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On Degenerate Elliptic-Parabolic Operators of Second Order and Their Associated Diffusions*

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1. Introduction

This paper consists of two parts. In the first part we extend our earlier results [10] on the strong maximum principle to a broader class of operators, namely degenerate parabolic operators

$$\frac{\partial}{\partial t} + L_t ,$$

where $L_t = \frac{1}{2}\nabla \cdot (a(t, x)\nabla) + b(t, x) \cdot \nabla$ with a and b suitably smooth. This leads to a generalization of the results of M. Bony [1] that was sought by C. D. Hill [3]. It is also related to a recent result of M. Redheffer [8].

The second part of the paper is devoted to the study of the first boundary value problem for degenerate elliptic operators

$$L = \frac{1}{2}\nabla \cdot (a(x)\nabla) + b(x) \cdot \nabla - k(x)$$

in smooth regions. For example we show that if G is a smooth region in R^d and if g is a bounded continuous function on $\Sigma_2 \cup \Sigma_3$ (see [2], [4] or [5] for the definition of Σ_2 and Σ_3), then there is a unique $u \in L^\infty(G)$ such that

$$\text{(i)} \qquad \int uL^*v \, dx = 0 \quad \text{for all} \quad v \in C_0^\infty(G) ,$$

$$\text{(ii)} \qquad \operatorname*{ess\,lim}_{\substack{x \to a \\ x \in G}} u(x) = g(a) , \qquad a \in \Sigma_2 \cup \Sigma_3 ,$$

provided $\inf_{x \in G} k(x) > 0$. Under an additional assumption on G, a and b, we need assume only that $k(x) \geqq 0$. Moreover, there is a set F of measure zero such that for $x \notin F$ the solution $u(x)$ is the limit of solutions of perturbed problems. One can use a wide class of perturbations and the set F of measure zero depends

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only on the coefficients a, b and the region G. We also give sufficient conditions which ensure that F is empty.

The second half of Section 2 contains a description of the main results from the analytical as well as the probabilistic points of view.

The first part, consisting of Sections 3 and 4, is almost independent of Sections 5 through 8, which constitute the second part. Either part can be read independently of the other. Results from the first part are used only twice in the second part, and there in a rather weak form. Remarks following Theorem 5.1 and Theorem 6.3 explain how the results from the first part are used.

2. Background and Summary

Let $\Omega = C[[0, \infty), R^d]$ be the space of R^d-valued continuous functions on $[0, \infty)$. The value of the function $\omega \in \Omega$ at t is $x(t, \omega) = x_t(\omega) = \omega(t)$. M_t^s is the σ-field generated by the functions $x_u(\cdot)$ for $s \leqq u \leqq t$. M^s is the smallest σ-field containing M_t^s for all $t \geqq s$. M_t denotes M_t^0. Given

$$a:[0, \infty) \times R^d \to S_d \quad \text{and} \quad b:[0, \infty) \times R^d \to R^d,$$

we define

$$L_t = \frac{1}{2} \sum_{i,j=1}^{d} a^{ij}(t, x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b^i(t, x) \frac{\partial}{\partial x_i}. \tag{2.1}$$

A solution to the martingale problem for L_t starting at $(s, x) \in [0, \infty) \times R^d$ is a probability measure P on $\langle \Omega, M^s \rangle$ satisfying

$$P[x(s) = x] = 1, \tag{2.2}$$

and

$$f(x(t)) - \int_s^t (L_u f)(x(u))\, du \tag{2.3}$$

is a P-martingale for all $f \in C_0^\infty(R^d)$. We shall also refer to P as a solution to the martingale problem for a and b starting at (s, x).

THEOREM 2.1. *Let $a:[0, \infty) \times R^d \to S_d$ and $b:[0, \infty) \times R^d \to R^d$ be bounded measurable functions. Assume that $a \in C^{1,2}[[0, \infty) \times R^d]$ and that b is uniformly Lipschitz continuous with respect to the space variables on $[0, \infty) \times R^d$. Let L_t be as in (2.1). Then, for each $(s, x) \in [0, \infty) \times R^d$, there is exactly one probability* *measure* $P_{s,x}$ *on* $\langle\Omega, M^s\rangle$ *satisfying* (2.2) *and* (2.3). *Moreover, the family*

$$\{P_{s,x}: (s, x) \in [0, \infty) \times R^d\}$$

is a Feller continuous strong Markov process. Finally there is an enlargement $\hat{\Omega}$ *of* Ω *and an extension* $\hat{P}_{s,x}$ *of* $P_{s,x}$ *to* $\hat{\Omega}$ *such that*

$$x(t) = x + \int_s^t a^{1/2}(u, x(u))\, d\hat{\beta}(u) + \int_s^t b(u, x(u))\, du \qquad a.s.\ P_{s,x},$$

where $\hat{\beta}(t)$ *is a* $\hat{P}_{s,x}$*-Brownian motion.*

Theorem 2.1 is proved in [10]. Another fact proved in [10] which we want to use is

THEOREM 2.2. *If* a *and* b *are as in Theorem* 2.1 *and* $c:[0, \infty) \times R^d \to R^d$ *is bounded and measurable, then there is exactly one solution* $Q_{s,x}$ *to the martingale problem for* a *and* $b + a^{1/2} c$ *starting at* (s, x). *Moreover, the support of* $Q_{s,x}$ *is independent of* c.

COROLLARY 2.1. *Suppose* G *is an open subset of* $[0, \infty) \times R^d$. *Let* $a: G \to S_d$ *and* $b: G \to R^d$ *be bounded measurable functions such that* $a \in C^{1,2}(G)$ *and* b *satisfies a Lipschitz condition with respect to* x *uniformly on compact subsets of* G. *Given a bounded measurable function* $c: G \to R^d$, *set*

$$L_u = \tfrac{1}{2}\nabla \cdot a\nabla + (b + a^{1/2} c) \cdot \nabla ;$$

then for each $(s, x) \in G$ *there is a unique probability measure* $Q_{s,x}$ *on* $\langle\Omega, M^s_{\tau_s}\rangle$, *where* $\tau_s = \inf\{t \geqq s:\ (t, x(t)) \notin G\}$, *such that*

$$Q_{s,x}[x(s) = x] = 1, \tag{2.4}$$

and

$$f(t \wedge \tau_s, x(t \wedge \tau_s)) - \int_s^{t\wedge\tau_s} L_u f(u, x(u))\, du \tag{2.5}$$

is a $Q_{s,x}$ *martingale for all* $f \in C_b^{1,2}(G)$.

In [10] we proved (cf. Section 5) that if $\beta(t)$ is a W-Brownian motion on R^d and if

$$\eta(t) = x + \int_s^t \sigma(u, \eta(u))\, d\beta(u) + \int_s^t b(u, \eta(u))\, du,$$

then under certain assumptions on σ,

$$W\left(\sup_{s\leqq t\leqq T}\left|\int_s^t \sigma(u,\eta(u))\,d\beta(u) + \tfrac{1}{2}\int_s^t (\sigma'\sigma)(u,\eta(u))\,du\right| \geqq \varepsilon \;\Big|\; |||\beta(\cdot)|||_s^T < \delta\right)$$

tends to 0 as $\delta \to 0$ for each $\varepsilon > 0$, where

$$(\sigma'\sigma)^i = \sigma^{il}_{\ \ ,j}\,\sigma^{lj}\,,$$

and $|||\beta(\cdot)|||_s^T = \max_{1\leqq i\leqq d}\sup_{s\leqq t\leqq T}|\beta_i(t)|$. By an easy extension of the same argument we can prove the following result.

THEOREM 2.3. *Let* $\sigma, \tilde{\sigma}:[0,\infty)\times R^d \to R^d \otimes R^d$ *and* $b:[0,\infty)\times R^d\to R^d$ *be bounded measurable functions. Assume that* $\sigma \in C_b^{1,2}([0,\infty)\times R)$ *and suppose that*

$$\eta(t) = x + \int_s^t \sigma(u,\eta(u))\,d\beta(u) + \int_s^t \tilde{\sigma}(u,\eta(u))\,d\tilde{\beta}(u) + \int_s^t b(u,\eta(u))\,du\,,$$

where $\beta(\cdot)$ *and* $\tilde{\beta}(\cdot)$ *are independent W-Brownian motions on* R^d. *Then, for all* $\varepsilon > 0$ *and* $T > s$,

$$\lim_{\delta\to 0} W\left(\sup_{s\leqq t\leqq T}\left|\int_s^t \sigma(u,\eta(u))\,d\beta(u) + \tfrac{1}{2}\int_s^t (\sigma'\sigma)(u,\eta(u))\,du\right| \geqq \varepsilon \;\Big|\; |||\beta(\cdot)|||_s^T < \delta\right) = 0.$$

THEOREM 2.4. *Let* $a_i:[0,\infty)\times R^d\to S_d$ *and* $b_i:[0,\infty)\times R^d \to R^d$ *satisfy the conditions of Theorem* 2.1 *for* $i = 1, 2$. *Suppose* G *is an open set in* $[0,\infty)\times R^d$ *such that* $a_1 \equiv a_2$ *and* $b_1 \equiv b_2$ *in* G. *Denote by* $P^{(i)}_{s,x}$, $i = 1, 2$, *the solution to the martingale problem for* a_i *and* b_i *starting from* $(s, x) \in \bar{G}$. *Define*

$$\tau_s' = \inf\{t \geqq s : (t, x(t)) \notin \bar{G}\}\,.$$

Then

$$P^{(1)}_{s,x}[A] = P^{(2)}_{s,x}[A]$$

for all $A \in M^s_{\tau_s'}$.

Proof: The proof of the theorem is an easy application of Theorem 3.4 in [9]. Indeed let Q be the measure which results from patching $P^{(1)}_{s,x}$ and $P^{(2)}_{\tau_s',x(\tau_s')}$ together at time τ_s' as in Lemma 3.6 of [9]. One must show that $Q = P^{(2)}_{s,x}$. To do this let $\{G_n\}_1^\infty$ be a decreasing sequence of open sets containing $\bar{G}$ such that $\bar{G} = \bigcap_1^\infty G_n$. For each $n \geqq 1$, define

$$\tau^{(n)} = \inf\{t \geqq s : (t, x(t)) \notin G_n\}$$

and let $Q^{(n)}$ be the result of patching $P^{(1)}_{s,x}$ and $P^{(2)}_{\tau^{(n)},x(\tau^{(n)})}$ at time $\tau^{(n)}$. Then $\tau^{(n)} \downarrow \tau'$ and it is easy to check that $Q^{(n)}$ tends to both Q and $P^{(2)}_{s,x}$ as $n \to \infty$.

We shall now give a summary of the results contained in this paper.

In [10] we proved that the support in function space of the solution to the martingale problem $P_{s,x}$ corresponding to

$$L_t = \tfrac{1}{2}\sigma^*(t, x)\nabla \cdot \sigma^*(t, x)\nabla + b(t, x) \cdot \nabla$$

starting from (s, x) consists of the closure of the set of trajectories of the form

$$\phi(t) = x + \int_s^t \sigma(u, \phi(u))\psi(u)\, du + \int_s^t b(u, \phi(u))\, du\,, \qquad t \geqq s\,,$$

where ψ runs over smooth functions. In Section 3 this result is extended to operators of the form

$$L_t = \tfrac{1}{2}\nabla \cdot (a(t, x)\nabla) + b(t, x) \cdot \nabla \tag{2.6}$$

and it is shown (Theorem 3.2) that the support of the corresponding solution $P_{s,x}$ consists of the closure of the set of trajectories of the form

$$\phi(t) = x + \int_s^t a(u, \phi(u))\psi(u)\, du + \int_s^t b(u, \phi(u))\, du\,, \qquad t \geqq s\,, \tag{2.7}$$

where ψ runs over smooth functions. This enables us to prove a general version of the strong maximum principle (Theorem 4.1). Namely, if L_t is as in (2.6) and v is a function on $G \subset [0, \infty) \times R^d$ such that

$$\frac{\partial v}{\partial t} + L_t v \geqq 0 \tag{2.8}$$

and

$$v(s, x) = \sup_{(t,y)\in G} v(t, y)\,,$$

then $v(s, x) = v(s', x')$ for all $(s', x') \in A_{s,x}$. $A_{s,x}$ is the set in G containing (s, x) which is described as follows:

$$A_{s,x} = \text{closure}\,\{(t, \phi(t)) : t \geqq s,\ \phi(u) \in G \quad \text{for all} \quad s \leqq u \leqq t\}\,,$$

ϕ being as in (2.7) with ψ running over smooth functions. We also prove that $A_{s,x}$ is maximal. For this purpose we have to construct counterexamples. These are in general only generalized solutions of (2.8). From the probabilistic point of view the natural notion of a generalized solution is in terms of martingales. Solving the equation

$$\frac{\partial u}{\partial t} + L_t u = f$$

should be the same as finding a function u such that

$$u(t, x(t)) - \int_s^t f(\sigma, x(\sigma))\, d\sigma$$

is a $P_{s,x}$ martingale for all s, x. In Sections 4 and 5 we adopt this as our notion of a solution. Later in Section 8, we prove that it is equivalent to extending L_t by certain closure operations starting from L_t acting on smooth functions. See for instance Theorem 8.1.

The rest of the paper is devoted to the study of the elliptic first boundary value problem:

$$Lu - ku = -f \quad \text{in} \quad G,$$
$$u = g \quad \text{in} \quad \delta G,$$

where

$$L = \tfrac{1}{2} \sum a^{ij}(x) \frac{\partial^2}{\partial x_i\, \partial x_j} + \sum b^j(x) \frac{\partial}{\partial x_j},$$

$G \subset R^d$, and δG is the boundary of G. From the probabilistic point of view, once the problem is properly formulated one would expect the solution to be given by the formula

(2.9)
$$u(x) = E^{P_x}\left[\int_0^\tau \exp\left\{-\int_0^t k(x(s))\, ds\right\} f(x(t))\, dt + \exp\left\{-\int_0^\tau k(x(s))\, ds\right\} g(x(\tau))\right],$$

where $\tau(\omega)$ is the first exit time of the path $x(\cdot, \omega)$ from G. This is because (2.9) holds in the nondegenerate case. It is easy to verify that, under suitable assumptions, $u(x)$ is well defined by (2.9) and satisfies

$$Lu - ku = -f$$

in the martingale sense. The manner in which the boundary value is taken on is more difficult to describe. Obviously the boundary function has to be specified only at those points at which $x(\cdot, \omega)$ can exit. An additional difficulty is that although g, k and f are smooth functions, u need not even be continuous, due to the discontinuous nature of the functional $\tau(\omega)$.

Actually, instead of considering τ one can consider τ', the first exit time of the path $x(\cdot, \omega)$ from $\bar{G}$. There is no *a priori* reason why one should prefer one to the other. We have a similar formula for the solution

(2.10)
$$v(x) = E^{P_x}\left[\int_0^{\tau'} \exp\left\{-\int_0^t k(x(s))\, ds\right\} f(x(t))\, dt + \exp\left\{-\int_0^{\tau'} k(x(s))\, ds\right\} g(x(\tau'))\right]$$

when one uses τ'. The function $v(x)$ has as much right to be a solution as $u(x)$. In the nondegenerate case, $P_x[\tau = \tau'] = 1$ for all $x \in G$ so that the two solutions coincide. In the general case, this need not be true. If one wants the boundary value to be taken on continuously at a point $x_0 \in \delta G$, one needs

$$\lim_{\substack{x \to x_0 \\ x \in G}} P_x[\tau' \geqq \varepsilon] = 0 \quad \text{for any} \quad \varepsilon > 0 . \tag{2.11}$$

If one defines Γ as the set of points x_0 on δG such that (2.11) is true, one can show that, for g which are continuous and bounded,

$$\lim_{x \to x_0} v(x) = g(x_0) \qquad \text{for} \qquad x_0 \in \Gamma .$$

In order to show that it is enough to prescribe g on Γ, one has to show that

$$P_x[x(\tau') \notin \Gamma] = 0 \quad \text{for all} \quad x \in G .$$

This is also relatively easy. The analogous properties for τ are not true in general. Thus the τ' problem is better behaved than the τ problem.

It is nonetheless possible to formulate both problems in analytical terms so that the solutions are $u(x)$ and $v(x)$, respectively. This is done in Section 5.

If $\Sigma_2 \cup \Sigma_3 \subset \delta G$ is defined as in [2], [4] or [5], then we have

$$\Sigma_2 \cup \Sigma_3 \subset \Gamma \subset \overline{\Sigma_2 \cup \Sigma_3} .$$

Sections 6 and 7 deal with the above inclusions and give some criteria which help determine the set Γ precisely. We also prove in Section 7 that $P_x[\tau = \tau'] = 1$ for almost all x in G. This means that although $u(x)$ and $v(x)$ can be different they must agree almost everywhere. Moreover, at any point x where

$$P_x[\tau = \tau'] = 1$$

both u and v must be continuous and the solution is stable with respect to various perturbations. Even if one cannot determine Γ exactly, it is still enough to prescribe the boundary function on $\Sigma_2 \cup \Sigma_3$. This is made possible by the fact that

$$P_x[\tau = \tau' \quad \text{and} \quad x(\tau) \in \Sigma_2 \cup \Sigma_3] = 1$$

for almost all x in G. Therefore, $\Sigma_2 \cup \Sigma_3$ is a big enough set on the boundary to determine u almost everywhere. This is the basis of the uniqueness proved in Theorem 8.2 and Corollary 8.2.

3. The Support

It was proved in [10] that the support of the diffusion process starting from (t_0, x_0) corresponding to the operator

$$L_t = \tfrac{1}{2}\sigma^*(t, x)\nabla \cdot \sigma^*(t, x)\nabla + b(t, x) \cdot \nabla \tag{3.1}$$

consists of the closure of the set of trajectories having the form

$$\phi(t) = x_0 + \int_{t_0}^{t} \sigma(s, \phi(s))\psi(s)\, ds + \int_{t_0}^{t} b(s, \phi(s))\, ds, \tag{3.2}$$

where ψ runs over nice functions. Of course, if L_t is given in the form

$$L_t = \tfrac{1}{2} \sum a^{ij}(t, x) \frac{\partial^2}{\partial x_i\, \partial x_j} + \sum b^j(t, x) \frac{\partial}{\partial x_j}, \tag{3.3}$$

then one cannot always write it in the form (3.1) with a smooth choice of $\sigma(t, x)$. The purpose of this section is to extend the result proved in [10] for operators in the form (3.1) to those in the form (3.3).

The main idea behind the proof is that we notice first that σ need not be a $d \times d$ matrix in (3.1). It could be any $m \times d$ matrix so long as $\sigma^*\sigma = a$. While it is not true that an arbitrary operator with smooth coefficients in the form (3.3) can be always written in the form (3.1) with m equal d or with any other finite m, we prove that it can be written in the form (3.1) with smooth coefficients provided one takes $m = \infty$. This of course introduces technical difficulties regarding convergence. The approach we take overcomes these problems.

We start with a few lemmas.

LEMMA 3.1. *Assume that* $\Omega^{(i)} = C[[0, \infty), R^{d_i}]$, $i = 1, 2$, *and let us identify*

$$\Omega = C[[0, \infty), R^{d_1+d_2}]$$

with $\Omega^{(1)} \times \Omega^{(2)}$ (*i.e.*, $z(t) = (x(t), y(t))$); *let*

$$a:[0, \infty) \times \Omega \to S_{d_2} \quad \text{and} \quad b:[0, \infty) \times \Omega \to R^{d_2}$$

be bounded nonanticipating functions. Define

$$\bar{a}:[0, \infty) \times \Omega \to S_{d_1+d_2} \quad \text{and} \quad \bar{b}:[0, \infty) \times \Omega \to R^{d_1+d_2}$$

by

$$\langle \theta, \bar{a}\theta \rangle = |\lambda|^2 + \langle \mu, a\mu \rangle ,$$

$$\langle \theta, \bar{b} \rangle = \langle \mu, b \rangle ,$$

where $\theta = (\lambda, \mu)$ *is any element of* $R^{d_1+d_2}$. *Suppose* P *is a probability measure on* Ω *such that* $P[z(0) = 0] = 1$ *and*

$$Z_\theta(t) = \exp\left\{\left\langle \theta, z(t) - \int_0^t \bar{b}(u)\, du \right\rangle - \tfrac{1}{2}\int_0^t \langle \theta, \bar{a}(u)\theta \rangle\, du\right\}$$

is a P*-martingale for all* $\theta \in R^{d_1+d_2}$. *Denote by* P_w *the r.c.p.d. of* P *given the* σ*-field* $\mathscr{M}^{(1)}$ *generated by* $x(s)$ *for* $0 \leqq s < \infty$. *Then there is a set* $N \in \mathscr{M}^{(1)}$ *with* $P(N) = 0$ *such that, for* $w \notin N$,

(i) $$P_w[y(0) = 0] = 1 ,$$

(ii) *for all* $\mu \in R^{d_2}$,

$$Y(t) = \exp\left\{\left\langle \mu, y(t) - \int_0^t b(u)\, du \right\rangle - \tfrac{1}{2}\int_0^t \langle \mu, a(u)\mu \rangle\, du\right\}$$

is a P_w*-martingale.*

Proof: First we shall prove that for each $t_0 \geqq 0$

$$E^{P_w}[Y_\mu(t_0)] = 1 \qquad \text{a.s.}$$

Let us denote the function above by $H(w)$. We define a measure Q on $\mathscr{M}^{(1)}$ by

$$\frac{dQ}{dP} = H(w) .$$

Since $H(w)$ is obviously measurable with respect to $\mathscr{M}^{(1)}$ it suffices to show that Q on $\mathscr{M}^{(1)}$ is the same as P on $\mathscr{M}^{(1)}$, i.e., the Wiener measure. Since

$Q[x(0) = 0] = 1$, we have to establish that

$$X_\lambda(t) = \exp\{\langle\lambda, x(t)\rangle - \tfrac{1}{2}t|\lambda|^2\}$$

is a Q-martingale. Assume $0 \leqq t_1 \leqq t_2$ and let $A \in \mathscr{M}_{t_1}^{(1)}$. We have to prove that

$$E^Q[\chi_A\, X_\lambda(t_2)] = E^Q[\chi_A\, X_\lambda(t_1)]\,. \tag{3.4}$$

Case i. $0 \leqq t_1 \leqq t_2 \leqq t_0$.

$$\begin{aligned} E^Q[\chi_A\, X_\lambda(t_2)] &= E^P[\,Y_\mu(t_0)\,X_\lambda(t_2)\chi_A] \\ &= E^P[\,Y_\mu(t_0)\,X_\lambda(t_0)\chi_A] \\ &= E^P[Z_\theta(t_0)\chi_A] \\ &= E^P[Z_\theta(t_1)\chi_A] \\ &= E^P[X_\lambda(t_1)\,Y_\mu(t_1)\chi_A] \\ &= E^P[X_\lambda(t_1)\,Y_\mu(t_0)\chi_A] \\ &= E^Q[X_\lambda(t_1)\chi_A]\,. \end{aligned}$$

Here $\theta = (\lambda, \mu)$ and $X_\lambda(t)$, $Y_\mu(t)$ and $Z_\theta(t)$ are all P-martingales.

Case ii. $0 \leqq t_1 \leqq t_0 \leqq t_2$.

$$\begin{aligned} E^Q[\chi_A\, X_\lambda(t_2)] &= E^P[\chi_A\, X_\lambda(t_2)\, Y_\mu(t_0)] \\ &= E^P[\chi_A\, X_\lambda(t_0)\, Y_\mu(t_0)] \\ &= E^P[\chi_A\, Z_\theta(t_0)] \\ &= E^P[\chi_A\, Z_\theta(t_1)] \\ &= E^P[\chi_A\, X_\lambda(t_1)\, Y_\mu(t_1)] \\ &= E^P[\chi_A\, X_\lambda(t_1)\, Y_\mu(t_0)] \\ &= E^Q[\chi_A\, X_\lambda(t_1)]\,. \end{aligned}$$

Case iii. $0 \leqq t_1 \leqq t_0 \leqq t_2$.

$$\begin{aligned} E^Q[\chi_A\, X_\lambda(t_2)] &= E^P[\chi_A\, X_\lambda(t_2)\, Y_\mu(t_0)] \\ &= E^P[\chi_A\, X_\lambda(t_1)\, Y_\mu(t_0)] \\ &= E^Q[\chi_A\, X_\lambda(t_1)]\,. \end{aligned}$$

This proves (3.4) in all cases. Let P_w^s be the r.c.p.d. of P given $\mathscr{M}_s$. Then $X_\lambda(t)$, $Y_\mu(t)$ and $Z_\theta(t)$ are all martingales with respect to P_w^s for almost all w. Hence the same argument as before yields

$$E^{P_w^s}[Y_\mu(t) \mid \mathscr{M}^{(1)}] = Y_\beta(s) \qquad \text{for} \qquad t \geqq s\,.$$

By the usual argument one can end up with one exceptional set of measure zero which works for all times and μ simultaneously.

LEMMA 3.2. *Suppose $\beta(\cdot)$ and $\tilde{\beta}(\cdot)$ are two independent d-dimensional Q-Brownian motions and let σ, $\tilde{\sigma}:[0, \infty) \times R^d \to R^d \otimes R^d$ and $b:[0, \infty) \times R^d \to R^d$ be bounded measurable functions which are uniformly Lipschitz continuous with respect to space variables on $[0, \infty) \times R^d$. Further, assume that $\sigma \in C_b^{1,2}([0, \infty) \times R^d)$. Let $\eta(\cdot)$ be the solution of*

$$\eta(t) = x + \int_s^t \sigma(u, \eta(u))\, d\beta(u) + \int_s^t \tilde{\sigma}(u, \eta(u))\, d\tilde{\beta}(u) + \int_s^t b(u, \eta(u))\, du$$

for $s \leqq t \leqq T$, where $T > s$ is fixed. For any $\delta > 0$, let Q_δ be defined so that

$$Q_\delta(A) = \frac{Q(A \cap |||\beta|||_T^s < \delta)}{Q(|||\beta|||_T^s < \delta)} = Q(A \mid |||\beta|||_T^s < \delta)\,,$$

where

$$|||\beta|||_T^s = \sup_{\substack{s \leqq t \leqq T \\ 1 \leqq i \leqq d}} |\beta_i(t)|\,.$$

Denote by P_δ the distribution of $\eta(t)$ under Q_δ for $s \leqq t \leqq T$. Then

$$\lim_{\delta \to 0} P_\delta = P_0\,,$$

where P_0 is the unique solution of the martingale problem in $s \leqq t \leqq T$ corresponding to the coefficients $\tilde{a} = \tilde{\sigma}\tilde{\sigma}^*$ and $\tilde{b} = b - \frac{1}{2}\sigma'\sigma$, starting at time s from the point x.

Proof: There is no loss of generality in assuming that $x = 0$ and $s = 0$. We make the following definitions:

$$\xi(t) = \int_0^t \sigma(u, \eta(u))\, d\beta(u)\,,$$

$$\tilde{\xi}(t) = \int_0^t \tilde{\sigma}(u, \eta(u))\, d\tilde{\beta}(u)\,,$$

$y(t)$ is the solution of

$$y(t) = \int_0^t \tilde{\sigma}(u, y(u))\, d\tilde{\beta}(u) + \int_0^t \tilde{b}(u, y(u))\, du\,,$$

$$x(t) = -\tfrac{1}{2}\int_0^t (\sigma'\sigma)(u, y(u))\, du\,,$$

$$\tilde{x}(t) = \int_0^t \tilde{\sigma}(u, y(u))\, d\tilde{\beta}(u)\,,$$

and

$$\bar{x}(t) = -\tfrac{1}{2}\int_0^t (\sigma'\sigma)(u, \eta(u))\, du\,.$$

By Lemma 3.1, conditioning with respect to β does not affect the distributions of $y(\cdot)$ or $\tilde{\beta}(\cdot)$. Hence they have the same distribution under Q_δ as under Q. In particular, $y(\cdot)$ has P_0 for its distribution. It is therefore sufficient to prove that, for any $\varepsilon > 0$,

$$\lim_{\delta\to 0} Q_\delta(\|\eta - y\|_T^0 \geqq \varepsilon) = 0\,.$$

First let us observe that, by Theorem 2.3, for any $\varepsilon > 0$,

$$\lim_{\delta\to 0} Q_\delta(\|\xi - \bar{x}\|_T^0 \geqq \varepsilon) = 0\,.$$

Let us choose a large number M so that it dominates σ, $\tilde{\sigma}$, b, $\sigma'\sigma$ and their Lipschitz constants. Define

$$A_\delta(t) = E^{Q_\delta}[\{\min(\|\eta - y\|_T^0, M)\}^2]\,,$$

$$B_\delta(t) = E^{Q_\delta}[\{\min(\|\xi - x\|_T^0, M)\}^2]\,,$$

$$\tilde{B}_\delta(t) = E^{Q_\delta}(\{\min(\|\xi - \tilde{x}\|_T^0, M)\}^2]\,,$$

$$C_\delta(t) = E^{Q_\delta}[\{\min(\|\xi - \bar{x}\|_T^0, M)\}^2]\,.$$

Then

$$A_\delta(t) \leqq 8B_\delta(t) + 8\tilde{B}_\delta(t) + 8M^2 t\int_0^t A_\delta(u)\, du\,,$$

$$B_\delta(t) \leqq 2C_\delta(t) + 2M^2 t\int_0^t A_\delta(u)\, du\,,$$

$$\begin{aligned}\tilde{B}_\delta(t) &\leqq E^{Q_\delta}[\{\|\xi - \tilde{x}\|_T^0\}^2] \\ &\leqq 2E^{Q_\delta}[|\xi(\tau) - \tilde{x}(\tau)|^2] \\ &\leqq 2M^2\int_0^t A_\delta(u)\, du\,.\end{aligned}$$

Combining the above inequalities, we obtain

$$A_\delta(t) \leqq 16\, C_\delta(t) + k\int_0^t A_\delta(u)\, du\,,$$

where k is some constant. Since $C_\delta(t) \to 0$ as $\delta \to 0$, it follows that $A_\delta(t) \to 0$ as $\delta \to 0$. This concludes the proof.

LEMMA 3.3. *Let a, σ, $\tilde{\sigma}:[0, \infty) \times \Omega \to S_d$ and $b:[0, \infty) \times \Omega \to R^d$ be bounded nonanticipating functions. Assume that, for each t and w, $a(t, w)$, $\sigma(t, w)$ and $\tilde{\sigma}(t, w)$ commute. Further, let $a = \sigma^2 + \tilde{\sigma}^2$. Let P be a probability measure on Ω with the property that*

$$f(x(t)) - \int_0^t (L_s f)(x(s))\, ds$$

is a P-martingale for all $f \in C_0^\infty(R^d)$, where

$$L_s = \frac{1}{2}\sum_{i,j=1}^d a^{ij}(s)\frac{\partial^2}{\partial x_i\, \partial x_j} + \sum_{i=1}^d b^i(s)\frac{\partial}{\partial x_i}\,.$$

Then there is an enlargement $\hat{\Omega}$ of Ω and an extension $\hat{P}$ of P to $\hat{\Omega}$ such that

$$x(t) = x(0) + \int_0^t \sigma(s)\, d\beta(s) + \int_0^t \tilde{\sigma}(s)\, d\tilde{\beta}(s) + \int_0^t b(s)\, ds \qquad a.s.\ \ \hat{P}\,,$$

where β and $\tilde{\beta}$ are independent d-dimensional $\hat{P}$-Brownian motions.

Proof: There is an enlargement Ω' of Ω and an extension P' of P to Ω' such that

$$x(t) - x(0) = \int_0^t a^{1/2}(s)\, d\alpha(s) + \int_0^t b(s)\, ds\,,$$

where $\alpha(s)$ is a d-dimensional P'-Brownian motion. Define

$$\rho = \lim_{\varepsilon \downarrow 0} a^{1/2}(a + \varepsilon I)^{-1}\,.$$

Then $\rho a^{1/2} = a^{1/2}\rho = \pi_R$, where π_R is the orthogonal projection onto the range of a. Let $\pi_N = I - \pi_R$ be the orthogonal projection onto the null space of a. Define

$$U^{11} = U^{22} = \rho\sigma + \frac{1}{\sqrt{2}}\pi_N\,,$$

$$U^{12} = -U^{21} = \rho\tilde{\sigma} + \frac{1}{\sqrt{2}}\pi_N\,;$$

then the $2d \times 2d$ matrix

$$U = \begin{bmatrix} U^{11} & U^{12} \\ U^{21} & U^{22} \end{bmatrix}$$

is an orthogonal matrix. Let $\hat{\Omega} = \Omega' \times \tilde{\Omega}$ and $\hat{P} = P' \times \tilde{P}$, where $(\tilde{\Omega}, \tilde{P})$ admits the Brownian motion $\tilde{\alpha}(t)$. Let us define

$$\begin{bmatrix} \beta(t) \\ \tilde{\beta}(t) \end{bmatrix} = \int_0^t U(s) \begin{bmatrix} d\alpha(s) \\ d\tilde{\alpha}(s) \end{bmatrix}.$$

Then β and $\tilde{\beta}$ are independent $\hat{P}$-Brownian motions. Moreover,

$$\int_0^t \sigma(s)\, d\beta(s) = \int_0^t \sigma(s) U^{11}(s)\, d\alpha(s) + \int_0^t \sigma(s) U^{12}(s)\, d\tilde{\alpha}(s)\,,$$

$$\int_0^t \tilde{\sigma}(s)\, d\tilde{\beta}(s) = \int_0^t \tilde{\sigma}(s) U^{21}(s)\, d\alpha(s) + \int_0^t \tilde{\sigma}(s) U^{22}(s)\, d\tilde{\alpha}(s)\,.$$

Therefore,

$$\int_0^t \sigma(s)\, d\beta(s) + \int_0^t \tilde{\sigma}(s)\, d\tilde{\beta}(s)$$
$$= \int_0^t (\sigma U^{11} + \tilde{\sigma} U^{21})(s)\, d\alpha(s) + \int_0^t (\sigma U^{12} + \tilde{\sigma} U^{22})(s)\, d\tilde{\alpha}(s)$$
$$= \int_0^t a^{1/2}(s)\, d\alpha(s)\,.$$

Indeed, one need only note that

$$\sigma U^{11} + \tilde{\sigma} U^{21} = a^{1/2}$$

and

$$\sigma U^{12} + \tilde{\sigma} U^{22} = 0\,.$$

THEOREM 3.1. *Let a, σ, $\tilde{\sigma}:[0, \infty) \times R^d \to S_d$ and b, $c:[0, \infty) \times R^d \to R^d$ be bounded measurable functions which are uniformly Lipschitz continuous in the space variables on $[0, \infty) \times R^d$. Let us assume further that $a, \sigma \in C_b^{1,2}([0, \infty) \times R^d)$, that a, σ and $\tilde{\sigma}$ commute and that $a = \sigma^2 + \tilde{\sigma}^2$. Let $\tilde{a} = \tilde{\sigma}^2$ and $\tilde{b} = b + a^{1/2} c - \frac{1}{2}\sigma'\sigma$. If $P_{s,x}$ and $\tilde{P}_{s,x}$ are the solutions to the martingale problems for $[a, b]$ and $[\tilde{a}, \tilde{b}]$, respectively, starting from x at time s, then*

$$\operatorname{supp}(\tilde{P}_{s,x}) \subset \operatorname{supp}(P_{s,x})\,.$$

Proof: Since the support of $P_{s,x}$ is the same as the support of the solution $Q_{s,x}$ to the martingale problem for $[a, b + a^{1/2} c]$, we can assume that $c = 0$. In that case the theorem is an immediate consequence of Lemmas 3.2 and 3.3.

COROLLARY 3.1. *Let $a \in C_b^{1,2}([0, \infty) \times R^d, S_d)$ and let $b:[0, \infty) \times R^d \to R^d$ be bounded and uniformly Lipschitz continuous in space. Given a bounded measurable*

$$\psi:[0, \infty) \to R^d\,,$$

define $\phi(\cdot)$ for $t \geqq s$ by

$$\phi(t) = x + \int_s^t a(u, \phi(u))\psi(u)\, du + \int_s^t \bar{b}(u, \phi(u))\, du\,,$$

where $\bar{b} = b - \frac{1}{2}a'$. Then $\phi \in \operatorname{supp}(P_{s,x})$, where $P_{s,x}$ solves the martingale problem for $[a, b]$ starting from x at time s.

Proof: Clearly we can assume without loss of generality that the trace of a is at most $\frac{1}{2}$. Define $\sigma_0 = a$ and, for $n \geqq 1$,

$$\sigma_n = \sigma_{n-1} - \sigma_{n-1}^2 \quad \text{and} \quad b_n = b + a\psi - \tfrac{1}{2}\sum_0^{n-1} (\sigma_j^2)' .$$

Let $P_{s,x}^{(n)}$ be the solution to the martingale problem for $[\sigma_n, b_n]$ starting from x at time s. Then, by induction, we see from the preceding theorem that for all n

$$\text{supp}\,(P_{s,x}^{(n)}) \subset \text{supp}\,(P_{s,x}) .$$

Moreover, by Theorem 10.2 of the appendix, $\sigma_n \to 0$ and $b_n \to b + a\psi - \frac{1}{2}a'$ uniformly on compacts as $n \to \infty$. Therefore, $P_{s,x}^{(n)}$ converges weakly as $n \to \infty$ to the distribution degenerate at the path ϕ. In particular,

$$\phi \in \text{supp}\,(P_{s,x}) .$$

Given coefficients $[a, b]$ as in the preceding corollary, let us define $S_{a,b}(s, x)$ to be the set of $\phi:[s, \infty) \to R^d$ such that

$$\phi(t) = x + \int_s^t a(u, \phi(u))\psi(u)\, du + \int_s^t b(u, \phi(u))\, du , \qquad t \geqq s ,$$

for some bounded measurable function $\psi:[s, \infty) \to R^d$.

THEOREM 3.2. *Let $[a, b]$ satisfy the conditions of Corollary 3.1. Given a bounded measurable $c:[0, \infty) \times R^d \to R^d$, let*

$$L_t = \tfrac{1}{2}\nabla \cdot a\nabla + (b + a^{1/2} c) \cdot \nabla ,$$

and denote by $P_{s,x}$ the solution to the martingale problem for L_t starting from x at time s. Then

$$\text{supp}\,(P_{s,x}) = \overline{S_{a,b}(s, x)} .$$

Proof: We can assume that $c = 0$. The inclusion $\overline{S_{a,b}(s, x)} \subseteq \text{supp}\,(P_{s,x})$ follows at once from Corollary 3.1. To prove the opposite inclusion, let us assume that the trace of a is at most $\frac{1}{2}$ and define σ_n, for $n \geqq 0$, as in Corollary 3.1. Let $a_n = \sum_0^n \sigma_j^2$ and

$$L_t^{(n)} = \tfrac{1}{2}\nabla \cdot a_n \nabla + b \cdot \nabla .$$

If $P_{s,x}^{(n)}$ is the solution to the martingale problem corresponding to $L_t^{(n)}$, starting from x at time s, then we know that

$$\operatorname{supp}(P_{s,x}^{(n)}) \subset \overline{S_{a_n,b}(s, x)} .$$

Moreover,

$$\overline{S_{a_n,b}(s, x)} \subset \overline{S_{a,b}(s, x)} \quad \text{for all} \quad n .$$

Since $P_{s,x}^{(n)}$ converges weakly to $P_{s,x}$ as $n \to \infty$, we have

$$\operatorname{supp}(P_{s,x}) \subset \overline{S_{a,b}(s, x)} .$$

Remark 3.1. Suppose that G is an open set in $[0, \infty) \times R^d$ and that a, b and c are coefficients satisfying the conditions of Corollary 2.1. Let τ_s and $Q_{s,x}$, $(s, x) \in G$, be defined as in that corollary. Given $T > s$ such that

$$Q_{s,x}(\tau_s \geqq T) > 0 ,$$

let $Q_{s,x}^T$ denote the measure on $C([s, T], R^d)$ obtained by conditioning $Q_{s,x}$ with respect to $\{\tau_s \geqq T\}$. Then an easy consequence of the results of this section is that $\operatorname{supp}(Q_{s,x}^T)$ coincides with the closure in $C([s, T], R^d)$ of the paths $\phi:[s, T] \to R^d$ whose graph lies in G and for which there is a bounded measurable $\psi:[s, T] \to R^d$ such that

$$\phi(t) = x + \int_s^t a(u, \phi(u))\psi(u)\, du + \int_s^t b(u, \phi(u))\, du , \qquad t \in [s, T] .$$

4. The Strong Maximum Principle

Let G be an open set in $[0, \infty) \times R^d$ and suppose a, b and c are coefficients satisfying the conditions of Corollary 2.1. Define

$$L_u = \tfrac{1}{2}\nabla \cdot a\nabla + (b + a^{1/2} c) \cdot \nabla$$

and define τ_s and $Q_{s,x}$, $(s, x) \in G$, as in Corollary 2.1. Given $(s, x) \in G$, consider the measures

$$q(s, x;\ t, \Gamma) = Q_{s,x}(x(t) \in \Gamma ,\quad \tau_s > t). \tag{4.1}$$

Define $A(s, x)$ to be the closure in G of the points $(t, \phi(t))$, where $t \geqq s$ and $\phi:[s, t] \to R^d$ is a path whose graph lies in G and for which there is a bounded measurable $\psi:[s, t] \to R^d$ such that

$$\phi(u) = x + \int_s^u a(\alpha, \phi(\alpha))\psi(\alpha)\, d\alpha + \int_s^u b(\alpha, \phi(\alpha))\, d\alpha , \qquad s \leqq u \leqq t .$$

As a consequence of Remark 3.1, we see that $y \in \text{supp}\,(q(s, x;\; t, \cdot))$ if and only if $(t, y) \in A(s, x)$.

The purpose of the present section is to show that if u is a "subsolution" for $\partial/\partial t + L_t$ in G and if $u(s, x) = \sup_G u(t, x)$, then $u \equiv u(s, x)$ on $A(s, x)$. Moreover, we want to show that $A(s, x)$ is the largest subset of G having this property. Before proving this result, it is necessary to explore what we mean by a "subsolution". Certainly any function $u \in C_b^{1,2}(G)$ satisfying

$$\frac{\partial u}{\partial t} + L_t\, u \geqq 0 \qquad \text{in} \qquad G$$

must be covered by our definition. However, in order to show that $A(s, x)$ is maximal, we need to extend our notion of subsolution to include functions which, in the case $L_t = \frac{1}{2}\Delta$, are called subparabolic functions on G.

Let $k: G \to [0, \infty)$ be a bounded measurable function. Given a function $u: G \to R$, we shall say that u is a *k-subsolution* if u is bounded above, u is upper semicontinuous, and, for any $(s, x) \in G$,

$$u(t \wedge \tau_s, x(t \wedge \tau_s)) \exp\left\{-\int_0^{t\wedge\tau_s} k(u, x(u))\, du\right\}$$

is a $Q_{s,x}$-submartingale. We shall first show that if $u \in C_b^{1,2}(G)$ satisfies

$$\frac{\partial u}{\partial t} + L_t\, u - ku \geqq 0 \qquad \text{in} \qquad G\,, \tag{4.2}$$

then u is a k-subsolution.

LEMMA 4.1. *Let $\xi: [0, \infty) \times \Omega \to R$ and $\eta: [0, \infty) \times \Omega \to R$ be bounded nonanticipating functions. Assume that $\xi(t)$ is a P-martingale for some probability measure P on Ω, and define $\psi(t) = \int_0^t \eta(s)\, ds$. Then*

$$\xi(t)\psi(t) - \int_0^t \xi(s)\eta(s)\, ds$$

is again a P-martingale.

Proof: Note that

$$E\left[\xi(t)\phi(t) - \xi(s)\phi(s) - \int_s^t \xi(u)\eta(u)\, du \,\middle|\, M_s\right]$$
$$= E[(\xi(t) - \xi(s))\phi(s) \mid M_s] + E\left[\int_s^t (\xi(t) - \xi(u))\eta(u)\, du \,\middle|\, M_s\right].$$

The first term obviously vanishes. Moreover, if $A \in M_s$, then

$$E\left[\chi_A \int_s^t (\xi(t) - \xi(u))\eta(u)\, du\right] = \int_s^t E[\chi_A(\xi(t) - \xi(u))\eta(u)]\, du = 0\,,$$

and this completes the proof.

LEMMA 4.2. *If* $u \in C_b^{1,2}(G)$ *satisfies* (4.2), *then* u *is a k-subsolution.*

Proof: Let

$$\xi(t) = u(t \wedge \tau_s, x(t \wedge \tau_s)) - \int_0^{t\wedge\tau_s} \left(\frac{\partial u}{\partial \alpha} + L_\alpha u\right)(\alpha, x(\alpha))\, d\alpha\,,$$

and

$$\eta(t) = -\chi_{\tau_s > t} k(t, x(t)) \exp\left\{-\int_s^{t\wedge\tau_s} k(\alpha, x(\alpha))\, d\alpha\right\}.$$

Then an application of Lemma 4.1 shows that

$$u(t \wedge \tau_s, x(t \wedge \tau_s)) \exp\left\{-\int_s^{t\wedge\tau_s} k(\alpha, x(\alpha))\, d\alpha\right\} - \int_t^{t\wedge\tau_s} \left(\frac{\partial u}{\partial \alpha} + L_\alpha u - ku\right) \exp\left\{-\int_s^\alpha k(\gamma, x(\gamma))\, d\gamma\right\} d\alpha$$

is a $Q_{s,x}$-martingale. Since the integrand in the second term is non-negative, this proves that the first term is a $Q_{s,x}$-submartingale.

THEOREM 4.1. *If* u *is a k-subsolution for some non-negative bounded measurable* k *and if* $0 = u(s, x) = \sup_G u(t, y)$ *for some* $(s, x) \in G$, *then* $u \equiv u(s, x)$ *on* $A(s, x)$. *Conversely, if* $(t_0, y_0) \in G - A(s, x)$, *then there is a* 0-*subsolution* u *such that* $u \leqq 0$, $u(s, x) = 0$, *and* $u(t_0, y_0) < 0$.

Proof: Choose $\lambda \geqq \sup_G k(t, y)$. Then u is a λ-subsolution, and so

$$\begin{aligned} e^{-\lambda(t-s)} \int q(s, x; t, dy) u(t, y) &= E^{Q_{s,x}}[e^{-\lambda(t-s)} u(t, x(t))\chi_{\tau_s > t}] \\ &\geqq E^{Q_{s,x}}[e^{-\lambda(t\wedge\tau_s - s)} u(t \wedge \tau_s, x(t \wedge \tau_s))] \geqq u(s, x) = 0\,. \end{aligned}$$

Thus, since u is upper semicontinuous and at most 0, $u(t, \cdot) \equiv 0$ on

$$\operatorname{supp}(q(s, x;\ t, \cdot)) = A(s, x)\,.$$

To prove the converse, suppose that $(t_0, y_0) \in G - A(s, x)$ and choose $\psi \in C_0^\infty(G - A(s, x))$ such that $\psi \leqq 0$ and $\psi(t_0, y_0) = -1$. Define

$$u(t, y) = E^{Q_{t,y}}\left[\int_t^{\tau_t} \psi(\alpha, x(\alpha))\, d\alpha\right].$$

Then $u \leqq 0$ and $u(t_0, y_0) < 0$. Moreover, using Remark 3.1 and the fact that $\psi = 0$ on $A(s, x)$, one sees that $u(s, x) = 0$. Finally, the weak continuity of the $Q_{t,y}$ and standard Markov arguments imply that u is a 0-subsolution.

Remark 4.1. It is obvious from the proof of Theorem 4.1 that u need be a subsolution only on $A(s, x)$ and $0 = u(s, x) = \sup_{A(s,x)} u(t, y)$ in order for the conclusion to obtain.

Remark 4.2. The analytic meaning of the extension of $\partial/\partial t + L_t$ which is implicit in our definition of subsolutions will be discussed in Section 8. In particular, it will be described there in what sense the counterexample constructed in Theorem 4.1 satisfies $\partial u/\partial t + L_t u \geqq 0$ on G. However, it should be mentioned here that if $G = (t_1, t_2) \times R^d$, then our counterexample is essentially as smooth as the coefficients in L_t. This fact follows from the theorem of Oleinik proved in the appendix.

Remark 4.3. Because our notion of subsolution is basically a mean-value property, our class of subsolutions is closed under the same operations as the class of subparabolic functions. Hence it is considerably larger than the set of smooth functions satisfying (4.2).

5. The First Boundary Value Problem

Let $a: R^d \to S_d$ and $b: R^d \to R^d$ be bounded measurable functions such that $a \in C^2(R^d)$ and $b \in C^1(R^d)$. Let

$$L = \frac{1}{2}\sum_{i,j=1}^{d} a^{ij}(x)\frac{\partial^2}{\partial x_i\, \partial y_j} + \sum_{i=1}^{d} b^i(x)\frac{\partial}{\partial x_i},$$

and denote by P_x the solution to the martingale problem for L starting at x. Given an open set G in R^d, define

$$\tau = \inf\{t \geqq 0:\ x(t) \notin G\},$$

$$\tau' = \inf\{t \geqq 0:\ x(t) \notin \bar{G}\}.$$

If L is strictly elliptic and G is reasonably nice (e.g., satisfies an exterior cone condition), then $P_x(\tau \neq \tau') = 0$ for all $x \in G$. However, if L is degenerate, then, as we shall see, $P_x(\tau \neq \tau')$ need not vanish for all $x \in G$ no matter how smooth G is. This fact leads us to two formulations of the first boundary value problem, one corresponding to τ and the other to τ'. As will be shown later, these two formulations correspond analytically to approximating the solution from inside and from the outside, respectively.

We shall use Γ to denote the set of $x \in \delta G$ such that $P_x(\tau' > 0) = 0$. By the $0-1$ law, if $x \in \delta G$, then $x \notin \Gamma$ if and only if $P_x(\tau' > 0) = 1$.

LEMMA 5.1. *The set* Γ *is a* G_δ *subset of* δG. *Moreover, for all* $x \in G$,

$$P_x(x(\tau') \notin \Gamma\,, \quad \tau' < \infty) = 0\,.$$

Finally, if $x \in \Gamma$, *then* $\lim_{\substack{y \to x \\ y \in G}} P_y(\tau' \geqq \varepsilon) = 0$ *for all* $\varepsilon > 0$.

Proof: Note that $\{\tau' < \varepsilon\}$ is open. Since P_x is weakly continuous in x, $P_x(\tau' < \varepsilon) \leqq \underline{\lim}_{y \to x} P_y(\tau' < \varepsilon)$. Hence $\{x \in \delta G\colon\ P_x(\tau' < \varepsilon) > 0\}$ is open in δG, and so

$$\Gamma = \{x \in \delta G : P_x(\tau' = 0) > 0\} = \bigcap_1^\infty \{x \in \delta G : P_x(\tau' < 1/n) > \tfrac{1}{2}\}$$

is a G_δ.

To prove the second assertion, observe that $x(\tau') \in \delta G$ a.s. P_x for $x \in G$ since the paths are continuous. Hence,

$$P_x(x(\tau') \notin \Gamma, \tau' < \infty) = E_x(\chi_{\tau' < \infty} P_{x(\tau')}(\tau' > 0)] = P_x(\tau' > \tau') = 0\,,$$

by the strong Markov property.

Finally, suppose $x \in \Gamma$. Then, for all $\varepsilon > 0$, $P_x(\tau' \geqq \varepsilon) = 0$. Since $\{\tau' \geqq \varepsilon\}$ is closed, $\overline{\lim}_{y \to x} P_y(\tau' \geqq \varepsilon) \leqq P_x(\tau' \geqq \varepsilon) = 0$; q.e.d.

Given bounded measurable functions u and f on $\bar{G}$, we shall say that

$$\tilde{L}u = f \quad \text{in} \quad \bar{G}$$

if

$$u(x(t \wedge \tau')) - \int_0^{t \wedge \tau'} f(x(s))\, ds$$

is a P_x-martingale for all $x \in \bar{G}$.

Remark 5.1. If $u \in C^2(\bar{G})$ and $Lu = f$ on $\bar{G}$, then clearly $\tilde{L}u = f$. Hence $\tilde{L}$ is an extension of L. Moreover, it follows from Theorem 2.4 that the definition of $\tilde{L}$ really depends only on the restriction of the coefficients of L to G. Finally, although the operator $\tilde{L}$ is not strictly single-valued, nonetheless if $\tilde{L}u = f_i$ in $\bar{G}$, $i = 1, 2$, then $f_1 = f_2$ at common points of continuity.

We are now ready to state the first boundary value problem associated with τ'. Let k be a bounded, continuous non-negative function on $\bar{G}$. Given bounded measurable functions f and g on $\bar{G}$ and Γ, respectively, we shall say that u solves the *τ'-first boundary value problem* for f and g if

$$\begin{aligned} \tilde{L}u - ku &= f \quad \text{on} \quad \bar{G}, \\ u|_\Gamma &= g. \end{aligned} \tag{5.1}$$

THEOREM 5.1. *If either*

$$\sup_{x \in \bar{G}} E_x[\tau'] < \infty \tag{5.2}$$

or

$$\inf_{x \in \bar{G}} k(x) > 0, \tag{5.3}$$

the τ'-first boundary value problem has exactly one solution u for each f and g. Moreover, $\lim\limits_{\substack{y \to x \\ y \in G}} u(y) = g(x)$ *for $x \in \Gamma$ which are points of continuity of g.*

Proof: Standard Markov arguments show that

$$\begin{aligned} u(x) = E_x\Big[&-\int_0^{\tau'} \exp\Big\{-\int_0^t k(x(s))\,ds\Big\} f(x(t))\,dt \\ &+ \exp\Big\{-\int_0^{\tau'} k(x(s))\,ds\Big\} g(x(\tau'))\Big] \end{aligned} \tag{5.4}$$

is a solution (note that all the indicated integrals exist under either (5.2) or (5.3)).

Conversely, if u is a solution, then

$$u(x(t \wedge \tau')) - \int_0^{t \wedge \tau'} (ku + f)(x(s))\,ds$$

is a P_x-martingale for all $x \in \bar{G}$. Hence, by an application of Lemma 4.1,

$$u(x(t \wedge \tau')) \exp\Big\{-\int_0^{t\wedge\tau'} k(x(s))\,ds\Big\} - \int_0^{t\wedge\tau'} \exp\Big\{-\int_0^\theta k(x(s))\,ds\Big\} f(x(\theta))\,d\theta$$

is also a P_x-martingale for all $x \in \bar{G}$, and so

$$u(x) = E_x\left[-\int_0^{t\wedge\tau'} \exp\left\{-\int_0^\theta k(x(s))\,ds\right\} f(x(\theta))\,d\theta \right.$$
$$\left. + \exp\left\{-\int_0^{t\wedge\tau'} k(x(s))\,ds\right\} u(x(t\wedge\tau'))\right],$$

for all $t \geqq 0$ and $x \in G$. Using (5.2) or (5.3) and the fact that

$$u(x(t\wedge\tau')) = g(x(\tau')) \qquad \text{on} \qquad \{\tau' \leqq t\},$$

one sees upon letting $t \uparrow \infty$ that $u(x)$ is given by (5.4).

To prove that $\lim_{\substack{y\to x\\ y\in G}} u(y) = g(x)$ if $x \in \Gamma$ is a point of continuity of g, observe that by Lemma 5.1 and either (5.2) or (5.3)

$$\lim_{\substack{y\to x\\ y\in \bar{G}}} E_y\left[\int_0^{t\wedge\tau'} \exp\left\{-\int_0^t k(x(s))\,ds\right\} dt\right] = 0\,.$$

This is obvious if (5.3) obtains. To prove it under (5.2) note that

$$\overline{\lim_{\substack{y\to x\\ y\in G}}}\, E_y[\tau'] \leqq \varepsilon + \overline{\lim_{\substack{y\to x\\ y\in G}}}\, E_y[\tau', \tau' > \varepsilon] = \varepsilon + \overline{\lim_{\substack{y\to x\\ y\in G}}}\, E_y(\chi_{\tau'>\varepsilon}\, E_{x(\varepsilon)}[\tau'])$$
$$\leqq \varepsilon + c\,\overline{\lim_{\substack{y\to x\\ y\in G}}}\, P_y(\tau' > \varepsilon) = \varepsilon\,,$$

where $c = \sup_{x\in\bar{G}} E_x[\tau']$. This implies that

$$\lim_{\substack{y\to x\\ y\in G}} u(y) = \lim_{\substack{y\to x\\ y\in G}} E_y[g(x(\tau')), \tau' < \infty]\,.$$

Our assertion now follows from another application of Lemma 5.1 and use of the fact that

$$\lim_{\delta\searrow 0}\, \sup_{y\in\bar{G}} P_y\left(\sup_{0\leqq t\leqq\delta} |x(t) - y| \geqq \varepsilon\right) = 0$$

for all $\varepsilon > 0$.

Remark 5.2. The condition (5.2) is difficult to verify in general. However, one sufficient condition for (5.2) to hold can be obtained from our characterization of supp (P_x). Indeed, suppose $\bar{G}$ is compact and that there is a T such that for each $x \in \bar{G}$ there is a $\phi \in S_{a,b}(x)$ such that $\phi(T) \notin \bar{G}$. Then $P_x(\tau' < T) > 0$ for all $x \in \bar{G}$. Hence, since $P_x(\tau' < T)$ is lower semi-continuous,

$$\inf_{x \in \bar{G}} P_x(\tau' < T) \geqq 1 - \alpha > 0 .$$

Thus by induction and the strong Markov property, $P_x(\tau' \geqq nT) \leqq \alpha^n$ for all $x \in \bar{G}$. Hence, in this case, we even have $\sup_{x \in \bar{G}} E_x[e^{\lambda \tau'}] < \infty$ for λ such that $\alpha e^{\lambda T} < 1$.

We now want to describe the first boundary problem associated with τ. To do this we must first say what extension of L we are going to use. Given bounded measurable functions u and f on G, we write

$$\hat{L}u = f \quad \text{in} \quad G ,$$

if

$$u(x(t \wedge \tau)) - \int_0^{t \wedge \tau} f(x(s))\, ds$$

is a P_x-martingale for all $x \in G$. Observe that $\hat{L}u = f$ on G if and only if for every compact $K \subset G$, $\hat{L}u = f$ on K. In order to describe how we want the boundary data to be taken on, we introduce the "barrier" function,

$$\psi(x) = 1 - E_x[e^{-\tau}]$$

and define

$$\Gamma_0 = \left\{ x \in \delta G : \varliminf_{\substack{y \to x \\ y \in G}} \psi(y) = 0 \right\} .$$

Let k be a non-negative bounded continuous function on G. Given a bounded measurable function f on G and a bounded continuous function g on Γ_0, we say that a bounded measurable function u solves the *τ-first boundary value problem* for f and g if

$$\begin{aligned} \hat{L}u - ku &= f \quad \text{in} \quad G , \\ u(y) &\xrightarrow{\psi} g(x) \quad \text{as} \quad y \to x \in \Gamma_0 , \end{aligned} \tag{5.5}$$

where $u(y) \xrightarrow{\psi} g(x)$ means that $u(y_n) \to u(x)$ if $\langle y_n \rangle_1^\infty \subseteq G$ tends to x and $\lim_{n\to\infty} \psi(y_n) = 0$.

Remark 5.3. It is easy to check that $\hat{L}\psi = \psi - 1$ in G. However, ψ is lower semicontinuous and not in general upper semicontinuous. It is this fact that prevents ψ from being a good barrier and forces us to give such a clumsy description of the way in which u takes on its boundary data. If one were to take the lower semicontinuous regularization of ψ, then for nice regions G one would end up with $1 - E_x[e^{-\tau'}]$, which is the appropriate barrier for the τ'-first boundary value problem but not for the τ-problem. In this connection, it should be noted that if $G_n \nearrow G$ and $\bar{G}_n \subset G$, then $1 - E_x[e^{-\tau'_n}] \searrow \psi(x)$ on G.

LEMMA 5.2. *For all* $x \in G$, $\chi_{\tau<\infty}\psi(x(t \wedge \tau)) \to 0$ *a.s.* P_x *as* $t \nearrow \infty$.

Proof: Clearly it is enough to show that $\xi(t) = e^{-t\wedge\tau}\psi(x(t \wedge \tau)) \to 0$ a.s. P_x as $t \to \infty$. Using Lemma 4.1, one can easily check that $\xi(t)$ is a P_x-supermartingale, and therefore, since it is bounded, $\xi(t)$ converges a.s. P_x. Thus, to prove that $\xi(t) \to 0$ a.s. P_x, it is enough to show that $E_x[\xi(t)] \to 0$ as $t \to \infty$. But

$$\begin{aligned} E_x(e^{-t\wedge\tau}\psi(x(t \wedge \tau))] &= E_x(e^{-t\wedge\tau}E_{x(t\wedge\tau)}[1 - e^{-\tau}]] \\ &= E_x(e^{-t\wedge\tau}(1 - e^{-(\tau - t\wedge\tau)})] \\ &= E_x(e^{-t}(1 - e^{-(\tau-t)})\chi_{\tau>t}] \to 0 \qquad \text{as} \qquad t \nearrow \infty . \end{aligned}$$

Remark 5.4. As a consequence of Lemma 5.2, we see that

$$P_x(x(\tau) \notin \Gamma_0 , \tau < \infty) = 0 , \qquad x \in G .$$

In the next section we shall show that $P_x(x(\tau) \notin \bar{\Gamma}, \tau < \infty) = 0, x \in G$. From this it follows that $\Gamma_0 \subseteq \bar{\Gamma}$. Indeed, if $x \in \Gamma_0 - \bar{\Gamma}$, then we can find an $\varepsilon > 0$ so that $|y - x| \leqq \varepsilon$ implies $y \notin \bar{\Gamma}$. But $\varliminf_{\substack{y\to x\\ y\in G}} P_x(\tau \geqq t) = 0$ for all $t > 0$, and so there is a $y \in G$ such that $P_y(|x(\tau) - x| < \varepsilon, \tau < \infty) > 0$, which contradicts $P_y(x(\tau) \in \bar{\Gamma}, \tau < \infty) = 0$. Also notice that, for $x \in \Gamma$,

$$\lim_{\substack{y\to x\\ y\in G}} P_y(\tau \geqq t) \leqq \lim_{\substack{y\to x\\ y\in G}} P_y(\tau' \geqq t) = 0$$

for $t > 0$ and so $\Gamma \subseteq \Gamma_0$. Hence the only points where the bad convergence may occur lie in $\bar{\Gamma} - \Gamma$.

THEOREM 5.2. *Let* k *be a non-negative continuous bounded function on* G. *Assume that either*

$$\sup_{x\in G} E_x[\tau] < \infty \tag{5.6}$$

or

$$\inf_{x\in G} k(x) > 0\,. \tag{5.7}$$

Then for each bounded measurable function f on G and bounded continuous function g on Γ_0 there is exactly one bounded measurable u which solves the τ-first boundary value problem for f and g.

Proof: To prove existence, set

$$\begin{aligned} u(x) = E_x\Big[-\int_0^\tau \exp\Big\{-\int_0^t k(x(\alpha))\,d\alpha\Big\} f(x(t))\,dt \\ + \exp\Big\{-\int_0^\tau k(x(\alpha))\,d\alpha\Big\} g(x(\tau))\Big]\,. \end{aligned} \tag{5.8}$$

Then, by the same type of argument as that given in Theorem 5.1, one can show that $\hat{L}u - ku = f$ in G and that $u(y_n) \to g(x)$ if $\{y_n\}_1^\infty \subseteq G$ tends to x in such a way that $\psi(y_n) \to 0$.

To prove uniqueness, note that if u is a solution, then

$$\begin{aligned} u(x) = E_x\Big[-\int_0^{t\wedge\tau} \exp\Big\{-\int_0^s k(x(\alpha))\,d\alpha\Big\} f(x(s))\,ds \\ + \exp\Big\{-\int_0^{t\wedge\tau} k(x(\alpha))\,d\alpha\Big\}\, u(x(t\wedge\tau))\Big]\,. \end{aligned}$$

By Lemma 5.2, $u(x(t\wedge\tau)) \to g(x(\tau))$ a.s. P on $\{\tau < \infty\}$, and so we see, upon letting $t \nearrow \infty$, that u must be given by (5.8).

Remark 5.5. Although the τ-first boundary value problem has many drawbacks, it has one major advantage over the τ'-problem. Consider coefficients $a \in C^2(G)$ and $b \in C^1(G)$ which cannot be extended smoothly outside of G. (For example, let $G = (0, \infty)$ and suppose $a(x) = x$ near 0.) Then one can no longer discuss the τ'-problem, but the τ-problem still makes sense and Theorem 5.2 remains true.

6. Infinitesimal Processes and Conditions for Regularity

Let $G \subset R^d$ be a region with $\phi(x)$ as its defining function. That is,

$$G = \{x\colon\ \phi(x) > 0\}\,,$$
$$G = \{x\colon\ \phi(x) = 0\}\,,$$
$$|\nabla\phi| \neq 0 \qquad \text{on} \qquad \delta G\,.$$

Let $x^0 \in \delta G$. We would like to determine, in terms of the behavior of the coefficients of L and the function ϕ, when $x^0 \in \Gamma$, the set of τ' regular points described in Section 5. For this purpose we look at the process $x(t)$ corresponding to L, starting at time 0 from the point x^0. Let $\xi(t) = \phi(x(t))$. We have to decide when $\xi(t)$ becomes immediately negative. Let us define

$$\begin{aligned}\alpha(t) &= E\xi(t)\,,\\ \beta(t) &= E\xi^2(t)\,,\\ s(t) &= [\beta(t) - \alpha^2(t)]^{1/2}\,;\end{aligned}$$

$s(t)$ is of course non-negative and is the standard deviation of $\xi(t)$.

We shall assume that

$$\alpha(t) = \alpha t^r + o(t^r) \qquad \text{as} \qquad t \downarrow 0\,,$$

and

$$\beta(t) = \beta t^s + o(t^s) \qquad \text{as} \qquad t \downarrow 0\,,$$

where α, β are different from zero and r, s are some integers. Clearly, $s \leqq 2r$ and if $s = 2r$, then $\beta \geqq \alpha^2$. Three cases arise naturally in trying to compare $\alpha(t)$ with $s(t)$.

Case 1: $s < 2r$.

In this case, $s(t)$ is much larger than $\alpha(t)$. So the diffusion must dominate and this indicates that $\xi(t)$ takes on immediately positive and negative values. Therefore, the point in question is probably regular.

Case 2: $s = 2r$, $\beta = \alpha^2$.

In this case, $s(t)$ is negligible compared to $\alpha(t)$. So the behavior of $\alpha(t)$ must depend on the sign of α; $\alpha > 0$ should imply irregularity and $\alpha < 0$ should imply regularity.

Case 3: $s = 2r$, $\beta > \alpha^2$.

In this case, $s(t)$ and $\alpha(t)$ are of the same order of magnitude. The situation is unclear and one must look more carefully.

To illustrate these cases let us consider some examples.

EXAMPLE 1. If, at a boundary point, $\langle a\nabla\phi, \nabla\phi\rangle > 0$, then that point must be regular. In this case, $s = 1$ and no matter what r is we always have $s < 2r$.

EXAMPLE 2. Consider the region $G \subset R^2$ consisting of

$$0 < x < 1\,, \qquad -\infty < y < \infty\,.$$

Let the operator be

$$\frac{\partial}{\partial y^2} + y\frac{\partial}{\partial x}\,.$$

Take the boundary point (0, 0). For the function $\phi(x) = x$, $s = 3$ and $r = \infty$. So the origin is regular.

EXAMPLE 3. If the operator is purely of first order or at least if the second order terms have a strong enough zero at the point under consideration, then we are in Case 2, so that the regularity is determined purely by the sign of $E\phi(x(t))$ for small t.

EXAMPLE 4. Consider the operator

$$\frac{1}{2}\frac{\partial^2}{\partial x^2} + x\frac{\partial^2}{\partial x\,\partial y} + \tfrac{1}{2}x^2\frac{\partial}{\partial y^2} + (\alpha + \tfrac{1}{2})\frac{\partial}{\partial y}$$

in either $y > 0$ or $y < 0$. A solution starting from (0, 0) is easily seen to be

$$y(t) = \tfrac{1}{2}\beta^2(t) + \alpha t\,,$$

where $\beta(t)$ is Brownian motion. If the region is $y > 0$, we have regularity of (0, 0) provided $\alpha < 0$. If the region is $y < 0$ we have regularity for all α. All these cases come under Case 3 with $s = 2$ and $r = 1$.

Theorem 6.1 deals with Case 1 and under some additional assumptions proves what is indicated. Theorem 6.2 treats Case 2 and establishes what was suggested before. Theorem 6.3 looks in detail at Case 3 when $r = 1$ and $s = 2$. Each theorem is preceded by a lemma or two which cover some preliminary material.

Remark. If the coefficients and the function ϕ are sufficiently smooth, we can determine α, β, r and s by the relations

$$(L^k\phi)(x^0) = 0 \qquad \text{for} \qquad k = 1, 2, \cdots, r-1\,,$$
$$(L^r\phi)(x^0) = (r!)\alpha\,,$$
$$(L^k\phi^2)(x^0) = 0 \qquad \text{for} \qquad k = 1, 2, \cdots, s-1\,,$$
$$(L^r\phi^2)(x^0) = (s!)\beta\,.$$

We first consider Case 1. Here we have to assume in addition that $a = \sigma\sigma^*$, where $\sigma \in C^\infty[R^d, R^d \otimes R^N]$ for some N.

LEMMA 6.1. *If* $a = \sigma\sigma^*$, *where* $\sigma \in C^\infty[R^d, R^d \otimes R^N]$, *then for* $f \in C_d^\infty[R^d]$

$$E^{P_{x^0}}[f^2(x(t))] \leqq Ct^n \quad \textit{for} \quad \textit{small} \quad t$$

implies that

$$E^{P_{x^0}}[f^4(x(t))] \leqq C't^{2n} \quad \textit{for} \quad \textit{small} \quad t\,.$$

Proof: We shall drop P_{x^0} and denote the expectations by just E. We shall prove by induction on k that if $g = \sum_1^m f_i^2$, where $f_i \in C_b^\infty[R^d]$, for any arbitrary m, then

$$Eg(x(t)) \leqq Ct^k \quad \text{implies} \quad \text{that} \quad Eg^2(x(t)) \leqq C't^{2k}\,.$$

The assertion is trivial for $k = 0$. Suppose it is true for $k \leqq n$ and let $g = \sum_1^m f_i^2$ satisfy $Eg(x(t)) \leqq Ct^{n+1}$. Then $Ef_i^2(x(t)) \leqq Ct^{n+1}$ for $1 \leqq i \leqq m$. This means that $L^r f_i^2(x^0) = 0$ for $0 \leqq r \leqq n$. In particular, $L^r < \nabla f_i\,,\, a\nabla f_i > (x^0) = 0$ for $0 \leqq r \leqq n-1$ (by Theorem 10.3 of the appendix) and $L^r(Lf)^2(x^0) = 0$ for $0 \leqq r \leqq n-2$. Hence,

$$\begin{aligned}
E[f_i^4(x(t))] &= E\int_0^t (Lf_i^4)(x(s))\,ds \\
&= 6E\int_0^t f_i^2\langle\nabla f_i\,,\, a\nabla f_i\rangle(x(s))\,ds + 4E\int_0^t f_i^3(Lf_i)(x(s))\,ds \\
&\leqq 6\int_0^t [Ef_i^4(x(s))]^{1/2}(E\,|\sigma^*\nabla f_i|^4\,(x(s))]^{1/2}\,ds \\
&\quad + 4\int_0^t (Ef_i^4(x(s))]^{3/4}(E(Lf)^4(x(s))]^{1/4}\,ds\,.
\end{aligned}$$

Note that

$$E[|\sigma^*\nabla f_i|^2(x(t))] \leqq Ct^n\,,$$

and

$$E[(Lf_i)^2(x(t))] \leqq Ct^{n-1}\,.$$

Since $|\sigma^*\nabla f_i|^2$ is a sum of squares, the induction hypothesis yields

$$E[|\sigma^*\nabla f_i|^4(x(t))] \leqq C't^{2n}\,,$$

and

$$E[(Lf_i)^4(x(t))] \leqq C' t^{2n-2}.$$

Thus

$$E[f_i^4(x(t))] \leqq A\int_0^t [Ef_i^4(x(s))]^{1/2} s^n\, ds + B\int_0^t [Ef_i^4(x(s))]^{3/4} s^{(2n-2)/4}\, ds.$$

We can now show that $E[f_i^4(x(t))] \leqq Ct^{2n+2}$. Indeed, if $L^r f_i^4(x^0) = 0$ for all r, there is nothing to prove. Let r_0 be the first r for which $L^r f_i^4(x^0)$ is nonzero. Then there are positive numbers λ_1 and λ_2 such that

$$\lambda_1 t^{r_0} \leqq E[f^4(x(t))] \leqq \lambda_2 t^{r_0}.$$

But this means that

$$\begin{aligned}\lambda_1 t^{r_0} &\leqq \lambda_2^{1/2} A\int_0^t s^{r_0/2+n}\, ds + \chi_2^{3/4} B\int_0^t s^{(3r_0/4)+(2n-2)/4}\, ds \\ &\leqq A' t^{(r_0+2n+2)/2} + B' t^{(3r_0+2n+2)/4},\end{aligned}$$

and this is possible only if $r_0 \geqq 2n+2$. It follows that

$$E[g^2(x(t))] \leqq C\sum_1^m E[f^4(x(t))] \leqq C'\ {}^{2n+2}.$$

LEMMA 6.2. *Let X be a random variable with mean zero. Let $\alpha = EX^2$ and $\beta^4 = EX^4$. Then*

$$P\left[X^\pm \geqq \frac{\alpha^{3/2}}{4\beta^2}\right] \geqq \frac{\alpha^2}{64\beta^4}.$$

Proof: By Hölder's inequality,

$$\alpha = E[|X|^2] \leqq E[|X|]^{2/3} E[|X|^4]^{1/3},$$

and therefore $E[|X|] \geqq \alpha^{3/2}/\beta^2$

Since $E[X^+] = E[X^-]$, we now have that $E[X^\pm] \geqq \alpha^{3/2}/2\beta^2$. Next observe that

$$\frac{\alpha^{3/2}}{2\beta^2} \leqq E[X^\pm] \leqq \frac{\alpha^{3/2}}{4\beta^2} + RP\left(X^\pm \geqq \frac{\alpha^{3/2}}{4\beta^2}\right) + \frac{\alpha}{R}$$

for all $R > 0$. Hence,

$$P\left(X^{\pm} \geqq \frac{\alpha^{3/2}}{4\beta^2}\right) \geqq \frac{\alpha^{3/2}}{4\beta^2}\frac{1}{R} - \frac{\alpha}{R^2}.$$

Taking $R = 8\beta^2/\alpha^{1/2}$, we get our result.

THEOREM 6.1. *Assume that* $a = \sigma\sigma^*$, *where* $\sigma \in C^\infty(R^d, R^d \otimes R^N)$. *Suppose that* $L^r \phi^2(x^0) \neq 0$ *for some* r *and let* n *denote the first such* r. *If* $L^r \phi(x^0) = 0$ *for* $r \leqq \frac{1}{2}n$, *then* $P_{x^0}(\tau' > 0) = 0$.

Proof: Let

$$X(t) = \frac{\phi(x(t)) - E[\phi(x(t))]}{t^{n/2}}.$$

Then there is a $\lambda > 0$ such that $1/\lambda \leqq E[|X(t)|^2] \leqq \lambda$ and there is a $C < \infty$ such that $E[|X(t)|^4] \leqq C$. Hence, we can find positive numbers ε, γ such that $P(X^-(t) \geqq \varepsilon) \geqq \gamma$. Since $E[\phi(x(t))]/t^{n/2} \to 0$ as $t \to 0$, this proves that, for small t,

$$P_{x^0}(\phi(x(t)) \leqq -\tfrac{1}{2}\varepsilon t^{n/2}) \geqq \gamma.$$

Hence $P_{x^0}(\tau' \leqq t) \geqq \gamma$ for all $t > 0$, and so, by the $0-1$ law, $P_{x^0}(\tau' > 0) = 0$. We next consider Case 2.

THEOREM 6.2. *If* $L^r \phi(x^0) = 0$ *for* $r \leqq n - 1$, $L^r \phi^2(x^0) = 0$ *for* $r \leqq 2n - 1$ *and*

$$\frac{1}{(2n)!} L^{2n} \phi^2(x^0) = \left(\frac{1}{n!} L^n \phi(x^0)\right)^2 \neq 0.$$

Then $P_{x^0}(\tau' > 0) = 0$ *if and only if* $L^n \phi(x^0) < 0$.

Proof: Let $\psi(t, x) = \phi(x) - (t^n/n!)L^n \phi(x^0)$ and $\bar{L} = \partial/\partial t + L$. Then clearly $\bar{L}^r \psi(0, x^0) = 0$ for $r \leqq n$, and $\bar{L}^r \psi^2(0, x^0) = 0$ for $r \leqq 2n$. Consequently,

$$E^{P_{x^0}}\left[\left|\frac{\phi(x(t))}{t^n} - \frac{(L^n \phi)(x^0)}{n!}\right|^2\right] = o(t) \qquad \text{as} \qquad t \to 0.$$

It follows that $\phi(x(t))/t^n$ converges to $L^n \phi(x^0)/n!$ in mean-square as $t \searrow 0$. We want to show now that this convergence is almost sure. By enlarging the sample space, if necessary, we can produce a one-dimensional Brownian motion

$\beta(\cdot)$ such that

$$\psi(t, x(t)) = \int_0^t \langle \nabla\phi, a\nabla\phi\rangle^{1/2}(x(s))\, d\beta(s) + \int_0^t L\psi(s, x(s))\, ds \qquad \text{a.s. } P_{x^0}\,.$$

Thus,

$$\frac{\phi(x(t))}{t^n} - \frac{\phi(x(s))}{s^n} = \int_s^t \frac{\langle \nabla\phi, a\nabla\phi\rangle^{1/2}(x(u))}{u^n}\, d\beta(s) + \int_s^t \frac{L\psi(u, x(u))}{u^n}\, du - n\int_s^t \frac{\psi(u, x(u))}{u^{n+1}}\, du \qquad \text{a.s. } P\,.$$

Observe that

$$E^{P_{x^0}}\left[\left(\int_s^t \frac{\langle \nabla\phi, a\nabla\phi\rangle^{1/2}(x(u))}{u^n}\, d\beta(u)\right)^2\right] = \int_s^t \frac{E^{P_{x^0}}[\langle \nabla\phi, a\nabla\phi\rangle(x(u))]}{u^{2n}}\, du \leqq C(t-s)\,,$$

$$E^{P_{x^0}}\left[\int_s^t \left|\frac{L\psi(u, x(u))}{u^n}\right| du\right] \leqq \int_s^t \frac{E^{P_{x^0}}[|\psi(u, x(u))|^2]^{1/2}}{u^n}\, du \leqq C\int_s^t u^{-1/2}\, du\,,$$

$$E^{P_{x^0}}\left[\int_s^t \left|\frac{\psi(u, x(u))}{u^{n+1}}\right| du\right] \leqq \int_s^t \frac{E^{P_x}[|\psi(u, x(u))|^2]^{1/2}}{u^{n+1}}\, du \leqq C\int_s^t u^{-1/2}\, du\,,$$

where we have used the facts that

$$\begin{aligned} L^r\langle \nabla\phi, a\nabla\phi\rangle(x^0) &= 0\,, & r &\leqq 2n-1\,,\\ L^r\psi^2(0, x^0) &= 0\,, & r &\leqq 2n\,,\\ L^r(L\psi)^2(0, x^0) &= 0\,, & r &\leqq 2n-1\,. \end{aligned}$$

Using these facts together with Doob's martingale inequality, we see that

$$P_{x^0}\left(\sup_{0<s<t<\delta}\left|\frac{\phi(x(t))}{t^n} - \frac{\phi(x(s))}{s^n}\right| \geqq \varepsilon\right) = o(1)$$

as $\delta \searrow 0$ for all $\varepsilon > 0$. Hence, $\phi(x(t))/t^n \to L^n\phi(x^0)/n!$ a.s. P_{x^0}. From here it is easy to see that $P_{x^0}(\tau' > 0) = 0$ if $L^n\phi(x^0) < 0$ and $P_{x^0}(\tau' > 0) = 1$ if $L^n\phi(x^0) > 0$.

We turn finally to Case 3.

LEMMA 6.3. *Let* $a \in C_b^2[R^d, S_d]$ *and* $b \in C_b^1[R^d, R^d]$. *Assume that* $a^{ii}(0) = 0$ *for* $1 \leqq i \leqq r$. *Suppose that* P *solves the martingale problem for* a *and* b *starting at* 0. *For each* $\varepsilon > 0$ *let* $P^{(\varepsilon)}$ *be the distribution of the process*

$$x^\varepsilon(t) = \left(\frac{x_1(\varepsilon t)}{\varepsilon}, \cdots, \frac{x_r(\varepsilon t)}{\varepsilon}, \frac{x_{r+1}(\varepsilon t)}{\varepsilon^{1/2}}, \cdots, \frac{x_d(\varepsilon t)}{\varepsilon^{1/2}}\right),$$

where $x(t) = (x_1(t), \cdots, x_n(t))$ *has* P *for its distribution. Then* P^ε *is the solution to the martingale problem for* a_ε *and* b_ε *starting at* 0, *where*

$$a_\varepsilon^{ij}(x) = \frac{a^{ij}(\varepsilon x_1, \cdots, \varepsilon x_r, \varepsilon^{1/2} x_{r+1}, \cdots, \varepsilon^{1/2} x_d)}{\varepsilon^{\alpha_i+\alpha_j-1}},$$

$$b_\varepsilon^i(x) = \frac{b^i(\varepsilon x_1, \cdots, \varepsilon x_r, \varepsilon^{1/2} x_{r+1}, \cdots, \varepsilon^{1/2} x_d)}{\varepsilon^{\alpha_i-1}},$$

and $\alpha_i = 1$ *if* $1 \leqq i \leqq r$ *and* $\alpha_i = \frac{1}{2}$ *if* $r < i \leqq d$. *Moreover,* $P^{(\varepsilon)}$ *converges weakly as* $\varepsilon \to 0$ *to* $P^{(0)}$ *which is the solution to the martingale problem for* a_0 *and* b_0, *starting from* 0, *where*

$$\begin{aligned}
a_0^{ij}(x) &= \tfrac{1}{2}\sum_{k,l>r} a^{ij}_{.kl}(0)x_k x_l && \textit{for} \quad i,j \leqq r,\\
a_0^{ij}(x) &= \sum_{k>r} a^{ij}_{.k}(0)x_k && \textit{for} \quad i \leqq r < j,\\
a_0^{ij}(x) &= a^{ij}(0) && \textit{for} \quad i,j > r,\\
b_0^i(x) &= b^i(0) && \textit{for} \quad i \leqq r,\\
b_0^i(x) &= 0 && \textit{for} \quad i > r.
\end{aligned}$$

Proof: Note that

$$x_\theta(t) = \exp\left[\left\langle \theta, x(t) - \int_0^t b(x(s))\, ds\right\rangle - \tfrac{1}{2}\int_0^t \langle \theta, a(x(s))\theta\rangle\, ds\right]$$

is a P-martingale for all $\theta \in R$. Hence if

$$\theta^{(\varepsilon)} = \left(\frac{\theta_1}{\varepsilon}, \cdots, \frac{\theta_r}{\varepsilon}, \frac{\theta_{r+1}}{\varepsilon^{1/2}}, \cdots, \frac{\theta_d}{\varepsilon^{1/2}}\right),$$

then

$$X_\theta^{(\varepsilon)}(t) = X_{\theta^{(\varepsilon)}}(\varepsilon t)$$
$$= \exp\left\{\left\langle \theta, x^{(\varepsilon)}(t) - \frac{1}{\varepsilon}\int_0^{\varepsilon t} b_\varepsilon\left(x^{(\varepsilon)}\left(\frac{s}{\varepsilon}\right)\right) ds\right\rangle - \frac{1}{2\varepsilon}\int_0^{\varepsilon t}\left\langle \theta, a_\varepsilon\left(x^{(\varepsilon)}\left(\frac{s}{\varepsilon}\right)\right)\theta\right\rangle ds\right\}$$
$$= \exp\left\{\left\langle \theta, x^{(\varepsilon)}(t) - \int_0^t b_\varepsilon(x^{(\varepsilon)}(s))\right\rangle ds - \tfrac{1}{2}\int_0^t \langle \theta, a_\varepsilon(x^{(\varepsilon)}(s))\theta\rangle\, ds\right\}$$

is a P-martingale. This proves that the distribution of $x^{(\varepsilon)}(\cdot)$ under P is $P^{(\varepsilon)}$.

Finally, since $a_\varepsilon \to a_0$ and $b_\varepsilon \to b_0$ uniformly on compacts, we shall see that $P^{(\varepsilon)} \Rightarrow P^{(0)}$ once we have shown that $\{P^{(\varepsilon)}\}_{\varepsilon>0}$ is relatively compact. Note that $\operatorname{tr} a_\varepsilon(x) \leqq C|x|^2$. Hence, if $a_{\varepsilon,R}(x) = a_\varepsilon(x \wedge R)$ and $P^{(\varepsilon,R)}$ solves the martingale problem for $a_{\varepsilon,R}$ and b_ε starting at 0, then $\{P^{(\varepsilon,R)}\}_{\varepsilon>0}$ is relatively compact for each $R > 0$. Moreover, $P^{(\varepsilon)}|_{M_{\tau_R}} = P^{(\varepsilon,R)}|_{M_{\tau_R}}$, where τ_R is the first exit time from $B(0, R)$. Thus it suffices to show that

$$\lim_{R\to\infty}\sup_{\varepsilon>0} P^{(\varepsilon)}\,(\tau_R \leqq t) = 0$$

for all $t \geqq 0$. Observe that

$$E^{P^{(\varepsilon)}}[|x(t)|^2] \leqq 2CE^{P^{(\varepsilon)}}\left[\int_0^t |x(s)|^2\, ds\right] + 2\,\|b\|_\infty^2\, t^2,$$

and therefore

$$E^{P^{(\varepsilon)}}[|x(t)|^2] \leqq 2\,\|b\|_\infty^2\left(t^2 + 2C\int_0^t e^{2C(t-s)}\, s^2\, ds\right).$$

Since

$$x(t) - \int_0^t b_\varepsilon(x(s))\, ds$$

is a $P^{(\varepsilon)}$-martingale, it follows from this that

$$\lim_{R\to\infty}\sup_{\varepsilon>0} P^{(\varepsilon)}(\tau_R \leqq t) = 0.$$

LEMMA 6.4. *Let*

$$L = \tfrac{1}{2}\sum_{i,j=1}^d a^{ij}(x)\frac{\partial^2}{\partial x_i\,\partial x_j} + \sum_1^d b^i(x)\frac{\partial}{\partial x_i}$$

and assume $\phi \in C_b^2(R^d)$. *If* $\langle \nabla\phi, a\nabla\phi\rangle(x^0) = 0$, *then* $\langle b - \frac{1}{2}a', \nabla\phi\rangle$ *is an invariant scalar and*

$$aH_{\langle\nabla\phi, a\nabla\phi\rangle}a(x^0) \equiv \left(\left(a^{ik}\frac{\partial^2}{\partial x_k\,\partial x_l}\,a^{lj}\right)\right)_{1\le i,j\le d},$$

$$a(\nabla(a\nabla\phi))^*(x^0) \equiv ((a^{ik}(a^{lj}\,\phi_{,l})_{,k}))_{1\le i,j\le d},$$

and

$$(\nabla(a\nabla\phi))a(\nabla(a\nabla\phi))^*(x^0) \equiv (((a^{li}\,\phi_{,l})_{,k}\,a^{kk'}(a^{l'j}\phi_{,l'})_{,k'}))_{1\le i,j\le d}$$

are invariant as elements of $T_{x^0}\otimes T_{x^0}$.

Proof: Let $F\colon R^d \to R^d$ be a C^2 diffeomorphism and set $y = F(x)$. Then $L(f\circ F) = (\bar{L}f)\circ F$, where

$$\bar{L} = \tfrac{1}{2}\sum_{i,j=1}^{d}\bar{a}^{ij}\frac{\partial}{\partial y_i\,\partial y_j} + \sum_{i=1}^{d}\bar{b}^i\frac{\partial}{\partial y_i},$$

$$\bar{a}^{ij}\circ F = F^i_{,k}\,a^{kl}\,F^j_{,l},$$

$$\bar{b}^i\circ F = F^i_{,k}\,b^k + \tfrac{1}{2}F^i_{,kl}\,a^{kl}.$$

Hence,

$$\begin{aligned}\bar{a}^{ij}_{,j}\circ F &= (F^{-1})^r_{,j}\circ F(F^i_{,k}\,a^{kl}\,F^j_{,l})_{,r}\\ &= (F^{-1})^r_{,j}\circ FF^i_{,kr}\,a^{kl}\,F^j_{,l} + (F^{-1})^r_{,j}\circ FF^i_{,k}\,a^{kl}_{,r}\,F^j_{,l}\\ &\quad + (F^{-1})^r_{,j}\circ FF^i_{,k}\,a^{kl}\,F^j_{,lr}\\ &= F^i_{,kr}\,a^{kr} + F^i_{,k}\,a^{kr}_{,r} + (F^{-1})^r_{,j}\circ FF^i_{,k}\,a^{kl}F^j_{,lr}.\end{aligned}$$

Thus

$$(\bar{b}^i - \tfrac{1}{2}\bar{a}^{i,j}_{,j})\circ F = F^i_{,k}\,b^k - \tfrac{1}{2}F^i_{,k}\,a^{kr}_{,r} - \tfrac{1}{2}(F^{-1})^r_{,j}\circ FF^i_{,k}\,a^{kl}F^j_{,lr}.$$

Since $(\nabla\bar{\phi})^i\circ F = (F^{-1})^r_{,i}\circ F\phi_{,r}$,

$$\langle\bar{b} - \tfrac{1}{2}\bar{a}', \nabla\bar{\phi}\rangle\circ F = \langle b - \tfrac{1}{2}a', \nabla\phi\rangle - \tfrac{1}{2}(F^{-1})^r_{,j}\,F^j_{,lr}\,a^{rl}\phi_{,r}.$$

In particular, if $\langle\nabla\phi, a\nabla\phi\rangle(x^0) = 0$, then

$$\langle\bar{b} - \tfrac{1}{2}\bar{a}', \nabla\bar{\phi}\rangle(y^0) = \langle b - \tfrac{1}{2}a', \nabla\phi\rangle(x^0).$$

Since a transforms like an element of $T_{x^0}\otimes T_{x^0}$, it suffices to prove that $H_{\langle\nabla\phi, a\nabla\phi\rangle}(x^0) \in T^*_{x^0}\otimes T^*_{x^0}$ in order to show that $aH_{\langle\nabla\phi, a\nabla\phi\rangle}a(x^0) \in T_{x^0}\otimes T_{x^0}$.

But

$$\frac{\partial^2 \langle \nabla \bar{\phi}, \bar{a} \nabla \bar{\phi} \rangle}{\partial y_i \partial y_j} \circ F = (F^{-1})^k_{.ij} \circ F(\langle \nabla \phi, a \nabla \phi \rangle)_{.k}$$

$$+ (F^{-1})^k_{.j} \circ F(\langle \nabla \phi, a \nabla \phi \rangle)_{.kl} (F^{-1})^k_{.i} \circ F ,$$

and $(\langle \nabla \phi, a \nabla \phi \rangle)_{.k}(x^0) = 0$, since x^0 is a minimum of $\langle \nabla \phi, a \nabla \phi \rangle$.

Next note that

$$(\nabla(\bar{a}\nabla\bar{\phi}))^i_j \circ F = (F^{-1})^r_{.j} \circ F(F^i_{.k} a^{lk} \phi_{.l})_{.r}$$

$$= F^i_{.k}(a^{lk}\phi_{.l})_{.r}(F^{-1})^r_{.j}$$

when $\langle \nabla \phi, a \nabla \phi \rangle = 0$. Hence,

$$(\bar{a}(\nabla(\bar{a}\nabla\phi))^*)^{ij}(y^0) = F^i_{.k} a^{kr}(a^{lk'}\phi_{.l})_{.r} F^j_{.k'}(x^0)$$

$$= F^i_{.k}(a(\nabla(a\nabla\phi))^*)^{kk'} F^j_{.k'}(x^0) .$$

Finally,

$$((\nabla(\bar{a}\nabla\bar{\phi}))\bar{a}(\nabla\bar{a}\nabla\bar{\phi})^*)^{ij}(y^0) = F^i_{.k}(a^{lk}\phi_{.l})_{.r} a^{rs}(a^{l'k'}\phi_{.l'})_{.s} F^j_{.k'}(x^0)$$

$$= F^i_{.k}[(\nabla a(a\nabla\phi))a(\nabla a(a\nabla\phi))^*]^{kk'} F^j_{.k'}(x^0) .$$

THEOREM 6.3. *Assume $x^0 \in \delta G$ is given; then for $P_{x^0}(\tau' > 0) = 1$ to hold it is necessary that each of the following conditions be satisfied:*

(i) $\langle \nabla \phi, a \nabla \phi \rangle(x^0) = 0$,

(i′) $L\phi^2(x^0) = 0$,

(ii) $\langle b - \frac{1}{2}a', \nabla \phi \rangle(x^0) \geqq 0$,

(iii) $L\phi(x^0) \geqq 0$,

(iv) $a(\nabla(a\nabla\phi))^*(x^0) \in S_d$,

(v) $(\nabla(a\nabla\phi))a(\nabla(a\nabla\phi))^*(x^0) = \frac{1}{2}aH_{\langle\nabla\phi, a\nabla\phi\rangle}a(x^0)$.

Proof: First observe that $L\phi^2(x^0) = \langle \nabla \phi, a \nabla \phi \rangle(x^0)$. Hence (i) and (i′) are equivalent. Next note that conditions (i)–(v) are invariant under C^2 changes of coordinates. Therefore we shall assume that $x^0 = 0$ and that, near 0, $\phi(x) = x_1$. Further, we can choose the coordinates so that $((a^{ij}(0)))_{2 \leq i, j \leq d}$

is diagonal. In such a coordinate system our conditions become:

(i) $a^{11}(0) = 0$,

(ii) $b^1(0) - a^{1j}_{,j}(0) \geqq 0$,

(iii) $b^1(0) \geqq 0$,

(iv) $((a^{ii}a^{1j}_{,i}(0)))_{2 \leqq i, j \leqq d} \in S_{d-1}$,

(v) $a^{1i}_{,k}\, a^{kk}\, a^{1j}_{,k}(0) = \frac{1}{2} a^{ii} a^{11}_{,ij}\, a^{jj}(0)$, $\quad 2 \leqq i, j \leqq d$.

To prove (i), let $Q^{(\varepsilon)}$ be the distribution of $x(\varepsilon t)/\varepsilon^{1/2}$ under P_0. Then $Q^{(\varepsilon)}$ tends weakly to $Q^{(0)}$, the solution of the martingale problem for $a(0)$ and 0 starting at 0. In particular, $x_1(\cdot)$ is distributed under $Q^{(0)}$ like $(a^{11}(0))^{1/2}\beta(\cdot)$, where $\beta(\cdot)$ is a one-dimensional Brownian motion. Since $P_0(\tau' > 0) = 1$ implies $Q^{(0)}(\tau' = \infty) = 1$, it follows that $a^{11}(0) = 0$. Condition (iii) is proved by observing that $x_1(\cdot)$ is distributed under P_0 like a process $\xi(\cdot)$ having the form

$$\xi(t) = \int_0^t \alpha(s)\, d\beta(s) + \int_0^t b^1(\xi(s))\, ds,$$

where $\alpha : [0, \infty) \times \Omega \to R$ is a bounded nonanticipating function relative to the one-dimensional Brownian motion $\beta(\cdot)$. Thus,

$$P_0\left(x(t) = \int_0^t b^1(x(s))\, ds \text{ for some } t \in (0, \delta) \text{ and every } \delta > 0\right) = 1.$$

We turn now to the proof of (ii), (iv) and (v). Let $P^{(\varepsilon)}$ denote the distribution of

$$x^{(\varepsilon)}(t) = \frac{x_1(\varepsilon t)}{\varepsilon}, \frac{x_2(\varepsilon t)}{\varepsilon^{1/2}}, \dots, \frac{x_d(\varepsilon t)}{\varepsilon^{1/2}}$$

under P_0. Since $a^{11}(0) = 0$, $P^{(\varepsilon)}$ tends weakly to the solution of the martingale problem for a_0 and b_0 starting at 0, where

$$a^{11}(x) = \tfrac{1}{2} \sum_{k,l \geqq 2} a^{11}_{,kl}(0) x_k\, x_l,$$

$$a_0^{1i}(x) = \sum_{k \geqq 2} a^{1i}_{,k}\, x_k, \qquad i \geqq 2,$$

$$a_0^{ij}(x) = a^{ij}(0), \qquad i, j \geqq 2,$$

$$b_0^1(x) = b^1(0) \quad \text{and} \quad b_0^i(x) = 0, \qquad i \geqq 2.$$

In particular, supp $(P^0) = \overline{S_{a_0, b_0 - \frac{1}{2}a'_0}(0)}$. Let

$$A^{ij} = a^{ij}(0)\,,$$
$$B_j^i = a^{1i}_{,j}(0)\,,$$
$$H_{ij} = a^{11}_{,ij}(0)\,,$$

for $2 \leqq i, j \leqq d$. Then $P_0(\tau' > 0) = 1$ implies that, for $\psi \in C([0, \infty), R^d)$ and all $t \geqq 0$,

$$\tfrac{1}{2}\int_0^t \langle \tilde{\eta}(s), H\tilde{\eta}(s)\rangle \psi_1(s)\, ds + \int_0^t \langle \tilde{\psi}(s), B\tilde{\eta}(s)\rangle\, ds + \int_0^t (b - \tfrac{1}{2}a')'(0)\, ds \geqq 0\,,$$

where

$$\tilde{\eta}(t) = \int_0^t B\tilde{\eta}(s)\psi_1(s)\, ds + A\int_0^t \tilde{\psi}(s)\, ds$$

and $\tilde{x} = (x_2, \cdots, x_d)$. Taking $\psi \equiv 0$, we see that $(b - \frac{1}{2}a')'(0) \geqq 0$. Taking $\psi_1 \equiv 0$, we have

$$\int_0^t \langle \tilde{\psi}(s), BA\dot{\tilde{\psi}}(s)\rangle\, ds + \int_0^t (b - \tfrac{1}{2}a')'(0)\, ds \geqq 0$$

for all $\tilde{\psi} \in \tilde{\Psi} = \{f \in C^1([0, \infty), R^{d-1}) : f(0) = 0\}$. Thus,

$$\int_0^t \langle \tilde{\psi}(s), BA\dot{\tilde{\psi}}(s)\rangle\, ds \geqq 0$$

for all $\tilde{\psi} \in \tilde{\Psi}$. Now suppose that $\tilde{\psi} \in \tilde{\Psi}$ such that $\tilde{\psi}(1) = 0$ and

$$\int_0^1 \langle \tilde{\psi}(s), BA\dot{\tilde{\psi}}(s)\rangle\, ds > 0\,.$$

Then,

$$\int_0^1 \langle \tilde{\psi}(1 - s), BA\dot{\tilde{\psi}}(1 - s)\rangle\, ds < 0\,,$$

which is a contradiction. It follows that the vector field $BA\tilde{x}$ is an exact differential, and therefore $BA = (BA)^* = AB^*$. In particular,

$$0 \leqq \int_0^t \langle \tilde{\psi}(s), BA\dot{\tilde{\psi}}(s)\rangle\, ds = \langle \tilde{\psi}(t), BA\tilde{\psi}(t)\rangle$$

for all $\tilde{\psi} \in \tilde{\Psi}$, and so $BA \in S_{d-1}$. This proves (iv).

To prove (v), suppose $w \in R^d$ is given and define $\psi_1(t) = w_1$ and

$$\tilde{\psi}(t) = \tilde{w} - tw_1 B^* \tilde{w} .$$

Then, using the fact that $BA = AB^*$, one can easily show that the corresponding $\tilde{\eta}(\cdot)$ is given by $\tilde{\eta}(t) = tA\tilde{w}$. Thus,

$$\tfrac{1}{2}w_1 \int_0^t s^2 \langle \tilde{w}, AHA\tilde{w}\rangle\, ds + \int_0^t s\langle \tilde{w}, BA\tilde{w}\rangle\, ds - w_1 \int_0^t s^2 \langle B^*\tilde{w}, BA\tilde{w}\rangle\, ds + \int_0^t (b - \tfrac{1}{2}a')'(0)\, ds \geqq 0$$

for all $w \in R^d$ and $t \geqq 0$. Clearly this is possibly only if

$$\tfrac{1}{2}\langle \tilde{w}, AHA\tilde{w}\rangle = \langle \tilde{w}, B^2 A\tilde{w}\rangle$$

for all $\tilde{w} \in R^{d-1}$. Since $B^2 A = BAB^*$, we have now proved (v).

Remark. It is interesting to note that although in the proof of Theorem 6.3 we used the characterization of supp $(P^{(0)})$, we could have proved the theorem without our previous result. Indeed, the stochastic integral equations associated with a_0 and b_0 are sufficiently simple to prove (i) and (ii) directly. The importance of this observation is that Theorem 6.3 can be used to give an alternate derivation of our characterization of the support of a degenerate diffusion. This alternate route would resemble the method used by Bony [1], Hill [3] and Redheffer [8] in their work on the strong maximum principle. In their scheme, Theorem 6.3 would play the role of their lemma which states that if x^0 is a characteristic boundary point of a ball and if the Fichera drift at x^0 points into the ball, then a "barrier" exists for x^0. With minor modifications, our method can be made to cover the operators studied by Redheffer. Namely, if

$$L = \tfrac{1}{2} \sum_{i,j=1}^{n} a^{ij}(t, x) X_i X_j + Y ,$$

where $X_1, \cdots, X_n$, Y are d-dimensional C^1-vector fields and $((a^{ij}(t, x)))_{1 \leqq i,j \leqq n}$ is bounded, continuous, and uniformly positive definite, then the support of any solution to the martingale problem for L starting at (s, x) is independent of $((a^{ij}(t, x)))_{1 \leqq i,j \leqq n}$. Unfortunately, unless $((a^{ij}(t, x)))$ is smooth enough, we have no uniqueness result for the martingale problem.

7. On the Set of Regular Points

Let us define the following sets on the boundary δG of G:

$$\begin{aligned}
\Sigma_3 &= \{x \in \delta G : \langle \nabla\phi, a\nabla\phi\rangle(x) > 0\}, \\
\Sigma_2 &= \{x \in \delta G - \Sigma_3 : \langle b - \tfrac{1}{2}a', \nabla\phi\rangle(x) < 0\}, \\
\Sigma_2^* &= \{x \in \delta G - \Sigma_3 : (L\phi)(x) < 0\}, \\
\Sigma_0 &= \{x \in \delta G - \Sigma_3 : \langle b - \tfrac{1}{2}a', \nabla\phi\rangle(x) = 0\}, \\
\Sigma_1 &= \{x \in \delta G - \Sigma_3 : \langle b - \tfrac{1}{2}a', \nabla\phi\rangle(x) > 0\}.
\end{aligned}$$

Clearly, the sets $\Sigma = \Sigma_2 \cup \Sigma_3$ and $\Sigma^* = \Sigma_2^* \cup \Sigma_3$ are both open sets of δG. Moreover, in view of Theorem 6.3, the set Γ of τ'-regular points contains Σ and Σ^*.

From the earlier works of Fichera, Kohn-Nirenberg and Oleinik it is clear that the set of τ'-regular points Γ and the set Σ cannot be essentially different.

We prove in this section that $\Gamma \subset \bar{\Sigma}$ so that one always has

$$\Sigma \cup \Sigma^* \subset \Gamma \subset \bar{\Sigma} = \overline{\Sigma^*}.$$

Moreover, one should also try to answer the following questions: Are the τ and τ' solutions for the boundary value problems really different? If the boundary value is specified only on Σ (or Σ^*), which can be described more easily than Γ, is the problem meaningful? It is proved in this section that

1. the two solutions corresponding to τ and τ' agree almost everywhere, that is,
$$P_x[\tau = \tau'] = 1 \quad \text{a.e. } x \quad \text{in} \quad G;$$

2. it is sufficient to provide the boundary values only on Σ (or Σ^*), if one is satisfied with having the solution only almost everywhere, that is,

$$\begin{aligned}
P_x[x(\tau) \in \Sigma \cap \Sigma^*, \tau < \infty] &= 1 \quad \text{a.e. } x \quad \text{in} \quad G, \\
P_x[x(\tau') \in \Sigma \cap \Sigma^*, \tau' < \infty] &= 1 \quad \text{a.e. } x \quad \text{in} \quad G.
\end{aligned}$$

LEMMA 7.1. *The sets* Σ_0, Σ_1, Σ_2, Σ_2^* *and* Σ_3 *are all invariantly defined and* $\bar{\Sigma} = \overline{\Sigma^*}$.

Proof: The invariance is obvious from the earlier results. To prove that $\bar{\Sigma} = \overline{\Sigma^*}$, it suffices to show that $L\phi(x^0) = \langle b - \tfrac{1}{2}a', \nabla\phi\rangle(x^0)$ for $x^0 \in \delta G - \bar{\Sigma}_3$. Given $x^0 \in \delta G - \overline{\Sigma_3}$, choose coordinates in such a way that $x^0 = 0$ and

$x_1 = \phi(x)$ near the point x^0. In terms of this coordinate system, we have to prove that $a^{1j}_{,j}(0) = 0$. But $a^{11}(0, x_2, \cdots, x_d) = 0$ for x near 0. Thus, $a^{ij}(0, x_2, \cdots, x_d) = 0$ for x near 0 and so $a^{1j}_{,l}(0) = 0$ for $j \geqq 2$ and $l \geqq 2$. Moreover, a^{11} is a non-negative function vanishing at 0. Thus $a^{11}_{,1} = 0$.

LEMMA 7.2. *For any* $x \in G$, $P_x(x(\tau) \notin \bar{\Sigma}, \tau < \infty) = 0$.

Proof: Assume that $x^0 = 0$ and that $\phi(x) = x_1$ in some neighborhood $|x| \leqq R$. Choose $0 < 2\delta < R$ so that $\{x : |x| \leqq 2\delta\} \cap \bar{\Sigma} = \phi$. Then

$$a^{11}(0, x_2, \cdots, x_d) = 0 \qquad \text{and} \qquad b^1(0, x_2, \cdots, x_d) \geqq 0$$

for $|x| \leqq 2\delta$. Hence, there is a $\lambda > 0$ such that for $|x| \leqq 2\delta$

$$a^{11}(x) \leqq \lambda x_1^2 ,$$

and

$$b^1(x) \geqq -\lambda |x_1| .$$

Given $0 < \varepsilon < \delta$, define $V_\varepsilon(x) = \varepsilon/x_1$. Then

$$LV_\varepsilon(x) = \varepsilon a^{11}(x) \frac{1}{x_1^3} - \varepsilon b^1(x) \frac{1}{x_1^2} \leqq 2\lambda V_\varepsilon(x) \qquad \text{if} \qquad |x| \leqq 2\delta .$$

Thus if

$$\sigma_T = (T \wedge \inf\{t \geqq 0 : |x(t)| \geqq 2\delta\})$$

and

$$\tau_\varepsilon = \inf\{t \geqq 0 : x_1(t) = \varepsilon\} ,$$

then

$$\begin{aligned} 1 &= E^{P_x}\left[\frac{V_\varepsilon(x(\tau_\varepsilon \wedge \sigma_T))}{V_\varepsilon(x)} \exp\left\{-\int_0^{\tau_\varepsilon \wedge \sigma_T} \frac{LV_\varepsilon}{V_\varepsilon}(x(s))\, ds\right\}\right] \\ &\geqq E^{P_x}\left[\frac{V_\varepsilon(x(\tau_\varepsilon \wedge \sigma_T))}{V_\varepsilon(x)} \exp\{-2\lambda\tau_\varepsilon \wedge \sigma_T\}\right] \\ &\geqq \frac{x_1}{\varepsilon} E^{P_x}[e^{-2\lambda\tau_\varepsilon}, \tau_\varepsilon \leqq \sigma_T] \end{aligned}$$

for $|x| \leqq \delta$ such that $x_1 \geqq \varepsilon$. Letting $\varepsilon \to 0$ and then letting $T \to \infty$, we conclude that

$$P_x[\tau < \sigma_\infty] = 0$$

for x such that $x_1 > 0$ and $|x| < \delta$. Now define

$$\tau_0 = \inf\{t \geqq 0 : |x(t)| \leqq \delta\}, \cdots, \tau_{2n+1} = \{\inf t \geqq \tau_{2n} : |x(t)| \geqq 2\delta\}$$

and

$$\tau_{2n} = \inf\{t \geqq \tau_{2n-1} : |x(t)| \leqq \delta\}.$$

Then,

$$\begin{aligned} P_x(|x(\tau)| < \delta, \tau < \tau_{2n+1}) &= \sum_{k=0}^{n} P_x(|x(\tau)| < \delta, \tau_{2k} \leqq \tau < \tau_{2k+1}) \\ &= \sum_{k=0}^{n} E^{P_x}[x_{\tau \geqq \tau_{2k}} P_{x(\tau_{2k})}(\tau < \tau_1)] = 0 \end{aligned}$$

for all n. Since $\tau_{2n+1} \to \infty$ as $n \to \infty$, we see that

$$P_x(|x(\tau)| < \delta, \ \tau < \infty) = 0.$$

LEMMA 7.3. *If $x^0 \in \delta G - \overline{\Sigma}$, then there is an $\alpha > 0$ such that*

$$\varliminf_{\substack{x \to x_0 \\ x \in G}} P_x[\tau \geqq \alpha] \geqq \alpha.$$

In particular, $\Gamma \subset \overline{\Sigma}$.

Proof: If $x^0 \in \delta G - \overline{\Sigma}$, then there is a $\delta > 0$ such that, for $|x - x^0| \leqq \delta$, $\{y : |y - x| \leqq \delta\} \cap \overline{\Sigma} = \varnothing$. Thus, by standard estimates and Lemma 7.2, there is an $\alpha > 0$ such that

$$P_x[\tau \geqq \alpha] \geqq P_x\left[\sup_{0 \leqq t \leqq \alpha} |x(t) - x| < \delta\right] \geqq \alpha \qquad \text{if} \qquad |x - x^0| < \delta.$$

Finally, if $x^0 \in \Gamma$, then $P_{x^0}(\tau' > 0) = 0$. Hence $P_{x^0}(\tau' \geqq t) = 0$ for any $t > 0$. But $[\tau' \geqq t]$ is a closed set and therefore

$$\lim_{\substack{x \to x^0 \\ x \in G}} P_x[\tau' \geqq t] \leqq P_{x^0}[\tau' \geqq t] = 0 \quad \text{for all} \quad t > 0.$$

Since $\tau \leqq \tau'$, this proves that

$$\lim_{\substack{x \to x^0 \\ x \in G}} P_x[\tau' \geqq t] = 0 \quad \text{for all} \quad t > 0.$$

In particular, $x^0 \in \overline{\Sigma}$.

LEMMA 7.4. *Let* $\Delta = \delta G - \Sigma_3$ *and* $\Delta_1 = \{x \in \Delta : L\langle \nabla\phi, a\nabla\phi\rangle(x) \neq 0\}$. *Then* $L\phi(x) = \langle b - \frac{1}{2}a', \nabla\phi\rangle(x)$ *for* $x \in \Delta - \Delta_1$ *and* Δ_1 *has zero boundary measure.*

Proof: First suppose that $x^0 \in \Delta - \Delta_1$ and choose a coordinate system so that $x^0 = 0$, $\phi(x) = x_1$ near x^0, and $((a^{ij}(0)))_{i,j\leq 2}$ is diagonal. Then

$$0 = La^{11}(0) = \tfrac{1}{2}\operatorname{tr}(a(0)H_{a^{11}}(0)) .$$

Since $H_{a^{11}}(0)$ is non-negative definite, $a^{il}(0)a^{11}_{,ij}(0) = 0$ for $1 \leqq i, j \leqq d$. In particular, $a^{ii}(0) \neq 0$ implies $a^{11}_{,ii}(0) = 0$ for $2 \leqq i \leqq d$. From this and the inequality $|a^{1i}(x)|^2 \leqq a^{11}(x)a^{ii}(x)$, it follows that $a^{1i}_{,i}(0) = 0$, $2 \leqq i \leqq d$. Since $a^{11}_{,i}(0) = 0$, we see that $a_1'(0) = 0$.

Next suppose that $x^0 \in \Delta_1$. Again choose coordinates such that $x^0 = 0$ and $\phi(x) = x_1$ near x^0. Then, locally on Δ_1, $La^{11} \neq 0$. Since a^{11} is zero, we must have $a^{11}_{,ij} \neq 0$ for some $i, j \geqq 2$.

If A is any open set in a Euclidean space and f is C^2 and non-negative, then on the set A_0 where $f = 0$ we must also have $D^2 f = 0$ a.e.

This means that Δ_1 must be of boundary measure zero.

LEMMA 7.5. *Let* $\Delta_2 = \{x \in \Delta - \Delta_1 : L\phi(x) = 0\}$. *Then there is a set* $E \subseteq G$ *of zero Lebesgue measure such that* $P_x(x(\tau) \in \Delta_1 \cup \Delta_2, \tau < \infty) = 0$ *for all* $x \notin E$.

Proof: Given a non-negative $g \in C_0^\infty(G)$, choose $\lambda > 0$ large enough so that $(\lambda - L^*)u = g$ has a C_b^2-solution. Let $f \in C_0^\infty(R^d)$ be non-negative. Then,

$$\begin{aligned}\int u(x)f(x)\,dx &= \int g(x)E^{P_x}\left[\int_0^\infty e^{-\lambda t}f(x(t))\,dt\right]dx \\ &\geqq \int g(x)E^{P_x}\left[e^{-\lambda\tau}E^{P_{x(\tau)}}\left[\int_0^\infty e^{-\lambda t}f(x(t))\,dt\right]\right]dx \\ &= \int_{\partial G}\Pi_g^\lambda(dz)E^{P_z}\left[\int_0^\infty e^{-\lambda t}f(x(t))\,dt\right],\end{aligned}$$

where

$$\Pi_g^\lambda(F) = \int g(x)E^{P_x}(e^{-\lambda\tau}\chi_F(x(\tau))]\,dx\,, \qquad F \in B[\delta G].$$

Suppose $x^0 \in \Delta$ and choose coordinates so that $x^0 = 0$ and $\phi(x) = x_1$ for x in a neighborhood N of x^0. Then

$$\int u(x)f(x)\,dx \geqq \int_{\Delta\cap T}\Pi_g^\lambda(dz)E^{P_z}\left[\int_0^\infty e^{-\lambda t}f(x(t))\,dt\right].$$

Making the obvious substitution, one now gets

$$\int u(\varepsilon x_1, x_2, \cdots, x_d) f(x)\, dx \geqq \int_{\Delta \cap T} \Pi_g^\lambda(dz) E^{P_z^{(0)}}\left[\int_0^\infty e^{-\lambda\varepsilon t} f(x(t))\, dt\right],$$

where $P_z^{(\varepsilon)}$ denotes the distribution of $[x_1(\varepsilon t)/\varepsilon, x_2(\varepsilon t), \cdots, x_d(\varepsilon t)]$ under P_z. Using our earlier result, Lemma 6.3, one can easily see that, for $z \in \Delta \cap N$, $P_z^{(\varepsilon)}$ tends weakly to a measure $P_z^{(0)}$ under which $x_i(0) \equiv z_i$, $2 \leqq i \leqq d$. Moreover, $x_i(\cdot) \equiv z_i$ a.s. $P_z^{(0)}$ if $z \in \Delta_2 \cap N$. Hence,

$$\int u(0, x_2, \cdots, x_d) f(x)\, dx \geqq \int_{\Delta \cap N} \Pi_g^\lambda(dz) E^{P_z^{(0)}}\left[\int_0^\infty f(x_1(t), z_2, \cdots, z_d)\, dt\right],$$

where $x_1(\cdot) \equiv z_1$ a.s. $P_z^{(0)}$ for $z \in \Delta_2 \cap N$. Since this is true for all non-negative $f \in C_0^\infty(R^d)$, it is also true for $f(x) = h(x_1)\bar{h}(x_2, \cdots, x_d)$, where h and $\bar{h}$ are bounded, non-negative measurable functions having compact support. Thus

$$\left(\int h(x_1)\, dx_1\right) \int u(0, \bar{x}) \bar{h}(z)\, dz \geqq \int_{\Delta \cap T} \Pi_g^\lambda(dz) \bar{h}(z) E^{P_z^{(0)}}\left[\int_0^\infty h(x_1(t))\, dt\right].$$

But this is possible only if $\Pi_g^\lambda|_{\Delta \cap N}$ is absolutely continuous with respect to Lebesgue measure on $\Delta \cap N$ and if $\Pi_g^\lambda(\Delta_2 \cap N) = 0$. Hence,

$$\Pi_g^\lambda((\Delta_1 \cup \Delta_2) \cap N) = 0.$$

We see now that

$$\int g(x) P_x(x(\tau) \in \Delta_1 \cup \Delta_2, \tau < \infty)\, dx = 0$$

for all non-negative $g \in C_0^\infty(G)$, and our assertion follows from this.

THEOREM 7.1. *There is a set $E \subseteq G$ of zero Lebesgue measure such that*

$$P_x(x(\tau) \notin \Sigma, \tau < \infty) = P(x(\tau) \notin \Sigma^*, \tau < \infty) = 0$$

for $x \notin E$. In particular, $P_x(\tau < \tau') = 0$ for $x \notin E$.

Proof: Since $\delta G - \Sigma^* \subseteq \Delta_1 \cup \Delta_2 \cup \Delta_3$, where

$$\Delta_3 = \{x \in \Delta - \Delta_1 : L\phi(x) > 0\},$$

and $(\Sigma - \Sigma^*) \cup (\Sigma^* - \Sigma) \subseteq \Delta_1$, we need only show that

$$P_x(x(\tau) \in \Delta_3, \tau < \infty) = 0 \quad \text{a.e.} \quad \text{in} \quad G.$$

To do this, let V be a non-negative, bounded, smooth function which is positive off $\bar{G}$. Given a non-negative $g \in C_0^\infty(G)$, choose $\mu > 0$ in such a way that $(\mu + \lambda V - L^*)u_\lambda = g$ has a C_b^2-solution u_λ for all $\lambda > 0$. Then for any non-negative $f \in C_0^\infty(R^d)$ we have

$$\int u_\lambda(x)f(x)\,dx = \int g(x)E^{P_x}\left[\int_0^\infty e^{-\mu t}\exp\left\{-\lambda\int_0^t V(x(s))\,ds\right\}f(x(t))\,dt\right]dx$$
$$\geqq \int_{\partial G}\Pi_g^\mu(dz)E^{P_z}\left[\int_0^\infty e^{-\mu t}\exp\left\{-\lambda\int_0^t V(x(s))\,ds\right\}f(x(t))\,dt\right],$$

where Π_g^μ is defined as in the proof of Lemma 7.5. Letting $\lambda \to \infty$, we now get

$$\int u_\infty(x)f(x)\,dx \geqq \int_{\partial G}\Pi_g^\mu(dz)E^{P_z}\left[\int_0^{\tau'} e^{-\mu t}f(x(t))\,dt\right],$$

where u_∞ is the non-increasing limit of the u_λ. The rest of the proof parallels that of Lemma 7.5. Given $x^0 \in \Delta_3$, choose coordinates so that $x^0 = 0$ and $\phi(x) = x$ in a neighborhood N of x^0. Substituting as in Lemma 7.5, we find that

$$\int u_\infty(\varepsilon x_1, x_2, \cdots, x_d)f(x)\,dx \geqq \int_{\Delta_3\cap N}\Pi_g^\mu(dz)E^{P_z^{(\varepsilon)}}\left[\int_0^{\tau'} e^{-\mu\varepsilon t}f(x(t))\,dt\right],$$

where $P_z^{(\varepsilon)}$ is as in Lemma 7.5. Next observe that, for $z \in \Delta_3$, $L\phi(z) > 0$, $L\phi^2(z) = 0$ and $L^2\phi^2(z) = 2(L\phi)^2(z)$. Hence $P_z(\tau' > 0) = 1$ for $z \in \Delta_3$; and so for every $z \in \Delta_3$ and $0 < \alpha < 1$ there is a $t > 0$ such that

$$P_z^{(\varepsilon)}(\tau' > t/\varepsilon) \geqq 1 - \alpha$$

for all $\varepsilon > 0$. Therefore,

$$\lim_{\varepsilon\searrow 0}\int u(\varepsilon x_1, x_2, \cdots, x_d)f(x)\,dx$$
$$\geqq \int_{\Delta_3\cap N}\Pi_g^\mu(dz)E^{P_z^{(0)}}\left[\int_0^\infty f(x_1(t), z_2, \cdots, z_d)\,dt\right].$$

This inequality holds for all bounded, non-negative, measurable f with compact support. In particular, if $f(x) = h(x_1)\bar{h}(x_2, \cdots, x_d)$, where $\operatorname{supp}\bar{h} \subseteq \Delta_3 \cap N$, then, since points in Δ_3 are regular for the L^*-process, we find that

$$\int_{\Delta_3\cap N}\Pi_g^\mu(dz)E^{P_z^{(0)}}\left[\int_0^\infty h(x_1(t))\,dt\right]\bar{h}(z) = 0,$$

and therefore $\Pi_g^\mu(\Delta_3 \cap N) = 0$. We have used here the facts that u_∞ vanishes at regular points of the L^*-process and that u_∞ is upper semi-continuous. This proves the first part of the theorem.

To prove the second part, note that if $P_x(x(\tau) \notin \Sigma, \tau < \infty) = 0$, then

$$P_x(\tau < \tau') = E^{P_x}[\chi_{\tau<\infty} P_{x(\tau)}(\tau' > 0)] = 0 ,$$

since $P_x(\tau' > 0) = 0$ for $x \in \Sigma$.

8. On Generalized Solutions

Let G be an open set in R^d and $\bar{G}$ its closure. In connection with the solution of the boundary value problem we introduced, in Section 5, the extension $\tilde{L}$ of L. Although from the probabilistic point of view this extension is very natural, from the analytical point of view it is rather obscure. The first part of this section provides a characterization of $\tilde{L}$ in analytic terms.

There is also the standard notion of a weak solution to the equation $Lu = f$. This uses twice differentiable test functions. The second part of the section is devoted to a study of the relation of weak solutions to our solutions of the boundary value problem.

According to the definition in Section 5, we say that $\tilde{L}u = f$ in $\bar{G}$ if

$$u(x(\tau' \wedge t)) - \int_0^{\tau' \wedge t} f(x(s))\, ds$$

is a P_x-martingale for all $x \in \bar{G}$.

Let us denote by X the Banach space of pairs of bounded measurable functions (u, f) on $\bar{G}$. We say that a sequence (u_n, f_n) in X is $*$-convergent to a point (u, f) in X if

$$\lim_{n\to\infty} u_n(x) = u(x) \quad \text{for each} \quad x \in \bar{G},$$

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \text{for each} \quad x \in \bar{G},$$

$$\sup_n |u_n(x)| < \infty ,$$

and

$$\sup_n |f_n(x)| < \infty .$$

We denote by $Y \subset X$ pairs (u, f) such that u is smooth and $Lu = f$. We denote by $\bar{Y}$ the smallest $*$-closed subset of X containing Y. We then have the following characterization of $\tilde{L}$.

THEOREM 8.1. *$\tilde{L}u = f$ in $\bar{G}$ if and only if $(u, f) \in \bar{Y}$.*

The above theorem will be proved in several steps, each of which is stated as a lemma.

LEMMA 8.1. *If $(u, f) \in \bar{Y}$, then $\tilde{L}u = f$ in $\bar{G}$.*

Proof: Clearly if $(u, f) \in Y$, then by Itô's formula $\tilde{L}u = f$. The set of pairs (u, f) such that $\tilde{L}u = f$ is $*$-closed because limits of martingales are again martingales. Therefore, $\tilde{L}u = f$ if $(u, f) \in \bar{Y}$.

For technical reasons it is easier to work with the operator $L_1 \equiv L - I$ rather than L itself. By the relation

$$\tilde{L}_1 u = f$$

we mean that

$$e^{-t\wedge\tau'} u(x(t \wedge \tau')) - \int_0^{t\wedge\tau'} e^{-s} f(x(s))\, ds$$

is a P_x-martingale for all $x \in G$. Letting Y_1 be the pairs $(u, L_1 u)$ with smooth u, we take $\bar{Y}_1$ to be the $*$-closure of Y_1.

LEMMA 8.2. *$\tilde{L}u = f$ if and only if $\tilde{L}_1 u = f - u$, and $(u, f) \in \bar{Y}$ if and only if $(u, f - u) \in \bar{Y}_1$.*

Proof: Let $\tilde{L}u = f$. Set

$$\xi(t) = u(x(t \wedge \tau')) - \int_0^{t\wedge\tau'} f(x(s))\, ds\,.$$

We apply Lemma 4.1 with $\eta(t) = -\chi_{\tau'>t}\, e^{-t}$ and conclude that

$$\xi(t) e^{-t\wedge\tau'} - \int_0^t \xi(s)\eta(s)\, ds$$

is a P_x-martingale for all $x \in G$. After simplification this yields

$$\tilde{L}_1 u = f - u\,.$$

The converse is proved by retracing the steps. To prove the second part, we note that $(u, f) \in Y$ if and only if $(u, f - u) \in Y_1$. Hence $(u, f) \in \bar{Y}$ if and only if $(u, f - u) \in \bar{Y}_1$.

It is now clear that in order to prove Theorem 8.1 we need show only that $\tilde{L}_1 u = f$ if and only if $(u, f) \in \bar{Y}_1$.

LEMMA 8.3. *If* $\tilde{L}_1 u = f$, *then*

$$u(x) = E^{P_x}\left[e^{-\tau'} u(x(\tau')) - \int_0^{\tau'} e^{-s} f(x(s))\, ds\right].$$

Proof: From the fact that

$$e^{-t\wedge\tau'} u(x(t \wedge \tau')) - \int_0^{t\wedge\tau'} e^{-s} f(x(s))\, ds$$

is a P_x-martingale we know that

$$u(x) = E^{P_x}\left[e^{-t\wedge\tau'} u(x(t \wedge \tau')) - \int_0^{t\wedge\tau'} e^{-s} f(x(s))\, ds\right]$$

for $t \geqq 0$. By letting $t \to \infty$ we get the desired result. There is no difficulty regarding convergence because

$$\left| e^{-t\wedge\tau'} u(x(t \wedge \tau')) - \int_0^{t\wedge\tau'} e^{-s} f(x(s))\, ds \right| \leqq \|u\|_\infty + \|f\|_\infty .$$

We do not need any continuity regarding u because if $\tau' < \infty$, then

$$u(x(t \wedge \tau')) = u(x(\tau'))$$

for $t \geqq \tau'$ and if $\tau' = \infty$, then $e^{-t\wedge\tau'} \to 0$ as $t \to \infty$.

LEMMA 8.4. *If* f *and* g *are bounded measurable functions on* R^d *and* u *is given by*

$$u(x) = E^{P_x}\left[\bar{e}^{\tau'} g(x(\tau')) - \int_0^{\tau'} \bar{e}^{s} f(x(s))\, ds\right],$$

then $(u, f) \in \bar{Y}_1$.

Proof: Observe that the class of functions f, g for which $(u, f) \in \bar{Y}_1$ is $*$-closed. Hence we can and do assume that f and g are in $C_b^\infty[R^d]$. Let $V \in C_b^2[R^d]$ be a non-negative function which is zero on $\bar{G}$ and is strictly positive off $\bar{G}$. Define

$$w_n(t, x) = E^{P_x}\left[\int_0^t (ngV - f)(x(s)) \exp\left\{-s - n\int_0^s V(x(\sigma))\, d\sigma\right\} ds\right].$$

Using Oleinik's result (cf. Theorem 10.1 of the appendix), one can easily check that $w_n \in C^{1,2}([0, \infty) \times R^d)$ and that

$$\frac{\partial w_n}{\partial t} = (L_1 - nV)w_n + ngV - f.$$

Clearly,

$$\begin{aligned}\lim_{t\to\infty} w_n(t, x) &= w_n(x)\\ &= E^{P_x}\left[\int_0^\infty (ngV - f)(x(s)) \exp\left\{-s - n\int_0^s V(x(\sigma))\, d\sigma\right\} ds\right]\end{aligned}$$

uniformly in x. Moreover, $\partial w_n(t, x)/\partial t \to 0$ uniformly in x as $t \to \infty$. Hence, $(w_n, f) \in \bar{Y}_1$ for each n. But it is clear that $w_n(x) \to u(x)$ boundedly for $x \in \bar{G}$ and therefore $(u, f) \in \bar{Y}_1$.

This completes the proof of Theorem 8.1.

Corollary 8.1. *If $\bar{L}u = f$, then, for each $x \in \bar{G}$, the P_x-martingale*

$$u(x(t \wedge \tau')) - \int_0^{t\wedge\tau'} f(x(s))\, ds$$

is a.s. continuous. In particular, $u(x(t \wedge \tau'))$, $0 \leqq t < \infty$, is a.s. continuous.

Proof: The assertion is obvious if $u \in C(\bar{G})$. Moreover, by Doob's inequality, if (u, f) is the $*$-limit of $\{(u_n, f_n)\}_1^\infty \subseteq \bar{Y}$ and

$$u_n(x(t \wedge \tau')) - \int_0^{t\wedge\tau'} f_n(x(s))\, ds$$

is a.s. continuous, then

$$u(x(t \wedge \tau')) - \int_0^{t\wedge\tau'} f(x(s))\, ds$$

is a.s. continuous. Hence

$$u(x(t \wedge \tau')) - \int_0^{t \wedge \tau'} f(x(s))\, ds$$

is a.s. continuous for all $(u, f) \in \bar{Y}$; q.e.d.

We now consider the standard notion of a weak solution. Given $u, f \in L_\infty(G)$, we say that

$$Lu \overset{W}{=} f \quad \text{in} \quad G$$

if

$$\int uL^*\phi\, dx = \int f\phi\, dx$$

for all $\phi \in C_0^\infty(G)$. One way of formulating the boundary value problem is to demand that

$$Lu \overset{W}{=} f \quad \text{in} \quad G,$$

$$u = g \quad \text{on} \quad \Sigma_2 \cup \Sigma_3 .$$

We proceed to prove that the above formulation can be reduced to the earlier one.

LEMMA 8.5. *Suppose $\psi \in C_b^2[R^d]$ is a non-negative function which is strictly positive on G and vanishes off G. Denote by L^ψ the operator ψL and let P_x^ψ denote the solution to the martingale problem for L^ψ starting at x. Then $P_x^\psi[\tau < \infty] = 0$ for $x \in G$. Define*

$$\sigma_t = \inf\left\{s \geqq 0 : \int_0^s \frac{du}{\psi(x(u))} \geqq t\right\},$$

and let Q_x be the distribution of $x(\sigma_t)$, $t \geqq 0$, under P_x. Then $P_x^\psi = Q_x$. In particular, $P_x(\sigma_t < \tau) = 1$ for all $t \geqq 0$ and $x \in G$. Finally, $P_x\left(\sigma_\infty = \lim_{t \nearrow \infty} \sigma_t = \tau\right) = 1$ for all x.

Proof: That $P_x^\psi(\tau < \infty) = 0$, $x \in G$, is an immediate consequence of the fact that $\Sigma_2 \cup \Sigma_3 = \varnothing$ for L^ψ. The equality $P_x^\psi = Q_x$ follows from the uniqueness of P_x^ψ. To see that $P_x(\sigma_t < \tau) = 1$ for $x \in G$ and $t \geqq 0$, note that

$$P_x(\sigma_t < \tau) = P_x(x(\sigma_s) \in G, 0 \leqq s \leqq t) = Q_x(\tau > t) = 1 .$$

Finally,

$$P_x(\sigma_\infty < \tau) = P_x\left(\int_0^{\tau - \varepsilon} \frac{1}{\psi(x(s))}\, ds = \infty,\ \varepsilon > 0\right) = 0 .$$

LEMMA 8.6. *If V is a non-negative function in $C_b^2(R^d)$ and $u \in L^\infty(R^d)$ satisfies*

$$\int u(L - V)^* \phi \, dx = 0 \tag{8.1}$$

for all $\phi \in C_0^\infty(R^d)$, then

$$E^{P_g}\left[u(x(t)) \exp\left\{-\int_0^t V(x(s)) \, ds\right\} dt\right] = \int g \, u \, dx \, , \qquad t \geqq 0 \, , \tag{8.2}$$

for all non-negative $g \in L^1(R^d)$.

Proof: It suffices to prove (8.2) when $g \in C_0^\infty(R)$. First note that (8.1) must remain true when $\phi \in C_b^2(R^d)$ and $(L - V)^*\phi \in L^1(R^d)$. In particular, one can use Oleinik's result to show that for large λ there is a ϕ_λ such that

$$\lambda \int u \phi_\lambda \, dx = \int g \, u \, dx \, .$$

Moreover,

$$\int u \phi_\lambda \, dx = \int_0^\infty e^{-t} E^{P_g}\left[\exp\left\{-\int_0^t V(x(s)) \, ds\right\} u(x(t))\right] dt \, ,$$

and $E^{P_g}\left[\exp\left\{-\int_0^t V(x(s))\right\} ds \, u(x(t))\right]$ is continuous in t. Hence, by the uniqueness of the Laplace transform, (8.2) follows.

We now have to describe the manner in which u takes on the boundary value g on $\Sigma_2 \cup \Sigma_x$. Let us suppose that

$$\begin{gathered} Lu \overset{W}{=} f , \\ \operatorname*{ess\,lim}_{x \to a} u(x) = g(a) \qquad \text{for} \qquad a \in \Sigma_2 \cup \Sigma_3 \, . \end{gathered} \tag{8.3}$$

It is easy to check that g is continuous on $\Sigma_2 \cup \Sigma_3$ and u can be changed on a set of measure zero so that

$$\begin{gathered} Lu \overset{W}{=} f , \\ \lim_{x \to a} u(x) = g(a) \qquad \text{for} \qquad a \in \Sigma_2 \cup \Sigma_3 \, . \end{gathered} \tag{8.4}$$

THEOREM 8.2. *Let* u, f *and* g *satisfy* (8.4). *Then there are bounded measurable versions* u' *and* f' *of* u *and* f *so that*

$$\bar{L}u' = f'$$

and

$$u'(a) = g(a) = \lim_{\substack{x \to a \\ x \in G}} u'(x) \qquad \textit{for} \qquad a \in \Sigma_2 \cup \Sigma_3 .$$

Proof: Let $\tilde{f}$ be a bounded measurable version of f. Let $\bar{g}$ be a bounded measurable extension of g from $\Sigma_2 \cup \Sigma_3$ to Γ. Define

$$v(x) = E^{P_x}\left[e^{-\tau'} \bar{g}(x(\tau')) - \int_0^{\tau'} e^{-s} (\tilde{f} - u)(x(s))\, ds\right].$$

Then, by Lemma 8.4, $(v, \tilde{f} - u) \in \bar{Y}_1$. Hence, for $\phi \in C_0^\infty(G)$,

$$\int v(L - I)^* \phi \, dx = \int (\tilde{f} - u)\phi \, dx .$$

Also for such ϕ

$$\int u(L - I)^* \phi \, dx = \int (\tilde{f} - u)\phi \, dx .$$

Therefore, if $w = v - u$,

$$\int w(L - I)^* \phi \, dx = 0 \tag{8.5}$$

for $\phi \in C_0^2(G)$. It is easily seen that (8.5) holds for $\phi \in C_0^2(R^d)$ which are zero off G. In particular, if $\psi \in C_b^2(R^d)$ is non-negative and zero off G, then

$$\int w(L^\psi - \psi)^* \phi \, dx = 0$$

for all $\phi \in C_0^\infty(R^d)$. Thus by Lemmas 8.5 and 8.6, if $g \in L^1(R^d)$ is non-negative and is supported in G, then

$$\int g w \, dx = E^{P_g}[e^{-\sigma_t} w(x(\sigma_t))] \qquad \text{for} \qquad t \geqq 0 ,$$

where σ_t is as in Lemma 8.6. We let $t \to \infty$; then $\sigma_t \to \tau$. We know that for almost all paths with respect to P_g, $\tau = \tau'$ and $x(\tau) \in \Sigma_2 \cup \Sigma_3$. But in such a case, $w(x(\sigma_t)) \to 0$ because both v and u approach $g(a)$ as $x \to a$. All this is when $\tau < \infty$. Otherwise, $e^{-\sigma_t} \to 0$. So in any case $\int gw\,dx = 0$ for all $g \in L^1(R^d)$ which are supported in G. Therefore,

$$v(x) = u(x) \qquad \text{a.e.} \qquad \text{in} \qquad G\,.$$

If we now take $u' = v$ and $f' = f + w$, then it is easily checked that

$$\tilde{L}u' = f'$$

and

$$u'(a) = \lim_{\substack{x \to a \\ x \in G}} u'(x) = g(a) \qquad \text{for} \qquad a \in \Sigma_2 \cup \Sigma_3\,.$$

COROLLARY 8.2. *Let k be a bounded non-negative function on G and assume that either $\sup_{x \in G} E_x[\tau'] < \infty$ or that k is uniformly positive. Given*

$$f \in L^\infty(G) \qquad \textit{and} \qquad g \in L^\infty(\Gamma) \cap C(\Sigma_2 \cup \Sigma_3)\,,$$

the function

$$u(x) = E^{P_x}\left[g(x(\tau')) \exp\left\{-\int_0^{\tau'} k(x(s))\,ds\right\} - \int_0^{\tau'} \exp\left\{-\int_0^t k(x(s))\,ds\right\} f(x(t))\,dt\right]$$

satisfies

(i) $$Lu \overset{W}{=} ku + f\,,$$

(ii) $$\lim_{\substack{x \to a \\ x \in G}} u(x) = g(a) \qquad \textit{for} \qquad a \in \Sigma_2 \cup \Sigma_3\,.$$

Moreover, if $v \in L^\infty(G)$ satisfies (i) *and* (ii), *then $v = u$ a.e. in G.*

Proof: That u satisfies (i) and (ii) is obvious. Suppose v is any solution of (i) and (ii). Then by Theorem 8.2 we can find v', f' such that they agree almost everywhere with v, f,

$$\tilde{L}v' = kv' + f'\,,$$

$$v'(a) = \lim_{\substack{x \to a \\ x \in G}} v'(x) = g(a) \qquad \text{for} \qquad a \in \Sigma_2 \cup \Sigma_3\,.$$

But from Section 5, we know that

$$v'(x) = E^{P_x}\left[v'(x(\tau')) \exp\left\{-\int_0^{\tau'} k(x(s))\,ds\right\} - \int_0^{\tau'} \exp\left\{-\int_0^t k(x(s))\,ds\right\} f'(x(t))\,dt\right].$$

Since $f = f'$ almost everywhere and $v' = g$ on $\Sigma_2 \cup \Sigma_3$, it follows that $v' = u$ a.e.

9. Some Additional Remarks

Consider once again the first boundary value problem

$$\tilde{L}u - ku = f \quad \text{in} \quad \bar{G},$$
$$u = g \quad \text{on} \quad \Gamma,$$

under the assumption that k is either uniformly positive or that $\sup_{x \in \bar{G}} E_x[\tau'] < \infty$. We saw in Section 5 that

$$u(x) = E_x[\xi(\omega)], \tag{9.1}$$

where

$$\xi(\omega) = g(x(\tau')) \exp\left\{-\int_0^{\tau'} k(x(s))\, ds\right\} - \int_0^{\tau'} \exp\left\{-\int_0^t k(x(s))\, ds\right\} f(x(t))\, dt. \tag{9.2}$$

Now we want to use this representation of u to study certain stability questions associated with the first boundary value problem. In the ensuing discussion we shall assume that k and f are continuous on $\bar{G}$ and that g is bounded and continuous on Γ. There are three closely related questions to which we seek the answer:

A. Is $u(x)$ continuous as a function of x?

B. Is u continuous as a function of the coefficients of L?

C. Can u be approximated by the solution of a suitable scheme of difference equations?

It is easy to check that the measures P_x are weakly continuous under the various kinds of convergence associated with A, B and C. Hence what we need to study are conditions under which $\xi(\omega)$ is a.s. continuous.

LEMMA 9.1. *If* $P_x[\tau = \tau'] = 1$, *then* $\xi(\omega)$ *is continuous almost surely with respect to* P_x.

Proof: τ is upper semicontinuous and τ' is lower semicontinuous; so if $P_x[\tau = \tau'] = 1$, then τ' is almost surely continuous. Therefore, $\xi(\omega)$ is continuous almost surely.

LEMMA 9.2. *Let* Γ *be closed. Then, for any* $x \in G$, $P_x[\tau = \tau'] = 1$.

Proof: We know that $\Gamma = \overline{\Sigma_2 \cup \Sigma_3}$. Therefore, for any $x \in G$,

$$P_x[x(\tau) \in \Gamma] = 1 .$$

This in turn implies that $P_x[\tau = \tau'] = 1$. Combining Lemmas 9.1 and 9.2 we have

THEOREM 9.1. *Let* Γ *be closed. Then the answers to questions* A, B, *and* C *raised earlier are in the affirmative.*

Remark 9.1. It is now obviously important to know when Γ is closed. This means that we want to know when the boundary points of $\Sigma_2 \cup \Sigma_3$ are regular. The results of Section 6 are relevant in this context.

If Γ is not closed or if we do not know that it is closed, we consider the set

$$E = [x \in G : P_x[x(\tau) \in \Sigma_2 \cup \Sigma_3] = 1] .$$

$G - E$ is a set of measure zero. For $x \in E$, $P_x[\tau = \tau'] = 1$. Consequently, the answers to questions A, B, C are in the affirmative for points $x \in E$. We combine all of this in the following theorem.

THEOREM 9.2. *There is a set* F *of measure zero in* G *such that*

(i) $u(x)$ *is continuous at* $x \in G - F$;

(ii) *if the operator* L *is approximated, then the solutions converge continuously on* $G - F$;

(iii) *if the operator* L *is approximated by suitable difference schemes, then the solutions converge continuously at points of* $G - F$;

(iv) *if* Γ *is closed, then* F *is empty.*

Remark 9.2. One consequence of Theorem 9.1 is that our solution to the first boundary value problem is almost everywhere the limit of the solution to the corresponding problem for the operators $\varepsilon\Delta + L$ as $\varepsilon \searrow 0$.

10. Appendix

We begin this section with a lemma of Phillips and Sarason (cf. Lemma 1.1 in [7]) and some variations on their result.

LEMMA 10.1. *Let* $a : R \to S_d$ *and assume* $a^{ij} \in C_b^2(R)$, $1 \leqq i, j \leqq d$. *Let* $a^{1/2}$ *denote the non-negative definite square root of* a. *Then* $\alpha = a^{1/2}$ *is uniformly Lipschitz continuous with a Lipschitz constant depending only on the bounds on the second derivatives of the* a^{ij}. *Furthermore, there is a constant* C *depending only on the bounds on* *the second derivatives of the* a^{ij} *such that, for all* $\lambda, \mu \in R^d$,

$$|\langle \lambda, \dot{a}(x)\mu\rangle| \leqq C(|\lambda| \langle \mu, a(x)\mu\rangle^{1/2} + |\mu| \langle \lambda, a(x)\lambda\rangle^{1/2}) , \tag{10.1}$$

and for all symmetric $((u_{ij})) \in R^d \otimes R^d$:

$$(\mathrm{tr}\,(\dot{a}u))^2 \leqq C\,\mathrm{tr}\,(u\,a\,u) . \tag{10.2}$$

Proof: Our proof mimics that of Phillips and Sarason. Clearly we can assume that $a(\cdot)$ is uniformly positive definite. Further, we may assume that $a(x^0)$ is diagonal. Then Loewner's formula tells us that

$$\dot{\alpha}^{ij}(x^0) = \frac{\dot{a}^{ij}(x^0)}{(a^{ii}(x^0))^{1/2} + (a^{jj}(x^0))^{1/2}} .$$

Given $\lambda \in R^d$, note that $\langle \lambda, a(\cdot)\lambda\rangle$ is a non-negative C^2-function. Hence,

$$\langle \lambda, \dot{a}(x)\lambda\rangle^2 \leqq 2\,\|\langle \lambda, \ddot{a}(\cdot)\lambda\rangle\|_\infty \langle \lambda, a(x)\lambda\rangle .$$

Applying this fact to the vectors e_i, and $(e_i + e_j)$, where e_i is the i-th coordinate vector, we obtain

$$|\dot{a}^{ij}(x^0)| \leqq A(a^{ii}(x^0) + a^{jj}(x^0))^{1/2} ,$$

where A depends only on the bounds on the $\ddot{a}^{ij}$. This together with Loewner's formula proves the first assertion. To prove (10.1), observe that

$$\begin{aligned} |\langle \lambda, \dot{a}(x^0)\mu\rangle|^2 &= (\lambda_i \dot{a}^{ij}(x^0)\mu_j)^2 \leqq d^4 A^2 \lambda_i^2 (a^{ii}(x^0) + a^{jj}(x^0))\mu_j^2 \\ &= d^4 A^2 (|\mu|^2 \langle \lambda, a(x^0)\lambda\rangle + |\lambda|^2 \langle \mu, a(x^0)\mu\rangle) \\ &\leqq d^4 A^2 (|\mu| \langle \lambda, a(x^0)\lambda\rangle^{1/2} + |\lambda| \langle \mu, a(x^0)\mu\rangle^{1/2})^2 . \end{aligned}$$

Finally, (10.2) follows from

$$\begin{aligned} (\mathrm{tr}\,(\dot{a}\,u)(x^0))^2 &= (\dot{a}^{ij}(x^0)u_{ij})^2 \leqq d^4 A^2 (a^{ii}(x^0) + a^{jj}(x^0))\,u_{ij}^2 \\ &= 2d^4 A^2\,\mathrm{tr}\,(u\,a(x^0)u) . \end{aligned}$$

We turn now to the proof of a theorem due to Oleinik [6].

Let $a : [0, \infty) \times R^d \to S_d$, $b : [0, \infty) \times R^d \to R^d$, and $k : [0, \infty) \times R^d \to R$ be bounded smooth functions. Assume for the present that a is uniformly positive definite and let u be the unique smooth solution of

$$(10.3)\qquad \begin{aligned} &u_t + a^{ij}u_{,ij} + b^i u_{,i} + ku = f, \qquad 0 \leqq t < T,\\ &\lim_{t \nearrow T} u(t,\cdot) = g, \end{aligned}$$

where f and g are bounded smooth functions. We want to derive estimates on the derivatives of u which are independent of the ellipticity of a.

Let $D_x^\alpha = \partial^{|\alpha|}/\partial x_1 \cdots \partial x_d$ and set $u^\alpha = D^\alpha u$, $f^\alpha = D^\alpha f$, and $g^\alpha = D^\alpha g$. Then

$$u_t^\alpha + a^{ij}u^\alpha_{,ij} + b^i u^\alpha_{,i} + \sum_{l=1}^d \alpha_l a^{ij}_{,l} u^{\hat\alpha^l}_{,ij} + \sum_{|\beta| \leqq |\alpha|} c_\beta u^\beta = f,$$
$$\lim_{t \nearrow T} v^\alpha(t,\cdot) = g^\alpha,$$

where $\hat{a}^l$ is defined so that

$$\frac{\partial}{\partial x_l} D^{\hat\alpha^l} = D^\alpha$$

and the c_β are linear combinations of the coefficients a, b, and k and their spacial derivatives up to order $|\alpha|$. Define $w = \sum_{|\alpha|=n} (u^\alpha)^2$. Then

$$\begin{aligned} w_t + a^{ij}w_{,ij} + b^i w_{,i} &- 2\sum_{|\alpha|=n} \langle \nabla_x u^\alpha, a\nabla_x u^\alpha\rangle \\ &+ 2\sum_{l=1}^d \sum_{|\alpha|=n} \alpha_l u^\alpha a^{ij}_{,l} u^{\hat\alpha^l}_{,ij} + 2\sum_{|\beta|\leqq n} c_\beta u^\alpha u^\beta = 2\sum_{|\alpha|=n} u^\alpha f^\alpha, \\ &\lim_{t\nearrow T} w(t,\cdot) = \sum_{|\alpha|=n} (g^\alpha)^2. \end{aligned}$$

Put $A = \sum_{|\alpha|=n} \langle \nabla_x u^\alpha, a\nabla_x u^\alpha\rangle$. Then, applying first Schwartz's inequality and then (10.2), we have

$$\begin{aligned} \left| \sum_{|\alpha|=n} \alpha_l u^\alpha a^{ij}_{,l} u^{\hat\alpha^l}_{,ij} \right| &\leqq n w^{1/2} \Big(\sum_{|\alpha|=n} (a^{ij}_{,l} u^{\hat\alpha^l}_{,ij})^2 \Big)^{1/2} \\ &\leqq C_1 w^{1/2} A^{1/2}, \end{aligned}$$

where C_1 depends only on the second spacial derivatives of the a^{ij}. Hence,

$$w_t + a^{ij}w_{,ij} + (C_1 w^{1/2}A^{1/2} - 2A) + 2\sum_{|\beta|\leqq n} \sum_{|\alpha|=n} C_\beta u^\alpha u^\beta - 2\sum_{|\alpha|=n} u^\alpha f^\alpha \geqq 0.$$

First note that $C_1 w^{1/2} A^{1/2} - 2A \leqq C_1^2 w/8$. Also

$$\left| 2 \sum_{|\beta|=n} \sum_{|\alpha|=n} c_\beta u^\alpha u^\beta \right| \leqq C_2 w \,,$$

$$\left| 2 \sum_{|\beta|<n} \sum_{|\alpha|=n} c_\beta u^\alpha u^\beta \right| \leqq C_3(1 + w) \, \|u\|^{(n-1)} \,,$$

$$\left| 2 \sum_{|\alpha|=n} u^\alpha f^\alpha \right| \leqq C_4(1 + w) \, \|f\|^{(n)} \,,$$

where the constants depend only on the bounds on the derivatives, up to order n, of the coefficients in equation (10.3), and $\|h\|^{(m)} = \sum_{|\alpha| \leqq m} \|D_x^\alpha h\|_\infty$. Hence there exist constants C_5 and C_6, depending only on $C_1 - C_4$, such that

$$\begin{aligned} w_t + a^{ij} w_{,ij} + b^i w_{,i} + C_5(1 + \|u\|^{(n-1)} + \|f\|^{(n)}) w & \\ + C_6(1 + \|u\|^{(n-1)} + \|f\|^{(n)}) &\geqq 0 \,. \end{aligned}$$

Using induction on n and the weak maximum principle, one arrives at the following theorem.

THEOREM 10.1 (Oleinik). *Let*

$$a : [0, \infty) \times R^d \to S_d \,, \qquad b : [0, \infty) \times R^d \to R^d \,, \qquad k : [0, \infty) \times R^d \to R$$

be bounded continuous functions. Assume that

$$a^{ij} \in C_b^{0,m}([0, \infty) \times R^d) \,, \qquad b^i \in C^{0,n}([0, \infty) \times R^d) \,, \qquad k \in C^{0,n}([0, \infty) \times R^d) \,.$$

Let $f \in C^{0,m\wedge n}([0, T) \times R^d)$ *and* $g \in C^{m \wedge n}(R^d)$, *and suppose* u *solves* (10.3). *If* $m \geqq 2$, *then for* $l \leqq m \wedge n$

$$\|u(t, \cdot)\|^{(l)} \leqq A(1 + \|f\|^{(l)} + \|g\|^{(l)}) e^{B(T-t)} \,, \tag{10.4}$$

where A *and* B *depend only on the bounds on the second spacial derivatives of the* a^{ij} *and the spacial derivatives up to order* l *of all the coefficients in* (10.3).

COROLLARY 10.1. *If* m *and* n *in Theorem* 10.1 *are at least* 2, *then, for each* $T > 0$, $f \in C_b^{0,2}([0, T) \times R^d)$ *and* $g \in C_b^2(R^d)$, *there is a unique* u *satisfying* (10.3). *Moreover, if the coefficients in* (10.3) *are independent of* t *and if* $l = m$, $n \geqq 2$, *then,* *for each* $f \in C_b(R^d)$, *there is a* $\lambda_0 > 0$ *such that, for all* $\lambda \geqq \lambda_0$,

$$\lambda u - a^{ij} u_{,ij} - b^i u_{,i} - ku = f \tag{10.5}$$

has a unique solution in $C_b^l(R^d)$.

THEOREM 10.2. *Let* $a : [0, \infty) \times R^d \to S_d$ *and assume that* $\operatorname{tr} a(t, x) \leqq \frac{1}{2}$ *for all* (t, x). *If* $a^{ij} \in C_b^{1,2}([0, \infty) \times R^d)$, $1 \leqq i, j \leqq d$, *then* $\sum_0^\infty \sigma_n^2$ *converges to* a *uniformly on compacts, where* $\sigma_0 = a$ *and* $\sigma_n = \sigma_{n-1} - \sigma_{n-1}^2$. *Moreover,* $\sum_0^\infty (\sigma_n^2)'$ *converges to* a' *uniformly on compacts.*

Proof: Define σ_n, $n \geqq 0$, as indicated and note that $\sigma_n \leqq \sigma_{n-1}$, in the sense of non-negative definite matrices, for all $n \geqq 1$. Moreover, we know that $\sum_0^N \sigma_n^2 = a - \sigma_{N+1} \leqq a$. Hence, for any $\theta \in R^d$, $\sum_0^N \langle \theta, \sigma_n^2 \theta \rangle \leqq \langle \theta, a\theta \rangle$, and so $\sum_0^\infty \langle \theta, \sigma_n^2 \theta \rangle$ is absolutely convergent. But this means $\langle \theta, \sigma_{N+1}^2 \theta \rangle \searrow 0$ as $N \to \infty$. Hence, by Dini's theorem, $\langle \theta, \sigma_{N+1}^2 \theta \rangle \searrow 0$ uniformly on compacts. Using the fact that the σ_n are non-negative definite, one sees that $\sigma_n \searrow 0$ uniformly on compacts. The first assertion is now proved.

To prove the second assertion, note that

$$\dot{\sigma}_{n+1} = \dot{\sigma}_n - \dot{\sigma}_n \sigma_n - \sigma_n \dot{\sigma}_n = \tfrac{1}{2}[\dot{\sigma}_n(I - 2\sigma_n) + (I - 2\sigma_n)\dot{\sigma}_n],$$

where the dot indicates that one takes a generic spacial derivative. Continuing by induction, one arrives at

$$\dot{\sigma}_{n+1} = \frac{1}{2^{n+1}} \sum_{A \in \mathscr{P}_n} (I - 2\sigma)_A \, \dot{a}(I - 2\sigma)_{A^c},$$

where $\mathscr{P}_n$ denotes the set of subsets of $\{0, \cdots, n\}$ and $(I - 2\sigma)_A = \prod_{j \in A}(I - 2\sigma_j)$. Hence, if $\lambda, \mu \in R^d$, then

$$|\langle \lambda, \dot{\sigma}_{n+1}\mu \rangle| \leqq \frac{1}{2^{n+1}} \sum_{A \in \mathscr{P}_n} |\langle (I - 2\sigma)_A \lambda, \dot{a}(I - 2\sigma)_{A^c}\mu \rangle|.$$

Using (10.1), we have

$$\begin{aligned} &|\langle (I - 2\sigma)_A \lambda, \dot{a}(I - 2\sigma)_{A^c}\mu \rangle| \\ &\qquad \leqq C(|(I - 2\sigma)_A \lambda| \, \langle (I - 2\sigma)_{A^c}\mu, a(I - 2\sigma)_{A^c}\mu \rangle^{1/2} \\ &\qquad\qquad + |(I - 2\sigma)_{A^c}\mu| \, \langle (I - 2\sigma)_A \lambda, a(I - 2\sigma)_A \lambda \rangle^{1/2}), \end{aligned}$$

where C depends on the bounds on a and its first two spacial derivatives.

Note that $|(I-2\sigma)_A\lambda| \leqq |\lambda|$. Thus it remains only to estimate quantities of the form

$$I_n = \frac{1}{2^{n+1}} \sum_{A\in\mathscr{S}_n} \langle (I-2\sigma)_A\lambda, a(I-2\sigma)_A\lambda\rangle^{1/2}.$$

By Schwarz's inequality,

$$I_n^2 \leqq \frac{1}{2^{n+1}} \sum_{A\in\mathscr{S}_n} \langle (I-2\sigma)_A\lambda, a(I-2\sigma)_A\lambda\rangle = \frac{1}{2^{n+1}} \sum_{A\in\mathscr{S}_n} \langle \lambda, a(I-2\sigma)_A^2\lambda\rangle .$$

But

$$\frac{1}{2^{n+1}} \sum_{A\in\mathscr{S}_n} (I-2\sigma)_A^2 = \prod_{j=0}^{n} \frac{I+(I-2\sigma_j)}{2},$$

and so

$$I_n^2 \leqq \left\langle \lambda, a \prod_{j=0}^{n} \frac{I+(I-2\sigma_j)}{2}\lambda \right\rangle = \left\langle \lambda, a \prod_0^n ((I-\sigma_j)^2+\sigma_j^2)\lambda \right\rangle$$
$$\leqq \left\langle \lambda, a \exp\left\{2\sum_0^n \sigma_j^2\right\} \prod_0^n (I-\sigma_j)^2\lambda \right\rangle .$$

Observe that $\sigma_{n+1} = a\prod_0^n (I-\sigma_j)$, and therefore

$$a\prod_0^n (I-\sigma_j)^2 = \sigma_{n+1}\prod_0^n (I-\sigma_j) \leqq a\prod_0^n (I-\sigma_j) = \sigma_{n+1}.$$

Hence,

$$I_{n-1}^2 \leqq \langle \lambda, e^{2a}\sigma_n\lambda\rangle \to 0$$

uniformly on compacts; q.e.d.

Let L be a second order elliptic-parabolic operator of the form

$$Lu = \tfrac{1}{2}\sum a^{ij}(x)\frac{\partial^2 u}{\partial x_i\,\partial x_j} + \sum b^j(x)\frac{\partial u}{\partial x_j}$$

with smooth coefficients. Let ϕ be a smooth function.

LEMMA 10.2. *For any integer* $m > 0$,

$$L^m\phi^2 = \sum_{\substack{p+q\leqq m-1\\ p\geqq 0\\ q\geqq 0}} \binom{p+q}{p} L^{m-p-q-1}\langle a\nabla L^p\phi, \nabla L^q\phi\rangle + \sum_{0\leqq\alpha\leqq m}\binom{m}{\alpha}(L^\alpha\phi)(L^{m-\alpha}\phi).$$

Proof: This Lemma is proved easily by induction on m.

THEOREM 10.3. *Let* ϕ *be such that, at a certain point* x^0,

$$L^r \phi^2 = 0 \qquad \text{for} \qquad r = 0, 1, 2, \cdots, n.$$

Then at the same point x^0

$$\text{(i)} \qquad L^s \phi = 0 \qquad \text{for} \qquad 0 \leqq s \leqq \left[\frac{n}{2}\right],$$

$$\text{(ii)} \qquad L^r \langle a \nabla L^p \phi, \nabla L^q \phi \rangle = 0 \qquad \text{if} \qquad p + q + r \leqq n - 1 .$$

Proof: First we prove (i). Since L generates a non-negative semigroup, our assumption implies that

$$(T_t \phi^2)(x^0) = o(t^n) \qquad \text{as} \qquad t \to 0 .$$

Moreover, $T_t \phi^2 \geqq (T_t \phi)^2$ by the Cauchy Schwartz inequality. Therefore,

$$(T_t \phi)(x^0) = o(t^{n/2}) \qquad \text{as} \qquad t \to 0 .$$

This implies that $\phi, L\phi, \cdots, L^{[n/2]}\phi$ are zero at x^0. We now turn to the proof of (ii). First, in view of Lemma 10.1 and the first part of the theorem, we have

$$\sum_{\substack{p+q \leqq m-1 \\ p \geqq 0 \\ q \geqq 0}} \binom{p+q}{q} L^{m-p-q-1} \langle a \nabla L^p \phi, \nabla L^q \phi \rangle = 0 \tag{10.6}$$

for $m = 1, 2, \cdots, n$. Let

$$\psi_{pq}(x) = \langle a \nabla L^p \phi, \nabla L^q \phi \rangle , \quad 0 \leqq p \leqq n, \quad 0 \leqq q \leqq n;$$

$\psi_{pq}(x)$ is a positive semidefinite matrix for each x. Therefore,

$$A_{pq}(t) = (T_t \psi_{pq})(x^0)$$

is a positive semidefinite matrix for each $t \geqq 0$. We shall prove our result by induction on $k = p + q + r$. We note that, for $k = 0$, (10.6) yields

$$\langle a \nabla \phi, \nabla \phi \rangle = 0 \qquad \text{at} \qquad x^0 ,$$

with $m = 1$. Let us suppose that (ii) is valid for $p + q + r$ equal to $0, 1, 2, \cdots, l - 1$. This implies that at x^0

$$L^r \langle a \nabla L^p \phi, \nabla L^q \phi \rangle = 0 \qquad \text{for} \qquad 0 \leqq r \leqq l - p - q - 1$$

or

$$A_{pq}(t) = \frac{a_{pq}}{(l - p - q)\,!}\, t^{l-p-q} + o(t^{l-p-q}) ,$$

where

$$a_{p,q} = (L^{l-p-q} \langle a \nabla L^p \phi, \nabla L^q \phi \rangle)(x^0) .$$

Since $A_{pq}(t)$ is a positive semidefinite matrix for every $t \geqq 0$, so is the matrix

$$B_{pq}(t) = A_{p,q}(t) t^{p+q-l} .$$

In particular, by taking $t = 0$, we see that

$$b_{p,q} = \begin{cases} \dfrac{a_{pq}}{(l - p - q)\,!} & \text{for} \quad p + q \leqq l , \\ 0 & \text{for} \quad p + q > l , \end{cases}$$

is a positive semidefinite matrix. Our assumptions imply that

$$\sum \binom{p+q}{p} (l - p - q)\,!\, b_{pq} = 0 ,$$

and what we want to conclude is that $b_{pq} = 0$ for $p + q \leqq l$. For this it is sufficient to prove that

$$\rho_{pq} = \binom{p+q}{p} (l - p - q)\,!$$

is positive definite. We know that $\rho_{pq} = (p + q)\,!\,(l - p - q)\,!\,/p\,!\,q\,!$ and so it is positive definite if $(r!)(l - r)!$ is a moment sequence. We have by computation

$$(r\,!)(l - r)\,! = c_l \int_0^1 \left(\frac{u}{1 - u} \right)^r (1 - u)^l \, du .$$

This completes the proof.

Bibliography

[1] Bony, J. M., *Principe due maximum, inégalité de Harnack et unicité du probleme de Cauchy pour les operateurs elliptiques dégénérés*, Ann. Inst. Fourier, XIX, No. 1, 1969, pp. 277–304.

[2] Fichera, G., *Sulla equazioni differenziali lineari ellittico-paraboliche del secondo ordine*, Atti. Naz. Lincei Mem., Ser. 8, Vol. 5, 1956, pp. 1–30.

[3] Hill, C. D., *A sharp maximum principle for degenerate elliptic parabolic equations*, Indiana Univ. Math. Jour., Vol. 20, No. 3, 1970, pp. 213–230.

[4] Kohn, J. J., and Nirenberg, L., *Degenerate elliptic-parabolic equations of second order*, Comm. Pure Appl. Math., Vol. 20, 1967, pp. 797–872.

[5] Oleinik, O. A., *A problem of Fichera*, Doklady Acad. Nauk SSSR, Vol. 157, 1964, pp. 1297–1301. (In Russian.) English translation in Soviet Math., Vol. 5, 1964, pp. 1129–1133.

[6] Oleinik, O. A., *Alcuni risultati sulle equazioni lineari e quasi lineari ellitico-paraboliche a derivate parziali del secondo ordine*, Rend. Classe, Sci. Fis. Mat. Nat. Acad. Naz. Lincei, Ser. 8, Vol. 40, 1966, pp. 775–784.

[7] Phillips, R. S., and Sarason, L., *Elliptic-parabolic equations of the second order*, J. Math. Mech. 17, 1967, pp. 891–917.

[8] Redheffer, R., *The sharp maximum principle for nonlinear inequalities*. (Preprint.)

[9] Stroock, D. W., and Varadhan, S. R. S., *Diffusion processes with continuous coefficients*, I, Comm. Pure Appl. Math., Vol. 22, 1969, pp. 345–400.

[10] Stroock, D. W., and Varadhan, S. R. S., *On the support of diffusion processes, with applications to the strong maximum principle*, to appear in the Proceedings of the Sixth Berkeley Symposium in Probability and Statistics.

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