\gamma(\omega_1,\omega_2)\right]^{\frac{1}{2}}, \tag{83}
$$

where $\Pi(\mu,\nu) := \{\gamma \subset \mathcal{M}(C_\rho\times C_\rho) : \gamma(A\times C([0,T];\mathbb{R}^n)) = \mu(A)$ and $\gamma(C([0,T];\mathbb{R}^n)\times A) = \nu(A), A \in \mathcal{B}(C([0,T];\mathbb{R}^n))\}$. Note that the metric space $\mathcal{M}_\rho := (\mathcal{M}(C_\rho), D_T^\rho)$ is also Polish. Let $\mathcal{M}_\rho^\beta \subset \mathcal{M}_\rho$ denote the set of stochastic measures which are Hölder continuous with exponent $0 < \beta < 1$; see [35, Definition 6.1] for the definition of Hölder continuity for random measures. Let $\mu_t(\omega)$, $0 \le t \le T$, be a fixed stochastic measure in the set $\mathcal{M}_\rho^\beta$. The analysis is based on designing a contraction mapping $\mathsf{T} : \mathcal{M}_\rho^\beta \to \mathcal{M}_\rho^\beta$ which takes $\mu_{(\cdot)}(\omega)$ as its parameter. We introduce the following assumption.

(A9) For any $q \in \mathbb{R}^n$, $\mu_{(\cdot)}$, and $\mu^0_{(\cdot)} := \delta_{z_0^o(\cdot)} \in \mathcal{M}_\rho^\beta$, the sets

$$
S_0 := \arg\inf_{u_0\in U_0} H_0^{u_0}[t,\omega,x,u_0,q] \text{ and } S := \arg\inf_{u\in U} H^u\left[t,\omega^o,x,p,u,q\right],
$$

with $H_0^{u_0}$ and H^u defined as in (42) and (78), respectively, are singletons and the resulting u_0 is a.s. continuous in t and Lipschitz in (x,q), uniformly with respect to t and (μ,μ^0), and, similarly, u is a.s. continuous in t and Lipschitz in (x,p,q), in the appropriate norms, uniformly with respect to t and $(\mu,\mu^0) \in \mathcal{M}_\rho^\beta \times \mathcal{M}_\rho^\beta$.

5.5. Analysis of the major agent's SMFG. Since the major agent has complete observation of its own state, its SMFG analysis follows directly from [35]. We herein provide only the relevant steps for the sake of completeness. We start with the following proposition.

PROPOSITION 5 (see [35, Proposition 6.2]). *Assume* (A3) *holds. Let* $\mu_t(\omega)$, $0 \le t \le T$, *be a fixed stochastic measure in the set* $\mathcal{M}_\rho^\beta$, *for* $0 < \beta < 1$, *such that* $\mu_0(dx) := dF(x)$. *Define*

$$
\begin{aligned}
&f_0^*\left(t, z_0, u_0, \omega\right) := f_0\left[t, z_0, u_0, \mu_t(\omega)\right], \ \sigma_0^*\left(t, z_0, \omega\right) := \sigma_0\left[t, z_0, \mu_t(\omega)\right], \\
&L_0^*\left(t, z_0, u_0, \omega\right) := L_0\left[t, z_0, u_0, \mu_t(\omega)\right].
\end{aligned} \tag{84}
$$

Then the following hold:

(i) *Under* (A4), *the functions* $f_0^*(t, z_0, u_0, \omega)$ *and* $\sigma_0^*(t, z_0, \omega)$ *and their first order derivatives (with respect to* z_0*) are a.s. continuous and bounded on* $[0,T] \times \mathbb{R}^n \times U_0$ *and on* $[0,T] \times \mathbb{R}^n$. $f_0^*(t, z_0, u_0, \omega)$ *and* $\sigma_0^*(t, z_0, \omega)$ *are a.s. Lipschitz continuous in* z_0. *In addition,* $f_0^*(t,0,0,\omega)$ *and* $\sigma_0^*(t,0,\omega)$ *are in the spaces* $L^2_{\mathcal{F}_t}([0,T];\mathbb{R}^n)$ *and* $L^2_{\mathcal{F}_t}([0,T];\mathbb{R}^{n\times m})$, *respectively.*

(ii) *Under* (A4), *the function* $f_0^*(t, z_0, u_0, \omega)$ *is a.s. Lipschitz continuous in* $u_0 \in U_0$, *i.e., there exits a constant* $c > 0$ *such that*

$$
\sup_{t\in[0,T],\ z_0\in\mathbb{R}^n} \left|f_0^*\left(t, z_0, u_0, \omega\right) - f_0^*\left(t, z_0, u_0', \omega\right)\right| \le c(\omega)|u_0 - u_0'| \ (a.s.). \tag{85}
$$

(iii) *Under* (A6), *the functions* $L_0^*(t, z_0, u_0, \omega)$ *and its first order derivative (with respect to* z_0*) are a.s. continuous and bounded on* $[0,T]\times\mathbb{R}^n\times U_0$. *In addition,* $L_0^*(t,0,0,\omega)$ *is a.s. Lipschitz continuous in* z_0 *and is in* $L^2_{\mathcal{F}_t}([0,T];\mathbb{R}_+)$.

(iv) *Under* (A9), *the set of minimizers*

$$
\arg\inf_{u_0\in U_0} \left\{\left\langle f_0^*\left(t, z_0, u_0, \omega\right), p\right\rangle + L_0^*\left(t, z_0, u_0, \omega\right)\right\}
$$

is a singleton for any $p \in \mathbb{R}^n$, *and the resulting* u_0 *as a function of* (t, z_0, p, ω) *is a.s. continuous in* t, *a.s. Lipschitz continuous in* (z_0, p), *uniformly with respect to* t. *In addition,* $u_0(t,0,0,\omega)$ *is in the space* $L^2_{\mathcal{F}_t}([0,T];\mathbb{R}^n)$.

Assume that $\mu_{(\cdot)}(\omega) \in \mathcal{M}_\rho^\beta$, $0 < \beta < 1$, is given. Hence, if (A3)–(A7) hold, then the SHJB for the major agent (41) has a unique solution $\big(\phi_0(t,\omega,x),\ \psi_0(t,\omega,x)\big) \in \big(L^2_{\mathcal{F}_t}([0,T];\mathbb{R}), L^2_{\mathcal{F}_t}([0,T];\mathbb{R}^m)\big)$. Assume further that $(\phi_0, \psi_0)(t,x)$ satisfies the following: (i) for each x, (ϕ_0, ψ_0) is a $C^2(\mathbb{R}^n)$ map from $\mathbb{R}^n$ into $\mathbb{R}^n \times \mathbb{R}^m$, (ii) for each x, (ϕ_0, ψ_0) and $\big(D_x\phi_0, D^2_{xx}\phi_0, D_x\psi_0\big)$ are continuous $\mathcal{F}_t^{w_0}$-adapted stochastic processes. Then we obtain the best response control process

$$
\begin{aligned}
u_0^o(t,\omega,x) \ &\equiv\ u_0^o\big(t, x|(\mu_s(\omega))_{0\le s\le T}\big) \\
&:= \arg\inf_{u_0\in U_0} \big\{\left\langle f_0\left[t, x, u_0, \mu_t(\omega)\right], D_x\phi_0(t,\omega,x)\right\rangle + L_0[t, x, u_0, \mu_t(\omega)]\big\}.
\end{aligned}
$$

The following assumption is next introduced [35]:

(A10) For any $\mu_{(\cdot)}(\omega) \in \mathcal{M}_\rho^\beta$, $0 < \beta < 1$, $u_0^o(t,\omega,x)$ is a.s. continuous in (t,x) and a.s. Lipschitz continuous in x.

Let us denote by $C_{\mathrm{Lip}(x)}([0,T]\times\Omega\times\mathbb{R}^n;A)$ the class of continuous functions from $[0,T]\times\Omega\times\mathbb{R}^n$ to a vector space A which are a.s. Lipschitz continuous in x. We now define the following well-defined map:

$$
\begin{aligned}
&\mathsf{T}_0^{\mathsf{SHJB}} : \mathcal{M}_\rho^\beta \to C_{\mathrm{Lip}(x)}\left([0,T]\times\Omega\times\mathbb{R}^n; U_0\right), \ 0 < \beta < 1, \\
&\mathsf{T}_0^{\mathsf{SHJB}}(\mu_{(\cdot)}(\omega)) = u_0^o(t,\omega,x) \equiv u_0^o\big(t, x|(\mu_s(\omega))_{0\le s\le T}\big).
\end{aligned} \tag{86}
$$

Next, we analyze the SMV equation for the major agent (43) under the condition that $\mu_{(\cdot)}(\omega) \in \mathcal{M}_\rho^\beta$ and $u_0^o(t,\omega,x) \in C_{\mathrm{Lip}(x)}([0,T]\times\Omega\times\mathbb{R}^n;U_0)$ satisfy (A10). It now follows that (43) has a unique solution, $z_0^o(t,\omega)$, and under the assumptions (A3)–(A7), (A9), and (A10) the probability measure obtained by setting $\mu_t^0(\omega) := \delta_{z_0^o(t,\omega)}$ satisfies $\mu_{(\cdot)}^0(\omega) \in \mathcal{M}_\rho^\gamma$, $0 < \gamma < 1/2$ [35, Theorems 6.4 and 6.5].

We conclude the major agent's SMFG system analysis by introducing the following map:

$$\begin{aligned}&\mathsf{T}_0^{\mathsf{SMV}} : \mathcal{M}_\rho^\beta \times C_{\mathrm{Lip}(x)}\left([0,T]\times\Omega\times\mathbb{R}^n;U_0\right) \to \mathcal{M}_\rho^\gamma,\ 0<\gamma<1/2,\ 0<\beta<1,\\ &\mathsf{T}_0^{\mathsf{SMV}}\left(\mu_{(\cdot)}(\omega), u_0^o(t,\omega,x)\right) = \mu_t^0(\omega) := \delta_{z_0^o(t,\omega)}.\end{aligned} \tag{87}$$

5.6. Analysis of the minor agent's SMFG system with partial observation. Let $\mu_{(\cdot)}(\omega)$, $0 \le t \le T$, be the fixed stochastic measure in $\mathcal{M}_\rho^\beta$ with $0<\beta<1$ assumed in section 5.1. Based on the analysis in the previous section, we also assume $\mu_{(\cdot)}^0(\omega) \in \mathcal{M}_\rho^\gamma$, $0<\gamma<1/2$, which is the unit mass random measure concentrated at $z_0^o(t,\omega), 0 \le t \le T$ (i.e., $\mu_t^0(\omega) = \delta_{z_0^o(t,\omega)}$), obtained via the composite map

$$\begin{aligned}&\mathsf{T}_0 : \mathcal{M}_\rho^\beta \to \mathcal{M}_\rho^\gamma,\ 0<\gamma<1/2,\ 0<\beta<1,\\ &\mathsf{T}_0\left(\mu_{(\cdot)}(\omega)\right) = \mathsf{T}_0^{\mathsf{SMV}}\left(\mu_{(\cdot)}(\omega), \mathsf{T}_0^{\mathsf{SHJB}}\left(\mu_{(\cdot)}(\omega)\right)\right) = \mu_{(\cdot)}^0(\omega),\end{aligned} \tag{88}$$

where $\mathsf{T}_0^{\mathsf{SHJB}}$ and $\mathsf{T}_0^{\mathsf{SMV}}$ are given in (86) and (87), respectively.

In the analysis of the minor agent's SMFG system, we follow arguments analogous to those of the previous section and also take the effect of the partial observation into the account. Hence, we need to consider the existence and uniqueness of solutions to the SHJB equation for the generic minor agent (62). Notice, however, that the generic minor agent's state process is infinite-dimensional, and consequently such an analysis requires developing a theory of BSPDEs with and infinite-dimensional domain. We remark that the existence and uniqueness of the solution of BSDEs with some infinite-dimensional space-valued state process has been studied before. Indeed, the SHJB equation obtained in [36], which provides the SHJB equations of the major and minor agents in [35], is shown to have a unique solution based on a Sobolev space technique by considering the value function and its associated semimartingale coefficient as a stochastic process taking values in some function spaces. In the partially observed case, the domain of the value function consists of a function space ($G_k \subset L^1$), and hence the value function in this case is a stochastic process taking values in a function space defined on another function space. This entails BSPDEs on function spaces; the value function is the process $V : [0,T]\times\Omega\times G_k\times\mathbb{R}^n \to \mathbb{R}$. As we mentioned in Theorem 4, we proceed on the assumption that a unique solution exists for such BSPDEs, and we sketch a possible path to develop such a theory.

We first need to use a more detailed characterization of a solution to (31). Define the Sobolev space $W^{1,2}(\mathbb{R}^n)$ by $W^{1,2}(\mathbb{R}^n) := \big\{f \in L^2(\mathbb{R}^n) : \frac{\partial f}{\partial x_i} \in L^2(\mathbb{R}^n), i = 1,\dots,n\big\}$ with the norm $\|f\|_{W^{1,2}(\mathbb{R}^n)} := \big\{\int_{\mathbb{R}^n}\big(|f(x)|^2 + \sum_{i=1}^n \big|\frac{\partial f(x)}{\partial x_i}\big|^2\big)dx\big\}^{1/2}$. Hence, the solution of (31) is $W^{1,2}(\mathbb{R}^n)$-valued [42]. Consider now the pair $(x,p) \in \mathbb{R}^n \times W^{1,2}(\mathbb{R}^n)$, which is a Hilbert space under the max norm. Note that $\phi : [0,T]\times\Omega\times\mathbb{R}^n\times W^{1,2}(\mathbb{R}^n) \to \mathbb{R}$, and hence we shall consider a subset of functions on $\mathsf{W} := (\mathbb{R}^n \times W^{1,2}(\mathbb{R}^n))$ so that they form a Hilbert space similar to the Sobolev space defined on finite-dimensional spaces. In this context we note that Sobolev space characterization of functions defined on arbitrary metric spaces as well as on infinite-dimensional spaces with differentiable measures has been studied in [17] and [5], respectively. Furthermore, Sobolev spaces on abstract Wiener spaces, of which we provide more details below, are discussed in great detail; for the earliest contributions see [31] and [40].

Let Θ be a separable Banach space into which H is continuously embedded, and let (Θ, H, ν) be an abstract Wiener space whose Borel structure is given by $\overline{\mathcal{B}(\Theta)^\nu}$, i.e., the completion of the topological σ-field $\mathcal{B}(\Theta)$ with respect to ν. Let E be a separable Hilbert space with norm $\|\|_E$, and for $1 \leq p < \infty$ define $L^p(E) = L^p(\Theta; E) \equiv \{f : \Theta \to E; \overline{\mathcal{B}(\Theta)^\nu}|\mathcal{B}(E)\text{-measurable and } \int_\Theta |f(\theta)|_E^p \nu(d\theta) < \infty\}$. Hence,

$$\|f\|_{p;E} \equiv \left(\int_\Theta |f(\theta)|_E^p \nu(dw) < \infty\}\right)^{1/p}, \ f \in L^p(E), \tag{89}$$

is a norm of $L^p(E)$ and $L^p(E)$ is a Banach space with this norm. Furthermore, set $\mathcal{H}(E) = \mathcal{H}(H; E) \equiv \{V : H \to E; V \text{ is a linear operator of Hilbert–Schmidt type}\}$. Notice that $\mathcal{H}(E)$ is a Hilbert space whose norm is the Hilbert–Schmidt norm. Note also that $\mathcal{H}^n(E) = \mathcal{H}(\mathcal{H}^{n-1}(E))$, $n = 2, 3, \dots$, with $\mathcal{H}^1(E) = \mathcal{H}(E)$.

Definition 6 (see [30]).

(i) *A measurable mapping $f : \Theta \to E$ is said to be ray absolutely continuous (RAC) if for every $h \in H$ there exists a measurable mapping $\tilde{f}_h : \Theta \to E$ such that $f(\theta) = \tilde{f}(\theta)$, ν-a.e. $\theta \in \Theta$, and for any $\theta \in \Theta$, $\tilde{f}_h(\theta + th)$, $t \in \mathbb{R}$, is absolutely continuous in t.*

(ii) *A measurable mapping $f : \Theta \to E$ is said to be stochastically Gâteaux differentiable (SGD) if there exists a measurable mapping $F : \Theta \to \mathcal{H}(E)$ such that for any $h \in H$, $\frac{1}{t}(f(\theta + th) - f(\theta))$ converges to $F[\theta](h)$ in probability with respect to ν as $t \to 0$. When it exists such an F is unique in the ν-a.e. sense, and in that case it is denoted by $\widetilde{\mathsf{D}}f$. Higher order SGD can be defined similarly: $\widetilde{\mathsf{D}}^n f \equiv \widetilde{\mathsf{D}}(\tilde{\mathsf{D}}^{n-1})f$.*

Definition 7 (Kusuoka–Stroock Sobolev spaces [31]). *Let $1 < p < \infty$. Define the space*

$$\widetilde{\mathbf{D}}_{p,1}(E) \equiv \{f \in L^p(E); f \text{ is RAC and SGD}, \widetilde{\mathsf{D}}f \in L^p(\mathcal{H}(E))\}$$

and endow it with a norm

$$\|f\|_{p,1;E} \equiv \|f\|_{p;E} + \|\widetilde{\mathsf{D}}f\|_{p;\mathcal{H}(E)}, \ f \in \widetilde{\mathbf{D}}_{p,1}(E). \tag{90}$$

Then for $n = 2, 3, \dots$, we define the spaces $\tilde{\mathbf{D}}_{p,n}(E)$ inductively by

$$\widetilde{\mathbf{D}}_{p,n}(E) \equiv \{f \in \widetilde{\mathbf{D}}_{p,n-1}(E); \widetilde{\mathsf{D}}f \in \widetilde{\mathbf{D}}_{p,n-1}(\mathcal{H}(E))\}$$

and endow them with the following norms:

$$\|f\|_{p,n;E} \equiv \|f\|_{p;E} + \|\widetilde{\mathsf{D}}^n f\|_{p;\mathcal{H}^n(E)}, \ f \in \widetilde{\mathbf{D}}_{p,n}(E). \tag{91}$$

It is known that the normed spaces $(\widetilde{\mathbf{D}}_{p,n}(E), \|\cdot\|_{p,n;E})$, $n = 1, 2, \dots$, are complete. Recall now that the infinite-dimensional BSDE setup in [36] takes values in the rigged Hilbert space $H^1 \hookrightarrow H^0 \hookrightarrow H^{1'}$, where $H^{1'}$ is the dual space of H^1. Therefore, since the semilinearity of SHJB equations holds in the infinite-dimensional domain situation, one shall next construct a specific Hilbert space in a Gaussian Sobolev setup defined via Definition 7 in order to prove the existence and uniqueness of solutions to the BSDEs given by (62).

We assume the following:

(A11) The SHJB equation for the generic minor agent (62) has a unique solution $(\phi_i(t, x, p), \psi_i(t, x, p))$.

For the probability measure flow $\mu_{(\cdot)}(\omega) \in \mathcal{M}_\rho^\beta$, $0 < \beta < 1$, and $\mu_{(\cdot)}^0(\omega) \in \mathcal{M}_\rho^\gamma$, $0 < \gamma < 1/2$, we assume that the unique solution $(\phi_i(t,x,p), \psi_i(t,x,p))$ satisfies the regularity properties defined in Theorem 3 so that $\phi_i(t,x,p)$ coincides with the value function (59), and under (A9) we get the best response control process:

$$\begin{aligned} u_i^o\big(t,\omega^o,x,p\big) &\equiv u_i^o\big(t,\omega^o,x,p|\mu_s(\omega^o)_{0\le s\le T}\big) \\ &:= \arg\inf_{u\in U} H^u\left[t,\omega^o,x,p,u,D_x\phi_i(t,\omega^o,x,p)\right], \end{aligned} \tag{92}$$

where $(t,x,p) \in [0,T] \times \mathbb{R}^n \times G_k$. We introduce the following assumption (see [35, (A9)] and [21, (H6)]):

(A12) For any $\mu_{(\cdot)}(\omega) \in \mathcal{M}_\rho^\beta$, $0 < \beta < 1$, and $\mu_{(\cdot)}^0(\omega) \in \mathcal{M}_\rho^\gamma$, $0 < \gamma < 1/2$, the best response control process $u_i^o\left(t,\omega^o,x,p\right)$ is a.s. continuous in t and a.s. Lipschitz continuous in (x,p).

Note that in (A12), u_i^o depends on $\mu_{(\cdot)}(\omega)$ through $\mu_{(\cdot)}(\omega^o)$. We now define the following map for the generic minor agent i:

$$\begin{aligned} &\mathsf{T}^{\mathsf{NLF}} : \mathcal{M}_\rho^\beta \times \mathcal{M}_\rho^\gamma \to C\left([0,T]\times\mathbb{R}^n\times\Omega;\mathbb{R}\right),\ 0<\gamma<1/2,\ 0<\beta<1, \\ &\mathsf{T}^{\mathsf{NLF}}\Big(\mu_{(\cdot)}(\omega),\mu_{(\cdot)}^0(\omega)\Big) = \varphi_i\left(t,x,\omega^o\right). \end{aligned} \tag{93}$$

Consequently, we can define the following composition map for the generic minor agent:

$$\mathsf{T}^{\mathsf{SHJB}} : \mathcal{M}_\rho^\beta \times \mathcal{M}_\rho^\gamma \to C_{\mathrm{Lip}(x,p)}\left([0,T]\times\Omega\times\mathbb{R}^n\times G_k;U\right),\ 0<\gamma<1/2,\ 0<\beta<1,$$

$$\begin{aligned} &\mathsf{T}^{\mathsf{SHJB}}\Big(\mu_{(\cdot)}(\omega),\mathsf{T}^{\mathsf{NLF}}\Big(\mu_{(\cdot)}(\omega),\mu_{(\cdot)}^0(\omega)\Big)\Big) \\ &:= \mathsf{T}^{\mathsf{SHJB}}\Big(\mu_{(\cdot)}(\omega),\mu_{(\cdot)}^0(\omega)\Big) = u_i^o(t,\omega^o,x,p) \\ &\equiv u_i^o\left(t,x,p|\{\mu_s(\omega^o)\}_{0\le s\le T}\right). \end{aligned} \tag{94}$$

Consider now the generic minor agent's closed loop SMV SDE where the control policy depends on the filtering process $\varphi_i(t)$:

$$dz_i^o(t,\omega,\omega',\omega^o) = f\left[t,z_i^o,u_i^o\left(t,\omega^o,z_i^o,\varphi_i\right),\mu_t(\omega)\right]dt + \sigma\left[t,z^o,\mu_t(\omega)\right]dw(t,\omega'), \tag{95}$$

where $z_i^o(0) = z_i(0)$ and $u_i^o\left(t,\omega^o,z_i^o(t),\varphi_i(t)\right) \in C_{\mathrm{Lip}(x,p)}\big([0,T]\times\Omega\times\mathbb{R}^n\times G_k;U\big)$, which is obtained via $\mathsf{T}^{\mathsf{SHJB}}$. The pair $\left(z_i^o(\cdot),\mu_{(\cdot)}(\omega)\right)$ is called a consistent solution to (95) if $z_i^o(\cdot)$ is a solution to (95) and $\mu_{(\cdot)}(\omega)$ is the law of the process $z_i^o(\cdot)$. We define Λ as the map which associates to $\mu_{(\cdot)}(\omega) \in \mathcal{M}_\rho^\beta$, $0<\beta<1/2$, the law of the process $z_i^o\left(\cdot,\omega,\omega',\omega^o\right)$ in (95):

$$\begin{aligned} z_i^o(t) = z_i(0) &+ \int_0^t \int_{\mathbb{R}^n} f\left(s,z_i^o,u_i^o,y\right)d\mu_s(\omega)(y)ds \\ &+ \int_0^t \int_{\mathbb{R}^n} \sigma\left(s,z_i^o,y\right)d\mu_s(\omega)(y)dw_i(s). \end{aligned} \tag{96}$$

We shall show that there exists a unique $\mu_{(\cdot)}(\omega) \in \mathcal{M}_\rho^\beta$, $0<\beta<1/2$, which satisfies $\mu(\omega) = \Lambda\left(\mu(\omega)\right)$. Such an analysis has been made in [35] in the completely observed case which does not include the existence of the conditional density in the control law. Therefore, there is no formal result that we can refer to, and hence we obtain the following theorem.

THEOREM 8. *Assume* (A2)–(A8) *and* (A12) *hold. Let* $\mu_{(\cdot)}(\omega), \mu^0_{(\cdot)}(\omega) \in \mathcal{M}^\beta_\rho \times \mathcal{M}^\gamma_\rho$, $0 < \beta < 1$, $0 < \gamma < 1/2$, *and let* $u^o_i(t, \omega^o, x, p)$ *be given in* (92). *Then there exists a unique consistent solution pair* $\big(z^o_i(\cdot, \omega, \omega', \omega^o), \mu_{(\cdot)}(\omega)\big)$ *to the generic minor agent's SMV equation given by* (95).

In the proof of the above theorem, we need an intermediate result which guarantees that $\mu_{(\cdot)}(\omega) \in \mathcal{M}^\beta_\rho$, $0 < \beta < 1$.

THEOREM 9. *Assume* (A2)–(A7) *and* (A12) *hold. Let* $\big(\mu_{(\cdot)}(\omega), \mu^0_{(\cdot)}(\omega)\big) \in \mathcal{M}^\beta_\rho \times \mathcal{M}^\gamma_\rho$, $0 < \beta < 1$, $0 < \gamma < 1/2$. *Given* $u^o_i(t, \omega^o, x, p)$ *in* (92), *let* $\big(z^o_i(\cdot), \mu_{(\cdot)}(\omega)\big)$ *be the consistent solution pair to the generic minor agent's SMV in* (95). *Then the probability measure* $\mu_{(\cdot)}(\omega^o)$ *is in the class* $\mathcal{M}^\beta_\rho$, $0 < \beta < 1$.

Proof of Theorem 9. The proof is similar to that of [35, Theorem 6.8]. Indeed, the existence of a consistent solution to (95) as well as the Lipschitz continuity and the boundedness of f is sufficient for the result to hold. ∎

Proof of Theorem 8. The proof is based on a fixed point argument in the space of random probability measures and follows similarly along the lines of [35, Theorem 6.7]. The only difference is that the filtering equation depends on the state of the major agent, which consequently depends on the law of the generic minor agent. Notice, however, that $\mu^0_{(\cdot)}(\omega)$ is fixed, which implies that the fixed point argument is insensitive to the state of the major agent. Indeed, let $\varphi_i(t)$ denote the filtering process for $\mu^0_{(\cdot)}(\omega)$ and let $\omega \in \Omega$ be fixed. For given $\mu_{(\cdot)}(\omega) \in \mathcal{M}^\beta_\rho$, $0 < \beta < 1$, the law of $z^o_i(\cdot, \omega, \omega', \omega^o)$ given in (96), $\Lambda\big(z^o_i(\cdot, \omega, \omega', \omega^o)\big)$, belongs to $\mathcal{M}^\beta_\rho$, $0 < \beta < 1$. Let $\mu_{(\cdot)}(\omega)$, $\nu_{(\cdot)}(\omega) \in \mathcal{M}^\beta_\rho$, $0 < \beta < 1$. Let $z^o_i(\cdot, \omega, \omega', \omega^o)$ and $x^o(\cdot, \omega, \omega', \omega^o)$ be defined by (96) with measures $\mu_{(\cdot)}(\omega)$, $\nu_{(\cdot)}(\omega)$, respectively. We have

$$
\begin{aligned}
\mathbb{E}_{\mathcal{F}^{w_0}_t} \sup_{0 \le s \le t} |z^o_i(s, \omega) - x^o_i(s, \omega)|^2 &\le 2t \int_0^t \bigg| \int_{\mathbb{R}^n} f\left(s, z^o_i, u^o_i\left(s, z^o_i, \varphi_i\right), y\right) d\mu_s(\omega)(y) \\
&\qquad - \int_{\mathbb{R}^n} f\left(s, x^o_i, u^o_i(s, x^o, \varphi_i), y\right) d\nu_s(\omega)(y) \bigg|^2 ds \\
\le 2 \int_0^t \bigg| \int_{\mathbb{R}^n} \sigma\left(s, z^o_i, y\right) d\mu_s(\omega)(y) - \int_{\mathbb{R}^n} \sigma\left(s, x^o_i, y\right) d\nu_s(\omega)(y) \bigg|^2 ds.
\end{aligned}
$$

However,

$$
\begin{aligned}
&\left| f\left(s, z^o_i, u^o_i\left(s, z^o_i, \varphi_i\right), y\right) d\mu_s(\omega)(y) - f\left(s, x^o_i, u^o_i(s, x^o_i, \varphi_i), y\right) d\nu_s(\omega)(y) \right|^2 \\
&\quad \le 2C \bigg(|z^o_i(s) - x^o_i(s)|^2 + \int_{C_\rho \times C_\rho} |z_s(\omega_1) - z_s(\omega_2)|^2 \, d\gamma(\omega_1, \omega_2) \bigg),
\end{aligned}
\tag{97}
$$

where C is obtained from the boundedness and Lipschitz continuity of f and u^o_i, and $\gamma \in \mathcal{M}\left(C_\rho \times C_\rho\right)$ is any coupling of $\mu_{(\cdot)}$ and $\nu_{(\cdot)}$ where $\gamma\left(A \times C([0,T]; \mathbb{R}^n)\right) = \mu(A)$, $\gamma\left(C([0,T]; \mathbb{R}^n) \times A\right) = \hat{\mu}(A)$ for any $A \in \mathcal{B}\left(C([0,T]; \mathbb{R}^n)\right)$. Notice that φ vanishes in (97), and hence the proof follows from [35, Theorem 6.7]. ∎

By Theorems 8 and 9, we can now define the following map for generic minor agent: For $0 < \gamma < 1/2$, $0 < \beta < 1$,

$$
\begin{aligned}
&\mathsf{T}^{\mathsf{SMV}} : \mathcal{M}^\beta_\rho \times \mathcal{M}^\gamma_\rho \times C_{\mathrm{Lip}(x,p)}\left([0,T] \times \Omega \times \mathbb{R}^n \times G_k; U\right) \to \mathcal{M}^\beta_\rho, \\
&\mathsf{T}^{\mathsf{SMV}}\left(\mu_{(\cdot)}(\omega), \mathsf{T}^{\mathsf{NLF}}\left(\mu_{(\cdot)}(\omega), \mu^0_{(\cdot)}(\omega)\right), u^o_i\left(t, \omega^o, x, p\right)\right) = \mu_{(\cdot)}(\omega).
\end{aligned}
\tag{98}
$$

We remark that since it is assumed that the dynamics of the minor agents do not explicitly depend upon the state of the major agent, in (98) the major agent's state appears only through the filtering equations in the SHJB equation.

5.7. The joint analysis of major and minor agents' SMFG systems. In the previous sections, we have analyzed the major and generic minor agents' SMFG system starting with a fixed measure $\mu_{(\cdot)}(\omega) \in \mathcal{M}_\rho^\beta$, $0 < \beta < 1$, and obtained the following map:

$$
\begin{aligned}
&\mathsf{T}^{\mathsf{SMFG}} : \mathcal{M}_\rho^\beta \to \mathcal{M}_\rho^\beta,\\
&\mathsf{T}^{\mathsf{SMFG}}\left(\mu_{(\cdot)}(\omega)\right)\\
&\quad = \mathsf{T}^{\mathsf{SMV}}\left(\mu_{(\cdot)}(\omega), \mathsf{T}^{\mathsf{NLF}}\left(\mathsf{T}_0\left(\mu_{(\cdot)}(\omega)\right)\right), \mathsf{T}^{\mathsf{SHJB}}\left(\mu_{(\cdot)}(\omega), \mathsf{T}^{\mathsf{NLF}}\left(\mathsf{T}_0\left(\mu_{(\cdot)}(\omega)\right)\right)\right)\right)\\
&\quad := \mathsf{T}\left(\left(\mu_{(\cdot)}(\omega)\right)\right),
\end{aligned}
\tag{99}
$$

which is the composition of maps $\mathsf{T}^{\mathsf{SMV}}, \mathsf{T}^{\mathsf{SHJB}}, \mathsf{T}^{\mathsf{NLF}}$, and T_0 introduced in (86), (87), (88), (93), (94), and (98). Therefore, the existence and uniqueness analysis of the SMFG system reduces to a fixed point argument with random parameters for the composition map T on the Polish space, $\mathcal{M}_\rho^\beta$, $0 < \beta < 1$.

Following [35], we introduce the following assumption:

(A13) The diffusion coefficient σ_0 in (1) does not depend on z_i^N, $0 \le i \le N$.

In the completely observed MM-MFG, the sensitivity analysis of the SMV and SHJB equations with respect to different parameters plays an essential role in the fixed point argument. In the partially observed setup, we require some further arguments on the sensitivity analysis in order to accommodate the filtering equations. We first present a result which characterizes the sensitivity of the filtering process with respect to the signal process.

Theorem 10. *Assume* (A2)–(A10) *hold. Let* $\mu^0(\omega) := \delta_{z_0^{o,\mu}}$, $\nu^0(\omega) := \delta_{z_0^{o,\nu}} \in \mathcal{M}_\rho^\beta$, $0 < \gamma < 1/2$, *are induced by the map* $\mathsf{T}_0^{\mathsf{SMV}}$ *in* (87) *using the measures* $\mu_{(\cdot)}(\omega)$, $\nu_{(\cdot)}(\omega) \in \mathcal{M}_\rho^\beta$, $0 < \beta < 1$. *Let* φ^μ *and* φ^ν *denote the solutions to the filtering equations associated to the processes* $z_0^{o,\mu}, z_0^{o,\nu}$, *respectively. Then there exists a constant* c_7 *such that*

$$
\|\varphi^\mu - \varphi^\nu\|_{G_k} \le c_7 \left(D_T^\rho\left(\mu^0(\omega), \nu^0(\omega)\right)\right) \ a.s. \tag{100}
$$

Proof. The proof depends on the robustness analysis of nonlinear filtering theory, and we follow the presentation in [1] where the results are attributed to [15]. Let

$$
\begin{aligned}
dy^\mu(t) &= h\left(t, z_0^{o,\mu}(t)\right) dt + dv(t),\\
dy^\nu(t) &= h\left(t, z_0^{o,\nu}(t)\right) dt + dv(t),
\end{aligned}
\tag{101}
$$

where $v(t)$ is Brownian motion independent of $w_0(t)$. By robust representation of nonlinear filtering, given $\ell \in C(\mathbb{R}^n)$ with $\mathbb{E}\left[\ell\left(z(t)\right)\right] < \infty$, there exists a function $\hat{f}^\ell : C([0,t];\mathbb{R}) \to \mathbb{R}$, such that $\hat{f}^\ell\big(y^\mu_{(\cdot)}\big)$, or $\hat{f}^\ell\big(y^\nu_{(\cdot)}\big)$, is a version of $\mathbb{E}_{\mathcal{F}_t^{y^\mu}}\left[\ell\left(z_0^{o,\mu}(t)\right)\right]$, or of $\mathbb{E}_{\mathcal{F}_t^{y^\nu}}\left[\ell\left(z_0^{o,\nu}(t)\right)\right]$, i.e.,

$$
\mathbb{E}_{\mathcal{F}_t^{y^\mu}}\left[\ell\left(z_0^{o,\mu}(t)\right)\right] = \hat{f}^\ell\big(y^\mu_{(\cdot)}\big) \quad P\text{-a.s.}, \tag{102}
$$

where path-valued process $y^\mu_{(\cdot)}$ is defined by $y^\mu_{(\cdot)} : \Omega \to C([0,t];\mathbb{R})$, $y^\mu_{(\cdot)}(\omega) = \big(y^\mu(s,\omega),\ 0 \le s \le t\big)$. It is further known that the function $\hat{f}^\ell$ is locally Lipschitz continuous [1, Theorem 5.7] in the sup-norm. Note also that for $0 \le t \le T$

$$
\begin{aligned}
|y^\mu(t) - y^\nu(t)|^2 &\le \left| \int_0^t \int_{\mathbb{R}^n} h(s,x) d\mu_s^0(\omega)(x) - \int_{\mathbb{R}^n} h(x) d\nu_s^0(\omega)(x) ds \right|^2 \\
&\le \int_0^t \left[\int_{C_\rho \times C_\rho} h(s, z_{0,s}(\omega_1)) d\gamma^0(\omega_1,\omega_2) - \int_{C_\rho \times C_\rho} h(s, z_{0,s}(\omega_2)) d\gamma^0(\omega_1,\omega_2) \right]^2 ds \\
&\le K \int_0^t \left[\int_{C_\rho \times C_\rho} |z_{0,s}(\omega_1) - z_{0,s}(\omega_2)|^2 d\gamma^0(\omega_1,\omega_2) \right] ds,
\end{aligned}
$$

where $\gamma^0 \in \mathcal{M}(C_\rho \times C_\rho)$ is any coupling of the two measures μ^0 and ν^0, where $\gamma^0\big(A \times C([0,T];\mathbb{R}^n)\big) = \mu^0(A)$ and $\gamma^0\big(C([0,T];\mathbb{R}^n); A\big) = \nu^0(A)$ for any Borel set $A \in C([0,T];\mathbb{R}^n)$ and the last step is due to the Lipschitz continuity of $h(\cdot)$. Therefore,

$$
\sup_{0\le s\le t} |y^\mu(s) - y^\nu(s)| \le Kt \left(D_t^\rho \left(\mu^0(\omega), \nu^0(\omega) \right) \right), \tag{103}
$$

where K depends on the Lipschitz and the boundedness coefficient of h. Let $\ell = (1+|x|^k)$ and consider the densities associated to the distributions φ^μ, φ^ν. We have

$$
\begin{aligned}
&\|\varphi^\mu(t) - \varphi^\nu(t)\|_{G_k} \\
&= \int_{\mathbb{R}} \left((1+|x|^k) \left(\varphi^\mu(t,x) - \varphi^\nu(t,x) \right) \right) dx = \hat{f}^{1+|x|^k}\big(y^\mu_{(\cdot)}\big) - \hat{f}^{1+|x|^k}\big(y^\nu_{(\cdot)}\big) \ P\text{-a.s.}
\end{aligned} \tag{104}
$$

Finally, notice that $\hat{f}^{1+|x|^k}$ is only locally Lipschitz, however, since $y^\mu_{(\cdot)}$ and $y^\nu_{(\cdot)}$ takes values in $C([0,T];\mathbb{R}^n)$, there exists $R' > 0$ such that $\|y_{(\cdot)}\| \le R'$ for all $\omega \in \Omega$. Hence, let $R > R'$ and consequently, for $\|y^\mu_{(\cdot)}\| \le R$ and $\|y^\nu_{(\cdot)}\| \le R$, there exists a constant $C_R > 0$ which combining with (102)-(104) yields

$$
\|\varphi^\mu(t) - \varphi^\nu(t)\|_{G_k} \le C_R K t \left(D_t^\rho \left(\mu^0(\omega), \nu^0(\omega) \right) \right)^2, \tag{105}
$$

which completes the proof with $c_7 := C_R K t$. ∎

The following lemma generalizes the sensitivity analysis of the laws of state variables obtained via SMV SDEs to the case where the filtering process is also involved.

LEMMA 11.

(i) *Assume* (A3)–(A7) *and* (A13) *hold. Let* $\mu_{(\cdot)}(\omega)$ *be in the set* $\mathcal{M}_\rho^\beta$ *where* $0 < \beta < 1$. *Then, for given* $u_0, u_0' \in C_{\mathrm{Lip}(x)}([0,T] \times \Omega \times \mathbb{R}^n; U_0)$ *there exists a constant* c_0 *such that*

$$
\left(D_T^\rho \left(\mu^0(\omega), \nu^0(\omega) \right) \right)^2 \le c_0 \sup_{(t,x)\in[0,T]\times\mathbb{R}^n} |u_0(t,\omega,x) - u_0'(t,\omega,x)|^2 \quad a.s.,
$$

where $\mu^0(\omega)$, $\nu^0(\omega) \in \mathcal{M}_\rho^\gamma$, $0 < \gamma < 1/2$, *are induced by the map* $\mathsf{T}_0^{\mathsf{SMV}}$ *in* (87) *using the two control processes* u_0 *and* u_0'.

(ii) *Assume* (A3)–(A7) *and* (A13) *hold. Let* $u_0^o \in C_{\mathrm{Lip}(x)}([0,T] \times \Omega \times \mathbb{R}^n; U_0)$. *Then, for given* $\mu_{(\cdot)}(\omega), \nu_{(\cdot)}(\omega) \in \mathcal{M}_\rho^\beta$ *where* $0 < \beta < 1$, *there exists a constant* c_1 *such that*

$$
\left(D_T^\rho \left(\mu^0(\omega), \nu^0(\omega) \right) \right)^2 \le c_1 \left(D_T^\rho \left(\mu(\omega), \nu(\omega) \right) \right)^2 \quad a.s., \tag{106}
$$

where $\mu^0(\omega)$, $\nu^0(\omega) \in \mathcal{M}_\rho^\beta$, $0 < \gamma < 1/2$, *are induced by the map* $\mathsf{T}_0^{\mathsf{SMV}}$ *in* (87) *using the two measures* $\mu(\omega)$ *and* $\nu(\omega)$, *respectively.*

(iii) *Assume* (A3)–(A7) *and* (A13) *hold. Let* $\mu^0_{(\cdot)}(\omega)$ *be in the set* $\mathcal{M}^\gamma_\rho$ *where* $0 < \gamma < 1/2$. *Then, for given* $u, u' \in C_{\mathrm{Lip}(x,p)}\left([0,T] \times \Omega \times \mathbb{R}^n \times G_k; U\right)$, *there exists a constant* c_2 *such that*

$$\begin{aligned}&\left(D^\rho_T\big(\mu(\omega), \nu(\omega)\big)\right)^2 \\ &\leq c_2 \sup_{(t,x,p)\in[0,T]\times\mathbb{R}^n\times G_k} |u(t,\omega^o,x,p) - u'(t,\omega^o,x,p)|^2 \quad a.s.,\end{aligned} \tag{107}$$

where $\mu_{(\cdot)}(\omega) \in \mathcal{M}^\beta_\rho$ *and* $\nu_{(\cdot)}(\omega) \in \mathcal{M}^\beta_\rho$, $0 < \beta < 1$, *are induced by the map* $\mathsf{T}^{\mathsf{SMV}}$ *in* (98) *using the two control processes* u *and* u', *respectively.*

(iv) *Assume* (A3)–(A8) *hold. Let* $u^o \in C_{\mathrm{Lip}(x,p)}\left([0,T] \times \Omega \times \mathbb{R}^n \times G_k; U\right)$. *Then, for given* $\mu^0_{(\cdot)}(\omega)$, $\nu^0_{(\cdot)}(\omega) \in \mathcal{M}^\gamma_\rho$ *where* $0 < \gamma < 1/2$, *there exists a constant* c_3 *such that*

$$\left(D^\rho_T\big(\mu(\omega), \nu(\omega)\big)\right)^2 \leq c_3 \left(D^\rho_T\left(\mu^0(\omega), \nu^0(\omega)\right)\right)^2 \quad a.s., \tag{108}$$

where $\mu_{(\cdot)}(\omega) \in \mathcal{M}^\beta_\rho$ *and* $\nu(\omega) \in \mathcal{M}^\beta_\rho$, $0 < \beta < 1$, *are induced by the map* $\mathsf{T}^{\mathsf{SMV}}$ *in* (98) *using the stochastic measures* $\mu^0(\omega)$ *and* $\nu^0(\omega)$, *respectively.*

The proof is available at Appendix C.

In the next step, we shall investigate the sensitivity analysis of the best response control processes that are obtained via SHJB equations with respect to the stochastic measures $\left(\mu_{(\cdot)}(\omega), \mu^0_{(\cdot)}(\omega)\right)$. We first define the Gâteaux derivative of $F(t,x,\mu)$ with respect to the measure μ as follows:

$$\partial_{\mu(y)} := \lim_{\epsilon\to 0} \frac{F(t,x,\mu+\epsilon\delta(y)) - F(t,x,\mu)}{\epsilon}. \tag{109}$$

We have the following assumption.

(A14) In (40), the Gâteaux derivative of f_0 and of L_0 with respect to μ exists, is $C^\infty(\mathbb{R}^n)$, and is a.s. uniformly bounded. In (49) and (50) the partial derivatives of f, σ, and $\mathbf{L}$ with respect to μ exist, are $C^\infty(\mathbb{R}^n \times G_k)$, and are a.s. uniformly bounded.

The proof of the following lemma extends the sensitivity analysis of the SHJB equations to the stochastic measures $\mu_{(\cdot)}(\omega)$ developed in [35] to the case where the state is infinite-dimensional.

Lemma 12.

(i) *Assume* (A3)–(A7) *for* U_0, f_0, σ_0, *and* L_0 *and* (A14) *hold. Let the pair* $\big(\phi_0(t,\omega,x), \psi_0(t,\omega,x)\big)$ *be the unique solution pair to* (41) *which is* $C^\infty(\mathbb{R}^n)$ *and is a.s. uniformly bounded. In addition, we assume* (A9) *holds for* S_0 *and the resulting* u_0 *is also a.s. Lipschitz continuous in* μ. *Then, for* $\mu_{(\cdot)}(\omega)$ *and* $\nu_{(\cdot)}(\omega)$ *in the set* $\mathcal{M}^\beta_\rho$, $0 < \beta < 1$, *there exists a constant* c_4 *such that*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^n} |u_0(t,\omega,x) - u_0'(t,\omega,x)|^2 \leq c_4 \left(D^\rho_T\left(\mu(\omega), \nu(\omega)\right)\right)^2 \quad a.s., \tag{110}$$

where $u_0, u_0' \in C_{\mathrm{Lip}(x)}\left([0,T] \times \Omega \times \mathbb{R}^n; U_0\right)$ *are induced by the map* $\mathsf{T}^{\mathsf{SHJB}}_0$ *in* (86) *using two stochastic measures* $\mu_{(\cdot)}(\omega)$ *and* $\nu_{(\cdot)}(\omega)$, *respectively.*

(ii) *Assume* (A3)–(A7) *for* U, f, σ, *and* L *and* (A14) *hold. Let* $\big(\phi(t,\omega^o,x,p), \psi(t,\omega^o,x,p)\big)$ *be the unique solution pair to* (62) *which is* $C^\infty\left(\mathbb{R}^n \times G_k\right)$ *and is a.s. uniformly bounded. In addition, we assume* (A9) *holds for* S *and the* *resulting u is also a.s. Lipschitz continuous in its parameters. Then, for $\mu_{(\cdot)}(\omega)$ in the set $\mathcal{M}_\rho^\beta$, $0 < \beta < 1$, and $\mu^0_{(\cdot)}(\omega)$ and $\nu^0_{(\cdot)}(\omega)$ in the set $\mathcal{M}_\rho^\gamma$, $0 < \gamma < 1/2$, there exists a constant c_6 such that*

$$\sup_{(t,x,p)\in[0,T]\times\mathbb{R}^n\times G_k} |u(t,\omega^o,x,p) - u'(t,\omega^o,x,p)|^2 \le c_6 \left(D_T^\rho\left(\mu^0(\omega),\nu^0(\omega)\right)\right)^2 \quad a.s., \tag{111}$$

where $u, u' \in C_{\mathrm{Lip}(x,p)}\left([0,T]\times\Omega\times\mathbb{R}^n\times G_k;U\right)$ are induced by the map $\mathsf{T}^{\mathsf{SHJB}}$ in (94) using two densities $\varphi^{\mu^0}(\cdot)$ and $\varphi^{\nu^0}(\cdot)$, respectively.

Proof. The proof for part (i) is identical to the proof of [35] as the major agent has complete observation of its own state. For (ii), due to assumption (A9), u and u' are Lipschitz continuous in p and the claim follows by Theorem 10 and by the fact that $D_t^\rho(\cdot)$ is increasing in t. ∎

Remark 5.4. The regularity properties of the feedback control laws of the minor agents with respect to the mean field term, $\mu_{(\cdot)}$, is derived in [35] using the sensitivity analysis of SHJB equations which was first developed in [28]. On the other hand such a regularity condition is assumed in [21]. In this work, since the domain of the BSPDEs associated to SHJB equations are infinite-dimensional, the sensitivity analysis is more involved and requires extending the results of [18] so that the existence and uniqueness analysis of BSPDEs with infinite-dimensional domain is established. Such a program is under development and we currently assume the following.

(A15) Assume (A3)–(A7) for U, f, σ, and L and (A12) hold. Let $(\phi(t,x,p),\psi(t,x,p))$ be the unique solution pair to (62) which is a.s. uniformly bounded. In addition, we assume (A9) holds for S and the resulting u is also a.s. Lipschitz continuous in $\mu_{(\cdot)}$. Then, for $\mu^0(\omega) \in \mathcal{M}_\rho^\gamma$, $0 < \gamma < 1/2$, and $\mu_{(\cdot)}(\omega)$ and $\nu_{(\cdot)}(\omega)$ in the set $\mathcal{M}_\rho^\beta$, $0 < \beta < 1$, there exists a constant c_5 such that

$$\sup_{\substack{(t,x,p)\in\\ [0,T]\times\mathbb{R}^n\times G_k}} |u(t,\omega^o,x,p) - u'(t,\omega^o,x,p)|^2 \le c_5 \left(D_T^\rho\left(\mu^m(\omega^o),\nu^m(\omega^o)\right)\right)^2 \quad a.s., \tag{112}$$

where $u, u' \in C_{\mathrm{Lip}(x,p)}\left([0,T]\times\Omega\times\mathbb{R}^n\times G_k;U\right)$ are induced by $\mathsf{T}^{\mathsf{SHJB}}$ in (94) using two stochastic measures $\mu_{(\cdot)}(\omega)$ and $\nu_{(\cdot)}(\omega)$, respectively.

We can now present the main result of the partially observed MM-MFG.

THEOREM 13. *Let the assumptions* (A1)–(A15) *hold. If the constants $\{c_i; 0 \le i \le 7\}$ satisfy the condition that*

$$\max\left\{c_2c_5, c_2c_6c_0c_4, c_2c_6c_1, c_1c_3, c_0c_3c_4\right\} < 1, \tag{113}$$

then there exists a unique solution for the map T and hence a unique solution to the partially observed MM-MFG system.

Proof. The proof is based upon designing a contraction operator for the map given in Figures 1–2 and consequently follows from the Banach fixed point theorem on the Polish space $\mathcal{M}_\rho^\beta$, $0 < \beta < 1$. The fixed point argument can be completed by the gain condition provided in (113) combined with (106)–(108), (110)–(111), and (112). ∎

$$
\begin{array}{ccccc}
\mu_{(\cdot)}(\omega)) & & \overset{\mathsf{T}_0^{\mathrm{SHJB}}}{\longrightarrow} & & u_0(\cdot,\omega,x) \\
\uparrow \mathsf{T}^{\mathrm{SMV}} & & & & \downarrow \mathsf{T}_0^{\mathrm{SMV}} \\
u(\cdot,\omega^o,x,p) & \overset{\mathsf{T}^{\mathrm{SHJB}}}{\longleftarrow} & \varphi(\cdot) & \overset{\mathsf{T}^{\mathrm{NLF}}}{\longleftarrow} & \mu^0_{(\cdot)}(\omega) := \delta_{z_0^o(\cdot,\omega)}
\end{array}
$$

FIG. 2. *Partially observed MM-MFG composition map.*

6. ϵ-Nash equilibrium property of the partially observed SMFG control laws. Let

$$
\begin{aligned}
&\big(\phi_0(t,\omega,x), \psi_0(t,\omega,x), u_0^o(t,\omega,x), z_0^o(t,\omega), \\
&\qquad \phi(t,\omega^o,x,p), \psi(t,\omega^o,x,p), u^o(t,\omega^o,x,p), z^o(t,\omega^o), \varphi(t)\big)
\end{aligned} \tag{114}
$$

be the unique solution of the partially observed MM-MFG system such that the best response processes $u_0^o(t,\omega,x)$, $u^o(t,\omega^o,x,p)$ are a.s. continuous in (t,x), a.s. Lipschitz continuous in x, and $u^o(t,\omega^o,x,p)$ is also a.s. Lipschitz continuous in p. Define

$$
\begin{aligned}
dz_0^o(t) &= f_0\left[t, z_0^o, u_0^o(t,z_0^o), \mu_t(\omega)\right] dt + \sigma_0\left[t, z_0^o, \mu_t(\omega)\right] dw_0(t), \\
dz_i^o(t) &= f\left[t, z_i^o, u^o(t, z_i^o, \varphi_i(t)), \mu_t(\omega)\right] dt + \sigma\left[t, z_i^o, \mu_t(\omega)\right] dw_i(t), \\
dy_i(t) &= h\left(t, z_0^o(t)\right) dt + d\nu_i(t,\omega_i),\ 1 \le i \le N,\ 0 \le t \le T,
\end{aligned} \tag{115}
$$

where $\mu_t(\omega) := \mathcal{L}\left(z_i^o | \mathcal{F}_t^{w_0}\right)$ and $\varphi_i(\cdot)$ is given by (31) for the generic minor agent i.

Let us apply the SMFG best responses $u_0^o(t,\omega,x)$ and $u^o(t,\omega^o,x,p)$ into a finite $N+1$ partially observed major/minor population given by (1)–(2). We obtain the following closed loop individual dynamics:

$$
\begin{aligned}
dz_0^{o,N}(t) &= \frac{1}{N}\sum_{j=1}^N f_0\left(t, z_0^{o,N}(t), u_0^o(t, z_0^{o,N}(t)), z_j^{o,N}(t)\right) dt \\
&\qquad + \frac{1}{N}\sum_{j=1}^N \sigma_0\left(t, z_0^{o,N}(t), z_j^{o,N}(t)\right) dw_0(t),\ z_0^{o,N}(0) = z_0(0), \\
dz_i^{o,N}(t) &= \frac{1}{N}\sum_{j=1}^N f\left(t, z_i^{o,N}(t), u^o(t, z_i^{o,N}(t), \varphi_i^{o,N}(t)), z_j^{o,N}(t)\right) dt \\
&\qquad + \frac{1}{N}\sum_{j=1}^N \sigma\left(t, z_i^{o,N}(t), z_j^{o,N}(t)\right) dw_i(t),\ z_i^{o,N}(0) = z_i(0), \\
dy_i^{o,N}(t) &= h\left(t, z_0^{o,N}(t)\right) dt + d\nu_i(t,\omega_i),\ 1 \le i \le N,\ 0 \le t \le T,
\end{aligned} \tag{116}
$$

where $\varphi_i^{o,N}(\cdot)$ denotes the filtering process (see (31)) corresponding to the observation process $y_i^{o,N}(\cdot)$ for the generic minor agent i. By the Lipschitz continuity of $h(t,x)$ in x and the approximation result in Theorem 1, one can show that

$$
\sup_{i=1,2,\ldots,N} \sup_{0 \le t \le T} \mathbb{E}\left| y_i^{o,N}(t) - y_i(t) \right| = O\left(\frac{1}{\sqrt{N}}\right). \tag{117}
$$

Since in the rest of this section we show that the best response control policies $u_0^o(t,\omega,x)$ and $u^o(t,\omega^o,x,p)$ generate approximate Nash equilibria when applied to a finite population game, we first recall the definition of ϵ-Nash equilibria [21], [35].

DEFINITION 14. *Given $\epsilon > 0$, the admissible control laws $(u_0^o, \dots, u_N^o)$ for $N+1$ agents generate an ϵ-Nash equilibrium with respect to the costs J_i^N, $i = 0, \dots, N$, if $J_i^N(u_i^o; u_{-i}^o) - \epsilon \le \inf_{u_j i \in \mathcal{U}_i} J_i^N(u_i; u_{-i}^o) \le J_i^N(u_i^o; u_{-i}^o)$ for any $0 \le i \le N$ and where $\mathcal{U}_i$ is the set of admissible control laws for agent j.*

Let $\mathcal{F}_t^{z_i, y_i} := \sigma\{z_i(s), y_i(s) : 0 \le s \le t\}$, $1 \le i \le N$, and $\mathcal{F}_t^z := \sigma\{z_i(s) : 0 \le i \le N, 0 \le s \le t\}$. Define

$$\mathcal{U}_0^l := \Big\{ u_0(\cdot, \omega) \in U := u_0\left(\cdot, \omega, z_0(\cdot), \dots, z_N(\cdot)\right) \in C_{\mathrm{Lip}(z_0, \dots, z_N)} : u_0(t) \text{ is}$$

$$\mathcal{F}_t^{w_0}\text{-measurable and } \mathcal{F}_t^z\text{-adapted such that } \mathbb{E} \int_0^T |u(t)|^2 dt < \infty \Big\}, \tag{118}$$

$$\mathcal{U}_i^l := \Big\{ u_i(\cdot, \omega_i) \in U := u_i\left(\cdot, \omega_i, z_i(\cdot), y_i(\cdot)\right) \in C_{\mathrm{Lip}(z_i, y_i)} : u_i(t) \text{ is}$$

$$\mathcal{F}_t^{w_i}\text{-measurable and } \mathcal{F}_t^{z_i, y_i}\text{-adapted such that } \mathbb{E} \int_0^T |u_i(t)|^2 dt < \infty \Big\} \tag{119}$$

as the set of feedback control processes. Notice that whereas the admissible set for the major agent is the full information admissible control, which is not decentralized, the admissible set for the generic minor agent depends only its local information. This is in contrast to the situation where the minor agents have complete observation of the state of the major agent.

THEOREM 15. *Assume* (A1)–(A14) *hold and there exists a unique solution to the partially observed MM-SMFG system given by* (114) *such that the best response control process $u_0^o(t, \omega, x)$ is a.s. continuous in t and a.s. Lipschitz continuous in x, and $u^o(t, x, p)$ is a.s. continuous in t and a.s. Lipschitz continuous in (x, p). Then $(u_0^o, u_1^o, \dots, u_N^o)$, where $u_i^o = u^o$, $0 \le i \le N$, generates an $O\big(\epsilon_N + \frac{1}{\sqrt{N}}\big)$-Nash equilibrium, where $\lim_{N \to \infty} \epsilon_N = 0$, with respect to the cost functions defined in* (4) *such that*

$$J_j^N\left(u_j^o; u_{-j}^o\right) - O\left(\epsilon_N + \frac{1}{\sqrt{N}}\right) \le \inf_{u_j \in \mathcal{U}_j^l} J_j^N\left(u_j; u_{-j}^o\right) \le J_j^N\left(u_j^o; u_{-j}^o\right). \tag{120}$$

Proof. Without loss of generality, assume that the first minor agent changes its best response control strategy from $u^o(t, x, p)$ to $u_1(t, x, y) \in \mathcal{U}_1^l$, while the major agent and other minor agents apply u_0^o and u^o, respectively. Consequently, we obtain

$$\begin{aligned}
dz_0^N(t) &= \frac{1}{N} \sum_{j=1}^N f_0\left(t, z_0^N(t), u_0^o(t, z_0^N(t)), z_j^N(t)\right) dt \\
&\qquad + \frac{1}{N} \sum_{j=1}^N \sigma_0\left(t, z_0^N(t), z_j^N(t)\right) dw_0(t), \;\; z_0^N(0) = z_0(0), \\
dy_1^N(t) &= h\left(t, z_0^N(t)\right) dt + d\nu_1(t, \omega_1), \;\; y_1^N(t) = y_1(0), \\
dz_1^N(t) &= \frac{1}{N} \sum_{j=1}^N f\left(t, u_1(t, z_1^N(t), y_1^N(t)), z_1^N(t), z_j^N(t)\right) dt \\
&\qquad + \frac{1}{N} \sum_{j=1}^N \sigma\left(t, z_1^N(t), z_j^N(t)\right) dw_1(t), \;\; z_1^N(0) = z_1(0),
\end{aligned}$$

$$dz_i^N(t) = \frac{1}{N}\sum_{j=1}^{N} f\left(t, u^o(t, z_i^N(t), \varphi_i^N(t)), z_i^N(t), z_j^N(t)\right) dt$$
$$+ \frac{1}{N}\sum_{j=1}^{N} \sigma\left(t, , z_i^N(t), z_j^N(t)\right) dw_i(t),\ \ z_i^N(0) = z_i(0),\ \ 2 \le i \le N. \tag{121}$$

We define

$$dz_0^o(t) = f_0\left[t, z_0^o, u_0^o(t, z_0^o), \mu_t(\omega)\right] dt + \sigma_0\left[t, z_0^o, \mu_t(\omega)\right] dw_0(t),$$
$$dz_i^o(t) = f\left[t, z_i^o, u^o(t, z_i^o, \varphi_i), \mu_t(\omega)\right] dt + \sigma\left[t, z_i^o, \mu_t(\omega)\right] dw_i(t),$$
$$dy_i^o(t) = h\left(t, z_0^o(t)\right) dt + d\nu_i(t, \omega_i) \tag{122}$$

with initial conditions, $z_i^o(0) = z_i(0)$, $i = 0, \ldots, N$, $y_i^o(t) = y_i(0)$, $i = 1, \ldots, N$, where $\left(\varphi_i(t)\right)_{0 \le t \le T}$ is the solution of (31) for agent i. Following the same approach as that of Theorem 1 and using the boundedness and Lipschitz continuity of h, f, and u^o, one can show that

$$\sup_{i=0,2,\ldots,N}\ \sup_{0\le t\le T} \mathbb{E}\left|z_i^{o,N}(t) - z_i^N(t)\right| = O\left(1/\sqrt{N}\right), \tag{123}$$

$$\sup_{i=0,2,\ldots,N}\ \sup_{0\le t\le T} \mathbb{E}\left|z_i^o(t) - z_i^N(t)\right| = O\left(1/\sqrt{N}\right), \tag{124}$$

$$\sup_{i=0,1,\ldots,N}\ \sup_{0\le t\le T} \mathbb{E}\left|z_i^{o,N}(t) - z_i^o(t)\right| = O\left(1/\sqrt{N}\right), \tag{125}$$

$$\sup_{i=2,\ldots,N}\ \sup_{0\le t\le T} \mathbb{E}\left|y_i^{o,N}(t) - y_i^o(t)\right| = O\left(1/\sqrt{N}\right), \tag{126}$$

$$\sup_{i=2,\ldots,N}\ \sup_{0\le t\le T} \mathbb{E}|y_i^N(t) - y_i^o(t)| = O\left(1/\sqrt{N}\right). \tag{127}$$

Let us now consider the following dynamics

$$d\hat{z}_1^N(t) = \frac{1}{N}\sum_{j=1}^{N} f\left(t, \hat{z}_1^N(t), u_1(t, \hat{z}_1^N(t), y_1^N(t)), z_j^o(t)\right) dt$$
$$+\frac{1}{N}\sum_{j=1}^{N} \sigma\left(t, \hat{z}_1^N(t), z_j^o(t)\right) dw_1(t),\ \ \hat{z}_1^N(0) = z_1(0),\ \ 0 \le t \le T,$$

and by Theorem 1 and Gronwall's lemma, we obtain

$$\sup_{0\le t\le T} \mathbb{E}\left|\hat{z}_1^N(t) - z_1^N(t)\right| = O\left(\frac{1}{\sqrt{N}}\right). \tag{128}$$

Finally, let us define the McKean–Vlasov dynamics

$$d\hat{z}_1(t) = f\left[t, \hat{z}_1, u_1(t, \hat{z}_1(t), y_1^N(t)), \mu_t(\omega)\right] dt + \sigma\left[t, \hat{z}_1, \mu_t(\omega)\right] dw_1(t),\ \ \hat{z}_1(0) = z_1(0),$$

and once again, by Theorem 1 and Gronwall's lemma, we have

$$\sup_{0\le t\le T} \mathbb{E}\left|\hat{z}_1^N(t) - \hat{z}_1(t)\right| = O\left(\frac{1}{\sqrt{N}}\right). \tag{129}$$

Let $z(0) = \int_{\mathbb{R}^n} x dF(x)$ be the mean value of the minor agents' initial states where $F(\cdot)$ is the asymptotic distribution given in (A2), and define

$$(\epsilon_N)^2 = \left| \int_{\mathbb{R}^n} x^T x dF_N(x) - 2z^T(0) \int_{\mathbb{R}^n} x dF_N(x) + z^T(0)z(0) \right|. \tag{130}$$

Observe now that

$$\begin{aligned}
&J_1^N(u_1, u_{-1}^o) \\
&= \mathbb{E} \int_0^T \frac{1}{N} \sum_{j=2}^{N} L\left(t, z_1^N(t), u_1(t, z_1^N(t), y_1^N(t)), z_0^N(t), z_j^N(t)\right) dt \\
&\overset{(123)}{\geq} \mathbb{E} \int_0^T \frac{1}{N} \sum_{j=2}^{N} L\left(t, z_1^N(t), u_1(t, z_1^N(t), y_1^N(t)), z_0^{o,N}(t), z_j^{o,N}(t)\right) dt - O\left(1/\sqrt{N}\right) \\
&\overset{(125)}{\geq} \mathbb{E} \int_0^T \frac{1}{N} \sum_{j=2}^{N} L\left(t, z_1^N(t), u_1(t, z_1^N(t), y_1^N(t)), z_0^o(t), z_j^o(t)\right) dt - O\left(\epsilon_N + 1/\sqrt{N}\right) \\
&\overset{(128)}{\geq} \mathbb{E} \int_0^T \frac{1}{N} \sum_{j=2}^{N} L\left(t, \hat{z}_1^N(t), u_1(t, \hat{z}_1^N(t), y_1^N(t)), z_0^o(t), z_j^o(t)\right) dt - O\left(\epsilon_N + 1/\sqrt{N}\right) \\
&\overset{(129)}{\geq} \mathbb{E} \int_0^T \frac{1}{N} \sum_{j=2}^{N} L\left(t, \hat{z}_1(t), u_1(t, \hat{z}_1(t), y_1^N(t)), z_0^o(t), z_j^o(t)\right) dt - O\left(\epsilon_N + 1/\sqrt{N}\right) \\
&\geq \mathbb{E} \int_0^T L\left[t, \hat{z}_1(t), u_1(t, \hat{z}_1(t), y_1^N(t)), z_0^o(t), \mu_t(\omega)\right] dt - O\left(\epsilon_N + 1/\sqrt{N}\right) \\
&\overset{(127)}{\geq} \mathbb{E} \int_0^T L\left[t, \hat{z}_1(t), u_1(t, \hat{z}_1(t), y_1^o(t)), z_0^o(t), \mu_t(\omega)\right] dt - O\left(\epsilon_N + 1/\sqrt{N}\right),
\end{aligned} \tag{131}$$

where in the second inequality ϵ_N appears due to the fact that the sequence of minor agents' initials $\{z_i^o, 1 \leq i \leq N\}$ in the SMV system (122) is generated by independent randomized observations on the distribution F defined in (A2), and (131) is due to Theorem 1. Notice now that by the construction of minor agents' SMFG (SMV-SHJB equations), we have

$$\begin{aligned}
&\mathbb{E} \int_0^T L\left[t, \hat{z}_1(t), u_1(t, \hat{z}_1(t), y_1^o(t)), z_0^o(t), \mu_t(\omega)\right] dt \geq \\
&\qquad\qquad \mathbb{E} \int_0^T \mathbf{L}\left[t, z_1^o(t), \varphi_1(t), u^o\left(t, z_1^o(t), \varphi_1(t)\right), \mu_t(\omega)\right] dt.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&J_1^N(u_1, u_{-1}^o) \\
&\geq \quad \mathbb{E} \int_0^T \mathbf{L}\left[t, z_1^o(t), \varphi_1(t), u^o\left(t, z_1^o(t), \varphi_1(t)\right), \mu_t(\omega)\right] dt - O\left(\epsilon_N + 1/\sqrt{N}\right) \\
&\overset{(48)}{=} \quad \mathbb{E} \int_0^T L\left[t, z_1^o(t), u^o\left(t, z_1^o(t), \varphi_1(t)\right), z_0^o(t), \mu_t(\omega)\right] dt - O\left(\epsilon_N + 1/\sqrt{N}\right)
\end{aligned}$$

$$
\begin{aligned}
&\overset{(i)}{\geq} \quad \mathbb{E}\int_0^T \frac{1}{N}\sum_{j=1}^N L\left(t, z_1^o(t), u^o\left(t, z_1^o(t), \varphi_1(t)\right), z_0^o(t), z_j^o(t)\right) dt - O\left(\epsilon_N + 1/\sqrt{N}\right)\\
&\overset{(125)}{\geq} \quad \mathbb{E}\int_0^T \frac{1}{N}\sum_{j=1}^N L\left(t, z_1^{o,N}(t), u^o\left(t, z_1^{o,N}(t), \varphi_1(t)\right), z_0^{o,N}(t), z_j^{o,N}(t)\right) dt\\
&\qquad\qquad -O\left(\epsilon_N + 1/\sqrt{N}\right)\\
&\overset{(104),(117)}{\geq} \mathbb{E}\int_0^T \frac{1}{N}\sum_{j=2}^N L\left(t, z_1^{o,N}(t), u^o\left(t, z_1^{o,N}(t), \varphi_1^N(t)\right), z_0^{o,N}(t), z_j^{o,N}(t)\right) dt\\
&\qquad\qquad -O\left(\epsilon_N + 1/\sqrt{N}\right)\\
&\equiv \quad J_1^N\left(u_1^o, u_{-1}^o\right) - O\left(\epsilon_N + 1/\sqrt{N}\right),
\end{aligned}
$$

where (i) is due to Theorem 1. The above analysis holds for any $u_1 \in \mathcal{U}_1^l$, and hence $J_1^N\left(u_1^o, u_{-1}^o\right) - O\left(\epsilon_N + 1/\sqrt{N}\right) \leq \inf_{u_1\in\mathcal{U}_1^l} J_1^N(u_1, u_{-1}^o)$, which proves the desired result. The analysis for the major agent follows similar arguments. ∎

7. Conclusions. In this work we have extended MM-MFG theory to the case where the minor agents partially observe the state of the major agent in a distributed manner. For such an extension, we first derived nonlinear filtering equations for the major agent's state. Next, the generic partially observed stochastic optimal control problem (SOCP) that the minor agents need to solve is converted into a fully observed one by the application of the nonlinear filtering theory and the separation principle.

The completely observed stochastic optimal control problem have two distinct features that makes it nonstandard. On the one hand, due to the existence of a stochastic measure in the cost function, the control problem belongs to the class of SOCP with random parameters, which is studied by Peng [36]; on the other hand, the state process consists of an infinite-dimensional object. An initial analysis of such systems is provided in section 5.2, where we also discussed the BSDE that the value function satisfies under the optimal control.

By obtaining sufficient conditions on the system such that a map from the random measures of minor agents back to itself is a contraction operator on the space of random probability measures, we obtain the existence and uniqueness of solutions to the stochastic MM-MFG system. Finally, it is demonstrated that the best response control processes obtained via stochastic MM-MFG systems generates an approximate Nash equilibria for the finite population game.

Appendix A. Proof of Theorem 1.

Proof. The proof is an extension of [35, Theorem 3.1] to the case where control laws are $\zeta_i(\cdot)$ dependent. Observe first that due to the observation modeling, the $\zeta_i(t)$, $1 \leq i \leq N$, terms are assumed to be conditionally independent given $\mathcal{F}_t^{w_0}$. Let us start with the major agent. We first have

$$
\begin{aligned}
&\mathbb{E}\left|\hat{z}_0^N(t) - \bar{z}_0(t)\right|^2\\
&\leq 2t\mathbb{E}\left(\int_0^t \left|\frac{1}{N}\sum_{j=1}^N f_0\left(s, \hat{z}_0^N(s), \eta_0(s, \hat{z}_0^N), \hat{z}_j^N(s)\right) - f_0[s, \bar{z}_0(s), \eta_0(s, \bar{z}_0), \mu_s(\omega)]\right|^2 ds\right)\\
&\quad + 2\mathbb{E}\left(\int_0^t \left|\frac{1}{N}\sum_{j=1}^N \sigma_0(s, \hat{z}_0^N(s), \hat{z}_j^N(s)) - \sigma_0[s, \bar{z}_0(s), \mu_s(\omega)]\right|^2 ds\right).
\end{aligned} \tag{132}
$$

We first consider the first expression on the right-hand side of (132). We have

$$
\begin{aligned}
&\frac{1}{N}\sum_{j=1}^{N} f_0\left(s, \hat{z}_0^N(s), \eta_0(s,\hat{z}_0^N), \hat{z}_j^N(s)\right) - f_0[s, \bar{z}_0(s), \eta_0(s,\bar{z}_0), \mu_s(\omega)] \\
&\qquad = \frac{1}{N}\sum_{j=1}^{N} f_0\left(s, \hat{z}_0^N(s), \eta_0(s,\hat{z}_0^N), \hat{z}_j^N(s)\right) - \frac{1}{N}\sum_{j=1}^{N} f_0[s, \bar{z}_0(s), \eta_0(s,\bar{z}_0), \hat{z}_j^N(s)] \\
&\qquad + \frac{1}{N}\sum_{j=1}^{N} f_0\left(s, \bar{z}_0, \eta_0(s,\bar{z}_0), \hat{z}_j^N(s)\right) - \frac{1}{N}\sum_{j=1}^{N} f_0[s, \bar{z}_0(s), \eta_0(s,\bar{z}_0), \bar{z}_j(s)] \\
&\qquad + \frac{1}{N}\sum_{j=1}^{N} f_0\left(s, \bar{z}_0, \eta_0(s,\bar{z}_0), \bar{z}_j(s)\right) - f_0[s, \bar{z}_0(s), \eta_0(s,\bar{z}_0), \mu_s(\omega)], \qquad (133)
\end{aligned}
$$

and since f_0 and η_0 are Lipschitz continuous, we have

$$
\begin{aligned}
&\mathbb{E}\biggl(\int_0^t \left|\frac{1}{N}\sum_{j=1}^{N} f_0\left(s, \hat{z}_0^N(s), \eta_0(s,\hat{z}_0^N), \hat{z}_j^N(s)\right) - f_0[s, \bar{z}_0(s), \eta_0(s,\bar{z}_0), \mu_s(\omega)]\right|^2 ds\biggr) \\
&\leq 3C\int_0^t \mathbb{E}\left|\hat{z}_0^N(s) - \bar{z}_0(s)\right|^2 ds + 3C\int_0^t \mathbb{E}\left|\frac{1}{N}\sum_{j=1}^{N} \hat{z}_j^N(s) - \bar{z}_j(s)\right|^2 ds \\
&\quad + 3C\int_0^t \mathbb{E}\left|\frac{1}{N}\sum_{j=1}^{N} f_0\left(s, \bar{z}_0(s), \eta_0(s,\bar{z}_0), \bar{z}_j(s)\right) - f_0\left[s, \bar{z}_0(s), \eta_0(s,\bar{z}_0), \mu_s(\omega)\right]\right|^2 ds, \qquad (134)
\end{aligned}
$$

where C is a constant independent of N. Considering the last term, for $0 \leq s \leq t$, let

$$
g_0\left(s, \bar{z}_j(s)\right) := f_0\left(s, \bar{z}_0(s), \eta_0(s,\bar{z}_0), \bar{z}_j(s)\right) - f_0\left[s, \bar{z}_0(s), \eta_0(s,\bar{z}_0), \mu_s(\omega)\right]
$$

and observe that for $j \neq k$, $\zeta_j(s)$ and $\zeta_k(s)$—and consequently $\bar{z}_j(s)$ and $\bar{z}_k(s)$—are conditionally independent given $\mathcal{F}_s^{w_0}$. This implies $\mathbb{E}\left[g_0\left(s, \bar{z}_j(s)\right) g_0\left(s, \bar{z}_k(s)\right)\right] = \mathbb{E}\bigl(\mathbb{E}\left[g_0\left(s, \bar{z}_j(s)\right) g_0\left(s, \bar{z}_k(s)\right) | \mathcal{F}_s^{w_0}\right]\bigr) = 0$. Further, since $\mu_s(\omega) := \mathcal{L}(\bar{z}_k(s)|\mathcal{F}_s^{w_0}) = \mathcal{L}(\bar{z}_l(s)|\mathcal{F}_s^{w_0})$, there are no cross terms in the last term of (134), and since f_0 is bounded

$$
3C\int_0^t \mathbb{E}\left|\frac{1}{N}\sum_{j=1}^{N} f_0\left(s, \bar{z}_0(s), \eta_0(s,\bar{z}_0), \bar{z}_j(s)\right) - f_0\left[s, \bar{z}_0(s), \eta_0(s,\bar{z}_0), \mu_s(\omega)\right]\right|^2 ds \leq \frac{k_1(t)}{N},
$$

where $k_1(t)$ is an increasing function of t and is independent of N. For the second term in (133), note that $(\sum_{i=1}^N x_i)^2 \leq N(\sum_{i=1}^N x_i^2)$; we obtain

$$
\begin{aligned}
&\mathbb{E}\biggl(\int_0^t \left|\frac{1}{N}\sum_{j=1}^{N} f_0\left(s, \hat{z}_0^N(s), \eta_0(s,\hat{z}_0^N), \hat{z}_j^N(s)\right) - f_0\left[s, \bar{z}_0(s), \eta_0(s,\bar{z}_0), \mu_s(\omega)\right]\right|^2\biggr) \\
&\leq 3C\int_0^t \mathbb{E}\left|\hat{z}_0^N(s) - \bar{z}_0(s)\right|^2 ds + \frac{3C}{N}\int_0^t \sum_{j=1}^{N} \mathbb{E}\left|\hat{z}_j^N(s) - \bar{z}_j(s)\right|^2 ds + \frac{k_1(t)}{N}. \qquad (135)
\end{aligned}
$$

Following similar steps, we obtain

$$\mathbb{E}\biggl(\int_0^t \Bigl|\frac{1}{N}\sum_{j=1}^N \sigma_0(s,\hat{z}_0^N(s),\hat{z}_j^N(s)) - \sigma_0[s,\bar{z}_0(s),\mu_s(\omega)]\Bigr|^2 ds\biggr)$$

$$\leq 3C\int_0^t \mathbb{E}\bigl|\hat{z}_0^N(s)-\bar{z}_0(s)\bigr|^2 ds + \frac{3C}{N}\int_0^t \mathbb{E}\Bigl|\sum_{j=1}^N \hat{z}_j^N(s)-\bar{z}_j(s)\Bigr|^2 ds + \frac{k_1(t)}{N}. \tag{136}$$

By (133), (135), and (136), we have

$$\sup_{0\leq t\leq T}\mathbb{E}\bigl|\hat{z}_0^N(t)-\bar{z}_0(t)\bigr|^2 \leq 6C(T+1)\int_0^T \mathbb{E}\bigl|\hat{z}_0^N(s)-\bar{z}_0(s)\bigr|^2 ds$$

$$+\frac{6C(T+1)}{N}\int_0^T\sum_{j=1}^N \mathbb{E}\bigl|\hat{z}_j^N(s)-\bar{z}_j(s)\bigr|^2 ds + \frac{2(T+1)k_1(T)}{N}. \tag{137}$$

Consider now the ith minor agent. We first have

$$\mathbb{E}\bigl|\hat{z}_i^N(t)-\bar{z}_i(t)\bigr|^2$$

$$\leq 2t\mathbb{E}\biggl(\int_0^t \Bigl|\frac{1}{N}\sum_{j=1}^N f\left(s,\hat{z}_i^N,\eta(s,\hat{z}_i^N,\zeta_i),\hat{z}_j^N\right) - f\left[s,\eta(s,\bar{z}_i,\zeta_i),\bar{z}_i,\mu_s(\omega)\right]\Bigr|^2 ds\biggr)$$

$$+2\mathbb{E}\biggl(\int_0^t \Bigl|\frac{1}{N}\sum_{j=1}^N \sigma(s,\hat{z}_i^N,\hat{z}_j^N) - \sigma\left[s,\bar{z}_i,\mu_s(\omega)\right]\Bigr|^2 ds\biggr). \tag{138}$$

To continue, consider the first term in (138) and notice that

$$\frac{1}{N}\sum_{j=1}^N f\left(s,\hat{z}_i^N,\eta(s,\hat{z}_i^N,\zeta_i),\hat{z}_j^N\right) - f\left[s,\bar{z}_i,\eta(s,\bar{z}_i,\zeta_i),\mu_s(\omega)\right]$$

$$= \frac{1}{N}\sum_{j=1}^N f\left(s,\hat{z}_i^N,\eta(s,\hat{z}_i^N,\zeta_i),\hat{z}_j^N\right) - \frac{1}{N}\sum_{j=1}^N f\left(s,\bar{z}_i,\eta(s,\bar{z}_i,\zeta_i),\hat{z}_j^N\right)$$

$$+\frac{1}{N}\sum_{j=1}^N f\left(s,\bar{z}_i,\eta(s,\bar{z}_i,\zeta_i),\hat{z}_j^N\right) - \frac{1}{N}\sum_{j=1}^N f\left(s,\bar{z}_i,\eta(s,\bar{z}_i,\zeta_i),\bar{z}_j\right)$$

$$+\frac{1}{N}\sum_{j=1}^N f\left(s,\bar{z}_i,\eta(s,\bar{z}_i,\zeta_i),\bar{z}_j\right) - f\left[s,\bar{z}_i,\eta(s,\bar{z}_i,\zeta_i),\mu_s(\omega)\right]. \tag{139}$$

Due to the Lipschitz continuity of the functions $f(\cdot)$ and $\eta(\cdot)$, we have

$$\mathbb{E}\biggl(\int_0^t \Bigl|\frac{1}{N}\sum_{j=1}^N f\left(s,\hat{z}_i^N,\eta(s,\hat{z}_i^N,\zeta_i),\hat{z}_j^N\right) - f\left[s,\bar{z}_i,\eta(s,\bar{z}_i,\zeta_i),\mu_s(\omega)\right]\Bigr|^2 ds\biggr)$$

$$\leq 3C\int_0^t \mathbb{E}\bigl|\hat{z}_i^N(s)-\bar{z}_i(s)\bigr|^2 ds + 3C\int_0^t \mathbb{E}\Bigl|\frac{1}{N}\sum_{j=1}^N \hat{z}_j^N(s)-\bar{z}_j(s)\Bigr|^2 ds$$

$$+3C\int_0^t \mathbb{E}\Bigl|\frac{1}{N}\sum_{j=1}^N f\left(s,\bar{z}_i,\eta(s,\bar{z}_i,\zeta_i),\bar{z}_j\right) - f\left[s,\bar{z}_i,\eta(s,\bar{z}_i,\zeta_i),\mu_s(\omega)\right]\Bigr|^2 ds, \tag{140}$$

and as in the major agent's analysis, the critical step is to show that there are no cross terms in the last term in (140). Let

$$\begin{aligned}\mathcal{G}_s &= \sigma\{w_i(\tau), w_j(\tau), w_k(\tau), w_0(\tau), \nu_i(\tau), \nu_j(\tau), \nu_k(\tau);\ 0 \le \tau \le s\},\\ g\left(s, \bar{z}_j(s)\right) &:= f\left(s, \bar{z}_i, \eta(s, \bar{z}_i, \zeta_i), \bar{z}_j\right) - f\left[s, \bar{z}_i, \eta(s, \bar{z}_i, \zeta_i), \mu_s(\omega)\right]\end{aligned}$$

and observe that for $j \neq k$

$$\begin{aligned}&\mathbb{E}\left[g\left(s, \bar{z}_j(s)\right) g\left(s, \bar{z}_k(s)\right)\right] = \mathbb{E}\left[\mathbb{E}_{\mathcal{G}_s}\left[g\left(s, \bar{z}_j(s)\right) g\left(s, \bar{z}_k(s)\right)\right]\right]\\ &\quad= \mathbb{E}\bigg[\mathbb{E}_{\mathcal{G}_s}\left[f\left(s, \bar{z}_i, \eta(s, \bar{z}_i, \zeta_i), \bar{z}_j\right) f\left(s, \bar{z}_i, \eta(s, \bar{z}_i, \zeta_i), \bar{z}_k\right)\right]\\ &\qquad - \mathbb{E}_{\mathcal{G}_s}\left[f\left(s, \bar{z}_i, \eta(s, \bar{z}_i, \zeta_i), \bar{z}_j\right) f\left[s, \bar{z}_i, \eta(s, \bar{z}_i, \zeta_i), \mu_s(\omega)\right]\right]\\ &\qquad - \mathbb{E}_{\mathcal{G}_s}\left[f\left(s, \bar{z}_i, \eta(s, \bar{z}_i, \zeta_i), \bar{z}_k\right) f\left[s, \bar{z}_i, \eta(s, \bar{z}_i, \zeta_i), \mu_s(\omega)\right]\right]\\ &\qquad + \mathbb{E}_{\mathcal{G}_s}\left[f\left[s, \bar{z}_i, \eta(s, \bar{z}_i, \zeta_i), \mu_s(\omega_i)\right] f\left[s, \bar{z}_i, \eta(s, \bar{z}_i, \zeta_i), \mu_s(\omega)\right]\right]\bigg]\\ &\quad= \mathbb{E}\bigg[\mathbb{E}_{\mathcal{G}_s}\left[f\left(s, \bar{z}_i, \eta(s, \bar{z}_i, \zeta_i), \bar{z}_j\right)\right] \mathbb{E}_{\mathcal{G}_s}\left[f\left(s, \bar{z}_i, \eta(s, \bar{z}_i, \zeta_i), \bar{z}_k\right)\right]\\ &\qquad - f\left[s, \bar{z}_i, \eta(s, \bar{z}_i, \zeta_i), \mu_s(\omega)\right] \mathbb{E}_{\mathcal{G}_s}\left[f\left(s, \bar{z}_i, \eta(s, \bar{z}_i, \zeta_i), \bar{z}_j\right)\right]\\ &\qquad - f\left[s, \bar{z}_i, \eta(s, \bar{z}_i, \zeta_i), \mu_s(\omega)\right] \mathbb{E}_{\mathcal{G}_s}\left[f\left(s, \bar{z}_i, \eta(s, \bar{z}_i, \zeta_i), \bar{z}_k\right)\right]\\ &\qquad + f\left[s, \bar{z}_i, \eta(s, \bar{z}_i, \zeta_i), \mu_s(\omega)\right] f\left[s, \bar{z}_i, \eta(s, \bar{z}_i, \zeta_i), \mu_s(\omega)\right]\bigg] = 0,\end{aligned}\tag{141}$$

where (141) is valid since $f\left[s, \eta(s, \bar{z}_i, \zeta_i), \bar{z}_i, \mu_s(\omega)\right]$ is $\mathcal{G}_s$-measurable and $\bar{z}_j$ and $\bar{z}_k$ are conditionally independent given $\mathcal{G}_s$. To verify the last step, note first that

$$\mathbb{E}_{\mathcal{G}_s}\left[f\left(s, \bar{z}_i, \eta(s, \bar{z}_i, \zeta_i), \bar{z}_k\right)\right] = \mathbb{E}_{\mathcal{G}_s}\left[f\left(s, \bar{z}_i, \eta(s, \bar{z}_i, \zeta_i), \bar{z}_i\right)\right], \tag{142}$$

since z_i and z_k have the same laws. Moreover, since $\mu_s(\omega) := \mathcal{L}(\bar{z}_i(s)|\mathcal{F}_s^{w_0})$,

$$\mathbb{E}_{\mathcal{G}_s}\left[f\left[s, \bar{z}_i, \eta(s, \bar{z}_i, \zeta_i), \mu_s(\omega)\right]\right] = \mathbb{E}_{\mathcal{G}_s}\left[f\left(s, \bar{z}_i, \eta(s, \bar{z}_i, \zeta_i), \bar{z}_i\right)\right],$$

which gives (141). To continue, recall that f is bounded, and therefore

$$4C\int_0^t \mathbb{E}\left|\frac{1}{N}\sum_{j=1}^N f\left(s, \bar{z}_i, \eta(s, \bar{z}_i, \zeta_i), \bar{z}_j\right) - f\left[s, \bar{z}_i, \eta(s, \bar{z}_i, \zeta_i), \mu_s\right]\right|^2 ds \le \frac{k(t)}{N}, \tag{143}$$

where $k(t) > 0$ independent of N and increasing in time. Combining (139)–(143) as well as the analysis for the diffusion term, we obtain that for the ith minor agent

$$\begin{aligned}\sup_{0\le t\le T} \mathbb{E}\left|\hat{z}_i^N(t) - \bar{z}_i(t)\right|^2 &\le 8C(T+1)\int_0^T \mathbb{E}\left|\hat{z}_i^N(s) - \bar{z}_i(s)\right|^2 ds + \frac{k(T)}{N}\\ &+8C(T+1)\left(\frac{1}{N}\int_0^T \sum_{j=1}^N \mathbb{E}\left|\hat{z}_j^N(s) - \bar{z}_j(s)\right|^2 ds\right).\end{aligned}\tag{144}$$

The proof now follows from (137) and (144) and Gronwall's lemma. ∎

Appendix B. Proof of the Itô–Kunita lemma for density-valued processes. To prove Theorem 3, we need the following lemma, which considers the regularity of stochastic integrals for functions whose domain is infinite-dimensional.

LEMMA 16. *Let $M(t)$ be a real-valued continuous local martingale, and let Y be a G_k-valued $\mathcal{F}_0$-measurable random variable. Let $\Delta : [0,T] \times G_k \to \mathbb{R}$ such that $\Delta(s,\lambda)$ is continuous in (s,λ) and continuously differentiable in λ. Then it holds that*

$$\int_0^t \Delta(s,\lambda)dM(s)\big|_{\lambda=Y} = \int_0^t \Delta(s,Y)dM(s). \tag{145}$$

Proof. Notice that by Taylor expansion

$$\int_0^t \Delta(s,\lambda)dM(s)\big|_{\lambda=Y} = \int_0^t \big(\Delta(s,Y) + \mathsf{D}_\lambda\Delta(s,\lambda+\varrho(\lambda-Y))\cdot(\lambda-Y)\big)dM(s)\big|_{\lambda=Y}$$

for some $0 < \rho < 1$, where by Riesz representation there exists a kernel $D_\lambda\Delta(s,\lambda)$ which is continuous in its arguments and satisfies that

$$\mathsf{D}_\lambda\Delta(s,\lambda+\varrho(\lambda-Y))\cdot(\lambda-Y) = \int_{\mathbb{R}^n} D_\lambda\Delta(s,\lambda+\varrho(\lambda-Y))(z)(\lambda-Y)(z)dz. \tag{146}$$

Consequently,

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^n} D_\lambda\Delta(s,\lambda+\varrho(Y-\lambda))(z)(\lambda-Y)(z)dzdM(s)\big|_{\lambda=Y} \\
&\quad = \int_{\mathbb{R}^n} (\lambda-Y)(z) \int_0^t D_\lambda\Delta(s,\lambda+\varrho(Y-\lambda))(z)dM(s)dz\big|_{\lambda=Y} \\
&\quad = 0 \quad \text{a.s.,}
\end{aligned} \tag{147}$$

where the first step follows from Fubini's theorem and the last step completes the proof by following [29, Corollary 7.9]. ∎

Proof of Theorem 3. For simplicity, we take $m = 1$. Following the proof of Itô's lemma for finite-dimensional spaces (see, e.g., [27, Theorem 3.3]), we first employ a localization procedure so that one can restrict f_i, κ_i, ζ_i, and hence α_i, $i = 1,\dots,d$, to be bounded. Hence, let $M_t^1 = \int_0^t \kappa_i(s)dW(s)$, $M_t^2 = \int_0^t \zeta_i(s)dB(s)$, and $N_t = \int_0^t f(s)ds$. We introduce a stopping time for each n as follows:

$$T_n := \begin{cases} 0 & \text{if } |\alpha(0)| \geq n, \\ \inf\big\{t \geq 0; |M_t^1| \geq n \text{ or } |M_t^2| \geq n \text{ or } \check{N}_t \geq n \text{ or } \langle M^1\rangle_t \geq n \\ \qquad\qquad \text{or } \langle M^1\rangle_t \geq n\big\} & \text{if } |\alpha(0)| < n, \\ \infty & \text{if } |\alpha(0)| < n \text{ and } \{\cdots\} = \varnothing, \end{cases} \tag{148}$$

where $\check{N}_t$ denotes the total variation of the process N_t and $\langle M^i\rangle_t$, $i = 1,2$, denotes the quadratic variations of the process M_t^i, respectively, in $[0,t]$. The resulting sequence is nondecreasing with $\lim_{n\to\infty} T_n = \infty$ P-a.s. Hence, if (58) can be proven for the stopped process $\alpha^n(t) := \alpha(t \wedge T_n)$, $t \geq 0$, then the desired result can be obtained upon letting $n \to \infty$. Consequently, one can assume that $\alpha(0)$, M_t^1, M_t^2, $\check{N}_t$, $\langle M^1\rangle_t$, and $\langle M^2\rangle_t$ are all bounded by a common constant K which implies $|\alpha(t)| < 4K$, $0 \leq t \leq T$.

Now take a partition $\Pi = \{0 = t_0 < t_1 < \cdots < t_n = t\}$ of the interval $[0,t]$, denote its mesh by $\|\Pi\| := \max_{0\leq i\leq n-1} |t_{i+1} - t_i|$, and write

$$\begin{aligned}
&\Delta\big(t,\alpha(t),\beta(t,z)\big) - \Delta\big(0,\alpha(0),\beta(0,z)\big) \\
&= \sum_{i=0}^{n-1} \Delta\big(t_{i+1},\alpha(t_i),\beta(t_i,z)\big) - \Delta\big(t_i,\alpha(t_i),\beta(t_i,z)\big)
\end{aligned}$$

$$
\begin{aligned}
&\quad +\Delta\big(t_{i+1},\alpha(t_{i+1}),\beta(t_{i+1},z)\big)-\Delta\big(t_{i+1},\alpha(t_i),\beta(t_i,z)\big)\\
&=\sum_{i=0}^{n-1}\Delta\big(t_{i+1},\alpha(t_i),\beta(t_i,z)\big)-\Delta\big(t_i,\alpha(t_i),\beta(t_i,z)\big)\\
&+\sum_{i=0}^{n-1}\mathsf{D}_{xp}\Delta\left(t_{i+1},\alpha(t_i),\beta(t_i,z)\right)\cdot\big((\alpha(t_{i+1}),\beta(t_{i+1},z))-(\alpha(t_i),\beta(t_i,z))\big)\\
&+\frac{1}{2}\sum_{i=1}^{n-1}\mathsf{D}^2_{xpxp}\Delta\big(t_{i+1},(\alpha(t_i),\beta(t_i,z))+\varrho_i\left((\alpha(t_{i+1}),\beta(t_{i+1},z))-(\alpha(t_i),\beta(t_i,z))\right)\big)\\
&\quad\cdot\big[\big((\alpha(t_{i+1}),\beta(t_{i+1},z))-(\alpha(t_i),\beta(t_i,z))\big),\big((\alpha(t_{i+1}),\beta(t_{i+1},z))-(\alpha(t_i),\beta(t_i,z))\big)\big]
\end{aligned}
\tag{149}
$$

$$
\begin{aligned}
&=\sum_{i=0}^{n-1}\Delta\big(t_{i+1},\alpha(t_i),\beta(t_i,z)\big)-\Delta\big(t_i,\alpha(t_i),\beta(t_i,z)\big)\\
&+\sum_{i=0}^{n-1}\big(\mathsf{D}_{xp}\Delta\left(t_{i+1},\alpha(t_i),\beta(t_i,z)\right)-\mathsf{D}_{xp}\Delta\left(t_i,\alpha(t_i),\beta(t_i,z)\right)\big)\\
&\qquad\qquad\cdot\big((\alpha(t_{i+1}),\beta(t_{i+1},z))-(\alpha(t_i),\beta(t_i,z))\big)\\
&+\sum_{i=0}^{n-1}\mathsf{D}_{xp}\Delta\left(t_i,\alpha(t_i),\beta(t_i,z)\right)\cdot\big(\alpha(t_{i+1})-\alpha(t_i),\beta(t_{i+1},z)-\beta(t_i,z)\big)\\
&+\frac{1}{2}\sum_{i=0}^{n-1}\mathsf{D}^2_{xpxp}\Delta\big(t_{i+1},(\alpha(t_i),\beta(t_i,z))+\varrho_i\left((\alpha(t_{i+1}),\beta(t_{i+1},z))-(\alpha(t_i),\beta(t_i,z))\right)\big)\\
&\quad\cdot\big[\big((\alpha(t_{i+1}),\beta(t_{i+1},z))-(\alpha(t_i),\beta(t_i,z))\big),\big((\alpha(t_{i+1}),\beta(t_{i+1},z))-(\alpha(t_i),\beta(t_i,z))\big)\big],
\end{aligned}
\tag{150}
$$

where $\varrho_i\in(0,1)$ and (149) follows from Taylor's formula for normed vector spaces. Set $\chi^i_{\alpha,\beta}:=\big((\alpha(t_{i+1}),\beta(t_{i+1},z))-(\alpha(t_i),\beta(t_i,z))\big)$, $\chi^i_\alpha:=\big(\alpha(t_{i+1})-\alpha(t_i)\big)$, $\chi^i_\beta:=\big(\beta(t_{i+1},z)-\beta(t_i,z)\big)$, and $\chi^i_t=t_{i+1}-t_i$. Using the properties of the Fréchet derivative,

$$
\begin{aligned}
&\mathsf{D}_{xp}\Delta\left(t_{i+1},\alpha(t_i),\beta(t_i,z)\right)\cdot\chi^i_{\alpha,\beta}=D_x\Delta\left(t_{i+1},\alpha(t_i),\beta(t_i,z)\right)\chi^i_\alpha\\
&\qquad\qquad\qquad\qquad\qquad\qquad+\mathsf{D}_p\Delta\left(t_{i+1},\alpha(t_i),\beta(t_i,z)\right)\cdot\chi^i_\beta,\\
&\mathsf{D}^2_{xpxp}\Delta\big(t_{i+1},(\alpha(t_i),\beta(t_i,z))+\varrho_i\chi^i_{\alpha,\beta}\big)\cdot\left[\chi^i_{\alpha,\beta},\chi^i_{\alpha,\beta}\right]\\
&\qquad\qquad=D^2_{xx}\Delta\big(t_{i+1},(\alpha(t_i),\beta(t_i,z))+\varrho_i\chi^i_{\alpha,\beta}\big)\cdot[\chi^i_\alpha,\chi^i_\alpha]\\
&\qquad\qquad+D_x\mathsf{D}_p\Delta\big(t_{i+1},(\alpha(t_i),\beta(t_i,z))+\varrho_i\chi^i_{\alpha,\beta}\big)\cdot[\chi^i_\alpha,\chi^i_\beta]\\
&\qquad\qquad+\mathsf{D}_pD_x\Delta\big(t_{i+1},(\alpha(t_i),\beta(t_i,z))+\varrho_i\chi^i_{\alpha,\beta}\big)\cdot[\chi^i_\beta,\chi^i_\alpha]\\
&\qquad\qquad+\mathsf{D}^2_{pp}\Delta\big(t_{i+1},(\alpha(t_i),\beta(t_i,z))+\varrho_i\chi^i_{\alpha,\beta}\big)\cdot[\chi^i_\beta,\chi^i_\beta].
\end{aligned}
\tag{151}
$$

Hence, let

$$
\begin{aligned}
\mathcal{I}_1\left(\|\Pi\|\right)&:=\sum_{i=1}^{n-1}\Delta\big(t_{i+1},\alpha(t_i),\beta(t_i,z)\big)-\Delta\big(t_i,\alpha(t_i),\beta(t_i,z)\big),\\
\mathcal{I}_2\left(\|\Pi\|\right)&:=\sum_{i=1}^{n-1}\big(D_x\Delta\left(t_{i+1},\alpha(t_i),\beta(t_i,z)\right)-D_x\Delta\left(t_i,\alpha(t_i),\beta(t_i,z)\right)\big)\chi^i_\alpha,
\end{aligned}
$$

$$\mathcal{I}_3\left(\|\Pi\|\right) := \sum_{i=1}^{n-1} \big(\mathsf{D}_p\Delta\left(t_{i+1},\alpha(t_i),\beta(t_i,z)\right) - \mathsf{D}_p\Delta\left(t_i,\alpha(t_i),\beta(t_i,z)\right)\big)\cdot\chi^i_\beta,$$

$$\mathcal{I}_4\left(\|\Pi\|\right) := \sum_{i=1}^{n-1} D_x\Delta\left(t_i,\alpha(t_i),\beta(t_i,z)\right)\chi^i_\alpha,$$

$$\mathcal{I}_5\left(\|\Pi\|\right) := \sum_{i=1}^{n-1} \mathsf{D}_p\Delta\left(t_i,\alpha(t_i),\beta(t_i,z)\right)\cdot\chi^i_\beta,$$

$$\mathcal{I}_6\left(\|\Pi\|\right) := \frac{1}{2}\sum_{i=1}^{n-1} D^2_{xx}\Delta\big(t_{i+1},(\alpha(t_i),\beta(t_i,z)) + \varrho_i\chi^i_{\alpha,\beta}\big)\cdot[\chi^i_\alpha,\chi^i_\alpha],$$

$$\mathcal{I}_7\left(\|\Pi\|\right) := \frac{1}{2}\sum_{i=1}^{n-1} D_x\mathsf{D}_p\Delta\big(t_{i+1},(\alpha(t_i),\beta(t_i,z)) + \varrho_i\chi^i_{\alpha,\beta}\big)\cdot[\chi^i_\alpha,\chi^i_\beta],$$

$$\mathcal{I}_8\left(\|\Pi\|\right) := \frac{1}{2}\sum_{i=1}^{n-1} \mathsf{D}_p\mathsf{D}_x\Delta\big(t_{i+1},(\alpha(t_i),\beta(t_i,z)) + \varrho_i\chi^i_{\alpha,\beta}\big)\cdot[\chi^i_\beta,\chi^i_\alpha],$$

$$\mathcal{I}_9\left(\|\Pi\|\right) := \frac{1}{2}\sum_{i=1}^{n-1} \mathsf{D}^2_{pp}\Delta\big(t_{i+1},(\alpha(t_i),\beta(t_i,z)) + \varrho_i\chi^i_{\alpha,\beta}\big)\cdot[\chi^i_\beta,\chi^i_\beta], \tag{152}$$

and consequently we will investigate each of the summations above as $\|\Pi\|\to 0$. To start with, consider $\mathcal{I}_1\left(\|\Pi\|\right)$ and note that it directly follows that

$$\lim_{\|\Pi\|\to 0}\sum_{i=1}^{n-1}\int_{t_i}^{t_{i+1}} Q\big(s,x,p\big)ds\big|_{(x=\alpha(t_i),p=\beta(t_i,z))} = \int_0^t Q\big(s,\alpha(s),\beta(s,z)\big)ds \tag{153}$$

since the term in the summation has bounded variation. Consider now the local martingale term and note that

$$\int_{t_i}^{t_{i+1}}\Sigma\big(s,x,p\big)dW(s) = \int_{t_i}^{t_{i+1}}\mathsf{D}_{xp}\Sigma\big(s,(\alpha(t_i),\beta(t_i,z)) + \varrho(x-\alpha(t_i),p-\beta(t_i,z))\big)$$

$$\cdot(x-\alpha(t_i),p-\beta(t_i,z))dW(s) + \int_{t_i}^{t_{i+1}}\Sigma\big(s,\alpha(t_i),\beta(t_i,z)\big)dW(s) \tag{154}$$

for some $0<\varrho<1$. Considering the terms in (154), we observe that

$$\begin{aligned}&\mathsf{D}_{xp}\Sigma\big(s,\alpha(t_i),\beta(t_i,z)\big)\cdot(x-\alpha(t_i),p-\beta(t_i,z))\\ &\quad = D_x\Sigma\big(s,\alpha(t_i),\beta(t_i,z)\big)\left(x-\alpha(t_i)\right) + \mathsf{D}_p\Sigma\big(s,\alpha(t_i),\beta(t_i,z)\big)\cdot(p-\beta(t_i,z))\end{aligned}$$

and consequently

$$\int_{t_i}^{t_{i+1}} D_x\Sigma\big(s,\alpha(t_i),\beta(t_i,z)\big)(x-\alpha(t_i))dW(s)\big|_{x=\alpha(t_i)} = 0 \quad \text{a.s.,} \tag{155}$$

$$\int_{t_i}^{t_{i+1}} \mathsf{D}_p\Sigma\big(s,\alpha(t_i),\beta(t_i,z)\big)\cdot(p-\beta(t_i,z))dW(s)\big|_{p=\beta(t_i,z)} = 0 \quad \text{a.s.,} \tag{156}$$

where (155) and (156) follow from the regularity of Itô integrals derived in [29, Corollary 7.9] and Lemma 16, and therefore we have that

$$\sum_{i=1}^{n-1}\int_{t_i}^{t_{i+1}}\Sigma\big(s,x,p\big)dW(s)\big|_{(x=\alpha(t_i),p=\beta(t_i,z))} = \sum_{i=1}^{n-1}\int_{t_i}^{t_{i+1}}\Sigma\big(s,\alpha(t_i),\beta(t_i,z)\big)dW(s).$$

Let $T_n(s) := \Sigma\big(s, \alpha(t_i), \beta(t_i, z)\big)$ and note that by the continuity of Σ, α, and β we have $T_n(s) \xrightarrow[\|\Pi\|\to 0]{\text{a.s.}} T(s) := \Sigma\big(s, \alpha(s), \beta(s, z)\big)$ for almost all $s \in [0, t]$. Consequently, $\int_0^t |T_n(s) - T(s)|^2 ds \to 0$ in probability as $\|\Pi\| \to 0$, and therefore

$$\lim_{\|\Pi\|\to 0} \mathcal{I}_1\left(\|\Pi\|\right) = \int_0^t Q\big(s, \alpha(s), \beta(s, z)\big)ds + \int_0^t \Sigma\big(s, \alpha(s), \beta(s, z)\big)dW(s). \tag{157}$$

Before proceeding further, we note that in the rest of the proof we only consider the local martingale part of the process $\Delta(\cdot)$, that is, $\int_0^t \Sigma(s)dW(s)$, since the analysis of the finite-variation part is standard. Hence, we let $Q(t, x, p) = 0$ for simplicity of notation.

Consider next $\mathcal{I}_2\left(\|\Pi\|\right)$, which can be written as

$$\mathcal{I}_2\left(\|\Pi\|\right) = \sum_{i=1}^{n-1}\sum_{j=1}^{d}\Big(\partial_{x_j}\Delta\left(t_{i+1}, \alpha(t_i), \beta(t_i, z)\right) - \partial_{x_j}\Delta\left(t_i, \alpha(t_i), \beta(t_i, z)\right)\Big)\chi_\alpha^i. \tag{158}$$

Observe that for every $p \in G_k$ it holds that

$$\partial_{x_j}\Delta\left(t_{i+1}, x, p\right) - \partial_{x_j}\Delta\left(t_i, x, p\right) = \int_{t_i}^{t_{i+1}} \partial_{x_j}\Sigma(s, x, p)dW(s). \tag{159}$$

Furthermore, since $\partial_{x_j}\Sigma(s, x, p)$ is continuous in (s, x, p), hence one can use the regularity lemmas to obtain

$$\partial_{x_j}\left(\Delta\left(t_{i+1}, \alpha(t_i), \beta(t_i, z)\right) - \Delta\left(t_i, \alpha(t_i), \beta(t_i, z)\right)\right) = \int_{t_i}^{t_{i+1}} \partial_{x_j}\Sigma(s, \alpha(t_i), \beta(t_i, z))dW(s),$$

and as a result $\mathcal{I}_2\left(\|\Pi\|\right) = \mathcal{I}_{2,1}\left(\|\Pi\|\right) + \mathcal{I}_{2,2}\left(\|\Pi\|\right) + \mathcal{I}_{2,3}\left(\|\Pi\|\right)$ with

$$\begin{aligned}
\mathcal{I}_{2,1}\left(\|\Pi\|\right) &:= \sum_{i=1}^{n-1}\sum_{j=1}^{d}\int_{t_i}^{t_{i+1}} \partial_{x_j}\Sigma(s, \alpha(t_i), \beta(t_i, z))dW(s)\left[\int_{t_i}^{t_{i+1}} f_j(s)ds\right],\\
\mathcal{I}_{2,2}\left(\|\Pi\|\right) &:= \sum_{i=1}^{n-1}\sum_{j=1}^{d}\int_{t_i}^{t_{i+1}} \partial_{x_j}\Sigma(s, \alpha(t_i), \beta(t_i, z))dW(s)\left[\int_{t_i}^{t_{i+1}} \kappa_j(s)dW(s)\right],\\
\mathcal{I}_{2,3}\left(\|\Pi\|\right) &:= \sum_{i=1}^{n-1}\sum_{j=1}^{d}\int_{t_i}^{t_{i+1}} \partial_{x_j}\Sigma(s, \alpha(t_i), \beta(t_i, z))dW(s)\left[\int_{t_i}^{t_{i+1}} \zeta_j(s)dB(s)\right].
\end{aligned} \tag{160}$$

Following standard steps, one can show that $\lim_{\|\Pi\|\to 0}\mathcal{I}_{2,k}\left(\|\Pi\|\right) = 0$, $k = 1, 3$, and that

$$\lim_{\|\Pi\|\to 0}\mathcal{I}_2\left(\|\Pi\|\right) = \lim_{\|\Pi\|\to 0}\mathcal{I}_{2,2}\left(\|\Pi\|\right) := \sum_{j=1}^{d}\int_0^t \partial_{x_j}\Sigma(s, \alpha(t_i), \beta(t_i, z))\kappa_j(s)ds. \tag{161}$$

Next consider $\mathcal{I}_3\left(\|\Pi\|\right) + \mathcal{I}_5\left(\|\Pi\|\right)$. We have

$$\begin{aligned}
\mathcal{I}_3\left(\|\Pi\|\right) + \mathcal{I}_5\left(\|\Pi\|\right) &= \sum_{i=1}^{n-1}\mathsf{D}_p\Delta\left(t_{i+1}, \alpha(t_i), \beta(t_i, z)\right)\cdot\chi_\beta^i\\
&= \sum_{i=1}^{n-1}\mathsf{D}_p\Delta\left(t_{i+1}, \alpha(t_i), \beta(t_i, z)\right)\cdot\left[\int_{t_i}^{t_{i+1}} g(s, z)ds + \int_{t_i}^{t_{i+1}} \varsigma(s, z)dY(s)\right],
\end{aligned} \tag{162}$$

where $\mathsf{D}_p\Delta(s,x,p)\cdot\eta := \int_{\mathbb{R}^n} D_p\Delta(s,x,p)(z)\eta(z)dz,\ \eta(\cdot)\in G_{k-1}$, which is assumed to satisfy $|D_p\Delta(s,x,p)(z)| \le \gamma_p(s,x,\|p\|_l)(1+|z|^l)$ for some γ_p, which is continuous on $[0,T]\times\mathbb{R}^n\times\mathbb{R}_+$, and therefore, by [2, Theorem 6.1],

$$\sum_{i=1}^{n-1}\mathsf{D}_p\Delta\left(t_{i+1},\alpha(t_i),\beta(t_i,z)\right)\cdot\left[\int_{t_i}^{t_{i+1}} g(s,z)ds+\int_{t_i}^{t_{i+1}}\varsigma(s,z)dY(s)\right]\xrightarrow{\|\Pi\|\to 0}$$

$$\int_0^t \mathsf{D}_p\Delta\left(s,\alpha(s),\beta(s,z)\right)\cdot g(s,z)ds+\int_0^t \mathsf{D}_p\Delta\left(s,\alpha(s),\beta(s,z)\right)\cdot\varsigma(s,z)dY(s) \tag{163}$$

in probability. The analysis of $\mathcal{I}_4\left(\|\Pi\|\right)$ and $\mathcal{I}_6\left(\|\Pi\|\right)$ follows immediately from the Itô–Kunita lemma [29, Theorem 8.1] for finite-dimensional spaces, and we have

$$\mathcal{I}_4\left(\|\Pi\|\right)\xrightarrow{\|\Pi\|\to 0}\sum_{j=1}^{d}\int_0^t\frac{\partial}{\partial x_j}\Delta\left(s,\alpha(s),\beta(s,z)\right)\big(f_j(s)ds+\kappa_j(s)dW(s)+\zeta_j(s)dB(s)\big)$$

$$\mathcal{I}_6\left(\|\Pi\|\right)\xrightarrow{\|\Pi\|\to 0}\frac{1}{2}\sum_{j=1}^{d}\int_0^t\frac{\partial^2}{\partial x_j x_j}\Delta\left(s,\alpha(s),\beta(s,z)\right)\big(\kappa_j^2(s)+\zeta_j^2(s)\big)ds \tag{164}$$

in quadratic variation.

We now analyze the last three terms. Consider first $\mathcal{I}_7\left(\|\Pi\|\right)$ and note that

$$\begin{aligned}
\mathcal{I}_7\left(\|\Pi\|\right) &= \frac{1}{2}\sum_{i=1}^{n-1}\sum_{j=1}^{d}\frac{\partial}{\partial x_j}\left(\mathsf{D}_p\Delta\big(t_{i+1},(\alpha(t_i),\beta(t_i,z))+\varrho_i\chi^i_{\alpha,\beta}\big)\cdot\chi^i_\beta\right)\chi^i_\alpha\\
&= \frac{1}{2}\sum_{i=1}^{n-1}\sum_{j=1}^{d}\frac{\partial}{\partial x_j}\left(\int_{\mathbb{R}^n} D_p\Delta\big(t_{i+1},(\alpha(t_i),\beta(t_i,z))+\varrho_i\chi^i_{\alpha,\beta}\big)(\acute{z})\chi^i_\beta(\acute{z})d\acute{z}\right)\chi^i_\alpha \qquad (165)\\
&= \frac{1}{2}\sum_{i=1}^{n-1}\sum_{j=1}^{d}\frac{\partial}{\partial x_j}\left(\int_{\mathbb{R}^n} D_p\Delta\big(t_{i+1},(\alpha(t_i),\beta(t_i,z))+\varrho_i\chi^i_{\alpha,\beta}\big)(\acute{z})\right.\\
&\qquad\qquad\left.\times\left[\int_{t_i}^{t_{i+1}} g(s,\acute{z})ds+\int_{t_i}^{t_{i+1}}\varsigma(s,\acute{z})dY(s)\right]d\acute{z}\right)\chi^i_\alpha,
\end{aligned}$$

where (165) is due to (52). Let $\mathcal{I}_7\left(\|\Pi\|\right) := \sum_{j=1}^d \mathcal{I}^j_{7,1}+\mathcal{I}^j_{7,2}$, where

$$\mathcal{I}^j_{7,1} := \frac{1}{2}\sum_{i=1}^{n-1}\frac{\partial}{\partial x_j}\int_{\mathbb{R}^n} D_p\Delta\big(t_{i+1},(\alpha(t_i),\beta(t_i,z))+\varrho_i\chi^i_{\alpha,\beta}\big)(\acute{z})\int_{t_i}^{t_{i+1}} g(s,\acute{z})dsd\acute{z}\chi^i_\alpha,$$

$$\mathcal{I}^j_{7,2} := \frac{1}{2}\sum_{i=1}^{n-1}\frac{\partial}{\partial x_j}\int_{\mathbb{R}^n} D_p\Delta\big(t_{i+1},(\alpha(t_i),\beta(t_i,z))+\varrho_i\chi^i_{\alpha,\beta}\big)(\acute{z})\int_{t_i}^{t_{i+1}}\varsigma(s,\acute{z})dY(s)d\acute{z}\chi^i_\alpha.$$

We further partition the above sums. Let $\mathcal{I}^j_{7,1}\left(\|\Pi\|\right) := 2(\mathcal{I}^{j,a}_{7,1}+\mathcal{I}^{j,b}_{7,1}+\mathcal{I}^{j,c}_{7,1})$, where $\mathcal{I}^{C,j}_{7,1} := \sum_{i=1}^{n-1}\frac{\partial}{\partial x_j}\int_{\mathbb{R}^n} D_p\Delta\big(t_{i+1},(\alpha(t_i),\beta(t_i,z))+\varrho_i\chi^i_{\alpha,\beta}\big)(\acute{z})\int_{t_i}^{t_{i+1}} g(s,\acute{z})dsd\acute{z}$ and $\mathcal{I}^{j,a}_{7,1} := \mathcal{I}^{C,j}_{7,1}\int_{t_i}^{t_{i+1}} f_j(\acute{s})d\acute{s}$, $\mathcal{I}^{j,b}_{7,1} := \mathcal{I}^{C,j}_{7,1}\int_{t_i}^{t_{i+1}}\kappa_j(\acute{s})dW(\acute{s})$ and $\mathcal{I}^{j,c}_{7,1} := \mathcal{I}^{C,j}_{7,1}\int_{t_i}^{t_{i+1}}\zeta_j(\acute{s})dB(\acute{s})$. Consider now $\mathcal{I}^{j,a}_{7,1}$ and observe that

$$\left|\mathcal{I}^{j,a}_{7,1}\right| \le \sum_{i=1}^{n-1}\int_{\mathbb{R}^n}\left|\frac{\partial}{\partial x_j}D_p\Delta\big(t_{i+1},(\alpha(t_i),\beta(t_i,z))+\varrho_i\chi^i_{\alpha,\beta}\big)(\acute{z})\right|$$

$$\times \int_{t_i}^{t_{i+1}} g(s,\acute{z})dsd\acute{z} \int_{t_i}^{t_{i+1}} f_j(\acute{s})d\acute{s} \tag{166}$$

$$\overset{(53)}{\leq} \sum_{i=1}^{n-1} \int_{\mathbb{R}^n} \left|\gamma_{xp}\big(t_{i+1}, \alpha(t_i) + \varrho_i \chi^i_{t,\alpha}, \|\beta(t_i) + \varrho_i \chi^i_{t,\beta}\|_l\big)\right| (1 + |\acute{z}|^l)$$

$$\times \int_{t_i}^{t_{i+1}} g(s,\acute{z})dsd\acute{z} \int_{t_i}^{t_{i+1}} f_j(\acute{s})d\acute{s} \tag{167}$$

$$\leq K\|\Pi\| \max_{0\leq s\leq t} \gamma_{xp}\big(s, \alpha(s), \|\beta(s,z)\|_l\big) \int_0^t \int_{\mathbb{R}^n} (1+|\acute{z}|^l)\,|g(s,\acute{z})|\,d\acute{z}ds \tag{168}$$

$$\xrightarrow[\|\Pi\|\to 0]{P} 0 \tag{169}$$

where (168) follows by Fubini's theorem and by the localization step in (148) and, finally, (169) follows from the assumption that $g(t,\acute{z}) \in \mathcal{M}_{l,1}[\mathcal{F}_T]$. In a similar way,

$$\begin{aligned}
\left|\mathcal{I}^{j,b}_{7,1}\right| &\overset{(53)}{\leq} \sum_{i=1}^{n-1} \int_{\mathbb{R}^n} \left|\gamma_{xp}\big(t_{i+1}, \alpha(t_i) + \varrho_i \chi^i_{t,\alpha}, \|\beta(t_i) + \varrho_i \chi^i_{t,\beta}\|_l\big)\right| \big(1 + |\acute{z}|^l\big) \\
&\quad \times \int_{t_i}^{t_{i+1}} |g(s,\acute{z})|\,dsd\acute{z} \left|\int_{t_i}^{t_{i+1}} \kappa_j(\acute{s})dW(\acute{s})\right| \\
&\leq \max_{0\leq s\leq t} \gamma_{xp}\big(s, \alpha(s), \|\beta(s,z)\|_l\big) \sum_{i=1}^{n-1} \int_{\mathbb{R}^n} (1+|\acute{z}|^l) \int_{t_i}^{t_{i+1}} |g(s,\acute{z})|\,dsd\acute{z}
\end{aligned}$$

$$\times \left|\int_{t_i}^{t_{i+1}} \kappa_j(\acute{s})dW(\acute{s})\right| \tag{170}$$

$$\left|\mathcal{I}^{j,b}_{7,1}\right|^2 \leq \|\Pi\| \max_{0\leq s\leq t} \gamma_{xp}\big(s, \alpha(s), \|\beta(s,z)\|_l\big)^2 \int_0^t \left(\int_{\mathbb{R}^n} (1+|\acute{z}|^l)\,|g(s,\acute{z})|\,d\acute{z}\right)^2 ds$$

$$\times \sum_{i=1}^{n-1} \left|\int_{t_i}^{t_{i+1}} \kappa_j(\acute{s})dW(\acute{s})\right|^2 \xrightarrow[\|\Pi\|\to 0]{P} 0 \tag{171}$$

where (171) follows from Fubini's theorem and by the fact that the term $\sum_{i=1}^{n-1} \big|\int_{t_i}^{t_{i+1}} \kappa_j(\acute{s})dW(\acute{s})\big|^2$ is bounded in probability, uniformly over all subdivisions of $[0,t]$. One can similarly show that $\lim_{\|\Pi\|\to 0} \mathcal{I}^{j,c}_{7,1} = 0$ in probability.

We now consider $\mathcal{I}^j_{7,2}$ and let $\mathcal{I}^j_{7,2}(\|\Pi\|) = 2(\mathcal{I}^{j,a}_{7,2} + \mathcal{I}^{j,b}_{7,2} + \mathcal{I}^{j,c}_{7,2})$, where

$$\begin{aligned}
&\mathcal{I}^{C,j}_{7,2} := \int_{\mathbb{R}^n} D_p\Delta\big(t_{i+1}, (\alpha(t_i), \beta(t_i,z)) + \varrho_i \chi^i_{\alpha,\beta}\big)(\acute{z}) \int_{t_i}^{t_{i+1}} \varsigma(s,\acute{z})dY(s)d\acute{z}, \\
&\mathcal{I}^{j,a}_{7,2} := \sum_{i=1}^{n-1} \frac{\partial}{\partial x_j} \mathcal{I}^{C,j}_{7,2} \int_{t_i}^{t_{i+1}} f_j(\acute{s})d\acute{s}, \quad \mathcal{I}^{j,b}_{7,2} := \sum_{i=1}^{n-1} \frac{\partial}{\partial x_j} \mathcal{I}^{C,j}_{7,2} \int_{t_i}^{t_{i+1}} \kappa_j(\acute{s})dW(\acute{s}),
\end{aligned}$$

$$\mathcal{I}^{j,c}_{7,2} := \sum_{i=1}^{n-1} \frac{\partial}{\partial x_j} \mathcal{I}^{C,j}_{7,2} \int_{t_i}^{t_{i+1}} \zeta_j(\acute{s})dB(\acute{s}). \tag{172}$$

We start with $\mathcal{I}^{j,a}_{7,2}$. Note that

$$\left|\mathcal{I}^{j,a}_{7,2}\right| = \sum_{i=1}^{n-1} \frac{\partial}{\partial x_j} \mathcal{I}^{C,j}_{7,2} \int_{t_i}^{t_{i+1}} f_j(\acute{s})d\acute{s} \leq \|\Pi\| K \sum_{i=1}^{n-1} \frac{\partial}{\partial x_j} \mathcal{I}^{C,j}_{7,2} \xrightarrow[\|\Pi\|\to 0]{P} 0, \tag{173}$$

where (173) follows by the localization step and by considering the fact that

$$\sum_{i=1}^{n-1} \mathcal{I}_{7,2}^{C,j} \xrightarrow[\|\Pi\|\to 0]{P} \int_0^t \mathsf{D}_p\big(s,\alpha(s),\beta(s,z)\big)\cdot[\zeta(s,z)]dY(s), \tag{174}$$

which follows from [2, Theorem 6.1]. For the analysis of the term $\mathcal{I}_{7,2}^{j,b}$, we note that

$$\begin{aligned}\left|\mathcal{I}_{7,2}^{j,b}\right| &\le \max_{0\le s\le t}\gamma_{xp}\big(s,\alpha(s),\|\beta(s,z)\|_l\big)\\ &\qquad\times\left|\sum_{i=1}^{n-1}\int_{t_i}^{t_{i+1}}\kappa_j(\acute{s})dW(\acute{s})\int_{\mathbb{R}^n}\big(1+|\acute{z}|^l\big)\int_{t_i}^{t_{i+1}}\varsigma(s,\acute{z})dY(s)d\acute{z}\right|\\ &\le K\max_{0\le s\le t}\gamma_{xp}\big(s,\alpha(s),\|\beta(s,z)\|_l\big)\sum_{i=1}^{n-1}\Bigg\{\\ &\quad\int_{\mathbb{R}^n}\big(1+|\acute{z}|^l\big)\left|\int_{t_i}^{t_{i+1}}dW(\acute{s})\int_{t_i}^{t_{i+1}}\varsigma(s,\acute{z})dY(s)\right|d\acute{z}\Bigg\}\xrightarrow[\|\Pi\|\to 0]{P} 0\end{aligned} \tag{175}$$

where (175) follows from Fubini's theorem and from the assumption that $\varsigma(t,z)\in\mathcal{M}_{l,2}[\mathcal{F}_t]$. Following similar steps, one can show that $\mathcal{I}_{7,2}^{j,c}\xrightarrow[\|\Pi\|\to 0]{P} 0$, and this completes the proof that

$$\mathcal{I}_7\left(\|\Pi\|\right)\xrightarrow[\|\Pi\|\to 0]{P} 0. \tag{176}$$

We now analyze $\mathcal{I}_8\left(\|\Pi\|\right)$, which can be written as

$$\mathcal{I}_8\left(\|\Pi\|\right)=\frac{1}{2}\sum_{i=1}^{n-1}\sum_{j=1}^{d}\chi_\alpha^i\mathsf{D}_p\left(\frac{\partial}{\partial x_j}\Delta\big(t_{i+1},(\alpha(t_i),\beta(t_i,z))+\varrho_i\chi_{\alpha,\beta}^i\big)\right)\cdot\chi_\beta^i, \tag{177}$$

and consequently the analysis in the term $\mathcal{I}_7\left(\|\Pi\|\right)$ follows and we obtain

$$\mathcal{I}_8\left(\|\Pi\|\right)\xrightarrow[\|\Pi\|\to 0]{P} 0. \tag{178}$$

We finally analyze $\mathcal{I}_9\left(\|\Pi\|\right)$. Notice first that the kernel for the second order Fréchet derivative, $D_p^2(t,x,p)(\acute{z},\bar{z})$, is continuous in its arguments and satisfies (51) and (54). Therefore, by [2, Theorem 6.1],

$$\begin{aligned}\mathcal{I}_9\left(\|\Pi\|\right)&=\sum_{i=1}^{n-1}\mathsf{D}_{pp}^2\Delta\big(t_{i+1},(\alpha(t_i),\beta(t_i,z))+\varrho_i\chi_{\alpha,\beta}^i\big)\cdot[\chi_\beta^i,\chi_\beta^i]\\ &\xrightarrow[\|\Pi\|\to 0]{P}\int_0^t\mathsf{D}_{pp}^2\Delta\big(s,(\alpha(s),\beta(s,z)\big)\cdot[\varsigma(s,z),\varsigma(s,z)]ds.\end{aligned} \tag{179}$$

The proof is now complete by (153), (157), (161), (163), (164), (176), (178), and (179). ∎

Appendix C. Proof of Lemma 11.

Proof. The proofs of the claims in (i) and (ii) directly follow from [35, Lemma 6.9] since the major agent has complete state observation. Hence, one can prove the claims with $c_0=c_1=2C_1T^2\exp(2C_0T)$ for some C_0 and C_1 which are obtained by the

Lipschitz coefficients of f_0. In order to prove (iii), note first that for any bounded and Lipschitz continuous function ϕ on $\mathbb{R}^n$ with a Lipschitz constant $K > 0$, we have

$$\begin{aligned}\mathbb{E}\left|\int_{\mathbb{R}^n}\phi(x)\mu_t(\omega,dx)-\int_{\mathbb{R}^n}\phi(x)\mu_s(\omega,dx)\right| \\ &= \mathbb{E}\left|\mathbb{E}_\omega\left(\phi\left(z^o(t,\omega,\omega',\omega^o)\right)-\phi\left(z^o(s,\omega,\omega',\omega^o)\right)\right)\right| \\ &\le K\mathbb{E}\left|\mathbb{E}_\omega\left(z^o(t,\omega,\omega',\omega^o)-z^o(s,\omega,\omega',\omega^o)\right)\right|, \end{aligned} \tag{180}$$

where $\left(\mu_{(\cdot)}, z^o(\cdot)\right)$ is a unique consistent solution. In this case, we also have

$$\mathbb{E}_\omega\left(z^o(t,\omega,\omega',\omega^o)-z^o(s,\omega,\omega',\omega^o)\right)=\int_s^t f[\tau,z^o,u^o,\mu_\tau(\omega)]d\tau$$

due to the properties of Itô integral. Now let $\mu^0_{(\cdot)}(\omega) \in \mathcal{M}^\gamma_\rho$, where $0 < \gamma < 1/2$ and $(\mu(\omega),\nu(\omega)) \in \mathcal{M}^\beta_\rho \times \mathcal{M}^\beta_\rho$, $0 < \beta < 1$, are induced by the map $\mathsf{T}^{\mathsf{SMV}}$ in (98) using u and $u' \in C_{\mathrm{Lip}(x,p)}\left([0,T]\times\Omega\times\mathbb{R}^n\times G_k;U\right)$, respectively. Denote the filtering equation associated to $\mu^0_{(\cdot)}$ by $\varphi^\mu(t)$. We have

$$\begin{aligned} z^o\left(t,\omega,\omega',\omega^o\right) &= z^o(0)+\int_0^t\left(\int_{\mathbb{R}^n} f(s,z^o,u(s,z^o,\varphi^\mu),y)d\mu_s(\omega)(y)\right)ds \\ &\quad+\int_0^t\left(\int_{\mathbb{R}^n}\sigma(s,z^o,y)d\mu_s(\omega)(y)\right)dw(s,\omega'), \\ x^o\left(t,\omega,\omega',\omega^o\right) &= z^o(0)+\int_0^t\left(\int_{\mathbb{R}^n} f(s,x^o,u'(s,x^o,\varphi^\mu),y)d\nu_s(\omega)(y)\right)ds \\ &\quad+\int_0^t\left(\int_{\mathbb{R}^n}\sigma(s,x^o,y)d\nu_s(\omega)(y)\right)dw(s,\omega'), \end{aligned}$$

where $z^o(\cdot)$ and $x^o(\cdot)$ denote the SMV equations induced by the map $\mathsf{T}^{\mathsf{SMV}}$ in (98) using the two control processes u and u', respectively. By the Lipschitz continuity of f and σ, there exist positive constants C_0, C_1, and C_2 such that

$$\begin{aligned}\mathbb{E}_\omega|z^o\left(s,\omega,\omega',\omega^o\right)-x^o\left(s,\omega,\omega',\omega^o\right)|^2 &\le \left(4C_0s+4C_1\right)\mathbb{E}_\omega\int_0^s|z^o\left(\tau\right)-x^o\left(\tau\right)|^2d\tau \\ &+\left(4C_0s+4C_1\right)\mathbb{E}_\omega\int_0^s D^\rho_\tau\left(\mu(\omega),\nu\left(\omega\right)\right)^2d\tau \\ &+4C_2s^2\mathbb{E}_\omega\sup_{(s,x,p)\in[0,T]\times\mathbb{R}^n\times G_k}|u(s,x,p)-u'(s,x,p)|^2. \end{aligned} \tag{181}$$

By Gronwall's lemma,

$$\begin{aligned}\rho_t\left(z^o\left(t,\omega\right),x^o\left(t,\omega\right)\right) &\le \left(4C_0t+4C_1\right)\exp\left(4C_0t+4C_1\right)\int_0^t\left(D^\rho_\tau\left(\mu(\omega),\nu\left(\omega\right)\right)\right)^2d\tau \\ &+4C_2t^2\exp\left(4C_0t+4C_1\right)\sup_{(t,x,p)\in[0,T]\times\mathbb{R}^n\times G_k}|u(t,\omega^o,x,p)-u'(t,\omega^o,x,p)|^2. \end{aligned} \tag{182}$$

By the definition of Wasserstein metric $D^\rho_{(\cdot)}$ this yields

$$D^\rho_T\left(\mu(\omega),\nu\left(\omega\right)\right)^2 \le K(T)\int_0^T\left(D^\rho_\tau\left(\mu(\omega),\nu\left(\omega\right)\right)\right)^2d\tau$$

$$+ K'(T) \sup_{(t,x,p)\in[0,T]\times\mathbb{R}^n\times G_k} |u(t,\omega^o,x,p) - u'(t,\omega^o,x,p)|^2, \tag{183}$$

with $K(T) = 4(C_0T + C_1)\exp(4(C_0T + C_1))$, $K'(T) = 4C_2T^2 \exp\ (4C_0T + 4C_1)$. Applying Gronwall's lemma one more time yields

$$\begin{aligned} &D_T^\rho(\mu(\omega), \nu(\omega))^2 \\ &\leq K'(T)\exp(K(T)) \sup_{(s,x,p)\in[0,T]\times\mathbb{R}^n\times G_k} |u(s,\omega^o,x,p) - u'(s,\omega^o,x,p)|^2. \end{aligned} \tag{184}$$

Consequently, we have $c_2 = K'(T)\exp(K(T))$ for some C_0, C_1, and C_2 which are obtained by the Lipschitz coefficients of f and σ.

We now prove the last item. Recall that $\mu_t^0(\omega) = \delta_{z_0^o(t,\omega)}$ and $\nu_t^0(\omega) = \delta_{z_0^{o'}(t,\omega)}$, where $z_0^o(t,\omega)$ and $z_0^{o'}(t,\omega)$ are the solutions of the major agent's SMV. Denote the filtering processes associated to $\mu_{(\cdot)}^0$ and $\nu_{(\cdot)}^0$ by $\varphi^\mu(\cdot)$ and $\varphi^\nu(\cdot)$, respectively. We have

$$\begin{aligned} z^o(t,\omega,\omega',\omega^o) &= z^o(0) + \int_0^t \left(\int_{\mathbb{R}^n} f(s,z^o,u^o(s,z^o,\varphi^\mu),y)d\mu_s(\omega)(y)\right)ds \\ &\quad + \int_0^t \left(\int_{\mathbb{R}^n} \sigma(s,z^o,y)d\mu_s(\omega)(y)\right)dw(s,\omega'), \\ x^o(t,\omega,\omega',\omega^o) &= z^o(0) + \int_0^t \left(\int_{\mathbb{R}^n} f(s,x^o,u^o(s,x^o,\varphi^\nu),y)d\nu_s(\omega)(y)\right)ds \\ &\quad + \int_0^t \left(\int_{\mathbb{R}^n} \sigma(s,x^o,y)d\nu_s(\omega)(y)\right)dw(s,\omega), \end{aligned}$$

where $z^o(\cdot)$ and $x^o(\cdot)$ correspond to the solution of SMV given by (44) associated to $\mu^0(\omega)$ and $\nu^0(\omega)$, $\mu_t(\omega) = \mathcal{L}(z^o(t)|\mathcal{F}_t^{w_0})$ and $\nu_t(\omega) = \mathcal{L}(x^o(t)|\mathcal{F}_t^{w_0})$. We next have

$$\begin{aligned} &\mathbb{E}_\omega \sup_{0\leq s\leq t} |z^o(s,\omega,\omega^o,\omega') - x^o(s,\omega,\omega^o,\omega')|^2 \\ &\quad \leq 2t\int_0^t \left|\int_{\mathbb{R}^n} f(s,z^o,u^o,y)\,d\mu_s(\omega)(y) - \int_{\mathbb{R}^n} f(s,x^o,u^o,y)\,d\nu_s(\omega)(y)\right|^2 ds \\ &\quad + 2\int_0^t \left|\int_{\mathbb{R}^n} \sigma(s,z^o,y)\,d\mu_s(\omega)(y) - \int_{\mathbb{R}^n} \sigma(s,x^o,y)\,d\nu_s(\omega)(y)\right|^2 ds. \end{aligned} \tag{185}$$

Furthermore,

$$\begin{aligned} &\left|\int_{\mathbb{R}^n} f(s,z^o,u^o,y)\,d\mu_s(\omega)(y) - \int_{\mathbb{R}^n} f(s,x^o,u^o,y)\,d\nu_s(\omega)(y)\right|^2 \\ &\leq \Bigg|K_1(|z^o(s) - x^o(s)| + \|\varphi^\mu(s) - \varphi^\nu(s)\|_{G_k}) \\ &\qquad + K_2\int_{C_\rho\times C_\rho} (z_s(\omega_1) - z_s(\omega_2))\,d\gamma(\omega_1,\omega_2)\Bigg|^2 \qquad (186) \\ &\leq 3K_1^2\left(|z^o(s) - x^o(s)|^2 + \|\varphi^\mu(s) - \varphi^\nu(s)\|_{G_k}^2\right) \\ &\qquad + 3K_2^2\int_{C_\rho\times C_\rho} |z_s(\omega_1) - z_s(\omega_2)|^2\,d\gamma(\omega_1,\omega_2), \qquad (187) \end{aligned}$$

where in (186) K_1 and K_2 follow by the Lipschitz continuity of f and u^o, and $\gamma \in \mathcal{M}(C_\rho\times C_\rho)$ is any coupling of the two measures μ and ν where $\gamma\big(A\times C([0,T];\mathbb{R}^n)\big) =$

$\mu(A)$ and $\gamma\big(C\left([0,T];\mathbb{R}^n\right);A\big) = \nu(A)$ for any Borel set $A \in C\left([0,T];\mathbb{R}^n\right)$. Taking the infimum over all such $\gamma(\cdot)$ and following similar steps for the diffusion term, we obtain that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} f\left(s, z^o, u^o, y\right) d\mu_s(\omega)(y) - \int_{\mathbb{R}^n} f\left(s, x^o, u^o, y\right) d\nu_s(\omega)(y)\right|^2 \\
&\qquad \leq 3K_1^2\left(\rho_s\left(z^o(s), x^o(s)\right) + \|\varphi^\mu(s) - \varphi^\nu(s)\|^2_{G_k}\right) + 3K_2^2 D_s^\rho(\mu,\nu)^2, \\
&\left|\int_{\mathbb{R}^n} \sigma\left(s, z^o, y\right) d\mu_s(\omega)(y) - \int_{\mathbb{R}^n} \sigma\left(s, x^o, y\right) d\nu_s(\omega)(y)\right|^2 \\
&\qquad \leq 2C_1\big(\rho_s\left(z^o(s), x^o(s)\right) + D_s^\rho(\mu,\nu)^2\big).
\end{aligned} \tag{188}$$

It now follows from (185)–(188) that

$$\begin{aligned}
&\mathbb{E}_\omega \rho_t\big(z^o\left(\omega\right), x^o\left(\omega\right)\big) = \mathbb{E}_\omega \sup_{0\leq s\leq t} |z^o\left(s,\omega\right) - x^o\left(s,\omega\right)|^2 \wedge 1 \\
&\quad \leq 6K_{\max}t\left(\int_0^t \left(\rho_s\left(z^o(s), x^o(s)\right) + \|\varphi^\mu(s) - \varphi^\nu(s)\|^2_{G_k} + D_s^\rho\left(\mu,\nu\right)^2\right) ds\right) \\
&\qquad\qquad + 4C_1\left(\int_0^t \left(\rho_s\left(z^o(s), x^o(s)\right) + D_s^\rho\left(\mu,\nu\right)^2\right) ds\right),
\end{aligned} \tag{189}$$

where $K_{\max} = \max\{K_1^2, K_2^2\}$. Applying Gronwall's lemma to (189) yields

$$\begin{aligned}
&D_t^\rho\big(\mu,\nu\big)^2 \leq \mathbb{E}_\omega \rho_t\big(z^o\left(\omega\right), x^o\left(\omega\right)\big) \leq \big(6K_{\max}t + 4C_1\big)\exp\left(6K_{\max}t + 4C_1\right) \\
&\left(\big(6K_{\max}t + 4C_1\big)\int_0^t D_s^\rho\left(\mu,\nu\right)^2 ds + 6K_{\max}t\int_0^t \|\varphi^\mu(s) - \varphi^\nu(s)\|^2_{G_k} ds\right).
\end{aligned} \tag{190}$$

Let $K_{\max}^{C_1}(t) := 6K_{\max}t + 4C_1$. By Theorem 10, we obtain that

$$\begin{aligned}
&D_t^\rho(\mu(\omega),\nu(\omega))^2 \leq \exp\left(K_{\max}^{C_1}(T)\right)\left(K_{\max}^{C_1}(T)\int_0^t D_s^\rho\left(\mu(\omega),\nu(\omega)\right)^2 ds\right. \\
&\qquad\qquad\left. + 6K_{\max}t\left(\int_0^t C_R K s D_s^\rho\left(\mu^0(\omega),\nu^0(\omega)\right)^2\right) ds\right),
\end{aligned} \tag{191}$$

where (191) follows by (105). Applying Gronwall's lemma a second time to (191), we obtain

$$\begin{aligned}
&D_T^\rho(\mu(\omega),\nu(\omega))^2 \\
&\leq \exp\left(K_{\max}^{C_1}(T)\right) 6K_{\max}T^3 C_R K \exp\left(\exp\left(K_{\max}^{C_1}(t)\right) K_{\max}^{C_1}(T)\right) D_T^\rho\left(\mu^0(\omega),\nu^0(\omega)\right).
\end{aligned} \tag{192}$$

Hence, setting

$$c_3 = \exp\left(K_{\max}^{C_1}(T)\right) 6K_{\max}T^3 C_R K \exp\left(\exp\left(K_{\max}^{C_1}(t)\right) K_{\max}^{C_1}(T)\right)$$

completes the proof. ∎

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