On The Existence of Value Functions of Two-Player, Zero-Sum Stochastic Differential Games

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Introduction. In this paper we investigate the question of the existence of the value of zero-sum, two-player stochastic differential games. We formulate a notion of value functions (upper and lower) which satisfy the dynamic programming principle and are the unique (viscosity) solutions of the associated Bellman-Isaacs partial differential equations. In the case that the Isaacs' condition holds this implies that the game has a value. Finally, we establish the convergence of fully discrete approximations to the upper and lower value functions. For a general introduction to the subject of differential games we refer to R. Isaacs [15]. For a more expanded discussion of the scope of the theory as well as a brief description of the results presented here we refer to W. H. Fleming and P. E. Souganidis [12].

In what follows for the sake of definiteness we consider the case of a finite-horizon stochastic differential game, which in the sequel will be referred to as (SDG), with the state variable in $\mathbf{R}^N$ and horizon $T > 0$. Other cases (e.g., state variable in bounded domains, infinite horizon problems, etc.) can be treated in a similar way. The (SDG) consists of the *dynamics*, i.e., a stochastic differential equation

$$dX_s = f(s,X_s,Y_s,Z_s)\,ds + \sigma(s,X_s,Y_s,Z_s)\,dW_s \qquad (s \in [t,T]), \tag{0.1}$$

with initial condition

$$X_t = x \qquad (x \in \mathbf{R}^N), \tag{0.2}$$

and the *pay-off*,

$$J(x,t;\,Y.,Z.) = E_{xt}\left\{\int_t^T h(s,X_s,Y_s,Z_s)\,ds + g(X_T)\right\}. \tag{0.3}$$

Here $W.$ is a standard M-dimensional brownian motion, $Y.$ and $Z.$ are stochastic processes taking values in some compact subsets $\mathcal{Y}$ and $\mathcal{Z}$ or $\mathbf{R}^p$ and $\mathbf{R}^q$ respectively, for some positive integers p and q. Precise assumptions on $Y.$ and

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$Z.$ as well as on the functions $g : \mathbf{R}^N \to \mathbf{R}$, $f : [0,T] \times \mathbf{R}^N \times \mathcal{Y} \times \mathcal{Z} \to \mathbf{R}^N$, $h : [0,T] \times \mathbf{R}^N \times \mathcal{Y} \times \mathcal{Z} \to \mathbf{R}$ and the $N \times M$-matrix $\sigma : [0,T] \times \mathbf{R}^N \times \mathcal{Y} \times \mathcal{Z} \to \mathcal{M}^{N \times M}$ are given in Section 1. Finally, $E_{xt}\{G\}$ denotes the expected value of G.

The intuitive idea is that there are two players I and II. Player I controls $Y.$ and wishes to maximize J over all choices of $Z.$. On the other hand, player II controls $Z.$ and tries to minimize J over all choices of $Y.$. The main difficulty lies in the fact that, although at any time $s \in [t,T]$ both players know X_r, Y_r, Z_r and W_r for $r \in [t,s)$, instantaneous switches of $Y.$ and $Z.$ are possible in continuous time. To overcome this problem in the deterministic case ($\sigma \equiv 0$) or in the special stochastic cases $\big(\sigma \equiv \sigma(X)\big)$, W. H. Fleming [8], [9], A. Friedman [13], [14], R. Elliott and N. J. Kalton [5], [6], and N. N. Subbotina, A. I. Subbotin, and V. E. Tret'jakov [27] introduced two approximate games, namely, a *lower* and an *upper* one. In the lower game, player II is allowed to know Y_s before choosing Z_s, while in the upper game player I chooses Y_s knowing Z_s. Using this as well as a number of suitable approximations (discretization in time, white noise, etc.) in combination with several probabilistic arguments, the above mentioned authors established the existence of a lower and an upper value for the deterministic as well as the special stochastic case noted above. In the case that the Isaacs' condition holds, they showed that the game has a value. When $\sigma \equiv 0$, these results were later greatly simplified by N. Barron, L. C. Evans and R. Jensen [1], L. C. Evans and P. E. Souganidis [7] and P. E. Souganidis [25], [26] via the use of arguments from the theory of viscosity solutions of Hamilton-Jacobi equations introduced by M. G. Crandall and P.-L. Lions [3]. Their method extended in one way or another an observation of P.-L. Lions [20], [21] concerning the relation between the dynamic programming principle for deterministic and stochastic optimal control problems and the notion of the viscosity solution of the Bellman equations.

We next turn our attention to a few basic facts from the theory of scalar, fully nonlinear, possibly degenerate elliptic or parabolic partial differential equations, which can be written as

$$F(D^2u,Du,u,\xi) = 0 \qquad \text{in } \mathcal{O}, \tag{0.4}$$

where $\mathcal{O}$ is an open subset of $\mathbf{R}^k$ for some positive integer k, F and $u = u(\xi)$ are continuous functions of their arguments and D^2u (resp. Du) stands for the matrix (resp. vector) of the second (resp. first) partial derivatives of u. Special cases of (0.4) are the Bellman-Isaacs equations, heretofore referred to as (BI) equations, corresponding to the (SDG) (0.1)–(0.3). These are the equations

$$\begin{cases} u_t + H^-(D^2u,Du,x,t) = 0 & \text{in } \mathbf{R}^N \times [0,T], \\ u = g & \text{on } \mathbf{R}^N \times \{T\}, \end{cases} \tag{0.5}$$

and

$$\begin{cases} u_t + H^+(D^2u,Du,x,t) = 0 & \text{in } \mathbf{R}^N \times [0,T], \\ u = g & \text{on } \mathbf{R}^N \times \{T\}, \end{cases} \tag{0.6}$$

where, for $A \in S^N$ (symmetric $N \times N$- matrices), p, $x \in \mathbf{R}^N$ and $t \in [0,T]$,

$$(0.7) \quad H^-(A,p,x,t) = \max_{y \in Y} \min_{z \in Z} \left[\text{trace} \left(\frac{1}{2} a(t,x,y,z) A \right) + f(t,x,y,z) \cdot p + h(t,x,y,z) \right]$$

and

$$(0.8) \quad H^+(A,p,x,t) = \min_{z \in Z} \max_{y \in Y} \left[\text{trace } \frac{1}{2} \left(a(t,x,y,z) A \right) + f(t,x,y,z) \cdot p + h(t,x,y,z) \right],$$

with $a = \sigma\sigma'$, σ' denoting the transpose of σ. Equation (0.5) (resp. (0.6)) corresponds to the lower (resp. upper) version of the (SDG) (0.1)–(0.3). It is well known that equations (0.4) do not have global smooth solutions in general. The derivation of (0.5) and (0.6) is heuristic and it is justified in the classical sense only when the value functions are smooth (cf. W. H. Fleming and R. Rishel [11] for the case of stochastic control with only one controller). On the other hand non-smooth solutions of (0.4) (for example Lipschitz continuous solutions) are highly non-unique (cf. [3], [20]). To address this problem for the first-order version of (0.4) ($F \equiv F(Du,u,\xi)$ or $a \equiv 0$ in (0.5) and (0.6)), i.e., the Hamilton-Jacobi equations, M. G. Crandall and P.-L. Lions [3] introduced the notion of viscosity solutions and proved general uniqueness results. (For an easy introduction to the subject we refer to M. G. Crandall, L. C. Evans and P.-L. Lions [2], for a general view of the scope of the theory as well as some of the bibliography we refer to the book by P.-L. Lions [20] and the review papers by M. G. Crandall and P. E. Souganidis [4] and P.-L. Lions [22]). The notion of viscosity solution has a natural extension to second- order problems. This was first exhibited by P.-L. Lions [20] for the case of the Bellman equations (F convex or concave in (0.4)) via a combination of probabilistic and analytic arguments. More recently, the general equation (0.4) was treated by purely analytic techniques. This was started by R. Jensen [18] whose work was later simplified and generalized by R. Jensen, P.-L. Lions and P. E. Souganidis [19]. The most general results concerning (0.4) are due to H. Ishii [16] and H. Ishii and P.-L. Lions [17]. (For an outline of the uniqueness results as well as a general review, we refer to P.-L. Lions and P. E. Souganidis [23].)

In this note we adapt the methods of R. Elliott and N. J. Kalton [5] and L. C. Evans and P. E. Souganidis [7] to deal with the general case of the (SDG) (0.1)–(0.3). Once the notion of an upper and a lower value function is introduced, the main step is to establish that these functions are viscosity solutions of the associated Bellman-Isaacs' equations. This would follow immediately, had we been able to show that these functions satisfy the principle of dynamic programming. We encounter, however, some measurability problems which seem to be, at least to us, non trivial; we are thus able to establish only half of the desired equalities for some restrictions of the value functions which we call sub- and super-optimality principle of dynamic programming. (This is done in Section 1, where we also give the general setting, precise statements and assumptions.) It turns out, however, to be sufficient. Indeed, combining these inequalities with a semidiscretization argument (developed in Section 2), motivated from the work of M. Nisio [24], and the uniqueness results of H. Ishii [16], we are able to show that the lower and upper value functions are the unique viscosity solutions of (0.5) and (0.6) respectively and that they satisfy the principle of dynamic programming. This is described in Section 3. In Section 4 we recall an alternative approach to the (SDG) (0.1)–(0.3) due to W. H. Fleming [10] which involves full discretization in time and space. By adapting and extending the main ideas of P. E. Souganidis [26], we show that the approximations converge to the value functions.

Section 1. We begin by making the following assumptions which will hold throughout the paper.

(1.1) The spaces $\mathcal{Y}$, $\mathcal{Z}$ are compact metric spaces. The functions f, σ and h are bounded, uniformly continuous and Lipschitz continuous with respect to (x,t) uniformly in $(y,z) \in \mathcal{Y} \times \mathcal{Z}$. Finally, the function g is bounded and Lipschitz continuous.

Before we continue, we remark that we have made no attempt to present the weakest possible assumptions on the data. In particular, it is possible that some of the assumptions (1.1) may be relaxed.

We next begin our way towards formulating a notion of upper and lower value for the (SDG) (0.1)–(0.3). As mentioned in the Introduction we will adapt the methods used by R. Elliott and N. J. Kalton [5] and L. C. Evans and P. E. Souganidis [7]. Our notation follows closely that of [7]. We first introduce the concepts of *admissible control* and *admissible strategy.* To this end, we recall the notion of *progressively measurable process*.

Definition 1.1. Let $(\Omega,\mathcal{F},P)$ be a probability space, $\{G_s\}$ an increasing family of σ-algebras of subsets of Ω, and $\mathcal{X}$ a complete separable metric space. An $\mathcal{X}$-valued stochastic process $\xi.$ on the sample space Ω and the time interval $[t,T]$ is G_s-progressively measurable, if the mapping $[t,s] \times \Omega \ni (r,\omega) \to \xi(\omega)_r$ is $(\mathcal{F}^0_{t,s} \otimes G_s, \mathcal{F}^0_{\mathcal{X}})$ measurable for every $s \in [t,T]$ where $\mathcal{F}^0_{t,s}$ is the Borel σ-algebra on $[t,s]$ and $\mathcal{F}^0_{\mathcal{X}}$ is the Borel σ-algebra on $\mathcal{X}$.

The sample space for (0.1) is defined as follows: For every $t \in [0,T]$ let

$$\Omega^\omega_t = \{\omega \in C([t,T];\mathbf{R}^M) : \omega_t = 0\}, \tag{1.2}$$

where M is the dimension of the brownian motion in (0.1). Then let $\mathcal{F}^\omega_{t,s}$ denote the σ- algebra generated by paths up to time s in Ω^ω_t. When provided with the Wiener measure P^ω_t on $\mathcal{F}^\omega_{t,T}$, Ω^ω_t becomes the *canonical sample space* of (0.1). Occasionally we will also be using the space

$$\Omega^\omega_{t,s} = \{\omega \in C([t,s];\mathbf{R}^M) : \omega_t = 0\} \tag{1.3}$$

for $0 \leq t < s \leq T$. Note that with the above notation $\Omega_t^\omega = \Omega_{t,T}^\omega$. If $s \in (t,T)$ and $\omega \in \Omega_t^\omega$ let

$$\begin{cases} \omega_1 = \omega|_{[t,\tau]}, \\ \omega_2 = \omega - \omega_\tau|_{[\tau,T]} \quad \text{and} \\ \pi\omega = (\omega_1,\omega_2). \end{cases} \tag{1.4}$$

The map $\pi : \Omega_t^\omega \to \Omega_{t,\tau}^\omega \times \Omega_\tau^\omega$ induces the identification

$$\Omega_t^\omega = \Omega_{t,\tau}^\omega \times \Omega_\tau^\omega; \tag{1.5}$$

moreover, $\omega = \pi^{-1}(\omega_1,\omega_2)$, i.e., the inverse of π is defined in the evident way. Finally, $P_t^\omega = P_{t,\tau}^\omega \otimes P_\tau^\omega$, where $P_{t,\tau}^\omega$ and P_τ^ω are the Wiener measures on $\Omega_{t,\tau}^\omega$ and Ω_τ^ω respectively.

Definition 1.2. An admissible control process $Y.$ (resp. $Z.$) for player I (resp. II) on $[t,T]$ is an $\mathcal{F}_{t,s}^\omega$-progressively measurable process taking values in $\mathcal{Y}$ (resp. $\mathcal{Z}$). The set of all admissible controls for player I (resp. II) on $[t,T]$ is denoted by $M(t)$ (resp. $N(t)$).

Before we introduce the concept of admissible strategies we need to agree on a way to identify admissible controls. We say that the controls $Y.$, $\tilde{Y}.$ in $M(t)$ are the same on $[t,s]$ and we write $Y. \approx \tilde{Y}.$ on $[t,s]$, if $P_t^\omega\big(Y. = \tilde{Y}. \text{ a.e. in } [t,s]\big) = 1$. A similar convention is assumed to hold for elements of $N(t)$.

Finally, if $Y. \in M(t)$ then for every $s \in [t,T]$ there exists $Y.^s : [t,s] \times \Omega_{t,s}^\omega \to \mathcal{Y}$ such that

$$Y(\omega)_r = Y^s(\omega^s)_r \qquad \big(r \in [t,s],\, \omega \in \Omega_t^\omega,\, \omega^s = \omega|_{[t,s]}\big) \tag{1.6}$$

where $Y.^s$ is $\mathcal{F}_{t,r}^\omega$ progressively measurable, and $\mathcal{F}_{t,r}^\omega$ is the Borel σ-algebra of $\Omega_{t,r}^\omega$. Similar statements hold for $Z \in N(t)$.

Definition 1.3. An admissible strategy α (resp. β) for player I (resp. II) on $[t,T]$ is a mapping $\alpha : N(t) \to M(t)$ (resp. $\beta : M(t) \to N(t)$) such that if $Z. \approx \tilde{Z}.$ (resp. $Y. \approx \tilde{Y}.$) on $[t,s]$, then $\alpha[Z.]. \approx \alpha[\tilde{Z}.].$ (resp. $\beta[Y.]. \approx \beta[\tilde{Y}.].$) on $[t,s]$ for every $s \in [t,T]$. The set of admissible strategies of player I (resp. II) on $[t,T]$ is denoted by $\Gamma(t)$ (resp. $\Delta(t)$).

The definitions of admissible controls and strategies given above are consistent with the intuitive idea that controls should depend only on the past of the brownian motion and that strategies should only depend on the past of controls. In addition, if $Z. = \beta[Y.].$ where $Y. \in M(t)$ and $\beta \in \Delta(t)$, then (1.1) and Definition 1.3 yield that the stochastic differential equation (0.1) with initial condition (0.2) has a unique pathwise solution $X.$. A similar fact holds if $Y. = \alpha[Z.].$ for $Z. \in N(t)$ and $\alpha \in \Gamma(t)$.

We are now ready to define the notions of the lower and upper value for the (SDG) (0.1)–(0.3). The intuitive idea is that in choosing the controls at time s, the player who moves first (maximizing player for the lower game, minimizing player for the upper game) is allowed to use the past of the brownian motion $W.$ driving (0.1) up to time s, while the player with the advantage (player II for the lower game, player I for the upper game) is allowed to use the past of both $W.$ and the other player's control.

Definition 1.4. The lower value of the (SDG) (0.1)–(0.3) with initial data (x,t) is given by

$$V(x,t) = \inf_{\beta \in \Delta(t)} \sup_{Y. \in M(t)} J(x,t;\, Y.,\beta). \tag{1.7}$$

The upper value of (SDG) (0.1)-(0.3) is given by

$$U(x,t) = \sup_{\alpha \in \Gamma(t)} \inf_{Z. \in N(t)} J(x,t;\, \alpha,Z.). \tag{1.8}$$

The properties of the stochastic differential equation (0.1) and assumption (1.1) yield the following lemma which we state without proof.

Lemma 1.5.

(a) *For every $Y. \in M(t)$, $Z. \in N(t)$, $\alpha \in \Gamma(t)$, $\beta \in \Delta(t)$, the functions $(x,t) \to J(\cdot,\cdot,\alpha,Z.)$ and $(x,t) \to J(\cdot,\cdot,Y.,\beta)$ are bounded Lipschitz continuous with respect to x and Hölder continuous with respect to t uniformly in $\alpha, Z.$ and $Y.,\beta$ respectively.*

(b) *The value functions U and V are bounded Lipschitz continuous with respect to x and Hölder continuous with respect to t.*

As we mentioned in the Introduction, we are really interested in coming up with value functions which satisfy the dynamic programming principle. It turns out that the functions U and V do so.

Theorem 1.6. *Let t, $\tau \in [0,T]$ be such that $t < \tau$. For every $x \in \mathbf{R}^N$ we have*

$$V(x,t) = \inf_{\beta \in \Delta(t)} \sup_{Y. \in M(t)} E_{xt} \left\{ \int_t^\tau h\big(s,X_s,Y_s,\beta[Y.]_s\big)\, ds + V(X_\tau,\tau) \right\}. \tag{1.9}$$

where $X.$ is the solution of (0.1), (0.2) *with $Z. = \beta[Y.]$ for $Y. \in M(t)$, and*

$$U(x,t) = \sup_{\alpha \in \Gamma(t)} \inf_{Z. \in N(t)} E_{xt} \left\{ \int_t^T h\big(s,X_s,\alpha[Z.]_s,Z_s\big)\, ds + V(X_\tau,\tau) \right\}, \tag{1.10}$$

where, for $Z. \in N(t)$, $X.$ is the solution of (0.1), (0.2) *with $Y. = \alpha[Z.]$.*

One would expect that Theorem 1.6 would have a tedious but nevertheless straight forward proof, which would more or less follow the lines of the corresponding results of [7] for the deterministic case. Working along these lines we encountered serious technical problems related to measurability issues, which we were unable to overcome. We were, however, able to work around them. To this end, we need to introduce to use a more restrictive class of admissible strategies which we will call in the sequel *r-strategies* (for restrictive). We need the following fact which is an immediate consequence of Definition 1.2. For $0 \leq \bar{t} < t < T$, $Y. \in M(\bar{t})$ and $P^{\omega}_{\bar{t},t}$—a.e. $\omega_1 \in \Omega^{\omega}_{\bar{t},t}$, the map $Y(\omega_1) : [t,T] \times \Omega^{\omega}_t \to \mathcal{Y}$ given by $Y(\omega_1)(\omega_2)_r = Y(\omega_1,\omega_2)_r$ is an admissible control for player I, i.e., $Y(\omega_1) \in M(t)$. A similar observation holds for admissible controls for player II.

Definition 1.7. An r-strategy β for player II on $[t,T]$ is an admissible strategy with the following additional properties: For every $\bar{t} < t < \tau$ and $Y. \in M(\bar{t})$ the map $(r,\omega) \to \beta[Y(\omega_1)](\omega_2)_r$ is $\left(\mathcal{F}^0_{t,\tau} \otimes \mathcal{F}^{\omega}_{\bar{t},\tau}, \mathcal{F}^0_{\mathcal{Z}}\right)$ measurable. The set of r-strategies of player II on $[t,T]$ is denoted by $\Delta_1(t)$. Similarly, we define r-strategies for player I with their collection denoted by $\Gamma_1(t)$.

Next using admissible controls and r-strategies we define r-lower and upper values.

Definition 1.8. The r-lower and the r-upper value of the (SDG) (0.1)–(0.3) with initial data (x,t) are given by

$$V_1(x,t) = \inf_{\beta\in\Delta_1(t)} \sup_{Y.\in M(t)} J(x,y;\, Y.,\beta) \tag{1.11}$$

and

$$U_1(x,t) = \sup_{\alpha\in\Gamma_1(t)} \inf_{Z.\in N(t)} J(x,t;\, \alpha, Z.) \tag{1.12}$$

The following is immediate.

Lemma 1.9.

(a) *The r-value functions V_1 and U_1 are bounded, Lipschitz continuous in x and Hölder continuous in t.*

(b) *For every $(x,t) \in \mathbf{R}^N \times [0,T]$,*

$$V(x,t) \leq V_1(x,t) \quad \textit{and} \quad U_1(x,t) \leq U(x,t). \tag{1.13}$$

The r-value functions turn out to satisfy two inequalities related to (1.9) and (1.10).

Proposition 1.10. *Under the assumptions of Theorem 1.6 we have*

$$(1.14)\qquad V_1(x,t) \le \inf_{\beta\in\Delta_1(t)} \sup_{Y.\in M(t)} E_{xt}\left\{\int_t^\tau h\big(s,X_s,Y_s,\beta[Y.]_s\big)\,ds + V_1(X_\tau,\tau)\right\}$$

and

$$(1.15)\qquad U_1(x,t) \ge \sup_{\alpha\in\Gamma_1(t)} \inf_{Z.\in N(t)} E_{xt}\left\{\int_t^\tau h\big(s,X_s,\alpha[Z.]_s,Z_s\big)\,ds + U_1(X_\tau,\tau)\right\}.$$

We refer to inequality (1.14) as the *sub-optimality dynamic programming principle* and to inequality (1.15) as the *super-optimality dynamic programming principle.*

Before we prove Proposition 1.10 it is necessary to make some preliminary remarks which will be used extensively in the sequel. Given $Y. \in M(t)$ and $\beta \in \Delta(t)$, let $\gamma(\omega)_s = \big(Y(\omega)_s,\beta[Y](\omega)_s\big)$. It is immediate that γ is $\mathcal{F}^\omega_{t,s}$-progressively measurable. Let $\tilde\gamma = \gamma\pi^{-1}$. For $s \in (\tau,T)$, (0.1) can be rewritten as

$$(1.16)\qquad X_s = X_\tau + \int_\tau^s f\big(r,X_r,\tilde\gamma(\omega_1,\omega_2)_r\big)\,dr + \int_\tau^s \sigma\big(r,X_r,\tilde\gamma(\omega_1,\omega_2)_r\big)\,dW_{2,r},$$

where W_2 is the *canonical* brownian motion on Ω^ω_τ. Let us write $P^\omega_1 = P^\omega_{t,\tau}$ and $P^\omega_2 = P^\omega_\tau$. Recall that with the identification (1.5), $P^\omega_t = P^\omega_1 \otimes P^\omega_2$. For P^ω_1-almost all ω_1, the process $X.$ in (1.16) coincides with the unique solution of the same stochastic differential equation on Ω^ω_τ with initial condition $X_\tau(\omega_1)$. For the latter to be true, we need to know that

$$\int_\tau^s \sigma\big(r,X_r,\tilde\gamma(\omega_1,\omega_2)_r\big)\,dW_r = \int_\tau^s \sigma\big(r,X_r,\tilde\gamma(\omega_1,\omega_2)_r\big)\,dW_{2,r}.$$

This is immediate, however, for it holds for step processes which converge to the above integrals P^ω_2-almost surely for P^ω_1—almost all ω_1. The above together with a similar analysis for $\alpha \in \Gamma(t)$, $Z. \in N(t)$ and $\tilde\delta = \delta\pi^{-1}$ where $\delta(\omega)_s = \big(\alpha[Z.](\omega)_s,Z(\omega)_s\big)$, as well as the fact that

$$(1.17)\qquad E^{P^\omega_1\times P^\omega_2}\big[\psi(\omega_1,\omega_2)|\mathcal{F}^\omega_{t,\tau}\big] = E^{P^\omega_2}\psi(\omega_1,\omega_2),\quad P^\omega_1\text{—almost surely}$$

yield the following technical lemma.

Lemma 1.11. *For any bounded continuous function φ and any $s \in [\tau,T]$ we have*

$$(1.18)\qquad E_{xt}\big[\varphi\big(X_s,\gamma(\omega)_s\big)|\mathcal{F}^\omega_{t,\tau}\big] = E_{X_\tau\tau}\big(\varphi\big(X_s,\tilde\gamma(\omega_1,\omega_2)_s\big)\big),\quad P^\omega_1\text{—almost surely}$$

and

$$(1.19)\qquad E_{xt}\big[\varphi\big(X_s,\delta(\omega)_s\big)|\mathcal{F}^\omega_{t,\tau}\big] = E_{X_\tau\tau}\big(\varphi\big(X_s,\tilde\delta(\omega_1,\omega_2)_s\big)\big),\quad P^\omega_1\text{—almost surely.}$$

Proof of Proposition 1.10. Here we only prove (1.14), as the proof of (1.15) is similar. Let (x,t) be fixed. Following L. C. Evans and P. E. Souganidis [7, pp. 777–779] we set

$$W_1(x,t) \equiv \inf_{\beta\in\Delta_1(t)} \sup_{Y.\in M(t)} E_{xt}\left\{\int_t^\tau h\big(r,X_r,Y(\omega)_r,\beta[Y.](\omega)_r\big)\,dr + V_1(X_\tau,\tau)\right\}$$

and fix $\varepsilon > 0$. Then there exists $\delta \in \Delta_1(t)$ such that

$$W_1(x,t) \geq E_{xt}\left\{\int_t^\tau h\big(r,X_r,Y(\omega)_r,\beta[Y.](\omega)_r\big)\,dr + V_1(X_\tau,\tau)\right\} - \varepsilon \tag{1.20}$$

for every $Y. \in M(t)$. Also, for each $\xi \in \mathbf{R}^N$

$$V_1(\xi,\tau) = \inf_{\beta\in\Delta_1(\tau)} \sup_{Y.\in M(\tau)} J\big(\xi,\tau;\, Y.,\beta[Y.].\big);$$

thus there exists $\delta_\xi \in \Delta_1(\tau)$ for which

$$V_1(\xi,\tau) \geq \sup_{Y.\in M(\tau)} J\big(\xi,\tau;\, Y.,\delta_\xi[Y.].\big) - \varepsilon. \tag{1.21}$$

Next let $\{A_i : i = 1,2,\ldots\}$ be a partition of $\mathbf{R}^N$ by Borel sets and choose $\xi_i \in A_i (i = 1,2,\ldots)$. If the diameter of the A_i's is sufficiently small then for $i = 1,2,\ldots$, and $w \in A_i$ Lemmata 1.5(a) and 1.9(a) yield

$$|J(w,\tau;\, Y.,\beta) - J(\xi_i,\tau;\, Y.,\beta)| < \varepsilon \qquad \text{for every } Y. \in M(\tau) \text{ and } \beta \in \Delta(\tau) \tag{1.22}$$

and

$$|V_1(w,\tau) - V_1(\xi_i,\tau)| < \varepsilon. \tag{1.23}$$

Now we use the strategies δ and δ_{ξ_i}, $i = 1,2,\ldots$, to construct a new admissible strategy $\beta \in \Delta_1(t)$ as follows: for $(r,\omega) \in [t,T]\times\Omega_t^\omega$ and $Y. \in M(t)$ we define

$$\beta[Y](\omega)_r = \begin{cases} \delta[Y.](\omega)_r & \text{if } r < \tau \\ \sum\limits_{i=1}^{\infty} \chi_{A_i}(X_\tau)\delta_{\xi_i}\big[Y(\omega_1).\big](\omega_2)_r & \text{if } r \geq \tau, \end{cases}$$

where χ_B is the indicator function of the set B, $\omega = (\omega_1,\omega_2) \in \Omega_{t,\tau}^\omega \times \Omega_\tau^\omega$ and $Y(\omega_1). \in M(\tau)$ is given by $Y(\omega_1)(\omega_2)_r = Y(\omega_1,\omega_2)_r$. It is immediate that $\beta \in \Delta_1(t)$, since all the required properties are built in the definition of δ and δ_{ξ_i}. Note that it is exactly here where we need these r-strategies.

Consequently for any $Y. \in M(t)$, using (1.20), (1.22) and (1.23) we obtain

$$\begin{aligned} W(x,t) &\geq E_{xt}\left\{\int_t^\tau h\big(r,X_r,Y(\omega)_r,\beta[Y.](\omega)_r\big)\,dr + \sum_{i=0}^{n}\chi_{A_i}(X_\tau)V_1(X_\tau,\tau)\right\} - \varepsilon \\ &\geq E_{xt}\left\{\int_t^\tau h\big(r,X_r,Y(\omega)_r,\beta[Y.](\omega)_r\big)\,dr + \sum_{i=0}^{n}\chi_{A_i}(X_\tau)V_1(\xi_i,\tau)\right\} - 2\varepsilon. \end{aligned}$$

On the other hand, for $X_\tau \in A_i$, $i = 1,2,\ldots$, and $Y. \in M(\tau)$, (1.21) and (1.22) yield

$$V_1(\xi_i,\tau) \geq J\big(\xi_i,\tau;\, Y(\omega_1).,\delta_{\xi_i}\big) - \varepsilon \geq J\big(X_\tau,\tau;\, Y(\omega_1).,\delta_{\xi_i}\big) - 2\varepsilon.$$

Combining the above inequalities and using again (1.22) we get

$$\begin{aligned} W(x,t) \geq E_{xt}\bigg\{ \int_t^\tau & h\big(r,X_r,Y(\omega)_r,\beta[Y.](\omega)_r\big)\, dr \\ & + \sum_{i=1}^{\infty} \chi_{A_i}(X_\tau) E_{X_\tau \tau} \int_\tau^T h\big(r,X_r,Y(\omega)_r,\beta[Y](\omega_r)\big)\, dr \\ & + g(X_T)\bigg\} - 4\varepsilon \end{aligned}$$

Finally, Lemma 1.11 yields

$$W(x,t) \geq J(x,t;\, Y.,\beta) - 4\varepsilon$$

which implies

$$W(x,t) \geq V_1(x,t) - 4\varepsilon;$$

thus the result. □

The next step is to show that V_1 and U_1 are viscosity sub- and super-solutions, respectively, of the lower- and upper-Bellman Isaacs' equations (0.5) and (0.6) respectively. For the definition of viscosity solutions we refer to the Appendix. The proof of the next proposition follows very closely the proof of Theorem 4.1 in L. C. Evans and P. E. Souganidis [7, pp. 782–786].

Proposition 1.12. *The r-lower value function V_1 (resp. r-upper value function U_1) is a viscosity subsolution (resp. supersolution) of* (0.5) *(resp.* (0.6)*).*

Proof. We only prove here the assertion for U_1. The result for V_1 follows in a similar way. Let φ be smooth and suppose that $V_1 - \varphi$ attains a local minimum at $(x_0,t_0) \in \mathbf{R}^N \times [0,T)$. We must prove that

$$\varphi_t(x_0,t_0) + H^+\big(D^2\varphi(x_0,t_0), D\varphi(x_0,t_0),x_0,t_0\big) \leq 0.$$

Should this fail, there would exist some $\vartheta > 0$ so that

$$\varphi_t(x_0,t_0) + H^+\big(D^2\varphi(x_0,t_0),D\varphi(x_0,t_0),x_0,t_0\big) \geq \vartheta > 0. \tag{1.24}$$

Set

$$\begin{aligned} \Lambda(x,t,y,z) \equiv \varphi_t(x,t) &+ f(t,x,y,z) \cdot D\varphi(x,t) \\ & + \frac{1}{2}\sum_{i,j} a_{ij}(t,x,y,z)\varphi_{x_i x_j}(x,t) + h(t,x,y,z). \end{aligned}$$

According to (1.24)

$$\min_{z\in\mathcal{Z}}\max_{y\in\mathcal{Y}}\Lambda(x_0,t_0,y,z)\geq\vartheta>0.$$

Hence for each $z\in\mathcal{Z}$ there exists $y=y(z)\in\mathcal{Y}$ such that

$$\Lambda(x_0,t_0,y,z)\geq\vartheta.$$

Since Λ is uniformly continuous we have in fact

$$\Lambda(x_0,t_0,y,\xi)\geq\frac{3\vartheta}{4}$$

for all $\xi\in B(z,r)\cap\mathcal{Z}$ and some $r=r(z)>0$. Because $\mathcal{Z}$ is compact there exist finitely many distinct points $z_1,\ldots,z_n\in\mathcal{Z}$, $y_1,\ldots,y_n\in\mathcal{Y}$ and $r_1,\ldots,r_n>0$ such that

$$\mathcal{Z}\subset\bigcup_{i=1}^{n}B(z_i,r_i)$$

and

$$\Lambda(x_0,t_0,y_i,\xi)\geq\frac{3\vartheta}{4}\qquad\text{for }\xi\in B(z_i,r_i).$$

Define $\psi:\mathcal{Z}\to\mathcal{Y}$ by setting $\psi(z)=y_k$ if

$$z\in B(z_k,r_k)\setminus\bigcup_{i=1}^{k-1}B(z_i,r_i)\qquad(k=1,\ldots,n).$$

Thus

$$\Lambda\big(x_0,t_0,\psi(z),z\big)\geq\frac{3\vartheta}{4}\qquad\text{for all }z\in\mathcal{Z}.$$

Since Λ is uniformly continuous there exists an $R>0$ such that

$$\Lambda\big(x,t,\psi(z),z\big)\geq\frac{\vartheta}{2}\qquad\text{for all }z\in\mathcal{Z}\text{ and }|t-t_0|,\ |x-x_0|\leq R. \tag{1.25}$$

The map $\psi:\mathcal{Z}\to\mathcal{Y}$ gives rise to an r-strategy α^* for the first player on $[t_0,T]$ as follows: For $Z.\in M(t_0)$ and $(r,\omega)\in[t_0,T]\times\Omega_{t_0}^{\omega}$ let

$$\alpha^*[Z.](\omega)_r=\psi\big(Z(\omega)_r\big)$$

Since ψ is $(\mathcal{F}_{\mathcal{Z}}^0,\mathcal{F}_{\mathcal{Y}}^0)$ measurable, it is immediate that $\alpha^*\in\Delta_1(t_0)$.

On the other hand the super-optimality dynamic programming principle (1.15), the choice of (x_0,t_0) and Itô's formula yield

$$\inf_{Z.\in N(t_0)}E_{x_0t_0}\int_{t_0}^{T}\Lambda\big(\tilde{X}_r,r,\alpha^*[Z.](\omega)_r,Z(\omega)_r\big)\,dr\leq 0,$$

where $\tilde{X}.$ is the solution of (0.1) and (0.2) with $Y. = \alpha^*[Z]$.

Given $\varepsilon > 0$ we choose $Z_1. \in N(t_0)$ such that

$$E_{x_0 t_0} \int_{t_0}^{\tau} \Lambda\big(\tilde{X}_r, r, \alpha^*[Z_1.](\omega)_r, Z_1(\omega)_r\big)\, dr \le \varepsilon(\tau - t_0).$$

But if $\| \ \|_\tau$ denotes the sup-norm on $[t_0,\tau]$, then by (1.25) we have

$$-CP^{\omega}_{t_0}\big(\|\tilde{X}. - x_0\|_\tau > R\big) + \frac{\vartheta}{2}(\tau - t_0)P^{\omega}_{t_0}\big(\|\tilde{X}. - x_0\|_\tau \le R\big) \le \varepsilon(\tau - t_0)$$

where C depends on φ and the bounds from (1.1). For $\tau - t_0$ small, however, there exists $a > 0$ such that

$$P^{\omega}_{t_0}\big(\|\tilde{X}. - x_0\|_\tau > R\big) \le P^{\omega}_{t_0}\big(\|\zeta.\|_\tau > aR\big),$$

where $\zeta_t = \int_{t_0}^{t} \sigma\big(r, \tilde{X}_r, \alpha^*[Z_1](\omega)_r, Z_1(\omega)_r\big)\, dW_r$. Finally, a standard martingale inequality gives

$$P_{t_0}\big(\|\zeta.\|_\tau > aR\big) \le \frac{C}{(aR)^4} E_{x_0 t_0}[\zeta_\tau^4] \le \frac{K(\tau - t_0)^2}{aR^4},$$

for some positive constants C and K. Combining all the above we obtain a contradiction for $\tau - t_0$ small. □

If v and u are the unique viscosity solutions of (0.5) and (0.6) respectively, then Theorem A.1 (see Appendix), and Proposition (1.12) yield

Proposition 1.13. *For every $(x,t) \in \mathbf{R}^N \times [0,t]$ we have*

$$V_1(x,t) \le v(x,t) \quad \textit{and} \quad u(x,t) \le U_1(x,t).$$

In the next section we will show that $V \ge v$ and $U \le u$, thus establishing that the lower- and upper-value are the unique viscosity solutions of (0.5) and (0.6) respectively.

Section 2. We now consider an approximation procedure in which time is discretized but not the state dynamics. This formulation is similar to the one of [24], where, however, the concept of strategy is not introduced explicitly. Related but somewhat different discretizations were introduced earlier in [9] and [13]. The development in the section follows rather closely the abstract arguments of [26]. Finally, here we only consider the case of the lower value, the upper value case being treated similarly.

Let $\pi = \{0 = t_0 < t_1 < \cdots < t_m = T\}$ be a partition of $[0,T]$ and denote by $\|\pi\| = \max_{1 \le i \le m}(t_i - t_{i-1})$ its mesh. We need to introduce to notions of *π-admissible controls* and *π-admissible strategies.*

Definition 2.1. A π-admissible control $Y.$ for player I on $[t,T]$ is an admissible control with the following additional property: If $i_0 \in \{0,\dots,m-1\}$ is such that $t \in [t_{i_0}, t_{i_0+1})$, then $Y_s = y$ for $s \in [t, t_{i_0+1})$ with $y \in \mathcal{Y}$ and $Y_s = Y_{t_k}$ is $\mathcal{F}^{\omega}_{t,t_k}$-measurable for $s \in [t_k, t_{k+1})$ for $k = i_0+1,\dots,m-1$. The set of π-admissible controls for player I on $[t,T]$ is denoted by $M_\pi(t)$. The π-admissible controls $Z.$, for player II are defined in a similar way and their collection is denoted by $N_\pi(t)$.

Definition 2.2. A π-admissible strategy α (resp. β) for player I (resp. II) on $[t,T]$ is an $\alpha \in \Gamma(t)$ (resp. $\beta \in \Delta(t)$) such that $\alpha[N(t)] \subset M_\pi(t)$ (resp. $\beta[M(t)] \subset N_\pi(t)$) with the further property that if $t \in [t_{i_0}, t_{i_0+1})$ then for every $Z. \in N(t)$ (resp. $N. \in M(t)$) $\alpha[Z.]|_{[t,t_{i_0+1})}$ (resp. $\beta[Y.]|_{[t,t_{i_0+1})}$) does not depend on $Z.$ (resp. $Y.$) and if $Z. \approx \tilde{Z}.$ (resp. $Y. \approx \tilde{Y}.$) on $[t,t_k]$, then $\alpha[Z.]_{t_k} = \alpha[\tilde{Z}.]_{t_k}, P^{\omega}_t$—a.s. (resp. $\beta[Z.]_{t_k} = \beta[\tilde{Z}.]_{t_k}, P^{\omega}_t$—a.s.) for every $k \in \{i_0+1,\dots,m\}$. The collection of π-admissible strategies for player I (resp. II) on $[t,T)$ is denoted by $\Gamma_\pi(t)$ (resp. $\Delta_\pi(t)$).

Let $C_b^{0,1}(\mathbf{R}^N)$ denote the space of bounded, Lipschitz continuous functions on $\mathbf{R}^N$. For every $\varphi \in C_b^{0,1}(\mathbf{R}^N)$, $t \in [0,T)$ and $\tau \in (t,T]$, let

$$F(t,\tau)\varphi(x) = \sup_{y\in Y} \inf_{Z.\in N(t,\tau)} E_{xt}\left\{\varphi(X_\tau) + \int_t^\tau h(\sigma, X_\sigma, y, Z_\sigma)\, d\sigma\right\} \tag{2.1}$$

where $N(t,\tau)$ denotes the set of admissible controls for player II on $[t,\tau)$ and $X.$ is the solution of (0.1) and (0.2) on $[t,\tau)$ with $Y. \equiv y$. It is easily shown that $F(t,\tau)$ is a self mapping of $C_b^{0,1}(\mathbf{R}^N)$. Therefore the function $v_\pi : \mathbf{R}^N \times [0,T]$ given by $v_\pi(x,T) = g(x)$ and

$$v_\pi(x,t) = F(t,t_{i_0+1}) \prod_{k=i_0+2}^{m} F(t_{k-1},t_k)g(x) \qquad \text{if } t \in [t_{i_0}, t_{i_0+1}) \tag{2.2}$$

is well defined.

Proposition 2.3. *For every $(x,t) \in \mathbf{R}^N \times [0,T]$ we have*

$$v_\pi(x,t) = \inf_{\beta\in\Delta(t)} \sup_{Y.\in M_\pi(t)} J(x,t;\, Y.,\beta). \tag{2.3}$$

Proof. This stochastic game characterization of v_π is an immediate consequence of the following fact:

(2.4) For every $(x,t) \in (\mathbf{R}^N) \times [0,T]$ and every $\varepsilon > 0$ there exist $\alpha_\varepsilon \in \Gamma_\pi(t)$ and $\beta_\varepsilon \in \Delta(t)$ such that for all $Y. \in M_\pi(t)$ and $Z. \in N(t) \cdot J(x,t;\, Y.,\beta_\varepsilon) - \varepsilon \le v_\pi(x,t) \le J(x,t;\, \alpha_\varepsilon, Z.) + \varepsilon$.

The left hand inequality of (2.4) implies the $\geq$ inequality in (2.3). On the other hand, for any $\beta \in \Delta(t)$ the pair of strategies $\alpha_\varepsilon \in \Gamma_\pi(t)$ and $\beta \in \Delta(t)$ define a pair of controls $Y_\cdot^\varepsilon \in M_\pi(t)$ and $Z_\cdot^\varepsilon \in N(t)$ such that

$$J(x,t;\, \alpha_\varepsilon, Z_\cdot^\varepsilon) = J(x,t;\, Y_\cdot^\varepsilon, \beta). \tag{2.5}$$

Once we have (2.5), the other inequality in (2.4) is immediate. The controls $Y_\cdot^\varepsilon$ and $Z_\cdot^\varepsilon$ are defined as follows. Let $t \in [t_{i_0}, t_{i_0+1})$ and $z_0 \in \mathcal{Z}$. To simplify the rather cumbersome (but actually quite natural) construction of the $Y_\cdot^\varepsilon$ and $Z_\cdot^\varepsilon$ in the sequel we will assume that $t = t_{i_0}$. (If $t \in (t_{i_0}, t_{i_0+1})$, the construction is changed in the obvious way.) Moreover, to simplify the notation we will write Y_j for $Y_{j\cdot}$ and Z_j for $Z_{j\cdot}$. To this end, let $Y_{i_0} = \alpha_\varepsilon[z_0]$, $Z_{i_0} = \beta[Y_{i_0}]$, and define $Z_k \in N(t)$, $k = i_0+1, \ldots, m$ and $Y_k \in M_\pi$, $k = i_0+1, \ldots, m$, by

$$Z_k = \beta[Y_k] \quad \text{and} \quad Y_k = \alpha_\varepsilon[Z_{k+1}].$$

We must check that $Y_{k+1} \approx Y_k$ and $Z_{k+1} \approx Z_k$ on $[t_{i_0}, t_k]$ for $k = i_0+1, \ldots, m-1$. We proceed inductively. For $k = i_0+1$, it is immediate from the definitions of α_ε and β; more precisely, from the fact that on $[t_{i_0}, t_{i_0+1}]\alpha_\varepsilon$ is independent of the choice of a z- strategy. Next we assume that $Y_k \approx Y_{k-1}$ and $Z_k \approx Z_{k-1}$ on $[t_{i_0}, t_{k-1}]$. But $Y_{k+1} = \alpha_\varepsilon[Z_k]$ and $Y_k = \alpha_\varepsilon[Z_{k-1}]$, therefore (see Definition 2.1) $(Y_{k+1})_{t_{k-1}} = (Y_k)_{t_{k-1}}$ and, since Y_{k+1}, Y_k are constant on $[t_{k-1}, t_k)$, $Y_{k+1} \approx Y_k$ on $[t_{i_0}, t_k]$. Finally, $Z_k = \beta[Y_k]$ and $Z_{k+1} = \beta[Y_{k+1}]$ yields $Z_{k+1} \approx Z_k$ on $[t_{i_0}, t_k]$.

We conclude the proof by establishing (2.4). Here for simplicity we take $h \equiv 0$ in (0.3). For $G \in C_b^{0,1}(\mathbf{R}^N)$, $y \in \mathcal{Y}$, $t \in [0,T]$ and $\tau \in (t,T]$, let

$$\psi(x,y,t,\tau,G) = \inf_{Z_\cdot \in N(t,\tau)} E_{xt} G(X_\tau), \tag{2.6}$$

where $X_\cdot$ is as in (0.1) with $Y_s \equiv y$. Then $\psi(\,\cdot\,,\,\cdot\,,t,\tau,G) \in C_b^{0,1}(\mathbf{R}^N \times \mathcal{Y})$ and

$$F(t,\tau)G(x) = \sup_{y \in \mathcal{Y}} \psi(x,y,t,\tau,G).$$

If $t \in [t_{i_0}, t_{i_0+1})$ for $i_0 \in \{0,1,\ldots,m-1\}$, let $G_m = g$, $G_j = F(t_j, t_{j+1})G_{j+1}$ for, $j = i_0+1, \ldots, m-1$ and $G_{i_0} = F(t, t_{i_0+1})G_{i_0+1}$. Thus

$$G_{i_0}(x) = v_\pi(x,t).$$

Following [24, Lemma 1] we partition $\mathbf{R}^N$ and $\mathcal{Y}$ into Borel sets $\{A_k : k = 1,2,\ldots\}$ and $\{B_\ell : \ell = 1,\ldots,L\}$ respectively of diameter less than δ, where $\delta > 0$ is to be specified later, and we choose $x_k \in A_k$ and $y_\ell \in B_\ell$. Given $\gamma > 0$ we can choose δ small enough and $y_{kj}^* = y_{\ell_{(j,k)}} \in \mathcal{Y}$ for $k = 1,2,\ldots, j = i_0+1, \ldots, m$, such that

$$\psi(x_k, y_{kj}^*, t_{j-1}, t_j, G_j) > F(t_{j-1}, t_j)G_j(x_k) - \gamma. \tag{2.7}$$

We also choose $Z_{k\ell j} \in N(t_{j-1},t_j)$ such that, for $Z = Z_{k\ell j}$ and $Y_s \equiv y_\ell$

$$E_{x_k t_{j-1}} G_j(X_{t_j}^{k\ell j}) < \psi(x_k,y_\ell,t_{j-1},t_j,G_j) + \gamma, \tag{2.8}$$

where for $j = i_0 + 1$ we replace t_{i_0} by t. Here the superscripts indicate the dependence of the solution $X_s^{k\ell j}$ of (0.1) and (0.2) on the initial data (x_k,t_{j-1}) and on y_ℓ.

We need to introduce more notation. As before (cf. beginning of Section 1) we identify $\omega \in \Omega_t^\omega$ with the pair $(\omega_{1j},\omega_{2j})$ for $j = i_0+2,\dots,m$ where $\omega_{1j} = \omega\big|_{[t,t_{j-1}]}$ and $\omega_{2j} = \omega - \omega_{t_{j-1}}\big|_{[t_{j-1},T]}$. With this identification the Wiener measure P_t^ω on Ω_t^ω can be regarded as the product measure $P_{1j}^\omega \otimes P_{2j}^\omega$ of the Weiner measures P_{1j}^ω and P_{2j}^ω on $\Omega_{t,t_{j-1}}^\omega$ and $\Omega_{t_{j-1}}^\omega$ respectively. In view of this identification, we will be writing

$$E^{P_{2j}} \equiv E_{x_k t_{j-1}}. \tag{2.9}$$

The strategies α_ε and β_ε are defined as follows: Let $(x,t) \in \mathbf{R}^N \times [0,T)$ be fixed. For $Z. \in N(t)$ we define

$$\alpha_\varepsilon[Z]_r = \chi_{[t,t_{i_0+1})}(r)\sum_k y_{ki_0}^* \chi_{A_k}(x) + \sum_{j=i_0+1}^{m-1} \chi_{[t_j,t_{j+1})}(r)\sum_k y_{kj}^* \chi_{A_k}(X_{t_j}), \tag{2.10}$$

where the random variable, X_t, is defined successively on intervals $[t,t_{i_0+1}]$, $[t_j,t_{j+1}]$, $j = i_0+1,\dots,m-1$ as the solution to (0.1) and (0.2) with $Y_r = \alpha_\varepsilon[Z.]_r$. For $Y. \in M(t)$ we define

$$\beta_\varepsilon[Y.]_r = \chi_{[t,t_0+1)}(r)\sum_{k,\ell}(\widehat{Z}_{kli_0})_r \chi_{A_k}(x)\chi_{B_\ell}(Y_r) + \sum_{j=i_0+1}^{m-1}\sum_{k,\ell}\chi_{[t_j,t_{j+1})}(r)(\widehat{Z}_{klj})_r\chi_{A_k}(X_{t_j})\chi_{B_\ell}(Y_r), \tag{2.11}$$

where again X_t is defined on successive intervals as the solution to (0.1) and (0.2) with $Z_r = \beta_\varepsilon[Y.]_r$. and where $\widehat{Z}_{k\ell j}(\omega) = Z_{k\ell j}(\omega_{2j})$.

For any $Z. \in N(t)$ and $Y. = \alpha_\varepsilon[Z.]$. or $Y. \in M_\pi(t)$ and $Z. \in \beta_\varepsilon[Y.]$ we have

$$\begin{aligned} v_\pi(x,t) - J &= G_{i_0}(x) - E^{P_t} g(X_t) \\ &= \sum_{j=i_0+1}^{m} \left[E^{P_t} G_{j-1}(X_{t_{j-1}}) - E^{P_t} G_j(X_{t_j})\right] \\ &= E^{P_t} \sum_{j=i_0+1}^{m} \left[G_{j-1}(X_{t_{j-1}}) - E^{P_t}\{G_j(X_{t_j}) | \mathcal{F}_{t,t_{j-1}}^\omega\}\right], \end{aligned}$$

where J stands for either $J(x,t;\, \alpha_\varepsilon, Z.)$ or $J(x,t;\, Y., \beta_\varepsilon)$. To obtain (2.4) it will suffice to show that the following statements hold:

$$\text{(2.12a)}\quad \left|\begin{array}{l} \text{For any } Z. \in N(t) \text{ and } Y. = \alpha_\varepsilon[Z.]. \\[1em] \quad G_{j-1}(X_{t_{j-1}}) \le E^{P_t}\left[G_j(X_{t_j}) | \mathcal{F}^\omega_{t,t_{j-1}}\right] + \varepsilon(t_j - t_{j-1}), \\[1em] P^\omega_t\text{—almost surely,} \end{array}\right.$$

and

$$\text{(2.12b)}\quad \left|\begin{array}{l} \text{for any } Y. \in M_\pi(t) \text{ and } Z. = \beta_\varepsilon[Y.]. \\[1em] \quad E^{P_t}\left[G_j(X_{t_j}) | \mathcal{F}^\omega_{t,t_{j-1}}\right] \le G_{j-1}(X_{t_{j-1}}) + \varepsilon(t_j - t_{j-1}), \\[1em] P^\omega_t\text{—almost surely.} \end{array}\right.$$

We recall that by (1.17), the conditional expectations in (2.12) can be replaced by expectations with respect to P^ω_{2j}. Moreover, by taking $\tau = t_{j-1}$ and $\omega_2 = \omega_{2j}$ in (1.16) we have that $X(\omega_{1j}, \cdot)_s$ is a solution to the stochastic differential equation (1.16) for P^ω_{1j}-almost all ω_1. Finally, for $X_{t_{j-1}} \in A_k$ and $Y_{t_{j-1}} \in B_\ell$ we have

$$\max\Big\{ |X_{t_{j-1}} - x_k|,\ E^{P_{2j}}|X_{t_j} - X^{k\ell j}_{t_j}|,\ |G_{j-1}(X_{t_{j-1}}) - G(x_k)|,$$
$$|E^{P_{2j}} G_j(X_{t_j}) - E^{P_{2j}} G_j(X^{k\ell j}_{t_j})| \Big\} < C\delta^{1/2},$$

where C is some constant which depends on (1.1). Then by (2.7) for each k and $Z. \in N(t_{j-1})$ we have

$$G_{j-1}(x_k) < E^{P_{2j}} G_j(X^k_{t_j}) + \gamma$$

where $X^k_\cdot = X^{k\ell(j,k)j}_\cdot$. By noting that any $Z. \in N(t)$ gives rise to

$$Z(\omega_{1j}, \cdot)\big|_{[t_{j-1},T]} \in N(t_{j-1}),$$

we obtain (2.12a) provided $2C\delta^{1/2} + \gamma < \varepsilon(t_j - t_{j-1})$. A similar argument yields (2.12b) once we observe that for $Y. \in M_\pi(t_{j-1})$ and $Y_s = Y_{t_{j-1}} \in B_k$ for $s \in [t_{j-1}, t_j]$ (2.8) yields

$$E^{P_{2j}} G_j(X^{k\ell j}_{t_j}) < G_{j-1}(x_k) + \gamma.$$

Our goal is to pass to the limit as $\|\pi\| \to 0$. To this end we need some kind of compactness for the family $\{V_\pi\}$. This follows from the following lemma which is stated without proof as it is more or less immediate.

Lemma 2.4. *There exists a constant K which only depends on* (1.1) *such that*

$$|v_\pi(x,t)| \le K \quad \text{and} \quad |v_\pi(x,t) - v_\pi(\hat{x},\hat{t})| \le K\big(|x-\hat{x}| + |t-\hat{t}|^{1/2}\big)$$

for all x, $\hat{x} \in \mathbf{R}^N$ and t, $\hat{t} \in [0,T]$. □

In view of Lemma 2.4, the functions v_π converge locally uniformly as $\|\pi\| \to 0$ along subsequences to bounded uniformly continuous functions which turn out to be viscosity solutions of (0.5). The uniqueness results concerning viscosity solutions then yield that the whole family v_π converges.

Proposition 2.5. *Assume* (1.1). *If v_π is given by* (2.2), *then the limit $v = \lim_{\|\pi\|\to 0} v_\pi$ exists locally uniformly and it is the unique viscosity solution of* (0.5).

Proof. The fact that the limit v exists and it is the unique viscosity solution of (0.5) follows from the uniqueness of viscosity solutions (cf. Appendix) and Lemma 2.4, once we establish that any possible limit of the v_π's is a viscosity solution. To this end, let v be a local uniform limit of the v_π's. Here we only argue that v is a viscosity subsolution of (0.5) as the supersolution case is very similar. The argument used here as well as a related one in Section 3 is very similar to the ones in P. E. Souganidis [25], [26]. If φ is a smooth function and $v - \varphi$ has a strict local maximum at (x_0,t_0) we want to show that

$$\varphi_t(x_0,t_0) + H^-\big(D^2\varphi(x_0,t_0), D\varphi(x_0,t_0), x_0, t_0\big) \ge 0. \tag{2.13}$$

Since $v_\pi \to v$ locally uniformly as $\|\pi\| \to 0$, there exist (x_π,t_π) such that $(x_\pi,t_\pi) \to (x_0,t_0)$ as $\|\pi\| \to 0$ and $v_\pi - \varphi$ attains a local maximum at (x_π,t_π) which without any loss of generality may be assumed to be global. On the other hand, (2.2) yields $v_\pi(x_\pi,t_\pi) = F(t_\pi,t^\pi_{i_0+1})v_\pi(\,\cdot\,,t^\pi_{i_0+1})(x_\pi)$, if $t_\pi \in [t^\pi_{i_0}, t^\pi_{i_0+1})$. Thus

$$\varphi(x_\pi,t_\pi) \le F(t_\pi,t^\pi_{i_0+1})\varphi(\,\cdot\,,t^\pi_{i_0+1})(x_\pi). \tag{2.14}$$

We conclude by showing that (2.14) implies (2.13). This follows immediately from the following fact that if φ be any smooth function, then

$$\lim_{s\downarrow t}\left\| \frac{F(t,s)\varphi(\,\cdot\,) - \varphi(\,\cdot\,)}{s-t} - H^-(D^2\varphi, D\varphi, \,\cdot\,, t)\right\| = 0,$$

which is a direct consequence of Itô's formula. (A similar argument is used in Lemma 3.2 below.) □

We are now ready to state and prove our main result concerning lower and upper values of the (SDG) (0.1)–(0.3) as well as the existence of the value.

Theorem 2.6. *Assume* (1.1). *The lower value V (resp. upper value U) of the (SDG)* (0.1)–(0.3) *defined by* (1.9) *(resp.* (1.10)*) is the unique viscosity solution of the lower (resp. upper) Bellman-Isaacs equation* (0.5) *(resp.* (0.6)*). Moreover, if the Isaacs' condition* (0.9) *holds, then the (SDG)* (0.1)–(0.3) *has a value.*

Proof. We only argue about the lower value. Relations (1.9) and (2.3) immediately yield that for every partition π of $[0,T]$ we have $v_\pi \leq V$ on $\mathbf{R}^N \times [0,T]$, and, in view of Proposition 2.5, $v \leq V$ on $\mathbf{R}^N \times [0,T]$. But $V \leq v$ on $\mathbf{R}^N \times [0,T]$, thus the result. If the Isaacs' condition holds, then equations (0.5) and (0.6) coincide, therefore V and U must be identical by the uniqueness of viscosity solutions (cf. Appendix). □

We conclude this section by showing that U and V satisfy the principle of dynamic programming.

Proof of Theorem 1.6. Here we prove only (1.9), and (1.10) follows in exactly the same way. Let $\tau \in (0,T]$ be fixed. For $t \in [0,T]$ let $\tilde{V}(x,t)$ denote the right hand side of (1.9). It is immediate that it suffices to consider in (1.9) controls $Y.$ and strategies β defined on $[t,\tau]$ instead of $[t,T]$. By Theorem 2.6 it follows that $\tilde{V}$ is the unique viscosity solution of (0.5) in $\mathbf{R}^N \times [0,\tau]$ with $\tilde{V}(x,\tau) = V(x,\tau)$. Since V is also a viscosity solution of the Cauchy problem, the uniqueness yields $\tilde{V} = V$. □

Section 3. In this section we follow a different approach about obtaining the value functions of the (SDG) (0.1)–(0.3) and the viscosity solutions of (0.5) and (0.6). This approach amounts to fully discretizing time and the dynamics (0.1). By combining the methods of [10] and [26] we show that the viscosity solutions to the Bellman-Isaacs' equations (0.5), (0.6) can be obtained as limits of solutions to their discrete counterparts. Here we only consider the lower stochastic differential game; the case of the upper stochastic differential game is treated similarly. Finally, for simplicity, we assume that f, h and σ in (0.1), (0.3) are s-independent. Let $\pi = \{0 = t_0 < \cdots < t_m = T\}$ be a partition of $[0,T]$. The discrete analogue of (0.1), (0.2) is the difference equation

$$
\begin{cases} X_{k+1} = X_k + \rho_k f(X_k,Y_k,Z_k) \\ \qquad\qquad + \; \rho_k^{1/2}\sigma(X_k,Y_k,Z_k)\eta_k \quad (k = i_0,\dots,m-1) \\ X_t = x, \end{cases} \tag{3.1}
$$

where $i_0 \in \{0,\dots,m-1\}$ is such that $t \in [t_{i_0},t_{i_0+1})$, $\rho_{i_0} = t_{i_0+1} - t$, $\rho_k = t_{k+1} - t_k$ for $k = i_0+1,\dots,m-1$ and $\eta_0,\dots,\eta_{m-1}$ are independent identically distributed random vectors of dimension M with $E\eta_\ell = 0$, $E\eta_\ell\eta_\ell' =$ identity. To avoid any measurability issues, we assume that each η_ℓ has finitely many values. For example, the components $\eta_{\ell j}$, $j = 1,\dots,M$, may be independent random variables each equal to 1 or -1 with probability $\frac{1}{2}$. The controls Y_k, Z_k at the k^{th} step are chosen knowing the opponents' previous choices and the previous random inputs. The minimizing player also knows Y_k before choosing Z_k.

For any $\psi \in C_b^{0,1}(\mathbf{R}^N)$, $\rho > 0$ and random variable η with the same distribution as the η_k we consider the function

$$\tilde{F}(\rho)\psi(x) = \max_{y\in\mathcal{Y}} \min_{z\in\mathcal{Z}} \Big\{ E\psi\big(x+\rho f(x,y,z)+\rho^{1/2}\sigma(x,y,z)\eta\big) + \rho h(x,y,z)\Big\}. \tag{3.2}$$

It is clear that, in view of (1.1), $\tilde{F}(\rho)$ is a self map of $C_b^{0,1}(\mathbf{R}^N)$. Following [26] we define the function $\tilde{v}_\pi : \mathbf{R}^N \times [0,T] \to \mathbf{R}$ by

$$\tilde{v}_\pi(x,t) = \tilde{F}(t_{i_0+1}-t) \prod_{k=t_{i_0+1}}^{m-1} \tilde{F}(t_{k+1}-t_k) g(x). \tag{3.3}$$

It is not difficult to show that $\tilde{v}_\pi$ is the value of the lower discretized game [10]. Equation (3.3) expresses the dynamic programming principle for this game. It is then more or less immediate by either probabilistic or analytic methods that there exists a constant K such that for every x, $\hat{x} \in \mathbf{R}^N$ and t, $\hat{t} \in [0,T]$

$$|\tilde{v}_\pi(x,t)| \le K \quad \text{and} \quad |\tilde{v}_\pi(x,t) - \tilde{v}_\pi(\hat{x},\hat{t})| \le K\big(|x-\hat{x}| + |t-\hat{t}|^{1/2}\big); \tag{3.4}$$

therefore the family $\{\tilde{v}_\pi\}$ is precompact.

Theorem 3.1. *The limit $v = \lim_{|\pi|\to 0} \tilde{v}_\pi$ exists locally uniformly in $\mathbf{R}^N[0,T]$ and it is the unique viscosity solution of* (0.5).

The proof of Theorem 3.1 follows very closely the proof of the analogous results of [26] and Proposition 2.5. It is based on the following Lemma which establishes the fact that $\tilde{F}$ behaves as a generator. Here we only prove this Lemma, as the Theorem follows in exactly the same way as Proposition 2.5.

Lemma 3.2. *For any φ smooth and H^- given by* (0.7), *we have*

$$\lim_{\rho\to 0} \left\| \frac{\tilde{F}(\rho)\varphi - \varphi}{\rho} - H^-(D^2\varphi, D\varphi, \cdot) \right\| = 0.$$

Proof. We expand $\varphi(x+\xi)$ about $\xi = 0$ by Taylor's formula and set $\xi = \rho f(x,y,z) + \rho^{1/2}\sigma(x,y,z)\eta$. Since $E\eta = 0$, $E\eta\eta' = I$, we obtain

$$E\varphi(x+\xi) = \varphi(x) + \frac{\rho}{2}\operatorname{trace}\big(a(x,y,z)D^2\varphi(x)\big) + \rho f(x,y,z)D\varphi(x) + o(\rho)$$

where $\rho^{-1}o(\rho) \to 0$ as $\rho \to 0$ uniformly with respect to x,y and z. We then have

$$\tilde{F}(\rho)\varphi(x) = \varphi(x) + \rho H^-\big(D^2\varphi(x), D\varphi(x), x\big) + o(\rho). \qquad \square$$

Appendix. Here we recall the definition of viscosity solution as well as some very recent comparison and uniqueness results. For definiteness we consider the problem

$$\begin{cases} u_t + H(D^2u, Du, x, t) = 0 & \text{in } \mathbf{R}^N \times [0,T) \\ u = g & \text{on } \mathbf{R}^N \times \{T\} \end{cases}. \tag{A1}$$

In this note, we take either $H = H^-$ or $H = H^+$, defined by (0.7) or (0.8) respectively.

Definition. A continuous function $U : \mathbf{R}^N \times [0,T] \to \mathbf{R}$ is a viscosity subsolution (resp. supersolution) of (A1) if

$$u \le g \qquad \text{on } \mathbf{R}^N \times \{T\} \tag{A2}$$

(resp. (A3))
$$u \ge g \qquad \text{on } \mathbf{R}^N \times \{T\}$$

and

$$\varphi_t(x,t) + H\big(D^2\varphi(x,t), D\varphi(x,t), x, t\big) \ge 0 \tag{A4}$$

(resp. (A5))
$$\varphi_t(x,t) + H\big(D^2\varphi(x,t), D\varphi(x,t), x, t\big) \le 0$$

for every smooth function φ and any local maximum (resp. minimum) (x,t) of $U - \varphi$. We say that U is a viscosity solution of (A1) if it is both sub- and supersolution of (A1).

The most general uniqueness result for viscosity solutions of (0.5) and (0.6) is an immediate consequence of the following comparison principle:

Theorem A.1 ([16], [17]). *Assume that the functions σ, f, h and g are bounded and Lipschitz continuous. If v and $\tilde v$ (resp. u and $\tilde u$) are a viscosity subsolution and supersolution of* (0.5) *(resp.* (0.6)*) with initial data g and $\tilde g$ and if $g \le \tilde g$ on $\mathbf{R}^N \times \{T\}$, then $v \le \tilde v$ (resp. $u \le \tilde u$) on $\mathbf{R}^N \times [0,T]$.*

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