A Rate of Convergence for Monotone Finite Difference Approximations to Fully Nonlinear, Uniformly Elliptic PDEs

LUIS A. CAFFARELLI
The University of Texas at Austin

AND

PANAGIOTIS E. SOUGANIDIS
The University of Texas at Austin

Abstract

We obtain an algebraic rate of convergence for monotone and consistent finite difference approximations to Lipschitz-continuous viscosity solutions of uniformly elliptic partial differential equations. © 2007 Wiley Periodicals, Inc.

0 Introduction

We obtain an algebraic rate of convergence for monotone and consistent finite difference approximations to Lipschitz-continuous viscosity solutions of fully nonlinear, uniformly elliptic partial differential equations of the form

$$F(D^2u) = f \quad \text{in } U \tag{0.1}$$

with boundary condition

$$u = g \quad \text{on } \partial U, \tag{0.2}$$

where

(0.3) U is an open subset of $\mathbb{R}^n$ with regular boundary,

(0.4) F is uniformly elliptic with ellipticity constants λ and Λ such that $\lambda \leq \Lambda$,

(0.5) $f \in C^{0,1}(\bar{U})$, and

(0.6) $g \in C^{1,\eta'}(\partial U)$ for some $\eta' \in (0, 1]$.

In a forthcoming paper, we obtain algebraic error estimates for monotone and consistent approximations to equations that also depend on the gradient. The arguments are more complicated and require some additional ideas.

A key step in our analysis is a regularity result for Lipschitz-continuous viscosity solutions of (0.1). Roughly speaking, it asserts that, outside sets of arbitrarily small measure, Lipschitz-continuous solutions of (0.1) have pointwise second-order expansions with an error that is controlled by the size of the exceptional set

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and the quadratic expansion or that, at a given scale, solutions have uniform polynomial approximations. This result is of independent interest and can be used in other contexts. For example, it is used in [7] to establish a rate of convergence for the homogenization of fully nonlinear, uniformly elliptic, second-order PDEs in random environments.

The regularity result is:

THEOREM A *Assume* (0.4) *and* (0.5) *and, for* $r \in (0,1)$*, let* u *be a Lipschitz-continuous solution of* (0.1) *in* $B(\hat{x}, 2r)$*. There exist positive constants* σ*,* t_0*, and* C *depending only on* λ*,* Λ*, and* n *such that, for all* $t > t_0$*, there exists an open set* $A_t \subset B(\hat{x}, 2r)$ *such that*

$$|(B(\hat{x},2r) \setminus A_t) \cap B(\hat{x}, r)| \leqq Cr^{n-1}(\|Du\|_\infty + \|Df\|_{L^n(B(\hat{x},2r))})t^{-\sigma},$$

and, for all $x_0 \in A_t \cap B(\hat{x}, r)$*, there exists a quadratic polynomial* $P^t_{x_0}$ *of opening* t *such that*

$$F(D^2 P^t_{x_0}) = f(x_0),$$

and, for all $x \in B(\hat{x}, 2r)$*,*

$$|u(x) - u(x_0) - P^t_{x_0}(x - x_0)| \leqq Cr^{-1}t|x - x_0|^3 \ .$$

The monotone approximation schemes we are considering here can be written (following the notation of Barles and Souganidis [3] and Kuo and Trudinger [22]) in the form

$$\begin{cases} F_h[u_h] = f & \text{in } U_h, \\ u_h = g & \text{on } \partial U_h, \end{cases} \tag{0.7}$$

where h is the mesh size and F_h and U_h are the approximate nonlinearity and domain, respectively.

Note that the notation here has been considerably simplified to facilitate the statement of the error estimate. The details are given in Section 2. Moreover, throughout the paper, we use C to denote generic constants that may change from line to line. Finally, we say that a constant is *universal* if it depends only on λ, Λ, and n.

The key assumptions are that the schemes are *monotone*, which means that

$$\text{if } u_h^1 \text{ and } u_h^2 \text{ solve (0.7) and } u_h^1 \leqq u_h^2 \text{ on } \partial U_h, \text{ then } u_h^1 \leqq u_h^2 \text{ in } U_h, \tag{0.8}$$

and *consistent with an error estimate*; i.e., there exists $C > 0$ such that, for any smooth ϕ,

$$|F(D^2\phi) - F_h(\phi_h)| \leqq C_3(1 + \|D^3\phi\|h). \tag{0.9}$$

The main result is:

THEOREM B *Assume* (0.3), (0.4), (0.5), *and* (0.6)*. Let* $u \in C^{0,1}(\bar{U})$ *be the solution* (0.1), (0.2) *and* u_h *the solution of* (0.7) *that is monotone and consistent with an* *error estimate given by* (0.9). *There exist positive constants C and α_* depending on λ, Λ, n, and $\|u\|_{C^{0,1}(\bar{U})}$ such that, for all $\alpha \in (0, \alpha_*)$,*

$$\sup_{U_h} |u - u_h| \leqq C h^{\alpha} \ .$$

We remark that the Lipschitz continuity of u in $\bar{U}$ follows from (0.6). If one assumes only $g \in C^{0,1}(\partial U)$, then, in general, $u \in C^{0,1}_{\mathrm{loc}}(U)$. In this paper to keep things simple we will always assume that we are given $u \in C^{0,1}(\bar{U})$.

The convergence of and error estimates for monotone and consistent approximations to fully nonlinear, first-order PDEs were established a while ago by Crandall and Lions [10] and Souganidis [24]. The convergence, without error, of monotone and consistent approximations for fully nonlinear, possibly degenerate second-order PDEs was first proved in Barles and Souganidis [3]. In a series of papers Kuo and Trudinger [21, 22, 23] also looked in great detail at the issues of regularity and existence of such approximations for uniformly elliptic equations. Finding a rate of convergence has been, however, a longstanding open problem. The main difficulty has been the lack of appropriate regularizations of viscosity solutions yielding control on derivatives higher than 2 except for either convex or concave F. Notice that the classical inf- and sup-convolution approximations, which are widely used in the theory of viscosity solutions, only give (one-sided) control on second derivations.

The first result in the convex/concave case, with $h = \frac{1}{27}$, was obtained by Krylov [18, 19] using the stochastic control interpretation of the equation that is available in view of the convexity/concavity of F. Later Barles and Jakobsen [1, 2] improved the error to $h = \frac{1}{5}$ by purely PDE techniques using switching-control-type approximations—once again the convexity/concavity plays a crucial role. More recently, Krylov [20], always in the convex/concave but degenerate case, improved the rate to $h = \frac{1}{2}$ again using stochastic control considerations. Some other related papers that deal with particular/restricted versions of the problem (special equations and/or dimensions) are due to Jakobsen [12, 13] and Bonnans, Maroso, and Zidani [4]. The rate established in Theorem B is the first known for fully nonlinear, uniformly elliptic equations without any concavity/convexity assumptions. The exponent σ in Theorem A is not optimal. As a result, the rate of convergence obtained here, which depends strongly on σ, is not optimal either.

The paper is organized as follows. In Section 1 we prove Theorem A. We also revisit the classical inf- and sup-convolution approximations of viscosity solutions and prove some additional results about them. In Section 2 we introduce the approximations, discuss all the assumptions, and recall some of their important properties. Theorem B is proved in Section 3.

1 The Regularity Result and Inf- and Sup-Convolution Approximations

The regularity of viscosity solutions of fully uniformly elliptic PDEs has attracted a lot of attention. In the case that F is convex/concave, it has been shown by Evans [11] and Krylov [16, 17] that the solutions are actually smooth ($C^{2,\alpha}$). In the general case, the best-known regularity is $C^{1,\eta}$ for some $\eta \in (0,1)$ (see Trudinger [25], Wang [26], and Cabre and Caffarelli [6]). Recently Nadirashvili presented a particular example of a solution that is $C^{1,1}$ but not C^2. A different kind of regularity result of maximal type, which controls the size of the set where solutions are differentiable with some estimate on the size of the gradient, was proven in [5] and is also presented in [6]. This result is the basis of (the proof of) Theorem A, which is presented next.

PROOF OF THEOREM A: We prove the claim first in $B_1 = B(0,1)$.

The special form of the equation (separated dependence on D^2u and x) yields that, for each $i = 1, \dots, n$, u_{x_i} solves, at least formally, the linear uniformly elliptic equation

$$\operatorname{tr} D_X F D^2 u_{x_i} = f_{x_i} \quad \text{in } B_1,$$

which satisfies the assumptions of lemma 7.6 of [6].

It follows that there exist positive constants σ, t_0, and C, depending only on λ, Λ, and n, such that, for all $t > t_0$, there exists an open set $A_t \subset B_1$ such that

$$|(B_1 \setminus A_t) \cap B_{1/2}| \leqq C(\|Du\|_\infty + \|D_x f\|_{L^n(B_1)})t^{-\sigma},$$

and, for each $x_0 \in A_t \cap B_{1/2}$, there exist quadratic polynomials $\underline{P}^{i,t}_{x_0}$ and $\overline{P}^{i,t}_{x_0}$ with opening t such that

$$u_{x_i}(x_0) = \underline{P}^{i,t}_{x_0}(x_0) = \overline{P}^{i,t}_{x_0}(x_0) \quad \text{and} \quad \underline{P}^{i,t}_{x_0} \leqq u_{x_i} \leqq \overline{P}^{i,t}_{x_0} \quad \text{in } B_1.$$

It is immediate that, for each $i = 1, \dots, n$, u_{x_i} is differentiable at every $x_0 \in A_t \cap B_{1/2}$,

$$Du_{x_i}(x_0) = D\underline{P}^{i,t}_{x_0}(x_0) = D\overline{P}^{i,t}_{x_0}(x_0),$$

and, for a universal constant C,

$$|u_{x_i}(x) - u_{x_i}(x_0) - Du_{x_i}(x_0)\cdot(x - x_0)| \leqq Ct|x - x_0|^2.$$

Integrating the above inequalities yields the claim. The coefficients of $P^t_{x_0}$ are provided by the tangent planes to Du at x_0. The fact that $F(D^2 P^t_{x_0}) = f(x_0)$ is an immediate consequence of the definition of viscosity solutions.

The claim for general balls follows by a simple scaling argument. Since the assertion is translation invariant, it is enough to work with the balls $B_{2r} = B(0, 2r)$

and $B_r = B(0,r)$. The result now follows by applying the claim above to the function $\hat{u} \in C^{0,1}(B_1)$ given by

$$\hat{u}(x) = (2r)^{-2} u(2rx),$$

using the special form of the equation (independent of Du), and the facts that

$$r \in (0,1), \quad \|DU\| = (2r)^{-1}\|Du\|, \quad \text{and} \quad \|Df(r\cdot)\|_{L^n(B_1)} = \|Df\|_{L^n(B_{2r})}.$$

□

Next we recall the well-known sup- and inf-convolution regularizations of $u \in C^{0,1}(\bar{U})$, given, respectively, for $\delta > 0$ by

$$\begin{aligned} u_\delta^+(x) &= \sup_{\bar{U}} \left[u(y) - \frac{1}{2\delta}|x-y|^2 \right], \\ u_\delta^-(x) &= \inf_{\bar{U}} \left[u(y) + \frac{1}{2\delta}|x-y|^2 \right]. \end{aligned} \tag{1.1}$$

Let

$$U^\delta = \{x \in U : d(x, \partial U) \geqq \delta \|Du\|\},$$

and, for $f \in C^{0,1}(\bar{U})$ and $x \in \overline{U}^\delta$, define

$$f_\delta^-(x) = \sup_{|y-x| \leqq \delta\|Du\|} f(y) \quad \text{and} \quad f_\delta^+(x) = \inf_{|y-x| \leqq \delta\|Du\|} f(y).$$

The following technical proposition summarizing the essential properties of $u_\delta^\pm$ is classical in the theory of viscosity solutions. We refer to Jensen, Lions, and Souganidis [15] and Crandall, Ishii, and Lions [6, 9] for its proof.

PROPOSITION 1.1 *Assume $u \in C^{0,1}(\bar{U})$. Then*

(i) *$u_\delta^\pm \in C^{0,1}(\bar{U}^\delta)$ and, as $\delta \to 0$ and uniformly, $u_\delta^+ \downarrow u$ and $u_\delta^- \uparrow u$.*

(ii) *For all $u \in C(\bar{U}^\delta)$, there exist concave (respectively, convex) paraboloids of opening δ^{-1} that touch u_δ^+ (respectively, u_δ^-) from below (respectively, above), and, in the sense of distributions, $D^2 u_\delta^+ \geqq -\delta^{-1} I$ and $D^2 u_\delta^- \leqq \delta^{-1} I$. Moreover, $u_\delta^\pm$ is twice differentiable a.e. in U^δ.*

(iii) *If u is a viscosity solution of (0.1), then u_δ^+ (respectively, u_δ^-) is a subsolution (respectively, supersolution) of*

$$F(D^2 w) = f_\delta^+ \quad (\textit{respectively, } F(D^2 w) = f_\delta^-) \quad \textit{in } U^\delta.$$

Proposition 1.1 is very general and the last property holds for all degenerate second-order equations. The first and third claims and first part of the second claim are a direct consequence of the definition, while the second part of claim (ii) (the differentiability properties) requires some real analysis. Uniform ellipticity is not necessary for claim (iii).

It turns out, however, that, when the equation is uniformly elliptic, the inf- and sup-convolutions enjoy more regularity. In particular, the regularity claimed in Theorem A carries over to $u_\delta^\pm$.

We have:

PROPOSITION 1.2 *Assume* (0.4) *and let* $u \in C^{0,1}(\bar{U})$ *be a solution of* (0.1) *with* $U = B(\hat{x}, 2r)$. *Then*

(i) *For each* $x \in B^{\delta}_{2r}(\hat{x}) = B(\hat{x}, 2r - \delta\|Du\|)$, *let* $y^{+}_{\delta}(x)$ (*respectively,* $y^{-}_{\delta}(x)$) $\in B(\hat{x}, 2r)$ *be a point where the maximum* (*respectively, minimum*) *is achieved in the definition of* u^{+}_{δ} (*respectively,* u^{-}_{δ}). *There exists* $C > 0$, *depending on* λ, Λ, *and* n *but not* δ, *such that, for all* $x_1, x_2 \in B^{\delta}_{2r}$,

$$|x_1 - x_2| \leqq C|y^{\pm}_{\delta}(x_1) - y^{\pm}_{\delta}(x_2)|.$$

(ii) *There exists positive constants* t_0 *and* σ, *depending only on* λ, Λ, *and* n *and not* δ, *such that, for every* $t > t_0$, *there exist open sets* $A^{\delta,\pm}_{t} \subset B^{\delta}_{2r}(\hat{x})$ *such that*

$$|(B^{\delta}_{2r}(\hat{x}) \setminus A^{\delta,\pm}_{t}) \cap B^{\delta/2}_{r}(\hat{x})| \leqq Cr^{n-1}(\|Du\|_{\infty} + \|Df\|_{L^n(B(\hat{x},2r))})t^{-\sigma},$$

and, for every $x_0 \in A^{\delta,\pm}_{t} \cap B^{\delta/2}_{r}(\hat{x})$, *there exist paraboloids* $P^{t,\delta,\pm}_{x_0}$ *of opening* t *such that, for* $x \in B^{\delta}_{2r}(\hat{x})$,

$$u^{+}_{\delta}(x) \leqq u^{+}_{\delta}(x_0) + P^{t,\delta,+}_{x_0}(x - x_0) + Cr^{-1}t|x - x_0|^3$$

and

$$u^{-}_{\delta}(x) \geqq u^{-}_{\delta}(x_0) + P^{+,\delta,-}_{x_0}(x - x_0) - Cr^{-1}t|x - x_0|^3.$$

(iii) *Let* P *be a paraboloid that touches* u^{+}_{δ} (*respectively,* u^{-}_{δ}) *from above* (*respectively, below*) *at* $x_0 \in B^{\delta}_{2r}(\hat{x})$. *Then* u *is touched at* $y^{+}_{\delta}(x_0)$ (*respectively,* $y^{-}_{\delta}(x_0)$) *from above* (*respectively, below*) *by a paraboloid* $P^{+,\delta}$ (*respectively,* $P^{-,\delta}_{x_0}$) *and, for a uniform constant* $C > 0$, *at* x_0

$$D^2u^{+}_{\delta}(x) \geqq D^2u + C\delta^2|D^2u|^2 \quad \text{and} \quad D^2u^{-}_{\delta}(x) \leqq D^2u + C\delta^2|D^2u|^2.$$

PROOF: We present the arguments only for u^{+}_{δ}. The proof for u^{-}_{δ} follows the same exact lines. Moreover, to simplify the notation we write B^{δ}_{2r} instead of $B^{\delta}_{2r}(\hat{x})$.

At any point where the maximum is achieved in the definition of u^{+}_{δ}, u is obviously touched from above by a paraboloid of opening δ^{-1}. Since u solves a uniformly elliptic equation with bounded right-hand side, the (Krylov-Safanov) Harnack inequality yields that, at such points, u is also touched from below by a paraboloid of opening $C\delta^{-1}$ with $C > 0$ depending only on Λ, λ, and n.

In particular, for every $x \in B^{\delta}_{2r}$, u is differentiable at $y^{+}_{\delta}(x)$ and

$$x = y^{+}_{\delta}(x) - \delta Du(y^{+}_{\delta}(x)).$$

Moreover, at $y^{+}_{\delta}(x)$, u has a C^1 contact from above and below with, respectively, the convex and concave envelopes of paraboloids with opening $C\delta^{-1}$. In view of this observation, for all $x_1, x_2 \in B^{\delta}_{2r}$,

$$|Du(y^{+}_{\delta}(x_1)) - Du(y^{+}_{\delta}(x_2))| \leqq \delta^{-1}|y^{+}_{\delta}(x_1) - y^{+}_{\delta}(x_2)|,$$

and, hence, from (ii) of Proposition 1.1,

$$|x_1 - x_2| \leqq (1 + C)|y^{+}_{\delta}(x_1) - y^{+}_{\delta}(x_2)|.$$

The second claim is a direct consequence of Theorem A and the Lipschitz estimate in (i).

The third claim follows from a direct computation. Indeed, for $A \in \mathbb{S}^n$, the space of $n \times n$ symmetric matrices, $\ell \in \mathbb{R}^n$, and $c \in \mathbb{R}$, let

$$P(x) = (Ax, x) + (\ell, x) + c,$$

and, assume that, for some $x_0 \in B_{2r}^{\delta}$,

$$u_\delta^+(x) \leqq P(x) \quad \text{and} \quad u_\delta^+(x_0) = P(x_0).$$

It follows from the definition of u_δ^+ that

$$u(y) \leqq P(x) + \frac{|x-y|^2}{2\delta} \quad \text{and} \quad u(y_0) = P(y_0) + \frac{|x_0 - y_0|^2}{2\delta},$$

where, to simplify the notation, we write $y_0 = y_\delta^+(x_0)$. Hence

$$u(y) \leqq Q(y) = \inf_x \left[P(x) + \frac{|x-y|^2}{2\delta} \right] \quad \text{and} \quad u(y_0) = Q(y_0).$$

A straightforward calculation yields, for some $\tilde{\ell}_\delta \in \mathbb{R}^n$ and $\tilde{c}_\delta \in \mathbb{R}$,

$$Q(y) = (A(I + 2\delta A)^{-1} y, y) + (\tilde{\ell}_\delta, y) + \tilde{c}_\delta.$$

The result now follows from a direct computation. $\square$

Instead of the inf- and sup-convolutions, it is possible to use the parallel surface regularizations that were employed by Jensen [14] in the proof of uniqueness of viscosity solutions of second-order PDEs and Caffarelli to obtain regularity properties of viscosity solutions (see, for example, [8]).

2 Monotone Finite Difference Approximations

We introduce the class of monotone and consistent finite difference approximations of (0.1) and discuss the properties needed for the error estimate. For notation and setting, we borrow extensively from [3, 21, 23] and especially [22]. To keep things simple, we do not present any particular examples of numerical schemes. Instead we refer to the aforementioned papers for many such schemes as well as general discussions about what is needed to construct them.

To simplify the presentation, in what follows we rewrite (0.1) as

$$F[u] = F(D^2 u) - f(x),$$

and we talk about the second-order differential operator F.

Let

$$E = \mathbb{Z}_h^n = \{mh : m \in \mathbb{Z}^n\}$$

denote the orthogonal lattice (mesh) with mesh length h. A function $u : E \to \mathbb{R}$ is called a *mesh function*. A general difference operator is written as

$$F_h[u] = F_h(Tu, u) - f(x), \tag{2.1}$$

where
$$Tu(x) = \{u(x+y) : y \in E' = E \setminus \{0\}\}$$
is the set of nontrivial translates of u and $F_h : \mathbb{R}^{E'} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is given.

It is always assumed that $F_h[u]$ is independent of $u(x+y)$ for

$$|y|_\infty = \sup |y_i| > Nh$$

for some fixed $N \in \mathbb{N}$ that depends only on the particular scheme. In view of this, the lattice E may be replaced by the cube

$$Y = \{y \in E' : |y|_\infty \leqq Nh\}.$$

The difference operator F_h is *monotone* if, for all $x \in U$, $z, \tau \in \mathbb{R}$, and $q, \eta \in \mathbb{R}^Y$ such that $0 \leqq \eta_y \leqq \tau$, for all $y \in Y$,

$$F_h(q+\eta, z, x) \geqq F_h(q, z, x) \geqq F_h(q+\eta, z+\tau, x). \tag{2.2}$$

If F_h is differentiable with respect to (q, z), (2.2) is equivalent, for all $y \in Y$, to

$$\frac{\partial F_h}{\partial q_y} \geqq 0 \quad \text{and} \quad \frac{\partial F_h}{\partial z} + \sum_{y \in Y} \frac{\partial F_h}{\partial q_y} \leqq 0. \tag{2.3}$$

The family of difference operators $(F_h)_{0<h\leqq h_0}$, for some $h_0 > 0$, is called *consistent with F in $U \subset \mathbb{R}^n$*, if, for each $\phi \in C^2(U)$, as $h \to 0$,

$$F_h[\phi] \to F[\phi] \ \text{ in } \ C(U).$$

To obtain a rate of convergence, it is necessary to quantify the above limit. To this end, we assume there exists $C > 0$ such that, for all $\phi \in C^3(U)$,

$$|F_h[\phi] - F[\phi]| \leqq C(1 + \|D^3\phi\|) \quad \text{in } U. \tag{2.4}$$

The difference schemes considered here are constructed using the standard first- and second-order difference operators

$$\delta u = \{\delta_y u : y \in Y\} \quad \text{and} \quad \delta^2 u = \{\delta_y^2 u : y \in Y\},$$

where, for $y \in Y$,

$$\begin{cases} \delta_y^+ u(x) = \frac{1}{|y|}(u(x+y) - u(x)), \quad \delta_y^- u(x) = \frac{1}{|y|}(u(x) - u(x-y)), \\ \delta_y u(x) = \frac{1}{2}(\delta_y^+ + \delta_y^-)u(x) = \frac{1}{2|y|}(u(x+y) - u(x-y)), \\ \delta_y^2 u(x) = \delta_y^+ \delta_y^- u(x) = \frac{1}{|y|^2}(u(x+y) - 2u(x) + u(x-y)). \end{cases}$$

The difference operators F_h are of the form

$$F_h[u] = \mathcal{F}(\delta^2 u, u, x) \tag{2.5}$$

for some given $\mathcal{F} : \mathbb{R}^Y \times \mathbb{R} \times U \to \mathbb{R}$. Denoting points in $\mathbb{R}^Y \times \mathbb{R} \times U$ by (s, z, x), $\mathcal{F}$ is assumed symmetric with respect to $s_{\pm y}$ for all $y \in Y$. It follows from this notation that F_h is monotone if $\mathcal{F}$ is nondecreasing with respect to s_y for each $y \in E_N$ and nonincreasing in z. If $\mathcal{F}$ is differentiable, then (2.3) is equivalent to

$$\frac{\partial \mathcal{F}}{\partial s_y} \geqq 0 \ \text{ for all } y \in E' \quad \text{and} \quad \frac{\partial \mathcal{F}}{\partial z} \leqq 0. \tag{2.6}$$

It turns out, as explained in [22], that, given a uniformly elliptic second-order operator F with ellipticity constants λ and Λ, it is possible to find monotone and consistent families $(F_h)_{0 \leqq h \leqq h_0}$ of the form (2.5) satisfying, in addition to (2.6), the stronger monotonicity condition

$$\lambda_0 \leqq \frac{\partial \mathcal{F}}{\partial s_y} \leqq \Lambda_0 \quad \text{and} \quad -\frac{\partial F}{\partial z} \leqq c_0 \tag{2.7}$$

for some $\lambda_0, \Lambda_0 > 0$ related to λ and Λ and all $y = y' = he_i$ $(i = 1, \dots, n)$.

Next we incorporate the boundary conditions. To this end, let

$$U_h = \overline{U} \cap \mathbb{Z}_h^N$$

denote the subset of mesh points in U. Given a difference operator F_h like (2.1), U_h is divided into interior and boundary sets relative to F_h. The interior set U_h^i consists of those points $x \in U_h$ such that, for any mesh function u, $F_h[u](x)$ depends only on translates of u at points in U_h. It follows that

$$\operatorname{dist}(U_{h_1}^i, \partial U) \geqq Nh. \tag{2.8}$$

The boundary set U_h^b is defined as

$$U_h^b = U_h \setminus U_h^i.$$

We review now some of the key results about the finite difference approximations. The first one concerns their stability. Before we state it, we recall the definitions of the half limits u^* and u_*, which are a basic tool in the theory of viscosity solutions. Let $(u_m)_{m \in \mathbb{N}}$ be a sequence of bounded functions. The upper- and lower-half limits are given by

$$u^*(x) = \limsup_{\substack{m \to \infty \\ y \to x}} u_m(y) \quad \text{and} \quad u_*(x) = \liminf_{\substack{m \to \infty \\ y \to x}} u_m(y).$$

The following result was first proved in [3].

THEOREM 2.1 *Let F_h be a family of monotone difference operators that are consistent with the differential operator F in the domain U. Let $(u_m)_{m \in \mathbb{N}}$ be a bounded sequence of mesh functions corresponding to mesh lengths $h_m \to 0$ satisfying the differential inequalities*

$$F_m[u_m] \geqq 0 \quad (\textit{respectively, } F_m[u_m] \leqq 0) \quad \textit{in } U_m^i,$$

where $F_m = F_{h_m}$ and $U_m^i = U_{h_m}^i$. Assume that $\sup_{m, U_m^i} \|u_m\| < \infty$. Then

$$F[u^*] \leqq 0 \quad (\textit{respectively, } F[u_*] \geqq 0) \quad \textit{in } U.$$

The next result, which is presented in this general form in [22], concerns the existence and the regularity of solutions of the approximate boundary value problems

$$F_h[u_h] = 0 \quad \text{in } U_h^i, \qquad u_h = g \quad \text{in } U_h^b. \tag{2.9}$$

In addition to the existence of solutions, it is proved in [22] that the approximate solution u_h of the discrete problem (2.9) satisfies some additional a priori estimates—Hölder continuity—which play a crucial role in obtaining the error estimate in this paper.

We have:

PROPOSITION 2.2 *Assume* (0.3)*,* (0.4)*,* (0.5)*, and* (0.6) *and let F_h be a monotone and consistent approximation of* (0.1)*. The approximate boundary value problem* (2.9) *admits a unique solution u_h. Moreover, there exists $\eta \in (0,1)$ and $C > 0$ depending on the data such that, for $y, z \in U_h$ and $h \in (0,1)$,*

$$|u_h(y)| \leqq C \quad \textit{and} \quad |u_h(y) - u_h(z)| \leqq C|y - z|^{\eta}. \tag{2.10}$$

We conclude with discussion about sup- and inf-type regularizations of u_h. To this end, for $\delta > 0$, define

$$\begin{aligned} u_{h,\delta}^{+}(x) &= \sup_{U_h}\left[u_h(y) - \frac{1}{2\delta}|x-y|^2\right],\\ u_{h,\delta}^{-}(x) &= \inf_{U_h}\left[u_h(y) + \frac{1}{2\delta}|x-y|^2\right]. \end{aligned} \tag{2.11}$$

The next lemma summarizes of the basic properties of $u_{h,\delta}^{\pm}$ that will be used in the proof of Theorem B. The constants appearing throughout the statement depend on the bounds in (2.10).

PROPOSITION 2.3 *Let u_h be the solution of* (2.9) *satisfying* (2.10)*. Then*:

(i) $u_{h,\delta}^{\pm} \in C^{0,1}(U)$ *and, for some* $C > 0$,

$$\begin{cases} \|u_{h,\delta}^{\pm}\| \leqq \|u_h\| + (2\delta)^{-1}h^2, & \|Du_{h,\delta}^{\pm}\| \leqq (2\|u_{h,\delta}^{\pm}\|\delta^{-1})^{1/2},\\ 0 \leqq u_{h,\delta}^{+} - u_h \leqq C\delta^{\eta/(2-\eta)}, & 0 \leqq u_h - u_{h,\delta}^{-} \leqq C\delta^{\eta/(2-\eta)} \textit{ in } U_h. \end{cases} \tag{2.12}$$

(ii) *There exist concave* (*respectively, convex*) *paraboloids of opening δ^{-1} touching $u_{h,\delta}^{+}$* (*respectively, $u_{h,\delta}-$*) *from below* (*respectively, above*) *and, in the sense of distributions, $D^2u_{h,\delta}^{+} \geqq -\delta^{-1}I$* (*respectively, $D^2u_{h,\delta}^{-} \leqq \delta^{-1}I$*)*. Moreover, $u_{h,\delta}^{\pm}$ is a.e. twice differentiable.*

(iii) *Assume that for $x \in U$ the* max (*respectively,* min) *in the definition of $u_{h,\delta}^{+}(x)$* (*respectively, $u_{h,\delta}^{-}(x)$*) *is achieved at $y_{h,\delta}^{+}(x)$* (*respectively, $y_{h,\delta}^{-}(x)$*) $\in U_h$*. There exists $C > 0$ such that*

$$|x - y_{h,\delta}^{\pm}(x)| \leqq \omega(\delta,h) = C\left(\delta^{\frac{1}{2-\eta}} + h^{\eta}\delta^{\frac{1-\eta}{2-\eta}} + h^2\delta^{-\frac{1}{2-\eta}}\right). \tag{2.13}$$

(iv) *There exists $C > 0$ such that if, for $x \in U$, $|\bar{x} - x| = \mathrm{dist}(x, \partial U)$, then*

$$|u^{\pm}_{h,\delta}(x) - g(\bar{x})| \leqq C(|x - \bar{x}|^{\eta} + h^2\delta^{-1} + (\omega(\delta, h))^{\eta}). \tag{2.14}$$

(v) *Let*

$$U^h_\delta = \{x \in U : \mathrm{dist}(x, \partial U) \geqq \omega(\delta, h) + Nh\}. \tag{2.15}$$

There exists $C > 0$, depending on $\|Df\|$, such that

$$F_h[u^+_{h,\delta}] \geqq -C\omega(\delta, h) \quad \textit{in } U^h_\delta \quad \textit{and} \quad F_h[u^-_{h,\delta}] \leqq C\omega(\delta, h) \quad \textit{in } U^h_\delta. \tag{2.16}$$

PROOF: We only present the proof for $u^+_{h,\delta}$. The arguments for $u^-_{h,\delta}$ are exactly the same.

For $x \in U$ and for all $y \in U_h$,

$$u^+_{h,\delta}(x) = u_h(y^+(x)) - \frac{|x - y^+_{h,\delta}(x)|^2}{2\delta} \geqq u_h(y) - \frac{|x - y|^2}{2\delta}. \tag{2.17}$$

Since for all $x \in U$ there exists $y \in U_h$ such that

$$|x - y| \leqq h,$$

the asserted bound on the $\|u^+_{h,\delta}\|$ is immediate.

It also follows from (2.17) that, for all $x \in U$, $\chi \in \mathbb{R}^n$ such that $|\chi| = 1$ and $\varepsilon > 0$,

$$u^+_{h,\delta}(x + \varepsilon\chi) - 2u^+_{h,\delta}(x) + u^+_{h,\delta}(x - \varepsilon\chi) \geqq -\frac{\varepsilon^2}{\delta};$$

hence the second claim follows as usual. The Lipschitz bound and its form is standard for bounded semiconvex functions.

If $x \in U_h$, (2.17) yields

$$u_h(y^+_\delta(x)) - \frac{|x - y^+_\delta(x)|^2}{2\delta} = u^+_{h,\delta}(x) \geqq u_h(x).$$

Therefore

$$\frac{|x - y^+_\delta(x)|^2}{2\delta} \leqq u_h(y^+_\delta(x)) - u_h(x) \leqq 2C|x - y^+_\delta(x)|^{\eta}$$

and hence

$$|x - y^+_\delta(x)| \leqq (2C\delta)^{\frac{1}{2-\eta}}.$$

It follows from (2.17) that

$$0 \leqq u^+_{h,\delta}(x) - u_h(x) \leqq u_h(y^+_\delta(x)) - u_h(x) \leqq C\delta^{\frac{\eta}{2-\eta}}.$$

When $x \in U \setminus U_h$, again let $y \in U_h$ be such that

$$|x - y| \leqq h.$$

Starting from

$$u_h(y^+_\delta(x)) - \frac{|x - y^+_\delta(x)|^2}{2h} \geqq u_h(y) - \frac{|x - y|^2}{2\delta} \geqq u_h(y) - \frac{h^2}{2\delta}$$

and employing a straightforward variant of the previous argument, we find

$$\frac{|x - y^{+}_{h,\delta}(x)|^2}{2\delta} \leqq C|y^{+}_{h,\delta}(x) - y|^{\eta} + \frac{h^2}{2\delta} \leqq C(|x - y^{+}_{h,\delta}(x)|^{\eta} + h^{\eta}) + \frac{h^2}{2\delta}.$$

An elementary calculation yields (2.13) and (2.14).

Finally, in view of (2.17), for $x \in U^{h}_{\delta}$, it is always the case that $y^{+}_{h,\delta}(x) \in U^{i}_{h}$. Therefore

$$F_h[u_h](y^{+}_{\delta,h}(x)) = 0,$$

and the claim then follows from (2.17) and the facts that F_h is monotone, f is Lipschitz-continuous, and, for all $y \in Y$,

$$\delta^2_y u^{+}_{h,\delta}(y^{+}_{h,\delta}(x)) \leqq \delta^2_y u_h(x). \qquad \square$$

We conclude this section with a recasting of some of the properties of $u^{\pm}_{h,\delta}$ in the particular case that $\delta = h^{2\theta}$ for some $\theta \in (0, 1)$.

To simplify the statements we write

$$u^{+}_{h} = u^{+}_{h,h^{2\theta}} \quad \text{and} \quad u^{-}_{h} = u^{-}_{h,h^{2\theta}}. \tag{2.18}$$

We have:

PROPOSITION 2.4 *Assume that* $h < 1$ *and* $\theta \in (0, \frac{1}{2})$. *There exists* $C > 0$ *such that*

(i) $\|u^{\pm}_{h}\| \leqq C$, $\|Du^{+}_{h}\| \leqq Ch^{-\theta}$, *and* $|u^{\pm}_{h} - u_h| \leqq Ch^{2\theta\eta/(2-\eta)}$ *in* U_h.

(ii) *If, for* $x \in U$, $|\bar{x} - x| = \operatorname{dist}(x, \partial U)$, *then*

$$|u^{\pm}_{h}(x) - g(\bar{x})| \leqq C(|x - \bar{x}|^{\eta} + h^{\frac{2\theta\eta}{2-\eta}}).$$

(iii) *Let* $\tilde{U}^h = \{x \in U : \operatorname{dist}(x, \partial U) \geqq Ch^{2\theta/(2-\eta)}\}$; *then*

$$F_h[u^{+}_{h}] \geqq -Ch^{\frac{2\theta}{2-\eta}} \quad \text{and} \quad F_h[u^{-}_{h}] \leqq Ch^{\frac{2\theta}{2-\eta}}. \tag{2.19}$$

Since the proof is based on simply comparing powers of δ for $h \in (0, 1)$, we omit it.

3 The Error Estimate

The precise statement of the error estimate is:

THEOREM 3.1 *Assume* (0.3), (0.4), (0.5), (0.6), (2.2), *and* (2.4). *Let* $u \in C^{0,1}(\bar{U})$ *and* $u_h \in C^{0,\eta}(U_h)$, *respectively, be the solutions of* (0.1)–(0.2), *and* (2.9) *for* $h \in (0, 1)$. *There exist positive constants* C *and* $\alpha_* \in (0, 1)$ *depending only on* λ, Λ, n, U, $\|f\|_{C^{0,1}}$, *and* $\|g\|_{C^{1,\eta'}}$ *such that, for all* $\alpha \in (0, \alpha_*)$,

$$\sup_{U_h} |u - u_h| \leqq Ch^{\alpha}. \tag{3.1}$$

The proof involves the choice of several approximations and perturbations. The main objective is to show that the sup-convolution of the solution (respectively, discrete approximation) stays below the inf-convolution of the discrete approximation (respectively, solution).

Since it does not seem plausible to obtain an explicit rate of convergence, we can choose a priori a very small power of h, h^α, and then any perturbation we make introducing an error of order less than h^α will be acceptable.

We discuss next the main approximations and steps in the proof. We begin by moving the boundary data and change the right-hand side of the equation by h^α to have some room in our calculations (this way, we further "separate" u from u_h). Then we regularize u and u_h by sup- and inf-convolution. This makes u and u_h semiconvex (concave) in the right direction and allows us to control D^2u and D^2u_h (always up to an error controlled in terms of h^α). These two approximations allow us to compare the modified u and u_h except on a small set.

To control the remaining part, we use the Alexandrov-Bakelman-Pucci (ABP) method by constructing the convex envelope $\Gamma(w)$ of the difference w of the modified u and u_h. Here comes into play the regularity theorem (Theorem A) that gives a controlled second-order expansion with small error in most of the set, forcing the contact set $\{\Gamma(w) = w\}$, where the support of $\det D^2\Gamma(w)$ is concentrated, to be small. The estimate on the Hessian of the approximation then yields that, even in this exceptional set, the quantity $\det D^2\Gamma(w)|\{\Gamma(w) = w\}|$ falls within the h^α margin of error.

PROOF: We prove here that

$$\sup_{U_h}(u - u_h) \leqq Ch^\alpha\ ; \tag{3.2}$$

the other inequality follows in exactly the same way. We begin introducing the several layers of the above discussed approximations for u, u_h, and U_h.

The first step is to create some "room" in the equation by considering, for some $\alpha \in (0, 1)$ to be chosen later, the approximate boundary value problem

$$\begin{cases} F(D^2u^h) = f + h^\alpha & \text{in } U, \\ u^h = g & \text{on } \partial U. \end{cases} \tag{3.3}$$

The comparison and regularity theories of viscosity solutions as well as the assumptions on F, f, and g yield the existence of a positive constant C such that

$$\|u - u^h\| \leqq Ch^\alpha \quad \text{and} \quad \sup_h \|Du^h\| \leqq C. \tag{3.4}$$

Next we consider the sup-convolution $u^{h,+}_\delta$ and inf-convolution $u^-_{h,\delta}$ regularizations of u^h and u_h, respectively. At the expense of some greater generality, since the rate may be slightly better if we work with a general δ, here we let

$$\delta = h^{2\theta} \quad \text{with } \theta \in (0, \tfrac{1}{2}), \tag{3.5}$$

and we write

$$u^{h,+} = u^{h,+}_{h,h^{2\theta}}, \quad u^-_h = u^-_{h,h^{2\theta}} \quad \text{and} \quad \tilde{U}_h = \tilde{U}^h \cap \mathbb{Z}^n_h. \tag{3.6}$$

The regularity of u and u_h, the choice of $\tilde{U}_h$, and Proposition 2.4(ii) yield, for an appropriate C that depends on $\|u\|_{C^{0,1}}$ and $\|u_h\|_{C^{0,\eta}}$,

$$\begin{aligned} \sup_{U_h}(u - u_h) &= \max\Big(\sup_{U_h \setminus \tilde{U}_h}(u - u_h), \\ \sup_{\tilde{U}_h}(u - u_h)\Big) &\leqq \max\big(Ch^\gamma, \sup_{\tilde{U}_h}(u - u_h)\big) \end{aligned} \tag{3.7}$$

with

$$\gamma = \frac{2\theta\eta}{2 - \eta}.$$

Moreover, Propositions 1.1 and 2.3—recall that $\tilde{U}_h$ is a subset of the lattice—yield

$$\sup_{\tilde{U}_h}(u - u_h) \leqq \sup_{\tilde{U}_h}(u^{h,+} - u^-_h). \tag{3.8}$$

Using the Lipschitz continuity of $u^{h,+}$ (Proposition 2.1) and u^-_h (Proposition 2.4), for some $C > 0$, we have

$$\begin{aligned} \sup_{\tilde{U}_h}(u^{h,+} - u^-_h) &\leqq \sup_{\tilde{U}^h}(u^{h,+} - u^+_h) + C(h + h^{1-\theta}) \\ &\leqq \sup_{\tilde{U}^h}(u^{h,+} - u^-_h) + Ch^{1-\theta}. \end{aligned} \tag{3.9}$$

Finally, once again the Lipschitz continuity of $u^{h,+}$ and u^-_h, the Lipschitz continuity of u^h, the Hölder continuity of u_h, and the fact that $u^h = g$ on ∂U and $u_h = g$ on U^b_h give yet another positive constant C such that

$$\bar{u}^h = u^{h,+} - Ch^\gamma \leqq u^-_h \quad \text{on } \partial\tilde{U}^h. \tag{3.10}$$

We summarize all the previous estimates in the inequality

$$\sup_{U_h}(u - u_h) \leqq Ch^\gamma + \sup_{\tilde{U}^h}(\bar{u}^h - u^-_h). \tag{3.11}$$

We concentrate on the sup in the right-hand side of the inequality above. To this end, recall that

$$F[\bar{u}^h] \geqq -\|Df\| h^{\frac{2\theta}{2-\eta}} \quad \text{and} \quad F_h[\underline{u}_h] \leqq Ch^{\frac{2\theta}{2-\eta}} \quad \text{in } \tilde{U}_h. \tag{3.12}$$

Let

$$w = u^{h,+} - u^-_h,$$

and consider the concave envelope Γ_w of w^+ in B_{2R} (with $U \subset B_R$). On the contact set $\{w = \Gamma_w\}$, we always have

$$D^2\Gamma_w \leqq 0.$$

Since $u^{h,+}$ is semiconvex and u_h^- is semiconcave (recall $D^2u^{h,+} \geqq h^{-2\theta}I$ and $D^2u_h^- \leqq h^{-2\theta}I$), it follows that, on the contact set,

$$|D^2\Gamma_w| \leqq Ch^{-2\theta}. \tag{3.13}$$

The classical Alexandrov-Bakelman-Pucci estimate (lemma 3.5 in [6]) yields, for some uniform constant C, the estimate

$$\sup_{\tilde{U}_h} w \leqq C\Bigg(\int\limits_{\{\Gamma_w=w\}} \det D^2\Gamma_w\Bigg)^{\frac{1}{n}} \leqq Ch^{-2\theta}|\{\Gamma_w = w\}|^{\frac{1}{n}}. \tag{3.14}$$

We argue next by contradiction assuming that, for some appropriate constant $C > 0$ that is independent of h,

$$Ch^{\alpha} \leqq \sup_{\tilde{U}_h} w. \tag{3.15}$$

It then follows from the last three inequalities that, for some other constant $C > 0$,

$$h^{(\alpha+2\theta)n} \leqq C|\{\Gamma_w = w\}|. \tag{3.16}$$

Consider next the collection

$$\{B(x, h^{\gamma}/2) : x \in \tilde{U}^h\}$$

of open balls that clearly cover $\overline{\tilde{U}^h}$ and satisfy, in view of the definition of $\tilde{U}^h$, for each $x \in \tilde{U}^h$,

$$B(x, h^{\gamma}) \subset U^h.$$

Since $\overline{\tilde{U}^h}$ is compact, there exist M balls $B(x_i, \frac{1}{2}h^{\gamma})$ $(i = 1, \dots, M)$ such that

$$\bar{U}^h \subset \bigcup_{i=1}^{M} B(x_i, \tfrac{1}{2}h^{\gamma}) \quad \text{with } M \approx h^{-n\gamma}.$$

In view of (3.16), there must exist $i \in \{1, \dots, M\}$ such that

$$|B(x_i, \tfrac{1}{2}h^{\gamma}) \cap \{\Gamma_w = w\}| \geqq CM^{-1}h^{(\alpha+2\theta)n} \geqq Ch^{(\alpha+2\theta+\gamma)n}. \tag{3.17}$$

Next we apply Proposition 1.2 in $B(x_i, h^{\gamma})$ with t such that

$$(\{\Gamma_w = w\} \cap B(x_i, h^{\gamma})) \cap A_t \neq \varnothing.$$

It suffices to choose, for an appropriate $C > 0$,

$$t = Ch^{-\frac{1}{\sigma}((\alpha+2\theta)n+\gamma)}.$$

Hence the contact set $\{w = \Gamma_w\}$ contains points where u^h has second-order expansion P of opening t with error of order $th^{-\gamma}$.

At any such point, if P denotes the tangent quadratic, we have

$$F(D^2P) \geqq f - Ch^{2\theta} + h^{\alpha},$$

while the monotonicity and consistency of the scheme yield the estimate

$$f + C(1 + h^{-\gamma}t)h \geqq F(D^2P).$$

Combining the last two inequalities, we find that with the above choice of t and the assumption (3.15) on $\sup_{\tilde{U}_h} w$, we must have

$$h^{2\theta} + C(1 + h^{-\gamma-\frac{1}{\sigma}((\alpha+2\theta)n+\gamma)})h \geqq h^{\alpha},$$

which is not possible, for small h, if

$$\alpha < \min\left(2\theta, \frac{\sigma - 2\theta n - (1+\sigma)\gamma}{n+\sigma}\right).$$

For any such α we then have

$$\sup_{U_h}(u - u_h) \leqq C(h^{\gamma} + h^{\gamma}) \leqq Ch^{\alpha}$$

for all $\alpha \in (0, \alpha_*)$ with

$$\alpha_* = \min\left(\gamma, \frac{\sigma(2-\eta) - 2\theta(n(2-\eta) + 1 + \sigma)}{(\sigma+n)(2-\eta)}\right).$$

□

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LUIS A. CAFFARELLI
University of Texas at Austin
Department of Mathematics
1 University Station - C1200
Austin, TX 78712-0257
E-mail: caffarel@math.utexas.edu

PANAGIOTIS E. SOUGANIDIS
University of Texas at Austin
Department of Mathematics
1 University Station - C1200
Austin, TX 78712-0257
E-mail: souganid@math.utexas.edu

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