Multidimensional Symmetric Stable Processes

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Abstract

This paper surveys recent remarkable progress in the study of potential theory for symmetric stable processes. It also contains new results on the two-sided estimates for Green functions, Poisson kernels and Martin kernels of discontinuous symmetric α-stable process in bounded $C^{1,1}$ open sets. The new results give explicit information on how the comparing constants depend on parameter α and consequently recover the Green function and Poisson kernel estimates for Brownian motion by passing $\alpha \uparrow 2$. In addition to these new estimates, this paper surveys recent progress in the study of notions of harmonicity, integral representation of harmonic functions, boundary Harnack inequality, conditional gauge and intrinsic ultracontractivity for symmetric stable processes. Here is a table of contents.

Keywords and phrases: Symmetric stable processes, Green function, Poisson kernel, Martin kernel, Martin boundary, integral representation of harmonic functions, boundary Harnack principle, Feynman-Kac semigroup, conditional gauge theorem, logarithmic Sobolev inequality, and intrinsic ultracontractivity.

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1 Introduction

A symmetric α-stable process X on $\mathbf{R}^n$ is a Lévy process whose transition density function $p(t, x-y)$ relative to Lebesgue measure is uniquely determined by its Fourier transform

$$\int_{\mathbf{R}^n} e^{ix\cdot\xi} p(t,x)dx = e^{-t|\xi|^\alpha}.$$

Here α must be in the interval (0, 2]. When $\alpha = 2$, we get a Brownian motion running with a time clock twice as fast as the standard one. Brownian motion has been intensively studied due to its central role in modern probability theory and its numerous important applications in other scientific areas including many other branches of mathematics. In the sequel, symmetric stable processes refer to the case when $0 < \alpha < 2$.

Similar to Brownian motion, symmetric α-stable process has the scaling property but of index α; that is, if X is a symmetric α-stable process in $\mathbf{R}^n$ starting from $x \in \mathbf{R}^n$, then for any $k > 0$, $\{k^{-1/\alpha}X_{kt},\, t \geq 0\}$ is a symmetric α-stable process in $\mathbf{R}^n$ starting from $k^{-1/\alpha}x$. Unlike the generator Δ of Brownian motion whose time clock is twice as fast as the standard one, the generator of a symmetric α-stable process with $0 < \alpha < 2$ is the fractional Laplacian $-(-\Delta)^{\alpha/2}$, an integro-differential operator. A symmetric stable process has discontinuous sample paths and heavy tails, while Brownian motion has continuous sample paths and exponential decay tails. The transition density function $p(t, x-y)$ for symmetric α-stable process X is approximately $c|x-y|^{-(n+\alpha)}$ when $|x-y|$ is large. So X_t has infinite variance and when $0 < \alpha \leq 1$, $|X_t|$ even has infinite mean.

During the last thirty years, there has been an explosive growth in the study of physical and economic systems that can be successfully modeled with the use of stable processes. Stable processes are now widely used in physics, operations research, queuing theory, mathematical finance and risk estimation. For these and more applications of stable processes, see the interesting book [33] by Janicki and Weron and the references therein and the recent article [34] by Klafter, Shlesinger and Zuomofen. In order to make precise predictions about natural phenomena and to better cope with these widespread applications, there is a need to study the fine properties of symmetric stable processes, just as for the Brownian motion case. Although a lot is known about symmetric stable processes and their potential theory (see, e.g., [5]-[10], [17], [23]-[25], [27]-[28], [31]-[34], [37]-[42], [48] and the references therein), the study of some fine properties related to symmetric stable processes and Riesz potential theory is quite recent (see [11]–[16], [35]-[36] and [43]). The purpose of this paper is to survey the recent progress on the study of potential theoretic properties for symmetric stable processes made during the last three years as well as to present some new results on the estimates of Green function, Poisson and Martin kernels for symmetric stable processes in bounded $C^{1,1}$ open sets in $\mathbf{R}^n$ with $n \geq 2$. The new estimates are given in Section 2 and in Theorem 3.6 and Remark 3.3(2) of Section 3.

Throughout this paper, the dimension $n \geq 2$. In the sequel, c_0 and c_1 denote positive constants depending on n while c_2 and c_3 are positive constants depending only on D, and $c(\cdot)$ denote constants depending on the variables inside the parentheses; the values of these constants may change from one line to another. We use $X = \{X_t, t \geq 0\}$ to denote the symmetric α-stable process on $\mathbf{R}^n$. For an open set D in $\mathbf{R}^n$, let $\delta_D(x) = \text{dist}(x, \partial D)$ denote the Euclidean distance between x and ∂D.

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2 Green Function and Poisson Kernel Estimates

For $n \geq 2$ and $0 < \alpha < 2$, the symmetric α-stable process X on $\mathbf{R}^n$ is transient. It is known that the Green function of X is given by

$$G(x,y) = \int_0^\infty p(t,x,y)dt = 2^{-\alpha}\pi^{-\frac{n}{2}}\Gamma\left(\frac{n-\alpha}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)^{-1}|x-y|^{\alpha-n} \tag{2.1}$$

(see, for example, [9]). Here Γ is the Gamma function defined by $\Gamma(\lambda) = \int_0^\infty t^{\lambda-1}e^{-t}dt$ for $\lambda > 0$. Since $\Gamma(\lambda+1) = \lambda\Gamma(\lambda)$ and $\Gamma(1) = 1$, there is a constant $c > 1$ such that

$$\frac{1}{\lambda\, c} \leq \Gamma(\lambda) \leq \frac{c}{\lambda} \quad \text{for } 0 < \lambda \leq 1. \tag{2.2}$$

For an open set $D \subset \mathbf{R}^n$, define

$$\tau_D = \inf\{t > 0 : X_t \notin D\}.$$

Adjoin a cemetery point ∂ to D and set

$$X_t^D(\omega) = \begin{cases} X_t(\omega) & \text{if } t < \tau_D(\omega), \\ \partial & \text{if } t \geq \tau_D(\omega). \end{cases}$$

X^D is a strong Markov process with state space $D_\partial = D \cup \{\partial\}$, which is called the subprocess of the symmetric α–stable process X killed upon leaving D, or simply the symmetric α–stable process in D. It is well known that there is a continuous symmetric function $G_D(\cdot,\cdot)$ defined on $D \times D$ except along the diagonal such that for any Borel measurable function $f \geq 0$ on D,

$$E_x\left[\int_0^{\tau_D} f(X_s)ds\right] = \int_D G_D(x,y)f(y)dy \quad \text{for } x \in D.$$

G_D is the Green function of X^D, also called the Green function of X in D. We set $G_D = 0$ off $D \times D$. It follows from the scaling property of the symmetric stable process X that

$$G_D(x,y) = a^{\alpha-n}G_{D/a}(x/a,\, y/a)\,, \quad x, y \in D.$$

where $a > 0$ is a constant. When $D = B(0, r)$ is a ball centered at the origin with radius r, Blumenthal, Getoor and Ray [10] showed that

$$G_{B(0,r)}(x, y) = 2^{-\alpha}\pi^{-n/2}\Gamma\left(\frac{\alpha}{2}\right)^{-2}\Gamma\left(\frac{n}{2}\right)\int_0^z (u+1)^{-n/2}u^{\alpha/2-1}du\,|x-y|^{\alpha-n} \tag{2.3}$$

where $z = (r^2 - |x|^2)(r^2 - |y|^2)|x - y|^{-2}$. However except for some special domains, the explicit formula for G_D is in general unknown.

In the Brownian motion case, the Poisson kernel in a bounded $C^{1,1}$ domain is the normal derivative of the Green function. In the case of a symmetric α-stable process, this kind of relationship can not be expected to hold. For $0 < \alpha < 2$, the symmetric α-stable process has discontinuous sample paths and therefore the exit distribution of X_{τ_D} under P_x does not concentrate on the boundary ∂D. In fact, we have the following

Theorem 2.1 *For every bounded open set D in $\mathbf{R}^n$ satisfying uniform exterior cone condition, there is a function $K_D(x, z)$ defined on $D \times D^c$ such that*

$$E_x[\varphi(X_{\tau_D})] = \int_{D^c} K_D(x, z)\varphi(z)dz, \quad x \in D$$

for every $\varphi \geq 0$ on D^c. Furthermore

$$K_D(x, z) = A(n, \alpha)\int_D \frac{G_D(x, y)}{|y - z|^{n+\alpha}}dy, \quad x \in D,\ z \in \overline{D}^c,$$

where

$$A(n, \alpha) = \frac{\alpha 2^{\alpha-1}\Gamma(\frac{\alpha+n}{2})}{\pi^{n/2}\Gamma(1 - \frac{\alpha}{2})}. \tag{2.4}$$

The kernel $K_D(x, z)$ is called the Poisson kernel for the symmetric α-stable process in D. Recall that an open set D in $\mathbf{R}^n$ is said to satisfy the uniform exterior cone condition if there exist constants $\eta > 0$, $r > 0$ and a cone $\mathcal{C} = \{x = (x_1, \dots, x_n) \in \mathbf{R}^n : 0 < x_n, (x_1^2 + \cdots + x_{n-1}^2)^{1/2} < \eta x_n\}$ such that for every $z \in \partial D$, there is a cone $\mathcal{C}_z$ with vertex z, isometric to $\mathcal{C}$ and satisfying $\mathcal{C}_z \cap B(z, r) \subset D^c$.

Proof. Using Lévy system for X, it is easy to see that for any bounded measurable $\phi \geq 0$ on D^c (see, e.g., [31]),

$$E_x[\phi(X_{\tau_D}); X_{\tau_D} \neq X_{\tau_D -}] = A(n, \alpha)\int_{D^c}\phi(z)dz\int_D \frac{G_D(x, y)}{|y - z|^{n+\alpha}}dy.$$

The theorem now follows from Lemma 6 of [11] which says $P_x(X_{\tau_D} \in \partial D) = 0$ for all $x \in D$. ∎

It follows from the scaling property of symmetric stable process that for $a > 0$,

$$K_D(x,z) = a^{-n} K_{D/a}(x/a,\, z/a) \quad \text{for any } x \in D \text{ and } z \in \overline{D}^c. \tag{2.5}$$

When $D = B(0,r)$, Riesz [40] showed that

$$K_{B(0,r)}(x,z) = \frac{\Gamma(n/2)\sin\frac{\pi\alpha}{2}}{\pi^{\frac{n}{2}+1}} \frac{(r^2 - |x|^2)^{\alpha/2}}{(|z|^2 - r^2)^{\alpha/2}\,|x-z|^n}, \tag{2.6}$$

for $|x| < r$ and $|z| > r$. Except for some special domains, the explicit formula of K_D is in general unknow. Thus it will be very useful if one can have good two-sided bounds on the Green functions G_D and Poisson kernels K_D.

Suppose that $f \geq 0$ is a function on $\mathbf{R}^n$ such that $f(x) = E_x[f(X_{\tau_{B(a,r)}})]$ for every $x \in B(a,r)$; that is, $f \geq 0$ is a harmonic function in $B(a,r)$ with respect to X according to Definition 3.1 below. Then by (2.6), there is a constant $c = c\,(n) > 1$ that depends only on dimension n such that

$$f(x) \leq c f(y) \quad \text{for } x, y \in B(a,\, r/2). \tag{2.7}$$

The above property is called Harnack inequality and will be used in the sequel.

2.1 Estimates on balls

Though the exact formula (2.3) for Green functions is known on balls, it is not very informative and so it is desirable to have more transparent two-sided bounds on them in terms of the distance functions δ_B.

Lemma 2.2 *Let B be a ball of radius r in $\mathbf{R}^n$ with $n \geq 2$ and let $\delta(x) = \mathrm{dist}(x, \partial B)$. Then there is a constant $c_0 > 1$ depending only on n such that*

$$\frac{\alpha}{c_0} \min\left\{ |x-y|^{\alpha-n},\, \frac{\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}}{|x-y|^n} \right\} \leq G_B(x,y) \leq \frac{c_0\,\alpha}{n-\alpha} \min\left\{ |x-y|^{\alpha-n},\, \frac{\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}}{|x-y|^n} \right\} \tag{2.8}$$

and

$$G_B(x,y) \leq c_0\,\alpha \min\left\{ \frac{\delta(x)^{\alpha/2}}{|x-y|^{n-\frac{\alpha}{2}}},\, \frac{\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}}{|x-y|^n} \right\} \tag{2.9}$$

for every $x, y \in B$.

Proof. Without loss of generality, assume that $B = B(0,r)$ is centered at the origin. It is proved in Chen-Song [13] (see Lemma 2.2 and 6.6 there) that there is a constant $c = c\,(D,\alpha) > 1$

$$c^{-1} \min\left\{ |x-y|^{\alpha-n},\, \frac{\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}}{|x-y|^n} \right\} \leq G_D(x,y) \leq c \min\left\{ |x-y|^{\alpha-n},\, \frac{\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}}{|x-y|^n} \right\}$$

This lemma makes the dependence of the comparing constants on α more precise and explicit. In the following, constant c_0 depends only on n whose value may change from line to line. In view of the explicit formula (2.3) for $G_{B(0,r)}$, the key is to estimate the integral there. First assume $r = 1$. Recall that in this case, $z = (1-|x|^2)(1-|y|^2)|x-y|^{-2}$.

$$\begin{aligned}
&\int_0^z (u+1)^{-n/2} u^{\alpha/2-1} du \\
= \ & z^{\alpha/2} \int_0^1 (1+vz)^{-n/2} v^{\alpha/2-1} dv \\
= \ & (1-|x|)^{\alpha/2}(1-|y|)^{\alpha/2}(1+|x|)^{\alpha/2}(1+|y|)^{\alpha/2}|x-y|^{-\alpha} \\
& \qquad \cdot \int_0^1 (1+vz)^{-n/2} v^{\alpha/2-1} dv \\
\leq \ & \left(\int_0^1 (1+vz)^{-n/2} v^{\alpha/2-1} dv \right) 2^\alpha \delta_B(x)^{\alpha/2} \delta_B(y)^{\alpha/2} |x-y|^{-\alpha}.
\end{aligned}$$

Since

$$\int_0^1 (1+vz)^{-n/2} v^{\alpha/2-1} dv < \int_0^1 v^{\alpha/2-1} dv = \frac{2}{\alpha} < \infty$$

we have, by (2.2) and (2.3), that there is a constant c_0 that only depends on n such that

$$G_{B(0,1)}(x,y) \leq c_0 \, \alpha \, \frac{\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}}{|x-y|^n} \quad \text{for } x, y \in B(0,1), \tag{2.10}$$

On the other hand, it follows from (2.1),

$$G_{B(0,1)}(x,y) \leq G(x,y) \leq \frac{c_0 \, \alpha}{n-\alpha} |x-y|^{\alpha-n} \quad \text{for } x, y \in B(0,1) \tag{2.11}$$

Combining with (2.10), we have for $x, y \in B(0,1)$,

$$G_{B(0,1)}(x,y) \leq \frac{c_0 \, \alpha}{n-\alpha} \min\left\{ \frac{1}{|x-y|^{n-\alpha}}, \ \frac{\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}}{|x-y|^n} \right\}.$$

To prove (2.9), note that if $\delta(y) < 2|x-y|$, then by (2.10),

$$G_{B(0,1)}(x,y) \leq c_0 \, \alpha \, \frac{\delta(x)^{\alpha/2}}{|x-y|^{n-\frac{\alpha}{2}}};$$

So assume that $\delta(y) \geq 2|x-y|$ and set $z_1 = (1-|x|^2)|x-y|^{-1}$, $z_2 = (1-|y|^2)|x-y|^{-1}$. Since $z_2 \geq 2$ and $\delta(x) \geq \delta(y) - |x-y| \geq |x-y|$, we have $z_1 \geq 1$ and $z_1 \geq (z_2-1)/2 \geq z_2/4$. Now

$$\begin{aligned}
&\int_0^z (u+1)^{-n/2} u^{\alpha/2-1} du \\
= \ & z_1^{\alpha/2} \int_0^{z_2} (1+z_1 v)^{-n/2} v^{\alpha/2-1} dv \\
\leq \ & z_1^{\alpha/2} \left(\int_0^1 v^{\alpha/2-1} dv + \int_1^{z_2} (z_1 v)^{-n/2} v^{\alpha/2-1} dv \right) \\
= \ & z_1^{\alpha/2} \left(\frac{2}{\alpha} + z_1^{-n/2} \frac{2}{n-\alpha} \left(1 - z_2^{-(n-\alpha)/2}\right) \right)
\end{aligned}$$

By the mean-value theorem, $(1-z_2^{-x})/x \le \log z_2 \le \log(4z_1)$ for $x>0$; while it is known from calculus that there is a constant $c>0$ such that $\log x \le cx^{n/2}$ for all $x \ge 2$. Hence

$$\int_0^z (u+1)^{-n/2} u^{\alpha/2-1} du \le z_1^{\alpha/2}\,(2/\alpha + c) \le c\alpha^{-1}\delta(x)^{\alpha/2}|x-y|^{-\alpha/2}.$$

Thus by (2.2) and (2.3), we have

$$G_{B(0,1)}(x,y) \le c_0\,\alpha\,\frac{\delta(x)^{\alpha/2}}{|x-y|^{n-\frac{\alpha}{2}}} \quad \text{for } x,y\in B(0,1), \tag{2.12}$$

which together with (2.10) yields inequality (2.9).

For the lower bound, if $z \ge 1$,

$$\int_0^z (u+1)^{-n/2} u^{\alpha/2-1} du \ge \int_0^1 (u+1)^{-n/2} u^{\alpha/2-1} du \ge \frac{2^{1-n/2}}{\alpha}.$$

If $z<1$, then

$$\begin{aligned}
&\int_0^z (u+1)^{-n/2} u^{\alpha/2-1} du \\
= \;& z^{\alpha/2}\int_0^1 (1+vz)^{-n/2} u^{\alpha/2-1} dv \\
\ge \;& z^{\alpha/2}\int_0^1 (1+v)^{-n/2} u^{\alpha/2-1} dv \\
\ge \;& (1-|x|)^{\alpha/2}(1-|y|)^{\alpha/2}|x-y|^{-\alpha}\int_0^1 (1+v)^{-n/2} v^{\alpha/2-1} dv \\
\ge \;& \frac{2^{1-n/2}}{\alpha}\frac{\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}}{|x-y|^2}.
\end{aligned}$$

Thus from (2.2)-(2.3), we have

$$G_{B(0,1)}(x,y) \ge \frac{\alpha}{c_0}\min\left\{\frac{1}{|x-y|^{n-\alpha}},\ \frac{\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}}{|x-y|^n}\right\}.$$

This proves the Lemma for unit ball $B(0,1)$. For $B=B(0,r)$ with radius r, the two-sided estimates follows immediately from the scaling property. ∎

Lemma 2.3 *Let B be a ball of radius r in $\mathbf{R}^n$ with $n\ge 2$ and $\delta(x)=\mathrm{dist}(x,\partial B)$. There is a constant $c_1>1$ depending only on n such that*

$$\frac{\alpha(2-\alpha)\,\delta(x)^{\alpha/2}}{c_1\,\delta(z)^{\alpha/2}(1+r^{-1}\delta(z))^{\alpha/2}|x-z|^n} \le K_{B(0,r)}(x,z) \le \frac{c_1\alpha(2-\alpha)\,\delta(x)^{\alpha/2}}{\delta(z)^{\alpha/2}(1+r^{-1}\delta(z))^{\alpha/2}|x-z|^n}.$$

for all $|x|<r$ and $|z|>r$.

Proof. This follows directly from (2.6). ∎

2.2 Estimates on bounded $C^{1,1}$ domains

Now we can present Green function estimates for bounded $C^{1,1}$ domains. Recall that an open set D is $C^{1,1}$ means that for every $z \in \partial D$, there exist a $r > 0$ such that $B(z,r) \cap D$ is the subdomain in $B(z,r)$ that lies above the graph of a function whose first derivatives are Lipschitz. In particular, a bounded $C^{1,1}$ open set can only have finitely many components, sharing no common boundary points. A bounded $C^{1,1}$ open set D has the following geometric property (see Zhao [47]). there exist constants $r_0 > 0$ and $s_0 > 0$ depending only on D such that for any $z \in \partial D$ and $0 < r \le r_0$, there exist two balls $B_1^z(r)$ and $B_2^z(r)$ of radius r such that $B_1^z(r) \subset D$, $B_2^z(r) \subset \mathbf{R}^n \setminus \overline{D}$ and $\{z\} = \partial B_1^z(r) \cap \partial B_2^z(r)$. For each $z \in \partial D$, there is an inward unit normal vector $\mathbf{n}_z$ and that for any $z, w \in \partial D$, $|\mathbf{n}_z - \mathbf{n}_w| \le s_0 |z-w|$.

Theorem 2.4 *Let D be a bounded $C^{1,1}$ domain in $\mathbf{R}^n$ with $n \ge 2$ and $\delta(x) = \mathrm{dist}(x, \partial D)$. There exists a constant $c_2 = c_2(D) > 1$ that only depends on D such that*

$$\frac{\alpha}{c_2} \min\left\{ |x-y|^{\alpha-n}, \frac{\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}}{|x-y|^n} \right\} \le G_D(x,y) \le \frac{c_2\,\alpha}{n-\alpha} \min\left\{ |x-y|^{\alpha-n}, \frac{\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}}{|x-y|^n} \right\} \tag{2.13}$$

and

$$G_D(x,y) \le c_2 \alpha \min\left\{ \frac{\delta(x)^{\alpha/2}}{|x-y|^{n-\frac{\alpha}{2}}}, \frac{\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}}{|x-y|^n} \right\} \tag{2.14}$$

for every $x, y \in D$. By (2.3), the constant $c_2 = c_2(D)$ can be so chosen that it is invariant under the rigid translation and dilation of D.

Proof. It is proved independently by Chen-Song [13] for $n \ge 2$ and by Kulczycki [35] for $n \ge 3$ that there is a constant $c = c(D, \alpha) > 1$ such that

$$c^{-1} \min\left\{ |x-y|^{\alpha-n}, \frac{\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}}{|x-y|^n} \right\} \le G_D(x,y) \le c \min\left\{ |x-y|^{\alpha-n}, \frac{\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}}{|x-y|^n} \right\}$$

The novelty here is the more precise and explicit information on how the comparing constants depend on parameter α. This can then be used to recover the Green function estimates for Brownian motion in $D \subset \mathbf{R}^n$ with $n \ge 3$ by passing $\alpha \uparrow 2$ (see Remark 2.2 below). In [13], the upper bound estimate is obtained by using inversion with respect to spheres along with the explicit formulae for Green functions (2.3) and Poisson kernels (2.6) on balls, while in [35], it is obtained through establishing an induction inequality. The proofs for lower bound estimates in [13] and [35] both adopted the approach in Zhao [47] for Brownian motions but with substantial changes. While the approach in [13] for the upper bound estimates is simpler than that in [35], the proof for the lower bound estimates in [35] is shorter than that in [13].

The current upper bounds in (2.13)-(2.14) follow from the proof of (1.4) and (1.5) in Theorem 1.1 of [13], by keeping track of all the constants and using at the corresponding places the refined estimate for balls stated in Lemmas 2.2 and 2.3 above. So we omit the details here. However we are going to spell out the proof for the lower bound estimates. For clarity, the current lower bound will be proved through four lemmas. The proof here adopts some ideas from the approach in [47] and [35] but additional features are needed to get the more precise and explicit information on how the comparing constants depend on α.

Lemma 2.5 *There are constants $0 < \delta_0 < r_0/16$ and $c = c(D, \delta_0) > 0$ such that $D_{\delta_0} = \{x \in D : \delta(x) > \delta_0\}$ is a (connected) domain and*

$$G_D(x,y) \geq c\,\alpha\,|x-y|^{\alpha-n} \quad \text{for all } x,y \in D_{\delta_0}.$$

Proof. Clearly there is a constant $0 < \delta_0 < r_0/16$ such that D_{δ_0} is still a domain. Let $W = \{Q_j\}$ be a Whitney decomposition of D. This is a decomposition of D into closed cubes Q with the following three properties (see [44] for details).

(1) for $j \neq k$, the interior of Q_j and the interior of Q_k are disjoint;

(2) if Q_j and Q_k touchs, then

$$\frac{1}{4} \leq \frac{\operatorname{diam}(Q_j)}{\operatorname{diam}(Q_k)} \leq 4;$$

(3) for any j,

$$1 \leq \frac{\operatorname{dist}(Q_j, \partial D)}{\operatorname{diam}(Q_j)} \leq 4.$$

For each cube Q_j with diameter $\operatorname{diam}(Q_j)$, it contains a ball with radius $\operatorname{diam}(Q_j)/(2\sqrt{n})$ and is contained by a concentric ball with radius $\operatorname{diam}(Q_j)/2$. Note that

$$D_{\delta_0} \subset \bigcup \{Q_j : \operatorname{diam}(Q_j) > \delta_0/8\}$$

and there are only finite many Q_j, say N, that have diameter $\operatorname{diam}(Q_j) > \delta_0/8$.

Let $\{Q(j) = Q_{n_j}\}_{j=1}^{k}$, $1 \leq k \leq N$, be the minimal number of cubes such that $x \in Q(1)$, $y \in Q(k)$, $Q(i) \in W$ for all i, $Q(i)$ and $Q(i+1)$ have touching edges for all i. Dividing each $Q(j)$ into 16^n equal subcubes $\{Q(j,i)\}_{i=1}^{16^n}$. Denote the center of $Q(k)$ by e and $\frac{3}{2}\operatorname{diam}(Q(k))$ by s.

Case 1: $|x-e| < 5\operatorname{diam}(Q(k))/4$. Then by Lemma 2.2

$$G_D(x,y) \geq G_{B(e,s)}(x,y) \geq c\,\alpha|x-y|^{\alpha-n}.$$

Case 2: $|x-e| \geq 5\operatorname{diam}(Q(k))/4$. In this case, x does not fall into $Q(k)$ nor any those of $Q(k-1,i)$'s that are adjacent to $Q(k)$, Let $\{S_j\}_{j=1}^{m}$ be the minimal number of cubes such that $x \in S_1$, $y \in S_m$, where $S_i \in \{Q(j,i)\}_{1\le j\le k-1, 1\le i\le 16^n}$ for all $1 \le i \le m-1$ and $S_m = Q(k)$, S_i and S_{i+1} have touching edges for all i. Clearly $m \le 16^n N$. By the minimality of $\{S_i\}_{1\le i\le m}$ and Properties (2) and (3) above for the Whitney cubes, we see that for $1 \le i \le m-2$, the distance between S_j and $Q(k)$ is no less than $\mathrm{diam}(S_j)$ and that $3S_j \subset D\setminus\{y\}$. Here we abuse the notation a little bit and use $3S_j$ to denote a cube that is obtained from S_i by dilating 3 times of it from the center of S_j. Let v_0 be a point on the touching edge between S_{m-2} and S_{m-1}. Clearly $|v_0 - y| \le 5\,\mathrm{diam}(Q(k))/4 \le 2|x-y|$. Note that, $G_D(u,y) = E_u[G_D(X^D_{\tau_{B(a,r)}}, y)]$ as long as $u \in B(a,r) \subset D\setminus\{y\}$. Thus for each subcubes S_i with $1 \le i \le m-2$, by the Harnack inequality (2.7) there is a universal constant $0 < c = c\,(n) < 1$ that depends only on dimension n such that $G_D(u,y) \ge c\, G_D(v,y)$ for u, v in such $\overline{S}_i$. Therefore we have $G_D(x,y) \ge c^{16^n N} G_D(v_0,y)$. Since $G_D(v_0,y) \ge G_{B(e,s)}(v_0,y)$, it follows from Lemma 2.2 that there is a constant c, which only depends on D and δ_0, such that

$$G_D(x,y) \ge c\,\alpha\,|v_0 - y|^{\alpha-n} \ge c\,\alpha\,|x-y|^{\alpha-n}$$

This proves the lemma. ∎

Remark 2.1. In fact, we showed that the lemma holds for any bounded domain D.

Lemma 2.6 *There is a constant $c = c\,(D)$ such that for $x, y \in D$ with $\delta(x) \le \min\{\delta(y),\, \delta_0\}$ and $y \notin B_1^{x^*}(r_0/2)$,*

$$G_D(x,y) \ge c\,\alpha\,\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}.$$

Here x^* is the point on ∂D such that $|x - x^*| = \delta(x)$.

Proof. For simplicity, let $B_x = B_1^{x^*}(r_0/2) = B(o_x, r_0/2)$. Since $y \notin B_x$, then by (2.6)

$$\begin{aligned} G_D(x,y) &= E_x[G_D(X_{\tau_{B_x}}, y)] = \int_{D\cap B_x^c} G_D(u,y) K_{B_x}(x,u)du \\ &\ge c\,\delta(x)^{\alpha/2}\int_{D\cap B_x^c} G_D(u,y)K_{B_x}(o_x,u)du = c\,\delta(x)^{\alpha/2} G_D(o_x,y). \end{aligned} \tag{2.15}$$

If $\delta(y) \ge \delta_0$, then by Lemma 2.5 that $G_D(x,y) \ge c\,\alpha\,\delta(x)^{\alpha/2} \ge c\,\alpha\,\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}$. So assume from now on that $\delta(y) < \delta_0$. Let $y^* \in \partial D$ be such that $|y - y^*| = \delta(y)$ and $B_y = B_1^{y^*}(r_0/2) = B(o_y, r_0/2)$.

Case 1: $o_x \notin B_y$. In this case, by applying (2.15) twice and then Lemma 2.5,

$$G_D(x,y) \ge c\,\delta(x)^{\alpha/2} G_D(o_x,y) \ge c\,\delta(x)^{\alpha/2}\delta(y)^{\alpha/2} G_D(o_x,o_y) \ge c\,\alpha\,\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}.$$

Case 2: $o_x \in B_y$. In this case, since $|o_x - o_y| < r_0/2$, $B(o_x, r_0/4) \cap B(o_y, r_0/4) \ne \emptyset$. Take $e \in B(o_x, r_0/4) \cap B(o_y, r_0/4)$. Note that since $\delta(y) < \delta_0 < r_0/16$ and $\delta(o_x) = r_0/2$, $y \notin B(o_x, \frac{3r_0}{8})$. Thus by the Harnack inequality (2.7) and Lemma 2.2

$$G_D(o_x,y) \ge c\, G_D(e,y) \ge c\, G_{B(o_y, r_0/2)}(e,y) \ge c\,\alpha\,\delta(y)^{\alpha/2}.$$

This together with (2.15) proves the lemma. ∎

Lemma 2.7 *There is a constant $c = c(D) > 0$ such that*

$$G_D(x,y) \geq c\,\alpha\,\delta(x)^{\alpha/2}\delta(y)^{\alpha/2} \quad \text{for } x, y \in D.$$

Proof. By symmetry of $G_D(x,y)$, we may assume that $\delta(x) \leq \delta(y)$.

When $\delta(x) \geq \delta_0$, it follows from Lemma 2.5.

When $\delta(x) < \delta_0$ and $y \notin B_x = B_1^{x^*}(r_0/2)$, it follows from Lemma 2.6.

When $\delta(x) < \delta_0$, $y \in B_x = B(o_x, r_0/2)$ but $d(y, \partial B_x) \geq \delta(y)/4$, by Lemma 2.2,

$$\begin{aligned} G_D(x,y) &\geq G_{B_x}(x,y) \\ &\geq c\,\alpha\,\min\left\{|x-y|^{\alpha-n},\, \delta(x)^{\alpha/2}\delta_{B_x}(y)^{\alpha/2}|x-y|^{-n}\right\} \\ &\geq c\,\alpha\,\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}. \end{aligned}$$

For the remaining case of $\delta(x) < \delta_0$, $y \in B_x = B(o_x, r_0/2)$ but $d(y, \partial B_x) < \delta(y)/4$, note that $\delta(y) \leq |y - x^*| < r_0$. Thus by Lemma 2.2 and Lemma 2.3,

$$\begin{aligned} G_D(x,y) &\geq E_x\left[G_D(X_{\tau_{B_x}}, y)\right] = \int_{D\cap B_x^c} G_D(u,y)K_{B_x}(x,u)du \\ &\geq \int_{B_x^c\cap B(y,\delta(y)/2)} G_{B(y,\delta(y))}(u,y)K_{B_x}(x,u)du \\ &\geq c\int_{B_x^c\cap B(y,\delta(y)/2)} \frac{\alpha}{|u-y|^{n-\alpha}} \frac{\alpha(2-\alpha)\delta(x)^{\alpha/2}}{\delta_{B_x}(u)^{\alpha/2}(1+r_0^{-1}\delta_{B_x}(u))^{\alpha/2}|x-u|^n}du \\ &\geq c\,\alpha\,\delta(x)^{\alpha/2}\int_{B_x^c\cap B(y,\delta(y)/2)} \frac{\alpha(2-\alpha)}{\delta_{B_x}(u)^{\alpha/2}|u-y|^{n-\alpha}}du. \end{aligned}$$

The last inequality holds because for $u \in B_x^c \cap B(y, \delta(y)/2)$, $|x-u| \leq r_0/2 + \delta(y) < 3r_0/2$. Using polar coordinate at y, a change of radial coordinate $r = 2s/\delta(y)$, and co-area formula (see, e.g., Theorem 3.2.11 of Federer [26]), it is easy to see that there is a constant $c = c(D) > 0$ that only depends on D such that

$$\int_{B_x^c\cap B(y,\delta(y)/2)} \frac{\alpha(2-\alpha)}{\delta_{B_x}(u)^{\alpha/2}|u-y|^{n-\alpha}}du \geq c\,\delta(y)^{\alpha/2}.$$

This completes the proof of the lemma. ∎

Lemma 2.8 *There is a constant $c = c(D) > 0$ that depends only on D such that*

$$G_D(x,y) \geq c\,\alpha\,\frac{\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}}{|x-y|^n}$$

for all $x, y \in D$ with $\frac{1}{2}\max\{\delta(x), \delta(y)\} \leq |x-y| \leq \frac{r_0}{10(1+r_0 s_0)}$.

Proof. Recall that x^* and y^* are points in ∂D such that $|x-x^*|=\delta(x)$ and $|y-y^*|=\delta(y)$ respectively. Define $B_x=B_1^{x^*}(r_0)=B(o_x,r_0)$ and $B_y=B_1^{y^*}(r_0)=B(o_y,r_0)$. Note that $|x^*-y^*|\le|x-y|+\delta(x)+\delta(y)\le 5|x-y|$, Therefore

$$|o_x-o_y|\le|x^*-y^*|+r_0|\mathbf{n}_{x^*}-\mathbf{n}_{y^*}|\le(1+r_0s_0)|x^*-y^*|\le r_0/2. \tag{2.16}$$

Since $\max\{\delta(x),\delta(y)\}\le r_0/5$, $B(x,\delta(x))\subset B_x$ and $B(y,\delta(y))\subset B_y$. Recall also $\delta_{B_x}(u)=\mathrm{dist}(u,\partial B_x)$ and $\delta_{B_y}(u)=\mathrm{dist}(u,\partial B_y)$.

Case 1: $y\in B_x$ with $\delta_{B_x}(y)\ge\delta(y)/4$. Since $\delta_{B_x}(x)=\delta(x)\le 2|x-y|$ and $\delta(x)/4\le\delta_{B_x}(y)\le|x-y|$, we have by Lemma 2.2 that

$$G_D(x,y)\ge G_{B_x}(x,y)\ge c\,\alpha\,\frac{\delta(x)^{\alpha/2}\delta(x)^{\alpha/2}}{|x-y|^n}.$$

Case 2: $y\notin B_x$ with $\delta_{B_x}(y)\ge\delta(y)/4$. Since $|x-y|<r_0/10$, by Lemma 2.2 and Lemma 2.3

$$\begin{aligned}
&G_D(x,y)\\
=\;&E_x[G_D(X_{\tau_{B_x}},y)]=\int_{D\cap B_x^c}K_{B_x}(x,u)G_D(u,y)du\\
\ge\;&\int_{B_x^c\cap B_y\cap B(x,2|x-y|)}K_{B_x}(x,u)G_{B_y}(u,y)du\\
\ge\;&\frac{c\,\alpha\,\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}}{|x-y|^n}\int_{B_x^c\cap B_y\cap B(x,2|x-y|)}\frac{\alpha(2-\alpha)}{\delta_{B_x}(u)^{\alpha/2}}\min\left\{\frac{1}{\delta(y)^{\alpha/2}|y-u|^{n-\alpha}},\,\frac{\delta_{B_y}(u)^{\alpha/2}}{|u-y|^n}\right\}du.
\end{aligned}$$

Using dilation $u=\delta(y)w$ and co-area formula (cf. Federer [26]), it is easy to see there is a constant $c=c\,(D)>0$ that depends only on D such that

$$\int_{B_x^c\cap B_y\cap B(x,2|x-y|)}\frac{\alpha(2-\alpha)}{\delta_{B_x}(u)^{\alpha/2}}\min\left\{\frac{1}{\delta(y)^{\alpha/2}|y-u|^{n-\alpha}},\,\frac{\delta_{B_y}(u)^{\alpha/2}}{|u-y|^n}\right\}du\ge c.$$

This proves the lemma under Case 2.

Case 3: $\delta_{B_x}(y)<\delta(y)/4$. Note that

$$G_D(x,y)\ge\int_{D\cap B_x^c}K_{B_x}(x,u)G_D(u,y)du\ge\int_{B_x^c\cap B(y,\,\delta(y)/2)}K_{B_x}(x,u)G_{B(y,\delta(y))}(u,y)du. \tag{2.17}$$

For $u\in B_x^c\cap B(y,\,\delta(y)/2)$,

$$\begin{aligned}
\delta_{B_y}(y)&=\delta(y)>2|u-y|;\\
\delta_{B_y}(u)&\ge\delta_{B_y}(y)-|u-y|>\delta(y)/2>|u-y|;,\\
|x-u|&\le|x-y|+|y-u|\le|x-y|+\frac{1}{2}\delta(y)<2|x-y|;\\
\delta_{B_x}(u)&\le\delta_{B_x}(y)+|u-y|\le\frac{1}{4}\delta(y)+\frac{1}{2}\delta(y)<\delta(y)<r_0.
\end{aligned}$$

Thus by (2.17), Lemmas 2.2 and 2.3,

$$\begin{aligned} G_D(x,y) &\geq c\int_{B_x^c\cap B(y,\,\delta(y)/2)} \frac{\alpha(2-\alpha)\delta(x)^{\alpha/2}}{\delta_{B_x}(u)^{\alpha/2}|x-u|^n}\,\frac{\alpha}{|u-y|^{n-\alpha}}\,du \\ &\geq \frac{c\,\alpha\,\delta(x)^{\alpha/2}}{|x-y|^n}\int_{B_x^c\cap B(y,\,\delta(y)/2)} \frac{\alpha(2-\alpha)}{\delta_{B_x}(u)^{\alpha/2}|u-y|^{n-\alpha}}\,du. \end{aligned}$$

Using polar coordinates at point y, a change of radial coordinate $r=\delta(y)s$ and co-area formula (cf. [26]), it is easy to see that there is a constant $c>0$ that depends only on D,

$$\int_{B_x^c\cap B(y,\,\delta(y)/2)} \frac{\alpha(2-\alpha)}{\delta_{B_x}(u)^{\alpha/2}|u-y|^{n-\alpha}}\,du. \geq c\delta(y)^{\alpha/2}.$$

Thus we have $G_D(x,y)\geq c\,\alpha\,\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}|x-y|^{-n}$. ∎

The lower bound estimate in Theorem 2.4 follows immediately from the following theorem.

Theorem 2.9 *There is a constant $c_2=c_2(D)>1$ that depends only on D such that for any $x,y\in D$,*

$$G_D(x,y)\geq \begin{cases} c_2^{-1}\,\alpha\,|x-y|^{\alpha-n} & \text{if} \quad |x-y|\leq \frac12\max\{\delta(x),\,\delta(y)\}, \\ c_2^{-1}\,\alpha\,\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}\,|x-y|^{-n} & \text{if} \quad |x-y|> \frac12\max\{\delta(x),\,\delta(y)\}. \end{cases}$$

Proof. If $|x-y|\leq\max\{\delta(x),\,\delta(y)\}/2$, say $|x-y|\leq\delta(x)/2$, then $G_D(x,y)\geq G_{B(x,\delta(x))}(x,y)$. Since $\delta_{B(x,\delta(x))}(x)=\delta(x)$ and $\delta_{B(x,\delta(x))}(y)\geq\delta_{B(x,\delta(x))}(x)-|x-y|\geq|x-y|$, by the lower bound in Lemma 2.2,

$$G_D(x,y)\geq\frac{\alpha}{c_2\,|x-y|^{n-\alpha}}.$$

If $\frac12\max\{\delta(x),\,\delta(y)\}<|x-y|\leq\frac{r_0}{10(1+r_0s_0)}$, the lower bound comes from Lemma 2.8.

If $\frac12\max\{\delta(x),\,\delta(y)\}<|x-y|$ with $|x-y|>\frac{r_0}{10(1+r_0s_0)}$, then by Lemma 2.7

$$G_D(x,y)\geq c\,\alpha\,\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}\geq\frac{\alpha\,\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}}{c_2\,|x-y|^n}$$

∎

Remark 2.2: When $n\geq3$, the same estimates in Theorem 2.4 hold for Brownian motion with 2 in place of α, with the upper bound being obtained by Widman [45] and lower bound by Zhao [47]. In fact, since the symmetric α-stable process converges weakly to a Brownian motion W with infinitesimal generator Δ as $\alpha\uparrow2$ in $D([0,\infty),\mathbf{R}^n)$, their corresponding Green functions converge pointwise to that of Brownian motion W (cf., e.g., [25]). Here $D([0,\infty),\mathbf{R}^n)$ is the space of right continuous functions on $[0,\infty)$ taking values in $\mathbf{R}^n$ having left limits, equipped with Skorokhod topology. Thus the results in Theorem 2.4 give another (and unified) proof for two-sided estimates for Green functions of Brownian motion by passing $\alpha \uparrow 2$.

It is well known that bounded $C^{1,1}$ domains satisfy the uniform exterior cone condition, therefore the Theorem 2.1 holds in particular for bounded $C^{1,1}$ domains. Through it and the bounds for the Green functions in Theorem 2.4, one could get two–sided bounds on the Poisson kernels.

Theorem 2.10 *Let D be a bounded $C^{1,1}$ domain in $\mathbf{R}^n$ with $n \geq 2$ and $\delta(x) = \mathrm{dist}(x, \partial D)$. There exists a constant $c_3 = c_3(D) > 1$ that depends only on D such that*

$$\frac{\alpha(2-\alpha)\,\delta(x)^{\alpha/2}}{c_3\,\delta(z)^{\alpha/2}(1+r_0^{-1}\delta(z))^{\alpha/2}|x-z|^n} \leq K_D(x,z) \leq \frac{c_3\alpha(2-\alpha)\,\delta(x)^{\alpha/2}}{\delta(z)^{\alpha/2}(1+r_0^{-1}\delta(z))^{\alpha/2}|x-z|^n}.$$

for all $x, \in D$ and $z \in \overline{D}^c$. Here r_0 is the characteristic radius for $C^{1,1}$ domain D defined in the paragraph proceeding Theorem 2.4. By (2.5), the constant $c_3 = c_3(D)$ here can be so chosen that it is invariant under the rigid translation and dilation of D.

Proof. It is proved by Chen and Song in [13] that there is a constant $c = c\,(D, \alpha) > 1$ such that

$$\frac{\delta(x)^{\alpha/2}}{c\,\delta(z)^{\alpha/2}(1+\delta(z))^{\alpha/2}|x-z|^n} \leq K_D(x,z) \leq \frac{c\,\delta(x)^{\alpha/2}}{\delta(z)^{\alpha/2}(1+\delta(z))^{\alpha/2}|x-z|^n}.$$

for all $x \in D$ and $z \in \overline{D}^c$. The novelty here is the more precise and explicit information on how the comparing constants depend on parameter α. This can then be used to recover the Poisson kernel estimates for Brownian motion in $D \subset \mathbf{R}^n$ with $n \geq 2$ by passing $\alpha \uparrow 2$ (see Remark 2.3 below).

The proof for the upper bound is the same as that for Theorem 3.3 in [13], except using at appropriate places the refined Green function upper bound estimates (2.14) instead.

The proof for the lower bound is the same as that for Theorem 3.4 in [13], except using at appropriate places the refined Green function lower bound estimates (2.13) instead. In the proof, we also need to use the following (refined) elementary result in place of Lemma 3.3(2) in [13]. *Suppose that $\mathbf{0} \in \overline{D}$, $r > 0$ and $0 < \alpha < 2$. Then there is a constant $c = c\,(D, r)$ depending only on r, α, and the Lipschitz characteristic constants of D such that*

$$\int_{D \cap B(\mathbf{0}, r)} |y|^{\alpha - n} \left(1 \wedge \frac{\delta(y)^{\alpha/2}}{|y|^\alpha}\right) dy \geq \alpha\, c.$$

∎

Proposition 2.11 *Measure $\alpha(2-\alpha)\delta(z)^{-\alpha/2}(1+r_0^{-1}\delta(z))^{-\alpha/2}|x-z|^{-n}\,dz$ converges weakly on $\overline{D}$ to the measure $\gamma(z)\,|x-z|^{-n}\sigma(dz)$, where σ is the normalized surface measure on ∂D and $\gamma(z)$ is a measurable function defined on ∂D such that $c^{-1} \leq \gamma \leq c$ on ∂D for some constant $c > 1$.*

Proof. This just follows from the co-area formula (see, e,g, Federer [26]). ∎

Remark 2.3: For Brownian Motion case (that is, for $\alpha = 2$), it is well known that for a bounded $C^{1,1}$ domain domain D and $x \in D$, $P_x(X_{\tau_D} \in dz) = K_D(x,z)\sigma(dz)$, with

$$K_D(x,z) = \frac{\partial}{\partial n_z} G_D(x,z)$$

and

$$\frac{\delta(x)}{c_2|x-z|^n} \le K_D(x,z) \le \frac{c_2\delta(x)}{|x-z|^n} \quad \text{for } x \in D,\ z \in \partial D.$$

Here n_z is the inward unit normal vector of D at $z \in \partial D$. The above upper bound was obtained by Widman in [45] and lower bound was obtained by Zhao in [46]. Since the symmetric α-stable process converges weakly to Brownian motion W with infinitesimal generator Δ in $D([0,\infty), \mathbf{R}^n)$ as $\alpha \uparrow 2$, the corresponding exit distribution (or harmonic measure) from D converges to that of Brownian motion W (cf., e.g., [25]). Thus the results in Theorem 2.10 and Proposition 2.11 yield another (and unified) proof for a two-sided estimate for the Poisson kernels of Brownian motion by letting $\alpha \uparrow 2$.

2.3 Estimates on bounded $C^{1,1}$ open sets

One can also get two-sided estimates on G_D for bounded $C^{1,1}$ open set D (not necessarily connected). Kulczycki showed in [35] that when dimension $n \ge 3$ there is a constant $c = c(D,\alpha) > 1$ such that

$$c^{-1} \min\left\{ |x-y|^{\alpha-n},\ \frac{\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}}{|x-y|^n} \right\} \le G_D(x,y) \le c \min\left\{ |x-y|^{\alpha-n},\ \frac{\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}}{|x-y|^n} \right\}$$

In fact, we have the following.

Theorem 2.12 *Let D be a bounded $C^{1,1}$ open set in $\mathbf{R}^n$ with $n \ge 2$ and $\delta(x) = \mathrm{dist}(x, \partial D)$. Then there is a constant $c_2 = c_2(D) > 1$ that depends only on D such that the estimates in (2.13)-(2.14) hold for x, y in a same component of D, and*

$$\frac{\alpha(2-\alpha)\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}}{c_2\,|x-y|^n} \le G_D(x,y) \le \frac{c_2\,\alpha(2-\alpha)\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}}{|x-y|^n} \tag{2.18}$$

for x and y in two different components of D. By (2.3), the constant $c_2 = c_2(D)$ can be so chosen that it is invariant under the rigid translation and dilation of D.

Proof. When x, y are in the same component of D, it is easy to see that estimates (2.13)-(2.14) hold by slightly modifying the argument as that in the proof of Theorem 2.4. So we assume now that x and y are in two different components of D. Let D_x and D_y denote the two different components of D that contain x and y, respectively. In the following, c_2 is a constant that depends only on D but its value may change from line to line. By the lower bound for $G_D(u,y)$ with $u \in D_x$ and Theorem 2.10

$$
\begin{aligned}
G_D(x,y) &= E_x[G_D(X_{\tau_{D_x}}, y)] \\
&\geq \int_{D_y} K_{D_x}(x,u) G_D(u,y) du \\
&\geq c_2^{-1}\, \alpha(2-\alpha)\delta(x)^{\alpha/2} \int_{D_y} \frac{G_D(u,y)}{\delta_{D_x}(u)^{\alpha/2}(1+r_0^{-1}\delta_{D_x}(u))^{\alpha/2}|x-u|^n}\, du \\
&\geq c_2^{-1}\, \alpha(2-\alpha)\delta(x)^{\alpha/2}\delta(y)^{\alpha/2} \int_{D_y} \alpha \min\left\{ \frac{1}{\delta(y)^{\alpha/2}|u-y|^{n-\alpha}},\, \frac{\delta(u)^{\alpha/2}}{|u-y|^n} \right\} du \\
&\geq c_2^{-1}\alpha(2-\alpha)\delta(x)^{\alpha/2}\delta(y)^{\alpha/2} \\
&\geq \frac{\alpha(2-\alpha)\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}}{c_2\, |x-y|^n}
\end{aligned}
$$

For the upper bound, note that

$$
\begin{aligned}
G_D(x,y) &= E_x[G_D(X_{\tau_{D_x}}, y)] \\
&= \int_{D_x^c \cap D_y} K_{D_x}(x,u) G_D(u,y) du + \int_{D \cap D_x^c \setminus D_y} K_{D_x}(x,u) G_D(u,y) du
\end{aligned}
\tag{2.19}
$$

For $u \in D_x^c \cap D_y$ and $v \in D \cap D_y^c$, by Theorem 2.10

$$
K_{D_x}(x,u) \leq c_2\, \alpha(2-\alpha)\delta(x)^{\alpha/2} \quad \text{and} \quad K_{D_y}(y,v) \leq c_2\, \alpha(2-\alpha)\delta(y)^{\alpha/2}. \tag{2.20}
$$

Therefore by inequalities (2.14) and (2.20)

$$
\begin{aligned}
&\int_{D_x^c \cap D_y} K_{D_x}(x,u) G_D(u,y) du \\
\leq\ & c_2\, \alpha(2-\alpha)\delta(x)^{\alpha/2} \int_{D_x^c \cap D_y} G_D(u,y) du \\
\leq\ & c_2\, \alpha(2-\alpha)\delta(x)^{\alpha/2} \int_{D_x^c \cap D_y} \alpha \min\left\{ \frac{\delta(y)^{\alpha/2}}{|u-y|^{n-\frac{\alpha}{2}}},\, \frac{\delta(y)^{\alpha/2}\delta(u)^{\alpha/2}}{|u-y|^n} \right\} du \\
\leq\ & c_2\, \alpha(2-\alpha)\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}
\end{aligned}
$$

Similarly, by (2.1)-(2.2) and (2.20),

$$
\begin{aligned}
&\int_{D \cap D_x^c \setminus D_y} K_{D_x}(x,u) G_D(u,y) du \\
=\ & \int_{D \cap D_x^c \setminus D_y} K_{D_x}(x,u) \left(\int_{D \cap D_y^c} K_{D_y}(y,v) G_D(u,v) dv \right) du \\
\leq\ & c_2\, \alpha(2-\alpha)\delta(x)^{\alpha/2}\delta(y)^{\alpha/2} \int_{D \cap D_x^c \setminus D_y} \left(\int_{D \cap D_y^c} \alpha(2-\alpha) G_{\mathbf{R}^n}(u,v) dv \right) du
\end{aligned}
$$

$$
\begin{aligned}
&\le c_2\, \alpha(2-\alpha)\delta(x)^{\alpha/2}\delta(y)^{\alpha/2} \int_{D\cap D_x^c\setminus D_y} \left(\int_{D\cap D_y^c} \frac{\alpha^2(2-\alpha)}{(n-\alpha)|u-v|^{n-\alpha}} dv \right) du \\
&\le c_2\, \alpha(2-\alpha)\delta(x)^{\alpha/2}\delta(y)^{\alpha/2} \int_{D\cap D_x^c\setminus D_y} \alpha\, du \\
&\le c_2\, \alpha(2-\alpha)\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}.
\end{aligned}
$$

Hence by (2.19)

$$
G_D(x,y) \le c_2\, \alpha(2-\alpha)\delta(x)^{\alpha/2}\delta(y)^{\alpha/2} \le \frac{c_2\, \alpha(2-\alpha)\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}}{|x-y|^n}.
$$

for $x, y \in D$ in two different components. ∎

Remark 2.4: Since there is a constant $c = c(D) > 1$ that depends only on D such that $1/c \le |x-y| \le c$ for any x, y in two different components of D, thus for such x and y in D,

$$
\frac{\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}}{|x-y|^n} \le c^{2\alpha}|x-y|^{\alpha-n}
$$

and therefore (2.18) can also be rewritten as

$$
\begin{aligned}
\frac{\alpha(2-\alpha)}{c_2} \min\left\{ |x-y|^{\alpha-n},\, \frac{\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}}{|x-y|^n} \right\} &\le G_D(x,y) \\
\le c_2\, \alpha(2-\alpha) \min\left\{ |x-y|^{\alpha-n},\, \frac{\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}}{|x-y|^n} \right\}& \qquad (2.21)
\end{aligned}
$$

It follows in particular that $\lim_{\alpha\uparrow 2} G_D(x,y) = 0$ for x, y in two different components of D. This is expected since as we noted in Remark 2.2 that the Green function of the symmetric α-stable process in D converges to that of Brownian motion W in D.

One can also get two-sided estimates on Poisson kernel K_D for bounded $C^{1,1}$ open set D.

Theorem 2.13 *Let D be a bounded $C^{1,1}$ open set in $\mathbf{R}^n$ with $n \ge 2$ and $\delta(x) = \mathrm{dist}(x, \partial D)$. There is a constant $c_3 = c_3(D) > 1$ that depends only on D such that*

$$
c_3^{-1} h(x,z) \le K_D(x,z) \le c_3\, h(x,z) \quad \text{for } x \in D,\ z \in \overline{D}^c, \qquad (2.22)
$$

where

$$
h(x,z) = \frac{\alpha(2-\alpha)\,\delta(x)^{\alpha/2}}{|x-z|^n} \left(\frac{1}{\delta_{D_x}(z)^{\alpha/2}(1+r_0^{-1}\delta_{D_x}(z))^{\alpha/2}} + \frac{2-\alpha}{\delta(z)^{\alpha/2}(1+r_0^{-1}\delta(z))^{\alpha/2}} \right).
$$

Here D_x is the component of D that contains x. By (2.5), the constant $c_3 = c_3(D)$ here can be so chosen that it is invariant under the rigid translation and dilation of D.

Proof. The above estimates follow from Theorem 2.1 and Theorem 2.12, in a much the same way as that in the proof of Theorem 2.10. We omit the details here. ■

Remark 2.5: In view of Remark 2.3, for each $x \in D$ the exit distribution $K_D(x,z)dz$ of the symmetric α-stable process in D converges weakly on $\overline{D}$ to the exit distribution $\mu_x(dz)$ of Brownian motion from the bounded $C^{1,1}$ open set D as $\alpha \uparrow 2$. By passing $\alpha \uparrow 2$ in Theorem 2.13, we have by Proposition 2.11 that $\mu_x(dz) = K_D(x,z)\sigma_{D_x}(dz)$, where D_x is the component of D that contains x and σ_{D_x} is the Lebesgue surface measure on ∂D_x; furthermore

$$\frac{\delta(x)}{c_3\,|x-z|^n} \le K_D(x,z) \le \frac{c_3\delta(x)}{|x-z|^n} \quad \text{for } x \in D,\ z \in \partial D_x.$$

The following is a direct consequence of Green function and Poisson kernel estimates.

Corollary 2.14 *(3G Theorem) For a bounded $C^{1,1}$ domain D in $\mathbf{R}^n$ with $n \ge 2$, there is a constant $c = c\,(D) > 0$ that depends only on D such that*

$$\frac{G_D(x,y)G_D(y,w)}{G_D(x,w)} \le \frac{c\,\alpha}{n-\alpha}\,\frac{|x-w|^{n-\alpha}}{|x-y|^{n-\alpha}|y-w|^{n-\alpha}} \tag{2.23}$$

and

$$\frac{G_D(x,y)K_D(y,z)}{K_D(x,z)} \le \frac{c\,\alpha}{n-\alpha}\,\frac{|x-z|^{n-\alpha}}{|x-y|^{n-\alpha}|y-z|^{n-\alpha}}. \tag{2.24}$$

Proof. It follows from (2.14) in Theorem 2.4 that

$$G_D(x,y) \le \frac{c_2\alpha\,\delta(x)^{\alpha/2}}{\delta(y)^{\alpha/2}|x-y|^{n-\alpha}}\,\min\left\{\frac{\delta(y)^{\alpha/2}}{|x-y|^{\alpha/2}},\, \frac{\delta(y)^{\alpha}}{|x-y|^{\alpha}}\right\} \le \frac{c_2\alpha\,\delta(x)^{\alpha/2}}{\delta(y)^{\alpha/2}|x-y|^{n-\alpha}}.$$

Hence by the symmetry of $G_D(x,y)$, we have

$$G_D(x,y) \le c_2\,\alpha\,\min\left\{\frac{\delta(x)^{\alpha/2}}{\delta(y)^{\alpha/2}|x-y|^{n-\alpha}},\, \frac{\delta(y)^{\alpha/2}}{\delta(x)^{\alpha/2}|x-y|^{n-\alpha}}\right\}. \tag{2.25}$$

The rest of proof is the same as that of Theorem 1.6 in [13], by using estimates in Theorem 2.4, (2.25), and Theorem 2.10. So we omit the details here. ■

By Theorems 2.12-2.13, one can similarly get $3G$ estimates on bounded $C^{1,1}$ open sets. While the Green functions and Poisson kernel estimates in Theorems 2.4, 2.10 and 2.12-2.13 can no longer hold on general bounded Lipschitz domains or open sets, the above $3G$ estimates can be extended to more general open sets with constant $c\,\alpha/(n-\alpha)$ being replaced by a constant $c = c\,(D,\alpha)$ that depends on D and α. It was extended to bounded Lipschitz domain in [16] and to bounded κ-fat open sets (see below for its definition) in [43], using boundary Harnack principle. The feature in above corollary is that it gives comparing constants with explicit information on how they depend on parameter α. The following definition is taken from [43].

Definition 2.1 *Let $\kappa \in (0,1)$. An open set D in $\mathbf{R}^n$ is said to be κ-fat if there exists $r_0 > 0$ such that for each $z \in \partial D$ and $r \in (0, r_0)$, $D \cap B(z,r)$ contains a ball $B(A_r(z), \kappa r)$ with radius κr.*

Note that any bounded Lipschitz domain is κ-fat for some $\kappa > 0$.

3 Harmonic Functions and Integral Representation

Martin boundary and integral representation for harmonic functions of diffusions processes (or of elliptic differential operators) are well studied. However there was little detailed analysis of these for Markov processes with jumps (or for integro-differential operators) until very recently. In this section we take a closer look at an important class of discontinuous Markov processes—symmetric α-stable processes (with $0 < \alpha < 2$), and survey the notion and integral representation of harmonic functions for these processes studied in [15] and in [11], where some new phenomena arise.

3.1 Two notions of harmonicity

Unlike the Brownian case, there are two different kinds of harmonicity with respect to symmetric stable processes, one kind are functions harmonic in D with respect to the subprocess X^D killed upon leaving D; the other are functions harmonic in D with respect to the global process X, which is used in analysis (cf. Landkof [38]). The precise definitions of these two kinds of harmonic functions are as follows.

Definition 3.1 *Let D be an open set in $\mathbf{R}^n$. A locally integrable function f defined on D taking values in $(-\infty, \infty]$ and satisfying the condition $\int_{\{|x|>1\}\cap D} |f(x)||x|^{-(n+\alpha)}dx < \infty$ is said to be*

1) harmonic with respect to X^D if f is continuous in D and for each $x \in D$ and each ball $B(x,r)$ with $\overline{B(x,r)} \subset D$,

$$f(x) = E_x[f(X_{\tau_{B(x,r)}}); \tau_{B(x,r)} < \tau_D];$$

2) superharmonic respect to X^D if f is lower semicontinuous in D and for each $x \in D$ and each ball $B(x,r)$ with $\overline{B(x,r)} \subset D$,

$$f(x) \geq E_x[f(X_{\tau_{B(x,r)}}); \tau_{B(x,r)} < \tau_D].$$

The next definition is taken from Landkof [38].

Definition 3.2 *Let D be an open set in $\mathbf{R}^n$. A locally integrable function f defined on $\mathbf{R}^n$ taking values in $(-\infty, \infty]$ and satisfying the condition $\int_{\{|x|>1\}} |f(x)||x|^{-(n+\alpha)}dx < \infty$ is said to be*

1) harmonic in D with respect to X if f is continuous in D and for each $x \in D$ and each ball $B(x,r)$ with $\overline{B(x,r)} \subset D$, $f(x) = E_x[f(X_{\tau_{B(x,r)}})]$;

2) superharmonic in D with respect to X if f is lower semicontinuous in D and for each $x \in D$ and each ball $B(x,r)$ with $\overline{B(x,r)} \subset D$, $f(x) \geq E_x[f(X_{\tau_{B(x,r)}})]$.

Remark 3.1 (1) If f is a lower semicontinuous function defined on D taking values in $(-\infty, \infty]$, then f is bounded from below on any subdomain whose closure is contained in D. Thus for such kind of function f which is locally integrable and satisfying $\int_{\{|x|>1\}} |f(x)||x|^{-(n+\alpha)}dx < \infty$, the expectations in Definitions 3.1 and 3.2 are well defined.

(2) For a function f which is (super)harmonic with respect to X^D, if we extend it to be zero off the open set D, then the resulting function is (super)harmonic in D with respect to X.

(3) Conversely, if f is non-negative superharmonic in D with respect to X, then clearly it is a superharmonic with respect to X^D.

The following four theorems were proved in Chen and Song [15] (see Theorems 2.1-2.5 and Remark 2.2 there). We refer readers to [15] for the proofs which in fact work for bounded open set D.

Theorem 3.1 *Suppose that D is a bounded open set in $\mathbf{R}^n$. If h is superharmonic in D with respect to X, then for any open set $D_1 \subset \overline{D_1} \subset D$, $E_x[h^-(X_{\tau_{D_1}})] < \infty$ and*

$$h(x) \geq E_x[h(X_{\tau_{D_1}})] \quad \textit{for every } x \in D_1.$$

If h is harmonic in D with respect to X, then for any open set $D_1 \subset \overline{D_1} \subset D$, $h(X_{\tau_{D_1}})$ is P_x-integrable and

$$h(x) = E_x[h(X_{\tau_{D_1}})] \quad \textit{for every } x \in D_1.$$

Similarly, we have the following result for functions harmonic with respect to X^D.

Theorem 3.2 *Suppose that D is a bounded open set in $\mathbf{R}^n$. If h is superharmonic in D with respect to X^D, then for any open set $D_1 \subset \overline{D_1} \subset D$, $E_x[h^-(X^D_{\tau_{D_1}})] < \infty$ and*

$$h(x) \geq E_x[h(X^D_{\tau_{D_1}})] \quad \textit{for every } x \in D_1.$$

If h is harmonic in D with respect to X^D, then for any open set $D_1 \subset \overline{D_1} \subset D$, $h(X^D_{\tau_{D_1}})$ is P_x-integrable and

$$h(x) = E_x[h(X^D_{\tau_{D_1}})] \quad \textit{for every } x \in D_1.$$

Theorem 3.3 *Suppose either (a) D is a bounded open set and h is harmonic in D with respect to X and continuous on $\overline{D}$ or (b) D is a bounded open set satisfying the uniform exterior cone condition. Then $h(X_{\tau_D})$ is P_x-integrable and*

$$h(x) = E_x[h(X_{\tau_D})], \qquad \textit{for each } x \in D.$$

Obviously there are plenty of bounded functions which are harmonic in D with respect to the global processes X. The following results says that, when D is a bounded domain satisfying the uniform exterior cone condition, the only bounded function which is harmonic in D with respect to X^D is constant zero. This is a truthful reflection of the fact that for such domain D, $P_x(X_{\tau_D} \in \partial D) = 0$ for $x \in D$.

Theorem 3.4 *Suppose that D is a bounded domain in $\mathbf{R}^n$ satisfying the uniform exterior cone condition. If h is a bounded function harmonic in D with respect to X^D, then h must be identically zero.*

3.2 Martin kernel and Martin boundary

Superharmonic and harmonic functions with respect to X^D have been studied in the context of general theory of Markov processes and their potential theory (see, for instance, Kunita-Watanabe [37]).

Fix $x_0 \in D$ and set

$$M_D(x, y) = \frac{G_D(x, y)}{G_D(x_0, y)}, \qquad x, y \in D.$$

There is a compactification D^M of D, unique up to a homeomorphism, such that $M_D(x, y)$ has a continuous extension to $D \times (D^M \setminus \{x_0\})$ and $M_D(\cdot, z_1) = M_D(\cdot, z_2)$ if and only if $z_1 = z_2$ (see, for instance, [37]). The set $\partial_M D = D^M \setminus D$ is called the Martin boundary of D.

A nonnegative harmonic function u on D with respect to X^D is minimal if, whenever $u \geq v$ where v is a nonnegative harmonic function on D with respect to X^D, then $u = cv$ on D for some constant $c \geq 0$. The set of points $z \in \partial^M D$ such that $M_D(\cdot, z)$ is minimal harmonic in D with respect to X^D is called the minimal Martin boundary of D.

The following theorem was proved independently in Bogdan [12] and in Chen-Song [15]. The key to the proof is a boundary Harnack principle on bounded Lipschitz domains due to Bogdan [11] (see Theorem 3.11 below).

Theorem 3.5 *Let D be a bounded Lipschitz domain in $\mathbf{R}^n$ with $n \geq 2$. Then*
(1) $M_D(x, y) = G_D(x, y)/G_D(x_0, y)$ converges to a minimal harmonic function $M_D(x, z)$ in $x \in D$ with respect to X^D when $y \to z \in \partial D$. Furthermore, $M_D(\cdot, z_1) \neq M_D(\cdot, z_2)$ if $z_1 \neq z_2$.

(2) Both the Martin boundary and the minimal Martin boundary of D can be identified with its Euclidean boundary ∂D. Thus any positive superharmonic function f with respect to X^D has a unique representation

$$f(x) = \int_D G_D(x,y)\nu(dy) + \int_{\partial D} M_D(x,z)\mu(dz)$$

where ν and μ are finite measures on D and ∂D respectively.

Remark 3.2: (1) Bogdan showed in [12] that for $x \in D$ and $z \in \partial D$,

$$M_D(x,z) = \lim_{\overline{D}^c \ni w \to z} \frac{K_D(x,w)}{K_D(x_0,w)}.$$

(2) Theorem 3.5 in fact holds for bounded Lipschitz open sets as well. Song and Wu [43] extended the above theorem to bounded κ-fat open sets after they established Theorem 3.13.

From the explicit expression of Green function on ball $B(0,r)$ in (2.3) and the L'Hôpital's rule, it is easy to see that if we select $x_0 = 0$, then

$$M_{B(0,r)}(x,z) = \frac{(r^2 - |x|^2)^{\alpha/2}}{|x-z|^n} \quad \text{for } |x| < r \text{ and } |z| = r.$$

Theorem 3.6 *Let D be a bounded $C^{1,1}$ domain in $\mathbf{R}^n$ with $n \geq 2$ and $\delta(x) = \mathrm{dist}(x, \partial D)$.*
(1) There is a constant $c = c(D, x_0)$ that depends only on D and x_0 such that for $x \in D$ and $z \in \partial D$,

$$\frac{\delta(x)^{\alpha/2}}{c\,|x-z|^n} \leq M_D(x,z) \leq \frac{c\,\delta(x)^{\alpha/2}}{|x-z|^n}; \tag{3.1}$$

(2) There is a constant $c = c(D, x_0) > 0$ that depends only on D and x_0 such that

$$\frac{G_D(x,y)M_D(y,z)}{M_D(x,z)} \leq \frac{c\,\alpha}{n-\alpha}\,\frac{|x-z|^{n-\alpha}}{|x-y|^{n-\alpha}|y-z|^{n-\alpha}}$$

for $x, y \in D, z \in \partial D$.

Proof. (1) For $x \in D$ and $z \in \partial D$, by the lower bound in (2.13) and upper bound in (2.14),

$$M_D(x,y) = \lim_{y \to z,\, y \in D} \frac{G_D(x,y)}{G_D(x_0,y)} \leq c_2^2\,\frac{\delta(x)^{\alpha/2}|z-x_0|^n}{|x-z|^n}.$$

Again, by the lower bound in (2.13) and upper bound in (2.14),

$$M_D(x,y) = \lim_{y \to z,\, y \in D} \frac{G_D(x,y)}{G_D(x_0,y)} \geq c_2^{-2}\,\frac{\delta(x)^{\alpha/2}|z-x_0|^n}{|x-z|^n}.$$

This proves (3.1).

(2) follows immediately from Corollary 2.14, the $3G$ estimate for Green functions. ∎

Remark 3.3: (1) It is proved in Chen and Song [15] that for any bounded $C^{1,1}$ domain, there is a constant $c = c\,(D, \alpha)$ such that

$$\begin{aligned}\frac{\delta(x)^{\alpha/2}}{c\,|x-z|^n} \;&\leq\; M_D(x,z) \;\leq\; \frac{c\,\delta(x)^{\alpha/2}}{|x-z|^n};\\ \frac{G_D(x,y)M_D(y,z)}{M_D(x,z)} \;&\leq\; \frac{c\,|x-z|^{n-\alpha}}{|x-y|^{n-\alpha}|y-z|^{n-\alpha}}\end{aligned}$$

for $x, y \in D, z \in \partial D$.

(2) Suppose that D is a bounded $C^{1,1}$ open set. Let $x_0 \in D$ be the reference point in defining the Martin kernel $M_D(x,z)$ and denote D_{x_0} the component of D that contains x_0. It follows from Theorem 2.12 that there is a constant $c = c\,(D, x_0)$ such that for $x \in D_{x_0}$,

$$\frac{\delta(x)^{\alpha/2}}{c\,|x-z|^n} \;\leq\; M_D(x,z) \leq \frac{c\,\delta(x)^{\alpha/2}}{|x-z|^n} \quad \text{for } z \in \partial D_{x_0}, \tag{3.2}$$

$$\frac{(2-\alpha)\,\delta(x)^{\alpha/2}}{c\,|x-z|^n} \;\leq\; M_D(x,z) \leq \frac{c\,(2-\alpha)\,\delta(x)^{\alpha/2}}{|x-z|^n} \quad \text{for } z \in \partial D \setminus \partial D_{x_0}. \tag{3.3}$$

One can also get $3G$ estimates on bounded $C^{1,1}$ open set D by using estimates in Theorem 2.12.

(3) The $3G$ theorem in Theorem 3.6 (2) was extended to bounded Lipschitz domain in [16] and to bounded κ-fat open sets in [43] using boundary Harnack principle, with constant $c\,\alpha/(n-\alpha)$ being replaced by a constant $c = c\,(\alpha, D, x_0)$ that depends on α, D and x_0.

3.3 Integral representation and uniqueness

The Martin boundary theory in Theorem 3.5 above only gives an integral representation for positive functions harmonic in a domain D with respect to the killed process X^D. Now we study the integral representations of positive functions harmonic in a domain D with respect to the global process X.

From Definition 3.2 for harmonic function f in D with respect to X, its value on D^c is part of the definition of harmonicity. It is shown in Chen and Song [15] that the value of a harmonic function f on D^c is in fact uniquely determined by its value in D.

Theorem 3.7 *(Theorem 4.2 of [15]) Suppose that D is a bounded domain in $\mathbf{R}^n$. If f and g are both harmonic in D with respect to X with $f = g$ in an open subset of D, then $f = g$ in $\mathbf{R}^n$.*

Proof. There is a ball $B(x_0,r) \subset \overline{B(x_0,r)} \subset D$ such that $f = g$ on $B(x_0,r)$. From Theorem 3.1 we have

$$E_x[(f-g)(X_{\tau_{B(x_0,r)}})] = 0 \quad \text{ for } x \in B(x_0,r).$$

By Theorem 2.1, for all $x \in B(x_0,r)$,

$$\begin{aligned} E_x[(f-g)(X_{\tau_{B(x_0,r)}})] &= \int_{B(x_0,r)^c} K_{B(x_0,r)}(x,z)(f-g)(z)dz \\ &= A(n,\alpha)\int_{B(x_0,r)^c}\left(\int_{B(x_0,r)} \frac{G_{B(x_0,r)}(x,y)}{|y-z|^{n+\alpha}}dy\right)(f-g)(z)dz \\ &= A(n,\alpha)\int_{B(x_0,r)} G_{B(x_0,r)}(x,y)\left(\int_{B(x_0,r)^c} \frac{(f-g)(z)}{|y-z|^{n+\alpha}}dz\right)dy. \end{aligned}$$

Therefore by general potential theory (see Section 5.2 of [17], for instance),

$$\int_{B(x_0,r)^c} \frac{(f-g)(z)}{|y-z|^{n+\alpha}}dz = 0$$

for almost every and therefore for each $y \in B(x_0,r)$. Using induction, it can be shown that for all integers $0 \le m \le k$,

$$\int_{B(x_0,r)^c} \prod_{j=1}^{m}(x_{i_j}-y_{i_j})|x-y|^{-(n+\alpha+2k)}\left(f(y)-g(y)\right)dy = 0, \qquad x \in B(x_0,r), \tag{3.4}$$

where for each j, $1 \le i_j \le n$. (For details, see the proof of Lemma 4.1 in [15].) Evaluate identity (3.4) at $x = x_0$, we get that for any non-negative integer k, and any multi–index $\beta = (\beta_1, \cdots, \beta_n)$ with $|\beta| = \beta_1 + \cdots + \beta_n \le k$,

$$\int_{B(x_0,r)^c} (y-x_0)^\beta |y-x_0|^{-2k} \frac{f(y)-g(y)}{|y|^{n+\alpha}}dy = 0.$$

Since the linear span of the set $\{(y-x_0)^\beta|y-x_0|^{-2k} : |\beta| \le k\}$ is an algebra of real–valued continuous functions on $B(x_0,r)^c$ which separates points in $B(x_0,r)^c$ and vanishes at infinity, by the Stone–Weierstrass Theorem the linear span of $\{(y-x_0)^\beta|y-x_0|^{-2k} : |\beta| \le k\}$ is dense in $C_\infty(B(x_0,r)^c)$ with respect to the uniform topology. Here $C_\infty(B(x_0,r)^c)$ is the space of continuous functions on $B(x_0,r)^c$ which vanishes at infinity. Thus for all $\phi \in C_\infty(B(x_0,r)^c)$,

$$\int_{B(x_0,r)^c} \phi(y)\frac{f(y)-g(y)}{|y|^{n+\alpha}}dy = 0$$

which implies that $f - g = 0$ almost everywhere on $B(x_0,r)^c$. ∎

Theorem 3.8 *(Theorem 4.3 of [15]) Suppose D is a bounded Lipschitz domain in $\mathbf{R}^n$. If f is a non-negative harmonic function in D with respect to X, then there exists a unique finite measure μ on ∂D such that the restriction of f to D can be written as*

$$f(x) = \int_{D^c} K_D(x,z)f(z)dz + \int_{\partial D} M_D(x,z)\mu(dz), \qquad \forall x \in D. \tag{3.5}$$

Proof. Using Fatou's lemma and the fact that $P_x(X_{\tau_D} \in \partial D) = 0$ for $x \in D$, it can be shown that $f(x) - E_x[f(X_{\tau_D})] \geq 0$ for $x \in D$. (For details, see the proof of Theorem 4.3 in [15].) Thus $f(x) - \int_{D^c} K_D(x,z)f(z)dz$ is a nonnegative harmonic function in $x \in D$ with respect to X^D. Hence by Theorem 3.5(2), there is a unique finite measure μ on ∂D such that $f(x) - \int_{D^c} K_D(x,z)f(z)dz = \int_{\partial D} M_D(x,z)\mu(dz)$ for $x \in D$. ∎

From the above theorem we can easily get the following

Theorem 3.9 *(Theorem 4.4 of [15]) If D is a bounded Lipschitz domain, then the restriction to D of any non-negative function f which is superharmonic in D with respect to X can be written as*

$$f(x) = \int_{D^c} K_D(x,z)f(z)dz + \int_D G_D(x,y)\nu(dy) + \int_{\partial D} M_D(x,z)\mu(dz),$$

where ν and μ are finite measures on D and ∂D respectively.

Proof. Similar to the above proof of Theorem 3.8, we have that the function

$$f(x) - E_x[f(X_{\tau_D})]$$

is a non-negative function which vanishes outside D and is superharmonic in D with respect to X. Hence it is a non-negative function which is harmonic in D with respect to X^D. Now the Theorem follows from Theorem 3.5. ∎

However, the above decomposition is not unique anymore. This non-uniqueness is due to Theorem 2.1 so one can absorb the first term on the right hand side above, or part of it, into the second term.

3.4 Boundary Harnack principle

When D is a bounded $C^{1,1}$ in $\mathbf{R}^n$, the Poisson kernel estimate in Theorem 2.10 immediately gives the following boundary Harnack principle for nonnegative harmonic functions in D with respect to X.

Theorem 3.10 *(Theorem 1.7 of [13]) Suppose that D is a bounded $C^{1,1}$ domain in $\mathbf{R}^n$ with $n \geq 2$, V is an open set of $\mathbf{R}^n$ and K is a compact subset of V. Then there is a constant $c = c(D,V,K) > 1$ such that for any two nonnegative harmonic functions u, v in D with respect to the symmetric α-stable process X, which are strictly positive and bounded on $V \cap D$, and vanish on $V \cap D^c$, we have*

$$u(x) \leq cu(y) \quad \text{for any } x, y \in K \cap D; \tag{3.6}$$

In particular,

$$\frac{u(x)}{v(x)} \leq c^2 \, \frac{u(y)}{v(y)}, \quad \text{for any } x, y \in K \cap D.$$

Proof. One can find a $C^{1,1}$ domain D_1 which is relatively compact in $D \cap V$ such that $K \cap D \subset D_1$. Then by Theorem 3.3(b),

$$u(x) = E_x u(X_{\tau_{D_1}}) = \int_{D_1^c \cap (D^c \cap V)^c} u(z) K_{D_1}(x, z) dz.$$

for $x \in D_1$. However for $x, y \in K \cap D_1$ and $z \in D_1^c \cap (D^c \cap V)^c$, by Theorem 2.10 there is a constant $c = c\,(D, V, K) > 1$ such that

$$K_{D_1}(x, z) \le c\, K_{D_1}(y, z).$$

This implies $u(x) \le c\, u(y)$ for any $x, y \in K \cap D$. ■

In [11], Bogdan proved the following important result.

Theorem 3.11 *Suppose that D is a bounded Lipschitz domain in $\mathbf{R}^n$ with $n \ge 2$, V is an open set of $\mathbf{R}^n$ and K is a compact subset of V. Then there is a constant $c = c\,(\alpha, D, V, K) > 1$ such that for any two nonnegative harmonic functions u, v in D with respect to the symmetric α-stable process X, which are strictly positive and bounded on $V \cap D$, and vanish on $V \cap D^c$ satisfying $u(x_0) = v(x_0)$ for some $x_0 \in K \cap D$,*

$$c^{-1} u(x) \le v(x) \le c u(x) \quad \text{for any } x \in K \cap D;$$

Moreover there is a constant $\nu = \nu(\alpha, D, V, K) > 0$ such that the function $u(x)/v(x)$ is ν-Hölder continuous in $K \cap D$. In particular, for every $z \in \partial D \cap V$, $\lim u(x)/v(x)$ exists as $D \ni x \to z$.

In the process of proving above theorem, Bogdan showed the following local version of boundary Harnack inequality. In the statement below, a function h is said to be regular α-harmonic in an open set $U \subset \mathbf{R}^n$ if h is a harmonic function in U with respect to symmetric α-stable process X with

$$h(x) = E_x[h(X_{\tau_U})], \qquad \forall x \in U.$$

Theorem 3.12 *(Lemmas 13 and 16 of [11]) Suppose that D is a bounded Lipschitz domain in $\mathbf{R}^n$ with $n \ge 2$. Then there are positive constants $r_0 = r_0(D)$, $c = c\,(D, \alpha) > 1$ and $\nu = \nu(D, \alpha) \in (0,\, 1)$ such that for any $z \in \partial D$, $0 < r < r_0$, and functions $u,\, v \ge 0$ on $\mathbf{R}^n$ that are strictly positive and regular α-harmonic in $D \cap B(z, 2r)$ vanishing in $D^c \cap B(z, 2r)$,*

$$\frac{u(x)}{v(x)} \le c\, \frac{u(y)}{v(y)} \quad \text{for } x, y \in D \cap B(z, r).$$

Furthermore, the limit $g(z) = \lim_{D \ni x \to z} u(x)/v(x)$ exists and if $u(A_r(z)) = v(A_r(z)) > 0$, then

$$\left| \frac{u(x)}{v(x)} - g(z) \right| \le c\, |x - z|^\nu \quad \text{for } x \in D \cap B(z, r).$$

Recall that bounded Lipschitz domain D is κ-fat for some $\kappa > 0$ and $A_r(z)$ is a point in $D \cap B(z, r)$ such that $B(A_r(z), \kappa r) \subset D \cap B(z, r)$. Very recently, Song and Wu [43] extended above theorem and obtained the following remarkable result.

Theorem 3.13 *(1) Suppose that D is an open set in $\mathbf{R}^n$ with $n \geq 2$, $z \in \partial D$, $r > 0$ and that $B(A, \kappa r)$ is a ball in $D \cap B(z, r)$. Then there is a constant $c = (n, \alpha)$ such that for any functions $u, v \geq 0$ in $\mathbf{R}^n$, positive regular α-harmonic in $D \cap B(z, 2r)$ vanishing in $D^c \cap B(z, 2r)$,*

$$\frac{u(x)}{v(x)} \leq c\kappa^{-n-\alpha} \frac{u(y)}{v(y)} \qquad \text{for } x, y \in D \cap B\left(z,\, r/2\right).$$

(2) Suppose further that D is a bounded open set which is κ-fat. Then there exist positive constants $r_0 = r_0(D)$, $c = c\,(n, \alpha, \kappa)$ and $\nu = \nu(n, \alpha, \kappa)$ such that for all $z \in \partial D$ and $r \in (0, r_0)$, functions $u, v \geq 0$ in $\mathbf{R}^n$, regular α-harmonic in $D \cap B(z, 2r)$ vanishing on $D^c \cap B(z, 2r)$ and satisfying $u(A_r(z)) = v(A_r(z)) > 0$, the limit $g(z) = \lim_{D \ni x \to z} u(x)/v(x)$ exists and

$$\left| \frac{u(x)}{v(x)} - g(z) \right| \leq c\,|x - z|^{\nu} \qquad \text{for } x \in D \cap B(z, r).$$

Boundary Harnack inequality (in the form of Theorem 3.12 and of Theorem 3.13 for bounded κ-fat open sets) has important implications. For example, from it follows easily the $3G$ estimates and the Martin boundary identification.

3.5 Conditional process and its limiting behavior

Suppose that $h > 0$ is a positive superharmonic function with respect to X^D. Note that by Theorem 3.2 above, one has (see, e.g., page 11 of Dynkin [24])

$$h(x) \geq E_x[h(X_t^D)].$$

Define

$$p_D^h(t, x, y) = h(x)^{-1} p_D(t, x, y) h(y), \qquad t > 0, x, y \in D,$$

where p_D is the transition density function of killed symmetric stable process X^D in D. It is easy to check that p_D^h is a transition density function and it determines a Markov process on the state space $D_\partial = D \cup \{\partial\}$. This process is called the h–conditioned symmetric stable process. Its distribution for process starting from x will be denoted as P_x^h.

When D is bounded Lipschitz, Green functions $G_D(\cdot, y)$ and Poisson kernel $K_D(\cdot, z)$ are positive superharmonic function in D with respect to X^D, while Martin kernels $M_D(\cdot, w)$ are positive minimal harmonic function in D with respect to X^D. So one can take h to be $G_D(\cdot, y)$, $K_D(\cdot, z)$ and $M_D(\cdot, w)$. The resulting distribution is denoted as P_x^y, P_x^z and P_x^w, respectively.

The following corollary is an immediate consequence of the $3G$ theorem (Corollary 2.14 and Theorem 3.6(2)).

Corollary 3.14 *(Conditional Lifetimes) For a bounded $C^{1,1}$ domain D in $\mathbf{R}^n$ with $n \geq 2$, there is a constant $c = c(D) > 0$ that depends only on D such that*

$$\sup_{x \in D,\, z \in \mathbf{R}^n \setminus \{x\}} E_x^z(\tau_D) \leq \frac{c\,\alpha}{n-\alpha}$$

As we will see from Corollary 4.4 below, the conditional lifetimes theorem holds on bounded open sets. The feature of this theorem is giving an upper bound with explicit information on its dependence of parameter α. The following theorem, which describes the limiting behavior of the conditional process, is proved in Chen and Song [15] as Theorems 3.17 and 3.18. See also Section 12 of Kunita and Watanabe [37].

Theorem 3.15 *Suppose D is a bounded Lipschitz domain.*
(1) For every $x \in D$ and $z \in \partial D$,

$$P_x^z\left(\tau_D < \infty \quad \text{and} \quad \lim_{t \uparrow \tau_D} X_t = z\right) = 1.$$

(2) Let h be positive harmonic function h in D with respect to X^D. Then for any Borel measurable subset $A \subset \partial D$,

$$P_x^h\left(\lim_{t \uparrow \tau_D} X_t \in A\right) = \frac{1}{h(x)} \int_A M_D(x,z)\mu(dz).$$

Here μ is a finite positive measure on ∂D such that $h(x) = \int_{\partial D} M_D(x,z)\mu(dz)$ for $x \in D$.

4 Conditional Gauge and Intrinsic Ultracontractivity

Conditional gauge theorem is of great importance in the study of Schrödinger type operators $L + q$, where L is the generator of the process. See [20] and [21] for the history and uses of the conditional gauge theorem for Brownian motions. The study of conditional gauge for discontinuous process is very recent. In [14], Chen and Song showed that the conditional gauge theorem holds for discontinuous symmetric α-stable process on bounded $C^{1,1}$ domains, conditioned according to Green function $G_D(\cdot, y)$ with $y \in D$ and Poisson kernel $K_D(\cdot, z)$ with $z \in \overline{D}^c$. This result was subsequently extended to conditional process conditioned according to Martin kernel $M_D(\cdot, z)$ with $z \in \partial D$, in [15] and to bounded Lipschitz domains in [16]. It was further extended to bounded κ-fat open sets in [43]. To state the conditional gauge theorem, we need first to introduce the notion of Kato class functions.

Definition 4.1 *A Borel function q on $\mathbf{R}^n$ is in the Kato class $\mathbf{K}_{n,\alpha}$ if*

$$\lim_{r \downarrow 0} \sup_{x \in \mathbf{R}^n} \int_{|x-y| \le r} \frac{|q(y)|}{|x-y|^{n-\alpha}} dy = 0.$$

Zhao [48] has the following probabilistic characterization of Kato class functions.

Theorem 4.1 *A Borel measurable function q is in the Kato class $\mathbf{K}_{n,\alpha}$ if and only if*

$$\lim_{t \downarrow 0} \sup_{x \in \mathbf{R}^n} E^x \left[\int_0^t |q(X_s)| ds \right] = 0.$$

where X is the symmetric α-stable process in $\mathbf{R}^n$.

For $q \in \mathbf{K}_{n,\alpha}$, set

$$e_q(t) = \exp\left(\int_0^t q(X_s) ds \right).$$

The function g defined by $g(x) = E_x[e_q(\tau_D)]$ is called the gauge function of (D, q) (for process X). Chung and Rao [19] showed that either $g \equiv \infty$ or g is bounded on D. In the latter case, (D, q) is said to be *gaugeable*.

The conditional gauge function of (D, q) is the expected value of $e_q(\tau_D)$ under the law of conditional process (X, P_x^z), with $x \in D$ and $z \in \mathbf{R}^n \setminus \{x\}$. Recall that when $z \in D \setminus \{x\}$, P_x^z is the law of the $G_D(\cdot,\, z)$-conditioned process starting from x; when $z \in \partial D$, P_x^z is the law of the $M_D(\cdot,\, z)$-conditioned process starting from x; and when $z \in \overline{D}^c$, P_x^z is the law of the $K_D(\cdot,\, z)$-conditioned process starting from x.

Theorem 4.2 *Let D be a bounded Lipschitz domain in $\mathbf{R}^n$ with $n \ge 2$ and $q \in \mathbf{K}_{n,\alpha}$. If (D, q) is gaugeable, then there exists $c = c\,(D, q, \alpha) > 1$ such that*

$$c^{-1} \le \inf_{x \in D,\, z \in \mathbf{R}^n \setminus \{x\}} E_x^z[e_q(\tau_D)] \le \sup_{x \in D,\, z \in \mathbf{R}^n \setminus \{x\}} E_x^z[e_q(\tau_D)] \le c.$$

Idea of Proof. Let L^D denote the infinitesimal generator of X^D. First it can be shown that the semigroup of $L^D + q$ is intrinsically ultracontractive (see below for its definition). This together with $3G$ estimates imply

$$c^{-1} V_q \le G_D \le c\, V_q \quad \text{on } D \times D$$

for some constant $c > 1$, which then leads to the conditional gauge theorem. Here V_q is the 0-resolvent for $L^D + q$. For details of the proof, we refer interested readers to [16]. ∎

Now let us recall the definition of intrinsic ultracontractivity, due to Davies and Simon [22]. Suppose that H is a semibounded self-adjoint operator on $L^2(D)$ with D being a domain in $\mathbf{R}^n$ and that $\{e^{Ht},\, t > 0\}$ is an irreduc ible positivity-preserving semigroup with integral kernel $a(t,x,y)$. Assume that the top of the spectrum μ_0 of H is an eigenvalue. In this case, μ_0 has multiplicity one and the corresponding eigenfunction φ_0, normalized by $\|\varphi_0\|_2 = 1$, can be chosen to be positive almost everywhere on D. φ_0 is called the ground state of H.

Let U be the unitary operator U from $L^2(D, \varphi_0^2(x)dx)$ to $L^2(D)$ given by $Uf = \varphi_0 f$ and define $\widetilde{H}$ on $L^2(D, \varphi_0^2(x)\,dx)$ by

$$\widetilde{H} = U^{-1}(H - \mu_0)\,U.$$

Then $e^{\widetilde{H}t}$ is an irreducible symmetric Markov semigroup on $L^2(D, \varphi_0^2(x)\,dx)$ whose integral kernel with respect to the measure $\varphi_0^2(x)dx$ is given by

$$\frac{e^{-\mu_0 t} a(t,x,y)}{\varphi_0(x)\varphi_0(y)}.$$

Definition 4.2 *H is said to be* **ultracontractive** *if e^{Ht} is a bounded operator from $L^2(D)$ to $L^\infty(D)$ for all $t > 0$. H is said to be* **intrinsically ultracontractive** *if $\widetilde{H}$ is ultracontractive; that is, $e^{\widetilde{H}t}$ is a bounded operator from $L^2(D, \varphi_0^2(x)\,dx)$ to $L^\infty(D, \varphi_0^2(x)\,dx)$ for all $t > 0$.*

Ultracontractivity is connected to the logarithmic Sobolev inequalities. The connection between the logarithmic Sobolev inequalities and L^p to L^q bounds of semigroups was first given by L. Gross [29] in 1975. E. Davies and B. Simons [1] adopted L. Gross's approach to allow $q = \infty$ and therefore established the connection between the logarithmic Sobolev inequalities and ultracontractivity. (For an updated survey on the subject of the logarithmic Sobolev inequalities and contractivity properties of semigroups, see [2], [30].) In [3], R. Bañuelos proved the intrinsic ultracontractivity for Shrödinger operators on uniformly Hölder domains of order $\alpha \in (0,2)$ using the logarithmic Sobolev inequality characterization. In [16], the same strategy is used to establish the intrinsic ultracontractivity for $L^D + q$, where L^D is the generator for symmetric stable processes in some rough domains D that include bounded Lipschitz domains.

In a recent beautiful paper by Kulczycki [36], it is showed that L^D is intrinsically ultracontractive for any bounded open set D in $\mathbf{R}^n$. His proof can be extended to show that the Schrödinger type operator $L^D + q$ is intrinsically ultracontractive for any bounded open set D in $\mathbf{R}^n$ and $q \in \mathbf{K}_{n,\alpha}$.

When D is a bounded open set, it is easy to see that $\sup_{x\in D} E_x[\tau_D] < \infty$ (see, e.g., [18]). and therefore $(D, 0)$ is gaugeable. Hence the first eigenvalue μ_0 of L^D is negative. Let ϕ_0 be the ground state of L^D. Recall that P_D is the transition density function for the killed symmetric stable process X^D. Note also that L^D is intrinsically ultracontractive due to [36]. Similar to Corollary 1 of Bañuelos [3], one has

Theorem 4.3 *Let D be a bounded domain in $\mathbf{R}^n$. There is a constant $c > 0$ such that*

(1) $e^{\mu_0 t}\phi_0(x)\phi_0(y) \le p^D(t,x,y) \le ce^{\mu_0 t}\phi_0(x)\phi_o(y)$ *for all* $x, y \in D$ *and* $t > 1$*;*

(2) Let SH^+ *denote all non-trivial nonnegative superharmonic functions in* D *with respect to* X^D*. Then*

$$\sup_{x\in D,\, h\in \mathrm{SH}^+} E_x^h[\tau_D] < \infty;$$

(3) For $h \in \mathrm{SH}^+$*,*

$$\lim_{t\to\infty} e^{-\mu_0 t} P_x^h(\tau_D > t) = \frac{\phi_0(x)}{h(x)} \int_D \phi_0(y)h(y).$$

In particular, $\lim_{t\to\infty} \frac{1}{t} \log P_x^h(\tau_D > t) = \mu_0$*.*

Note that $G_D(x,y)$ is a superharmonic functions in $x \in D$ with respect to X^D for each fixed $y \in D$. When D is a bounded Lipschitz domain, $G_D(x,y)$, $M_D(x,z)$ and $K_D(x,w)$ are superharmonic functions in $x \in D$ with respect to X^D for each fixed $y \in D$, $z \in \partial D$ and $w \in \overline{D}^c$, respectively. The above theorem in particular implies that

Corollary 4.4 *(Conditional Lifetimes) For any bounded open set* D *in* $\mathbf{R}^n$*,*

$$\sup_{x\in D,\, y\in D\setminus\{x\}} E_x^z[\tau_D] < \infty.$$

If D *is a bounded Lipschitz open set, then*

$$\sup_{x\in D,\, z\in \mathbf{R}^n\setminus\{x\}} E_x^z[\tau_D] < \infty.$$

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