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THE DIRICHLET PROBLEM FOR SEMILINEAR SECOND-ORDER DEGENERATE ELLIPTIC EQUATIONS AND APPLICATIONS TO STOCHASTIC EXIT TIME CONTROL PROBLEMS

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Introduction.

The aim of this article is to study the Dirichlet problem for second-order semilinear degenerate elliptic PDEs

$$-\frac{1}{2}\sum_{i,j} a_{i,j}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + H(x,u,Du) = 0 \quad \text{in } \Omega\,, \tag{1}$$

and the connections of these problems with stochastic exit time control problems.

Here and below Ω stands for a smooth bounded domain in $\mathbb{R}^n$, the solution u is a real-valued function defined on $\overline{\Omega}$, the closure of Ω, its gradient is denoted by $Du = (\frac{\partial u}{\partial x_1},\dots,\frac{\partial u}{\partial x_n})$ and we use also below the notation D^2u for the Hessian matrix of u: $D^2u = \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)_{i,j}$. The nonlinearity H that we will often call the "Hamiltonian", using that way a terminology coming from control theory, is a continuous real function defined on $\overline{\Omega} \times I\!R \times I\!R^n$; precise assumptions on the regularity of $\partial\Omega$ and on H will be made later on.

In the classical theories for such types of second-order elliptic equations (cf. for example, D. Gilbarg and N.S Trudinger [24]), the basic assumption is that the matrix $a(x) = (a_{i,j}(x))_{i,j}$ satisfies

$$\sum_{i,j} a_{i,j}(x)\xi_i\xi_j \geq \nu|\xi|^2 ,$$

for some constant $\nu > 0$ and for all $\xi = (\xi_1, \ldots, \xi_n) \in I\!R^n$ and $x \in \Omega$. We want to investigate here degenerate cases when one allows ν to be equal to zero. For reasons explained below, we will assume the degeneracy of a to be of a particular type: in the case when a is symmetric, this assumption takes the simple form

$$a(x) = \sigma(x)\sigma^T(x) , \tag{2}$$

where $\sigma(.)$ is a $n \times p$ matrix with a Lipschitz continuous dependence in x $(p \leq n)$. If $p < n$, it is clear enough that a being at most of rank p cannot be positive definite.

In the litterature, there is a lot of articles dealing with degenerate elliptic equations but, most of the time, the degeneracy comes from the dependence in Du of the diffusion as in the p-laplacian. Few articles deals with such type of degeneracy in x.

In 1951, M.V Keldysh[36] first pointed out the fact that, in the case of degenerate elliptic equations, defaults of the boundary condition might occur. The systematic study of degenerate elliptic and parabolic <u>linear</u> equations with degeneracy in x, and of the associated Dirichlet problem, was initiated by G. Fichera[18, 19]: he showed that the Dirichlet boundary condition can be imposed only on a suitable part of the boundary and he obtained the existence of weak solutions in the distribution sense, by the mean of elliptic regularizations.

Later on, J.J Kohn and L. Nirenberg [37] and O.A Oleinik[46, 47, 48, 50] generalized these results and obtained, in some cases, the existence of smooth solutions. Then O.A Oleinik[49] gave also certain conditions under which weak solutions are strong and was able in this context to derive the uniqueness of solutions. R.S Phillips and L. Sarason [52], using the results of J.J Kohn and L. Nirenberg, as well as the machinery of symmetric systems of first-order partial differential equations, were able to improve the results of O.A Oleinik, and proved a uniqueness result for weak solutions in an appropriate Hilbert space. Most of these uniqueness results made use of assumptions of "smooth continuability" of the coefficients beyond the boundary. We refer the reader to the book of O.A Oleinik and E.V Radkevic[51] where these results and some of their generalizations are presented.

The Dirichlet problem for again <u>linear</u> equations was also attacked by using probabilistic methods and, in particular, by Feynman-Kac type formulas; we refer the reader to M.I Freidlin[22], D.W Stroock and S.R.S Varadhan[55] and references therein for results in this direction. We end this short list of references (which, of course, is not complete!) by the celebrated works of L. Hörmander[25] on the hypoellipticity of such linear equations. To the best of our knowledge, the only results in the nonlinear case were obtained by A. V. Ivanov[31] under very particular structure conditions on the diffusion and on the nonlinearity.

Recently, questions related to degenerate elliptic equations, even in the more general context of fully nonlinear equations, were considered in a completely different way by using the theory of viscosity solutions. We recall that the notion of viscosity solutions was introduced in 1981 by M.G Crandall and P.L Lions[14] for solving problems related to first-order equations (which are obviously the most degenerate ones!) The first definition and the related proofs were simplified later in a work of M.G Crandall, L.C Evans and P.L Lions[12] (See also P.L Lions[41]). In these works, this notion of solutions was proved to have nice properties in the context of first-order equations: existence, uniqueness, passages to the limit, connections with the applications...etc

Its natural extension to second-order fully nonlinear equations, first given in P.L Lions[42], faced a long time the difficulty connected to the lack of uniqueness results. R. Jensen[32, 33] first broke this difficulty and showed under certain conditions that the Maximum Principle holds for semicontinuous viscosity solutions. Then different Maximum Principle type results for various kind of structure conditions on the equations and of boundary conditions were proved. We refer to the "User's guide" of M.G Crandall, H. Ishii and P.L Lions[13] and to the book of W.H Fleming and H.M Soner[20] for a complete description of all this kind of results but also for a modern presentation of this notion of solution.

Before coming back to the description of some of these results we will use here, we want to point out that the theory of viscosity solutions led to a better understanding of what the "natural" boundary conditions for degenerate elliptic PDEs are. For example, in our case, it is well-known that one cannot solve (1) in general together with a "classical" Dirichlet boundary condition

$$u = \varphi \quad \text{on } \partial\Omega \ , \tag{3}$$

where φ is a continuous function. This condition has in fact to be relaxed in the following way

$$\max\left(-\frac{1}{2}\sum_{i,j} a_{i,j}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + H(x,u,Du), u-\varphi\right) \geq 0 \quad \text{on } \partial\Omega \ , \tag{4}$$

and

$$\min\left(-\frac{1}{2}\sum_{i,j} a_{i,j}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + H(x,u,Du), u-\varphi\right) \leq 0 \quad \text{on } \partial\Omega \ . \tag{5}$$

Of course these inequalities have to be understood in the viscosity sense and we will recall their precise meanings in section I. Roughly speaking, one can say that the equation has to hold up to the boundary where the solution does not assume the boundary data φ.

In order to make these boundary data a bit more natural, we come back now to some properties of viscosity solutions. One of the key result in the theory is a so-called "stability" result that we describe now on a particular example: assume that we have a smooth solution u_ε of the elliptic regularized equation

$$-\varepsilon\Delta u_\varepsilon - \frac{1}{2}\sum_{i,j} a_{i,j}(x)\frac{\partial^2 u_\varepsilon}{\partial x_i \partial x_j} + H(x, u_\varepsilon, Du_\varepsilon) = 0 \quad \text{in } \Omega\,, \tag{6}$$

which satisfies the Dirichlet boundary condition (3) in the classical sense and assume that the u_ε for $0 < \varepsilon \leq 1$ are uniformly bounded in L^∞. Then define $\underline{u}$ and $\overline{u}$ respectively by

$$\underline{u}(x) = \liminf_{\substack{y\to x\\ \varepsilon\to 0}} u_\varepsilon(y)\,,$$

and

$$\overline{u}(x) = \limsup_{\substack{y\to x\\ \varepsilon\to 0}} u_\varepsilon(y)\,.$$

The stability result says that the functions $\underline{u}$ and $\overline{u}$ are respectively viscosity super- and subsolution of (1)-(4) and of (1)-(5). This so-called "half-relaxed limits" method which leads to general stability results of this type for fully nonlinear elliptic PDEs, was introduced by B. Perthame and the first author[3] and developed in a series of works [4, 5].

The advantage of such stability result is to allow passages to the limit in a nonlinear elliptic PDE with just an L^∞ estimate on the u_ε, but it is really useful only if we can connect $\underline{u}$ and $\overline{u}$. The standard way to do it is through a **Strong Comparison Result** i.e. a Maximum Principle type result for semicontinuous viscosity solutions. Indeed, if such a result exists, $\underline{u}$ and $\overline{u}$ being super and subsolution of the same problem, we have

$$\overline{u} \leq \underline{u} \quad \text{in } \Omega\,. \tag{7}$$

But, by their very definition, $\underline{u} \leq \overline{u}$ in Ω, and therefore (7) immediately implies $\overline{u} = \underline{u}$ in Ω. Finally it is a simple exercise to show that this equality implies the local uniform convergence in Ω of u_ε to the continuous function $u := \overline{u} = \underline{u}$ [1], which turn out to be the unique bounded solution of (1)-(4)-(5) by using again such type of Strong Comparison Result.

So a Strong Comparison Result which gives us the inequality (7) is a key result for performing such kind of passage to the limit. This is a Maximum

[1]Notice that $\overline{u}$ is usc and $\underline{u}$ is lsc.

Principle type result since it says that a subsolution of the problem has to be below a supersolution. The term "strong" refers to the comparison of discontinuous sub and supersolutions, the comparison between continuous ones being far easier although being not completely trivial. The difficulty to prove it comes both from the discontinuity of the sub and supersolution to be compared but also from the (admittedly) strange form of the boundary conditions.

The main new result of this paper is an optimal Strong Comparison Result for the Dirichlet problem (1)-(4)-(5) and it is also the key point of our approach: the proof we sketch above shows how it gives existence and uniqueness of a continuous solution through an approximation by an elliptic regularization; existence can also be obtained by the Perron's method introduced in the theory of viscosity solutions by H. Ishii[26]. Moreover, any suitable approximation of the Dirichlet problem would be proven to be convergent by the same type of arguments (See G. Barles and P.E Souganidis[6] for the convergence of numerical schemes).

Notice that because of the a priori existence of a boundary layer, the inequality (7) cannot hold in general up to the boundary $\partial\Omega$, and therefore the combination of the stability result and the Strong Comparison Result has to take care of this boundary layer.

For first-order equations, optimal Strong Comparison Result exists for any type of boundary conditions: the case of Dirichlet boundary conditions was obtained in [5] and the case of Neumann boundary conditions, even the case of nonlinear Neumann boundary conditions, is treated in P.L Lions and the first author[2].

In the case of second-order equations, the case of nonlinear Neumann boundary conditions is also well understood: H. Ishii[29] and the first author[1] obtained complementary results under different types of conditions on the equation and on the regularity of $\partial\Omega$. Since we will meet below these boundary conditions, we will compare these two results more precisely later. Results on oblique derivatives problems in domains with corners were obtained by P. Dupuis and H. Ishii[15, 16]. It is worth mentionning that, for Neumann boundary conditions the above results are proved for fully nonlinear equations, but for Dirichlet boundary conditions, the result of this paper is the first one in the second-order case which really allows losses of boundary datas.

Of course the proof of this type of results requires – in the case of Dirichlet boundary conditions – some conditions on the nonlinearity H, especially near the boundary, which are natural from a stochastic control point of view. So we turn now to the presentation of stochastic exit time control problems.

We are given a system whose state is described by the solution $(X_t)_t$ of the controlled stochastic differential equation

$$dX_t = b(X_t, \alpha_t)dt + \sigma(X_t)dW_t \ , \ X_0 = x \in \Omega \ , \tag{8}$$

where $(\alpha_t)_t$, the control, is some progressively measurable process with respect to the filtration associated to the Brownian Motion $(W_t)_t$, with values in a compact metric space $\mathcal{A}$, and b, σ are continuous functions defined respectively on $\overline{\Omega} \times \mathcal{A}$ and on $\overline{\Omega}$.

Then we define the value function of the exit time control problem by

$$u(x) = \inf_{(\alpha_s)_s} I\!E_x \left[\int_0^\tau f(X_t, \alpha_t) e^{-\lambda t} dt + \varphi(X_\tau) e^{-\lambda \tau} \right] , \tag{9}$$

where $I\!E_x$ denotes the conditional expectation with respect to the event $\{X_0 = x\}$, τ is the first exit time of the trajectory $(X_t)_t$ from Ω, $\lambda > 0$ and f, φ are continuous, real-valued functions defined respectively on $\overline{\Omega} \times \mathcal{A}$ and on $\partial\Omega$. Precise assumptions on the data will be given later on.

We refer the reader interested in stochastic control problems to A. Bensoussan[7], A. Bensoussan and J.L Lions[8, 9] where classical PDE approaches are described and to N. El Karoui[17], N.V Krylov[38] and E.D Sontag[54] where these problems are considered from a probabilistic point of view. The more recent approach by viscosity solutions was first introduced in P.L Lions[42, 43, 44] and is presented in the book of W.H Fleming and H.M Soner[20].

According to optimal control theory, it is natural to think u as being a solution (in fact, the "right solution") of (1)-(4)-(5) where a is given by (2) and

$$H(x,t,p) = \sup_{\alpha \in \mathcal{A}} \{-b(x,\alpha).p + \lambda t - f(x,\alpha)\} , \tag{10}$$

(cf. for example P.L Lions[42, 43, 44] or W.H Fleming and H.M Soner[20]).

If one knows a priori that u is <u>continuous</u>, it is now standard in the theory of viscosity solutions to prove that u is the unique continuous (viscosity) solution of the generalized Dirichlet problem by combining Dynamic Programming Principle type arguments of P.L Lions[42, 43, 44] (cf. also [20]) and the uniqueness argument of H.M Soner[53]. But, in general, it is not known how to prove the continuity of u except in few cases: in the deterministic case, H.M Soner[53] proved this continuity in the State Constraint case (the case when $\varphi \equiv +\infty$) and H. Ishii[27] extended the method to treat any exit time problem. Recently, in the stochastic case, M. Katsoulakis[35] obtained a nice but still not completely general result.

But these proofs of continuity are tedious and difficult to extend to the general cases we want to handle. To avoid them, B. Perthame and the first author[3, 4, 5] used the notion of semicontinuous viscosity sub and supersolutions. The strategy was the following: to prove that the lower semicontinuous envelope of u, denoted by u_*, is a supersolution of (1)-(4)-(5) and to prove that the upper semicontinuous envelope of u, denoted by u^* [2], is a subsolution of (1)-(4)-(5). Then the Strong Comparison Result implies

[2] $u^*(x) = \limsup_{y \to x} u(y)$ and $u_*(x) = \liminf_{y \to x} u(y)$

$$u^* \leq u_* \quad \text{in}\, \Omega\ ,$$

and therefore it turns out that $u = u^* = u_*$ is continuous in Ω and is the unique solution of (1)-(4)-(5). Again the Strong Comparison Result is the key point of this approach.

Concerning the conditions we require on H on the boundary, we recall that it was shown in [5] in the deterministic control case (i.e. $\sigma \equiv 0$) that non-uniqueness features for the associated problem (1)-(4)-(5) come from the role played by vector fields b tangent to the boundary $\partial\Omega$. To avoid this difficulty, H was assumed in [5] to satisfy a "non degeneracy condition" which can be explained in the determistic control case, i.e. for H given by (10), in the following way: if, at $x \in \partial\Omega$, there exists a vector field $b(x,\alpha)$ which is tangent to the boundary then there exists a vector field $b(x,\alpha_1)$ pointing strictly inside Ω and a vector field $b(x,\alpha_2)$ pointing strictly outside Ω. We use here the same control ideas which have to be properly adapted to the stochastic case but it is worth pointing out that our "translation" on how these conditions have to be read on H and σ are completely different.

The plan of this paper is as follows: in section I, we recall basic facts on viscosity solutions in order to be as self-contained as possible; this section is entirely expository. Section II is devoted to state precisely our Strong Comparison Result and to give "PDE type " applications (existence, uniqueness, convergence of approximations...). The proof of this result is given in section III. In section IV, we consider the stochastic exit time control problem; we give two different methods to show that the value function is a solution of the Dirichlet problem: the first one mimics exactly the strategy described above by using a Dynamic Principle type result of V.S Borkar[10] (See also W.H Fleming and H.M Soner[20]) while the second one relies on approximation arguments.

1 Viscosity Solutions of the Dirichlet Problem.

This section is entirely expository: it is written both for the sake of completeness and for the convenience of the reader who is not familiar with viscosity solutions theory.

We first recall the definition of viscosity solutions for fully nonlinear elliptic equations and for general boundary conditions. Then we describe some basic properties of these solutions in the case of the Dirichlet problem for semilinear degenerate elliptic equations (passage to the limit in the approximation by elliptic regularization, conditions for losses of boundary data, solvability of the "classical" Dirichlet problem...). Of course, we refer to the "User's guide" of M.G Crandall, H. Ishii and P.L Lions[13] and to the book of W.H Fleming and H.M Soner[20] for a more detailed introduction and for a wider scope.

We first consider the fully nonlinear elliptic equations

$$F(x,u,Du,D^2u)=0 \quad \text{in } \Omega\ , \tag{11}$$

where F is a continuous real-valued function. The equation (11) is said to be (degenerate) elliptic if F satisfies

$$F(x,u,p,M)\le F(x,u,p,N) \quad \text{if } M\ge N\ , \tag{12}$$

for any $x\in\overline{\Omega}$, $u\in I\!R$, $p\in I\!R^n$ and $M,N\in\mathcal{S}^n$ where $\mathcal{S}^n$ is the space of $n\times n$ symmetric matrices endowed with the usual partial ordering. We complement (11) with a general boundary condition

$$B(x,u,Du)=0 \quad \text{on } \partial\Omega\ , \tag{13}$$

where B is a continuous function.

We choose this general presentation of the definition for the sake of simplicity of notations but also because we are going to use below different types of boundary conditions.

In the case of Dirichlet problem, one has

$$\begin{cases} -\frac{1}{2}\mathrm{Tr}\left[a(x)D^2u\right]+H\left(x,u,Du\right) &= 0 \quad \text{in } \Omega\ , \\ u &= \varphi \quad \text{in } \partial\Omega\ , \end{cases} \tag{14}$$

where Tr denotes the trace operator and therefore F and B are given by

$$F(x,u,p,M)=-\frac{1}{2}\mathrm{Tr}\left[a(x)M\right]+H\left(x,u,p\right)\ ,$$

and

$$B(x,u,p)=u-\varphi(x)\ .$$

The equation is degenerate elliptic when the $n\times n$ matrix a satisfies

$$a(x)\xi.\xi\ge 0 \quad \text{for all } \xi\in I\!R^n\ , \tag{15}$$

for all $x\in\overline{\Omega}$.

The definition is the following

Definition 1.1 : *A bounded usc function u is a viscosity subsolution of* (11)-(13) *iff, for any $\phi\in C^2(\overline{\Omega})$ and for any local maximum point x_0 of $u-\phi$ on $\overline{\Omega}$, one has*

$$\begin{cases} F\left(x_0,u(x_0),D\phi(x_0),D^2\phi(x_0)\right)\le 0 \quad \text{if } x_0\in\Omega\ , \\ \min\left(F\left(x_0,u(x_0),D\phi(x_0),D^2\phi(x_0)\right),B(x_0,u(x_0),D\phi(x_0))\right)\le 0 \quad \text{if } x_0\in\partial\Omega. \end{cases} \tag{16}$$

A bounded lsc function v is a viscosity supersolution of (11)-(13) *iff, for any $\phi\in C^2(\overline{\Omega})$ and for any local minimum point x_0 of $v-\phi$ on $\overline{\Omega}$, one has*

$$\begin{cases} F\left(x_0, v(x_0), D\phi(x_0), D^2\phi(x_0)\right) \geq 0 & \text{if } x_0 \in \Omega\,, \\ \max\left(F\left(x_0, v(x_0), D\phi(x_0), D^2\phi(x_0)\right), B(x_0, v(x_0), D\phi(x_0))\right) \geq 0 & \text{if } x_0 \in \partial\Omega. \end{cases} \tag{17}$$

□

The definition of viscosity <u>solution</u> of (11)-(13) depends slightly on the authors: for some of them it is a *continuous* function which satisfies (16) and (17) while for others it is a possibly *discontinuous* function whose usc and lsc envelopes satisfy respectively (16) and (17).

The following result gives a justification of this definition in the case of the Dirichlet problem

Theorem 1.1 : *Assume that for* $0 < \varepsilon \leq 1$, *there exists a classical solution* $u_\varepsilon \in C^2(\Omega) \cap C(\overline{\Omega})$ *of the Dirichlet problem*

$$\begin{cases} -\varepsilon\Delta u_\varepsilon - \frac{1}{2}\mathrm{Tr}\left[a(x)D^2 u_\varepsilon\right] + H_\varepsilon\left(x, u_\varepsilon, Du_\varepsilon\right) &= 0 \text{ in } \Omega \\ u_\varepsilon &= \varphi \text{ in } \partial\Omega\,. \end{cases} \tag{18}$$

If $||u_\varepsilon||_{L^\infty(\Omega)}$ *is uniformly bounded and if the sequence of continuous functions* $(H_\varepsilon)_\varepsilon$ *converges locally uniformly to* H. *Then the functions* $\overline{u}$ *and* $\underline{u}$ *respectively defined by*

$$\overline{u}(x) = \limsup_{\substack{y \to x \\ \varepsilon \to 0}} \; u_\varepsilon(y)\;,$$

and

$$\underline{u}(x) = \liminf_{\substack{y \to x \\ \varepsilon \to 0}} \; u_\varepsilon(y)\;,$$

are respectively viscosity sub and supersolutions of (14). □

Since the elliptic regularization (18) is the natural way to approximate (14) in order to get the existence of solutions, (16) and (17) appear as being the right properties satisfied by the solution of (14), in particular on the boundary. Theorem 1.1 is again only a particular case of a general method of passage to the limit introduced in G. Barles and B. Perthame[3] and we refer the reader to [13] for a proof in a more general context (See also G. Barles and P.E Souganidis[6]).

We are now going to analyze the possible default of the "classical" Dirichlet boundary condition. To do so, we assume the boundary of Ω to be of class C^2. This implies, in particular, that there exists a positive C^2 function d defined on $\overline{\Omega}$ that agrees with the distance to the boundary in the neighborhood of $\partial\Omega$. We denote by $n(x) = -Dd(x)$, even if x is not on the boundary. If $x \in \partial\Omega$, $n(x)$ is just the unit outward normal to $\partial\Omega$ at x.

To simplify the exposure, we remark that eventhough a is not assumed to be symmetric, we may reduce to the symmetric case. Indeed, for any symmetric matrix M, thanks to the properties of the Trace operator, one has

$$\mathrm{Tr}\,[a(x)M] = \mathrm{Tr}\left[(a(x)M)^T\right] = \mathrm{Tr}\left[Ma^T(x)\right] = \mathrm{Tr}\left[a^T(x)M\right] .$$

Hence

$$\mathrm{Tr}\,[a(x)M] = \mathrm{Tr}\left[\frac{(a(x)+a^T(x))}{2}M\right] .$$

So the equation (1) is exactly the same for a and for $\frac{1}{2}(a+a^T)$ and from now on we will assume a to be symmetric.

We also introduce the following assumptions

(H1) For every $R > 0$, there exist γ_R, $\delta_R > 0$, such that, for every $x \in \overline{\Omega}$, $p \in I\!R^n$, and $-R \leq v \leq u \leq R$

$$\gamma_R(u-v) \leq H(x,u,p) - H(x,v,p) \leq \delta_R(u-v) .$$

(H2) For every $R > 0$, there exists a modulus $m_R : I\!R^+ \to I\!R$ such that $m_R(t) \to 0$ when $t \to 0^+$ and

$$|H(x,u,p) - H(y,u,p)| \leq m_R\left(|x-y|(1+|p|)\right) ,$$

for every x, $y \in \overline{\Omega}$, $p \in I\!R^n$ and $|u| \leq R$.

(H3) For every $R > 0$, there exists a modulus $\tilde{m}_R : I\!R^+ \to I\!R$ such that $\tilde{m}_R(t) \to 0$ when $t \to 0^+$ and

$$|H(x,u,p) - H(x,u,q)| \leq \tilde{m}_R\left(|p-q|\right) ,$$

for every $x \in \overline{\Omega}$, $|u| \leq R$ and p, $q \in I\!R^n$.

(H4) $a(x) = \sigma(x)\sigma^T(x)$ for any $x \in \overline{\Omega}$, where σ is a $n \times p$ matrix, Lipschitz continuous in $\overline{\Omega}$.

The result is the

Proposition 1.1 : *Assume that $\varphi \in C(\partial\Omega)$ and that* **(H1)-(H4)** *hold. Let u be an usc subsolution of problem* (14). *If $u(x_0) > \varphi(x_0)$ at $x_0 \in \partial\Omega$, then*

$$\left\{\begin{array}{c} a(x_0)n(x_0).n(x_0) = 0 \\ \textit{and} \\ -\frac{1}{2}\mathrm{Tr}\left[a(x_0)D^2d(x_0)\right] + \limsup\limits_{\lambda\to+\infty} \dfrac{1}{\lambda}H(x,0,-\lambda n(x_0)) \leq 0 . \end{array}\right. \tag{19}$$

Let v be an lsc supersolution of problem (14). *If $v(x_0) < \varphi(x_0)$ at $x_0 \in \partial\Omega$, then*

$$\left\{\begin{array}{c} a(x_0)n(x_0).n(x_0) = 0 \\ \textit{and} \\ \frac{1}{2}\mathrm{Tr}\left[a(x_0)D^2d(x_0)\right] + \liminf\limits_{\lambda\to+\infty} \dfrac{1}{\lambda}H(x,0,\lambda n(x_0)) \geq 0 . \end{array}\right. \tag{20}$$

□

We now consider two examples where we translate conditions (19) and (20). The first one is the case of linear equations

$$-\frac{1}{2}\mathrm{Tr}\left[a(x)D^2u\right] - b(x).Du + \lambda u - f(x) = 0 \quad \text{in } \Omega .$$

Assumptions **(H1)**-**(H3)** are satisfied as soon as $\lambda > 0$, b is Lipschitz continuous on $\overline{\Omega}$ (hence bounded) and f is continuous on $\overline{\Omega}$. In this case, (19) and (20) read

$$\left\{ \begin{array}{c} a(x_0)n(x_0).n(x_0) = 0 \\ \text{and} \\ -\frac{1}{2}\mathrm{Tr}\left[a(x_0)D^2d(x_0)\right] + b(x_0).n(x_0) \leq 0 . \end{array} \right. \tag{21}$$

It is known in the probabilistic literature that the points $x \in \partial\Omega$ satisfying either

$$a(x)n(x).n(x) > 0 ,$$

or $a(x)n(x).n(x) = 0$ and

$$-\frac{1}{2}\mathrm{Tr}\left[a(x)D^2d(x)\right] + b(x).n(x) > 0 ,$$

are *regular points* i.e. one can prescribe the Dirichlet boundary condition at x (cf. for example M.I Freidlin[22]). The condition (21) which says that losses of boundary conditions can only occur in the complement of such points is therefore sharp.

Now we turn to the nonlinear case and, more precisely, to the case of stochastic control. Since H is given by (10), (19) reads

$$\left\{ \begin{array}{c} a(x_0)n(x_0).n(x_0) = 0 \\ \text{and} \\ \frac{1}{2}\mathrm{Tr}\left[a(x_0)D^2d(x_0)\right] - b(x_0,\alpha).n(x_0) \geq 0 , \end{array} \right. \tag{22}$$

for any $\alpha \in \mathcal{A}$ while (20) reads

$$\left\{ \begin{array}{c} a(x_0)n(x_0).n(x_0) = 0 \\ \text{and} \\ \frac{1}{2}\mathrm{Tr}\left[a(x_0)D^2d(x_0)\right] - b(x_0,\alpha_{x_0}).n(x_0) \geq 0 , \end{array} \right. \tag{23}$$

for some $\alpha_{x_0} \in \mathcal{A}$.

Remark : Assumption **(H4)** is obviously not satisfied for any Lipschitz continuous function a with value in the set of positive semi-definite symmetric matrices. However, the following Lemma allows us to raise a sufficient condition for **(H4)** to hold

Lemma 1.1 : *Assume a is a $W^{2,\infty}$ function on Ω (resp. $\overline{\Omega}$), then $x \to \sqrt{a(x)}$ is uniformly Lipschitz continuous on any compact subset of Ω (resp. $\overline{\Omega}$).* □

For a proof of this result, the reader may refer to R.S Phillips and L. Saranson[52] or M.I Freidlin[22, 23]. Assumption **(H4)** holds thanks to the above Lemma. It is worth noticing that the assumption of "smooth continuability beyond the boundary" is made by many authors, (see [18, 19], [46]-[50], [51]).

Now we turn to the **Proof of Proposition 1.1 :** we first prove (19). Let x_0 be a point on $\partial\Omega$ such that $u(x_0) > \varphi(x_0)$. We define

$$\phi_\varepsilon(y) = u(y) - \frac{|y-x_0|^4}{\varepsilon^4} - \frac{1}{\varepsilon^2}d(y) + \frac{1}{2\varepsilon^3}d(y)^2.$$

Since ϕ_ε is upper semicontinuous on $\overline{\Omega}$, there exists a maximum point y_ε of ϕ_ε in $K_\varepsilon = \{y,\ d(y) \leq \varepsilon\}$. Since $y_\varepsilon \in K_\varepsilon$, it satisfies

$$\frac{1}{\varepsilon^2}d(y_\varepsilon) - \frac{1}{2\varepsilon^3}d(y_\varepsilon)^2 \geq 0. \tag{24}$$

and the maximum point property implies in particular

$$u(x_0) = \phi_\varepsilon(x_0) \leq \phi_\varepsilon(y_\varepsilon) \leq u(y_\varepsilon) - \frac{|y_\varepsilon - x_0|^4}{\varepsilon^4} - \frac{1}{\varepsilon^2}d(y_\varepsilon) + \frac{1}{2\varepsilon^3}d(y_\varepsilon)^2. \tag{25}$$

Since u is bounded, we deduce from (25) that the term

$$\frac{|y_\varepsilon - x_0|^4}{\varepsilon^4} + \frac{1}{\varepsilon^2}d(y_\varepsilon) - \frac{1}{2\varepsilon^3}d(y_\varepsilon)^2$$

is also bounded. But using (24) this implies also that the term $\dfrac{|y_\varepsilon - x_0|^4}{\varepsilon^4}$ is bounded. Hence y_ε tends to x_0 as ε tends to 0. Moreover, thanks again to (24), (25) implies $u(x_0) \leq u(y_\varepsilon)$ and this property together with the upper-semicontinuity of u yields

$$\lim_{\varepsilon\to 0} u(y_\varepsilon) = u(x_0).$$

Now we take the liminf in (25): using again (24), we obtain at the same time that

$$\frac{|y_\varepsilon - x_0|^4}{\varepsilon^4} \to 0,$$

and

$$\frac{1}{\varepsilon^2}d(y_\varepsilon) - \frac{1}{2\varepsilon^3}d(y_\varepsilon)^2 \to 0.$$

This last property implies $d(y_\varepsilon) = o(\varepsilon^2)$, so that, for ε small enough, the maximum cannot be achieved for $d(y_\varepsilon) = \varepsilon$. So y_ε is also a point of local maximum of ϕ_ε in $\overline{\Omega}$.

Now $u(x_0) > \varphi(x_0)$ implies $u(y_\varepsilon) > \varphi(y_\varepsilon)$ for ε small enough if $y_\varepsilon \in \partial\Omega$ since φ is continuous. As a consequence, the equation holds for ε small enough, wherever y_ε lies. We get

$$-\frac{1}{2}\mathrm{Tr}\Big[a(y_\varepsilon)\{\frac{4}{\varepsilon^4}|y_\varepsilon - x_0|^2 Id + \frac{8}{\varepsilon^4}(y_\varepsilon - x_0)\otimes(y_\varepsilon - x_0) +$$
$$\frac{1}{\varepsilon^2}D^2 d(y_\varepsilon) - \frac{1}{\varepsilon^3}n(y_\varepsilon)\otimes n(y_\varepsilon) - \frac{1}{\varepsilon^3}d(y_\varepsilon)D^2 d(y_\varepsilon)\}\Big] +$$
$$H\left(y_\varepsilon, u(y_\varepsilon), \frac{4}{\varepsilon^4}|y_\varepsilon - x_0|^2(y_\varepsilon - x_0) - \frac{1}{\varepsilon^2}n(y_\varepsilon) + \frac{1}{\varepsilon^3}d(y_\varepsilon)n(y_\varepsilon)\right) \le 0 .$$

We denote by

$$p_\varepsilon = \frac{4}{\varepsilon^4}|y_\varepsilon - x_0|^2(y_\varepsilon - x_0) + \frac{1}{\varepsilon^3}d(y_\varepsilon)n(y_\varepsilon) .$$

If m and $\tilde{m}$ are the modulus given in the statements of assumptions **(H2)** and **(H3)** for $R = ||u||_\infty$, we may assume without loss of generality that they satisfies

$$m(t)\,,\, \tilde{m}(t) \le C(1+t)\,,$$

for all $t \ge 0$ for some constant $C > 0$. Therefore, using also **(H1)**, we have, changing C if necessary,

$$\varepsilon^2\Big|H(y_\varepsilon, u(y_\varepsilon), p_\varepsilon - \frac{1}{\varepsilon^2}n(y_\varepsilon)) - H(x_0, 0, -\frac{1}{\varepsilon^2}n(x_0))\Big| \le$$
$$C\varepsilon^2\Big(|p_\varepsilon - \frac{1}{\varepsilon^2}n(y_\varepsilon) + \frac{1}{\varepsilon^2}n(x_0)| + |y_\varepsilon - x_0|(1 + \frac{1}{\varepsilon^2} + |p_\varepsilon|) + |u(y_\varepsilon)| + 1\Big) .$$

Since $|y_\varepsilon - x_0| = o(\varepsilon)$ and $d(y_\varepsilon) = o(\varepsilon^2)$, we get $|p_\varepsilon| = o(\frac{1}{\varepsilon^2})$. Hence, using the Lipschitz continuity of n, we deduce that the right-hand side of the above inequality vanishes as ε tends to zero. Coming back to the viscosity inequality, using again $|y_\varepsilon - x_0| = o(\varepsilon)$ and the regularity of $\partial\Omega$, we obtain

$$\frac{1}{2}\mathrm{Tr}\Big[a(x_0)[\frac{1}{\varepsilon}n(x_0)\otimes n(x_0) - D^2 d(x_0)]\Big] + \varepsilon^2 H(x_0, 0, -\frac{1}{\varepsilon^2}n(x_0)) \le o(1) .$$

We first multiply this inequality by ε and we let ε tends to 0, we obtain $a(x_0)n(x_0).n(x_0) = 0$. Then

$$-\frac{1}{2}\mathrm{Tr}\left[a(x_0)D^2 d(x_0)\right] + \limsup_{\varepsilon\to 0}\ \varepsilon^2 H(x_0, 0, -\frac{1}{\varepsilon^2}n(x_0)) \le 0 ,$$

and the proof is complete.

This proof for (20) is entirely anologous to the previous one. If $x_0 \in \partial\Omega$ satisfies $v(x_0) < \varphi(x_0)$, it is enough to set

$$\phi_\varepsilon(y) = v(y) + \frac{|y - x_0|^4}{\varepsilon^4} + \frac{1}{\varepsilon^2}d(y) - \frac{1}{2\varepsilon^3}d(y)^2 ,$$

and consider a point of local minimum of ϕ_ε in the neighborhood of x_0. Running through the same kind of arguments as in the proof of (19), we conclude $a(x_0)n(x_0).n(x_0) = 0$ and

$$\frac{1}{2}\mathrm{Tr}\left[a(x_0)D^2d(x_0)\right]+\liminf_{\varepsilon\to 0}\ \varepsilon^2 H(x_0,0,\frac{1}{\varepsilon^2}n(x_0))\geq 0\ .$$

□

We conclude this section by a result concerning the solvability of the "classical" Dirichlet problem: this means that we are looking here for a viscosity solution of (14) which assumes the boundary data φ continuously. Hence the term "classical" refers to the boundary data and not to the equation inside Ω.

In view of Proposition 1.1, we first define

$$\Sigma_n=\left\{x\in\partial\Omega,\ a(x)n(x).n(x)>0\right\}\ ,$$

$$\Sigma_d=\left\{x\in\partial\Omega,\ a(x)n(x).n(x)=0\right\}\ ,$$

and then we set

$$\Sigma_d^1=\left\{x\in\Sigma_d,\ \frac{1}{2}\mathrm{Tr}\left[a(x)D^2d(x)\right]+\liminf_{\lambda\to\infty}\ \frac{1}{\lambda}H(x,0,\lambda n(x))\geq 0\right\}\ ,$$

$$\Sigma_d^2=\left\{x\in\Sigma_d,\ -\frac{1}{2}\mathrm{Tr}\left[a(x)D^2d(x)\right]+\limsup_{\lambda\to\infty}\ \frac{1}{\lambda}H(x,0,-\lambda n(x))\leq 0\right\}\ .$$

We also use the following additional assumption

(H5) There exists $M>0$ such that $H(x,-M,0)\leq 0\leq H(x,M,0)$ on $\overline{\Omega}$.

The result is the

Theorem 1.2 : *Assume that $\varphi\in C(\partial\Omega)$ and that* **(H1)-(H5)** *hold. If $\Sigma_d^1=\Sigma_d^2=\emptyset$, there exists a unique solution of the "classical" Dirichlet problem in $C(\overline{\Omega})$.* □

Proof : We first prove the existence part. Since φ is continuous on the boundary, it is bounded. We set $M'=\max(M,||\varphi||_\infty)$. Thanks to **(H1)** and **(H5)**, $-M'$ and M' are respectively viscosity sub and supersolutions of (14). Using the Perron's method which was introduced in the context of viscosity solutions by H. Ishii[26] (See also [13]), we obtain the existence of a (posssibly discontinuous) solution u of (14). Using Proposition 1.1 and the fact that $\Sigma_d^1=\Sigma_d^2=\emptyset$, we have

$$u^*\leq\varphi\leq u_*\quad\text{ on }\partial\Omega\ . \tag{26}$$

Indeed u^* and u_* are respectively viscosity sub and supersolutions of (14) and (19) and (20) are satisfied by no point on the boundary because Σ_d^1 and Σ_d^2 are both empty.

Then we use the Maximum Principle for semicontinuous sub- and supersolutions (cf. [13]): since u^* and u_* are respectively viscosity sub and supersolutions of (1) and since $u^*\leq u_*$ on $\partial\Omega$, we have

$$u^* \le u_* \quad \text{on } \Omega\ . \tag{27}$$

But, by the very definition of u^* and u_*, $u_* \le u \le u^*$ on $\overline{\Omega}$. Therefore $u = u^* = u_*$ on $\overline{\Omega}$ and this means that u, being both usc and lsc, is continuous on $\overline{\Omega}$ and (26) implies that $u = \varphi$ on $\partial\Omega$. So we have proved the existence part. The uniqueness part relies again on the Maximum Principle since by Proposition 1.1 any viscosity subsolution v and any supersolution w of (14) satisfy

$$v \le \varphi \le w \quad \text{on } \partial\Omega\ , \tag{28}$$

then the Maximum Principle yields

$$v \le w \quad \text{in } \Omega\ , \tag{29}$$

and the proof is complete. □

In this proof, the key point is the Maximum Principle which enables us to get (27) and (29): the first inequality yields to the existence of a continuous solution while the second one gives the uniqueness. The aim of our main result in the next section is to provide (27) and (29) without knowing a priori (26) and (28) but assuming only (4) and (5) to hold.

2 The main result and its consequences.

The aim of this section is to state our main result and to present its consequences for the Dirichlet problem. Throughout this section, Ω stands for a bounded domain of $\mathbb{R}^n$ with a C^2 boundary.

We introduce the following additional "non-degeneracy" assumptions on a and H.

(H6) Σ_d^1 and Σ_d^2 are open.

(H7) $$\lambda \mapsto \frac{1}{2}\lambda \mathrm{Tr}\left[a(x)D^2 d(x)\right] + H(x, u, p + \lambda n(x))\ ,$$
is an increasing function as soon as $\lambda \ge C_R(1 + |p|)$, where $C_R > 0$, for every x in a neighborhood of Σ_d^1, $p \in \mathbb{R}^n$ and $|u| \le R$ ($\forall R > 0$).

(H8) $$\lambda \mapsto -\frac{1}{2}\lambda \mathrm{Tr}\left[a(x)D^2 d(x)\right] + H(x, u, p - \lambda n(x))\ ,$$
is a decreasing function as soon as $\lambda \ge C_R(1 + |p|)$, where $C_R > 0$, for every x in a neighborhood of Σ_d^2, $p \in \mathbb{R}^n$ and $|u| \le R$ ($\forall R > 0$).

The main theorem is now the following comparison result

Theorem 2.1 : *Assume that* **(H1)**-**(H4)** *and* **(H6)**-**(H8)** *hold. If u is a bounded usc subsolution of* (14) *and if v is a bounded lsc supersolution of* (14), *then*

$$u \le v \quad in\, \Omega\ .$$

Moreover if we define $\tilde{u}$ and $\tilde{v}$ on $\overline{\Omega}$ by setting

$$\tilde{u}(x) = \begin{cases} \limsup\limits_{\substack{y \to x \\ y \in \Omega}} u(y) & on\ \Sigma_d^1 \cap \{u(x) \le \varphi(x)\}\ , \\ u(x) & otherwise. \end{cases} \tag{30}$$

and

$$\tilde{v}(x) = \begin{cases} \liminf\limits_{\substack{y \to x \\ y \in \Omega}} v(y) & on\ \Sigma_d^2 \cap \{v(x) \ge \varphi(x)\}\ , \\ v(x) & otherwise. \end{cases} \tag{31}$$

then $\tilde{u}$ and $\tilde{v}$ are still respectively an usc subsolution and a lsc supersolution of (14) *and*

$$\tilde{u} \le \tilde{v} \quad on\, \overline{\Omega}\ .$$

□

This result is optimal: we will provide in the section devoted to the study of stochastic exit time control problem a counter-example given in G.Barles and B. Perthame[5] showing that it is optimal even for first-order equations.

We want to emphasize that there is no way to get a comparison result up to the boundary: indeed if, for example, u is a continuous solution of (14) and if $u(x) < \varphi(x)$ at some point $x \in \partial\Omega$, one can consider the function $\tilde{u}$ which is equal to u for any point of $\overline{\Omega}$ except at x where we set $\tilde{u}(x) = \varphi(x)$. $\tilde{u}$ is an usc subsolution of (14) and $\tilde{u}$ is not less than u on $\overline{\Omega}$. Of course, an analogous remark is true for supersolutions v.

(H6) implies in fact that both Σ_d^1 and Σ_d^2 are unions of connected components of $\partial\Omega$: indeed they are closed by their very definitions and by our assumptions on a, H and $\partial\Omega$ (which implies that $D^2 d$ is continuous on $\partial\Omega$). In the same way, the complement of $\Sigma_d^1 \cup \Sigma_d^2$, the so-called *non characteristic boundary*, is also a union of connected components of $\partial\Omega$.

We postpone the proof of Theorem 2.1 to the next section and we first give a few corollaries.

We first consider the so-called State Constraints problem i.e.

$$\begin{cases} -\frac{1}{2}\mathrm{Tr}\left[a(x)D^2u(x)\right] + H(x, u(x), Du(x)) & \ge & 0 \ \text{ in } \overline{\Omega} \\ -\frac{1}{2}\mathrm{Tr}\left[a(x)D^2u(x)\right] + H(x, u(x), Du(x)) & \le & 0 \ \text{ in } \Omega \end{cases} \tag{32}$$

The terminology will be clear when we will study stochastic control problems. The result is the

Corollary 2.1 : *Assume that* **(H1)-(H4)** *and* **(H6)-(H8)** *hold with $\Sigma_d^1 = \Sigma_d = \partial\Omega$. If u is a bounded usc subsolution of* (32) *and if v is a bounded lsc supersolution of* (32) *then $u \le v$ in Ω.* □

Proof of Corollary 2.1 : We first remark that, since there is no property satisfied by the subsolution u on the boundary, u can take any value on $\partial\Omega$ compatible with its upper semi-continuity. Hence again no comparison argument can work up to the boundary and we start by redefining u on $\partial\Omega$ by setting

$$u(x) = \limsup_{\substack{y\to x\\ y\in\Omega}} \; u(y)\ .$$

We still denote by u this new usc function which is still a subsolution of (32). We now remark that u and v are respectively sub- and supersolutions of the Dirichlet problem (14) with $\varphi \equiv C$ if the constant C satisfies $C > \max\,(||u||_\infty, ||v||_\infty)$. The corollary is then a straightforward application of Theorem 2.1. □

The next result concerns the existence of continuous solutions

Corollary 2.2 : *Under the assumptions of* Theorem 2.1, *if* **(H5)** *holds, there exists a unique solution of* (14) *in* $C(\overline{\Omega})$. □

Since the proof of Corollary 2.2 is exactly the same as the proof of Theorem 1.2, the only change being that Theorem 2.1 replaces the Maximum Principle, we leave it to the reader.

Now we turn to the singular perturbation problem

$$\begin{cases} -\varepsilon\Delta u_\varepsilon - \frac{1}{2}\mathrm{Tr}\,[a(x)D^2u_\varepsilon] + H(x,u_\varepsilon,Du_\varepsilon) &= \;\; 0 \;\text{ in } \Omega \\ \hfill u_\varepsilon &= \;\; \varphi \;\text{ in } \partial\Omega \end{cases} \tag{33}$$

Corollary 2.3 : *Let u_ε be the unique solution of problem* (33). *Then, under the assumptions of* Corollary 2.2, *$(u_\varepsilon)_\varepsilon$ converges to the unique solution of problem* (14) *in $C(\Omega)$, as $\varepsilon\to 0$.* □

It is worth noticing that we cannot expect a uniform convergence on $\overline{\Omega}$. Indeed, $u_\varepsilon = \varphi$ on $\partial\Omega$ whereas some losses of the boundary condition can occur in (14). These losses of boundary data generate boundary layers for u_ε.

Proof of Corollary 2.3 : Theorem 1.2 provides us with a unique solution u_ε of problem (33) in $C(\overline{\Omega})$. The only thing we have to check to be in position to use the stability result of Theorem 1.1 is the boundedness of $(u_\varepsilon)_\varepsilon$. We take $M' = \max\,(M, ||\varphi||_\infty)$; $-M'$ and M' are respectively a sub and a supersolution of (33). By the Maximum Principle, we deduce

$$||u_\varepsilon||_\infty \le M'\ .$$

As in the introduction, we consider $\underline{u}$ and $\overline{u}$ respectively defined by

$$\underline{u}(x) = \liminf_{\substack{y\to x\\ \varepsilon\to 0}} \; u_\varepsilon(y)\ ,$$

and

$$\overline{u}(x) = \limsup_{\substack{y \to x \\ \varepsilon \to 0}} u_\varepsilon(y) .$$

By Theorem 1.1, $\underline{u}$ and $\overline{u}$ are respectively super- and subsolution of (14).
Then we use Theorem 2.1 to get

$$\overline{u} \leq \underline{u} \quad \text{in } \Omega .$$

But $\underline{u} \leq \overline{u}$ on $\overline{\Omega}$ by their very definition and therefore the equality $\overline{u} = \underline{u}$ holds in Ω. It is now an elementary exercise to check that this equality implies the uniform convergence of $(u_\varepsilon)_\varepsilon$ to $u := \underline{u} = \overline{u}$ on each compact subset of Ω, that is in $C(\Omega)$. □

The same result holds and the same method applies to *any* type of approximation for the Dirichlet problem. This fact has to be emphasized, since it has also practical consequences, in particular in the field of the numerical analysis of such degenerate elliptic equations (cf. G. Barles and P.E Souganidis[6]).

3 Proof of Theorem 2.1.

We first prove the

Lemma 3.1 : *$\tilde{u}$ and $\tilde{v}$ defined in the statement of* Theorem 2.1 *are respectively an usc subsolution and a lsc supersolutions of* (14). □

Proof : We only treat the case of $\tilde{u}$, the case of $\tilde{v}$ being analogous. Since $\tilde{u}$ differs from u only on the set

$$A = \left\{ x \in \Sigma_d^1 \, , u(x) \leq \varphi(x) \right\} ,$$

it is clear enough that the problem comes only from the points in $\overline{A}$, the closure of A. Moreover $\tilde{u} \leq u \leq \varphi$ on A since $u(x) = \limsup_{\substack{y \to x \\ y \in \Omega}} u(y)$ and therefore the only remaining question concerns the points in $\overline{A} - A$. Let $x \in \overline{A} - A$ then $u(x) > \varphi(x)$: indeed our assumptions clearly imply that Σ_d^1 is closed and therefore $x \in \Sigma_d^1$. Moreover, $\tilde{u}(x) = u(x)$ since $x \notin A$. These two properties imply

$$\limsup_{\substack{y \to x \\ y \in A}} \tilde{u}(y) < \tilde{u}(x) .$$

Therefore, if ϕ is a smooth function, the properties "x is a local maximum point if $u - \phi$" and "x is a local maximum point if $\tilde{u} - \phi$" are equivalent since the points of A play essentially no role. And the conclusion follows easily. □

In the remainder of the proof, we are going to show that $\tilde{u} \leq \tilde{v}$ on $\overline{\Omega}$. For simplicity of notations, we are going to drop the "$\tilde{}$" on $\tilde{u}$ and on $\tilde{v}$. So we consider u and v being respectively a subsolution and a supersolution of (14) which satisfy

$$u(x) = \limsup_{\substack{y \to x \\ y \in \Omega}} u(y) \quad \text{if } x \in \Sigma_d^1 \,,\, u(x) \leq \varphi(x) \,, \tag{34}$$

and

$$v(x) = \limsup_{\substack{y \to x \\ y \in \Omega}} v(y) \quad \text{if } x \in \Sigma_d^2 \,,\, v(x) \geq \varphi(x) \,. \tag{35}$$

We set $M = \max\limits_{x \in \overline{\Omega}} (u(x) - v(x))$: our aim is to show that $M \leq 0$. We argue by contradiction assuming that $M > 0$. Since discontinuous viscosity solutions satisfy the Maximum Principle (See [13]), M is at some point x_0 on the boundary. In what follows, we work only in a neighborhood $\mathcal{V}$ of x_0. Now three possibilities can occur

- $u(x_0) \leq \varphi(x_0)$ and $v(x_0) < \varphi(x_0)$.
- $u(x_0) > \varphi(x_0)$ and $v(x_0) \geq \varphi(x_0)$.
- $u(x_0) > \varphi(x_0)$ and $v(x_0) < \varphi(x_0)$.

Of course the fourth possibility $u(x_0) \leq \varphi(x_0)$ and $v(x_0) \geq \varphi(x_0)$ cannot hold since we assume $M > 0$.

We consider the first case. Since u and v are bounded, we set $R = \max(||u||_\infty, ||v||_\infty)$. Throughout the proof, we will denote by γ, C, m and $\tilde{m}$ the constants γ_R, C_R and the modulus m_R and $\tilde{m}_R$ which are provided by assumptions **(H1)**, **(H2)**, **(H3)**, **(H6)** and **(H7)**.

Because of Proposition 1.1, $x_0 \in \Sigma_d^1$ and therefore by Lemma 3.1, we have (34).

The next lemma is the essential tool for completing the proof.

In the sequel, for a smooth function ϕ and for x in a neighborhood of $\partial\Omega$, we denote by $D_T\phi(x)$ the tangential derivative of ϕ at x, i.e.

$$D_T\phi(x) = D\phi(x) - \frac{\partial \phi}{\partial n}(x) n(x) \,.$$

Lemma 3.2 : *Under the assumptions of* Theorem 2.1, *if u is an usc subsolution of* (1) *then u is a subsolution of* (1) *and*

$$\operatorname{Min}\left(-\frac{1}{2}\operatorname{Tr}\left[a(x)D^2u\right] + H(x,u,Du), \frac{\partial u}{\partial n} - C(1 + |D_T u|) \right) = 0 \text{ on } K \,, \tag{36}$$

where

$$K = \left\{ x \in \Sigma_d^1 \,,\, (34) \; holds \right\} \,,$$

and C is the constant C_R appearing in **(H6)** *for $R = ||u||_\infty$.* □

We first have to make precise what we mean by (36) since we defined above sub and supersolutions for a boundary condition which holds on an entire boundary whereas K may be here only a part of it; (36) holds if

for any $\phi \in C^2(\overline{\Omega})$ and for any local maximum point $x_0 \in K$ of $u - \phi$ on $\overline{\Omega}$, one has

$$\min \Big(-\frac{1}{2}\mathrm{Tr}\left[a(x_0)D^2\phi(x_0)\right] + H(x_0, u(x_0), D\phi(x_0)),$$

$$\frac{\partial \phi(x_0)}{\partial n} - C(1 + |D_T\phi(x_0)|)\Big) \leq 0 \tag{37}$$

In fact, (37) is only a particular case of the general definition with

$$B(x,u,p) = p.n(x) - C(1 + |p - p.n(x)n(x)|) \,. \tag{38}$$

Nonlinear Neumann type boundary conditions were first considered in G. Barles and P.L Lions[2] for first-order equations. As mentioned in the introduction two kinds of Strong Comparison Results exist in the case of second-order equations: the first one obtained by H. Ishii[29] requires only the boundary $\partial\Omega$ to be of class C^1 but B has to be the restriction to the boundary of a $W^{2,\infty}$ function still denoted by B which satisfies

$$|D_x B(x,u,p)| \leq C_R(1 + |p|) \quad , \quad |D_p B(x,u,p)| \leq C_R \,, \tag{39}$$

$$|D^2_{xx}B(x,u,p)| \leq C_R(1+|p|)\,,\, |D^2_{xp}B(x,u,p)| \leq C_R\,,\, |D^2_{pp}B(x,u,p)| \leq \frac{C_R}{1+|p|} \tag{40}$$

for some constant C_R, for $|u| \leq R$, $p \in I\!\!R^n$ and for any x in a neighborhood of $\partial\Omega$ ($\forall R > 0$).

Of course, the main restrictive assumption here is the last requirement in (40) which excludes in particular a dependence of B in $p - p.n(x)n(x)$ as in (38) (even if we replace the norm by a smooth function). The second type of results proved by the first author in [1] requires only B to satisfy (39) but the boundary $\partial\Omega$ has to be of class $W^{3,\infty}$; this result applies to (38).

Fortunately, due to particular context in which the Neumann boundary condition occurs here we can avoid below the restrictions of both types of results and it is worth mentionning that our C^2 regularity assumption on $\partial\Omega$ is used only to give a simple form to the analysis of the default of boundary conditions and to the assumptions on H and on a on Σ^1_d and on Σ^2_d. The improvement of the regularity assumption on $\partial\Omega$ will be considered in a future work.

Proof of Lemma 3.2 : Let ϕ be a C^2 function on $\overline{\Omega}$ and let $x_0 \in K$ be a maximum point of $u - \phi$. If

$$\frac{\partial \phi}{\partial n}(x_0) \leq C(1 + |D_T \phi(x_0)|)$$

we are done; hence we may assume that

$$\frac{\partial \phi}{\partial n}(x_0) > C(1 + |D_T \phi(x_0)|) \,. \tag{41}$$

Changing if it is needed ϕ into $\phi + |. - x_0|^4$ which leaves its first and second derivatives at x_0 unchanged, we may assume x_0 to be a strict local maximum point of $u - \phi$. We set

$$\psi_\varepsilon(x) = u(x) - \phi(x) - \frac{\varepsilon}{d(x)} \,,$$

for $x \in \Omega$. Since x_0 is a strict local maximum point of $u - \phi$, there exists a sequence $(x_\varepsilon)_\varepsilon$ of local maximum points of ψ_ε such that $\lim_{\varepsilon \to 0} x_\varepsilon = x_0$ and $\lim_{\varepsilon \to 0} \psi_\varepsilon(x_\varepsilon) = u(x_0) - \phi(x_0)$. Since $x_\varepsilon \in \Omega$, the equation holds for u at x_ε and we obtain

$$-\frac{1}{2}\mathrm{Tr}\Big[a(x_\varepsilon)(D^2\phi(x_\varepsilon) - \frac{\varepsilon}{d(x_\varepsilon)^2}D^2 d(x_\varepsilon) + 2\frac{\varepsilon}{d(x_\varepsilon)^3}n(x_\varepsilon)\otimes n(x_\varepsilon))\Big] +$$
$$H(x_\varepsilon, u(x_\varepsilon), D\phi(x_\varepsilon) + \frac{\varepsilon}{d(x_\varepsilon)^2}n(x_\varepsilon)) \leq 0 \,.$$

We write now $p_\varepsilon := D_T\phi(x_\varepsilon)$ and $\lambda_\varepsilon := \dfrac{\partial \phi}{\partial n}(x_\varepsilon)$. For ε small enough, since ϕ is smooth and satisfies (41), one clearly has

$$\lambda_\varepsilon \geq C(1 + |p_\varepsilon|) \,.$$

Then the viscosity subsolution inequality can be written as

$$-\frac{1}{2}\mathrm{Tr}\left[a(x_\varepsilon)D^2\phi(x_\varepsilon)\right] + \frac{\varepsilon}{2d(x_\varepsilon)^2}\mathrm{Tr}\left[a(x_\varepsilon)D^2 d(x_\varepsilon)\right] +$$
$$H(x_\varepsilon, u(x_\varepsilon), p_\varepsilon + \lambda_\varepsilon n(x_\varepsilon) + \frac{\varepsilon}{d(x_\varepsilon)^2}n(x_\varepsilon)) \leq \frac{\varepsilon}{d(x_\varepsilon)^3}\mathrm{Tr}\left[a(x_\varepsilon)n(x_\varepsilon)\otimes n(x_\varepsilon)\right] \,.$$

We apply now assumption **(H7)**: using that $D\phi(x_\varepsilon) = p_\varepsilon + \lambda_\varepsilon n(x_\varepsilon)$, we deduce

$$-\frac{1}{2}\mathrm{Tr}\left[a(x_\varepsilon)D^2\phi(x_\varepsilon)\right] + H(x_\varepsilon, u(x_\varepsilon), D\phi(x_\varepsilon)) \leq -\frac{1}{2}\mathrm{Tr}\left[a(x_\varepsilon)D^2\phi(x_\varepsilon)\right] +$$
$$\frac{\varepsilon}{2d(x_\varepsilon)^2}\mathrm{Tr}\left[a(x_\varepsilon)D^2 d(x_\varepsilon)\right] + H(x_\varepsilon, u(x_\varepsilon), D\phi(x_\varepsilon) + \frac{\varepsilon}{d(x_\varepsilon)^2}n(x_\varepsilon))$$
$$\leq \frac{\varepsilon}{d(x_\varepsilon)^3}\mathrm{Tr}\left[a(x_\varepsilon)n(x_\varepsilon)\otimes n(x_\varepsilon)\right] \,.$$

Finally using **(H6)** together with the Lipschitz continuity of σ provides us with

$$|\mathrm{Tr}\,[a(x_\varepsilon)n(x_\varepsilon)\otimes n(x_\varepsilon)]\,| = a(x_\varepsilon)n(x_\varepsilon).n(x_\varepsilon) = |\sigma^T(x_\varepsilon)n(x_\varepsilon)|^2 \leq \|\sigma\|^2_{W^{1,\infty}} d(x_\varepsilon)^2$$

if ε is small enough. Indeed $x_\varepsilon \to x_0 \in \Sigma^1_d \subset \Sigma_d$ and Σ^1_d is open. Hence we are led to

$$-\frac{1}{2}\mathrm{Tr}\left[a(x_\varepsilon)D^2\phi(x_\varepsilon)\right] + H(x_\varepsilon, u(x_\varepsilon), D\phi(x_\varepsilon)) \leq \frac{\varepsilon}{d(x_\varepsilon)}\|\sigma\|^2_{W^{1,\infty}} \;.$$

and since $\dfrac{\varepsilon}{d(x_\varepsilon)}$ vanishes as ε tends to zero, we conclude, letting ε to zero

$$-\frac{1}{2}\mathrm{Tr}\left[a(x_0)D^2\phi(x_0)\right] + H(x_0, u(x_0), D\phi(x_0)) \leq 0 \;.$$

□

We now come back to the proof of Theorem 2.1: we introduce the following test function

$$\psi_\varepsilon(x,y) = u(x) - v(y) - \phi_\varepsilon(x,y) \;,$$

where

$$\phi_\varepsilon(x,y) = \frac{|x-y|^4}{\varepsilon} + A(\frac{|x-y|^3}{\varepsilon}+1)(x-y).n(x_0) + B\frac{[(x-y).n(x_0)]^4}{\varepsilon} + |x-x_0|^4 \;,$$

and the constants A and B are to be chosen such that $B >> A^4$ and $A >> C$.

Finally, we set

$$M_\varepsilon = \max_{\overline{\Omega}\times\overline{\Omega}} \psi_\varepsilon(x,y) \;.$$

Since ψ_ε is upper semi-continuous, its maximum is achieved at some point $(\overline{x}_\varepsilon, \overline{y}_\varepsilon)$. Here and below, we drop the dependence of $\overline{x}$ and $\overline{y}$ in ε for the sake of simplicity of notations. We first need the

Lemma 3.3 : *We have*

1. $M_\varepsilon \to M$ *as* $\varepsilon \to 0$.

2. $u(\overline{x}) \to u(x_0)$, $v(\overline{y}) \to v(x_0)$, $\overline{x}$, $\overline{y} \to x_0$ *and* $\dfrac{|\overline{x}-\overline{y}|^4}{\varepsilon} \to 0$ *as* $\varepsilon \to 0$.

3. *We set* $\chi_\varepsilon(x) = \phi_\varepsilon(x, \overline{y})$. *If* $\overline{x} \in \partial\Omega$ *and if* ε *is small enough, then*

$$\frac{\partial \chi_\varepsilon}{\partial n}(\overline{x}) > C(1 + |D_T\chi_\varepsilon(\overline{x})|) \;.$$

□

We postpone the proof of the lemma, and first use this result to conclude the proof.

By Theorem 3.2 of [13], we know that, for any $\eta > 0$, there exist symmetric matrices X and Y such that

$$(D_x\phi_\varepsilon(\overline{x},\overline{y}), X) \in \overline{D}^{2,+}u(\overline{x}) ,$$
$$(- D_y\phi_\varepsilon(\overline{x},\overline{y}), Y) \in \overline{D}^{2,-}v(\overline{y}) ,$$

and

$$-(\frac{1}{\eta} + \|D^2\phi_\varepsilon(\overline{x},\overline{y})\|) \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq D^2\phi_\varepsilon(\overline{x},\overline{y}) + \eta(D^2\phi_\varepsilon(\overline{x},\overline{y}))^2 .$$

We claim that we have, for ε small enough

$$-\frac{1}{2}\mathrm{Tr}\,[a(\overline{x})X] + H(\overline{x}, u(\overline{x}), D_x\phi_\varepsilon(\overline{x},\overline{y})) \leq 0 , \tag{42}$$

and

$$-\frac{1}{2}\mathrm{Tr}\,[a(\overline{y})Y] + H(\overline{y}, v(\overline{y}), -D_y\phi_\varepsilon(\overline{x},\overline{y})) \geq 0 . \tag{43}$$

Indeed if $\overline{x} \in \Omega$, (42) obviously holds since u is a subsolution of (1) in Ω. If $\overline{x} \in \partial\Omega$ then either $u(\overline{x}) > \varphi(\overline{x})$ and (42) is a consequence of (16) or $\overline{x} \in K$ (recall again that Σ_d^1 is open); by Lemma 3.2, (37) holds for $\phi = \chi_\varepsilon$, $\overline{x}$ playing here the role of x_0. But, because of Lemma 3.3, $D\chi_\varepsilon(\overline{x}) = D_x\phi_\varepsilon(\overline{x},\overline{y})$ satisfies condition (41) and therefore (42) holds.

For the inequality (43): if $\overline{y} \in \Omega$, one uses only that v is a supersolution of (1); if $\overline{y} \in \partial\Omega$, since $v(x_0) < \varphi(x_0)$ then $v(\overline{y}) < \varphi(\overline{y})$ if ε is small enough since, by Lemma 3.3, $v(\overline{y}) \to v(x_0)$ and (43) is a consequence of (17).

The remainder of the proof is classical but we provide it for the convenience of the reader. We subtract the above inequalities and we write

$$H(\overline{x}, u(\overline{x}), D_x\phi_\varepsilon(\overline{x},\overline{y})) - H(\overline{y}, v(\overline{y}), -D_y\phi_\varepsilon(\overline{x},\overline{y})) \leq \frac{1}{2}\mathrm{Tr}\,[a(\overline{x})X] - \frac{1}{2}\mathrm{Tr}\,[a(\overline{y})Y] .$$

To simplify a bit the estimate on the second order terms, we are going to use the above matrix inequality for $\eta = 0$. A thorough proof would consist here in obtaining the estimate for $\eta > 0$ and then in letting η to zero.

Straightforward estimates on the second derivatives of $D^2\phi_\varepsilon$ and the fact that ϕ_ε is basically a function of $x - y$ allow us to deduce from the above the matrix inequality that we have

$$Xp.p - Yq.q \leq \frac{C|\overline{x} - \overline{y}|^2}{\varepsilon}|p - q|^2 + 12|\overline{x} - x_0|^2|p|^2 ,$$

for some constant C independent of ε.

We use this inequality with $p_i = \sigma(\overline{x})e_i$ and $q_i = \sigma(\overline{y})e_i$, where $(e_i)_i$ is an orthonormal basis in $\mathbb{R}^p$. Summing over i and using the Lipschitz continuity of σ together with the fact that $\overline{x}$ converges to x_0 as $\varepsilon \to 0$, we get

$$\sum_{i=1}^{p} Xp_i.p_i - Yq_i.q_i = \sum_{i=1}^{p} \sigma^T(\overline{x})X\sigma(\overline{x})e_i.e_i - \sigma^T(\overline{y})Y\sigma(\overline{y})e_i.e_i =$$

$$\mathrm{Tr}\,[a(\overline{x})X] - \mathrm{Tr}\,[a(\overline{y})Y] \leq O(\frac{|\overline{x}-\overline{y}|^4}{\varepsilon}) + o(1)\;.$$

Therefore, by Lemma 3.3, the right-hand side vanishes as ε tends to zero since $\frac{|\overline{x}-\overline{y}|^4}{\varepsilon} \to 0$.

Coming back to the viscosity inequality, we obtain

$$H(\overline{x}, u(\overline{x}), D_x\phi_\varepsilon(\overline{x},\overline{y})) - H(\overline{y}, v(\overline{y}), -D_y\phi_\varepsilon(\overline{x},\overline{y})) \leq o(1)\;.$$

We now observe that $u(\overline{x}) - v(\overline{y}) \to u(x_0) - v(x_0) = M$ as ε tends to zero by Lemma 3.3. Moreover, easy calculations show that

$$|D_x\phi(\overline{x},\overline{y}) + D_y\phi(\overline{x},\overline{y})| = 4|\overline{x} - x_0| = o(1)\;,$$

and

$$|D_y\phi(\overline{x},\overline{y})| \leq c\frac{|\overline{x}-\overline{y}|^3}{\varepsilon}\;,$$

for some constant c independent of ε. Putting these estimates together and using assumptions **(H1)**, **(H2)**, **(H3)**, we get

$$o(1) \geq H(\overline{x}, u(\overline{x}), D_x\phi_\varepsilon(\overline{x},\overline{y})) - H(\overline{y}, v(\overline{y}), -D_y\phi_\varepsilon(\overline{x},\overline{y})) \geq$$
$$\gamma M - \tilde{m}(4|\overline{x} - x_0|^3) - m\left(c|\overline{x}-\overline{y}|(1 + \frac{|\overline{x}-\overline{y}|^3}{\varepsilon})\right) + o(1)\;.$$

It is now enough to use the fact that $\frac{|\overline{x}-\overline{y}|^4}{\varepsilon} \to 0$ as $\varepsilon \to 0$ to raise the contradiction

$$\gamma M \leq o(1)\;,$$

and to complete the proof of the first case.

Before we give the proof of Lemma 3.3, we complete the proof of Theorem 2.1. The second case we have to consider is the case $u(x_0) > \varphi(x_0)$ and $v(x_0) \geq \varphi(x_0)$: we skip the proof since it is entirely analogous to the proof above exchanging the roles of u and v.

The last case is $u(x_0) > \varphi(x_0)$ and $v(x_0) < \varphi(x_0)$. We introduce the test function

$$\psi_\varepsilon(x,y) = u(x) - v(y) - \phi_\varepsilon(x,y)\;,$$

where

$$\phi_\varepsilon(x,y) = \frac{|x-y|^4}{\varepsilon} + |x - x_0|^4\;.$$

We argue as above: by classical arguments, the statements 1 and 2 in Lemma 3.3 are true and one has

$$u(\overline{x}) \to u(x_0) \text{ and } v(\overline{y}) \to v(x_0) .$$

In particular, if $\overline{x} \in \partial\Omega$, $u(\overline{x}) > \varphi(\overline{x})$ for ε small enough and in the same way if $\overline{y} \in \partial\Omega$, $v(\overline{y}) < \varphi(\overline{y})$ for ε small enough: this implies that the viscosity inequalities (42) and (43) corresponding to the equation hold for u and v and the proof follows as in the first case. This completes the proof of Theorem 2.1. □

Proof of Lemma 3.3: Notice first that, by Young inequality

$$A\frac{|x-y|^3}{\varepsilon}|(x-y).n(x_0)| \leq \frac{3}{4}\frac{|x-y|^4}{\varepsilon} + \frac{1}{4}A^4\frac{|(x-y).n(x_0)|^4}{\varepsilon} ,$$

so that, taking $B \geq \frac{1}{4}A^4$, we get

$$\phi_\varepsilon(x,y) \geq \frac{1}{4}\frac{|x-y|^4}{\varepsilon} + |x-x_0|^4 \quad \text{on } \overline{\Omega} \times \overline{\Omega} . \tag{44}$$

Taking $(x, y) = (x_0, x_0)$ in the formula defining ψ_ε, we obtain $M_\varepsilon \geq M$. Since u and v are bounded, it is clear enough that the penalisation terms $\phi_\varepsilon(\overline{x}, \overline{y})$ are also bounded. So $|\overline{x} - \overline{y}| \to 0$. We now use (44) and write

$$M \leq M_\varepsilon = \psi_\varepsilon(\overline{x},\overline{y}) \leq u(\overline{x}) - v(\overline{y}) - \frac{1}{4}\frac{|\overline{x}-\overline{y}|^4}{\varepsilon} - |\overline{x}-x_0|^4 \leq u(\overline{x}) - v(\overline{y}) . \tag{45}$$

But since $|\overline{x} - \overline{y}| \to 0$ and since u is usc and v is lsc, we have

$$\limsup_{\varepsilon\to 0}\ [u(\overline{x}) - v(\overline{y})] \leq M .$$

Combining these two inequalities, we first conclude $M_\varepsilon \to M$.

Then, by taking the liminf in inequality (45), one easily sees that $u(\overline{x}) - v(\overline{y}) \to M$ and that $\frac{1}{4}\frac{|\overline{x}-\overline{y}|^4}{\varepsilon} + |\overline{x} - x_0|^4 \to 0$. Therefore $\overline{x}, \overline{y} \to x_0$ and since u is usc and v is lsc, the convergence of $u(\overline{x}) - v(\overline{y})$ to $M = u(x_0) - v(x_0)$ necessarely implies $u(\overline{x}) \to u(x_0)$ and $v(\overline{y}) \to v(x_0)$.

Finally, we have to show that property (41) holds for $D\chi_\varepsilon(\overline{x}) = D_x\phi_\varepsilon(\overline{x}, \overline{y})$ if $\overline{x} \in \partial\Omega$ and if ε is small enough. We first remark that d being C^2, hence C^1

$$d(\overline{x}) - d(\overline{y}) = -(\overline{x}-\overline{y}).n(x_0) + |\overline{x}-\overline{y}|o(1) ,$$

and therefore if $\overline{x} \in \partial\Omega$

$$(\overline{x}-\overline{y}).n(x_0) \geq |\overline{x}-\overline{y}|o(1) . \tag{46}$$

Now we compute $D\chi_\varepsilon(\overline{x}).n(\overline{x}) - C\left(|D_T\chi_\varepsilon(\overline{x})| + 1\right)$. Since $\overline{x} \to x_0$ as $\varepsilon \to 0$ and since n is continuous, easy computations yield

$$D\chi_\varepsilon(\overline{x}).n(\overline{x}) - C\left(|D_T\chi_\varepsilon(\overline{x})| + 1\right) \geq$$
$$\left\{A\left(\frac{|\overline{x}-\overline{y}|^3}{\varepsilon} + 1\right) + 4B\frac{[(\overline{x}-\overline{y}).n(x_0)]^3}{\varepsilon}\right\}(1-o(1)) -$$
$$4C\frac{|\overline{x}-\overline{y}|^3}{\varepsilon} - 3AC\frac{|\overline{x}-\overline{y}|^2}{\varepsilon}|(\overline{x}-\overline{y}).n(x_0)| - 4BC\frac{|(\overline{x}-\overline{y}).n(x_0)|^3}{\varepsilon}o(1)$$
$$-4C|\overline{x}-x_0|^3 - C\ .$$

From now on two cases: if $(\overline{x}-\overline{y}).n(x_0) \leq 0$, we use (46) which yields $(\overline{x}-\overline{y}).n(x_0) = |\overline{x}-\overline{y}|o(1)$ and it is clear enough that, for any choice of B, the first term

$$A\left(\frac{|\overline{x}-\overline{y}|^3}{\varepsilon} + 1\right),$$

controls all the other terms if A is large enough and if ε is small enough.

If $(\overline{x}-\overline{y}).n(x_0) > 0$, we can take advantage of the (now) "good" term

$$4B\frac{[(\overline{x}-\overline{y}).n(x_0)]^3}{\varepsilon}\ .$$

Using Young inequality, we have

$$\frac{|\overline{x}-\overline{y}|^2}{\varepsilon}|(\overline{x}-\overline{y}).n(x_0)| \leq \frac{2\lambda}{3}\frac{|\overline{x}-\overline{y}|^3}{\varepsilon} + \frac{1}{3\lambda^2}\frac{|(\overline{x}-\overline{y}).n(x_0)|^3}{\varepsilon},$$

for any $\lambda > 0$. To conclude, we first choose λ small enough; then for a choice of A large enough and then of B large enough, the result now easily follows.

$\square$

Remark 3.1 : *In the results of* [5], *for first-order equations* ($a \equiv 0$), *the assumptions* **(H7)** *and* **(H8)** *were replaced respectively by*

(H7)' $\qquad H(x,u,p+\lambda n(x)) > 0\ ,$ *if* $\lambda \geq C_R(1+|p|)$,
where $C_R > 0$, *for every* x *in a neighborhood of* Σ_d^1, $p \in I\!R^n$ *and* $|u| \leq R$.

(H8)' $\qquad H(x,u,p-\lambda n(x)) < 0\ ,$ *if* $\lambda \geq C_R(1+|p|)$,
where $C_R > 0$, *for every* x *in a neighborhood of* Σ_d^2, $p \in I\!R^n$ *and* $|u| \leq R$.

So it seems that the result of [5] *and our result are very different in that context. But in the case of deterministic control where* H *is given by* (10) *these assumptions have exactly the same consequences i.e. respectively*

(H7)" $\qquad b(x,\alpha_x).n(x) < 0$
for some control $\alpha_x \in \mathcal{A}$ *and for every* x *in* Σ_d^1.

(H8)" $\qquad b(x,\alpha).n(x) < 0$
for any control $\alpha \in \mathcal{A}$ *and for every* x *in* Σ_d^2.

4 Applications to Stochastic Exit Time Control Problems.

We consider as in the introduction a system whose state is described by the stochastic differential equation (8) and we introduce the cost function

$$J(x,(\alpha_s)_s) = I\!\!E_x\left[\int_0^{\tau} f(X_t,\alpha_t)\exp(-\lambda t)dt + \varphi(X_\tau)\exp(-\lambda\tau)\right] , \qquad (47)$$

where τ is the first exit time of the trajectory $(X_t)_t$ from the bounded open set Ω, i.e.

$$\tau = \inf\{t \geq 0;\ X_t \notin \Omega\}.$$

We are interested in the so-called value function u given by (9) which can be written using the cost function J as

$$u(x) = \inf_{(\alpha_s)_s} J(x,(\alpha_s)_s).$$

We first make the following classical assumptions

(H9) $\sigma(.,\alpha)$, $b(.,\alpha)$ and $f(.,\alpha)$ are $W^{1,\infty}$ functions defined on $\overline{\Omega}$ for any $\alpha \in \mathcal{A}$; moreover

$$\sup_{\alpha\in\mathcal{A}} ||\phi(.,\alpha)||_{W^{1,\infty}} < \infty,$$

for $\phi = \sigma_{i,j}$, b_i, f and c $(1 \leq i \leq n,\ 1 \leq j \leq p)$.

(H10) $\lambda > 0$.

(H11) $\varphi \in C(\partial\Omega)$.

Assumptions **(H9)**-**(H11)** are classical: in particular, **(H9)** ensures existence and uniqueness for the dynamics $(X_t)_t$ and **(H10)** together with the boundedness of f and φ ensures that J and u are well-defined. Since we want to treat cases when u does not necessarely assume the boundary data φ continuously, we need the following additional assumptions which are the analogue of **(H6)**-**(H8)**.

(H12) The subset Σ_d and Σ_n of $\partial\Omega$ defined by

$$\Sigma_d = \{x \in \partial\Omega \text{ s.t } \sigma^T(x)n(x) = 0\} ,$$

and

$$\Sigma_n = \{x \in \partial\Omega \text{ s.t } \sigma^T(x)n(x) \neq 0\} ,$$

are unions of connected components of $\partial\Omega$.

(H13) For all $x \in \Sigma_d$, if there is some $\alpha \in \mathcal{A}$ such that

$$\frac{1}{2}\mathrm{Tr}\left[a(x)D^2d(x)\right] - b(x,\alpha).n(x) = 0 ,$$

then there exists $\alpha_1, \alpha_2 \in \mathcal{A}$ such that

$$\frac{1}{2}\mathrm{Tr}\left[a(x)D^2d(x)\right] - b(x,\alpha_1).n(x) > 0 ,$$

$$\frac{1}{2}\mathrm{Tr}\left[a(x)D^2d(x)\right] - b(x,\alpha_2).n(x) < 0 .$$

where we have set $a(x) = \sigma(x)\sigma^T(x)$.

Our result is the

Theorem 4.1 : *Under the assumptions* **(H9)-(H13)**, *the value function u is continuous in Ω and it is the unique continuous solution of the Hamilton-Jacobi-Bellman problem*

$$\begin{cases} -\frac{1}{2}\mathrm{Tr}\left[a(x)D^2u\right] + H(x,u,Du) &= 0 \text{ in } \Omega , \\ u &= \varphi \text{ in } \partial\Omega. \end{cases} \tag{48}$$

where

$$H(x,u,p) = \sup_{\alpha\in\mathcal{A}} \{-b(x,\alpha).p + \lambda u - f(x,\alpha)\} .$$

Moreover, if u_1 and u_2 are respectively an usc subsolution of (48) *and a lsc supersolution of* (48) *then*

$$u_1 \le u \le u_2 \quad in\ \Omega .$$

□

This result is optimal as it is shown by the following example given in G. Barles and B. Perthame[4]. The domain $\Omega \subset \mathbb{R}^2$ is given by

$$\Omega = \{(x,y);\ |x|, |y| < 1 \ \text{ and } \ x < 0 \ \text{ or } \ y < 0\} .$$

The boundary of Ω is not smooth but this is not relevant here. The space of controls $\mathcal{A}$ is

$$\mathcal{A} = \{\alpha = (\alpha_1,\alpha_2) \in \mathbb{R}^2;\ \alpha_1 \ge 0 , \alpha_2 \le 0 , \alpha_1^2 + \alpha_2^2 \le 1\} .$$

We consider a deterministic control problem, hence $\sigma \equiv 0$, and b is given by $b(x,\alpha) = \alpha$. Finally we take $\lambda = 0$ (but this is again not relevant for the example), $f \equiv 0$ and $\varphi = 0$ except on $\Gamma = \{(x,0);\ 0 < x \le 1\}$ where we set $\varphi(x,0) = 4x(x-1)$. Then we introduce the function u^- defined on $\overline{\Omega}$ by

$$u^-(x) = \inf \left[\varphi(X_\theta)e^{-\lambda\theta} \ ;\ ((\alpha_s)_s, \theta) \text{ s.t } X_\theta \in \partial\Omega\right] .$$

The computation of u is straightforward since no trajectory can exit through Γ and therefore $u \equiv 0$. In the same way, $u^- \equiv 0$ for $y < 0$. But if $y \geq 0$, one sees easily that the optimal trajectory consists in first choosing the control $(0,-1)$ until the trajectory reaches the line $y = 0$ and then to choose the control $(1,0)$ until the trajectory reaches the point $(1/2,0)$. Therefore $u^-(x,y) \equiv -1$ for $(x,y) \in \Omega$, $y \geq 0$ is very different from u in this part of Ω. But it is proved in [4] that u^- is also a solution of (48) and we have an example of non-uniqueness. The point here is that the optimal trajectory for u^- and for $y \geq 0$ reach the point $(1/2,0)$ by being tangent and cannot be approximated by trajectories staying in Ω until they exit at $(1/2,0)$. The problem is at the point $(1/2,0)$ where **(H13)** fails. It is worth remarking also that we are in a non-uniqueness feature despite u is continuous.

We now consider the so-called State-Constraint problems. For this type of problems, the dynamics is still given by the stochastic differential equation (8) but the value function is now defined by

$$u(x) = \inf_{\mathcal{C}_x} I\!E_x \left[\int_0^{+\infty} f(X_t, \alpha_t) \exp(-\lambda t) dt \right] , \tag{49}$$

where $\mathcal{C}_x$, the set of admissible controls, is the set

$$\mathcal{C}_x = \{(\alpha_s)_s \; ; \; X_t \in \overline{\Omega} \text{ a.s, } \forall t > 0\} .$$

In this setting, the value function is obtained through a minimisation process under the "constraint" on the "state" X_t of the system to stay in $\overline{\Omega}$ a.s for all time $t > 0$; this justifies the terminology.

Of course, one needs assumptions to ensure that $\mathcal{C}_x$ is a non-empty set for all $x \in \overline{\Omega}$; for this reason, but also to prove the continuity of u, we introduce the following assumption which is the analogue of **(H13)** in this context

(H14) $\Sigma_d = \partial\Omega$ and for all $x \in \partial\Omega$, there is some $\alpha \in \mathcal{A}$ s.t

$$\frac{1}{2}\mathrm{Tr}\left[a(x)D^2 d(x)\right] - b(x,\alpha).n(x) > 0 .$$

Our result is the

Theorem 4.2 : *Under the assumptions* **(H9)**, **(H10)** *and* **(H14)**, *the value function u of the State-Constraint problem is well-defined and continuous on $\overline{\Omega}$; u is the unique solution of the Hamilton-Jacobi-Bellman problem*

$$\begin{cases} -\frac{1}{2}\mathrm{Tr}\left[a(x)D^2u + H(x,u,Du)\right] &= 0 \text{ in } \Omega , \\ -\frac{1}{2}\mathrm{Tr}\left[a(x)D^2u\right] + H(x,u,Du) &\geq 0 \text{ in } \partial\Omega. \end{cases} \tag{50}$$

where

$$H(x,u,p) = \sup_{\alpha \in \mathcal{A}} \{-b(x,\alpha).p + \lambda u - f(x,\alpha)\} .$$

Moreover, if u_1 *and* u_2 *are respectively an usc subsolution of* (50) *and a lsc supersolution of* (50) *then*

$$u_1 \leq u \leq u_2 \quad in\, \Omega \ .$$

As we mentioned it in the introduction, State-Constraint problems in the deterministic case were first considered by H.M Soner[53]: Soner introduced the assumption corresponding to **(H14)** with $\sigma \equiv 0$ and he was able to show that u is continuous directly by control arguments. He also proved that u is the unique continuous solution of (50). More recently, M. Katsoulakis[35] was able to extend in certain cases similar types of arguments to treat stochastic State Constraint problems.

Remark 4.1 : *The second part of* Theorem 4.2 *shows that* u *is in fact the "absolute" maximal subsolution (and solution) of the equation* (1) *in* Ω*: indeed, for any subsolution* u_1 *of* (1) *(or equivalently of* (50)*),* $u_1 \leq u$ *in* Ω.

We turn to the **Proof of Theorem 4.1.** We provide here two proofs. The first one is a classical one since it makes use of the Dynamic Programing Principle. However, the Dynamic Programing Principle is not that straightforward in this context since we do not have any regularity properties for the value function u. This is why we give an alternate proof, which uses approximate problems and relies on a stability result for discontinuous viscosity solutions.

Both proofs consist in two steps: first we show that our assumptions allow us to apply the uniqueness result of Theorem 2.1 and then we prove that u solves problem (48) indeed. The first step being the same for both proofs, we give it now.

It is clear that assumptions **(H1)**-**(H4)** hold in our context. To be able to apply the results of the previous section, we only have to pay attention to the boundary conditions. We have the

Lemma 4.1 : *If* v *is a bounded usc subsolution of* (48) *and if* w *is a bounded lsc supersolution of* (48), *then*

- *For* $x \in \partial\Omega$, *if* $v(x) > \varphi(x)$, *then* $\sigma^T(x)n(x) = 0$ *and for every* $\alpha \in \mathcal{A}$

$$\frac{1}{2}\mathrm{Tr}\left[a(x)D^2 d(x)\right] - b(x,\alpha).n(x) \geq 0 \ .$$

- *For* $x \in \partial\Omega$, *if* $w(x) < \varphi(x)$, *then* $\sigma(x)n(x) = 0$, *and there is some* $\alpha_x \in \mathcal{A}$ *such that*

$$\frac{1}{2}\mathrm{Tr}\left[a(x)D^2 d(x)\right] - b(x,\alpha_x).n(x) \geq 0 \ .$$

□

These results are the equivalent, in the context of Stochastic Optimal Control, of Proposition 1.1. Hence, we do not provide any proof for Lemma 4.1 and we refer to the proof of this Proposition.

Let us prove now that assumptions **(H6)**, **(H7)** and **(H8)** hold. Getting back to the definition of the sets Σ_d^1 and Σ_d^2 of the previous section, it appears that, in this new setting, we have

$$\Sigma_d^1 = \{x \in \Sigma_d,\ \frac{1}{2}\mathrm{Tr}\left[a(x)D^2d(x)\right] + \sup_\alpha\{-b(x,\alpha).n(x)\} \geq 0\}\ ,$$

and

$$\Sigma_d^2 = \{x \in \Sigma_d,\ -\frac{1}{2}\mathrm{Tr}\ \left[a(x)D^2d(x)\right] + \sup_\alpha\{b(x,\alpha).n(x)\} \leq 0\}\ .$$

On an other hand, the assumption **(H13)** allows us to divide Σ_d into

1. $\partial\Omega_1$: the set of points $x \in \Sigma_d$ for which there are $\alpha,\ \alpha' \in \mathcal{A}$, with $\frac{1}{2}\mathrm{Tr}\left[a(x)D^2d(x)\right] - b(x,\alpha).n(x) < 0\ ,\ \frac{1}{2}\mathrm{Tr}\left[a(x)D^2d(x)\right] - b(x,\alpha')n(x) > 0$
2. $\partial\Omega_2$: the set of points $x \in \Sigma_d$ that satisfy, for all $\alpha \in \mathcal{A}$
$$\frac{1}{2}\mathrm{Tr}\left[a(x)D^2d(x)\right] - b(x,\alpha).n(x) > 0\ ,$$
3. $\partial\Omega_3$: the set of points $x \in \Sigma_d$ that satisfy, for all $\alpha \in \mathcal{A}$
$$\frac{1}{2}\mathrm{Tr}\left[a(x)D^2d(x)\right] - b(x,\alpha).n(x) < 0\ .$$

It is clear enough that $\partial\Omega_1$, $\partial\Omega_2$ and $\partial\Omega_3$ are disjoint and open; moreover, using again **(H13)** they are also closed and therefore each of them is a union of connected components of $\partial\Omega$. Finally it is easy to see that

$$\Sigma_d^1 = \partial\Omega_1 \cup \partial\Omega_2\ ,$$

and

$$\Sigma_d^2 = \partial\Omega_2\ .$$

And therefore **(H6)** holds.

Now we turn to the checking of **(H7)**. From **(H13)**, we know that, for each $x \in \Sigma_d^1$

$$\frac{1}{2}\mathrm{Tr}\left[a(x)D^2d(x)\right] + \sup_\alpha\{-b(x,\alpha).n(x)\} > 0\ .$$

The function $x \mapsto \frac{1}{2}\mathrm{Tr}\left[a(x)D^2d(x)\right] + \sup_\alpha\{-b(x,\alpha).n(x)\}$ being continuous in a neighborhood of $\partial\Omega$ and $\Sigma_d^1 \subset \partial\Omega$ being compact, it follows that there is some $\nu > 0$ such that, for all $x \in \Sigma_d^1$,

$$\frac{1}{2}\mathrm{Tr}\left[a(x)D^2d(x)\right] + \sup_\alpha\{-b(x,\alpha).n(x)\} \geq \nu\,. \tag{51}$$

Changing ν if needed, we see that this inequality holds in the neighborhood V of Σ_d^1 as well. To check that **(H7)** holds, we consider $\mu_2 \geq \mu_1 \geq C_R(1+|p|)$ for some constant C_R chosen later, $x \in V$ and $|u| \leq R$. If

$$H(x,u,p+\mu_1 n(x)) = -b(x,\alpha).(p+\mu_1 n(x)) + \lambda u - f(x,\alpha)\,,$$

it is clear enough using (51) that, for C_R large enough, the supremum which gives H is achieved for α satisfying

$$-b(x,\alpha).n(x) \geq \frac{\nu}{2} - \frac{1}{2}\mathrm{Tr}\left[a(x)D^2d(x)\right]\,,$$

and therefore

$$\begin{aligned}\mu_2\frac{1}{2}\mathrm{Tr}\left[a(x)D^2d(x)\right] + H(x,u,p+\mu_2 n(x)) - \mu_1\frac{1}{2}\mathrm{Tr}\left[a(x)D^2d(x)\right]&\\ -H(x,u,p+\mu_1 n(x)) \geq&\\ (\mu_2-\mu_1)\left(\frac{1}{2}\mathrm{Tr}\left[a(x)D^2d(x)\right] + -b(x,\alpha).n(x)\right) \geq \frac{\nu}{2}(\mu_2-\mu_1)&\end{aligned}$$

and **(H7)** holds. **(H8)** can be checked with the same kind of arguments.

We have checked every assumption of Theorem 2.1. The remainder of the proof consists now in proving that the value function u solves problem (48).

4.1 The direct proof by the Dynamic Programming Principle.

We use the following theorem taken from V.S Borkar[10] (See also W.H Fleming and H.M Soner[20]).

Theorem 4.3 : Dynamic Programming Principle
The value function u satisfies

$$u(x) = \inf_{(\alpha_s)_s} \mathbb{E}_x\Bigg[\int_0^{\theta\wedge\tau} f(X_t,\alpha_t)\exp(-\lambda t)dt + 1_{\{\theta<\tau\}}u(X_\theta)\exp(-\lambda\theta) + \\ 1_{\{\theta\geq\tau\}}\varphi(X_\tau)\exp(-\lambda\tau)\Bigg],$$

for every $x \in \overline{\Omega}$ and for every stopping-time θ measurable w.r.t the filtration associated to $(W_t)_t$. □

We now prove

Theorem 4.4 : *Under the assumptions of* Theorem 4.1, *the value function* u *solves the Dirichlet problem for the Hamilton-Jacobi-Bellman equation* (48).□

Proof: We recall that the usc and lsc envelopes of u are respectively given by

$$u^*(x) = \limsup_{y \to x} u(y) , \qquad u_*(x) = \liminf_{y \to x} u(y) .$$

We have to check that they are respectively sub- and supersolutions of (48). We give the proof for u_* but we skip the details in the subsolution case since the proof in this situation is more classical and the result somehow easier to derive.

Let ϕ be a C^2 function defined on $\overline{\Omega}$ and let $x_0 \in \overline{\Omega}$ be a local minimum point of $u_* - \phi$. By changing ϕ but not its first and second derivatives at x_0, we may assume that $u_*(x_0) = \phi(x_0)$ and that x_0 is, in fact, a global minimum point of $u_* - \phi$ on $\overline{\Omega}$.

Throughout the proof, H_α stands for the function defined on $\overline{\Omega} \times I\!R \times I\!R^n$ by

$$H_\alpha(x, u, p) := -b(x, \alpha).p + \lambda u - f(x, \alpha) , \tag{52}$$

so that $H(x, u, p) = \sup_{\alpha \in \mathcal{A}} H_\alpha(x, u, p)$.

Case 1: $x_0 \in \Omega$.

For the sake of simplicity, we can change again ϕ outside a neighborhood of x_0 in order to have $\phi \le \varphi$ on $\partial\Omega$. Since

$$u_*(x_0) = \liminf_{x \to x_0} u(x) ,$$

there exists a sequence $(x_\varepsilon)_\varepsilon$ in Ω, converging to x_0 such that

$$\lim_{\varepsilon \to 0} u(x_\varepsilon) = u_*(x_0) .$$

Then we define $h_\varepsilon > 0$ by

$$h_\varepsilon^2 = |u(x_\varepsilon) - \phi(x_\varepsilon)| . \tag{53}$$

We know that $h_\varepsilon \to 0$ as $\varepsilon \to 0$ since $u(x_\varepsilon) \to u_*(x_0)$ and since $\phi(x_\varepsilon) \to \phi(x_0) = u_*(x_0)$ because ϕ is continuous and $x_\varepsilon \to x_0$.

We are now going to use the Dynamic Programming Principle: we use Theorem 4.4 for $x = x_\varepsilon$, that is

$$u(x_\varepsilon) = \inf_{(\alpha_s)_s} I\!E_{x_\varepsilon}\Big[\int_0^{h_\varepsilon \wedge \tau_\varepsilon} f(X_s^\varepsilon, \alpha_s) e^{-\lambda s} ds + 1_{\{h_\varepsilon < \tau_\varepsilon\}} u(X_{h_\varepsilon}^\varepsilon) e^{-\lambda h_\varepsilon} + \\ 1_{\{h_\varepsilon \ge \tau_\varepsilon\}} \varphi(X_{\tau_\varepsilon}^\varepsilon) e^{-\lambda \tau_\varepsilon} \Big],$$

where X^ε is the solution of the equation (8) with the initial condition $X_0^\varepsilon = x_\varepsilon$ and τ_ε is the first exit time of Ω for X^ε.

Our choice of h_ε ensures that $\phi(x_\varepsilon) = u(x_\varepsilon) + o(h_\varepsilon)$. Moreover, since x_0 is a global minimum point of $u_* - \phi$ on $\overline{\Omega}$ and since $u_*(x_0) = \phi(x_0)$, one has $u(x) \geq u_*(x) \geq \phi(x)$ on $\overline{\Omega}$. Therefore

$$\phi(x_\varepsilon) \geq u(x_\varepsilon) + o(h_\varepsilon) \geq \inf_{(\alpha_s)_s} I\!E_{x_\varepsilon}\Big[\int_0^{h_\varepsilon\wedge\tau_\varepsilon} f(X_s^\varepsilon,\alpha_s)e^{-\lambda s}ds + \\ 1_{\{h_\varepsilon<\tau_\varepsilon\}}\phi(X_{h_\varepsilon}^\varepsilon)\, e^{-\lambda h_\varepsilon} + 1_{\{h_\varepsilon\geq\tau_\varepsilon\}}\varphi(X_{\tau_\varepsilon}^\varepsilon)e^{-\lambda\tau_\varepsilon}\Big] + o(h_\varepsilon).$$

Then, using that $\phi \leq \varphi$ on $\partial\Omega$, we can write this inequality in the following way

$$\phi(x_\varepsilon) \geq \inf_{(\alpha_s)_s} I\!E_{x_\varepsilon}\Big[\int_0^{h_\varepsilon\wedge\tau_\varepsilon} f(X_s^\varepsilon,\alpha_s)e^{-\lambda s}ds + \phi(X_{h_\varepsilon\wedge\tau_\varepsilon}^\varepsilon)\, e^{-\lambda h_\varepsilon\wedge\tau_\varepsilon}\Big] + o(h_\varepsilon)\ . \quad (54)$$

We now apply Itô's Lemma to the C^2 function ϕ and to the process X^ε on $[0, h_\varepsilon \wedge \tau_\varepsilon]$: we get

$$\phi(X_{h_\varepsilon\wedge\tau_\varepsilon}^\varepsilon)e^{-\lambda h_\varepsilon\wedge\tau_\varepsilon} = \phi(x_\varepsilon) + \int_0^{h_\varepsilon\wedge\tau_\varepsilon} e^{-\lambda s}D\phi(X_s^\varepsilon).\sigma(X_s^\varepsilon)dW_s + \\ \int_0^{h_\varepsilon\wedge\tau_\varepsilon}\Big[\frac{1}{2}\mathrm{Tr}[\sigma\sigma^T(X_s^\varepsilon)D^2\phi(X_s^\varepsilon)] + b(X_s^\varepsilon,\alpha_s).D\phi(X_s^\varepsilon) - \lambda\phi(X_s^\varepsilon)\Big]e^{-\lambda s}ds\ .$$

Plugging this into the inequality (54) and using the definition of H_α, we obtain

$$\sup_{(\alpha_s)_s} I\!E_{x_\varepsilon}\Big[\int_0^{h_\varepsilon\wedge\tau_\varepsilon}\Big(-\frac{1}{2}\mathrm{Tr}[a(X_s^\varepsilon)D^2\phi(X_s^\varepsilon)] + H_{\alpha_s}(X_s^\varepsilon,\phi(X_s^\varepsilon),D\phi(X_s^\varepsilon))\Big)e^{-\lambda s}ds\Big] \\ \geq o(h_\varepsilon)\ .$$

But using that $H_\alpha \leq H$ for any $\alpha \in \mathcal{A}$ and using also the C^2 regularity of ϕ together with the continuity of H and a, we deduce

$$\sup_{(\alpha_s)_s} I\!E_{x_\varepsilon}\Big[\int_0^{h_\varepsilon\wedge\tau_\varepsilon}\Big(-\frac{1}{2}\mathrm{Tr}[a(x_\varepsilon)D^2\phi(x_\varepsilon)] + H(x_\varepsilon,\phi(x_\varepsilon),D\phi(x_\varepsilon))\Big)e^{-\lambda s}ds\Big] \geq o(h_\varepsilon)\,.$$

Finally the integrand on the left-hand side being independent of $(\alpha_s)_s$, of s and of the probability variable, we get

$$\Big(-\frac{1}{2}\mathrm{Tr}[a(x_\varepsilon)D^2\phi(x_\varepsilon)] + H(x_\varepsilon,\phi(x_\varepsilon),D\phi(x_\varepsilon))\Big)\sup_{(\alpha_s)_s} I\!E_{x_\varepsilon}\Big[\frac{1-e^{-\lambda h_\varepsilon\wedge\tau_\varepsilon}}{\lambda}\Big] \\ \geq o(h_\varepsilon)\ . \quad (55)$$

Now we claim that

$$\frac{1}{h_\varepsilon}\sup_{(\alpha_s)_s} I\!E_{x_\varepsilon}\Big[\frac{1-e^{-\lambda h_\varepsilon\wedge\tau_\varepsilon}}{\lambda}\Big] \to 1\,,$$

as $\varepsilon \to 0$. In order to prove this claim, we are going to show that

$$\sup_{(\alpha_s)_s} I\!P[\tau_\varepsilon \le h_\varepsilon] \to 0 \ ,$$

as $\varepsilon \to 0$. We set $\delta = \frac{1}{2}d(x_0, \partial\Omega)$. For any $x \in B(x_0, \delta)$, thanks to the regularity of b and σ, we may apply martingale inequalities which yield, for any $0 < h \le 1$ and for any control $(\alpha_s)_s$

$$I\!E_x\Big[\sup_{s\in[0,h]} |X_s - x|^4\Big] \le Ch^2 \ ,$$

for some constant C independent of x, h and $(\alpha_s)_s$. Therefore

$$I\!P[\tau_x \le h] \le I\!P\Big[\sup_{s\in[0,h]} |X_s - x| \ge \delta\Big] \le \frac{1}{\delta^4} I\!E_x\Big[\sup_{s\in[0,h]} |X_s - x|^4\Big] \le \frac{Ch^2}{\delta^4} \ .$$

As a consequence, for ε small enough, we may use this estimate for $h = h_\varepsilon$, $x = x_\varepsilon$ and for an arbitrary control $(\alpha_s)_s$. We write

$$\begin{aligned} I\!E_{x_\varepsilon}\left[\frac{1 - e^{-\lambda h_\varepsilon \wedge \tau_\varepsilon}}{\lambda}\right] &= \frac{1 - e^{-\lambda h_\varepsilon}}{\lambda} I\!P[\tau_\varepsilon > h_\varepsilon] + O(h_\varepsilon) I\!P[\tau_\varepsilon \le h_\varepsilon] \ , \\ &= \frac{1 - e^{-\lambda h_\varepsilon}}{\lambda} + O(h_\varepsilon) I\!P[\tau_\varepsilon \le h_\varepsilon] \ , \end{aligned}$$

since $I\!P[\tau_\varepsilon > h_\varepsilon] = 1 - I\!P[\tau_\varepsilon \le h_\varepsilon]$. And the claim is proved.

Dividing (55) by h_ε and letting ε go to zero, one gets

$$-\frac{1}{2}\mathrm{Tr}[a(x_0)D^2\phi(x_0)] + H(x_0, \phi(x_0), D\phi(x_0)) \ge 0 \ ,$$

and the conclusion follows since $\phi(x_0) = u_*(x_0)$.

Case 2: $x_0 \in \partial\Omega$.

If $u_*(x_0) \ge \varphi(x_0)$ we are done; so we have just to consider the case when $u_*(x_0) < \varphi(x_0)$. Changing ϕ again if needed without changing its derivatives at x_0, we may assume that $\phi < \varphi$ on $\partial\Omega$. Here again, by the very definition of u_*, there exists a sequence $(x_\varepsilon)_\varepsilon$ converging to x_0 in $\overline{\Omega}$ with

$$u_*(x_0) = \lim_{\varepsilon\to 0} u(x_\varepsilon) \ ,$$

we still set

$$h_\varepsilon^2 = |u(x_\varepsilon) - \phi(x_\varepsilon)| \ ,$$

and in the same way as in case 1, $h_\varepsilon \to 0$. Notice also that we can suppose $x_\varepsilon \in \Omega$ for every ε. Otherwise, there would be a sequence $(x_{\varepsilon'})_{\varepsilon'}$ in $\partial\Omega$ with $u_*(x_0) = \lim_{\varepsilon\to 0} u(x_{\varepsilon'})$. But by the very definition of u, $u(x) = \varphi(x)$ on $\partial\Omega$ and since φ is continuous, this would contradict $u_*(x_0) < \varphi(x_0)$.

Using the Dynamic Programming Principle at x_ε, we get

$$u(x_\varepsilon) = \inf_{(\alpha_s)_s} I\!E_{x_\varepsilon}\Big[\int_0^{h_\varepsilon\wedge\tau_\varepsilon} f(X_s^\varepsilon,\alpha_s)e^{-\lambda s}ds + 1_{\{h_\varepsilon<\tau_\varepsilon\}}u(X_{h_\varepsilon}^\varepsilon)e^{-\lambda h_\varepsilon}+ 1_{\{h_\varepsilon\geq\tau_\varepsilon\}}\varphi(X_{\tau_\varepsilon}^\varepsilon)e^{-\lambda\tau_\varepsilon}\Big].$$

We then consider a control $(\alpha_s^\varepsilon)_s$ which is h_ε^2-optimal, that is

$$u(x_\varepsilon) \geq I\!E_{x_\varepsilon}\Big[\int_0^{h_\varepsilon\wedge\tau_\varepsilon} f(X_s^\varepsilon,\alpha_s^\varepsilon)e^{-\lambda s}ds + 1_{\{h_\varepsilon<\tau_\varepsilon\}}u(X_{h_\varepsilon}^\varepsilon)e^{-\lambda h_\varepsilon}+ 1_{\{h_\varepsilon\geq\tau_\varepsilon\}}\varphi(X_{\tau_\varepsilon}^\varepsilon)e^{-\lambda\tau_\varepsilon}\Big] - h_\varepsilon^2, \tag{56}$$

where $(X_s^\varepsilon)_s$ is the solution of (8) associated to $(\alpha_s^\varepsilon)_s$ and τ_ε stands for its first exit time of Ω.

Then we repeat the same arguments as above and using Itô's Lemma, we obtain

$$\Big(-\frac{1}{2}\mathrm{Tr}[a(x_\varepsilon)D^2\phi(x_\varepsilon)] + H(x_\varepsilon,\phi(x_\varepsilon),D\phi(x_\varepsilon))\Big)\, I\!E_{x_\varepsilon}\Big[\frac{1-e^{-\lambda h_\varepsilon\wedge\tau_\varepsilon}}{\lambda}\Big] \geq o(h_\varepsilon)\ .$$

which is the analogue of the inequality (55). To conclude as in case 1, we have just to show that

$$I\!E_{x_\varepsilon}\Big[\frac{1-e^{-\lambda h_\varepsilon\wedge\tau_\varepsilon}}{\lambda}\Big] \to 1\ ,$$

as $\varepsilon\to 0$ or equivalently that

$$I\!P[\tau_\varepsilon \leq h_\varepsilon] \to 0\ ,$$

as it can easily be seen by the arguments of case 1.

To do so, we come back to (56). We use the fact that $u \geq u_* \geq \phi$ since x_0 is a global minimum point of $u_*-\phi$ and $u_*(x_0)=\phi(x_0)$ and after subtraction by $\phi(x_\varepsilon)$, we obtain

$$u(x_\varepsilon)-\phi(x_\varepsilon) \geq I\!E_{x_\varepsilon}\Big[\int_0^{h_\varepsilon\wedge\tau_\varepsilon} f(X_s^\varepsilon,\alpha_s^\varepsilon)e^{-\lambda s}ds + 1_{\{h_\varepsilon<\tau_\varepsilon\}}(\phi(X_{h_\varepsilon}^\varepsilon)e^{-\lambda h_\varepsilon}-\phi(x_\varepsilon))+ 1_{\{h_\varepsilon\geq\tau_\varepsilon\}}(\varphi(X_{\tau_\varepsilon}^\varepsilon)e^{-\lambda\tau_\varepsilon}-\varphi(x_\varepsilon))\Big] + I\!P[\tau_\varepsilon\leq h_\varepsilon](\varphi(x_\varepsilon)-\phi(x_\varepsilon)) - h_\varepsilon^2\ .$$

We consider all the terms successively. One clearly has

$$I\!E_{x_\varepsilon}\Big[\int_0^{h_\varepsilon\wedge\tau_\varepsilon} f(X_s^\varepsilon,\alpha_s^\varepsilon)e^{-\lambda s}ds\Big] = o_\varepsilon(1)\ .$$

Moreover, ϕ and φ being continuous

$$I\!E_{x_\varepsilon}\Big[1_{\{h_\varepsilon<\tau_\varepsilon\}}(\phi(X_{h_\varepsilon}^\varepsilon)e^{-\lambda h_\varepsilon}-\phi(x_\varepsilon))\Big] = o_\varepsilon(1)\ ,$$

and
$$\mathbb{E}_{x_\varepsilon}\left[1_{\{h_\varepsilon \geq \tau_\varepsilon\}}(\varphi(X^\varepsilon_{\tau_\varepsilon})e^{-\lambda\tau_\varepsilon} - \varphi(x_\varepsilon))\right] = o_\varepsilon(1) .$$

Furthermore these estimates are uniform w.r.t. the control processes $(\alpha^\varepsilon_s)_s$. Gathering them, the above inequality now reads

$$u(x_\varepsilon) - \phi(x_\varepsilon) \geq \mathbb{P}[\tau_\varepsilon \leq h_\varepsilon](\varphi(x_\varepsilon) - \phi(x_\varepsilon)) + o_\varepsilon(1) .$$

We finally recall that $u(x_\varepsilon) - \phi(x_\varepsilon) \to 0$ and that $\phi(x_0) = u_*(x_0) < \varphi(x_0)$. We can then conclude
$$\limsup_{\varepsilon\to 0} \mathbb{P}[\tau_\varepsilon \leq h_\varepsilon] = 0 .$$
And the proof is complete. □

We now sketch the proof in the subsolution case. We consider $\phi \in C^2(\overline{\Omega})$ and $x_0 \in \overline{\Omega}$ a maximum point of $u^* - \phi$. Arguing as above, we may assume without loss of generality that $u^*(x_0) = \phi(x_0)$, that x_0 is a global maximum point of $u^* \leq \phi$ (and therefore that $u^* \leq \phi$ on $\overline{\Omega}$).

If $x_0 \in \Omega$, we can choose a sequence $(x_\varepsilon)_\varepsilon \in \Omega$ converging to x_0 with $u(x_\varepsilon) \to u^*(x_0)$. In the case $x_0 \in \partial\Omega$, we may assume $u^*(x_0) > \varphi(x_0)$ and, for the same reasons as above, there is a sequence $(x_\varepsilon)_\varepsilon \in \Omega$ with $u(x_\varepsilon) \to u^*(x_0)$ as well. We still define $h_\varepsilon > 0$ by $h^2_\varepsilon = |u(x_\varepsilon) - \phi(x_\varepsilon)|$. In both cases, we may furthermore assume that $\phi \geq \varphi$ on $\partial\Omega$.

We recall that we want to show that

$$-\frac{1}{2}\mathrm{Tr}[a(x_0)D^2\phi(x_0)] + \sup_{\alpha\in\mathcal{A}} H_\alpha(x_0, u^*(x_0), D\phi(x_0)) \leq 0 . \tag{57}$$

As in the supersolution case, the main tool of the proof is the Dynamic Programing Principle. Using it together with Itô's Lemma and adapting the above arguments, we get an inequality similar to (55), namely

$$\sup_{(\alpha_s)_s} \mathbb{E}_{x_\varepsilon}\left[\int_0^{h_\varepsilon\wedge\tau_\varepsilon}\Big(-\frac{1}{2}\mathrm{Tr}[a(x_\varepsilon)D^2\phi(x_\varepsilon)] + H_{\alpha_s}(x_\varepsilon, \phi(x_\varepsilon), D\phi(x_\varepsilon))\Big)e^{-\lambda s}ds\right] \leq o(h_\varepsilon).$$

In contrast with the supersolution case, this inequality is sufficent to derive the result, either if $x_0 \in \Omega$ or if $x_0 \in \partial\Omega$. Indeed, taking constant control processes $\alpha_s \equiv \alpha \in \mathcal{A}$ in the above inequality, we are lead to

$$\Big(-\frac{1}{2}\mathrm{Tr}[a(x_\varepsilon)D^2\phi(x_\varepsilon)] + H_\alpha(x_\varepsilon, \phi(x_\varepsilon), D\phi(x_\varepsilon))\Big)\mathbb{E}_{x_\varepsilon}\left[\frac{1 - e^{-\lambda h_\varepsilon\wedge\tau_\varepsilon}}{\lambda}\right] \leq o_\varepsilon(1) ,$$

for any $\alpha \in \mathcal{A}$.

Again to conclude, it is enough to show

$$\frac{1}{h_\varepsilon}\mathbb{E}_{x_\varepsilon}\left[\frac{1 - e^{-\lambda h_\varepsilon\wedge\tau_\varepsilon}}{\lambda}\right] \to 1 ,$$

as $\varepsilon \to 0$ or equivalently that

$$\mathbb{P}[\tau_\varepsilon \leq h_\varepsilon] \to 0 \ .$$

The proof of these claims, for $x_0 \in \Omega$ or for $x_0 \in \partial\Omega$, are obtained by applying readily the same arguments as in case 1.

Dividing the above inequality by h_ε and letting ε go to zero yields

$$-\frac{1}{2}\mathrm{Tr}[a(x_0)D^2\phi(x_0)] + H_\alpha(x_0, \phi(x_0), D\phi(x_0)) \leq 0 \ ,$$

for any $\alpha \in \mathcal{A}$. And the proof is complete. □

We now turn to an alternate proof making use of adequate approximate problems.

4.2 A proof by approximation arguments.

In the sequel, for any function g defined on $\mathbb{R}^n$, g^* (resp. g_*) will stand for the usc (resp. lsc) envelope of g on $\mathbb{R}^n$. We will still denote by σ, b, f and φ suitable extensions of these functions to $\mathbb{R}^n$ such that **(H9)** still holds with $\overline{\Omega}$ being replaced by $\mathbb{R}^n$ for σ, b, f and φ being bounded and uniformly continuous in $\mathbb{R}^n$. We introduce the functions

$$\overline{\psi}(x) \quad = \quad \begin{cases} C & \text{if } x \in \Omega \ , \\ \varphi(x) & \text{otherwise} \ , \end{cases}$$

and

$$\underline{\psi}(x) \quad = \quad \begin{cases} -C & \text{if } x \in \Omega \ , \\ \varphi(x) & \text{otherwise} \ , \end{cases}$$

where C denotes here and below a large constant to be chosen later. Since Ω is smooth, it is easy to check that

$$\begin{array}{lll} (\underline{\psi}_*)^* = \underline{\psi}^* \ , & (\underline{\psi}^*)_* = \underline{\psi}_* & \text{in } \mathbb{R}^n \ , \\ (\overline{\psi}_*)^* = \overline{\psi}^* \ , & (\overline{\psi}^*)_* = \overline{\psi}_* & \text{in } \mathbb{R}^n \ . \end{array}$$

The key-point is the following result

Theorem 4.5 : *A function w, bounded on $\overline{\Omega}$, is a subsolution (resp. a supersolution) of* (48) *if and only if its extension $\tilde{w}$ to $\mathbb{R}^n$ defined by $\tilde{w}(x) = \varphi(x)$ if $x \notin \overline{\Omega}$ is a subsolution (resp. supersolution) of the variational inequality*

$$\max\Big(w - \overline{\psi}, \min(-\frac{1}{2}\mathrm{Tr}\left[a(x)D^2w\right] + H(x, w, Dw), w - \underline{\psi})\Big) = 0 \quad \text{in } \mathbb{R}^n \ , \quad (58)$$

for $C > \max(\|w\|_\infty, \|\varphi\|_\infty)$. □

The proof of this result is simple and left to the reader. We now introduce the following sequences of functions in $BUC(\mathbb{R}^n)$[3] $(\underline{\phi}_\varepsilon)_\varepsilon$, $(\overline{\phi}_\varepsilon)_\varepsilon$, $(\underline{\chi}_\varepsilon)_\varepsilon$ and $(\overline{\chi}_\varepsilon)_\varepsilon$. We assume $(\underline{\phi}_\varepsilon)_\varepsilon$ and $(\overline{\phi}_\varepsilon)_\varepsilon$ to be increasing sequences and to satisfy

$$\sup_\varepsilon \underline{\phi}_\varepsilon = \underline{\psi}_* \ , \quad \sup_\varepsilon \overline{\phi}_\varepsilon = \overline{\psi}_* \ ,$$

while $(\underline{\chi}_\varepsilon)_\varepsilon$ and $(\overline{\chi}_\varepsilon)_\varepsilon$ are assumed to be decreasing sequences and to satisfy

$$\inf_\varepsilon \underline{\chi}_\varepsilon = \underline{\psi}^* \ , \quad \inf_\varepsilon \overline{\chi}_\varepsilon = \overline{\psi}^* \ .$$

We consider the variational inequalities

$$\max\left(v - \overline{\phi}_\varepsilon, \min(-\frac{1}{2}\mathrm{Tr}\left[a(x)D^2v\right] + H(x,v,Dv), v - \underline{\phi}_\varepsilon)\right) = 0 \quad \text{in } \mathbb{R}^n \ , \tag{59}$$

and

$$\max\left(v - \overline{\chi}_\varepsilon, \min(-\frac{1}{2}\mathrm{Tr}\left[a(x)D^2v\right] + H(x,v,Dv), v - \underline{\chi}_\varepsilon)\right) = 0 \quad \text{in } \mathbb{R}^n \ . \tag{60}$$

Finally we set

$$\underline{u}_\varepsilon(x) = \sup_{\theta_2} \inf_{(\alpha_s)_s,\theta_1} \mathbb{E}_x\Big[\int_0^{\theta_1\wedge\theta_2} f(X_s,\alpha_s)e^{-\lambda s}ds + 1_{\{\theta_1\le\theta_2\}}\overline{\phi}_\varepsilon(X_{\theta_1})e^{-\lambda\theta_1} + 1_{\{\theta_1>\theta_2\}}\underline{\phi}_\varepsilon(X_{\theta_2})e^{-\lambda\theta_2}\Big] \ ,$$

and

$$\overline{u}_\varepsilon(x) = \sup_{\theta_2} \inf_{(\alpha_s)_s,\theta_1} \mathbb{E}_x\Big[\int_0^{\theta_1\wedge\theta_2} f(X_s,\alpha_s)e^{-\lambda s}ds + 1_{\{\theta_1\le\theta_2\}}\overline{\chi}_\varepsilon(X_{\theta_1})e^{-\lambda\theta_1} + 1_{\{\theta_1>\theta_2\}}\underline{\chi}_\varepsilon(X_{\theta_2})e^{-\lambda\theta_2}\Big] \ ,$$

for $x \in \mathbb{R}^n$, the supremum and infimum being taken over all stopping-times θ_1 and θ_2 measurable with respect to the filtration associated with $(W_t)_t$. We have the

Proposition 4.1 : *$(\underline{u}_\varepsilon)_\varepsilon$ and $(\overline{u}_\varepsilon)_\varepsilon$ are respectively the unique solutions to the variational inequalities* (59) *and* (60). *Moreover, if C is large enough, one has*

$$\underline{u}_\varepsilon \le u \le \overline{u}_\varepsilon \quad on \ \overline{\Omega} \ ,$$

for every ε, where u is the value function of the exit time control problem. □

We postpone the proof of the Proposition 4.1 and we first detail the remainder of the proof of Theorem 4.1. The essential tool is the stability result

[3] $BUC(\mathbb{R}^n)$ is the space of bounded uniformly continuous functions in $\mathbb{R}^n$

for discontinuous viscosity solutions in a general form that we are going to give now. To this end, we introduce the following notations: for a sequence of locally uniformly bounded functions $(w_\varepsilon)_\varepsilon$ on a subset K of some space $\mathbb{R}^p$, we set

$$\limsup{}^* \; w_\varepsilon(x) = \limsup_{\substack{y\to x\\ \varepsilon\to 0}} w_\varepsilon(y) \; ,$$

and

$$\liminf{}_* \; w_\varepsilon(x) = \liminf_{\substack{y\to x\\ \varepsilon\to 0}} w_\varepsilon(y) \; .$$

The result is the following

Theorem 4.6 : *Let $(u^\varepsilon)_\varepsilon$ be a sequence of uniformly locally bounded viscosity solutions of the equations*

$$G^\varepsilon(x,u,Du,D^2u) = 0 \quad \text{in } \mathbb{R}^n \; ,$$

where $(G^\varepsilon)_\varepsilon$ is a sequence of uniformly locally bounded functions defined in $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n$ and satisfying

$$G^\varepsilon(x,u,p,M) \le G^\varepsilon(x,u,p,N) \quad \text{if } M \ge N \; ,$$

for any $x \in \overline{\Omega}$, $u \in \mathbb{R}$, $p \in \mathbb{R}^n$, $M, N \in \mathcal{S}^n$. Then $\overline{u} = \limsup^* \; u^\varepsilon$ (resp. $\underline{u} = \liminf_* \; u^\varepsilon$) is a viscosity subsolution (resp. supersolution) of $\underline{G} = 0$ (resp. $\overline{G} = 0$), where $\underline{G} = \liminf_* \; G^\varepsilon$ (resp. $\overline{G} = \limsup^* \; G^\varepsilon$). □

We just need to apply this result in our setting. It can be easily checked that

$$\liminf{}_* \; \overline{\phi}_\varepsilon = \sup_\varepsilon \overline{\phi}_\varepsilon = \overline{\psi}_* \; , \quad \liminf{}_* \; \underline{\phi}_\varepsilon = \sup_\varepsilon \underline{\phi}_\varepsilon = \underline{\psi}_* \; ,$$

and that the same properties hold if we replace respectively $\overline{\phi}_\varepsilon$ and $\underline{\phi}_\varepsilon$ by $\overline{\chi}_\varepsilon$ and $\underline{\chi}_\varepsilon$. Denoting respectively by $\mathcal{H}_\varepsilon^-$ and by $\mathcal{H}_\varepsilon^+$ the Hamiltonians of the equations (59) and (60), it turns out that

$$\limsup{}^* \; \mathcal{H}_\varepsilon^- = \mathcal{H}^* \; , \quad \liminf{}_* \; \mathcal{H}_\varepsilon^+ = \mathcal{H}_* \; ,$$

where $\mathcal{H}$ is the Hamiltonian of equation (58). Applying Theorem 4.6, we deduce that $\underline{u} = \liminf_* \; \underline{u}_\varepsilon$ and $\overline{u} = \limsup^* \; \overline{u}_\varepsilon$ are respectively super- and subsolutions of the variational inequality (58). Moreover, it can easily be seen that $\underline{u}_\varepsilon \equiv \overline{u}_\varepsilon \equiv \varphi$ in $\mathbb{R}^n - \overline{\Omega}$; therefore $\overline{u} = \underline{u} = \varphi$ in $\mathbb{R}^n - \overline{\Omega}$ and thanks to Theorem 4.5, the restrictions of $\underline{u}$ and $\overline{u}$ to $\overline{\Omega}$ are respectively super- and subsolutions of problem (48).

Applying the Strong Comparison Result of Theorem 2.1 on $\overline{\Omega}$, we immediately obtain

$$\overline{u}(x) \le \underline{u}(x) \quad \text{in } \Omega \; .$$

Meanwhile, Proposition 4.1 provides us with $\underline{u} \le u \le \overline{u}$ in Ω. We now conclude $\underline{u} = u = \overline{u}$ in Ω, i.e. the value function u is continuous in Ω and, again by

Theorem 2.1, its continuous extension to $\overline{\Omega}$ is the unique solution to problem (48). □

Proof of Proposition 4.1: $\overline{u}_\varepsilon$ and $\underline{u}_\varepsilon$ are continuous by standard arguments since the problem is set in $I\!R^n$. The use of the Dynamic Programming Principle shows that they are solutions to the variational inequalities (60) and (59). As far as uniqueness is concerned, standard results in $I\!R^n$ apply. We notice further that, if we switch the *sup* and the *inf* in the formulae defining $\overline{u}_\varepsilon$ and $\underline{u}_\varepsilon$, we still have solutions to these variational inequalities. As a consequence, uniqueness ensures that the corresponding functions are equal. We now justify the fact that $\underline{u}_\varepsilon(x) \le u(x)$. Switching the *sup* and the *inf*, and taking $\theta_1 = \tau$, we obtain, for $x \in \overline{\Omega}$

$$\underline{u}_\varepsilon(x) \le \inf_{(\alpha_s)_s} \sup_{\theta_2} I\!E_x\Big[\int_0^{\tau\wedge\theta_2} f(X_s,\alpha_s)e^{-\lambda s}ds + 1_{\{\tau\le\theta_2\}}\overline{\phi}_\varepsilon(X_\tau)e^{-\lambda\tau} + \\ 1_{\{\tau>\theta_2\}}\underline{\phi}_\varepsilon(X_{\theta_2})e^{-\lambda\theta_2}\Big]\,.$$

Now choose $C > 2\left(\|\varphi\|_\infty + \frac{1}{\lambda}\|f\|_\infty\right)$. If $\theta_2 < \tau$, since $\underline{\phi}_\varepsilon \le \underline{\psi}_*$ and since $\underline{\psi}_* \equiv -C$ in Ω, one has

$$-\int_{\theta_2}^{\tau} f(X_s,\alpha_s)e^{-\lambda s}ds + \underline{\phi}_\varepsilon(X_{\theta_2})e^{-\lambda\theta_2} < C\left(e^{-\lambda\theta_2} - e^{-\lambda\tau}\right) + \underline{\phi}_\varepsilon(X_{\theta_2})e^{-\lambda\theta_2} \\ \le C\left(e^{-\lambda\theta_2} - e^{-\lambda\tau}\right) - Ce^{-\lambda\theta_2} = -Ce^{-\lambda\tau} < \varphi(X_\tau)e^{-\lambda\tau} = \overline{\phi}_\varepsilon(X_\tau)e^{-\lambda\tau}\,,$$

and so

$$1_{\{\tau>\theta_2\}}\left\{\int_0^{\theta_2} f(X_s,\alpha_s)e^{-\lambda s}ds + \underline{\phi}_\varepsilon(X_{\theta_2})e^{-\lambda\theta_2}\right\} \\ < 1_{\{\tau\le\theta_2\}}\left\{\int_0^{\tau} f(X_s,\alpha_s)e^{-\lambda s}ds + \overline{\phi}_\varepsilon(X_\tau)e^{-\lambda\tau}\right\}\,.$$

Therefore

$$\underline{u}_\varepsilon(x) \le \inf_{(\alpha_s)_s} I\!E_x\Big[\int_0^{\tau} f(X_s,\alpha_s)e^{-\lambda s}ds + \varphi(X_\tau)e^{-\lambda\tau}\Big] = u(x)\,.$$

The inequality for $\overline{u}_\varepsilon$ can be obtained in the same way, so we leave the details to the reader. □

4.3 Proof of Theorem 4.2.

We first consider the stochastic exit time control problem whose dynamics is given by (8) and the value function v is given by

$$v(x) = \inf_{(\alpha_s)_s} I\!E_x\left[\int_0^{\tau} f(X_t,\alpha_t)\exp(-\lambda t)dt + C\exp(-\lambda\tau)\right]\,, \tag{61}$$

where $C > 0$ is a large positive constant chosen later. We have therefore taken $\varphi \equiv C$ as exit cost in (9).

By the arguments of the proof of Theorem 4.1, v^* and v_* are respectively sub- and supersolutions of

$$\begin{cases} -\frac{1}{2}\mathrm{Tr}\left[a(x)D^2v\right] + H(x, v, Dv) & = & 0 \text{ in } \Omega\,, \\ v & = & C \text{ in } \partial\Omega. \end{cases} \tag{62}$$

We are going to prove a slightly more general result here than Theorem 4.2, namely that, for C large enough, v is a solution of (50) and that $v = u$ in Ω. The key result is the

Lemma 4.2 : *For any $x \in \Omega$, there exists a control $(\alpha_s)_s$ such that the trajectory given by (8) starting from x satisfies $X_t \in \Omega$ for all $t > 0$ a.s.* □

We postpone the proof of Lemma 4.2 to the Appendix and we first conclude the proof of Theorem 4.2.

The first consequence of Lemma 4.2 is that u is well-defined in Ω; moreover, from the very definition of u, it is clear that $u \le M$ in Ω.

Since the last statement of Theorem 4.2 is also a straightforward consequence of Corollary 2.1, the only remaining point is to prove that $v = u$ in Ω if $C > M$.

From the definitions of u and v, we already know that $v \le u$ in Ω; in particular $v \le M$ in Ω.

To prove the converse inequality, we use for u the Dynamic Programming argument of the proof of Theorem 4.1: it implies that u^* is a subsolution of (50). Moreover, since $v \le M$ in Ω, $v_* \le M < C$ on $\partial\Omega$. Therefore, v being a supersolution of (62), v is also a supersolution of (50). Then the inequality we want is just a consequence of Corollary 2.1. And the proof is complete. □

Appendix: Proof of Lemma 4.2.

We first want to point out that the key idea of this proof was suggested to us by N. El Karoui. Of course, any possible error in the proof is ours.

We recall that we want to prove that, for any $x \in \Omega$, one can build a control process $(\alpha_t)_t$ such that the associated solution of (8) satisfies $X_t \in \overline{\Omega}$ for all $t \ge 0$ a.s. Unfortunately, even if the result is rather intuitive, our construction turns out to be a bit technical.

We denote below by d a C^2 function on $\overline{\Omega}$ which agrees with the distance to the boundary in a neighborhood of $\partial\Omega$ and which satisfies $0 < d(x) \le \frac{1}{2}$ in Ω. From assumption **(H14)**, we know that, for any $x \in \partial\Omega$, $\sigma^T(x)n(x) = 0$ and there is some $\alpha \in \mathcal{A}$ s.t

$$\frac{1}{2}\text{Tr}\left[a(x)D^2d(x)\right] - b(x,\alpha).n(x) > 0\ . \tag{63}$$

Using the compactness of $\overline{\Omega}$ together with the continuity of a, b, n and D^2d, one can find $\varepsilon > 0$ such that (63) still holds for $x \in \overline{\Omega}_{2\varepsilon} = \{y \in \overline{\Omega}\,;\ d(y, \partial\Omega) \leq 2\varepsilon\}$ and for some control $\alpha \in \mathcal{A}$. Then one can also find an integer N, some points $(x_i)_{i=1,..,N}$ on $\overline{\Omega}_{2\varepsilon}$ and some controls $(\alpha_i)_{i=1,..,N}$ in $\mathcal{A}$ such that

$$\overline{\Omega}_{2\varepsilon} \subset \bigcup_{i=1}^{N} B(x_i, \varepsilon)\ ,$$

and, for all $i = 1,..,N$ and $x \in \overline{B}(x_i, 2\varepsilon) \cap \overline{\Omega}$

$$\frac{1}{2}\text{Tr}\left[a(x)D^2d(x)\right] - b(x,\alpha_i).n(x) \geq 0\ , \tag{64}$$

where $\overline{B}$ denotes the closed ball. In the sequel, we shall denote $B_i = B(x_i, \varepsilon)$ and $D_i = B(x_i, 2\varepsilon)$ for $i = 1,..,N$ and we set

$$\overline{B}_0 = \overline{\Omega} \backslash \bigcup_{i=1}^{N} B_i\ .$$

We pick some $\alpha_0 \in \mathcal{A}$. We set for $x \in \overline{\Omega}$

$$\alpha(x) := \sum_{i=0}^{N} \alpha_i 1_{\overline{B}_i}(x) \prod_{j=0}^{i-1} (1 - 1_{\overline{B}_j}(x))\ .$$

This is nothing but associating with each $x \in \overline{\Omega}$ the control α_i where i is the least integer such that $x \in \overline{B}_i$.

The key-point is the following Lemma

Lemma 4.3 : *Let $Z \in \Omega$ be a random variable and let α denote $\alpha(Z)$ defined as above. If $(X_t)_t$ is the solution of (8) associated with α and with the initial data $X_0 = Z$ a.s., we define a stopping-time θ by*

$$\theta = \inf\ \{0 \leq t \leq 1;\ |X_t - Z| \geq \varepsilon\}\ .$$

Then $\tau > \theta$ a.s., where τ is the first exit time of Ω for the process $(X_t)_t$. □

Proof : We apply Itô's Lemma to the function $\phi(x) = -\ln\left[d(x) + \eta\right]$ for some $0 < \eta < \frac{1}{2}$

$$-\ln\left[d(X_t) + \eta\right] = -\ln\left[d(Z) + \eta\right] + $$
$$\int_0^t \left[\frac{-1}{[d(X_s) + \eta]}\left(\frac{1}{2}\text{Tr}\left[a(X_s)D^2d(X_s)\right] - b(X_s,\alpha).n(X_s)\right) + \right.$$
$$\left.\frac{1}{2}\frac{|\sigma^T(X_s)n(X_s)|^2}{[d(X_s) + \eta]^2}\right] ds - \int_0^t \frac{1}{[d(X_s) + \eta]}\sigma^T(X_s)n(X_s).dW_s\ .$$

Now we claim that, for $s < \theta$, the quantity Q_s given by

$$\left[\frac{-1}{[d(X_s)+\eta]}\left(\frac{1}{2}\mathrm{Tr}\left[a(X_s)D^2d(X_s)\right] - b(X_s,\alpha).n(X_s)\right) + \frac{1}{2}\frac{|\sigma^T(X_s)n(X_s)|^2}{[d(X_s)+\eta]^2}\right],$$

is bounded from above independently of $\eta > 0$ a.s. Indeed, for $s \leq \theta$, one has

$$|X_s - Z| \leq \varepsilon \quad \text{a.s.}$$

So, if ω stands for the probability variable, for almost all ω, either $Z(\omega) \in B_0$ (therefore $d(Z(\omega)) \geq 2\varepsilon$) and then $d(X_s(\omega)) \geq \varepsilon$ for $s \leq \theta$. In this case, $Q_s(\omega) \leq \frac{C}{\varepsilon}$ (recall that, here, ε is fixed). Or $Z(\omega) \in B_i$ for some $i = 1,..,N$ but, in this case, $X_s \in D_i$ for $s \leq \theta$ and (64) holds, i.e.

$$\frac{1}{2}\mathrm{Tr}\left[a(X_s)D^2d(X_s)\right] - b(X_s,\alpha).n(X_s) \geq 0\,. \tag{65}$$

Moreover, since σ^T and n are Lipschitz continuous, using assumption **(H14)**, we have $|\sigma^T(x)n(x)| \leq Cd(x)$ for all $x \in \overline{\Omega}$, so that

$$\frac{|\sigma^T(X_s)n(X_s)|^2}{[d(X_s)+\eta]^2} \leq C\,.$$

And our claim is proved.

Using again that $|\sigma^T(x)n(x)| \leq Cd(x)$ for all $x \in \overline{\Omega}$, we know

$$M_t = \int_0^t \frac{1}{[d(X_s)+\eta]}\sigma^T(X_s)n(X_s).dW_s$$

is a square integrable martingale uniformly w.r.t η. Write now

$$-\ln[d(X_t)+\eta] + \ln[d(Z)+\eta] = A_t + M_t\,.$$

From what we just showed, we have $A_t \leq Ct$ a.s., for some non-negative constant C and M_t is a square integrable martingale; therefore, since we have by definition $\theta \leq 1$ a.s.

$$I\!E\left[\left((-\ln[d(X_{\tau\wedge\theta})+\eta] + \ln[d(Z)+\eta])^+\right)^2\right] \leq \tilde{C}\,,$$

for some constant $\tilde{C}$ independent of η. Using that $d(Z) \leq d(Z)+\eta$ a.s. together with the Monotone Convergence Theorem, we get

$$I\!E\left[(-\ln d(X_{\tau\wedge\theta}) + \ln(d(Z)))^+\right]^2 \leq \tilde{C}\,.$$

Since we chose d such that $d \leq \frac{1}{2}$ on $\overline{\Omega}$, $-\ln(d) \geq 0$ on $\overline{\Omega}$ and we conclude that $-\ln d(X_{\tau\wedge\theta}) < \infty$ almost surely, since $Z \in \Omega$ a.s. Therefore, $\tau > \theta$ a.s. and the Lemma is proved. □

Now we consider $x \in \Omega$. By induction, we build a sequence of stopping times $(\theta^k)_k$, a sequence of processes $(X_t^k)_k$ defined for $\theta^k \le t \le \theta^{k+1}$ and a sequence of some random variables $(\alpha^k)_k$ in the following way: we first set $\alpha^0 = \alpha(x)$, $\theta^0 \equiv 0$ and $(X_t^0)_t$ is the solution of

$$dX_t^0 = b(X_t^0, \alpha^0)dt + \sigma(X_t^0)dW_t \ ,$$

with initial data $X_0^0 = x$.

For $k \ge 1$, we are given α^{k-1}, θ^{k-1} and the process $(X_t^{k-1})_t$ defined for $t \ge \theta^{k-1}$ solution of the SDE

$$dX_t^{k-1} = b(X_t^{k-1}, \alpha^{k-1})dt + \sigma(X_t^{k-1})dW_t \ ,$$

with initial data $X_{\theta^{k-1}}^{k-1} = X_{\theta^{k-1}}^{k-2}$. We define θ^k and α^k by

$$\theta^k = \inf \{\theta^{k-1} \le t \le \theta^{k-1} + 1; \ |X_t^{k-1} - X_{\theta^{k-1}}^{k-1}| \ge \varepsilon\} \ .$$

and

$$\alpha^k = \alpha(X_{\theta^{k-1}}^{k-1}) \ .$$

Then $(X_t^k)_t$ is the solution, for $t \ge \theta^k$, of the SDE

$$dX_t^k = b(X_t^k, \alpha^k)dt + \sigma(X_t^k)dW_t \ ,$$

with initial data $X_{\theta^k}^k = X_{\theta^k}^{k-1}$.

Arguing by induction and applying successively Lemma 4.3 to $Z = X_{\theta^k}^k$ together with the strong Markov property, one proves easily that $X_t^k \in \Omega$ for all t such that $\theta^k \le t \le \theta^{k+1}$ and for any $k \in \mathbb{N}$ since it implies $\theta^{k+1} < \tau^k$ a.s where τ^k is the first exit time of Ω of X_t^k.

We now define θ^∞ by $\theta^\infty := \lim_{k\to\infty} \theta^k$ and we define a control process $(\tilde\alpha_t)_t$ for $t \le \theta^\infty$ by $\tilde\alpha_t = \alpha^k$ on $]\theta^{k-1}, \theta^k]$. If $\tilde X$ is the solution of (8) associated with $\tilde\alpha$, one has $\tilde X_t = X_t^k$ for $t \in]\theta^{k-1}, \theta^k]$ a.s. Moreover, thanks to the above remarks, if $\tilde\tau$ is the first exit time of $\tilde X$ from Ω, we have $\tilde\tau \ge \theta^\infty$ a.s. Hence it is enough to prove that $\theta^\infty = \infty$ a.s. to complete the proof of Lemma 4.2.

To this aim, notice that, thanks to the definition of these stopping times θ^k, one has for all k either $\theta^k = \theta^{k-1} + 1$ or

$$|\tilde X_{\theta^k} - \tilde X_{\theta^{k-1}}| \ge \varepsilon \quad \text{a.s.}$$

As a consequence, using standard estimates on the process $\tilde X$, we obtain for all $k \in \mathbb{N}$ and $0 < h < 1$

$$\mathbb{P}[\theta^k - \theta^{k-1} \le h] \le Ch^2 \ ,$$

for some non-negative constant C depending only on b, σ and ε. For all $k \ge 1$, we set $F_k = \{\theta^k - \theta^{k-1} \le \frac{1}{k}\}$. Since $\mathbb{P}[F_k] \le \frac{C}{k^2}$, we may apply the Borel-

Cantelli Lemma, and get that for almost all ω, ω stands in at most a finite number of F_k. This yields $\theta^k(\omega) - \theta^{k-1}(\omega) > \frac{1}{k}$ for all k large enough. This implies that the sum $\sum_{k\in\mathbb{N}} (\theta^k - \theta^{k-1})$ is divergent a.s. and therefore $\theta^\infty = +\infty$ a.s. and the proof is complete. □

Remark : *It can easily be seen from the above proof that an analogous result holds if, for any $x \in \partial\Omega$, $\sigma^T(x)n(x) = 0$ and there is some $\alpha \in \mathcal{A}$ s.t.*

$$\frac{1}{2}\mathrm{Tr}\left[a(x)D^2d(x)\right] - b(x,\alpha).n(x) \geq 0 \ ,$$

but assuming, this time, d to be of class $W^{3,\infty}$ in a neighborhood of $\partial\Omega$ (i.e. D^2d is now Lipschitz continuous). Indeed, in this case we have only

$$\frac{1}{2}\mathrm{Tr}\left[a(x)D^2d(x)\right] - b(x,\alpha).n(x) \geq Cd(x) \ ,$$

for x in a neighborhood of $\partial\Omega$ and for some $\alpha \in \mathcal{A}$ but this property is enough to prove Lemma 4.3 which is the key point in the proof of Lemma 4.2.

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