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On the Hamilton–Jacobi–Bellman Equations

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Abstract. We consider general problems of optimal stochastic control and the associated Hamilton–Jacobi–Bellman equations. We recall first the usual derivation of the Hamilton–Jacobi–Bellman equations from the Dynamic Programming Principle. We then show and explain various results, including (i) continuity results for the optimal cost function, (ii) characterizations of the optimal cost function as the maximum subsolution, (iii) regularity results, and (iv) uniqueness results. We also develop the recent notion of viscosity solutions of Hamilton–Jacobi–Bellman equations.

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0. Introduction

In many areas (engineering, economy, management theory, etc.) can be found the following problem: one wants to control, in an optimal way, systems, the state of which is governed by the solution of the following stochastic differential equation:

$$\frac{\mathrm{d}X_t}{\mathrm{d}t} = \sigma(X_t, \alpha_t)\zeta_t + b(X_t, \alpha_t) \quad \text{for } t \geqslant 0,\ X_0 = x \in \mathbb{R}^N, \tag{0}$$

where ζ_t is the famous 'white noise'; σ and b are respectively matrix-valued and vector-valued functions defined on $\mathbb{R}^N \times A$; A is a given separable metric space; and α_t, called the *control process*, is a stochastic process taking its values in A. To be rigorous, Equation (0) has to be written in the following way:

$$X_t = x + \int_0^t \sigma(X_s, \alpha_s)\cdot \mathrm{d}B_s + \int_0^t b(X_s, \alpha_s)\mathrm{d}s \quad \text{for } t \geqslant 0; \tag{1}$$

where B_t is a m-dimensional Brownian motion that is a continuous process with independent increments such that $E(B_t) = 0, E(B_t^2) = t$; and where the quantity $\int_0^t \sigma(X_s, \alpha_s)\mathrm{d}B_s$ is a stochastic integral. Later on we will make smoothness assumptions on σ, b which ensures the existence and uniqueness of X_t for each given control process α_t.

One then defines a *cost function* $J(x, \alpha_t)$ by:

$$J(x, \alpha_t) = E\int_0^\infty f(X_t, \alpha_t)\exp\left(-\int_0^t c(X_s, \alpha_s)\mathrm{d}s\right) \tag{2}$$

where E denotes the expectation; $f(x, \alpha)$, $c(x, \alpha)$ are given real valued functions on $\mathbb{R}^N \times A$; the function c is often called the *discount factor*. Of course, in order to be sure that (2) is meaningful, we will often require $c(x, \alpha) \geqslant \lambda > 0$. In the next sections, we will see many variants of cost functions: the above choice corresponds to the so-called *infinite–horizon problem.*

Then the purpose of optimal stochastic control theory is to determine the *optimal cost function* (also called the *value function*, or the *criterion*) u given by:

$$u(x) = \inf\{J(x, \alpha_t)/\alpha_t \text{ stochastic process with values in } A\} \tag{3}$$

(We will be more specific about the measurability assumptions on α_t below) and to determine *optimal controls* α_t, i.e., controls α_t realizing the infimum. An important class of optimal controls are the so-called *feedback controls* or *Markonian controls* α_t which are of the form $\alpha_t = \alpha(X_t)$, that is, controls whose values at time t are determined only by the knowledge of the state of the system at time t.

A fundamental tool to deal with u is given by the *Dynamic Programming Principle* introduced by Bellman [3]. As we will see in Section 2 below, this principle indicates that u should, in some way, be the solution of the following partial differential equation:

$$\sup_{\alpha\in A}\{A_\alpha u(x) - f_\alpha(x)\} = 0 \text{ in } \mathbb{R}^N \tag{4}$$

where $f_\alpha(\cdot) = f(\cdot, \alpha)$, $A_\alpha = -a_{ij}(x, \alpha)\partial_{ij} - b_i(x, \alpha)\partial_i + c(x, \alpha)$ and $a(x, \alpha) = \frac{1}{2}\sigma(x, \alpha) \times \sigma^T(x, \alpha)$. (Here and everywhere below we will use the implicit summation convention on repeated indices.) Equation (4) is called the *Hamilton–Jacobi–Bellman equation* associated with the control problem (3) (HJB in short). For readers unfamiliar with stochastic integration, let us just indicate that the derivation of (4) is very much akin to the classical theory of Hamilton–Jacobi equations for deterministic problems or calculus of variation problems. Let us also mention that often, in engineering literature equation (4) is called the Bellman equation.

In Section 2, we indicate how (4) is obtained by a heuristic argument, and we then show a rigorous classical treatment valid only if u is of class C^2. But, of course, in general, one does not know beforehand that $u\in C^2(\mathbb{R}^N)$ and actually, this is false! On the other hand, in Section 2 we will see that by an easy verification argument, if one knows a solution $\tilde{u}\in C_b^2(\mathbb{R}^N)$ of (4), then of necessity, $u \equiv \tilde{u}$ in $\mathbb{R}^N$.

Unfortunately, solving (4) is not an easy task, since (4) is a *fully nonlinear second-order elliptic equation* which is possibly degenerate (if $\sigma \equiv 0$, for example) and, in general, such a $\tilde{u}$ does not exist!

We refer the reader interested in those 'classical' statements to Fleming and Rishel [16], Krylov [21], and Bensoussan and Lions [4]. Let us finally indicate (cf. also Section 2) that if u solves (4) and u is smooth (say C^2), then using (4) one has a formal rule to build optimal feedbacks.

In this paper we will present several ways of deriving (4) (in some sense) and of characterizing u:

(i) We will show in Section 3 that u is the maximum element of the set of subsolutions v satisfying: $A_\alpha v \leqslant f_\alpha$ in $\mathbb{R}^N$ for all $\alpha \in A$.

(ii) We will recall in Section 5 the notion of *viscosity solutions* of (4) together with its main properties: (in particular, we will show that u is always *the viscosity solution of* (4)).

(iii) In Section 6, we present various regularity results obtained under 'optimal' conditions which, combined with the notion of viscosity solution, immediately yield that (4) holds in a standard sense.

(iv) Finally, Section 7 is devoted to uniqueness results for the solutions of (4). Let us also mention that since we will need in Section 5 that u is continuous, we will show in Section 4 very general results ensuring the continuity of u. Section 8 is devoted to a list of questions which can be treated by methods or approaches similar to those presented here.

Let us finally warn the reader that we will deal only with complete observation problems and that we will not develop the question of the existence of optimal controls. Let us also mention that most of the results presented here have been obtained by the author.

1. Notations and Assumptions

Most of the time we will make the following assumptions:

$$\sup_{\alpha \in A} \| \varphi(\cdot, \alpha) \|_{W^{2,\infty}(\mathbb{R}^N)} < \infty, \quad \varphi(x, \cdot) \in C(A) \ \text{ for all } x \in \mathbb{R}^N \tag{5}$$

for $\varphi = \sigma_{ij}$ $(1 \leqslant i \leqslant N,\ 1 \leqslant j \leqslant m)$, $= b_i$ $(1 \leqslant i \leqslant N)$, $= c, f$;

$$\inf\{c(x, \alpha) \mid x \in \mathbb{R}^N, \alpha \in A\} = \lambda > 0. \tag{6}$$

With those assumptions we may now rigorously define the optimal control problem considered, and for technical reasons, it will be convenient to consider, instead of a given probability space and a given Brownian motion, *admissible systems* defined as follows: an admissible system $\mathscr{A}$ is the collection of (i) a probability space (Ω, F, F^t, P) with an increasing right-continuous filtration of complete subalgebras of F, (ii) an adapted m-dimensional Brownian motion B_t, (iii) a progressively measurable process α_t taking its values in a compact set of A (which may depend on α_t, of course). Because of (5), it is well known that there exists a unique continuous process solution of (1). Of course, this process X_t depends on x and we will sometimes write X_t^x or $X(x, t)$.

We now define more general cost functions than (2):

$$J(x, \mathscr{A}) = E \int_0^\tau f(X_t, \alpha_t) \exp\left(- \int_0^t c(X_s, \alpha_s)\, ds \right) + \psi(X_\tau) \exp\left(- \int_0^\tau c(X_t, \alpha_t)\, dt \right) \tag{7}$$

where $\tau = \inf(t \geqslant 0, X_t \notin \bar{\mathcal{O}})$ $(= +\infty$ if $X_t \in \bar{\mathcal{O}}$ for all $t \geqslant 0)$ and ψ is a given continuous function on $\Gamma = \partial\mathcal{O}$. Finally, $\mathcal{O}$ will always be a smooth open set in $\mathbb{R}^N$ (possibly $\mathbb{R}^N$ itself). (We will also consider similar cost functions where τ is replaced by

$\tau' = \inf(t \geqslant 0, X_t \in \mathcal{O})$.) The optimal cost function is then defined by:

$$u(x) = \inf_{\mathscr{A}} J(x, \mathscr{A}) \tag{8}$$

where the infimum is taken over all admissible systems.

We will also consider *time-dependent problems*: let $T > 0$ be fixed – T is the so-called *horizon* – for each admissible system and for all $(x, t) \in Q = \mathcal{O} \times (0, T)$, X_μ is the solution of

$$X_s = x + \int_0^s \sigma(X_\mu, t + \mu, \alpha_\mu) \mathrm{d}B_\mu + \int_0^s b(X_\mu, t + \mu, \alpha_\mu) \mathrm{d}\mu, \tag{9}$$

where we assume

$$\sup_{\alpha \in A} \Big\| \sum_{|\beta| \leqslant 2} |D_x^\beta \varphi(\cdot, \alpha)| \Big\|_{L^\infty(\mathbb{R}^N \times (0, T))} < \infty, \ \varphi(x, t, \cdot) \in C(A) \quad \text{for } (x, t) \in \mathbb{R}^N \times [0, T]$$

$$\varphi(x, \cdot, \alpha) \in C([0, T]) \quad \text{uniformly for } x \in \mathbb{R}^N, \alpha \in A;$$

for $\varphi = \sigma_{ij}, b_i, c, f$.

We then define the corresponding cost functions $J(x, t, \mathscr{A})$ and the optimal cost function $u(x, t)$:

$$J(x, t, \mathscr{A}) = E \int_0^{\tau \wedge (T - t)} f(X_s, t + s, \alpha_s) \exp\left\{ - \int_0^s c(X_\mu, t + \mu, \alpha_\mu) \mathrm{d}\mu \right\} + \tag{7'}$$

$$+ \psi(X_{\tau \wedge (T - t)}, t + \tau \wedge (T - t)) \exp\left\{ - \int_0^{\tau \wedge (T - t)} c(X_s, t + s, \alpha_s) \, \mathrm{d}s \right\}$$

$$u(x, t) = \inf_{\mathscr{A}} J(x, t, \mathscr{A}). \tag{8'}$$

Here and everywhere below ψ is a given continuous function on $\partial_0 Q = (\Gamma \times [0, T]) \cup (\bar{\mathcal{O}} \times \{T\})$. Let us mention that the associated HJB equation is then:

$$-\frac{\partial u}{\partial t} + \sup_{\alpha \in A} \{A_\alpha u(x, t) - f_\alpha(x, t)\} = 0 \quad \text{in } Q \tag{4'}$$

with $f_\alpha(\cdot) = f(\cdot, \alpha)$, $A_\alpha = -a_{ij}(x, t, \alpha)\partial_{ij} - b_i(x, t, \alpha)\partial_i + c(x, t, \alpha)$ and $a = \frac{1}{2}\sigma\sigma^T$.

REMARK 1. It is worth noting that the time-dependent problem is in fact a special case of the preceding problem. Indeed, if we consider the process $\tilde{X}_s = (X_s, \psi_s)$ in Q with $\psi_s = t + s$, then (9) may be written:

$$\tilde{X}_s = (x, t) + \left(\int_0^s \sigma(\tilde{X}_\mu, \alpha_\mu) \, \mathrm{d}B_\mu, 0 \right) + \left(\int_0^s b(\tilde{X}_\mu, \alpha_\mu) \, \mathrm{d}\mu, \int_0^s 1 \, \mathrm{d}\mu \right)$$

and $\tau \wedge (T - t)$ is the first exit time from $\bar{Q}$ of $\tilde{X}_s$. ■

2. The Classical Approach

To avoid technical difficulties, we will begin with the simpler situation where $\mathcal{O} = \mathbb{R}^N$ (and $\tau = +\infty$).

PROPOSITION 1. *Let $\mathcal{O} = \mathbb{R}^N$ and let us assume (for example)* (5), (6). *Then if u, given by* (8), *is of class C_b^2 on $\mathbb{R}^N$ then the HJB equation holds:*

$$\sup_{\alpha\in A} \{A_\alpha u(x) - f(x, \alpha)\} = 0 \quad in\ \mathbb{R}^N. \tag{4}$$

REMARK 2. Of course this is only an example and similar results hold for general $\mathcal{O}$ or for time-dependent problems. ■

REMARK 3. Let us first give an intuitive proof of (4) based upon a heuristic form of the dynamic programming principle. Let us consider a very short time ε, then in order to obtain the minimal cost function $u(x)$, we may choose any of the 'strategies α' and solve (1) during the time interval $[0, \varepsilon]$, then at time ε we may choose any of the strategies starting at time 0 at the point X_ε. This yields the following 'relation':

$$u(x) \simeq \inf_{\alpha\in A} \left\{ E \int_0^\varepsilon f(X_t, \alpha)\, dt + E\{u(X_\varepsilon)\} \right\} \tag{10}$$

(to simplify the discussion, we took the discount factor $c \equiv 0$) where X_t is the solution of (1) corresponding to $\alpha_t \equiv \alpha$. Now if ε is small, X_ε is close to x and thus

$$Eu(X_\varepsilon) \simeq u(x) + \nabla u(x)\cdot E(X_\varepsilon - x) + \tfrac{1}{2} E\{(D^2u(x)\cdot(X_\varepsilon - x), X_\varepsilon - x)\} + Eo(|X_\varepsilon - x|^2)$$

To simplify matters even more, assume that σ does not depend on x, thus

$$X_\varepsilon = x + \sigma(\alpha)\cdot B_\varepsilon + \int_0^\varepsilon b(X_s, \alpha)\, ds$$

and from the properties of the Brownian motion we deduce:

$$Eu(X_\varepsilon) \simeq u(x) + \varepsilon b(x, \alpha)\cdot \nabla u(x) + \tfrac{1}{2}\varepsilon a_{ij}(\alpha)\partial_{ij} u(x) + 0(\varepsilon).$$

If we go back to (10), dividing by ε, this yields

$$\sup_{\alpha\in A} \left\{ - a_{ij}(\alpha)\partial_{ij} u(x) - b_i(x, \alpha)\partial_i u(x) - \frac{1}{\varepsilon} E \int_0^\varepsilon f(X_t, \alpha) dt \right\} \simeq 0$$

and letting ε go to 0, we conclude. ■

The rigorous proof is based upon the following rigorous version of the dynamic programming principle. This lemma is due to Krylov [20, 21] (see also Nisio [58], Itô [18]) and is easily deduced from straightforward considerations on stochastic differential equations:

LEMMA 1. *Under assumptions* (5) *and* (6), *for all* $T > 0$, *we have:*

$$u(x) = \inf_{\mathscr{A}} \left\{ E \int_0^T f(X_t, \alpha_t) \exp\left(- \int_0^t c \right) + u(X_T) \exp\left(- \int_0^T c \right) \right\}. \tag{10'}$$

Next, we use Itô's formula and we obtain easily:

$$\sup_{\mathscr{A}} \frac{1}{T} E \int_0^T \{ A_{\alpha_t} u(X_t) - f_{\alpha_t}(X_t) \} \exp\left(- \int_0^t c \right) = 0$$

and if we let $T \to 0+$, the above equality gives the HJB equation (4). For more details, we refer the reader to [20, 21, 58, 32]. ∎

Conversely, we have the:

PROPOSITION 2. *Let* $\mathcal{O} = \mathbb{R}^N$ *and let us assume* (*for example*) (5) *and* (6). *Let* $\tilde{u} \in C_b^2(\mathbb{R}^N)$ *be a solution of* (4), *then* $\tilde{u} \equiv u$ *in* $\mathbb{R}^N$.

Proof. Let $R > 0$ be fixed. Then for all $\varepsilon > 0$, one can find $n \geqslant 1$; $\alpha_1, \ldots, \alpha_n \in A$; $\mathcal{O}_1, \ldots, \mathcal{O}_n$ open sets such that: $\bar{B}_R \subset \bigcup_{i=1}^n \mathcal{O}_i$,

$$0 \geqslant A_{\alpha_i} u(x) - f_{\alpha_i}(x) \geqslant -\varepsilon \text{ in } \{x \in \mathbb{R}^N, \operatorname{dist}(x, \bar{\mathcal{O}}_i) \leqslant \varepsilon\}.$$

Next let $m \geqslant 1$, we consider the following processes (X_t^m, α_t^m) solution of (1) defined by: if $0 \leqslant t < 1/m$, $\alpha_t^m = \alpha_i$ if $x \in \mathcal{O}_i$, if $1/m < t < 2/m$, $\alpha_t^m = \alpha_i$ if $X_{1/m}^m \in \mathcal{O}_i$; if $2/m \leqslant t < 3/m$, $\alpha_t^m = \alpha_i$ if $X_{2/m}^m \in \mathcal{O}_i$. Those processes are well-defined on $[0, m]$ and taking any constant value for α_t for $t \geqslant m$, we may consider they are defined on $\mathbb{R}_+$. In view of Itô's formula, we have:

$$\tilde{u}(x) = E \int_0^m \{ A_{\alpha_t^m} \tilde{u}(X_t^m) \} \exp\left(- \int_0^t c \right) dt + \tilde{u}(X_m^m) \exp\left(- \int_0^m c \right)$$

for $x \in \bar{B}_R$. We thus consider the event B_m:

$$B_m = \{ w / \exists t \in [0, m] A_{\alpha_t^m} \tilde{u}(X_t^m) - f_{\alpha_t^m}(X_t^m) \leqslant -\varepsilon \}.$$

If we show that $P(B_m) \to 0$ as $m \to \infty$, we deduce easily:

$$\tilde{u} \geqslant u \text{ in } \mathbb{R}^N \text{ (take } m \to \infty, \varepsilon \to 0).$$

The converse being a trivial consequence of Itô's formula and of the inequalities: $A_\alpha \tilde{u} \leqslant f_\alpha$ in $\mathbb{R}^N$ for all $\alpha \in A$; we conclude by proving that $P(B_m) \to 0$ as $m \to \infty$. In view of the choice of α_t^m, it is clearly enough to show that uniformly in $k \in \{0, 1, \ldots, m^2 - 1\}$ we have:

$$P\{ \sup_{k/m \leqslant t \leqslant (k+1)/m} |X_t^m - X_{k/m}^m| \geqslant \varepsilon \} = 0(1/m^2).$$

But we have:

$$P\{ \sup_{k/m \leqslant t \leqslant (k+1)/m} |X_t^m - X_{k/m}^m| \geqslant \varepsilon \} \leqslant \frac{1}{\varepsilon^5} E[\sup_{k/m \leqslant t \leqslant (k+1)/m} |X_t^m - X_{k/m}^m]^5] \leqslant \frac{K}{\varepsilon^5} \left(\frac{1}{m} \right)^{5/2}$$

in view of the standard estimates on stochastic integrals. ∎

REMARK 4. This argument is taken from Safonov [63].

We now define *ε-optimal Markonian controls* (or feedbacks): let $\varepsilon \geqslant 0$, a Borel function $\alpha(x)$ – with values in A – defines an *ε-optimal Markonian control* (optimal if $\varepsilon = 0$) if there exists for all $x \in \bar{\mathcal{O}}$, a solution X_t in some probability space (Ω, F, F^t, P, B_t)

$$X_t = x + \int_0^t \sigma(X_s, \alpha(X_s))\,\mathrm{d}B_s + \int_0^t b(X_s, \alpha(X_s))\,\mathrm{d}s \quad \text{for } t \geqslant 0 \tag{11}$$

such that:

$$E\int_0^\tau f(X_t, \alpha(X_t))\exp\left(-\int_0^t c\right) + \psi(X_\tau)\exp\left(-\int_0^\tau c\right) \leqslant u(x) + \varepsilon, \quad \forall x \in \bar{\mathcal{O}}.$$

Observe that if we denote by $\mathscr{A}_x$ the admissible system defined with $\alpha_t^x = \alpha(X_t)$ the left-hand side of the preceding inequality is nothing but $J(x, \mathscr{A}_x)$ and thus we have $u(x) \leqslant J(x, \mathscr{A}_x) \leqslant u(x) + \varepsilon, \forall x \in \mathcal{O}$.

Now if we know that $u \in C_b^2$ and thus solves the HJB equation:

$$\sup_{\alpha \in A} \{A_\alpha u(x) - f(x, \alpha)\} = 0 \text{ in } \mathbb{R}^N$$

(with $\mathcal{O} = \mathbb{R}^N$ for example), clearly for all $\varepsilon > 0$ (and for $\varepsilon = 0$ if A is compact) there exists a Borel function $\alpha(x)$ such that:

$$A_{\alpha(x)} u(x) \geqslant f(x, \alpha(x)) - \varepsilon \text{ in } \mathbb{R}^N.$$

Then *if we can solve the stochastic differential Equation* (11), this scheme yields ε-optimal Markonian controls (or even optimal controls). Unfortunately, since the coefficients are Borel measurable in Equation (11), it is, in general, very difficult to solve. Let us mention two cases where it can be achieved:

(i) $\alpha(x) \in C(\bar{\mathcal{O}}, A)$,

(ii) the matrices $a(x, \alpha)$ are definite positive uniformly on $\bar{\mathcal{O}} \times A$.

This is a consequence of results due to Krylov [22] (see also Stroock and Varadhan [65]) concerning the existence of diffusion processes.

To summarize, let us point out that we showed that when u is C^2, or if there exists a C^2 solution of the HJB equation, we have a characterization of the optimal cost function as the unique solution (in C_b^2) of the HJB equation (with prescribed boundary conditions if $\mathcal{O} \neq \mathbb{R}^N$) and we have a procedure to build ε-optimal feedbacks. In particular, in case (ii) above, one deduces the existence of ε-optimal feedbacks or even optimal feedbacks from the existence of a smooth solution of the HJB equation. As in the following, we will prove, in particular, such existence results, we could state easy results for the existence of ε-optimal feedbacks, but we will not do so.

As we said in the introduction, the main question with the above 'classical approach' lies with the fact that u is not, in general, C^2:

EXAMPLE. Take $\mathcal{O} = \mathbb{R}^N$, $\sigma \equiv 0$, $b \equiv 0$, $c \equiv 1$ and if smooth such that $\inf_{\alpha \in A} f(x, \alpha) \notin$

$\notin C^1(\mathbb{R}^N)$. Then in this case, it is clear that u is given by: $u(x) = \inf_{\alpha \in A} f(x, \alpha)$; and u is not C^2 (and not even C^1).

3. Maximum Subsolution

A major tool for the study of the optimal cost function is given by the following extension of Lemma 1. Let us recall that we now consider the general case of u given by (8) (for a general domain $\mathcal{O}$ in $\mathbb{R}^N$).

LEMMA 2. *We assume* (5) *and* (6) *and for each admissible system* $\mathscr{A}$ *we take a stopping time* $\theta(\theta = \theta(\mathscr{A}))$. *Then we have:*

(i) $M_t = u(X_{t \wedge \tau}) \exp(-\int_0^{t \wedge \tau} c) + \int_0^{t \wedge \tau} f(X_s, \alpha_s) \exp(-\int_0^s c)$ *is a* F^t *submartingale and we have*:

$$u(x) \leqslant E[M_\theta] \leqslant E[M_\infty] = J(x, \mathscr{A}), \quad \forall x \in \bar{\mathcal{O}}.$$

(ii) *In particular we have for all* $x \in \bar{\mathcal{O}}$:

$$u(x) = \inf_{\mathscr{A}} \left\{ E\left[\int_0^{\theta \wedge \tau} f(X_t, \alpha_t) \exp\left(- \int_0^t c \right) \mathrm{d}t + u(X_{\theta \wedge \tau}) \exp\left(- \int_0^{\theta \wedge \tau} c \right) \right] \right\} \tag{12}$$

(iii) *In addition for all* $\mathscr{A}$ *and for all* $x \in \bar{\mathcal{O}}$:

$$1_{(\tau < \infty)} u(X_\tau) \leqslant 1_{(\tau < \infty)} \psi(X_\tau) \quad a.s. \tag{13}$$

REMARK 5. This result is due to Itô [18] in its full generality and extends various previous versions due to Lions and Menaldi [52–54]. ■

REMARK 6. Of course, similar results hold for time-dependent problems. The interpretation of (13) is quite clear, since if $\tau < \infty$, $X_\tau \in \partial \mathcal{O}$ and f_n 'the control α_t often τ', the system starting of X_τ immediately exists from $\bar{\mathcal{O}}$ and the corresponding cost function is $\psi(X_\tau)$. Thus (13) holds. ■

Clearly u is bounded and in addition u is Borel measurable. Then using Lemma 2, one deduces easily (cf. Lions [32] where the following result is proved)

THEOREM 1: *We assume* (5) *and* (6). *Then* u *satisfies*:

$$A_\alpha u \leqslant f_\alpha \quad \text{in } \mathscr{D}'(\mathcal{O}), \text{ for all } \alpha \in A; \tag{14}$$

and for any open set ω *such that* $\bar{\omega}$ *is compact and contained in* $\mathcal{O}$ *we have*:

$$\sup_{\alpha \in A} \| \sigma^T(\cdot, \alpha) \cdot \nabla u \|_{L^2(\omega)} < \infty. \tag{15}$$

REMARK 7. Clearly (14) shows that $\mu_\alpha = A_\alpha u - f_\alpha$ is a nonpositive Radon measure on $\mathcal{O}$ and, is addition, it is easy to check that $(-\mu_\alpha)$ are uniformly bounded on compact sets of $\mathcal{O}$. Therefore, we may consider $\mu = \sup_{\alpha\in A} \mu_\alpha$; μ is a nonpositive Radon measure on $\mathcal{O}$. We conjecture that the following form of HJB equations is always true:

$$\mu = \sup_{\alpha\in A} \mu_\alpha = 0 \quad \text{in} \mathscr{D}'(\mathcal{O}). \tag{16}$$

The only partial result along this line is due to Coron and Lions [7] in the case $N = 1$. ■

REMARK 8. In the time-dependent case, under assumption (5) we have:

$$-\frac{\partial u}{\partial t} + A_\alpha u \leq f_\alpha \text{ in } D'(Q), \quad \text{for all } \alpha\in A$$

and:

$$\sup_{\alpha\in A} \| \sigma^T(\cdot,\alpha)\cdot\nabla u \|_{L^2(\omega\times(0,T))} < \infty.$$

We conjecture that the analogue of (16) holds in that case. ■

REMARK 9. Applications of (14), (15) are given in [32]. ■

Theorem 1 implies that u is a subsolution of the HJB equation, the following result shows that, roughly speaking, u is always the maximum one (with prescribed values on $\partial\mathcal{O}$). Of course, for C^2 subsolutions, the following result is only a consequence of Itô's formula:

THEOREM 2. *We assume* (5) *and* (6). *Let* $v\in C(\bar{\mathcal{O}})$ *satisfy* (14). *Denoting by* $\mathcal{O}_\delta = \{x\in\mathcal{O}\cap B_{1/\delta}, \operatorname{dist}(x,\partial\mathcal{O}) > \delta\}$ *and by* τ_δ *the first exit time from* $\bar{\mathcal{O}}_\delta$ *for* $\delta > 0$, *we have for all* $\delta > 0$, *for all* $x\in\bar{\mathcal{O}}_\delta$:

$$v(x) \leq \inf_{\mathscr{A}} E\left[\int_0^{\tau_\delta} f(X_t,\alpha_t)\exp\left(-\int_0^t c\right)dt + v(X_{\tau_\delta})\exp\left(-\int_0^{\tau_\delta} c\right)\right]. \tag{16}$$

REMARK 10. This result is proved in Lions [32] (see also [34]) and extends particular cases treated in [33, 49, 52, 53]. Let us also point out that if v is only upper-semi-continuous on $\bar{\mathcal{O}}$ (u.s.c. in short), then for all $\delta\to 0$, for all $\mathscr{A}$ we have

$$v(x) \leq \inf_{\mathscr{A}} E\left[\int_0^{\tau_\delta} f(X_t\alpha_t)\exp\left(-\int_0^t c\right)dt + v(X_{\tau_\delta})\exp\left(-\int_0^{\tau_\delta} c\right)\right], \quad \text{a.e. in } \mathcal{O}_\delta. \tag{16'}$$

And the inequality holds at every Lebesgue point of continuity of v. In addition we may replace in (16) (or in (16')) τ_δ by τ'_δ the first exit time from $\mathcal{O}_\delta$. ■

REMARK 11. Let $\Gamma' \subset \Gamma = \partial\mathcal{O}$ be such that:

$$\forall\mathscr{A}, \quad \forall x\in\mathcal{O}, \quad P(\tau' < \infty, X_{\tau'}\notin\Gamma') = 0.$$

Then if $v\in C(\mathcal{O})\cap L^\infty(\mathcal{O})$ and if: $\forall x\in\Gamma', \limsup_{y\in\mathcal{O}, y\to x} v(y) \leq u(x)$ we deduce from (16):

$$v(x) \leqslant \inf_{\mathscr{A}} E \int_0^{\tau'} f(X_t, \alpha_t) \exp\left(- \int_0^t c \right) dt + u(X_{\tau'}) \exp\left(- \int_0^{\tau'} c \right) = u(x) \text{ in } \mathcal{O}.$$

In particular, if $\mathcal{O} = \mathbb{R}^N$ and $v \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ satisfies (14), then: $v \leqslant u$ in $\mathbb{R}^N$. It is even enough to assume that $v \in C(\mathbb{R}^N)$ satisfies (14) and:

$$v(x) \leqslant C(|x|^p + 1) \quad \text{in } \mathbb{R}^N \text{ for some } C, p \geqslant 0. \blacksquare$$

We will not prove Theorems 1 and 2 (we refer the reader to [32, 34]). Let us only explain the main points in their proofs: let $\alpha \in A$ be fixed and let us denote by X_t the diffusion process corresponding to A_α, then

(i) if u is bounded on $\mathcal{O}$ and satisfies:

$$M_t = u(X_{t \wedge \tau}) \exp\left(- \int_0^{t \wedge \tau} c \right) + \int_0^{t \wedge \tau} f(X_s, \alpha) \exp\left(- \int_0^s c \right) ds$$

is a F^t-submartingale for all $x \in \mathcal{O}$, then we have:

$$A_\alpha u \leqslant f_\alpha \text{ in } \mathscr{D}'(\mathcal{O}), \qquad \sigma^T(\cdot, \alpha) \nabla u \in L^2_{\text{loc}}(\mathcal{O});$$

(ii) if $v \in C(\mathcal{O})$ and satisfies: $A_\alpha v \leqslant f_\alpha$ in $\mathscr{D}'(\mathcal{O})$, then for all $\delta > 0$ and for all $x \in \bar{\mathcal{O}}_\delta$:

$$v(X_{t \wedge \tau_\delta}) \exp\left(- \int_0^{t \wedge \tau_\delta} c \right) + \int_0^{t \wedge \tau_\delta} f(X_s, \alpha) \exp\left(\int_0^s c \right) ds \text{ is a } F^t\text{-submartingale.}$$

4. Continuity of the Optimal Cost Function

To simplify the presentation, we will first treat the case $\psi = 0$ on Γ. It is easy to see on deterministic examples that if no assumptions are made on the coefficients σ, b, c, f, u may not be continuous in $\mathcal{O}$, nor l.s.c. or u.s.c. in $\mathcal{O}$. The following assumption, in view of the results in [49], seem to be pretty close from an optimal condition for insuring that u is u.s.c. in $\mathcal{O}$: we will assume that there exist a closed set, possibly empty $\Gamma_+ \subset \Gamma$, w Borel bounded function on $\bar{\mathcal{O}}$ satisfying:

$$\forall \mathscr{A}, \quad \forall x \in \bar{\mathcal{O}}; \quad P[\tau < \infty, X_\tau \notin \Gamma_+] = 0, \quad 1_{(\tau < \infty)} w(X_\tau) \leqslant 0 \text{ a.s.}$$

$$\forall x \in \Gamma_+, \ \liminf_{y \in \bar{\mathcal{O}}, y \to x} w(y) \geqslant 0,$$

$$\forall \mathscr{A}, \forall x \in \bar{\mathcal{O}}, \ N_t = w(X_{t \wedge \tau}) \exp\left(- \int_0^{t \wedge \tau} c \right) + \int_0^{t \wedge \tau} f \exp\left(- \int_0^s c \right) ds$$

is a F_t-submartingale satisfying for any bounded stopping times $\theta_1 \leqslant \theta_2$:

$$E[N_{\theta_2}] \geqslant E[N_{\theta_1}]. \tag{17}$$

THEOREM 3. *We assume* (5), (6) *and* (17). *Then we have*:

(i) *For all* $\mathscr{A}$, $J(\cdot, \mathscr{A})$ *is u.s.c. on* $\bar{\mathcal{O}}$ *and thus* u *is u.s.c. on* $\bar{\mathcal{O}}$

(ii) $u \geqslant w$ *in* $\bar{\mathcal{O}}$, $u \geqslant 0$ *on* Γ_+ *and for all* $\mathscr{A}$, *for all* $x \in \bar{\mathcal{O}}$: $1_{(\tau < \infty)} u(X_\tau) = 0$ *a.s.*

REMARK 11. We give below various ways of checking (17). Let us also mention (cf. Lions [32]) that under assumption (17) one can show that u is not changed if one restricts the infimum over admissible systems where the probability space and the Brownian motion are fixed. ■

This result is proved in Lions [32]: let us only explain the idea behind condition (17). The difficulty with the continuity of J or u lies in the fact that τ depends on x but τ is not, in general, continuous with respect to x: τ is only u.s.c. in x. Therefore, if $\mathscr{A}$ is fixed and $x_n \in \mathcal{O} \underset{n}{\to} x$ it is quite clear that we have:

$$J(x_n, \mathscr{A}) - E\int_0^{\tau_{x_n}\wedge\tau_x} f\exp\left(-\int_0^t c\right)\mathrm{d}t \underset{n}{\to} 0.$$

Now we deduce from (17)

$$E\int_{\tau_{x_n}\wedge\tau_x}^{\tau_x} f(X_t, \alpha_t)\exp\left(-\int_0^t c\right)\mathrm{d}t \geqslant E\left[w(X_{\tau_{x_n}\wedge\tau_x})\exp\left(-\int_0^{\tau_{x_n}\wedge\tau_x} e\right)\right]$$

But since $X^{x_n}_{\tau_{x_n}} \in \Gamma_+$ a.s. on the event $(\tau_{x_n} < \infty)$, we easily deduce:

$$\liminf_n E\int_{\tau_{x_n}\wedge\tau_x}^{\tau_x} f\exp\left(-\int_0^t c\right)\mathrm{d}t \geqslant 0$$

and this shows that J is u.s.c. on $\bar{\mathcal{O}}$. Roughly speaking, (17) is used in order to make sure that if τ_{x_n} does not converge to τ_x, then the missing part in the cost function eventually becomes nonnegative.

We now explain how one can check (17): we first assume that Γ may be divided into two smooth, closed, disjoint points Γ_+ and Γ_- (possibly empty).

PROPOSITION 3. *We assume* (5), (6) *and that there exists* $w \in W^{1,\infty}(\mathcal{O})$ *satisfying*:

$$A_\alpha w \leqslant f_\alpha \text{ in } D'(\mathcal{O}), \quad \forall\alpha\in A;\, w = 0 \text{ on } \Gamma_+ \tag{18}$$

In addition we assume:

$$a_{ij}(x,\alpha)n_i(x)n_j(x) = 0; \qquad b_i(x,\alpha)n_i(x) - a_{ij}(x,\alpha)\partial_{ij}\mathrm{d}(x) \leqslant 0 \quad \text{on } \Gamma_- \times A \tag{19}$$

where n denotes the unit outward normal and $\mathrm{d}(x) = \mathrm{dist}(x, \Gamma)$. *Then* (17) *holds and thus the conclusions of Theorem* 3 *are valid.*

REMARK 12. If $\mathcal{O} = \mathbb{R}^N$, we may choose $w \equiv \inf\{f(x,\alpha)c(x,\alpha)^{-1} \mid x \in \mathbb{R}^N, \alpha \in A\}$. Of course if $f_\alpha \geqslant 0 \, \forall\alpha\in A$, we can take $w \equiv 0$. Other examples are given in Lions [32, 35] and follow from the next result. ■

COROLLARY 1. *We assume* (5), (6), (19) *and*:

$$\exists\theta \geqslant 0, \exists\nu > 0, \theta a_{ij}(x,\alpha)n_i(x)n_j(x) + b_i(x,\alpha)n_i(x) - a_{ij}(x,\alpha)\partial_{ij}\mathrm{d}(x) \geqslant \nu \quad \text{on } \Gamma_+ \times A. \tag{20}$$

Then there exists w satisfying the conditions of Proposition 3 and thus the conclusions of Theorem 3 are valid.

All these results are proved in Lions [32] and extend previous results due to Lions and Menaldi [52, 53].

We now turn to conditions ensuring the continuity of u. Let us first observe that, in general, even if (17) holds u needs not to be continuous: take $A = \{\alpha_0\}$, $\sigma \equiv 0$, $c \equiv 1, f \equiv 1$ then (17) holds with $\Gamma_+ = \Gamma$, $w \equiv 0$ and $u(x) = 1 - e^{-t_x}$ where t_x is the first exit time from $\bar{\mathcal{O}}$ of the deterministic process X_t. Then for some geometries, t_x is not continuous but only u.s.c.

We then denote by $\bar{\Gamma} = \{x \in \Gamma_+ / u(x) = 0\}$, clearly if $u \in C(\bar{\mathcal{O}})$ then $\bar{\Gamma}$ is closed; and the following result shows that the converse is true:

THEOREM 4. *We assume* (5), (6) *and* (17).

(i) *Then* $u \in C_b(\bar{\mathcal{O}})$ *if and only if* $\bar{\Gamma}$ *is closed. In particular if* $\bar{\Gamma} = \Gamma_+$ *then* $u \in C_b(\bar{\mathcal{O}})$.

(ii) *We assume in addition*:

$$\exists C \geqslant 0, \quad \exists \beta \in (0, 1], \quad |u(x)| \leqslant C\{\operatorname{dist}(x, \bar{\Gamma})\}^\beta \quad \forall x \in \bar{\mathcal{O}}. \tag{21}$$

Then denoting by:

$$\lambda_0 = \sup\left\{\frac{1}{2}\operatorname{Tr}\left\{\frac{(\sigma(x,\alpha) - \sigma(x',\alpha))\cdot(\sigma^T(x,\alpha) - \sigma^T(x',\alpha))}{|x - x'|^2}\right\} + \right.$$
$$\left. + \frac{(b(x,\alpha) - b(x',\alpha), x \cdot x')}{|x - x'|^2}\right|_{x \neq x',\, x, x' \in \mathcal{O},\, \alpha \in A} \Bigg\}$$

we have

$$u \in C^{0,\gamma}(\bar{\mathcal{O}}) \quad \textit{with } \gamma = \min\left(\frac{\lambda}{\lambda_0}, \beta\right) \quad \textit{if } \lambda \neq \lambda_0, = \beta \textit{ if } \lambda = \lambda_0, \beta < 1 \tag{22}$$

and γ *is arbitrary in* $(0, 1)$ *if* $\lambda = \lambda_0, \beta = 1$.

REMARK 13. In [32], it is shown that the above exponent γ is, in general, the best possible. We give below some examples where it can be proved that $\bar{\Gamma}$ is closed and that (21) holds; other examples are to be found in [32]. Of course if $\mathcal{O} = \mathbb{R}^N$, then $\bar{\Gamma} = \Gamma_+ = \Gamma = \emptyset$ and all conditions are automatically satisfied and thus $u \in C^{0,\gamma}(\mathbb{R}^N)$ with $\gamma = \lambda/\lambda_0$ if $\lambda \neq \lambda_0$. ■

COROLLARY 2. *We assume* (5), (6), (18), (19) *and*

$$\exists C \geqslant 0; \quad \forall x_0 \in \Gamma_+, \quad \exists \mathscr{A} \quad J(x, \mathscr{A}) \leqslant C|x - x_0| \quad \textit{in } \bar{\mathcal{O}} \tag{23}$$

Then (21) *holds with* $\beta = 1$ *and the conclusion of Theorem* 4 *is valid with* $\beta = 1$. *In particular* (23) *holds if we assume*:

$$\exists \nu > 0, \quad \exists \theta \geqslant 0, \quad \forall x \in \Gamma_+, \exists \alpha \in A, \quad \theta a_{ij} n_i n_j + b_i n_i - a_{ij} \partial_{ij} d \geqslant \nu. \tag{24}$$

All these results are proved by elementary methods in [32]. Let us mention that one has, if $\mathcal{O} = \mathbb{R}^N$

$$E\{|X_t^x - X_t^{x'}|^2 e^{-2\lambda_0 t}\} \leqslant |x - x'|, \quad \forall x, x' \in \mathbb{R}^N, \forall t \geqslant 0$$

and thus, taking $c \equiv \lambda$ to simplify, for all $T \geqslant 0$ we have

$$\begin{aligned} |J(x, \mathscr{A}) - J(x', \mathscr{A})| &\leqslant E \int_0^T |f(X_t^x, \alpha_t) - f(X_t^{x'}, \alpha_t)| e^{-\lambda t} \, dt + C e^{-\lambda T} \\ &\leqslant C \int_0^T e^{(\lambda_0 - \lambda)t} \, dt + C e^{-\lambda T} \end{aligned}$$

and we conclude easily taking the infimum over $T \geqslant 0$.

Let us conclude this section by pointing out that similar results hold when we consider $\psi \not\equiv 0$ or time-dependent problems. For example, if $\psi \not\equiv 0$, one has to replace in (17), (18), $\bar{\Gamma}$, (21), (23) 0 by ψ and the results are preserved. But checking (21) or (23) then requires some regularity of ψ on Γ_+ (see [32] for more details).

5. Viscosity Solutions of HJB Equations

We denote by F the function on $S^N \times \mathbb{R}^N \times \mathbb{R} \times \bar{\mathcal{O}}$ defined by:

$$F(\xi, p, t, x) = \sup_{\alpha \in A} \{ -a_{ij}(x, \alpha)\xi_{ij} - b_i(x, \alpha)p_i + c(x, a)^t - f(x, \alpha) \}.$$

where S^N stands for the space of symmetric, $N \times N$ matrices.

Of course, the HJB equation may be written in the following way:

$$F(D^2u, Du, u, x) = 0 \quad \text{in } \mathcal{O}. \tag{25}$$

Recently Crandall and Lions [9, 10] introduced the notion of viscosity solutions of first-order Hamilton–Jacobi equations (i.e., equations of type (25) when F does not depend on $\xi = D^2u$): with this notion, the existence, uniqueness and stability questions were settled (cf. [8–10, 49]). In [34–36] this notion was generalized to the case of general second-order equations (possibly degenerate). We briefly recall the definition of viscosity solutions of (25). To this end we need a few notations: let $\Psi \in C(\mathcal{O})$, for all $x \in \mathcal{O}$ we set

$$D_{2,1}^+ \Psi(x) = \left\{ (\xi, p) \in S^N \times \mathbb{R}^N \,\middle|\, \limsup_{y \to x,\, y \in \mathcal{O}} \frac{\{\Psi(y) - \Psi(x) - (p, y - x) - \frac{1}{2}\xi_{ij}(y_i - x_i)(y_j - x_j)\}}{|y - x|^2} \leqslant 0 \right\},$$

$$D_{2,1}^- \Psi(x) = \left\{ (\xi, p) \in S^N \times \mathbb{R}^N \,\middle|\, \liminf_{y \to x,\, y \in \mathcal{O}} \frac{\{\Psi(y) - \Psi(x) - (p, y - x) - \frac{1}{2}\xi_{ij}(y_i - x_i)(y_j - x_j)\}}{|y - x|^2} \geqslant 0 \right\}.$$

REMARK 14. $D^+_{2,1}\Psi(x)$ (resp. $D^-_{2,1}\Psi(x)$) is a closed convex set in $S^N \times \mathbb{R}^N$, possibly empty; but each one of these sets is non-empty on a dense subset of $\mathcal{O}$ (depending on Ψ). Clearly if $(\xi, p) \in D^+_{2,1}\Psi(x)$ then $(\xi', p) \in D^+_{2,1}\Psi(x)$ for all $\xi' \geqslant \xi$ (a similar remark holds for $D^-_{2,1}$).

Finally if Ψ satisfies nearby x:

$$\Psi(y) = \Psi(x) + (p, y - x) + \tfrac{1}{2}\xi_{ij}(y_i - x_j)(y_j - x_j) + o(|y - x|^2)$$

then $D^+_{2,1}\Psi(x) = \{(\xi', p) \in S^N \times \mathbb{R}^N / \xi' \geqslant \xi\}$, $D^-_{2,1}\Psi(x) = \{(\xi', p) \in S^N \times \mathbb{R}^N / \xi' \leqslant \xi\}$. This is, of course, the case for all x if $\Psi \in C^2(\mathcal{O})$. ■

We now give the definition of viscosity solutions of (25):

DEFINITION. Let $u \in C(\mathcal{O})$; u is said to be a viscosity solution of (25) if the following inequalities hold:

$$\begin{aligned} &F(\xi, p, u(x), x) \leqslant 0, \quad \forall(\xi, p) \in D^+_{2,1}u(x), \forall x \in \mathcal{O} \\ &F(\xi, p, u(x), x) \geqslant 0, \quad \forall(\xi, p) \in D^-_{2,1}u(x), \forall x \in \mathcal{O}. \end{aligned} \tag{26}$$

In the above references, it is shown that an equivalent definition is given by:

EQUIVALENT FORMULATION. Let $u \in C(\mathcal{O})$; u is a viscosity solution of (25) if and only if we have for all $\varphi \in C^2(\mathcal{O})$:

$$\begin{aligned} &\text{at each local maximum point } x_0 \text{ of } u - \varphi, \text{ we have:} \\ &\quad F(D^2\varphi(x_0), D\varphi(x_0), u(x_0), x_0) \leqslant 0; \\ &\text{at each local minimum point } x_0 \text{ of } u - \varphi, \text{ we have:} \\ &\quad F(D^2\varphi(x_0), D\varphi(x_0), u(x_0), x_0) \geqslant 0. \end{aligned} \tag{26'}$$

Other equivalent formulations are obtained by replacing $\varphi \in C^2(\mathcal{O})$ by $\varphi \in C^\infty(\mathcal{O})$ and local maximum by either local strict maximum, or global maximum, or global strict maximum (and similar changes for the minima).

REMARK 15. In the case of the time-dependent problem, one gives similar definitions by considering the time variable just as another space variable. ■

An immediate application of the second formulation is the *stability of viscosity solutions*: we will not give precise results (see [34–36]) but let us indicate that if u_n is a viscosity solution corresponding to nonlinearities F_n converging uniformly to F on compact subsets of $S^N \times \mathbb{R}^N \times \mathbb{R} \times \mathcal{O}$ and if u_n converges uniformly on compact sets to u, then u is also a viscosity solution of (25).

A first illustration of the relevance of this notion to optimal control theory is the following result:

THEOREM 5. (i) *Under assumptions* (5), (6) *and if u, given by* (8), *is continuous on*

$\mathcal{O}$, then u is a viscosity solution of the HJB equation:

$$F(D^2u, Du, u, x) = 0 \quad \text{in } \mathcal{O}.$$

(ii) *Under assumption* (5′) *and if u, given by* (8′), *is continuous on* $\mathcal{O} \times (0, T)$, *then u is a viscosity solution of the HJB equation:*

$$-\frac{\partial u}{\partial t} + F(D^2u, Du, u, x, t) = 0 \quad \text{in } \mathcal{O} \times (0, T); \tag{25′}$$

where $F(\xi, p, s, x, t) = \sup_{\alpha\in A}\{-a_{ij}(x, t, \alpha)\xi_{ij} - b_i(x, t, \alpha)p_i + c(x, t, \alpha)s - f(x, t, \alpha)\}$.

Proof. We will prove only (i), since the proof of (ii) is similar. The proof is an *immediate consequence of the Dynamic Programming Principle*: indeed let us check (26′), for example it is enough to consider x_0 global minimum point of $u - \varphi$ where $\varphi \in C_b^2(\bar{\mathcal{O}})$. Let $t > 0$, recall that by Lemma 2 we have:

$$u(x_0) = \inf_{\mathscr{A}} E\int_0^{\tau\wedge t} f(X_s, \alpha_s)\exp\left(-\int_0^s c\right)ds + u(X_{\tau\wedge t})\exp\left(-\int_0^{\tau\wedge t} c\right).$$

Without loss of generality we may assume that $u(x_0) - \varphi(x_0) = 0$. Then we have:

$$E\left\{u(X_{\tau\wedge t})\exp\left(-\int_0^{\tau\wedge t} c\right)\right\} \geqslant E\left\{\varphi(X_{\tau\wedge t})\exp\left(-\int_0^{\tau\wedge t} c\right)\right\}$$

and we obtain:

$$\varphi(x_0) \geqslant \inf_{\mathscr{A}} E\left\{\int_0^{\tau\wedge t} f(X_s, \alpha_s)\exp\left(-\int_0^s c\right)ds + \varphi(X_{\tau\wedge t})\exp\left(-\int_0^{\tau\wedge t} c\right)\right\}.$$

Then arguing exactly as in Section 2, that is following the traditional derivation of the HJB equation which can now be performed since $\varphi \in C^2(\mathcal{O})$, we deduce:

$$\sup_{\alpha\in A}\{-a_{ij}(x_0, \alpha)\,\partial_{ij}\varphi(x_0) - b_i(x_0, \alpha)\,\partial_i\varphi(x_0) + c(x_0, \alpha)\varphi(x_0) - f(x_0, \alpha)\} \geqslant 0,$$

that is:

$$F(D^2\varphi(x_0), D\varphi(x_0), u(x_0), x_0) \geqslant 0. \quad \blacksquare$$

Let us give now an example which will be useful in several places below:

EXAMPLE. Take $A = \{p \in \mathbb{R}^N, |p| \leqslant 1\}$, $\sigma \equiv 0$, $b(x, \alpha) = \alpha$, $c(x, \alpha) \equiv \lambda > 0$, $f(x, \alpha) \equiv 1$. Then clearly: $u \equiv 1/\lambda$ and the HJB equation is:

$$|Du| + \lambda u = 1 \quad \text{in } \mathbb{R}^N.$$

But they are infinitely many other 'reasonable' solutions of the above equation: for example for $\beta > 0$ $u_\beta(x) = (1/\lambda)\{1 - \beta e^{-\lambda|x|}\}$ is smooth except at 0, $u_\beta \in W^{1,\infty}(\mathbb{R}^N)$ and the equation holds in $\mathbb{R}^N - \{0\}$. However, if $\beta > 0$ u_β is not a viscosity solution of the HJB equation: indeed $D^-_{2,1}u_\beta(0) = \{(\xi, p)/\xi \geqslant 0, |p| \leqslant 1\}$ and

$$F(\xi, p, u_\beta(0), 0) = |p| + 1 - \beta < 1 \quad \text{for } p \text{ small enough.}$$

Let us also point out that if $(x_n)_{n \geqslant 1}$ is a dense sequence in $\mathbb{R}^N$, then $u^n_\beta(x) = \inf_{1 \leqslant j \leqslant n} u_\beta(x - x_j)$ is a Lipschitz, piecewise analytic solution of the HJB equation and when $n \to \infty$, u^n_β converges uniformly on compact sets to $\bar{u}_\beta \equiv (1/\lambda)(1 - \beta)$ which is *not* a solution of the HJB equation! ■

We have seen with the above result that we have achieved our first goal, that is, to satisfy the HJB equation in some way: we now show that the notion of viscosity solutions of the HJB equations extracts all the information from the Dynamic Programming Principle since, in view of the following result, it characterizes optimal cost functions:

THEOREM 6. (i) *We assume* (5), (6). *If $\tilde{u} \in C(\mathcal{O})$ is a viscosity solution of* (25) *then for all $\delta > 0$, for all $x \in \bar{\mathcal{O}}_s$ we have:*

$$\tilde{u}(x) = \inf_{\mathscr{A}} E \int_0^{\tau_\delta} f(X_t, \alpha_t) \exp\left(- \int_0^t c \right) \mathrm{d}t + \tilde{u}(X_{\tau_\delta}) \exp\left(- \int_0^{\tau_\delta} c \right).$$

(ii) *We assume* (5′). *If $\tilde{u} \in C(\mathcal{O}_x(0, T))$ is a viscosity solution of* (25′) *then for all $\delta_{1>0}, \delta_{2>0}$, for all $(x, t) \in \bar{\mathcal{O}}_{\delta_1} \times (\delta_2, T - \delta_2)$ we have:*

$$\tilde{u}(x, t) = \inf_{\mathscr{A}} \left\{ E \int_0^{\tau_{\delta 1} \wedge (T - \delta_2 - t)} f(X_s, t + s, \alpha_s) \times \right.$$
$$\times \exp\left(- \int_0^s c \right) \mathrm{d}s + \tilde{u}(X_{\tau_{\delta 1} \wedge (T - \delta_2 - t)}, t + \tau_{\delta_1} \wedge (T - \delta_2 - t)) \times$$
$$\left. \times \exp\left(- \int_0^{\tau_{\delta 1} \wedge (T - \delta_2 - t)} c \right) \right\}.$$

REMARK 16. It is, of course, very easy to deduce from the above result various uniqueness results: for example if $u, v \in C_b(\bar{\mathcal{O}})$ are viscosity solutions of (25), then we have:

$$\sup_{\bar{\mathcal{O}}} (u - v)^+ \leqslant \sup_{\Gamma'} (u - v)^+. \quad ■$$

Theorem 6 is proved in [34] (see also [38]) by a combination of probabalistic and analytical arguments; various extensions and applications are also given there. Let us mention that Crandall and Lions [9, 10] have previously shown uniqueness results for general first-order Hamilton–Jacobi equations, including the Isaacs equations of deterministic games by totally different methods. We conjecture that similar uniqueness results hold for general second-order elliptic equations, including the Isaacs equations for stochastic differential games.

Let us conclude this section by stating a useful result:

THEOREM 7. (i) *Let* $u \in W^{2,N}_{\mathrm{loc}}(\mathcal{O})$ *satisfy:*

$$F(D^2u, Du, u, x) = 0 \quad \text{a.e. in } \mathcal{O} \tag{25}$$

then u is a viscosity solution of Equation (25).

(ii) *Let* $u \in C(\mathcal{O})$ *be a viscosity solution of the HJB Equation* (25) *satisfying:*

$$\forall \delta > 0, \quad \exists C_\delta \geqslant 0, \quad \forall \xi \in \mathbb{R}^N : |\xi| = 1, \quad \partial_\xi^2 u \leqslant C_\delta \text{ in } D'(\mathcal{O}_\delta). \tag{27}$$

Then $u \in W^{1,\infty}_{\mathrm{loc}}(\mathcal{O})$ *and* $A_\alpha u \in L^\infty_{\mathrm{loc}}(\mathcal{O})$ *for all* $\alpha \in A$; *in addition we have:*

$$\sup_{\alpha \in A} \| A_\alpha u \|_{L^\infty(\mathcal{O}_\delta)} \leqslant C'_\delta, \quad \text{for any } \delta > 0 \tag{28}$$

$$\sup_{\alpha \in A} \{ A_\alpha u(x) - f(x, \alpha) \} = 0 \quad \text{a.e. in } \mathcal{O}. \tag{29}$$

REMARK 17. Similar results hold in the time-dependent case. ■

Part (i) of Theorem 7 is proved in [39]. Notice that if u is C^2, then part (i) is obvious, and using the example given above, one sees that $W^{2,N}_{\mathrm{loc}}(\mathcal{O})$ is the 'best possible' Sobolev space since $D^2 u_\beta \in L^p(\mathbb{R}^N)$ for $p < N$ (and even $D^2 u_\beta \in M^N(\mathbb{R}^N)$).

Part (ii) is shown in [34] and [35] and is an easy consequence of a famous differentiability result due to Alexandrov [1] and Busemann [6].

6. Regularity Results

We will denote by:

$$\lambda_1 = \sup\{2|\partial_i \sigma^T \cdot \xi \xi_i|^2 + \mathrm{Tr}(\partial_i \sigma \cdot \partial_i \sigma^T \xi_i^2) + 2\partial_i b \cdot \xi \xi_i |_{x \in \mathcal{O}, \alpha \in A, |\xi| = 1}\} \tag{30}$$

and we will assume that Γ may be divided into three disjoint closed smooth parts, possibly empty: Γ_-, Γ_1, Γ_2. We will use the following assumption:

$$\begin{aligned} &\exists \nu > 0, \quad \forall (x, \alpha) \in \Gamma_1 \times A, \quad a_{ij}(x, \alpha) n_i(x) n_j(x) \geqslant \nu > 0, \\ &\exists \nu > 0, \quad \forall (x, \alpha) \in \Gamma_2 \times A, \quad \sigma(x, \alpha) = 0, \quad b_i(x, \alpha) n_i(x) \geqslant \nu > 0. \end{aligned} \tag{31}$$

THEOREM 8. *We assume* (5), (6), (19) *and* (31). *Then; if* $\lambda > \lambda_1$ *and if* $\psi \in W^{3,\infty}(\Gamma_1 \cup \Gamma_2)$, *the optimal cost function u* (*given by* (8)) *satisfies:*

$$u \in W^{1,\infty}(\mathcal{O}), \quad u = \psi \text{ on } \Gamma_1 \cup \Gamma_2 \tag{32}$$

$$\exists C > 0, \quad \forall \xi \in \mathbb{R}^N : |\xi| = 1, \quad \partial_\xi^2 u \leqslant C \text{ in } D'(\mathcal{O}). \tag{33}$$

Of course, if $\mathcal{O} = \mathbb{R}^N$, (19) and (31) become vacuous. Combining this result and Theorem 7, we find:

COROLLARY 3. *Under the assumptions of Theorem* 8, *the optimal cost function* *satisfies:*

$$A_\alpha u \in L^\infty(\mathcal{O}) \quad \textit{for all } \alpha \in A, \quad \textit{and} \quad \sup_{\alpha\in A} \| A_\alpha u \|_{L^\infty(\mathcal{O})} < \infty. \tag{34}$$

In addition the HJB equation holds:

$$\sup_{\alpha\in A} \{ A_\alpha u(x) - f_\alpha(x) \} = 0 \quad \textit{a.e. in } \mathcal{O}. \tag{35}$$

Let us also mention without proof, the following local regularity result derived from (31)–(32):

COROLLARY 4. *Under the assumptions of Theorem* 8, *and if we assume that there exists an open set $\omega \subset \mathcal{O}$, an integer $p \in \{1, \ldots, N\}$, a constant $\gamma > 0$ such that:*

$$\forall x \in \omega, \quad \exists n \geqslant 1, \quad \exists \alpha_1, \ldots, \alpha_n \in A, \quad \exists \theta_1, \ldots, \theta_n \in]0, 1[\quad \textit{such that:}$$

$$\sum_i \theta_i = 1, \qquad \sum_i \theta_i a_{kl}(x, \alpha_i) \xi_k \xi_l \geqslant \nu \sum_{k=1}^{p} \xi_k^2 \tag{36}$$

for all $\xi \in \mathbb{R}^N$, then $\partial_{ij} u \in L^\infty(\omega)$ for $1 \leqslant i, j \leqslant p$.

REMARK 18. The above results are taken from [37] and [38] (see also [40–42]) where variants and extensions are also considered. From the well-known linear case (i.e., the case when A reduces to a single point) it is well-known that some assumption like (31) is needed. Furthermore, the assumption that $\lambda > \lambda_1$ is, in general, necessary (except in the uniformly elliptic case, see the next result) as it is shown by an example due to Genis and Krylov [17].

THEOREM 9. *We assume* (5), (6) *if $\mathcal{O}$ is unbounded and $\lambda \geqslant 0$ if $\mathcal{O}$ is bounded. We assume in addition that $\psi \in BUC(\Gamma)$:*

$$\exists \nu > 0, \quad \forall (x, \alpha) \in \bar{\mathcal{O}} \times A, \quad a(x, \alpha) \geqslant \nu I_N. \tag{37}$$

Then the optimal cost function $u \in W^{2,\infty}(\mathcal{O}) \cap C^{2,\gamma}(\mathcal{O})$ (for some $\gamma \in (0, 1)$ depending only on $\mathcal{O}$, ν, $\| a \|_\infty$) and the HJB equation holds.

REMARK 19. Many authors have studied the HJB equations and the first general results were obtained by Krylov [20, 21, 23] in the case of partially-nondegenerate diffusions and with $\mathcal{O} = \mathbb{R}^N$. Then, Brèzis and Evans [5] solved the case of a general domain $\mathcal{O}$ with $A = \{\alpha_1, \alpha_2\}$ by a variational approach specific for that particular case (see also [43]). Safonov studied the case of $\mathcal{O} \subset \mathbb{R}^2$ with nondegeneracy ([63, 64]). Simultaneously, the nondegenerate case with matrices $a(x, \alpha)$ independent of x was solved by Evans and Friedman [14] and Lions and Menaldi [54]. Then, independently, the case $\mathcal{O} > \mathbb{R}^N$ was solved by Krylov [24, 25] and Lions [44, 33] by purely probabilistic methods. The case of a general $\mathcal{O}$ with nondegenerate diffusions (assumption (37)) was settled by Lions [45, 46] (see also [15, 47, 31]) by purely PDE methods. The deep results concerning *the* $C^{2,\gamma}$ *regularity* stated in Theorem 9 *are due to Evans* [12, 13].

We will not prove Theorem 8. Let us first indicate how the assumption $\lambda > \lambda_1$ naturally comes into the proof of (33) in the special case $\mathcal{O} = \mathbb{R}^N$. To simplify, assume the coefficients σ, b, c, f to be smooth and then by standard results on stochastic differential equations, we have:

$$\partial_\xi^2 E\left[f(X_t^x)\exp\left(-\int_0^t c\right)\right] \leqslant K\exp((\lambda_1 - \lambda)t)$$

for all $\mathscr{A}$, for all $x \in \mathbb{R}^N$, for all $\xi \in \mathbb{R}^N : |\xi| = 1$ and for all $t \geqslant 0$. Here K depends only on $\sup_\alpha\{\|\sigma\|_{W^{2,\infty}} + \|b\|_{W^{2,\infty}} + \|f\|_{W^{2,\infty}} + \|c\|_{W^{2,\infty}}\}$. Therefore we deduce easily if $\lambda > \lambda_1$:

$$\begin{aligned}\partial_\xi^2 J(x, \mathscr{A}) &= \int_0^\infty \partial_\xi^2 E\left[f(X_t^x)\exp\left(-\int_0^t c\right)\right]\mathrm{d}t \leqslant \\ &\leqslant K\int_0^\infty \exp((\lambda_1 - \lambda)t)\,\mathrm{d}t \\ &\leqslant \frac{K}{\lambda - \lambda_1} = C.\end{aligned}$$

In particular, $J(x, \mathscr{A}) - \frac{1}{2}C|x|^2$ is concave and thus:

$$u(x) - \tfrac{1}{2}C|x|^2 = \inf_{\mathscr{A}}\{J(x, \mathscr{A}) - \tfrac{1}{2}C|x|^2\}$$

is also concave; and (33) is proved. ■

We conclude this section by stating the results corresponding to Theorem 8 and Corollary 3 in the time-dependent case.

THEOREM 10. *We assume that* (5′) *holds, that* (19) *and* (31) *hold uniformly for* $t \in [0, T]$. *Then if* $\psi(\cdot, T) \in W^{2,\infty}(\mathcal{O})$, $D_x^\beta \psi \in L^\infty((\Gamma_1 \cup \Gamma_2) \times (0, T))$ *for all* $|\beta| \leqslant 3$, *the optimal cost function u* (*given by* (8′)) *satisfies:*

$$u \in C(\bar{\mathcal{O}}), \qquad D_x u \in L^\infty(Q), \quad u = \psi \quad \textit{on}\ \{(\Gamma_1 \cup \Gamma_2) \times [0, T]\} \cup \{\bar{\mathcal{O}} \times \{T\}\}; \quad (38)$$

$$\exists C > 0, \quad \forall \xi \in \mathbb{R}^N; \qquad \xi = 1, \qquad \partial_\xi^2 u \leqslant C \quad \textit{in}\ D'(\mathcal{O}); \quad (33')$$

$$\frac{\partial u}{\partial t} \in L^2(\mathcal{O}_S \times (0, T)); \quad -\frac{\partial u}{\partial t} \leqslant C \quad \textit{in}\ D'(Q); \quad -\frac{\partial u}{\partial t} - C\Delta u \geqslant -C \quad \textit{in}\ D'(Q); \quad (39)$$

$$A_\alpha u \in L^2(\mathcal{O}_\delta \times (0, T)), \quad -C \leqslant A_\alpha u \leqslant C + \frac{\partial u}{\partial t} \quad \textit{in}\ D'(Q), \quad \forall \alpha \in A; \quad (40)$$

furthermore the HJB equation holds:

$$-\frac{\partial u}{\partial t}+\sup_{\alpha\in A}\{A_\alpha u(x,t)-f_\alpha(x,t)\}=0 \quad \textit{a.e. in } Q. \tag{41}$$

Finally if σ, b, c, f, ψ are Lipschitz in t uniformly for $x\in\bar{\mathcal{O}}, \alpha\in A$, then $\partial u/\partial t\in L^\infty(Q)$ and thus: $\sup_{\alpha\in A}\|A_{\alpha u}\|_{L^\infty(Q)}<\infty$. ∎

7. Uniqueness Results

We have seen in the preceding section that under general assumptions, the optimal cost function u satisfies (32) and (34) and that regularity makes the HJB equation (35) meaningful. But by the example given in Section 5, there may be many solutions with this regularity. In this section we want to characterize the optimal cost function among all these possible solutions of the HJB equations. Of course, in Section 5 we have already given one answer to this question by the use of viscosity solutions. In this section, we give another approach by taking into account conditions like (33), which are satisfied by the optimal cost function.

THEOREM 11. *We assume* (5) *and* (6). *Let $\tilde{u}\in C(\mathcal{O})\cap W^{1,N}_{\mathrm{loc}}(\mathcal{O})$ satisfy:*

$$A_\alpha\tilde{u}\leqslant f_\alpha \quad \textit{in } D'(\mathcal{O}), \quad \forall\alpha\in A \tag{14}$$

$$\Delta\tilde{u}\leqslant g \quad \textit{in } D'(\mathcal{O}), \quad \textit{where } g\in L^N_{\mathrm{loc}}(\mathcal{O}). \tag{42}$$

In particular, (14) *implies that $\mu_\alpha=A_\alpha\tilde{u}-f_a$ is a nonpositive Radon measure on $\mathcal{O}$ and we assume that the HJB equation holds in the sense of measures:*

$$\sup_{\alpha\in A}\mu_\alpha=0 \quad \textit{in } D'(\mathcal{O}). \tag{43}$$

Then we have for all $\delta\in 0$, for all $x\in\bar{\mathcal{O}}_\delta$:

$$\tilde{u}(x)=\inf_{\mathscr{A}} E\left[\int_0^{\tau_\delta} f(X_t,\alpha_t)\exp\left(-\int_0^t c\right)+\tilde{u}(X_{\tau_\delta})\exp\left(-\int_0^{\tau_\delta} c\right)\right].$$

Let us give the analogue of Theorem 11 for time-dependent problems:

THEOREM 12. *We assume* (5′). *Let $\tilde{u}\in C(Q)$ satisfy: $D_x\tilde{u}\in L^{N+1}_{\mathrm{loc}}(Q)$ and*

$$-\frac{\partial u}{\partial t}+A_\alpha u\leqslant f_\alpha \quad \textit{in } D'(Q), \quad \forall\alpha\in A; \tag{14′}$$

$$\exists C>0, \quad \exists g\in L^{N+1}_{\mathrm{loc}}(Q), \qquad -\frac{\partial u}{\partial t}-C\Delta u\geqslant g \quad \textit{in } D'(Q); \tag{42′}$$

$$\sup_{\alpha\in A}\mu_\alpha=0 \quad \textit{in } D'(Q) \tag{43′}$$

where $\mu_\alpha=-\partial u/(\partial t)+A_\alpha u-f_\alpha$. Then for all $\delta_1>0, \delta_2>0$ and for all $(x,t)\in\bar{\mathcal{O}}_{\delta_1}\times$

$\times [\delta_2, T-\delta_2]$ *we have:*

$$\tilde{u}(x,t) = \inf_{\mathscr{A}}\left\{ E \int_0^{\tau_{\delta_1}\wedge(T-\delta_2-t)} f \exp\left(-\int_0^s c\right) ds + \right.$$
$$+ \tilde{u}(X_{\tau_{\delta_1}\wedge(T-\delta_2-t)}, t+\tau_{\delta_1}\wedge(T-\delta_2-t)) \times$$
$$\left. \times \exp\left(-\int_0^{\tau_{\delta_1}\wedge(T-\delta_2-t)} c\right)\right\}.$$

REMARK 20. These results are proved in [37] and [38] and heavily use the deep inequalities on stochastic integrals due to Krylov [26–28, 21]. Let us also point out that in (42) and (42′) Δ may be replaced by any uniformly elliptic operator with smooth coefficients. ■

REMARK 21. Clearly, the above results imply that in Theorems 8–10, the optimal cost function is the unique function satisfying the properties listed in those results. ■

REMARK 22. If we consider the example given in Section 5, we see that it is not possible to replace L^N in (42) by L^p (or even M^N) for $p<N$. Indeed, $u_\beta \in W^{1,\infty}(\mathbb{R}^N)$, $A_\alpha u_\beta \in L^\infty(\mathbb{R}^N)$, the equation holds in $\mathbb{R}^N - \{0\}$ thus a.e. and $\Delta u_\beta \in L^p(\mathbb{R}^N)$ for $p<N$. ■

8. Related Problems

In this section, we give a list of related questions where similar methods and results can be obtained.

I. UNBOUNDED COEFFICIENTS

The first possible extension of the above results consists of assuming that bounds are uniform in α but only for x bounded. More precisely, all the preceding results are easily adapted if we assume in the time-independent case:

$$\sup_{\alpha\in A} \|\varphi(\cdot,\alpha)\|_{W^{2,\infty}(B_R)} < \infty, \quad \forall R<\infty;\ \varphi(x,\alpha)\in C(A), \quad \forall x\in\mathbb{R}^N \text{ for } \varphi = \sigma, b, c, f;$$
$$\sup_{\alpha\in A} \|D_x\varphi(\cdot\alpha)\|_{L^\infty(\mathbb{R}^N)} < \infty \quad \text{for } \varphi=\sigma, b;$$
$$\forall\varepsilon>0, \quad \exists C_\varepsilon>0, \quad \forall(x,\alpha)\in\mathbb{R}^N\times A,\ \|\sigma\| + |b| \leqslant \varepsilon|x| + C_\varepsilon;$$
$$\exists C, q\geqslant 0, \quad \forall(x,\alpha)\in\mathbb{R}^N\times A,\ \|D_x^2\sigma\| + \|D^2 b\| +$$
$$+ \sum_{|\beta|\leqslant 2} |D_x^\beta c| + \sum_{|\beta|\leqslant 2} |D_x^\beta f| \leqslant C(1+|x|^q);$$
$$\inf\{c(x,\alpha)/x\in\mathbb{R}^N, \alpha\in A\} > 0.$$

and in the time-dependent case:

$$\sup_{\substack{\alpha\in A\\ t\in[0,T]}} \|\varphi(\cdot,t,\alpha)\|_{W^{2,\infty}(B_R)} < \infty, \quad \forall R<\infty;\ \varphi(x,t,\cdot)\in C(A),\quad \forall (x,t)\in\mathbb{R}^N\times[0,T]$$
$$\text{for } \varphi=\sigma,b,c,f;$$

$$\sup_{\substack{\alpha\in A\\ t\in[0,T]}} \|D_x\varphi(\cdot,t,\alpha)\|_{L^\infty(\mathbb{R}^N)} < \infty \quad \text{for } \varphi=\sigma,b;$$

φ is continuous on $[0,T]$ uniformly for $|x|\leqslant R, \alpha\in A$ for $\varphi=\sigma,b,c,f$;

$$\exists C,q\geqslant 0, \forall(x,t,\alpha)\in\mathbb{R}^N\times[0,T]\times A, \|D_x^2\sigma\|+\|D_x^2 b\|+ + \sum_{|\beta|\leqslant 2}|D_x^\beta c|+\sum_{|\beta|\leqslant 2}|D_x^\beta f|\leqslant C(1+|x|^q);$$

$$\inf\{c(x,t,\alpha)/x\in\mathbb{R}^N, t\in[0,T], \alpha\in A\} > -\infty.$$

For more details we refer the reader to [32, 34, 37].

Another type of extension consists of relaxing the assumption that the coefficients σ, b are bounded uniformly in α, then two types, essentially, of results are possible. First, if the Hamiltonian F makes sense, then the HJB equation still holds. Next, following Krylov [23], in general, one needs to renormalize the HJB equation (see [21, 23, 37]).

II. OTHER PROBLEMS

By similar techniques, one can also treat problems like: (i) those combining (8) and optimal stopping problems [21, 31, 32, 34, 37]; (ii) the control of reflecting diffusion processes [32, 37]; (iii) the control of jump diffusion processes [29, 31]; (iv) impulse control problems [61] and (v) control problems with switching costs [2, 37].

III. APPROXIMATION

We refer the reader interested to various ways of approximating the HJB equations to Evans and Friedman [14] and Jensen and Lions [19]. Concerning the numerical approximation, we refer to Quadrat [62], Lions and Mercier [55], and Crandall and Lions [11].

IV. STATE CONSTRAINTS PROBLEMS

A class of problems important for applications is the class of state constraint problems; that is, problems where one considers only strategies α_t such that X_t lies a.s. in $\bar{\mathcal{O}}$ for all $t\geqslant 0$. The setting is straightforward for degenerate problems like deterministic ones (cf. [48]) but in the nondegenerate case, it requires the possibility of 'unbounded action'. Indeed, it takes an unbounded action to 'force the Brownian motion to stay in a bounded domain'. The appropriate setting and the solution of those problems are given in [30] and [51].

V. NONLINEAR SEMIGROUPS

Nisio (cf. [59–61]) built and studied the nonlinear semigroups ω responding to the Cauchy problem for HJB equations. Using the notion of viscosity solutions, a general uniqueness result for nonlinear semigroups is given in [56]. This type of results is useful for studying asymptotic problems (cf. [57]).

VI. PERRON'S METHOD FOR HJB EQUATIONS

In [50], Lions proved that one can directly built a maximum subsolution for the HJB equations for very general data, which corresponds to the optimal cost function as soon as the data is regular. It is also proved that the HJB equation holds in the sense of measures and various variational interpretations are given. This kind of question corresponds, roughly speaking, to the control of densities.

VII. FULLY NONLINEAR ELLIPTIC SECOND-ORDER EQUATIONS

Of course, solving the HJB equations can be interpreted as solving some particular class of fully nonlinear elliptic second-order equations and it might be useful to decide what is the exact class of equations one can solve with the above results and methods [37, 66, 67].

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