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EXIT PROBLEMS AS THE GENERALIZED SOLUTIONS OF DIRICHLET PROBLEMS*

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Abstract. This paper investigates sufficient conditions for a Feynman–Kac functional up to an exit time to be the generalized viscosity solution of a Dirichlet problem. The key ingredient is to find out the continuity of an exit operator under the Skorokhod topology, which reveals the intrinsic connection between the overfitting Dirichlet boundary and fine topology. As an application, we establish the sub- and supersolutions for a class of nonstationary Hamilton–Jacobi–Bellman (HJB) equations with fractional Laplacian operator via Feynman–Kac functionals associated to α-stable processes, which lead to the existence of a strong solution to the original HJB equation.

Key words. stochastic control problem, HJB equation, Dirichlet boundary, generalized viscosity solution, a-stable process, fractional Laplacian operator, fine topology

AMS subject classifications. 60H30, 47G20, 93E20, 60J75, 49L25, 35J60, 35J66

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1. Introduction. In this paper we investigate the solvability of a Dirichlet partial differential equation (PDE) given by

$$- \mathcal{L}u(x) + \lambda u(x) - \ell(x) = 0 \text{ on } O \text{ with } u = g \text{ on } O^c, \tag{1.1}$$

where $\mathcal{L}$ is the infinitesimal generator associated to some Feller semigroup $\{P_t : t \geq 0\}$ and O is a connected bounded open set in $\mathbb{R}^d$ for some positive integer d. (See Assumption 2.1 below for more detail.) We will adopt the "verification" approach and characterize the solution to (1.1) by the associated stochastic representation $v(x)$ given by the Feynman–Kac functional:

$$v(x) := \mathbb{E}^x \left[\int_0^\zeta e^{-\lambda s} \ell(X_s) ds + e^{-\lambda \zeta} g(X_\zeta) \right], \tag{1.2}$$

where X is càdlàg Feller with generator $\mathcal{L}$, denoted by $X \sim \mathcal{L}$, and ζ is the exit time from the closure of the domain $\bar{O}$, denoted by $\zeta = \tau_{\bar{O}}(X)$.

The scope of a generator $\mathcal{L}$ associated to a Feller process covers many well-known operators. For instance, the gradient operator ∇ corresponds to a uniform motion, the Laplacian Δ corresponds to a Brownian motion, the fractional Laplacian $-(-\Delta)^{\alpha/2}$ corresponds to a symmetric α-stable process, and any linear combination of the above operators corresponds to a Feller process.

Due to this versatility of the operators, the study of an elliptic or parabolic PDE and its interplay with the corresponding stochastic representation has a wide range of applications and many successful connections to other disciplines outside

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of mathematics. For instance, in mathematical finance, the general approach to the derivative pricing is either given by the solution of a Cauchy problem or derived via the so-called martingale approach (see [22]). The most well-known and practical tool in this direction is the Feynman–Kac formula (see Chapter 8 of [23]). However, a rigorous verification that connects the PDE and the stochastic representation is a difficult task in general, due to the subtle boundary behavior of the underlying random process. In the case of Laplacian operator $\mathcal{L} = \Delta$, [9, sections 4.4 and 4.7] shows that the Feynman–Kac functional v of (1.2) solves (1.1). With $\mathcal{L}$ being a second order differential operator, and thus almost surely continuous $X \sim \mathcal{L}$, the relation between the Feynman–Kac functional and the Dirichlet problem is discussed in [4, 15, 16, 18, 21, 19] and the references therein.

If $\mathcal{L}$ is a nonlocal operator corresponding to a Lévy jump diffusion, which has become popular in the recent development in financial modeling (see [10, 13, 24]), the discontinuity of the random path $X \sim \mathcal{L}$ brings extra difficulty in studying the boundary behavior (see [5, 17, 29, 34]). To the best of our knowledge, the verification for a Feynman–Kac functional to be a solution, or even a generalized viscosity solution (as discussed in this paper) of the Dirichlet PDE, has not been thoroughly studied for jump diffusion in the extant literature. A few closely related papers, such as [11] for one-dimensional nonstationary problems and [3] for multidimensional stationary problems, provide the following partial answer: v of (1.2) is the (strong, hence a generalized) viscosity solution of (1.1) if *all points on the boundary ∂O are regular.* (See the definition of the regularity in section 2.4.)

However, this sufficient condition is not always satisfied, and a simple example below (see section 2.1 with $\epsilon = 0$) provides an explicit calculation for a Feynman–Kac functional, which is not a (strong) viscosity solution but only a generalized viscosity solution. In this paper we focus on the sufficient conditions for v of (1.2) to be a generalized solution of (1.1), which turn out to be much more involved than that for a strong viscosity solution (see Theorem 2.6 for details).

Next in section 2, we present the precise setup, definitions of the (strong) viscosity solution and generalized viscosity solution, and the main result. To avoid unnecessary confusion, we emphasize that "strong" vs. "generalized" are in contrast for the classification of viscosity solutions according to its boundary behavior[1] (see Definitions 2.3 and 2.4), and in this paper, we only focus on the generalized viscosity solution. Section 3 provides the analysis leading to the sufficient conditions for the existence of the generalized viscosity solution for a class of Dirichlet problems, which proves the main theorem.

As part of the motivation for this paper, the study of the solvability of (1.1) is also closely related to the solvability of a class nonlinear PDEs, for example, Hamilton–Jacobi–Bellman (HJB) equations. One of the major existing analytical approaches to solving HJB equations, in the sense of generalized solutions, is a combination of comparison principle (CP) and Perron's method (PM) (see [12] and [2], and the references therein). Such an approach successfully establishes the unique solvability under the assumption that *there exists a supersolution and a subsolution.* In other words, it reduces the solvability question of nonlinear PDE into that of a class of linear PDEs of the type (1.1). However, the answer to the latter is not trivial and was proposed as an open question for the general case in Example 4.6 of [12]. In section 4, by applying the main result (Theorem 2.6) in this paper, we are able to prove the solvability of a class of linear equations, which in turn serve as sub- and

[1] Another possible classification is "classical" vs. "viscosity" or "weak" solution by its smoothness.

supersolutions to nonlinear equations with fractional Laplacian operators and help establish the existence of strong solutions to the latter. At the end, we include a brief summary and some technical results are relegated to appendices.

2. Problem setup and definitions. In this section, we start from the definition of an appropriate filtration, under which v in (1.2) can be characterized in terms of a stochastic exit problem. Then we formally define the generalized viscosity solution to the Dirichlet problem and state the main result of this paper—a sufficient condition for v of (1.2) to be a generalized viscosity solution of (1.1), which is proved in the next section. To motivate the analysis of the generalized solution, we first provide an example of a Dirichlet problem, for which the associated Feynman–Kac functional is not its strong solution, but only a generalized solution.

2.1. An example. For illustration purposes, consider a Dirichlet problem of (1.1) with the following simplified setup, parameterized by $\epsilon \geq 0$:

$$O = (0,1),\ \ell \equiv 1,\ \mathcal{L}^\epsilon u = \frac{1}{2}\epsilon^2 u'' + u',\ \lambda = 1, g \equiv 0. \tag{2.1}$$

Then, (1.1) becomes a second order ordinary differential equation (ODE)

$$-u' - \frac{1}{2}\epsilon^2 u'' + u - 1 = 0 \text{ on } (0,1), \text{ and } u(x) = 0 \text{ for } x \geq 1 \text{ and } x \leq 0. \tag{2.2}$$

If $\epsilon > 0$, then there exists a unique $C^2(O) \cap C(\bar{O})$ solution

$$u(x) = 1 + \frac{(1 - e^{\lambda_1})e^{\lambda_2 x} + (e^{\lambda_2} - 1)e^{\lambda_1 x}}{e^{\lambda_1} - e^{\lambda_2}}, \tag{2.3}$$

where

$$\lambda_1 = \frac{\sqrt{1+2\epsilon^2} - 1}{\epsilon^2}, \quad \text{and } \lambda_2 = \frac{-\sqrt{1+2\epsilon^2} - 1}{\epsilon^2}.$$

If $\epsilon = 0$, then ODE (2.2) has no solution. However, if one removes the boundary condition imposed on 0, ODE (2.2) has a unique solution $u(x) = -e^{-1+x} + 1$.

On the other hand, from a probabilistic perspective, let $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}, \{\hat{\mathcal{F}}_t : t \geq 0\})$ be a filtered probability space satisfying the usual conditions with a standard Brownian motion W, and X be a stochastic process defined by

$$X_t = x + t + \epsilon W_t,$$

of which the generator is $\mathcal{L}^\epsilon$ above. The corresponding Feynman–Kac functional of the form (1.2) is

$$v_\epsilon(x) := \hat{\mathbb{E}}\left[\int_0^\zeta e^{-s} ds \middle| X_0 = x\right] \tag{2.4}$$

with ζ being the exit time from the closure of the domain $\bar{O}$, of which the distribution can be explicitly computed, and $\hat{\mathbb{E}}$ being the expectation under $\hat{\mathbb{P}}$. Note the following:

- If $\epsilon > 0$, then v_ϵ of (2.4) coincides with (2.3) and is the unique solution of (1.1).
- If $\epsilon = 0$, then $v_0(x) = -e^{-1+x} + 1$ is not a solution of (2.2), because it does not meet the boundary condition at $x = 0$.

Note that, if $\epsilon = 0$ and the boundary condition imposed on the point 0 is dropped, then v_0 is identical to the solution of the new ODE. This solution is called the generalized solution to the original equation (2.2). In this paper, we investigate the sufficient conditions under which the associated Feynman–Kac functional is a solution to the Dirichlet PDE in the above general sense (by relaxing the boundary condition). As we will show in the following, v_0 is a generalized viscosity solution of (2.2) according to Definition 2.4.

2.2. Setup. Let $\Omega = \mathbb{D}^d$ be the space of càdlàg functions from $[0, \infty)$ to $\mathbb{R}^d$ with Skorokhod metric d_o. X is the coordinate mapping process, i.e.,

$$X_t(\omega) = \omega(t) \ \forall \omega \in \Omega.$$

Denote the natural filtration generated by X as

$$\mathcal{F}_t^0 = \sigma\{X_s : s \leq t\} \ \forall t \geq 0, \text{ and } \mathcal{F}^0 = \sigma(X_s : 0 \leq s < \infty\}.$$

Denote as $C_0^m(\mathbb{R}^d)$ and $C_0^{m,\alpha}(\mathbb{R}^d)$ the space of functions on $\mathbb{R}^d$ with continuous and locally α-Hölder continuous derivatives, respectively, up to the m^{th} order, which vanish at infinity, and for $C_0^m(\mathbb{R}^d)$, the superscript is dropped if $m = 0$. Let $\{P_t : t \geq 0\}$ be a Feller semigroup on $C_0(\mathbb{R}^d)$ (see Definition III.2.1 of [27]). With $\mathcal{P}(\mathbb{R}^d)$ being the collection of probability measures on $\mathbb{R}^d$, by the Daniell–Kolmogorov theorem and standard path regularization, for any $\nu \in \mathcal{P}(\mathbb{R}^d)$, there exists probability measure $\mathbb{P}^\nu$ on $(\Omega, \mathcal{F}^0)$ with its transition function identical to the given Feller semigroup $\{P_t : t \geq 0\}$ and initial distribution $X_0 \sim \nu$ (see section III.7 of [28]). Denote as $\mathbb{E}^\nu$ the expectation under $\mathbb{P}^\nu$ for $\nu \in \mathcal{P}(\mathbb{R}^d)$, and $\mathbb{E}^x = \mathbb{E}^{\delta_x}$ for the Dirac measure δ_x with $x \in \mathbb{R}^d$. Moreover, $\mathcal{F}$ is the augmented filtration from $\mathcal{F}^0$ satisfying usual conditions. We make the following assumptions for the rest of the paper without further mentioning.

Assumption 2.1.
1. $\{P_t : t \geq 0\}$ is a Feller semigroup with its infinitesimal generator $\mathcal{L}$ satisfying $C_0^\infty(\mathbb{R}^d) \subset \mathcal{D}(\mathcal{L})$, where $\mathcal{D}(\mathcal{L})$ is the domain of $\mathcal{L}$;
2. $(\Omega, \mathcal{F}, \{X_t : t \geq 0\}, \{\mathbb{P}^\nu : \nu \in \mathcal{P}(\mathbb{R}^d)\})$ is the canonical setup of a Feller process associated to the Feller semigroup $\{P_t, t \geq 0\}$, denoted by $X \sim \mathcal{L}$;
3. O is a connected bounded open set in $\mathbb{R}^d$;
4. g and ℓ are Lipschitz continuous functions vanishing at infinity.
5. $\lambda > 0$.

Given a Borel set B in $\mathbb{R}^d$ and a sample path $\omega \in \mathbb{D}^d$, define the exit time $\tau_B(\omega)$ and the exit point $\Pi_B(\omega)$ as

$$\tau_B(\omega) = \inf\{t > 0, \omega_t \notin B\}, \quad \Pi_B(\omega) = \omega(\tau_B(\omega)) = X_{\tau_B(\omega)}(\omega) \quad \forall \omega \in \mathbb{D}^d, \tag{2.5}$$

and for notational convenience, denote

$$\zeta := \tau_{\bar{O}}, \ \Pi := \Pi_{\bar{O}} \text{ and } \hat{\zeta} := \tau_O, \ \hat{\Pi} := \Pi_O. \tag{2.6}$$

In general, τ_B is not necessarily an $\mathcal{F}_t^0$-stopping time, because the set $\{\tau_B \leq t\} \in \mathcal{F}_{t+}^0 = \cap_{s>t} \mathcal{F}_t^0$, and the natural filtration is not always right continuous, i.e., $\mathcal{F}_{t+}^0 \neq \mathcal{F}_t^0$. Thus we modify the natural filtration, based on the following heuristic observation: (1) the natural filtration does not depend on any probability; (2) $\mathcal{F}_0^0$ contains only "deterministic" events and $\mathcal{F}_0^0 \neq \mathcal{F}_{0+}^0$; (3) but the gap between $\mathcal{F}_0^0$ and

$\mathcal{F}^0_{0+}$ consists of only those "almost deterministic events." For example, if we focus on canonical Wiener measure on the path space, then the Blumenthal 0-1 law implies that any event of $A \in \mathcal{F}^0_{0+} \setminus \mathcal{F}^0_0$ happens only with probability one or zero. This motivates us to reshuffle the natural filtration by moving the "almost deterministic" sets to the set of "past information sets," i.e., the σ-algebra at $t = 0$.

DEFINITION 2.2. *For each $\nu \in \mathcal{P}(\mathbb{R}^d)$, denote the $\mathbb{P}^\nu$-augmentation of the natural filtration as $\{\mathcal{F}^\nu_t : t \geq 0\}$, i.e., $\mathcal{F}^\nu_t = \sigma(\mathcal{F}^0_t, \mathcal{N}^\nu)$, where $\mathcal{N}^\nu$ is the collection of all $\mathbb{P}^\nu$-null sets. Let $\mathcal{F}_t = \bigcap_{\nu \in \mathcal{P}(\mathbb{R}^d)} \mathcal{F}^\nu_t$ for each $t \geq 0$.*

In the above definition, we adopt the usual augmentation by manipulating negligible sets, and move the *universally almost deterministic sets*[2] to $\mathcal{F}_0$, without changing the value of $\mathbb{E}^x[F|\mathcal{F}^0_t]$ for any $\mathcal{F}^0$-measurable random variable F. Furthermore, the Feller property asserts that (a) the filtration $\{\mathcal{F}_t : t \geq 0\}$ (also $\{\mathcal{F}^\nu_t : t \geq 0\}$ for each ν) is right continuous, (b) Blumenthal's 0-1 law holds, and (c) τ_B is an $\mathcal{F}_t$-stopping time (the debut theorem; see Proposition III.2.10 and Theorems III.2.15 and III.2.17 of [27]). Last but not least, the strong Markov property of X holds with the filtration $\{\mathcal{F}_t : t \geq 0\}$ (see Theorem III.3.1 of [27]).

For the rest of the paper, we work with the filtration $\{\mathcal{F}_t : t \geq 0\}$ in the classical setting defined above, and then v of (1.2) in the stochastic exit problem can be written as $v(x) := \mathbb{E}^x[F]$, where $F : \mathbb{D}^d \mapsto \mathbb{R}$ is

$$F(\omega) = \int_0^{\zeta(\omega)} e^{-\lambda s} \ell(\omega_s) ds + e^{-\lambda \zeta(\omega)} g \circ \Pi(\omega) \ \forall \omega \in \mathbb{D}^d.$$

We recall the simplified setup in 2.1, where the associated process X is defined as a function of a Brownian motion in a filtered probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}, \{\hat{\mathcal{F}}_t : t \geq 0\})$. To see the equivalence between this setup and the above definition with coordinate mapping process, for X in section 2.1, one can first induce a family of probabilities $\mathbb{P}^\nu$ on the space $\mathbb{D}^d$ associated to initial distribution $\nu \in \mathcal{P}(\mathbb{R}^d)$, i.e., $\mathbb{P}^\nu(A) = \hat{\mathbb{P}}(X \in A | X_0 \sim \nu)$ for any Borel set $A \in \mathbb{D}^d$. The distribution of coordinate mapping is identical to the distribution of X (see section II.28 of [28]). Then starting from a natural filtration $\{\mathcal{F}^0_t : t \geq 0\}$ of the coordinate mapping, one can generate a family of $\{\mathcal{F}^\nu_t : t \geq 0\}$ with $\nu \in \mathcal{P}(\mathbb{R}^d)$, and $\{\mathcal{F}_t : t \geq 0\}$ as defined in Definition 2.2 above.

2.3. Dirichlet problems and viscosity solutions. In this section, we give the definition of the generalized viscosity solution. For simplicity, denote

$$G(\phi, x) = -\mathcal{L}\phi(x) + \lambda \phi(x) - \ell(x), \tag{2.7}$$

and (1.1) becomes

$$G(u, x) = 0 \text{ on } O \text{ and } u = g \text{ on } O^c. \tag{2.8}$$

Note that the Dirichlet boundary data g is given to the entire O^c. The reason is that, if the generator $\mathcal{L}$ is nonlocal, X_ζ may fall anywhere in O^c, and the above definition of G makes sure that for $(u, x) \in C^\infty_0(\mathbb{R}^d) \times \mathbb{R}^d$, the value $G(u, x)$ is well-defined. To generalize the definition of (2.8) to a possibly nonsmooth function with domain $\bar{O}$, we use the following test functions in place of u:[3]

[2] Deterministic with respect to $\mathbb{P}^\nu$ for all $\nu \in \mathcal{P}(\mathbb{R}^d)$.

[3] $f \in USC(\bar{O})$ means f is upper semicontinuous in $\bar{O}$, and $f \in LSC(\bar{O})$ means $-f \in USC(\bar{O})$. Moreover, f^* and f_* are USC and LSC envelopes of f, respectively. $I_A(\cdot)$ is the indicator function of the set A.

1. For a given $u \in USC(\bar{O})$ and $x \in \bar{O}$, the space of supertest functions is

$$J^+(u,x) = \{\phi \in C_0^\infty(\mathbb{R}^d),\ \text{s.t. } \phi \geq (uI_{\bar{O}} + gI_{\bar{O}^c})^* \text{ and } \phi(x) = u(x)\}.$$

2. For a given $u \in LSC(\bar{O})$ and $x \in \bar{O}$, the space of subtest functions is

$$J^-(u,x) = \{\phi \in C_0^\infty(\mathbb{R}^d),\ \text{s.t. } \phi \leq (uI_{\bar{O}} + gI_{\bar{O}^c})_* \text{ and } \phi(x) = u(x)\}.$$

In the above, although the state space is restricted to $\bar{O}$, we shall define the test function ϕ on the whole space $\mathbb{R}^d$, so that the integral operator in the equation makes sense.

We say that a function $u \in USC(\bar{O})$ satisfies the viscosity subsolution property at some $x \in \bar{O}$ if the following inequality holds for all $\phi \in J^+(u,x)$:

$$G(\phi, x) \leq 0. \tag{2.9}$$

Similarly, a function $u \in LSC(\bar{O})$ satisfies the viscosity supersolution property at some $x \in \bar{O}$ if the following inequality holds for all $\phi \in J^-(u,x)$:

$$G(\phi, x) \geq 0. \tag{2.10}$$

In the following we define the (strong) viscosity solution of (1.1). Note that it does not require the viscosity property at any point $x \in \partial O$. However, the viscosity solution property at $x \in \partial O$ will be needed in the definition of the generalized viscosity solution introduced later.

Definition 2.3.

1. *$u \in USC(\bar{O})$ is a viscosity subsolution of* (1.1) *if* (a) *u satisfies the viscosity subsolution property at each $x \in O$ and* (b) *$u(x) \leq g(x)$ at each $x \in \partial O$.*
2. *$u \in LSC(\bar{O})$ is a viscosity supersolution of* (1.1) *if* (a) *u satisfies the viscosity supersolution property at each $x \in O$ and* (b) *$u(x) \geq g(x)$ at each $x \in \partial O$.*
3. *$u \in C(\bar{O})$ is a viscosity solution of* (1.1) *if it is a viscosity subsolution and supersolution simultaneously.*

Recalling the setup (2.1) above, the associated stochastic representation v_ϵ of (2.4) is a viscosity (indeed a classical) solution of (1.1) if $\epsilon > 0$. It is not anymore for $\epsilon = 0$ due to the loss of the boundary $v_0(0) > 0$. The next is the definition of generalized viscosity solution, as discussed in [2].

Definition 2.4.

1. *$u \in USC(\bar{O})$ is a generalized viscosity subsolution of* (1.1) *if (a) u satisfies the viscosity subsolution property at each $x \in O$ and (b) u satisfies at the boundary*

$$\min\{-\mathcal{L}\phi(x) + \lambda\phi(x) - \ell(x), u(x) - g(x)\} \leq 0\ \forall x \in \partial O \textit{ and } \forall \phi \in J^+(u,x).$$

2. *$u \in LSC(\bar{O})$ is a generalized viscosity supersolution of* (1.1) *if (a) u satisfies the viscosity supersolution property at each $x \in O$ and (b) u satisfies at the boundary*

$$\max\{-\mathcal{L}\phi(x) + \lambda\phi(x) - \ell(x), u(x) - g(x)\} \geq 0\ \forall x \in \partial O \textit{ and } \ \forall \phi \in J^-(u,x).$$

3. *$u \in C(\bar{O})$ is a generalized viscosity solution of* (1.1) *if it is a generalized viscosity subsolution and supersolution simultaneously.*

2.4. Main result. Our main objective is to identify the sufficient condition which guarantees that v of (1.2) is a generalized viscosity solution of (1.1), and the results are summarized in Theorem 2.6 below. As a preparation, we briefly recall some basic definitions on regular points and the induced fine topology (see more details in section 3.4 of [9]).

DEFINITION 2.5.

1. *A point x is said to be regular (with respect to $\mathcal{L}$) for the set B if and only if $\mathbb{P}^x(\tau_{B^c} = 0) = 1$. Let*
$$\partial_0 O = \left\{ x \in \partial O : x \text{ is regular for } \bar{O}^c \right\},$$
and $\partial_1 O = \partial O \setminus \partial_0 O$.
2. *For any set B, the set of all regular points for B is be denoted by B^r and $B^* = B \cup B^r$ is called the fine closure of B. A set B is finely closed if $B = B^*$, and B^c is said to be finely open. The collection of all finely open sets generates the fine topology.*

By the above definition, if x is not regular for $\bar{O}^c$, then $\mathbb{P}^x(\tau_{\bar{O}} = 0) < 1$, which implies that $\mathbb{P}^x(\tau_{\bar{O}} = 0) = 0$ due to the Blumenthal 0-1 law. In addition, since the process X has right continuous paths and O is an open set, any point $x \in \bar{O}^c$ is regular for $\bar{O}^c$ and any point $x \in O$ is not regular for $\bar{O}^c$. Thus, $\bar{O}^{c,r}$, the set of all regular points to $\bar{O}^c$, satisfies $\bar{O}^c \subset \bar{O}^{c,r} = \bar{O}^{c,*} \subset O^c$. Therefore, $\partial_0 O = \bar{O} \cap \bar{O}^{c,*}$ and $\partial_1 O = \bar{O} \setminus \bar{O}^{c,*}$.

THEOREM 2.6. *Suppose there exists a neighborhood*[4] *N_1 of $\partial_1 O$ satisfying*
$$\mathbb{P}^x \left(\hat{\Pi} \in \bar{N}_1 \right) = 0 \ \forall x \in \bar{O}.$$
Then v of (1.2) is a generalized viscosity solution of (1.1). Moreover,[5]
$$\partial_0 O \subset \{x \in \partial O : v = g\} := \Gamma_{out}[v].$$

Before the proof of Theorem 2.6 in the next section, the following are some immediate applications. Further applications of Theorem 2.6 on nonstationary problems are provided in section 4.

According to Theorem 2.6, the sufficient condition is closely related to the distribution of $X(\hat{\zeta})$, the exit point of X from O. Consider a special case where every ∂O is regular for $\bar{O}^c$ with respect to $\mathcal{L}$, for example, $\mathcal{L} = -(-\Delta)^{\alpha/2}$ with $\alpha \geq 1$. Then $\partial_0 O = \partial O$ and $\partial_1 O = \emptyset$. Hence, one can simply take $N_1 = \emptyset$ as the neighborhood of $\partial_1 O = \emptyset$, which fulfills the condition of Theorem 2.6, because $X(\hat{\tau})$ does not fall in the empty set $\bar{N}_1$. Furthermore, $v(x) = g(x)$ for every $x \in \partial O = \partial O_0$, which recovers the result of [3].

COROLLARY 2.7. *If every $x \in \partial O$ is regular to $\bar{O}^c$, then v of (1.2) is a strong viscosity solution of (1.1).*

As an example, in the setup (2.1), if $\epsilon > 0$, then $\partial O = \partial_0 O = \{0, 1\}$, and v_ϵ is the strong solution by Corollary 2.7. If $\epsilon = 0$, then $\mathbb{P}^x$ is the probability induced from the uniform motion $X(t) = x + t$. Thus $\partial_1 O = \{0\}$ and $\partial_0 O = \{1\}$. By taking that $N_1 = (-1/2, 1/2)$, N_1 satisfies the assumptions in Theorem 2.6 and v_0 is a generalized solution.

[4] A neighborhood of $\partial_1 O$ is an open set $N_1 \in \mathbb{R}^d$ such that $\partial_1 O \subset N_1$.

[5] By definition Γ_{out} depends on the function g, and should be denoted as $\Gamma_{out}[v, g]$, and the argument g is omitted in the rest of the paper, unless ambiguity arises.

3. Identification of Feynman–Kac functional as a generalized solution. This section is devoted to the proof of Theorem 2.6. As the first step, we show that the generalized viscosity solution property requires that on the boundary points, either the boundary condition or the viscosity solution property is satisfied.

3.1. The set of points losing the boundary condition. In section 2.1, v_0 of the setup of (2.1) with $\epsilon = 0$ is not a viscosity solution of (1.1) due to the loss of the boundary at $x = 0$. One can actually directly verify that v_0 is the generalized solution according to Definition 2.4 by checking its viscosity solution property at $x = 0$ (the argument for other points on [0,1] is straightforward):

1. The space of supertest functions satisfies

$$J^+(v_0, 0) \subset \{\phi \in C_0^\infty(\mathbb{R}) : \phi(0) = v_0(0) = 1 - e^{-1},\ \phi'(0) \geq v_0'(0+) = -e^{-1}\}.$$

   Thus, v_0 satisfies the subsolution property at $x = 0$ according to (2.9), and therefore, the inequality of Definition 2.4 (1) holds.
2. The space of subtest functions $J^-(v_0, 0)$ is an empty set, because $v_0(0) > 0$ and $(v_0 I_{\bar{O}})_*(0) = 0$, and it automatically implies its supersolution property at $x = 0$.

The above argument shows that although v_0 violates the boundary condition at $x = 0$, it satisfies the viscosity solution property according to the definitions (2.9)–(2.10) at $x = 0$. The next proposition generalizes the above observation: A generalized solution shall satisfy either the viscosity solution property or the boundary condition at every boundary point.

Proposition 3.1. *Suppose u is a continuous function up to the boundary, i.e., $u \in C(\bar{O})$. Then, u is a generalized viscosity solution of* (1.1) *if and only if u satisfies the viscosity solution property at each $x \in \bar{O} \setminus \Gamma_{out}[u]$.*

Proof. The sufficiency is straightforward by checking Definition 2.4. For the necessity, if u is a generalized viscosity solution, then according to Definition 2.4, it satisfies the viscosity solution property at every $x \in O$. For $x \in \partial O \setminus \Gamma_{out}[u]$, if $u(x) < g(x)$, then since $u \in USC(\bar{O})$ and $J^+(u, x) = \emptyset$, it implies the viscosity subsolution property at x. On the other hand, since $u \in LSC(\bar{O})$, and therefore $\max\{-\mathcal{L}\phi(x) + \lambda\phi(x) - \ell(x), u(x) - g(x)\} \geq 0$ for every $\phi \in J^-(u, x)$, it implies that the viscosity supersolution property is satisfied. The case of $u(x) > g(x)$ follows similar arguments. ∎

Concerning whether v of (1.2) a generalized viscosity solution of (1.1), the question can now be divided into the following two subquestions, to which the answers are provided in the next sections:

(Q1) Does v satisfy the condition of Proposition 3.1? That is, is $v \in C(\bar{O})$?

(Q2) If yes, where does v meet its boundary? That is, what is $\Gamma_{out}[v]$?

In the following, we can first answer (Q2) given that (Q1) holds. Notice that it is standard that, by using Ito's formula on test functions, the function v of (1.2) satisfies the viscosity solution property at any interior point x of the domain $\bar{O}$ given that v of (1.2) is continuous. Next, Lemma 3.2 shows that the same statement holds as long as x is an interior point of $\bar{O}$ in the fine topology, i.e., x is not regular for $\bar{O}^c$.

Lemma 3.2. *If $v \in C(\bar{O})$, then v of* (1.2) *is a generalized viscosity solution of* (1.1) *with*

$$\Gamma_{out}[v] \supset \partial_0 O.$$

Proof. We discuss the interior point and boundary point separately (recall that $G(\phi, x) = -\mathcal{L}\phi(x) + \lambda\phi(x) - \ell(x)$).

1. v's interior viscosity solution property. First, fixing an arbitrary $x \in O$, we show v satisfies the viscosity supersolution property, i.e.,

$$G(\phi, x) \geq 0 \text{ for every } \phi \in J^-(v, x). \tag{3.1}$$

To the contrary, assume $G(\phi, x) < 0$ for some $\phi \in J^-(v, x)$. By the continuity of $x \mapsto G(\phi, x)$ for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\sup_{|y-x|<\delta} G(\phi, y) < -\epsilon/2. \tag{3.2}$$

Since X is a càdlàg process and $x \in O$, $\mathbb{P}^x(\zeta > 0) = 1$. By the strong Markov property of X, we can rewrite the function v as, for any stopping time $h \in (0, \zeta]$,

$$v(x) = \mathbb{E}^x\left[e^{-\lambda h}v(X_h) + \int_0^h e^{-\lambda s}\ell(X_s)ds\right],$$

which, with the fact of $\phi \in J^-(v, x)$, implies that

$$\phi(x) \geq \mathbb{E}^x\left[e^{-\lambda h}\phi(X_h) + \int_0^h e^{-\lambda s}\ell(X_s)ds\right].$$

Moreover, Dynkin's formula on ϕ gives

$$\mathbb{E}^x[e^{-\lambda h}\phi(X_h)] = \phi(x) + \mathbb{E}^x\left[\int_0^h e^{-\lambda s}(\mathcal{L}\phi(X_s) - \lambda\phi(X_s))ds\right].$$

Adding up the above two (in)equalities, it yields

$$\mathbb{E}^x\left[\int_0^h e^{-\lambda s}G(\phi, X_s)ds\right] \geq 0.$$

Then, take $h = \inf\{t > 0 : X(t) \notin \bar{B}_\delta(x)\} \wedge \zeta$, where[6] $B_r(x)$ denotes the open ball with radius r centered at x. Since $h > 0$ almost surely under $\mathbb{P}^x$, it leads to a contradiction to (3.2) and implies the supersolution property at x. The interior subsolution property can be similarly obtained.

2. v's generalized boundary condition. For any $x \in \partial O$, by the Blumenthal 0-1 law, either $\mathbb{P}^x(\zeta = 0) = 1$ or $\mathbb{P}^x(\zeta > 0) = 1$. If $\mathbb{P}^x(\zeta = 0) = 1$, then $v(x) = g(x)$ by its definition (1.2) and hence $\Gamma_{out} \supset \partial_0 O$ holds. On the other hand, if $\mathbb{P}^x(\zeta > 0) = 1$, then we shall examine its viscosity solution property.

For the viscosity supersolution property, assume (3.2) holds for some $\phi \in J^-(v, x)$. Since $\mathbb{P}^x(\zeta > 0) = 1$, we can follow exactly the same argument above for the interior viscosity solution property to find a contradiction, which justifies the supersolution property. The subsolution property can be obtained in a similar way. ∎

Remark 3.3. From the definition of the generalized solution, $\Gamma_{out}[v]$ can be treated as part of the solution. Therefore, for the characterization of the unknown set $\Gamma_{out}[v]$, it seems not satisfactory to have "$\supset$" instead of "$=$" as its conclusion in Theorem 2.6 and Lemma 3.2. However, it is indeed a full characterization by noting that the

[6]The argument x is dropped in the rest of the paper if $x = 0$.

left-hand side $\Gamma_{out}[v]$ depends on the boundary value g, while the right-hand side $\partial_0 O$ is invariant of g. More precisely, one can show that under some mild conditions,

$$\cap_{g\in C_0^{0,1}(\mathbb{R}^d)}\Gamma_{out}[v,g] = \partial_0 O$$

holds (see Appendix A.1). □

3.2. Sufficient conditions—I. Proposition 3.1 demands the continuity up to the boundary as the sufficient condition. In general, Assumption 2.1 cannot guarantee this continuity, and hence v of (1.2) may not be a generalized solution of (1.1), as shown in Example 3.4 below. In this subsection, Lemma 3.6 points out a sufficient condition:

$$\zeta : \mathbb{D}^d \mapsto \mathbb{R} \text{ and } \Pi : \mathbb{D}^d \mapsto \mathbb{R}^d \text{ are continuous almost surely under } \mathbb{P}^x.$$

The condition of v's continuity up to the boundary in Lemma 3.2 may not be true in general. The next is an example for v of (1.2) being discontinuous even in the interior of the domain.

Example 3.4. Consider a problem on the two-dimensional domain of

$$O = (-1,1)\times(0,1),\ \mathcal{L}u(x) = \partial_{x_1}u(x) + 2x_1\partial_{x_2}u(x),\ \lambda = 1, \ell\equiv 1,\ \text{and } g\equiv 0. \tag{3.3}$$

Then, PDE (1.1) becomes

$$-\partial_{x_1}u(x) - 2x_1\partial_{x_2}u(x) + u(x) - 1 = 0,\ \text{on } O,\ \text{and } u(x) = 0 \text{ on } O^c.$$

In fact, the process $X \sim \mathcal{L}$ with initial value $x = (x_1, x_2)^T$ has the following deterministic parametric representation:

$$X_{1t} = x_1 + t,\ X_{2t} = x_2 - x_1^2 + X_{1t}^2.$$

Therefore, the lifetime ζ is also a deterministic number depending on its initial state x, which will be denoted as ζ^x:

$$\zeta^x = \begin{cases} -x_1 + \sqrt{1 - x_2 + x_1^2}, & x \in O_1 := \{x_2 \geq x_1^2\}\cap\bar{O}, \\ 1 - x_1, & x\in O_2 := \{x_2 < x_1^2, x_1 > 0\}\cap\bar{O}, \\ -x_1 - \sqrt{-x_2 + x_1^2}, & x\in O_3 := \{x_2 < x_1^2, x_1 < 0\}\cap\bar{O}. \end{cases}$$

The mapping $x \mapsto \zeta^x$ is discontinuous at every point on the curve $\partial O_1 \cap \partial O_3$, and so is v of (1.2), which can be rewritten as

$$v(x) = \int_0^{\zeta^x} e^{-s}ds = 1 - e^{-\zeta^x}.$$

In this example,

$$\begin{aligned} \partial_0 O = \quad & \{(x_1,x_2) : x_1 = 1, 0 \leq x_2 \leq 1\}\cup, \\ & \{(x_1,x_2) : x_2 = 1, 0 \leq x_1 \leq 1\}\cup, \\ & \{(x_1,x_2) : x_2 = 0, -1 \leq x_1 < 0\}, \end{aligned}$$

and $\partial_1 O = \partial O \setminus \partial_0 O$ is not open relative to ∂O. □

Example 3.4 together with Lemma 3.2 lead us to investigate sufficient conditions for the continuity of the function v, and the next lemma shows that it depends on the continuity of ζ and Π.

DEFINITION 3.5. *For a given function $\phi : \mathbb{D}^d \mapsto \mathbb{R}^m$ for some positive integer m,*

1. *ϕ is continuous at some $\omega \in \mathbb{D}^d$ if*

$$\lim_{n\to\infty} \phi(\omega_n) = \phi(\omega), \text{ whenever } \lim_{n\to\infty} d_o(\omega_n, \omega) = 0,$$

2. *denote as $C_\phi = \{\omega \in \mathbb{D}^d : \phi \text{ is continuous at } \omega\}$ the continuity set of ϕ. For a given probability $\mathbb{Q}$ on a σ-algebra of $\mathbb{D}^d$, ϕ is said to be continuous almost surely under $\mathbb{Q}$ if $\mathbb{Q}(C_\phi) = 1$.*

Note that if ϕ is a Borel measurable mapping, then C_ϕ is a Borel set in $\mathbb{D}^d$. Example 3.4 implies that C_ζ can be a proper subset of $\mathbb{D}^d$, and thus ζ may not be continuous everywhere. Lemma 3.6 below indicates that, for the continuity of v at x, it suffices that the sets C_ζ and C_Π are big enough so that ζ and Π are continuous almost surely under $\mathbb{P}^x$, i.e., $\mathbb{P}^x(C_\zeta \cap C_\Pi) = 1$.

LEMMA 3.6. *Let $x \in \bar{O}$. If $\zeta : \mathbb{D}^d \mapsto \mathbb{R}$ and $\Pi : \mathbb{D}^d \mapsto \mathbb{R}^d$ are continuous in Skorokhod topology almost surely under $\mathbb{P}^x$, then v of* (1.2) *is continuous at x relative to $\bar{O}$, i.e., $\lim_{\bar{O}\ni y\to x} v(y) = v(x)$.*

Proof. If $\bar{O} \ni y \to x$, then $\mathbb{P}^y$ converges to $\mathbb{P}^x$ weakly by Theorem 17.25 of [20]. By the continuous mapping theorem (Theorem 2.7 of [7]), together with the uniform boundedness of F, v is continuous at x if F is continuous almost surely under $\mathbb{P}^x$, i.e., $\mathbb{P}^x(C_F) = 1$ for the continuity set C_F of $F : \mathbb{D}^d \mapsto \mathbb{R}$. Then it suffices to show that $C_\zeta \cap C_\Pi \subset C_F$.

Rewrite F as $F = F_1 + F_2$, where

$$F_1(\omega) = \int_0^{\zeta(\omega)} e^{-\lambda s} \ell(\omega_s) ds, \quad F_2(\omega) = e^{-\lambda \zeta(\omega)} g \circ \Pi(\omega).$$

It is straightforward to see that F_2 is continuous at a given ω if ζ and Π are continuous at the same ω. For F_1, consider an arbitrary sequence $\omega_n \to \omega$ in Skorokhod metric, and ζ and Π are continuous at ω. $|F_1(\omega_n) - F_1(\omega)|$ can be approximated by

$$\begin{aligned} &|F_1(\omega_n) - F_1(\omega)| \\ &\quad \le K \int_0^{\zeta(\omega)\wedge\zeta(\omega_n)} e^{-\lambda s} |\omega_n(s) - \omega(s)| ds + K|\zeta(\omega_n) - \zeta(\omega)| \\ &\quad \le K \int_0^{\infty} e^{-\lambda s} |\omega_n(s) - \omega(s)| I_{(0,\zeta(\omega)\wedge\zeta(\omega_n))}(s) ds + K|\zeta(\omega_n) - \zeta(\omega)| \\ &\quad := K \cdot Term1_n + K \cdot Term2_n, \end{aligned} \tag{3.4}$$

where $K = \max_{x\neq y} |\frac{\ell(x)-\ell(y)}{x-y}| + \max_{\bar{O}} |\ell(x)|$ is a constant independent to n, $Term1_n = \int_0^\infty e^{-\lambda s} |\omega_n(s) - \omega(s)| I_{(0,\zeta(\omega)\wedge\zeta(\omega_n))}(s) ds$, and $Term2_n = |\zeta(\omega_n) - \zeta(\omega)|$. Observe the following:

- $\omega_n \to \omega$ in Skorokhod metric implies that $\zeta(\omega_n) \to \zeta(\omega)$ due to the continuity of ζ. Therefore, $Term2_n$ goes to zero as n goes to infinity.
- $\omega_n \to \omega$ in Skorokhod metric implies that $\omega_n(t) \to \omega(t)$ holds for all $t \in C_\omega$, where C_ω is the continuity set of the function $\omega : [0, \infty) \mapsto \mathbb{R}^d$ (see p. 124 of [7]). Since there are countably many discontinuities of the mapping $\omega : t \mapsto \mathbb{R}^d$ for any càdlàg path ω,

$$\lim_{n\to\infty} |\omega_n(t) - \omega(t)| = 0$$

almost everywhere in Lebesgue measure. Therefore, the integrand in $Term1_n$

$$|\omega_n(s) - \omega(s)| I_{(0,\zeta(\omega)\wedge\zeta(\omega_n))}(s)$$

converges to zero as $n \to \infty$ for almost every s in Lebesgue measure. Together with its uniform boundedness by $2\max_{\bar{O}} |x|$, the dominated convergence theorem implies that $Term1_n$ converges to zero as n goes to infinity.

Hence, each term of the right-hand side of (3.4) goes to zero and is uniformly bounded. Therefore, the limit of $|F_1(\omega_n) - F_1(\omega)|$ is also zero and F is continuous at ω. ∎

3.3. Sufficient conditions—II. Lemmas 3.2 and 3.6 lead us to investigate the continuity of ζ and Π, which is not always the case, as illustrated by Example 3.4. Furthermore, it's not an easy task to check almost continuity of a mapping on a Skorohod space. The main result (Proposition 3.9) of this section indicates that ζ and Π are almost continuous under $\mathbb{P}^x$ for all $x \in \bar{O} \setminus \partial_1 O$ if the following additional condition holds:[7]

(C) X exits from O and $\bar{O}$ at the same time almost surely.

Note that condition (C) is violated in Example 3.4: for $(x_1, x_2) \in \bar{O}_1 \cap \bar{O}_3$, $x_2 = x_1^2$ and $x_1 < 0$. Thus ζ (the exit time of $\bar{O}$) is $-x_1 + \sqrt{1 - x_2 + x_1^2}$, while $\hat{\zeta}$ (the exit time of O) is $-x_1 + \sqrt{-x_2 + x_1^2}$.

To proceed, we introduce the following notions. For a path $\omega \in \mathbb{D}^d$, denote ω^- as a càglàd version of ω,

$$\omega_0^- = \omega_0, \text{ and } \omega_t^- = \lim_{s \uparrow t^-} \omega_s \text{ for } t > 0,$$

and the associated exit time operator,

$$\tau_B^-(\omega) = \inf\{t > 0, \omega_t^- \notin B\}. \tag{3.5}$$

If ω is continuous, then $\omega = \omega^-$ and $\tau_B(\omega) = \tau_B^-(\omega)$. However, we shall not casually expect an equality or even an inequality between τ_B and τ_B^- in general, as demonstrated in the following example.

Example 3.7. Let $B = (0, 3)$ and a càdlàg path $\omega_t = |t - 1| + I_{[0,1)}(t)$,

$$\tau_B(\omega) = 1 < \tau_B^-(\omega) = 4.$$

On the other hand, for another càdlàg path $\omega_t = 1 - tI_{[0,1)}(t)$,

$$\tau_B(\omega) = \infty > \tau_B^-(\omega) = 1.$$ ∎

To discuss the continuity of the lifetime ζ, define

$$\zeta^-(\omega) = \tau_O^-(\omega). \tag{3.6}$$

By definition, the following inequality holds:

$$\max\{\hat{\zeta}(\omega), \zeta^-(\omega)\} \leq \zeta(\omega) \ \forall \omega \in \mathbb{D}^d. \tag{3.7}$$

Furthermore, though Example 3.7 shows that neither $\hat{\zeta} \geq \zeta^-$ nor $\hat{\zeta} \leq \zeta^-$ is generally true, interestingly, for the càdlàg Feller process X, the inequality $\zeta^- \geq \hat{\zeta}$ holds almost surely under $\mathbb{P}^x$.

[7]The $\mathbb{P}^x$-continuity ζ and Π for $x \in \partial_1 O$ is discussed in section 3.4 separately.

PROPOSITION 3.8. *For any $x \in \bar{O}$, the following identities hold:*

$$\mathbb{P}^x\left(\omega^-(\zeta^-) \in \partial O, \omega^-(\zeta^-) \neq \omega(\zeta^-)\right) = 0 \text{ and } \mathbb{P}^x\left(\hat{\zeta} \leq \zeta^- \leq \zeta\right) = 1.$$

Proof. Let $\zeta_1^-(\omega) = \zeta^-(\omega)$ if $\omega^-(\zeta^-) \in \partial O$, and infinity otherwise. Then, ζ_1^- is an $\mathcal{F}_{t-}$-stopping time, and hence a predictable stopping time.

If ω is discontinuous at ζ^-, then ζ^- is a totally inaccessible stopping time due to the jump by Meyer's theorem (see Theorem III.4 of [26]). According to Theorem III.3 of [26], the set of predictable stopping times has no overlap with the set of totally inaccessible stopping times almost surely. Hence, $\mathbb{P}^x\left(\omega^-(\zeta_1^-) \neq \omega(\zeta_1^-); \zeta_1(\omega) < \infty\right) = 0$, which is equivalent to $\mathbb{P}^x\left(\omega^-(\zeta^-) \in \partial O, \omega^-(\zeta^-) \neq \omega(\zeta^-)\right) = 0$.

Thus whenever $\omega^-(\zeta^-) \in \partial O$, $\omega(\zeta^-) = \omega^-(\zeta^-) \in \partial O$ almost surely, and therefore $\hat{\zeta} \leq \zeta^-$ by definition, i.e.,[8]

$$\mathbb{P}^x\left(\hat{\zeta} \leq \zeta^- | \omega^-(\zeta^-) \in \partial O\right) = 1.$$

On the other hand, if $\omega^-(\zeta^-) \notin \partial O$, then by the left-continuity of ω^-, $\omega^-(\zeta^-) \in O$. In this case, there must be a jump at ζ^-, and there exists a sequence $t_n \downarrow \zeta^-$ as n goes to infinity, such that $\omega^-(t_n) \in O^c$ for every n. By the right continuity of ω, $\omega(\zeta^-) = \lim_{n\to\infty} \omega^-(t_n) \in O^c$ due to the closedness of O^c. Hence, $\hat{\zeta} \leq \zeta^-$ whenever $\omega^-(\zeta^-) \in O$, and therefore

$$\mathbb{P}^x\left(\hat{\zeta} \leq \zeta^- | \omega^-(\zeta^-) \in O\right) = 1.$$

Since $\mathbb{P}^x(\{\omega^-(\zeta^-) \in \partial O\} \cup \{\omega^-(\zeta^-) \in O\}) = 1$, $\mathbb{P}^x(\hat{\zeta} \leq \zeta^-) = 1$. The other inequality $\zeta^- \leq \zeta$ holds by the definition. ∎

Next, we establish the almost sure continuity of ζ and Π if it starts from $x \notin \partial_1 O$.

PROPOSITION 3.9. *If $x \in \mathbb{R}^d \setminus \partial_1 O$ and $\mathbb{P}^x(\hat{\zeta} = \zeta) = 1$, then both ζ and Π are almost surely continuous under $\mathbb{P}^x$.*

Proof. If $x \in \bar{O}^c$, then any ω with its initial state x satisfies $\zeta(\omega) \equiv 0$ and $\Pi(\omega) \equiv x$ being constant mappings. ζ and Π are both continuous at ω with its initial $x \in \bar{O}^c$.

If $x \in O$, a slight modification of the proof of Theorem 3.1 and Proposition 2.4 of [3] implies that the mappings $\zeta : \mathbb{D}^d \mapsto \mathbb{R}$ and $\Pi : \mathbb{D}^d \mapsto \mathbb{R}^d$ are both continuous at any

$$\omega \in \Gamma := \{\omega : \omega(0) \in O\} \cap \Gamma_1 \setminus \Gamma_2$$

in Skorokhod topology, where $\Gamma_1 = \{\omega : \zeta^- = \hat{\zeta} = \zeta\}$ and $\Gamma_2 = \{\omega : \omega^-(\zeta^-) \in \partial O, \omega^-(\zeta^-) \neq \omega(\zeta^-)\}$. Proposition 3.8 and the condition $\mathbb{P}^x(\hat{\zeta} = \zeta) = 1$ imply that $\mathbb{P}^x(\Gamma_1) = 1$ and $\mathbb{P}^x(\Gamma_2) = 0$. Therefore, $\mathbb{P}^x(\Gamma) = 1$ and we conclude almost sure continuity of ζ and Π for this case.

Finally, if $x \in \partial O_0$, using exactly the same approach of Theorem 3.1 and Proposition 2.4 of [3], we know that the mappings $\zeta : \mathbb{D}^d \mapsto \mathbb{R}$ and $\Pi : \mathbb{D}^d \mapsto \mathbb{R}^d$ are both continuous at any

$$\omega \in \Gamma := \{\omega : \omega(0) \in \partial O, \hat{\zeta} = \zeta = 0\}$$

in Skorokhod topology. Since x is regular for $\bar{O}^c$, $\mathbb{P}^x(\Gamma) = 1$ and we conclude almost sure continuity of ζ and Π for this case. ∎

[8] To avoid ambiguity, let $\mathbb{P}(A|B) = 1$ whenever $\mathbb{P}(B) = 0$.

3.4. Sufficient conditions—III. For $x \in \partial_1 O$, the following example shows that $\mathbb{P}^x(\hat{\zeta} = \zeta) = 1$ does not guarantee the almost sure continuity of ζ and Π under $\mathbb{P}^x$.

Example 3.10. Consider $O = (0,1)$ and $X_t = t$. In other words, $\mathbb{P}^0(\omega_0) = 1$ for $\omega_0(t) = t$. Then, $0 \in \partial_1 O$ and $\mathbb{P}^0(\hat{\zeta} = \zeta = 1) = 1$. In particular, recall from the definition of $\hat{\zeta}$ in (2.6) that $\hat{\zeta} = 1$ instead of $\hat{\zeta} = 0$, because it is defined as hitting time $\inf\{t > 0 : \dots\}$ instead of entrance time $\inf\{t \geq 0 : \dots\}$. However, the sequence of paths $\{\omega_n\}_{n \geq 1}$ with

$$\omega_n(t) = \begin{cases} n^{-1} - 2t, & t \in [0, n^{-1}), \\ t, & t \geq n^{-1}, \end{cases}$$

satisfies $\lim_{n\to\infty} d_o(\omega_n, \omega_0) = 0$, while $\lim_{n\to\infty} \zeta(\omega_n) \to 0 \neq \zeta(\omega_0)$. Moreover, one shall note that $\mathbb{P}^0\{\omega_0\} > 0$. Hence, ζ cannot be continuous almost surely in $\mathbb{P}^0$. ∎

Example 3.10 implies that for $x \in \partial_1 O$, ζ is not almost surely continuous under $\mathbb{P}^x$, and we need to pursue other sufficient conditions for the continuity of v. In the following discussion, the idea is to consider a larger domain $O_1 \supset O$ to which Proposition 3.9 applies, and we assume on the regularity structure of O, which guarantees that $\mathbb{P}^x(\zeta = \tau_{\bar{O}_1}) = 1$ and hence $v(x) = v_1(x)$[9] for all $x \in \bar{O}$, and it turns out that this is the only assumption in addition to Assumption 2.1 for the main theorem to hold.

As a preparation, define the shift operator $\theta_t : \mathbb{D}^d \mapsto \mathbb{D}^d$ as

$$\theta_t \omega(s) = \omega(t+s) \ \forall s \geq 0.$$

This implies that $(X_s \circ \theta_t)(\omega) = X_s(\theta_t \omega) = \theta_t \omega(s) = \omega(t+s) = X_{t+s}(\omega)$.

PROPOSITION 3.11. *If $h \in [0, \hat{\zeta}(\omega)]$, then $\zeta \circ \theta_h(\omega) = \zeta(\omega) - h$ for all $\omega \in \mathbb{D}^d$.*

Proof. From the definition of θ,

$$\zeta \circ \theta_h(\omega) = \inf\{t > 0 : \omega(t+h) \notin \bar{O}\} = \inf\{t' > h : \omega(t') \notin \bar{O}\} - h.$$

Therefore, it suffices to show that $\inf\{t' > h : \omega(t') \notin \bar{O}\} = \inf\{t > 0 : \omega(t) \notin \bar{O}\}$. Observe that $\omega(t) \in O$ for all $t \in [0, h)$ due to $h \leq \hat{\zeta}$. Therefore,

1. if $\omega(h) \in \bar{O}$, then $\inf\{t' > h : \omega(t') \notin \bar{O}\} = \inf\{t > 0 : \omega(t) \notin \bar{O}\}$ by the definition of infimum;
2. if $\omega(h) \notin \bar{O}$, then $\inf\{t > 0 : \omega(t) \notin \bar{O}\} = h$. On the other hand, $\inf\{t' > h : \omega(t') \notin \bar{O}\} = h$ by the right continuity of ω. ∎

LEMMA 3.12. *Let $x \in \bar{O}$. If $\mathbb{P}^x\left(\hat{\Pi} \in \bar{O}^{c,*}\right) = 1$, then $\mathbb{P}^x\left(\hat{\zeta} = \zeta\right) = 1$.*

Proof. By Proposition 3.11, $\zeta = \hat{\zeta} + \zeta \circ \theta_{\hat{\zeta}}$. Furthermore, if $X(\hat{\zeta}) \in \bar{O}^{c,*}$, then $\mathbb{P}^{X(\hat{\zeta})}(\zeta = 0) = 1$. Therefore, since $\mathbb{P}^x(X(\hat{\zeta}) \in \bar{O}^{c,*}) = 1$, we have

$$\mathbb{P}^x\left(\zeta = \hat{\zeta}\right) = \mathbb{P}^x\left(\zeta \circ \theta_{\hat{\zeta}} = 0\right) = \mathbb{E}^x\left[\mathbb{P}^{X(\hat{\zeta})}(\zeta = 0)\right] = 1$$

and the result follows. ∎

[9]See (3.9) for the definition of v_1.

PROPOSITION 3.13. *Suppose there exists a neighborhood* N_1 *of* $\partial_1 O$ *such that* $\mathbb{P}^x(\hat{\Pi} \in \bar{N}_1) = 0$ *for all* $x \in \bar{O}$. *Let* $O_1 = N_1 \cup O$, *and accordingly define* $\zeta_1 = \tau_{\bar{O}_1}, \hat{\zeta}_1 = \tau_{O_1}, \Pi_1 = X(\zeta_1), \hat{\Pi}_1 = X(\hat{\zeta}_1)$. *Then, for all* $x \in \bar{O}$

$$\mathbb{P}^x\left(\Pi = \hat{\Pi} = \Pi_1 = \hat{\Pi}_1, \zeta = \hat{\zeta} = \zeta_1 = \hat{\zeta}_1\right) = 1$$

and ζ_1 *and* Π_1 *are almost surely continuous under* $\mathbb{P}^x$.

Proof. Since $\bar{O}_1 \supset O_1 \supset \bar{O} \supset O$,

$$\mathbb{P}^x\left(\hat{\zeta} \le \zeta \le \hat{\zeta}_1 \le \zeta_1\right) = 1. \tag{3.8}$$

We first show that $\mathbb{P}^x(\hat{\zeta} = \zeta_1) = 1$. Since $\mathbb{P}^x(\hat{\Pi} \in \bar{N}_1) = 0$ and $\mathbb{P}^x(\hat{\Pi} \in O) = 0$ due to the right continuity of X, the latter can be rewritten as

$$\mathbb{P}^x\left(\hat{\Pi} \in \bar{O}^c \setminus \bar{N}_1\right) + \mathbb{P}^x\left(\hat{\Pi} \in \partial_0 O \setminus \bar{N}_1\right) = 1.$$

Thus it suffices to discuss the following two cases:

- If $\omega \in \{\hat{\Pi} \in \bar{O}^c \setminus \bar{N}_1\}$, then since $\bar{O}^c \setminus \bar{N}_1 \subset \bar{O}_1^c$, $\hat{\zeta}(\omega) = \zeta_1(\omega)$.
- If $\omega \in \{\hat{\Pi} \in \partial_0 O \setminus \bar{N}_1\}$, then since $\partial_0 O \setminus \bar{N}_1$ is open relative to ∂O, there exists $r > 0$ such that $B_r(\hat{\Pi}(\omega)) \cap \bar{N}_1 = \emptyset$. In addition, since $\hat{\Pi}(\omega) \in \partial_0 O \subset \bar{O}^{c,*}$, Lemma 3.12 implies that there exists a sequence $h_n \downarrow 0$ as n goes to infinity, such that $\omega(\hat{\zeta} + h_n) \notin \bar{O}$ for all n. Together with right continuity of ω, we have
$$\omega\left(\hat{\zeta} + h_n\right) \in B_r\left(\hat{\Pi}(\omega)\right) \setminus \bar{O} = B_r(\hat{\Pi}(\omega)) \setminus \bar{O}_1$$
and
$$\lim_{n\to\infty} \omega\left(\hat{\zeta} + h_n\right) = \hat{\Pi}\left(\omega\right).$$
Thus, $\hat{\zeta}(\omega) = \zeta_1(\omega)$ also holds.

Then the above two cases together with (3.8) imply that

$$\mathbb{P}^x\left(\hat{\zeta} = \zeta = \hat{\zeta}_1 = \zeta_1\right) = 1,$$

and $\mathbb{P}^x(\hat{\Pi} = \Pi = \Pi_1 = \hat{\Pi}_1) = 1$ also holds. Since either $x \in \partial_0 O \subset \partial_0 O_1$ (note that $\mathbb{P}^x(\hat{\Pi} \in \bar{N}_1) = 0$) or $x \in \bar{O} \setminus \partial_0 O \subset O_1$ holds, we can apply Proposition 3.9 to $x \in \bar{O}$ with respect to the expanded domain O_1 and conclude that (ζ_1, Π_1) is continuous almost surely in $\mathbb{P}^x$. ∎

3.5. The proof of the main theorem. By wrapping up all the above outcomes together, we can provide the proof of the main result on the sufficient conditions for the stochastic representation $v = \mathbb{E}[F]$ in (1.2) to be a generalized viscosity solution of (1.1), stated in Theorem 2.6 in section 2.4.

Proof of Theorem 2.6. As in Proposition 3.13, we expand the domain O into O_1 and set the corresponding operators $(\zeta_1, \hat{\zeta}_1, \Pi_1, \hat{\Pi}_1)$. Consider the O_1-associated value function v_1 in the form of (1.2), i.e.,

$$v_1(x) := \mathbb{E}^x\left[\int_0^{\zeta_1} e^{-\lambda s}\ell(X_s)ds + e^{-\lambda\zeta_1}g(\Pi_1)\right]. \tag{3.9}$$

Then $v_1 = v$ on $\bar{O}$ because $\mathbb{P}^x(\Pi = \hat{\Pi} = \Pi_1 = \hat{\Pi}_1, \zeta = \hat{\zeta} = \zeta_1 = \hat{\zeta}_1) = 1$. Proposition 3.13 also implies that (ζ_1, Π_1) is continuous under $\mathbb{P}^x$ for all $x \in \bar{O}$. Therefore, v_1 is continuous in $\bar{O}$ due to Lemma 3.6, and so is v. Finally, Lemma 3.2 concludes the main result. ∎

Remark 3.14. Notice that in the definition of v in (1.2), we adopt the random time $\zeta = \tau_{\bar{O}}$, instead of possible alternative choices $\hat{\zeta} = \tau_O$ or $\bar{\zeta}(\omega) = \inf\{t \geq 0 : \omega_t \notin O\}$. All three are stopping times, and the main difference can be summarized as follows: $\bar{\zeta}$ is an entrance time to O^c, while $\hat{\zeta}$ and ζ are hitting times to O^c and $\bar{O}^c$, respectively. From the proof of Theorem 2.6, under the assumptions made, Proposition 3.13 tells us that

$$\mathbb{P}^x \left(\zeta = \hat{\zeta}, \Pi = \hat{\Pi}\right) = 1 \ \forall x \in \bar{O}$$

always holds. Therefore, if we denote as $\hat{v}$ and $\bar{v}$ the Feynman–Kac functionals with the random time ζ being replaced by $\hat{\zeta}$ and $\bar{\zeta}$ in (1.2), respectively, then $v = \hat{v}$ on $\bar{O}$. Hence, Theorem 2.6 still holds with v replaced by $\hat{v}$ without extra efforts. The main reason to adopt ζ is for convenience throughout the presentation.

On the other hand, Theorem 2.6 does not hold anymore, if v is replaced by $\bar{v}$. Indeed, under the same assumptions of Theorem 2.6,

$$\mathbb{P}^x \left(\zeta = \hat{\zeta} = \bar{\zeta}, \Pi = \hat{\Pi} = \bar{\Pi}\right) = 1,$$

and therefore $v = \hat{v} = \bar{v}$, but only for $x \in O \cup \partial_0 O$. As an example, consider the stochastic exit example with $\epsilon = 0$ given in section 2.1. It is a straightforward calculation that

$$v(x) = \hat{v}(x) = \bar{v}(x) \ \forall x \in (0, 1],$$

while

$$v(0) = \hat{v}(0) = 1 - e^{-1} \neq \bar{v}(0) = 0.$$

The next is an immediate consequence of Theorem 2.6, which prepares us for the nonstationary problems discussed in the following section.

COROLLARY 3.15. *Let O be a cylinder set of the form $O = (0, 1) \times A$ for some open set A. If* (1) *$\bar{A}^{c,*} = A^c$ with respect to $X_{-1} := (X_2, \ldots, X_d)$; and* (2) *$X_1$ is a subordinate process, then v of* (1.2) *is a generalized viscosity solution of* (1.1) *with $\Gamma_{out} \supset \{1\} \times A$.*

Proof. Since X is a Feller process, both X_1 and X_{-1} are Feller processes, and $\bar{O}^{c,*} = \bar{O} \setminus (\{0\} \times A)$. To apply Theorem 2.6, we can take $N_1 = \cup_{x \in A} N(x)$, where $N(x)$ is the neighborhood of $(0, x)$ given by

$$N(x) = \left(-\frac{\rho_x}{2}, \frac{\rho_x}{2}\right) \times B_{\frac{\rho_x}{2}}(x)$$

with $\rho_x = dist(x, \partial A) \wedge 1$. ∎

4. Applications to nonstationary problems. In this section, we apply the main results in Theorem 2.6 to consider the strong solutions of two nonstationary equations involving fractional Laplacian operators, one being linear and the other nonlinear. Given the state space O, its nonstationary (parabolic) domain Q_T and its nonstationary boundary $\mathcal{P}Q_T$ are defined by

$$Q_T := (0, T) \times O, \quad \mathcal{P}Q_T := (0, T] \times \mathbb{R}^d \setminus Q_T.$$

Given an operator $G(u, t, x)$, the viscosity solution of nonstationary problem

$$G(u, t, x) = 0 \text{ on } Q_T \text{ and } u = 0 \text{ on } \mathcal{P}Q_T \tag{4.1}$$

can be defined similarly as in Definition 2.4 for the stationary problem.

Definition 4.1.

1. *Given $u \in USC(\bar{Q}_T)$ and $(t,x) \in \bar{Q}_T$, the space of supertest functions is*

$$J^+(u,t,x) = \{\phi \in C_0^\infty(\mathbb{R}^{d+1}),\ \textit{s.t.}\ \phi \geq (uI_{\bar{Q}_T})^* \textit{ and } \phi(t,x) = u(t,x)\}.$$

We say u satisfies the viscosity subsolution property at (t,x) if $G(\phi,t,x) \leq 0$ for all $\phi \in J^+(u,t,x)$.

2. *Given $u \in LSC(\bar{Q}_T)$ and $(t,x) \in \bar{Q}_T$, the space of subtest functions is*

$$J^-(u,t,x) = \{\phi \in C_0^\infty(\mathbb{R}^{d+1}),\ \textit{s.t.}\ \phi \leq (uI_{\bar{Q}_T})_* \textit{ and } \phi(t,x) = u(t,x)\}.$$

We say u satisfies the viscosity supersolution property at (t,x) if $G(\phi,t,x) \geq 0$ for all $\phi \in J^-(u,t,x)$.

3. *A function $u \in C(\bar{Q}_T)$ is a viscosity solution (of* (4.1)*) if* (i) *u satisfies both the viscosity subsolution and supersolution properties at each $(t,x) \in Q_T$ and* (ii) *$u \equiv 0$ on $\mathcal{P}Q_T \cap \partial Q_T$.*

The nonlinear equation we are interested in is

$$-\partial_t u - |\nabla_x u|^\gamma + (-\Delta_x)^{\alpha/2} u + 1 = 0 \text{ on } Q_T \text{ and } u = 0 \text{ on } \mathcal{P}Q_T, \tag{4.2}$$

where for a function ϕ on $(t,x) \in \mathbb{R} \times \mathbb{R}^d$, the fractional Laplacian operator

$$(-\Delta_x)^{\alpha/2}\phi(t,x) = (-\Delta)^{\alpha/2}\phi(t,\cdot)(x),$$

and for a function $\tilde{\phi}$ on $x \in \mathbb{R}^d$,

$$-(-\Delta)^{\alpha/2}\tilde{\phi}(x) = C_d \int_{\mathbb{R}^d\setminus\{0\}} [\tilde{\phi}(x+y) - \tilde{\phi}(x) - y \cdot D\tilde{\phi}(x) I_{B_1}(y)] \frac{dy}{|y|^{d+\alpha}}$$

with some normalization constant C_d, and the index of the fractional Laplacian operator $\alpha \in (0,2)$.

Such a form of equations naturally arises in many applications. If $\gamma = 1$, then (4.2) becomes an HJB equation with $-|\nabla_x u| = \inf_{b \in B_1}(b \cdot \nabla_x u)$ (see [12] and its important roles in stochastic control problems in [16, 25, 32, 33]); if $\gamma > 1$, then (4.2) becomes a deterministic KPZ equation (see [1]). It can also be regarded as an HJB equation because $-|\nabla_x u|^\gamma = \inf_{b\in\mathbb{R}^d}(-b\cdot\nabla_x u + L(b))$ with $L(b) = \sup_{p\in\mathbb{R}^d}(p\cdot b - H(p))$ being the Legendre transform of the function $H(p) = |p|^\gamma$ (see section 3.3 of [14]).

4.1. Linear equation. To analyze the solvability of (4.2), we first consider a linear equation of a slightly more general form

$$\partial_t u + \mathbf{b} \cdot \nabla_x u - |\sigma|^\alpha (-\Delta_x)^{\alpha/2} u + \ell = 0 \text{ on } Q_T \text{ and } u = 0 \text{ on } \mathcal{P}Q_T, \tag{4.3}$$

where $\mathbf{b}$ is a Lipschitz continuous vector field $\mathbb{R}^d \mapsto \mathbb{R}^d$ known as a drift, σ is a constant in $\mathbb{R}^d \times \mathbb{R}^d$ known as a volatility, and l is Lipschitz continuous function $\mathbb{R}^{d+1} \mapsto \mathbb{R}$. Define the associated stochastic process X as

$$dX_t = \mathbf{b}(X_t)dt + \sigma dJ_t, \tag{4.4}$$

where J is an isotropic α-stable process for some $\alpha \in (0,2)$ with its generating triplets (see notions of Levy process in [30] or [6])

$$A = 0,\ \nu(dy) = \frac{1}{|y|^{d+\alpha}}dy,\ b = 0.$$

There exists a unique strong solution for (4.4), and X is a Feller process. It has a càdlàg version, with its generator $\mathcal{L}$ satisfying that its domain $D(\mathcal{L}) \supset C^2(\mathbb{R}^d)$. In particular, if $\phi \in C^2(\mathbb{R}^d)$, then $\mathcal{L}$ is consistent to the following integro-differential operator:

$$\mathcal{L}\phi(x) = \mathbf{b}(x) \cdot \nabla\phi(x) - |\sigma|^\alpha (-\Delta)^{\alpha/2}\phi(x). \tag{4.5}$$

With obvious extension of $\mathcal{L}\phi(x)$ to partial operator given by $\mathcal{L}_x u(t,x) = \mathcal{L}u(t,\cdot)(x)$, PDE (4.3) becomes

$$\partial_t u + \mathcal{L}_x u + \ell = 0 \text{ on } Q_T \text{ and } u = 0 \text{ on } \mathcal{P}Q_T.$$

Next, we solve the above nonstationary PDE via the solution of a stationary PDE and its associated random process: if (4.3) has a smooth solution u in $\bar{Q}_T$, then the change of variable of

$$y = (t,x) \in \mathbb{R}^{d+1}, \quad w(y) = e^{\lambda t} u(t,x) \tag{4.6}$$

with a given constant $\lambda > 0$ implies that w satisfies the following stationary equation with the domain in $\mathbb{R}^{d+1}$:

$$-\mathcal{L}_1 w(y) + \lambda w(y) - \ell_1(y) = 0 \text{ on } Q_T \quad \text{and } w(y) = 0 \text{ on } \mathcal{P}Q_T \cap \partial Q_T, \tag{4.7}$$

where $\mathcal{L}_1 w(y) = (\partial_{y_1} w + \mathcal{L}_{y_{-1}} w)(y)$, $\ell_1(y) = e^{\lambda y_1}\ell(y_1, y_{-1})$, and $y_{-1} = [y_2, \dots, y_{d+1}]^T$ is a d-dimensional column vector with elements of the vector y except the first scalar y_1. In particular, $\mathcal{L}_1$ is the generator of $\mathbb{R}^{d+1}$-valued Markov process $s \mapsto Y_s = (t+s, X_{t+s})$ for X of (4.4), which follows the following dynamics:

$$dY_s = \mathbf{b}_1(Y_s)dt + \sigma_1 dJ_t, \; Y_0 := y = (t, X_t), \tag{4.8}$$

where $\mathbf{b}_1(y) = \left[\begin{smallmatrix} 1 \\ \mathbf{b}(t,x) \end{smallmatrix}\right]$ and $\sigma_1 = \left[\begin{smallmatrix} 0_{1\times d} \\ I_d \end{smallmatrix}\right]\sigma$ with $d \times d$ identity matrix I_d and d-dimensional zero row vector $0_{1\times d}$. Theorem 2.6 can be applied to check if the Feynman–Kac functional associated to the random process (4.8) is a generalized viscosity solution of the stationary PDE (4.7).

Furthermore, with additional regularity conditions, we show in the following that the generalized viscosity solution coincides with the viscosity solution in the sense of Definition 4.1. As a preparation, we define the exterior cone condition.

DEFINITION 4.2. *For $y \in \mathbb{R}^d \setminus \{0\}$ and $\theta \in (0,\pi)$, define the cone $C(y,\theta)$ with the direction y and aperture θ as*

$$C(y,\theta) = \{x \in \mathbb{R}^d : x \cdot y > |x| \cdot |y| \cdot \cos\theta\}.$$

Denote as $C_r(y,\theta)$ the truncated cone by B_r, i.e., $C_r(y,\theta) = C(y,\theta) \cap B_r$. O satisfies the exterior cone condition with $C_{r(x)}(\mathbf{v}_x, \theta_x)$ if there exists $r(x) : \mathbb{R}^d \to \mathbb{R}^+$, $\mathbf{v}_x : \mathbb{R} \to \mathbb{R}^d \setminus \{0\}$, and $\theta_x : \mathbb{R}^d \to (0,\pi)$, such that for each $x \in \partial O$, its associated truncated exterior cone $x + C_{r(x)}(\mathbf{v}_x, \theta_x) \subset O^c$.

COROLLARY 4.3. *Let $\mathbf{b}$ be Lipschitz and σ be a constant, and O be a bounded open set satisfying the exterior cone condition. If $(\mathbf{b},\sigma)$ satisfies one of the following conditions* (A1)–(A3),

(A1) $|\sigma| > 0$ *and* $\alpha \geq 1$;

(A2) $|\sigma| > 0$ *and* $\mathbf{b} \equiv 0$;

(A3) $\mathbf{b}(x) \cdot \mathbf{v}_x > 0$ *for all* $x \in \partial O$,
then the function v_1 *defined by*

$$v_1(t,x) = \mathbb{E}^{t,x}\left[\int_t^{\zeta\wedge T} \ell(s, X_s)ds\right] \tag{4.9}$$

is a viscosity solution of (4.3), *where* ζ *is defined as lifetime* $\tau_{\bar{O}}(X)$ *for* X *in* (4.4).

Proof. For $w(t,x) = e^{\lambda t} v_1(t,x)$ and $r = s - t$,

$$w(t,x) = e^{\lambda t}\mathbb{E}^{t,x}\left[\int_t^{\zeta\wedge T} \ell(s, X_s)ds\right] = e^{\lambda t}\mathbb{E}^{t,x}\left[\int_0^{\zeta\wedge T-t} \ell(r+t, X_{r+t})dr\right].$$

With $Y_s = (t+s, X_{t+s})$ as a $d+1$ dimensional process, and ζ_1 as the lifetime of Y in the state space $\bar{Q}_T$, Y follows the dynamic of (4.8) with initial state $Y_0 = (t, X_t)$, and ζ_1 satisfies

$$\zeta_1 := \tau_{\bar{Q}_T}(Y) = \zeta \wedge T - t.$$

Therefore, w can be represented in terms of Y:

$$w(t,x) = e^{\lambda t}\mathbb{E}^{t,x}\left[\int_0^{\zeta_1} \ell(Y_r)dr\right].$$

Since $Y_1(r) = t + r$, a further substitution of $\ell_1(y) = e^{\lambda t}\ell(y)$ leads to

$$w(t,x) = \mathbb{E}^{t,x}\left[\int_0^{\zeta_1} e^{-\lambda r}e^{\lambda(t+r)}\ell(Y_r)dr\right] = \mathbb{E}^{y}\Big[\int_0^{\zeta_1} e^{-\lambda r}\ell_1(Y_r)dr\Big].$$

Since O satisfies the exterior cone condition, and one of the conditions (A1)–(A3) holds, Proposition A.2 of section A.2 shows that every point of ∂O is regular to $\bar{O}^c$. Then by Corollary 3.15, w is a generalized viscosity solution of (4.7), and $w(t,x) = 0$ if either $t = T$ or $x \in O^c$. Therefore, according to Definition 4.1, v_1 is the viscosity solution of (4.3). ∎

4.2. Nonstationary nonlinear equation. Back to the nonlinear equation (4.2),

$$\begin{cases} -\partial_t u - |\nabla_x u|^\gamma + (-\Delta_x)^{\alpha/2}u + 1 = 0 & \text{on } Q_T; \\ u = 0 & \text{on } \mathcal{P}Q_T. \end{cases}$$

As a starting point, we recall the following result about its sovability (see also [2, 12]), which will be referred to as (CP + PM) in the rest of this section:

> (CP + PM) Suppose the comparison principle holds and Perron's method is valid. If there exist sub- and supersolutions, then (4.2) is uniquely solvable.

To concentrate on the application of the Feynman–Kac functional as a generalized viscosity solution, we will not pursue the validity of (CP+PM) and take it as granted in the discussion below. The next proposition shows that our results about the linear equation (4.3) above help establish the semisolutions of (4.2), as a preparation for (CP+PM) argument.

Proposition 4.4. *Let* O *be a bounded open set satisfying the exterior cone condition. If* $\gamma \geq 1$ *and* $\alpha \in (0,2)$, *then there exist viscosity sub- and supersolutions of* (4.2).

Proof. First $u = 0$ is supersolution. On the other hand, Corollary 4.3 confirms that the stochastic representation v_1 of (4.9) with $X \sim -(-\Delta_x)^{\alpha/2}$ is the viscosity solution for

$$\begin{cases} -\partial_t u + (-\Delta_x)^{\alpha/2} u + 1 = 0 & \text{on } Q_T := (0,T) \times O; \\ u = 0 & \text{on } \mathcal{P}Q_T := (0,T] \times \mathbb{R}^d \setminus Q_T. \end{cases}$$

By the nonnegativity of $|\nabla_x u|^\gamma$, v_1 is also a viscosity subsolution of (4.2). ◻

5. Summary. In this paper, we provide the sufficient condition for v of (1.2) to be the generalized viscosity solution of (1.1) in Theorem 2.6. To the best of our knowledge, this is the first result for the verification of the Feynman–Kac functional as the generalized viscosity solution of the Dirichlet problem in the presence of jump diffusion. We also provide Example 3.4, where the assumptions in Theorem 2.6 do not hold and the Feynman–Kac functional fails to be continuous. Not to distract readers from the main idea, but we have rather strong assumptions (Assumption 2.1) on g, ℓ, and λ. However, these conditions could be appropriately relaxed with some mild integrability conditions.

Although the proof of Theorem 2.6 is mainly probabilistic, it gives an alternative constructive proof for the existence of a generalized viscosity solution on integro-differential equations with Dirichlet boundary, which could be utilized for the solvability of nonlinear equation together with the comparison principle and Perron's method. In other words, Theorem 2.6 together with the probabilistic regularity, e.g., as in Proposition A.2, yields a purely analytical result on the solvability of the Dirichlet problem. As an application, we considered an $\mathbb{R}^{d+1}$-valued process on a cylinder domain $Q_T = (0,T) \times O$ (see Corollary 4.3). If X_1 is uniform motion in time (i.e., $dX_1(t) = dt$) and $X_{-1} = (X_2, \dots, X_{d+1})$ is an $\mathbb{R}^d$-valued process with each point of ∂O regular for $\bar{O}^c$, then the corresponding Feynman–Kac functional is easily verified as the generalized viscosity solution of the stationary problem (4.7). Moreover, if one replaces the uniform motion X_1 by a subordinate process, assumptions of Theorem 2.6 can be verified analogously.

It is desirable to check if the value of associated stochastic control problem (or nonlinear Feynman–Kac functional) coincides with the solution of (4.2) constructed from semisolutions and Perron's method. On the other hand, relaxing the assumption of $\lambda > 0$ may result in an extension to gauge theory (see [8] and [31]). Both are interesting topics for our future work.

Appendix A. Some auxiliary reusults.

A.1. Characterization of $\boldsymbol{\Gamma_{out}}$. From the definitions of v in (1.2) and of $\Gamma_{out} := \Gamma_{out}[v] = \{x \in \partial O : v = g\}$, Γ_{out} depends on the function g via v, and we explicitly write it as $\Gamma_{out}[v, g]$, or briefly as $\Gamma_{out}[g]$ in this section with fixed function v.

LEMMA A.1. *If $\mathbb{P}^x(\zeta < \infty) = 1$ for every x, then $\cap_{g \in C_0^{0,1}(\mathbb{R}^d)} \Gamma_{out}[g] = \partial_0 O$.*

Proof. Lemma 3.2 implies that

$$\cap_{g \in C_0^{0,1}(\mathbb{R}^d, \mathbb{R})} \Gamma_{out}[g] \supset \partial_0 O.$$

On the other hand, for any $x_0 \in \partial_1 O$, take

$$g(x) = e^{-|x - x_0|} \frac{\|\ell\|_\infty / \lambda + 1}{1 - p(x_0)},$$

where $p(x_0) = \mathbb{E}^{x_0}[e^{-\lambda\zeta}]$. Since $x_0 \in \partial_1 O$, $\mathbb{P}^{x_0}(\zeta > 0) > 0$, and $\mathbb{P}^{x_0}(\zeta < \infty) = 1$ by assumption, $p(x_0) \in (0,1)$ and g is a well-defined strictly positive function in $C_0^{0,1}(\mathbb{R}^d)$. Furthermore, (1.2) yields an estimate of v:

$$v(x) < 1 + \frac{\|\ell\|_\infty}{\lambda} + \|g\|_\infty p(x_0) = 1 + \frac{\|\ell\|_\infty}{\lambda} + \frac{\|\ell\|_\infty/\lambda + 1}{1 - p(x_0)} p(x_0) \tag{A.1}$$

$$= \frac{\|\ell\|_\infty/\lambda + 1}{1 - p(x_0)} = g(x_0). \tag{A.2}$$

Thus $v(x_0) \neq g(x_0)$ and $x_0 \notin \cap_{g \in C_0^{0,1}(\mathbb{R}^d)} \Gamma_{out}[g]$. By the arbitrariness of $x_0 \in \partial_1 O$, $\cap_{g \in C_0^{0,1}(\mathbb{R}^d)} \Gamma_{out}[g] = \partial_0 O$. ∎

A.2. Regularity under the exterior cone condition. In this section, we prove the regularity condition used in Corollary 4.3 for the diffusion X satisfying

$$dX_t = \mathbf{b}(X_t)dt + \sigma dJ_t.$$

PROPOSITION A.2. *Let $\mathbf{b}$ be Lipschitz and σ be a constant, and O be a bounded open set satisfying the exterior cone condition with $C_{r(x)}(\mathbf{v}_x, \theta_x)$. In addition, assume that $(\mathbf{b}, \sigma)$ satisfies one of the conditions of* (A1)–(A3). *Then, any $x \in O^c$ is regular for the set $\bar{O}^c$ with respect to the process* (4.4), *i.e., $O^c = \bar{O}^{c,r} = \bar{O}^{c,*}$.*

Proof. By the right continuity of the sample path, $\bar{O}^c \subset \bar{O}^{c,r}$ and $O \cap \bar{O}^{c,r} = \emptyset$. Therefore, it suffices to verify that $\partial O \subset \bar{O}^{c,r}$.

Fix $x \in \partial O$, and let $Y = X \cdot \mathbf{v}_x$ be the projection of the process X of (4.4) on the unit vector $\mathbf{v}_x$ pointing to the direction of the exterior cone. Then, Y has a representation of

$$dY_t = \hat{\mathbf{b}}(X_t)dt + \hat{\sigma} d\hat{J}_t, \ Y_0 = x \cdot \mathbf{v}_x,$$

where $\hat{\mathbf{b}}(x) = \mathbf{b}(x) \cdot \mathbf{v}_x$, $\hat{\sigma} = |\mathbf{v}_x' \sigma|$, and $\hat{J}$ is an isotropic one-dimensional α-stable process with its generating triplets $A = 0$, $\nu(dz) = \frac{1}{|z|^{1+\alpha}} dz$, $b = 0$. To see that $\hat{J}$ is indeed an α-stable process, notice that the characteristic function of J_1 is

$$\mathbb{E}[\exp\{iu \cdot J_1\}] = e^{-c_0|u|^\alpha} \ \forall u \in \mathbb{R}^d$$

for some normalizing constant c_0. Therefore, the characteristic function of $\hat{J}_1$ is

$$\mathbb{E}[\exp\{iu \cdot \hat{J}_1\}] = \mathbb{E}[\exp\{iu\mathbf{v}_x \cdot J_1\}] = e^{-c_0|u\mathbf{v}_x|^\alpha} = e^{-c_0|u|^\alpha} \ \forall u \in \mathbb{R},$$

and hence $\hat{J}$ is an α-stable process.

By the definition of the exterior cone condition, the regularity of x for $\bar{O}^c$ with respect to the process X can be implied by the regularity of $y = x \cdot \mathbf{v}_x$ for the open line segment $(y, y + r_x)$ with respect to the process Y. Moreover, due to the right continuity of the sample path, it is equivalent to check the regularity of y with respect to the half line (y, ∞), i.e., $\mathbb{P}^y\left(\tau_{(-\infty,y]}(Y) = 0\right) = 1$.

- If $|\sigma| > 0$ and $\alpha \geq 1$, then consider

$$\hat{Y}_t = y - \sup_{x \in \bar{O}} |\mathbf{b}(x)| t + \hat{\sigma} \hat{J}_t.$$

  Note that $\hat{Y}_t \leq Y_t$, but $\hat{Y}$ is a Type C process by [30] and $\mathbb{P}^y(\tau_{(-\infty,y]}(\hat{Y}) = 0) = 1$. Therefore, $\mathbb{P}^y(\tau_{(-\infty,y]}(Y) = 0) = 1$.
- If $|\sigma| > 0$ and $\mathbf{b} \equiv 0$, then X is simply an isotropic Levy process and $\mathbb{P}^y\left(\tau_{(-\infty,y]}(Y) = 0\right) = 1$.

- If $\hat{\mathbf{b}}(x) = \mathbf{b}(x) \cdot \mathbf{v}_x > 0$, then define $h := \inf\{t \geq 0 : \hat{\mathbf{b}}(X_t) < \frac{1}{2}\hat{\mathbf{b}}(x)\}$. Due to the right continuity of $t \mapsto \hat{\mathbf{b}}(X_t)$, $h > 0$ $\mathbb{P}^x$-almost surely. Consider

$$\hat{Y}_t = y + \frac{1}{2}\hat{\mathbf{b}}(x)t + \hat{\sigma}\hat{J}_t,$$

and then $Y_t \geq \hat{Y}_t$ on $(0, h)$. Moreover, by Theorem 47.5 of [30], $\hat{Y}$ is a Type B process with $\frac{1}{2}\hat{\mathbf{b}}(x) > 0$, and $\mathbb{P}^y(\tau_{(-\infty,y]}(\hat{Y}_t) = 0) = 1$. Therefore, we obtain

$$\mathbb{P}^y\left(\tau_{(-\infty,y]}(Y) = 0\right) = 1.$$

This completes the proof. ∎

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