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OPTIMAL CONTROL OF DIFFUSION PROCESSES AND HAMILTON-JACOBI-BELLMAN EQUATIONS

PART 2 : VISCOSITY SOLUTIONS AND UNIQUENESS

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Abstract : We consider general optimal stochastic control problems and the associated Hamilton-Jacobi-Bellman equations. We develop a general notion of weak solutions - called viscosity solutions - of the Hamilton-Jacobi-Bellman equations that is stable and we show that the optimal cost functions of the control problems are always solutions in that sense of the Hamilton-Jacobi-Bellman equations. We then prove general uniqueness results for viscosity solutions of the Hamilton-Jacobi-Bellman equations.

Résumé : Nous considérons des problèmes généraux de contrôle optimal stochastique et les équations de Hamilton-Jacobi-Bellman qui leur sont associées. Nous développons une notion de solutions faibles - appelées solutions de viscosité - des équations de Hamilton-Jacobi-Bellman qui est stable et nous montrons que les fonctions coût optimum des problèmes de contrôle sont toujours solutions en ce sens des équations de Hamilton-Jacobi-Bellman. Nous démontrons ensuite des résultats d'unicité généraux pour les solutions de viscosité des équations de Hamilton-Jacobi-Bellman.

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Mots-clés : Contrôle optimal stochastique, équations de Hamilton-Jacobi-Bellman, Principe de la programmation dynamique, solutions de viscosité.

Introduction :

This paper is the second one of a series concerning the optimal control of diffusion processes and the associated Hamilton-Jacobi-Bellman equations. In particular we will keep the notations (and main assumptions) of Part 1 (P.L. Lions [19]) that we recall briefly below.

An admissible system $\mathcal{A}$ will be the collection of i) a probability space (Ω,F,F_t,P) with the usual assumptions, ii) a Brownian motion in R^m adapted to F_t, iii) a progressively measurable process α_t - the control - with values in a compact set of a given separable metric space A. The state of the system is given by the solution of the following stochastic differential equation :

$$(1) \qquad dX_t = \sigma(X_t,\alpha_t)dB_t + b(X_t,\alpha_t)dt \qquad \text{for } t \geq 0 \text{ , } X_0 = x$$

where $\sigma(x,\alpha)$ and $b(x,\alpha)$ are given vector fields satisfying assumptions detailed in section I.1. We then introduce for each admissible system $\mathcal{A}$ a cost function $J(x,\mathcal{A})$ given by :

$$(2) \qquad J(x,\mathcal{A}) = E\left\{\int_0^T f(X_t,\alpha_t)\exp\left[-\int_0^t c(X_s,\alpha_s)ds\right] + \right.$$
$$\left. + \varphi(X_T)\exp\left[-\int_0^T c(X_t,\alpha_t)dt\right]\right\}$$

where $f(x,\alpha)$, $c(x,\alpha)$, φ are given real-valued functions satisfying appropriate conditions detailed in section I.1 below. And τ is the first exit time of the process X_t from $\bar{\mathcal{O}}$ ($\tau = \inf (t \geq 0 \,,\, X_t \notin \bar{\mathcal{O}})$).

Finally we consider the optimal cost function :

$$(3) \qquad u(x) = \inf_{\mathcal{A}} J(x,\mathcal{A})$$

As we explained in Part 1, the heuristic dynamic programming principle - due to R. Bellman [2] - indicates that u should satisfy the following equation called Hamilton-Jacobi-Bellman equation (HJB in short) :

$$(4) \qquad \sup_{\alpha} \, (A_\alpha u - f(x,\alpha)) = 0 \qquad \text{in } \mathcal{O}$$

where $A_\alpha = -a_{ij}(x,\alpha)\partial_{ij} - b_i(x,\alpha)\partial_i + c(x,\alpha)$ and $a = \frac{1}{2}\sigma\sigma^T$ - we will always use the usual convention on repeated indices. This equation indeed holds if we know a priori that $u \in C^2(\mathcal{O})$ - or if (4) has a smooth solution $\tilde{u}$ with φ as boundary conditions on Γ and then $u = \tilde{u}$ - : we refer the reader to W.H. Fleming and R. Rishel [13], A. Bensoussan and J.L. Lions [3], N.V. Krylov [15].

The main questions related with the derivation of (4) are the following : i) since in many examples, u is not C^2 and sometimes not even continuous, is u a weak solution of (4) and in which sense ?, ii) Once a weak solution of (4) is given, is u the unique solution in this sense of (4) with appropriate boundary conditions ? In other words, does (4) characterize u ?

In Part 1, we already gave one possible characterization of u or one interpretation of (4) : u was proved to be the maximum subsolution of (4),

In this paper, we answer the above questions with the help of the notion of viscosity solutions of second-order nonlinear elliptic equations. This notion was introduced - in the case of first-order Hamilton-Jacobi-Bellman equations - by M.G. Crandall and the author [6], [7] - see also M.G. Crandall, L.C. Evans and P.L. Lions [5], P.L. Lions [20]. We recall the definition and elementary properties in sections I.2-3 : let us just mention for the moment that this gives a notion of solutions of (4) which are required to be only continuous in $\mathcal{O}$; moreover this notion is stable with respect to the topology of uniform convergence over compact sets. Two essential properties that we prove are :
i) as soon as u given by (3) is continuous in $\mathcal{O}$, then u is a viscosity solution of (4) (see section I.4); ii) If we have some regularity properties of a viscosity solution of (4), then (4) holds in a usual way - see section I.5 for precise statements. Let us also recall that Part 1 was essentially devoted to the study of the continuity of u and we refer to [19] for conditions insuring that $u \in C(\mathcal{O})$.

The second section is devoted to the study of the uniqueness of viscosity solutions of (4) ; we prove for example that if $\tilde{u} \in C(\bar{\mathcal{O}})$ is a viscosity solution of (4) then $\tilde{u}$ is given by (3) where, in the definition of J, we replace φ by $\tilde{u}$. In particular there is at most one viscosity solution $\tilde{u} \in C_b(\bar{\mathcal{O}})$ of (4) with prescribed values on Γ . We also mention a few applications and extensions of this uniqueness result in particular to the time-dependent case and the optimal stopping problem.

Let us point out that the results of this paper which provides complete answers to the questions raised above will be also used in the next parts to give very short proofs and extensions of stability results due to N.V. Krylov [16], [17] and also to show that (4) holds in usual ways as soon as regularity results are proved for u.

Most of the results proved here were announced in P.L. Lions [21], [22].

Let us also indicate that it is possible to extend the methods below to treat the optimal control of jump diffusion processes or the impulse control problems but we shall not do so here.

SUMMARY

I. VISCOSITY SOLUTIONS OF HJB EQUATIONS.

I.1. Notations and assumptions.

We will first assume that the coefficients σ, b, c satisfy :

$$(5)\qquad \begin{cases} \sup_{\alpha \in A} \|\psi(\cdot,\alpha)\|_{W^{2,\infty}(\mathbb{R}^N)} < \infty \text{ , if } \psi = \sigma_{ij}, b_i, c \\ \qquad\qquad\qquad\qquad (1 \leqslant i \leqslant N \text{ , } 1 \leqslant j \leqslant m) \text{ ,} \\ \psi(x,\cdot) \text{ is continuous on } A \text{ for each } x \in \mathbb{R}^N \text{ ;} \end{cases}$$

and that the functions (defining the cost) f, φ satisfy :

$$(6)\qquad \begin{cases} f(\cdot,\alpha) \text{ is bounded uniformly continuous on } \mathbb{R}^N, \\ \qquad\qquad \text{uniformly in } \alpha \in A \text{ ,} \\ f(x,\cdot) \text{ is continuous on } A \text{ for all } x \in \mathbb{R}^N \text{ ;} \end{cases}$$

$$(7)\qquad \varphi \in C_b(\mathbb{R}^N) \text{ .}$$

Obviously in view of (5), for each admissible system $\mathcal{A}$ and for each $x \in \mathbb{R}^N$ there exists a unique continuous process X_t solution of (1) : we will denote $X_t = X_t^x$ or $X(t,x)$ to recall the dependence upon x.

Next, the cost function given by (2) is surely well defined if we assume :

$$(8)\qquad \lambda = \inf \{c(x,\alpha) \;/\; x \in \mathbb{R}^N \text{ , } \alpha \in A\} > 0 \text{ ;}$$

we agree that $\tau = +\infty$ if $X_t \in \bar{\mathcal{O}}$ for all $t \geqslant 0$.

In all that follows $\mathcal{O}$ will be an open set in $\mathbb{R}^N$ and we will denote by Γ its boundary, $d(x) = \mathrm{dist}(x,\Gamma)$ and by $\mathcal{O}_\delta$ the open set given by : $\mathcal{O}_\delta = \{x \in \mathcal{O} \text{ , } d(x) > \delta \text{ , } |x| < 1/\delta\}$.

In the next sections, we will consider weak solutions of fully nonlinear elliptic equations of the form :

(9) $$F(D^2u,Du,u,x) = 0 \quad \text{in } \mathcal{O}$$

where F is a given continuous function on $S^N \times R^N \times R \times \mathcal{O}$ – where S^N is the space of symmetric N x N matrices – and where the ellipticity of (9) is expressed by the following condition :

(10) $$F(A,p,t,x) \geq F(B,p,t,x) \quad \text{if } A \leq B \text{ ; } A,B \in S^N \text{ , } p \in R^N \text{,}$$
$$t \in R \text{ , } x \in \mathcal{O} \text{ .}$$

Remark I.1 : If F is of class C^1, (10) reduces to the usual ellipticity condition :

$$\frac{\partial F}{\partial \xi_{ij}}(\xi,p,t,x) \leq 0 \text{ , } \forall (\xi,p,t,x) \in S^N \times R^N \times R \times \mathcal{O} \text{ .}$$

Remark I.2 : Of course (4) is a special case of (9) where F is given by :

$$F(\xi,p,t,x) = \sup_{\alpha \in A} \{-a_{ij}(x,\alpha)\xi_{ij} - b_i(x,\alpha)p_i + $$
$$+ c(x,\alpha)t - f(x,\alpha)\} \quad ;$$

and (10) is obviously satisfied since $a_{ij}(x,\alpha) \geq 0$ on $R^N \times A$.

Let us finally remark that the above setting contains the case of parabolic equations :

$$\frac{\partial u}{\partial t} + F(D^2u,Du,u,x,t) = 0 \quad \text{in } \mathcal{O} \times (0,T)$$

by considering $y = (x,t) \in R^{N+1}$ and

$$\tilde{F}(\tilde{\xi},\tilde{p},u,y) = p_{N+1} + F(\xi,p,u,x,t)$$

for all $\tilde{\xi} \in S^{N+1}$, $\tilde{p} \in R^{N+1}$, $u \in R$, $y = (x,t) \in R^{N+1}$ and where $\xi \in S^N$, $\xi_{ij} = \tilde{\xi}_{ij}$ for $1 \leq i,j \leq N$; $p \in R^N$, $\tilde{p}_i = p_i$ for $1 \leq i \leq N$.

I.2 Definitions :

In order to define viscosity solutions of (9), we need a few notations concerning continuous functions in $\mathcal{O}$: let $\psi \in C(\mathcal{O})$, we consider the following sets for each $x \in \mathcal{O}$:

$$D^{+}_{2,1}\,\psi(x) = \Big\{(A,p) \in S^N \times R^N \;/\; \limsup_{y \to x,\ y \in \mathcal{O}} \Big\{ [\psi(y)-\psi(x)-(p,y-x)-\tfrac{1}{2}(A(y-x),y-x)]\,|y-x|^{-2}\Big\} \leqslant 0\Big\}$$

$$D^{-}_{2,1}\,\psi(x) = \Big\{(A,p) \in S^N \times R^N \;/\; \liminf_{y \to x,\ y \in \mathcal{O}} \Big\{ [\psi(y)-\psi(x)-(p,y-x)-\tfrac{1}{2}(A(y-x),y-x)]\,|y-x|^{-2}\Big\} \geqslant 0\Big\}$$

Remark I.3 : $D^{+}_{2,1}\,\psi(x)$ (resp. $D^{-}_{2,1}\,\psi(x)$) is a closed, convex set in $S^N \times R^N$, possibly empty.

Remark I.4 : It is easy to show that $D^{+}_{2,1}\,\psi$ (resp. $D^{-}_{2,1}\,\psi$) is non-empty on a dense subset of $\mathcal{O}$ (depending on ψ).

Remark I.5 : If $(A,p) \in D^{+}_{2,1}\,\psi(x)$ (resp. $D^{-}_{2,1}\,\psi(x)$) and if $B \geqslant A$ (resp. $B \leqslant A$) then $(B,p) \in D^{+}_{2,1}\,\psi(x)$ (resp. $D^{-}_{2,1}\,\psi(x)$).

Remark I.6 : If ψ satisfies

$$\psi(y) = \psi(x) + (p,y-x) + \frac{1}{2}(A(y-x),y-x) + o(|y-x|^2), \quad \forall\, y \in \mathcal{O}$$

for some $(p,A) \in R^N \times S^N$, then

$$D^{+}_{2,1}\,\psi(x) = \{(B,p) \in S^N \times R^N \;/\; B \geqslant A\}$$

$$D^{-}_{2,1}\,\psi(x) = \{(B,p) \in S^N \times R^N \;/\; B \leqslant A\}$$

This is the case in particular if $u \in C^2(\mathcal{O})$. ∎

We may now define viscosity solutions of (9) :

Definition I.1 : Let $u \in C(\mathcal{O})$; u is said to be a viscosity solution of (9) if the following holds:

(11) $$F(\xi,p,u(x),x) \leq 0 \quad , \quad \forall\ (\xi,p) \in D^+_{2,1}u(x) \ , \ \forall\ x \in \mathcal{O} \ ;$$

(12) $$F(\xi,p,u(x),x) \geq 0 \quad , \quad \forall\ (\xi,p) \in D^-_{2,1}u(x) \ , \ \forall\ x \in \mathcal{O} \ .$$

Remark I.7 : One defines in a similar way viscosity subsolutions (resp. supersolutions) of (9) as elements of $C(\mathcal{O})$ satisfying (11) (resp. (12)). All properties proved below are easily extended to sub or supersolutions.

Remark I.8 : Of course (11) or (12) are to be checked only at points where $D^+_{2,1}u$ or $D^-_{2,1}u$ are non-empty. ■

Let us immediately give an equivalent definition :

Proposition I.1 : *Let* $u \in C(\mathcal{O})$. *Then* u *is a viscosity solution of* (9) *if and only if we have for all* $\psi \in C^2(\mathcal{O})$:

(11') $$\begin{cases} \text{for each local maximum point } x_o \text{ of } u-\psi \text{ , we have :} \\ \quad F(D^2\psi(x_o),D\psi(x_o),u(x_o),x_o) \leq 0 \ ; \end{cases}$$

(12') $$\begin{cases} \text{for each local minimum point } x_o \text{ of } u-\psi \text{ , we have :} \\ \quad F(D^2\psi(x_o),D\psi(x_o),u(x_o),x_o) \geq 0 \ . \end{cases}$$

Remark I.9 : Equivalent definitions are obtained by replacing in the above statement $\varphi \in C^2(\mathcal{O})$ by $\varphi \in C^\infty(\mathcal{O})$ and local maximum point by local strict or global or global strict maximum point. The proofs are exactly the same as the one sketched below.

Remark I.10 : Under this form, one may get the feeling of the relations between the notion of viscosity solutions of (9) and accretivity theory (see M.G. Crandall and P.L. Lions [7], P.L. Lions [10] for further explanations in this direction). This also indicates the relations with some arguments introduced by L.C. Evans [8], [9].

Remark I.11 : From Proposition I.1 and Remark I.9 above, it is easy to check that, if F does not depend on D^2u (that is if (9) reduces to first order Hamilton-Jacobi equations), then the above definition is equivalent to the one used in [7], [5].

Since the proof of Proposition I.1 is a simple modification of similar proofs in [7], [5]; we only explain the key point in the proof : the proof is based upon the following elementary Lemma, proved exactly as a similar result is proved in L.C. Evans [8] :

Lemma I.1 : *Let* $u \in C(\mathcal{O})$, $x_o \in \mathcal{O}$. *Then* $(\xi,p) \in D^+_{2,1}u(x_o)$ *(resp.* $D^-_{2,1}u(x_o)$) *if and only if there exists* $\psi \in C^2(\mathcal{O})$ *such that :*

$$\begin{cases} u(x_o) = \psi(x_o) \quad , \quad p = D\psi(x_o) \quad , \quad \xi = D^2\psi(x_o) \quad ; \\ u(x) - \psi(x) < 0 \qquad \forall\, x \in \mathcal{O}\ ,\ x \neq x_o \qquad (\text{resp. } >) . \end{cases}$$

In the case when (9) has the structure of a nonlinear parabolic equation :

$$(9') \qquad \frac{\partial u}{\partial t} + F(D^2_x u, D_x u, u, x, t) = 0 \qquad \text{in } \mathcal{O} \times (0,T)$$

(where F is continuous and satisfies (10)), the following remark is useful.

Proposition I.2 : *Let* $u \in C(\mathcal{O} \times (0,T\,])$. *If* u *is a viscosity solution of* (9') *then for all* $\psi \in C^2(\mathcal{O} \times (0,T\,])$ *(or* $C^{2,1}(\mathcal{O} \times (0,T\,])$) *we have :*

$$(11'')\quad \begin{cases} \textit{at each local maximum point } (x_o,t_o) \textit{ in } \mathcal{O} \times (0,T\,] \textit{ of} \\ u-\psi\ , \textit{ we have } \dfrac{\partial\psi}{\partial t}(x_o,t_o) + F(D^2\psi,D\psi,u,x_o,t_o) \leqslant 0\ ; \end{cases}$$

$$(12'')\quad \begin{cases} \textit{at each local minimum point } (x_o,t_o) \textit{ in } \mathcal{O} \times (0,T\,] \textit{ of} \\ u-\psi\ , \textit{ we have } \dfrac{\partial\psi}{\partial t}(x_o,t_o) + F(D^2\psi,D\psi,u,x_o,t_o) \geqslant 0\ . \end{cases}$$

<u>Proof :</u> Let us prove (11") for example. Of course the only new case is when $t_o = T$. In addition, by replacing ψ by $\tilde{\varphi}(x,t) = \psi(x,t) + |x-x_o|^4 + (t-t_o)^2$, we may assume without loss of generality that $u-\psi$ has a local strict maximum point on $\mathcal{O} \times (0,T\,]$ at (x_o,T). Then, let $\delta > 0$ and let $\alpha_\delta > 0$ be given by : $\alpha_\delta = \inf \{(u-\psi)(x_o,T) - (u-\psi)(x,t)\ /\ |x-x_o| \leqslant \delta\ ,\ t = T-\delta$ or $|x-x_o| = \delta\ ,\ T-\delta \leqslant t \leqslant T\}$. We then set :

$$\psi_\varepsilon(x,t) = \psi(x,t) + \frac{\varepsilon}{T-t} \qquad \forall\ (x,t) \in \mathcal{O} \times (0,T)\ .$$

For ε small enough, the maximum of $u-\psi_\varepsilon$ over $Q_\delta = \{(x,t)\ /\ |x-x_o| \leqslant \delta\ ,\ T-\delta \leqslant t \leqslant T\}$ is attained at some interior point (x_δ,t_δ). Applying (11') we find at the point (x_δ,t_δ) :

$$\frac{\partial\psi}{\partial t}(x_\delta,t_\delta) + \frac{\varepsilon}{(T-t_\delta)^2} + F(D^2\psi,\ D\psi,u,x_\delta,t_\delta) \leqslant 0\ ,$$

in particular we have at the point (x_δ,t_δ) :

$$\frac{\partial\psi}{\partial t} + F(D^2\psi,D\psi,u,x_\delta,t_\delta) \leqslant 0$$

and sending δ to 0, we obtain (11"). ∎

I.3 <u>Elementary properties</u> :

Most of the properties described in M.G. Crandall and P.L. Lions [7] are easily adapted to the case of viscosity solutions of (9) (like invariance by change of unknown, change of variables, ...). The main property that we emphasize in this section is the stability with respect to the uniform convergence topology - this property is by no means obvious in view of the examples and counter-examples which already hold in the first-order case ; see [7], [20].

Proposition I.3 : *Let* $\varepsilon > 0$, *let* (F_ε) *be continuous functions on* $S^N \times R^N \times R \times \mathcal{O}$ *satisfying* (10) *and let* u_ε *be viscosity solutions of*

$$(9\text{-}\varepsilon) \qquad F\,(D^2u_\varepsilon, Du_\varepsilon, u_\varepsilon, x) = 0 \qquad \text{in } \mathcal{O}\,.$$

We assume that F_ε *converges on compact subsets of* $S^N \times R^N \times R \times \mathcal{O}$ *to some function* F *and that* u_ε *converges on compact subsets of* $\mathcal{O}$ *to some* u . *Then* u *is a viscosity solution of* (9).

Proof : As we remarked in Remark I.9, it is enough to check (11')-(12') for local strict extrema of $u-\psi$. Let us then check (11') for some $\psi \in C^2(\mathcal{O})$ such that : $u-\psi$ has a local strict maximum at $x_0 \in \mathcal{O}$. Let $\delta > 0$ be small enough such that $\overline{B(x_0,\delta)} \subset \mathcal{O}$, then

$$(u-\psi)(x_0) > \max_{\partial B(x_0,\delta)} (u-\psi)(x)$$

therefore by continuity for ε small enough (depending on δ) :

$$\max_{\overline{B(x_0,\delta)}} u_\varepsilon - \psi > \max_{\partial B(x_0,\delta)} (u_\varepsilon - \psi)\,.$$

Therefore there exists $x_\delta \in B(x_0,\delta)$ such that

$$\max_{\overline{B(x_0,\delta)}} (u_\varepsilon - \psi) = (u_\varepsilon - \psi)\,(x_\delta)$$

and $\varepsilon = \varepsilon(\delta) \to 0$ if $\delta \to 0_+$.

Since u_ε is a viscosity solution of (9-ε), we apply (11') to find :

$$F_\varepsilon(D^2\psi(x_\delta), D\psi(x_\delta), u_\varepsilon(x_\delta), x_\delta) \leqslant 0 \quad .$$

Now $x_\delta \to x_o$ if $\delta \to 0_+$ and $u_\varepsilon(x_\delta) \to u(x_o)$; since ψ is C^2, $D^2\psi(x_\delta) \to D^2\psi(x_o)$, $D\psi(x_\delta) \to D\psi(x_o)$ as $\delta \to 0_+$; and finally since F_ε converges uniformly on compact sets to F, we find sending δ to 0 :

$$F\ (D^2\psi(x_o), D\psi(x_o), u(x_o), x_o) \leqslant 0 \quad .$$

I.4 Dynamic programming and viscosity solutions.

So far we developped - independently of stochastic control considerations - a (stable) notion of weak solutions of general fully nonlinear elliptic equations and we explained how (4) is a very particular case of (9). The following fundamental and simple result explains the intrinsic relations between this notion and HJB equations ; and furthermore, as it will be apparent from the proof, this seems to be the right way of deriving (4) from the dynamic programming principle. We keep the notations of section I.1.

Theorem I.1 : *Let* u *be given by* (3) *and assume that* $u \in C(\mathcal{O})$; *then* u *is a viscosity solution of* (4) .

Remark I.12 : As we recalled in the introduction, we refer the reader to Part 1 [19] for many results insuring the continuity of u.

Remark I.13 : Of course this is only an example of the results one can obtain by similar methods : in particular the same result holds for the time-dependent case. ∎

Proof of Theorem I.1 : In order to illustrate the idea of the proof we begin by the classical derivation of (4) when $u \in C^2(\mathcal{O})$.

First case : $u \in C^2(\mathcal{O})$.

We want to derive (4) and we thus fix $x_o \in \mathcal{O}$. The main ingredient in the derivation of (4) is the mathematical formulation of the dynamic programming principle : we have (Cf. Theorem A of Part 1 [19], see K. Itô [14]) for all $h > 0$:

$$u(x_o) = \inf_{A} \Big\{ E \int_o^{\tau \wedge h} f(X_t, \alpha_t) e^{-\lambda t} dt + u(X_h) e^{-\lambda h} 1_{(h < \tau)} + \varphi(X_\tau) e^{-\lambda \tau} (1_{\tau \leq h}) \Big\} \;;$$

where to simplify notations we assumed $c \equiv \lambda$.

Since $x_o \in \mathcal{O}$, it is easy to derive from the uniform boundedness of the coefficients σ, b - as we did in [19] :

$$\sup_{A} P[\tau \leq h] \leq C h^2$$

and combining this inequality together with Itô's formula (valid since $u \in C^2(\mathcal{O})$) we deduce from the above inequality :

$$\sup_{A} E \frac{1}{h} \int_o^h \Big\{ A_{\alpha_t} u(X_t) - f(X_t, \alpha_t) \Big\} dt = \varepsilon(h) \underset{h \to 0_+}{\to} 0 .$$

Using the C^2 regularity of u, one obtains easily :

$$\sup_{A} E \frac{1}{h} \int_o^h \Big\{ A_{\alpha_t} u(x_o) - f(x_o, \alpha_t) \Big\} dt = \varepsilon'(h) \underset{h \to 0_+}{\to} 0 .$$

Choosing $\alpha_t \equiv \alpha \in A$, we find : $A_\alpha u(x_o) - f(x_o, \alpha) \leq 0$ and thus : $\sup_{\alpha \in A} \{A_\alpha u(x_o) - f(x_o, \alpha)\} \leq 0$. On the other hand if

$\lambda_\alpha = A_\alpha u(x_o) - f(x_o,\alpha)$ and $C = \{\lambda_\alpha \ / \ \alpha \in A\}$ we deduce from the above limit :

$$\sup_{\lambda \in \overline{co}(C)} \lambda \geq 0$$

and we conclude since : $\sup_{\lambda \in C} \lambda = \sup_{\lambda \in \overline{co}(C)} \lambda$. ∎

Second case : $u \in C(\mathcal{O})$.

We have to check (for example) (12) : thus let $x_o \in \mathcal{O}$ be such that $D^-_{2,1}u(x_o) \neq \emptyset$ and let $(A,p) \in D^-_{2,1}u(x_o)$. In view of Lemma I.1, we may find $\psi \in C^2(\mathcal{O}) \cap C_b(\overline{\mathcal{O}})$ such that :

$$\begin{cases} u(x_o) = \psi(x_o) \ , \ p = D\psi(x_o) \ , \ D^2\psi(x_o) = A \\ u(x) - \psi(x) \geq 0 \qquad \text{for } x \in \overline{\mathcal{O}} \ . \end{cases}$$

Writing again the dynamic programming principle, we deduce using the above properties of ψ that for all $h > 0$ we have :

$$\psi(x_o) = u(x_o) \geq \inf_{A} \left\{ E \int_0^{\tau \wedge h} f(X_t,\alpha_t)e^{-\lambda t}dt + \right.$$
$$\left. + \psi(X_h)e^{-\lambda h}1_{(h < \tau)} + \varphi(X_\tau)e^{-\lambda\tau}1_{(\tau \leq h)} \right. ;$$

and rewriting the above proof, we immediately get :

$$\sup_{\alpha \in A} \{A_\alpha\psi(x_o) - f(x_o,\alpha)\} \geq 0 \ .$$

And we conclude since we have in view of the choice of ψ :

$$\begin{aligned} F(A,p,u(x_o),x_o) &= F(D^2\psi(x_o),D\psi(x_o),\psi(x_o),x_o) \\ &= \sup_{\alpha \in A} \{A_\alpha\psi(x_o) - f(x_o,\alpha)\} \ . \end{aligned}$$

∎

I.5 Validity of the equations.

In this section we want to investigate the following questions : i) if u is a viscosity solution of (9) (or (4)) and if we know that u "has some regularity, does it imply that (9) (or (4)) holds in a more usual way ?; ii) if u is a solution of (9) in some sense, is u a viscosity solution of (9) ?

In order to give some answers to these questions, we begin by a few remarks :

<u>Remark I.14</u> : Let u be a viscosity solution of (9) and assume that near some point $x_o \in \mathcal{O}$ we have for some $p^o \in R^N$, $\xi^o \in S^N$:

$$u(x) = u(x_o) + (p^o, x-x_o) + \frac{1}{2}(\xi^o \cdot (x-x_o), x-x_o) + o(|x-x_o|^2)$$

then from (11)-(12) we deduce :

$$F(\xi^o, p^o, u(x_o), x_o) = 0 \quad .$$

<u>Remark I.15</u> : If $u \in C^2(\mathcal{O})$ satisfies (9) then we claim that u is a viscosity solution of (9) : indeed let us check for example (11') and let $x_o \in \mathcal{O}$ be such that $u-\psi$ has a local maximum at x_o for some $\psi \in C^2(\mathcal{O})$. By the necessary conditions for maxima, we find : $Du(x_o) = D\psi(x_o)$, $(D^2u(x_o)) \leqslant (D^2\psi(x_o))$ and thus because of (10), this proves (11'). ■

We now proceed to extend these remarks. First of all, recalling (Cf. E. Stein [39]) that if $u \in W^{2,p}_{loc}(\mathcal{O})$ for some $p > N$ then the situation described in Remark I.14 occurs for almost all x^o, we deduce immediately the :

<u>Proposition I.4</u> : *Let* u *be a viscosity solution of* (9) *and assume that* $u \in W^{2,p}_{loc}(\mathcal{O})$ *for some* $p > N$, *then we have :*

$$F(D^2u, Du, u, x) = 0 \qquad \text{a.e. in } \mathcal{O} \; .$$

Remark I.16 : With very little assumptions on F, it is possible to extend the above result to the case when $u \in W^{2,N}_{loc}(\mathcal{O})$. In the case when (9) has the special parabolic structure, it is enough to take $u \in W^{2,1,p}_{loc}(\mathcal{O} \times (0,T))$ for some $p > N+1$.

In the case when (9) reduces to (4), it is useful to give more specific results :

Proposition I.5 : *Let* u *be given by* (3). *Assume that* $u \in K - K$ *where* K *denotes the cone in* $C(\mathcal{O})$ *given by* :

$$K = \Big\{ v \in C(\mathcal{O}) \ / \ \exists \varphi \in W^{2,p}_{loc}(\mathcal{O}) \quad \textit{for some } p > N , \\ (D^2 v) \leq (D^2 \varphi) \quad \textit{in } \mathcal{D}'(\mathcal{O}) \Big\}$$

Then $u \in W^{1,\infty}_{loc}(\mathcal{O})$, $\partial_{ij} u \in M(\mathcal{O})$ (for $1 \leq i,j, \leq N$) *and if we denote by* u_{ij} *its Radon-Nykodim derivative with respect to the Lebesgue measure we have* :

$$\text{(13)} \qquad \sup_{\alpha \in A} \Big\{ -a_{ij}(x,\alpha) u_{ij}(x) - b_i(x,\alpha) \partial_i u(x) + \\ + c(x,\alpha) u(x) - f(x,\alpha) \Big\} = 0 \quad \textit{a.e.}$$

In particular if $A_\alpha u \in L^1_{loc}(\mathcal{O})$ ($\forall \alpha \in A$) *then* (4) *holds a.e.*.

Remark I.17 : In the time-dependent case, K has to be replaced by (for example) :

$$K = \Big\{ v \in C(\mathcal{O}\times(0,T)) \ / \quad \varphi \in W^{2,1,\infty}_{loc}(\mathcal{O}\times(0,T)), \ (D^2_x v) \leq (D^2_x \varphi) \text{ in } \\ \mathcal{D}'(\mathcal{O}\times(0,T)) \ , \ \frac{\partial v}{\partial t} \leq \frac{\partial \varphi}{\partial t} \text{ in } \mathcal{D}'(\mathcal{O}\times(0,T)) \Big\} .$$

■

Corollary I.1 : *Let* u *be given by* (3) : *assume that* u *is continuous and semi-concave in* $\mathcal{O}$ *i.e.* :

(14) $\qquad \forall\, \delta > 0\ ,\ \exists C_\delta \geq 0\ ,\ (D^2u) \leq C_\delta\, I_N \quad in\ \ \mathcal{D}'(\mathcal{O}_\delta)$.

Then we have : $\sup_{\alpha \in A} \|A_\alpha u\|_{L^\infty(\mathcal{O}_\delta)} \leq C'_\delta < \infty\ ,\ \forall\, \delta > 0$.

And (4) holds a.e..

<u>Remark I.18</u> : If C_δ is bounded in δ and if $u \in W^{1,\infty}(\mathcal{O})$ then we have : $\sup_{\alpha \in A} \|A_\alpha u\|_{L^\infty(\mathcal{O})} \leq C' < \infty$. ■

We first prove Corollary I.1 using Proposition I.5 : in view of Theorem I.1, u is a viscosity solution of (4) and lies in K (in each open set $\mathcal{O}_\delta$). Furthermore one deduces easily from (14) (Cf. P.L. Lions [26]) that $A_\alpha u \in L^\infty(\mathcal{O}_\delta)$ ($\forall\, \alpha \in A\ ,\ \forall\, \delta > 0$) and $\sup_{\alpha \in A} \|A_\alpha u\|_{L^\infty(\mathcal{O}_\delta)} < \infty$. Therefore we may apply Proposition I.5 on each $\mathcal{O}_\delta$ and we conclude : (4) holds a.e. in $\mathcal{O}$.

<u>Proof of Proposition I.5</u> : Remarking that if $v \in K$ then $\varphi - v$ is locally convex (i.e. convex on each convex open subset of $\mathcal{O}$ completely contained in $\mathcal{O}$), we deduce from the well-known Lipschitz properties of convex functions that $v-\varphi$ and thus $v \in W^{1,\infty}_{loc}(\mathcal{O})$. Next, if $\xi \in \mathbb{R}^N$, $|\xi| = 1$ and if $v \in K$ we have :

$$\partial^2_\xi v \leq \partial^2_\xi \varphi \qquad \text{in}\ \ \mathcal{D}'(\mathcal{O})$$

and thus $\partial^2_\xi v \in M(\mathcal{O})$ for all $|\xi| = 1$ in $\mathbb{R}^N$.

This shows that if $u \in K - K$, then $u \in W^{1,\infty}_{loc}(\mathcal{O})$, $\partial_{ij} u \in M(\mathcal{O})$ ($1 \leq i,j \leq N$). The fact that (13) holds is a consequence of Remark I.14, Theorem I.1 and of the following fact : let v be a convex function continuous on $\mathbb{R}^N$, then for almost all $x^o \in \mathbb{R}^N$ we have : v is differentiable at x^o and

$$\ddot{v}(x) = v(x^o) + (\nabla v(x^o), x-x^o) + \frac{1}{2} v_{ij}(x_i-x_i^o)(x_j-x_j^o) + + o(|x-x^o|^2) ,$$

in addition $v_{ij}(x^o)$ is the Radon-Nykodim (or a representant) derivative of the measure $\partial_{ij}v$ with respect to the Lebesgue measure.

The fact that such an expansion holds for almost all $x^o \in R^N$ is a classical result on the differentiability properties of convex functions due to Alexandrov [1], H. Busemann [4] and the characterization of v_{ij} is also proved in N.V. Krylov [18].

An alternative proof is the following : let $\partial v(x)$ be the subdifferential of v, ∂v is a maximal monotone operator on R^N and if we denote by $\partial_o v$ its principal section $\left(\partial_o v(x) = \mathrm{Proj}_{\partial v(x)}(0)\right)$ we have, by a general result due to F. Mignot [29] on monotone operators in R^N : $\partial_o v$ is almost everywhere differentiable ; and this implies obviously the above expansion. Next to identify v_{ij}, observe that - if $v_\varepsilon = v \star \rho_\varepsilon$ where $\rho_\varepsilon = \frac{1}{\varepsilon^N} \rho\left(\frac{\cdot}{\varepsilon}\right)$, $\rho \in \mathcal{D}_+(R^N)$, $\int_{R^N} \rho \, dx = 1$ - we have :

$$\partial_{ij} v_\varepsilon(x^o) = v \star \partial_{ij} \rho_\varepsilon = v_{ij}(x^o) + \delta(\varepsilon)$$

where $\delta(\varepsilon) \to 0$ if $\varepsilon \to 0_+$.

Therefore $\partial_{ij}v_\varepsilon$ converges a.e. to v_{ij}. But clearly $\partial_{ij}v_\varepsilon = (\partial_{ij}v) \star \rho_\varepsilon$ and if we decompose the measure $\mu = \partial_{ij}v$ in its absolutely continuous part μ_c (with respect to the Lebesgue measure) and its singular part μ_s, we deduce since $\mu_c \in L^1_{loc}$ and meas (Supp μ_s) = 0 :

$$\begin{cases} \mu \star \rho_\varepsilon = (\mu_c \star \rho_\varepsilon) + (\mu_s \star \rho_\varepsilon) \\ \mu_c \star \rho_\varepsilon \underset{\varepsilon}{\to} \mu_c \quad \text{a.e.} \\ |\mu_s \star \rho_\varepsilon| \leq |\mu_s| \, (B(x,K\varepsilon)) \underset{\varepsilon}{\to} 0 \quad \text{a.e.} \end{cases}$$

and this yields : $\mu_c = v_{ij}$ a.e..

Next recall (Cf. Part 1 [19]) that $\mu_\alpha = A_\alpha u - f(\cdot,\alpha)$ is a non-positive measure on $\mathcal{O}$ and since $u \in W^{1,\infty}_{loc}(\mathcal{O})$ the quantity given by $(-a_{ij}u_{ij} - b_i\partial_i u + cu - f)$ is the Radon-Nykodim derivative of μ_α and thus coïncides with μ_α if we know that $\mu_\alpha \in L^1_{loc}(\mathcal{O})$. This concludes the proof of the Proposition. ∎

Remark I.19 : In Proposition I.5 and Corollary I.1, we considered the viscosity solution u of (4) given by formula (3). As we will see in section II, we did not lose any generality since we will see in section II that if u is a viscosity solution of (4), then u is given by (3) where φ is replaced by u and where τ is now the first exit time of X_t from $\overline{\mathcal{O}}_\delta$ (for each $\delta > 0$). ∎

We now turn to the second question mentionned in the beginning of this section and we give one result extending Remark I.15 : this result is proved in P.L. Lions [23] :

Theorem I.2 : *Let* $u \in W^{2,N}_{loc}(\mathcal{O})$ *satisfy* :

$$F(D^2u,Du,u,x) = 0 \qquad \text{a.e. in } \mathcal{O}$$

then u *is a viscosity solution of* (9).

Remark I.20 : As it is explained in [23], this result is in general optimal since $u(x) = |x| \in W^{2,p}_{loc}(R^N)$ for $p < N$ (and $D^2u \in M^N(R^N)$) and u solves obviously : $1 - |Du| = 0$ a.e. in R^N ; but u is not a viscosity solution of (9) with $F = 1 - |p|$ since we have :

$$D^-_{2,1}u(0) = \emptyset \quad ,$$

$$D^+_{2,1}u(0) = \{(A,p) \in S^N \times R^N \ / \ |p| \leq 1 \ , \ A \geq 0\} \ ;$$

and thus we do not have : $F(A,p,u(0),0) \leq 0$ for $(A,p) \in D^+_{2,1}u(0)$. ■

In the next parts (Parts 3 and 4) we will see various uniqueness results which yield in particular results of the same type than Theorem I.2 but adapted to the specific equation (4).

II. UNIQUENESS RESULTS.

II.1 Main uniqueness result.

Let $\mathcal{O}$ be an open set in R^N : recall that $\mathcal{O}_\delta = \{x \in \mathcal{O},$ $|x| < 1/\delta$, $d(x) > \delta\}$ and that τ_δ is the first exit time of X_t from $\bar{\mathcal{O}}_\delta$.

Theorem II.1 : *We assume* (5) , (6) *and* (8) . *Let* $u \in C(\mathcal{O})$ *be a viscosity solution of* (4) , *then we have for all* $x \in \bar{\mathcal{O}}_\delta$, *for all* $\delta > 0$:

$$u(x) = \inf_A \left\{E\int_0^{\tau_\delta} f(X_t,\alpha_t)\exp\left(-\int_0^t c\right)dt + u(X_{\tau_\delta})\exp\left(-\int_0^{\tau_\delta} c\right)\right\}. \tag{15}$$

Remark II.1 : We will see in section II.3 how (5)-(6) may be relaxed and what is the exact role of assumption (8). ■

Of course we deduce immediately from the above result the

Corollary II.1 : *Let* $u,v \in C_b(\mathcal{O} \cup \Gamma_+)$ *be viscosity solutions of* (4) *where* Γ_+ *is a closed subset of* $\Gamma = \partial\mathcal{O}$ *such that* :

$$\forall A \,,\ \forall x \in \mathcal{O} \,,\quad P(\tau' < \infty \,,\ X_{\tau'} \notin \Gamma_+) = 0 \tag{16}$$

and τ' *is the first exit time of* X_t *from* $\mathcal{O}$. *Then, under assumptions* (5) , (6) *and* (8), *we have* :

$$\sup_{\mathcal{O} \cup \Gamma_+} (u-v)^+ \leq \sup_{\Gamma_+} (u-v)^+ .$$

Corollary II.2 : *Let* $u,v \in C_b(R^N)$ *be viscosity solutions of* (4) *in* R^N *then* : $u \equiv v$ *in* R^N .

Let us only indicate how, for example, Corollary II.2 is deduced from Theorem II.1 ; indeed if $\mathcal{O} = R^N$, τ_δ is the first exit time of X_t from $\bar{B}_{1/\delta}$ and thus, by standard arguments using the uniform boundedness of σ, b - Cf. section II.3 -, we deduce :

$$\sup_A E[e^{-\lambda\tau_\delta}] \underset{\delta\to 0}{\to} 0 \text{ , uniformly for x in bounded sets.}$$

Therefore from (15), we deduce :

$$|u(x)-v(x)| \leqslant \sup_A E\left\{|u(X_{\tau_\delta})-v(X_{\tau_\delta})|\, e^{-\lambda\tau_\delta}\right\}$$

$$\leqslant \|u-v\|_{L^\infty(R^N)} \sup_A E\{e^{-\lambda\tau_\delta}\} , \qquad \blacksquare$$

We will give in section II.3 many extensions and (or) variants of the above results. We now make some preliminary reductions in order to prove Theorem II.1.

Let $\xi \in \mathcal{D}_+(\mathcal{O})$, $\xi \equiv 1$ in $\bar{\mathcal{O}}_\delta$, $0 \leqslant \xi \leqslant 1$ in $\mathcal{O}$ and let $\varphi \in \mathcal{D}_+(\mathcal{O})$, $\varphi \equiv 1$ on a neighborhood of Supp ξ , $0 \leqslant \varphi \leqslant 1$ in $\mathcal{O}$.

We claim that $\tilde{u} = \varphi u$ in $\mathcal{O}$, $= 0$ in $R^N - \mathcal{O}$ is a viscosity solution of

$$\text{(17)} \qquad \sup_\alpha [\tilde{A}_\alpha \tilde{u}(x) - \tilde{f}(x,\alpha)] = 0 \quad \text{in } R^N$$

where $\tilde{A}_\alpha = -\xi^2 a_{ij}(\alpha)\partial_{ij} - \xi^2 b_i(\alpha)\partial_i + \xi^2 c(\alpha)$, $\tilde{f}(\alpha) = \xi^2 f(x)$.

Next, let $\gamma > 0$, $\tilde{u}$ is also a viscosity solution of

$$\text{(17')} \qquad \sup_{\alpha\in A} [(\tilde{A}_\alpha+\gamma)\tilde{u} - (\tilde{f}+\gamma\tilde{u})] = 0 \quad \text{in } R^N .$$

In the next section, we are going to prove that this implies :

(18) $$\tilde{u}(x) = \inf_{A} \tilde{\mathcal{J}}(x,A)$$

where $\tilde{\mathcal{J}}(x,A) = E\int_0^\infty \tilde{f}(\tilde{X}_t,\alpha_t)\exp\left(-\int_0^t \tilde{c}(\tilde{X}_s,\alpha_s)ds\right)dt$

with $\tilde{f}(x,\alpha) = \xi^2(x)\ f(x,\alpha) + \gamma\ \tilde{u}(x)$, $\tilde{c}(x,\alpha) = \xi^2(x)\ c(x,\alpha) + \gamma$ and $\tilde{X}_t$ is the solution of (1) (for a given admissible system A) with σ replaced by $\xi\sigma$, b replaced by $\xi^2 b$. We set $\tilde{\sigma} = \xi\sigma$, $\tilde{b} = \xi^2 b$, $\tilde{a} = \xi^2 a$; remark that (5) still holds for $\psi = \tilde{\sigma}_{ij}$, $\tilde{b},\tilde{c}$; (6) and (8) still holds for $\tilde{f}$, $\tilde{c}$ and in addition $\tilde{\sigma}$, $\tilde{b}$, $\tilde{f}$, $\tilde{u}$ have a uniform compact support. Finally we will denote by $\bar{u}$ the right-hand side member of (18), we are going to prove in the next section that $\tilde{u} \equiv \bar{u}$.

But (in view of N.V. Krylov [15] or Part 1 , Theorem B [19], for example) this will imply :

$$\tilde{u}(x) = \bar{u}(x) = \inf_{A}\left\{E\left\{\int_0^{\tilde{\tau}_\delta} \tilde{f}(\tilde{X}_t,\alpha_t)\exp\left(-\int_0^t \tilde{c}\right)dt + \right.\right.$$
$$\left.\left. + \bar{u}(\tilde{X}_{\tilde{\tau}_\delta})\exp\left(-\int_0^{\tilde{\tau}_\delta}\tilde{c}\right)\right\}\right\}$$

where $\tilde{\tau}_\delta$ is the first exit time of $\tilde{X}_t$ from $\mathcal{O}_\delta$. But since $\xi \equiv 1$ in $\mathcal{O}_\delta$ and $\varphi \equiv 1$ on $\mathcal{O}_\delta$, we have : $\tau_\delta = \tilde{\tau}_\delta$ a.s. and

$$\sup_{0 \leqslant t \leqslant \tau_\delta} |X_t - \tilde{X}_t| = 0 \quad \text{a.s.} \quad ;$$

and we obtain :

$$u(x) = \inf_{A} E\left\{\int_0^{\tau_\delta} f(X_t,\alpha_t)\exp\left\{-\gamma t - \int_0^t c(X_s,\alpha_s)ds\right\} + \right.$$

$$+ u(X_{\tau_\delta}) \exp\left\{-\gamma\tau_\delta - \int_0^{\tau_\delta} c(X_t,\alpha_t)dt\right\}\Big\}$$

but $\gamma > 0$ is arbitrary and we obtain (15) sending γ to 0 . ∎

II 2 Proof of Theorem II.1 :

In view of the arguments given in the end of the preceding section, we just have to prove : $\tilde{u}(x) = \bar{u}(x)$, $\forall\, x \in \mathbb{R}^N$ - where we keep the notations of the preceding section.

Step 1 : $\tilde{u} \geq \bar{u}$ in $\mathbb{R}^N$.

Let $\varepsilon > 0$, we introduce $\sigma_\varepsilon(x,\alpha) = (\tilde{\sigma}(x,\alpha)\ ,\ \varepsilon I_N)$ - σ_ε is a N x (m+N) matrix - and $a_\varepsilon = \frac{1}{2}\ \sigma_\varepsilon\ \sigma_\varepsilon^T = \frac{\varepsilon^2}{2}\ I_N + \tilde{a}$. In view of the results of P.L. Lions [26], there exists $\lambda_o > 0$ (depending only on $\tilde{\sigma},\tilde{b}$) such that the following holds : let $f_\mu(x,\alpha)$ satisfy (5) and be such that :

$$\begin{cases} \sup\limits_{\alpha} \|f_\mu(\cdot,\alpha) - (\tilde{f}(\cdot,\alpha) + \lambda_o\bar{u}(\cdot))\|_{L^\infty(\mathbb{R}^N)} \underset{\mu\to 0}{\to} 0 \\ \mathrm{Supp}\ f_\mu(\cdot,\alpha) \subset B_R \quad , \quad \forall\, \alpha \in A \end{cases}$$

for some $R < \infty$ (such (f_μ) exist since $\bar{u} \in B \cup C(\mathbb{R}^N)$ and $\bar{u}$ has compact support). Then

$$u_\varepsilon^\mu(x) = \inf_A E \int_0^\infty f_\mu(X_t,\alpha_t) \exp\left\{-\int_0^t \tilde{c}(X_s^\varepsilon,\alpha_s)ds - \lambda_o t\right\} dt$$

belongs to $W^{2,\infty}(\mathbb{R}^N)$ and is the unique solution in $W^{2,\infty}(\mathbb{R}^N)$ of :

$$-\frac{\varepsilon^2}{2}\ \Delta u_\varepsilon^\mu + \sup_{\alpha\in A}\left[(\tilde{A}_\alpha+\gamma+\lambda_o)u_\varepsilon^\mu(x) - f_\mu(x,\alpha)\right] = 0 \text{ a.e. in } \mathbb{R}^N$$

In addition - see P.L. Lions [26] - :

$$(D^2 u^\mu_\varepsilon) \leqslant C_\mu \, I_N \quad \text{in } \mathcal{D}'(R^N) \quad , \quad \forall \, \varepsilon > 0 \quad .$$

Furthermore we have easily : $u^\mu_\varepsilon \in C_o(R^N)$ and :

$$\| u^\mu_\varepsilon - u^\mu \|_{L^\infty(R^N)} \to 0 \qquad \text{as } \varepsilon \to 0_+ \ ;$$

where $u^\mu(x) = \inf\limits_{A} E \int_o^\infty f_\mu(\tilde{X}_t, \alpha_t) \exp\left(-\int_o^t \tilde{c}(\tilde{X}_s, \alpha_s) - \lambda_o t\right) dt$.

In addition since $\bar{u}$ satisfies the following identity (see for example N.V. Krylov [15], P.L. Lions [26]) :

$$\bar{u}(x) = \inf_{A} E \int_o^\infty \left(\tilde{f}(\tilde{X}_t, \alpha_t) + \lambda_o \bar{u}(\tilde{X}_t)\right) \cdot$$
$$\cdot \ \exp\left(-\int_o^t \tilde{c}(\tilde{X}_s, \alpha_s) - \lambda_o t\right) dt \ ,$$

we have :

$$|(\bar{u} - u^\mu)(x)| \leqslant \frac{1}{\lambda_o + \gamma} \sup_{R^N \times A} |f_\mu(x,\alpha) - (\tilde{f}(x,\alpha) + \lambda_o \bar{u}(x))|$$

and thus : $\| \bar{u} - u^\mu \|_{L^\infty(R^N)} \to 0$ as $\mu \to 0_+$.

The final remark we want to make is that $u^\mu_\varepsilon \in C^2(R^N)$ in view of the regularity results due to L.C. Evans [10], [11] - see also N.S. Trudinger [38].

We can now show easily that we have : $\tilde{u} \geqslant \bar{u}$ in R^N. Indeed if this were not the case, we would have : $\min\limits_{R^N} (\tilde{u} - u^\mu_\varepsilon) < 0$, for μ, ε small enough. Let $x_o \in R^N$ be such that

$$\tilde{u}(x_o) - u^\mu_\varepsilon(x_o) = \min_{R^N} (\tilde{u} - u^\mu_\varepsilon) < 0$$

- such an x_o exists since $\tilde{u}, u^\mu_\varepsilon \in C_o(R^N)$ - .

We may apply (12') since $u_\varepsilon^\mu \in C^2(\mathbb{R}^N)$ and this yields :

$$\sup_{\alpha \in A} \Big\{ -\tilde{a}_{ij}(x_o,\alpha)\partial_{ij}u_\varepsilon^\mu(x_o) - \tilde{b}_i(x_o,\alpha)\partial_i u_\varepsilon^\mu(x_o) + \\ + (\tilde{c}(x_o,\alpha)+\lambda_o)\tilde{u}(x_o) - (\tilde{f}(x_o,\alpha)+\lambda_o\bar{u}(x_o)) \Big\} \geq 0 .$$

Using the equation and the estimates satisfied by u_ε^μ , this yields :

$$\begin{aligned} 0 &\geq (\lambda_o+\gamma)(\tilde{u}(x_o)-u_\varepsilon^\mu(x_o)) \\ &\geq -\frac{\varepsilon^2}{2}\,\Delta u_\varepsilon^\mu(x_o) - \sup_{\alpha \in A} \|f_\mu(\cdot,\alpha) - (\tilde{f}(\cdot,\alpha)+\lambda_o\bar{u}(\cdot))\|_{L^\infty(\mathbb{R}^N)} \\ &\geq - C_\mu\,\varepsilon^2 - \delta(\mu) \end{aligned}$$

where $\delta(\mu) \to 0$ as $\mu \to 0_+$.

We deduce :

$$\|(u_\varepsilon^\mu - \tilde{u})^+\|_{L^\infty(\mathbb{R}^N)} \leq (C_\mu\,\varepsilon^2 + \delta(\mu))\ (\lambda_o + \gamma)^{-1}$$

and if we send ε and then μ to 0 , we conclude : $\bar{u} \leq \tilde{u}$. ■

<u>Step 2 : $\tilde{u} \leq \bar{u}$ in $\mathbb{R}^N$</u>.

In view of the results of N.V. Krylov [15], it is enough to show that, for each α fixed in A and each $T > 0$, we have :

$$\tilde{u}(x) \leq u(x,T) \qquad \text{in } \mathbb{R}^N$$

where $u(x,T) = E\int_0^T \tilde{f}(\tilde{X}_t,\alpha)\exp\left(-\int_0^t \tilde{c}\right) dt + \tilde{u}(\tilde{X}_T)\exp\left(-\int_0^T \tilde{c}\right)$

and where $\tilde{X}_t$ is the diffusion process corresponding to $\tilde{A}_\alpha$ (or equivalently the solution of (1) with $\alpha_t \equiv \alpha$). Again, we introduce $f_\mu(x)$, $u_\mu(x) \in \mathcal{D}(\mathbb{R}^N)$ and satisfying :

$$\|f_\mu(\cdot) - \tilde{f}(\cdot,\alpha)\|_{L^\infty(R^N)} + \|u_\mu - \tilde{u}\|_{L^\infty(R^N)} \underset{\mu \to 0_+}{\to} 0 \quad .$$

From well-known differentiability properties of X_t with respect to the initial point x, we deduce that

$$u_\mu(x,t) = E \int_0^t f_\mu(X_s) \exp\left(-\int_0^s \tilde{c}\right) ds + u_\mu(X_t) \exp\left(-\int_0^t \tilde{c}\right)$$

lies in $W^{2,1,\infty}(R^N \times (0,T))$ - see [37] for example.

Moreover regularizing σ, b, c if necessary (in such a way that (5) holds uniformly) we may assume that $u_\mu \in C^3(R^N \times [0,T])$ and :

$$\begin{cases} \dfrac{\partial u_\mu}{\partial t} + (\tilde{A}_\alpha + \gamma) u_\mu = f_\mu + g_\mu & \text{in } R^N \times [0,T] \\ u_\mu(x,0) = u_\mu(x) & \text{in } R^N \end{cases}$$

where $g_\mu \to 0$ in $L^\infty(R^N \times [0,T])$ as $\mu \to 0_+$. Clearly :

$$\sup_{R^N \times [0,T]} |u_\mu(x,t) - u(x,t)| \to 0 \quad \text{as } \mu \to 0_+ \ .$$

We now show the desired inequality remarking that $\tilde{u}$ is a viscosity subsolution of :

$$\frac{\partial \tilde{u}}{\partial t} + (\tilde{A}_\alpha + \gamma)\tilde{u} \leqslant \tilde{f}(\cdot,\alpha) \qquad \text{in } R^N \times (0,T) .$$

Then if $\max\limits_{R^N \times [0,T]} [\tilde{u} - u]^+ > 0$, for μ small enough we still have : $\max\limits_{R^N \times [0,T]} (\tilde{u} - u_\mu)^+ > 0$. Let $(x_o,t_o) \in R^N \times [0,T]$ be such that : $(\tilde{u}(x_o) - u_\mu(x_o,t_o)) = \max\limits_{R^N \times [0,T]} (\tilde{u} - u_\mu)^+$ - such an (x_o,t_o) exists since $u_\mu(x,t) \to 0$ as $|x| \to \infty$ uniformly in $t \in [0,T]$.

If $t_o = 0$, we deduce :

$$(\tilde{u}(x_o) - u_\mu(x_o,t_o)) = \max_{R^N \times [0,T]} (\tilde{u} - u_\mu)^+$$

$$\leq \|\tilde{u} - u_\mu\|_{L^\infty(R^N)} .$$

If $t_o > 0$, we deduce from (12') (or (12") if $t_o = T$) :

$$\frac{\partial u_\mu}{\partial t}(x_o,t_o) - \tilde{a}_{ij}(x_o,\alpha)\partial_{ij}u_\mu(x_o,t_o) - \tilde{b}_i(x_o,\alpha)\partial_i u_\mu(x_o,t_o)$$

$$+ (\tilde{c}(x_o,\alpha)+\gamma)\tilde{u}(x_o) - \tilde{f}(x_o,\alpha) \leq 0$$

and using the equation satisfied by u_μ, this yields :

$$\max_{R^N \times [0,T]} (\tilde{u} - u_\mu)^+ \leq \frac{1}{\gamma}\left\{\|g_\mu\|^+_{L^\infty(R^N \times (0,T))} + \|f_\mu - \tilde{f}(\cdot,\alpha)\|_{L^\infty} \right\} ;$$

and we may conclude, in all cases, sending μ to 0.

<u>Remark II.2</u> : We actually proved in step 2, that u being a viscosity subsolution of : $\tilde{A}_\alpha \tilde{u} \leq \tilde{f}(\cdot,\alpha)$ in R^N, we have :

$$\tilde{u}(\tilde{X}_t)\exp\left(-\int_0^t \tilde{c}\right) + \int_0^t \tilde{f}(\tilde{X}_s,\alpha)\exp\left(-\int_0^s \tilde{c}\right) ds$$

is a submartingale. This a consequence of the Markov property and of the inequality proved in Step 2 (see for a similar argument Part 1 [19]). In particular we proved that if u is a viscosity subsolution of : $A_\alpha u \leq f_\alpha$ in $\mathcal{O}$, then for all $\delta > 0$ and for all $x \in \bar{\mathcal{O}}_\delta$:

$$u(X_{\tau_\delta \wedge t})\exp\left(-\int_0^{t\wedge\tau_\delta} c\right) + \int_0^{t\wedge\tau_\delta} f_\alpha(X_s)\exp\left(-\int_0^s c\right) ds$$

is a $F_{t\wedge\tau_\delta}$-submartingale. ∎

II.3 Extensions.

We first want to give a result where we relax assumptions (5) and (6). We will use the following assumptions :

$$(19)\quad \begin{cases} \sup\limits_{\alpha\in A}\left\{\sum\limits_{ij}\|\sigma_{ij}\|_{W^{2,\infty}(B_R)}+\sum\limits_{i}\|b_i\|_{W^{1,\infty}(B_R)}+\|c\|_{L^\infty(B_R)}\right\}<\infty, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall R<\infty, \\ c,f \text{ are continuous in } \overline{B_R}, \text{ uniformly in } \alpha\in A, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for all } R<\infty\ ; \\ \sigma_{ij},\ b_i,\ c,\ f \text{ are continuous in } \alpha,\ \text{for all } x\in\mathbb{R}^N \end{cases}$$

$$(20)\quad \begin{cases} \forall\varepsilon>0,\ \exists C_\varepsilon>0,\ \ \|\sigma\|+|b|\leqslant\varepsilon|x|+C_\varepsilon \quad \forall x\in\mathbb{R}^N,\ \forall\alpha\in A\ ; \\ \exists C>0,\ \exists p\geqslant 0,\ \ |f|\leqslant C(1+|x|^p)\quad \forall x\in\mathbb{R}^N,\ \forall\alpha\in A. \end{cases}$$

Then we have :

Theorem II.2 : *We assume (19) and (8).*

i) *Let* $u\in C(\mathcal{O})$ *be a viscosity solution of (4), then we have for all* $x\in\overline{\mathcal{O}}_\delta$ *and for all* $\delta>0$:

$$(15)\qquad u(x)=\inf_A E\int_0^{\tau_\delta} f(X_t,\alpha_t)\exp\left(-\int_0^t c\right)dt+u(X_{\tau_\delta})\exp\left(-\int_0^{\tau_\delta} c\right).$$

ii) *Let* $u,v\in C(\mathcal{O}\cup\Gamma_+)$ *(where* Γ_+ *satisfies the conditions of Corollary II.1) be viscosity solutions of (4) and, if* $\mathcal{O}$ *is unbounded, assume (20) and*

$$(21)\qquad \exists C>0,\ \exists m\geqslant 0,\ \ |u|+|v|\leqslant C(1+|x|^m)\qquad \forall x\in\mathcal{O}.$$

Then we have :

$$\sup_{\mathcal{O}\cup\Gamma_+} (u-v)^+ \leq \sup_{\Gamma_+} (u-v)^+ .$$

iii) *Let* $u,v \in C(\mathbb{R}^N)$ *be viscosity solutions of* (4) *in* $\mathbb{R}^N$. *We assume* (20) *and* (21). *Then* : $u \equiv v$ *in* $\mathbb{R}^N$.

We will prove only i) and iii) since the proof of ii) is totally similar to the one of iii).

Proof of i) : Since in the proof of Theorem II.1 we only used the local regularity of σ,b,c, the main new point in i) lies in the fact that b is assumed to be only Lipschitz and c merely continuous. Keeping the notations of sections II.1-2, we have to prove : $\tilde{u} \equiv \bar{u}$ in $\mathbb{R}^N$ - remark in addition that $\tilde{\sigma} \in W^{2,\infty}(\mathbb{R}^N)$, $\tilde{b} \in W^{1,\infty}(\mathbb{R}^N)$, $\tilde{c} \in B \cup C(\mathbb{R}^N)$ uniformly in $\alpha \in A$ -. To explain how we modify the proof given in section II.2, we prove Part 2. Let $\mu,\nu > 0$: we introduce f_μ, u_μ, c_μ satisfying :

$$\begin{cases} \|f_\mu - \tilde{f}(\cdot,\alpha)\|_{L^\infty(\mathbb{R}^N)} + \|u_\mu - \tilde{u}\|_{L^\infty(\mathbb{R}^N)} + \|c_\mu - \tilde{c}\|_{L^\infty(\mathbb{R}^N)} \underset{\mu\to 0}{\to} 0 \\ f_\mu, u_\mu \in \mathcal{D}(\mathbb{R}^N) \ ; \ c_\mu \in C_b^\infty(\mathbb{R}^N) \ , \ c_\mu \geq \gamma \text{ in } \mathbb{R}^N \ ; \end{cases}$$

we also introduce $b_\nu \in \mathcal{D}(\mathbb{R}^N;\mathbb{R}^N)$ such that $\|b_\nu - \tilde{b}\|_{W^{1,\infty}(\mathbb{R}^N)} \underset{\nu\to 0}{\to} 0$.

Remarking that : $u_\mu^\nu(x,t) = E\left\{\int_0^t f_\mu(X_s^\nu) \exp\left(-\int_0^s c_\mu\right) ds + u_\mu(X_t^\nu) \exp\left(-\int_0^t c_\mu\right)\right\}$ satisfies : $\|u_\mu^\nu\|_{W^{1,\infty}(\mathbb{R}^N \times (0,T))} \leq C_\mu$ (ind. of ν) ; we obtain easily the existence of $v_\mu^\nu \in C^2(\mathbb{R}^N \times [0,T])$ with compact support in $\mathbb{R}^N \times [0,T]$ satisfying : $\|v_\mu^\nu - u\|_{L^\infty(\mathbb{R}^N \times (0,T))} \to 0$ as $\mu,\nu \to 0$.

$$\begin{cases} \dfrac{\partial v_\mu^\nu}{\partial t} + (\tilde{A}_\alpha + \gamma) v_\mu^\nu = f_\mu + g_\mu^\nu & \text{in } R^N \times [0,T] \\ v_\mu^\nu(x,0) = u_\mu(x) & \text{in } R^N \end{cases} ;$$

where $g_\mu^\nu \underset{\nu \to 0}{\to} g_\mu$ in $L^\infty(R^N \times (0,T))$ for $\mu > 0$ and $g_\mu \underset{\mu \to 0}{\to} 0$ in $L^\infty(R^N \times (0,T))$.

Then exactly as in Step 2, we deduce :

$$\max_{R^N \times [0,T]} (\tilde{u} - v_\mu^\nu)^+ \leqslant \max \left\{ \|u_\mu - \tilde{u}\|_{L^\infty(R^N)} , \right.$$
$$\left. \frac{1}{\gamma} \|g_\mu^\nu\|_{L^\infty(R^N \times (0,T))} + \frac{1}{\gamma} \|f_\mu - \tilde{f}(\cdot,\alpha)\|_{L^\infty(R^N)} \right\}$$

and we conclude sending ν to 0 and then μ to 0. ■

<u>Proof of iii)</u> : Because of i), we have for all $|x| \leqslant \frac{1}{\delta}$, $\delta > 0$:

$$u(x) = \inf_A E \left\{ \int_0^{\tau_\delta} f(X_t,\alpha_t) \exp\left(-\int_0^t c\right) dt + u(X_{\tau_\delta}) \exp\left(-\int_0^{\tau_\delta} c\right) \right\}$$

$$v(x) = \inf_A E \left\{ \int_0^{\tau_\delta} f(X_t,\alpha_t) \exp\left(-\int_0^t c\right) dt + v(X_{\tau_\delta}) \exp\left(-\int_0^{\tau_\delta} c\right) \right\}$$

where τ_δ is the first exit time of the process X_t from $\bar{B}_{1/\delta}$.

Let $R_0 > 0$, take $x \in \bar{B}_{R_0}$. If we show that, uniformly in A,

$$(22) \qquad E \int_{\tau_\delta}^{+\infty} (1 + |X_t|^m) e^{-\lambda t} dt \to 0 \qquad \text{as } \delta \to 0_+$$

$$(23) \qquad (1 + \delta^{-m}) E(e^{-\lambda \tau_\delta}) \to 0 \qquad \text{as } \delta \to 0_+$$

for any $m > 0$; we will conclude easily since in view of (20)-(21) we will be able to take $\delta \to 0_+$ and obtain :

$$u(x) = v(x) = \inf_{A} E \int_o^\infty f(X_t,\alpha_t)\, \exp\left(-\int_o^t c(X_s,\alpha_s)ds\right)$$

for $|x| \leqslant R_o$ and $R_o > 0$ is arbitrary.

Next, let $p \geqslant \max(2,1+2m)$, we deduce easily from (20) that we have for $|x| \leqslant R_o$:

$$\text{(24)} \qquad \begin{cases} \forall\, \varepsilon > 0\ ,\ \exists K_\varepsilon = K(\varepsilon,R_o,p) > 0\ , \\ E\,[\sup_{[0,t]} |X_s|^p\,] \leqslant K_\varepsilon e^{\varepsilon t}\ . \end{cases}$$

And this yields :

$$P\,[\tau_\delta \leqslant t] \leqslant P\,[\sup_{[0,\,t]} |X_s|^p \geqslant \delta^{-p}] \leqslant \delta^p K_\varepsilon e^{\varepsilon t}\ ;$$

hence we have :

$$E(e^{-\lambda\tau_\delta}) = \int_o^\infty \lambda\, e^{-\lambda t}\, P\,[\tau_\delta \leqslant t]\, dt \leqslant \delta^p\ K'$$

if ε is choosen smaller than λ . This proves (23) since $p > m$. On the other hand, using (24), we have for all $T > 0$:

$$E\int_{\tau_\delta}^{+\infty} |X_t|^m\, e^{-\lambda t}dt \leqslant E\int_T^\infty |X_t|^m\, e^{-\lambda t}dt\ +$$

$$+\ E\left\{\left(\int_{\tau_\delta}^T |X_t|^m\, e^{-\lambda t}dt\right)\, 1_{(\tau_\delta \leqslant T)}\right\}$$

$$\leqslant \int_T^\infty e^{-\lambda t}(K_\varepsilon e^{\varepsilon t})^{m/p}dt + \frac{1}{\sqrt{\lambda}}\, P(\tau_\delta \leqslant T)^{1/2} \left\{E\int_o^T |X_t|^{2m} e^{-\lambda t}dt\right\}^{1/2}$$

$$\leqslant K'_\varepsilon\, e^{-(\lambda-\varepsilon)T} + \frac{1}{\sqrt{\lambda}}\left(K'_\varepsilon\, e^{\frac{\varepsilon T}{2}}\, \delta^{p/2}\right) \left\{\int_o^T (K_\varepsilon e^{\varepsilon t})^{\frac{2m}{p}}\, e^{-\lambda t}\, dt\right\}^{1/2}$$

$$\leqslant K'_\varepsilon\, e^{-(\lambda-\varepsilon)T} + K''_\varepsilon\, e^{\varepsilon T/2}\, \delta^{p/2}$$

where we choose again $\varepsilon \in (0,\lambda)$.

Then if $\delta \to 0_+$ and then $T \to +\infty$, (22) is proved. ■

We next turn to time-dependent problems and to viscosity solutions of parabolic equations : we consider coefficients σ, b, c, f which depend on $t \in [0,T]$ where $T > 0$ is given. We will assume :

$$(19') \begin{cases} \sup_{\alpha \in A,\, t \in [0,T]} \left\{ \sum_{i,j} \|\sigma_{ij}(\cdot,t,\alpha)\|_{W^{2,\infty}(B_R)} + \right. \\ \qquad \left. + \sum_i \|b_i(\cdot,t,\alpha)\|_{W^{1,\infty}(B_R)} \right\} < \infty \quad , \forall\, R < \infty ; \\ c,f \text{ are bounded continuous in } \bar{B}_R \text{ uniformly in } \alpha \in A , \\ \qquad t \in [0,T],\ \text{for all } R < \infty , \\ \sigma_{ij}, b_i, c, f \text{ are continuous in } t, \text{ uniformly in } \alpha \in A , \\ \qquad \text{for all } x \in R^N ; \\ \sigma_{ij}, b_i, c, f \text{ are continuous in } \alpha , \text{ for all } (x,t) \in R^N \times [0,T]; \end{cases}$$

$$(20') \begin{cases} \exists\, C > 0 , \ \|\sigma\| + |b| \leq C(1+|x|) , \ \forall\, (x,t,\alpha) \in R^N \times [0,T] \times A; \\ \exists\, C > 0 , \ \exists\, p \geq 0 , \ |f| \leq C(1+|x|^p) \\ \qquad \forall\, (x,t,\alpha) \in R^N \times [0,T] \times A ; \\ \exists\, \lambda \in R , \ c(x,t,\alpha) \geq \lambda , \ \forall\, (x,t,\alpha) \in R^N \times [0,T] \times A . \end{cases}$$

For any admissible system $\mathcal{A}$ and for all $(x,t) \in R^N \times [0,T]$, we denote by X_s the solution of the following stochastic differential equation :

(25) $\quad dX_s = \sigma(X_s, t+s, \alpha_s)dB_s + b(X_s, t+s, \alpha_s)ds \quad , \quad X_0 = x .$

We denote by $Q = \mathcal{O} \times (0,T)$, the associated HJB equations (for similar minimization problems to (3)) are now :

(26) $$-\frac{\partial u}{\partial t} + \sup_{\alpha \in A} \{A_\alpha u(x,t) - f(x,t,\alpha)\} = 0 \quad \text{in } Q$$

where $A_\alpha = - a_{ij}(x,t,\alpha)\partial_{ij} - b_i(x,t,\alpha)\partial_i + c(x,t,\alpha)$ – notice that time is reversed if we compare (26) with usual parabolic equations–. We set $Q_{\delta_1,\delta_2} = \mathcal{O}_{\delta_1} \times (\delta_2, T-\delta_2)$ for any $\delta_1 > 0$, $\delta_2 > 0$.

Results corresponding to Theorems II.1 – II.2 are given in

the

Theorem II.3 : *We assume* (19') .

i) *Let* $u \in C(Q)$ *be a viscosity solution of* (26). *Then, we have for all* $\delta_1,\delta_2 > 0$ *and for all* $(x,t) \in \bar{Q}_{\delta_1,\delta_2}$:

(27) $$u(x,t) = \inf_{A} \left\{E \int_0^{(T-\delta_2-t)\wedge\tau_{\delta_1}} f(X_s,t+s,\alpha_s)\exp\left(-\int_0^s c\right)ds + \right.$$

$$\left. + u\left(X_{(T-t-\delta_2)\wedge\tau_{\delta_1}}, t+(T-t-\delta_2)\wedge\tau_{\delta_1}\right)\exp\left(-\int_0^{(T-t-\delta_2)\wedge\tau_{\delta_1}} c\right)\right\};$$

where τ_{δ_1} *is the first exit time of* X_s *from* $\bar{\mathcal{O}}_{\delta_1}$.

ii) *Let* $u,v \in C(\bar{Q})$ *be viscosity solutions of* (26). *Assume, if* $\mathcal{O}$ *is unbounded that* (20') *holds. Then we have* :

$$\sup_{\bar{Q}} \{(u-v)^+ e^{-\lambda t}\} \leq \sup_{\partial_o Q} \{(u-v)^+ e^{-\lambda t}\} \leq +\infty$$

where $\partial_o Q = (\bar{\mathcal{O}} \times \{T\}) \cup (\Gamma \times [0,T])$.

iii) *Let* $u,v \in C(R^N \times (0,T])$ *be viscosity solutions of* (26) *in* $R^N \times (0,T)$. *Assume that* (20') *holds and that we have* :

(21') $$\exists C > 0,\ \exists m \geq 0,\quad |u|+|v| \leq C(1+|x|^m)$$
$$\forall (x,t) \in R^N \times (0,T)$$

then we have :

$$\sup_{R^N x(0,T)} \{(u-v)^+ e^{-\lambda t}\} \leqslant \sup_{R^N} \{(u(T)-v(T))^+ e^{-\lambda T}\} .$$

Remark II.3 : If we introduce as in Part 1 [19], $\tilde{X}_s = (X_s, Y_s)$ where Y_s is given by : $dY_s = 1\, ds$, $Y_o = t$ ($Y_s = t+s$), then $(T-t-\delta_2)\wedge\tau_{\delta_1}$ is nothing but the first exit time of $\tilde{X}_s$ from $\overline{Q}_{\delta_1,\delta_2}$. Therefore, in this context, Theorem II.3 is a special case of Theorem II.2 - except for the assumptions made upon σ, b : indeed we did not assume that σ has uniformly bounded first and second derivatives in t and that b has uniformly bounded first derivatives in t -. Thus, because of the special structure of (26), we may relax the smoothness assumption in t. It is possible to give general results including both Theorems II.2 and II.3 but we will skip this kind of extensions since that would be too technical.

Remark II.4 : As in Theorem II.2, it is possible to extend ii) by looking at Γ_+ "the closure of the exit points of X_t". ∎

We just have to prove i) since ii) and iii) are then proved exactly as in Theorem II.2 (letting $\delta_2 \to 0_+$ and then $\delta_1 \to 0_+$). In addition to prove i), we remark that replacing T by $T-\delta_2$ if necessary we may assume $u \in C(\mathcal{O} \times [0,T])$ and that by the same arguments as in section II.1, we may reduce the proof of i) to the case when $\mathcal{O} = R^N$, b,c,f,u have uniform compact supports in $R^N \times [0,T]$ - and thus $u \in B \cup C(R^N \times [0,T])$. We then have to prove for $(x,t) \in R^N \times [0,T]$:

$$u(x,t) = \inf_{A} E\left\{\int_0^{T-t} f(X_s,t+s,\alpha_s) \exp\left(-\int_0^s c\right) ds + \right.$$

$$+ u(X_{T-t},T) \exp\left(-\int_0^{T-t} c\right)\Bigg\} \quad .$$

We denote by $\bar{u}(x,t)$ the right-hand side member. By regularization arguments similar to those performed in section II.2 and in the proof of Theorem II.2, we may assume that $\sigma,b,c,f,u(T)$ are smooth in x, uniformly for $(t,\alpha) \in [0,T] \times A$. The proof of Step 2 is easily adapted to the above situation and we get : $u \leqslant \bar{u}$ in $R^N \times [0,T]$.

Let us prove the reversed inequality : we see immediately (Cf. P.L. Lions [26] proof, N.V. Krylov [15]) that $\bar{u}$ satisfies :

(28) $\quad (D_x^2\bar{u}) \leqslant C\, I_N \quad$ in $\mathcal{D}'(R^N \times (0,T))$

(29) $\quad D_x\bar{u} \in L^\infty(R^N \times (0,T)) \quad .$

Then for $\varepsilon > 0$, we introduce $\sigma_\varepsilon(x,t,\alpha)$, $b_\varepsilon(x,t,\alpha)$, $c_\varepsilon(x,t,\alpha)$, $f_\varepsilon(x,t,\alpha)$ satisfying :

$$\left\{\begin{array}{l} \sigma_\varepsilon, b_\varepsilon, c_\varepsilon, f_\varepsilon \in C_b^{2,1}(R^N \times [0,T]), \text{ uniformly in } \alpha \in A\ ; \\ \|D_x^2\, \psi_\varepsilon\|_{L^\infty(R^N)} \leqslant C \text{ for all } (t,\alpha) \in [0,T] \times A\ , \\ \qquad\qquad \text{for } \psi = \sigma_{ij}, b_i, c, f \\ \sigma_\varepsilon, b_\varepsilon, c_\varepsilon, f_\varepsilon \text{ have uniform compact supports (for } \alpha,\varepsilon,t) \\ \sigma_\varepsilon \underset{\varepsilon}{\to} \sigma\ ,\ b_\varepsilon \underset{\varepsilon}{\to} b\ ,\ c_\varepsilon \underset{\varepsilon}{\to} c\ ,\ f_\varepsilon \underset{\varepsilon}{\to} f \quad \text{uniformly in} \\ \qquad\qquad\qquad R^N \times [0,T] \times A \end{array}\right.$$

and denoting by $a_\varepsilon = \frac{1}{2}\, \sigma_\varepsilon \sigma_\varepsilon^T$:

(30) $\quad \varepsilon I_N \leqslant a_\varepsilon(x,t,\alpha) \leqslant a(x,t,\alpha) + 2\varepsilon\, I_N$

$$\forall\, (x,t,\alpha) \in R^N \times [0,T] \times A\ ,$$

Then, if we set for $(x,t) \in R^N \times [0,T]$:

$$u_\varepsilon(x,t) = \inf \left\{ E \int_0^{T-t} f_\varepsilon(X_s, t+s, \alpha_s) \exp\left(-\int_0^s c_\varepsilon\right) ds + \right.$$

$$\left. + u(X_{T-t}, T) \exp\left(-\int_0^{T-t} c_\varepsilon\right)\right\} ,$$

where X_s^ε corresponds to $\sigma_\varepsilon, b_\varepsilon$, we have (see for example N.V. Krylov [15] and P.L. Lions [26]) : $u_\varepsilon \in W^{2,1,\infty}(R^N \times (0,T))$ is the unique solution of

$$\begin{cases} -\dfrac{\partial u_\varepsilon}{\partial t} + \sup\limits_{\alpha \in A} [A_\alpha^\varepsilon u_\varepsilon(x,t) - f_\varepsilon(x,t,\alpha)] = 0 \quad \text{a.e. in } R^N \times (0,T) \\ u_\varepsilon(x,T) = u(x,T) \text{ in } R^N \; ; \; u_\varepsilon \in C_0(R^N) \; , \; \forall\, t \in [0,T] \\ (D_x^2 u_\varepsilon) \leq C\, I_N \text{ in } R^N \times (0,T) \; , \; |D_x u_\varepsilon| \leq C \quad \text{in } R^N \times (0,T) \end{cases}$$

where C does not depend on ε .

Adapting easily the regularity results due to L.C. Evans [10], [11], we find : $u_\varepsilon \in C^{2,1}(R^N \times [0,T])$ - another possible way to argue is to regularize u_ε by a standard convolution proce dure.

But because of (30), we see that :

$$-\frac{\partial u_\varepsilon}{\partial t} + \sup_{\alpha \in A} [A_\alpha u_\varepsilon(x,t) - f(x,t,\alpha)] \geq$$

$$\geq \; - C\varepsilon - C\|a_\varepsilon - a\|_\infty - C\|f_\varepsilon - f\|_\infty - C\|b_\varepsilon - b\|_\infty - C\|c_\varepsilon - c\|_\infty .$$

It is now easy to conclude using Proposition I.2 (recall that time is reversed) and the fact that $u_\varepsilon \underset{\varepsilon}{\to} \bar{u}$ in $L^\infty(R^N \times (0,T))$. ∎

Let us indicate an application of Theorem II.3 :

Corollary II.3 : *Assume that $\mathcal{O}$ is bounded, that* (19) *holds. We set* $\lambda = \inf_{\substack{\alpha \in A \\ x \in \mathbb{R}^N}} c(x,\alpha)$ *and we assume :*

$$\text{(31)} \qquad \sup_{x \in \overline{\mathcal{O}}} \sup_{A} E\,[e^{-\lambda\tau}] < +\infty \qquad \textit{if} \quad \lambda < 0$$

and we set, if $\lambda < 0$, $C_o = \sup_{x \in \overline{\mathcal{O}}} \sup_{A} E\,[e^{-\lambda\tau}]$.

Let $u,v \in C(\overline{\mathcal{O}})$ *be viscosity solutions of* (4), *then we have :*

$$\sup_{\overline{\mathcal{O}}} (u-v)^+ \leqslant C_o \sup_{\Gamma} (u-v)^+ \quad .$$

Proof : Remarking that, for all $T > 0$, u,v are viscosity solutions of :

$$-\frac{\partial u}{\partial t} + \sup_{\alpha \in A} (A_\alpha u(x,t) - f_\alpha(x,\alpha)) = 0 \qquad \text{in } \mathcal{O} \times (0,T)$$

we deduce from Theorem II.3, letting $\delta_1, \delta_2 \to 0_+$:

$$(u(x)-v(x))^+ \leqslant \sup_{\Gamma} (u-v)^+ \cdot E\,[e^{-\lambda T \wedge \tau}] \quad ,$$

If we let $T \to +\infty$, we obtain the inequality we want. ■

Remark II.4 : The above result is just an illustration of a method of proof and thus admits many variants that we skip. ■

Let us finally conclude this section by stating without proof a result concerning optimal stopping problems. We only state the part corresponding to part i) of Theorem II.3 but parts ii) and iii) have obvious analogues.

<u>Theorem II.4</u> : *Let* $u, \psi \in C(\mathcal{O})$. *We assume that* (19), (8) *hold and that* u *is a viscosity solution of* :

$$\max \left\{ u-\psi \, , \, \sup_{\alpha \in A} \left(A_\alpha u(x) - f(x,\alpha) \right) \right\} = 0 \qquad \text{in } \mathcal{O}$$

then we have for all $\delta > 0$ *and for all* $x \in \overline{\mathcal{O}}_\delta$:

$$u(x) = \inf_{A,\theta} E\left\{ \int_0^{\tau_\delta \wedge \theta} f(X_t,\alpha_t) \exp\left[-\int_0^t c\right] dt \, + \right.$$

$$\left. + \, 1_{(\tau_\delta \leqslant \theta)} u(X_{\tau_\delta}) \exp\left[-\int_0^{\tau_\delta} c\right] + 1_{(\tau_\delta > \theta)} \psi(X_\theta) \exp\left[-\int_0^\theta c\right] \right\} ;$$

where the infimum is taken over all admissible systems A *and all stopping times* θ .

II.4 Applications.

We want to recall in this section a uniqueness result concerning nonlinear semi-groups (corresponding to HJB equations) due to M. Nisio and the author [27], which is a direct application of the above uniqueness results. We will slighty extend it without giving the most general version. We will assume :

$$(32) \quad \begin{cases} \sup_{\alpha \in A} \left\{ \sum_{i,j} \|\sigma_{ij}\|_{W^{2,\infty}(R^N)} + \sum_i \|b_i\|_{W^{1,\infty}(R^N)} + \|c\|_{L^\infty(R^N)} \right\} < \infty ; \\ c,f \text{ are uniformly continuous, bounded in } R^N, \\ \qquad\qquad \text{uniformly in } \alpha \in A \, ; \\ \sigma_{ij}, b_i, c, f \text{ are continuous in } \alpha, \text{ for all } x \in R^N . \end{cases}$$

We will say that a semi-group $S(t)$ on $C_b(R^N)$ is continuous if for all $u \in C_b(R^N)$, $S(t)u \in C_b(R^N \times [0,\infty))$.

<u>Theorem II,5</u> : *We assume* (32)*. Let* $S(t)$ *be a continuous semigroup on* $C_b(R^N)$ *, preserving order i.e.* :

$$S(t)u \leqslant S(t)v \quad \text{in } R^N \ , \ \forall\, t \geqslant 0 \ \text{ if } \ u \leqslant v \text{ in } R^N \ ;$$

and satisfying for all $\psi \in C_b^\infty(R^N)$, $\varphi \in C_b^\infty(R^N)$:

$$\frac{1}{t}\left\{\psi - S(t)(\psi+t\varphi)\right\} \underset{t \to 0_+}{\to} -\varphi + \sup_{\alpha \in A}\left[A_\alpha\psi - f(\cdot,\alpha)\right], \quad \forall\, x \in R^N .$$

Then we have for all $u \in C_b(R^N)$:

$$S(t)u\,(x) = \inf_{A} E\left[\int_0^t f(X_s,\alpha_s)\exp\left(-\int_0^s c\right) + u(X_t)\exp\left(-\int_0^t c\right)\right] \tag{33}$$

<u>Remark II,5</u> : The semi-group $S^o(t)$ given by the right-hand side member was studied by M. Nisio [30], [31], [32], [33], [34]. It is easy to check that $S^o(t)$ is a continuous semi-group on $C_b(R^N)$, preserving order and satisfying :

$$\frac{1}{t}\{\psi - S^o(t)\psi\} \underset{t \to 0_+}{\to} \sup_{\alpha \in A}\{A_\alpha\psi - f(\cdot,\alpha)\} \qquad \text{in } L^\infty(R^N)$$

uniformly for ψ bounded in $C_b^{2,\alpha}(R^N)$ (for example) with $\alpha > 0$.

And this implies the weaker assumption on $S(t)$ above.

<u>Remark II,6</u> : This kind of result is useful in asymptotic problems, see, for example, P.L. Lions, G. Papanicolaou and S.R.S. Varadhan [28]. ∎

<u>Proof of Theorem II,5</u> : In view of Theorem II,3, it is enough to show that for all $u \in C_b(R^N)$, $S(t)u\,(x) = u(x,t)$ is a viscosity solution of :

$$\frac{\partial u}{\partial t} + \sup_{\alpha \in A} [A_\alpha u(x,t) - f(x,\alpha)] = 0 \qquad \text{in } R^N \times (0,\infty)$$

- indeed, use the homogeneity in t of the coefficients -.

To this end, it is enough to consider a global maximum point (x_o,t_o) of $u-\psi$ where $\psi \in C_b^\infty(R^N \times [0,T])$ ($\forall\, T < \infty$) and to prove (11') : now, remark that we have for $h > 0$ small :

$$u(x_o,t_o) = \psi(x_o,t_o) + \Delta = S(h)\, u(t_o-h)(x_o) + \Delta$$

where $\Delta = u(x_o,t_o) - \psi(x_o,t_o)$. Remark also that without loss of generality we may take $\Delta = 0$ and thus using the fact that $S(t)$ preserves order, we find :

$$\psi(x_o,t_o) \leqslant S(h)\, \psi(\cdot,t_o-h)\,(x_o) \quad .$$

Next, let $\varepsilon > 0$, for h small enough we have on R^N :

$$\psi(\cdot,t_o-h) \leqslant \psi(\cdot,t_o) - h\frac{\partial\psi}{\partial t}(\cdot,t_o) + \varepsilon h \quad .$$

Using again the order-preserving property, we deduce :

$$\frac{1}{h}\left\{\psi(x_o,t_o) - S(h)\left\{\psi(\cdot,t_o)+h\left[-\frac{\partial\psi}{\partial t}(\cdot,t_o)+\varepsilon\right]\right\}(x_o)\right\} \leqslant 0$$

and if we let $h \to 0_+$, this yields :

$$\frac{\partial\psi}{\partial t}(x_o,t_o) - \varepsilon + \sup_{\alpha \in A} \{A_\alpha\psi(x_o,t_o)-f(x_o,\alpha)\} \leqslant 0$$

which implies (11') since $\varepsilon > 0$ is arbitrary. ∎

It is worth pointing out that the above proof is very similar to the proof of Theorem I.1.

Appendix : Equivalence of the various notions of degenerate elliptic inequalities.

Let $\sigma_{ij} \in W^{2,\infty}_{loc}(\mathcal{O})$ $(1 \leqslant i \leqslant N\ ,\ 1 \leqslant j \leqslant m)$, let $b \in W^{1,\infty}_{loc}(\mathcal{O})$, let $c, f \in C(\mathcal{O})$. We denote by $a = \frac{1}{2}\sigma\sigma^T$ and by A the operator

$$A = -a_{ij}\,\partial_{ij} - b_i\,\partial_i + c\ .$$

We want to mention in the following result (which admits many variants or extensions) the relations between the various notions of inequalities like :

$$Au \leqslant f \qquad \text{in } \mathcal{O}\ .$$

Theorem : *Let* $u \in C(\mathcal{O})$ *. Then the following are equivalent*

i) u *is a viscosity subsolution of* : $Au \leqslant f$ *in* $\mathcal{O}$;

ii) *For all* $x \in \overline{\mathcal{O}}_\delta$, *for all* $\delta > 0$:

$$M_{t\wedge\tau_\delta} = u(X_{t\wedge\tau_\delta})\exp\left(-\int_0^{t\wedge\tau_\delta} c\right) + \int_0^{t\wedge\tau_\delta} f(X_s)\exp\left(-\int_0^s c\right)$$

is a submartingale - where X_t *is the diffusion process associated with* A *and* τ_δ *is the first exit time from* $\overline{\mathcal{O}}_\delta$ *of* X_t - ;

iii) u *satisfies* : $Au \leqslant f$ *in* $\mathcal{D}'(\mathcal{O})$.

Remarks :

i) As we explained in Part 1 [19] after Theorem I,2, the above result can be used to extend Theorem I,2 : one may suppress the assumption $\sigma^T\cdot\nabla v \in L^2_{loc}(\mathcal{O})$ (or $a^{1/2}\cdot\nabla v \in L^2_{loc}$) in Theorem I,2.

ii) In D,W, Stroock and S,R,S, Varadhan [36] a similar result is proved concerning notions ii) and iii) for elliptic (degenerate) equations,

Proof : The fact that i) implies ii) has been pointed out in Remark II,2 ; while the fact that ii) implies i) follows from the proof of Theorem I,1, In addition we proved in Part 1 [19] that ii) implies iii) (Proof of Theorem I,1 in [19]), We thus have to prove that iii) implies ii),

Exactly as in the proof of Theorem II,1, we reduce first the general case to the case when $\mathcal{O} = R^N$, u,f,b,c,σ have compact supports and because of the Markov property (Cf, Remark II,2 and [19]) it is enough to show : $u(x) \leqslant u(x,t)$, $\forall\, x \in R^N$, $\forall\, t \geqslant 0$ where $u(x,t) = E\int_0^t f(X_s)\exp\left(-\int_0^s c\right) + u(X_t)\exp\left(-\int_0^t c\right)$ is the solution of :

$$\frac{\partial u}{\partial t} + Au = f \quad \text{in } R^N x(0,\infty) \quad , \quad u|_{t=0} = u \quad \text{in } R^N .$$

Therefore we just need to show that if $v \in C([0,T\,]; L^2(R^N))$ satisfies

$$\frac{\partial v}{\partial t} + Av \geqslant 0 \quad \text{in } \mathcal{D}'(R^N \times (0,T)) \quad , \quad v|_{t=0} \geqslant 0 \quad \text{in } R^N$$

then $v \geqslant 0$ in $R^N \times (0,T)$, Multiplying by $\varphi \in C^{2,1}_{comp}(R^N \times [0,T])$, $\varphi \geqslant 0$ we obtain integrating by parts :

$$(v(T),\varphi(T))_{L^2(R^N)} + \left(v\, ,\, -\frac{\partial\varphi}{\partial t} + A^\star\varphi\right)_{L^2(R^N \times (0,T))} \geqslant 0$$

where $A^\star = -\, a_{ij}\partial_{ij} + (b_i - 2\partial_j a_{ij})\partial_i + (c - \partial_{ij}a_{ij} + \partial_i b_i)$.

Now, as in the proof of Theorem II,2, if $f \in \mathcal{D}_+(R^N \times (0,T))$, regularizing first $(c - \partial_{ij}a_{ij} + {}_i b_i)$ and then $(b_i - 2\partial_j a_{ij})$, we can find $\varphi_\varepsilon \in C^{2,1}_b(R^N \times [0,T\,]) \cap C([0,T\,]; L^2(R^N))$ such that

$$\begin{cases} -\dfrac{\partial\varphi_\varepsilon}{\partial t} + A^\star\varphi_\varepsilon \underset{\varepsilon \to 0}{\to} f & \text{in } L^2(R^N \times (0,T)) \\ \varphi_\varepsilon(T) = 0 & \text{in } R^N . \end{cases}$$

Therefore we obtain for all $f \in \mathcal{D}_+(R^N \times (0,T))$:

$$(v,f)_{L^2(R^N \times (0,T))} \geqslant 0$$

and we conclude : $v \geqslant 0$ in $R^N \times (0,T)$. ■

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