SIAM J. CONTROL OPTIM.
Vol. 51, No. 5, pp. 3558–3591

LONG TIME AVERAGE OF MEAN FIELD GAMES WITH A NONLOCAL COUPLING*

P. CARDALIAGUET†, J.-M. LASRY†, P.-L. LIONS‡, AND A. PORRETTA§

Abstract. We study the long time average, as the time horizon tends to infinity, of the solution of a mean field game system with a nonlocal coupling. We show an exponential convergence to the solution of the associated stationary ergodic mean field game. Proofs rely on semiconcavity estimates and smoothing properties of the linearized system.

Key words. mean field games, long time behavior, parabolic equations

AMS subject classifications. 35B40, 35K55

DOI. 10.1137/120904184

1. Introduction. Mean field game systems have been introduced by Lasry and Lions in a series of papers [6, 7, 8, 9] to analyze large population stochastic differential games. When the horizon T of the game is finite, the problem takes the form of a backward/forward system:

$$
(1.1)\qquad \begin{cases} \text{(i)}\ \ -u^T_t - \Delta u^T + \frac{1}{2}|Du^T|^2 = F(x, m^T(t)),\\ \text{(ii)}\ \ m^T_t - \Delta m^T - \operatorname{div}(m^T Du^T) = 0,\\ \text{(iii)}\ \ m^T(0) = m_0,\ u^T(T) = u_f, \end{cases}
$$

where the scalar unknowns u^T, m^T are defined on $[0, T] \times \mathbb{R}^d$ and F is a coupling between the two equations. In the above systems, $m^T = m^T(t, x)$ is, for any t, the probability density of the population of players. The function u^T is understood as the value function—for a typical player—of an optimal control problem in which the density m^T enters as a data.

When the horizon T tends to infinity, it is expected that the above system simplifies into the ergodic mean field game problem

$$
(1.2)\qquad \begin{cases} \text{(i)}\ \ \bar\lambda - \Delta \bar u + \frac{1}{2}|D\bar u|^2 = F(x, \bar m),\\ \text{(ii)}\ \ -\Delta \bar m - \operatorname{div}(\bar m D\bar u) = 0,\\ \qquad \int_Q \bar u\, dx = 0, \quad \int_Q \bar m\, dx = 1, \end{cases}
$$

in which the unknowns are now the ergodic constant $\bar\lambda$, and the functions $\bar u$, $\bar m$ are defined on $\mathbb{R}^d$. This convergence has been analyzed in [4] for discrete mean field game systems and in [3] for the above system with a local coupling $F := \tilde F(x, m^T(t, x))$,

*Received by the editors January 2, 2013; accepted for publication (in revised form) June 17, 2013; published electronically September 19, 2013. This work was partially supported by the Agence Nationale de la Recherche under grants ANR JEUDY, ANR-10-BLAN 0112, and ANR HJnet ANR-12-BS01-0008-01 and by EU under the 7th Framework Programme Marie Curie Initial Training Network "FP7-PEOPLE-2010-ITN," SADCO project, GA 264735-SADCO.

http://www.siam.org/journals/sicon/51-5/90418.html

†Ceremade, Université Paris-Dauphine, 75775 Paris cedex 16 France (cardaliaguet@ceremade.dauphine.fr, 2007lasry@gmail.com).

‡College de France, Paris 75005, France, and Ceremade, Université Paris-Dauphine, 75775 Paris cedex 16, France (lions@ceremade.dauphine.fr).

§Dipartimento di Matematica, Università di Roma "Tor Vergata," 00133 Roma, Italy (porretta@axp.mat.uniroma2.it).

where $\tilde{F} : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$. We proved in [3] that the convergence holds when $\tilde{F}$ is increasing with respect to m, and has actually an exponential rate as soon as $\tilde{F}$ is strongly increasing: $\frac{\partial \tilde{F}}{\partial m} \geq \gamma > 0$. We also refer to [3] for a more general comparison between the techniques used for the long time behavior of Hamilton–Jacobi equations and those for mean field games.

Here we consider the nonlocal, smoothing coupling $F : \mathbb{R}^d \times L^1_\sharp(\mathbb{R}^d) \to \mathbb{R}$, where $L^1_\sharp(\mathbb{R}^d)$ is the space of locally integrable, periodic maps. In this framework, we prove similar results: the convergence, in a suitable sense, when F is nondecreasing, and the exponential convergence rate under a stronger smoothing condition. If the method for proving the convergence is close to that of [3], the mechanism towards the exponential rate is completely different. For the convergence, the starting point is an energy equality, established in [6, 7, 8]:

$$
\begin{aligned}
(1.3) \qquad & \int_0^T \int_Q \frac{(m^T + \bar{m})}{2} |Du^T - D\bar{u}|^2 + (F(x, m^T) - F(x, \bar{m}))(m^T - \bar{m}) \; dxdt \\
& \quad = - \left[\int_Q (u^T - \bar{u})(m^T - \bar{m}) dx \right]_0^T .
\end{aligned}
$$

Since $\bar{m}$ and m^T are positive and F is nondecreasing, the right-hand side of the equality governs the convergence of Du^T to $D\bar{u}$ and of m^T to $\bar{m}$. In [3], for local equations, the main ingredient for bounding the right-hand side comes from the Hamiltonian structure of the system. Under the assumption $\frac{\partial \tilde{F}}{\partial m} \geq \gamma > 0$ this structure also entails the exponential decay of the quantity $\int_Q (u^T(t) - \bar{u})(m^T(t) - \bar{m})$, which, in turn, implies the exponential convergence of Du^T to $D\bar{u}$ and of m^T to $\bar{m}$. Note that, in this approach, the diffusion plays basically no role. In contrast, for the nonlocal system studied here, we do not assume that the system has a Hamiltonian structure (which amounts to saying that F is not necessarily the derivative of a potential) and, above all, we cannot expect the map F to be "strictly increasing" with respect to m. Therefore the previous approach fails completely. To overcome this difficulty, we rely, on the one hand, on a uniform Lipschitz estimate in space for u^T, derived from classical semiconcavity properties of solutions of Hamilton–Jacobi equations. The exponential rate, on the other hand, is based on the regularizing properties of the diffusion and on some algebraic properties of the linearized system.

The paper is organized in the following way: In section 2 we state the main notation and assumptions used in the paper. Section 3 is devoted to the convergence result, while the exponential rate is proved through sections 4, 5, and 6. In section 4 we explain the mechanism of this rate by the analysis of the linearized system near the solution $(\bar{u} - \bar{\lambda}(t - T), \bar{m})$. Then, in section 5, we show the exponential rate for sufficiently small initial data. The case of general data is treated in section 6. In the appendix, we recall several decay estimates for some linear equations which are used throughout the paper.

2. Notation and assumptions.

Notation. Throughout the paper we work with periodic (in space) functions $f : \mathbb{R}^d \to \mathbb{R}$: $f(x + k) = f(x)$ for any $x \in \mathbb{R}^d$, $k \in \mathbb{Z}^d$. We denote by Q the unit cube $[0, 1]^d$ and often use the notation $\langle f \rangle = \int_Q f$. The spaces $C_\sharp$, $C^1_\sharp$, etc... are the spaces of periodic functions which are in C^0, C^1, etc.... In the same way, $L^2_\sharp(\mathbb{R}^d)$ denotes the space of $L^2_{\text{loc}}(\mathbb{R}^d)$ functions which are periodic. Moreover, $L^2_{\sharp,0}$ denotes the subset of $L^2_\sharp$ consisting of maps with zero average on Q. We set $M_\sharp$ as the subset of $L^2_\sharp(\mathbb{R}^d)$

consisting in probability densities on Q: namely, $m \in M_\sharp$ if $m \in L^2_\sharp(\mathbb{R}^d)$, $m \geq 0$ a.e., and $\int_Q m = 1$.

We use the notation $C^{\alpha/2,\alpha}$, with norms $\|\cdot\|_{C^{\alpha/2,\alpha}}$, for the parabolic Hölder spaces as defined in [5], and we often use their periodic-in-space versions $C^{\alpha/2,\alpha}_\sharp$.

Throughout the proofs, we denote by C possibly different constants depending on the data F, u_f, m_0, but not on T. Note that quantities depending on $\bar{m}$ and $\bar{u}$ are written as depending on F.

Assumptions. Throughout the paper, we assume that the nonlocal coupling $F : \mathbb{R}^d \times M_\sharp \to \mathbb{R}$, the initial condition m_0, and the terminal condition u_f satisfy the following hypotheses:

1. (Regularity.) The map $m \to F(\cdot, m)$ has a bounded range in $\mathcal{C}^2_\sharp(\mathbb{R}^d)$.
2. (Continuity.) The map $m \to F(\cdot, m)$ is continuous from $M_\sharp$ into $C^0_\sharp(\mathbb{R}^d)$ for the $L^2_\sharp$ topology.
3. (Monotonicity.)

$$\int_Q (F(x, m_1) - F(x, m_2))(m_1 - m_2)dx \geq 0 \qquad \forall m_1, m_2 \in M_\sharp. \tag{2.1}$$

4. The map $m_0 : \mathbb{R}^d \to \mathbb{R}$ is continuous, $\mathbb{Z}^d$-periodic, with $m_0 > 0$ and $\int_Q m_0 = 1$.
5. The map $u_f : \mathbb{R}^d \to \mathbb{R}$ is of class C^2 and $\mathbb{Z}^d$-periodic.

Let us recall that under the above conditions, system (1.1) has a unique solution and that this solution is classical (see [8]). Moreover $m^T(t, x) > 0$ for all $(t, x) \in [0, T] \times \mathbb{R}^d$ and $\int_Q m^T(t, x)dx = 1$ for all $t \in [0, T]$. In the same way, system (1.2) has a unique solution, with $\bar{m} = \frac{e^{-\bar{u}}}{\int_Q e^{-\bar{u}}}$ and $D\bar{m} = -\bar{m}D\bar{u}$.

For the exponential rate, we will need to strengthen the regularity assumption on F:

$$\|F(x, m_1) - F(x, m_2)\|_{\mathcal{C}^{1+\alpha}} \leq \bar{C}_F \|m_1 - m_2\|_{H^{-1}_{\sharp,0}} \qquad \forall m_1, m_2 \in M_\sharp \tag{2.2}$$

for some $\alpha \in (0, 1)$.

A typical example of map F satisfying the above conditions is

$$F(x, m) = [\Phi(\cdot, (\rho \star m)(\cdot)) \star \rho]\,(x),$$

where $\star$ denotes the usual convolution product, $\Phi : \mathbb{R}^2 \to \mathbb{R}$ is smooth and nondecreasing with respect to the second variable, and ρ is a smooth, even function with compact support.

3. Convergence. In this section, we consider a solution (u^T, m^T) of (1.1) and prove its convergence to the solution $(\bar{u}, \bar{m})$ of (1.2) as $T \to +\infty$. Recall that the nonlocal coupling satisfies the assumptions stated in section 2, and in particular the monotonicity condition (2.1) on F (but we do not assume (2.2) here). To explain our convergence result, it is convenient to introduce the scaled functions

$$U^T(t, x) := u^T(tT, x), \qquad M^T(t, x) := m^T(tT, x). \tag{3.1}$$

THEOREM 3.1. *As $T \to +\infty$,*
- *(U^T/T) converges uniformly to $(1-t)\bar{\lambda}$ in $[0, 1] \times \mathbb{R}^d$;*
- *$(U^T - \int_Q U^T(t))$ converges to $\bar{u}$ in $L^2(Q)$ uniformly on compact intervals of $(0, 1)$;*
- *$M^T(t)$ converges to $\bar{m}$ in $L^2(Q)$, uniformly on compact intervals of $(0, 1)$.*

The key ingredient of the proof is the estimate (1.3). To control the right-hand side of this equality, we use a uniform Lipschitz continuity property of the map $u^T(t,\cdot)$, which comes from classical estimates for solutions of the Hamilton–Jacobi equation: this easily implies the convergence of DU^T to $D\bar{u}$ in L^2 and the weak convergence of M^T to $\bar{m}$. The stronger convergences in the theorem are mostly due to the regularizing properties of the equation.

3.1. Preliminary estimates.

LEMMA 3.2. *The map u^T is bounded by CT for some constant C. Moreover, u^T is uniformly semiconcave in space and, therefore, uniformly Lipschitz continuous in space, with a constant depending only on F and on $\|u_f\|_{\mathcal{C}^2}$.*

Proof. The bounds on u^T are standard. Let us check the uniform semiconcavity. Let $\xi \in \mathbb{R}^d$ with $|\xi| = 1$, and let $w(t,x) = D^2_{\xi\xi}u^T(t,x)$ be the second order derivative of u in the direction ξ. Then w satisfies

$$-w_t - \Delta w + |D_\xi Du^T|^2 + Du^T Dw = D^2_{\xi\xi}F(x,m^T) \qquad \text{in } (0,T)\times\mathbb{R}^d.$$

In particular, if (t,x) is a maximum point of w where this maximum is positive, then either $t = T$, in which case $\|w_+\|_\infty \le \|D^2u_f(x)\|_\infty \le C$, or

$$\|w_+\|_\infty^2 = |w(t,x)|^2 \le |D_\xi Du^T(t,x)|^2 \le \|D^2_{\xi\xi}F\|_\infty \le C.$$

In conclusion, we have proved that $D^2u^T \le C$. Since u^T is $\mathbb{Z}^d$-periodic, this semiconcavity estimate readily implies that $\|Du^T\|_\infty \le \sqrt{N}C$. ∎

The following fundamental equality is proved in [8].

LEMMA 3.3. *For any $0 \le t_1 < t_2 \le T$ we have*

$$\left[\int_Q (u^T - \bar{u})(m^T - \bar{m})dx\right]_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_Q \frac{(m^T + \bar{m})}{2}|Du^T - D\bar{u}|^2 + (F(x,m^T) - F(x,\bar{m}))(m^T - \bar{m})\;dxdt = 0.$$

Combining Lemma 3.2 with Lemma 3.3 we obtain the following.

LEMMA 3.4. *For any $R > 0$ there is a constant $C = C(R,F)$ such that if $\|u_f\|_{\mathcal{C}^2} \le R$,*

$$\int_0^T\int_Q \frac{(m^T + \bar{m})}{2}|Du^T - D\bar{u}|^2 + (F(x,m^T) - F(x,\bar{m}))(m^T - \bar{m})\;dxdt \le C. \tag{3.2}$$

In particular

$$\frac{1}{T}\int_0^T\int_Q m^T\,|Du^T|^2\,dxdt \le C \tag{3.3}$$

and

$$\lim_{T\to+\infty}\frac{1}{T}\int_0^T\int_Q |Du^T - D\bar{u}|^2\;dxdt = 0. \tag{3.4}$$

Proof. According to Lemma 3.3, it is enough to estimate

$$\left|\left[\int_Q (u^T - \bar{u})(m^T - \bar{m})dx\right]_0^T\right|.$$

Since $u^T(x,T) = u_f(x)$, where u_f is bounded by R, we have

$$\left| \int_Q (u^T(x,T) - \bar{u}(x))(m^T(x,T) - \bar{m}(x))dx \right| \le 2(\|u_f\|_\infty + \|\bar{u}\|_\infty) \le C.$$

On another hand, from the Lipschitz continuity of u^T given by Lemma 3.2 (which depends only on R and F), we have

$$\begin{aligned}
&\left| \int_Q (u^T(x,0) - \bar{u}(x))(m^T(x,0) - \bar{m}(x))dx \right| \\
&\quad \le \left| \int_Q (u^T(x,0) - u^T(0,0))(m^T(x,0) - \bar{m}(x))dx \right| + \left| \int_Q \bar{u}(x)(m^T(x,0) - \bar{m}(x))dx \right| \\
&\quad \le 2(\|Du^T\|_\infty + \|\bar{u}\|_\infty) \ \le \ C.
\end{aligned}$$

So (3.2) holds. Moreover, using assumption (2.1) and since $\bar{m} > 0$, we deduce

$$\int_0^T \int_Q m^T \left| Du^T - D\bar{u} \right|^2 dxdt \le C.$$

Since

$$\int_0^T \int_Q m^T \left| D\bar{u} \right|^2 dxdt \le \|D\bar{u}\|_\infty^2 \, T$$

we also get (3.3). The last part of the lemma follows from (3.2) using that $\bar{m}$ is bounded from below by a positive constant. ▯

3.2. Use of the regularizing properties of the equation. Note that Lemma 3.4 implies that Du^T converges to $D\bar{u}$: more precisely, in terms of scaled functions, DU^T converges to $D\bar{u}$ in $L^2([0,1] \times Q)$. In order to prove the convergence of m^T and obtain a stronger convergence for u^T, we have to use the regularizing properties of the equations. For this it will be convenient to introduce new notation: let

$$v^T = u^T - (\bar{u} + \bar{\lambda}(T-t)) \qquad \text{and} \qquad \mu^T = m^T - \bar{m}$$

and

$$\mu_0 = m_0 - \bar{m}, \ v_f = u_f - \bar{u}.$$

Then (v^T, μ^T) solves the system

$$\text{(3.5)} \qquad \begin{cases} \text{(i)} \ \ -v_t^T - \Delta v^T + D\bar{u}Dv^T + \frac{1}{2}|Dv^T|^2 = F(x, \bar{m} + \mu^T(t)) - F(x,\bar{m}), \\ \text{(ii)} \ \ \mu_t^T - \Delta\mu^T - \operatorname{div}(\mu^T D\bar{u}) - \operatorname{div}(\bar{m}Dv^T) - \operatorname{div}(\mu^T Dv^T) = 0, \\ \text{(iii)} \ \ \mu^T(0) = \mu_0, \ v^T(T) = v_f. \end{cases}$$

Our starting point is the analysis of the behavior of μ^T. The idea is that the solutions to

$$z_t - \Delta z - \operatorname{div}(zDu^T) = 0$$

have an exponential decay (here we use that Du^T is uniformly bounded), while the residual term $-\operatorname{div}(\bar{m}Dv^T)$ in (ii) is controlled by the energy estimate. This yields a uniform bound on $\|\mu^T(t)\|_2$ (Lemma 3.5) and eventually a smallness for this term in intervals of the form $[\delta T, (1-\delta T)]$ (Lemma 3.6).

LEMMA 3.5. *Let $\omega > 0$ be defined by Lemma 7.1. For any $R > 0$, there exists a constant $C = C(R, F)$ such that if $\|v_f\|_{\mathcal{C}^2}, \|\mu_0\|_{L^2(Q)} \le R$, then for any $0 \le t_0 \le t \le T$ we have*

$$\|\mu^T(t)\|_2 \le C\left\{\|\mu^T(t_0)\|_2 e^{-\omega(t-t_0)} + \left(\int_{t_0}^t \int_Q |Dv^T|^2\right)^{\frac{1}{2}}\right\} \tag{3.6}$$

and

$$\sup_{t\in[0,T]} \|\mu^T(t)\|_2 \le C. \tag{3.7}$$

Proof. Since μ^T solves

$$\mu^T_t - \Delta\mu^T - \operatorname{div}(\mu^T Du^T) - \operatorname{div}(\bar{m}Dv^T) = 0,$$

applying Lemma 7.6, on account of the uniform bound on Du^T, we deduce (3.6). Inequality (3.7) is easily obtained by choosing $t_0 = 0$ in (3.6) and recalling that, by (3.2), $\int_0^T \int_Q |Dv^T|^2 \le C$. ∎

The integral estimate then implies the smallness of μ^T on intervals of the form $(\delta T, (1-\delta)T)$ for $\delta \in (0, 1)$.

LEMMA 3.6. *For any $R > 0$, there exists a constant $C = C(R, F)$ such that if $\|v_f\|_{\mathcal{C}^2}, \|\mu_0\|_{L^2(Q)} \le R$, then for any $\delta \in (0, 1)$ we have*

$$\int_{\delta T}^{(1-\delta)T} \int_Q |Dv^T|^2 \le \frac{C}{(\delta T)^{\frac{1}{2}}} \qquad \text{and} \qquad \sup_{t\in[\delta T,(1-\delta)T]} \|\mu^T(t)\|_2 \le \frac{C}{(\delta T)^{\frac{1}{4}}}.$$

Proof. By Lemma 3.4, we have $\int_0^T \int_Q |Dv^T|^2 \le C$. Therefore there exist $t_0 \in [0, (\delta/2)T]$ and $t_1 \in [T(1-\delta/2), T]$ such that

$$\int_Q |Dv^T(t_0)|^2 \le \frac{2C}{\delta T} \qquad \text{and} \qquad \int_Q |Dv^T(t_1)|^2 \le \frac{2C}{\delta T}.$$

Going back to the full energy estimate (Lemma 3.3), we have

$$\int_{t_0}^{t_1} \int_Q |Dv^T|^2 \le \int_Q v^T(t_0)\mu^T(t_0) - \int_Q v^T(t_1)\mu^T(t_1).$$

Setting $\tilde{v}^T(t, x) = v^T(t, x) - \langle v^T(t)\rangle$, we have, by the Poincaré–Wirtinger inequality,

$$\left|\int_Q v^T(t_0)\mu^T(t_0)\right| = \left|\int_Q \tilde{v}^T(t_0)\mu^T(t_0)\right| \le \|\tilde{v}^T(t_0)\|_2\|\mu^T(t_0)\|_2 \le C\|Dv^T(t_0)\|_2 \le \frac{C}{(\delta T)^{\frac{1}{2}}}$$

because μ^T is bounded in L^2 according to Lemma 3.5. Estimating the term $\int_Q v^T(t_1)\mu^T(t_1)$ in the same way, we get

$$\int_{t_0}^{t_1} \int_Q |Dv^T|^2 \le \frac{C}{(\delta T)^{\frac{1}{2}}}, \tag{3.8}$$

and therefore the first estimate in the statement. Plugging this inequality into Lemma 3.5 gives

$$\begin{aligned}\sup_{t\in[\delta T,(1-\delta)T]} \|\mu^T(t)\|_2 &\le C\left\{\|\mu^T(t_0)\|_2 e^{-\omega\delta T/2} + \left(\int_{t_0}^{t_1} \int_Q |Dv^T|^2\right)^{\frac{1}{2}}\right\} \\ &\le Ce^{-\omega\delta T/2} + \tfrac{C}{(\delta T)^{\frac{1}{4}}};\end{aligned}$$

hence we conclude the proof. ∎

We are now ready to complete the proof of Theorem 3.1.

Proof of Theorem 3.1. Recall that the scaled functions (U^T, M^T) are defined by (3.1). As observed before, (3.4) already implies that DU^T converges to $D\bar{u}$ in $L^2((0,1)\times Q)$, and therefore in $L^p((0,1)\times Q)$ for every $p \geq 1$, since the gradient is uniformly bounded. The convergence of the map $(t,x) \to U^T(t,x) - \int_Q U^T(t,y)dy$ in $L^2((0,1)\times Q)$ is a straightforward consequence of the Poincaré inequality.

Moreover, Lemma 3.6 implies that $M^T(t,\cdot)$ converges in $L^2(Q)$ to $\bar{m}$ uniformly on compact intervals of $(0,1)$. Next we check the convergence of $\tilde{U}^T(t) := U^T(t) - \int_Q U^T(t)$. For $\epsilon > 0$ and $\delta \in (0, 1/2)$, let $\bar{T}$ be so large that $\sup_{t\in[\delta T,(1-\delta)T]} \|\mu^T(t)\|_2 \leq \epsilon$ and $\int_{\delta T}^{(1-\delta)T} \|Dv^T\|_2^2 \leq \epsilon^2$ for $T \geq \bar{T}$. In particular, there is $\bar{t} \in [(1-2\delta)T, (1-\delta)T]$ such that $\|Dv^T(\bar{t})\|_2 \leq \epsilon/(\delta T)^{\frac{1}{2}}$. Since F is continuous from $M_\sharp$ into $C^0_\sharp$, we can choose $\bar{T}$ large enough so that $\|F(\cdot, \bar{m} + \mu^T(t)) - F(\cdot, \bar{m})\|_\infty \leq \epsilon$ for $t \in [\delta T, (1-\delta)T]$. If we set $B(t) = F(\cdot, \bar{m} + \mu^T(t)) - F(\cdot, \bar{m})$, we have, from Lemma 7.4 in the appendix,

$$\begin{aligned}
\|\tilde{v}^T(t)\|_2 &\leq Ce^{-\omega(\bar{t}-t)}\|\tilde{v}^T(\bar{t})\|_2 + C\int_t^{\bar{t}} e^{-\omega(s-t)}\left(\frac{1}{2}\|Dv^T(s)\|_2^2 + \|B(s)\|_2\right)ds \\
&\leq C\|D\tilde{v}^T(\bar{t})\|_2 + C\epsilon \;\leq\; \frac{\epsilon}{(\delta T)^{\frac{1}{2}}} + C\epsilon
\end{aligned}$$

for any $t \in [\delta T, (1-2\delta)T]$. This shows that $\tilde{v}^T(t)$ converges to zero in $L^2(Q)$ uniformly in $[\delta T, (1-2\delta)T]$. So $\tilde{U}^T$ converges uniformly to $\bar{u}$ on compact intervals of $(0,1)$.

In order to show the convergence of $\frac{U^T}{T}$, let us first prove that $(1/T)\int_Q U^T(t,x)dx$ converges to $(1-t)\bar{\lambda}$ for any $t \in [0,1]$. For this, let us note that the map $(t,x) \to F(x, M^T(t))$ converges locally uniformly to $(t,x) \to F(x,\bar{m})$ on $(0,1)\times \mathbb{R}^N$ because F is continuous on $M_\sharp$. Since F is also bounded, the map $(t,x) \to F(x, M^T(t))$ also converges in $L^1((0,1)\times Q)$. Let us now integrate the equation satisfied by U^T on $[t,1]\times Q$. We have

$$\frac{1}{T}\left(\int_Q U^T(t,x)dx - \int_Q u_f(x)dx\right) + \frac{1}{2}\int_t^1\int_Q |DU^T|^2 dxds = \int_t^1\int_Q F(x, M^T(s))dxds.$$

Recall that u_f is uniformly bounded and that $DU^T \to D\bar{u}$ in $L^2([0,1]\times Q)$ by Lemma 3.4. Therefore we have

$$\lim_{T\to+\infty} \frac{1}{T}\int_Q U^T(t,x)dx = (1-t)\int_Q -\frac{1}{2}|D\bar{u}|^2 + F(x,\bar{m})dx = (1-t)\bar{\lambda},$$

the last equality coming from the equation satisfied by $\bar{u}$. Note that this convergence is uniform with respect to t. Since, by Lemma 3.2, $U^T(t,\cdot)$ is Lipschitz continuous (uniformly in T and t), we can also conclude that U^T/T converges uniformly to $(1-t)\bar{\lambda}$ in $\mathbb{R}^d \times [0,1]$. ∎

4. Properties of the linearized system. In the previous section, we have explained that the solution (u^T, m^T) of the mean field game system (1.1) converges to the solution $(\bar{u}, \bar{m})$ of the ergodic problem (1.2). The rest of the paper is devoted to quantifying this convergence. In this section, we analyze the linearized system near the solution $(\bar{u} - \bar{\lambda}(t-T), \bar{m})$. It has a very special structure (see Lemmas 4.1 and 4.2) which entails an exponential decay for the solutions (see Lemma 4.5). Although, strictly speaking, we do not work here with the exact linearized system but directly with the full equation, we consider throughout the section that the remaining terms are small perturbation of the other ones. Quantifying more carefully their size is quite technical and is postponed to the last two sections.

4.1. The "linearized problem."

Let (u^T, m^T) be a solution to (1.1) and let us set $(v^T, \mu^T) = (u^T, m^T) - (\bar{u} + \bar{\lambda}(T-t), \bar{m})$. Then (v^T, μ^T) solves system (3.5), which we rewrite, for simplicity, as

$$\begin{cases} \text{(i)} \ \ \frac{d}{dt} v^T = \Omega^* v^T - B + R_2, \\ \text{(ii)} \ \ \frac{d}{dt}\mu^T = -\Omega \mu^T - A v^T - R_1, \\ \text{(iii)} \ \ \mu^T(0) = \mu_0, \ v^T(T) = v_f, \end{cases} \tag{4.1}$$

where we have set

$$\mu_0 = m_0 - \bar{m}, \quad v_f = u_f - \bar{u},$$
$$\Omega\mu = -\Delta\mu - \operatorname{div}(\mu D\bar{u}), \qquad \Omega^* v = -\Delta v + D\bar{u}Dv,$$
$$Av = -\operatorname{div}(\bar{m}Dv), \quad B = F(x, \bar{m} + \mu(t)) - F(x, \bar{m}),$$
$$R_1 = -\operatorname{div}(\mu^T Dv^T), \quad R_2 = \frac{1}{2}|Dv^T|^2.$$

We note for later use that $\bar{m} + \mu^T = m^T \geq 0$, while $\int_Q \mu^T(t)dx = 0$ for any $t \in [0, T]$.

In the rest of the paper we consider Ω, Ω^*, and A as unbounded operators on $L^2_\sharp$. In particular, A is symmetric and coercive in $L^2_{\sharp,0}$. Its inverse, A^{-1}, is bounded and, for $f \in L^2_{\sharp,0}$, $A^{-1}f$ is nothing but the unique solution v of the problem $-\operatorname{div}(\bar{m}Dv) = f$, $\int_Q v = 0$. We denote by $A^{-\frac{1}{2}}$ and $A^{\frac{1}{2}}$ the respective square roots of A^{-1} and A. Note that $A^{-\frac{1}{2}}$ is bounded and injective, $A^{\frac{1}{2}}$ being its inverse. Note also that, since $A^{-1} = A^{-\frac{1}{2}}A^{-\frac{1}{2}}$, the range of A^{-1} is contained in the range of $A^{-\frac{1}{2}}$, so that $C^2_\sharp(\mathbb{R}^d)$, which is contained in the range of A^{-1}, is also contained in the range of $A^{-\frac{1}{2}}$ and therefore in the domain of $A^{\frac{1}{2}}$.

LEMMA 4.1. *We have $A = \bar{m}\Omega^* = \Omega\bar{m}$. In particular, $A\Omega^* = \Omega A$. Moreover*

$$A^{-1}Az = z - \langle z \rangle \qquad \forall z \in \mathcal{C}^2_\sharp,$$

while

$$A^{-1}\Omega z = \frac{z}{\bar{m}} - \left\langle \frac{z}{\bar{m}} \right\rangle \qquad \forall z \in \mathcal{C}^2_\sharp.$$

Proof. For any smooth map v

$$Av = -\operatorname{div}(\bar{m}Dv) = -\bar{m}\Delta v - D\bar{m}Dv,$$

where $D\bar{m} = -\bar{m}D\bar{u}$. Hence $A = \bar{m}\Omega^*$. Moreover, since A is symmetric, $A = (\bar{m}\Omega^*)^* = \Omega\bar{m}$. Equality $A\Omega^* = \Omega\bar{m}\Omega^* = \Omega A$ then follows.

Next let $z \in \mathcal{C}^2$ and set $w = Az$. Then $y := z - \langle z \rangle$ is the unique solution to $-\operatorname{div}(\bar{m}Dy) = w$, which has zero average. So $A^{-1}Az = A^{-1}w = z - \langle z \rangle$. Since $\Omega = A\frac{1}{\bar{m}}$, we get immediately that

$$A^{-1}\Omega z = A^{-1}A\frac{z}{\bar{m}} = \frac{z}{\bar{m}} - \left\langle \frac{z}{\bar{m}} \right\rangle \qquad \forall z \in \mathcal{C}^2. \qquad \square$$

We immediately deduce the following.

LEMMA 4.2 (equation for μ^T). *We have*

$$\frac{d^2}{dt^2}\mu^T = \Omega^2\mu^T + AB + \Omega R_1 - AR_2 - \frac{d}{dt}R_1.$$

Proof. Indeed

$$\begin{aligned}
\frac{d^2}{dt^2}\mu^T &= -\Omega\left(\frac{d}{dt}\mu^T\right) - A\left(\frac{d}{dt}v^T\right) - \frac{d}{dt}R_1 \\
&= -\Omega\left(\frac{d}{dt}\mu^T\right) - A\left[\Omega^* v^T - B + R_2\right] - \frac{d}{dt}R_1 \\
&= -\Omega\left(\frac{d}{dt}\mu^T\right) - \Omega A v^T + AB - AR_2 - \frac{d}{dt}R_1 \\
&= \Omega^2\mu^T + \Omega R_1 + AB - AR_2 - \frac{d}{dt}R_1. \qquad \square
\end{aligned}$$

LEMMA 4.3. *For any $f \in L^2_{\sharp,0}(\mathbb{R}^d)$ we have, for some constant $C > 0$,*

$$\frac{1}{C}\|A^{-\frac{1}{2}}f\|_2 \leq \|f\|_{H^{-1}_{\sharp,0}} \leq C\|A^{-\frac{1}{2}}f\|_2.$$

Proof. Let us set $g = A^{-1}f$. Then

$$\|A^{-\frac{1}{2}}f\|_2^2 = \langle A^{-1}f, f\rangle = \langle g, -\mathrm{div}(\bar{m}Dg)\rangle = \int_Q \bar{m}|Dg|^2.$$

If $\phi \in H^1_{\sharp,0}$, then we have

$$\begin{aligned}
\langle f, \phi\rangle = \int_Q \bar{m}\langle Dg, D\phi\rangle &\leq \left(\int_Q \bar{m}|Dg|^2\right)^{\frac{1}{2}}\left(\int_Q \bar{m}|D\phi|^2\right)^{\frac{1}{2}} \\
&\leq C\|A^{-\frac{1}{2}}f\|_2\|D\phi\|_2 \;=\; C\|\phi\|_{H^1_{\sharp,0}}\|A^{-\frac{1}{2}}f\|_2.
\end{aligned}$$

So $\|f\|_{H^{-1}_{\sharp,0}} \leq C\|A^{-\frac{1}{2}}f\|_2$. Conversely, if we choose $\phi = g/\|Dg\|_2$, then $\|\phi\|_{H^1_{\sharp,0}} = 1$ and

$$\|f\|_{H^{-1}_{\sharp,0}} \geq \langle f, \phi\rangle = \int_Q \bar{m}\langle Dg, D\phi\rangle = \int_Q \bar{m}\frac{|Dg|^2}{\|Dg\|_2} = \frac{\|A^{-\frac{1}{2}}f\|_2^2}{\|Dg\|_2} \geq c\,\|A^{-\frac{1}{2}}f\|_2. \qquad \square$$

4.2. An estimate on the linearized system. Let us first recall classical decay estimates—the proofs of which are given in the appendix—for the solution ($\mu(t) := e^{-\Omega t}\mu_0$) of

$$\begin{cases} \frac{d}{dt}\mu(t) = -\Omega\mu(t), \\ \mu(0) = \mu_0 \end{cases}$$

and for the solution ($v(t) := e^{-\Omega^*(T-t)}v_f$) of the backward equation

$$\begin{cases} \frac{d}{dt}v(t) = \Omega^* v(t), \\ v(T) = v_f, \end{cases}$$

where

$$\Omega\mu = -\Delta\mu - \mathrm{div}(\mu D\bar{u}) \qquad \text{and} \qquad \Omega^* v = -\Delta v + D\bar{u}Dv.$$

LEMMA 4.4. *There are constants $\omega > 0$ and $C_0 > 0$ such that*

$$\|e^{-\Omega t}\mu_0\|_2 \leq C_0\|\mu_0\|_2 e^{-\omega t} \qquad \forall t \geq 0, \qquad \forall \mu_0 \in L^2_{\sharp,0},$$

$$\left\|e^{-\Omega^*(T-t)}v_f - \langle e^{-\Omega^*(T-t)}v_f\rangle\right\|_2 \leq C_0\|\tilde{v}_f\|_2 e^{-\omega(T-t)} \qquad \forall t \leq T, \qquad \forall v_f \in L^2_\sharp,$$

where $\tilde{v} = v - \langle v\rangle$.

Now let (v^T, μ^T) be the solution to the system (4.1). Let us set $w^T = A^{-\frac{1}{2}}\mu^T$. Since μ^T is smooth and has zero average, it belongs to the domain of $A^{-\frac{1}{2}}$.

LEMMA 4.5. *There are constants $\omega_0 > 0$ and $C_0 > 0$ such that*

$$\frac{d^2}{dt^2}\frac{1}{2}\|w^T(t)\|_2^2 \geq 2\omega_0^2\|w^T(t)\|_2^2 - C_0\left\|A^{-1}\left(\Omega R_1 - AR_2 - \frac{d}{dt}R_1\right)\right\|_2^2.$$

Remark 4.6. For the truly linearized system (i.e., when $R_1 = R_2 = 0$), the above estimate implies the exponential decay

$$\|w^T(t)\|_2^2 \leq \max\{\|w^T(0)\|_2^2, \|w^T(T)\|_2^2\}\left(e^{-2\omega_0 t} + e^{-2\omega_0(T-t)}\right).$$

Proof. In view of Lemma 4.2, we have

$$\begin{aligned}\frac{d^2}{dt^2}\frac{1}{2}\|w^T(t)\|_2^2 &= \left\|\frac{d}{dt}w^T(t)\right\|_2^2 + \left\langle \frac{d^2}{dt^2}w^T(t), w^T(t)\right\rangle \geq \left\langle A^{-1}\left(\frac{d^2}{dt^2}\mu^T(t)\right), \mu^T(t)\right\rangle \\ &\geq \left\langle A^{-1}\left(\Omega^2\mu^T(t) + AB + \Omega R_1 - AR_2 - \frac{d}{dt}R_1\right), \mu^T(t)\right\rangle,\end{aligned}$$

where, by Lemma 4.1,

$$\langle A^{-1}\Omega^2\mu^T(t), \mu^T(t)\rangle = \left\langle \frac{\Omega\mu^T(t)}{\bar{m}} - \left\langle\frac{\Omega\mu^T(t)}{\bar{m}}\right\rangle, \mu^T(t)\right\rangle = \left\langle \frac{1}{\bar{m}}A\frac{\mu^T(t)}{\bar{m}}, \mu^T(t)\right\rangle$$

(because $\mu^T(t)$ has mean zero), while

$$\langle A^{-1}AB, \mu^T(t)\rangle = \langle B - \langle B\rangle, \mu^T(t)\rangle = \langle B, \mu^T(t)\rangle.$$

So, from assumption (2.1),

$$\langle A^{-1}AB, \mu^T(t)\rangle \; = \; \langle B, \mu^T(t)\rangle = \int_Q (F(x, \bar{m} + \mu(t)) - F(x, \bar{m}(t)))\mu^T(t) \geq 0,$$

and we get

$$\frac{d^2}{dt^2}\frac{1}{2}\|w^T(t)\|_2^2 \; \geq \; \left\langle \frac{1}{\bar{m}}A\frac{1}{\bar{m}}\mu^T(t), \mu^T(t)\right\rangle + \left\langle A^{-1}(\Omega R_1 - AR_2 - \frac{d}{dt}R_1), \mu^T(t)\right\rangle. \tag{4.2}$$

We wish to show now that the first term on the right-hand side is coercive with respect to $\|w^T\|_2$. For this purpose, let us note that

$$\left\langle \frac{1}{\bar{m}}A\frac{1}{\bar{m}}\mu^T(t), \mu^T(t)\right\rangle = \int_Q \bar{m}\left|D\left(\frac{\mu^T}{\bar{m}}\right)\right|^2. \tag{4.3}$$

We also recall that

$$\|\mu^T\|_2^2 = \langle A^{\frac{1}{2}}w^T, A^{\frac{1}{2}}w^T\rangle = \langle Aw^T, w^T\rangle,$$

and since A is coercive on $L^2_{\sharp,0}$ we deduce

$$\|w^T\|_2^2 \leq c\|\mu^T\|_2^2. \tag{4.4}$$

On the other hand, if we set $f = A^{-1}\mu^T$, i.e., $Af = \mu^T$, we have

$$\|w^T\|_2^2 = \langle A^{-1}\mu^T, \mu^T\rangle = \int_Q f\,\mu^T dx = \int_Q \bar m\,|Df|^2\,dx. \tag{4.5}$$

Multiplying equality $Af = \mu^T$ by $\frac{\mu^T}{\bar m}$ we also have

$$\int_Q \bar m Df D\left(\frac{\mu^T}{\bar m}\right)\,dx = \int_Q \frac{(\mu^T)^2}{\bar m} dx.$$

Hence, for $\epsilon > 0$ to be chosen later,

$$\|\mu^T\|_2^2 \le c\int_Q \frac{(\mu^T)^2}{\bar m} dx \le \varepsilon \int_Q \bar m\,|Df|^2 + C_\varepsilon \int_Q \left|D\left(\frac{\mu^T}{\bar m}\right)\right|^2\,dx.$$

Together with (4.4) and (4.5) we deduce

$$\|w^T\|_2^2 \le c\|\mu^T\|_2^2 \le c\,\varepsilon\|w^T\|_2^2 + C_\varepsilon\int_Q \left|D\left(\frac{\mu^T}{\bar m}\right)\right|^2\,dx. \tag{4.6}$$

Choosing ε suitably and recalling (4.3) we obtain

$$\|w^T\|_2^2 \le C\int_Q \left|D\left(\frac{\mu^T}{\bar m}\right)\right|^2\,dx \le C\int_Q \bar m\left|D\left(\frac{\mu^T}{\bar m}\right)\right|^2 = C\left\langle A\frac{1}{\bar m}\mu^T(t), \frac{1}{\bar m}\mu^T(t)\right\rangle, \tag{4.7}$$

and in turn from (4.6) we also get

$$\|\mu^T\|_2^2 \le c\left\langle A\frac{1}{\bar m}\mu^T(t), \frac{1}{\bar m}\mu^T(t)\right\rangle. \tag{4.8}$$

Since we have

$$\left|\left\langle A^{-1}\left(\Omega R_1 - AR_2 - \frac{d}{dt}R_1\right), \mu^T(t)\right\rangle\right| \le \left\|A^{-1}\left(\Omega R_1 - AR_2 - \frac{d}{dt}R_1\right)\right\|_2 \|\mu^T\|_2,$$

we deduce from (4.2) and (4.8)

$$\frac{d^2}{dt^2}\frac{1}{2}\|w^T(t)\|_2^2 \;\ge\; \frac{1}{2}\left\langle A\frac{1}{\bar m}\mu^T(t), \frac{1}{\bar m}\mu^T(t)\right\rangle - C\left\|A^{-1}\left(\Omega R_1 - AR_2 - \frac{d}{dt}R_1\right)\right\|_2^2,$$

which implies, thanks to (4.7),

$$\frac{d^2}{dt^2}\frac{1}{2}\|w^T(t)\|_2^2 \;\ge c\|w^T(t)\|_2^2 - C\left\|A^{-1}\left(\Omega R_1 - AR_2 - \frac{d}{dt}R_1\right)\right\|_2^2. \qquad \square$$

5. The exponential convergence for small initial data. The aim of this section is to prove an exponential decay for sufficiently small initial data. The exponential rate will be of order $\bar\omega := \omega \wedge \omega_0$, where ω is the constant which appears in Lemma 7.1, while ω_0 is given by Lemma 4.5.

As in the previous section, if (u^T, m^T) is a solution to (1.1), we set $(v^T, \mu^T) = (u^T, m^T) - (\bar u + \bar\lambda(T-t), \bar m)$. Then (v^T, μ^T) solves (4.1). We also normalize v^T into

$\tilde{v}^T$ by setting $\tilde{v}^T(t) = v^T(t) - \langle v^T(t)\rangle$, where $\langle v^T(t)\rangle = \int_Q v^T(t)$. In the same way, $\tilde{v}_f = v_f - \langle v_f\rangle$, where $\langle v_f\rangle = \int_Q v_f$.

THEOREM 5.1. *Let us assume that* (2.1), (2.2) *hold for some* $\alpha \in (0,1)$. *Let* $\lambda \in (0, \frac{\bar{\omega}}{2})$. *Then there exists* $\eta = \eta(F,\lambda) > 0$ *and* $C = C(F,\lambda)$ *such that if*

$$\|\mu_0\|_{C^{2+\alpha}} + \|\tilde{v}_f\|_{C^{3+\alpha}} \leq \eta, \tag{5.1}$$

then the solution (μ^T, v^T) *of* (4.1) *satisfies the exponential decay*

$$\|\mu^T\|_{C^{\frac{2+\alpha}{2},2+\alpha}([t-1,t]\times Q)} + \|\tilde{v}^T\|_{C^{\frac{3+\alpha}{2},3+\alpha}([t-1,t]\times Q)} \leq C\left(e^{-2\lambda t} + e^{-2\lambda(T-t)}\right) \tag{5.2}$$

for any $t \in [1,T]$.

Remark 5.2. We will show in the next section that one can get an exponential decay also for large initial data in time intervals of the form $[M, T-M]$ for $M > 0$ large.

The main reason Theorem 5.1 holds comes from Lemma 4.5. The technical difficulties arise when handling the quadratic perturbations R_1, R_2. The proof is based on a fixed point argument. Let X_T be the set defined by

$$X_T = \left\{(\mu, v) \in C_\sharp^{\frac{1+\alpha}{2},1+\alpha}([0,T]\times\mathbb{R}^d) \times C_\sharp^{\frac{2+\alpha}{2},2+\alpha}([0,T]\times\mathbb{R}^d)\right.$$
$$\left.\text{with } \int_Q \mu(t) = 0 \ \forall t \in [0,T]\right\}.$$

We consider the map $\mathcal{T} : X_T \to X_T$ defined by $\mathcal{T}(\hat{\mu}^T, \hat{v}^T) = (\mu^T, v^T)$, where (μ^T, v^T) is the solution to

$$\begin{cases} \text{(i)} \ \frac{d}{dt}v^T = \Omega^* v^T - B + \hat{R}_2, \\ \text{(ii)} \ \frac{d}{dt}\mu^T = -\Omega\mu^T - Av^T - \hat{R}_1, \\ \text{(iii)} \ \mu^T(0) = \mu_0, \ v^T(T) = v_f, \end{cases} \tag{5.3}$$

and where we have set

$$\hat{R}_1 = -\mathrm{div}(\hat{\mu}^T D\hat{v}^T), \ \hat{R}_2 = \frac{1}{2}|D\hat{v}^T|^2 \tag{5.4}$$

and B is, as before, given by

$$B = F(x, \bar{m} + \mu^T(t)) - F(x, \bar{m}).$$

For $\alpha \in (0,1)$ given in assumption (2.2) and $\lambda \in (0, \bar{\omega}/2)$ and $M > 0$ to be chosen later, we consider the convex subset $\mathcal{C}_T$ of X_T consisting of maps $(\mu, v) \in X_T$ such that $\mu \in C^1([0,T]\times Q)$, $v \in C^2([0,T]\times Q)$ and such that

(5.5)
$$\|\mu\|_{C^{\frac{1+\alpha}{2},1+\alpha}([t-1,t]\times Q)} + \|\tilde{v}\|_{C^{\frac{2+\alpha}{2},2+\alpha}([t-1,t]\times Q)} \leq M\left(e^{-\lambda t} + e^{-\lambda(T-t)}\right) \quad \forall t \in [1,T],$$
$$\|\mu_t(t)\|_\infty + \|Dv_t\|_\infty \leq M\left(e^{-\lambda t} + e^{-\lambda(T-t)}\right) \quad \forall t \in [0,T],$$

where we have set $\tilde{v} = v - \langle v\rangle$ with $\langle v\rangle = \int_Q v$. Our aim is to prove that $\mathcal{T} : \mathcal{C}_T \to \mathcal{C}_T$ for M sufficiently small and suitably small boundary data (μ_0, v_f), and that $\mathcal{T}$ admits a fixed point in $\mathcal{C}_T$.

5.1. Preliminaries to the fixed point argument. We first observe that $(\hat\mu^T, \hat v^T) \in \mathcal{C}_T$ implies the following:

$$\hat\mu^T(t) \in C^{1+\alpha}(Q),\ \|\hat\mu^T(t)\|_{C^{1+\alpha}(Q)} \le M\left(e^{-\lambda t} + e^{-\lambda(T-t)}\right) \qquad \forall t \in [0,T]$$

and

$$\hat v^T(t) \in C^{2+\alpha}(Q),\ \|\hat v^T(t) - \langle \hat v^T(t)\rangle\|_{C^{2+\alpha}(Q)} \le M\left(e^{-\lambda t} + e^{-\lambda(T-t)}\right) \qquad \forall t \in [0,T].$$

Note also that $\hat R_1$ and $\hat R_2$ depend only on $\hat\mu^T$ and on the normalized function $\hat v^T(t) - \langle \hat v^T(t)\rangle$. As a consequence of the above spatial estimates, we deduce then

$$\|\hat R_1(t)\|_\infty + \|\hat R_2(t)\|_\infty + \|D\hat R_2(t)\|_\infty \le c\, M^2\left(e^{-2\lambda t} + e^{-2\lambda(T-t)}\right) \qquad \forall t \in [0,T]. \tag{5.6}$$

Moreover, the requirement on the time derivatives also implies that, whenever $(\hat\mu^T, \hat v^T) \in \mathcal{C}_T$,

$$\left\|\frac{d}{dt}(\hat\mu^T D\hat v^T)(t)\right\|_\infty \le c\, M^2\left(e^{-2\lambda t} + e^{-2\lambda(T-t)}\right) \qquad \forall t \in [0,T]. \tag{5.7}$$

Finally, we also deduce from the definition of $\hat R_2$, $\hat R_1$ that, whenever $(\hat\mu^T, \hat v^T) \in \mathcal{C}_T$, we have

$$\hat R_1 \in C^{\frac{\alpha}{2},\alpha}, \quad \|\hat R_1\|_{C^{\frac{\alpha}{2},\alpha}([t-1,t]\times Q)} \le cM^2\left(e^{-2\lambda t} + e^{-2\lambda(T-t)}\right) \qquad \forall t \in [1,T] \tag{5.8}$$

and

$$\hat R_2 \in C^{\frac{1+\alpha}{2},1+\alpha}, \quad \|\hat R_2\|_{C^{\frac{1+\alpha}{2},1+\alpha}([t-1,t]\times Q)} \le cM^2\left(e^{-2\lambda t} + e^{-2\lambda(T-t)}\right) \qquad \forall t \in [1,T]. \tag{5.9}$$

5.2. First step. Throughout this section, (μ^T, v^T) is the solution to (5.3) with $\hat R_1$ and $\hat R_2$ defined by (5.4), where $(\hat\mu^T, \hat v^T) \in \mathcal{C}_T$. Let us set

$$\phi(t) = A^{-1}\left(\Omega \hat R_1 - A\hat R_2 - \frac{d}{dt}\hat R_1\right).$$

LEMMA 5.3. *Let $(\hat\mu^T, \hat v^T) \in \mathcal{C}_T$. Then there is a constant $C_1 > 0$ such that*

$$\|\phi(t)\|_2 \le C_1\, M^2\left(e^{-2\lambda t} + e^{-2\lambda(T-t)}\right) \qquad \forall t \in [0,T].$$

Proof. Following Lemma 4.1, we have

$$A^{-1}\Omega \hat R_1 = \frac{\hat R_1}{\bar m} - \left\langle \frac{\hat R_1}{\bar m}\right\rangle, \qquad A^{-1}A\hat R_2 = \hat R_2 - \langle \hat R_2\rangle.$$

Therefore, we deduce

$$\|A^{-1}\Omega \hat R_1\|_2 \le 2\left\|\frac{\hat R_1}{\bar m}\right\|_2 \le c\|\hat R_1\|_2$$

and

$$\|A^{-1}A\hat{R}_2\|_2 \leq 2\|\hat{R}_2\|_2.$$

Finally, since A^{-1} is continuous from $H^{-1}_{\sharp,0}$ into $L^2_{\sharp,0}$, we have

$$\left\|A^{-1}\left(\frac{d}{dt}\hat{R}_1\right)\right\|_2 \leq c\left\|\frac{d}{dt}\hat{R}_1\right\|_{H^{-1}_{\sharp,0}} \leq c\left\|\frac{d}{dt}(\hat{\mu}^T D\hat{v}^T)\right\|_2.$$

Therefore, we conclude that

$$\|\phi(t)\|_2 \leq c\left\{\|\hat{R}_1(t)\|_2 + \|\hat{R}_2(t)\|_2 + \left\|\frac{d}{dt}(\hat{\mu}^T D\hat{v}^T)(t)\right\|_2\right\}.$$

The conclusion follows from (5.6), (5.7). ∎

The above estimate is needed for using the fundamental Lemma 4.5. We deduce the following.

COROLLARY 5.4. *Let $w^T = A^{-\frac{1}{2}}\mu^T$. Then there is a constant $C_2 > 0$ such that*

$$\|w^T(t)\|_2^2 \leq C_2\left(\|\mu_0\|_2^2 + \|w^T(T)\|_2^2 + M^4\right)\left(e^{-4\lambda t} + e^{-4\lambda(T-t)}\right) \qquad \forall t \in [0,T]. \tag{5.10}$$

Proof. From Lemma 4.5, we are given $\omega_0 > 0$ and a constant C_0 such that

$$\frac{d^2}{dt^2}\frac{1}{2}\|w^T(t)\|_2^2 \geq 2\omega_0^2\|w^T(t)\|_2^2 - C_0\|\phi(t)\|_2^2.$$

So, from Lemma 5.3, $\|w^T(t)\|_2^2$ is a subsolution to

$$-\frac{d^2}{dt^2}\|w^T(t)\|_2^2 + 4\omega_0^2\|w^T(t)\|_2^2 - \tilde{C}_0\, M^4\left(e^{-4\lambda t} + e^{-4\lambda(T-t)}\right) \leq 0$$

for some $\tilde{C}_0 > 0$. Since the map

$$\rho(t) := \|w^T(0)\|^2 e^{-2\omega_0 t} + \|w^T(T)\|^2 e^{-2\omega_0(T-t)} + \frac{\tilde{C}_0 M^4}{4\omega_0^2 - 16\lambda^2}\left(e^{-4\lambda t} + e^{-4\lambda(T-t)}\right)$$

satisfies

$$\frac{d^2}{dt^2}\rho(t) = 4\omega_0^2\rho(t) - \tilde{C}_0\, M^4\left(e^{-4\lambda t} + e^{-4\lambda(T-t)}\right)$$

with $\rho(0) \geq \|w^T(0)\|_2^2$, $\rho(T) \geq \|w^T(T)\|_2^2$, we get, by the comparison principle,

$$\|w^T(t)\|_2^2 \leq \rho(t) \qquad \forall t \in [0,T].$$

Using the fact that $A^{-\frac{1}{2}}$ is bounded, we have $\|w^T(0)\|_2 \leq \|\mu_0\|_2$; and since $2\lambda < \omega_0$, from the definition of ρ we obtain (5.10). ∎

Our next purpose is to show that $\|w^T(T)\|_2$ can be dropped in the estimate (5.10). More precisely, we are going to estimate $\|\mu^T(t)\|_2$ in terms of $\|\mu_0\|_2 + \|\tilde{v}_f\|_2 + M^2$.

PROPOSITION 5.5. *Recall that $w^T = A^{-\frac{1}{2}}\mu^T$ and $\tilde{v}^T = v^T - \langle v^T\rangle$. There is a constant $C_3 > 0$ such that*

$$\|\mu^T(t)\|_2 \leq C_3\left(\|\mu_0\|_2 + \|\tilde{v}_f\|_2 + M^2\right) \qquad \forall t \in [0,T],$$

while

(5.11)
$$\|w^T(t)\|_2+\|\tilde{v}^T(t)\|_2 \le C_3\left(\|\mu_0\|_2+\|\tilde{v}_f\|_2+M^2\right)\left(e^{-2\lambda t}+e^{-2\lambda(T-t)}\right) \qquad \forall t\in[0,T].$$

In order to prove Proposition 5.5, we define

$$K := \sup_{t\in[0,T]} \|\mu^T(t)\|_2$$

and proceed in two steps.

LEMMA 5.6. *For any $\varepsilon>0$, there exists a constant C_ε such that we have*

$$\int_0^T\int_Q \bar{m}\,|Dv^T|^2 dxdt \le \varepsilon K^2 + C_\varepsilon(M^4+\|\mu_0\|_2^2+\|\tilde{v}_f\|_2^2) \tag{5.12}$$

and

$$\|\tilde{v}^T(t)\|_2 \le e^{-\omega(T-t)}\|\tilde{v}_f\|_2 + c\left(\|\mu_0\|_2+K+M^2\right)\left(e^{-2\lambda t}+e^{-2\lambda(T-t)}\right). \tag{5.13}$$

Proof.
Step 1. From assumption (2.2), Lemma 4.3, and Corollary 5.4, we have

$$\begin{aligned}\|B(t)\|_2 &\le \bar{C}_F\|w(t)\|_2\\ &\le C\left(\|\mu_0\|_2+\|w^T(T)\|_2+M^2\right)\left(e^{-2\lambda t}+e^{-2\lambda(T-t)}\right).\end{aligned}$$

Hence, since $\|w^T(T)\|_2 \le c\|\mu^T(T)\|_2 \le cK$, we have

$$\|B(t)\|_2 \le C\left(\|\mu_0\|_2+K+M^2\right)\left(e^{-2\lambda t}+e^{-2\lambda(T-t)}\right).$$

On the other hand, one has (from Lemma 7.4)

$$\|\tilde{v}^T(t)\|_2 \le e^{-\omega(T-t)}\|\tilde{v}_f\|_2 + \int_t^T e^{-\omega(s-t)}\left(\|B(s)\|_2+\|\hat{R}_2\|_2\right)ds,$$

and so we obtain, due also to the estimate (5.6) on $\hat{R}_2$,

$$\|\tilde{v}^T(t)\|_2 \le e^{-\omega(T-t)}\|\tilde{v}_f\|_2 + c\left(\|\mu_0\|_2+K+M^2\right)\int_t^T e^{-\omega(s-t)}\left(e^{-2\lambda s}+e^{-2\lambda(T-s)}\right),$$

which implies (5.13) since $2\lambda<\omega$.

Step 2. We multiply now (5.3)(ii) by v^T and (5.3)(i) by μ^T, add and integrate in space-time to get

$$\int_Q \mu^T(T)v_f - \int_Q \mu_0 v^T(0) = -\int_0^T\int_Q (Av^Tv^T+\hat{R}_1v^T+B\mu^T-\hat{R}_2\mu^T).$$

From assumption (2.1), $\int_Q B\mu^T\ge 0$. Hence

$$\int_0^T\int_Q \bar{m}\,|Dv^T|^2dxdt \le \left|\int_Q \mu^T(T)v_f\right| + \left|\int_Q \mu_0v^T(0)\right| + \left|\int_0^T\int_Q \hat{R}_1v^T\right| + \left|\int_0^T\int_Q \hat{R}_2\mu^T\right|.$$

Of course we have, by definition of K,

$$\left|\int_Q \mu^T(T)v_f\right| = \left|\int_Q \mu^T(T)\tilde{v}_f\right| \le K\,\|\tilde{v}_f\|_2$$

and

$$\left|\int_0^T\int_Q \hat{R}_2\mu^T\right| \le \int_0^T \|\hat{R}_2(t)\|_2\|\mu^T(t)\|_2\,dt \le K\int_0^T \|\hat{R}_2(t)\|_2 dt.$$

Moreover, since μ_0 and $\hat{R}_1$ have zero average, we have

$$\left|\int_Q \mu_0 v^T(0)\right| = \left|\int_Q \mu_0\tilde{v}^T(0)\right| \le \|\mu_0\|_2\|\tilde{v}^T(0)\|_2$$

and, similarly,

$$\left|\int_0^T\int_Q \hat{R}_1 v^T\right| \le \int_0^T \|\hat{R}_1(t)\|_2\|\tilde{v}^T(t)\|_2\,dt.$$

Therefore we get

$$\int_0^T\int_Q \bar{m}\,|Dv^T|^2 dxdt$$
$$\le K\|\tilde{v}_f\|_2 + \|\mu_0\|_2\|\tilde{v}^T(0)\|_2 + K\int_0^T \|\hat{R}_2\|_2 dt + \int_0^T \|\hat{R}_1\|_2\|\tilde{v}^T(t)\|_2 dt.$$

Now we use (5.6) to bound $\|\hat{R}_2\|_2$ and $\|\hat{R}_1\|_2$, and (5.13) to bound $\|\tilde{v}^T(t)\|_2$; then by Young's inequality we conclude (5.12). $\square$

Proof of Proposition 5.5. We use Lemma 7.6 for the equation of μ^T, with $f = \hat{R}_1$ and $F = \bar{m}Dv^T$, obtaining

$$\|\mu^T(t)\|_2 \le C\left\{e^{-\omega t}\|\mu_0\|_2 + \int_0^t e^{-\omega(t-s)}\|\hat{R}_1(s)\|_2 ds + \left(\int_0^t\int_Q \bar{m}|Dv^T|^2\right)^{\frac{1}{2}}\right\}.$$

Using (5.12) and the bound on $\hat{R}_1$, we deduce

$$\|\mu^T(t)\|_2 \le C\left\{e^{-\omega t}\|\mu_0\|_2 + \sqrt{\varepsilon}K + C_\varepsilon(M^2 + \|\mu_0\|_2 + \|\tilde{v}_f\|_2)\right\}$$

for every $t > 0$. Taking sup, and choosing ε suitably small, we conclude that $K \le c(M^2 + \|\mu_0\|_2 + \|\tilde{v}_f\|_2)$, hence the estimate for μ^T.

Thanks to the new bound on K, the estimate on w^T and $\tilde{v}^T$ follows from (5.10) and (5.13). $\square$

5.3. Second step. We now wish to conclude the fixed point argument on account of the estimate (5.11). First of all, we notice that assumption (2.2), Lemma 4.3, and (5.11) imply

$$\|B(t)\|_{\mathcal{C}^{1+\alpha}} \le \bar{C}_F\|w^T(t)\|_2 \le C\left(\|\mu_0\|_2 + \|\tilde{v}_f\|_2 + M^2\right)\left(e^{-2\lambda t} + e^{-2\lambda(T-t)}\right) \tag{5.14}$$

for every $t \in [0,T]$.

We now can estimate v^T in stronger norms. As a first step, we estimate Dv^T and D^2v^T.

LEMMA 5.7. *We have*

$$\|Dv^T(t)\|_2 + \|D^2v^T(t)\|_2 \\ \leq C\left(\|\mu_0\|_2 + \|\tilde v_f\|_2 + M^2\right)\left(e^{-2\lambda t} + e^{-2\lambda(T-t)}\right) \qquad \forall t < T-2.$$

In addition, if $v_f \in H^1_\sharp(Q)$ (respectively, $v_f \in H^2_\sharp(Q)$), we also have

$$\|Dv^T(t)\|_2 \leq C\left(\|\mu_0\|_2 + \|\tilde v_f\|_{H^1} + M^2\right) \qquad \forall t \in [0,T], \tag{5.15}$$

respectively,

$$\|D^2v^T(t)\|_2 \leq C\left(\|\mu_0\|_2 + \|\tilde v_f\|_{H^2} + M^2\right) \qquad \forall t \in [0,T]. \tag{5.16}$$

Proof. In a first step we establish the estimate for Dv^T and D^2v^T on $[0, T-2]$, and then we show that (5.15) and (5.16) hold.

Step 1. Let us apply Lemma 7.5 in the appendix to the equation of v^T in the interval $(t, t+1)$ with $f = B - \hat R_2$: we get

$$\int_Q |Dv^T(t)|^2\,dx \leq C\left\{\int_Q |\tilde v^T(t+1)|^2 dx + \int_t^{t+1}\int_Q \left(|B|^2 + |\hat R_2|^2 + |\tilde v^T|^2\right) dxds\right\}.$$

Using (5.6), (5.11), and (5.14), we conclude that

$$\|Dv^T(t)\|_2 \leq C\left(\|\mu_0\|_2 + \|\tilde v_f\|_2 + M^2\right)\left(e^{-2\lambda t} + e^{-2\lambda(T-t)}\right) \tag{5.17}$$

for every $t < T-1$.

We proceed similarly for the estimate on D^2v^T. Taking one derivative on the equation satisfied by v^T, and setting $v_i^T = \frac{\partial v^T}{\partial x_i}$ for $i = 1, \ldots, d$, implies

$$(v_i^T)_t - \Delta v_i^T + Dv_i^T\, D\bar u = f_{x_i} + Dv^T \cdot D\bar u_{x_i} \qquad \text{in } (0,T)\times\mathbb{R}^d,$$

where, as before, $f = B - \hat R_2$. Applying again Lemma 7.5 in $(t, t+1)$ we get

$$\int_Q |Dv_i^T|^2\,dx \leq C\left\{\int_Q |\tilde v_i^T(t+1)|^2 dx + \int_t^{t+1}\int_Q \left(|f_{x_i}|^2 + |Dv^T|^2 + |\tilde v_i^T|^2\right) dxds\right\},$$

where c depends on $\|D^2\bar u\|_\infty$. Since v_i^T has zero average and $|v_i^T| \leq |Dv^T|$, we obtain

$$\int_Q |D^2v^T(t)|^2\,dx \\ \leq C\left\{\int_Q |Dv^T(t+1)|^2 dx + \int_t^{t+1}\int_Q \left(|DB|^2 + |D\hat R_2|^2 + |Dv^T|^2\right) dxds\right\}.$$

We use now the previous estimate (5.17) on $\|Dv^T(t)\|_2$, estimate (5.14), and (5.9). Overall, we conclude for $\|D^2v^T(t)\|_2$ estimate (5.17) as well for $t < T-2$.

Step 2. We can get a bound in $(T-2, T)$ using a global version of Lemma 7.5. Namely, proceeding as in the lemma, any solution of (7.10) satisfies

$$\int_Q |Dv^T(t)|^2\,dx \leq C\left\{\int_Q (|Dv^T(t_0)|^2 + |\tilde v^T(t_0)|^2) dx + \int_{t_0}^t\int_Q (|f|^2 + |\tilde v^T|^2) dsdx\right\}.$$

Applied to the equation of v^T in $[T-2,T]$ we obtain

$$\int_Q |Dv^T(t)|^2\,dx \le C\left\{\int_Q (|Dv_f|^2+|\tilde v_f|^2)dx + \int_t^T\int_Q (|B|^2+|\hat R_2|^2+|\tilde v^T|^2)dsdx\right\},$$

and then we get

$$\|Dv^T(t)\|_2 \le C\left(\|\mu_0\|_2+\|\tilde v_f\|_{H^1}+M^2\right) \qquad \forall t\in[T-2,T].$$

Similarly we get on D^2v

$$\int_Q |D^2v^T(t)|^2\,dx$$
$$\le C\left\{\int_Q |D^2v_f|^2dx + \int_Q |Dv_f|^2\,dx + \int_t^T\int_Q \left(|DB|^2+|D\hat R_2|^2+|Dv^T|^2\right)dxds\right\},$$

and hence we conclude (5.16). ∎

Thanks to the previous lemma, we conclude the estimate for $\|\mu^T\|_2$.

COROLLARY 5.8. *We have*

$$\|\mu^T(t)\|_2 \le C\left(\|\mu_0\|_2+\|\tilde v_f\|_2+M^2\right)\left(e^{-2\lambda t}+e^{-2\lambda(T-t)}\right) \tag{5.18}$$

for every $t\in[0,T]$.

Proof. Recalling the equation satisfied by μ^T, we have, in view of Lemma 7.6,

$$\|\mu^T(t)\|_2 \le C\,e^{-\omega t}\|\mu_0\|_2 + C\int_0^t e^{-\omega(t-s)}(\|Av^T(s)\|_2+\|\hat R_1(s)\|_2)ds.$$

But $Av^T = -\bar m\Delta v^T - D\bar m\cdot Dv^T$, so thanks to Lemma 5.7, and since $\bar m\in C^1$, we have

$$\|Av^T(t)\|_2 \le C\left(\|\mu_0\|_2+\|\tilde v_f\|_2+M^2\right)\left(e^{-2\lambda t}+e^{-2\lambda(T-t)}\right) \qquad \forall t<T-2.$$

Using also (5.6), and recalling that $2\lambda<\omega$, we conclude (5.18) at least for $t<T-2$. However, on account of Proposition 5.5, the same estimate clearly holds in $[T-2,T]$ as well, and we conclude the proof. ∎

As in Lemma 5.7, we now use the regularizing property of the μ^T equation.

LEMMA 5.9. *For every $t\in[1,T-2]$, we have*

$$\|D\mu^T(t)\|_2+\|\mu_t^T(t)\|_{H^{-1}(Q)} \le C\left(\|\mu_0\|_2+\|\tilde v_f\|_2+M^2\right)\left(e^{-2\lambda t}+e^{-2\lambda(T-t)}\right). \tag{5.19}$$

In addition, if $m_0\in H^1(Q)$ and $v_f\in H^2(Q)$, we also have

$$\begin{aligned} &\|D\mu^T(t)\|_2+\|\mu_t^T(t)\|_{H^{-1}(Q)} \le C\left(\|\mu_0\|_{H^1(Q)}+\|\tilde v_f\|_2+M^2\right) \quad \forall t\in(0,1),\\ &\|D\mu^T(t)\|_2+\|\mu_t^T(t)\|_{H^{-1}(Q)} \le C\left(\|\mu_0\|_2+\|\tilde v_f\|_{H^2}+M^2\right) \quad \forall t\in(T-2,T). \end{aligned} \tag{5.20}$$

Proof.

Step 1. Since μ^T satisfies

$$\mu_t^T - \Delta\mu^T - D\bar u D\mu^T = \mu^T\Delta\bar u + D\bar m Dv^T + \bar m\Delta v^T - \hat R_1,$$

applying Lemma 7.5 with $f = \mu^T \Delta\bar{u} + D\bar{m}Dv^T + \bar{m}\Delta v^T - \hat{R}_1$, we obtain
(5.21)

$$\|D\mu^T(t)\|_2^2 \le C\left\{\|\mu^T(t-1)\|_2^2 + \int_{t-1}^t \|f(s)\|_2^2\, dsdx + \int_{t-1}^t \|\mu^T(s)\|_2^2\, dsdx\right\} \quad \forall t > 1,$$

where C depends on $\|D\bar{u}\|_\infty$. On account of (5.6), and thanks to Lemma 5.7 and Corollary 5.8, we have

$$\|f(s)\|_2 \le C\left(\|\mu_0\|_2 + \|\tilde{v}_f\|_2 + M^2\right)\left(e^{-2\lambda s} + e^{-2\lambda(T-s)}\right)$$

for every $s < T-2$. Therefore, since $\|\mu^T(t)\|_2$ is estimated for every t due to (5.18), from (5.21) we deduce

$$\|D\mu^T(t)\|_2 \le C\left(\|\mu_0\|_2 + \|\tilde{v}_f\|_2 + M^2\right)\left(e^{-2\lambda t} + e^{-2\lambda(T-t)}\right) \qquad \forall t \in [1, T-2].$$

Now, the equation satisfied by μ^T implies

$$\|\mu_t^T(t)\|_{H^{-1}(Q)} \le C\left(\|D\mu^T(t)\|_2 + \|\mu^T(t)\|_2 + \|Dv^T(t)\|_2 + \|\hat{\mu}^T(t)\, D\hat{v}^T(t)\|_2\right).$$

Putting together the estimate obtained on $\|D\mu^T(t)\|_2$ with Lemma 5.7, Corollary 5.8, and the fact that $(\hat{\mu}^T, \hat{v}^T) \in \mathcal{C}_T$, we complete (5.19).

Step 2. Similarly as was done for v^T, a global bound is also possible on the μ^T equation. Proceeding as before, and using $m_0 \in H^1(Q)$, we get

$$\|D\mu^T(t)\|_2 \le C\left(\|\mu_0\|_{H^1} + \|\tilde{v}_f\|_2 + M^2\right) \qquad \forall t \in [0, 1],$$

which in turn implies a similar estimate for μ_t^T in $H^{-1}(Q)$ as was done before.

Similarly, in $[T-2, T]$ we have, now using (5.15) and (5.16),

$$\begin{aligned}
&\int_Q |D\mu^T(t)|^2\, dx \\
&\quad \le C\left\{\|\mu^T(T-2)\|_{H^1}^2 + \int_{T-2}^t \int_Q (|Dv^T|^2 + |D^2v^T|^2 + |\hat{R}_1|^2 + |\mu^T|^2) dsdx\right\} \\
&\quad \le C\left(\|\mu_0\|_2 + \|\tilde{v}_f\|_2 + M^2\right)^2 + C\left(\|\mu_0\|_2 + \|\tilde{v}_f\|_{H^2} + M^2\right)^2 \qquad \forall t \in [T-2, T].
\end{aligned}$$

Therefore we have in this case

$$\|\mu_t^T(t)\|_{H^{-1}(Q)} \le C\left(\|\mu_0\|_2 + \|\tilde{v}_f\|_{H^2} + M^2\right) \qquad \forall t \in [T-2, T],$$

and we conclude with (5.20). ☐

We now have all the ingredients to complete the estimates.

PROPOSITION 5.10. *Let $\alpha \in (0,1)$ be given by* (2.2). *There exists $C > 0$ such that we have*

$$\begin{aligned}
&\|\tilde{v}^T\|_{C^{\frac{3+\alpha}{2}, 3+\alpha}([t-1,t]\times Q)} \\
&\qquad \le C\left(\|\mu_0\|_2 + \|\tilde{v}_f\|_2 + M^2\right)\left(e^{-2\lambda t} + e^{-2\lambda(T-t)}\right) \qquad \forall t \in (2, T-2),
\end{aligned}$$

and globally

$$\|\tilde{v}^T\|_{C^{\frac{3+\alpha}{2}, 3+\alpha}([0,T]\times Q)} \le C\left(\|\mu_0\|_{H^1} + \|\tilde{v}_f\|_{C^{3+\alpha}} + M^2\right).$$

In particular, we have

$$\|\tilde v^T(t)\|_{C^{3+\alpha}} + \|\tilde v_t^T(t)\|_{C^{1+\alpha}} \le C\left(\|\mu_0\|_{H^1} + \|\tilde v_f\|_{C^{3+\alpha}} + M^2\right)\left(e^{-2\lambda t} + e^{-2\lambda(T-t)}\right) \qquad \forall t\in[0,T]. \tag{5.22}$$

Proof.

Step 1. First of all, we recall that

$$\|\tilde v^T(t)\|_2 \le C\left(\|\mu_0\|_2 + \|\tilde v_f\|_2 + M^2\right)\left(e^{-2\lambda t} + e^{-2\lambda(T-t)}\right) \qquad \forall t\in[0,T]. \tag{5.23}$$

Now, assumption (2.2) implies that

$$\begin{aligned}\|B(s') - B(s)\|_{\mathcal{C}^{1+\alpha}} &= \|F(x,\bar m + \mu^T(s')) - F(x,\bar m+\mu^T(s))\|_{\mathcal{C}^{1+\alpha}}\\ &\le c\|\mu^T(s') - \mu^T(s)\|_{H^{-1}_{\sharp,0}(Q)}.\end{aligned} \tag{5.24}$$

Thanks to estimate (5.19), for every $t\in(2,T-2)$ we have

$$\begin{aligned}&\|\mu^T(s') - \mu^T(s)\|_{H^{-1}(Q)}\\ &\quad\le c\left(\|\mu_0\|_2 + \|\tilde v_f\|_2 + M^2\right)\left(e^{-2\lambda t} + e^{-2\lambda(T-t)}\right)|s-s'| \qquad \forall s,s'\in[t-1,t].\end{aligned}$$

Therefore, on account of (5.14) as well, we conclude that

$$\|B\|_{C^{\frac{1+\alpha}{2},1+\alpha}([t-1,t]\times Q)} \le c\left(\|\mu_0\|_2 + \|\tilde v_f\|_2 + M^2\right)\left(e^{-2\lambda t} + e^{-2\lambda(T-t)}\right).$$

On the other hand, we know (see (5.9)) that $\hat R_2$ satisfies

$$\|\hat R_2\|_{C^{\frac{1+\alpha}{2},1+\alpha}([t-1,t]\times Q)} \le c\, M^2\left(e^{-2\lambda t} + e^{-2\lambda(T-t)}\right)$$

for every $t\in(2,T-2)$. Finally, we recall that

$$-\tilde v_t^T - \Delta\tilde v^T + D\bar u D\tilde v^T = f + \frac{d}{dt}\langle v^T\rangle,$$

where $f = B - \hat R_2$ and

$$\frac{d}{dt}\langle v^T\rangle = -\int_Q f(t)\,dy - \int_Q \tilde v^T(t)\,\Delta\bar u\,dy.$$

We set $h(t) := \frac{d}{dt}\langle v^T\rangle$ and claim that it is bounded in some Hölder norm. Indeed, due to (5.14) and (5.23), we have

$$\left\|\frac{d}{dt}\langle v^T\rangle\right\|_\infty \le c\left(\|\mu_0\|_2 + \|\tilde v_f\|_2 + M^2\right)\left(e^{-2\lambda t} + e^{-2\lambda(T-t)}\right) \qquad \forall t\in[0,T]. \tag{5.25}$$

Moreover, since

$$h(s) - h(s') = \int_Q (f(s') - f(s))\,dy + \int_Q (\tilde v^T(s') - \tilde v^T(s))\,\Delta\bar u\,dy,$$

for every $s,s'\in[t-1,t]$ we deduce

$$|h(s) - h(s')| \le \|f(s') - f(s)\|_\infty + c\sup_{[t-1,t]}\|\tilde v_t^T\|_{L^1(Q)}\,|s'-s|. \tag{5.26}$$

Since the equation implies

$$\|\tilde v_t^T(t)\|_2 \le c\left\{\|\Delta v^T(t)\|_2 + \|Dv^T(t)\|_2 + \|B\|_2 + \|\hat R_2\|_2 + \left\|\frac{d}{dt}\langle v^T\rangle\right\|_2\right\}, \tag{5.27}$$

using also Lemma 5.7 we conclude that

$$\|\tilde v_t^T(t)\|_2 \le c\left(\|\mu_0\|_2 + \|\tilde v_f\|_2 + M^2\right)\left(e^{-2\lambda t} + e^{-2\lambda(T-t)}\right)$$

for every $t < T-2$. Therefore, from (5.26), and from the estimates obtained so far on f, we conclude that

$$\|h\|_{C^{\frac{1+\alpha}{2}}([t-1,t])} \le c\left(\|\mu_0\|_2 + \|\tilde v_f\|_2 + M^2\right)\left(e^{-2\lambda t} + e^{-2\lambda(T-t)}\right)$$

for every $t \in (2, T-2)$.

Step 2. We apply now the classical regularity theory for the equation of $\tilde v^T$ in the interval $(t-1,t)$ (see [5, Theorem 10.1, p. 351] or [10, Theorems 5.1.3–5.1.8]): there exists a constant C such that

$$\begin{aligned}&\|\tilde v^T\|_{C^{\frac{3+\alpha}{2},3+\alpha}([t-1,t]\times Q)}\\ &\quad\le C\left(\|\mu_0\|_2 + \|\tilde v_f\|_2 + M^2\right)\left(e^{-2\lambda t} + e^{-2\lambda(T-t)}\right) + C\|\tilde v^T\|_{L^\infty([t-1,t]\times Q)}.\end{aligned}$$

Since the regularizing property of the equation implies that

$$\|\tilde v^T\|_{L^\infty([t-1,t]\times Q)} \le C\left(\|\tilde v^T(t+1)\|_2 + \left\|B - \hat R_2 + \frac{d}{dt}\langle v^T\rangle\right\|_{L^\infty([t-1,t+1]\times Q)}\right),$$

using (5.23), (5.25), and the usual estimates on B and $\hat R_2$ we conclude that

$$\|\tilde v^T\|_{C^{\frac{3+\alpha}{2},3+\alpha}([t-1,t]\times Q)} \le C\left(\|\mu_0\|_2 + \|\tilde v_f\|_2 + M^2\right)\left(e^{-2\lambda t} + e^{-2\lambda(T-t)}\right) \tag{5.28}$$

for every $t \in (2, T-2)$.

Step 3. We now complete the estimate for $t \in [T-2, T]$ and $t \in [0,2]$.

We can estimate the Hölder norm of B thanks to (5.20) and to (5.24). Using (5.27) and (5.15)–(5.16), we estimate $\|\tilde v_t^T\|_2$ and so the Hölder norm of h. Overall we get

$$\|B\|_{C^{\frac{1+\alpha}{2},1+\alpha}([T-2,T]\times Q)} + \|h\|_{C^{\frac{1+\alpha}{2}}([T-2,T])} \le c\left(\|\mu_0\|_2 + \|\tilde v_f\|_{H^2} + M^2\right),$$

and using a global version of the regularity theorem (see [5, Theorem 5.1, p. 320] or [10, Theorem 5.1.8]) we deduce

$$\|\tilde v^T\|_{C^{\frac{3+\alpha}{2},3+\alpha}([T-2,T]\times Q)} \le C\left(\|\mu_0\|_2 + \|\tilde v_f\|_{C^{3+\alpha}} + M^2\right). \tag{5.29}$$

In $[0,2]$ we have, by previous estimates and by using (5.27),

$$\|\tilde v_t^T(t)\|_2 \le c\left(\|\mu_0\|_2 + \|\tilde v_f\|_2 + M^2\right) \qquad \forall t \in [0,2],$$

while the H^{-1}-norm of μ^T is estimated from (5.20). Hence now we estimate

$$\|B\|_{C^{\frac{1+\alpha}{2},1+\alpha}([0,2]\times Q)} + \|h\|_{C^{\frac{1+\alpha}{2}}([0,2])} \le c\left(\|\mu_0\|_{H^1} + \|\tilde v_f\|_2 + M^2\right)$$

and the regularity theorem implies

$$
\begin{aligned}
\|\tilde v^T\|_{C^{\frac{3+\alpha}{2},3+\alpha}([0,2]\times Q)} &\le C\left(\|\mu_0\|_{H^1}+\|\tilde v_f\|_2+M^2\right)+C\|\tilde v^T(2)\|_{C^{3+\alpha}} \\
&\le C\left(\|\mu_0\|_{H^1}+\|\tilde v_f\|_2+M^2\right).
\end{aligned} \tag{5.30}
$$

Putting together (5.28)–(5.30), we obtain in particular (5.22). ∎

Thanks to the previous result, the source term in the equation of μ^T is estimated, and we can deduce the estimates on μ^T as well.

Proposition 5.11. *Let $\alpha\in(0,1)$ be given by (2.2). There exists $C>0$ such that we have*

$$
\begin{aligned}
&\|\mu^T\|_{C^{\frac{2+\alpha}{2},2+\alpha}([t-1,t]\times Q)} \\
&\quad\le C\left(\|\mu_0\|_2+\|\tilde v_f\|_2+M^2\right)\left(e^{-2\lambda t}+e^{-2\lambda(T-t)}\right) \qquad \forall t\in(3,T-2),
\end{aligned}
$$

and globally

$$
\|\mu^T\|_{C^{\frac{2+\alpha}{2},2+\alpha}([0,T]\times Q)} \le C\left(\|\mu_0\|_{C^{2+\alpha}}+\|\tilde v_f\|_{C^{3+\alpha}}+M^2\right).
$$

In particular, we have

$$
\begin{aligned}
&\|\mu^T(t)\|_{C^{2+\alpha}}+\|\mu^T_t(t)\|_{C^{\alpha}} \\
&\quad\le C\left(\|\mu_0\|_{C^{2+\alpha}}+\|\tilde v_f\|_{C^{3+\alpha}}+M^2\right)\left(e^{-2\lambda t}+e^{-2\lambda(T-t)}\right) \qquad \forall t\in[0,T].
\end{aligned} \tag{5.31}
$$

Proof.

Step 1. First we use the regularizing effect

$$
\|\mu^T(t)\|_\infty \le c\left(\|f\|_{L^\infty((t-1,t)\times Q)}+\|\mu^T(t-1)\|_2\right),
$$

where $f=D\bar m Dv^T+\bar m\Delta v^T+\hat R_1$. In view of Proposition 5.10 we get

$$
\|f\|_{L^\infty((t-1,t)\times Q)} \le C\left(\|\mu_0\|_2+\|\tilde v_f\|_2+M^2\right)\left(e^{-2\lambda t}+e^{-2\lambda(T-t)}\right) \qquad \forall t\in(2,T-2).
$$

Hence we deduce

$$
\|\mu^T(t)\|_\infty \le C\left(\|\mu_0\|_2+\|\tilde v_f\|_2+M^2\right)\left(e^{-2\lambda t}+e^{-2\lambda(T-t)}\right) \qquad \forall t\in(2,T-2).
$$

Moreover, we have

$$
\begin{aligned}
&\|f\|_{C^{\frac{\alpha}{2},\alpha}([t-1,t]\times Q)} \\
&\quad\le c\left\{\|Dv^T\|_{C^{\frac{\alpha}{2},\alpha}([t-1,t]\times Q)}+\|D^2v^T\|_{C^{\frac{\alpha}{2},\alpha}([t-1,t]\times Q)}+\|\hat R_1\|_{C^{\frac{\alpha}{2},\alpha}([t-1,t]\times Q)}\right\}.
\end{aligned}
$$

Hence from (5.8) and Proposition 5.10 we get

$$
\|f\|_{C^{\frac{\alpha}{2},\alpha}([t-1,t]\times Q)} \le c\left(\|\mu_0\|_2+\|\tilde v_f\|_2+M^2\right)\left(e^{-2\lambda t}+e^{-2\lambda(T-t)}\right).
$$

Now, for $t\in(3,T-2)$ we apply as before the regularity theorem [5, Theorem 10.1, p. 351], obtaining

$$
\begin{aligned}
\|\mu^T\|_{C^{\frac{2+\alpha}{2},2+\alpha}([t-1,t]\times Q)} &\le C\left\{\|f\|_{C^{\frac{\alpha}{2},\alpha}([t-1,t]\times Q)}+\|\mu^T\|_{L^\infty([t-1,t]\times Q)}\right\} \\
&\le C\left(\|\mu_0\|_2+\|\tilde v_f\|_2+M^2\right)\left(e^{-2\lambda t}+e^{-2\lambda(T-t)}\right)
\end{aligned}
$$

for every $t\in(3,T-2)$.

Step 2. Using now the global estimate on v^T, we also have a similar estimate in $[0,3]$ and in $[T-2,T]$ with constants depending on stronger norms. That is, we obtain

$$\|\mu^T\|_{C^{\frac{2+\alpha}{2},2+\alpha}([0,T]\times Q)} \le C\left(\|\mu_0\|_{C^{2+\alpha}} + \|\tilde v_f\|_{C^{3+\alpha}} + M^2\right) \qquad \forall t\in[0,T]. \qquad \square$$

5.4. Conclusion of the fixed point argument. Combining Propositions 5.10 and 5.11 we have obtained that

$$\begin{aligned}\|\mu^T\|_{C^{\frac{2+\alpha}{2},2+\alpha}([t-1,t]\times Q)} &+ \|\tilde v^T\|_{C^{\frac{3+\alpha}{2},3+\alpha}([t-1,t]\times Q)}\\ &\le c\left(\|\mu_0\|_{C^{2+\alpha}} + \|\tilde v_f\|_{C^{3+\alpha}} + M^2\right)\left(e^{-2\lambda t} + e^{-2\lambda(T-t)}\right) \quad \forall t\in[1,T]\end{aligned}$$

and

$$\begin{aligned}\|\mu_t^T(t)\|_\infty &+ \|Dv_t^T\|_\infty\\ &\le c\left(\|\mu_0\|_{C^{2+\alpha}} + \|\tilde v_f\|_{C^{3+\alpha}} + M^2\right)\left(e^{-2\lambda t} + e^{-2\lambda(T-t)}\right) \quad \forall t\in[0,T].\end{aligned}$$

Therefore, comparing with (5.5), if we choose M such that $cM^2 \le M/2$ and the boundary conditions (μ_0, v_f) so small that $c(\|\mu_0\|_{C^{2+\alpha}} + \|\tilde v_f\|_{C^{3+\alpha}}) \le M/2$, we have that (μ^T, v^T) belongs to $\mathcal{C}^T$. Thus $\mathcal{C}_T$ is a convex invariant set for the map $\mathcal{T}$. Since, moreover, the range of $\mathcal{T}$ is contained in a relatively compact set of X_T, we conclude the existence of a fixed point. Note that this fixed point satisfies the above inequalities, which shows that (5.2) holds.

6. The exponential rate for general data.

We now state and prove the exponential convergence rate for general data: this is essentially a combination of the convergence results of section 3 with the exponential rate for small initial conditions of section 5. Recall that the convergence rate is of order $\bar\omega := \omega \wedge \omega_0$, where ω is the constant which appears in Lemma 7.1, while ω_0 is given by Lemma 4.5.

THEOREM 6.1. *Let us assume that* (2.1), (2.2) *hold for some* $\alpha\in(0,1)$, *and let* $\lambda\in(0,\frac{\bar\omega}{2})$. *For any* $R>0$, *there are* $\bar M$, $C>0$ *(depending on* F *and on* λ*) such that if*

$$\|\mu_0\|_{C^0} + \|v_f\|_{C^2} \le R,$$

then the solution (μ^T, v^T) *of* (3.5) *satisfies the exponential decay*

$$\begin{aligned}\|\mu^T\|_{C^{\frac{2+\alpha}{2},2+\alpha}([t-1,t]\times Q)} &+ \|\tilde v^T\|_{C^{\frac{3+\alpha}{2},3+\alpha}([t-1,t]\times Q)}\\ &\le C\left(e^{-2\lambda t} + e^{-2\lambda(T-t)}\right) \qquad \forall t\in[\bar M, T-\bar M],\end{aligned}$$

where $\tilde v^T(t) = v^T(t) - \langle v^T(t)\rangle$ *and* $\langle v^T(t)\rangle = \int_Q v^T(t)$.

The proof is mostly an application of Theorem 5.1 to initial data inside the domain. It requires as usual some preliminary estimates. From section 3, Lemma 3.6, we already know that there is a constant $C = C(R,F)$ such that, for $\delta\in(0,1)$ and $\|v_f\|_{\mathcal{C}^2}, \|\mu_0\|_{\mathcal{C}^0} \le R$,

$$\int_{\delta T}^{(1-\delta)T}\int_Q |Dv^T|^2 \le \frac{C}{(\delta T)^{\frac12}} \qquad \text{and} \qquad \sup_{t\in[\delta T,(1-\delta)T]} \|\mu^T(t)\|_2 \le \frac{C}{(\delta T)^{\frac14}}. \tag{6.1}$$

Our aim is to obtain these inequalities in stronger norms in order to apply Theorem 5.1 to the initial condition $\mu^T(\delta T)$ and the terminal condition $v^T((1-\delta)T)$.

LEMMA 6.2. *There is a constant $C = C(R, F)$ such that, for $\delta \in (0,1)$ and $\|v_f\|_{\mathcal{C}^2}, \|\mu_0\|_{\mathcal{C}^0} \le R$, we have*

$$\|\tilde{v}(t)\|_{C^{3+\alpha}(Q)} + \|\mu^T(t)\|_{C^{2,\alpha}(Q)} \le \frac{C}{(\delta T)^{\frac{1}{4}}} \qquad \forall t \in (\delta T + 1, (1-\delta)T - 1).$$

Proof.

Step 1. We start with the estimate on v^T. Let t_0, t_1 be as in the proof of Lemma 3.6. From Lemma 3.5 and (3.8), we have, for every $t \in (\delta T, t_1)$,

$$\|\mu^T(t)\|_2 \le C\left\{\|\mu(t_0)\|_2 e^{-\frac{\omega\,\delta T}{2}} + \frac{1}{(\delta T)^{\frac{1}{4}}}\right\}. \tag{6.2}$$

On the other hand, from Lemma 7.4 we have

$$\|\tilde{v}^T(t)\|_{L^\infty(Q)} \le Ce^{-\omega(t_1-t)}\|\tilde{v}^T(t_1)\|_{L^\infty(Q)} + C\int_t^{t_1}\|B(s)\|_\infty e^{-\omega(s-t)}ds, \tag{6.3}$$

where, for $p > N$, we have

$$\|\tilde{v}^T(t_1)\|_{L^\infty(Q)} \le c\|Dv^T(t_1)\|_{L^p(Q)} \le C\|Dv^T\|_\infty^{\frac{p-2}{p}}\|Dv^T(t_1)\|_{L^2(Q)}^{\frac{2}{p}} \le \frac{C}{(\delta T)^{\frac{1}{2p}}}$$

by the choice of t_1. From (6.2) and assumption (2.2), we deduce that

$$\|B(s)\|_{C^1} \le C\|\mu^T(s)\|_2 \le \frac{C}{(\delta T)^{\frac{1}{4}}} \qquad \forall s \in (\delta T, t_1). \tag{6.4}$$

Therefore, we obtain from (6.3)

$$\|\tilde{v}^T(t)\|_{L^\infty(Q)} \le \frac{C}{(\delta T)^{\frac{1}{2p}}}e^{-\omega(t_1-t)} + \frac{C}{(\delta T)^{\frac{1}{4}}}.$$

We deduce, in particular, that

$$\|\tilde{v}^T(t)\|_{L^\infty(Q)} \le \frac{C}{(\delta T)^{\frac{1}{4}}} \qquad \forall t \in (\delta T, (1-\delta)T). \tag{6.5}$$

Next we estimate $\|Dv^T(t,\cdot)\|_\infty$. Setting $v_i^T = \frac{\partial v^T}{\partial x_i}$ for $i = 1, \dots, d$, we have

$$-(v_i^T)_t - \Delta v_i^T + D\bar{u}Dv_i^T = -D\bar{u}_iDv^T - \left(\frac{1}{2}|Dv^T|^2\right)_{x_i} + B_{x_i}.$$

Combining the Lipschitz estimate on v^T with (6.1) and (6.4), we have for any $t \in (\delta T, (1-\delta)T - 1)$

$$\int_t^{t+1}\int_Q (|D\bar{u}_iDv^T|^p + |Dv^T|^p + |B_{x_i}|^p) \;\le\; \frac{C}{(\delta T)^{\frac{1}{4}}}$$

for any p sufficiently large. As $\|v_i^T\|_{L^2((t,t+1)\times Q)} \le \frac{C}{(\delta T)^{\frac{1}{4}}}$, [5, Theorem 8.1, p. 192] gives a bound on $\|v_i\|_\infty$; hence

$$\|D^Tv(t)\|_\infty \le \frac{C}{(\delta T)^{\frac{1}{4}}} \tag{6.6}$$

in any interval contained in $(\delta T, (1-\delta)T-1)$. We now consider the equation of $\tilde v^T$, that is,

$$-\tilde v^T_t - \Delta \tilde v^T + \left(D\bar u + \frac{1}{2} Dv^T\right) D\tilde v^T = B + \frac{d}{dt}\langle v^T\rangle,$$

where

$$\frac{d}{dt}\langle v^T\rangle = -\int_Q B(t)\,dy - \int_Q \tilde v^T(t)\,\Delta \bar u\,dy + \frac{1}{2}\int_Q |D\tilde v^T|^2.$$

From (6.4), (6.5), and (6.6), we know that

$$\left\| B + \frac{d}{dt}\langle v^T\rangle\right\|_\infty \le \frac{C}{(\delta T)^{\frac14}}$$

in any interval contained in $(\delta T, (1-\delta)T-1)$. Applying, e.g., [5, Theorem 11.1, p. 211], we deduce that $\|\tilde v^T(t)\|_{C^{1,\gamma}(Q)} \le \frac{C}{(\delta T)^{\frac14}}$ for some $\gamma \in (0,1)$. We bootstrap now the regularity of $\tilde v^T$ by considering it as a solution of a linear equation with smooth (time-independent) coefficients:

$$-\tilde v^T_t - \Delta \tilde v^T + D\bar u\, D\tilde v^T = f, \tag{6.7}$$

where $f = B - \frac12 |Dv^T|^2 + \frac{d}{dt}\langle v^T\rangle$. We obtained so far an estimate of f in $C^{0,\gamma}(Q)$ for some $\gamma < 1$, and we deduce that $\tilde v^T(t) \in C^{2,\gamma}(Q)$, say for $t < (1-\delta)T - \frac12$, by applying, e.g., [10, Theorem 5.1.4]. In particular, this gives an estimate of Dv^T in $C^1(Q)$, and, on account of (6.4), now we have f in $C^{0,\gamma}(Q)$ for any $\gamma < 1$, in particular for $\gamma = \alpha$, the exponent given by (2.2). Repeating the previous step, we then have $\tilde v^T(t) \in C^{2,\alpha}(Q)$, say for $t < (1-\delta)T - \frac34$. Hence, we deduce that f belongs actually to $C^{1,\alpha}(Q)$, and in a further step we end up with an estimate of $\tilde v^T$ in $C^{3+\alpha}(Q)$, namely,

$$\|\tilde v^T(t)\|_{C^{3+\alpha}(Q)} \le \frac{C}{(\delta T)^{\frac14}} \qquad \forall t \in (\delta T, (1-\delta)T-1).$$

Step 2. Let us now estimate μ^T. Let $(t, t+1)$ be any interval contained in $(\delta T, (1-\delta)T-1)$. Using the regularizing property of the equation, we have

$$\left\|\mu^T\left(t+\frac12\right)\right\|_\infty \le C\left\{\|\mu^T(t)\|_2 + \|\mathrm{div}(\bar m Dv^T)\|_{L^\infty((t,t+1)\times Q)}\right\}.$$

Hence from (6.2) and the estimate obtained in the first step, we deduce that $\|\mu^T(t+\frac12)\|_\infty \le \frac{C}{(\delta T)^{\frac14}}$. Applying to μ^T [5, Theorem 11.1, p. 211], we deduce that $\|\mu^T(t)\|_{C^{1,\gamma}(Q)} \le \frac{C}{(\delta T)^{\frac14}}$ for some $\gamma \in (0,1)$. Then μ^T solves an equation of the same type as (6.7), and we proceed as before with a bootstrap argument, to conclude that

$$\|\mu^T(t)\|_{C^{2,\alpha}(Q)} \le \frac{C}{(\delta T)^{\frac14}} \qquad \forall t \in (\delta T+1, (1-\delta)T-1). \qquad \square$$

Proof of Theorem 6.1. We just need to apply Theorem 5.1 on the time interval $[\delta T+1, (1-\delta)T-1]$ with the initial conditions $\mu^T(\delta T+1)$ and $v^T((1-\delta)T-1)$, which, in view of Lemma 6.2, satisfy assumption (5.1) for $\delta\, T$ large enough. $\square$

7. Appendix. In this part we recall and refine several classical decay estimates for the heat equation with convection or advection. For $V \in L^\infty_\sharp(\mathbb{R}\times\mathbb{R}^d)$ and $\rho_0 \in L^1_{\sharp,0}(\mathbb{R}^d)$, we consider the solution $\rho \in L^1_{loc}([0,+\infty), L^1_{\sharp,0}(\mathbb{R}^d))$ to

$$\begin{cases} \rho_t - \Delta\rho - \operatorname{div}(\rho V) = 0 & \text{in } (0,\infty)\times Q, \\ \rho(0) = \rho_0,\ \int_Q \rho_0 = 0. \end{cases} \tag{7.1}$$

Recall that problem (7.1) has a unique weak solution, i.e., a unique $\rho \in L^1_{\sharp,loc}([0,+\infty)\times\mathbb{R}^d)$ such that

$$-\int_Q \rho_0\,\varphi(0)\,dx + \int_0^T\int_Q \rho\,\{-\varphi_t - \Delta\varphi + D\bar{u}D\varphi\}\,dxdt = 0 \qquad \forall\varphi \in C^\infty_c([0,T)\times Q).$$

LEMMA 7.1. *There are constants $\omega > 0$ and $C > 0$, depending only on $\|V\|_{L^\infty_\sharp(\mathbb{R}\times\mathbb{R}^d)}$, such that if ρ is a solution of (7.1), then*

$$\|\rho(t)\|_{L^1(Q)} \le Ce^{-\omega t}\|\rho_0\|_{L^1(Q)} \qquad \forall t\ge 0, \tag{7.2}$$

$$\|\rho(t)\|_{L^2(Q)} \le Ce^{-\omega t}\|\rho_0\|_{L^2(Q)} \qquad \forall t\ge 0, \tag{7.3}$$

while

$$\|\rho(t)\|_{L^\infty(Q)} \le Ce^{-\omega t}\|\rho_0\|_{L^1(Q)} \qquad \forall t\ge 1. \tag{7.4}$$

The proof of this fact requires several preliminary lemmas.

LEMMA 7.2. *If $\rho \in L^2_{loc}([0,+\infty); H^1_\sharp(Q))$ is a weak solution of (7.1), then the map $t \to \|\rho(t)\|_{L^1(Q)}$ is nonincreasing. Moreover, if this map is constant in time on a nonempty, open, interval, then ρ must be equal to zero.*

Proof. Take $\frac{1}{\varepsilon}T_\varepsilon(\rho)$ as a test function, where $T_\varepsilon(s) = \max(-\varepsilon, \min(s,\varepsilon))$; we have

$$\int_Q \left(\int_0^{\rho(T)} \frac{1}{\varepsilon}T_\varepsilon(r)dr\right)dx + \int_0^T\int_Q \frac{1}{\varepsilon}|DT_\varepsilon(\rho)|^2\,dxdt + \int_0^T\int_Q \frac{1}{\varepsilon}\rho\,DT_\varepsilon(\rho)V\,dxdt$$
$$= \int_Q \left(\int_0^{\rho(0)} \frac{1}{\varepsilon}T_\varepsilon(r)dr\right)dx.$$

Since $|\rho| \le \varepsilon$ whenever $DT_\varepsilon(\rho) \ne 0$, we have

$$\left|\int_0^T\int_Q \frac{1}{\varepsilon}\rho\,DT_\varepsilon(\rho)V\,dxdt\right| \le \frac{1}{2}\int_0^T\int_Q \frac{1}{\varepsilon}|DT_\varepsilon(\rho)|^2\,dxdt + \frac{\varepsilon}{2}\int_0^T\int_Q |V|^2\,dxdt,$$

and hence we deduce

$$\int_Q \left(\int_0^{\rho(T)} \frac{1}{\varepsilon}T_\varepsilon(r)dr\right)dx + \frac{1}{2}\int_0^T\int_Q \frac{1}{\varepsilon}|DT_\varepsilon(\rho)|^2\,dxdt \le CT\varepsilon + \int_Q \left(\int_0^{\rho(0)} \frac{1}{\varepsilon}T_\varepsilon(r)dr\right)dx. \tag{7.5}$$

As $\varepsilon \to 0$, we have

$$\int_Q \left(\int_0^{\rho(t)} \frac{1}{\varepsilon}T_\varepsilon(r)dr\right)dx \to \|\rho(t)\|_{L^1(Q)}$$

so we get

$$\|\rho(T)\|_{L^1(Q)} \leq \|\rho(0)\|_{L^1(Q)}.$$

In particular, $\|\rho(t)\|_{L^1(Q)}$ is nonincreasing. In addition, if $\|\rho(T)\|_{L^1(Q)} = \|\rho(0)\|_{L^1(Q)}$, we get from (7.5)

$$\lim_{\varepsilon\to 0} \int_0^T \int_Q \frac{1}{\varepsilon} |DT_\varepsilon(\rho)|^2 \, dxdt = 0.$$

Now we take $\frac{1}{\varepsilon}T_\varepsilon(\rho)\varphi$ as a test function, where $\varphi \in C^\infty([0,T]\times\overline{Q})$, $\varphi(T)=0$. We get

$$\begin{aligned}
&-\int_0^T \int_Q \left(\int_0^{\rho(t)} \frac{1}{\varepsilon} T_\varepsilon(r)dr\right)\varphi_t \, dx \, dt - \int_Q \left(\int_0^{\rho(0)} \frac{1}{\varepsilon}T_\varepsilon(r)dr\right)\varphi(0)\, dx \\
&\quad + \int_0^T \int_Q \frac{1}{\varepsilon}|DT_\varepsilon(\rho)|^2 \, \varphi \, dxdt + \int_0^T \int_Q \frac{1}{\varepsilon} T_\varepsilon(\rho) D\rho D\varphi \, dxdt \\
&\quad + \int_0^T \int_Q \frac{1}{\varepsilon}\rho \, DT_\varepsilon(\rho) V \, \varphi \, dxdt + \int_0^T \int_Q \frac{1}{\varepsilon} T_\varepsilon(\rho)\, \rho \, D\varphi V \, dxdt = 0.
\end{aligned}$$

Now we know that

$$\lim_{\varepsilon\to 0} \int_0^T \int_Q \frac{1}{\varepsilon} |DT_\varepsilon(\rho)|^2 \, \varphi \, dxdt = 0,$$

and similarly we deduce

$$\left| \int_0^T \int_Q \frac{1}{\varepsilon} \rho \, DT_\varepsilon(\rho) V \, \varphi \, dxdt \right| \leq C \left(\int_0^T \int_Q |DT_\varepsilon(\rho)|^2 \, dxdt \right)^{\frac{1}{2}} \to 0.$$

Therefore, passing to the limit as $\varepsilon \to 0$, we get

$$-\int_0^T \int_Q |\rho| \, \varphi_t \, dxdt - \int_Q |\rho(0)| \varphi(0)\, dx + \int_0^T \int_Q D|\rho| \, D\varphi \, dxdt + \int_0^T \int_Q |\rho| \, D\varphi \, V \, dxdt = 0.$$

So we find that $|\rho|$ is a solution of the equation in (7.1); since $\int_Q \rho(t) = 0$, for any t there is a point x_0 in Q such that $\rho(t, x_0) = 0$. Therefore, $|\rho|$ is a positive solution which has a minimum point in Q where it vanishes. By the Harnack inequality [1], this implies that $|\rho|$ must be constant, and then $|\rho| = 0$. $\square$

LEMMA 7.3. *Let $\rho_0 \in L^1(Q)$, $V \in L^\infty_\sharp(\mathbb{R}\times\mathbb{R}^d)$ and let $\rho \in L^1((0,T)\times Q))$ be a weak solution of (7.1). Then we have*

$$\|\rho(t)\|_{L^r(Q)} \leq C \, t^{-\frac{N(r-1)}{2r}} \|\rho_0\|_{L^1(Q)} \quad \forall t \leq 1 \tag{7.6}$$

for every $r \in (1, \infty]$, where $C = C(\|V\|_\infty)$.

Moreover, for any $M > 0$, the family of solutions ρ of (7.1) for V such that $\|V\|_{L^\infty_\sharp(\mathbb{R}\times\mathbb{R}^d)} \leq M$ and $\|\rho_0\|_{L^1(Q)} \leq 1$ is relatively compact in $L^1((T_1,T_2); L^1_\sharp)$ for any $0 < T_1 < T_2$.

Proof. Assume for the moment that ρ is a bounded weak solution. Multiply by $|\rho|^{r-2}\rho$ the equation ($r \geq 2$) to get

$$\frac{1}{r}\frac{d}{dt}\left(\int_Q |\rho(t)|^r\,dx\right) + (r-1)\int_Q |D\rho|^2\,|\rho|^{r-2}\,dx = (r-1)\int_Q V\,D\rho\,|\rho|^{r-2}\rho\,dx,$$

which yields

$$\frac{1}{r}\frac{d}{dt}\left(\int_Q |\rho(t)|^r\,dx\right) + \frac{(r-1)}{2}\int_Q |D\rho|^2\,|\rho|^{r-2}\,dx \leq \frac{(r-1)}{2}\|V\|_\infty^2\int_Q |\rho|^r\,dx. \tag{7.7}$$

Since ρ has zero average, we have

$$\|\rho\|_{L^{2^*}(Q)} \leq \frac{1}{S}\|D\rho\|_{L^2(Q)},$$

where 2^* is given by Sobolev embedding ($2^* = \frac{2N}{N-2}$ if $N > 2$ or, if $N = 2$, any value $r > 2$). Since

$$\int_Q |\rho|^2\,dx \leq \left(\int_Q |\rho|\,dx\right)^{\frac{4}{N+2}}\left(\int_Q |\rho|^{2^*}\,dx\right)^{\frac{(N-2)}{N+2}} \leq \|\rho_0\|_1^{\frac{4}{N+2}}\left(\int_Q |\rho|^{2^*}\,dx\right)^{\frac{(N-2)}{N+2}}$$

we get

$$\left(\int_Q |\rho|^2\,dx\right)^{\frac{N+2}{2N}} \leq \frac{\|\rho_0\|_{L^1(Q)}^{\frac{2}{N}}}{S}\|D\rho\|_2.$$

Hence for $r = 2$ we obtain

$$\frac{d}{dt}\left(\int_Q |\rho(t)|^2\,dx\right) + \frac{c_0}{\|\rho_0\|_{L^1(Q)}^{\frac{4}{N}}}\left(\int_Q |\rho|^2\,dx\right)^{1+\frac{2}{N}} \leq c_1\|V\|_\infty^2\int_Q |\rho|^2\,dx.$$

We deduce (see, e.g., [11, Lemma 2.6]) that

$$\int_Q |\rho(t)|^2\,dx \leq c\,t^{-\frac{N}{2}}\|\rho_0\|_{L^1(Q)}^2\,e^{c_1\|V\|_\infty^2 t}.$$

Once ρ is bounded in $L^2(Q)$, one can use, e.g., [5, Theorem 8.1, p. 192] to deduce a further regularizing effect into $L^\infty(Q)$. We notice, however, that the regularizing effect in $L^\infty(Q)$ may follow directly from (7.7); indeed, it implies

$$\frac{1}{r}\frac{d}{dt}\left(\int_Q |\rho(t)|^r\,dx\right) + \frac{2(r-1)}{r^2}\int_Q |D(|\rho|^{\frac{r}{2}}|)|^2\,dx \leq \frac{(r-1)}{2}\|V\|_\infty^2\int_Q |\rho|^r\,dx,$$

and by Sobolev embedding

$$\begin{aligned}\frac{1}{r}\frac{d}{dt}\left(\int_Q |\rho(t)|^r\,dx\right) &+ \frac{2(r-1)}{r^2}\frac{1}{S}\left(\int_Q |\rho|^{\frac{rN}{N-2}}\,dx\right)^{\frac{N-2}{N}}\\ &\leq \left\{\frac{(r-1)}{2}\|V\|_\infty^2 + \frac{2(r-1)}{r^2}\right\}\int_Q |\rho|^r\,dx.\end{aligned}$$

Therefore, for every $r \geq 2$ we have

$$\frac{d}{dt}\left(\int_Q |\rho(t)|^r\,dx\right) + c_0\left(\int_Q |\rho|^{\frac{rN}{N-2}}\,dx\right)^{\frac{N-2}{N}} \leq c_1\,r^2 \int_Q |\rho|^r\,dx$$

for constants c_0, c_1 independent of r. With the same proof as in [11, Proposition 2.7], standing on the Moser iteration method, one deduces estimate (7.6) for every r including $r = \infty$. Let us further note that, once ρ is proved to be bounded, then by standard results it is a strong solution (in particular, it belongs to $L^p(0,T;W^{1,p}(Q))$ for any $p \geq 1$).

As far as compactness is concerned, we have the following estimates (see, e.g., [2, Lemma 3.1]):

$$\|\rho\|_{L^\infty(0,T;L^1(Q))} + \|\log(1+|\rho|)\|^2_{L^2(0,T;H^1)} \leq c\left(\|V\|_2^2 + \|\rho_0\|_{L^1(Q)}\right). \tag{7.8}$$

In particular, setting $v = \log(1+|\rho|)\,\mathrm{sign}\rho$, we have that v is bounded in $L^2(0,T;H^1(Q))$ and v_t is bounded in $L^2(0,T;H^{-1}(Q)) + L^1((0,T)\times Q)$. Therefore, if ρ_0 lies in a bounded set of $L^1(Q)$, then v lies in a compact set of $L^1((0,T)\times Q))$. Combining this compactness with the regularizing effect, one deduces that ρ is compact in $L^p((\tau,T)\times Q)$ for every $p < \infty$ and any $\tau > 0$. Standard arguments also yield compactness in $C^0([\tau,T];L^p(Q))$ for any $\tau > 0$.

Recalling that problem (7.1) has a unique weak solution, by approximating V and ρ_0 in L^1 with sequences of smooth initial data, and using the compactness properties, we extend such properties to the unique weak solution. $\square$

Proof of Lemma 7.1. We start with the proof of (7.2). Fix $M > 0$. For $V \in L^\infty_\sharp(\mathbb{R}\times\mathbb{R}^d)$, $\xi \in L^1_\sharp(Q)$, $\int_Q \xi = 0$, let $\rho_{\xi,V}$ denote the (periodic) solution of

$$\begin{cases} \rho_t - \Delta\rho - \mathrm{div}(\rho V) = 0 \quad \text{in } (0,\infty)\times Q, \\ \rho(0) = \xi. \end{cases}$$

Observe that $\int_Q \rho_{\xi,V} = \int_Q \xi = 0$, and we have, by Lemma 7.2,

$$\frac{d}{dt}\int_Q |\rho_{\xi,V}(t)|dt \leq 0.$$

We set

$$\mathcal{M}(t) := \sup\left\{\|\rho_{\xi,V}(t)\|_{L^1(Q)}\ ;\ \|V\|_{L^\infty_\sharp(\mathbb{R}\times\mathbb{R}^d)} \leq M,\ \|\xi\|_{L^1(Q)} \leq 1\right\}.$$

Then $\mathcal{M}(t)$ is nonincreasing and

$$\exists\ l := \lim_{t\to\infty} \mathcal{M}(t). \tag{7.9}$$

We claim that $l = 0$. To see this, let $\xi_n \in L^1_{\sharp,0}(\mathbb{R}^d)$, $V_n \in L^\infty_\sharp(\mathbb{R}\times\mathbb{R}^d)$ with $\|\xi_n\|_{L^1(Q)} \leq 1$, and $\|V_n\|_{L^\infty_\sharp(\mathbb{R}\times\mathbb{R}^d)} \leq M$ satisfy $\|\rho_{\xi_n,V_n}(n)\|_{L^1(Q)} \geq \mathcal{M}(n) - \frac{1}{n}$, and consider the sequence $\rho_n(t,x) = \rho_{\xi_n,V_n}(t+n-1,x)$, defined for $t > 1-n$. Since $\|\rho_n(t)\|_{L^1(Q)} \leq 1$, (ρ_n) is a sequence of solutions (corresponding to the field $\tilde V_n = V_n(t+n-1,x)$) which is bounded in $L^1(Q)$ uniformly with respect to t. Lemma 7.3 then implies that it is relatively compact in any compact interval of $(0,+\infty)$: up to a subsequence, it converges in L^1 to some $\bar\rho$ locally in time, while $\tilde V_n$ converges weakly in L^2_{loc} to some map $\bar V \in L^\infty_\sharp(\mathbb{R}\times\mathbb{R}^d)$ with $\|\bar V\|_{L^\infty_\sharp(\mathbb{R}\times\mathbb{R}^d)} \le M$. Note that $\bar\rho$ is a solution of (7.1) associated with $\bar V$. We have, for $t\in[0,1]$,

$$\|\rho_{\xi_n,V_n}(n)\|_{L^1(Q)} \le \|\rho_n(t)\|_{L^1(Q)} \le \|\rho_{\xi_n,V_n}(n-1)\|_{L^1(Q)} \le \mathcal{M}(n-1).$$

Hence, by choice of ξ_n, V_n

$$\mathcal{M}(n) - \frac{1}{n} \le \|\rho_n(t)\|_{L^1(Q)} \le \mathcal{M}(n-1),$$

so that $\|\rho_n(t)\|_{L^1(Q)} \to l$ for every $t\in[0,1]$. We conclude that $\bar\rho$ is a solution of (7.1) in $[0,1]$ (with $\bar V$ instead of V) for which $\|\bar\rho(t)\|_{L^1(Q)} = l$ for every $t\in[0,1]$. Lemma 7.2 then implies that $\bar\rho = 0$. Therefore $l=0$.

We have proved so far that

$$\sup\left\{\|\rho_{\xi,V}(t)\|_{L^1(Q)} \mid \|V\|_{L^\infty_\sharp(\mathbb{R}\times\mathbb{R}^d)} \le M,\ \|\xi\|_{L^1(Q)} \le 1\right\} \overset{t\to\infty}{\to} 0.$$

Now let t_0 be such that

$$\sup\left\{\|\rho_{\xi,V}(t_0)\|_{L^1(Q)} \mid \|V\|_{L^\infty_\sharp(\mathbb{R}\times\mathbb{R}^d)} \le M,\ \|\xi\|_{L^1(Q)} \le 1\right\} \le \frac{1}{2}.$$

Assume that $\|\rho_0\|_{L^1(Q)} = 1$, which we can always suppose, and let ρ be a solution of (7.1) for some $V\in L^\infty_\sharp(\mathbb{R}\times\mathbb{R}^d)$ with $\|V\|_{L^\infty_\sharp(\mathbb{R}\times\mathbb{R}^d)} \le M$. Then $\|\rho(t_0)\|_{L^1(Q)} \le \frac{1}{2}$. Applying this inequality to $\rho(t_0+\cdot)$ (which is a solution with the time-shifted vector field $V(t_0+\cdot,\cdot)$), we get

$$\|\rho(2t_0)\|_{L^1(Q)} \le \frac{1}{2}\|\rho(t_0)\|_{L^1(Q)} \le \frac{1}{4}\ .$$

Iterating, we obtain $\|\rho(nt_0)\|_{L^1(Q)} \le \frac{1}{2^n}$ for any n, which implies the exponential decay in L^1-norm.

Next we show (7.3). If $\rho_0 \in L^2_{\sharp,0}(\mathbb{R}^d)$, then, by classical arguments,

$$\|\rho(t)\|_{L^2(Q)} \le C\|\rho_0\|_{L^2(Q)} \qquad \forall t\in[0,1],$$

while, by Lemma 7.3 and (7.2),

$$\|\rho(t)\|_{L^2(Q)} \le C\|\rho(t-1)\|_{L^1(Q)} \le Ce^{-\omega(t-1)}\|\rho_0\|_{L^1(Q)} \le Ce^{-\omega t}\|\rho_0\|_{L^2(Q)} \qquad \forall t\ge 1,$$

where C depends only on $\|V\|_\infty$. Combining the two above inequalities gives (7.3). Finally, (7.4) can be proved by similar arguments. ☐

Now let $V\in L^\infty_\sharp(\mathbb{R}\times\mathbb{R}^d)$ and $v_0\in L^\infty_\sharp(\mathbb{R}^d)$. We consider the (periodic) solution v of the problem

$$\begin{cases} v_t - \Delta v + Dv\, V = f & \text{in } (0,\infty)\times Q,\\ v(0) = v_0. \end{cases} \tag{7.10}$$

We easily deduce the following by duality.

LEMMA 7.4. *Let $\omega>0$ be as in Lemma 7.1. If v is the solution of* (7.10) *and $\tilde v := v - \int_Q v\,dx$, then*

$$\|\tilde v(t)\|_{L^\infty(Q)} \le Ce^{-\omega t}\|\tilde v_0\|_{L^\infty(Q)} + C\int_0^t \|f(s)\|_{L^\infty(Q)} e^{-\omega(t-s)}ds \qquad \forall t\ge 0 \tag{7.11}$$

and

$$\|\tilde v(t)\|_{L^2(Q)} \le Ce^{-\omega t}\|\tilde v_0\|_{L^2(Q)} + C\int_0^t \|f(s)\|_{L^2(Q)} e^{-\omega(t-s)}ds \qquad \forall t\ge 0,$$

where $C = C(\|V\|_\infty)$.

Proof. Let ρ be the solution of

$$\begin{cases} -\rho_t - \Delta\rho - \operatorname{div}(\rho V) = 0 & \text{in } (0,t)\times Q,\\ \rho(t) = \xi - \int_Q \xi\, dx. \end{cases}$$

Then we have

$$\int_Q v(t)\left(\xi - \int_Q \xi\, dy\right) dx = \int_Q v_0\,\rho(0)\, dx + \int_0^t \int_Q f\,\rho(s) ds,$$

and hence

$$\int_Q \tilde v(t)\xi\, dx = \int_Q \tilde v_0\,\rho(0)\, dx + \int_0^t \int_Q f\,\rho(s) ds.$$

Lemma 7.1 implies

$$\left|\int_Q \tilde v(t)\xi\, dx\right| \le \|\tilde v_0\|_\infty\, Ce^{-\omega t}\|\xi\|_{L^1(Q)} + \int_0^t \|f(s)\|_\infty\, Ce^{-\omega(t-s)}\|\xi\|_{L^1(Q)}\, ds,$$

and since this is true for every $\xi\in L^1(Q)$ we deduce the L^∞ decay in (7.11). Using (7.3) we get the same for the L^2-norm. ∎

With standard energy methods, we can also bound the derivatives in L^2.

LEMMA 7.5. *Let v be a solution of* (7.10). *For every $0<t_0<t$, we have*

(7.12)

$$(t-t_0)\,\|Dv(t)\|^2_{L^2(Q)} \le c[(t-t_0)+1]\left\{\|\tilde v(t_0)\|^2_{L^2(Q)} + \|f\|^2_{L^2((t_0,t)\times Q)} + \|\tilde v\|^2_{L^2((t_0,t)\times Q)}\right\},$$

where c depends only on $\|V\|_\infty$.

Proof. First of all, observe that $\tilde v$ satisfies the equation

$$\tilde v_t - \Delta\tilde v + D\tilde v\, V = f - \langle v\rangle_t \qquad \text{in } (0,\infty)\times Q,$$

where

$$\langle v\rangle_t = \int_Q f - \int_Q D\tilde v\, V.$$

Since V is bounded, we have

$$\begin{aligned}
&\int_{t_0}^t\int_Q |D\tilde v|^2 ds dx\\
&\quad\le \int_Q |\tilde v(t_0)|^2 dx + \int_{t_0}^t\int_Q |V|\,|D\tilde v|\,|\tilde v|\, dsdx + \int_{t_0}^t\int_Q (|f| + |\langle v\rangle_t|)|\tilde v|\, dsdx\\
&\quad\le \int_Q |\tilde v(t_0)|^2 dx + \frac12\int_{t_0}^t\int_Q |D\tilde v|^2 dsdx\\
&\qquad + C\int_{t_0}^t\int_Q (|f|^2 + |\tilde v|^2)\, dxds + \int_{t_0}^t\int_Q |\langle v\rangle_t|\,|\tilde v|\, dsdx.
\end{aligned}$$

On the other hand, the value of $\langle v\rangle_t$ and the fact that V is bounded imply

$$\int_{t_0}^t\int_Q |\langle v\rangle_t|\,|\tilde v|\,dsdx \le C\int_{t_0}^t\int_Q(|f|^2+|\tilde v|^2)\,dxds+\frac14\int_{t_0}^t\int_Q |D\tilde v|^2 dsdx,$$

so we conclude that

$$\int_{t_0}^t\int_Q |Dv|^2 dsdx \le c\int_Q |\tilde v(t_0)|^2dx + c\int_{t_0}^t\int_Q(|f|^2+|\tilde v|^2)dsdx \tag{7.13}$$

for any solution of (7.10).

To show (7.12), we notice that $v_i := \frac{\partial v}{\partial x_i}$ (for $i=1,\dots,d$) satisfies

$$(v_i)_t - \Delta v_i + (Dv\, D\bar u)_{x_i} = f_{x_i}.$$

After multiplication by $(t-t_0)v_i$ and integration by parts we get

$$\begin{aligned}(t-t_0)&\int_Q \frac{|v_i|^2}{2}\,dx + \int_{t_0}^t\int_Q (s-t_0)|Dv_i|^2dxds\\ &\le \int_{t_0}^t\int_Q (s-t_0)\left(|Dv\cdot D\bar u|+|f|\right)\left|\frac{\partial v_i}{\partial x_i}\right| dxds + \int_{t_0}^t\int_Q\frac{|v_i|^2}{2}dsdx,\end{aligned}$$

which implies

$$\begin{aligned}(t-t_0)&\int_Q \frac{|v_i|^2}{2}\,dx + \frac12\int_{t_0}^t\int_Q (s-t_0)|Dv_i|^2dxds\\ &\le C\int_{t_0}^t\int_Q (s-t_0)\left(|Dv|^2+|f|^2\right) dxds + \int_{t_0}^t\int_Q\frac{|Dv|^2}{2}dsdx.\end{aligned}$$

On account of (7.13), we deduce

$$(t-t_0)\int_Q |v_i|^2\,dx \le c[(t-t_0)+1]\left\{\int_Q |\tilde v(t_0)|^2dx + \int_{t_0}^t\int_Q\left(|f|^2+|\tilde v|^2\right)dxds\right\}$$

for every v_i, which gives (7.12). $\square$

Finally, we conclude with a lemma which is useful once we consider the coupling of the above two equations as in the mean field game (MFG) system.

LEMMA 7.6. *Let $V\in L^\infty_\sharp(\mathbb{R}\times\mathbb{R}^d)$, $F\in L^2_{loc}(0,T;L^2_\sharp(\mathbb{R}^d))$, and $f\in C^0([0,+\infty)$, $L^2_{\sharp,0}(\mathbb{R}^d))$. If $\rho\in L^1_{loc}([0,+\infty),L^1_{\sharp,0}(\mathbb{R}^d))$ is a solution to*

$$\rho_t - \Delta\rho - \operatorname{div}(\rho V) = \operatorname{div}(F) + f \qquad \text{in } (0,\infty)\times Q, \tag{7.14}$$

then we have, for every $t>t_0\ge 0$,

$$\begin{aligned}&\|\rho(t)\|_{L^2(Q)}\\ &\quad\le C\left\{\|\rho(t_0)\|_{L^2(Q)}e^{-\omega(t-t_0)} + \int_{t_0}^t e^{-\omega(t-s)}\|f(s)\|_{L^2(Q)}\,ds + \left(\int_{t_0}^t\int_Q |F|^2\right)^{\frac12}\right\}\end{aligned} \tag{7.15}$$

for some $C=C(\|V\|_\infty)$, where ω is defined by Lemma 7.1.

Proof. Let $\bar\xi \in L^2_\sharp(\mathbb{R}^d)$ and let ξ be the solution to the backward equation

$$\begin{cases} -\xi_t - \Delta\xi + D\xi\, V = 0 & \text{in } (0,t)\times\mathbb{R}^d, \\ \xi(t) = \bar\xi. \end{cases} \tag{7.16}$$

From Lemma 7.4 we have

$$\|\tilde\xi(s)\|_2 \le C\|\tilde{\bar\xi}\|_2 e^{-\omega(t-s)}, \qquad s\in[0,t]. \tag{7.17}$$

Moreover, since $\xi(t-s)$ satisfies (7.10) with $f=0$ and $v_0=\bar\xi$, from (7.13) we also deduce

$$\int_0^t\int_Q |D\xi|^2 \le c\int_Q |\bar\xi|^2\,dx + c\int_0^t\int_Q |\tilde\xi|^2\,dxds \le C\,\|\bar\xi\|_2^2. \tag{7.18}$$

Since ρ solves (7.14), we have

$$\frac{d}{dt}\int_Q \xi\rho = \int_Q f\,\xi - \int_Q F\,D\xi.$$

Therefore, since ρ and f have zero average, for any $t_0\in[0,t]$ we have

$$\begin{aligned} \int_Q \rho(t)\bar\xi &= \int_Q \rho(t_0)\tilde\xi(t_0) + \int_{t_0}^t\int_Q f\,\tilde\xi - \int_{t_0}^t\int_Q F\,D\xi \\ &\le C\|\rho(t_0)\|_2\|\tilde\xi(t_0)\|_2 + C\int_{t_0}^t \|f(s)\|_2\,\|\tilde\xi(s)\|_2\,ds \\ &\quad + C\left(\int_{t_0}^t\int_Q |F|^2\right)^{\frac12}\left(\int_{t_0}^t\int_Q |D\xi(t)|^2\right)^{\frac12}. \end{aligned}$$

Hence, due to (7.17) and (7.18), we deduce

$$\int_Q \rho(t)\bar\xi \le C\left\{\ \|\rho(t_0)\|_2 e^{-\omega(t-t_0)} + \int_{t_0}^t e^{-\omega(t-s)}\,\|f(s)\|_2\,ds + \left(\int_{t_0}^t\int_Q |F|^2\right)^{\frac12}\right\}\ \|\bar\xi\|_2,$$

which gives (7.15). □

REFERENCES

[1] D. G. Aronson and J. Serrin, *Local behavior of solutions of quasilinear parabolic equations*, Arch. Rational Mech. Anal., 25 (1967) pp. 81–122.

[2] L. Boccardo, L. Orsina, and A. Porretta, *Some noncoercive parabolic equations with lower order terms in divergence form*, J. Evol. Equ., 3 (2003), pp. 407–418.

[3] P. Cardaliaguet, J.-M. Lasry, P.-L. Lions, and A. Porretta, *Long time average of mean field games*, Networks and Heterogeneous Media, 7 (2012), pp. 279–301.

[4] D. A. Gomes, J. Mohr, and R. Souza, *Discrete time, finite state space mean field games*, J. Math. Pures Appl. (9), 93 (2010), pp. 308–328.

[5] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural'ceva, *Linear and Quasilinear Equations of Parabolic Type*, Trans. Math. Monogr. 23, AMS, Providence, RI, 1967.

[6] J.-M. Lasry and P.-L. Lions, *Jeux à champ moyen.* I. *Le cas stationnaire*, C. R. Math. Acad. Sci. Paris, 343 (2006), pp. 619–625.

[7] J.-M. Lasry and P.-L. Lions, *Jeux à champ moyen.* II. *Horizon fini et contrôle optimal*, C. R. Math. Acad. Sci. Paris, 343 (2006), pp. 679–684.

[8] J.-M. Lasry and P.-L. Lions, *Mean field games*, Japan. J. Math. (N.S.), 2 (2007), pp. 229–260.

[9] J.-M. Lasry, P.L. Lions, *In Cours au Collège de France*, www.college-de-france.fr.
[10] A. Lunardi, *Analytic Semigroups and Optimal Regularity in Parabolic Problems*, Progr. Nonlinear Differential Equations Appl. 16, Birkhäuser Verlag, Basel, 1995.
[11] A. Porretta, *Local existence and uniqueness of weak solutions for nonlinear parabolic equations with superlinear growth and unbounded initial data*, Adv. Differential Equations, 6 (2001), pp. 73–128.