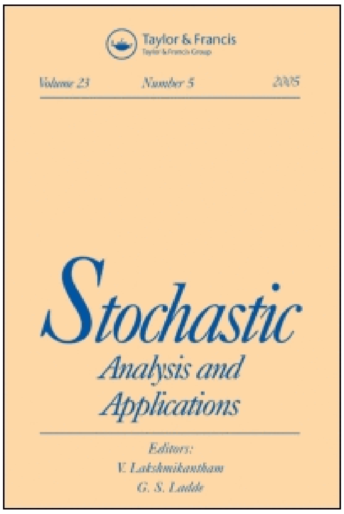

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Optimal Reinsurance and Dividend Strategies Under the Markov-Modulated Insurance Risk Model

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In this article, we consider the optimal reinsurance and dividend strategy for an insurer. We model the surplus process of the insurer by the classical compound Poisson risk model modulated by an observable continuous-time Markov chain. The object of the insurer is to select the reinsurance and dividend strategy that maximizes the expected total discounted dividend payments until ruin. We give the definition of viscosity solution in the presence of regime switching. The optimal value function is characterized as the unique viscosity solution of the associated Hamilton–Jacobi–Bellman equation and a verification theorem is also obtained.

Keywords Compound Poisson model; Dividend strategy; HJB equation; Regime switching; Reinsurance; Viscosity solution.

Mathematics Subject Classification Primary 93E20; Secondary 91B70, 60H30.

1. Introduction

With the application of control theory, the optimal dividend problem has been investigated by many authors in the literature. Under the diffusion model, this problem was studyed by [2, 12] in the case of no reinsurance, by [14] in the

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case of proportional reinsurance, and by [3] and [8] in the case of excess-of-loss reinsurance. Under the classical compound Poisson model, it was discussed by [4] via the viscosity approach, by [13, 16, 22].

Recently, the Markov-modulated insurance risk model becomes popular. This model was proposed by [1] in which the claim inter-arrivers and claim sizes are influenced by an external environment process $\{J(t)\}_{t\geq 0}$. This model can capture the feature that insurance policies may need to change if economical or political environment changes. Under the regime-switching compound Poisson model, [17] considered the constant barrier strategy, but they did not address the optimal dividend strategy. Sotomayor and Cadenillas [25] and Jiang and Pistorius [15] discussed the optimal dividend strategy under the regime-switching diffusion model. In this article, we study the optimal reinsurance and dividend strategy under the regime-switching compound Poisson model via the viscosity approach, and generalize some results of [4].

The notion of viscosity solution was introduced by [9, 18]. Nowadays, it is a standard tool for studying the Hamilton–Jacobi–Bellman equations (see, e.g., [5, 11, 20, 21, 23, 24].

Suppose that $\{J(t)\}_{t\geq 0}$ is a homogenous continuous-time Markov chain taking values in a finite set $\mathbb{J} = \{1, 2, \ldots, N\}$ with generator $\mathbf{Q} = (q_{ij})_{N\times N}$ where $-q_{ii} = q_i$, $i = 1, 2, \ldots, N$. We further assume that $\{J(t)\}_{\geq 0}$ is irreducible.

At time t, given $J(t) = i$, the premium rate is c_i, claims arrive according to Poisson process with rate λ_i, and the size of the claim which arrives at time t follows the distribution F_i with density f_i and mean μ_i. Furthermore, we assume the premium rate c_i is calculated using the expected value principle with relative safety loading $\theta_i > 0$; that is, $c_i = (1 + \theta_i)\lambda_i\mu_i$.

In the case of no control, suppose the initial surplus is $u \geq 0$, the corresponding surplus process $\{U(t)\}_{t\geq 0}$ of the insurer is given by

$$U(t) = u + \int_0^t c_{J(s)} ds - \int_0^t \int_{(0,\infty)} x N_{J(s)}(ds \times dx), \quad t \geq 0,$$

where $N_i(dt \times dx)$ is a Poisson random measure with intensity measure $\lambda_i f_i(x) dt\, dx$.

Now, we assume that the insurer has the possibility of reinsurance in which each individual claim of X is dividend between the insurer and the reinsurer according to a reinsurance strategy $R(\cdot) : [0\infty) \to [0\infty)$ such that $0 \leq R(X) \leq X$: the insurer pays $R(X)$, the reinsurer pays $X - R(X)$. For this, the insurer pays a reinsurance premium to the reinsurer. We restrict our analysis to the case of cheap reinsurance, which means that the relative safety loading of the reinsurer is the same with the insurer's. Then given the state i, the distribution of the claim sizes paid by the insurer is

$$F_i(R, x) = \int_0^\infty \mathbf{1}_{\{R(y)\leq x\}} dF_i(y),$$

where $\mathbf{1}_{\{\cdot\}}$ is the indicator function. We define $\mu_i(R)$ as its expectation. Then the premium rate received by the insurer is

$$c_i(R) = (1 + \theta_i)\lambda_i\mu_i(R).$$

Since $R(X) \leq X$, we get that $\mu_i(R) \leq \mu_i$ and $c_i(R) \leq c_i$.

Considering a family $\mathscr{R}$ of reinsurance policies, we are going to find a dynamic choice of both the reinsurance police $R \in \mathscr{R}$ and the dividend strategy which maximizes the cumulative expected discounted dividend payments.

Similar to [4], we consider the following families of reinsurance policies:

- $\mathscr{R}_0 = \{R_I\}$, where $R_I(X) = X$;
- $\mathscr{R}_P = \{R : R(X) = rX \text{ for some } r \in [0, 1]\}$ the set of proportional reinsurance policies;
- $\mathscr{R}_{XL} = \{R : R(X) = \min\{r, X\} \text{ for some } r \in [0, \infty)\}$ the set of excess-of-loss reinsurance policies;
- $\mathscr{R}_A$ the set of all reinsurance policies.

Suppose that the aforementioned random variables and stochastic processes are defined on the filtered probability space $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = \{\mathscr{F}_t, t \geq 0\}$ is the filtration generated by $\{U(t)\}_{t\geq 0}$ and $\{J(t)\}_{t\geq 0}$. We also assume that $\mathbb{F}$ satisfies the usual condition. Denote by $\mathbb{P}_{u,i}$ the measure $\mathbb{P}$ conditioned on $\{U(0) = u, J(0) = i\}$, and write $\mathbb{E}_{u,i}$ for the corresponding expectations.

A control strategy π is a two-dimensional stochastic process $\{R^\pi(t), L^\pi(t)\}_{t\geq 0}$, where $R^\pi(t) \in \mathscr{R}$ is the reinsurance policy and $L^\pi(t)$ is the cumulative amount of dividends paid out up to time t. Given such a control strategy π, the surplus of the insurer is given by

$$U^\pi(t) = u + \int_0^t c_{J(s)}(R^\pi(s))ds - \int_0^t \int_{(0,\infty)} xN_{J(s)}(R^\pi(s), ds \times dx) - L^\pi(t), \tag{1.1}$$

where $N_i(R^\pi(s), dt \times dx)$ is a Poisson random measure with intensity measure $\lambda_i dt dF_i(R^\pi(t), x)$.

We say that a control strategy π is admissible if

- the process $R^\pi(t)$ is predictable;
- the process $L^\pi(t)$ is predictable, nondecreasing, no-negative and càglàd;
- for any $t \geq 0$, the process $L^\pi(t)$ satisfies

$$L^\pi(t+) - L^\pi(t) \leq U^\pi(t). \tag{1.2}$$

We denote by Π the set of all admissible strategies.

Let $\tau^\pi = \inf\{t \geq 0 : U^\pi(t) < 0\}$ be the corresponding ruin time of the insurer, and

$$V_\pi(u, i) = \mathbb{E}_{u,i}\left[\int_0^{\tau^\pi} e^{-\delta t} dL^\pi(t)\right] \tag{1.3}$$

be the expected present value of all dividend until ruin given initial surplus u and initial state i, where δ is the force of interest. Note that, from the definition of the admissible strategy, the ruin can occur only by the arrival of a claim.

The value function of the insurer is

$$V(u, i) = \sup_{\pi\in\Pi} V_\pi(u, i). \tag{1.4}$$

The problem for the insurer is to identify a control strategy $\pi^* \in \Pi$ that satisfies $V(u, i) = V_{\pi^*}(u, i)$ for all $u \geq 0$ and $i \in \mathbb{J}$. However, this article discusses the viscosity property of the value function instead of construction of the optimal strategy.

The rest of the article is organized as follows. Section 2 gives some properties of the value function. In Section 3, we prove the Dynamic Programming Priciple and the Hamilton–Jacobi–Bellman equation. In Section 4, we give the definition of the viscosity solution in the case of regime swithcing; and show that the value function is an unique vicosity solution of the associated Hamilton–Jacobi–Bellman equation. In Section 5, we give a characterization of the value funtion and prove a verification theorem. The techniques used in Sections 4 and 5 are similar to [4]. In Section 6, the special case with exponential claim density is discussed.

2. Some Properties of the Value Function

In this section, we are going to show some properties of $V(u, i)$ for $u \geq 0$ and $i \in \mathbb{J}$. The techniques are standard [4, 22]. To simplify the notation, we denote

$$A_i = 1 + \frac{\delta}{\lambda_i}, \quad B_i = 1 + \theta_i \quad \text{and} \quad C_i = \frac{q_i}{\lambda_i}.$$

Proposition 2.1. *For $u \geq 0$ and $i \in \mathbb{J}$, the value function $V(u, i)$ satisfies*

(i) $V(u, i) \leq u + B_{i_0}\mu_{i_0}/(A_{i_0} - 1)$, *where $i_0 \in \mathbb{J}$ such that $c_{i_0} = \max_{i \in \mathbb{J}}\{c_i\}$;*
(ii) $V(u, i) \geq u + B_i\mu_i/(A_i + C_i)$.

Proof. (i) For any admissible strategy $\pi = (R^\pi(t), L^\pi(t))$, $L^\pi(t)$ cannot be greater than the sum of initial surplus and the total incoming premium without reinsurance up to time t (see also (1.1) and (1.2)). Then we have

$$L^\pi(t) \leq u + c_{i_0}t;$$

consequently,

$$V_\pi(u, i) \leq \mathbb{E}_{u,i}\left[\int_0^\infty e^{-\delta s} d(u + c_{i_0}s)\right] = u + \frac{B_{i_0}\mu_{i_0}}{A_{i_0} - 1}.$$

So $V(u, i) = \sup_{\pi \in \Pi} V_\pi(u, i)$ satisfies inequality (i).

(ii) Let T_1 and S_1 be the time at which the first transition of the environment process $\{J(t)\}_{t \geq 0}$ occurs and the first claim comes, respectively. Given the initial surplus u and initial state i, consider the admissible strategy π_0 that pays the whole reserve u and pays the incoming premium without reinsurance $c_i = (1 + \theta_i)\lambda_i\mu_i$ as dividend until $T_1 \wedge S_1$. Then we have

$$V_{\pi_0}(u, i) \geq u + (1 + \theta_i)\lambda_i\mu_i\mathbb{E}_{u,i}\left[\int_0^{T_1 \wedge S_1} e^{-\delta s} ds\right] = u + \frac{B_i\mu_i}{A_i + C_i}.$$

Then by the definition (1.4), we get the result. □

Proposition 2.2. *For all $i \in \mathbb{J}$, $0 \le u_1 < u_2$, the value function $V(u, i)$ satisfies*

(i) $V(u_2, i) - V(u_1, i) \ge u_2 - u_1$;
(ii) $V(u_2, i) - V(u_1, i) \le \left(e^{(A_i+C_i)(u_2-u_1)/(B_i\mu_i)} - 1\right) V(u_1, i)$.

Proof. (i) For each $\epsilon > 0$, take an admissible strategy π such that $V_\pi(u_1, i) \ge V(u_1, i) - \epsilon$. For $0 \le u_1 < u_2$, we define a new strategy π_0 as follows: pay $u_2 - u_1$ as dividends immediately, and then follow the strategy π. Then we get for each $\epsilon > 0$ that

$$V(u_2, i) \ge V_{\pi_0}(u_2, i) = V_\pi(u_1, i) + (u_2 - u_1) \ge V(u_1, i) - \epsilon + (u_2 - u_1).$$

The result follows from the arbitrariness of ϵ.

(ii) Given the initial surplus u_1 and initial state i, for each $\epsilon > 0$, consider an admissible strategy π such that $V_\pi(u_2, i) \ge V(u_2, i) - \epsilon$ for any $u_2 > u_1$. Take now the strategy π_0 that starting with initial surplus u_1 and initial state i, pays no dividends and take no reinsurance if $U^{\pi_0}(t) < u_2$ and follow strategy π after the current surplus reaches u_2. Given the event of no claims and no transition of the environment process $\{J(t)\}_{t\ge0}$, the surplus $U^{\pi_0}(t)$ reaches u_2 at time $t_0 = (u_2 - u_1)/(B_i\lambda_i\mu_i)$. Thus, we get

$$\begin{aligned} V(u_1, i) &\ge V_{\pi_0}(u_1, i) \ge V_\pi(u_2, i)e^{-\delta t_0}\mathbb{P}\left[(T_1 \wedge S_1) > t_0\right] \\ &\ge (V(u_2, i) - \epsilon)e^{-(A_i+C_i)(u_2-u_1)/(B_i\mu_i)} \end{aligned}$$

and then we obtain the result. □

Remark 2.1. From the above propositions, we know that the value function V is increasing and locally Lipschitz in $[0, \infty)$, and so for each $i \in \mathbb{J}$, $V'(\cdot, i)$ exists almost everywhere in $[0, \infty)$ and satisfies $1 \le V'(u, i) \le \frac{(A_i+C_i)}{B_i\mu_i} V(u, i)$.

3. The Hamilton–Jacobi–Bellman Equation

First, we will show the Dynamic Programming Principle for the value function V.

Theorem 3.1. *Given $u \ge 0$, $i \in \mathbb{J}$ and any stopping time τ, we have*

$$V(u, i) = \sup_{\pi\in\Pi} \mathbb{E}_{u,i}\left[\int_0^{\tau\wedge\tau^\pi} e^{-\delta s}dL^\pi(s) + e^{-\delta(\tau\wedge\tau^\pi)}V\left(U^\pi(\tau\wedge\tau^\pi), J(\tau\wedge\tau^\pi)\right)\right]. \tag{3.1}$$

Proof. First, we prove (3.1) for the case τ is a fixed time $T > 0$. Let

$$v(u, i, T) = \sup_{\pi\in\Pi} \mathbb{E}_{u,i}\left[\int_0^{T\wedge\tau^\pi} e^{-\delta s}dL^\pi(s) + e^{-\delta(T\wedge\tau^\pi)}V\left(U^\pi(T\wedge\tau^\pi), J(T\wedge\tau^\pi)\right)\right].$$

Next, we will show $V(u, i) \le v(u, i, T)$. For any admissible strategy $\pi = (R^\pi(t), L^\pi(t))$, we have

$$\begin{aligned} V_\pi(u, i) &= \mathbb{E}_{u,i}\left[\mathbf{1}_{\{\tau^\pi\le T\}}\int_0^{\tau^\pi} e^{-\delta s}dL^\pi(s)\right] \\ &\quad + \mathbb{E}_{u,i}\left[\mathbf{1}_{\{\tau^\pi> T\}}\left(\int_0^{T} e^{-\delta s}dL^\pi(s) + e^{-\delta T}\int_0^{\tau^\pi-T} e^{-\delta s}dL^\pi(s+T)\right)\right] \end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}_{u,i}\left[\int_0^{T\wedge\tau^\pi} e^{-\delta s} dL^\pi(s)\right] \\
&\quad + e^{-\delta T}\mathbb{E}_{u,i}\left[\mathbf{1}_{\{\tau^\pi > T\}}\mathbb{E}\left(\int_0^{\tau^\pi - T} e^{-\delta s} dL^\pi(s+T)\Big| U^\pi(T), J(T)\right)\right] \\
&\le \mathbb{E}_{u,i}\left[\int_0^{T\wedge\tau^\pi} e^{-\delta s} dL^\pi(s)\right] + e^{-\delta T}\mathbb{E}_{u,i}\left[\mathbf{1}_{\{\tau^\pi > T\}} V(U^\pi(T), J(T))\right] \\
&\le \mathbb{E}_{u,i}\left[\int_0^{T\wedge\tau^\pi} e^{-\delta s} dL^\pi(s)\right] + \mathbb{E}_{u,i}\left[e^{-\delta(T\wedge\tau^\pi)} V(U^\pi(T\wedge\tau^\pi), J(T\wedge\tau^\pi))\right] \\
&\le v(u,i,T).
\end{aligned}
$$

From (1.4) we get the result.

Now, we are in the position to prove $V(u,i) \ge v(u,i,T)$. For any $\pi \in \Pi$, denote $u^0 = U^\pi(T\wedge\tau^\pi)$ and $i^0 = J(T\wedge\tau^\pi)$. For each $\epsilon > 0$, we find an admissible strategy π_0 such that $V_{\pi_0}(u^0, i^0) \ge V(u^0, i^0) - \epsilon$. We define an admissible strategy $\tilde{\pi}$ as

$$
\tilde{\pi} = \begin{cases} \pi & \text{when } t \le T\wedge\tau^\pi; \\ \pi_0 & \text{when } t > T\wedge\tau^\pi. \end{cases}
$$

Then we have

$$
\begin{aligned}
V(u,i) &\ge V_{\tilde{\pi}}(u,i) \\
&= \mathbb{E}_{u,i}\left[\int_0^{T\wedge\tau^\pi} e^{-\delta s} dL^\pi(s)\right] + \mathbb{E}_{u,i}\left[\int_{T\wedge\tau^\pi}^{\tau^{\tilde{\pi}}} e^{-\delta s} dL^{\pi_0}(s)\right] \\
&= \mathbb{E}_{u,i}\left[\int_0^{T\wedge\tau^\pi} e^{-\delta s} dL^\pi(s)\right] + \mathbb{E}_{u,i}\left[e^{-\delta(T\wedge\tau^\pi)}\int_0^{\tau^{\tilde{\pi}}-(T\wedge\tau^\pi)} e^{-\delta s} dL^{\pi_0}(s + T\wedge\tau^\pi)\right] \\
&= \mathbb{E}_{u,i}\left[\int_0^{T\wedge\tau^\pi} e^{-\delta s} dL^\pi(s)\right] \\
&\quad + \mathbb{E}_{u,i}\left[e^{-\delta(T\wedge\tau^\pi)}\mathbb{E}\left(\int_0^{\tau^{\tilde{\pi}}-(T\wedge\tau^\pi)} e^{-\delta s} dL^{\pi_0}(s + T\wedge\tau^\pi)\Big| U^\pi(T\wedge\tau^\pi), J(T\wedge\tau^\pi)\right)\right] \\
&= \mathbb{E}_{u,i}\left[\int_0^{T\wedge\tau^\pi} e^{-\delta s} dL^\pi(s)\right] + \mathbb{E}_{u,i}\left[e^{-\delta(T\wedge\tau^\pi)} V_{\pi_0}(U^\pi(T\wedge\tau^\pi), J(T\wedge\tau^\pi))\right] \\
&\ge \mathbb{E}_{u,i}\left[\int_0^{T\wedge\tau^\pi} e^{-\delta s} dL^\pi(s)\right] + \mathbb{E}_{u,i}\left[e^{-\delta(T\wedge\tau^\pi)}\left(V(U^\pi(T\wedge\tau^\pi), J(T\wedge\tau^\pi)) - \epsilon\right)\right].
\end{aligned}
$$

Then the result follows from the arbitrariness of ϵ. Thus, we have $V(u,i) = v(u,i,T)$.

In the following, we are going to show (3.1) holds for any bounded stopping time τ by the standard methods (see [26]). With the assumption that $0 \le \tau \le T$, where $T > 0$ is a fixed time, we can define a sequence of stopping times

$$
\tau_n = \frac{k}{2^n}T, \quad \text{if } \frac{k-1}{2^n}T \le \tau \le \frac{k}{2^n}T.
$$

It is obvious that $\tau_n \to \tau$, *a.s.* For each $\tau_n = t_k := \frac{k}{2^n}T$, the Equation (3.1) holds, that is,

$$V(u,i) = \sup_{\pi\in\Pi} \mathbb{E}_{u,i}\left[\int_0^{t_k\wedge\tau^\pi} e^{-\delta s} dL^\pi(s) + e^{-\delta(t_k\wedge\tau^\pi)} V\left(U^\pi(t_k\wedge\tau^\pi), J(t_k\wedge\tau^\pi)\right)\right].$$

Multiplying by $\mathbb{P}[\tau_n = t_k]$ on both sides of the above equation and summing them up over all $1 \le k \le 2^n$, then we get

$$\begin{aligned} V(u,i) &= \sum_{k=1}^{2^n} \sup_{\pi\in\Pi} \mathbb{E}_{u,i}\left[\int_0^{t_k\wedge\tau^\pi} e^{-\delta s} dL^\pi(s) + e^{-\delta(t_k\wedge\tau^\pi)} V\left(U^\pi(t_k\wedge\tau^\pi), J(t_k\wedge\tau^\pi)\right)\right]\mathbb{P}[\tau_n = t_k] \\ &= \sum_{k=1}^{2^n} \sup_{\pi\in\Pi} \mathbb{E}_{u,i}\Bigg[\mathbb{E}\Bigg(\int_0^{t_k\wedge\tau^\pi} e^{-\delta s} dL^\pi(s) \\ &\qquad + e^{-\delta(t_k\wedge\tau^\pi)} V\big(U^\pi(t_k\wedge\tau^\pi), J(t_k\wedge\tau^\pi)\big) \,\Big|\, \tau_n = t_k\Bigg)\Bigg]\mathbb{P}[\tau_n = t_k], \end{aligned}$$

which implies that

$$V(u,i) = \sup_{\pi\in\Pi} \mathbb{E}_{u,i}\left[\int_0^{\tau_n\wedge\tau^\pi} e^{-\delta s} dL^\pi(s) + e^{-\delta(\tau_n\wedge\tau^\pi)} V\left(U^\pi(\tau_n\wedge\tau^\pi), J(\tau_n\wedge\tau^\pi)\right)\right].$$

Then by the continuity of the value function, letting $n \to \infty$ yields the Equation (3.1). For any stopping time τ, we consider the sequence of bounded stopping times $\hat{\tau}_n = \tau \wedge n$. Then the result follows from the Dominated Convergence Theorem. □

To give the Hamilton–Jacobi–Bellman Equation, we show a lemma first.

Lemma 3.1. *Given $i \in \mathbb{J}$, let $f(t,x,i)$ be a function which is continuously differentiable with respect to t and x. Then $f(t, U^\pi(t), J(t))$ is a finite variation process and*

$$\begin{aligned} &f(t, U^\pi(t), J(t)) - f(0, U^\pi(0), J(0)) \\ &\quad = \int_0^t \sum_{i\in\mathbb{J}} q_{J(s)i} f(s, U^\pi(s), i) ds + \int_0^t \sum_{i\in\mathbb{J}} f(s, U^\pi(s-), i) dm(s,i) \\ &\qquad + \int_0^t \left[f_t(s, U^\pi(s), J(s)) + f_x(s, U^\pi(s), J(s)) c_{J(s)}(R^\pi(s))\right] ds \\ &\qquad - \int_0^t f_x(s, U^\pi(s), J(s)) dL^{\pi c}(s) - \int_0^t \int_0^{L^\pi(s+)-L^\pi(s)} f_x(s, U^\pi(s)-y, J(s)) dy\, ds \\ &\qquad + \int_0^t \int_0^\infty [f(s, U^\pi(s-)-x, J(s)) - f(s, U^\pi(s-), J(s))] N_{J(s)}(R^\pi(s), ds\times dx), \end{aligned} \tag{3.2}$$

where $m(\cdot, i)$ is a martingale.

Proof. Obviously,

$$f(t, R_\pi(t), J(t)) = \sum_{i\in\mathbb{J}} f(t, R_\pi(t), i)\xi(t,i) \tag{3.3}$$

where $\xi(t, i) = \mathbf{1}_{\{J(t)=i\}}$. It is shown in [19] that

$$\xi(t, i) = \xi(0, i) + \int_0^t \sum_{j\in\mathbb{J}} q_{ji}\xi(s, j)ds + m(t, i), \quad i \in \mathbb{J}, \tag{3.4}$$

where $m(t, i)$ is a square integrable martingale with right-continuous paths. From Itô formula and Equation (1.1), we know that $f(t, U^\pi(t), i)$ is a finite variation process and

$$\begin{aligned}
&f(t, U^\pi(t), i) - f(0, U^\pi(0), i)\\
&\quad = \int_0^t f_t(s, U^\pi(s), i)ds + \int_0^t f_x(s, U^\pi(s), i)c_{J(s)}(R^\pi(s))ds - \int_0^t f_x(s, U^\pi(s), i)dL^{\pi c}(s)\\
&\qquad + \int_0^t \int_0^\infty [f(s, U^\pi(s-) - x, i) - f(s, U^\pi(s-), i)]\, N_{J(s)}(R^\pi(s), ds \times dx)\\
&\qquad - \int_0^t \int_0^{L^\pi(s+)-L^\pi(s)} f_x(s, U^\pi(s) - y, i)dy\, ds,
\end{aligned} \tag{3.5}$$

where $L^{\pi c}(t)$ is the continuous part of $L^\pi(t)$, that is, $L^\pi(t) = L^{\pi c}(t) + \sum_{0\le s\le t} \{L^\pi(s+) - L^\pi(s)\}$.

Since $\xi(\cdot, i)$ and $f(t, U^\pi(t), i)$ are a finite variation processes, then from integration by parts, we get

$$\begin{aligned}
f(t, U^\pi(t), i)\xi(t, i) &= f(0, U^\pi(0), i)\xi(0, i) + \int_0^t f(s, U^\pi(s-), i)d\xi(s, i)\\
&\quad + \int_0^t \xi(s, i)df(s, U^\pi(s), i).
\end{aligned} \tag{3.6}$$

The following equation is derived from Equations (3.4), (3.5), and (3.6).

$$\begin{aligned}
&f(t, U^\pi(t), i)\xi(t, i) - f(0, U^\pi(0), i)\xi(0, i)\\
&\quad = \int_0^t \sum_{j\in\mathbb{J}} q_{ji}\xi(s, j)f(s, U^\pi(s), i)ds + \int_0^t f(s, U^\pi(s-), i)dm(s, i)\\
&\qquad + \int_0^t \xi(s, i)\left[f_t(s, U^\pi(s), i) + f_x(s, U^\pi(s), i)c_{J(s)}(R^\pi(s))\right] ds\\
&\qquad - \int_0^t \xi(s, i)f_x(s, U^\pi(s), i)dL^{\pi c}(s) - \int_0^t \xi(s, i)\int_0^{L^\pi(s+)-L^\pi(s)} f_x(s, U^\pi(s) - y, i)dy\, ds\\
&\qquad + \int_0^t \xi(s, i)\int_0^\infty [f(s, U^\pi(s-) - x, i) - f(s, U^\pi(s-), i)]\, N_{J(s)}(R^\pi(s), ds \times dx).
\end{aligned} \tag{3.7}$$

Then Equation (3.2) follows from Equations (3.3) and (3.7). □

For all $i \in \mathbb{J}$ and $R \in \mathcal{R}$, given a continuous vector function $\mathbf{v}(\cdot) = (v(\cdot, 1), v(\cdot, 2), \ldots, v(\cdot, N))^T$, we define the operator

$$\begin{aligned}
\mathcal{L}_i(R, v)(u) &= v'(u, i)c_i(R) + \lambda_i \int_0^u v(u - x, i)dF_i(R, x)\\
&\quad - (\lambda_i + \delta)v(u, i) + \sum_{k\in\mathbb{J}} q_{ik}v(u, k).
\end{aligned}$$

Theorem 3.2. *For all $i \in \mathbb{J}$, assume the value function $V(u, i)$ defined by Equation (1.4) is continuously differentiable with respect to $u \in (0, \infty)$. Then $V(u, i)$ satisfies the Hamilton–Jacobi–Bellman equation*

$$\max\left\{1 - V'(u, i), \sup_{R\in\mathcal{R}} \mathcal{L}_i(R, V)(u)\right\} = 0. \tag{3.8}$$

Proof. From Equation (3.2), we have

$$\begin{aligned}
&e^{-\delta(\tau\wedge\tau^\pi)} V\left(U^\pi(\tau\wedge\tau^\pi), J(\tau\wedge\tau^\pi)\right) - V\left(U^\pi(0), J(0)\right)\\
&= \int_0^{\tau\wedge\tau^\pi} e^{-\delta s} \sum_{k\in\mathbb{J}} q_{J(s)k} V(U^\pi(s), k) ds + \int_0^{\tau\wedge\tau^\pi} e^{-\delta s} \sum_{k\in\mathbb{J}} V(U^\pi(s-), k) dm(s, k)\\
&\quad + \int_0^{\tau\wedge\tau^\pi} e^{-\delta s}\left[-\delta V(U^\pi(s), J(s)) + V'(U^\pi(s), J(s)) c_{J(s)}(R^\pi(s))\right] ds\\
&\quad - \int_0^{\tau\wedge\tau^\pi} e^{-\delta s} V'(U^\pi(s), J(s)) dL^{\pi c}(s) - \int_0^{\tau\wedge\tau^\pi}\int_0^{L^\pi(s+)-L^\pi(s)} e^{-\delta s} V'(U^\pi(s) - y, J(s)) dyds\\
&\quad + \int_0^{\tau\wedge\tau^\pi}\int_0^\infty e^{-\delta s}\left[V(U^\pi(s-) - x, J(s)) - V(U^\pi(s-), J(s))\right] N_{J(s)}(R^\pi(s), ds\times dx).
\end{aligned}$$

Applying conditional expectation to the above equation yields

$$\begin{aligned}
&\mathbb{E}_{u,i}\left[e^{-\delta(\tau\wedge\tau^\pi)} V\left(U^\pi(\tau\wedge\tau^\pi), J(\tau\wedge\tau^\pi)\right)\right] - V(u, i)\\
&= \mathbb{E}_{u,i}\left[\int_0^{\tau\wedge\tau^\pi} e^{-\delta s} \sum_{i\in\mathbb{J}} V(U^\pi(s-), i) dm(s, i)\right]\\
&\quad - \mathbb{E}_{u,i}\left[\int_0^{\tau\wedge\tau^\pi} e^{-\delta s} V'(U^\pi(s), J(s)) dL^{\pi c}(s)\right]\\
&\quad - \mathbb{E}_{u,i}\left[\int_0^{\tau\wedge\tau^\pi}\int_0^{L^\pi(s+)-L^\pi(s)} e^{-\delta s} V'(U^\pi(s) - y, J(s)) dyds\right]\\
&\quad + \mathbb{E}_{u,i}\left[\int_0^{\tau\wedge\tau^\pi} e^{-\delta s} \mathcal{L}_{J(s)}(R^\pi(s), V)(U^\pi(s)) ds\right]
\end{aligned} \tag{3.9}$$

Since $V(\cdot, i)$ is bounded and $m(t, i)$ is a martingale, it is easy to show

$$\mathbb{E}_{u,i}\left[\int_0^{\tau\wedge\tau^\pi} e^{-\delta s} \sum_{k\in\mathbb{J}} V(U^\pi(s-), k) dm(s, k)\right] = 0.$$

From Remark 2.1, we know that $V'(u, i) \geq 1$. Then

$$\begin{aligned}
&-\mathbb{E}_{u,i}\left[\int_0^{\tau\wedge\tau^\pi} e^{-\delta s} V'(U^\pi(s), J(s)) dL^{\pi c}(s)\right]\\
&\quad - \mathbb{E}_{u,i}\left[\int_0^{\tau\wedge\tau^\pi}\int_0^{L^\pi(s+)-L^\pi(s)} e^{-\delta s} V'(U^\pi(s) - y, J(s)) dyds\right]
\end{aligned}$$

$$\leq -\mathbb{E}_{u,i}\left[\int_0^{\tau\wedge\tau^\pi} e^{-\delta s} dL^{\pi c}(s)\right] - \mathbb{E}_{u,i}\left[\int_0^{\tau\wedge\tau^\pi}\int_0^{L^\pi(s+)-L^\pi(s)} e^{-\delta s} dy\, ds\right]$$
$$= -\mathbb{E}_{u,i}\left[\int_0^{\tau\wedge\tau^\pi} e^{-\delta s} dL^{\pi}(s)\right].$$

For any admissible strategy π and $h > 0$, let $\tau_\pi(h) = h \wedge \inf\{t : U^\pi(t) \in \mathbb{R}/(u-h, u+h)\}$. Then $\tau_\pi(h) < \infty$ a.s. and $\tau_\pi(h) \to 0$ a.s. Furthermore, if we choose $h < u$, then $\tau_\pi(h) \leq \tau^\pi$.

Thus, from the Dynamic Programming Principle (3.1) and Equation (3.9), for any $\epsilon > 0$, we can choose an admissible strategy π and $h < u$ such that

$$V(u,i) - \epsilon \leq \mathbb{E}_{u,i}\left[\int_0^{\tau_\pi(h)} e^{-\delta s} dL^\pi(s) + e^{-\delta(\tau_\pi(h))} V\left(U^\pi(\tau_\pi(h)), J(\tau_\pi(h))\right)\right]$$
$$\leq V(u,i) + \mathbb{E}_{u,i}\left[\int_0^{\tau_\pi(h)} e^{-\delta s} \mathscr{L}_{J(s)}(R^\pi(s), V)(U^\pi(s)) ds\right].$$

Because of the arbitrariness of ϵ, we have

$$\mathbb{E}_{u,i}\left[\int_0^{\tau_\pi(h)} e^{-\delta s} \mathscr{L}_{J(s)}(R^\pi(s), V)(U^\pi(s)) ds\right] \geq 0.$$

Dividing the above equation by $\mathbb{E}_{u,i}[\tau_\pi(h)]$ and then letting $h \to 0$, we get

$$\sup_{R\in\mathscr{R}} \mathscr{L}_i(R, V)(u) \geq 0.$$

Now, we consider an admissible strategy π_0 such that $R^{\pi_0}(t)$ can be any element $R \in \mathscr{R}$, $L^{\pi_0}(t) = 0$ for $t < \tau_{\pi_0}(h)$ and arbitrary for $t \geq \tau_{\pi_0}(h)$. Then from the Dynamic Programming Principle (3.1) and Equation (3.9), we have

$$V(u,i) \geq \mathbb{E}_{u,i}\left[e^{-\delta(\tau_{\pi_0}(h))} V\left(U^{\pi_0}(\pi_0(h)), J(\tau_{\pi_0}(h))\right)\right]$$
$$= V(u,i) + \mathbb{E}_{u,i}\left[\int_0^{\tau_{\pi_0}(h)} e^{-\delta s} \mathscr{L}_{J(s)}(R^{\pi_0}(s), V)(U^{\pi_0}(s)) ds\right].$$

Similarly, we have

$$\sup_{R\in\mathscr{R}} \mathscr{L}_i(R, V)(u) \leq 0.$$

Thus, we get the Hamilton–Jacobi–Bellman equation

$$\max\left\{1 - V'(u,i), \sup_{R\in\mathscr{R}} \mathscr{L}_i(R, V)(u)\right\} = 0. \qquad \square$$

In the above proof, we assume that the value function V is a sufficiently sooth function which is not the case in most of the examples. Thus, in the following we will study the viscosity solutions of the Hamilton–Jacobi–Bellman equation.

4. Viscosity Solution

Now we are going to study the viscosity property of the value function. Some techniques used in this section and the next one are similar to [4].

First we write the Hamilton–Jacobi–Bellman Equation (3.8) in the matrix form.

$$\max\left\{\mathbf{1} - \mathbf{V}'(u), \sup_{R\in\mathscr{R}} \mathscr{L}(R, \mathbf{V})(u)\right\} = \mathbf{0}, \tag{4.1}$$

where

$$\mathbf{V}(u) = (V(u,1), V(u,2), \ldots, V(u,N))^T$$

and

$$\mathscr{L}(R, \mathbf{V})(u) = (\mathscr{L}_1(R, V)(u), \mathscr{L}_1(R, V)(u), \ldots, \mathscr{L}_N(R, V)(u))^T.$$

Next we give the definition of viscosity solution. Note that our equation is derivative constrained and $\mathbf{V}$ makes sense only in $[0, \infty)$.

Definition 4.1. (i) We say that a continuous vector function (each element is continuous) $\underline{\mathbf{v}}(\cdot) = (\underline{v}(\cdot,1), \underline{v}(\cdot,2), \ldots, \underline{v}(\cdot,N))^T : [0,\infty) \to \mathbb{R}^N$ is a viscosity subsolution of (4.1) at $u \in (0, \infty)$, if any continuously differentiable vector function (each element is continuously differentiable) $\boldsymbol{\psi}(\cdot) = (\psi(\cdot,1), \psi(\cdot,2), \ldots, \psi(\cdot,N))^T : (0,\infty) \to \mathbb{R}^N$ with $\underline{\mathbf{v}}(u) = \boldsymbol{\psi}(u)$ such that for each $i \in \mathbb{J}$, $\underline{v}(\cdot, i) - \psi(\cdot, i)$ reaches the maximum at u satisfies

$$\max\left\{1 - \psi'(u,i), \sup_{R\in\mathscr{R}} \mathscr{L}_i(R, \psi)(u)\right\} \geq 0, \quad \forall i \in \mathbb{J}.$$

(ii) We say that a continuous vector function $\overline{\mathbf{v}}(\cdot) = (\overline{v}(\cdot,1), \overline{v}(\cdot,2), \ldots, \overline{v}(\cdot,N))^T : [0,\infty) \to \mathbb{R}^N$ is a viscosity supersolution of (4.1) at $u \in (0,\infty)$, if any continuously differentiable vector function $\boldsymbol{\phi}(\cdot) = (\phi(\cdot,1), \phi(\cdot,2), \ldots, \phi(\cdot,N))^T : (0,\infty) \to \mathbb{R}^N$ with $\overline{\mathbf{v}}(u) = \boldsymbol{\phi}(u)$ such that for each $i \in \mathbb{J}$, $\overline{v}(\cdot,i) - \phi(\cdot,i)$ reaches the minimum at u satisfies

$$\max\left\{1 - \phi'(u,i), \sup_{R\in\mathscr{R}} \mathscr{L}_i(R, \phi)(u)\right\} \leq 0, \quad \forall i \in \mathbb{J}.$$

(iii) We say that a continuous vector function $\mathbf{v}(\cdot) = (v(\cdot,1), v(\cdot,2), \ldots, v(\cdot,N))^T : [0,\infty) \to \mathbb{R}^N$ is a viscosity solution of (4.1) if it is both viscosity subsolution and viscosity supersolution at any $u \in (0,\infty)$.

Before showing that the value function $\mathbf{V}$ is the unique viscosity solution of (4.1), we give an equivalent definition of the viscosity solution. When $\mathbb{J}$ contains only one element, the next lemma is standard in the context of viscosity theory; see for example, [4, 10]; and the proof can be found in [6, 20, 24].

For all $i \in \mathbb{J}$, $R \in \mathscr{R}$, given a continuous vector function $\mathbf{v}(\cdot) = (v(\cdot, 1), v(\cdot, 2), \ldots, v(\cdot, N))^T$ and a continuously differentiable vector function $\mathbf{f}(\cdot) = (f(\cdot, 1), f(\cdot, 2), \ldots, f(\cdot, N))^T$, we define the operator

$$\mathscr{L}_i(R, v, f)(u) = f'(u, i)c_i(R) + \lambda_i \int_0^u v(u - x, i)dF_i(R, x) \\ - (\lambda_i + \delta)v(u, i) + \sum_{k \in \mathbb{J}} q_{ik} v(u, k).$$

Lemma 4.1.

(i) *A continuous vector function* $\underline{\mathbf{v}}(\cdot) = (\underline{v}(\cdot, 1), \underline{v}(\cdot, 2), \ldots, \underline{v}(\cdot, N))^T : [0, \infty) \to \mathbb{R}^N$ *is a viscosity subsolution of* (4.1) *at* $u \in (0, \infty)$, *if and only if any continuously differentiable vector function* $\boldsymbol{\psi}(\cdot) = (\psi(\cdot, 1), \psi(\cdot, 2), \ldots, \psi(\cdot, N))^T : (0, \infty) \to \mathbb{R}^N$ *such that for each* $i \in \mathbb{J}$, $\underline{v}(\cdot, i) - \psi(\cdot, i)$ *reaches the maximum at u satisfies*

$$\max\left\{1 - \psi'(u, i), \sup_{R \in \mathscr{R}} \mathscr{L}_i(R, \underline{v}, \psi)(u)\right\} \geq 0, \quad \forall i \in \mathbb{J}.$$

(ii) *A continuous vector function* $\overline{\mathbf{v}}(\cdot) = (\overline{v}(\cdot, 1), \overline{v}(\cdot, 2), \ldots, \overline{v}(\cdot, N))^T : [0, \infty) \to \mathbb{R}^N$ *is a viscosity supersolution of* (4.1) *at* $u \in (0, \infty)$, *if and only if any continuously differentiable vector function* $\boldsymbol{\phi}(\cdot) = (\phi(\cdot, 1), \phi(\cdot, 2), \ldots, \phi(\cdot, N))^T : (0, \infty) \to \mathbb{R}^N$ *such that for each* $i \in \mathbb{J}$, $\overline{v}(\cdot, i) - \phi(\cdot, i)$ *reaches the minimum at u satisfies*

$$\max\left\{1 - \phi'(u, i), \sup_{R \in \mathscr{R}} \mathscr{L}_i(R, \overline{v}, \phi)(u)\right\} \leq 0, \quad \forall i \in \mathbb{J}.$$

Proof. We only proof the statement (i), and (ii) can be proven similarly.

First, we prove the sufficiency. For any continuously differentiable vector function $\boldsymbol{\psi}(\cdot)$ with $\mathbf{v}(u) = \boldsymbol{\psi}(u)$ such that for each $i \in \mathbb{J}$, $\underline{v}(\cdot, i) - \psi(\cdot, i)$ reaches the maximum at u, we have

$$\max\left\{1 - \psi'(u, i), \sup_{R \in \mathscr{R}} \mathscr{L}_i(R, \overline{v}, \psi)(u)\right\} \geq 0, \quad \forall i \in \mathbb{J}.$$

Since $\psi(y, i) \geq v(y, i)$, we have $\sup_{R \in \mathscr{R}} \mathscr{L}_i(R, \psi)(u) \geq \sup_{R \in \mathscr{R}} \mathscr{L}_i(R, \overline{v}, \psi)(u)$. Thus we get

$$\max\left\{1 - \psi'(u, i), \sup_{R \in \mathscr{R}} \mathscr{L}_i(R, \psi)(u)\right\} \geq 0, \quad \forall i \in \mathbb{J}.$$

Next, we are going to show the necessity. For any continuously differentiable vector function $\boldsymbol{\psi}(\cdot)$ such that for each $i \in \mathbb{J}$, $\underline{v}(\cdot, i) - \psi(\cdot, i)$ reaches the maximum at u, we define $\psi_n(y, i)$ for each $i \in \mathbb{J}$ as follows

$$\psi_n(y, i) = \chi_n(y)\left[\psi(y, i) + \underline{v}(u, i) - \psi(u, i)\right] + (1 - \chi_n(y))\underline{v}(y, i) \quad \text{for } y \in [0, \infty),$$

where $\chi_n(\cdot)$ is a smooth function satisfying

$$0 \leq \chi_n(\cdot) \leq 1;$$

$$\chi_n(y) = 1 \quad \text{for } y \in \left(u - \frac{1}{n}, u + \frac{1}{n}\right);$$

$$\chi_n(y) = 0 \quad \text{for } y \in \mathbb{R} \setminus \left(u - \frac{2}{n}, u + \frac{2}{n}\right).$$

Observe that $\psi_n(\cdot, i)$ is continuously differentiable with $\psi_n(u, i) = \underline{v}(u, i)$, $\psi_n'(u, i) = \psi'(u, i)$ and

$$\begin{aligned}
\underline{v}(y, i) - \psi_n(y, i) &= \chi_n(y)\left[\underline{v}(y, i) - \psi(y, i) - \underline{v}(u, i) + \psi(u, i)\right] \\
&\leq 0 \\
&= \underline{v}(u, i) - \psi_n(u, i).
\end{aligned}$$

Since $\underline{\mathbf{v}}$ is a viscosity subsolution, we have

$$\max\left\{1 - \psi'(u, i), \sup_{R\in\mathcal{R}} \mathcal{L}_i(R, \psi_n)(u)\right\} \geq 0, \quad \forall i \in \mathbb{J}.$$

Letting $n \to \infty$ and using the Dominated Convergence Theorem, we conclude that

$$\max\left\{1 - \psi'(u, i), \sup_{R\in\mathcal{R}} \mathcal{L}_i(R, \underline{v}, \psi)(u)\right\} \geq 0, \quad \forall i \in \mathbb{J}.$$

□

Theorem 4.1. $\mathbf{V}(\cdot)$ *is a viscosity solution of* (4.1).

Proof. Let us prove first that $\mathbf{V}$ is a viscosity supersolution. Let $h > 0$, T_1 and S_1 be the time at which the first transition of the environment process $\{J(t)\}_{t\geq 0}$ occurs and the first claim comes, respectively.

We consider the following admissible strategy π_0: between the interval $[0, T_1 \wedge S_1 \wedge h]$, the reinsurance police R^{π_0} is any element in $\mathcal{R}$, dividends are paid at rate $l^{\pi_0} \geq 0$, that is, $L^{\pi_0}(t) = l^{\pi_0} t$, and thereafter, an optimal strategy is applied.

Given the initial surplus u and initial state i, denote $\eta(t) = (c_i(R^{\pi_0}) - l^{\pi_0})\, t$. By total probability formula, we see that the expectation of the present value of all dividends until ruin is

$$\begin{aligned}
I_1 + I_2 + I_3 = {} & e^{-(\lambda_i+q_i+\delta)h} V(u + \eta(h), i) \\
& + \int_0^h \lambda_i e^{-(\lambda_i+q_i+\delta)s} \int_0^{u+\eta(s)} V(u + \eta(s) - y, i) dF_i(R^{\pi_0}, y) ds \\
& + \int_0^h q_i e^{-(\lambda_i+q_i+\delta)t} \sum_{k\neq i} \frac{q_{ik}}{q_i} V(u + \eta(t), k) dt + \Pi(h) \\
\leq {} & V(u, i),
\end{aligned}$$

where

$$I_1 = \int_h^\infty q_i e^{-q_i t} \int_h^\infty \lambda_i e^{-\lambda_i s} \left[\int_0^h l^{\pi_0} e^{-\delta x} dx + e^{-\delta h} V(u + \eta(h), i)\right] ds\, dt;$$

$$\begin{aligned}
I_2 = {} & \left(\int_h^\infty \lambda_i e^{-\lambda_i s} \int_0^h q_i e^{-q_i t} + \int_0^h \lambda_i e^{-\lambda_i s} \int_0^s q_i e^{-q_i t}\right) \\
& \times \left[\int_0^t l^{\pi_0} e^{-\delta x} dx + e^{-\delta t} \sum_{k\neq i} \frac{q_{ik}}{q_i} V(u + \eta(t), k)\right] dt\, ds;
\end{aligned}$$

$$
\begin{aligned}
I_3 &= \left(\int_h^{\infty} q_i e^{-q_i t} \int_0^h \lambda_i e^{-\lambda_i s} + \int_0^h q_i e^{-q_i t} \int_0^t \lambda_i e^{-\lambda_i s} \right) \\
&\quad \times \left[\int_0^s l^{\pi_0} e^{-\delta x} dx + e^{-\delta s} \int_0^{u+\eta(s)} V(u+\eta(s)-y, i) dF_i(R^{\pi_0}, y) \right] ds\, dt; \\
I(h) &= \frac{l^{\pi_0}}{\delta}(1-e^{-\delta h}) e^{-(\lambda_i+q_i)h} + \int_0^h \frac{l^{\pi_0}}{\delta}(1-e^{-\delta s}) \lambda_i e^{-(\lambda_i+q_i)s} ds \\
&\quad + \int_0^h \frac{l^{\pi_0}}{\delta}(1-e^{-\delta t}) q_i e^{-(\lambda_i+q_i)t} dt.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&\frac{V(u+\eta(h), i) - V(u, i)}{h} - \frac{1-e^{-(\lambda_i+q_i+\delta)h}}{h} V(u+\eta(h), i) \\
&\quad + \frac{1}{h} \int_0^h \lambda_i e^{-(\lambda_i+q_i+\delta)s} \int_0^{u+\eta(s)} V(u+\eta(s)-y, i) dF_i(R^{\pi_0}, y) ds \\
&\quad + \frac{1}{h} \int_0^h q_i e^{-(\lambda_i+q_i+\delta)t} \sum_{k\neq i} \frac{q_{ik}}{q_i} V(u+\eta(t), k) dt + \frac{1}{h} I(h) \leq 0. \qquad (4.2)
\end{aligned}
$$

Let $\boldsymbol{\phi}(\cdot) = (\phi(\cdot, 1), \phi(\cdot, 2), \ldots, \phi(\cdot, N))^T : (0, \infty) \to \mathbb{R}^N$ be a continuously differentiable vector function such that for each $i \in \mathbb{J}$, $V(\cdot, i) - \phi(\cdot, i)$ reaches the minimum at u. From (4.2) we get

$$
\begin{aligned}
&\frac{\phi(u+\eta(h), i) - \phi(u, i)}{h} - \frac{1-e^{-(\lambda_i+q_i+\delta)h}}{h} V(u+\eta(h), i) \\
&\quad + \frac{1}{h} \int_0^h \lambda_i e^{-(\lambda_i+q_i+\delta)s} \int_0^{u+\eta(s)} V(u+\eta(s)-y, i) dF_i(R^{\pi_0}, y) ds \\
&\quad + \frac{1}{h} \int_0^h q_i e^{-(\lambda_i+q_i+\delta)t} \sum_{k\neq i} \frac{q_{ik}}{q_i} V(u+\eta(t), k) dt + \frac{1}{h} I(h) \leq 0.
\end{aligned}
$$

Letting $h \to 0$ in the above inequality gives

$$
\begin{aligned}
0 &\geq l^{\pi_0}(1-\phi'(u, i)) + c_i(R^{\pi_0})\phi'(u, i) + \lambda_i \int_0^u V(u-y, i) dF_i(R^{\pi_0}, y) \\
&\quad - (\lambda_i + \delta) V(u, i) + \sum_{k\in\mathbb{J}} q_{ik} V(u, k).
\end{aligned}
$$

Since this inequality holds for all $l^{\pi_0} \geq 0$, we have that $\phi'(u, i) \geq 1$, and taking $l^{\pi_0} = 0$ we get $\mathscr{L}_i(R^{\pi_0}, V, \phi)(u) \leq 0$, so

$$
\max\left\{ 1 - \phi'(u, i), \sup_{R\in\mathscr{R}} \mathscr{L}_i(R, V, \phi)(u) \right\} \leq 0
$$

and we have the result.

It remains to prove that $\mathbf{V}$ is a viscosity subsolution at any $u > 0$. Arguing by contradiction, we assume that $\mathbf{V}$ is not a viscosity subsolution of (4.1). Using the same method of [4], we can find $\epsilon > 0$, $h \in (0, u/2)$ and a continuously differentiable vector function $\boldsymbol{\psi}(\cdot) = (\psi(\cdot, 1), \psi(\cdot, 2), \ldots, \psi(\cdot, N))^T$ with $\mathbf{V}(u) = \boldsymbol{\psi}(u)$ such that for all $i \in \mathbb{J}$

$$1 - \psi'(y, i) \leq 0 \tag{4.3}$$

when $y \in [0, u + h]$,

$$\sup_{R \in \mathcal{R}} \mathcal{L}_i(R, \psi)(y) \leq -2\delta\epsilon \tag{4.4}$$

when $y \in [u - h, u + h]$ and also

$$V(y, i) \leq \psi(y, i) - 3\epsilon \tag{4.5}$$

when $y \in [0, u - h] \cup \{u + h\}$.

Since $\boldsymbol{\psi}$ is continuously differentiable we can find a positive constant C such that for all $i \in \mathbb{J}$

$$\sup_{R \in \mathcal{R}} \mathcal{L}_i(R, \psi)(y) \leq C \tag{4.6}$$

when $y \in [0, u + h]$. Since the value function $\mathbf{V}$ is continuous, we can find $\zeta > 0$ small enough such that for all $i \in \mathbb{J}$

$$\zeta < \left(\frac{\epsilon}{2C} \wedge \frac{1}{4\delta} \right) \tag{4.7}$$

and

$$V(y + c_{i_0}\zeta, i) \leq V(y, i) + \epsilon/2, \quad \psi(y + c_{i_0}\zeta, i) \geq \psi(y, i) - \epsilon/2 \tag{4.8}$$

for all $y \in [u - h - c_{i_0}\zeta, u - h]$.

Let us take any admissible strategy $\pi = (R^\pi(t), L^\pi(t))$, and define

$$\overline{\tau} = \inf\{t > 0 : U^\pi(t) \geq u + h\}, \quad \underline{\tau} = \inf\{t > 0 : U^\pi(t) \leq u - h\},$$

and $\hat{\tau} = \overline{\tau} \wedge (\underline{\tau} + \zeta)$. If ruin occurs between $[\underline{\tau}, \underline{\tau} + \zeta]$, we define $U^\pi(t) = 0$ for $t > \tau^\pi$.

It is obvious that $U^\pi(\overline{\tau}) = u + h$, $U^\pi(\underline{\tau}) \in [0, u - h + c_{i_0}\zeta]$ and $U^\pi(t) \in [0, u + h]$ for $t \in [0, \hat{\tau}]$. Thus, when $U^\pi(\hat{\tau}) \in [0, u - h] \cup \{u + h\}$, (4.5) yields

$$V(U^\pi(\hat{\tau}), J(\hat{\tau})) \leq \psi(U^\pi(\hat{\tau}), J(\hat{\tau})) - 2\epsilon. \tag{4.9}$$

When $U^\pi(\hat{\tau}) \in (u - h, u - h + c_{i_0}\zeta]$, we can write

$$U^\pi(\hat{\tau}) = y_0 + c_{i_0}\zeta,$$

where $y_0 \in (u - h - c_{i_0}\zeta, u - h]$. Then from (4.5) and (4.8), inequality (4.9) still holds.

From Lemma 3.1 we have

$$\begin{aligned}
&e^{-\delta\hat{\tau}}\psi\left(U^{\pi}(\hat{\tau}), J(\hat{\tau})\right) - \psi\left(U^{\pi}(0), J(0)\right)\\
&= \int_0^{\hat{\tau}} e^{-\delta s}\sum_{k\in\mathbb{J}} q_{J(s)k}\psi(U^{\pi}(s), k)ds + \int_0^{\hat{\tau}} e^{-\delta s}\sum_{k\in\mathbb{J}}\psi(U^{\pi}(s-), k)dm(s,k)\\
&\quad + \int_0^{\hat{\tau}} e^{-\delta s}\left[-\delta\psi(U^{\pi}(s), J(s)) + \psi'(U^{\pi}(s), J(s))c_{J(s)}(R^{\pi}(s))\right]ds\\
&\quad - \int_0^{\hat{\tau}} e^{-\delta s}\psi'(U^{\pi}(s), J(s))dL^{\pi c}(s) - \int_0^{\hat{\tau}}\int_0^{L^{\pi}(s+)-L^{\pi}(s)} e^{-\delta s}\psi'(U^{\pi}(s)-y, J(s))dy\,ds\\
&\quad + \int_0^{\hat{\tau}}\int_0^{\infty} e^{-\delta s}\left[\psi(U^{\pi}(s-)-x, J(s)) - \psi(U^{\pi}(s-), J(s))\right]N_{J(s)}(R^{\pi}(s), ds\times dx)
\end{aligned} \tag{4.10}$$

From (4.3) we get

$$\begin{aligned}
&-\int_0^{\hat{\tau}} e^{-\delta s}\psi'(U^{\pi}(s), J(s))dL^{\pi c}(s) - \int_0^{\hat{\tau}}\int_0^{L^{\pi}(s+)-L^{\pi}(s)} e^{-\delta s}\psi'(U^{\pi}(s)-y, J(s))dy\,ds\\
&\le -\int_0^{\hat{\tau}} e^{-\delta s}dL^{\pi c}(s) - \int_0^{\hat{\tau}}\int_0^{L^{\pi}(s+)-L^{\pi}(s)} e^{-\delta s}dy\,ds\\
&= -\int_0^{\hat{\tau}} e^{-\delta s}dL^{\pi}(s).
\end{aligned}$$

Thus,

$$\begin{aligned}
&e^{-\delta\hat{\tau}}\psi\left(U^{\pi}(\hat{\tau}), J(\hat{\tau})\right) - \psi\left(U^{\pi}(0), J(0)\right)\\
&\quad\le \int_0^{\hat{\tau}} e^{-\delta s}\mathscr{L}_{J(s)}(R^{\pi}(s), \psi)(U^{\pi}(s))ds + \int_0^{\hat{\tau}} e^{-\delta s}\sum_{k\in\mathbb{J}}\psi(U^{\pi}(s-), k)dm(s,k)\\
&\qquad - \int_0^{\hat{\tau}} e^{-\delta s}dL^{\pi}(s) + M(\hat{\tau}),
\end{aligned} \tag{4.11}$$

where

$$\begin{aligned}
M(t) &= \int_0^{t}\int_0^{\infty} e^{-\delta s}\left[\psi(U^{\pi}(s-)-x, J(s)) - \psi(U^{\pi}(s-), J(s))\right]N_{J(s)}(R^{\pi}(s), ds\times dx)\\
&\quad - \int_0^{t}\int_0^{\infty} e^{-\delta s}\left[\psi(U^{\pi}(s-)-x, J(s)) - \psi(U^{\pi}(s-), J(s))\right]\lambda_{J(s)}dF_{J(s)}(R^{\pi}(s), dx)ds
\end{aligned}$$

is a martingale.

Using the inequalities (4.4), (4.6), and (4.19), we get

$$\begin{aligned}
&\int_0^{\hat{\tau}} e^{-\delta s}\mathscr{L}_{J(s)}(R^{\pi}(s), \psi)(U^{\pi}(s))ds\\
&\quad= \int_0^{\overline{\tau}\wedge\underline{\tau}} e^{-\delta s}\mathscr{L}_{J(s)}(R^{\pi}(s), \psi)(U^{\pi}(s))ds + \int_{\overline{\tau}\wedge\underline{\tau}}^{\hat{\tau}} e^{-\delta s}\mathscr{L}_{J(s)}(R^{\pi}(s), \psi)(U^{\pi}(s))ds\\
&\quad\le -2\delta\epsilon\int_0^{\overline{\tau}\wedge\underline{\tau}} e^{-\delta s}ds + \frac{\epsilon}{2}.
\end{aligned} \tag{4.12}$$

Also from (4.19), we have

$$2\delta\epsilon\int_0^{\overline{\tau}\wedge\underline{\tau}} e^{-\delta s}ds = 2\delta\epsilon\int_0^{\hat{\tau}} e^{-\delta s}ds - 2\delta\epsilon\int_{\overline{\tau}\wedge\underline{\tau}}^{\hat{\tau}} e^{-\delta s}ds$$
$$\geq 2\delta\epsilon\int_0^{\hat{\tau}} e^{-\delta s}ds - \frac{\epsilon}{2}. \tag{4.13}$$

From (4.9), using (4.11), (4.12), and (4.13), it follows that

$$\begin{aligned} e^{-\delta\hat{\tau}}V\left(U^\pi(\hat{\tau}), J(\hat{\tau})\right) &\leq \left(\psi\left(U^\pi(0), J(0)\right) - e^{-\delta\hat{\tau}}2\epsilon\right) \\ &\quad + \left(e^{-\delta\hat{\tau}}\psi\left(U^\pi(\hat{\tau}), J(\hat{\tau})\right) - \psi\left(U^\pi(0), J(0)\right)\right) \\ &\leq \left(\psi\left(U^\pi(0), J(0)\right) - e^{-\delta\hat{\tau}}2\epsilon\right) - 2\delta\epsilon\int_0^{\hat{\tau}} e^{-\delta s}ds + \epsilon + M(\hat{\tau}) \\ &\quad + \int_0^{\hat{\tau}} e^{-\delta s}\sum_{k\in\mathbb{J}}\psi(U^\pi(s-), k)dm(s,k) - \int_0^{\hat{\tau}} e^{-\delta s}dL^\pi(s). \end{aligned} \tag{4.14}$$

For all $i \in \mathbb{J}$, since $\psi(y, i)$ is continuously differentiable, we have $\psi(y, i)$ is bounded for $y \in [0, u+h]$. Consequently,

$$\mathbb{E}_{u,i}\left[\int_0^{\hat{\tau}} e^{-\delta s}\sum_{k\in\mathbb{J}}\psi(U^\pi(s-), k)dm(s,k)\right] = 0.$$

Furthermore,

$$\mathbb{E}_{u,i}\left[1 - e^{-\delta\hat{\tau}}\right] = \delta\mathbb{E}_{u,i}\left[\int_0^{\hat{\tau}} e^{-\delta s}ds\right].$$

From (4.14) and the Dynamic Programming Principle (3.1), we have

$$\begin{aligned} V(u,i) &= \sup_{\pi\in\Pi}\mathbb{E}_{u,i}\left[\int_0^{\hat{\tau}} e^{-\delta s}dL^\pi(s) + e^{-\delta\hat{\tau}}V\left(U^\pi(\hat{\tau}), J(\hat{\tau})\right)\right] \\ &\leq \psi(u,i) - \epsilon \end{aligned}$$

and this contradicts the assumption that $\mathbf{V}(u) = \boldsymbol{\psi}(u)$. □

The following proposition gives us uniqueness of the viscosity solution with boundary condition $\mathbf{v}(0)$ among all the function $\mathbf{v}$ that satisfy the following conditions.

(C1)] For all $i \in \mathbb{J}$, $v(\cdot, i) : [0, \infty) \to \mathbb{R}$ is locally Lipschitz.
(C2)] For all $i \in \mathbb{J}$ and $0 \leq u_1 < u_2$, $v(u_2, i) - v(u_1, i) \geq u_2 - u_1$.
(C3)] For For all $i \in \mathbb{J}$ and $u_1 \geq 0$, there exists a constant $k > 0$ such that $v(u,i) \leq u + k$.

Proposition 4.1. *Assume*

$$\max_{i\in\mathbb{J}}\sum_{n\in\mathbb{J}}|q_{in}| < \delta; \tag{4.15}$$

and for all $u > 0$, $\underline{\mathbf{v}}(u)$ *and* $\overline{\mathbf{v}}(u)$ *are viscosity subsolution and supersolution of* (4.1), *respectively*; *and both satisfying conditions* (C1), (C2), *and* (C3). *If for all* $i \in \mathbb{J}$, $\underline{v}(0, i) \leq \overline{v}(0, i)$ *then for all* $i \in \mathbb{J}$, $\underline{v}(\cdot, i) \leq \overline{v}(\cdot, i)$ *in* $[0, \infty)$.

Proof. We consider $\overline{\mathbf{v}}_s(u) = s\overline{\mathbf{v}}(u)$ with $s > 1$. It is easy to verify that $\overline{\mathbf{v}}_s(u)$ is also a supersolution with $\forall i \in \mathbb{J}$, $\underline{v}(0, i) \leq \overline{v}_s(0, i)$. In fact, if $\boldsymbol{\phi}(u)$ is a continuously differentiable function such that for $i \in \mathbb{J}$ the minimum of $\overline{v}_s(u, i) - \phi(u, i)$ is attained at a point $u_0 > 0$, then

$$1 - \phi'(u_0, i) \leq 1 - s < 0 \quad \text{for all } i \in \mathbb{J}. \tag{4.16}$$

Arguing by contradiction, assume that

$$\mathcal{J} := \{i \in \mathbb{J} : \exists u_i \geq 0 \text{ such that } \underline{v}(u_i, i) > \overline{v}(u_i, i)\} \neq \emptyset.$$

For $i \in \mathcal{J}$, we can choose s_0 such that $\underline{v}(u_i, i) > \overline{v}_{s_0}(u_i, i)$. From conditions (C2) and (C3)we obtain for $i \in \mathbb{J}$

$$\underline{v}(u, i) - \overline{v}_{s_0}(u, i) \leq k + (1 - s_0)u.$$

Hence, we have

$$\underline{v}(u, i) - \overline{v}_{s_0}(u, i) \leq 0 \quad \text{for } u \geq b \text{ and } i \in \mathbb{J}, \tag{4.17}$$

where

$$b = \frac{k}{s_0 - 1} > 0.$$

For $i \in \mathbb{J}$ let

$$M_i = \sup_{u \geq 0} \left(\underline{v}(u, i) - \overline{v}_{s_0}(u, i)\right)$$

and

$$M = M_{i_0} := \max_{i \in \mathbb{J}} \{M_i\}.$$

Obviously, $i_0 \in \mathcal{J}$, and from (4.17) we have

$$0 < \underline{v}(u_{i_0}, i_0) - \overline{v}_{s_0}(u_{i_0}, i_0) \leq M = \sup_{u \in [0, b]} \left(\underline{v}(u, i_0) - \overline{v}_{s_0}(u, i_0)\right). \tag{4.18}$$

And let $u^* = \arg\sup_{u \in [0, b]} \left(\underline{v}(u, i_0) - \overline{v}_{s_0}(u, i_0)\right)$.

Since $\underline{v}(u, i_0)$ and $\overline{v}_{s_0}(u, i_0)$ satisfy (C1), there exist a constant $m > 0$ such that

$$\frac{\underline{v}(u_1, i_0) - \underline{v}(u_2, i_0)}{u_1 - u_2} \leq m, \quad \frac{\overline{v}_{s_0}(u_1, i_0) - \overline{v}_{s_0}(u_2, i_0)}{u_1 - u_2} \leq m \tag{4.19}$$

for $0 \leq u_2 \leq u_1 \leq b$.

Define the set

$$\mathscr{A} = \{(x, y) : 0 \leq x \leq y \leq b\}$$

and for all $a > 0$ the function

$$\Phi_a(x, y, i_0) = \underline{v}(x, i_0) - \overline{v}_{s_0}(y, i_0) - \frac{a}{2}(x - y)^2 - \frac{2m}{a^2(y - x) + a}. \tag{4.20}$$

Let $M_a = \max_{\mathscr{A}} \Phi_a(x, y, i_0)$ and $(x_a, y_a) = \arg\max_{\mathscr{A}} \Phi_a(x, y, i_0)$. Obviously, $y_a \geq x_a$ and

$$M_a \geq \Phi_a(u^*, u^*, i_0) = M - \frac{2m}{a},$$

and from (4.18) we get $M_a > 0$ for $a \geq 4m/M$ and

$$\liminf_{a \to \infty} M_a \geq M. \tag{4.21}$$

By the same argument of [4], we can show that there exist $a_0 > 0$ large enough such that for any $a \geq a_0$ we have that

$$(x_a, y_a) \notin \partial\mathscr{A}.$$

Here we omit the details of the proof.

Let

$$\psi(x, i_0) = \overline{v}_{s_0}(y_a, i_0) + \frac{a}{2}(x - y_a)^2 + \frac{2m}{a^2(y_a - x) + a}$$

and

$$\phi(y, i_0) = \underline{v}(x_a, i_0) - \frac{a}{2}(x_a - y)^2 - \frac{2m}{a^2(y - x_a) + a}.$$

It is obvious that $\psi'(x_a, i_0) = \phi'(y_a, i_0)$ and both of them are continuously differentiable. Furthermore, the maximum of $\underline{v}(x, i_0) - \psi(x, i_0)$ is attained at x_a and the minimum of $\overline{v}_{s_0}(y, i_0) - \phi(y, i_0)$ is attained at y_a. Therefore, we have the following inequalities

$$\max\left\{1 - \psi'(x_a, i_0), \sup_{R \in \mathscr{R}} \mathscr{L}_{i_0}(R, \underline{v}, \psi)(x_a)\right\} \geq 0 \tag{4.22}$$

and

$$\max\left\{1 - \phi'(y_a, i), \sup_{R \in \mathscr{R}} \mathscr{L}_{i_0}(R, \overline{v}_{s_0}, \phi)(y_a)\right\} \leq 0. \tag{4.23}$$

From (4.16) and $\psi'(x_a) = \psi'(y_a)$, we have from (4.22),

$$\sup_{R \in \mathscr{R}} \mathscr{L}_{i_0}(R, \underline{v}, \psi)(x_a) \geq 0. \tag{4.24}$$

Therefore, from (4.23) and (4.24) we get

$$\begin{aligned}\frac{\lambda_{i_0}+\delta}{\lambda_{i_0}}&\left[\underline{v}(x_a,i_0)-\overline{v}_{s_0}(y_a,i_0)\right]\\ &\leq \sup_{R\in\mathscr{R}}\left\{\int_0^{x_a}\underline{v}(x_a-x,i_0)dF_{i_0}(R,x)-\int_0^{y_a}\overline{v}_{s_0}(y_a-x,i_0)dF_{i_0}(R,x)\right\}\\ &\quad+\frac{1}{\lambda_{i_0}}\sum_{n\in\mathbb{J}}q_{i_0,n}(\underline{v}(x_a,n)-\overline{v}_{s_0}(y_a,n)).\end{aligned}\tag{4.25}$$

Using the inequality

$$\Phi_a(x_a,x_a,i_0)+\Phi_a(y_a,y_a,i_0)\leq 2\Phi_a(x_a,y_a,i_0),$$

we obtain that

$$a(x_a-y_a)^2\leq \underline{v}(x_a,i_0)-\underline{v}(y_a,i_0)+\overline{v}_{s_0}(x_a,i_0)-\overline{v}_{s_0}(y_a,i_0)+4m(y_a-x_a);$$

then from (4.19), we have

$$a(x_a-y_a)^2\leq 6m|x_a-y_a|.\tag{4.26}$$

We can find a sequence $a_n\to\infty$ such that $(x_{a_n},y_{a_n})\to(x^*,y^*)\in\mathscr{A}$. From (4.14), we get that

$$|\ x_{a_n}-y_{a_n}\ |\leq 6m/a_n$$

and this implies $x^*=y^*$. From (4.25) we have

$$\begin{aligned}\frac{\lambda_{i_0}+\delta}{\lambda_{i_0}}&\left[\underline{v}(x^*,i_0)-\overline{v}_{s_0}(x^*,i_0)\right]\\ &\leq \sup_{R\in\mathscr{R}}\left\{\int_0^{x^*}\underline{v}(x^*-x,i_0)dF_{i_0}(R,x)-\int_0^{x^*}\overline{v}_{s_0}(x^*-x,i_0)dF_{i_0}(R,x)\right\}\\ &\quad+\frac{1}{\lambda_{i_0}}\sum_{n\in\mathbb{J}}q_{i_0,n}(\underline{v}(x^*,n)-\overline{v}_{s_0}(x^*,n)).\end{aligned}\tag{4.27}$$

Since

$$\begin{aligned}\sum_{n\in\mathbb{J}}q_{i_0,n}(\underline{v}(x^*,n)-\overline{v}_{s_0}(x^*,n))&\leq\sum_{n\in\mathbb{J}}|\ q_{i_0,n}\ |\ (\underline{v}(x^*,n)-\overline{v}_{s_0}(x^*,n))\\ &\leq M\sum_{n\in\mathbb{J}}|\ q_{i_0,n}\ |,\end{aligned}$$

the inequality (4.27) yields

$$\left(\lambda_{i_0}+\delta\right)\left[\underline{v}(x^*,i_0)-\overline{v}_{s_0}(x^*,i_0)\right]\leq M\left(\lambda_{i_0}+\sum_{n\in\mathbb{J}}|q_{i_0,n}|\right).\tag{4.28}$$

From (4.26) we get that

$$\lim_{n\to\infty} a_n(x_{a_n} - y_{a_n})^2 = 0;$$

thus from (4.15), (4.20), (4.21), and (4.28), we obtain

$$\begin{aligned} M \le \liminf_{a\to\infty} M_a &\le \lim_{n\to\infty} M_{a_n} \\ &= \underline{v}(x^*, i_0) - \overline{v}_{s_0}(x^*, i_0) \\ &\le \frac{\lambda_{i_0} + \sum_{n\in\mathbb{J}} |q_{i_0,n}|}{\lambda_{i_0} + \delta} M < M \end{aligned}$$

which implies a contradiction. □

Note that the value function $\mathbf{V}$ satisfies the conditions (C1), (C2), and (C3). The following corollary is a direct consequence.

Corollary 4.1. *There is at most one viscosity solution of* (4.1) *with boundary condition* $\mathbf{V}(0) = \left(v_1^0, v_2^0, \ldots, v_N^0\right)^T$ *among all the functions that satisfy the conditions* (C1), (C2), *and* (C3).

5. Verification Theorem

In this section, we will prove a verification theorem; namely, if for some admissible strategy π, $\mathbf{V}_\pi$ is a supersolution, then it is the value function $\mathbf{V}$ and π is the optimal strategy. Using the same arguments of the proof of Lemma A.2 in [4], we can show the next lemma.

Lemma 5.1. *Let* $\overline{\mathbf{v}}$ *be an supersolution of* (4.1) *with absolution continuous elements satisfying the condition* (C3). *We can find a sequence of positive function* $\mathbf{v}_n$ *such that for each* $i \in \mathbb{J}$ *and* $y \in [0, \infty)$

(i) $v_n(y, i)$ *is continuously differentiable.*
(ii) $v_n(y, i)$ *satisfies the condition* (C3).
(iii) $1 \le v_n'(y, i) \le \frac{1}{c_i}\left[(\lambda_i + \delta)v_n(u, i) - \sum_{k\in\mathbb{J}} q_{ik} v_n(u, k)\right]$.
(iv) $v_n(y, i)$ *converges to* $\overline{v}(y, i)$ *uniformly on compact set and* $v_n'(y, i)$ *converges to* $\overline{v}'(y, i)$ *a.e.*

Theorem 5.1. *Let* $\overline{\mathbf{v}}(u)$ *be an supersolution of* (4.1) *with absolution continuous elements satisfying the condition* (C3), *then for each* $i \in \mathbb{J}$, $\overline{v}(u, i) \ge V(u, i)$.

Proof. Consider an admissible strategy $\pi = (R^\pi(t), L^\pi(t))$. Since the functions $v_n(y, i)$ defined in Lemma 5.1 are continuously differentiable, by Lemma 3.1 we have

$$\begin{aligned} &e^{-\delta(t\wedge\tau^\pi)} v_n\left(U^\pi(t\wedge\tau^\pi), J(t\wedge\tau^\pi)\right) - v_n\left(U^\pi(0), J(0)\right) \\ &\quad = \int_0^{t\wedge\tau^\pi} e^{-\delta s} \sum_{k\in\mathbb{J}} q_{J(s)k} v_n(U^\pi(s), k)\,ds + \int_0^{t\wedge\tau^\pi} e^{-\delta s} \sum_{k\in\mathbb{J}} v_n(U^\pi(s-), k)\,dm(s, k) \\ &\qquad + \int_0^{t\wedge\tau^\pi} e^{-\delta s}\left[-\delta v_n(U^\pi(s), J(s)) + v_n'(U^\pi(s), J(s)) c_{J(s)}(R^\pi(s))\right] ds \end{aligned}$$

$$-\int_0^{t\wedge\tau^\pi} e^{-\delta s} v_n'(U^\pi(s), J(s))dL^{\pi c}(s) - \int_0^{t\wedge\tau^\pi}\int_0^{L^\pi(s+)-L^\pi(s)} e^{-\delta s} v_n'(U^\pi(s)-y, J(s))dyds$$
$$+\int_0^{t\wedge\tau^\pi}\int_0^\infty e^{-\delta s}\left[v_n(U^\pi(s-)-x, J(s)) - v_n(U^\pi(s-), J(s))\right] N_{J(s)}(R^\pi(s), ds\times dx).$$

From Lemma 5.1, $v_n'(y,i)\geq 1$, similar to (4.11) we have

$$\begin{aligned}
&e^{-\delta(t\wedge\tau^\pi)} v_n\left(U^\pi(t\wedge\tau^\pi), J(t\wedge\tau^\pi)\right) - v_n\left(U^\pi(0), J(0)\right)\\
&\quad\leq \int_0^{t\wedge\tau^\pi} e^{-\delta s}\mathscr{L}_{J(s)}(R^\pi(s), v_n)(U^\pi(s))ds + \int_0^{t\wedge\tau^\pi} e^{-\delta s}\sum_{k\in\mathbb{J}} v_n(U^\pi(s-), k)dm(s,k)\\
&\qquad -\int_0^{t\wedge\tau^\pi} e^{-\delta s}dL^\pi(s) + M_n(t\wedge\tau^\pi),
\end{aligned}\tag{5.1}$$

where

$$\begin{aligned}
M_n(t) &= \int_0^t\int_0^\infty e^{-\delta s}\left[v_n(U^\pi(s-)-x, J(s)) - v_n(U^\pi(s-), J(s))\right] N_{J(s)}(R^\pi(s), ds\times dx)\\
&\quad -\int_0^t\int_0^\infty e^{-\delta s}\left[v_n(U^\pi(s-)-x, J(s)) - v_n(U^\pi(s-), J(s))\right]\lambda_{J(s)} dF_{J(s)}(R^\pi(s), dx)ds
\end{aligned}$$

is a martingale.

Using the Monotone Convergence Theorem, we get

$$\lim_{t\to\infty}\mathbb{E}_{u,i}\left[\int_0^{t\wedge\tau^\pi} e^{-\delta s}dL^\pi(s)\right] = \mathbb{E}_{u,i}\left[\int_0^{\tau^\pi} e^{-\delta s}dL^\pi(s)\right].\tag{5.2}$$

From Lemma 5.1(iii), we have

$$-(\lambda_{J(s)}+\delta)v_n(U^\pi(s), J(s)) \leq \mathscr{L}_{J(s)}(R^\pi(s), v_n)(U^\pi(s)) \leq \lambda_{J(s)} v_n(U^\pi(s), J(s)).$$

From Lemma 5.1(ii) and $U^\pi(s)\leq u + c_{i_0}s$, we get that

$$v_n(U^\pi(s), J(s)) \leq k_0 + U^\pi(s) \leq k_0 + u + c_{i_0}s\tag{5.3}$$

for some k_0. So, using the Bounded Convergence Theorem, we obtain

$$\begin{aligned}
&\lim_{t\to\infty}\mathbb{E}_{u,i}\left[\int_0^{t\wedge\tau^\pi} e^{-\delta s}\mathscr{L}_{J(s)}(R^\pi(s), v_n)(U^\pi(s))ds\right]\\
&\quad = \mathbb{E}_{u,i}\left[\int_0^{\tau^\pi} e^{-\delta s}\mathscr{L}_{J(s)}(R^\pi(s), v_n)(U^\pi(s))ds\right].
\end{aligned}\tag{5.4}$$

From (5.1), (5.2), and (5.4), we get

$$\begin{aligned}
&\lim_{t\to\infty}\mathbb{E}_{u,i}\left[e^{-\delta(t\wedge\tau^\pi)} v_n\left(U^\pi(t\wedge\tau^\pi), J(t\wedge\tau^\pi)\right)\right] - v_n(u,i)\\
&\quad\leq \mathbb{E}_{u,i}\left[\int_0^{\tau^\pi} e^{-\delta s}\mathscr{L}_{J(s)}(R^\pi(s), v_n)(U^\pi(s))ds\right] - V_\pi(u,i).
\end{aligned}\tag{5.5}$$

By Lemma 5.1(iv) and Dominated Convergence Theorem, we get

$$\lim_{n\to\infty} \mathbb{E}_{u,i}\left[\int_0^{\tau^\pi} e^{-\delta s}\mathcal{L}_{J(s)}(R^\pi(s), v_n)(U^\pi(s))ds\right]$$
$$= \mathbb{E}_{u,i}\left[\int_0^{\tau^\pi} e^{-\delta s}\mathcal{L}_{J(s)}(R^\pi(s), \overline{v})(U^\pi(s))ds\right] \le 0. \tag{5.6}$$

Since

$$\lim_{n\to\infty}\lim_{t\to\infty} \mathbb{E}_{u,i}\left[e^{-\delta(t\wedge\tau^\pi)} v_n\left(U^\pi(t\wedge\tau^\pi), J(t\wedge\tau^\pi)\right)\right] \ge 0,$$

from (5.5) and (5.6), we have

$$\overline{v}(u,i) = \lim_{n\to\infty} v_n(u,i) \ge V_\pi(u,i)$$

and the result holds. □

The following characterization of value function $\mathbf{V}$ is directly from Corollary 4.1 and Theorem 5.1.

Corollary 5.1. *The value function* $\mathbf{V}$ *can be characterized as the unique viscosity solution of* (4.1) *satisfying conditions* (C1), (C2), *and* (C3); *and for each* $i \in \mathbb{J}$, *the boundary condition*

$$V(0,i) = \inf\{v(0,i) : \mathbf{v} \text{ is viscosity solution of } (4.1) \text{ satisfying condition } (C3)\}$$

holds.

Theorem 5.2. *Let* π *be an admissible strategy such that* $\mathbf{V}_\pi$ *is an absolutely continuous supersolution of* (4.1), *then* $\mathbf{V} = \mathbf{V}_\pi$.

Proof. Since for all $i \in \mathbb{J}$ and $u \ge 0$, $V_\pi(u,i) \le V(u,i)$, and $\mathbf{V}$ satisfies the condition (C3), then $\mathbf{V}_\pi$ also satisfies the condition (C3) and the result follows from Theorem 5.1. □

Up to this point we do not know if there exists an admissible strategy π which satisfies the conditions of Theorem 5.2. In the next section, we are going to present an example in which such a strategy exists.

6. Case Study

In this section we consider the special case with $\mathbb{J} = \{1, 2\}$, no reinsurance and exponential claim density, that is, $f_i(x) = \gamma_i e^{-\gamma_i x}$, $i = 1, 2$. We consider the following strategy π which is called modulated barrier strategy in [15]. For two constants $b_i > 0$, $i = 1, 2$, when the state is i and the surplus $u < b_i$, pay out no dividend; when the state is i and the surplus $u \geq b_i$, pay out everything above b_i as dividends. By the standard method (see [12]), for $i = 1, 2$, we have

$$\begin{cases} 0 = c_i V_\pi'(u, i) + \lambda_i \gamma_i \int_0^u V_\pi(u-y, i) e^{-\gamma_i y} dy \\ \quad - (\lambda_i + q_i + \delta) V_\pi(u, i) + q_i V_\pi(u, 3-i), & \text{if } 0 \leq u < b_i, \\ V_\pi(u, i) = u - b_i + V_\pi(b_i, i), & \text{if } u \geq b_i, \end{cases} \tag{6.1}$$

where

$$\begin{aligned} V_\pi(b_i, i) = & \frac{c_i}{\lambda_i + q_i + \delta} + \frac{\lambda_i}{\lambda_i + q_i + \delta} \gamma_i \int_0^{b_i} V_\pi(u-y, i) e^{-\gamma_i y} dy \\ & + \frac{q_i}{\lambda_i + q_i + \delta} V_\pi(b_i, 3-i). \end{aligned} \tag{6.2}$$

In this section, we consider $b_1 \leq b_2$; the case $b_1 > b_2$ may be treated similarly. Here we have to consider three cases: $u \in [0, b_1)$, $u \in [b_1, b_2)$ and $u \in [b_2, \infty)$.

If $u \in [0, b_1)$, system (6.1) gives the following system of integro-differential equations:

$$\begin{cases} -c_1 V_\pi'(u, 1) = \lambda_1 \gamma_1 e^{-\gamma_1 u} \int_0^u V_\pi(y, 1) e^{\gamma_1 y} dy \\ \qquad - (\lambda_1 + q_1 + \delta) V_\pi(u, 1) + q_1 V_\pi(u, 2), \\ -c_2 V_\pi'(u, 2) = \lambda_2 \gamma_2 e^{-\gamma_2 u} \int_0^u V_\pi(y, 2) e^{\gamma_2 y} dy \\ \qquad - (\lambda_2 + q_2 + \delta) V_\pi(u, 2) + q_2 V_\pi(u, 1). \end{cases} \tag{6.3}$$

The right-hand sides of the both equations are differentiable. Therefore,

$$\begin{cases} c_1 V_\pi''(u, 1) = \lambda_1 \gamma_1^2 e^{-\gamma_1 u} \int_0^u V_\pi(y, 1) e^{\gamma_1 y} dy - \lambda_1 \gamma_1 V_\pi(u, 1) \\ \qquad + (\lambda_1 + q_1 + \delta) V_\pi'(u, 1) - q_1 V_\pi'(u, 2), \\ c_2 V_\pi''(u, 2) = \lambda_2 \gamma_2^2 e^{-\gamma_2 u} \int_0^u V_\pi(y, 2) e^{\gamma_2 y} dy - \lambda_2 \gamma_2 V_\pi(u, 2) \\ \qquad + (\lambda_2 + q_2 + \delta) V_\pi'(u, 1) - q_2 V_\pi'(u, 2). \end{cases}$$

Plugging in the original equations yields

$$\begin{cases} 0 = c_1 V_\pi''(u, 1) - (\lambda_1 + q_1 + \delta - c_1 \gamma_1) V_\pi'(u, 1) \\ \quad - \gamma_1 (q_1 + \delta) V_\pi(u, 1) + \gamma_1 q_1 V_\pi(u, 2) + q_1 V_\pi'(u, 2), \\ 0 = c_2 V_\pi''(u, 2) - (\lambda_2 + q_2 + \delta - c_2 \gamma_2) V_\pi'(u, 2) \\ \quad - \gamma_2 (q_2 + \delta) V_\pi(u, 2) + \gamma_2 q_2 V_\pi(u, 1) + q_2 V_\pi'(u, 1). \end{cases} \tag{6.4}$$

We conjecture that the solution is given by

$$\begin{cases} V_\pi(u, 1) = A_{11} e^{\alpha_1 (u - b_1)} + A_{12} e^{\alpha_2 (u - b_1)} + A_{13} e^{\alpha_3 (u - b_1)} + A_{14} e^{\alpha_4 (u - b_1)}, \\ V_\pi(u, 2) = A_{21} e^{\alpha_1 (u - b_1)} + A_{22} e^{\alpha_2 (u - b_1)} + A_{23} e^{\alpha_3 (u - b_1)} + A_{24} e^{\alpha_4 (u - b_1)}. \end{cases} \tag{6.5}$$

By plugging (6.5) in (6.4) and comparing the coefficients, we know that if we can show the system of equations

$$\begin{cases} c_1\alpha^2 - (\lambda_1 + q_1 + \delta - c_1\gamma_1)\alpha - \gamma_1(q_1 + \delta) + \gamma_1 q_1\beta + q_1\beta\alpha = 0, \\ c_2\alpha^2 - (\lambda_2 + q_2 + \delta - c_2\gamma_2)\alpha - \gamma_2(q_2 + \delta) + \gamma_2 q_2/\beta + q_2\alpha/\beta = 0 \end{cases} \tag{6.6}$$

has four roots (α, β), then we can determine $(\alpha_j, A_{2j}/A_{1j})$, $j = 1, 2, 3, 4$.

Lemma 6.1. *The system of equations* (6.6) *has four real roots* (α_j, β_j), $j = 1, 2, 3, 4$, *and* $\alpha_1 < \alpha_2 < 0 < \alpha_3 < \alpha_4$.

Proof. Let

$$g_i(\alpha) = c_i\alpha^2 - (\lambda_i + q_i + \delta - c_i\gamma_i)\alpha - \gamma_i(q_i + \delta), \quad i = 1, 2$$

and

$$G(\alpha) = g_1(\alpha)g_2(\alpha) - q_1q_2(\gamma_1 + \alpha)(\gamma_2 + \alpha).$$

Obviously, for $i = 1, 2$, $g_i(\alpha) = 0$ has two roots θ_{i1} and θ_{i2}, such that $\theta_{i1} < 0 < \theta_{i2}$. Since $G(\infty) = G(-\infty) = \infty$ and $G(0) > 0$, the proof is then completed by showing that there are θ_1 and θ_2, such that $\theta_1 < 0 < \theta_2$ and $G(\theta_i) < 0$, $i = 1, 2$. First, we have $\theta_{12} > 0$ and $G(\theta_{12}) < 0$. Second, for $i = 1, 2$, $g_i(-\gamma_i) > 0$, then $-\gamma_i < \theta_{i1} < 0$. Thus, we have $\theta_1 = \max\{\theta_{11}, \theta_{21}\}$. In fact, $G(\theta_1) = -q_1q_2(\gamma_1 + \theta_1)(\gamma_2 + \theta_1) < 0$. □

Hence, we identify $(\alpha_j, A_{2j}/A_{1j})$, $j = 1, 2, 3, 4$ as the four roots of (6.6). Hence for $j = 1, 2, 3, 4$, we have

$$A_{2j} = -\frac{g_1(\alpha_j)}{\gamma_1 q_1 + q_1\alpha_j}A_{1j} = -\frac{\gamma_2 q_2 + q_2\alpha_j}{g_2(\alpha_j)}A_{1j}. \tag{6.7}$$

Moreover, plugging (6.5) in (6.3) yields

$$\begin{cases} \frac{A_{11}}{\alpha_1 + \gamma_1}e^{-\alpha_1 b_1} + \frac{A_{12}}{\alpha_2 + \gamma_1}e^{-\alpha_2 b_1} + \frac{A_{13}}{\alpha_3 + \gamma_1}e^{-\alpha_3 b_1} + \frac{A_{14}}{\alpha_4 + \gamma_1}e^{-\alpha_4 b_1} = 0 \\ \frac{A_{21}}{\alpha_1 + \gamma_2}e^{-\alpha_1 b_1} + \frac{A_{22}}{\alpha_2 + \gamma_2}e^{-\alpha_2 b_1} + \frac{A_{23}}{\alpha_3 + \gamma_2}e^{-\alpha_3 b_1} + \frac{A_{24}}{\alpha_4 + \gamma_2}e^{-\alpha_4 b_1} = 0 \end{cases} \tag{6.8}$$

If $u \in [b_1, b_2)$, system (6.1) gives the following system of integro-differential equations:

$$\begin{aligned} V_\pi(u, 1) &= u - b_1 + V_\pi(b_1, 1), \\ -c_2V_\pi'(u, 2) &= \lambda_2\gamma_2 e^{-\gamma_2 u}\int_0^u V_\pi(y, 2)e^{\gamma_2 y}dy - (\lambda_2 + q_2 + \delta)V_\pi(u, 2) + q_2V_\pi(u, 1). \end{aligned} \tag{6.9}$$

$V_\pi(b_1, 1)$ can be calculated by (6.5) and similar to (6.4), we have

$$\begin{aligned}0 = c_2 V_\pi''(u,2) - (\lambda_2 + q_2 + \delta - c_2\gamma_2)V_\pi'(u,2) - \gamma_2(q_2+\delta)V_\pi(u,2) \\ + \gamma_2 q_2(u - b_1 + V_\pi(b_1,1)) + q_2.\end{aligned} \tag{6.10}$$

We conjecture that the solution of $V_\pi(u, 2)$ is given by

$$V_\pi(u,2) = B_1 e^{\tilde{\alpha}_1(u-b_2)} + B_2 e^{\tilde{\alpha}_2(u-b_2)} + \eta_1 u + \eta_2. \tag{6.11}$$

Plugging (6.11) in (6.10) and comparing the coefficients show that $\tilde{\alpha}_i$, $i = 1, 2$ are the root of the equation

$$c_2\tilde{\alpha}^2 - (\lambda_2 + q_2 + \delta - c_2\gamma_2)\tilde{\alpha} - \gamma_2(q_2+\delta) = 0, \tag{6.12}$$

and

$$\begin{cases} -(q_2+\delta)\eta_1 + \gamma_2 q_2 = 0, \\ -(\lambda_2 + q_2 + \delta - c_2\gamma_2)\eta_1 - \gamma_2(q_2+\delta)\eta_2 + \gamma_2 q_2[-b_1 + V_\pi(b_1,1)] + q_2 = 0 \end{cases} \tag{6.13}$$

Moreover, plugging the second equation of (6.5) and (6.11) into (6.9) yields

$$\sum_{i=1}^{4} \frac{A_{2i}}{\alpha_i + \gamma_2}[1 - e^{(\alpha_i - \gamma_2)b_1}] - \sum_{i=1}^{2} \frac{B_i}{\tilde{\alpha}_i + \gamma_2} e^{\tilde{\alpha}_i(b_1 - b_2)} - \frac{\eta_1 b_1 + \eta_2 - 1}{\gamma_2} = 0 \tag{6.14}$$

Finally, if $u \in [b_2, \infty)$, system (6.1) gives the following system of integro-differential equations:

$$\begin{cases} V_\pi(u,1) = u - b_1 + V_\pi(b_1,1), \\ V_\pi(u,2) = u - b_2 + V_\pi(b_2,2). \end{cases} \tag{6.15}$$

Here, we want to find a continuous version of $V_\pi(u, i)$. Thus, by letting $u \to b_i$ in (6.1) and noting that $V_\pi''(u, 2)$ exists at b_1, we have the following smooth-fit condition:

$$\begin{cases} V_\pi(b_1-,2) = V_\pi(b_1+,2), \\ V_\pi'(b_1-,i) = V_\pi'(b_1+,i), \quad \text{for } i = 1,2, \\ V_\pi'(b_2-,2) = 1, \\ V_\pi''(b_1-,2) = V_\pi''(b_1+,2). \end{cases} \tag{6.16}$$

The solution of the system of Equations (6.7), (6.8), (6.13), (6.14), and (6.16) will give us the values for b_1 and b_2, and also the values for A_{ij}, B_i, η_i, $i = 1, 2$, and $j = 1, 2, 3, 4$.

We can verify that $V_\pi(u, i)$ is a supersolution of (4.1) by the same method as in Section 4 in [25]. Although we get a modulated-barrier strategy in this special case, a modulated-barrier strategy maybe not the optimal strategy in general cases. Azcue and Muler [4] considers only one regime shows that the optimal strategy is a band strategy in general cases.

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