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Degenerate Elliptic-Parabolic Equations of Second Order*

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1. Introduction

In this paper we shall study the first boundary value problem ("Dirichlet problem") for a single second order equation

$$(1.1)\qquad Lu = a^{ij}u_{ij} + b^i u_i + cu = f \qquad \text{with} \qquad \sum a^{ij}\xi_i\xi_j \geqq 0\,.$$

Here $u(x)$ is a real function defined in a compact domain $\mathscr{M}$ in R^n (or on a manifold), with C^∞ boundary; $x = (x^1, \cdots, x^n)$ represent the coordinates, and we have used subscripts to denote differentiation; we have also used summation convention. The coefficients are real and of class C^∞ in $\bar{\mathscr{M}}$, the closure of $\mathscr{M}$. The first boundary value problem consists in prescribing the values of u on a certain portion of the boundary $\dot{\mathscr{M}}$. We wish to obtain unique solutions of the problem which are smooth up to and including the boundary. If the leading part is elliptic, $\Sigma\, a^{ij}\xi_i\xi_j > 0$ for $\xi = (\xi_1, \cdots, \xi_n) \neq 0$, we have the usual Dirichlet problem. Another well known example of (1.1) is the heat equation $u_{xx} - u_t = 0$. For this classical equation, however, certain aspects of the first boundary value problem have never been adequately studied. In Section 2 which is devoted to this equation (and also to second order parabolic equations in more independent variables) we consider some of these questions and present several new results, including unexpected compatibility conditions. We should remark that we have not succeeded in settling all the questions discussed there.

It is customary to call operators L, with $\Sigma\, a^{ij}\xi_i\xi_j \geqq 0$, degenerate elliptic-parabolic. The systematic study of the general class of such equations was initiated by Fichera [4], [5] who obtained estimates in L_p (and maximum) norms, and proved the existence of generalized solutions. Oleinik, in a series of papers [23]–[27] proved under certain conditions that "weak solutions are strong" and hence, in these cases, obtained generalized solutions in a class in which the solutions were

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unique. In [25], [26] she proved, under more conditions, that solutions are smooth up to the boundary. Oleinik also presented examples showing that, in general, the solution may not be smooth. Phillips and Sarason [28] have obtained further results on "weak equals strong". Degenerate elliptic-parabolic equations arise in certain problems in probability, and have been investigated with the aid of probabilistic methods. In this connection Freidlin has studied the question of existence and smoothness of solutions in [7], [8] (where other references may be found). We refer also to an interesting lecture of Kac [12] in which a degenerate equation and its physical significance are discussed, and to a treatment of the simple-looking equation $yu_x + u_{yy} = 0$ in the strip $0 < x < 1$ by Fleming [6].

Our interest in degenerate elliptic-parabolic equations developed in connection with our work [14] on systems of differential equations arising from an integral quadratic form which is positive semi-definite in its dependence on the highest order derivatives. The aim there was also to obtain solutions which are smooth in the closure $\bar{\mathscr{M}}$ of the domain. One of the crucial assumptions made was that the boundary of the domain is a noncharacteristic surface relative to the system. In order to acquire some insight into problems where this is not the case, we treated (in Section 9.2 of [14]) the first boundary value problem for (1.1) in a domain whose boundary may admit characteristic points. However, as in the work of Oleinik, we assumed that the equation could be extended, as a degenerate elliptic-parabolic equation, across the characteristic portions of the boundary. We then used the trick of modifying the equation in the extended region so as to make it elliptic there. The boundary of the enlarged region is then noncharacteristic, and our general results of [14] were applicable. (Recently Baouendi [1], [2] has treated elliptic equations which degenerate at the boundary, but for which the boundary is still noncharacteristic. Making very precise conditions on the nature of the degeneracy, he has derived rather sharp results concerning the existence and regularity of solutions.)

The device of extension is too limited; it is important to treat equations which cannot be so extended, and this paper is concerned with a more detailed study of (1.1) without any assumptions of extendibility. The results, being confined to one equation for one real unknown function, and subject to certain restrictions, are rather special. But even for this case our proofs are surprisingly complicated. We hope that the methods employed here may prove useful in treating more general systems.

We now describe the boundary value problem and the main existence theorem; the notation is more or less the same as in Section 9.2 of [14]. We use ν_j to denote the j-th component of the unit exterior normal at $\dot{\mathscr{M}}$. Following Fichera [4], [5], the boundary $\dot{\mathscr{M}}$ is divided into three portions, on two of which the boundary values of u will be given:

$\Sigma_3 =$ the set of noncharacteristic boundary points, i.e., those where

$$a^{ij}\nu_i\nu_j > 0 ,$$

$\Sigma_2 =$ the set of characteristic boundary points where

$$(b^i - a_i^{ij})\nu_i > 0\,,$$

$\Sigma_1 = \dot{\mathscr{M}} - \Sigma_2 - \Sigma_3$.

The "Dirichlet problem" is that of finding a solution of (1.1) which has given values on $\Sigma_2 + \Sigma_3$. After subtraction of a function with the same values, we may assume that the given boundary values on $\Sigma_2 \cup \Sigma_3$ are zero.

Under certain conditions we shall prove that this problem has a smooth solution in $\bar{\mathscr{M}}$—a solution belonging to class H_m, i.e., having square integrable derivatives up to order m.

Σ_1 is clearly closed. In [14] and in [26] (which are concerned with regularity) it was assumed that also both Σ_2 and Σ_3 are closed. This implies that, in fact, Σ_1, $\bar{\Sigma}_2$, $\bar{\Sigma}_3$ are disjoint. We shall now assume the weaker condition:

(a) $\qquad\qquad \Sigma_2 \cup \Sigma_3$ *is closed*.

This condition implies that Σ_1 is disjoint from $\overline{\Sigma_2 \cup \Sigma_3}$. Thus each component of $\dot{\mathscr{M}}$ belongs either to Σ_1 or to $\Sigma_2 \cup \Sigma_3$. There are simple examples (see [23], [25]) showing that, if Σ_1 touches Σ_2 or Σ_3, then the solutions need not be smooth. On the other hand, there are interesting cases (such as the heat equation) in which they do touch and where, nevertheless, the solutions are smooth; see Remark 9.2 in [14].

Our second condition (b)[1] concerns the last coefficient in (1.1); in Section 9 we give some conditions under which (b) is not needed.

(b) $-c$ *is very large positive compared to the other coefficients and their derivatives of second order, and to derivatives of order* $\leqq 3$ *of the* a^{ij}, *as well as to* γ^{-1} *defined in* (c).

The remaining conditions will refer to the behavior of the coefficients of (1.1) on Σ_2. Because of condition (a) it follows that Σ_2 is closed. In the following N is a fixed integer greater than one; we seek solutions in H_m with $2m$ approximately equal to N. We shall introduce certain invariants associated with our problem which are well defined on Σ_2.

Let x_0 be a point of Σ_2 and let ϕ be a function defined in a neighborhood of x_0 which vanishes identically on $\dot{\mathscr{M}}$ and such that grad $\phi(x_0) \neq 0$ and $\phi < 0$ in $\mathscr{M}$;

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[1] This formulation of (b) is an improvement of the one given on page 486 of [14] and is related to (but more stringent than) the conditions given by Oleinik [25], [26] in obtaining smooth solutions. In the proof we make use of arguments similar to those used in proving Theorem 2′ of [14] and its corollary, and our improvement of condition (b) will make use of the following improvement of Theorem 2′ which may be verified by a close examination of its proof:

Remark. In Theorem 2′ of [14] the constant ε^{-1} need only be large compared to the other coefficients and to the first and second derivatives of the coefficients $a_{ij}^{\alpha\beta}$ with $|\alpha + \beta| = 2$, as well as to the first derivatives of the $a_{ij}^{\alpha\beta}$ with $|\alpha + \beta| = 1$.

then, as we have remarked on page 488 of [14], on Σ_2 the values of

$$\beta = (b^i - a^{ij}_j)\phi_i\,, \qquad \alpha = (b^i - a^{ij}_j)(\phi_r\phi_s a^{rs})_i \tag{1.2}$$

are independent of the particular coordinate system.[2] By hypothesis, $\beta > 0$ on Σ_2 and, since the vector ϕ_i is parallel to ν_i and $\nu_i(\partial/\partial x^i)(\phi_r\phi_s a^{rs}) \leqq 0$ there, we verify easily that $\alpha \leqq 0$. Furthermore, α/β^2 is invariant under change of the function ϕ or under multiplication of the operator L by a positive factor; it is thus a *complete invariant*. Our third condition is

$$\text{(c)} \qquad \gamma = 1 + \tfrac{1}{2}N\frac{\alpha}{\beta^2} > 0 \ \ on\ \ \Sigma_2\,.$$

To formulate the last condition let ϕ be defined in a neighborhood of $x_0 \in \Sigma_2$ as above; then the function

$$a = a^{rs}\phi_r\phi_s \tag{1.3}$$

vanishes on Σ_2. Our last condition is expressed in terms of the leading part $L_0 = a^{ij}(\partial^2/\partial x^i\,\partial x^j)$ of the operator L:

(d) *For a constant ε_0 depending only on n we require*

$$\frac{\sqrt{L_0 a}}{\beta}(x_0) \leqq \varepsilon_0\frac{\gamma(x_0)}{N}\,. \tag{1.4}$$

We shall take $\varepsilon_0 = (100\,n^2 3^n)^{-1}$ (no doubt this absurd value of ε_0 is unnecessarily small). It is easily verified that the condition is independent of the choice of ϕ. In case x_0 is in the closure of the interior points (relative to $\dot{\mathscr{M}}$) of Σ_2, the left-hand side of (1.4) is zero. Thus condition (1.4) is really a condition only at the remaining points of Σ_2. Further interpretations of condition (d) will be given in Sections 2 and 3. We repeat that our conditions (c), (d) are completely invariant.

Under conditions (a) to (d) we shall find a smooth solution of the Dirichlet problem.

Our proof of regularity is based on a global argument; we cannot prove local regularity of the solution. We have made no hypotheses concerning the coefficients in the interior, except that $-c$ is large, and so if, for example, the leading coefficients a^{ij} vanish on an open set, and the coefficients b^i do not all vanish there, then there can be no local regularity theorem for the solution in this set. In that case one must follow the curves defined by the direction field $\dot{x}^i = b^i$; the global behavior of these curves will determine the smoothness of solutions. The problem

[2] We note that in [14] the term β is denoted by b.

of determining under what conditions the operator L is hypoelliptic has recently been settled by Hörmander [11].

Before describing our existence theorem we subdivide Σ_1 still further:

$$\Sigma_1 = \Sigma_{11} + \Sigma_{12},$$

where Σ_{12} is the union of those components of Σ_1 on which $(b^i - a^{ij}_j)\nu_i < 0$ everywhere; Σ_{12} may be empty. Our solution will have the property that near Σ_{11} and Σ_2 its derivatives in directions parallel to the boundary will be smoother than the normal derivatives. We now formulate this more precisely. Near $\dot{\mathscr{M}}$ let $-y$ denote the distance from the boundary; on the surfaces $y =$ constant we use $x^1, \cdots, x^{n-1}$ as local coordinates, taking $x^n = y$, after a change of coordinates. In these coordinates we shall denote a general derivative of order m by

$$D^{\alpha,k}u = \left(\frac{\partial}{\partial x^1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x^{n-1}}\right)^{\alpha_{n-1}} \left(\frac{\partial}{\partial y}\right)^k u, \qquad m - k = |\alpha| = \sum_{i=1}^{n-1} \alpha_i. \tag{1.5}$$

Here $\alpha = (\alpha_1, \cdots, \alpha_{n-1})$ is an $(n-1)$-tuple of integers, and we shall use δ_j to represent the $(n-1)$-tuple α with all $\alpha_i = 0$ except for $\alpha_j = 1$. For suitably chosen small disjoint neighborhoods in $\bar{\mathscr{M}}$ of Σ_2 and Σ_{11}, which we denote by $\mathscr{M}_2$ and $\mathscr{M}_1$, setting also $\mathscr{M} - \mathscr{M}_1 - \mathscr{M}_2 = \Omega$, we introduce the norms (here dV represents element of volume in $\mathscr{M}$):

$$\begin{aligned} \widetilde{\|u\|}_N^2 = &\int_{\mathscr{M}_2}\left[\sum_{|\alpha|\leqq N} |D^{\alpha,0}u|^2 + \sum_{\substack{|\alpha|+k\leqq N\\ 0<k}} |y|^{k-1}\, |D^{\alpha,k}u|^2\right] dV \\ &+ \int_{\mathscr{M}_1} \sum_{|\alpha|+k\leqq N} |y|^k\, |D^{\alpha,k}u|^2\, dV + \int_\Omega \sum_{|\alpha|\leqq N} |D^\alpha u|^2\, dV. \end{aligned} \tag{1.6}$$

We may now state our main existence theorem. We actually prove a sharper form, Theorem 1′, in Section 7.

THEOREM 1. *Assume that conditions* (a)–(d) *hold. For any $f \in H_N$ there exists a solution u of* (1.1) *vanishing on $\Sigma_2 \cup \Sigma_3$ with finite norm $\widetilde{\|u\|}_N$ and*

$$\widetilde{\|u\|}_N \leqq \text{constant} \cdot \|f\|_N, \tag{1.7}$$

the constant being independent of f. Consequently: (i) *In Ω the solution u lies in H_N.* (ii) *In $\mathscr{M}_2$, $u \in H_{N/2}$ for N even, and $u \in H_{(N+1)/2}$ for N odd.* (iii) *In $\mathscr{M}_1$, $u \in H_{N/2}$ for N even, and $u \in H_{(N-1)/2}$ for N odd.*

An analysis of the various conditions, and illustrations, will be given in Sections 2 and 3.

Unfortunately the equations described in [12] and [6] do not satisfy our conditions ($\Sigma_2 \cup \Sigma_3$ is not closed). For instance for the equation $yu_x + u_{yy}$ in $0 \leqq x \leqq 1$ the segment $y \geqq 0$ on the y-axis is in Σ_1 and $y < 0$ in Σ_2; while on $x = 1, y > 0$ is in Σ_2 and $y \leqq 0$ in Σ_1.

We remark here that Theorem 1 also holds in an unbounded domain $\mathscr{M}$ provided $\mathscr{M}$ satisfies the following conditions: $\dot{\mathscr{M}}$ is of class C^N, and there is a positive number d such that each point P in $\bar{\mathscr{M}}$ within a distance d of $\dot{\mathscr{M}}$ has a neighborhood U_P, containing the ball about P of radius $\frac{1}{2}d$, so that the set $\bar{U}_P \cap \bar{\mathscr{M}}$ can be mapped in a one-to-one way onto the closure of a hemisphere of radius d, with $\bar{U}_P \cap \dot{\mathscr{M}}$ mapping onto the flat part of the hemisphere, by a mapping T_0 which (together with its inverse) is of class C^N; furthermore, the derivatives up to order N of the mapping (and the inverse map) are to be bounded in absolute value by a constant κ independent of P. In addition we require that the derivatives of the coefficients of L be uniformly bounded, and that γ in condition (c) be bounded from below by a positive constant.

In Section 2 we apply Theorem 1 to the heat equation and prove several results for this, as well as the more general parabolic equation. Except for the reference to Theorem 1 we have tried to make this section self-contained.

In Section 3 we give some analysis of conditions (c), (d) and make some preliminary normalizations that will be useful in the proof of Theorem 1.

As we have indicated, the proof is complicated. It is based on *a priori* estimates which we have tried to localize as much as possible—working separately in the compact subset of $\mathscr{M}$, near Σ_1 and near Σ_2, Σ_3 (in Sections 4, 5 and 6). All the estimates are derived by applying certain operators to the equation, multiplying the result by a derivative of u (times a suitable weight factor, depending on which boundary we are near) and then integrating by parts, and again integrating by parts—we never cease to integrate by parts. Since the equation is not elliptic the error terms arising from the derivatives of the coefficients require careful treatment; if they are not handled in just the right way the proof falls apart. In Section 7 the estimates are all combined, and the full result, Theorem 1′, is stated.

Finally we must "regularize" the equation in order to get smooth approximating solutions to which these estimates may be applied. The regularization is given in the sketch of the proof described in Section 3.6. For the "regularized" equation it is necessary to modify and rederive some of the estimates near Σ_1. This is carried out in Section 8—where the proof is completed.

In Section 9 we point out some conditions under which the hypothesis (b), that $-c$ is large, is not needed. These remarks apply also to our general theorem of [14].

2. On the Heat Equation

2.1. The first boundary value problem. Consider the heat equation for one space variable

$$u_{xx} - u_t = 0 \tag{2.1}$$

in the domains (with C^∞ boundaries except at the corners shown). In the first

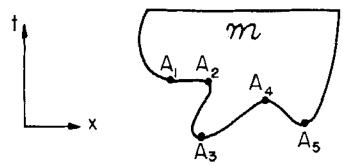

Figure 1

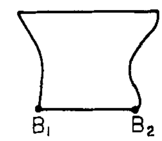

Figure 2

(Dirichlet) boundary value problem the value of the solution is prescribed on the whole boundary except for the top segment. There is an extensive literature devoted to this problem; however most of the research concerned with "smooth" (say C^∞) solutions in the closure of the domain treats only domains of the kind of Figure 2. At the corners B_1, B_2 the boundary values have to satisfy certain compatibility conditions. Concerning domains of the type of Figure 1, Levi in his paper [17] pointed out that the problem of the behavior of the solution at the characteristic points A_3, A_4, A_5, and the characteristic segment $\overline{A_1A_2}$ (all of which belong to Σ_2) is a very difficult one, and there has been little further study of this problem.

In the course of our work on degenerate elliptic-parabolic equations we found—and it was a great surprise to us, though easy enough to verify—that, if the solution is smooth in the closure of the domain $\mathscr{M}$ of Figure 1, then the boundary values of u may have to satisfy compatibility conditions at the point A_4 depending on the value of the curvature of the boundary there. Furthermore, if the curvature is not zero, the solution need not be C^∞ there, but the smaller the curvature the smoother is the solution at that point—it is C^∞ if the curvature vanishes.

It is to be expected that there might be non-smoothness at $A_4 = (x_0, t_0)$, since for $t < t_0$ on the two sides of A_4 the solution is determined by different data, and there may not be smooth matching at A_4. However, it is not *a priori* clear that it is just the curvature (in distinction to higher order derivatives of the boundary) which is the distinguishing factor.

At all other points, in particular the points A_3 and A_5, where the boundary curve is convex, the solution is C^∞; however, to our knowledge this result is not in the literature. Kondrat'ev [15] has studied general parabolic equations which are of first order in t and of arbitrary order in the space variables in domains like Figure 1. In the case of second order parabolic equations he has noted that at convex boundary points like A_3 and A_5, the smaller the curvature the smoother is the solution. He informed us in October 1966 that he could prove that for the heat equation the solution is C^∞ at convex boundary points $A_3 = (x_0, t_0)$ provided that at these points the boundary curve has the form

$$t - t_0 = k(x - x_0)^p,$$

where $k \geqq 0$ and $p \geqq 2$ is an integer.

We shall prove that $u \in C^\infty$ at points A_3, where the boundary has positive curvature; this is done for general second order parabolic equations in n dimensions in Section 2.5. Our proof, however, is not based on Theorem 1. It makes use of a singular transformation of variable not unlike one used by Kondrat'ev in [15] which "blows up" the point A_3.

In Section 2.2 we describe the compatibility conditions at A_4. In Section 2.3 we use our Theorem 1 to show that the solution is "smooth" there provided the curvature is small, and in the following section we present an example due to H. Weinberger showing that the solutions need not be C^∞. In Section 2.5 we treat general second order parabolic equations in domains of the type of Figure 1.

2.2. Compatibility. Consider the domain $\mathscr{M}$ near A_4 which we take to be the origin. Assume that the boundary through 0 has the local representation

$$t = g(x) = -Mx^2 + \text{higher order terms}\,; \tag{2.2}$$

in Figure 1 we have taken $M > 0$, but in the discussion below M may have any value. If u belongs to C^m in the intersection of $\bar{\mathscr{M}}$ with a neighborhood of 0, then we claim that for certain values of M the derivatives of the boundary values v satisfy compatibility relations at the origin. These are found when one tries to determine all derivatives of u at the origin. The easiest way to see this is to flatten the boundary locally by introducing as new variables

$$\begin{aligned} \tau &= t - g(x)\,, \\ \xi &= x\,. \end{aligned} \tag{2.3}$$

The equation then transforms to

$$u_{\xi\xi} - 2\dot{g}u_{\xi\tau} + u_{\tau\tau}\dot{g}^2 - (1 + \ddot{g})u_\tau = 0\,. \tag{2.4}$$

Here the dot denotes differentiation of $g(\xi)$ with respect to ξ. Since $u(\xi, 0) = v(\xi)$ and $\dot{g}(0) = 0$, we see first that at the origin

$$(1 - 2M)u_\tau = v_{\xi\xi}\,. \tag{2.5}$$

Thus we obtain as a necessary condition

$$v_{\xi\xi}(0) = 0 \text{ in case } M = \tfrac{1}{2}\,.$$

Now suppose $M \neq \frac{1}{2}$; differentiating (2.4) with respect to ξ we find with the aid of (2.5) that at the origin

$$v_{\xi\xi\xi} - (1 - 6M)u_{\xi\tau} - \frac{\dddot{g}\,v_{\xi\xi}}{1 - 2M} = 0\,.$$

Hence we obtain the compatibility condition: at the origin

$$(1 - 2M)v_{\xi\xi\xi} = \ddot{g}v_{\xi\xi} \qquad \text{if} \qquad M = \tfrac{1}{6}\,. \tag{2.6}$$

Next, assuming $M \neq \frac{1}{2}, \frac{1}{6}$, so that u_τ and $u_{\tau\xi}$ are known at the origin, we differentiate (2.4), in turn, with respect to τ and twice with respect to ξ, and find that at the origin the following expressions are determined, in terms of derivatives of v:

$$(2M - 1)u_{\tau\tau} + u_{\tau\xi\xi}\,,$$

$$8M^2u_{\tau\tau} + (10M - 1)u_{\tau\xi\xi}\,.$$

Thus again we shall obtain a compatibility condition on the derivatives of v at the origin in case the determinant of coefficients vanishes, i.e.,

$$\frac{3}{4} - \frac{3}{4M} + \left(\frac{1}{4M}\right)^2 = 0\,. \tag{2.7}$$

THEOREM 2. *Let v be the boundary values of a C^∞ solution of the heat equation in $t \geqq -Mx^2 + \cdots$ (in a neighborhood of the origin). There is a sequence $M_1, M_2, \cdots$ of positive numbers such that, if M is equal to one of these values, then v and its derivatives satisfy necessary compatibility conditions at the origin. For any other value of M there are no compatibility conditions. The distinguished values M_i are obtained as follows: the numbers $c_i = 1/(4M_i)$ are the roots of the Laguerre polynomials*

$$L_k^{(-1/2)}\,, \qquad L_k^{(1/2)}\,, \qquad\qquad k = 1, 2, \cdots.$$

These polynomials are defined, for instance, in Szegö [29], Section 5.1:

$$L_k^{(\alpha)}(c) = \sum_{j=0}^{k} \binom{k+\alpha}{k-j} \frac{(-c)^j}{j!}\,. \tag{2.8}$$

The function $y(c) = L_k^{(\alpha)}(c)$ satisfies the differential equation

$$c\ddot{y} + (\alpha + 1 - c)\dot{y} + ky = 0\,. \tag{2.9}$$

Proof: The distinguished values of M are obtained when one attempts to determine the values of derivatives of the form

$$D_\xi^{2j} D_\tau^{r-j} u\,, \qquad\qquad j = 0, \cdots, r\,, \tag{2.10}$$

and

$$D_\xi^{2j+1} D_\tau^{r-j} u\,, \qquad\qquad j = 0, \cdots, r\,, \tag{2.10$'$}$$

at the origin—successively for $r = 1, 2, \cdots$. Assuming that the derivatives of the form (2.10) and (2.10)′ have been determined at the origin for $r = 1, \cdots, k-1$, we find on applying the operators $D_\xi^{2j} D_\tau^{k-j-1}$ and $D_\xi^{2j+1} D_\tau^{k-j-1}$ to (2.4) that the following combinations of derivatives are determined there:

$$(2.11)\qquad \begin{aligned}&(8Mj + 2M - 1) D_\xi^{2j} D_\tau^{k-j} u + D_\xi^{2(j+1)} D_\tau^{k-j-1} u \\ &\qquad + 8M^2 j(2j-1) D_\xi^{2(j-1)} D_\tau^{k-j+1} u\,, \qquad j = 0, \cdots, k-1\,,\end{aligned}$$

and

$$(2.11)'\qquad \begin{aligned}&(8Mj + 6M - 1) D_\xi^{2j+1} D_\tau^{k-j} u + D_\xi^{2j+3} D_\tau^{k-j-1} u \\ &\qquad + 8M^2 j(2j+1) D_\xi^{2j-1} D_\tau^{k-j+1} u\,, \qquad j = 0, \cdots, k-1\,.\end{aligned}$$

The distinguished values of M, for which compatibility arises, are the roots of the determinants of the coefficient matrices of the systems (2.11) and (2.11)′.

Considering the system (2.11) as a system for the terms $(4M)^{k-j} D_\xi^{2j} D_\tau^{k-j} u$ we see, after multiplying (2.11) by $(4M)^{k-j-1}$, that its determinant has the form (with all elements off the three main diagonals vanishing)

$$D_k = \det \begin{vmatrix} \frac{1}{2} - \frac{1}{4M} & 1 & \bigcirc & & \bigcirc \\ \frac{1}{2} & \frac{5}{2} - \frac{1}{4M} & 1 & & \\ 0 & 3 & \frac{9}{2} - \frac{1}{4M} & 1 & \\ & & & \ddots & 1 \\ \bigcirc & & & (k-1)\left(k - \frac{3}{2}\right) & 2k - \frac{3}{2} - \frac{1}{4M} \end{vmatrix}$$

and hence, with $c = 1/(4M)$, satisfies the recursive relation

$$D_k = (2k - \tfrac{3}{2} - c) D_{k-1} - (k-1)(k - \tfrac{3}{2}) D_{k-2}\,.$$

Now

$$D_1(c) = \tfrac{1}{2} - c = L_1^{(-1/2)}(c)\,, \qquad D_2 = \tfrac{3}{4} - 3c + c^2 = 2L_2^{(-1/2)}(c)$$

(compare with (2.7)). Using the recursive formula (5.1.10) of [29] one verifies that the functions $k\,!\,L_k^{(-1/2)}(c)$ satisfy the same recursive relations as the D_k above, and hence

$$D_k(c) = k\,!\,L_k^{(-1/2)}(c)\,.$$

Thus the determinant of the system (2.11) is singular if and only if c is a root of $L_k^{(-1/2)}(c)$; from (2.8) we see that all real roots are positive (in fact, as shown in [29], the zeros are all real and positive).

In a similar way one sees that the determinant of the system (2.11)′ is singular if and only if c is a root of $L_k^{(1/2)}(c)$—again all such roots are positive.

Remark. Because of the relationship of the Laguerre polynomials $L_m^{(\pm 1/2)}$ to the Hermite polynomials H_k:

$$H_{2m}(s) = (-1)^m 2^{2m} m\,!\, L_m^{(-1/2)}(s^2)\,, \qquad H_{2m+1} = (-1)^m 2^{2m+1} m\,!\, s L_m^{(1/2)}(s^2)\,,$$

we see that the distinguished values M_i of Theorem 2 are such that $\frac{1}{2}M_i^{-1/2}$ are exactly the positive roots of the Hermite polynomials H_k, $k = 1, 2, \cdots$. These same values M_i were also found by Gross [30] in his determination of the possible level lines $t = -Mx^2 + \cdots$ of solutions of the heat equation.

2.3. Smooth solutions by means of Theorem 1. The existence of compatibility conditions suggests the following

QUESTION. *If the boundary values v satisfy all necessary compatibility conditions (if any), does there exist a smooth solution of the heat equation with these boundary values?*

It is well known (see for instance [9]) that in a neighborhood of the origin the solution of (2.1) in $\mathscr{M}: t > -Mx^2$ with smooth (say C^∞) boundary values is smooth (C^∞) up to the boundary with the possible exception of the origin where the boundary is characteristic. Let us see if our Theorem 1 yields solutions with some degree of smoothness near the origin. First of all, by the usual device of multiplying u by $e^{-\lambda t}$, we obtain a solution of $u_{xx} - u_t - \lambda u = 0$ in $\mathscr{M}$, which we continue to call u. The coefficient $-\lambda$ may be as large negative as we please. In applying our Theorem 1 we shall make use of the smoothness of u everywhere except at the origin (however in Section 2.5 we do not assume this). In the region $\mathscr{D}$ shown below, lying in $\mathscr{M}$, and having C^∞ boundary, let $\zeta \geqq 0$ be a C^∞ function which vanishes in $x^2 + t^2 < \varepsilon$ and is otherwise positive. In $\mathscr{D}$ the function u satisfies the equation (with $\Delta u = u_{xx} + u_{tt}$)

$$L'u = u_{xx} - u_t - \lambda u + \zeta \Delta u = \zeta \Delta u = f\,.$$

By the known smoothness of u the value of u on the boundary of $\mathscr{D}$ belongs to C^∞ and also f belongs to C^∞ in $\overline{\mathscr{D}}$. Since λ may be arbitrarily large, we see that L' satisfies condition (b) in $\mathscr{D}$, and for ε small the whole boundary of $\mathscr{D}$ belongs to Σ_3 except for the origin which is Σ_2.

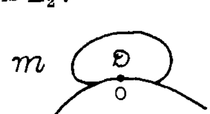

Figure 3

For $\phi = g(x) - t$ in (1.2) we see that at the origin

$$\beta = 1\,, \qquad \alpha = 0 \qquad \text{and hence} \qquad \gamma = 1\,.$$

Let us now consider condition (d); $a = a^{rs}\phi_r\phi_s = \dot{g}^2$, and at the origin $L_0 a = 8M^2$. Hence the left-hand side of (1.4) equals $\sqrt{8M^2}$; furthermore, $\varepsilon_0^{-1} = 3600$. Thus we are in a position to apply Theorem 1 and conclude that the solution u belongs to H_m in $\bar{\mathscr{D}}$ *provided that* $|3mM| \leqq 10^{-4}$ (here M may be positive or negative and we take $N = 2m$). Thus the smaller M^2, the larger m may be taken; in particular if $M = 0$, $u \in C^\infty(\bar{\mathscr{D}})$. However, no compatibility conditions enter into Theorem 1 and one might first think that this creates a contradiction. But indeed the coefficient matrices of the relations (2.11) and (2.11)′ are such that for $|M|$ small the main diagonal terms (in (2.11) say) $8Mj + 2M - 1, j = 0, \cdots, r - 1$, are close to -1 (provided we consider fixed bounded r), the next diagonal below it, with terms $8M^2 j(2j - 1)$, is close to zero, and all other terms below vanish. Thus for fixed r, and $|M|$ sufficiently small, these matrices have determinants close to ± 1, and therefore there are no compatibility relations to be satisfied for the derivatives up to this order (they might occur for higher order derivatives).

The C^∞ character of the solution at $\overline{A_1A_2}$ is proved in a similar manner, and at the points A_3 and A_4 it is contained in Theorem 4 of Section 2.5.

2.4. An example. Returning again to our question of Section 2.3 the answer is no! The following counterexample for $M > 0$ was proposed to us by H. Weinberger: For positive m the function

$$u(x, t) = \int_{-1}^{\min(0,t)} \frac{(-\tau)^m}{\sqrt{t - \tau}}\, e^{-x^2/[4(t-\tau)]}\, d\tau \tag{2.12}$$

is a solution of the heat equation (2.1) in $t > -1$ outside the segment $x = 0$, $-1 \leqq t \leqq 0$. A slightly tedious but straightforward analysis shows that in the region $t \geqq -Mx^2$, for any constant M, the derivatives of the form $D_t^k D_x^j u$, with $k + 2j \leqq m$, are continuous. However for $k \geqq m + \frac{1}{2}$, $|D_t^k u(0, t)| \to \infty$ as $t \to +0$.

LEMMA 2.1. (a) *For fixed m there is a finite number of distinguished positive values $M_1, M_2, \cdots$ such that the boundary value of u on $t = -Mx^2$ is in C^∞ if and only if M is equal to one of these values.*

(b) *For any positive number M there is a value of m for which this M is one of the distinguished values.*

As we shall see, for given m, the smallest such value of M is less than $(12m)^{-1}$ provided m is large enough. For this value of M we have then a solution u in $t \geqq -Mx^2$, with M small, its boundary values are in C^∞, but u is not in $C^{[m+1]}$

in $t \geqq -Mx^2$. Using Theorem 1 we showed that, if mM is bounded by some fixed small constant, then $u \in C^m$; however, this example shows that u is not in C^∞ in $t \geqq -Mx^2$.

Proof: We have

$$v(x) = u(x, -Mx^2) = \int_{-1}^{-Mx^2} \frac{(-\tau)^m}{\sqrt{-\tau - Mx^2}} e^{1/4(M+\tau x^2)} \, d\tau$$

or, setting $\tau = -Mx^2 s$, $c = 1/4M$,

$$v(x) = (Mx^2)^{m+1/2} \int_1^{1/Mx^2} \frac{s^m}{\sqrt{s-1}} e^{-c/(s-1)} \, ds \, .$$

If we now set $Mx^2 = y$ and $v(x) = w(y)$ we find

$$\text{(2.13)} \quad w(y) = y^{m+1/2} \int_1^{\frac{1}{y}} \frac{s^m}{\sqrt{s-1}} e^{-c/(s-1)} \, ds = y^{m+1/2} \int_0^{\frac{1}{y}-1} \frac{(s+1)^m}{\sqrt{s}} e^{-c/s} \, ds \, .$$

For a finite set of values of c we wish to show that $w(y)$ is a C^∞ function of y. For convenience we shall use the letter $h = h(y)$ to represent such functions of y.

We wish to study the behavior of the integral (2.13) as $y \to 0$. To this end we first subtract a finite number of terms of the Taylor expansion of $(s+1)^m s^{-1/2}$ for large s in order that it become integrable from zero to infinity: i.e., we make the decomposition

$$\frac{(s+1)^m}{\sqrt{s}} = s^{m-1/2}\left(1 + \frac{1}{s}\right)^m = \sum_{j=0}^{k} \binom{m}{j} s^{m-j-1/2} + \left(\frac{(s+1)^m}{\sqrt{s}} - \sum_0^k \binom{m}{j} s^{m-j-1/2}\right)$$

with $k > m - \frac{1}{2}$. Then

$$w(y) = w_1(y) + w_2(y) \, ,$$

$$\text{(2.14)} \qquad w_1(y) = y^{m+1/2} \int_0^{\frac{1}{y}-1} \sum_0^k \binom{m}{j} s^{m-j-1/2} e^{-c/s} \, ds \, ,$$

$$w_2(y) = y^{m+1/2} \int_0^{\frac{1}{y}-1} \left[\frac{(s+1)^m}{\sqrt{s}} - \sum_0^k \binom{m}{j} s^{m-j-1/2}\right] e^{-c/s} \, ds \, .$$

We first analyze $w_2(y)$. For large s the integrand is $O(s^{m-k-3/2})$ and is therefore integrable, and we may write

$$w_2(y) = y^{m+1/2} \int_0^\infty [\quad] e^{-c/s} \, ds + y^{m+1/2} \int_{\frac{1}{y}-1}^\infty \left[\frac{(s+1)^m}{\sqrt{s}} - \sum_0^k \binom{m}{j} s^{m-j-1/2}\right] e^{-c/s} \, ds \, .$$

It is straightforward to verify that the second integral is a C^∞ function of y for $y \geqq 0$, i.e.,

$$(2.15)\qquad w_2(y) = h(y) + y^{m+1/2}\int_0^\infty \left[\frac{(s+1)^m}{\sqrt{s}} - \sum_0^k \binom{m}{j} s^{m-j-1/2}\right] e^{-c/s}\, ds\,.$$

Consider now $w_1(y)$ and suppose first that $m + \frac{1}{2}$ is not an integer. After a change of variable and partial integration we see that, for $0 \leqq y \leqq \frac{1}{2}$ and $0 \leqq j \leqq k$,

$$\begin{aligned} y^{m+1/2}\int_0^{\frac{1}{y}-1} s^{m-j-1/2}e^{-c/s}\, ds &= y^{m+1/2}c^{m-j+1/2}\int_{\frac{cy}{1-y}}^\infty s^{j-3/2-m}e^{-s}\, ds \\ &= y^{m+1/2}\frac{c^{m-j+1/2}}{m+\frac{1}{2}-j}\left(\frac{cy}{1-y}\right)^{j-1/2-m} e^{-cy/(1-y)} \\ (2.16)\qquad &\quad + y^{m+1/2}\frac{c^{m-j+1/2}}{j-\frac{1}{2}-m}\int_{\frac{cy}{1-y}}^\infty s^{j-1/2-m}e^{-s}\, ds \\ &= h(y) + y^{m+1/2}\frac{c^{m-j+1/2}}{j-\frac{1}{2}-m}\int_{\frac{cy}{1-y}}^\infty s^{j-1/2-m}e^{-s}\, ds \\ = h + y^{m+1/2}&\frac{c^{m-j+1/2}}{(j-\frac{1}{2}-m)(j+\frac{1}{2}-m)\cdots(k-\frac{1}{2}-m)}\int_{\frac{cy}{1-y}}^\infty (s^{k-1/2-m}e^{-s})\, ds \end{aligned}$$

—after repeated integrations by parts. Now, in the last term,

$$y^{m+1/2}\int_{\frac{cy}{1-y}}^\infty (\quad)\, ds = y^{m+1/2}\int_0^\infty (\quad)\, ds - y^{m+1/2}\int_0^{\frac{cy}{1-y}} (s^{k-1/2-m}e^{-s})\, ds\,.$$

Again, since $k > m - \frac{1}{2}$, it is easily verified that the last integral is a C^∞ function of y on $0 \leqq y \leqq \frac{1}{2}$. Hence,

$$\begin{aligned} (2.17)\qquad & y^{m+1/2}\int_0^{\frac{1}{y}-1} s^{m-j-1/2}e^{-c/s}\, ds \\ &= h + y^{m+1/2}\frac{c^{m-j+1/2}}{(j-\frac{1}{2}-m)(j+\frac{1}{2}-m)\cdots(k-\frac{1}{2}-m)}\int_0^\infty s^{k-1/2-m}e^{-s}\, ds\,. \end{aligned}$$

Inserting this into (2.14), we find

$$(2.18)\qquad w_1 = h + y^{m+1/2}\sum_0^k \binom{m}{j}\frac{c^{m-j+1/2}}{(j-\frac{1}{2}-m)\cdots(k-\frac{1}{2}-m)}\int_0^\infty s^{k-1/2-m}e^{-s}\, ds\,.$$

Combining (2.15) and (2.18), and multiplying by $c^{-1/2}$, we find that $w(y)$ *is a* C^∞ *function on* $0 \leqq y < \frac{1}{2}$ *if and only if* c *is a root of*

$$\sum_0^k \binom{m}{j} \frac{c^{m-j}}{(j-\frac{1}{2}-m)\cdots(k-\frac{1}{2}-m)} \int_0^\infty s^{k-1/2-m}e^{-s}\,ds$$
$$+ c^{-1/2}\int_0^\infty \left[\frac{(s+1)^m}{\sqrt{s}} - \sum_0^k \binom{m}{j} s^{m-j-1/2}\right] e^{-c/s}\,ds ,$$

i.e., if and only if c satisfies

$$F_m(c) = \sum_0^k \binom{m}{j} \frac{c^{m-j}}{(j-\frac{1}{2}-m)\cdots(k-\frac{1}{2}-m)}$$
$$+ \left\{\int_0^\infty \left[\frac{(s+1)^m}{\sqrt{s}} - \sum_0^k \binom{m}{j} s^{m-j-1/2}\right] e^{-c/s}\frac{ds}{\sqrt{c}}\right\}\left\{\int_0^\infty s^{k-1/2-m}e^{-s}\,ds\right\}^{-1} = 0 .$$

By changing k to $k+1$ it is readily verified that the condition $F_m = 0$ is independent of k. Furthermore, F_m is a solution of the Laguerre differential equation (see (2.9))

$$c\ddot{F}_m + (\tfrac{3}{2} - c)\dot{F}_m + mF_m = 0 ; \tag{2.19}$$

this may be verified by direct (slightly tedious) computation.

We remark that if m is an integer, then the second integrand vanishes and $F(c)$ is a constant times the Laguerre polynomial $L_m^{(1/2)}(c)$. We first investigated this case, for which the relationship of $F(c)$ to the Laguerre polynomial was pointed out to us by A. Novikoff.

Consider now the case that $m + \frac{1}{2}$ is an integer. We may choose $k = m + \frac{1}{2}$. From (2.15) we see that w_2 is in C^∞, while

$$w_1(y) = y^k \int_0^{\frac{1}{y}-1} \sum_0^k \binom{m}{j} s^{k-j-1}e^{-c/s}\,ds . \tag{2.20}$$

According to (2.16),

$$\begin{aligned} y^k\int_0^{\frac{1}{y}-1} s^{k-j-1}e^{-c/s}\,ds &= y^k c^{k-j}\int_{\frac{cy}{1-y}}^\infty s^{j-1-k}e^{-s}\,ds \\ &= h + y^k\frac{c^{k-j}}{j-k}\int_{\frac{cy}{1-y}}^\infty s^{j-k}e^{-s}\,ds \qquad \text{for} \qquad j<k . \end{aligned} \tag{2.21}$$

After repeated integrations by parts we see as before that when $j < k$ the right-hand side of (2.21) is equal to

$$h + y^k \frac{(-c)^{k-j}}{(k-j)!} \int_{\frac{cy}{1-y}}^{\infty} s^{-1} e^{-s}\, ds \qquad \text{for} \qquad j < k\,.$$

Inserting this, and (2.21) for $j = k$, into (2.20), we find

$$w(y) = w_1(y) + h = h + y^k \sum_0^k \binom{k-\frac{1}{2}}{j} \frac{(-c)^{k-j}}{(k-j)!} \int_{\frac{cy}{1-y}}^{\infty} s^{-1} e^{-s}\, ds\,.$$

Since the integral behaves like $\log(1/y)$ for y small, it is clear that $w(y)$ is C^∞ if and only if

$$\sum_0^k \binom{k-\frac{1}{2}}{j} \frac{(-c)^{k-j}}{(k-j)!} = 0\,,$$

i.e., if and only if c is a root of the Laguerre polynomial

$$L_{m+1/2}^{(-1/2)}(c) = \sum_0^k \binom{m}{m+\frac{1}{2}-j} \frac{(-c)^j}{j!}\,. \tag{2.22}$$

Recall that the Laguerre polynomial $L_{m+1/2}^{(-1/2)}(c)$ satisfies the differential equation (2.9), i.e.,

$$c\ddot{y} + (\tfrac{1}{2} - c)\dot{y} + (m + \tfrac{1}{2})y = 0\,, \tag{2.23}$$

which is not the same differential equation as (2.19); however, setting $L_{m+1/2}^{(-1/2)}(c) = c^{1/2}F_m(c)$ in (2.23), we see that F satisfies the differential equation (2.19). It is not difficult to verify that $F_m(c)$, now defined for all positive m, varies continuously with m.

We may now complete the proof of Lemma 2.1. Consider $F_m(c)$ for $c > 0$. In case m is an integer the Laguerre function $F_m(c)$ has exactly m positive roots. Also if $m + \frac{1}{2} = k$ is an integer, then $F_m(c) = c^{-1/2}L_k^{(-1/2)}$ has exactly k positive roots. For general m, we see that since $F_m(c)$ is a solution of the differential equation (2.19), its roots are distinct and simple and vary continuously with m. Furthermore, via a standard separation theorem, the roots of $F_m(c)$ separate those of $F_j(c)$ for $m > j$; i.e., any closed interval whose end points are roots of F_j contains a root of F_m. Denoting by d_m the number of positive roots of F_m, we easily infer that if $[m] < m < [m] + \frac{1}{2}$, then $[m] - 1 \leqq d_m \leqq [m] + 2$, while if $[m] + \frac{1}{2} < m$, then $[m] \leqq d_m \leqq [m] + 2$. This proves part (a) of Lemma 2.1. To prove part (b) we make use of the fact that for large integral m the largest and next to largest roots of $L_m^{(1/2)}(c)$ exceed $3m$ (see formula (8.9.15) in [29]). Hence by the principle of separation of roots we see easily that for large m the largest root of $F_m(c)$ exceeds $3m$. Now $F_1(c)$ has only one root c_0; as m increases all the new roots must come out of the origin (a singular point of the differential equation (2.19)). The first root c_0 moves continuously with m and remains always the largest root. The next root, starting from the origin, remains from then on the second largest and it becomes arbitrarily large (greater than $3m$) for large m. Thus this second root traverses all positive values. Hence there is some m for which it takes on any given value—proving part (b); recall that $c = 1/4M$.

When m is an integer, the corresponding distinguished values of M are those for which $1/4M$ are roots of $L_m^{(1/2)}(c)$, and hence are values for which compatibility conditions may arise—see Theorem 2. One might then expect that the boundary values of u on the corresponding parabolas $t = -Mx^2$ do not satisfy the compatibility conditions, so that of course the function u given by (2.12) cannot be smooth. However for $m = 1$ (with the distinguished $M = \frac{1}{2}$) one readily verifies that the boundary values *do* in fact satisfy the necessary compatibility conditions for smoothness of derivatives up to order four. Nevertheless, the solution u defined by (2.12) is not of class C^2 in $t \geqq -\frac{1}{2}x^2$. Thus we have demonstrated that the answer to the question of Section 2.3 is in the negative.

A final remark: Consider u of (2.12) for some m having a distinguished value M for which there are no compatibility conditions. Then for any integer $k > 0$ we can find a polynomial w_k satisfying the heat equation and such that $w_k(x, -Mx^2) - u(x, -Mx^2)$ vanishes to order k at the origin. Thus, although the boundary value of the solution of the heat equation $w_k - u$ vanishes to arbitrary given order at the origin, the solution is not in $C^{[m]+1}$ at the origin.

2.5. General parabolic equations. Consider now the general second order parabolic equation with n space variables $x = (x^1, \cdots, x^n)$,

$$Lu = a^{ij}u_{x^ix^j} + b^iu_{x^i} + cu - u_t = f, \tag{2.24}$$

a^{ij} positive definite. We suppose that f and the coefficients belong to C^∞ in the closure of a bounded domain G with C^∞ boundary $\dot{G}$, and study the first boundary value problem in

$$G_T = G \cap \{t < T\}$$

—to find a solution u of (2.24) in G_T having given C^∞ boundary values on $\dot{G} \cap \bar{G}_T$. Any point of $\dot{G} \cap \dot{G}_T$ at which the inner normal to G is parallel to the t-axis will be called a "horizontal" point, and it is assumed that *at every horizontal point the inner normal to G points upward* (in the positive t direction). This implies that all boundary points of G_T with $t < T$ belong to Σ_3 except the horizontal points—which then belong to Σ_2.

At any horizontal point (x_0, t_0) the boundary $\dot{G}$ is represented locally by an equation

$$t - t_0 = \psi(x).$$

Setting $\phi = \psi(x) - t + t_0$, let us calculate the values of α, β, a, γ of (1.2), (1.3) and condition (c). We see easily that $\beta = 1$ and $\alpha = 0$, so $\gamma = 1$. Furthermore,

$$L_0 a = a^{kl} a^{ij} \psi_{ik} \psi_{jl} \; ; \tag{2.25}$$

subscripts denote, as usual, partial derivatives with respect to the space variables x^i.

We first present a preliminary result.

THEOREM 3. *Let m be a fixed positive integer. Assume that at every horizontal boundary point the value of $L_0 a$ as given by* (2.25) *satisfies*

$$m^2 L_0 a = m^2 a^{kl} a^{ij} \psi_{ik} \psi_{jl} \leqq \tfrac{1}{4}\varepsilon_0^2 = \tfrac{1}{4}[10(n+1)]^{-4} \cdot 3^{-2(n+1)} . \tag{2.26}$$

Then there exists a unique solution of the first boundary value problem in $\bar{G}_T$ which belongs to H_m.

Proof: First, by the usual device of multiplying u by a suitable exponential $e^{\lambda t}$, we may suppose that $-c$ is as large as we like. Then, we extend L to $G - \bar{G}_T$ so that it is elliptic there, and f and v to $\bar{G}$ in a C^∞ way. Applying Theorem 1 to the Dirichlet problem for the modified equation in $\bar{G}$, we obtain the desired solution.

In Theorem 3 we see that if the curvature of the boundary at all horizontal points vanishes, i.e., $\psi_{ij} = 0$ there, then the solution belongs to C^∞ in $\bar{G}_T$. However, if the principal curvatures at a horizontal point are all positive, i.e., the Hessian ψ_{ij} is positive definite there, then the curvatures need not be small—the solution is C^∞ there anyway. This is a purely local result for which we give an independent proof.

THEOREM 4. *Let $\psi(x)$ be a C^∞ function defined in a neighborhood of the origin with $\psi_i(0) = 0$ and positive definite Hessian $\psi_{ij}(0)$. For small T consider the first boundary value problem for* (2.24) *in*

$$A : \psi(x) \leqq t \leqq T$$

—with u equal to a given C^∞ function on $t = \psi(x)$. There exists a unique solution which belongs to C^∞ in A.

With the aid of this result we may prove the following improvement of Theorem 3.

THEOREM 3′. *Let m be a fixed positive integer. Assume that at every horizontal boundary point in $\bar{G}_T$ either the principal curvatures of the boundary are all positive, or* (2.26) *holds. Then the first boundary value problem has a unique solution in H_m. The solution belongs to C^∞ in $\bar{G}_T$ except possibly at the horizontal boundary points where not all the principal curvatures are positive.*

CONJECTURE. *Theorem 3′ holds if, in place of the condition that the principal curvatures be positive, one assumes that they are merely non-negative.*

One may go further than this conjecture and ask if the conclusion of Theorem 3′ holds under the condition that, at every horizontal point, m^2 times the principal curvatures are all greater than a small negative constant.

Proof of Theorem 3′: Let (x_0, t_0) be a horizontal point at which the principal curvatures of the boundary are all positive; necessarily, then, $t_0 < T$. Near (x_0, t_0) the domain G is described by

$$t - t_0 > \psi(x) ,$$

and the Hessian $\psi_{ij}(x_0)$ is positive definite. Then, according to Theorem 4, for some small positive ε, in a region $\mathscr{D}^\varepsilon : \psi(x) \leqq t - t_0 \leqq \varepsilon$, the solution u is uniquely determined by its boundary values on $t = \psi \leqq \varepsilon$, and belongs to C^∞. Let $\psi_0(x) \geqq \psi(x)$ be a C^∞ positive function which is identically equal to ψ when $\psi(x) \geqq \frac{1}{2}\varepsilon$ and is such that, whenever the first derivatives of ψ_0 vanish, its second derivatives also vanish. Choose ψ_0 so that, in addition, $\psi_0 < \varepsilon$ wherever $\psi_0 \neq \psi$. Then the region $\mathscr{D}_0^\varepsilon : \psi_0 \leqq t - t_0 \leqq \varepsilon$ is contained in $\mathscr{D}^\varepsilon$ and on its boundary where $t - t_0 = \psi_0$ the values of the solution u belong to C^∞. Now modify the region $\bar{G}_T$ by removing $\mathscr{D}^\varepsilon$ and replacing it by the smaller region $\mathscr{D}_0^\varepsilon$—taking as boundary values on the new portion of the boundary the values of the solution u determined in $\mathscr{D}^\varepsilon$.

If we repeat this process at every horizontal point where all the principal curvatures are positive, we obtain a slightly smaller domain $\tilde{G}_T$ with the property that at its horizontal points the conditions of Theorem 3 are satisfied—for, the horizontal points with positive principal curvatures have been replaced by horizontal points with vanishing principal curvatures. Applying Theorem 3 to $\tilde{G}_T$ (with the new boundary values) we obtain a solution in H_m. It is easily seen that this solution matches with the solutions already obtained in the regions $\mathscr{D}^\varepsilon$ in such a way as to provide a solution of our original problem in $\bar{G}_T$ of class H_m. The C^∞ character of the solution at all points but the horizontal ones with some principal curvature $\leqq 0$ is proved with the aid of Theorem 4 and the known local regularity theory of non-characteristic boundary points (as in [9]).

We turn now to Theorem 4. Our proof makes use of a simple form of Theorem 1 proved earlier in [14], but involves, in addition, a special device which has otherwise not entered our approach to degenerate elliptic-parabolic problems: a singular transformation of variables, (2.28) below, which is suggested by (but is not the same as) a transformation used by Kondrat'ev (formula (8) in [15]).

Proof of Theorem 4: We may suppose that u has zero boundary values on $t = \psi(x)$. The function ψ has the form

$$\psi(x) = \tfrac{1}{2}\psi_{ij}(0)x^i x^j + \text{higher order terms} ,$$

with $\psi_{ij}(0)$ positive definite. By the well known lemma of Morse (see [18], Lemma 2.2) we may make a change of space variables so that in the new variables, which we continue to denote by x, ψ has the form

$$\psi(x) = \sum x^{i^2} . \tag{2.27}$$

We shall continue to denote the transformed equation by (2.24).

For any positive integer m we shall show that (2.24) has a solution in C^m in A. By uniqueness the solution is in C^∞.

(a) Assume first that the inhomogeneous term $f(x, t)$ vanishes to order r at the origin, with r to be chosen later. We shall then reduce the general case to this one.

We introduce as new variables

$$y^i = \frac{x^i}{\sqrt{t}}, \qquad s = \sqrt{t} , \tag{2.28}$$

so that the set A is mapped onto the cylinder

$$\sum y^{i^2} \leqq 1 , \qquad\qquad 0 \leqq s \leqq \sqrt{T} . \tag{2.29}$$

Then $v(y, s) = u(ys, s^2)$ satisfies the transformed form of (2.24),

$$a^{ij}(sy, s^2)v_{y^iy^j} + (sb^i(sy, s^2) + \tfrac{1}{2}y^i)v_{y^i} - \tfrac{1}{2}sv_s + s^2cv = s^2f(sy, s^2) , \tag{2.30}$$

and vanishes on the side of the cylinder.

The coefficient of v is no longer large negative; hence in order to achieve this, we set

$$v = s^k w$$

with k large. The corresponding equation for w is

$$\begin{aligned} a^{ij}(sy, s^2)w_{y^iy^j} &+ (sb^i(sy, s^2) + \tfrac{1}{2}y^i)w_{y^i} \\ &- \tfrac{1}{2}sw_s + (s^2c - \tfrac{1}{2}k)w = s^{2-k}f(sy, s^2) . \end{aligned} \tag{2.31}$$

Thus for k large, the coefficient of w is as large negative as we please. We shall always assume $k \geqq m$. Since $f(sy, s^2)$ vanishes to order r for $s = 0$, we see that for $r > k$ the inhomogeneous term of (2.31) is regular at $s = 0$.

We seek a solution w of class C^m in the cylinder (2.29) and vanishing on the side. To this end we follow step 2 of the proof of Theorem 6 in [14] and extend the finite cylinder (2.29) on top and bottom by rounding it off there, so that the extended region $\mathscr{M}_0$ has C^∞ boundary; we denote the lower additional end by $\mathscr{M}_1$. Next we extend the equation (2.31) to the new ends of the cylinder, making it elliptic there. Since the side of the cylinder was noncharacteristic for (2.31), the entire boundary of $\mathscr{M}_0$ is noncharacteristic for our new equation. Finally we extend f in an arbitrary C^∞ way to the upper end, and as zero to the lower end. If we choose

$$r = k + m ,$$

the resulting extended function will be of class C^m in $\bar{\mathscr{M}}_0$.

For k large enough the coefficient of w is large negative. Hence we may apply the Corollary of Theorem 2′ of [14] (as in our proof of Theorem 6 in [14]) and infer that the new equation in the extended domain has a unique solution w of class C^m which vanishes on the boundary of $\mathscr{M}_0$. Consider this solution in the lower end $\mathscr{M}_1$. The base, $s = 0$, of the cylinder is part of the boundary of $\mathscr{M}_1$, and for $\mathscr{M}_1$ it corresponds to Σ_1. Hence, since the solution w is uniquely determined by its boundary values (namely zero) on the other parts of the boundary of $\mathscr{M}_1$ and by the values of f in $\mathscr{M}_1$ (also zero), it follows that $w = 0$ in $\mathscr{M}_1$. Thus $v = s^k w \in C^m$ is a solution of (2.30) in our cylinder (2.29) which vanishes on the side and bottom.

Let us analyze the behavior of derivatives of v in the cylinder as $s \to 0$. Since $v = s^k w$, and w vanishes together with its derivatives up to order m on the base of the cylinder, it follows that for any derivative D^j of order j we have

$$D^j v = o(s^{k+m-j}) \qquad \text{as} \qquad s \to 0 . \tag{2.32}$$

Our solution of (2.24) is finally given by

$$u(x, t) = v\left(\frac{x}{\sqrt{t}}, \sqrt{t}\right) . \tag{2.33}$$

It clearly belongs to C^m in A, except possibly at the origin. However with the aid of (2.32), with $k \geqq m$, it is easily established that the derivatives of u up to order m tend to zero as (x, t) approaches the origin. The solution given by (2.33) has therefore all the required properties.

(b) To reduce the general problem to that of case (a) we shall determine a polynomial $u^r(x, t)$ which vanishes on $t = \Sigma x^{i^2}$ and is such that

$$f - Lu^r$$

vanishes to order r at the origin. Then our desired solution will be of the form $u^r + z$, where z satisfies $Lz = f - Lu^r$ and hence z is determined from part (a). The fact that such a polynomial u^r may be found means simply that there are no compatibility conditions on f at the origin. But the proof of this is not completely trivial.

We shall use a special rearrangement of Taylor's series. For any integer $r > 0$ we may expand any C^∞ function g uniquely in the form

$$g = \sum_{k=0}^{r} (g_k + g_k') + O((t + |x|^2)^{r+1})$$

(here $|x|^2 = \Sigma x^{i^2}$), where g_k, g_k' are polynomials of the form

$$g_k = \sum_{\substack{|\alpha| \text{ even} \\ j + \frac{|\alpha|}{2} = k}} a_\alpha t^j x^\alpha , \tag{2.34}$$

$$g_k' = \sum_{\substack{1 \leq |\alpha| \text{ odd} \\ j + \frac{|\alpha| - 1}{2} = k}} b_\alpha t^j x^\alpha . \tag{2.35}$$

Denoting by $\mathscr{P}_k$, $\mathscr{R}_k$ the finite dimensional spaces of polynomials of the forms (2.34) and (2.35), respectively, we shall determine $u^r = g$ of the form

$$u^r = g = (t - |x|^2) \sum_{k=0}^{r} (p_k + q_k)$$

in such a way that

$$f - Lu^r = O((t + |x|^2)^{r+1}) ; \tag{2.36}$$

here $p_k \in \mathscr{P}_k$, $q_k \in \mathscr{R}_k$ are unique (and independent of r).

First we note the following

LEMMA 2.2. *The linear operator L':*

$$L'u = \left(a^{ij}(0) \frac{\partial^2}{\partial x^i \, \partial x^j} - \frac{\partial}{\partial t} \right) (t - |x|^2) u$$

is a bijective map of $\mathscr{P}_r$ to itself and $\mathscr{R}_r$ to itself.

Proof: It is clear that the operator L' maps $\mathscr{P}_r$ and $\mathscr{R}_r$ into themselves. Since these spaces are finite-dimensional, the map in each is bijective if it is injective. Suppose then that u is in one of the spaces and $L'u = 0$; then, $v = (t - |x|^2)u$ is a solution of the parabolic equation

$$a^{ij}(0) v_{ij} - v_t = 0$$

and vanishes on $t = |x|^2$. By uniqueness, via, say, the maximum principle, $v \equiv 0$ and so $u = 0$; q.e.d.

To find u^r we shall proceed by induction on k; assuming that p_k, q_k are determined for $k < r$, we show how to determine p_r, q_r. Setting $u^{r-1} = (t - |x|^2) \sum_0^{r-1} (p_k + q_k)$ we want u^r to satisfy (2.36). Since $Lu^{r-1} - f = O((t + |x|^2)^r)$, we see that we have to solve

$$[L(t - |x|^2)(p_r + q_r)]_r = f_r - [L(t - |x|^2)u^{r-1}]_r \tag{2.37}$$

and

$$[L(t - |x|^2)(p_r + q_r)]_r' = f_r' - [L(t - |x|^2)u^{r-1}]_r' . \tag*{(2.37)$'$}$$

We first solve (2.37) for p_r. The term q_r contributes something of higher order at the origin and in fact we may rewrite this equation as

$$\left(a^{ij}(0) \frac{\partial^2}{\partial x^i \partial x^j} - \frac{\partial}{\partial t}\right)(t - |x|^2)p_r = f_r - [L(t - |x|^2)u^{r-1}]_r , \tag{2.38}$$

since all the terms that have been discarded vanish to higher order at the origin. By Lemma 2.2, the polynomial p_r is uniquely determined.

Turning now to (2.37)$'$ we see in a similar way that q_r satisfies

$$\left(a^{ij}(0) \frac{\partial^2}{\partial x^i \partial x^j} - \frac{\partial}{\partial t}\right)(t - |x|^2)q_r = f_r' - [L(t - |x|^2)(u^{r-1} + p_r)]_r' \tag*{(2.38)$'$}$$

and is again uniquely determined by Lemma 2.2.

The polynomials p_0, q_0 are also determined in this way as solutions of the corresponding equations (2.38), (2.38)$'$ and thus the induction argument is complete.

3. Further Preliminary Comments

In Sections 3.1 to 3.4 we shall investigate conditions (c) and (d) of Section 1. We have already seen in the previous section that condition (d) cannot be omitted; this is also true for condition (c). We remark first that for the case treated in [14], in which Σ_2 and Σ_3 are closed, the term α of (1.2) vanishes on Σ_2 so that (c) automatically holds—because it is assumed that the equation can be extended across Σ_2 so as to remain elliptic-parabolic. In addition the term a of (1.3) is identically zero on Σ_2 and hence (1.4) is also fulfilled.

We have seen, in the case of the heat equation, that if the solution is smooth, then f may necessarily have to satisfy some local compatibility conditions. For the general equation (1.1) this may even occur at interior points. Consider for instance the following simple example for a function of one variable t:

$$Lu = tu_t + cu = f \quad \text{in} \quad -1 \leqq t \leqq 1 , \qquad c < 0 . \tag{3.1}$$

If c is a negative integer and if u is, say, in C^∞, then, differentiating $-c$ times we obtain the compatibility condition $D_t^{-c}f(0) = 0$. Although the equation is of first order, both end points $t = \pm 1$ belong to Σ_2, and hence the solution is uniquely determined only when both end values are given. Furthermore, if c is large negative the solution will be smooth but need not be in C^∞—even if the compatibility condition is satisfied. For, the solution of the homogeneous equation:

$$u = at^{-c} \quad \text{for} \quad t \leqq 0 \quad \text{and} \quad u = bt^{-c} \quad \text{for} \quad t \geqq 0 ,$$

with a and b arbitrary constants, belongs to $C^{[-c]}$ but is not in C^∞.

On the other hand, for the equation

$$(3.1)' \qquad Lu = -tu_t + cu = f \quad \text{on} \quad -1 \leqq t \leqq 1 , \qquad c < 0 ,$$

the end points are in Σ_1, so no boundary values need be given, and there is a unique smooth solution for a given $f \in C^\infty$ which is also in C^∞—as is easily verified.

Our conditions (b), (c), (d) ensure that there are no compatibility conditions on f—at least for the derivatives of the orders which we estimate; compatibility may arise for higher order derivatives. In Section 3.3 we describe the possible compatibility conditions at a point of Σ_2, and show how our conditions (c), (d) preclude them.

In Section 3.5 the uniqueness of the solution of our problem is proved, and in Section 3.6 a brief description of the structure of the proof of Theorem 1 is presented.

3.1. On condition (c). On page 489 of [14] we gave a one-dimensional example showing that, if $-\alpha\beta^{-2}$ is not small on Σ_2, then solutions need not be very smooth. We re-examine it in order to compare the results with Theorem 1. The example is $u = y^r e^{sy}$, $r, s > 0$, which on $0 < y < 1$ satisfies

$$yu_{yy} + (1 - r)u_y - (1 + r + sy)su = 0 .$$

The origin is Σ_2; $-c = s(1 + r + sy)$ is as large as we please for large s. For $\phi = -y$, $\beta = r$, $\alpha = -r$. Hence

$$1 + r\frac{\alpha}{\beta^2} = 0 .$$

If r is not an integer, u belongs to H_m if $m < r + \frac{1}{2}$ but not if $m > r + \frac{1}{2}$. The largest integer N such that $\gamma = 1 + \frac{1}{2}N\alpha/\beta^2$ is positive, is the largest integer N satisfying

$$(3.2) \qquad N < 2r .$$

Now suppose $r - [r] \leqq \frac{1}{2}$. Then u belongs to $H_{[r]}$ but not to $H_{[r]+1}$. The largest integer N satisfying (3.2) is $N = [2r]$, and Theorem 1 asserts that $u \in H_{[r]}$— the best result. If $r - [r] > \frac{1}{2}$, Theorem 1 again yields the best result; for, the largest N satisfying (3.2) is $N = 2[r] + 1$ and, according to the theorem, $u \in H_{[r]+1}$. In fact u does not belong to $H_{[r]+2}$. Thus condition (c) is essential, and, indeed, rather sharp.

Another comment: With $N = 2m$ in Theorem 1, we see that if $f \in H_{2m}$ and conditions (a), (b), (d), as well as condition (c):

$$1 + m\frac{\alpha}{\beta^2} > 0 \tag{3.3}$$

hold, then the solution u belongs to H_m. Thus, in particular, the conjecture formulated in Remark 9.3 on page 488–9 of [14] is proved. In fact in place of (3.3) the stronger assumption $1 + (m + 1)\alpha/\beta^2 > 0$ was made there.

3.2. Special coordinates. It is convenient for our discussion as well as for the proofs to introduce the special coordinate system of page 801 in a neighborhood of a boundary point in which the boundary lies in a hyperplane. Near the boundary we let $-y$ denote the distance from the boundary and use $x = (x^1, \cdots, x^{n-1})$ as local coordinates on $y =$ constant, with $x^n = y$; points are denoted by $(x, y) = (x^1, \cdots, x^{n-1}, y)$. The curves $x =$ constant are to correspond to geodesics in $\mathscr{M}$ orthogonal to the boundary (straight lines in Euclidean space). We shall denote the leading coefficients of L by a^{ij} only for $i, j = 1, \cdots, n - 1$; otherwise we set $a^{nn} = a$, $2a^{ni} = a^i$ for $i \neq n$, and $b^n = b$, so that L has the form

$$Lu = au_{yy} + bu_y + a^i u_{yi} + a^{ij}u_{ij} + b^i u_i + cu\,. \tag{3.4}$$

Here summation over i, j extends from 1 to $n - 1$; the subscripts i, j, y denote differentiation with respect to x^i and x^j, $i, j < n$, and y.

Let us suppose that, in this coordinate system, the origin (0, 0) belongs to Σ_2. Then

$$a(0, 0) = a_i(0, 0) = a^i(0, 0) = 0\,, \qquad i = 1, \cdots, n - 1\,; \tag{3.5}$$

the a^i vanish there since, in general, the positive semidefiniteness of a^{ij} implies

$$a^{i^2} \leqq 4aa^{ii}\,, \qquad i = 1, \cdots, n - 1\,. \tag{3.6}$$

If in (3.6) we divide by $(x^j)^2$ and let $(x, 0) \to 0$ along the x^j-axis, we find

$$|a^i_j|^2 \leqq 2a_{jj}a^{ii} \text{ at the origin}\,. \tag{3.6$'$}$$

We also have $a_y(0, 0) \leqq 0$.

Now taking $\phi = y$, let us consider conditions (c) and (d) of Section 1. We see first that our coefficient a agrees with (1.3). From (1.2) we have, using summation notation: at the origin,

$$\beta = b - \tfrac{1}{2}a_j^j - a_y > 0\,, \qquad \alpha = \beta a_y \leqq 0\,, \tag{3.7}$$

$$\gamma = 1 + \tfrac{1}{2}N\frac{a_y}{\beta}\,, \tag{3.8}$$

$$\frac{L_0 a}{\beta^2} = \frac{a^{ij}a_{ij}}{\beta^2} \geqq 0\,. \tag{3.9}$$

After a suitable rotation we may assume that the Hessian matrix a_{ij} is diagonal at the origin so that

$$\frac{L_0 a}{\beta^2} = \frac{a^{ii}a_{ii}}{\beta^2} \quad \text{at the origin}\,. \tag{3.10}$$

3.3. Compatibility conditions. As in Section 2 the function f may have to satisfy compatibility conditions but this can occur only at points of Σ_2 which are also on the boundary of Σ_3. Let us determine the conditions for compatibility at the origin. If u is a smooth solution of (3.4) vanishing on $y = 0$, then because of (3.5),

$$bu_y = (\beta + \tfrac{1}{2}a_j^j)u_y = f \quad \text{at the origin}\,.$$

Now $a^i \equiv 0$ on Σ_2; hence if the origin is in the closure of the interior (relative to $\dot{\mathscr{M}}$) of Σ_2, we have $a_j^i = 0$ so that $b = \beta \neq 0$ there. In general, if $b \neq 0$ there, then u_y is determined at the origin. However if $b(0, 0) = 0$, then a necessary compatibility condition on f is the condition $f(0, 0) = 0$. Supposing that $u_y(0, 0)$ has been determined, we try next to determine $u_{yi}(0, 0)$ by applying the operator $D_i = \partial/\partial x^i$ to (3.4). We find that at the origin the following combinations are determined:

$$bu_{yj} + a_j^i u_{yi}\,, \qquad j = 1, \cdots, n-1\,.$$

Hence there will be a compatibility condition on the derivatives of f if

$$\det\,(a_j^i + b\delta_j^i) = 0 \quad \text{at the origin}\,.$$

In a similar way we try to solve for derivatives (see (1.5)) of the form

$$D^{\alpha,j}u\,, \qquad |\alpha| \text{ even}, \; j + \tfrac{1}{2}\,|\alpha| = k\,, \tag{3.11}$$

and then for

$$D^{\alpha,j}u\,, \qquad |\alpha| \text{ odd}, \; j + \tfrac{1}{2}(|\alpha| - 1) = k\,, \tag{3.11$'$}$$

with fixed k. We obtain linear combinations of these derivatives, and if the corresponding matrix of coefficients is singular, then we are led to compatibility conditions. Applying $D^{\alpha,j-1}$, $j + \frac{1}{2}|\alpha| = k$, to (3.4) we find that the following combinations of the derivatives of the form (3.11) at the origin are determined:

$$
\begin{aligned}
& bD^{\alpha,j}u + \sum_{r\neq s} \frac{\alpha_r\alpha_s}{2}\, a_{rs}D^{\alpha-\delta_r-\delta_s,j+1}u \\
(3.12) \qquad & + \sum_r \frac{\alpha_r(\alpha_r - 1)}{2}\, a_{rr}D^{\alpha-2\delta_r,j+1}u \\
& + \sum_r \alpha_r a^i_r D^{\alpha-\delta_r+\delta_i,j}u + a^{is}D^{\alpha+\delta_i+\delta_s,j-1}u\,.
\end{aligned}
$$

If in (3.12) we then set $j + \frac{1}{2}(|\alpha| - 1) = k$, we obtain the combination for derivatives of the form (3.11)$'$. The coefficients in these combinations involve only the values of b, a^i_j, a^{ij} and a_{ij} at the origin. In particular, it is easily seen from (3.12) that the matrices are all nonsingular (and hence there are no compatibility conditions) in case the derivatives a_{ij} vanish at the origin—for then also a^i_j vanish, by (3.6)$'$. If the origin is in the closure of the interior of Σ_2, then all x-derivatives of a vanish there and hence there are no compatibility conditions. In addition if the terms a_{ij} are sufficiently small, so that also the a^i_j are small, then, by (3.6)$'$, a certain number of the matrices of coefficients will be nonsingular. As we shall see in Section 3.4 our condition (d) will ensure precisely this—that the a_{ij} are relatively small.

3.4. Normalization. It is useful to transform the equation and the local coordinates so as to normalize various terms. First of all, after fixing a metric, we shall always take as function ϕ defined near the boundary the function $\phi = y =$ $-$the distance to the boundary. Then we multiply the equation by a suitable factor so as to normalize the term β of (1.2): we fix

$$
\beta \equiv \tfrac{1}{8} \qquad \text{on} \qquad \Sigma_2\,. \tag{3.13}
$$

Having thus fixed the equation and ϕ, the function

$$
a = a^{ij}\phi_i\phi_j \tag{3.13$'$}
$$

is well defined in a neighborhood of the boundary. This function is the coefficient of u_{yy} in equation (3.4) in terms of our special local coordinates.

Now consider condition (d), (1.4), at the origin. From (3.10) we see that it takes the form

$$
N\sqrt{\textstyle\sum a^{ii}a_{ii}} \leqq \varepsilon_0\beta\gamma = \varepsilon_0(\beta + \tfrac{1}{2}Na_y) = \sigma\,, \tag{3.14}
$$

for convenience we denote the right-hand side by σ.

It is extremely useful to make a linear change of variable which normalizes the values of a^{ij} and a_{ij} in such a way that

$$N^2 |a_{ij}| , |a^{ij}| \leqq \varepsilon_0 \beta\gamma = \sigma \text{ for all } i,j \tag{3.15}$$

at the origin. To achieve this we recall that $a_{ij}(0, 0)$ is diagonal. After a reordering we may suppose that

$$N^2 a_{ii}(0,0) \geqq \sigma \qquad \text{for} \qquad i = 1, \cdots, n_1 ,$$

and $N^2 a_{ii}(0, 0) < \sigma$ for $i > n_1$. For the remainder of this subsection all coefficients and derivatives are to be evaluated at the origin even though we neglect to specify this everywhere. We recall that $|a^{ij}|$, $|a_{ij}| \leqq M$ for some constant M. Replace $x^1, \cdots, x^{n_1}$ by the new variables

$$\overline{x^i} = N\sqrt{\frac{a_{ii}}{\sigma}}\, x^i , \qquad\qquad i = 1, \cdots, n_1 ,$$

without changing the remaining variables. Then

$$\frac{1}{N}\sqrt{\frac{\sigma}{M}}\,\overline{x^i} \leqq x^i \leqq \overline{x^i} , \qquad\qquad i = 1, \cdots, n_1 . \tag{3.16}$$

Distinguishing the new coefficients and their derivatives with respect to the new variables by a bar we see that

$$\overline{a_{ii}} = a_{ii} \cdot \frac{\sigma}{N^2 a_{ii}} , \qquad\qquad i = 1, \cdots, n_1 ,$$

so that

$$N^2 \overline{a_{ii}} \leqq \sigma \text{ for all } i . \tag{3.17}$$

In addition we have (without summation)

$$\overline{a^{ii}} = a^{ii} \frac{N^2}{\sigma} a_{ii} \leqq \sigma \qquad \text{for} \qquad i = 1, \cdots, n_1 , \tag{3.18}$$

in virtue of (3.14).

For $i > n_1$ the coefficients a^{ii} (and a_{ii}) are unchanged. Suppose now

$$\begin{aligned} &a^{ii}(0,0) \geqq \sigma \qquad \text{for} \qquad i = n_1 + 1, \cdots, n_2 , \\ &a^{ii}(0,0) < \sigma \qquad \text{for} \qquad i > n_2 . \end{aligned} \tag{3.18$'$}$$

Then for $i = n_1 + 1, \cdots, n_2$ we replace the coordinates x^i by

$$\text{(3.16)}' \qquad \begin{aligned} \overline{x^i} &= \sqrt{\frac{\sigma}{a^{ii}}}\, x^i , & i &= n_1 + 1, \cdots, n_2 , \\ \overline{x^i} \leqq x^i &\leqq \sqrt{\frac{M}{\sigma}}\, \overline{x^i} , & i &= n_1 + 1, \cdots, n_2 . \end{aligned}$$

Identifying again the new coefficients with the aid of a bar, we find

$$\overline{a^{ii}} = a^{ii} \frac{\sigma}{a^{ii}} = \sigma , \qquad i = n_1 + 1, \cdots, n_2 ,$$

and we conclude from (3.17) and (3.18) that

$$\text{(3.19)} \qquad \overline{a^{ii}} \leqq \sigma \text{ for all } i .$$

Furthermore, although we have changed the a_{ii} for $i = n_1 + 1, \cdots, n_2$, we still have

$$\overline{a_{ii}} = a_{ii} \frac{a^{ii}}{\sigma} \leqq \frac{\sigma}{N^2} \qquad \text{for} \qquad i = n_1 + 1, \cdots, n_2 .$$

Thus, in terms of our new coordinates, (3.18) and (3.19) hold for all i. Since $\overline{a^{ij}}^2 \leqq \overline{a^{ii}}\,\overline{a^{jj}}$ we infer that (3.15) holds in these coordinates.

In the new coordinates (dropping the bar), the second derivatives $|a_{ij}|$ are very small if N is large and so are the $|a^i_j|^2 \leqq a^{ii} a_{jj}$. Thus, as pointed out in Section 3.3, compatibility conditions for f at the origin can occur only for high order derivatives of f.

It is clear from (3.16) and (3.16)′ that the derivatives of the new variables with respect to the old are bounded without regard to how small the a^{ij} or a_{ij} may be.

3.5. An integral formula. We shall make use of an integral identity which is based on Green's theorem and which yields a proof of the uniqueness of the solution of our boundary value problem. With the aid of Green's theorem (see (9.7) of [14]) one finds that, for $u, v \in C^\infty(\bar{\mathscr{M}})$,

$$\text{(3.20)} \qquad \begin{aligned} (Lu, v) = &\int_{\mathscr{M}} [-a^{ij} u_i v_j + \tfrac{1}{2}(b^i - a^{ij}_j)(u_i v - u v_i) + \tfrac{1}{2}(2c - b^i_i + a^{ij}_{ij}) uv] \, dV \\ &+ \int_{\dot{\mathscr{M}}} [a^{ij} u_i v \nu_j + \tfrac{1}{2}(b^i - a^{ij}_j) \nu_i uv] \, ds . \end{aligned}$$

Here dV and ds represent elements of volume on $\mathscr{M}$ and $\dot{\mathscr{M}}$; $(\ ,\ )$ is the L_2 scalar product in $\mathscr{M}$. If u, v vanish on $\Sigma_2 \cup \Sigma_3$, then, since $a^{ij}\nu_j = 0$ on Σ_1, the boundary integral equals

$$\int_{\Sigma_1} \tfrac{1}{2}\,(b^i - a^{ij}_j)\,\nu_i uv\, ds\,. \tag{3.21}$$

It is convenient to write the boundary integral (3.21) as an integral in $\mathscr{M}$. To this end let g be a smooth function in $\bar{\mathscr{M}}$ with support in a neighborhood of Σ_1 such that $g \equiv 1$ on Σ_1. Then the boundary integral (3.21) equals

$$\int \frac{1}{2}\frac{\partial}{\partial x^i}\,[(b^i - a^{ij}_j)uvg]\; dV\,.$$

Hence (3.20) may be written for functions u, v vanishing on $\Sigma_2 \cup \Sigma_3$ as

$$\begin{aligned}(Lu, v) &= \int_{\mathscr{M}} [-a^{ij}u_iv_j + \tfrac{1}{2}(b^i - a^{ij}_j)(u_iv - uv_i) + \tfrac{1}{2}(2c - b^i_i + a^{ij}_{ij})uv \\ &\qquad + \frac{1}{2}\frac{\partial}{\partial x^i}\,((b^i - a^{ij}_j)uvg)]\; dV \\ &= -Q(u, v)\,.\end{aligned} \tag{3.22}$$

We have denoted the integral bilinear form by $-Q(u, v)$; this differs slightly from our notation in (9.8) of [14]. We shall use the fact that, because of our condition (b), we have: if u vanishes on $\Sigma_2 \cup \Sigma_3$, then

$$-(Lu, u) = Q(u, u) \geqq \lambda(u, u)\,, \tag{3.23}$$

with $-\frac{1}{2}c \geqq \lambda$, a large constant. This implies of course that $u \equiv 0$ is the only smooth solution of the homogeneous problem.

3.6. The scheme of the proof. The proof is based on *a priori* estimates for the integral norms of the derivatives, as in (1.6) and (1.7). These are derived in several steps at different portions of $\bar{\mathscr{M}}$. In Section 4 we derive estimates in a compact subset of $\mathscr{M}$. The same method is applied in a boundary strip near Σ_3, and it is also used to estimate purely tangential derivatives in boundary patches around the entire boundary. In Sections 5 and 6 we then estimate other derivatives in boundary strips near Σ_1 and Σ_2. One has to take particular care in deriving the estimates near Σ_2. These estimates are all combined in Section 7.2 to give the global *a priori* estimates.

But then, as is so often the case, we are still left with the problem of constructing smooth solutions to which these estimates apply, by some regularizing procedure, and in the present case this is decidedly nontrivial. In [14] we used the so-called method of elliptic regularization. This involves adding ε times a second order elliptic operator to L, solving the corresponding equation for a smooth solution u_ε, and then deriving our *a priori* estimates for u_ε with constants independent of ε. Letting $\varepsilon \to 0$ we would infer the existence of a smooth solution of our original problem. Unfortunately we cannot use this method here in that form. The reason is simply that our given boundary values on $\Sigma_2 \cup \Sigma_3$ alone are not sufficient to determine the solution u_ε. (If the equation can be extended across Σ_1 so as to remain elliptic-parabolic, then this method may be used, as in Section 9 of [14].) We remark also that if we try to formulate the new, regularized, equation in connection with a quadratic integral form (3.22), say as a variational problem, without prescribing boundary values on Σ_1, then the solution u_ε will have to satisfy the "natural" or "free" boundary conditions: $\partial u_\varepsilon/\partial \nu =$ the normal derivative of $u_\varepsilon = 0$ on Σ_1. If we could then obtain estimates of derivatives of u_ε, independent of ε, this same "free" boundary condition would necessarily hold as $\varepsilon \to 0$ for the desired solution of (1.1); there is, of course, no reason why this should be the case.

We have been forced, consequently, to use a modified, somewhat awkward, form of elliptic regularization: We add to L another operator, which near $\Sigma_2 \cup \Sigma_3$ is ε times a second order elliptic operator, but which near Σ_1 is δ times an elliptic operator of order $2N$; ε and δ are small positive constants. On Σ_1 we impose the boundary condition

$$\left(\frac{\partial}{\partial \nu}\right)^j u = 0\,, \qquad j = N, \cdots, 2N-1\,, \tag{3.24}$$

(in reality this will arise as a "free" boundary condition). We then derive our L_2 *a priori* estimates for derivatives of u of order $\leqq N$, independent of ε and δ. We have no control on the higher order derivatives, and as ε and δ tend to zero the boundary conditions (3.24) can be lost. No doubt some boundary layer phenomenon occurs near Σ_1 but we are able to simply ignore it. Lions has informed us that in a joint work with Lattès [16] he has also used a higher order operator as regularizer in connection with some difference schemes so as to avoid trouble arising out of new boundary conditions.

We shall now give a more precise description of our regularization. Our regularized operator will be expressed in terms of an integral quadratic form as in (3.22). Let $\rho(y)$, $\sigma(y) \geqq 0$ be fixed C^∞ functions in $\bar{\mathscr{M}}$ having their respective supports (which are disjoint) in a neighborhood of Σ_1 and a neighborhood of $\Sigma_2 \cup \Sigma_3$, such that σ is identically equal to 1 in a smaller neighborhood $\mathscr{M}_{23}$ of $\Sigma_2 \cup \Sigma_3$, and that $\rho = 1$ on Σ_1. The function ρ will be linear in a strip $\mathscr{M}_1$, $0 \leqq -y < d'$, near Σ_1 with $\rho_y = -1$ there.

Let $g^{ij}(x)$ be a symmetric real tensor, $i, j = 1, \cdots, n-1$, defined on $\Sigma_2 \cup \Sigma_3$ such that as a matrix it is positive definite. Extending g^{ij} to a neighborhood by making it independent of y we define the bilinear form (in terms of our special coordinates (x, y))

$$Q_2(u, v) = \int (u_y v_y + g^{ij}(x) u_i v_j) \sigma \, dV . \tag{3.25}$$

On Σ_1 consider an invariantly defined symmetric bilinear form with C^∞ coefficients:

$$G[u, v] = \sum_{|\alpha|, |\beta| \leqq N} G^{\alpha\beta}(x) D_x^\alpha u D_x^\beta v$$

such that $G[u, u]$ is positive definite in all derivatives: $G[u, u] \geqq \sum_{|\alpha| \leqq N} |D_x^\alpha u|^2$. Extending this to a neighborhood of Σ_1 with coefficients independent of y, we define

$$Q_1(u, v) = \int (D_y^N u \cdot D_y^N v + G[u, v]) \rho \, d\tilde{V} . \tag{3.26}$$

Here the invariant element of volume $d\tilde{V}$ is chosen to be of the form $J(x)\, dx\, dy$ so that $J(x)\, dx$ is itself invariant.

Now we define, for small positive ε, δ,

$$Q_{\varepsilon,\delta}(u, v) = Q(u, v) + \delta Q_1(u, v) + \varepsilon Q_2(u, v) . \tag{3.27}$$

Our approximate solution $u_{\varepsilon,\delta}$ will be defined as the function $u_{\varepsilon,\delta}$ vanishing on $\Sigma_2 \cup \Sigma_3$ which satisfies

$$Q_{\varepsilon,\delta}(u_{\varepsilon,\delta}, v) = -(f, v) \tag{3.28}$$

for all C^∞ functions v vanishing on $\Sigma_2 \cup \Sigma_3$. In $\mathscr{M}_1$, where the corresponding differential equation is elliptic of order $2N$, the function $u_{\varepsilon,\delta}$ belongs to C^∞ and satisfies the usual "free" boundary conditions (3.24) corresponding to the Neumann problem. We shall show in Section 8 that *a priori* estimates hold also for the function $u_{\varepsilon,\delta}$—independent of ε and δ.

4. Estimates of Tangential Derivatives Near the Boundary and of all Derivatives near Σ_3 and in the Interior

These estimates are consequences of (3.23) and Corollary 3.2 on p. 461 of [14]. By examining the proof of this corollary we easily obtain the following:

LEMMA 4.1. *If ζ is a C^∞ function which has support in the domain $\mathscr{U}$ of a special coordinate system (described in Section 3.2), then we have*

$$Q(\zeta D^{\alpha,0} u, \zeta D^{\alpha,0} u) \leqq (-1)^m Q(u, D^{\alpha,0} \zeta^2 D^{\alpha,0}) + \|B^{m-1,1} u\|^2 + \|B^{m,0} u\|^2 \tag{4.1}$$

when $|\alpha| = m$, *and*

$$B^{m,k} = \sum_{\substack{|\gamma| \leqq m \\ j \leqq k}} b_{\gamma,j} D^{\gamma,j} \tag{4.2}$$

with the support of $b_{\gamma,j}$ *contained in the support of* ζ. *Furthermore, the coefficients of the terms of total order* m *in* $B^{m-1,1}$ *and* $B^{m,0}$ *depend only on the coefficients of second and first order in* L, *and on derivatives up to second order of the second order coefficients and first order of the first order coefficients.*

We denote by C constants which depend only on the bounds of the coefficients of order one and two of L and their derivatives up to first and second orders, respectively, and by K a constant which depends on the coefficients of L and their derivatives. In particular, C is independent of c and hence is also independent of λ. In the following, ζ has its support in a special coordinate neighborhood near the boundary—either Σ_1 or $\Sigma_2 \cup \Sigma_3$.

PROPOSITION 4.1. *We assume that condition* (a) *holds and that* ζ *is of the form described in Lemma* 4.1. *If* ζ' *is a function which is one on the support of* ζ, *whose support also lies in* $\mathscr{U}$, *then, with* $|\alpha| = m$, *we have*

$$\lambda \, \|\zeta D^{\alpha,0} u\|^2 \leqq C \sum_{\substack{|\beta|+r=m \\ r \leqq 1}} \|\zeta' D^{\beta,r} u\|^2 + K\left[\sum_{|\beta| \leqq m} \|\zeta' D^{\beta,0} f\|^2 + \sum_{|\beta| \leqq m-2} \|\zeta' D^{\beta,1} u\|^2\right], \tag{4.3}$$

for all u *and* f *in* $C^\infty(\bar{\mathscr{M}})$ *such that* $Lu = f$ *and* $u|_{\Sigma_2 \cup \Sigma_3} = 0$.

Proof: Condition (a) implies that, if u satisfies the boundary conditions, then $\zeta D^{\alpha,0} u$ also satisfies the boundary conditions; hence in (3.23) we can replace u by $\zeta D^{\alpha,0} u$. Combining this with Lemma 4.1 we obtain

$$\lambda \, \|\zeta D^{\alpha,0} u\|^2 \leqq (-1)^m Q(u, D^{\alpha,0} \zeta^2 D^{\alpha,0} u) + \|B^{m-1,1} u\|^2 + \|B^{m,0} u\|^2 .$$

Since $D^{\alpha,0} \zeta^2 D^{\alpha,0} u$ also satisfies the boundary conditions and since $Q(u, v) = -(Lu, v)$ for all v satisfying the boundary conditions, we obtain

$$\lambda \, \|\zeta D^{\alpha,0} u\|^2 \leqq \|\zeta D^{\alpha,0} f\|^2 + C \sum_{\substack{|\beta|+r=m \\ r \leqq 1}} \|\zeta' D^{\beta,r} u\|^2 + \|B^{m-2,1} u\|^2 + \|B^{m-1,0} u\|^2 . \tag{4.4}$$

The proof of the proposition then follows by induction on m. For $m = 0$ the desired result is an immediate consequence of (3.23), and for $m > 0$ the last two terms of (4.4) can be estimated by the right side of (4.3) due to the induction hypothesis.

Notation. We shall use the usual notation: if $\mathscr{A} \subset \mathscr{M}$, we set

$$(\|u\|_N^{\mathscr{A}})^2 = \int_{\mathscr{A}} \sum_{|\alpha| \leqq N} |D^\alpha u|^2 \, dV .$$

PROPOSITION 4.2. *If $\mathscr{K}$ is a compact subset of Σ_3, then there exist neighborhoods $\mathscr{M}_3(\mathscr{K}) = \mathscr{M}_3$ and $\mathscr{A}_3$ of $\mathscr{K}$ (in $\bar{\mathscr{M}}$) such that $\bar{\mathscr{M}}_3 \subset \mathscr{A}_3$ and such that*

$$\lambda^{1/2} \|u\|_N^{\mathscr{M}_3} \leqq C \|u\|_N^{\mathscr{A}_3} + K \|f\|_N^{\mathscr{A}_3} \tag{4.5}$$

for all u and f in $C^\infty(\bar{\mathscr{M}})$ with $Lu = f$ and $u|_{\Sigma_3} = 0$.

Proof: We shall show that given a C^∞ function ζ whose support lies in a domain of the special coordinates of Section 3.2 we have

$$\lambda \sum_{|\alpha|+r \leqq N} \|\zeta D^{\alpha,r} u\|^2 \leqq C \sum_{|\alpha|+r \leqq N} \|\zeta' D^{\alpha,r} u\|^2 + K \sum_{|\alpha|+r \leqq N} \|\zeta' D^{\alpha,r} f\|^2 , \tag{4.6}$$

where ζ' is one on the support of ζ and also has its support in the neighborhood. We can then choose finitely many ζ such that the sets where $\zeta = 1$ cover $\mathscr{K}$; (4.5) will follow (by adding the inequalities (4.6) over this set of ζ) when $\mathscr{M}_3$ consists of those points in which one of the ζ equals one and $\mathscr{A}_3$ is the interior of the supports of the ζ'.

Writing Q in terms of the special coordinates of (3.2), we have

$$(au_y , u_y) \leqq Q(u, u) + |(a^i u_i , u_y)| + |(a^{ij} u_i , u_j)| . \tag{4.7}$$

Substituting $\zeta D^{\beta,0} u$ with $|\beta| = N - 1$ for u in (4.7), applying Lemma 4.1 and choosing ζ so that on its support we have

$$a > \tfrac{1}{2} \max_{P \in \operatorname{supp} \zeta} a(P) , \tag{4.8}$$

we obtain, by induction on N,

$$\sum_{|\beta| \leqq N-1} \|\zeta D^{\beta,1} u\|^2 \leqq C \sum_{|\gamma|=N} \|\zeta D^{\gamma,0} u\|^2 + K \sum_{|\gamma| \leqq N-1} \|\zeta' D^{\gamma,0} f\|^2 . \tag{4.9}$$

Combining this with (4.3), we get (4.6) for $r \leqq 1$. To prove the inequality for $r > 1$ we apply $\zeta D^{\alpha,r-2}$ (with $|\alpha| = N - r$) to (3.4) and obtain

$$a\zeta D^{\alpha,r} u = -a^i \zeta D^{\alpha,r-2} u_{iy} - a^{ij} \zeta D^{\alpha,r-2} u_{ij} + \zeta D^{\alpha,r-2} f + A^{N-1} u , \tag{4.10}$$

where A^{N-1} is a differential operator of order $N - 1$ whose coefficients have supports contained in the support of ζ. Taking L_2 norms of (4.10) we find, by induction on r, using (4.8) and (4.9) (with a different ζ),

$$\sum_{|\alpha|=N-r} \|\zeta D^{\alpha,r} u\|^2 \leqq C \sum_{|\gamma|=N} \|\zeta D^{\gamma,0} u\|^2 + K \sum_{\substack{|\gamma|+j \leqq N-1 \\ j \leqq r-1}} \|\zeta D^{\gamma,j} f\|^2 + \|A^{N-1} u\|^2 . \tag{4.11}$$

Combining the above with (4.3), we obtain the desired estimate (4.6) by induction on N, thus completing the proof.

We remark that it is also possible to prove Proposition 4.2 by estimating the tangential derivatives in (4.3) and the purely normal derivatives in (4.10) with $\alpha = 0$, $r = N$. The mixed derivatives are then estimated with the help of the following well known lemma, which is also useful in the sections that follow.

LEMMA 4.2. *If $\mathscr{U}$ is a parallelepiped, then for any $\eta > 0$ there exists a constant $C(\eta)$ such that*

$$\int_{\mathscr{U}} \sum_{\substack{|\alpha|+k \leqq N \\ k<N}} |D^{\alpha,k}u|^2 \, dV \leqq \eta \int_{\mathscr{U}} |D_y^N u|^2 \, dV + C(\eta)\left[\int_{\mathscr{U}} \sum_{|\alpha|=N} |D^{\alpha,0}u|^2 \, dV + \int_{\mathscr{U}} |u|^2 \, dV\right]. \tag{4.12}$$

For the proof of this lemma see, for instance, Lemma 2.4 in [19].

The interior estimates are established by exactly the same argument as those in Proposition 4.1; since there are no "normal" derivatives in the interior, we obtain (4.3) for all derivatives, that is we have

$$\lambda \, \|\zeta D^\alpha u\|^2 \leqq C \sum_{|\beta| \leqq m} \|\zeta' D^\beta u\|^2 + K \sum_{|\beta| \leqq m} \|\zeta' D^\beta f\|^2, \tag{4.13}$$

where $|\alpha| = m$ and the support of ζ does not meet $\dot{\mathscr{M}}$. Choosing a suitable ζ and ζ' we obtain the following proposition.

PROPOSITION 4.3. *If $\mathscr{M}_0$ is an open subset of $\mathscr{M}$ such that $\bar{\mathscr{M}}_0 \subset \mathscr{M}$, then for any open subset $\mathscr{A}_0$, $\bar{\mathscr{M}}_0 \subset \mathscr{A}_0 \subset \bar{\mathscr{A}}_0 \subset \mathscr{M}$, we have*

$$\lambda^{1/2} \, \|u\|_N^{\mathscr{M}_0} \leqq C \, \|u\|_N^{\mathscr{A}_0} + K \, \|f\|_N^{\mathscr{A}_0}, \tag{4.14}$$

for all u and f in $C^\infty(\mathscr{M})$ with $Lu = f$.

We remark that with the aid of Corollary 3.2 of [14] it is possible to obtain global analogues of Proposition 4.1 in whole boundary regions near Σ_1 and $\Sigma_2 \cup \Sigma_3$ without having to go through such local arguments, as in the proposition which uses locally supported functions ζ. This may be done as follows. Let A be an elliptic differential operator in the x-variables of order m defined on a boundary component, with the property that A^* (the formal adjoint of A) differs from A by an operator of order less than m. Consider A as defined in a neighborhood of the boundary component with coefficients independent of y. Let $\zeta(y)$ be a C^∞ function having its support in this neighborhood, and identically equal to one in a smaller neighborhood of the boundary component. Inserting $v = A^*\zeta^2 Au$ in the relation $Q(u, v) = -(f, v)$ and applying Corollary 3.2 of [14], one may derive again the inequality (4.3) (which is now global), where $\zeta'(y)$ is a function with compact support which equals one on the support of ζ.

5. Estimates Near Σ_1

We shall work in domains of special local coordinates described in Section 3.2. Our *a priori* estimates will involve the following norms:

$$(5.1) \qquad (\overline{\|u\|}_N^{\mathcal{M}_1})^2 = \sum_{|\alpha|+k\leqq N} \int_{\mathcal{M}_1} |y|^k \, |D^{\alpha,k}u|^2 \, dV ,$$

where $\mathcal{M}_1$ is a neighborhood of Σ_1 in $\overline{\mathcal{M}}$.

We denote by C constants which depend on a bound of the coefficients of the second and first order derivatives in L and on bounds of their derivatives up to third and second order, respectively. K will denote constants that depend on coefficients of L and finitely many of their derivatives. Again we assume that $-c \geqq 2\lambda$ and that condition (a) holds.

Proposition 5.1. *There exist neighborhoods $\mathcal{M}_1$ and $\mathcal{A}_1$ of Σ_1 in $\overline{\mathcal{M}}$ with $\overline{\mathcal{M}}_1 \subset \mathcal{A}_1$ such that*

$$(5.2) \qquad \lambda^{1/2} \, \overline{\|u\|}_N^{\mathcal{M}_1} \leqq C \, \overline{\|u\|}_N^{\mathcal{A}_1} + K \, \overline{\|f\|}_N^{\mathcal{A}_1}$$

for all u and f in $C(\overline{\mathcal{M}})$ with $Lu = f$.

To prove this proposition we need the following well known lemmas which will also be useful in later sections.

Lemma 5.1. *If g is in C^2, non-negative, defined on $[a, b]$ and if $\delta > 0$, then for all $t_0 \in [a + \delta, b - \delta]$ we have*

$$(5.3) \qquad (g'(t_0))^2 \leqq 2 \max_{t\in[a,b]} |g''(t)| \, g(t_0) + \frac{4}{\delta^2} (g(t_0))^2 .$$

The proof of this lemma may be found, for instance, in [22] (page 341).

Lemma 5.2. *If v is a function supported in the domain of a special coordinate system (as defined in Section 3.2) and if τ is a function such that $(\tau^2)_y = -1$, then*

$$(5.4) \qquad \|\tau^p v\| \leqq \frac{2}{p+1} \|\tau^{p+2} v_y\| .$$

Proof: We have

$$(\tau^{2p+2}v^2)_y = -(p+1)\tau^{2p}v^2 + 2\tau^{2p+2}vv_y ;$$

integrating we obtain

$$(5.5)\qquad (p+1)\,\|\tau^p v\|^2 + \int_{\dot{\mathscr{M}}} (\tau^{p+1} v)^2\,dx \leqq 2\,\|\tau^p v\| \cdot \|\tau^{p+2} v_y\|\,,$$

from which (5.4) follows.

Proof of Proposition 5.1: Condition (a) (i.e., $\Sigma_2 \cup \Sigma_3$ is closed) implies that Σ_1 is a union of components of $\dot{\mathscr{M}}$ and hence the function a with all its x-derivatives vanishes on Σ_1. By the semi-definiteness we have (3.6):

$$(a^i)^2 \leqq 4a^{ii}a\,,$$

and thus a^i and all of its x-derivatives also vanish on Σ_1.

Defining τ near Σ_1 by

$$(5.6)\qquad \tau = |y|^{1/2}\,,$$

we see that second order tangential derivatives of a and the a^i are bounded by $C\tau^2$. Now we adopt the following notation (which we also use in later sections): for any function w defined on a neighborhood of $\dot{\mathscr{M}}$ we define $\dot{w}$ by $\dot{w}(P) = w(P^*)$, where P^* is the nearest point to P on $\dot{\mathscr{M}}$. In terms of our local coordinate system we have $\dot{w}(x, y) = w(x, 0)$. Near Σ_1 we define the function σ by

$$(5.7)\qquad \sigma = (-b + \dot{a}_y)^{1/2}\,.$$

Recall that, on Σ_1, $-b + a_y \geqq 0$ and that, since $a_i^i = 0$ on Σ_1, we have $\sigma = (-\dot{\beta})^{1/2}$.

We shall prove the following estimate which clearly implies the desired inequality (5.2). Given a point $O \in \Sigma_1$ and a function ζ with support contained in a small neighborhood of O, the inequality

$$(5.8)\qquad \begin{aligned} \lambda \sum_{|\alpha|+k \leqq N} \|\tau^k \zeta D^{\alpha,k} u\|^2 + \sum_{\substack{|\alpha|+k\leqq N \\ k \geqq 1}} \|\sigma \tau^{k-1} \zeta D^{\alpha,k} u\|^2 + \sum_{|\alpha| \leqq N} \int_{\dot{\mathscr{M}}} (\sigma \zeta D^{\alpha,0} u)^2\,dx \\ \leqq C(\overline{\|u\|_N^{\mathscr{A}_1}})^2 + K(\overline{\|f\|_N^{\mathscr{A}_1}})^2 \end{aligned}$$

holds.

We shall denote by $B_p^{m,k}$ operators of the form

$$(5.9)\qquad B_p^{m,k} = \sum_{\substack{|\alpha| \leqq m \\ j \leqq k}} b_{\alpha,j} D^{\alpha,j}\,,$$

where supp $(b_{\alpha,j}) \subset$ supp ζ and

$$(5.10)\qquad |b_{\alpha,j}| = O(|y|^p)\,,$$

and we set $B^{m,k} = B_0^{m,k}$ as in the previous section.

If we apply $\zeta D^{\alpha,k}$, $|\alpha| = m = N - k$, to (3.4), we find, after analyzing the contributions of various terms,

$$
\begin{aligned}
-c\zeta D^{\alpha,k}u - b\zeta D^{\alpha,k+1}u - \alpha^j b_j \zeta D^{\alpha-\delta_j,k+1}u \\
&= \zeta L_0 D^{\alpha,k}u + \alpha^j a_j \zeta D^{\alpha-\delta_j,k+2}u \\
&\quad + k a_y \zeta D^{\alpha,k+1}u + k\alpha^j a_{yj}\zeta D^{\alpha-\delta_j,k+1}u \\
&\quad + B_1^{m-2,k+2}u + B^{m,k}u + B^{m-2,k+1}u + B_1^{m,k+1}u \\
&\quad + B^{m+1,k}u + B^{m+2,k-1}u - \zeta D^{\alpha,k}f,
\end{aligned} \tag{5.11}
$$

where L_0 is the second order part of L, i.e.,

$$L_0 v = a v_{yy} + a^i v_{iy} + a^{ij} v_{ij}\,, \tag{5.12}$$

and the coefficients of the terms of order N in the B's are bounded by C and the first derivatives of the coefficients of the B's of terms of order $N + 1$ are bounded by C. Now we take inner products of (5.11) with $\tau^{2k}\zeta D^{\alpha,k}u$ and we set

$$
\begin{aligned}
I_1 &= (-c\zeta D^{\alpha,k}u, \tau^{2k}\zeta D^{\alpha,k}u)\,, \\
I_2 &= ((-b - k a_y)\zeta D^{\alpha,k+1}u, \tau^{2k}\zeta D^{\alpha,k}u)\,, \\
I_3 &= ((-\alpha^j b_j - k\alpha^j a_{yj})\zeta D^{\alpha-\delta_j,k+1}u, \tau^{2k}\zeta D^{\alpha,k}u)\,, \\
I_4 &= (\zeta L_0 D^{\alpha,k}u, \tau^{2k}\zeta D^{\alpha,k}u)\,, \\
I_5 &= (\alpha^j a_j \zeta D^{\alpha-\delta_j,k+2}u, \tau^{2k}\zeta D^{\alpha,k}u)\,, \\
I_6 &= (B_1^{m-2,k+2}u + B^{m,k}u + B^{m-2,k+1}u + B_1^{m,k+1}u + B^{m+1,k}u + B^{m+2,k-1}u \\
&\qquad - \zeta D^{\alpha,k}f, \tau^{2k}\zeta D^{\alpha,k}u)\,;
\end{aligned}
$$

we have

$$I_1 + I_2 + I_3 = I_4 + I_5 + I_6\,. \tag{5.13}$$

We shall estimate each of these terms. Since $-c \geqq 2\lambda$, we have

$$I_1 \geqq 2\lambda\, \|\tau^k \zeta D^{\alpha,k}u\|^2\,. \tag{5.14}$$

Using partial integration one verifies easily that

$$(5.15)\quad I_2 \geqq -C\,\|\tau^k\zeta D^{\alpha,k}u\|^2 + \begin{cases} -\frac{1}{2}k((\dot b + k\dot a_y)\zeta D^{\alpha,k}u,\ |y|^{k-1}\zeta D^{\alpha,k}u) & \text{if } k>0\,, \\ -\dfrac{1}{2}\displaystyle\int_{\dot{\mathcal{M}}} \dot b(\zeta D^{\alpha,0}u)^2\,dx & \text{if } k=0\,, \end{cases}$$

$$(5.16)\quad \begin{aligned} I_3 \geqq\ & -\alpha^j((\dot b_j + k\dot a_{yj})D^{\alpha-\delta_j,k+1}u,\ \tau^{2k}\zeta D^{\alpha,k}u) \\ & -C\sum_{|\beta|=m-1}\|\tau^{k+1}\zeta D^{\beta,k+1}u\|\sum_{|\gamma|=m}\|\tau^k D^{\gamma,k}u\|\,. \end{aligned}$$

Observe that the semi-definiteness of L_0 implies that, for $v = D^{\alpha,k}u$,

$$(5.17)\quad \begin{aligned} (\zeta^2 L_0 v,\ \tau^{2k}v) &\leqq -((\zeta^2 a\tau^{2k})_y v_y\,,\ v) - ((\zeta^2 a^i)_i v_y\,,\ \tau^{2k}v) - ((\zeta^2 a^{ij})_j v_i\,,\ \tau^{2k}v) \\ &\leqq -\frac{1}{2}\int_{\dot{\mathcal{M}}} a_y\,|\zeta\tau^k v|^2\,dx + \tfrac{1}{2}((\zeta^2 a\tau^{2k})_{yy}v,\ v) + C(\overline{\|u\|}{}_N^{\mathcal{A}_1})^2\,, \end{aligned}$$

by further integration by parts, and using the fact that $\zeta^2 a^i = 0$ in Σ_1.

Since

$$(5.18)\qquad a(x,y) = -\dot a_y\,|y| + O(|y|^2)\,,$$

we have

$$(5.19)\qquad (\zeta^2\tau^{2k}a)_{yy} = -k(k+1)\dot a_y\,|y|^{k-1} + O(|y|^k)\,,$$

and hence

$$(5.20)\quad I_4 \leqq C(\overline{\|u\|}{}_N^{\mathcal{A}_1})^2 + \begin{cases} -\frac{1}{2}k(k+1)(\dot a_y\zeta D^{\alpha,k}u,\ |y|^{k-1}\zeta D^{\alpha,k}u) & \text{if } k>0\,, \\ -\dfrac{1}{2}\displaystyle\int_{\dot{\mathcal{M}}} \dot a_y(\zeta D^{\alpha,0}u)^2\,dx\,, & \text{if } k=0\,. \end{cases}$$

Next, on integrating by parts with respect to y, we obtain

$$I_5 = -\alpha^j(\zeta^2\tau^{2k}a_j D^{\alpha-\delta_j,k+1}u,\ D^{\alpha,k+1}u) - \alpha^j((\zeta^2\tau^{2k}a_j)_y D^{\alpha-\delta_j,k+1}u,\ D^{\alpha,k}u)\,.$$

Integrating the first term on the right by parts, we see that it is bounded by $C(\overline{\|u\|}{}_N^{\mathcal{A}_1})^2$. If we also make use of the analogue of (5.18),

$$(5.21)\qquad a_j(x,y) = -\dot a_{yj}\,|y| + O(|y|^2)\,,$$

in the other term, we find

$$(5.22)\qquad I_5 \leqq -\alpha^j(k+1)(\dot a_{yj}\zeta D^{\alpha-\delta_j,k+1}u,\ \tau^{2k}\zeta D^{\alpha,k}u) + C(\overline{\|u\|}{}_N^{\mathcal{A}_1})^2\,.$$

We use the following lemmas to estimate the terms in I_6.

LEMMA 5.3. *If the support of ψ is contained in the support of ζ and if $\psi = 0$ on $\dot{\mathcal{M}}$, then*

$$
\begin{aligned}
&|(\psi D^{\beta,k+1}u,\ \tau^{2k}D^{\alpha,k}u)| \\
&\qquad \leqq C\Big[\sum_{|\gamma|=m-1}(\|\tau^{k+1}D^{\gamma,k+1}u\|^{\mathscr{A}_1})^2 + \sum_{|\sigma|=m}(\|\tau^k D^{\sigma,k}u\|^{\mathscr{A}_1})^2\Big],
\end{aligned}
\tag{5.23}
$$

where $|\beta| = |\alpha| = m$, and C depends only on the bounds of ψ and its first derivatives.

Proof: Let $\phi = \psi\tau^{2k}$; integrating by parts $2m$ times in the x-directions (so as to exchange the x-derivatives on either side of the scalar product), and once in the y-direction, we obtain

$$
\begin{aligned}
(\phi D^{\beta,k+1}u,\ D^{\alpha,k}u) = &-(\phi D^{\alpha,k}u,\ D^{\beta,k+1}u) \\
&+ \sum_i r_i(\phi_i D^{\gamma_i,k+1}u,\ D^{\sigma_i,k}u) \\
&- (\phi_y D^{\alpha,k}u,\ D^{\beta,k}u)\,,
\end{aligned}
$$

where $|\gamma_i| = m - 1$ and $|\sigma_i| = m$. Since $\phi = O(|y|^{k+1})$, we have $\phi_i = O(|y|^{k+1})$ and $\phi_y = O(|y|)^k)$. Hence we obtain the desired inequality.

By similar arguments based on integrations by parts we also obtain

LEMMA 5.4. *If the support of ψ is contained in the support of ζ, then*

$$
|(\psi D^{\beta,k}u,\ \tau^{2k}D^{\alpha,k}u)| \leqq C\sum_{|\gamma|=m}(\|\tau^k D^{\gamma,k}u\|^{\mathscr{A}_1})^2\,, \tag{5.24}
$$

with $|\beta| = m + 1$ and $|\alpha| = m$. Furthermore,

$$
\begin{aligned}
&|(\psi D^{\sigma,k-1}u,\ \tau^{2k}D^{\alpha,k}u)| \\
&\qquad \leqq C\Big[\sum_{|\gamma|=m}(\|\tau^k D^{\gamma,k}u\|^{\mathscr{A}_1})^2 + \sum_{|\gamma|=m+1}(\|\tau^{k-1}D^{\gamma,k-1}u\|^{\mathscr{A}_1})^2\Big],
\end{aligned}
\tag{5.25}
$$

where $|\sigma| = m + 2$ and $|\alpha| = m$. In both inequalities C depends only on bounds for ψ and its first derivatives.

Now the term corresponding to $B^{m-2,k+1}u$ can be estimated using Lemma 5.2, the term corresponding to $B_1^{m,k+1}$ by using Lemma 5.3, and the terms which correspond to $B^{m+1,k}$ and $B^{m+2,k-1}u$ using (5.24) and (5.25), respectively. We then obtain the following estimate of I_6 :

$$I_6 \leqq C(\overline{\|u\|_N^{\mathscr{A}_1}})^2 + C(\overline{\|f\|_N^{\mathscr{A}_1}})^2 + K(\overline{\|u\|_{N-1}^{\mathscr{B}_1}})^2 , \tag{5.26}$$

where supp $\zeta \subset \mathscr{B}_1 \subset \overline{\mathscr{B}}_1 \subset \mathscr{A}_1$.

We now prove (5.8) by induction on N. Applying the estimates of the I_j to (5.13) and using the induction hypothesis, we see that

$$\begin{aligned} &2\lambda \|\tau^k \zeta D^{\alpha,k} u\|^2 + \begin{cases} \frac{1}{2}k \, \|\sigma\tau^{k-1}\zeta D^{\alpha,k}u\|^2 & \text{if} \quad k > 0 , \\ \dfrac{1}{2}\displaystyle\int_{\dot{\mathscr{M}}} (\sigma\zeta D^{\alpha,0}u)^2 \, dx & \text{if} \quad k = 0 , \end{cases} \\ &\leqq \alpha^j((\dot{b} - \dot{a}_y)_j \zeta D^{\alpha-\delta_j,k+1}u, \tau^{2k}\zeta D^{\alpha,k}u) + C(\overline{\|u\|_N^{\mathscr{A}_1}})^2 + K(\overline{\|f\|_N^{\mathscr{A}_1}})^2 . \end{aligned} \tag{5.27}$$

Applying Lemma 5.1 to $\sigma^2 = g$ as a function of x, we have

$$|(\dot{b} - a_y)_j| \leqq C\sigma . \tag{5.28}$$

Therefore, the first term on the right in (5.27) can be estimated by

$$C \sum_{|\beta|=m-1} \|\sigma\tau^k\zeta D^{\beta,k+1}u\| \cdot \|\tau^k\zeta D^{\alpha,k}u\| . \tag{5.29}$$

Finally summing over α, k with $|\alpha| + k = N$, we see that this term can be absorbed in the left-hand side and we obtain the inequality (5.8) thus completing the proof of Proposition 5.1.

6. Estimates Near Σ_2

6.1. In order to describe these estimates, which are the most delicate, we shall introduce certain norms in a neighborhood of Σ_2. As before, near $\Sigma_2 \cup \Sigma_3$ we let $-y$ denote the distance from a point in $\mathscr{M}$ to the boundary. Setting $\phi = y$, the function a of (3.13)$'$ is well defined in a neighborhood of $\Sigma_2 \cup \Sigma_3$. For any point P in $\mathscr{U}$ we denote by $\dot{a}$, the value of a at the point of $\Sigma_2 \cup \Sigma_3$ closest to P, as in Section 5, and introduce the non-negative function

$$\tau = \sqrt{\dot{a} - y} . \tag{6.1}$$

In terms of our local coordinates of Section 3.2, the function a agrees with the coefficient of u_{yy} in (3.4), and $\tau = \sqrt{a(x, 0) - y}$. In a neighborhood $\mathscr{M}_2$ (in $\overline{\mathscr{M}}$) of Σ_2, we now introduce the norms, similar to (1.6),

$$(\widehat{\|u\|}_N^{\mathscr{M}_2})^2 = \int_{\mathscr{M}_2} \left[\sum_{|\alpha| \leqq N} |D^{\alpha,0}u|^2 + \sum_{\substack{|\alpha|+k \leqq N \\ k \leqq 0}} |\tau^{k-1} D^{\alpha,k} u|^2 \right] dV , \tag{6.2}$$

$$\widehat{|f|}_N^{\mathscr{M}_2} = \left\{ \int_{\mathscr{M}_2} \left[\sum_{|\alpha| \leqq N} |D^{\alpha,0}f|^2 + \sum_{|\alpha|+k \leqq N-1} \tau^k \, |D^{\alpha,k}f|^2 \right] dV \right\}^{1/2} . \tag{6.2$'$}$$

Because of some ambiguity in the choice of x-coordinates these norms are not precisely defined but they are easily made precise by fixing coordinates on a finite number of patches covering $\mathscr{M}_2$ and then summing over these patches, as below. For a domain $\mathscr{A}$ we use the usual notation

$$\|u\|_N^{\mathscr{A}} = \left(\int_{\mathscr{A}} \sum_{|\alpha|\leqq N} |D^\alpha u|^2\, dV\right)^{1/2}.$$

As always we assume that $-c$ is large,

$$-\tfrac{1}{2}c \geqq \lambda \tag{6.3}$$

for some large positive constant λ. As in Section 5, we use the letter K to denote various constants which depend on N and bounds on derivatives of our coefficients, while C denotes constants depending only on the derivatives of the leading (and first order) coefficients of orders not exceeding three (two) and on N and $\mathscr{M}$ *and a lower bound for* γ. Let $\mathscr{A}$ be a larger neighborhood of Σ_2 in $\overline{\mathscr{M}}$ such that $\overline{\mathscr{M}}_2$ is contained in the interior (relative to $\overline{\mathscr{M}}$) of $\mathscr{A}$. For suitable $\mathscr{M}_2$ and $\mathscr{A}$ we shall prove the following *a priori* estimate—assuming, always, that our conditions (a), $\cdots$, (d) hold.

Proposition 6.1. *Let u be a solution of* (1.1) *in a neighborhood of Σ_2 and vanishing on $\Sigma_2 \cup \Sigma_3$. There exists a constant λ,*[3] *such that if* (6.3) *holds, then*

$$\widehat{\|u\|}_N^{\mathscr{M}_2} \leqq K(\widehat{|f|}_N^{\mathscr{A}} + \|u\|_{N-1}^{\mathscr{A}-\mathscr{M}_2} + \|u\|_0^{\mathscr{M}}) + C\,\|u\|_N^{\mathscr{A}-\mathscr{M}_2}. \tag{6.4}$$

Furthermore, (6.4) *holds for the solution of our modified equation* (3.28) *after elliptic regularization, with the constants K, C independent of ε, δ.*

For $N = 0$ there is nothing to prove. We proceed by induction on N, and may therefore assume that (6.4) has been established for smaller values of N. We shall first reduce our proof to a local estimate: In $\overline{\mathscr{M}}$ a neighborhood $\mathscr{M}_2$ of Σ_2 (within a distance d' to the boundary) may be covered by a finite number of small neighborhoods $\mathscr{U}_i$ each of which may be described in terms of our special local coordinates as in Sections 3.2 and 3.4:

$$\begin{aligned} &|x^i - x_0^i| < d\,, \qquad\qquad i = 1, \cdots, n-1\,,\\ &0 \leqq -y < d'\,, \end{aligned} \tag{6.5}$$

for small constants d, d'. These neighborhoods $\mathscr{U}_j$ may be chosen in such a way that the enlarged region $\mathscr{U}_j'$: $|x^i - x_0^i| < 2d$, $i = 1, \cdots, n-1$, $0 \leqq -y < d'$, intersects at most 3^{n-1} of the other $\mathscr{U}_i$. Furthermore, for d, d' suitably small, on

[3] λ depends only on $\mathscr{M}$, N, on the derivatives up to the third order of the leading terms a^{ij}, on the derivatives up to second order of the coefficients of the $b_\nu{}^i$, and on $\sup \gamma^{-1}$.

the overlap, the changes from coordinates on $\mathscr{U}_j$ to coordinates on the other $\mathscr{U}_i$ have Jacobian matrices which are very close to the unit matrix. Each $\mathscr{U}_i$ contains a point of Σ_2 and in each $\mathscr{U}_i$ we use the local normalizing coordinates of Section 3.4. In each $\mathscr{U}_i$ we extend the functions β and γ, using (3.7) and (3.8),

$$\beta = b - \tfrac{1}{2}a^j_j - a_y\,, \qquad \beta\gamma = \beta + \tfrac{1}{2}Na_y\,. \tag{6.6}$$

If two $\mathscr{U}_i$ overlap, the corresponding values of $\beta\gamma$ given by (6.6) at the same point are very close.

With a suitable neighborhood $\mathscr{A}$ of $\mathscr{M}_2 = \cup\, \mathscr{U}_i$, and constants K, C as in Proposition 6.1, we shall show, using our induction hypothesis, that

$$\kappa = \max_{\substack{|\alpha|+k=N \\ j}} \frac{1}{k\,!}\int_{\mathscr{U}_j} \beta\gamma\tau^{2k-2}\,|D^{\alpha,k}u|^2\,dV \leqq H\,, \tag{6.7}$$

where for $k = 0$ the factor τ^{2k-2} is replaced by one. For convenience we denote by H (for harmless) any terms which can be bounded by

$$K[|\widehat{f}|_N^{\mathscr{A}} + \|u\|_{N-1}^{\mathscr{A}} + \|u\|_0^{\mathscr{M}}]^2 + C[\|u\|_N^{\mathscr{A}-\mathscr{M}_2}]^2\,.$$

Here we have included $(\|u\|_{N-1}^{\mathscr{A}})^2$ as harmless in virtue of our induction hypothesis. Inequality (6.7) yields (6.4). The factor $1/k!$ plays a role in technical considerations arising in the derivation of the estimate.

6.2. Proof of (6.7). Choose $\mathscr{U} = \mathscr{U}_j$ (described by (6.5)) for which the maximum κ occurs in (6.7). Let $\zeta \geqq 0$ be a C^∞ function which equals one on $\mathscr{U}$ and has support in

$$\mathscr{U}_j' = \mathscr{U}' : |x^i - x_0^i| < 2d\,, \qquad i = 1, \cdots, n-1\,, \quad 0 \leqq -y < 2d'\,;$$

ζ may be chosen independent of y for $-y \leqq d'$ and so that its x-derivatives satisfy

$$|\zeta_j| \leqq \frac{3}{d}\,. \tag{6.8}$$

(The domain $\mathscr{A}$ will be the union of these domains $\mathscr{U}_j'$.) We observe that for $|\alpha| + r = N$, $\|\zeta D^{\alpha,r}u\|^2$ involves integration over $\mathscr{U}'$ which is covered by a finite number of the $\mathscr{U}_j$ and by $\mathscr{A}$. Hence it is bounded by $C\kappa + H$, since β is close to $\frac{1}{8}$ and γ is bounded from below.

With $\mathscr{U}$ chosen as above we now define

$$\kappa'_{\alpha,k} = \frac{1}{k\,!}\left\{\int \zeta\beta\gamma\tau^{2k-2}\,|D^{\alpha,k}u|^2\,dV \right.$$
$$\left. + \int_{\dot{\mathscr{M}}} \zeta\tau^{2k}\,|D^{\alpha,k}u|^2\,dx + \frac{k-1}{2}\,(\dot{a}D^{\alpha,k}u,\ \zeta\tau^{2k-4}D^{\alpha,k}u)\right\}.$$

Clearly, $\kappa \leqq \kappa' = \max_{|\alpha|+k=N} \kappa'_{\alpha,k}$, and in proving (6.7) it is convenient to prove the stronger inequality[4]

$$\kappa' = \max_{|\alpha|+k=N} \kappa'_{\alpha,k} \leqq H. \tag{6.7'}$$

Let α, k be the values for which the maximum $\kappa'_{\alpha,k}$ occurs. If $k = 0$, then (6.7)$'$ follows directly from Proposition 4.1. Applying the proposition, with ζ' also having support in $\mathscr{U}'$, we find, as above, for $-c \geqq 2\lambda$ sufficiently large

$$\begin{aligned}\lambda\kappa' = \lambda \int \zeta\, |D^{\alpha,0}|^2\, dV &\leqq C \sum_{\substack{|\beta|+r=N\\ r\leqq 1}} \|\zeta' D^{\beta,r}u\|^2 + H \\ &\leqq C\kappa + H \leqq C\kappa' + H.\end{aligned}$$

For $\lambda > 2C$ the inequality (6.7)$'$ follows.

Thus we may assume $k > 0$. Now begins the delicate task—to establish the local estimate (6.7)$'$.

Apply the operator $D^{\alpha,k-1}$ to the equation (3.4) in our local coordinates. The result expresses $D^{\alpha,k-1}f$ as a sum of terms of which we must keep a number under careful control (we set $|\alpha| = m$):

$$\begin{aligned}D^{\alpha,k-1}au_{yy} = {} & aD^{\alpha,k+1}u + (k-1)a_y D^{\alpha,k}u + \alpha^j a_j D^{\alpha-\delta_j,k+1}u \\ & + \sum_{j\neq r} \alpha^j \alpha^r a_{jr} D^{\alpha-\delta_j-\delta_r,k+1}u \\ & + \sum_j \alpha^j(\alpha^j - 1)a_{jj} D^{\alpha-2\delta_j,k+1}u \\ & + B^{m,k-1}u + B^{m-1,k}u + B^{m-3,k+1}u,\end{aligned}$$

here we have used our notation (4.3) for the operators $B^{m,j}$;

$$\begin{aligned}D^{\alpha,k-1}bu_y &= bD^{\alpha,k}u + B^{m,k-1}u + B^{m-1,k}u, \\ D^{\alpha,k-1}a^i u_{iy} &= a^i D^{\alpha,k}u_i + \alpha^j a^i_j D^{\alpha-\delta_j,k}u_i + B^{m+1,k-1}u + B^{m-1,k}u, \\ D^{\alpha,k-1}a^{ij}u_{ij} &= D_j(a^{ij}D^{\alpha,k-1}u_i) + B^{m+1,k-1}u + B^{m+2,k-2}u, \\ D^{\alpha,k-1}(b^i u_i + cu) &= B^{m+1,k-1}u.\end{aligned}$$

If we now take the scalar product of this expression for $D^{\alpha,k-1}f$ with $\zeta\tau^{2k-2}D^{\alpha,k}u$, we obtain the equation

$$(\zeta D^{\alpha,k-1}f, \tau^{2k-2}D^{\alpha,k}u) = \sum_0^7 I_j, \tag{6.9}$$

[4] Our choice of κ' was arrived at only after much groping.

where

$$I_0 = ((b + (k-1)a_y)D^{\alpha,k}u,\ \zeta\tau^{2k-2}D^{\alpha,k}u)\ ,$$

$$I_1 = (aD^{\alpha,k+1}u,\ \zeta\tau^{2k-2}D^{\alpha,k}u)\ ,$$

$$I_2 = \alpha^j(a_jD^{\alpha-\delta_j,k+1}u,\ \zeta\tau^{2k-2}D^{\alpha,k}u)\ ,$$

$$I_3 = \left(\sum_{j\neq r}\alpha^j\alpha^r a_{jr}D^{\alpha-\delta_j-\delta_r,k+1}u + \sum_j \alpha^j(\alpha^j-1)a_{jj}D^{\alpha-2\delta_j,k+1}u,\ \zeta\tau^{2k-2}D^{\alpha,k}u\right)\ ,$$

$$I_4 = (a^iD^{\alpha,k}u_i\,,\ \zeta\tau^{2k-2}D^{\alpha,k}u)\ ,$$

$$I_5 = \alpha^j(a^i_jD^{\alpha-\delta_j,k}u_i\,,\ \zeta\tau^{2k-2}D^{\alpha,k}u)\ ,$$

$$I_6 = (D_j(a^{ij}D^{\alpha,k-1}u_i),\ \zeta\tau^{2k-2}D^{\alpha,k}u)\ ,$$

$$I_7 = (B^{m,k-1}u + B^{m-1,k}u + B^{m-3,k+1}u + B^{m+1,k-1}u + B^{m+2,k-2}u,\ \zeta\tau^{2k-2}D^{\alpha,k}u)\ .$$

We shall estimate all the terms on the right of (6.9) in a suitable way. Let us first dispose of the last term I_7. If d and d' are sufficiently small, we claim that

$$I_7 \leqq \varepsilon\kappa + H\,, \tag{6.10}$$

for any positive ε. Clearly,

$$\begin{aligned} I_7 \leqq\ & \varepsilon(\zeta\tau^{2k-2}D^{\alpha,k}u,\ D^{\alpha,k}u) + \frac{C}{\varepsilon}(B^{m-3,k+1}u,\ \zeta\tau^{2k-2}B^{m-3,k+1}u) + H \\ & + \frac{C}{\varepsilon}(B^{m+1,k-1}u,\ \zeta\tau^{2k-2}B^{m+1,k-1}u) + \frac{C}{\varepsilon}(B^{m+2,k-2}u,\ \zeta\tau^{2k-2}B^{m+2,k-2}u)\ . \end{aligned} \tag{6.11}$$

The contribution of the terms $B^{m,k-1}$, $B^{m-1,k}$ is harmless, since they are of lower order than N, and can be estimated by the induction hypothesis. The term $B^{m-3,k+1}$ is also of lower order; however, in order to use the induction hypothesis we would have to have a factor τ^{2k} in the term instead of τ^{2k-2} which actually occurs there. Nevertheless, the term

$$\frac{C}{\varepsilon}(B^{m-3,k+1}u,\ \zeta\tau^{2k-2}B^{m-3,k+1}u)$$

is easily estimated with the aid of Lemma 5.2. Applying the lemma with $v = \zeta^{1/2}B^{m-3,k+1}u$ (we may always assume that $\zeta^{1/2}\in C^\infty$), we find that this term is bounded by

$$\frac{C}{\varepsilon}(B^{m-3,k+2}u,\ \zeta\tau^{2k+2}B^{m-3,k+2}u) + H\,, \tag{6.11$'$}$$

since the term involving the derivative of $\zeta^{1/2}$ with respect to y has support outside $\mathscr{M}_2$ and so is of type H. By induction the estimate (6.11)$'$ so obtained is H. In each of the last two terms of (6.11), τ occurs with a power higher than necessary for our estimate. Thus since $\beta\gamma$ is bounded from below, these terms may be estimated by

$$\frac{C}{\varepsilon}\,\kappa \max_{\operatorname{supp}\zeta} \tau^2 = \frac{C}{\varepsilon}\,\kappa \max_{\operatorname{supp}\zeta} \, (a(x, 0) - y)$$

which can be estimated by $\varepsilon\kappa$ provided that d and d' are sufficiently small. Finally the first term on the right of (6.11) may be bounded by $C\varepsilon\kappa$ and thus (6.10) follows—after renaming ε.

6.3. Preliminary estimates. Before considering the other terms of (6.9) we make some preparatory remarks. On the boundary $y = 0$ of $\mathscr{U}$ there is a point $(x_1, 0)$ of Σ_2 where the a_i vanish and a^{ij} and a_{ij} satisfy the normalized conditions (3.15). Clearly then, in the region

$$\mathscr{U}'_j : |x^i - x_0^i| < 2d\,, \qquad i = 1, \cdots, n-1\,, \quad 0 \leqq -y < 2d', \tag{6.5$'$}$$

for d and d' sufficiently small we still have

$$\begin{aligned} N^2\,|a_{ij}(x, 0)| &\leqq 2\varepsilon_0 \min_{\mathscr{U}'} \beta\gamma\,, \\ |a^{ij}(x, y)| &\leqq 2\varepsilon_0 \beta\gamma\,. \end{aligned} \tag{6.12}$$

The size of d and d' will depend on the third derivatives of a (or at least the modulus of continuity of the second derivatives) and on the first (and second) derivatives of b (and a^i). Since $\beta(x_1, 0) = \frac{1}{8}$, so that $\beta\gamma = \beta + \frac{1}{2}Na_y \leqq \frac{1}{8}$ there, we have for d, d' small

$$\beta \leqq \tfrac{1}{4}\,, \qquad \beta\gamma = \beta + \tfrac{1}{2}Na_y \leqq \tfrac{1}{4} \qquad \text{in} \qquad \mathscr{U}'\,. \tag{6.13}$$

As a consequence we find, with the aid of the first inequality of (6.12),

$$\max_{\mathscr{U}'} a(x, 0) \leqq \frac{32n^2d^2\varepsilon_0}{N^2} \min_{\mathscr{U}'} \beta\gamma\,, \tag{6.14}$$

and hence

$$\max_{\mathscr{U}'} \tau^2 \leqq \frac{32n^2d^2\varepsilon_0}{N^2} + 2d'\,.$$

If now we take

$$d' = d^2\,, \tag{6.15}$$

we get

$$\max_{\mathscr{U}'} \tau^2 \leqq d^2(32n^2\varepsilon_0 + 2) \leqq 3d^2 , \tag{6.16}$$

since $32n^2\varepsilon_0 \leqq 1$. From now on we assume that $d' = d^2$.

Since $|a^i|^2 \leqq 4aa^{ii}$, we see with the aid of (6.12) and (6.14) that

$$|a^i| \leqq \frac{16nd}{N} \varepsilon_0\beta\gamma . \tag{6.17}$$

Next we have from (3.6)′: at $(x_1 , 0)$

$$\begin{aligned} |a^i_j|^2 &\leqq 2a^{ii}a_{jj} \\ &\leqq \frac{8\varepsilon_0^2\beta^2\gamma^2}{N^2} . \end{aligned}$$

Hence for d sufficiently small,

$$|a^i_j| \leqq \frac{3\varepsilon_0\beta\gamma}{N} \quad \text{in} \quad \mathscr{U}' . \tag{6.17$'$}$$

In $V: |x^i - x^i_0| < 3d$, $0 \leqq -y \leqq 2d^2$, we also apply Lemma 5.1 which, in virtue of (6.14) and (6.12), which we may also assume holds in V, yields the inequality

$$\begin{aligned} |a_i|^2 &\leqq a\left(2 \max_V |a_{ii}| + \frac{4}{d^2} a\right) \\ &\leqq a\left(2 \max_V |a_{ii}| + \frac{128n^2\varepsilon_0}{N^2} \beta\gamma\right) \\ &\leqq a \frac{\varepsilon_0\beta\gamma}{N^2} (128n^2 + 4n) \leqq \frac{130n^2}{N^2} a\varepsilon_0\beta\gamma \end{aligned} \tag{6.18}$$

in $\mathscr{U}'$. Furthermore,

$$\left| \sum_i a^i a_i \right| \leqq 2a \sum_i |a^{ii}|^{1/2}\left(2 \max_V |a_{ii}|^{1/2} + \frac{8n\varepsilon_0^{1/2}}{N} \sqrt{\beta\gamma}\right) .$$

From (3.14) we see that for d small

$$\begin{aligned} \sum |a^{ii}|^{1/2} \max_V |a_{ii}|^{1/2} &\leqq \sqrt{n^{1/2} \sum a^{ii} \max_V |a_{ii}|} \\ &\leqq \frac{2n^{1/2}}{N} \varepsilon_0\beta\gamma , \end{aligned}$$

and hence, using also (6.12), we find

$$\left| \sum_i a^i a_i \right| \leqq \frac{a\varepsilon_0}{N} (8n^{1/2} + 32n^2)\beta\gamma \leqq \frac{40n^2}{N} \varepsilon_0\beta\gamma a . \tag{6.19}$$

We need also the following statement: in view of (6.13), (6.12) and (6.18) imply

$$\left|\sum_j a^{ij}\dot{a}_j\right| \leqq 2\varepsilon_0\beta\gamma\sqrt{a}\left(2\sum_j \max_V |a_{jj}|^{1/2} + \frac{12n^2}{N}\sqrt{\varepsilon_0\beta\gamma}\right) \tag{6.20}$$

$$\leqq \frac{30n^2\varepsilon_0^{3/2}}{N}\sqrt{a}\,\beta\gamma\,.$$

Finally we observe that from (6.8) and (6.17) it follows that

$$|\sum \zeta_i a^i| \leqq 48n^2\varepsilon_0\beta\gamma\,, \tag{6.21}$$

and using also (6.16) and (6.12) we find

$$\left|\tau\sum_j a^{ij}\zeta_j\right| \leqq 12n\varepsilon_0\beta\gamma\,. \tag{6.21$'$}$$

6.4. With these tedious estimates, as well as some readers, behind us we may proceed to deal with I_0 to I_6. In doing so we remark that any terms involving ζ_y contribute H since the integration then is outside $\mathscr{M}_2$. Let us first take up the terms which make a positive contribution in (6.7)$'$; these are I_0, I_1 and I_4. Using Green's theorem, we have

$$I_1 = \frac{1}{2}\int_{\dot{\mathscr{M}}}\zeta a\tau^{2k-2}|D^{\alpha,k}u|^2\,dx - \tfrac{1}{2}(\zeta a_y D^{\alpha,k}u,\ \tau^{2k-2}D^{\alpha,k}u)$$

$$+\frac{k-1}{2}(\zeta a D^{\alpha,k}u,\ \tau^{2k-4}D^{\alpha,k}u) + H\,.$$

Now $a(x,y) = a(x,0) + ya_y(x,y) + O(|y|^2)$, and since, at $(x_1,0)$, $\beta + \frac{1}{2}Na_y > 0$, we see that, for $N \geqq 1$, $a_y \geqq -\frac{1}{4}$ there, and hence $a_y \geqq -\frac{1}{2}$ in $\mathscr{U}'$ for d small. Thus, since $y^2 \leqq \tau^4$,

$$\frac{k-1}{2}(\zeta a D^{\alpha,k}u,\ \tau^{2k-4}D^{\alpha,k}u) = \frac{k-1}{2}((\dot{a} + ya_y)D^{\alpha,k}u,\ \zeta\tau^{2k-4}D^{\alpha,k}u) + O(d'\kappa')$$

$$= -\frac{k-1}{2}(a_y(\dot{a} - y)D^{\alpha,k}u,\ \zeta\tau^{2k-4}D^{\alpha,k}u)$$

$$+\frac{k-1}{2}(\dot{a}(a_y + 1)D^{\alpha,k}u,\ \zeta\tau^{2k-4}D^{\alpha,k}u) + O(d'\kappa')$$

$$\geqq -\frac{k-1}{2}(a_y D^{\alpha,k}u,\ \zeta\tau^{2k-2}D^{\alpha,k}u)$$

$$+\frac{k-1}{4}(\dot{a}D^{\alpha,k}u,\ \zeta\tau^{2k-4}D^{\alpha,k}u) - \frac{k\,!}{20}\kappa'$$

for $d' = d^2$ sufficiently small. Here we have used the fact that $a_y + 1 \geqq \frac{1}{2}$ and that $\tau^2 = \dot{a} - y$. Inserting this estimate into the one above for I_1, we find, since $a = \tau^2$ on $y = 0$,

$$I_1 \geqq \frac{1}{2}\int_{\dot{\mathscr{M}}} \zeta\tau^{2k}\,|D^{\alpha,k}u|^2\,dx - \frac{k}{2}\,(a_y D^{\alpha,k}u,\ \zeta\tau^{2k-2}D^{\alpha,k}u)$$
$$(6.22) \qquad - \frac{1}{20}\,\kappa' + H + \frac{k-1}{4}\,(\dot{a}D^{\alpha,k}u,\ \zeta\tau^{2k-4}D^{\alpha,k}u)\,.$$

Turning next to I_4 we have, by Green's theorem,

$$I_4 = -\tfrac{1}{2}(a_i^i D^{\alpha,k}u,\ \zeta\tau^{2k-2}D^{\alpha,k}u) - \frac{k-1}{2}\,(a^i\dot{a}_i D^{\alpha,k}u,\ \zeta\tau^{2k-4}D^{\alpha,k}u)$$
$$-\tfrac{1}{2}(a^i\zeta_i D^{\alpha,k}u,\ \tau^{2k-2}D^{\alpha,k}u)\,.$$

Now since $k - 1 \leqq N$ and $a \leqq \tau^2$, we see from (6.19) that

$$\frac{k-1}{2}\,(a^i\dot{a}_i D^{\alpha,k}u,\ \zeta\tau^{2k-4}D^{\alpha,k}u) \leqq 20n^2\varepsilon_0(\zeta\beta\gamma D^{\alpha,k}u,\ \tau^{2k-2}D^{\alpha,k}u)\,.$$

Furthermore, since the support of ζ intersects at most 3^{n-1} regions $\mathscr{U}_i$ in each of which the values of $\beta\gamma$ and $D^{\alpha,k}u$ in the corresponding coordinate system are close to their values for the coordinate system of $\mathscr{U}$ (at the same point), we see with the aid of (6.21) that

$$\tfrac{1}{2}(a^i\zeta_i D^{\alpha,k}u,\ \tau^{2k-2}D^{\alpha,k}u) \leqq 24n^2 3^{n-1}\varepsilon_0(\beta\gamma D^{\alpha,k}u,\ \tau^{2k-2}D^{\alpha,k}u) + H$$
$$\leqq 25n^2\varepsilon_0 3^{n-1}k\,!\,\kappa + H \leqq 25n^2\varepsilon_0 3^{n-1}k\,!\,\kappa' + H\,.$$

The term H arises since the support of ζ may fall outside $\mathscr{M}_2$.

Inserting these estimates into the preceding expression for I_4, we find

$$I_4 \geqq H - \tfrac{1}{2}(a_i^i D^{\alpha,k}u,\ \zeta\tau^{2k-2}D^{\alpha,k}u)$$
$$- 20n^2\varepsilon_0(\zeta\beta\gamma D^{\alpha,k}u,\ \tau^{2k-2}D^{\alpha,k}u) - 15n^2\varepsilon_0 3^{n-1}k\,!\,\kappa'\,.$$

If we add this estimate to (6.22) and to the expression for I_0 and recall that

$$\beta\gamma = b - \tfrac{1}{2}a_i^i + (\tfrac{1}{2}N - 1)a_y \leqq b - \tfrac{1}{2}a_i^i + (\tfrac{1}{2}k - 1)a_y$$

at $(x_1, 0)$, so that

$$\tfrac{2}{3}\beta\gamma \leqq b - \tfrac{1}{2}a_i^i + (\tfrac{1}{2}k - 1)a_y \qquad \text{in} \qquad \mathscr{U}'\,,$$

we obtain the inequality

$$I_0 + I_1 + I_4 \geqq \tfrac{1}{2}(\zeta\beta\gamma D^{\alpha,k}u,\ \tau^{2k-2}D^{\alpha,k}u) + \tfrac{1}{2}\int_{\dot{\mathscr{U}}} \zeta\tau^{2k}\,|D^{\alpha,k}u|^2\,dx$$

$$+ H - (\tfrac{1}{20} + 15n^2\varepsilon_0\,3^{n-1})k\,!\,\kappa'$$

$$+ \frac{k-1}{4}\,(\dot{a}D^{\alpha,k}u,\ \zeta\tau^{2k-4}D^{\alpha,k}u)$$

for $\frac{1}{6} \geqq 20n^2\varepsilon_0$. Thus, dividing by $k\,!$, we have

$$\frac{1}{k\,!}\,(I_0 + I_1 + I_4) \geqq (\tfrac{1}{2} - \tfrac{1}{20} - 25n^2 3^{n-1}\varepsilon_0)\kappa' + H\,. \tag{6.23}$$

6.5. Estimates of I_2, I_3 and I_5. First, we have, since $a_j = \dot{a}_j + O(|y|) = \dot{a}_j + O(\tau^2)$,

$$I_2 = \alpha^j(\dot{a}_j D^{\alpha-\delta_j,k+1}u,\ \zeta\tau^{2k-2}D^{\alpha,k}u) + O(d\kappa')\,.$$

Hence, for d small, using (6.18) and the fact that $\Sigma\alpha_j \leqq N$, we see that

$$\frac{1}{k\,!}\,|I_2| \leqq \frac{1}{20}\,\kappa' + N\frac{12n\varepsilon_0}{N}\sum_j\left(\frac{\|\sqrt{\zeta\beta\gamma}\ \tau^{k-1}D^{\alpha,k}u\|}{\sqrt{k\,!}}\right)\left(\frac{\|\sqrt{\zeta}\ \tau^{k-1}D^{\alpha-\delta_j,k+1}u\|}{\sqrt{k\,!}}\right).$$

Now the last factor equals

$$\sqrt{\frac{2(k+1)}{k}}\sqrt{\frac{\frac{1}{2}k}{(k+1)\,!}}\ \|\sqrt{\zeta}\ \tau^{k-1}D^{\alpha-\delta_j,k+1}u\| \leqq 2\sqrt{\kappa'}\,.$$

Hence

$$\frac{1}{k\,!}\,|I_2| \leqq \frac{1}{20}\,\kappa' + 24n^2\varepsilon_0\kappa' \leqq \tfrac{1}{10}\kappa'\,. \tag{6.24}$$

Secondly, from (6.12), we find

$$\frac{1}{k\,!}\,|I_3| \leqq \frac{2\varepsilon_0}{N^2}\sum_{j,r}\alpha^j\alpha^r\,\frac{\|\sqrt{\zeta\beta\gamma}\ \tau^{k-1}D^{\alpha-\delta_j-\delta_r,k+1}u\|}{\sqrt{k\,!}}\ \frac{\|\sqrt{\zeta\beta\gamma}\ \tau^{k-1}D^{\alpha,k}u\|}{\sqrt{k\,!}}$$

$$\leqq \frac{2\varepsilon_0}{N^2}\sqrt{\frac{\kappa'}{k\,!}}\sum\alpha^j\alpha^r\,\|\sqrt{\zeta\beta\gamma}\ \tau^{k-1}D^{\alpha-\delta_j-\delta_r,k+1}u\|\,.$$

We apply Lemma 5.2, with $p = k - 1$ and $v = \sqrt{\zeta\beta\gamma}\, D^{\alpha-\delta_j-\delta_r,k+1}u$. For d small we find

$$\|\sqrt{\zeta\beta\gamma}\ \tau^{k-1}D^{\alpha-\delta_j-\delta_r,k+1}u\| \leqq \frac{2}{k}\,\|\sqrt{\zeta\beta\gamma}\ \tau^{k+1}D^{\alpha-\delta_j-\delta_r,k+2}u\| + H^{1/2}$$

$$\leqq \frac{2}{k}\sqrt{(k+2)\,!\,\kappa'} + H^{1/2}\,.$$

Inserting this into the preceding, we obtain

$$\begin{aligned}\frac{1}{k\,!}\,|I_3| &\leqq \frac{1}{20}\,\kappa' + \frac{4\varepsilon_0}{N^2}\,\kappa'\,\frac{\sqrt{(k+2)(k+2)}}{k}\cdot N^2 + H \\ &\leqq \tfrac{1}{10}\kappa' + H\,.\end{aligned} \tag{6.25}$$

And, if the reader is still with us, we obtain immediately, using (6.17)′ and the relation $\Sigma\alpha^j \leqq N$, the inequality

$$\frac{1}{k\,!}\,|I_5| \leqq 3\varepsilon_0\kappa\,. \tag{6.26}$$

6.6. Bound for I_6. By Green's theorem, we have

$$\begin{aligned}I_6 = &-(a^{ij}D^{\alpha,k-1}u_i\,,\ \zeta\tau^{2k-2}D^{\alpha,k}u_j) - (a^{ij}_jD^{\alpha,k-1}u_i\,,\ \zeta\tau^{2k-2}D^{\alpha,k}u) \\ &- (k-1)(a^{ij}\dot{a}_jD^{\alpha,k-1}u_i\,,\ \zeta\tau^{2k-4}D^{\alpha,k}u) \\ &- (a^{ij}\zeta_jD^{\alpha,k-1}u_i\,,\ \tau^{2k-2}D^{\alpha,k}u)\,.\end{aligned}$$

We observe that the second term on the right has one more τ factor than needed and so it is $O(d\kappa' k\,!)$. Hence we find, on integrating the first term by parts with respect to y,

$$\begin{aligned}I_6 = O(d\kappa' k\,!) &- \frac{1}{2}\int_{\mathscr{M}} a^{ij}D^{\alpha,k-1}u_i\zeta\tau^{2k-2}D^{\alpha,k-1}u_j\,dx \\ &- \frac{k-1}{2}\,(a^{ij}D^{\alpha,k-1}u_i\,,\ \zeta\tau^{2k-4}D^{\alpha,k-1}u_j) + H \\ &+ \tfrac{1}{2}(a^{ij}_yD^{\alpha,k-1}u_i\,,\ \zeta\tau^{2k-2}D^{\alpha,k-1}u_j) \\ &- (k-1)(a^{ij}\dot{a}_jD^{\alpha,k-1}u_i\,,\ \zeta\tau^{2k-4}D^{\alpha,k}u) \\ &- (a^{ij}\zeta_jD^{\alpha,k-1}u_i\,,\ \tau^{2k-2}D^{\alpha,k}u)\,,\end{aligned}$$

or

$$I_6 = O(d\kappa' k\,!) + H + J_1 + J_2 + J_3 + J_4 + J_5\,, \tag{6.27}$$

where we have denoted the integrals on the right by J_1 to J_5.

Using (6.12) and (6.13), we see that

$$\frac{1}{k\,!}|J_1| \leqq \tfrac{1}{4}\varepsilon_0\kappa'\,,$$

$$\frac{1}{k\,!}|J_2| \leqq \frac{k-1}{k}\,\frac{\varepsilon_0}{(k-1)\,!}\sum_{i,j}(\zeta\beta\gamma D^{\alpha,k-1}u_i\,,\ \tau^{2k-4}D^{\alpha,k-1}u_j) \leqq n^2\varepsilon_0\kappa'\,.$$

In J_3 there is an extra τ^2 factor, so

$$\frac{1}{k\,!}|J_3| \leqq O(d^2\kappa')\,.$$

To estimate J_4 we make use of (6.20):

$$\frac{1}{k\,!}|J_4| \leqq \frac{30n^2\varepsilon_0^{3/2}}{N}\sum_i(\sqrt{a}\ \zeta\beta\gamma D^{\alpha,k-1}u_i\,,\ \tau^{2k-4}D^{\alpha,k}u)\,\frac{k-1}{k\,!}\,.$$

Since $\sqrt{a} \leqq \tau$, we find

$$\frac{1}{k\,!}|J_4| \leqq 30n^3\varepsilon_0^{3/2}\kappa'\,.$$

The last term is bounded with the aid of (6.21)′; we recall that the support of ζ intersects at most 3^{n-1} of the $\mathscr{U}_i$; hence

$$\begin{aligned}\frac{1}{k\,!}|J_5| &\leqq \frac{12n\varepsilon_0}{k\,!}\sum_i\int_{\operatorname{supp}\zeta}[\beta\gamma D^{\alpha,k-1}u_i\tau^{2k-3}D^{\alpha,k}u]\,dV\\ &\leqq 12n3^{n-1}\varepsilon_0\kappa' + H\,,\end{aligned}$$

where the term H arises from that part of the support of ζ lying outside $\mathscr{M}_2$.

Inserting these estimates for J_1 to J_5 in (6.27), we have

$$\frac{1}{k\,!}|I_6| \leqq \tfrac{1}{10}\kappa' + H\,. \tag{6.28}$$

6.7. Completion of the proof of Proposition 6.1. Combining now (6.9) and our estimates for the I_j, (6.23)–(6.26), (6.28) and (6.10), we obtain the inequality

$$\begin{aligned}(\tfrac{1}{2} - \tfrac{1}{20} - 25n^2 3^{n-1}\varepsilon_0)\kappa' \leqq \tfrac{1}{10}\kappa' + \tfrac{1}{10}\kappa' + 3\varepsilon_0\kappa + \tfrac{1}{10}\kappa' + H + \tfrac{1}{10}\kappa'&\\ + C(\zeta D^{\alpha,k-1}f,\ \tau^{2k-2}D^{\alpha,k-1}f)&\end{aligned}$$

from which the desired inequality (6.7)′ follows.

We have proved the *a priori* estimate (6.4) for solutions of our boundary value problem, and we claim that the estimate also holds for the solution of our modified problem—obtained after elliptic regularization—with the estimate (6.4) independent of the parameters ε, δ. Indeed, we never used the fact that a vanishes on Σ_2. We only required that $a_i = 0$ on Σ_2 and that a satisfy our condition (d), (3.14), there. We *did* make use of the estimate (6.14) for $a(x, 0)$ which used the fact that $a(x_1, 0) = 0$. However (6.14) still holds provided ε, in the elliptic regularization, is sufficiently small. Thus Proposition 6.1 is completely proved.

7. Precise Formulation of Theorem 1

In this section we shall describe the remaining *a priori* estimates and our main result, the precise form of Theorem 1.

7.1. Estimates near Σ_{12}. Let $\mathcal{N} \subset \mathcal{N}'$ be neighborhoods in $\bar{\mathcal{M}}$ of Σ_{12} defined by

$$\mathcal{N}: 0 \leqq -y < d', \qquad \mathcal{N}': 0 \leqq -y < 2d',$$

with d' small. We use the same notation as in the preceding sections, and assume that conditions (a), (b) hold. Define

$$|f|_N^{\mathcal{N}'} = \left\{ \int_{\mathcal{N}'} \left[\sum_{|\alpha| \leqq N} |D^{\alpha,0} f|^2 + \sum_{|\alpha|+k \leqq N-1} |D^{\alpha,k} f|^2 \right] dV \right\}^{1/2}. \tag{7.1}$$

PROPOSITION 7.1. *The following a priori estimate holds for the solution u of the boundary value problem* (1.1), *provided d' is sufficiently small*:

$$\|u\|_N^{\mathcal{N}} \leqq K(|f|_N^{\mathcal{N}'} + \|u\|_{N-1}^{\mathcal{N}'} + \|u\|_0^{\mathcal{M}}) + C\,\|u\|^{\mathcal{N}'-\mathcal{N}}. \tag{7.2}$$

Proof: The proof is similar to that of Proposition 6.1 but much simpler. We shall merely indicate the main points. As before we use induction, assuming the result has been proved for smaller values of N, and denote by H terms that can be bounded by

$$K[|f|_N^{\mathcal{N}'} + \|u\|_{N-1}^{\mathcal{N}'} + \|u\|_0^{\mathcal{M}}]^2 + C[\|u\|_N^{\mathcal{N}'-\mathcal{N}}]^2.$$

On Σ_{12} we have, in terms of our special coordinates (with L given by (3.4)): $a = a^i = 0$ and $\beta = b - a_y < 0$. Since $a_y \leqq 0$ on Σ_{12} it follows that $b < 0$ there. After multiplication by a suitable factor we may assume that

$$b - \tfrac{1}{2}a_i^i + (k-1)a_y \leqq -1 \quad \text{in} \quad \mathcal{N}' \quad \text{for} \quad 0 \leqq k \leqq N. \tag{7.3}$$

As in Section 6, we cover $\mathscr{N}$ by a finite number of regions $\mathscr{U}_j$ of the form (6.5) but it is not necessary to take d small; we keep d fixed. We wish to prove that

$$\kappa = \max_{\substack{|\alpha|+k=N \\ j}} \int_{\mathscr{U}_j} |D^{\alpha,k}u|^2 \, dV \leqq H . \tag{7.4}$$

Choose $\mathscr{U} = \mathscr{U}_j$ as the one for which the maximum occurs.

According to Proposition 4.1 we may assert that

$$\lambda \int_{\mathscr{U}'} \sum_{|\alpha|=N} |D^{\alpha,0}u|^2 \, dV \leqq C\kappa + H , \tag{7.5}$$

where $\mathscr{U}'$ is the region $|x^i - x_0^i| < 2d$, $0 \leqq -y < 2d'$. Hence if we use Lemma 4.2, there exists for any $\varepsilon > 0$ a constant $C(\varepsilon)$ such that

$$\begin{aligned} \int_{\mathscr{U}} \sum_{\substack{|\alpha|+k=N \\ k<N}} |D^{\alpha,k}u|^2 \, dV \leqq \varepsilon \int_{\mathscr{U}} |D^{0,N}u|^2 \, dV \\ + C(\varepsilon) \int_{\mathscr{U}} \sum_{|\alpha|=N} |D^{\alpha,0}u|^2 \, dV + C(\varepsilon) \int_{\mathscr{U}} |u|^2 \, dV . \end{aligned} \tag{7.6}$$

We see that for λ large, (7.4) follows from (7.5) and the special case

$$\int_{\mathscr{U}} |D^{0,N}u|^2 \, dV \leqq H . \tag{7.4$'$}$$

Introducing ζ as in Section 6, we see that (7.4)$'$ will follow from the stronger inequality

$$\kappa' = \int \zeta \, |D^{0,N}u|^2 \, dV \leqq H . \tag{7.4$''$}$$

From (7.5) and (7.6) we have for λ large

$$\kappa \leqq 2\kappa' + H . \tag{7.4$'''$}$$

We now proceed as in Section 6: applying the operator $D^{\alpha,k-1}$ with $\alpha = 0$, $k = N - 1$ and $\tau = 1$, we obtain the representation (6.9), and we estimate the various I_j. We see first that

$$I_2 = I_3 = I_5 = 0 ,$$

while the estimate (6.10) holds for I_7.

The contribution from the principal terms I_0, I_2, I_4 now has the opposite sign to the one it had in Section 6. Since $\tau = 1$ and $a = 0$ on Σ_{12}, we have, on integrating by parts,

$$I_1 = -\tfrac{1}{2}(\zeta a_y D^{0,N}u, D^{0,N}u) + H ,$$

and

$$I_4 = -\tfrac{1}{2}(a_i^i \zeta D^{0,N}u, D^{0,N}u) - \tfrac{1}{2}(a^i \zeta_i D^{0,N}u, D^{0,N}u) .$$

Similarly, since $a^i = O(d')$ and $|(|\zeta_i|\, D^{0,N}u, D^{0,N}u)| \leqq C\kappa$, we see that for d' sufficiently small

$$I_4 \leqq -\tfrac{1}{2}(a_i^i \zeta D^{0,N}u, D^{0,N}u) + \tfrac{1}{20}\kappa .$$

Adding these estimates for I_4 and I_1 to the expression for I_0 we have, in virtue of (7.3),

$$\begin{aligned} I_0 + I_1 + I_4 &\leqq -(\zeta D^{\alpha,k}u, D^{\alpha,k}u) + \tfrac{1}{20}\kappa + H \\ &= -\kappa' + \tfrac{1}{20}\kappa + H . \end{aligned} \tag{7.7}$$

Finally, by Green's theorem, we have

$$\begin{aligned} I_6 = -(a^{ij}D^{0,N-1}u_i\, , \zeta D^{0,N}u_j) &- (a_j^{ij}D^{0,N-1}u_i\, , \zeta D^{0,N}u) \\ &- (a^{ij}\zeta_j D^{0,N-1}u_i\, , D^{0,N}u) . \end{aligned} \tag{7.8}$$

Using (7.6) which also holds when integrated over $\mathscr{U}'$ (containing the support of ζ), we see easily that the last two terms of (7.8) may be estimated by

$$\begin{aligned} &\tfrac{1}{40}\kappa + C\int_{\mathscr{U}'} \sum_{|\alpha|=N} |D^{\alpha,0}u|^2\, dV , \\ &\leqq \tfrac{1}{30}\kappa + H \end{aligned}$$

for large λ, in virtue of (7.5). Hence

$$I_6 \leqq -(a^{ij}D^{0,N-1}u_i\, , \zeta D^{0,N}u_j) + \tfrac{1}{10}\kappa + H .$$

Integrating by parts with respect to y, we find

$$\begin{aligned} I_6 \leqq -\frac{1}{2}\int_{\dot{\mathscr{M}}} &\zeta a^{ij} D^{0,N-1}u_i D^{0,N-1}u_j\, dx \\ &+ \tfrac{1}{2}(\zeta a_y^{ij} D^{0,N-1}u_i\, , D^{0,N-1}u_j) + \tfrac{1}{20}\kappa + H . \end{aligned} \tag{7.9}$$

With the aid of (7.6) in $\mathscr{U}'$ we find again that the second term on the right can be estimated by $\tfrac{1}{20}\kappa + H$. The first term on the right is nonpositive. Thus

$$I_6 \leqq \tfrac{1}{10}\kappa + H . \tag{7.10}$$

Inserting the estimates (7.7), (7.10) and the estimate (6.10) for I_7 into (6.9), and using (7.4)‴, we easily derive (7.4)″ and hence (7.2). We point out that the various constants d occurring in the derivation of the *a priori* estimates near the boundary are constants of the type C.

7.2. The complete a priori estimates. We shall now combine all the estimates which have been derived for different portions of $\mathscr{M}$. Let $\mathscr{M}_1$ be the neighborhood of Σ_{11} constructed in Section 5 and $\mathscr{M}_2$ that of Σ_2 of Section 6. Set $\mathscr{W} = \mathscr{M} - \mathscr{M}_1 - \mathscr{M}_2$. In $\mathscr{M}_1$ and $\mathscr{M}_2$ we have introduced the norms $\overline{\|u\|}_N^{\mathscr{M}_1}$, $\overline{|f|}_N^{\mathscr{M}_1}$ and $\widehat{\|u\|}_N^{\mathscr{M}_2}$, $\widehat{|f|}_N^{\mathscr{M}_2}$. Assuming, as usual, that conditions (a) to (d) hold, we have the following *a priori* estimate:

Proposition 7.2. *The solution u of our boundary value problem satisfies*

$$\overline{\|u\|}_N^{\mathscr{M}_1} + \widehat{\|u\|}_N^{\mathscr{M}_2} + \|u\|_N^{\mathscr{W}} \leqq K(\overline{|f|}_N^{\mathscr{M}_1} + \widehat{|f|}_N^{\mathscr{M}_2} + \|f\|_N^{\mathscr{W}}) \tag{7.11}$$

with some constant K.

Remark 7.1. It will be clear from Proposition 7.1 that (7.11) holds even if in the expression $\|f\|_N^{\mathscr{W}}$ we omit the normal derivatives $D_\nu^N f$ in a small neighborhood of Σ_{12}.

Proof: Using induction we may suppose that (7.11) has been established for smaller values of N. By (3.23) it clearly holds for $N = 0$.

Again we use the letter H to denote anything that may be estimated by the square of the right-hand side of (7.11). Let $\mathscr{N}$ be the neighborhood of Σ_{12} of Proposition 7.1 and set

$$\mathscr{M}_4 = \mathscr{W} - \mathscr{N}.$$

$\mathscr{W}$ consists of a compact subset of $\mathscr{M}$ plus a neighborhood of a compact subset of Σ_3. Applying the results of Section 4 to these sets, we see that if $\mathscr{M}_5$ is a neighborhood in $\bar{\mathscr{M}}$ of $\bar{\mathscr{M}}_4$, then

$$\lambda(\|u\|_N^{\mathscr{M}_4})^2 \leqq H + C(\|u\|_N^{\mathscr{M}_5 - \mathscr{M}_4})^2. \tag{7.12}$$

If $\mathscr{M}_5 - \mathscr{M}_4$ is very small, then the constant C may be very large.

Now we shall construct a suitable neighborhood $\mathscr{M}_5$. We shall divide each of the sets $\mathscr{M}_2$, $\mathscr{M}_1$ and $\mathscr{N}$ in two. Consider first $\mathscr{M}_2$; its boundary consists of a portion of $\Sigma_2 \cup \Sigma_3$ and a remaining part. Let d_0 denote the distance of the remaining part to Σ_2. Set

$$\mathscr{M}_2' = \text{set of points in } \mathscr{M}_2 \text{ within a distance } \tfrac{1}{2}d_0 \text{ of } \Sigma_2,$$

$$\mathscr{M}_2'' = \mathscr{M}_2 - \mathscr{M}_2'.$$

Similarly we write $\mathscr{M}_1 = \mathscr{M}_1' + \mathscr{M}_1''$, where $\mathscr{M}_1''$ does not touch Σ_{11}, and $\mathscr{N} = \mathscr{N}' + \mathscr{N}''$, where $\mathscr{N}''$ does not touch Σ_{12}. Now set

$$\mathscr{M}_5 = \mathscr{M}_4 + \mathscr{M}_2'' + \mathscr{M}_1'' + \mathscr{N}'' .$$

According to Proposition 6.1, and by our induction hypothesis, we have

$$(\|\hat{u}\|_N^{\mathscr{M}_2})^2 \leqq H + C(\|u\|_N^{\mathscr{M}_4})^2 . \tag{7.13}$$

In $\mathscr{M}_2''$ the term $\tau = \sqrt{a - y}$, whose power occurs as a weight factor in the norm, is bounded away from zero and therefore we may assert that

$$(\|u\|_N^{\mathscr{M}_2''})^2 \leqq H + C(\|u\|_N^{\mathscr{M}_4})^2 .$$

From Propositions 5.1 and 7.1 we also obtain the corresponding inequalities for $\mathscr{M}_1''$ and $\mathscr{N}''$ which may be combined with the preceding to give

$$(\|u\|_N^{\mathscr{M}_5-\mathscr{M}_4})^2 \leqq H + C(\|u\|_N^{\mathscr{M}_4})^2 ,$$

since $\mathscr{M}_2'' + \mathscr{M}_1'' + \mathscr{N}'' = \mathscr{M}_5 - \mathscr{M}_4$. If we now combine this with (7.12) for λ large, we obtain the inequality

$$(\|u\|_N^{\mathscr{M}_5})^2 \leqq H .$$

From (7.13), and the corresponding estimate in $\mathscr{M}_1$ and $\mathscr{N}$ the inequality (7.11) now follows; q.e.d.

7.3. The main existence theorem. The sharp form of Theorem 1 which we shall prove is

THEOREM 1′. *Assume that conditions* (a) *to* (d) *hold, for some given N. Let f be a function in $\mathscr{M}$ for which the sum of norms on the right-hand side of* (7.11) *is finite. Then there exists a unique solution u of* (1.1) *vanishing on $\Sigma_2 \cup \Sigma_3$ and satisfying* (7.11). *Consequently,*

(i) *in $\mathscr{W}$, u is contained in H_N,*
(ii) *in $\mathscr{M}_2$, u is contained in $H_{N/2}$ for N even, and in $H_{(N+1)/2}$ for N odd,*
(iii) *in $\mathscr{M}_1$, u is in $H_{N/2}$ for N even, and in $H_{(N-1)/2}$ for N odd.*

The properties (i), (ii), (iii) follow directly from (7.11) by repeated application of Lemma 5.2, and will not be discussed further.

Remark 7.2. To prove Theorem 1′ it suffices to prove it for operators L which are elliptic in the interior of $\mathscr{M}$, and for which Σ_1 consists entirely of Σ_{12}.

Proof of Remark 7.2: Let $\tilde{L}$ be a second order operator with real C^∞ coefficients in $\bar{\mathscr{M}}$ which is elliptic in $\mathscr{M}$ and such that the coefficients of the second-order terms and their first and second derivatives are all identically zero on $\dot{\mathscr{M}}$. Let $\tilde{L}$ be such that the corresponding $\tilde{\beta}$ is positive on Σ_2 and negative on Σ_1. For any constant $\eta > 0$ the operator

$$L_\eta = L + \eta\tilde{L}\,, \qquad 0 < \eta < 1\,,$$

is elliptic in the interior and satisfies the conditions (a) to (d). Furthermore, for L_η, Σ_1 consists entirely of Σ_{12}. Assuming that Theorem 1′ has been proved for such operators, we have a solution u_η for which (7.11) holds. For η small, (7.11) holds, in fact, with a constant K independent of η (here we ignore the fact that Σ_1 belongs entirely to Σ_{12} for L_η and simply apply (7.11) to the original $\Sigma_1 = \Sigma_{11} + \Sigma_{12}$). Letting $\eta \to 0$, we obtain in the usual way the solution of (1.1) satisfying (7.11); q.e.d.

This argument via L_η may be considered as a preliminary "regularization".

8. Proof of Theorem 1′

8.1. Regularization. In virtue of Remark 7.2 we may suppose that L is elliptic in $\mathscr{M}$ and that Σ_1 consists entirely of Σ_{12}. We may also suppose that $f \in C^\infty(\bar{\mathscr{M}})$; for, the given f may be approximated by such C^∞ functions f in the norm of the right side of (7.11).

We now consider our regularized problem (3.28): For $\varepsilon, \delta > 0$ we seek a real function $u_{\varepsilon,\delta}$ vanishing on $\Sigma_2 \cup \Sigma_3$ and satisfying

$$Q_{\varepsilon,\delta}(u_{\varepsilon,\delta}\,, v) = -(f, v) \tag{8.1}$$

for all real $v \in C^\infty(\bar{\mathscr{M}})$ which vanish on $\Sigma_2 \cup \Sigma_3$.

Denote by $\mathscr{K} = \mathscr{K}_{\varepsilon,\delta}$ the completion with respect to the norm

$$Q_{\varepsilon,\delta}(u, u)^{1/2} \tag{8.2}$$

of functions belonging to $C^\infty(\bar{\mathscr{M}})$ which vanish on $\Sigma_2 \cup \Sigma_3$. One readily verifies that, for functions u, v vanishing on $\Sigma_2 \cup \Sigma_3$,

$$|Q_{\varepsilon,\delta}(u, v)| \leqq \text{constant} \cdot \sqrt{Q_{\varepsilon,\delta}(u, u) \cdot Q_{\varepsilon,\delta}(v, v)}\ ;$$

and applying the well known Lax-Milgram lemma (see [3], p. 180), we see that there exists a unique function $u = u_{\varepsilon,\delta}$ in $\mathscr{K}$ satisfying (8.1).

In the region in $\bar{\mathscr{M}}$ where $\rho > 0$, $u = u_{\varepsilon,\delta}$ satisfies a corresponding elliptic equation of order $2N$, and hence (see, for instance, [21]) it belongs to C^∞ there and satisfies the "free" boundary conditions (3.24):

$$\left(\frac{\partial}{\partial y}\right)^j u = 0 \qquad \text{on} \qquad \Sigma_1\,, \qquad j = N, \cdots, 2N-1\,.$$

Furthermore, in $\bar{\mathscr{M}} - \operatorname{supp} \rho$, u satisfies a second order elliptic equation and so belongs to C^∞. But we do not assert that u belongs to C^∞ everywhere.

We shall derive estimates for u which are independent of δ, but may depend on ε. The corresponding constants will be denoted by $K(\varepsilon)$. Setting $v = u$ in (8.1) we have

$$Q(u, u) + \varepsilon Q_2(u, u) + \delta Q_1(u, u) \leqq \|f\| \cdot \|u\| , \tag{8.3}$$

and since λ is large, we infer that

$$\varepsilon Q_2(u, u) + \delta Q_1(u, u) \leqq K , \tag{8.4}$$

where K is a fixed constant independent of ε, δ. In addition we see from the ellipticity of L in $\mathscr{M}$ that

$$\|u_{\varepsilon,\delta}\|_1^{\mathscr{M}-\mathscr{M}_1} \leqq K(\varepsilon) , \tag{8.5}$$

where $\mathscr{M}_1$ is the strip $0 \leqq -y < d'$ near Σ_1. We shall also prove the estimate

$$\|u_{\varepsilon,\delta}\|_N^{\mathscr{M}_1} \leqq K(\varepsilon) . \tag{8.6}$$

First we show how to complete the proof of Theorem 1′ with the aid of (8.6). Take a sequence of values δ, δ_j tending to zero and set $u_{\varepsilon,\delta_j} = u_j$. Then from (8.4) we have

$$Q_1(u_j , u_j) \leqq \frac{K}{\delta_j} .$$

It follows that, for fixed v,

$$\begin{aligned} |\delta_j Q_1(u_j , v)| &\leqq \delta_j \sqrt{Q_1(u_j , u_j) \cdot Q_1(v, v)} \\ &\leqq \sqrt{\delta_j K Q_1(v, v)} \to 0 . \end{aligned} \tag{8.4}'$$

According to (8.5) and (8.6) we have

$$\|u_j\|_1 + \|u_j\|_N^{\mathscr{M}_1} \leqq K(\varepsilon) . \tag{8.7}$$

By the Banach-Saks theorem, there is then a subsequence of the u_j whose arithmetic means converge in the norm $\| \ \|_1 + \| \ \|_N^{\mathscr{M}_1}$ to a function u_ε for which (8.7) continues to hold. By (8.4)′ we see that u_ε satisfies

$$Q(u_\varepsilon , v) + \varepsilon Q_2(u_\varepsilon , v) = -(f, v) \tag{8.8}$$

for all $v \in C^\infty(\bar{\mathscr{M}})$ vanishing on $\Sigma_2 \cup \Sigma_3$.

The second order equation $L_\varepsilon u_\varepsilon = f$ corresponding to (8.8) is elliptic in $\mathscr{M} \cup \Sigma_2 \cup \Sigma_3$, and u_ε therefore belongs to C^∞ there. In addition, by (8.7), $\|u_\varepsilon\|_N < \infty$. Thus we are in a position to apply the *a priori* estimate (7.11) to u_ε as a solution of $L_\varepsilon u_\varepsilon = f$. As remarked in Section 6, for ε sufficiently small, Proposition 6.1 holds for solutions of such equations.

Finally we may let $\varepsilon \to 0$ and obtain by the usual argument our desired solution of (1.1) satisfying (7.11). Thus Theorem 1′ is proved, except for the proof of (8.6).

We remark that our method of "elliptic regularization" consists of three successive regularizations: first the one at the end of Section 7 which makes L elliptic in $\mathscr{M}$ and makes Σ_1 entirely of type Σ_{12}, then the regularization coming from (8.8) which renders the operator elliptic in $\mathscr{M} \cup \Sigma_2 \cup \Sigma_3$, and finally the last step involving the form (8.1).

8.2. Estimate for tangential derivatives. The proof of (8.6) requires further careful estimates and is tedious. We shall make use of the following lemma due to Friedrichs [10] (all functions below are real).

LEMMA 8.1. *Let*

$$P(u, v) = \int \sum_{|\alpha|,|\beta| \leqq m} P_{\alpha\beta}(x) D^\alpha u D^\beta v \, dV$$

be a formally positive elliptic bilinear form, i.e.,

$$\sum_{|\alpha|=|\beta|=m} P_{\alpha\beta} D^\alpha u D^\beta u > 0$$

for all $\{D^\alpha u\} \neq 0$, having C^∞ coefficients. Let $\zeta \geqq 0$ be a C^∞ function with compact support and $r \geqq 0$ an integer. There exist positive constants K_1, K_2 depending on P, ζ, r and k such that

$$(-1)^k P\Big(u, \zeta^{2(k+m+r)} \sum_{|\gamma|=k} D^{2\gamma} u\Big) + K_1 \|\zeta^r u\|^2 \geqq K_2 \sum_{|\alpha| \leqq k+m} \|\zeta^{r+|\alpha|} D^\alpha u\|^2 .$$

Lemma 8.1 is proved with the aid of Green's theorem and the following lemma (which we formulate in one x-dimension, and which is also proved using Green's theorem).

LEMMA 8.2. *Let $\zeta(x)$ be a C^∞ function with compact support and m a positive integer. For any $\eta > 0$ there exists a constant $A(\eta)$ depending on ζ, m and η such that*

$$\sum_{p<m} \|\zeta^p D^p u\|^2 \leqq \eta \|\zeta^m D^m u\|^2 + A(\eta) \int_{\operatorname{supp} \zeta} |u|^2 \, dx .$$

In $\mathscr{M}_1: 0 \leqq -y < d'$ we require that $\rho_y \equiv -1$. We also assume that $\rho \geqq 1$ in $\Omega_1: d' \leqq -y < 2d'$. Let $\psi \geqq 0$, $\psi \leqq 1$, be a C^∞ function with compact support in $\bar{\mathscr{M}}$ contained in the region $\rho > \frac{1}{2}$. With $v = \psi^{2(k+N)} \sum_{|\gamma|=k} D^{2\gamma} u$, formula (8.1) takes the form (see (3.27))

$$Q\left(u, \psi^{2(k+N)} \sum_{|\gamma|=k} D^{2\gamma} u\right) + \delta Q_1\left(u, \psi^{2(k+N)} \sum_{|\gamma|=k} D^{2\gamma} u\right) = -\left(f, \psi^{2(k+N)} \sum_{|\gamma|=k} D^{2\gamma} u\right).$$

Applying Lemma 8.1 to Q (with $r = N - 1$) and to Q_1 (with $r = 0$), we obtain the inequality

$$K_2\left(\sum_{|\alpha| \leqq k+1} \|\psi^{N-1+|\alpha|} D^\alpha u\|^2 + \delta \sum_{|\alpha| \leqq k+N} \|\psi^{|\alpha|} D^\alpha u\|^2\right)$$
$$\leqq 2K_1(\|u\|^{\mathscr{M}-\mathscr{M}_1})^2 + K(\varepsilon) \sum_{|\alpha| \leqq k} \|\psi^{N-1+|\alpha|} D^\alpha u\|^2 .$$

By (8.3) the first term on the right is bounded by $K(\varepsilon)$. Let us now require that $\psi \equiv 1$ on Ω_1. Then we can infer from the inequality that

$$\int_{\Omega_1} \left(\sum_{|\alpha| \leqq k+1} |D^\alpha u|^2 + \delta \sum_{|\alpha| \leqq k+N} \|D^\alpha u\|^2\right) dV \leqq K(\varepsilon), \qquad k \leqq 2N. \tag{8.9}$$

The constant $K(\varepsilon)$ which depends on the derivatives of ρ may be large.

Now turn to the region $\mathscr{M}_1$ abutting Σ_1. Because of the higher order term we have to re-do some of the work done in Section 7.1 with some care. We may cover $\mathscr{M}_1: 0 \leqq -y < d'$ by the usual regions $\mathscr{U}_j$ represented locally by

$$|x^i - x_0^i| < d, \qquad i = 1, \cdots, n-1,$$
$$0 \leqq -y < d'.$$

We introduce

$$\mathscr{U}': |x^i - x_0^i| < 2d, \quad 0 \leqq -y < 2d',$$
$$\mathscr{U}'': |x^i - x_0^i| < 3d, \quad 0 \leqq -y < 2d'.$$

Using induction on N we shall prove (assuming that it is true for smaller N)

$$\kappa = \max_j \left\{\int_{\mathscr{U}_j} \left[\sum_{|\alpha|+k=N} |D^{\alpha,k} u|^2 + \delta \sum_{|\alpha| \leqq N} |D^{\alpha,N-1} u|^2\right] dV\right\} \leqq K(\varepsilon). \tag{8.10}$$

Clearly (8.10), which holds for $N = 0$, implies (8.6).

Choose $\mathscr{U} = \mathscr{U}_j$ as the one for which the maximum in (8.10) occurs; introduce ζ_0 as in Section 6, with $\zeta_0 \equiv 1$ in $\mathscr{U}'$ and supp $\zeta_0 \subset \mathscr{U}''$, and with ζ_0 independent of y for $-y \leqq d'$. We first prove that

$$\lambda \sum_{|\alpha|=N} \|\zeta_0^N D^{\alpha,0} u\|^2 + \tfrac{1}{2}\delta \sum_{|\alpha|, k \leqq N} \|\zeta_0^N D^{\alpha,k} u\|^2 \leqq C\kappa + K(\varepsilon) + H. \tag{8.11}$$

We remark that any integrals over $d' \leqq -y \leqq 2d'$ can be estimated by (8.9) and so are bounded by $K(\varepsilon)$.

To establish (8.11) we set

$$v = (-1)^N \sum_{|\alpha|=N} D^{\alpha,0}\zeta_0^{2N} D^{\alpha,0} u$$

in (8.1):

$$(8.12)\quad \begin{aligned} &(-1)^N Q\Big(u, \sum_{|\alpha|=N} D^{\alpha,0}\zeta_0^{2N}D^{\alpha,0}u\Big) + (-1)^N \delta Q_1\Big(u, \sum_{|\alpha|=N} D^{\alpha,0}\zeta_0^{2N}D^{\alpha,0}u\Big) \\ &\qquad = (-1)^N\Big(f, \sum_{|\alpha|=N} D^{\alpha,0}\zeta_0^{2N}D^{\alpha,0}u\Big) . \end{aligned}$$

As in Section 4, it is convenient to employ Corollary 3.2 of [14] in treating the first term in (8.12). Applying this corollary to various terms (with $A = ((1/i)\zeta_0)^N D^{\alpha,0}$), we find that

$$\begin{aligned} &\Big| (-1)^N \sum_{|\alpha|=N} Q(u, D^{\alpha,0}\zeta_0^{2N}D^{\alpha,0}u) - \sum_{|\alpha|=N} Q(\zeta_0^N D^{\alpha,0}u, \zeta_0^N D^{\alpha,0}u) \Big| \\ &\qquad \leqq C\int_{\mathscr{U}'} \sum_{|\alpha|+k\leqq N} |D^{\alpha,k}u|^2\, dV \\ &\qquad \leqq C\kappa + K(\varepsilon) , \end{aligned}$$

by the definition of κ and by our induction hypothesis. In virtue of (3.23) we may conclude that

$$(8.12)'\quad (-1)^N \sum_{|\alpha|=N} Q(u, D^{\alpha,0}\zeta_0^{2N}D^{\alpha,0}u) \geqq \lambda \sum_{|\alpha|=N} \|\zeta_0^N D^{\alpha,0}u\|^2 - C\kappa - K(\varepsilon) .$$

The second term in (8.12) is handled just as in Lemma 8.1. We first observe, using (8.4), that

$$(8.4)''\quad \delta(\|u\|_N^{\mathscr{U}''})^2 \leqq K(\varepsilon) .$$

Consequently, using Green's theorem, we can prove that

$$\begin{aligned} \sum_{|\alpha|=N} \delta Q_1(u, D^{\alpha,0}\zeta_0^{2N}D^{\alpha,0}u) &= \delta \sum_{|\alpha|=N} \int [D_y^N u \cdot D_y^N(D^{\alpha,0}\zeta_0^{2N}D^{\alpha,0}u) \\ &\quad + \sum G^{\beta\gamma}(D_x^\beta u) D_x^\gamma (D^{\alpha,0}\zeta_0^{2N}D^{\alpha,0}u)]\rho\, \overline{dV} \\ &\quad + \delta\, O\Big(\sum_{|\alpha|=k=N} \|\zeta_0^N D^{\alpha,k}u\| \cdot \sum_{\substack{|\beta|\leqq r<N \\ j\leqq N \\ |\beta|+j>N}} \|\zeta_0^r D^{\beta,j}u\| + K(\varepsilon)\Big) . \end{aligned}$$

Applying Lemma 8.2 and (8.4)″ we see that, for $|\beta| \leqq r < N$, $|\beta| + j > N$, $j \leqq N$,

$$\delta \, \|\zeta_0^r D^{\beta,j} u\|^2 \leqq \eta\delta \sum_{|\alpha|=N} \|\zeta^N D^{\alpha,j} u\|^2 + A(\eta)K(\varepsilon)$$

for any $\eta > 0$, since $\zeta_0^r \leqq \zeta_0^{|\beta|}$. Hence we may infer easily, using the positive-definiteness of the integrand of Q_1, that

$$(-1)^N \delta Q_1\Big(u, \sum_{|\alpha|=N} D^{\alpha,0} \zeta_0^{2N} D^{\alpha,0} u\Big) \geqq \tfrac{1}{2}\delta \sum_{|\alpha|,k \leqq N} \|\zeta_0^N D^{\alpha,k} u\|^2 - K(\varepsilon) \, .$$

Inserting this and (8.12)′ in (8.12), and using Schwarz' inequality on the right-hand side, we obtain the inequality (8.11).

To estimate the other derivatives in (8.10) it will suffice to estimate purely normal ones. In fact in virtue of Lemma 4.2 we may assert that for any $\eta > 0$ there exists a constant $B(\eta)$ such that

$$\sum_{|\alpha|+k \leqq N} \int_{\mathscr{U}'} |D^{\alpha,k} u|^2 \, dV \leqq \eta \int_{\mathscr{U}'} |D^{0,N} u|^2 \, dV + B(\eta) \sum_{|\alpha|=N} \int_{\mathscr{U}'} (|D^{\alpha,0} u|^2 + |u|^2) \, dV \, .$$

Hence, with a different η and different $B(\eta)$, we find using (8.11) (recall that $\zeta = 1$ on $\mathscr{U}'$) for λ large

$$\text{(8.13)} \qquad \sum_{\substack{|\alpha|+k \leqq N \\ k \leqq N}} \int_{\mathscr{U}'} |D^{\alpha,k} u|^2 \, dV \leqq \Big(\eta + \frac{C}{\lambda} B(\eta)\Big)\kappa + B(\eta)K \, ,$$

where K is a bound for $\|u\|^2$.

With the aid of familiar interpolation inequalities, estimating derivatives in terms of higher and lower ones we have also

$$\sum_{|\beta| < N} \int_{\mathscr{U}'} |D^{\beta,k} u|^2 \, dV \leqq \eta \sum_{|\alpha|=N} \int_{\mathscr{U}'} |D^{\alpha,k} u|^2 \, dV + B(\eta) \int_{\mathscr{U}'} |D^{0,k} u|^2 \, dV$$

for any $\eta > 0$ and suitable constant $B(\eta)$. Consequently, we infer from (8.11) and (8.4)″ that for any $\eta > 0$ there is a (different) constant $B(\eta)$ such that

$$\text{(8.13)}' \qquad \delta \sum_{\substack{|\beta| < N \\ k \leqq N}} \int_{\mathscr{U}'} |D^{\beta,k} u|^2 \, dV \leqq \eta\kappa + B(\eta)K(\varepsilon) \, .$$

Similarly we find

$$\text{(8.13)}'' \qquad \delta \sum_{\substack{|\beta| \leqq N \\ k < N-1}} \int_{\mathscr{U}'} |D^{\beta,k} u|^2 \, dV \leqq \eta\kappa + B(\eta)K(\varepsilon) \, .$$

8.3. Normal derivatives. In terms of our special coordinates (x, y) near Σ_1, the quadratic $Q(u, v)$, for v with support in a special coordinate patch, has the form

$$Q(u, v) = \int \Big[a u_y v_y - \tfrac{1}{2}\beta(u_y v - u v_y) + a^{ij} u_i v_j + \beta^i (u_i v - u v_i) - c_0 uv - \frac{1}{2}\frac{\partial}{\partial y}(\beta u v g)\Big]\, dx\, dy\,, \tag{8.14}$$

where g is a C^∞ function with support near Σ_1 which equals one on Σ_1. Here a, b, etc., are suitable coefficients, and $\beta = (b - \frac{1}{2}a_i^i - a_y) < 0$ and $a = 0$, $a_y \leqq 0$ on Σ_1; as always $-c_0$ is large positive $\geqq 2\lambda$. We may suppose that (7.3) holds in any such coordinate patch:

$$\beta + k a_y = b - \tfrac{1}{2}a_i^i + (k-1)a_y \leqq -1 \qquad \text{for} \qquad 0 \leqq k \leqq N. \tag{8.15}$$

To prove (8.10) we now take up the main estimates—for normal derivatives. Let ζ be the function with support in $\mathscr{U}'$ which equals one on $\mathscr{U}$, as in Section 6. In (8.1) set

$$v = (-1)^{N-1}\zeta^2 D_y^{2N-1} u\,.$$

Then (8.1) takes the form

$$Q(u, v) + \delta(\rho D_y^N u, D_y^N v) + \delta \sum_{\alpha,\beta} (\rho G^{\alpha,\beta} D_x^\alpha u, D_x^\beta v) = -(f, v)\,, \tag{8.16}$$

where the element of volume in scalar products with δ as factor is of the form $\overline{dV} = J(x)\, dx\, dy$. Before estimating the various terms, we recall that any terms involving ζ_y are integrated only over Ω_1 and hence can be estimated by (8.9).

Consider first the term $Q(u, v)$, as given by (8.13). We shall estimate the various terms in (8.13). Integrating N times by parts with respect to y, we find first

$$\int a u_y v_y\, dV = (-1)^{N-1} \int a u_y D_y \zeta^2 D_y^{2N-1} u\, dV$$

$$= -\int D_y^N(a u_y) \cdot \zeta^2 D_y^N u\, dV + K(\varepsilon)\,,$$

(there are no boundary terms occurring because of the free boundary conditions (8.3))

$$= -\int a D_y^{N+1} u \cdot \zeta^2 D_y^N u\, dV - N \int a_y D_y^N u \cdot \zeta^2 D_y^N u\, dV$$

$$+ O(\|\zeta D_y^N u\|\, K(\varepsilon)) + K(\varepsilon)\,,$$

using the induction hypothesis to estimate the lower order terms. Hence, after another integration by parts, we have

$$\int a u_y v_y \, dV = (\tfrac{1}{2} - N) \int a_y \zeta^2 \, |D_y^N u|^2 \, dV + O(K(\varepsilon) \, \|\zeta D_y^N u\|) + K(\varepsilon) \; ; \tag{8.17}$$

no boundary term occurs because $a = 0$ on Σ_1.

Next, we may choose g to be identically one in $\mathcal{M}_1$. Then, in $\mathcal{M}_1$,

$$-\tfrac{1}{2}\beta(u_y v - u v_y) - \tfrac{1}{2} \frac{\partial}{\partial y} (\beta u v g) = -\beta u_y v - \tfrac{1}{2}\beta_y u v \, .$$

Hence, using partial integration, we find

(8.18)

$$-\frac{1}{2}\int \left[\beta(u_y v - u v_y) + \frac{\partial}{\partial y} \beta u v g \right] dV = (-1)^N \int [\beta u_y \zeta^2 D_y^{2N-1} u + \tfrac{1}{2}\beta_y u \zeta^2 D_y^{2N-1} u] \, dV$$

$$= -\int \beta \zeta^2 \, |D_y^N u|^2 \, dV + O(\|\zeta D_y^N u\| \, K(\varepsilon)) + K(\varepsilon) \; ;$$

no boundary terms occur because of (8.3).

Proceeding with the next term, and integrating by parts, we obtain

$$\int a^{ij} u_i v_j \, dV = (-1)^{N-1} \int a^{ij} u_i (\zeta^2 D_y^{2N-1} u)_j \, dV$$

$$\geqq \int a^{ij} D_y^{N-1} u_i \cdot \zeta^2 D_y^N u_j \, dV - \left(\eta + \frac{C}{\lambda} B(\eta) \right) \kappa - B(\eta) K(\varepsilon) \; ;$$

the errors arising from the terms involving derivatives of a^{ij} and ζ are estimated with the aid of (8.13),

$$\geqq \frac{1}{2} \int_{\mathcal{M}} a^{ij} D_y^{N-1} u \cdot D_y^{N-1} u_j \cdot \zeta^2 \, dV - \left(\eta + \frac{C}{\lambda} B(\eta) \right) \kappa - B(\eta) K(\varepsilon) \tag{8.19}$$

(using (8.13) again)

$$\geqq -\left(\eta + \frac{C}{\lambda} B(\eta) \right) \kappa - B(\eta) K(\varepsilon) \, . \tag{8.20}$$

Similarly one verifies easily that

$$\int \beta^i (u_i v - u v_i) - c_0 u v \geqq -\left(\eta + \frac{C}{\lambda} B(\eta) \right) \kappa - B(\eta) K(\varepsilon) \, .$$

Adding these estimates for the terms of (8.13) and using (8.15), we may conclude that

$$Q(u, v) \geqq \tfrac{1}{2} \|\zeta D_y^N u\|^2 - \left(\eta + \frac{C}{\lambda} B(\eta)\right) - B(\eta)K(\varepsilon) . \tag{8.21}$$

8.4. Normal derivatives (continued). The second term in (8.16) is

$$\begin{aligned}\delta(\rho D_y^N u, D_y^N v) &= (-1)^{N-1}\delta \int \rho D_y^N u \cdot D_y^N(\zeta^2 D_y^{2N-1} u) J(x)\, dx\, dy \\ &= -\delta(D_y^N(\rho D_y^N u), \zeta^2 D_y^{2N-1} u)\end{aligned}$$

with no boundary contribution because of (8.3),

$$\geqq -\delta(\rho D_y^{2N} u, \zeta^2 D_y^{2N-1} u) - N\delta(\rho_y D_y^{2N-1} u, \zeta^2 D_y^{2N-1} u) - K(\varepsilon)$$

since we have chosen ρ to be linear in $0 \leqq -y < d'$, and since outside this region we may use (8.9); integrating by parts once more (what else?) we see that this equals

$$-(N - \tfrac{1}{2})\delta(\rho_y D_y^{2N-1} u, \zeta^2 D_y^{2N-1} u) - K(\varepsilon) .$$

Now in $0 \leqq -y < d'$ we have $\rho_y \equiv -1$ and so we find

$$\delta(\rho D_y^N u, D_y^N v) \geqq (N - \tfrac{1}{2})\delta \|\zeta D_y^{2N-1} u\|^2 - K(\varepsilon) . \tag{8.22}$$

Finally, the third term in (8.16) is

$$\begin{aligned}I = \delta \sum (\rho G^{\alpha\beta} D_x^\alpha u, D_x^\beta v) &= (-1)^{N-1}\delta \sum (\rho G^{\alpha\beta} D_x^\alpha u, D_x^\beta \zeta^2 D_y^{2N-1} u) \\ &\geqq \delta \sum (D_y^{N-1}(\rho G^{\alpha\beta} D_x^\alpha u), D_x^\beta \zeta^2 D_y^N u) - K(\varepsilon) .\end{aligned}$$

Now $D_x^\beta \zeta^2 D_y^N u = \zeta^2 D_x^\beta D_y^N u + E$ and, by (8.13)$'$,

$$\delta \|E\|^2 \leqq \left(\eta + \frac{C}{\lambda} B(\eta)\right)\kappa + B(\eta)K(\varepsilon) .$$

Hence we may prove, using (8.13)$'$ again, that the third term satisfies, for any $\eta > 0$,

$$I \geqq \delta \sum (\zeta^2 D_y^{N-1}(\rho G^{\alpha,\beta} D_x^\alpha u), D^{\beta,N} u) - \left(\eta + \frac{C}{\lambda} B(\eta)\right)\kappa - B(\eta)K(\varepsilon) ,$$

$$\begin{aligned}I \geqq \delta \sum (\zeta^2 \rho G^{\alpha\beta} D^{\alpha,N-1} u, D^{\beta,N} u) + (N-1)\delta \sum (\zeta^2 \rho_y G^{\alpha\beta} D^{\alpha,N-2} u, D^{\beta,N} u) \\ - \left(\eta + \frac{C}{\lambda} B(\eta)\right)\kappa - B(\eta)K(\varepsilon) .\end{aligned}$$

Integrating by parts with respect to y, we see that the first term on the right of the last inequality equals

$$\tfrac{1}{2}\delta \sum \int_{\dot{\mathscr{M}}} \zeta^2 \rho G^{\alpha\beta} D^{\alpha,N-1}u \cdot D^{\beta,N-1}u J(x)\, dx$$
$$- \tfrac{1}{2}\delta \sum (\zeta^2 \rho_y G^{\alpha\beta} D^{\alpha,N-1}u,\, D^{\beta,N-1}u) + O(K(\varepsilon)) .$$

Since $\rho_y = -1$ for $-y < d'$, this is

$$\geqq \tfrac{1}{2}\delta \sum_{|\alpha| \leqq N} \int_{\dot{\mathscr{M}}} \zeta^2 \rho\, |D^{\alpha,N-1}u|^2\, J(x)\, dx + \tfrac{1}{2}\delta \sum_{|\alpha| \leqq N} \|\zeta D^{\alpha,N-1}u\|^2 - K(\varepsilon) .$$

Thus, for some constant $B(\eta)$ we find, on inserting this into the last estimate for I,

$$
\begin{aligned}
I \geqq \tfrac{1}{2}\delta \sum_{|\alpha| \leqq N} \int_{\dot{\mathscr{M}}} \zeta^2 \rho\, |D^{\alpha,N-1}u|^2\, J\, dx &+ \tfrac{1}{2}\delta \sum_{|\alpha| \leqq N} \|\zeta D^{\alpha,N-1}u\|^2 \\
&+ (N-1)\delta \sum (\zeta^2 \rho_y G^{\alpha\beta} D^{\alpha,N-2}u,\, D^{\beta,N}u) \\
&- \left(\eta + \frac{C}{\lambda} B(\eta)\right)\kappa - B(\eta) K(\varepsilon) .
\end{aligned}
\tag{8.23}
$$

We now integrate by parts the third term on the right of (8.23) and obtain an unfortunate boundary term:

$$
\begin{aligned}
I' &= (N-1)\delta \sum (\zeta^2 \rho_y G^{\alpha\beta} D^{\alpha,N-2}u,\, D^{\beta,N}u) \\
&= -(N-1)\delta \sum \int_{\dot{\mathscr{M}}} \zeta^2 G^{\alpha\beta} D^{\alpha,N-2}u \cdot D^{\beta,N-1}u\, J\, dx + O(K(\varepsilon)) .
\end{aligned}
$$

We may estimate this as follows, with some constant K:

$$|I'| \leqq \tfrac{1}{4}\delta \sum_{|\alpha| \leqq N} \int_{\mathscr{M}} \zeta^2 \rho\, |D^{\alpha,N-1}u|^2\, J\, dx$$
$$+ \delta K \int_{\dot{\mathscr{M}}} \zeta^2 \sum_{|\alpha| \leqq N} |D^{\alpha,N-2}u|^2\, J\, dx + K(\varepsilon) .$$

The boundary integral can be estimated with the aid of a simple well known result similar to Lemma 5.2 (we omit its proof).

Lemma 8.3. *For $g(y)$ defined on $-d' < y \leqq 0$,*

$$|g(0)|^2 \leqq \frac{1}{d'} \int_{-d'}^{0} 3\, |g|^2\, dy + d' \int_{-d'}^{0} |g_y|^2\, dy .$$

Applying this to the function $\zeta D^{\alpha,N-2}u$ and integrating also with respect to x, we find

$$\delta K \sum_{|\alpha|\leqq N}\int_{\mathscr{M}}\zeta^2\,|D^{\alpha,N-2}u|^2\,J(x)\,dx \leqq d'\delta K\sum_{\alpha}\|\zeta D^{\alpha,N-1}u\|^2 + K(\varepsilon)$$
$$+\frac{3}{d'}\,\delta K\sum_{\alpha}\|\zeta D^{\alpha,N-2}u\|^2\,.$$

If we require that d' be small, and then invoke (8.13)$''$, we see that this is

$$\leqq (\eta + d'K)\kappa + B(\eta)K(\varepsilon)\,.$$

Substituting this into the estimate for I', we find

$$|I'| \leqq \tfrac{1}{4}\delta \sum_{|\alpha|\leqq N}\int_{\mathscr{M}}\zeta^2\rho\,|D^{\alpha,N-1}u|^2\,J\,dx + (\eta + d'K)\kappa + B(\eta)K(\varepsilon)\,,$$

and inserting this for the third term in (8.23), we obtain the inequality

$$(8.24)\qquad \begin{aligned} I = \delta\sum(\rho G^{\alpha\beta}D^{\alpha}_{x}uD^{\beta}_{\alpha}v) &\geqq \tfrac{1}{2}\delta\sum_{|\alpha|\leqq N}\|\zeta D^{\alpha,N-1}u\|^2 \\ &\quad - \left(\eta + \frac{C}{\lambda}B(\eta) + d'K\right)\kappa - B(\eta)K(\varepsilon)\end{aligned}$$

for any $\eta > 0$ and some $B(\eta)$.

8.5. Completion of proof of (8.10). We now insert the estimates (8.21), (8.22) and (8.24) into (8.16) and obtain the inequality

$$(8.25)\qquad \begin{aligned} \|\zeta D^{N}_{y}u\|^2 + \delta(N-\tfrac{1}{2})\,\|\zeta D^{2N-1}_{y}u\|^2 + \delta\sum_{|\alpha|\leqq N}\|\zeta D^{\alpha,N-1}u\|^2 \\ \leqq \left(\eta + d'K + \frac{C}{\lambda}B(\eta)\right)\kappa + B(\eta)K(\varepsilon)\,.\end{aligned}$$

On $\mathscr{U}$ the function $\zeta \equiv 1$; hence if we add (8.25), (8.13) and (8.13)$'$ we find, from the definition of κ,

$$\kappa \leqq \left(\eta + d'K + \frac{C}{\lambda}B(\eta)\right)\kappa + B(\eta)K(\varepsilon)$$

for any $\eta > 0$. Choosing $\eta = \frac{1}{4}$ and then d', $1/\lambda$ sufficiently small, we obtain the desired inequality (8.10).

9. On Condition (b)

9.1. Ensuring condition (b). In general solutions of (1.1) will not be smooth if one simply omits the condition that $-c$ be large. This was indicated by the example (3.1). However, in many interesting cases this condition is not necessary. As we have seen, for parabolic equations it could always be achieved by multiplying the solution u by $e^{\lambda t}$ for some suitable constant λ. We shall first examine a more general situation in which this simple trick works.

Let $\mathscr{M}_0$ be the subset of $\bar{\mathscr{M}}$ in which the operator L is not elliptic. Since the solution is automatically smooth in the elliptic region the coefficient $-c$ is required to be large only in $\mathscr{M}_0$. With ψ a C^∞ function defined in a neighborhood of $\mathscr{M}_0$, and λ a positive constant, let us set

$$u = e^{\lambda\psi} v$$

in (1.1). Then in the region where ψ is defined, (1.1) takes the form

$$a^{ij}v_{ij} + (b^i + 2\lambda a^{ij}\psi_j)v_i + (c + \lambda a^{ij}\psi_{ij} + \lambda^2 a^{ij}\psi_i\psi_j + \lambda b^i\psi_i)v = fe^{-\lambda\psi} . \tag{9.1}$$

We want to make the coefficient of v large negative in $\mathscr{M}_0$ by taking λ large. Clearly we need $a^{ij}\psi_i\psi_j = 0$ in $\mathscr{M}_0$. Suppose then that there is a function ψ satisfying

$$a^{ij}\psi_i\psi_j \equiv 0 \qquad \text{in} \qquad \mathscr{M}_0 ; \tag{9.2}$$

then, since a^{ij} is semi-definite,

$$a^{ij}\psi_j = 0 \qquad \text{in} \qquad \mathscr{M}_0 , \tag{9.2$'$}$$

so that the coefficient of v_i in (9.1) is b^i. Then we still want

$$a^{ij}\psi_{ij} + b^i\psi_i < 0 \qquad \text{in} \qquad \mathscr{M}_0 . \tag{9.3}$$

We may conclude that the coefficient of v may be made arbitrarily large negative provided there exists a function satisfying (9.2) and (9.3) in $\mathscr{M}_0$.

In case $\mathscr{M}_0$ is the closure of its interior, (9.3) may be expressed in the form

$$(b^i - a^{ij}_j)\psi_i < 0 \qquad \text{in} \qquad \mathscr{M}_0 . \tag{9.3$'$}$$

For we find on differentiating (9.2)$'$ that

$$a^{ij}\psi_{ij} + a^{ij}_i\psi_j = 0 \qquad \text{in} \qquad \mathscr{M}_0 ,$$

which, together with (9.3), yields (9.3)$'$.

9.2. Regularity without condition (b). The device described above to render $-c$ large, works only in very restricted situations. Let us recall that we required $-c$ to be large in order to be able to dominate the contributions due to the derivatives of the coefficients when we integrated by parts in Sections 4, 5. Let us now see if we can impose some conditions which ensure that these error terms contribute something of the right sign. For first order equations conditions of this kind were imposed by Moser in [20] (see Sections 3, 4, 5 and, in particular, the last footnote on page 305).

Let us first consider the special case that at the boundary the operator L is elliptic—so as to avoid any difficulties there. Our conditions will be global and therefore instead of differentiating the equation (1.1) and then multiplying the result by a derivative of u which has been suitably cut down to have support in a neighborhood patch, we shall apply globally defined operators. We shall still assume that $-c$ is sufficiently large so that

$$Q(u, u) \geqq \|u\|^2 \tag{9.4}$$

for smooth u vanishing on $\Sigma_2 \cup \Sigma_3$.

Since we shall not make use of the results described here, our discussion will be rather sketchy, without proofs. Let

$$\alpha(x, \xi) = \alpha^{ij}\xi_i\xi_j \qquad \text{(positive definite)} \tag{9.5}$$

with C^∞ coefficients, be invariantly defined on the cotangent bundle (i.e., for any function ψ in $\mathscr{M}$, $\alpha^{ij}\psi_i\psi_j$ is well defined, independent of local coordinates). (In place of a positive definite quadratic one might find it useful in some case to work with a function $\alpha(x, \xi) > 0$ which is homogeneous in ξ of some positive degree, for $\xi \neq 0$.) Since the region near the boundary, where L is elliptic, is harmless, we may suppose that α is multiplied by some factor ζ which is one outside this region and has its support in $\mathscr{M}$. However we shall not bother to write this factor. In the equation $Q(u, v) = -(f, v)$ let us set $v = A^2u$, where A is the formally selfadjoint elliptic operator

$$A = -\frac{\partial}{\partial x^j}\left(\alpha^{ij}\frac{\partial}{\partial x^i}\right). \tag{9.5$'$}$$

We wish to estimate $Q(Au, Au)$. Integrating by parts, we have

$$-(Af, Au) = Q(u, A^2u) = Q(Au, Au) + \text{an error term}. \tag{9.6}$$

In Section 4 we estimated the error term with the aid of Corollary 3.2 of [14]. But we wish now to examine this term more closely. We have (ignoring boundary contributions with the aid of ζ)

$$\begin{aligned} Q(u, A^2u) &= -(A^2Lu, u), \\ Q(Au, Au) &= -(ALAu, u). \end{aligned} \tag{9.7}$$

Let L_s, L_a denote the symmetric and antisymmetric parts of L:

$$\begin{aligned} L_s &= \frac{\partial}{\partial x^j}\left(a^{ij}\frac{\partial}{\partial x^i}\right) + c - \tfrac{1}{2}(b^i_i - a^{ij}_{ij}) , \\ L_a &= (b^i - a^{ij}_j)\frac{\partial}{\partial x^i} + \tfrac{1}{2}(b^i - a^{ij}_j)_i . \end{aligned} \tag{9.8}$$

Then from the symmetry of L_s we have

$$\begin{aligned} (A^2L_su, u) = (L_sA^2u, u) &= \tfrac{1}{2}((A^2L_s + L_sA^2)u, u) \\ &= ((\tfrac{1}{2}A[A, L_s] - \tfrac{1}{2}[A, L_s]A)u, u) \\ &= \tfrac{1}{2}([A, [A, L_s]]u, u) , \end{aligned} \tag{9.9}$$

where [,] denotes the commutator.

In addition we have, by antisymmetry of L_a,

$$\begin{aligned} (A^2L_au, u) &= ((A^2L_a - AL_aA)u, u) \\ &= (A[A, L_a]u, u) . \end{aligned} \tag{9.10}$$

Inserting (9.9) and (9.10) into (9.7), we find

$$Q(u, A^2u) = Q(Au, Au) - \left(\{\tfrac{1}{2}[A, [A, L_s]] + A[A, L_a]\}u, u\right) . \tag{9.11}$$

Now the operator

$$\tfrac{1}{2}[A, [A, L_s]] + A[A, L_a]$$

is of fourth order. If one calculates the symbol of its highest order part, as in [13], it turns out to be

$$-\frac{1}{2}\left[\alpha_\xi\frac{\partial}{\partial x} - \alpha_x\frac{\partial}{\partial \xi}\right]^2 a^{ij}\xi_i\xi_j - \alpha\left[\alpha_\xi\frac{\partial}{\partial x} - \alpha_x\frac{\partial}{\partial \xi}\right](b^i - a^{ij}_j)\xi_i , \tag{9.12}$$

where we have set

$$\sum_k\left(\alpha_{\xi_k}\frac{\partial}{\partial x^k} - \alpha_{x^k}\frac{\partial}{\partial \xi_k}\right) = \alpha_\xi\frac{\partial}{\partial x} - \alpha_x\frac{\partial}{\partial \xi} .$$

Hence if this symbol is non-negative, the corresponding term will be harmless (by Gårding's inequality, see [31]). We observe however that $Q(Au, Au)$ controls not only Au (by (9.4)) but that we also have

$$Q(Au, Au) \geqq \|Au\|^2 + (a^{ij}(Au)_i , (Au)_j) . \tag{9.13}$$

With the aid of pseudo-differential operators, and [31], we may therefore see that the last term in (9.11) will contribute something harmless provided that the expression (9.12) is non-negative only for those ξ which satisfy

$$a^{ij}\xi_i\xi_j = 0 \, . \tag{9.14}$$

For other values of ξ the second term of (9.13) gives us good control. Using this function α one may prove

PROPOSITION 9.1. *Assume that L is elliptic at the boundary and that* (9.4) *holds. If there exists an invariant positive definite quadratic $\alpha = \alpha^{ij}\xi_i\xi_j$ in $\mathscr{M}$ such that the expression* (9.12) *is non-negative for all $\xi \neq 0$ satisfying $a^{ij}\xi_i\xi_j = 0$, then the solution u belongs to H_2 in $\bar{\mathscr{M}}$.*

We note that, whenever $a^{ij}(x)\xi_i\xi_j = 0$ in $\mathscr{M}$, the first derivatives vanish and the matrix of second derivatives of $a^{ij}\xi_i\xi_j$ is positive semi-definite there. Hence the first term of (9.12) is nonpositive at such a point. Thus the second term of (9.12) will have to be non-negative if (9.12) is to be.

Suppose now we want a solution u in H_{2m}; then we use A^m in place of A. The expression corresponding to (9.12) is then slightly different; we can prove:

PROPOSITION 9.2. *Assume that L is elliptic at the boundary and that* (9.4) *holds. Suppose that $\alpha = \alpha^{ij}\xi_i\xi_j$ of Proposition* 9.1 *is such that*

$$\tfrac{1}{2}m\left[\alpha_\xi \frac{\partial}{\partial x} - \alpha_x \frac{\partial}{\partial \xi}\right]^2 a^{ij}\xi_i\xi_j + \alpha\left[\alpha_\xi \frac{\partial}{\partial x} - \alpha_x \frac{\partial}{\partial \xi}\right][(b^i - a^{ij}_j)\xi_i] \leqq 0 \tag{9.15}$$

whenever $a^{ij}\xi_i\xi_j = 0$, $\xi \neq 0$. Then the solution u belongs to H_{2m}.

The proposition is proved by establishing the *a priori* estimates for

$$Q(A^j u, A^j u) \, , \qquad j = 1, \cdots, m \, , \tag{9.16}$$

(using induction). Then, using Theorem 1′, we may find smooth solutions of

$$Lu - \lambda u = f$$

for sufficiently large λ. To solve $Lu = f$ we write it as

$$(L - \lambda)u = -\lambda u + f$$

and solve this by iteration, using the *a priori* estimates (9.16)—as in Section 9.1 of [14].

Finally, what about the case that L is not elliptic at the boundary? We describe an analogous result. Near the boundary let $-y$ denote the distance from the boundary, as in Section 3.2; denote by $\mathscr{B}_\varepsilon$ the boundary strip $0 \leqq -y < \varepsilon$, and set $\mathscr{M}_\varepsilon = \mathscr{M} - \mathscr{B}_\varepsilon$. Then we impose the condition:

(e) there exists a quadratic α of (9.5) satisfying (9.15) which is positive definite in $\mathscr{M}_\varepsilon$, and which in $\mathscr{B}_\varepsilon$ satisfies $\alpha(x, \nu) = 0$ for a vector ν which is orthogonal to surfaces $y =$ constant, while $\alpha(x, \xi) > 0$ for any real vector ξ unequal to a multiple of ν.

PROPOSITION 9.3. *Assume that conditions* (a), (c), (d), (e) *hold with* $N = 4m$, *as well as* (9.4), *then there exists a unique solution of* (1.1) *vanishing on* $\Sigma_2 \cup \Sigma_3$ *belonging to* H_{2m}.

The proof makes use of pseudo-differential operators and results for coercive forms (Figueiredo [32]).

Propositions 9.1, 9.2 and 9.3 raise the question: *under what circumstances does there exist a function* α *satisfying the conditions that* (9.12) *be non-negative or that* (9.15) *hold when* $a^{ij}\xi_i\xi_j = 0$? These relations are global nonlinear inequalities for α which, we should point out, are invariant under coordinate change, and the conditions on the coefficients of L which ensure their solvability are not local. We do not understand the meaning of these conditions well, and can give no answer to the question. It is worth noting that, if we try to obtain C^∞ solutions with the aid of Proposition 9.2, then we want (9.15) to hold for all m. That means we would require

$$(915)' \quad \left(\alpha_\xi \frac{\partial}{\partial x} - \alpha_x \frac{\partial}{\partial \xi}\right)^2 a^{ij}\xi_i\xi_j = 0\,, \qquad \left(\alpha_\xi \frac{\partial}{\partial x} - \alpha_x \frac{\partial}{\partial \xi}\right)(b^i - a^{ij}_j)\xi_i \leqq 0\,,$$

wherever $a^{ij}\xi_i\xi_j = 0$, $\xi \neq 0$.

As an illustration of this result we present the following example without proof. Consider a bounded domain Ω in x-space, $x \in R^n$, with C^∞ boundary, and the finite cylinder $\mathscr{M}$ in x, t-space: $\mathscr{M} = \Omega \times (0, T)$. In $\bar{\mathscr{M}}$ consider the degenerate parabolic equation with C^∞ coefficients depending on x and t:

$$Lu = a^{ij}u_{ij} + b^iu_i + bu_t + cu = f\,, \qquad -c \ll 0\,,$$

where $a^{ij}u_{ij} + b^iu_i$ is an elliptic operator in the x-variable (with coefficients depending also on t). Assume that $b(x, 0) \geqq 0$ and $b(x, T) \leqq 0$; then the top and bottom of the cylinder belong to Σ_1, and we seek a C^∞ solution in $\bar{\mathscr{M}}$ with given values, say zero, on the side of the cylinder.

PROPOSITION 9.3′. *Assume that there exists a* C^∞ *function of t alone*, $g(t)$, *such that* $b(x, t)$ *satisfies*

$$(9.15)'' \qquad b_t(x, t) \leqq g(t)b(x, t)\,.$$

Then there exists a unique C^∞ *solution of the boundary value problem.*

The condition on b is not one for which hypoellipticity necessarily holds (see [11]). If $b_t \leqq 0$, then the condition is satisfied with $g = 0$; the result is then in accordance with our example (3.1)′. A function of the form $b(x, t) = (t_0 - t)\psi(x, t)$, with ψ a smooth positive function in $\bar{\mathcal{M}}$ and $0 \leqq t_0 \leqq T$, also satisfies the condition; we need only take $g = C(t_0 - t)$ with C a large constant.

Remark (added in proof). By direct treatment of the problem we can prove a more general result. In particular, in Proposition 9.3′ the function g may also depend on x.

9.3. Regularity in non-coercive problems of [14]. The remarks of the preceding section may also be used in obtaining smooth solutions for the boundary value problems of the type considered in [14]; these are associated with a bilinear form for a system of functions $u = (u^1, \cdots, u^N)$:

$$Q(u, v) = \int_{\mathcal{M}} \sum_{\substack{|\alpha|,|\beta| \leq 1 \\ i,j \leqq N}} a_{ij}^{\alpha\beta} D^\alpha u^i \cdot \overline{D^\beta v^j}\, dV .$$

We wish to describe a result related to Theorem 2′ and its corollary of [14], and we shall assume that the reader is familiar with the notation of [14]. Consider the boundary value problem (with m a fixed integer): Given $f \in H_m$, find $u \in B_m$ satisfying (1.4) of [14]:

$$Q(u, v) = (f, v) \text{ for all } v \in B . \tag{9.17}$$

We assume that Q satisfies conditions (i), (ii), (iii) of Section 1.2 of [14] and that all coefficients are C^∞ in $\bar{\mathcal{M}}$. In addition we shall assume that at the boundary the form $Q(u, u)$ is coercive under the given boundary conditions (see [32]). Clearly this avoids much of the difficulty. A corresponding result for the general case can also be given but it is much more complicated to state—it makes use of the general coercive results [32]—and will be omitted.

PROPOSITION 9.4. *Suppose that there exists a C^∞ function $\alpha(x, \xi)$, as in (9.5), which is positive definite in $\mathcal{M}$. Suppose furthermore that whenever $\xi = (\xi_1, \cdots, \xi_n) \neq 0$ is a real n-vector and $\eta = (\eta^1, \cdots, \eta^N) \neq 0$ a complex N-vector satisfying*

$$\sum_{\substack{\alpha=|\beta|=1 \\ i,j}} a_{ij}^{\alpha\beta}(x)\xi^{\alpha+\beta}\eta^i\overline{\eta^j} = 0 ,$$

then

$$m\left(\alpha_\xi \frac{\partial}{\partial x} - \alpha_x \frac{\partial}{\partial \xi}\right)^2 \sum_{\substack{|\alpha|=|\beta|=1 \\ i,j}} a_{ij}^{\alpha\beta}\xi^{\alpha+\beta}\eta^i\overline{\eta^j} - 2i\alpha\left(\alpha_\xi \frac{\partial}{\partial x} - \alpha_x \frac{\partial}{\partial \xi}\right) \times \sum_{\substack{|\alpha|+|\beta|=1 \\ i,j}} (a_{ij}^{\alpha\beta} - \overline{a_{ji}^{\alpha\beta}})\xi^{\alpha+\beta}\eta^i\overline{\eta^j} \leqq 0$$

for all $x \in \bar{\mathcal{M}}$. Then there exists a unique solution $u \in B$ of (9.17) which belongs to H_{2m}.

The proof makes use of the sharp Gårding inequality [31].

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