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# Diffusion Processes with Boundary Conditions*

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## 1. Introduction

In this paper we adapt the approach that we took in [8] to the study of diffusions with boundary conditions. In [8] the idea was to describe the diffusion $P$ corresponding to

$$L = \frac{1}{2}\sum_{i,j} a_{ij}(t, x)\frac{\partial^2}{\partial x_i\,\partial x_j} + \sum b_j(t, x)\frac{\partial}{\partial x_j}$$

by saying that $P$ is determined by an initial condition plus the condition that

$$f(t, x(t)) - \int_0^t \left(\frac{\partial f}{\partial u} + Lf\right)(u, x(u))\,du$$

is a $P$-martingale for all smooth $f$ with compact support. To explain the analogous description for a diffusion with boundary conditions, suppose we consider the reflecting Brownian motion in a smooth region $G$. Then we want to describe $P$ by an initial condition plus the condition that

$$f(t, x(t)) - \int_0^t \left(\frac{\partial f}{\partial u} + \frac{1}{2}\Delta f\right)(u, x(u))\,du$$

is a $P$-submartingale whenever $f$ is a smooth function whose inner normal derivative on $\partial G$ is non-negative.

The class of problems that we have treated by this method is as follows. The region $G$ is an open set in $R^d$ such that $G = \{x \in R^d : \phi(x) > 0\}$ and $\partial G = \{x \in R^d : \phi(x) = 0\}$, where $\phi$ is bounded, has two bounded continuous derivatives on $R^d$ and $|\nabla\phi| \geqq 1$ on $\partial G$. In $G$ we are given bounded continuous diffusion coefficients $a_{ij}(t, x)$, which are strictly elliptic, and bounded measurable drift coefficients $b_i(t, x)$. On $\partial G$ there is a non-negative, bounded, Lipschitz continuous function $\rho(t, x)$ and a bounded Lipschitz continuous vector field $\gamma(t, x)$ satisfying $\langle\gamma, \nabla\phi\rangle \geqq \beta > 0$. If $\rho$ is either identically zero

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or uniformly positive, then we can show that for each $x \in \bar{G}$ there is one and only one $P$ on $C([0, \infty), \bar{G})$ such that $P(x(0) = x) = 1$ and

$$f(t, x(t)) - \int_0^t \left(\frac{\partial f}{\partial u} + Lf\right)(u, x(u))\chi_G(x(u))\, du$$

is a $P$-submartingale whenever $f$ is a smooth function with compact support and satisfies

$$\rho \frac{\partial f}{\partial t} + \langle \gamma, \nabla f \rangle \geqq 0 \quad \text{on} \quad \partial G .$$

Moreover, if $P$ is such a measure, then there is a continuous non-decreasing process $\xi(t)$ which increases only when $x(t) \in \partial G$ and has the property that

$$f(t, x(t)) - \int_0^t \chi_G\left(\frac{\partial f}{\partial u} + Lf\right)(u, x(u))\, du - \int_0^t \left(\rho \frac{\partial f}{\partial u} + \langle \gamma, \nabla f \rangle\right)(u, x(u))\, d\xi(u)$$

is a $P$-martingale for all smooth $f$. In the case when the coefficients are independent of time, we can handle general non-negative $\rho$.

As a consequence of our uniqueness result, we are able to prove an invariance principle for Markov chains converging to diffusions with boundary conditions. The conditions for such convergence coincide with the intuitively natural ones.

## 2. Some Martingale Manipulations

This section contains an assortment of results which are in one way or another related to martingales.

LEMMA 2.1. *Let* $\langle \Omega, \mathcal{M}, P \rangle$ *be a probability space and let* $\{\mathcal{M}_t : t \geqq 0\}$ *be a non-decreasing family of sub σ-algebras of* $\mathcal{M}$. *Suppose* $\phi$ *and* $\psi$ *are complex-valued functions on* $[0, \infty) \times \Omega$ *which are non-anticipating with respect to the* $\mathcal{M}_t$. *Further, assume that*

(i) $\phi(t)$ *is a continuous martingale which is uniformly bounded on finite time intervals,*

(ii) $\psi(t)$ *is a continuous function of bounded variation such that for each* $t$

$$E[|\psi|\,(t)] < \infty ,$$

*where* $|\psi|\,(t)$ *stands for the total variation of* $\psi(s)$ *on* $[0, t]$.

*Under these conditions,*

$$\phi(t)\psi(t) - \int_0^t \phi(u)\, d\psi(u)$$

*is a martingale.*



Proof: Consider

$$\begin{aligned} E\Big[\phi(t)\psi(t) - \phi(s)\psi(s) - \int_s^t \phi(u)\psi(u) \mid \mathcal{M}_s\Big] &= E\Big[\int_s^t (\phi(t) - \phi(u))\, d\psi(u) \mid \mathcal{M}_s\Big] + E[(\phi(t) - \phi(s))\psi(s) \mid \mathcal{M}_s] \\ &= I_1 + I_2 . \end{aligned}$$

Clearly, $I_2 = 0$. To prove that $I_1 = 0$, we approximate

$$\int_s^t (\phi(t) - \phi(u))\, d\psi(u)$$

by Riemann sums of the form

$$\sum_1^n (\phi(t) - \phi(u_j))(\psi(u_j) - \psi(u_{j-1})) ,$$

where $s = u_0 < \cdots < u_n$; q.e.d.

From now on we assume that $\Omega$ is the space of continuous functions on $[0, \infty)$ with values in $R^d$. We denote by $x(t, \omega)$ the value of the function $\omega$ at time $t$. $\Omega$ will be viewed as a complete separable metric space with uniform convergence on compact time intervals defining the basic topology. $\mathcal{M}_t^s$ will be the σ-field generated by $x(u, \omega)$ for $s \leqq u \leqq t$. If $s = 0$, $\mathcal{M}_t^s$ will be denoted by $\mathcal{M}_t$. If $t = \infty$, $\mathcal{M}_t^s$ will be denoted by $\mathcal{M}^s$.

THEOREM 2.1. *Let* $a : [t_0, \infty) \times \Omega \to S_N$[1] *and* $b : [t_0, \infty) \times \Omega \to R^N$ *be bounded non-anticipating functions. Define*

$$K_u = \tfrac{1}{2} \sum_{i,j=1}^N a_{ij}(u) \frac{\partial^2}{\partial x_i\, \partial x_j} + \sum_{i=1}^N b_i(u) \frac{\partial}{\partial x_i} .$$

*Let* $P$ *be a probability measure on* $\langle \Omega, \mathcal{M}^{t_0} \rangle$, *and suppose that* $\xi : [t_0, \infty) \times \Omega \to R^1$ *is a continuous, non-decreasing non-anticipating function satisfying:*

(i) $\int_{t_0}^t \langle \theta, a(u)\theta \rangle\, d\xi(u)$ *is bounded for each* $t \geqq t_0$ *and* $\theta \in R^N$,

(ii) $E^P[e^{\lambda(\xi(t) - \xi(t_0))}] < \infty$ *for all* $\lambda \geqq 0$ *and* $t \geqq t_0$.

[1] $S_N$ denotes the class of symmetric, non-negative definite $N \times N$-matrices. $S_N^+$ is the subclass of $S_N$ consisting of positive definite matrices.

*Finally, let* $\alpha : [t_0, \infty) \times \Omega \to R^N$ *be a continuous, non-anticipating function. Then the following are equivalent*:

(a) *for all* $f \in C_0^\infty(R^N)$,[2]

$$f(\alpha(t)) - \int_{t_0}^t K_u f(\alpha(u))\, d\xi(u)$$

*is a P-martingale,*

(b) *for all* $f \in C_b^{1,2}([t_0, \infty] \times R^N)$,[3]

$$f(t, \alpha(t)) - \int_{t_0}^t f_u(u, \alpha(u))\, du - \int_{t_0}^t K_u f(u, \alpha(u))\, d\xi(u) \quad ^4$$

*is a P-martingale,*

(c) *for all* $f \in C_b^{1,2}([t_0, \infty) \times R^N)$ *which are uniformly positive,*

$$f(t, \alpha(t)) \exp\left\{-\int_{t_0}^t \frac{f_u(u, \alpha(u))}{f(u, \alpha(u))}\, du - \int_{t_0}^t \frac{K_u f(u, \alpha(u))}{f(u, \alpha(u))}\, d\xi(u)\right\}$$

*is a P-martingale,*

(d) *for all* $g \in C_b^{1,2}([t_0, \infty) \times R^N$ *and* $\theta \in R^N$,

$$\begin{aligned}\exp\Big\{g(t, \alpha(t)) - g(t_0, \alpha(t_0)) + \langle\theta, \alpha(t) - \alpha(t_0)\rangle - \int_{t_0}^t g_u(u, \alpha(u))\, du \\ - \tfrac{1}{2}\int_{t_0}^t \langle\nabla g(u, \alpha(u)), a(u)\nabla g(u, \alpha(u))\rangle\, d\xi(u) \\ - \int_{t_0}^t \langle\theta, a(u)\nabla g(u, \alpha(u))\rangle\, d\xi(u) \\ - \tfrac{1}{2}\int_{t_0}^t \langle\theta, a(u)\theta\rangle\, d\xi(u) - \int_{t_0}^t \langle\theta, b(u)\rangle\, d\xi(u)\Big\}\end{aligned}$$

*is a P-martingale,*

(e) *for all* $\theta \in R^N$, *the function* $X_\theta^{t_0}(t)$ *defined by*

$$X_\theta^{t_0}(t) = \exp\left\{\langle\theta, \alpha(t) - \alpha(t_0)\rangle - \tfrac{1}{2}\int_{t_0}^t \langle\theta, a(u)\theta\rangle\, d\xi(u) - \int_{t_0}^t \langle\theta, b(u)\rangle\, d\xi(u)\right\}$$

*is a P-martingale,*

[2] $C_0^\infty(R^N)$ is the class of infinitely differentiable functions on $R^N$ having compact support.

[3] $C_b^{1,2}([t_0, \infty) \times R^N)$ denotes the class of functions on $[t_0, \infty) \times R^N$ which together with their first $t$-derivative and first two $x$-derivatives are bounded and continuous.

[4] $f_u$ denotes $\partial f/\partial u$.



(f) *for all* $\theta \in R^N$, *the function* $X_{i\theta}^{t_0}$ *defined by*

$$X_{i\theta}^{t_0}(t) = \exp\left\{i\langle\theta, \alpha(t) - \alpha(t_0)\rangle + \tfrac{1}{2}\int_{t_0}^t \langle\theta, a(u)\theta\rangle\, d\xi(u) - i\int_{t_0}^t \langle\theta, b(u)\rangle\, d\xi(u)\right\}$$

*is a P-martingale.*

*Moreover,* (e) *implies that*

$$\text{(2.1)} \qquad P\left(\sup_{t_0 \leqq t \leqq T}\left|\alpha(t) - \alpha(t_0) - \int_{t_0}^t b(u)\, d\xi(u)\right| \geqq R\right) \leqq 2Ne^{-R^2/2A_T N^{1/2}},$$

*where*

$$A_T = \sup_{\substack{\theta \in R^N \\ \omega \in \Omega}} \frac{1}{|\theta|^2}\int_{t_0}^T \langle\theta, a(u, \omega)\theta\rangle\, d\xi(u).$$

Proof: We may take $t_0 = 0$. Assume (a) and observe that it suffices to prove (b) for $f \in C_0^\infty([0, \infty) \times R^N)$. Given such an $f$, we have

$$\begin{aligned}E[f(t, \alpha(t)) - f(s, \alpha(s)) \mid \mathscr{M}_s] \\ = E\left[\int_s^t f_u(u, \alpha(u))\, du + \int_s^t K_u f(u, \alpha(u))\, d\xi(u) \;\Big|\; \mathscr{M}_s\right] \\ + E\left[\int_s^t (f_u(u, \alpha(t)) - f_u(u, \alpha(u)))\, du \;\Big|\; \mathscr{M}_s\right] \\ + E\left[\int_s^t (K_u f(s, \alpha(u)) - K_u f(u, \alpha(u)))\, d\xi(u) \;\Big|\; \mathscr{M}_s\right] \\ = I_1 + I_2 + I_3.\end{aligned}$$

But

$$\begin{aligned}I_2 &= E\left[\int_s^t du \int_u^t d\xi(v) K_v f_u(u, \alpha(v)) \;\Big|\; \mathscr{M}_s\right] \\ &= E\left[\int_s^t d\xi(u)\int_s^u dv K_u f_v(v, \alpha(u)) \;\Big|\; \mathscr{M}_s\right] \\ &= E\left[\int_s^t K_u f(u, \alpha(u)) - K_u f(s, \alpha(u)) \;\Big|\; \mathscr{M}_s\right] \\ &= -I_3.\end{aligned}$$

This proves that $I_1 = E[f(t, \alpha(t)) - f(s, \alpha(s)) \mid \mathscr{M}_s^0]$, and therefore

$$f(t, \alpha(t)) - \int_0^t f_u(u, \alpha(u))\, du - \int_0^t K_u f(u, \alpha(u))\, d\xi(u)$$

is a $P$-martingale. Hence (a) implies (b).

That (b) implies (c) follows from Lemma 2.1. Indeed, simply take

$$\phi(t) = f(t, \alpha(t)) - \int_0^t f_u(u, \alpha(u))\, du - \int_0^t K_u f(u, \alpha(u))\, d\xi(u)$$

and

$$\psi(t) = \exp\left\{-\int_0^t \frac{f_u(u, \alpha(u))}{f(u, \alpha(u))}\, du - \int_0^t \frac{K_u f(u, \alpha(u))}{f(u, \alpha(u))}\, d\xi(u)\right\},$$

where $f \in C_b^{1,2}([0, \infty) \times R^N)$ is uniformly positive.

Assume (c) and let $\theta \in R^N$ and $g \in C_b^{1,2}([0, \infty) \times R^N)$ be given. Choose $\phi_n' \in C_0^\infty(R')$ so that $0 \leqq \phi_n' \leqq 1$, $\phi_n'(s) = 1$ for $|s| \leqq n$, and $\phi_n'(s) = 0$ for $|s| \geqq n + 1$. Take $\phi_n(t) = \int_0^t \phi_n'(s)\, ds$, define $\Phi_n$ to be the $R^N$-valued function whose $i$-th component is $\phi_n(x_i)$, and set

$$f_n(t, x) = \exp\{g(t, x) + \langle\theta, \Phi_n(x)\rangle\}.$$

Then $f_n \in C_b^{1,2}([0, \infty) \times R^N)$, $f_n$ is uniformly positive, and

$$\frac{K_u f_n(u, x)}{f_n(u, x)} = K_u g + \tfrac{1}{2}\langle\nabla g, a\nabla g\rangle + \langle\theta\Phi_n', a\nabla g\rangle + \tfrac{1}{2}\langle\theta\Phi_n', a\Phi_n'\rangle + \langle\theta\Phi_n', b\rangle,$$

$$\frac{K_u f_n^2(u, x)}{f_n^2(u, x)} = 2K_u g + 2\langle\nabla g, a\nabla g\rangle + 4\langle\theta\Phi_n', a\nabla g\rangle + 2\langle\theta\Phi_n', a\theta\Phi_n'\rangle + 2\langle\theta\Phi_n', b\rangle,$$

where $\theta\Phi_n'(x)$ is the $N$-vector whose $i$-th component is $\theta_i\Phi_n'(x_i)$. Thus,

$$\begin{aligned}
&\int_0^t \frac{(f_n^2)_u(u, \alpha(u))}{f_n^2(u, \alpha(u))}\, du + \int_0^t \frac{K_u f_n^2(u, \alpha(u))}{f_n^2(u, \alpha(u))}\, d\xi(u) \\
&\qquad - 2\int_0^t \frac{(f_n)_u(u, \alpha(u))}{f_n(u, \alpha(u))}\, du - \int_0^t \frac{K_u f_n(u, \alpha(u))}{f_n(u, \alpha(u))}\, d\xi(u) \\
&= \int_0^t \langle\nabla g, a\nabla g\rangle\, d\xi(u) + 2\int_0^t \langle\theta\Phi_n', a\nabla g\rangle\, d\xi(u) + \int_0^t \langle\theta\Phi_n', a\theta\Phi_n'\rangle\, d\xi(u) \\
&\leqq C,
\end{aligned}$$



where $C$ is independent of $n$. Using this fact, we now see that

$$\begin{aligned}
&E\left[\left(\frac{f_n(t, \alpha(t))}{f_n(0, \alpha(0))} \exp\left\{-\int_0^t \frac{(f_n)_u(u, \alpha(u))}{f_n(u, \alpha(u))}\, du - \int_0^t \frac{K_u f_n(u, \alpha(u))}{f_n(u, \alpha(u))}\, d\xi(u)\right\}\right)^2\right] \\
&\quad \leqq e^C E\left[\frac{f_n^2(t, \alpha(t))}{f_n^2(0, \alpha(0))} \exp\left\{-\int_0^t \frac{(f_n^2)_u(u, \alpha(u))}{f_n^2(u, \alpha(u))}\, du - \int_0^t \frac{K_u f_n^2(u, \alpha(u))}{f_n^2(u, \alpha(u))}\, d\xi(u)\right\}\right] \\
&\quad = e^C,
\end{aligned}$$

because the integrand in the last expression is a $P$-martingale. It follows that the sequence

$$X_n(t) = \frac{f_n(t, \alpha(u))}{f_n(0, \alpha(0))} \exp\left\{-\int_0^t \frac{(f_n)_u(u, \alpha(u))}{f_n(u, \alpha(u))}\, du - \int_0^t \frac{K_u f_n(u, \alpha(u))}{f_n(u, \alpha(u))}\, d\xi(u)\right\}$$

is uniformly $P$-integrable. Moreover, $X_n(t)$ tends to the expression in (d) point-wise, and therefore (d) follows from (c).

Obviously (e) is just the special case of (d) when $g \equiv 0$. If (e) obtains, then, by the argument used in Lemma 3.1 of [8], we get the estimate (2.1). From this it is clear that $E[X_\theta^0(t)Y]$ is an analytic function of $\theta \in C^N$ for each bounded measurable function $Y$. Thus the equality

$$E[X_\theta^0(t) \mid \mathscr{M}_s^0] = X_\theta^0(s) \qquad \text{a.s. } P$$

for $\theta \in R^N$ extends to all $\theta \in C^N$.

It remains to show that (f) implies (a). To that end, take $\phi(t) = X_{i\theta}^0(t)$ and

$$\psi(t) = \exp\left\{-\tfrac{1}{2}\int_0^t \langle\theta, a(u)\theta\rangle\, d\xi(u) + i\int_0^t \langle\theta, b(u)\rangle\, d\xi(u)\right\}.$$

Applying Lemma 2.1 to $\phi$ and $\psi$, we see that

$$\exp\{i\langle\theta, \alpha(t) - \alpha(0)\rangle\} + \int_0^t (\tfrac{1}{2}\langle\theta, a(u)\theta\rangle - i\langle\theta, b(u)\rangle) \exp\{i\langle\theta, \alpha(t) - \alpha(0)\rangle\}\, d\xi(u)$$

is a $P$-martingale for all $\theta \in R^N$. Hence,

$$f(\alpha(t)) - \int_0^t K_u f(\alpha(u))\, d\xi(u)$$

is a $P$-martingale for all $f$ which are finite linear combinations of functions of the form $\exp\{i\langle\theta, x\rangle\}$, $\theta \in R^N$. It is easy to pass from this to (a); q.e.d.

We continue using the notation of Theorem 2.1 with $t_0 = 0$. Assume that

$\xi(t)$ is such that

$$\int_0^t \langle \theta, a(u)\theta \rangle \, d\xi(u) = \int_0^t \langle \theta, a(u)\theta \rangle \, du , \qquad t \geqq 0 , \quad \theta \in R^1 , \tag{2.2}$$

and let $P$ satisfy (a). From (e) we see that

$$X_\theta(t) = \exp\left\{\langle \theta, \bar{\alpha}(t) - \bar{\alpha}(0)\rangle - \tfrac{1}{2}\int_0^t \langle \theta, a(u)\theta\rangle \, du\right\}$$

is a $P$-martingale for all $\theta \in R^N$, where

$$\bar{\alpha}(t) = \alpha(t) - \int_0^t b(u) \, d\xi(u) .$$

In view of Theorem 3.2 in [8], we can now define a stochastic integral with respect to $\bar{\alpha}(t)$. Indeed, given a non-anticipating function $\theta : [0, \infty) \times \Omega \to R^N$ satisfying $E\left[\int_0^t |\theta(u)|^2 \, du\right] < \infty$, the expression $\int_0^t \langle \theta(u), d\bar{\alpha}(u)\rangle$ is well-defined and enjoys all the properties listed in that theorem (for a summary of these properties see Theorem 2.2 below). We now define stochastic integrals with respect to $\alpha(t)$ by setting

$$\int_0^t \langle \theta(u), d\alpha(u)\rangle = \int_0^t \langle \theta(u), d\bar{\alpha}(u)\rangle + \int_0^t \langle \theta(u), b(u)\rangle \, d\xi(u)$$

when the terms on the right are defined. In particular, $\int_0^t \langle \theta(u), d\alpha(u)\rangle$ is well-defined for bounded non-anticipating $\theta : [0, \infty) \times \Omega \to R^N$.

We want next to develop an Itô formula for $d\alpha(u)$-stochastic integrals. The method which we employed in [8] was to relate $d\alpha(u)$ integrals to Brownian integrals (cf. Theorem 3.3 of [8]). However, that approach breaks down here because we do not want to assume that $a(u)$ is strictly positive definite. Thus, in order to construct a Brownian motion $\beta(t)$ such that $d\bar{\alpha}(t) = a^{1/2}(t) \, d\beta(t)$, we would be forced to go outside our original sample space (cf. Theorem 5.3. in Doob [2]). We could avoid such problems by invoking the extremely general results along these lines of H. Kunita and S. Watanabe [4]. However, the reasoning which seems most consistent with our overall philosophy is the following.

Let $g \in C_b^{1,2}([0, \infty) \times R^N)$, and define $\tilde{\alpha} : [0, \infty) \times \Omega \to R^{N+1}$ by the recipe

$$\tilde{\alpha}_i(t) = \bar{\alpha}_i(t) , \qquad 1 \leqq i \leqq N ,$$

$$\tilde{\alpha}_{N+1}(t) = g(t, \alpha(t)) - \int_0^t g_u(u, \alpha(u)) \, du - \int_0^t K_u \, g(u, \alpha(u)) \, d\xi(u) .$$



Define $\tilde{a} : [0, \infty) \times \Omega \to S_{N+1}$ by

$$\tilde{a}_{ij}(t) = a_{ij}(t) , \qquad 1 \leqq i, j \leqq N ,$$

$$\tilde{a}_{N+1,i}(t) = \tilde{a}_{i,N+1}(t) = (a(t)\nabla g(t, \alpha(t)))_i , \qquad 1 \leqq i \leqq N ,$$

$$\tilde{a}_{N+1,N+1}(t) = \langle \nabla f(t, \alpha(t)), a(t)\nabla f(t, \alpha(t))\rangle .$$

Then, by (e),

$$\tilde{X}_{\tilde{\theta}}(t) = \exp\left\{\langle \tilde{\theta}\, \tilde{\alpha}(t) - \tilde{\alpha}(0)\rangle - \tfrac{1}{2}\int_0^t \langle \tilde{\theta}, \tilde{a}(u)\tilde{\theta}\rangle \, du\right\}$$

is a $P$-martingale for all $\tilde{\theta} \in R^{N+1}$. Hence we can define $d\tilde{\alpha}(t)$-stochastic integrals, and, by Theorem 3.2 of [8], we know that

$$\exp\left\{\int_0^t \langle \tilde{\theta}(u), d\tilde{\alpha}(u)\rangle - \frac{1}{2}\int_0^t \langle \tilde{\theta}(u), \tilde{a}(u)\tilde{\theta}(u)\rangle \, du\right\}$$

is a $P$-martingale for bounded non-anticipating $\tilde{\theta} : [0, \infty) \times \Omega \to R^{N+1}$. In particular, if

$$\tilde{\theta}_i(u) = -\lambda \frac{\partial g}{\partial x_i}(u, \alpha(u))$$

for $1 \leqq i \leqq N$ and $\tilde{\theta}_{N+1}(u) = \lambda$, then

$$\exp\left\{\lambda(g(t, \alpha(t)) - g(0, \alpha(0)) - \int_0^t g_u(u, \alpha(u)) \, du - \int_0^t K_u \, g(u, \alpha(u)) \, d\xi(u) - \int_0^t \langle \nabla g(u, \alpha(u)), d\bar{\alpha}(u)\rangle)\right\}$$

is a $P$-martingale. Since this is true for all $\lambda \in R^1$, we conclude that

$$g(t, \alpha(t)) - g(0, \alpha(0)) = \int_0^t \langle \nabla g(u, \alpha(u)), d\bar{\alpha}(u)\rangle + \int_0^t g_u(u, \alpha(u)) \, du + \int_0^t K_u \, g(u, \alpha(u)) \, d\xi(u) . \tag{2.3}$$

If $K_u^0$ denotes the second order part of $K_u$, then (2.3) can be rewritten as

$$g(t, \alpha(t)) - g(0, \alpha(0)) = \int_0^t \langle \nabla g(u, \alpha(u)), d\alpha(u)\rangle + \int_0^t (g_n + K_u^0 g)(u, \alpha(u)) \, du . \tag{2.4}$$

For future reference, we summarize these findings and some of their consequences in the following theorem, using the same notation as in Theorem 2.1.

THEOREM 2.2. *In addition to the assumptions in Theorem* 2.1, *let us assume that*

$$\int_{t_0}^{t}\langle\theta, a(u)\theta\rangle\, d\xi(u) = \int_{t_0}^{t}\langle\theta, a(u)\theta\rangle\, du \qquad for \qquad t \geqq t_0\,, \quad \theta \in R^N\,, \tag{2.5}$$

*and that $P$ satisfies any of the equivalent martingale formulations* (a)–(f) *of that theorem. Then stochastic integrals with respect to $\alpha(t)$ are well-defined for non-anticipating $\theta : [t_0, \infty) \times \Omega \to R^N$ satisfying*

$$E\left[\int_{t_0}^{t}|\theta(u)|^2\, du + \int_{t_0}^{t}|\langle\theta(u), b(u)\rangle|\, d\xi(u)\right] < \infty\,. \tag{2.6}$$

*The $d\alpha(t)$-stochastic integrals have the following properties:*

(i) $\int_{t_1}^{t_2}\theta(u)\, d\alpha(u) + \int_{t_2}^{t_3}\theta(u)\, d\alpha(u) = \int_{t_1}^{t_3}\theta(u)\, d\alpha(u), \qquad t_0 \leqq t_1 \leqq t_2 \leqq t_3,$

(ii) $\int_{t_0}^{t}\theta(u)\, d\alpha(u)$ *is continuous in* $t \geqq t_0$ *and*

$$\int_{t_0}^{t}\theta(u)\, d\alpha(u) - \int_{t_0}^{t}\langle\theta(u), b(u)\rangle\, d\xi(u)$$

*is a P-martingale,*

(iii) $$E\left[\left(\int_{t_1}^{t_2}\theta(u)\, d\alpha(u) - \int_{t_1}^{t_2}\langle\theta(u), b(u)\rangle\, d\xi(u)\right)^2\right] = E\left[\int_{t_1}^{t_2}\langle\theta(u), a(u), \theta(u)\rangle\, du\right],$$

(iv) *if $\theta$ is bounded, then*

$$X_\theta^{t_0}(t) = \exp\left\{\int_{t_0}^{t}\theta(u)\, d\alpha(u) - \int_{t_0}^{t}\langle\theta(u), b(u)\rangle\, d\xi(u) - \tfrac{1}{2}\int_{t_0}^{t}\langle\theta(u), a(u)\theta(u)\rangle\, du\right\}$$

*is a P-martingale,*

(v) $\int_{t_1}^{t_2}\theta_1(u)\, d\alpha(u) + \int_{t_1}^{t_2}\theta_2(u)\, d\alpha(u) = \int_{t_1}^{t_2}(\theta_1(u) + \theta_2(u))\, d\alpha(u)\,,$

(vi) *if $g \in C_b^{1,2}([0, \infty) \times R^N)$, then the Itô formula given in* (2.4) *holds.*



*Finally, if $\tau$ is a $t_0$-stopping time*[5] *and $Q_\omega$ is the regular conditional probability distribution*[6] *of $P$ given $\mathscr{M}_\tau^{t_0}$ (cf. Theorem* 2.1 *of* [8]), *then there is an $\mathscr{N} \in \mathscr{M}_\tau^{t_0}$ such that $P(\mathscr{N}) = 0$ and, for $\omega \notin \mathscr{N}$, $Q_\omega$ satisfies* (a) *of Theorem* 2.1 *above with $\tau(\omega)$ replacing $t_0$.*

Proof: The only assertion that has not been explained is the last. However, since (a) is equivalent to (e) in Theorem 2.1, this assertion follows from Theorem 3.1 of [8].

THEOREM 2.3. *Let $a : [0, \infty) \times R^d \to S_d^+$ be bounded and continuous, and let $b : [0, \infty) \times R^d \to R^d$ and $\gamma : [0, \infty) \times R^d \to R^d$ be bounded and measurable. Suppose $P$ is a probability measure on $(\Omega, \mathscr{M}^{t_0})$ and that $\xi : [0, \infty) \times \Omega \to R^1$ is a continuous, non-decreasing, non-anticipating function which satisfies*

(i) $\xi(t) - \xi(t_0) = \int_{t_0}^{t}\chi_{\partial G}(x(s))\, d\xi(s) \qquad for \qquad t \geqq t_0\,,$

(ii) $E[\exp\{\lambda(\xi(t) - \xi(t_0))\}] < \infty \qquad for \qquad t \geqq t_0 \qquad and \qquad \lambda \geqq 0\,,$

*where $\partial G$ is the boundary of some open region $G \subseteq R^d$. Furthermore, assume that, for $f \in C_0^\infty(R^d)$,*

$$f(x(t)) - \int_{t_0}^{t}\chi_G L_u f(x(u))\, du - \int_{t_0}^{t}\langle\gamma(u, x(u)), \nabla f(x(u))\rangle\, d\xi(u)$$

*is a P-martingale, where*

$$L_u = \tfrac{1}{2}\sum_{i,j=1}^{d} a_{ij}(u, x)\frac{\partial^2}{\partial x_i\, \partial x_j} + \sum_{i=1}^{d} b_i(u, x)\frac{\partial}{\partial x_i}\,.$$

*Finally, let $\tau$ be a bounded $t_0$-stopping time, define*

$$\tau' = \inf\{t \geqq \tau : x(t) \notin G\}\,,$$

*and let $P_\omega$ be the r.c.p.d. of $P$ given $\mathscr{M}_\tau^{t_0}$. Then there is an $\mathscr{N} \in \mathscr{M}_\tau^{t_0}$ such that $P(\mathscr{N}) = 0$ and, for $\omega \notin \mathscr{N}$, $P_\omega$ equals $Q_{\tau(\omega), x(\tau(\omega), \omega)}$ on $\mathscr{M}_{\tau'_\omega}^{\tau(\omega)}$, where $Q_{s,x}$ is the unique solution to the martingale problem for $a$ and $b$ starting from $x$ at time $s$,*[7] *and $\tau'_\omega = \inf\{t \geqq \tau(\omega) : x(t) \notin G\}$.*

Proof: Choose $T \geqq \tau$ and let $P^T$ be the probability measure on $\langle\Omega, \mathscr{M}^{t_0}\rangle$ which equals $P$ on $\mathscr{M}_{\tau'}^{t_0}$ and whose r.c.p.d. given $\mathscr{M}_{\tau' \wedge T}^{t_0}$ is $Q_{\tau' \wedge T, x(\tau' \wedge T)}$ (cf. Lemma 3.6 of [8]). Let $P_\omega^T$ be the r.c.p.d. of $P^T$ given $\mathscr{M}_\tau^{t_0}$. Then, by

[5] A $t_0$-stopping time is a function $\tau : \Omega \to [t_0, \infty)$ such that $\{\tau \leqq t\} \in \mathscr{M}_t^{t_0}$ for $t \geqq t_0$.

[6] In the future we shall use r.c.p.d.

[7] This terminology is explained in [8].

Theorem 2.1 above and Theorem 3.6 of [8], for almost all $\omega$, $P_\omega^T$ solves the martingale problem for $a$ and $b$ starting from $x(\tau(\omega), \omega)$ at time $\tau(\omega)$. Hence, $P_\omega^T = Q_{\tau(\omega), x(\tau(\omega), \omega)}$ on $\mathscr{M}^{\tau(\omega)}$ a.s. $P^T$. Since $P_\omega = P_\omega^T$ a.s. $P$, we have $P_\omega = Q_{\tau(\omega), x(\tau(\omega), \omega)}$ on $\mathscr{M}^{\tau(\omega)}_{\tau'(\omega) \wedge T}$. By letting $T \uparrow \infty$, we get our result.

COROLLARY 2.1. *(The notation is the same as that in Theorem 2.3.) If*

$$f \in W_p^{1,2}([0, \infty) \times R^d)\,,^{8}$$

*where* $p > d + 2$, *and if* $f_u + L_u^0 f = g \in B([0, \infty) \times G)$,[9] *where* $L_u^0$ *is the principal part of* $L_u$, *then for* $T \geqq \tau$

$$f(\tau' \wedge T, x(\tau' \wedge T)) - f(\tau, x(\tau)) = \int_\tau^{\tau' \wedge T} g(u, x(u))\, du + \int_\tau^{\tau' \wedge T} \nabla_x f(u, x(u))\, dx(u) \qquad \text{a.s. } P\,.$$

Proof: In view of Theorem 2.3, it suffices to prove that

$$f(t, x(t)) - f(s, x(s)) = \int_s^t g(u, x(u))\, du + \int_s^t \nabla_x f(u, x(u))\, dx(u) \qquad \text{a.s. } Q_{s,x}\,.$$

This is done in the same way as Theorem 11.2 of [9] was proved.

Given an open set $G$ in $R^d$ and a probability measure $P$ on $\langle \Omega, \mathscr{M}^{t_0} \rangle$, we say that a non-anticipating function $X : [t_0, \infty) \times \Omega \to R^1$ is a *local P-martingale in* $G$ if, for every $t_0$-stopping time $\tau$,

$$X(t \wedge \tau') - X(t \wedge \tau)$$

is a $P$-martingale, where $\tau' = \inf\{t \geqq \tau : x(t) \notin G\}$.

We now introduce the general set-up with which we shall be concerned in the rest of this paper. Let $G$ be a non-empty open set in $R^d$. We assume that there is associated with $G$ a function $\phi$, called the defining function of $G$, such that

(i) $\phi \in C_b^2(R^d)$,

(ii) $G = \{x \in R^d : \phi(x) > 0\}$ and $\partial G = \{x \in R^d : \phi(x) = 0\}$,

(iii) $|\nabla \phi(x)| \geqq 1$ for all $x \in \partial G$.

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8 The space $W_p^{1,2}([0, \infty) \times R^d)$ is the completion of $C_0^\infty([0, \infty) \times R^d)$ with respect to the norm

$$|||f||| = \|f\|_p + \|f_t\|_p + \sum_{i=1}^d \|f_{x_i}\|_p + \sum_{i,j=1}^d \|f_{x_i x_j}\|_p,$$

where $\|\cdot\|_p$ is the ordinary $L_p$ norm.

9 $B([0, \infty) \times G)$ is the class of bounded measurable functions on $[0, \infty) \times G$.



The following quantities will also be given

(i′) $a : [0, \infty) \times G \to S_d^+$ which is bounded and continuous,

(ii′) $b : [0, \infty) \times G \to R^d$ which is bounded and measurable,

(iii′) $\gamma : [0, \infty) \times \partial G \to R^d$ which is bounded, continuous, and satisfies $\langle \gamma(t, x), \nabla \phi(x) \rangle \geqq \beta > 0$ for $t \geqq 0$ and $x \in \partial G$,

(iv′) $\rho : [0, \infty) \times \partial G \to [0, \infty)$ which is bounded and continuous.

Define

$$L_u = \tfrac{1}{2} \sum_{i,j=1}^d a_{ij}(u, x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(u) \frac{\partial}{\partial x_i}$$

and

$$J_u = \sum_{i=1}^d \gamma_i(u, x) \frac{\partial}{\partial x_i}\,.$$

We say that a probability measure $P$ on $\langle \Omega, \mathscr{M}^{t_0} \rangle$ solves the *sub-martingale problem on* $G$ *for coefficients* $a$, $b$, $\gamma$ *and* $\rho$ *if*

(I) $\qquad P(x(t) \in \bar{G}) = 1\,, \qquad t \geqq t_0\,,$

(II) $f(t, x(t)) - \int_{t_0}^t [\chi(f_u\,{}_G + L_u f)](u, x(u))\, du$ is a $P$-submartingale for any $f \in C_0^{1,2}([0, \infty) \times R^d)$ satisfying

$$\rho f_t + J_t f \geqq 0 \qquad \text{on} \qquad [t_0, \infty) \times \partial G\,.$$

Unless it is stated to the contrary, the $P$ in the following is assumed to be such a solution.

LEMMA 2.2. *If* $f \in C_b^{1,2}([0, \infty) \times R^d)$, *then*

$$X_f(t) = f(t, x(t)) - \int_{t_0}^t \chi_G(f_u + L_u f)(u, x(u))\, du$$

*is a local P-martingale on* $G$. *If* $\rho f_t + J_t f \geqq 0$ *on* $[t_0, \infty) \times \partial G$, $X_f(t)$ *is a P-submartingale.*

Proof: Suppose $f$ has compact support in $[t_0, \infty) \times G$. Then

$$\rho f_t + J_t f = 0$$

on $[t_0, \infty) \times \partial G$, and so $X_f(t)$ is a $P$-martingale. In general, choose a non-decreasing sequence of open sets $G_n$ such that

$$\bar{G}_n \subset G_{n+1} \subset \bar{G}_{n+1} \subset \subset G$$

and $G_n \uparrow G$.[10] Choose $\eta_n \in C_0^\infty([0, \infty) \times R^d)$ so that $\eta_n = 1$ on $[0, t_0 + n] \times G_n$ and $\eta_n \equiv 0$ off $[0, t_0 + n + 1] \times G_{n+1}$. Take $f_n = \eta_n f$. Then $X_{f_n}(t)$ is a $P$-martingale. Given a $t_0$-stopping time $\tau$, let

$$\tau_n = \tau \wedge (t_0 + n) \qquad \text{and} \qquad \tau_n' = \inf\{t \geqq \tau_n : x(t) \in G_n\} \wedge (t_0 + n)\,.$$

Then

$$X_{f_n}(t \wedge \tau_n') - X_{f_n}(t \wedge \tau_n)$$

is a $P$-martingale. But

$$X_f(t \wedge \tau_n') - X_f(t \wedge \tau_n) = X_{f_n}(t \wedge \tau_n') - X_{f_n}(t \wedge \tau_n)\,,$$

$\tau_n \wedge t \uparrow \tau \wedge t$, and $\tau_n'$ increases to

$$\tau' \wedge t = \inf\{s \geqq \tau : x(s) \in G\} \wedge t\,.$$

Hence, $X_f(t \wedge \tau') - X_f(t \wedge \tau)$ is a $P$-martingale.

Now suppose that $f \in C_b^{1,2}([0, \infty) \times R^d)$ and that $f_t + J_t f \geqq 0$ on $[t_0, \infty) \times \partial G$. Assume that $f(t, \cdot) \equiv 0$ for $t \geqq T$, where $T > t_0$. For each $N \geqq 1$, choose $\eta_N \in C_0^\infty(R^d)$ such that $0 \leqq \eta_N \leqq 1$, $\eta_N = 1$ on $\{x : |x| \leqq N\}$, $|J_t \eta_N| \leqq 1/N$ on $[0, \infty) \times \partial G$, and all derivatives of $\eta_N$ up to the second order are uniformly bounded. Let $\bar{\eta}_N$ be an element of $C_0^\infty(R^d)$ having the properties

$$0 \leqq \bar{\eta}_N \leqq 1\,,$$

$$\bar{\eta}_N \equiv 1 \qquad \text{on} \qquad \text{supp. } (\eta_N)\,,$$

and all derivatives up to the second order are uniformly bounded. Set

$$f_N = \eta_N \cdot f + \frac{\|f\|}{N} \bar{\eta}_N \cdot \phi\,.$$

Then $f_N$ is in $C_0^{1,2}([0, \infty) \times R^d)$ and $(\rho(\partial/\partial t) + J_t) f_N \geqq 0$ on $[0, \infty) \times \partial G$. Hence, $X_{f_N}$ is a $P$-submartingale. Clearly, $X_{f_N}(t) \to X_f(t)$ boundedly, and so $X_f(t)$ is a $P$-submartingale. Finally, we can drop the assumption that $f(t, \cdot) = 0$ for $t > T$, because we can do the same truncation argument in time; q.e.d.

[10] $K \subset\subset U$ means that $K$ is a compact subset of $U$.



Suppose that $f \in C_b^{1,2}([0, \infty) \times R^d)$ and that $\rho f_t + J_t f \geqq 0$ on $[t_0, \infty) \times \partial G$. Then $X_f(t)$ is a locally bounded, continuous $P$-submartingale. Hence, by the Doob-Meyer decomposition theorem (cf. [10]), there exists an integrable, non-decreasing, non-anticipating continuous function $\xi_f : [t_0, \infty) \times \Omega \to [0, \infty)$ such that $\xi_f(t_0) = 0$ and $X_f(t) - \xi_f(t)$ is a $P$-martingale. In general, if $f \in C_b^{1,2}([0, \infty) \times R^d)$, we can find an $\alpha$ such that

$$\rho \tilde{f}_t + J_t \tilde{f} \geqq 0 \qquad \text{on} \qquad [t_0, \infty) \times \partial G\,,$$

where $\tilde{f} = f + \alpha\phi$. Hence we can choose a $\xi_{\tilde{f}}$ for $\tilde{f}$. If we let $\xi_f = \xi_{\tilde{f}} - \alpha \xi_\phi$, then we see that $\xi_f(t)$ is an non-anticipating continuous function of bounded variation such that $\xi_f(t_0) = 0$, $E[|\xi_f|(t)] < \infty$ for $t \geqq t_0$, and $X_f(t) - \xi_f(t)$ is a $P$-martingale. These remarks show that $C_b^{1,2}([0, \infty) \times R^d)$ is contained in the class $F$ described below.

The class $F$ consists of those functions $f$ which satisfy:

(i) $f \in C_b([t_0, \infty) \times R^d) \cap C_b^{1,1}([t_0, \infty) \times \partial G)$,[11]

(ii) there is a function $Kf \in B([0, \infty) \times G)$ such that

$$X_f(t) = f(t, x(t)) - \int_{t_0}^t \chi_G\, Kf(u, x(u))\, du$$

is a local $P$-martingale in $G$,

(iii) there is a continuous non-anticipating function $\xi_f : [t_0, \infty) \times \Omega \to R$ such that

(a) $\xi_f(t_0) = 0$, $\xi_f(t)$ is of locally bounded variation, and $E[|\xi_f|(t)] < \infty$ for $t \geqq t_0$,

(b) $X_f(t) - \xi_f(t)$ is a $P$-martingale,

(iv) if $g \in C_b^{1,2}([0, \infty) \times R^d)$, then $\tilde{f} = f + g$, satisfies (i), (ii) and (iii), and if $\rho \tilde{f}_t + J_t \tilde{f} \geqq 0$ on $[t_0, \infty) \times \partial G$, then $\xi_{\tilde{f}}(t)$ can be chosen to be non-decreasing.

LEMMA 2.3. *For each $f \in F$, there is at most one $\xi_f$ satisfying* (iii). *Moreover,*

$$\int_{t_0}^{t_1} \chi_G(x(u))\, d|\xi_f|(u) = 0\,.$$

Proof: Suppose $\xi_1$ and $\xi_2$ satisfy (iii) for some $f \in F$. Then $\eta = \xi_1 - \xi_2$

[11] If $f \in C_b^{1,2}[(t_0, \infty) \times \partial G)$, we mean that $f_t$ and $f_{x_i}$, $1 \leqq i \leqq d$, exist and are continuous on $[t_0, \infty) \times \partial G$.

is an integrable, continuous martingale with paths of bounded variation. Hence, by an elementary argument, $\eta(t)$ is almost surely constant, and therefore almost surely zero.

Finally, let $f \in F$. We must show that $\int_{t_0}^{t_1} \chi_G(x(u))\, d\,|\xi_f|\,(u) = 0$. Since $\xi_f(t) = \xi_{f+\alpha\phi}(t) - \xi_{\alpha\phi}(t)$, by choosing large enough $\alpha$, it suffices to treat the case when $\rho f_t + J_t f \geqq 0$ on $[t_0, \infty) \times \partial G$. In this case $\xi_f(t)$ is non-decreasing and so we need only show that $\int_{t_0}^{t_1} \chi_K(x(u))\, d\xi_f(u) = 0$ for all compact $K \subset G$. For such $K$, define

$$\tau_0 = \inf\{t \geqq t_0 : x(t) \in K\} \wedge t_1\,,$$
$$\tau_{2n+1} = \inf\{t \geqq \tau_{2n} : x(t) \in \partial U\} \wedge t_1\,,$$
$$\tau_{2n} = \inf\{t \geqq \tau_{2n-1} : x(t) \in K\} \wedge t_1\,.$$

Then $X_f(\tau_{2n+1} \wedge t) - X_f(\tau_{2n} \wedge t)$ is a $P$-martingale, and therefore so is $\xi_f(\tau_{2n+1} \wedge t) - \xi_f(\tau_{2n} \wedge t)$. Thus, $\xi_f(\tau_{2n+1} \wedge t) = \xi_f(\tau_{2n} \wedge t)$ a.s. $P$, and from this we get

$$\int_{t_0}^{t_1} \chi_K(x(u))\, d\xi_f(u) = 0 \qquad \text{a.s. } P\,.$$

LEMMA 2.4. *Take $t_0 \leqq a < b$. If $f \in F$ and $U$ is an open neighborhood of a point $x \in \partial G$ such that $f \equiv 0$ on $(a, b) \times U$, then*

$$\int_a^b \chi_U(x(u))\, d\,|\xi_f|\,(u) = 0\,.$$

Proof: Let $K \subset\subset U$ and take $a < c < d < b$. Define

$$\tau_0 = \inf\{t \geqq c : x(t) \in K\} \wedge d\,,$$
$$\tau_{2n+1} = \inf\{t \geqq \tau_{2n} : x(t) \in \partial U\} \wedge d\,,$$
$$\tau_{2n} = \inf\{t \geqq \tau_{2n-1} : x(t) \in K\} \wedge d\,.$$

Then,

$$X_f(t \wedge \tau_{2n+1}) - X_f(t \wedge \tau_{2n}) = \int_{t\wedge\tau_{2n}}^{t\wedge\tau_{2n+1}} \chi_G(x(u)) Kf(u, x(u))\, du - (\xi_f(t \wedge \tau_{2n+1}) - \xi_f(t \wedge \tau_{2n}))$$

is a continuous integrable $P$-martingale of bounded variation. Hence,

$$\int_{t\wedge\tau_{2n}}^{t\wedge\tau_{2n+1}} \chi_G(x(u)) Kf(u, x(u))\, du = -(\xi_f(t \wedge \tau_{2n}) - \xi_f(t \wedge \tau_{2n+1}))\,.$$



Since $d\xi_f(t)$ is supported on $\{t : x(t) \in \partial G\}$, it follows that $\xi_f(t \wedge \tau_{2n+1}) = \xi_f(\tau_{2n})$ for $t \geqq \tau_{2n}$. Therefore,

$$\int_c^d \chi_K(x(u))\, d\xi_f(u) = \sum_0^\infty \int_{\tau_{2n}}^{\tau_{2n+1}} \chi_K(x(u))\, d\xi_f(u) = 0\,.$$

LEMMA 2.5. *Take $t_0 \leqq a < b$. Let $f \in F$ and let $U$ be a neighborhood of a point $x \in \partial G$ such that $\rho f_t + J_t f \geqq 0$ on $(a, b) \times (U \cap \partial G)$. Then*

$$\int_a^b \chi_U(x(u))\, d\xi_f(u) \geqq 0\,.$$

Proof: Let $a < c < d < b$ and let $N$ be an open set such that $\bar{N} \subset\subset U$. Choose $\eta \in C_b^\infty([0, \infty) \times R^d)$ such that $0 \leqq \eta \leqq 1$, $\eta = 0$ on $(c, d) \times N$, and $\eta = 1$ off $(a, b) \times U$. Define $\tilde{f} = f + \alpha\eta \cdot \phi$. Then, for larger $\alpha$, $\rho \tilde{f} + J_t \tilde{f} \geqq 0$ on $[t_0, \infty) \times \partial G$ and $\tilde{f} = f$ on $(c, d) \times N$. Thus

$$\int_c^d \chi_N(x(u))\, d\xi_f(u) = \int_c^d \chi_N(x(u))\, d\xi_{\tilde{f}}(u) \geqq 0\,.$$

THEOREM 2.4. *There is a unique, continuous, non-decreasing, non-anticipating function $\xi_0 : [t_0, \infty) \times \Omega \to [0, \infty)$ such that $\xi_0(t_0) = 0$, $E[\xi_0(t)] < \infty$,*

$$\xi_0(t) = \int_{t_0}^t \chi_{\partial G}(x(u))\, d\xi_0(u)$$

*and*

$$f(t, x(t)) - \int_{t_0}^t \chi_G\, Kf(u, x(u))\, du - \int_{t_0}^t (\rho f_u + J_u f)(u, x(u))\, d\xi_0(u)$$

*is a $P$-martingale for all $f \in F$.*

Proof: We first show that if $f \in F$, then $d\xi_f(t)$ is absolutely continuous with respect to $d\xi_\phi(t)$. Indeed, let $\tilde{f} = \alpha\phi - f$. Then, for large $\alpha$, $\rho \tilde{f}_t + J_t \tilde{f} \geqq 0$ on $[t_0, \infty) \times \partial G$, and so $\alpha d\xi_\phi(t) \geqq d\xi_f(t) \geqq -\alpha d\xi_\phi(t)$. Let $d\xi_f(t)/d\xi_\phi(t) = \alpha(t)$. Given $s \in (t_0, \infty)$ and $x \in \partial G$, let

$$\beta = \frac{(\rho f_s + J_s f)(s, x)}{\langle \gamma(s, x), \nabla\phi(x)\rangle}\,.$$

If $\varepsilon > 0$, choose $s \in (a, b) \subset [t_0, \infty)$ and an open set $U$ containing $x$ such that

$$(\beta - \varepsilon)\langle \gamma, \nabla\phi\rangle \leqq (\rho f_u + J_u f) \leqq (\beta + \varepsilon)\langle \gamma, \nabla\phi\rangle$$

in $(a, b) \times U$. Then

$$(\beta - \varepsilon)\int_{a'}^{b'} \chi_U(x(u))\, d\xi_\phi(u) \leqq \int_{a'}^{b'} \chi_U(x(u))\alpha(u)\, d\xi_\phi(u) \leqq (\beta + \varepsilon)\int_{a'}^{b'} \chi_U(x(u))\, d\xi_\phi(u)$$

for $(a', b') \subset (a, b)$ and $U' \subset U$. Hence,

$$(\beta - \varepsilon)\chi_U(x(u)) \leqq \chi_U(x(u))\alpha(u) \leqq (\beta + \varepsilon)\chi_U(x(u)) .$$

It follows that

$$\alpha(u) = \frac{\rho f_u + J_u f}{\langle \gamma, \nabla \phi \rangle}.$$

We now define

$$\xi_0(t) = \int_{t_0}^t \frac{1}{\langle \gamma, \nabla \phi \rangle}\, d\xi_\phi(u) .$$

Clearly $\xi_0(t)$ has the desired properties. Finally, if $\xi$ is any other function with these properties, then

$$\int_{t_0}^t \langle \gamma, \nabla \phi \rangle\, d\xi_0(u) = \xi_\phi(t) = \int_{t_0}^t \langle \gamma, \nabla \phi \rangle\, d\xi(u) ,$$

and so $\xi_0(t) = \xi(t)$.

LEMMA 2.6 *For all $t \geqq t_0$ and $\lambda > 0$,*

$$E[e^{\lambda \xi_0(t)}] < \infty .$$

Proof: Clearly, it suffices to prove that $E[e^{\lambda \xi_\phi(t)}] < \infty$. We know that

$$Z_\lambda(t) = e^{i\lambda\phi(x(t))} - i\lambda\int_{t_0}^t e^{i\lambda\phi(x(u))}\, d\bar{\xi}(u) + \tfrac{1}{2}\lambda^2\int_{t_0}^t \beta(u) e^{i\lambda\phi(x(u))}\, du$$

is a $P$-martingale, where

$$\bar{\xi}(t) = \int_{t_0}^t \chi_G\, L_u\, \phi(x(u))\, du + \xi_\phi(t)$$

and

$$\beta(u) = \chi_G(x(u))\langle \nabla\phi(x(u)), a(u, x(u))\, \nabla\phi(x(u))\rangle .$$



Take $\eta(t) = \phi(x(t)) - \bar{\xi}(t)$. We shall show that

$$\exp\left\{i\lambda\eta(t) + \tfrac{1}{2}\lambda^2\int_{t_0}^t \beta(u)\, du\right\}$$

is a $P$-martingale. Indeed, simply apply Lemma 2.1 to the martingale $Z_\lambda(t)$ and the function of bounded variation

$$\exp\left\{-i\lambda\bar{\xi}(t) + \tfrac{1}{2}\lambda^2\int_{t_0}^t \beta(u)\, du\right\}.$$

Hence, by Theorem 2.1,

$$P\left(\sup_{t_0 \leqq s \leqq t} |\eta(s)| \geqq R\right) \leqq 2e^{-R^2/2At} .$$

In particular, $E[e^{\lambda|\eta(t)|}] < \infty$ for all $t > t_0$ and $\lambda > 0$. Since $\phi \in C_b^2(R^d)$, it is immediate from this that $E[e^{\lambda \xi_\phi(t)}] < \infty$.

THEOREM 2.5. *The measure $P$ on $\langle \Omega, \mathscr{M}^{t_0}\rangle$ solves the submartingale problem for $a$, $b$, $\rho$ and $\gamma$ if and only if there exists a continuous, non-decreasing, non-anticipating function $\xi_0 : [t_0, \infty) \times \Omega \to [0, \infty)$ such that*

(i) $\xi_0(t_0) = 0$, $E[e^{\lambda\xi_0(t)}] < \infty$ *for all* $\lambda \geqq 0$ *and* $t \geqq t_0$,

(ii) $\xi_0(t) = \int_{t_0}^t \chi_{\partial G}(x(u))\, d\xi_0(u)$ *for* $t \geqq t_0$,

(iii) *for all* $\lambda \in R$ *and* $\theta \in R^d$,

$$X_{\lambda,\theta}^{t_0}(t) = \exp\left\{\langle \theta, x(t) - x(t_0)\rangle - \tfrac{1}{2}\int_{t_0}^t \chi_G\langle \theta, a\theta\rangle\, du - \int_{t_0}^t \chi_G\langle \theta, b\rangle\, du - \int_{t_0}^t \langle \theta, \gamma\rangle\, d\xi_0(u)\right\} \exp\left\{\lambda\left(\int_{t_0}^t \chi_{\partial G}(x(u))\, du - \int_{t_0}^t \rho\, d\xi_0(u)\right)\right\}$$

*is a $P$-martingale.*

*Furthermore, if $P$ is such a solution, then $\xi_0(t)$ is uniquely determined, up to $P$-equivalence, by the condition that*

$$\phi(x(t)) - \int_{t_0}^t \chi_G\, L_u\, \phi(u, x(u))\, du - \int_{t_0}^t \langle \nabla\phi, \gamma\rangle\, d\xi_0(u)$$

*is a $P$-martingale.*

*Finally, if $P$ is a solution, then we can define stochastic integrals with respect to $x(t)$.*

*The $dx(t)$ integrals have the following properties:*

1. $$X_\theta(t) = \exp\left\{\int_{t_0}^t \theta(u)\, dx(u) - \tfrac{1}{2}\int_{t_0}^t \chi_G\langle\theta(u), a\theta(u)\rangle\, du - \int_{t_0}^t \chi_G\langle\theta(u), b\rangle\, du - \int_{t_0}^t \langle\theta(u), \gamma\rangle\, d\xi_0(u)\right\}$$

*is a P-martingale for bounded non-anticipating* $\theta : [t_0, \infty) \times \Omega \to R^d$.

2. *If* $f \in C_b^{1,2}([0, \infty) \times R^d)$, *then, for* $t_0 \leqq s \leqq t$,

$$f(t, x(t)) - f(s, x(s)) = \int_s^t \nabla_x f(u, x(u))\, dx(u) + \int_s^t (f_u + \chi_G L_u^0 f)(u, x(u))\, du\,,$$

*where* $L_u^0$ *is the principal part of* $L_u$.

3. $$\int_{t_0}^t \chi_{\partial G}(x(u))\, dx(u) = \int_{t_0}^t \gamma(u, x(u))\, d\xi_0(u)\,, \qquad t \geqq t_0\,, \quad a.s.\ P\,.$$

4. *Moreover,*

$$\int_{t_0}^t \chi_{\partial G}(x(u))\, du = \int_{t_0}^t \rho(u, x(u))\, d\xi_0(u)\,, \qquad t \geqq t_0\,, \quad a.s.\ P\,.$$

5. *Finally, if* $\rho = 0$, *then*

$$P\left[\int_{t_0}^t \chi_{\partial G}(x(u))\, du > 0\right] = 0\,.$$

Proof: The first few assertions are immediate from Lemma 2.6 and Theorems 2.1, 2.2 and 2.3. To prove properly 3, take $\theta(u) = \chi_{\partial G}(x(u))\theta$, where $\theta \in R^d$. Then

$$\exp\left\{\left\langle \theta, \int_{t_0}^t \chi_{\partial G}(x(u))\, dx(u) - \int_{t_0}^t \gamma(u, x(u))\, d\xi_0(u)\right\rangle\right\}$$

is a $P$-martingale. Since this is true for all $\theta \in R^d$, property 3 follows immediately.

Corollary 2.2 *If* $P$ *solves the submartingale problem and if* $\tau$ *is a* $t_0$*-stopping time, then there is a set* $N \in \mathcal{M}_\tau^{t_0}$ *such that* $P(N) = 0$, *and for all* $\omega \notin N$ *the r.c.p.d.* $P_\omega$ *of* $P$ *given* $\mathcal{M}_\tau^{t_0}$ *is again a solution and* $P_\omega\{x(\tau(\omega)) = x(\tau(\omega), \omega)\} = 1$. *Moreover, if* $\tau'_\omega = \inf\{t > \tau(\omega) : x(t) \in \partial G\}$, *then for* $\omega \notin N$, $P_\omega = Q_{\tau(\omega), x(\tau(\omega), \omega)}$ *on* $\mathcal{M}_{\tau_\omega'}^{\tau(\omega)}$, *where* $Q_{t,x}$ *solves the martingale problem for* $a$ *and* $b$ *starting from* $x$ *at time* $t$.



Proof: These assertions are easy consequences of the preceding.

Corollary 2.3. *Let* $P$ *be a solution to the submartingale problem starting from* $x \in \partial G$ *at time* $t_0$. *Then* $P(\xi_0(t) > 0, t > t_0) = 1$.

Proof: By Theorem 2.1, we know that

$$X_\lambda(t) = \exp\left\{\lambda\phi(x(t)) - \lambda\int_{t_0}^t \chi_G\, L_u\, \phi(x(u))\, du - \lambda\int_{t_0}^t \langle\gamma, \nabla\phi\rangle\, d\xi_0(u) - \tfrac{1}{2}\lambda^2\int_{t_0}^t \chi_G\, \langle\nabla\phi, a\nabla\phi\rangle\, du\right\}$$

is a $P$-martingale relative to $\mathcal{M}_t^{t_0}$. Since $X_\lambda(t)$ is continuous and locally bounded, it is easy to see that $X_\lambda(t)$ is a $P$-martingale relative to $\mathcal{M}_{t+0}^{t_0} = \bigcap_{\varepsilon>0} \mathcal{M}_{t+\varepsilon}^{t_0}$. Define $\tau_0 = \sup\{s \geqq t_0 : \xi_0(s) = 0\}$. Then $\tau_0$ is a stopping time relative to the $\mathcal{M}_{t+0}^{t_0}$. Moreover, since

$$\int_{t_0}^{\tau_0} \chi_{\partial G}(x(u))\, du = \int_{t_0}^{\tau_0} \rho(u, x(u))\, d\xi_0(u) = 0\,,$$

we see that

$$X_\lambda(\tau_0 \wedge t) = \exp\left\{\lambda\phi(x(t \wedge \tau_0)) - \lambda\int_{t_0}^{t\wedge\tau_0} L_u\, \phi\, du - \tfrac{1}{2}\lambda^2\int_{t_0}^{t\wedge\tau_0} \langle\nabla\phi, a\,\nabla\phi\rangle\, du\right\}$$

is a $P$-martingale relative to $\mathcal{M}_{t\wedge\tau_0+0}^{t_0}$. Choose $\delta > 0$ so that

$$\langle\nabla\phi, a\nabla\phi\rangle \geqq \alpha > 0$$

in $B(x, \delta) \cap \bar{G}$ and let $\tau_\delta = \inf\{t \geqq t_0 : |x(t) - x| \geqq \delta\}$.[12] Then

$$E[X_\lambda(\tau_0 \wedge \tau_\delta \wedge t)] = 1$$

for $t \geqq t_0$. Let us fix $t > t_0$ and choose $\lambda < 0$. Then, since $\phi \geqq 0$ on $\bar{G}$, we have

$$E\left[\exp\left\{-\lambda^2\left(\tfrac{1}{2}\alpha - \frac{1}{\lambda}\beta\right)(\tau - t_0)\right\}\right] \geqq 1\,,$$

[12] $B(x, \delta) = \{y : |y - x| < \delta\}$.

where

$$\beta = \sup_{\substack{u \geqq t_0 \\ x \in G}} |L_u \phi(x)| \quad \text{and} \quad \tau = \tau_0 \wedge \tau_\delta \wedge t.$$

As $\lambda \to -\infty$, we obtain $P(\tau - t_0 = 0) = 1$. Since $\tau_\delta > t_0$ a.s. $P$, it follows that $P(\tau_0 = t_0) = 1$.

COROLLARY 2.4. *Suppose $a$ and $b$ are such that $a^{-1}b$ is uniformly bounded on $[t_0, \infty) \times G$. Define*

$$R_t^{t_0} = \exp\left\{\int_{t_0}^t \chi_G \langle b(u, x(u)), a^{-1}(u, x(u))\, dx(u)\rangle - \tfrac{1}{2}\int_{t_0}^t \chi_G \langle b(u, x(u)), a^{-1}(u, x(u)) b(u, x(u))\rangle\, du\right\}.$$

*If $P^0$ is a solution to the submartingale problem for $a$, $\rho$ and $\gamma$ (i.e., $b \equiv 0$), then $R_t^{t_0}$ is a $P^0$-martingale and the measure $P$ defined by*

$$\frac{dP}{dP^0} = R_t^{t_0} \quad on \quad \mathscr{M}_t^{t_0}, \qquad t \geqq t_0,$$

*is a solution to the submartingale problem for $a$, $b$, $\rho$, and $\gamma$. Conversely, if $P$ is a solution for $a$, $b$, $\rho$ and $\gamma$, then $(R_t^{t_0})^{-1}$ is a $P$-martingale and the measure $P^0$ defined by*

$$\frac{dP^0}{dP} = (R_t^{t_0})^{-1} \quad on \quad \mathscr{M}_t^{t_0}, \qquad t \geqq t_0,$$

*is a solution for $a$, $\rho$ and $\gamma$.*

Proof: By Theorem 2.5, $R_t^{t_0}$ is a $P^0$-martingale. Hence the measure $P$ is consistently defined by

$$\frac{dP}{dP^0} = R_t^{t_0} \quad \text{on} \quad \mathscr{M}_t^{t_0}.$$

Moreover, by setting $\theta(u) = \theta + [\chi_G a^{-1} b](u, x(u))$, we see from Theorem 2.5 that

$$\exp\left\{\langle \theta, x(t) - x(t_0)\rangle - \tfrac{1}{2}\int_{t_0}^t \chi_G \langle \theta, a\theta\rangle\, du - \int_{t_0}^t \chi_G \langle \theta, b\rangle\, du - \int_{t_0}^t \langle \theta, \gamma\rangle\, d\xi_0(u)\right\} \cdot R_t^{t_0} \exp\left\{\lambda\left(\int_{t_0}^t \chi_{\partial G}\, du - \int_{t_0}^t \rho\, d\xi_0(u)\right)\right\}$$

is a $P^0$-martingale. This is equivalent to saying that

$$\exp\left\{\langle \theta, x(t) - x(t_0)\rangle - \tfrac{1}{2}\int_{t_0}^t \chi_G \langle \theta, a\theta\rangle\, du - \int_{t_0}^t \chi_G \langle \theta, b\rangle\, du - \int_{t_0}^t \langle \theta, \gamma\rangle\, d\xi_0(u)\right\} \cdot \exp\left\{\lambda\left(\int_{t_0}^t \chi_{\partial G}\, du - \int_{t_0}^t \rho\, d\xi_0(u)\right)\right\}$$

is a $P$-martingale. The converse is proved in a similar fashion.

So far the only functions which we know to be in $F$ are functions in $C_b^{1,2}([0, \infty) \times R^d)$. Later on we shall need to know that $F$ also includes functions $f$ of the form

$$f(t, x) = E^{Q_{t,x}}\left[\int_t^{\tau_G} g(s, x(s))\, ds\right] + E^{Q_{t,x}}[h(\tau_G, x(\tau_G))]\ [13]$$

such that $f \in C_b^{1,1}([0, \infty) \times \partial G)$, where $Q_{t,x}$ is as in Corollary 2.2 and

$$g, h \in B_0([0, \infty) \times R^d).\ [14]$$

The rest of the present section is devoted to proving this inclusion.

LEMMA 2.7. *Let $P$ be any solution to the submartingale problem for $a$, $b$, $\rho$ and $\gamma$ starting from $x \in \partial G$ at time $t_0$. If $a$ is uniformly elliptic and if we define*

$$\tau_\delta = \inf\{t \geqq t_0 : |x(t) - x(t_0)| \geqq \delta\},$$

$$T_\delta = E\left[\int_{t_0}^{\tau_\delta} \chi_G(x(u))\, du\right], \qquad T'_\delta = E\left[\left(\int_{t_0}^{\tau_\delta} \chi_G(x(u))\, du\right)^2\right],$$

$$\Xi_\delta = E[\xi_0(\tau_\delta)], \qquad \Xi'_\delta = E[\xi_0^2(\tau_\delta)],$$

*then there exist positive numbers $C$, $\alpha$, and $\delta_0$ such that*

$$E[\tau_\delta - t_0] \leqq C\delta, \qquad E[(\tau_\delta - t_0)^2] \leqq C\delta^2, \qquad T_\delta \leqq C\delta^2, \qquad T'_\delta \leqq C\delta^4,$$

$$\alpha\delta \leqq \Xi_\delta \leqq C\delta, \qquad \Xi'_\delta \leqq C\delta^2,$$

*for $0 < \delta < \delta_0$. The quantities $C$, $\alpha$ and $\delta_0$ can be chosen independent of $x \in \partial G$, $t_0 \geqq 0$, and the particular choice of $P$.*

Proof: We use $C$ to denote "the existence of a constant such that". It is

[13] $\tau_G = \inf\{t > 0 : x(t) \notin G\}$.

[14] $B_0(\delta)$ is the set of bounded measurable functions having compact support in $\delta$.

easy to check that all the estimates obtained are uniform in the relevant parameters.

Denote by $T_{\delta,t}$ and $T'_{\delta,t}$ the quantities

$$E\left[\int_{t_0}^{\tau_\delta\wedge t}\chi_G(x(u))\,du\right] \quad\text{and}\quad E\left[\left(\int_{t_0}^{\tau_\delta\wedge t}\chi_G(x(u))\,du\right)^2\right],$$

respectively. We know that

$$\phi^2(x(t)) = 2\int_{t_0}^t \chi_G\,\phi\nabla\phi\,dx + 2\int_{t_0}^t \chi_G\,\phi L_u^0\,\phi\,du + \int_{t_0}^t \chi_G\langle\nabla\phi, a\nabla\phi\rangle\,du\,.$$

Since $|\nabla\phi| \geqq 1$ on $\partial G$, we have for small $\delta$

$$T_{\delta,t} \leqq C(E[\phi^2(x(t\wedge\tau_\delta))] + \delta T_{\delta,t})\,,$$

where $C$ is independent of $t \geqq t_0$. Hence

$$T_{\delta,t} \leqq CE[\phi^2(x(\tau_\delta\wedge t))] \leqq C\delta^2\,, \qquad t \geqq t_0\,. \tag{2.7}$$

Similarly

$$T'_{\delta,t} \leqq CE[\phi^4(x(t\wedge\tau_\delta))] + \delta^2 T'_{\delta,t} + \delta^2 T_{\delta,t}\,,$$

and so

$$T'_{\delta,t} \leqq C\delta^2 E[\phi^2(x(t\wedge\tau_\delta))] \leqq C\delta^4\,, \qquad t \geqq t_0\,. \tag{2.8}$$

Next we use (2.8) and the equation

$$\phi(x(t)) = \int_{t_0}^t \chi_G\,\nabla\phi\,dx + \int_{t_0}^t \chi_G\,L_u^0\,\phi\,du + \int_{t_0}^t \chi_G\langle\nabla\phi, \gamma\rangle\,d\xi_0 \tag{2.9}$$

to obtain

$$E[\xi_0(\tau_\delta)] \leqq C\delta\,.$$

Since

$$\tau_\delta = \int_0^{\tau_\delta}\chi_G(x(u))\,du + \int_0^{\tau_\delta}\rho(u, x(u))\,d\xi_0(u)\,,$$

we get $E[\tau_\delta] \leqq C\delta$. Also, we have from (2.9)

$$E[\xi_0^2(\tau_\delta)] \leqq C(E[\phi^2(x(\tau_\delta))] + T_\delta + T'_\delta)\,,$$

and so

$$E[\xi_0^2(\tau_\delta)] \leqq CE[\phi^2(x(\tau_\delta))] \leqq C\delta^2\,. \tag{2.10}$$



From (2.8) and (2.10) we obtain $E[\tau_\delta^2] \leqq C\delta^2$, and from (2.7) and (2.10),

$$\begin{aligned}\delta^2 = E[|x(\tau_\delta) - x|^2] &\leqq 2E\left[\left|\int_{t_0}^{\tau_\delta}\chi_G(x(u))\,dx(u)\right|^2 + \left|\int_{t_0}^{\tau_\delta}\gamma\,d\xi_0\right|^2\right] \\ &\leqq CE[\phi^2(x(\tau_\delta))]\,.\end{aligned}$$

Since $\phi(x(\tau_\delta)) \leqq \delta$, this implies that $E[\phi(x(\tau_\delta))] \geqq \alpha'\delta$ for some positive $\alpha'$. Combining the preceding with (2.7) and (2.9), we conclude that $E[\phi(\tau_\delta)] \geqq \alpha\delta$ for some $\alpha > 0$.

LEMMA 2.8. *The notation and hypotheses are the same as in Lemma 2.7. If $f \in C_b([0,\infty)\times\bar{G}) \cap C_b^{1,1}([0,\infty)\times\partial G)$ and $f_t + J_t f \geqq \beta$ for some $\beta > 0$, then there exists a $\delta(t_0, x)$ such that $0 < \delta(t_0, x) \leqq \delta_0$ and*

$$E[f(\tau_\delta, x(\tau_\delta)) - f(t_0, x_0)] \geqq 0$$

*for $0 < \delta < \delta(t_0, x)$. The choice of $\delta(t_0, x)$ depends only on $\delta_0$ and the modulus of continuity of $\rho, \gamma, f_u$ and $\nabla_x f$ at $(t_0, x)$. In particular, $\delta(t, x)$ can be chosen uniformly positive on compact sets.*

Proof: We have

$$\begin{aligned}E[f(\tau_\delta, x(\tau_\delta)) - f(t_0, x)] &= E[\langle\nabla_x f(\tau_\delta, x(t_0)), x(\tau_\delta) - x(t_0)\rangle] \\ &\quad + E\left[\int_{t_0}^{\tau_\delta} f_u(u, x(t_0))\,du\right] + o(\delta) \\ &= I_1 + I_2 + o(\delta)\,.\end{aligned}$$

Now

$$\begin{aligned}I_1 &= E[\langle\nabla_x f(\tau_\delta, x(t_0)), \gamma(\tau_\delta, x(t_0))\rangle\xi_0(\tau_\delta)] \\ &\quad + E\left[\left\langle\nabla_x f(\tau_\delta, x(t_0)), \int_{t_0}^{\tau_\delta}(\gamma(u, x(u)) - \gamma(\tau_\delta, x(t_0)))\,d\xi_0(u)\right\rangle\right] \\ &\quad + E\left[\left\langle\nabla_x f(t_0, x(t_0)), \int_{t_0}^{\tau_\delta}\chi_G(x(u))b(u, x(u))\,du\right\rangle\right] \\ &\quad + E\left[\left\langle\nabla_x f(\tau_\delta, x(t_0)) - \nabla_x f(t_0, x(t_0)), \int_{t_0}^{\tau_\delta}\chi_G(x(u))\,dx(u)\right\rangle\right] \\ &= I_{11} + I_{12} + I_{13} + I_{14}\,.\end{aligned}$$

Using the estimates from Lemma 2.7, we get

$$I_{12} \leqq \|\nabla_x f\| \, E\left[\int_{t_0}^{\tau_\delta} |\gamma(u, x(u)) - \gamma(\tau_\delta, x(t_0))| \, d\xi_0(u)\right] = o(\delta) ,$$

$$I_{13} \leqq \|\nabla_x f\| \, E\left[\int_{t_0}^{\tau_\delta} \chi_G(x(u)) \, |b(u, x(u))| \, du\right] = o(\delta) ,$$

and

$$I_{14} \leqq CE[|\nabla_x f(\tau_\delta, x(t_0)) - \nabla_x f(t_0, x(t_0))|^2]^{1/2} T_\delta^{1/2} = o(\delta) .$$

Also,

$$\begin{aligned} I_2 &= E[\rho(\tau_\delta, x(t_0)) f_t(\tau_\delta, x(t_0)) \xi_0(\tau_\delta)] \\ &\quad + E\left[\int_{t_0}^{\tau_\delta} [\rho(u, x(u)) f_u(u, x(t_0)) - \rho(\tau_\delta, x(t_0)) f_t(\tau_\delta, x(t_0))] \, d\xi_0(u)\right] \\ &\quad + E\left[\int_{t_0}^{\tau_\delta} \chi_G f_u(u, x(t_0)) \, du\right] \\ &= I_{21} + I_{22} + I_{23} . \end{aligned}$$

Arguing as above, we find that $I_{22}$ and $I_{23}$ are $o(\delta)$. Combining these estimates, we have

$$E[f(\tau_\delta, x(t_\epsilon)) - f(t_0, x(t_0))] = I_{11} + I_{12} + o(\delta) .$$

Since $I_{11} + I_{12} \geqq \alpha\beta\delta$ for $0 < \delta < \delta_0$, where $\alpha$ and $\delta_0$ are as in Lemma 2.7, our assertion follows.

Lemma 2.9. *We still use the notation and hypotheses of Lemma 2.7, except that here we do not assume that $P$ starts at a boundary point. The contention is:*

$$E^P[(\tau_\delta - t_0)^2] \leqq C\delta^2 \quad \textit{for} \quad 0 < \delta \leqq \delta_0 ,$$

*where $C$ is independent of the starting point and the particular solution $P$.*

Proof: If $P$ starts on $\partial G$, then this estimate follows from Lemma 2.7. In general, let

$$\tau = \inf\{t \geqq t_0 : x(t) \notin G\} \quad \text{and} \quad \tau'_{2\delta} = \inf\{t \geqq \tau : |x(t) - x(\tau)| \geqq 2\delta\} .$$



Then

$$\begin{aligned} E^P[(\tau_\delta - t_0)^2] &\leqq E^P[(\tau_\delta - t_0)^2 \chi_{\tau_\delta < \tau}] + E^P[(\tau_\delta - t_0)^2 \chi_{\tau_\delta \geqq \tau}] \\ &\leqq E^{Q_{t_0,x}}[(\tau_\delta - t_0)^2] + 2E^P[(\tau - t_0)^2 \chi_{\tau_\delta \geqq \tau}] \\ &\quad + 2E^P[(\tau'_{2\delta} - \tau)^2 \chi_{\tau_\delta \geqq \tau}] \\ &\leqq E^{Q_{t_0,x}}[(\tau_\delta - t_0)^2] + 2E^{Q_{t_0,x}}[(\tau_\delta \wedge \tau - t_0)^2] \\ &\quad + 2E^{Q_{t_0,x}}[\chi_{\tau_\delta \geqq \tau} E^{P^\tau_\omega}[(\tau'_{2\delta} - t_0)^2]] \\ &\leqq C\delta^2 , \end{aligned}$$

where $P^\tau_\omega$ is the r.c.p.d. of $P$ given $\mathscr{M}^{t_0}_\tau$ and $Q_{t_0,x}$ is as in Corollary 2.2.

Lemma 2.10. (*Notation and hypotheses are as in Lemma* 2.9.) *Assume that $g, h \in B_0([0, \infty) \times R^d)$ and $y \in \bar{G}$. Define*

$$f(s, y) = -E^{Q_{s,y}}\left[\int_s^{\tau_s} g(u, x(u)) \, du\right] + E^{Q_{s,y}}[h(\tau_s, x(\tau_s))] ,$$

*where $\tau_s = \inf\{t \geqq s : x(t) \notin G\}$ and $Q_{s,y}$ is as in Corollary 2.2. Let*

$$X_f(t) = f(t, x) - \int_{t_0}^t \chi_G \, g(u, x(u)) \, du .$$

*Then $X_f(t)$ is a local P-martingale in $G$. If, in addition,*

$$f \in C_b([0, \infty) \times \bar{G}) \cap C_b^{1,1}([0, \infty) \times \partial G),$$

*then, for any $\psi \in C_b^{1,2}([0, \infty) \times R^d)$ satisfying $(\rho \, \partial/\partial t + J_t)(f + \psi) \geqq \beta > 0$ on $[t_0, \infty) \times \partial G$,*

$$X_{\bar{f}}(t) = \bar{f}(t, x(t)) - \int_{t_0}^t \chi_G (g + \psi_u + L_u \psi)(u, x(u)) \, du$$

*is a P-submartingale, where $\bar{f} = f + \psi$.*

Proof: That $X_f(t)$ is a local $P$-martingale in $G$ is immediate from Corollary 2.2 and the definition of $f$.

To prove the second assertion, let $0 < r < 1$ and define

$$\begin{aligned} \tau_0^r &= t_0 , \\ \tau_{2n+1}^r &= \inf\{t \geqq \tau_{2n}^r : x(t) \notin G\} , \\ \tau_{2n}^r &= \inf\{t \geqq \tau_{2n-1}^r : |x(t) - x(\tau_{2n-1}^r)| \geqq r\delta(\tau_{2n-1}, x(\tau_{2n-1}))\} , \end{aligned}$$

where $\delta(t, x)$ is chosen for $\tilde{f}$ as in Lemma 2.9. Define

$$\sigma_r = \begin{cases} T & \text{if there is no } n \text{ such that } \tau_{2n}^r \leqq T < \tau_{2n-1}^r , \\ \tau_{2n}^r & \text{if } \quad \tau_{2n-1}^r \leqq T < \tau_{2n}^r , \end{cases}$$

and let

$$Y(t) = X_f(t) - \tilde{f}(t_0, x(t_0)) .$$

We shall show that the family $\{Y(\sigma_r)\}_{0<r<1}$ is uniformly integrable and that $E[Y(\sigma_r)] \geqq 0$ for $0 < r < 1$. Since $\sigma_r \to T$ as $r \downarrow 0$, it will follow that $E[Y(T)] \geqq 0$.

To show that $\{Y(\sigma_r)\}_{0<r<1}$ is uniformly integrable, it suffices to check that $E[(\sigma_r - t_0)^2] \leqq C < \infty$, $0 < r < 1$. But

$$\begin{aligned} E[(\sigma_r - t_0)^2] &\leqq 2(T - t_0)^2 + 2E[\sum \chi_{\tau_{2n-1}^r < T < \tau_{2n}^r} (\tau_{2n}^r - T)^2] \\ &= 2(T - t_0)^2 + 2E[\sum \chi_{\tau_{2n-1}^r \leqq T < \tau_{2n}^r}(\omega) E^{P_\omega^T}[\tau_{2n}^r - T)^2]] \\ &\leqq 2(T - t_0)^2 + 2E[\sum \chi_{\tau_{2n-1}^r \leqq T < \tau_{2n}^r}(\omega) E^{P_\omega^T}[(\tau_{2r\delta_0} - T)^2]] \\ &\leqq 2(T - t_0)^2 + 8(r\delta_0)^2 , \end{aligned}$$

where $\tau_{2r\delta_0} = \inf\{u \geqq T : |x(u) - x(T)| \geqq \{2r\delta_0\}$ and $P_\omega^T$ is the r.c.p.d. of $P$ given $\mathscr{M}_T^{t_0}$. To prove that $E[Y(\sigma_r)] \geqq 0$, note that $Y(\sigma_r) = \lim_{N\to\infty} Y(\sigma_r \wedge \tau_{2N}^r)$ and that $E[|Y(\sigma_r \wedge \tau_{2N}^r)|^2] \leqq C_1 + C_2 E[\sigma_r^2]$. Hence it suffices to show that $E[Y(\sigma_r \wedge \tau_{2N}^r)] \geqq 0$. But

$$\begin{aligned} E[Y(\sigma_r \wedge \tau_{2N}^r)] &= E\left[\sum_{n=0}^{N-1} \chi_{\tau_{2n}^r \leqq T}(X_f(\tau_{2n+1}^r \wedge T) - X_f(\tau_{2n}^r))\right] \\ &\quad + E\left[\sum_{n=1}^{N} \chi_{\tau_{2n-1}^r \leqq T}(X_{f'}(\tau_{2n}^r) - X_f(\tau_{2n-1}^r))\right] , \end{aligned}$$

$$E[\chi_{\tau_{2n}^r \leqq T}(X_f(\tau_{2n+1}^r \wedge T) - X_f(\tau_{2n}^r))] = 0 ,$$

and

$$\begin{aligned} &E[\chi_{\tau_{2n-1}^r \leqq T}(X_f(\tau_{2n}^r) - X_f(\tau_{2n-1}^r))] \\ &\qquad = E[\chi_{\tau_{2n-1}^r \leqq T}(\omega) E^{P_\omega^{\tau_{2n-1}^r}} [X_f(\tau_{2n}^r) - X_f(\tau_{2n}^r(\omega), \omega)]] \geqq 0 , \end{aligned}$$

where $P_\omega^{\tau_{2n-1}^r}$ is the r.c.p.d. of $P$ given $\mathscr{M}_{\tau_{2n-1}^r}^{t_0}$.

We know now that, for any solution of the submartingale problem starting at time $t_0$,

$$E[X_f(T)] \geqq E[X_f(t_0)] \qquad \text{for} \qquad T > t_0 .$$



Given $t_0 \leqq s < T$, apply this result to the r.c.p.d. $P_\omega^s$ of $P$ given $\mathscr{M}_s^{t_0}$. Then

$$E^{P_\omega^s}[X_f(T)] \geqq E^{P_\omega^s}[X_f(x)] \qquad \text{a.s. } P ,$$

since $P_\omega^s$ is almost surely a solution to the submartingale problem starting at time $s$. In other words, $X_f(t)$ is a $P$-submartingale.

THEOREM 2.6. *Let $P$ be a solution to the submartingale problem for $a$, $b$, $\rho$ and $\gamma$ starting at time $t_0$. Assume that $a$ is uniformly elliptic. Define $f$ as in Lemma* 2.10 *and suppose $f \in C_b([0, \infty) \times \bar{G}) \cap C_b^{1,1}([0, \infty) \times \partial G)$. Then $f \in F$.*

Proof: We must check that $f$ fulfills conditions (ii) and (iii) defining the class $F$, since condition (i) is part of the hypothesis.

From the preceding we know that

$$X_f(t) = f(t, x(t)) - \int_{t_0}^t \chi_G g(u, x(u))\, du$$

is a local $P$-martingale in $G$. Hence $f$ satisfies (ii) with $Kf = g$. To show that $f$ satisfies (iii), choose $\alpha$ such that $\rho \tilde{f}_t + J_t \tilde{f} \geqq \beta > 0$ on $[t_0, \infty) \times \partial G$, where $\tilde{f} = f + \alpha\phi$. Then, by the preceding,

$$\tilde{f}(t, x) - \int_{t_0}^t \chi_G (g + \alpha L_u \phi)(u, x(u))\, du$$

is a $P$-submartingale. Hence there is a $\xi_{\tilde{f}}$ satisfying (iii) for $\tilde{f}$. Clearly $\xi_f = \xi_{\tilde{f}} - \alpha\xi_\phi$ satisfies (iii) for $f$. Finally, if $\psi \in C_b^{1,2}([0, \infty) \times R^d)$ satisfying

$$(\rho \frac{\partial}{\partial t} + J_t)(f + \psi) \geqq 0 \quad \text{on} \quad [t_0, \infty) \times \partial G ,$$

then, for every $\beta > 0$,

$$(f + \psi + \beta\phi)(t, x(t)) - \int_{t_0}^t \chi_G (g + L_u \psi + \beta L_u \phi)(u, x(u))\, du$$

is a $P$-submartingale. Let $\beta \downarrow 0$, we see that

$$(f + \psi)(t, x(t)) - \int_{t_0}^t \chi_G (g + L_u \psi)(u, x(u))\, du$$

is also a $P$-submartingale. Therefore, $\xi_{f+\psi}(t)$ must be non-decreasing.

## 3. Existence

Let $G$ be an open subset of $R^d$ and assume there exists a $\phi \in C_b^2(R^d)$ such that $G = \{x : \phi(x) > 0\}$, $\bar{G} = \{x : \phi(x) \geqq 0\}$, $|\nabla\phi| \geqq 1$ on $\partial G$. Let

$$a : [0, \infty) \times \bar{G} \to S_d^+$$

be bounded and continuous, let $\gamma : [0, \infty) \times \partial G \to R^d$ be a bounded continuous function such that $\langle \gamma, \nabla\phi \rangle \geqq \beta > 0$ on $[0, \infty) \times \partial G$, and let $\rho : [0, \infty) \times \partial G \to R^1$ be a uniformly positive, bounded, continuous function. Given $t_0 \geqq 0$ and $x_0 \in \bar{G}$, we are going to construct a measure $P$ on $\langle \Omega, \mathscr{M}^{t_0} \rangle$ such that $P(x(t_0) = x_0) = 1$ and $P$ solves the submartingale problem for $a$, $\rho$ and $\gamma$. Since $\rho$ is bounded and uniformly positive, we may replace $\gamma$ by $\gamma/\rho$ and thereby reduce the problem to the case when $\rho = 1$. Our construction will make use of the following simple lemma.

LEMMA 3.1. *Let $P$ be a measure on $\langle \Omega, \mathscr{M}^{t_0} \rangle$. Let*

$$t_0 \equiv \tau_{-1} \leqq \tau_0 \leqq \tau_1 \leqq \cdots \leqq \tau_n \leqq \cdots$$

*be a non-decreasing sequence of $t_0$-stopping times such that $\tau_n \to \infty$ a.s. $P$. If $X : [t_0, \infty) \times \Omega \to R$ is a locally bounded non-anticipating function such that $X(t \wedge \tau_n) \vee \tau_{n-1})$ is almost surely a $P_\omega^{\tau_{n-1}}$-submartingale for each $n \geqq 0$; where $P_\omega^{\tau_{n-1}}$ is the r.c.p.d. of $P$ given $\mathscr{M}_{\tau_{n-1}}^{t_0}$, then the process $X(t)$ is a $P$-submartingale.*

Proof: Given $t_0 \leqq s < t$ and $A \in \mathscr{M}_s^{t_0}$, we have

$$\begin{aligned} E[(X(t) - X(s))\chi_A] &= E\left[\sum_0^\infty (X(\tau_n \vee s) \wedge t) - X((\tau_{n-1} \vee s) \wedge t))\chi_A\right] \\ &= \lim_{N \to \infty} \sum_0^N E[(X((\tau_n \vee s) \wedge t) - X((\tau_{n-1} \vee s) \wedge t))\chi_A] . \end{aligned}$$

But

$$\begin{aligned} E[(X((\tau_n \vee s) \wedge t) &- X((\tau_{n-1} \vee s) \wedge t))\chi_A] \\ &= E[(X(\tau_n \wedge t) - X((\tau_{n-1} \vee s) \wedge t))\chi_{A \cap \{\tau_n > s\}}] \\ &= E[(X(\tau_n \wedge t) - X(\tau_{n-1} \wedge t))\chi_{A \cap \{\tau_{n-1} > s\}}] \\ &\quad + E[(X(\tau_n \wedge t) - X(s))\chi_{A \cap \{\tau_n > s \geqq \tau_{n-1}\}}] \\ &= E[\chi_{A \cap \{s < \tau_{n-1} < t\}}(\omega) E^{P_\omega^{\tau_{n-1}}}[X(\tau_n \wedge t) - X(\tau_{n-1}(\omega))]] \\ &\quad + E[\chi_{\{\tau_{n-1} \leqq s\}}(\omega) E^{P_\omega^{\tau_{n-1}}}[(X(\tau_n \wedge t) - X(s))\chi_{A \cap \{\tau_n > s\}}]] \\ &\geqq 0 ; \end{aligned}$$

q.e.d.



Now choose $\delta_0 > 0$ such that $x + \delta\gamma(u, x) \in G$ for all $x \in \partial G$, $u \geqq 0$, and $0 \leqq \delta \leqq \delta_0$. Given $0 < \delta < \delta_0$, define

$$Q^\delta(s, x; t, \Gamma) = \begin{cases} \chi_\Gamma(x) e^{-(t-s)/\delta} + \dfrac{1}{\delta} \displaystyle\int_s^t e^{-(u-s)/\delta} \chi_\Gamma(x + \delta\gamma(u, x))\, du & \text{if } x \in \partial G , \\ \chi_\Gamma(x) & \text{if } x \in G . \end{cases}$$

Clearly there is a Markov process $Q_{s,x}$, $s \geqq 0$ and $x \in \bar{G}$, on $D([0, \infty), \bar{G})$ [15] for which $Q^\delta(s, x; t, \Gamma)$ is the transition probability. Moreover, if $f$ belongs to $C_b^{1,2}([0, \infty) \times R^d)$, then

$$f(t, x(t)) - \int_s^t \chi_{\partial G}(x(u)) K_\delta f(u, x(u))\, du - \int_s^t f_u(u; x(u))\, du$$

is a $Q_{s,x}^\delta$-martingale, where

$$K_\delta f(t, x) = \frac{1}{\delta}[f(t, x + \delta\gamma(t, x)) - f(t, x)] .$$

Let $P_{s,x}^{(a)}$, $s \geqq 0$ and $x \in R^d$, be the Markov process on $D([s, \infty), R^d)$ associated with the diffusion coefficients $a$. [16] Define

$$\begin{aligned} \tau_{-1} &= s , \\ \tau_{2n} &= \inf\{t \geqq \tau_{2n-1} : x(t) \notin G\} , \\ \tau_{2n+1} &= \inf\{t \geqq \tau_{2n} : |x(t) - x(\tau_{2n})| \geqq \tfrac{1}{2}\beta\delta\} . \end{aligned}$$

For each fixed $\delta > 0$ we shall define a family of measures $P_{s,x}^\delta$, $s \geqq 0$ and $x \in \bar{G}$. First we define a sequence $P_{s,x}^{(n)}$ as follows:

$$P_{s,x}^{(-1)} = P_{s,x}^{(a)} .$$

Having defined $P_{s,x}^{(2n-1)}$, $P_{s,x}^{(2n)}$ is defined to be equal to $P_{s,x}^{(2n-1)}$ on $\mathscr{M}_{\tau_{2n-1}}^s$, and the r.c.p.d. of $P_{s,x}^{(2n)}$ given $\mathscr{M}_{\tau_{2n-1}}^s$ to be equal to $P_{\tau_{2n-1}, x(\tau_{2n-1})}^{(a)}$. Having defined $P_{s,x}^{(2n)}$, $P_{s,x}^{(2n+1)}$ is defined to be equal to $P_{s,x}^{(2n)}$ on $\mathscr{M}_{\tau_{2n}}^s$, and the r.c.p.d. of $P_{s,x}^{(2n+1)}$ given $\mathscr{M}_{\tau_{2n}}^s$ to be equal to $Q_{\tau_{2n}, x(\tau_{2n})}^\delta$. Because of the consistency

[15] $D(0, T)$ is the space of functions with no discontinuities of the second kind. We formalize the elements of $D(0, T)$ to be right-continuous and endow the space with the Skorokhod metric.

[16] We may assume that the coefficients $a$ are globally defined.

of the sequence $P_{s,x}^{(n)}$, we can define $P_{s,x}^{\delta}$ to be equal to $P_{s,x}^{(n)}$ on $\mathcal{M}_{\tau_n}^{s}$. Moreover, by Lemma 3.1, if $f \in C_b^{1,2}([0, \infty) \times R^d)$, then

$$\text{(3.1)} \qquad \begin{aligned} X_f^\delta(t) = f(t, x(t)) &- \int_s^t \chi_G \cdot (f_u + L_u f)(u, x(u))\, du \\ &- \int_s^t \chi_{\partial G} \cdot (f_u + K_\delta f)(u, x(u))\, du \end{aligned}$$

is a $P_{s,x}^\delta$-martingale, where

$$L_u f(u, x) = \sum a_{ij}(u, x) \frac{\partial^2 f}{\partial x_i\, \partial x_j}.$$

We want to show that, for given $t_0$, $x_0$, there is a sequence $\{\delta_m\}$ decreasing to zero such that $P_{t_0,x_0}^{\delta_m}$ converges in $D[[t_0, \infty), \bar{G}]$ to a measure $P$ fulfilling our requirements.

LEMMA 3.2. *There exists a constant $C_T$, independent of $s$, $x$ and $\delta$, such that*

$$E^{P_{s,x}^\delta}[|x(t) - x(s)|^2] \leqq C_T(t - s), \qquad T \geqq t \geqq s,$$

$$E^{P_{s,x}^\delta}\left[\int_s^t \chi_{G_\varepsilon}(x(u))\, du\right] \leqq C_T\, \varepsilon, \qquad t_0 \leqq t \leqq T,$$

*where* $G_\varepsilon = \{y \in G : |y - \partial G| < \varepsilon\}$.

Proof: The first assertion is immediate from (3.1). To prove the second estimate, define

$$\eta_\varepsilon(r) = \begin{cases} \dfrac{1}{\varepsilon}(\varepsilon - r)^3 & \text{if } \quad |r| < \varepsilon, \\ 0 & \text{if } \quad |r| \geqq \varepsilon, \end{cases}$$

and take $\psi_\varepsilon = \eta_\varepsilon \circ \phi$. Then

$$L_u \psi_\varepsilon = \eta_\varepsilon'' \circ \phi \cdot \langle \nabla \phi, a \nabla \phi \rangle + \eta_\varepsilon'' \circ \phi L_u \phi, \qquad |K_\delta \psi_\varepsilon| \leqq C\varepsilon.$$

Hence, from (3.1),

$$E^{P_{s,x}}\left[\int_s^t [\chi_G \eta_\varepsilon'' \circ \phi \langle \nabla\phi, a \nabla \phi\rangle](u, x(u))\, du\right] \leqq \varepsilon^2 + C(t - s)\varepsilon.$$

Since $a$ is uniformly elliptic, $|\nabla \phi| \geqq 1$ in $\partial G$, and $\eta_\varepsilon'' \geqq \frac{1}{2}$ in $G_{\varepsilon/2}$, our estimate follows.

For each $\delta > 0$, the family $P_{s,x}^\delta$, $s \geqq 0$ and $x \in \bar{G}$, forms a Markov process. Consequently, the first estimate in Lemma 3.2 is enough to guarantee



weak compactness on $D([t_0, \infty), \bar{G})$ for the family $P_{t_0,x_0}^\delta$, $0 < \delta < \delta_0$. Choose $\delta_m \downarrow 0$ such that $P_{t_0,x_0}^{\delta_m}$ converges weakly on $D([t_0, \infty), \bar{G})$ to a measure $P$. Clearly, $P(x(t_0) = x_0) = 1$. Furthermore,

$$P((\exists t \geqq t_0)\ |x(t) - x(t-)| \geqq \delta_m \|\gamma\|) = 0$$

for all $m$, and therefore $P$ is concentrated on $C([t_0, \infty), \bar{G})$. Finally, we must show that $P$ solves the submartingale problem. Given $f \in C_0^{1,2}([0, \infty) \times R^d)$ such that $f_u + \langle \gamma, \nabla_x f \rangle \geqq \alpha > 0$ on $[t_0, \infty) \times \partial G$, choose $0 < \delta_1 \leqq \delta_0$ so that $f_u + K_\delta f \geqq 0$ on $[t_0, \infty) \times \partial G$ for $0 < \delta \leqq \delta_1$. Given $\varepsilon > 0$, choose $\eta_\varepsilon \in C_b^\infty(R^d)$ such that $0 \leqq \eta_\varepsilon \leqq 1$, $\eta_\varepsilon \equiv 1$ on $G - \bar{G}_\varepsilon$, and $\eta_\varepsilon \equiv 0$ off $G$. Let $t > s \geqq t_0$ and suppose $\Phi$ is a bounded, non-negative, continuous $\mathcal{M}_s^{t_0}$-measurable function. Then there is a constant $C$ such that

$$\begin{aligned} E^P\Big[\Phi\{(f(t, x(t)) &- f(s, x(s)) - \int_s^t [\chi_G \cdot (f_u + L_u f)](u, x(u))\, du)\}\Big] \\ &\geqq -C\varepsilon + E^P\Big[\Phi\{(f(t, x(t)) - f(s, x(s))) \\ &\qquad - \int_s^t [\eta_\varepsilon \cdot (f_u + L_u f)](u, x(u))\, du\}\Big] \\ &= -C\varepsilon + \lim_{m\to\infty} E^{P_{t_0,x_0}^{\delta_m}}\Big[\Phi\{(f(t, x(t)) - f(s, x(s))) \\ &\qquad - \int_s^t [\eta_\varepsilon \cdot (f_u + L_u f)](u, x(u))\, du\}\Big] \\ &\geqq -2C\varepsilon + \lim_{m\to\infty} E^{P_{t_0,x_0}^{\delta_m}}\Big[\Phi\{(f(t, x(t)) - f(s, x(s))) \\ &\qquad - \int_s^t [\chi_G \cdot (f_u + L_u f)](u, x(u))\, du\}\Big] \\ &\geqq -2C\varepsilon. \end{aligned}$$

Letting $\varepsilon \downarrow 0$, we see that

$$f(t, x(t)) - \int_{t_0}^t [\chi_G \cdot (f_u + L_u f)](u, x(u))\, du$$

is a $P$-submartingale. It is easy to pass from this to the fact that $P$ solves the submartingale problem.

We have now constructed a solution to the submartingale problem starting from $x_0$ at time $t_0$ in the case when $a$ is uniformly elliptic, $b \equiv 0$, and $\rho$ is uniformly positive. We next extend the construction to the case when $\rho$

is merely non-negative. To do this, choose uniformly positive $\rho_n$ such that $\rho_n \to \rho$ uniformly. For each $n$, let $P^n$ denote a solution starting from $x_0$ at time $t_0$ for $a$, $\rho_n$ and $\gamma$. We shall show that $\{P^n\}_{n\geqq 1}$ is weakly compact on $C([t_0, \infty), R^d)$ and that any limit point $P$ of $\{P^n\}_{n\geqq 1}$ solves the submartingale problem for $a$, $\rho$, and $\gamma$.

LEMMA 3.3. *There exists a constant $C_T$, independent of $n$, such that*

$$E^{P^n}[|x(t_3) - x(t_2)|^2 \, |x(t_2) - x(t_1)|^2] \leqq C_T(t_3 - t_1)^2 ,$$
$$t_0 \leqq t_1 \leqq t_2 \leqq t_3 \quad \textit{and} \quad t_3 - t_1 \geqq T ,$$
$$E^{P^n}\left[\int_{t_0}^t \chi_{G_\varepsilon}(x(u))\, du\right] \leqq C_T\, \varepsilon , \qquad t_0 \leqq t \leqq T ,$$

*where $G_\varepsilon = \{x \in G : |x - \partial G| < \varepsilon\}$.*

Proof: Let $P$ on $\langle \Omega, \mathscr{M}^s\rangle$ be a solution to the submartingale problem for $a$, $\rho$ and $\gamma$. Denote by $\xi$ the associated increasing function described in Theorem 2.4. Then,

$$\phi(x(t)) - \phi(x(s)) = \int_s^t \chi_G \nabla\phi\, dx(u) + \int_s^t \chi_G\, L_u\, \phi\, du + \int_s^t \langle \gamma, \nabla\phi\rangle\, d\xi(u)$$

and

$$\phi^2(x(t)) - \int_s^t \chi_G \langle \nabla\phi, a\, \nabla\phi\rangle - 2\int_s^t \chi_G\, \phi L_u\, \phi\, du$$

are $P$-martingales. From these two facts, we see that

$$\begin{aligned} E^P[(\phi(x(t)) - \phi(x(s)))^2] &= E^P[\phi^2(x(t)) - \phi^2(x(s))] - 2E^P[\phi(x(s))(\phi(x(t)) - \phi(x(s)))] \\ &= E^P\left[\int_s^t \chi_G \langle \nabla\phi, a\nabla\phi\rangle\, du\right] + 2E^P\left[\int_s^t \chi_G\, \phi L_u\, \phi\, du\right] \\ &\quad - 2E^P\left[\phi(x(s))\int_s^t \chi_G\, L_u\, \phi\, du\right] - 2E^P\left[\int_s^t \langle \gamma, \nabla\phi\rangle\, d\xi(u)\right] \\ &\leqq C(t - s) , \end{aligned}$$

since $\phi(x(s)) \displaystyle\int_s^t \langle \gamma, \nabla\phi\rangle\, d\xi(u) \geqq 0$. Substituting this back into the first relation, we get

$$E^P\left[\left(\int_s^t \langle \gamma, \nabla\phi\rangle\, d\xi(u)\right)^2\right] \leqq C_T(t - s) , \qquad 0 \leqq t - s \leqq T,$$



where $C_T$ does not depend on $\rho$ or the particular choice of $P$. Since $\langle \gamma, \nabla\phi\rangle \geqq \beta > 0$, this proves that $E[\xi^2(t)] \leqq C_T\, t$, $s \leqq t \leqq T$. Hence,

$$\begin{aligned} E^P[|x(t) - x(s)|^2] &\leqq 2E^P\left[\left|\int_s^t \chi_G(x(u))\, dx(u)\right|^2\right] \\ &\quad + 2E^P\left[\left|\int_s^t \gamma(u, x(u))\, d\xi(u)\right|^2\right] \leqq C_T(t - s) , \\ &\qquad\qquad 0 \leqq t - s \leqq T . \end{aligned}$$

Now take $t_0 \leqq t_1 \leqq t_2 \leqq t_3$, where $t_3 - t_1 \leqq T$. Applying the preceding result to the r.c.p.d. of $P^n$ given $\mathscr{M}_{t_2}^{t_0}$, we get

$$E^{P^n}[|x(t_3) - x(t_2)|^2 \mid \mathscr{M}_{t_2}^{t_0}] \leqq C_T(t_3 - t_2) \qquad \text{a.s. } P^n .$$

From this our first assertion is immediate.

The second assertion is proved in very much the same way as we proved the second estimate in Lemma 3.2. The only difference is that here one must use the estimate $E^{P^n}[\xi^{(n)}(t)] \leqq C_T(t - t_0)^{1/2}$, $0 \leqq t - t_0 \leqq T$, which has just been found in the preceding paragraph.

The first estimate in Lemma 2.3 shows that $\{P^n\}_{n\geqq 1}$ is compact on $D([t_0, \infty), \bar{G})$. Using the second estimate in Lemma 2.3 and arguing in just the same way as we did before, we see that every limit point of $\{P^n\}_{n\geqq 1}$ solves our problem. Combining this with Corollary 2.3, we have proved:

THEOREM 3.1. *Let $G$, $\phi$, $a$ and $\gamma$ be as described at the beginning of this section. Let $b : [0, \infty) \times \partial G \to R^d$ be bounded and measurable and $\rho : [0, \infty) \times \partial G \to [0, \infty)$ bounded and continuous. Then, for each $t_0 \geqq 0$ and $x_0 \in \bar{G}$, there is a solution $P$, starting from $x_0$ at time $t_0$, to the submartingale problem for $a$, $b$, $\rho$ and $\gamma$.*

*Remark.* If one applies the uniform estimate

$$P\left(\sup_{t_0 \leqq u \leqq T} |x(u) - x(t_0)| \geqq R\right) \leqq \frac{C(T - t_0)}{R^2} ,$$

one can replace the assumption of uniform ellipticity by strict ellipticity using an obvious piecing-together argument (cf. [8]).

## 4. Boundary Processes

Suppose that we are given a region $G$ in $R^d$ and coefficients $a$, $b$, $\rho$ and $\gamma$ of the type discussed previously. Consider only the coefficients $a$ and $b$. Let $Q_{t,x}$ be the unique solution to the martingale problem for $a$ and $b$, starting from $(t, x)$ up to the first exit time $\tau$ after $t$ from the region $G$. We call $Q_{t,x}$, defined on $\mathscr{M}_\tau^t$, the interior process.

We now proceed to define a process called the boundary process. The importance of this process is that the entire process is built out of the interior and the boundary processes in a concrete way.

Let $P$ be a solution to the submartingale problem corresponding to $a$, $b$, $\rho$ and $\gamma$, starting from a boundary point $x$ at time $t_0$. Then by Theorem 2.4, there exists a nondecreasing continuous function $\xi(t)$ with $\xi(t_0) = 0$, whose increases are only at those times when the trajectory $x(t)$ is at the boundary, such that, for smooth functions $u(t, x)$,

$$u(t, x(t)) - \int_{t_0}^t u_s(s, x(s))\, ds - \int_{t_0}^t (L_s u)(s, x(s))\chi_G(x(s))\, ds - \int_{t_0}^t (J_s u)(s, x(s))\, d\xi(s)$$

is a martingale. Let us define stopping times $\tau_\theta$ as follows:

$$\tau_\theta = [\sup s : s \geqq t_0 \text{ and } \xi(s) \leqq \theta] .$$

Let $T(\omega) = \lim_{t\to\infty} \xi(t)$. Then $\tau_\theta < \infty$ if and only if $T(\omega) > \theta$. Although $\tau_\theta$ is not strictly a stopping time, for any $\varepsilon > 0$, $\tau_\theta + \varepsilon$ is a stopping time. Or equivalently, $\tau_\theta$ is a stopping time relative to $\mathscr{M}_{t+0}^{t_0}$.[17] Moreover, since the starting point is on the boundary, we have $\xi(t) > 0$ for $t > 0$ almost surely (i.e., $\tau_0 = t_0$ with probability one (see Corollary 2.3)).

Also, if $\theta_n \downarrow \theta$ we have $\tau_{\theta_n} \downarrow \tau_\theta$. Since $\xi(\tau_\theta + \varepsilon) > \xi(\tau_\theta) = \theta$ for every $\varepsilon > 0$, $x(\tau_\theta)$ must be on the boundary. Assume, for simplicity, that $T(\omega) = \infty$ a.s. Then the process

$$y(\theta) = x(\tau_\theta)$$

is defined for all $0 \leqq \theta < \infty$, and it is a right continuous process living on the boundary. Moreover, the left limit exists for almost all trajectories. Also, $y(0) = x(t_0) = x$. We find it more convenient to work with the $(d + 1)$-dimensional process $(\tau_\theta, y(\theta))$. This is what we call the boundary process starting from $x$ at time $t_0$. It is defined relative to the $\sigma$-fields $\mathscr{M}_{\tau_\theta}^{t_0}$. (If $T(\omega) < \infty$ with positive probability, it may be convenient to adjoin a point at $\infty$ and define $(\tau(\theta), y(\theta)) = \infty$ for $\theta \geqq T(\omega)$.)

In order to compute explicitly the "generator" of the boundary process, we make certain assumptions on the coefficients $a$ and $b$ and the region $G$. For any two functions $f$ in $C_0^{1,2}([0, \infty) \times \partial G)$ and $h$ in $C_0([0, \infty) \times G)$, we

[17] $\mathscr{M}_{t+0}^{t_0} = \bigcap_{s>t} \mathscr{M}_s^t$.



assume that the function

$$u(t, x) = E^{Q_{t,x}}\left[\int_t^\tau h(s, x(s))\, ds + f(\tau, x(\tau))\right]$$

has bounded continuous spatial derivatives of the first order throughout $[0, \infty) \times \bar{G}$.

From certain results in partial differential equations (cf. [5]) it is possible to show that this assumption is justified. However, we shall use the results of this section only in special cases where the assumption will be verified without appealing to the general theory.

Let us take $h$ to be zero and set

$$u(t, x) = E^{Q_{t,x}}[f(\tau, x(\tau))] .$$

Since the spatial derivatives of the first order exist everywhere and since $u$ equals $f$ on the boundary,

$$(Kf)(s, x) = (J_s u)(s, x) + \rho(s, x)u_s(s, x) \tag{4.1}$$

is well defined as a bounded continuous function $[0, \infty) \times \partial G$.

THEOREM 4.1. *For any function $f(t, x)$ on the boundary in $C_0^{1,2}([t_0, \infty) \times \partial G)$,*

$$f(\tau(\theta), y(\theta)) - \int_0^\theta (Kf)(\tau(s), y(s))\, ds$$

*is a martingale relative to $\mathscr{M}_{\tau_\theta}^{t_0}$ and $P$. Moreover,*

$$P[(\tau(0), y(0)) = (t_0, x)] = 1 .$$

Proof: We know that $u \in F$. Therefore, since $u$ is harmonic in the interior,

$$u(t, x(t)) - \int_{t_0}^t [u_s(s, x(s))]\chi_{\partial G}(x(s))\, ds - \int_{t_0}^t (J_s u)(s, x(s))\, d\xi(s)$$

is a martingale. We can write

$$\chi_{\partial G}(x(s))\, ds = \rho(s, x(s))\, d\xi(s) ,$$

and therefore

$$u(t, x(t)) - \int_{t_0}^t (Kf)(s, x(s))\, d\xi(s)$$

is a martingale. Let us assume for the moment that we can justify substituting $\tau_\theta$ for $t$. Then,

$$u(\tau_\theta, x(\tau_\theta)) - \int_{t_0}^{\tau_\theta} (Kf)(s, x(s))\, d\xi(s)$$

is a martingale relative to $\mathscr{M}_{\tau_\theta}^{t_0}$. Clearly, $x(\tau_\theta) = y(\theta)$ is a boundary point and $u = f$ on the boundary. If we change variables in the integration from $s$ to $\tau_s$, then

$$f(\tau(\theta), y(\theta)) - \int_0^\theta (Kf)(\tau(s), y(s))\, ds$$

is a martingale relative to $\mathscr{M}_{\tau_\theta}^{t_0}$, and that will prove the theorem.

To justify the substitution of random stopping times, we need to take care of two points. First, $\tau_\theta$ is a stopping time only relative to $\mathscr{M}_{t+0}^{t_0}$, and second, the $\tau_\theta$ are unbounded. The first point is overcome by considering $\tau_\theta + \varepsilon$ and the second point by considering $(\tau_\theta + \varepsilon) \wedge N$. We first let $N \to \infty$ and then let $\varepsilon \to 0$. Up to $\tau_\theta$, $\xi(s)$ can be at most $\theta$. Therefore, for $t_0 \leqq t \leqq \tau_\theta + \varepsilon$, there exist constants $A$ and $B$ such that

$$\begin{aligned}\left| u(t, x(t)) - \int_0^t (Kf)(s, x(s))\, d\xi(s) \right| &\leqq A + B\xi(t) \\ &\leqq A + B[\theta + (\xi(\tau_\theta + \varepsilon) - \xi(\tau_\theta))] .\end{aligned}$$

$E^P[|\xi(\tau_\theta + \varepsilon) - \xi(\tau_\theta)|^2]$ can be easily seen to be bounded. In fact, by Lemma 2.6, $E\xi^2(t)$ is finite for all $t$ and therefore, for any stopping time $\tau$,

$$E(\xi(\tau + t) - \xi(\tau))^2$$

can be estimated by taking r.c.p.d. We have uniform integrability by the same estimate, and the martingale is right continuous. This completes the justification.

We say that the uniqueness theorem is valid for the boundary process if, for any given point $x$ on the boundary and for any starting time $t_0$, there is only one solution $Q$ to the problem of finding a measure on the space

$$D([0, \infty), [t_0, \infty) \times \partial G)\ ^{18}$$

such that

1. $Q[\tau(0) = t_0, y(0) = x] = 1$,
2. $\infty$ is an absorbing point,

[18] See footnote 15.



3. for $f(t, x)$ in $C_0^{1,2}([t_0, \infty) \times \partial G)$,

$$f(\tau(\theta), y(\theta)) - \int_0^\theta (Kf)(s, y(s))\, ds$$

is a martingale relative to the natural $\sigma$-algebras $\mathscr{M}_\theta$ in $D([0, \infty), [t_0, \infty) \times \partial G)$ and the measure $Q$.

*Remark* 1. Of course it is clear that conditions 1, 2 and 3 are borrowed from Theorem 4.1. In particular, the validity of the uniqueness theorem for the boundary process implies that if $P_1$ and $P_2$ are two solutions to the submartingale problem for the same set of coefficients $a$, $b$, $\rho$ and $\gamma$, and if $\xi_1(t)$ and $\xi_2(t)$ are defined accordingly, then

$$E^{P_1}\int_{t_0}^\infty f(s, x(s))\, d\xi_1(s) = E^{P_2}\int_{t_0}^\infty f(s, x(s))\, d\xi_2(s)$$

for any bounded function on $[t_0, \infty) \times \partial G$ with compact support.

*Remark* 2. The point $\infty$ is only a device to overcome the fact that $T(\omega) < \infty$ may occur with positive probability. This has to do with the fact that the harmonic measure may be substochastic (i.e., with positive probability the process may stay away from the boundary, with the result that the boundary process will be nonconservative). In Section 5, where we use this result to prove uniqueness for the submartingale problem, $T(\omega)$ will be $\infty$ with probability 1.

*Remark* 3. Let $y$ be a point on the boundary and $R$ a solution to the submartingale problem starting from $y$ at time $t$. By Remark 1, if the uniqueness theorem holds for the boundary process, then

$$E^R\int_t^\infty f(s, x(s))\, d\xi(s) = q(t, y)$$

is uniquely determined. Further, if $x$ in $\bar{G}$ is any point, on or off the boundary, and $P$ is any solution to the submartingale problem starting from $x$ at time $t_0$, then

$$\begin{aligned} E^P\left[\int_{t_0}^\infty f(s, x(s))\, d\xi(s)\right] &= E^P\left[\int_\tau^\infty f(s, x(s))\, d\xi(s)\right] \\ &= E^P\left[E^P\left[\int_\tau^\infty f(s, x(s))\, d\xi(s) \mid \mathscr{M}_\tau^{t_0}\right]\right] \\ &= E^P[q(\tau, x(\tau))] \\ &= E^{Q_{t_0,x}}[q(\tau, x(\tau))] .\end{aligned}$$

Here $\tau$ is the first exit time from $G$ and $Q_{t_0,x}$ is the interior process corresponding to the given coefficients starting from $x$ at time $t_0$. Hence the uniqueness of the boundary process implies that

$$E^P\left[\int_{t_0}^{\infty} f(s, x(s))\, d\xi(s)\right]$$

is uniquely determined for all starting points on or off the boundary.

THEOREM 4.2. *Let $a$, $b$, $\rho$ and $\gamma$ be coefficients such that the uniqueness theorem holds for the boundary process. Then the solution to the submartingale problem for any starting time and place is unique.*

Proof: Let $x_0$ be the starting point and let $t_0$ be the starting time. Let $P_1$, $P_2$ be two solutions. Let $Q_{t,x}$ be the process corresponding to $a$ and $b$ in the interior and let $h(t, x)$ be a bounded continuous function with compact support in $[t_0, \infty) \times G$. Set

$$u(t, x) = E^{Q_{t,x}}\left[\int_t^{\tau} h(s, x(s))\, ds\right],$$

where $\tau$ is the first exit time from $G$ after time $t$. We define

$$g(s, x) = (J_s u)(s, x) + \rho(s, x) u_s(s, x).$$

Then $u \in F$ and

$$u(t, x(t)) - \int_{t_0}^{t} h(s, x(s))\, ds - \int_{t_0}^{t} g(s, x(s))\, d\xi_i(s)$$

is a martingale relative to $P_i$.

Since $u(t, x)$ has compact support in $t$, we have

$$u(t_0, x_0) = E^{P_i}\left[\int_{t_0}^{\infty} h(s, x(s))\, ds + \int_{t_0}^{\infty} g(s, x(s))\, d\xi_i(s)\right].$$

In view of Remark 3,

$$E^{P_1}\left[\int_{t_0}^{\infty} h(s, x(s))\, ds\right] = E^{P_2}\left[\int_{t_0}^{\infty} h(s, x(s))\, ds\right].$$

Since this is true for all bounded continuous functions $h(s, x)$ with compact support in $[t_0, \infty) \times G$, it is also valid for all functions vanishing identically for large $t$ which are bounded and supported in $G$. For all bounded functions $g$ on the boundary with compact support,

$$E^{P_1}\left[\int_{t_0}^{\infty} g(s, x(s))\, d\xi_1(s)\right] = E^{P_2}\left[\int_{t_0}^{\infty} g(s, x(s))\, d\xi_2(s)\right].$$



Taking $g$ to be a function with support in $\{(s, x) \in [t_0, \infty) \times \partial G \cap \{\rho > 0\}\}$, we have

$$E^{P_1}\left[\int_{t_0}^{\infty} g(s, x(s))\rho(s, x(s))\, ds\right] = E^{P_2}\left[\int_{t_0}^{\infty} g(s, x(s))\rho(s, x(s))\, ds\right].$$

Therefore,

$$E^{P_1}\left[\int_{t_0}^{\infty} g(s, x(s))\, ds\right] = E^{P_2}\left[\int_{t_0}^{\infty} g(s, x(s))\, ds\right]$$

for functions $g(s, x)$ with support on the boundary. Hence,

$$E^{P_1}\left[\int_{t_0}^{\infty} F(s, x(s))\, ds\right] = E^{P_2}\left[\int_{t_0}^{\infty} F(s, x(s))\, ds\right]$$

for all $F(s, x)$ bounded and vanishing for all large $s$. This in turn implies that

$$P_1[x(t) \in A] = P_2[x(t) \in A] \quad \text{for all} \quad t > t_0.$$

Our result now follows by taking r.c.p.d.

## 5. Uniqueness

Let $\bar{D}$ be a closed half-space, $D$ its interior and $\partial D$ its boundary. That is,

$$\bar{D} = \{x_1, \cdots, x_d : x_1 \geqq 0\},$$
$$D = \{x_1, \cdots, x_d : x_1 > 0\},$$
$$\partial D = \{x_1, \cdots, x_d : x_1 = 0\}.$$

Let us consider $[0, \infty) \times \bar{D}$ and suppose $\{a_{ij}(t, x)\}$ is a symmetric, positive definite continuous coefficient matrix such that $a_{ij}(t, x) = \delta_{ij}$ for $(t, x)$ lying outside a bounded neighborhood of $(t_0, x_0)$ and

$$|a_{ij}(t, x) - \delta_{ij}| \leqq \varepsilon \quad \text{for all} \quad i, j, t, x,$$

where $\varepsilon > 0$ will be chosen later on. $Q_{t,x}$ will denote the interior process corresponding to $\{a_{ij}(t,x)\}$ starting from an interior point $x$ at time $t$. $Q^0_{t,x}$ is the similar process for $\{\delta_{ij}\}$ (i.e., Brownian motion in $D$). The harmonic extension operators corresponding to these processes are defined by

$$(Hf)(t, x) = E^{Q_{t,x}}\{f(\tau, x(\tau))\}$$

and

$$(H_0 f)(t, x) = E^{Q^0_{t,x}}\{f(\tau, x(\tau))\},$$

$f(\cdot\,, \cdot)$ is a function on $[0, \infty) \times \partial D$. Clearly, $Hf$ and $H_0 f$ are functions defined in $[0, \infty) \times \bar{D}$ and agree with $f$ on the boundary.

By $\nabla$ we denote the gradient in $D$ and by $\tilde{\nabla}$ the gradient with respect to the tangential components on the boundary. From Theorem 7.1 of the appendix we know that, for any $\delta > 0$ and $1 < p < \infty$, there is an $\varepsilon$ such that

$$\|(H - H_0)f\|_p^{(2)} \leqq \delta \, |||f|||_{\sim p}^{(0,1)} .$$

The definition of these norms are given in the appendix. We now introduce two boundary operators acting on functions $f$ on the boundary such that $|||f|||_{\sim p}^{(0,1)}$ and $|||\tilde{\nabla} f|||_{\sim p}^{(0,1)}$ are finite. We take $p > d + 2$ and fix it. Then, by Remark 3, following Theorem 7.3 of the appendix,

$$(Kf)(t, x) = \langle \gamma(t, x), \nabla Hf(t, x)\rangle \quad \text{for} \quad x \in \partial D$$

is well defined and so is

$$(K_0 f)(t, x) = \langle \gamma_0 , \nabla H_0 f(t, x)\rangle \quad \text{for} \quad x \in \partial D .$$

Here $\gamma_0$ is a constant vector with $\langle \gamma_0 , e_1\rangle = 1$, where $e_1$ is the vector $(1, 0, \cdots, 0)$ or the unit normal into $D$; $\gamma$ is also normalized so that

$$\langle \gamma(t, x), e_1\rangle = 1 .$$

THEOREM 5.1. *Let $\delta > 0$ and $p > d + 2$ be given. Suppose that*

$$|\gamma(t, x) - \gamma(t', x')| \leqq A(|t - t'| + |x - x'|) \tag{5.1}$$

*for some finite $A$. Then there is an $\varepsilon > 0$, depending on $\delta$, $p$, and $A$, such that the inequalities*

$$|a_{ij}(t, x) - \delta_{ij}| \leqq \varepsilon \tag{5.2}$$

*and*

$$|\gamma(t, x) - \gamma_0| \leqq \varepsilon \tag{5.3}$$

*imply that*

$$|||Kf - K_0 f|||_{\sim p} \leqq \delta \, |||f|||_{\sim p}^{(0,1)} .$$

Proof:

$$\begin{aligned} Kf - K_0 f &= \langle \gamma, \nabla Hf\rangle - \langle \gamma_0 , \nabla h_0 f\rangle \\ &= \langle \gamma_0 , \nabla (H - H_0) f\rangle - \langle \gamma - \gamma_0 , \nabla Hf\rangle . \end{aligned}$$

Therefore,

$$|||Kf - K_0 f|||_{\sim p} \leqq \|\gamma_0\| \, |||\nabla (H - H_0) f|||_{\sim p} + |||\langle \gamma - \gamma_0 , \nabla Hf\rangle|||_{\sim p} .$$



The first term is estimated by

$$|||\nabla (H - H_0) f|||_{\sim p} \leqq C_1 \, \|(H - H_0) f\|_p^{(2)} \leqq C_2 \, \varepsilon \, |||f|||_{\sim p}^{(0,1)} ,$$

because $(H - H_0)f$ is identically zero for large enough $t$. Moreover, since the first component of $\gamma - \gamma_0$ is identically zero and $\gamma - \gamma_0$ is Lipschitz continuous, by Theorem 7.3, there is an $\varepsilon > 0$ such that if $\|\gamma - \gamma_0\| \leqq \varepsilon$, then

$$|||\langle \gamma - \gamma_0 , \nabla Hf\rangle|||_{\sim p} \leqq \tfrac{1}{2}\delta \, |||\tilde{\nabla} f|||_{\sim p} ;$$

q.e.d.

For each function $g$ which is $C^\infty$ with compact support in $[0, \infty) \times \partial D$ and for each $\lambda > 0$, we know that the equation

$$\lambda f - K_0 f = g$$

has a solution $f$ which is bounded and is again $C^\infty$. Moreover, from Theorem 7.2,

$$|||f|||_{\sim p}^{(0,1)} \leqq C_\lambda \, |||g|||_{\sim p}$$

and

$$|||f|||_{\sim p} \leqq C_\lambda \, |||g|||_{\sim p} ,$$

where $C_\lambda$ is bounded for $\lambda \geqq 1$. Therefore, by a perturbation argument,

$$(\lambda - K)^{-1} = (\lambda - K_0)^{-1}[I + (K_0 - K)(\lambda - K_0)^{-1}]^{-1} ,$$

and for $\lambda \geqq 1$

$$|||(K_0 - K)(\lambda - K_0)^{-1} g|||_{\sim p} \leqq C_4 \, \delta \, |||g|||_{\sim p} .$$

Hence, for suitably small $\delta$, the equation

$$\lambda f - Kf = g$$

has a solution for $\lambda \geqq 1$ such that

$$|||f|||_{\sim p}^{(0,1)} \leqq C \, |||g|||_{\sim p}$$

and

$$|||f|||_{\sim p} \leqq C \, |||g|||_{\sim p} .$$

THEOREM 5.2. *Let, for a suitably small $\varepsilon$, $a_{ij}$ satisfy (5.2) and be equal to $\delta_{ij}$ outside a bounded set. Let $b = 0$ and $\rho = 0$. If $\gamma$ satisfies (5.1) and (5.3), then the uniqueness theorem holds for the corresponding boundary process.*

Proof: Let $t_0$, $x_0$ be a starting point with $x_0 \in \partial D$. We have to show that there is only one measure $R_{t_0,x_0}$ on $D([0, \infty), [t_0, \infty) \times \partial D)$ such that

$$f(\tau(t), y(t)) - \int_0^t (Kf)(\tau(s), y(s))\, ds$$

is a martingale with respect to $R_{t_0,x_0}$, with

$$R_{t_0,x_0}[\tau(0) = t_0, y(0) = x_0] = 1 .$$

The martingale property, which is valid to start with for smooth functions, is easily extended to functions for which $|||f|||_{\sim p}$ and $|||f|||_{\sim p}^{(0,1)}$ are finite, provided $p > d + 2$. This is because such functions can be approximated by smooth functions $f_n$ with compact support in such a manner that

$$f_n \to f \quad \text{and} \quad Kf_n \to Kf \quad \text{uniformly} .$$

(See Remark 4 after Theorem 7.3 of the appendix.) Hence, for any solution $R$,

$$E^R[f(\tau(t), y(t))] = f(t, x) + E^R\left[\int_0^t (Kf)(\tau(s), y(s))\, ds\right] .$$

From this one passes easily to

$$\int_0^\infty e^{-\lambda s} E^R[g(\tau(s), y(s))]\, ds = f(t, x) ,$$

provided

$$\lambda f - Kf = g .$$

Therefore,

$$\int_0^\infty e^{-\lambda s} E^R[g(\tau(s), y(s))]\, ds$$

is uniquely determined for $g$ in $W_p([t_0, \infty) \times \partial D)$.[19] If one uses stochastic continuity, the uniqueness theorem for Laplace transforms guarantees the uniqueness of the marginals of $R$. By taking r.c.p.d. one gets the uniqueness of $R$; q.e.d.

We now turn to the elastic case (i.e., when $\rho > 0$). $H_0$, $H$ are as before. The new operators $K_0$ and $K$ are defined by

$$(Kf)(t, x) = \langle \gamma(t, x), \nabla Hf(t, x)\rangle + \rho(t, x) f_t(t, x) ,$$
$$(K_0 f)(t, x) = \langle \gamma_0 , \nabla H_0 f(t, x)\rangle + \rho_0 f_t(t, x) .$$

[19] $W_p([t_0, \infty) \times \partial D)$ is defined before Lemma 7.2 in the appendix.



In addition to the earlier assumptions on $\gamma$ and $\gamma_0$, we now suppose that $\rho$ satisfies a Lipschitz condition and that

$$|\rho(t, x) - \rho_0| \leqq \varepsilon , \tag{5.4}$$

where $\rho_0 > 0$. Of course, $\varepsilon$ is so small that $\rho(t, x) > 0$. As before, we can estimate

$$|||(K - K_0)f|||_{\sim p} \leqq \delta\, |||f|||_{\sim p}^{(1,1)}$$

provided $\varepsilon$ is chosen small depending on $\delta$, the Lipschitz constant $A$ and $\rho_0 > 0$. We can also solve the equation

$$\lambda f - K_0 f = g$$

for $g$ such that $g \in W_p([t_0, \infty) \times \partial D)$, and $\lambda \geqq 1$. Moreover,

$$|||f|||_{\sim p}^{(1,1)} \leqq C\, |||g|||_{\sim p}$$

with $C$ independent of $\lambda$.

We can again use a perturbation argument and solve

$$\lambda f - Kf = g$$

for $g \in W_p$ with an $f$ such that

$$|||f|||_{\sim p}^{(1,1)} \leqq C\, |||g|||_{\sim p},$$

provided $\varepsilon$ is chosen small enough. Armed with this, we can prove the following theorem in exactly the same manner as Theorem 5.2.

THEOREM 5.3. *If $a$, $b$, and $\gamma$ are as in Theorem* 5.2 *and $\rho$ satisfies* (5.4), *then for small $\varepsilon$ the uniqueness theorem holds for the corresponding boundary process.*

THEOREM 5.4. *Let $a$, $b$, $\rho$, and $\gamma$ satisfy the assumptions of Theorem* 5.2 *or of Theorem* 5.3. *Then the solution to the submartingale problem for $a$, $b$, $\rho$, and $\gamma$ is unique and depends continuously on $t$ and $x$.*

Proof: The uniqueness follows from the preceding theorems and from Section 4. Now let $t_n \to t$ and $x_n \to x$. We consider $P_n = P_{t_n,x_n}$. The proof consists of establishing the compactness of $P_n$ and checking that any limit $P$ of a subsequence of the $P_n$ is a solution for the submartingale problem starting at $t$ and $x$. We need the uniqueness theorem to prove convergence. Compactness is elementary, because we have the inequalities

$$E^{P_n}[|x(t_2) - x(t_1)|^2 \, |x(t_3) - x(t_2)|^2] \leqq C\,|t_3 - t_1|^2 \quad \text{for} \quad 0 \leqq t_1 \leqq t_2 \leqq t_3 < \infty .$$

The only other estimate needed is that, for the strip $G_\delta = \{x : 0 < x_1 < \delta\}$,

$$\sup_n E^{P_n}\left[\int_0^T \chi_{G_\delta}(x(s))\, ds\right] \to 0 \quad \text{as} \quad \delta \to 0$$

for each $T < \infty$. This is proved in the same way as in Theorem 3.1. The rest of the proof also follows that of Theorem 3.1.

THEOREM 5.5 *Let $a$, $\rho$, and $\gamma$ satisfy the hypotheses of Theorem* 5.2 *or Theorem* 5.3 *and let $b$ be a bounded measurable function. Then the uniqueness theorem holds for the submartingale problem with these coefficients and, moreover, the solution $P_{t,x}$ depends measurably on $t$ and $x$.*

Proof: By Corollary 2.4, if $Q_{t,x}$ is the unique solution to the problem corresponding to $a$, $(b = 0)$, $\rho$ and $\gamma$, then $Q_{t,x}$ and $P_{t,x}$ are equivalent to one another and

$$\left.\frac{dP_{t,x}}{dQ_{t,x}}\right|_{\mathcal{M}_s^t} = \exp\left\{\int_t^s \langle b(u, x(u))a^{-1}(u, x(u))\chi_D(x(u)), dx(u)\rangle - \frac{1}{2}\int_t^s \langle b(u, x(u))a^{-1}(u, x(u)), b(u, x(u))\rangle \chi_D(x(u))\, du\right\}.$$

This proves uniqueness. The continuity of $Q_{t,x}$ in $(t, x)$ implies the measurability of $P_{t,x}$ in $t$ and $x$; q.e.d.

Let $t_0 > 0$ and $x_0 \in \partial D$. Let $S$ be a spherical neighborhood of $x_0$ in $\bar{D}$. Let $\varepsilon > 0$ be arbitrary and take $\tau$ to be the minimum of $t_0 + \varepsilon$ and the first exit time from $S$ after time $t_0$. A measure $P$ on $(\Omega, \mathcal{M}_\tau^{t_0})$ is said to solve the *submartingale problem for $a$, $b$, $\rho$, and $\gamma$ up to time $\tau$ starting from $(t_0, x_0)$ if*

(i) $P(x(t_0) = x_0) = 1$,

(ii) $f(t \wedge \tau, x(t \wedge \tau)) - \int_{t_0}^{t\wedge\tau} (f_u + Lf)(u, x(u))\chi_D(x(u))\, du$

is a $P$-submartingale for $f \in C_0^\infty([t_0, \infty) \times D)$ satisfying $\rho f_u + \langle \gamma, \nabla f\rangle \geqq 0$ on $[t_0, \infty) \times \partial D$.



THEOREM 5.6. *Let $a$, $b$, $\rho$, and $\gamma$ be defined on $[t_0, t_0 + \varepsilon] \times S$ and have extensions $a'$, $b'$, $\rho'$, and $\gamma'$ to $[0, \infty) \times \bar{D}$ satisfying the hypotheses of Theorem* 5.5. *Then the solution to the corresponding submartingale problem starting from $(t_0, x_0)$ up to time $\tau$ is unique.*

Proof: If $P$ is a solution up to time $\tau$ for $a$, $b$, $\rho$, and $\gamma$, then by adjoining to $P$ a solution for $a'$, $b'$, $\rho'$, and $\gamma'$, we can obtain a solution $P'$ for $a'$, $b'$, $\rho'$ and $\gamma'$ which agrees with $P$ on $\mathcal{M}_\tau^{t_0}$. Since $P'$ is unique, $P$ must be unique on $\mathcal{M}_\tau^{t_0}$; q.e.d.

We now turn to the region $G$ and coefficients $a$, $b$, $\rho$ and $\gamma$ defined on $[0, \infty) \times \bar{G}$. We suppose

(i) $\{a_{ij}(t, x)\}$ is bounded, continuous, symmetric and strictly positive definite for each $(t, x)$,

(ii) $b$ is bounded and measurable on $[0, \infty) \times \bar{G}$,

(iii) $\gamma$ is a locally Lipschitz, bounded vector function on $[0, \infty) \times \partial G$ and $\langle \gamma(t, x), \nabla\phi(x)\rangle \geqq \beta > 0$,

(iv) either $\rho \equiv 0$, or $\rho$ is a bounded locally Lipschitz function which is strictly positive at each point of $[0, \infty) \times \partial G$.

THEOREM 5.7. *Under the above assumptions, the solution to the submartingale problem for $a$, $b$, $\rho$ and $\gamma$ is unique for each starting point $(t, x)$.*

Proof: Let us suppose that the following conditions hold:

1. For each $t_0 \geqq 0$ and $x_0 \in \partial G$, there is an $\varepsilon > 0$ and a neighborhood $N_{x_0}$ of $x_0$ in $\bar{G}$ such that, if $\tau$ is the minimum of the exit time from $N_{x_0}$ and $t_0 + \varepsilon$, the solution to the submartingale problem for $a$, $b$, $\rho$ and $\gamma$ up to time $\tau$ starting from $(t_0, x_0)$ is unique.

2. $\varepsilon$ and the size of $N_{x_0}$ stay uniformly positive as $(t_0, x_0)$ varies over a compact subset of $[0, \infty) \times \partial G$.

Then obviously the uniqueness theorem is valid for all time. Indeed, since uniqueness is valid up to the first exit time from $G$, by taking r.c.p.d., we need only prove uniqueness for starting points on the boundary. Then we use the stopping time described in condition 1 and we get uniqueness up to $\tau$. Repeating this process indefinitely, we get a sequence of stopping times $\tau_1, \tau_2, \cdots, \tau_n, \cdots$ such that the r.c.p.d. of $P$ given $\mathcal{M}_{\tau_j}^t$ is unique on $\mathcal{M}_{\tau_{j+1}}^{\tau_j}$ for each $j$. We also have uniqueness on $\mathcal{M}_{\tau_1}^{t_0}$. Since condition 2 guarantees that $\lim_{j\to\infty} \tau_j = \infty$ a.s., we have uniqueness on $\mathcal{M}^{t_0}$.

We need only prove that (i) and (ii) are valid.

Let $x_0 \in \partial G$ be given. Then there exists a neighborhood $N'_{x_0}$ of $x_0$ which can be mapped by a twice continuously differentiable map $\Phi$ onto a spherical neighborhood $S_0$ in $\bar{D}$ of a point $y_0$ on $\partial D$. The transformed boundary is part of a hyperplane. By choosing a linear map, if necessary, we can always assume that the transformed coefficients $a'$, $b'$, $\rho'$ and $\gamma'$ are such that

$$a'_{ij}(t_0, y_0) = \delta_{ij} .$$

Moreover, since the ratio of $\rho'$ and $\gamma'$ is the only relevant quantity, we may always assume that

$$\langle \gamma', e_1 \rangle = 1 .$$

By computing $a'$, $b'$, $\rho'$ and $\gamma'$ in terms of the old coefficients, we check that they satisfy the hypotheses of Theorem 5.6. We may have to choose a smaller neighborhood $S$ and consider everything there. Since the basic transformation $\Phi$ takes solutions to the submartingale problem into solutions for the transformed coefficients in a one-to-one manner, we get uniqueness for $a$, $b$, $\rho$ and $\gamma$ up to the time $\tau$, which is the smaller of $t_0 + \varepsilon$ and the escape time from the inverse image $N_{x_0}$ of $S$.

That condition 2 is valid is obvious because these various regions are uniformly large, since, on compact sets, everything is uniform; q.e.d.

We now turn to the homogeneous case (i.e., we assume that $a$, $b$, $\rho$ and $\gamma$ are functions of $x$ alone). In this case, the uniqueness theorem can be strengthened by weakening the assumptions on $\rho$. We assume that $a$, $b$ and $\gamma$ satisfy the same assumptions as before, but $\rho(x)$ is only assumed to be non-negative, continuous and bounded on $\partial G$. Since the transformation taking a small neighborhood of a point on the boundary of the half-space is time independent, the time homogeneity is not lost by such a transformation. Therefore, we must prove uniqueness only in the case which is analogous to Theorems 5.2 and 5.3.

The clue is the fact that, in this case, $K$ commutes with time translations. Let us use the earlier notation and define

$$(Kf)(t, x) = \langle \gamma(x), (\nabla Hf)(t, x) \rangle + \rho(x) f_t(t, x) ,$$
$$(K'f)(t, x) = \langle \gamma(x), \nabla Hf(t, x) \rangle .$$

We have the uniqueness theorem for $K'$ and we want to prove the uniqueness theorem for $K$. Let $R_{t,x}$ be a solution to the martingale problem for $K$. Then, if $(\tau(s), y(s))$ is a trajectory, we define

$$\tau'(s) = \tau(s) - \int_0^s \rho(y(u))\, du$$

and consider the process $R'_{t,x}$ corresponding to $(\tau'(s), y(s))$. We show now that $R'_{t,x}$ is a solution for $K'$ and, therefore, is uniquely determined. Since



$$\tau(s) = \tau'(s) + \int_0^s \rho(y'(u))\, du$$

and

$$y(s) = y'(s) ,$$

this means that $R_{t,x}$ is also unique.

We start with a measure $R$ such that for suitable functions $f$,

$$f(\tau(t), y(t)) - \int_0^t (Kf)(\tau(s), y(s))\, ds$$

is a martingale. It is easy to generalize this formula to functions $g$ depending on time $t$ as well. In fact,

$$g(t, \tau(t), y(t)) - \int_0^t (g_s + Kg)(s, \tau(s), y(s))\, ds$$

is a martingale. We choose $g$ to be of the form

$$g(t, \tau, y) = f(\tau - \rho t, y) .$$

Since $K$ commutes with translations in the $\tau$ direction,

$$f(\tau(t) - \rho t, y(t)) - \int_0^t (Kf)(\tau(s) - \rho s, y(s))\, ds + \int_0^t f_s(\tau(s) - \rho s, y(s))\, ds$$

is a martingale. Since this is true for any constant $\rho$, it must also be true for simple non-anticipating functions $\rho(t)$. In particular,

$$f(\tau(t) - \eta(t), y(t)) - \int_0^t (Kf)(\tau(s) - \eta(s), y(s))\, ds + \int_0^t f_s(\tau(s) - \eta(s), y(s))\, ds$$

is a martingale, where

$$\eta(t) = \int_0^t \rho(s, \omega)\, ds ,$$

$$\rho(s, \omega) = \rho(y(\sigma_j)) \qquad \text{for} \qquad \sigma_j \leqq s < \sigma_{j+1} ,$$

and the $\sigma_j$ are the successive exit times from small neighborhoods. By letting these neighborhoods shrink, we obtain

$$\eta(t) = \int_0^t \rho(y(s))\, ds$$

and see that

$$f(\tau'(t), y(t)) - \int_0^t (K'f)(\tau'(s), y(s))\, ds$$

is a martingale. We have, therefore, proved

THEOREM 5.8. *If $a$, $b$, $\rho$ and $\gamma$ are time independent, then the following conditions are sufficient for the uniqueness of the solution to the submartingale problem:*

(i) *$a(x)$ is continuous, symmetric and positive definite at each point of $\bar{G}$,*

(ii) *$b(x)$ is bounded and measurable,*

(iii) *$\gamma$ is bounded, locally Lipschitz and $\langle \gamma(x), \nabla\phi(x)\rangle \geqq \beta > 0$ on $\partial G$,*

(iv) *$\rho(x)$ is bounded, continuous and non-negative.*

To complete this section, we wish to state a few results which can be proved in a routine manner (cf. [8]).

1. By taking r.c.p.d., one sees that any time uniqueness holds, the solutions are strong Markov processes.

2. The process $P_{t,x}$ corresponding to a given $a$, $b$, $\rho$ and $\gamma$ for which the uniqueness holds is a Feller process. This is proved by establishing compactness, taking a subsequence that converges, and identifying any limiting process as the unique solution corresponding to the limiting starting point.

3. A similar theorem could be proved on the convergence of $P^n_{t,x}$ to $P_{t,x}$ if $a_n$, $b_n$, $\rho_n$ and $\gamma_n$ converge suitably to $a$, $b$, $\rho$ and $\gamma$. Essentially, we need compactness and uniqueness for the coefficients.

## 6. An Invariance Principle

$G$ is a region in $R^d$ with boundary $\partial G$. $\bar{G}$ stands for $G \cup \partial G$. $G$ is such that there exists a function $\phi(x)$ in $C_b^2(R^d)$ with

$$G = [x : \phi(x) > 0]\,,$$
$$\partial G = [x : \phi(x) = 0]\,,$$
$$\|\nabla\phi(x)\| \geqq 1 \quad \text{on} \quad \partial G\,.$$

For each $h > 0$ we are given a Markov chain with $\bar{G}$ as its state space. The transitions of this Markov chain occur at times that are multiples of $h$.



Let $\Pi_j^h(x, dy)$ denote the transition probabilities of this chain at time $jh$. We assume that the chain is defined for $0 \leqq jh \leqq T$.

Let $P_z^h$ denote the measure corresponding to this chain starting from the point $z \in \bar{G}$ at time 0 on the space $\Omega = D([0, T], \bar{G})$. If a transition occurs at time $T$ it will be ignored. $P_z^h$ is therefore defined by the following relations:

$$P_z^h[x(0) = z] = 1\,,$$
$$P_z^h[x(t) = x(jh) \quad \text{for} \quad jh \leqq t < (j+h)h] = 1\,,$$
$$P_z^h[x((j+1)h) \in A \mid \mathscr{M}_{jh}] = \Pi_j^h(x(jh), A)\,.$$

Here $x(t)$ is the coordinate function at time $t$ and $\mathscr{M}_t$ stands for the $\sigma$-field generated by $x(s)$ for $0 \leqq s \leqq t$. The customary demand that $x(T-0)$ be equal to $x(T)$ for elements of $D([0, T], \bar{G})$ forces us to disregard any transition occurring at time $T$.

We consider the following functions defined in terms of the $\Pi_j^h(x, dy)$:

$$\Delta^h(jh, x) = \frac{1}{h}\int \|y - x\|^{2+\alpha}\, \Pi_j^h(x, dy)\,,$$

$$a^h(jh, x) = \frac{1}{h}\int (y - x) \otimes (y - x) \Pi_j^h(x, dy)\,,$$

$$b^h(jh, x) = \frac{1}{h}\int (y - x) \Pi_j^h(x, dy)\,.$$

Here $\alpha > 0$ is arbitrary but fixed. The $b^h$ are of course vectors and the $a^h$ are positive semi-definite matrices. We extend the definition of these functions from points of the form $jh$ to all times $t$ by defining

$$\Delta^h(t, x) = \Delta^h(jh, x) \quad \text{for} \quad jh \leqq t < (j+1)h\,,$$
$$a^h(t, x) = a^h(jh, x) \quad \text{for} \quad jh \leqq t < (j+1)h\,,$$
$$b^h(t, x) = b^h(jh, x) \quad \text{for} \quad jh \leqq t < (j+1)h\,.$$

We shall make the following assumptions regarding these functions:

A. $$\sup_{0 \leqq t \leqq T}\, \sup_{x \in \bar{G}} \Delta^h(t, x) = \Delta(h)\,,$$

$$\Delta(h) \to 0 \quad \text{as} \quad h \to 0\,.$$

B. There exists a constant $M < \infty$ and a constant $c > 0$ such that

$$\|b^h(t, x)\| > M$$

implies

$$\langle \nabla\phi(x), b^h(t, x)\rangle \geqq c \, \|b^h(t, x)\| .$$

C. For every $\delta > 0$, there exists a constant $M_\delta < \infty$ such that

$$\|b^h(t, x)\| > M_\delta$$

implies

$$\phi(x) < \delta .$$

D. There exists a constant $M < \infty$ such that

$$\sup_{0 \leqq t \leqq T} \sup_{x \in \bar{G}} \|a^h(t, x)\| \leqq M .$$

*Remark.* The norm for matrices shall be the square root of $d$ times the sum of squares of all the entries. In particular, $\operatorname{Tr} a \leqq \|a\|$. Conditions B and C guarantee that if there are large drifts at all, they must occur close to the boundary and be directed uniformly inward and never outward or tangentially.

THEOREM 6.1. *Under hypotheses* A–D, *for any compact set* $K \subset \bar{G}$, *the family*

$$\{P_z^h : h > 0, z \in K\}$$

*is conditionally compact in* $D([0, T], \bar{G})$. *Moreover, any weak limit of these is concentrated on the subset* $C([0, T], \bar{G}) \subseteq D([0, T], \bar{G})$.

The proof of Theorem 6.1 will follow after a few lemmas.

LEMMA 6.1. *There exists a constant* $A < \infty$ *such that, with respect to any* $P_z^h$,

$$h\sum_{j=0}^{k-1} \|b^h(jh, x(jh))\| - A\phi(x(kh)) - Akh$$

*is a supermartingale relative to the* $\sigma$*-algebras* $\mathscr{M}_{kh}$ .

Proof: From the relation of the process $P_z^h$ to the transition probabilities $\{\Pi_j^h(x, dy)\}$, it follows that

$$\phi(x(kh)) - \sum_{j=0}^{k-1} \int [\phi(y) - \phi(x(jh))]\Pi_j^h(x(jh), dy)$$

is a martingale relative to $\mathscr{M}_{kh}$ and $P_z^h$. Let

$$\theta(jh, x) = \int [\phi(y) - \phi(x)]\Pi_j^h(x, dy) .$$



Since $\phi \in C_b^2(\bar{G})$, there exists a $C < \infty$ such that

$$|\phi(y) - \phi(x) - \langle y - x, \nabla\phi\rangle| \leqq C_1 |y - x|^2 . \tag{6.1}$$

By integrating (6.1) with respect to $\Pi_j^h(x, dy)$, we get

$$\begin{aligned} |\theta(jh, x) - h\langle b^h(jh, x), \nabla\phi(x)\rangle| &\leqq C_1 h \operatorname{Tr} a^h(jh, x) \\ &\leqq C_2 h . \end{aligned}$$

Therefore,

$$\theta(jh, x) \geqq h\langle b^h(jh, x), \nabla\phi(x)\rangle - C_2 h . \tag{6.2}$$

If $\|b^h(jh, x)\| \leqq M$, we have

$$\langle b^h(jh, x), \nabla\phi(x)\rangle \geqq -MC , \tag{6.3}$$

since $|\nabla\phi(x)| \leqq C$. If $\|b^h(jh, x)\| > M$, then, by hypothesis, we have

$$\langle b^h(jh, x), \nabla\phi(x)\rangle \geqq c \, \|b^h(jh, x)\| . \tag{6.4}$$

Therefore, combining (6.3) and (6.4), there exists a constant $C_3$ such that

$$\langle b^h(jh, x), \nabla\phi(x)\rangle \geqq c \, \|b^h(jh, x)\| - C_3 .$$

Using this and (6.2), we can find a constant $A$ such that

$$A\theta(jh, x) \geqq h \, \|b^n(jh, x)\| - Ah .$$

Since

$$\sum_{j=0}^{k-1} \theta(jh, x(jh)) - \phi(x(kh))$$

is a martingale,

$$h\sum_{j=0}^{k-1} \|b^h(jh, x)\| - Ah - A\phi(x(kh))$$

is a supermartingale.

LEMMA 6.2. *For any* $\delta > 0$, *there exists a constant* $M_\delta < \infty$ *such that, relative to* $\mathscr{M}_{kh}$ *and any* $P_z^h$,

$$\phi^2(x(kh)) - \delta h\sum_{j=0}^{k-1} \|b^h(jh, x(jh))\| - M_\delta \, kh$$

*is a supermartingale.*

Proof: Let

$$\theta(jh, x) = \int [\phi^2(y) - \phi^2(x)]\Pi_j^h(x, dy) .$$

Since $\phi^2$ is in $C_b^2(\bar{G})$, there exists a constant $C_1$ such that

$$|\phi^2(y) - \phi^2(x) - 2\phi(x)\langle y - x, \nabla\phi(x)\rangle| \leqq C_1 |y - x|^2 .$$

Therefore, as in Lemma 6.1, we have

$$\theta(jh, x) \leqq C_2 h + 2\phi(x)\langle b^h(jh, x), \nabla\phi(x)\rangle h . \tag{6.5}$$

Since $\|\nabla\phi(x)\|$ is bounded, given $\delta > 0$, there exists $M'_\delta < \infty$ such that

$$\|b^h(jh, x)\| > M'_\delta \Rightarrow 2\phi(x)\ \|\nabla\phi(x)\| < \delta .$$

Now if $\|b^h(jh, x)\| \leqq M'_\delta$, we have

$$2\phi(x)\langle b^h(jh, x), \nabla\phi(x)\rangle \leqq 2C_3 M'_\delta , \tag{6.6}$$

since $2\phi(x)\ \|\nabla\phi(x)\| \leqq C_3$. On the other hand, if $\|b^h(jh, x)\| > M'_\delta$,

$$2\phi(x)\langle b^h(jh, x), \nabla\phi(x)\rangle \leqq \delta\ \|b^h(jh, x)\| . \tag{6.7}$$

In any case, combining (6.6) and (6.7), we have

$$2\phi(x)\langle b^h(jh, x), \nabla\phi(x)\rangle \leqq (1 + 2C_3)M'_\delta + \delta\ \|b^h(jh, x)\| . \tag{6.8}$$

Therefore, there is a constant $M_\delta < \infty$ such that

$$\theta(jh, x) \leqq M_\delta h + \delta h\ \|b^h(jh, x)\| .$$

Since

$$\phi^2(x(kh)) - \sum_{j=0}^{k-1}\theta(jh, x(jh))$$

is a martingale, the proof of the lemma is complete.

Proof of Theorem 6.1: We want to estimate the quantity

$$E^{P_z^h}[\|x(kh) - x(rh)\| \mid \mathscr{M}_{rh}] .$$

Let us define

$$y(kh) = x(kh) - \sum_{l=0}^{k-1} hb^h(lh, x(lh)) .$$



Then $y(kh)$ is a martingale. Hence,

$$\begin{aligned} E^{P_z^h}[\|y(kh) - y(rh)\|^2 \mid \mathscr{M}_{rh}] &= \sum_{l=r}^{k-1} E^{P_z^h}[\|y(l+1)h) - y(lh)\|^2 \mid \mathscr{M}_{rh}] \\ &\leqq \sum_{l=r}^{k-1} E^{P_z^h}[\|x(l+1)h) - x(lh)\|^2 \mid \mathscr{M}_{rh}] \\ &\leqq (k - r)Mh . \end{aligned}$$

Therefore,

$$E^{P_z^h}[\|y(kh) - y(rh)\| \mid \mathscr{M}_{rh}] \leqq [M(k - r)h]^{1/2} .$$

It follows from this that

$$\begin{aligned} &E^{P_z^h}[\|x(kh) - x(rh)\| \mid \mathscr{M}_{rh}] \\ &\quad \leqq [M(k - r)h]^{1/2} + E^{P_z^h}\Big[h\sum_{l=r}^{k-1}\|b^h(lh, x(lh))\|\Big] \\ &\quad \leqq [M(k - r)h]^{1/2} + [A(k - r)h] + AE^{P_z^h}[\phi(x(kh)) - \phi(x(rh)|\mathscr{M}_{rh}] . \end{aligned}$$

To complete the proof we estimate

$$\begin{aligned} E^{P_z^h}[\phi(x(kh)) \mid \mathscr{M}_{rh}] &\leqq \{E^{P_z^h}[\phi^2(x(kh) \mid \mathscr{M}_{rh}]\}^{1/2} \\ &\leqq \phi^2(x(rh)) + M_\delta(k - r)h + \delta C^{1/2} \\ &\leqq \phi(x(rh)) + M_\delta^{1/2}((k - r)h)^{1/2} + \delta^{1/2}C^{1/2} . \end{aligned}$$

Consequently, for any $\delta > 0$, there exists a constant $C_\delta$ such that

$$E^{P_z^h}[\|x(kh) - x(rh)\| \mid \mathscr{M}_{rh}] \leqq \delta + C_\delta[(k - r)h]^{1/2} .$$

According to Theorem 7.5 of the appendix, this implies compactness in $D([0, T], \bar{G})$. To check that the limits are in $C([0, T], \bar{G})$, we note that

$$P_z^h\Big[\sup_{0 \leqq jh \leqq T}\|x((j + 1)h) - x(jh)\| \geqq \varepsilon\Big] \leqq \left(\left[\frac{T}{h}\right] + 1\right)\frac{\Delta(h)}{\varepsilon^{2+\rho}} \to 0 \quad \text{as} \quad h \to 0 ,$$

by hypothesis.

THEOREM 6.2. *Let $f \in C_b^{1,2}([0, T] \times \bar{G})$. Define*

$$\begin{aligned} F_j^h(x) = \int [f((j + 1)h, y) - f(jh, x)]\Pi_j^h(x, dy) - hf_t(jh, x) \\ - h\langle b^h(jh, x) \cdot \nabla f(jh, x)\rangle - \tfrac{1}{2}h \operatorname{Tr}[a^h(jh, x) \cdot D^2 f(jh, x)] . \end{aligned} \tag{6.9}$$

[20]

[20] $D^2 f$ stands for the Hessian matrix $((\partial^2 f/\partial x_i\, \partial x_j))$.

*Then, for* $K \subset\subset G$,

$$\lim_{h\to 0}\sup_{z\in K} E^{P_z^h}\Big[\sum_{0\le jh\le T} |F_j^h(x(jh))|\Big] = 0 .$$

Proof: Since the family $P_z^h$, $h > 0$ and $z \in K$, is compact, we can find for any $\varepsilon > 0$ an $M$ such that

$$\sup_{z\in K} P_z^h\Big[\sup_{0\le t\le T} |x(t)| \geqq M\Big] \leqq \varepsilon .$$

Clearly,

$$F_j^h(x) = \int H_j^h(x, y)\Pi_j^h(x, dy) ,$$

where

$$H_j^h(x, y) = f((j+1)h, y) - f(jh, x) - hf_t(jh, x) - (y - x)\nabla f(jh, x) - \tfrac{1}{2}\operatorname{Tr}[(y-x)^2 \cdot D^2 f(jh, x)] .$$

From our hypotheses we know that

$$|H_j^h(x, y)| \leqq C_1 |x - y|^2 + C_2 h ,$$

and so

$$|F_j^h(x)| \leqq Ch .$$

To complete the proof of the theorem, it suffices to prove that

$$\sup_{\substack{x\in G\\ |x|\le M}} |F_j^h(x)| = o(h) \quad \text{as} \quad h \to 0 \quad \text{for every} \quad M < \infty .$$

By Taylor's theorem, $|H_j^h(x, y)| = o(h) + o(|x - y|^2)$ uniformly for $x \in \bar G$ and $|x| \leqq M$. Hence, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|H_j^h(x, y)| \leqq \varepsilon h + \varepsilon |x - y|^2 ,$$

provided $|x - y| \leqq \delta$ and $h \leqq \delta$. Therefore, if $h \leqq \delta$,

$$\begin{aligned}|F_j^h(x)| &\leqq \Big[\int_{|x-y|\le\delta} H_j^h(x, y)\Pi_j^h(x, dy)\Big] + \Big[\int_{|x-y|>\delta} H_j^h(x, y)\Pi_j^h(x, dy)\Big]\\ &\leqq \varepsilon\int |y - x|^2\,\Pi_j^h(x, dy) + \Big[\int_{|x-y|>\delta} |x - y|^2\,\Pi_j^h(x, dy)\Big] + o(h)\\ &\leqq \varepsilon Ch + o(h) .\end{aligned}$$



Thus,

$$\sup_{\substack{x\in G\\ |x|\le M}} |F_j^h(x)| = o(h) \qquad \text{as} \qquad h \to 0 ;$$

q.e.d.

We assume the following concerning the behavior of the coefficients $a^h(t, x)$ and $b^h(t, x)$ defined for $(t, x) \in [0, T] \times \bar G$ as $h \to 0$.

1. $a^h \to a$ and $b^h \to b$ uniformly on compact subsets $K \subset [0, T] \times G$, where $a$ and $b$ are continuous diffusion and drift coefficients defined on $[0, T] \times G$.

2. There is a bounded continuous function $\rho(t, y) \geqq 0$ and a bounded continuous vector function $\gamma(t,y)$ on $[0, T] \times \partial G$ such that $\langle \gamma(t,y), \nabla\phi(y)\rangle = 1$ on $[0, T] \times \partial G$. (This is a convenient normalization. It is only the ratio of $\gamma$ to $\rho$ which is relevant.) The $a^h$ and $b^h$ are related to $\rho$ and $\gamma$ as $h \to 0$ in the following manner. Let

$$J_0 = [(t,y) : \rho(t,y) = 0] \quad \text{and} \quad J_1 = [(t,y) : \rho(t,y) > 0] .$$

(i) Given $(t, y) \in J_1$ and $\varepsilon > 0$, there exist $h_0 > 0$, $\delta_0 > 0$ such that if $|t - s| < \delta_0$, $|x - y| < \delta_0$, $h < h_0$ and $\langle \nabla\phi(x), a^h(s, x)\nabla\phi(x)\rangle < \delta_0$, we have

$$\|a^h(s, x)\| < \varepsilon \qquad \text{and} \qquad \|b^h(s, x) - \rho^{-1}(t, y)\gamma(t, y)\| < \varepsilon .$$

(ii) Given $(t, y) \in J_1$, there exist $\delta_0 > 0$ and $M_0 < \infty$ such that, if $|s - t| < \delta_0$ and $|y - x| < \delta_0$, we have

$$\|b^h(s, x)\| \leqq M_0 \quad \text{for all} \quad h .$$

(iii) Given $(t, y) \in J_0$ and $M < \infty$, there exist $\delta_0 > 0$ and $h_0 > 0$ such that, if

$$|t - s| < \delta_0 , \quad |x - y| < \delta_0 , \quad h < h_0 \quad \text{and} \quad \langle \nabla\phi(x), a^h(s, x)\nabla\phi(x)\rangle < \delta_0 ,$$

we have

$$\|b^h(s, x)\| \geqq M .$$

(iv) Given $(t, y) \in J_0$ and $\varepsilon > 0$, there exist $\delta_0 > 0$, $h_0 > 0$ and $N_0 < \infty$ such that, if

$$|s - t| < \delta_0 , \quad |x - y| < \delta_0 , \quad h < h_0 \quad \text{and} \quad \|b^h(s, x)\| > N_0 ,$$

we have

$$\Big\| \frac{b^h(s, x)}{\langle b^h(s, x), \nabla\phi(x)\rangle} - \gamma(t, y) \Big\| < \varepsilon .$$

THEOREM 6.3. *Let $h_n \to 0$, $z_n \to z$ and let $P_{z_n}^{h_n}$ converge weakly to a limit $P$. Then $P$ has the following properties*:

(i) $P\{x(0) = z\} = 1$.

(ii) *For any function $f$ in $C_0^{1,2}([0, T] \times \bar{G})$ with $\rho(\partial f/\partial t) + \langle \gamma, \nabla f \rangle \geqq 0$ on $[0, T] \times \partial G$,*

$$f(t, x(t)) - \int_0^t (f_s + L_s f)(s, x(s))\chi_G(x(s))\, ds$$

*is a submartingale, where*

$$(L_s f) = \tfrac{1}{2} \sum_{i,j=1}^{d} a_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_{j=1}^{d} b_j \frac{\partial f}{\partial x_j}.$$

The proof of the theorem will be deferred until a few lemmas have been proved. We denote $P_{z_n}^{h_n}$ simply by $P_n$ and $a^{h_n}$ and $b^{h_n}$ by $a_n$ and $b_n$, respectively.

LEMMA 6.3. *There is a constant $C$ such that*

$$E^{P_n}\left[\int_0^T \|b_n(s, x(s))\|\, ds\right] \leqq C.$$

Proof: By Lemma 6.1, if $k_n$ is the largest integer such that $k_n h_n \leqq T$, then

$$h_n \sum_{j=0}^{k_n - 1} \|b_n(jh, x(jh))\| = \int_0^{k_n h_n} \|b_n(s, x(s))\|\, ds \qquad \text{a.s. } P_n,$$

and the expectation of this expression is bounded by that of

$$A\phi(x(k_n h_n)) + Ak_n h_n.$$

Since $\phi$ is bounded by $C_1$, we have

$$E^{P_n}\left[\int_0^{h_n k_n} \|b_n(s, x))\|\, ds\right] \leqq A(C_1 + T).$$

Moreover, it is obvious that

$$\int_{k_n h_n}^T \|b_n(s, x(s))\|\, ds \leqq h_n \sup_{s,x} \|b_n(s, x)\|.$$



Since $\int \|y - x\|^2\, \Pi_j^h(x, dy) \leqq Mh$, we also have

$$\left\| \int (y - x)\Pi_j^h(x, dy) \right\| \leqq \sqrt{Mh}.$$

Therefore,

$$\|b_n(s, x)\| \leqq \frac{C_2}{\sqrt{h_n}}.$$

This proves Lemma 6.3.

LEMMA 6.4. *Let $B$ be a compact subset of the boundary $[0, T] \times \partial G$, let $U$ be a neighborhood of $B$, and take $\delta > 0$. Define*

$$f_n(s, x) = \begin{cases} 1 & \text{if } \langle \nabla\phi(x), a_n(s, x)\nabla\phi(x)\rangle \geqq \delta, \\ 0 & \text{otherwise}. \end{cases}$$

*Then*

$$\lim_{U \downarrow B} \limsup_{n\to\infty} E^{P_n}\left[\int_0^T f_n(s, x(s))\chi_U(x(s))\, ds\right] = 0.$$

Proof: Let $\rho > 0$ be arbitrary. Consider the function $\psi_\rho(x)$ defined by

$$\psi_\rho(x) = \begin{cases} 0 & \text{for} \quad \phi(x) \geqq \rho, \\ \dfrac{1}{\rho}(\rho - \phi(x))^3 & \text{for} \quad \phi(x) \leqq \rho; \end{cases}$$

$\psi_\rho(x)$ is clearly in $C^2(\bar{G})$ and by differentiating we find that there is a constant $C$ such that

$$\|\nabla\psi_\rho(x)\| \leqq C\rho \tag{6.10}$$

and

$$\left\| D_i D_j \psi_\rho(x) - \left(1 - \frac{\phi(x)}{\rho}\right)(D_i \phi(x))(D_j \phi(x)) \right\| \leqq C\rho.$$

Also we have $0 \leqq \psi_\rho(x) \leqq \rho^2$, which implies that

$$E^{P_n}[\psi_\rho(x(T)) - \psi_\rho(x(0))] \leqq \rho^2.$$

From Theorem 6.2 and the definition of $P_n$, this implies that

$$\overline{\lim_{n\to\infty}}\, E^{P_n}\left[\int_0^T [\langle b_n(s, x(s)), \nabla\psi_\rho(x(s))\rangle + \operatorname{Tr} a_n(s, x(s)) \cdot D^2 \psi_\rho(x(s))]\, ds\right] \leqq \rho^2,$$

which in turn implies that there exists a constant $C_1$ with

$$\overline{\lim_{n\to\infty}} E^{P_n}\left[\int_0^T 6\left(1-\frac{\phi(x)}{\rho}\right)\langle\nabla\phi(x(s)),\, a_n(s, x(s))\,\nabla\phi(x(s))\rangle\, ds\right] \leqq \rho^2+\rho C_1 .$$

We have used here the bound (6.10) for $\|\nabla\psi_\rho(x)\|$, Lemma 6.3, and the bounds for $\|a_n(s, x)\|$. Since $\phi(x)=0$ on $\partial G$, $U\subset\{x:\phi(x)<\frac{1}{2}\rho\}$ implies that, on $U$,

$$6\left(1-\frac{\phi(x)}{\rho}\right)\geqq 3$$

and $\langle\nabla\phi(x),\, a_n(s, x(s))\,\nabla\phi(x)\rangle \geqq \delta f_n(x)$. Therefore we have

$$\overline{\lim_{n\to\infty}} E^{P_n}\left[\int_0^T f_n(s, x)\chi_U(x(s))\, ds\right] \leqq \frac{\rho^2+3C_1}{3\delta}$$

which proves the lemma.

Let $\Phi$ be a bounded non-negative continuous function on $D([0, T], \bar{G})$. Let us define the measures $Q_n$ and $Q$ by

$$dQ_n=\Phi\, dP_n$$

and

$$dQ=\Phi\, dP .$$

Then $Q_n$ converges to $Q$ weakly. Let us define the following measures $\mu_n$ and $\mu$ on $[t_1, t_2]\times\bar{G}$, where $0\leqq t_1<t_2\leqq T$, by setting

$$\int u(t, x)\, d\mu_n = E^{Q_n}\left[h_n\sum_{t_1\leqq jh<t_2} u(jh_n, x(jh_n))\right],$$

$$\int u(t, x)\, d\mu = E^{Q}\left[\int_{t_1}^{t_2} u(s, x(s))\, ds\right],$$

for $u\in C([t_1, t_2]\times\bar{G})$. Then clearly $\mu_n$ converges weakly to $\mu$. Given $f(t, x)$ in $C_0^{1,2}([0, T]\times\bar{G})$, we define the functions

$$k_n(s, x)=f_t(s, x)+\tfrac{1}{2}\operatorname{Tr}(a_n(s, x)\cdot D^2f(s, x))+\langle b_n(s, x), \nabla f(s, x)\rangle ,$$

$$k(s, x)=f_t(s, x)+\tfrac{1}{2}\operatorname{Tr}(a(s, x)\cdot D^2f(s, x))+\langle b(s, x), \nabla f(s, x)\rangle .$$

Then $k_n$ and $k$ vanish outside a fixed compact set $K\subset[0, T]\times\bar{G}$, and $k_n\to k$ uniformly on compact subsets of $[0, T]\times G$, and therefore on closed subsets of $[0, T]\times G$. Let $f(t, x)$ have the further property that

$$\rho(s, x)f_t(s, x)+\langle\gamma(s, x), \nabla f(s, x)\rangle\geqq 0$$

on $[0, T]\times\partial G$.



LEMMA 6.5. *Let $B$ be a compact subset of $[0, T]\times\partial G$ and $B_0=B\cap J_0$. Given any $\varepsilon>0$ and $L<\infty$, there exist $n_0<\infty$, $\delta_0>0$, $N_0>L$ and a neighborhood $U_0$ of $B_0$ such that*

(i) *if $(s, x)\in U_0$, $n\geqq n_0$ and $\langle\nabla\phi(x), a_n(s, x)\nabla\phi(x)\rangle<\delta_0$, then*

$$\|b_n(s, x)\|\geqq N_0 ,$$

(ii) *if $(s, x)\in U_0$, $n\geqq n_0$ and $\|b_n(s, x)\|\geqq N_0$, then*

$$\langle b_n(s, x), \nabla f(s, x)\rangle\geqq -\varepsilon\,\|b_n(s, x)\| .$$

Proof: Let $(t, y)\in B_0$ and $\varepsilon'>0$ be given. Then, by hypotheses, we can find $N_0'$, $n_0'$ and a neighborhood $U_0'$ of $(t, y)$ such that, if

$$(s, x)\in U_0', \qquad n\geqq n_0', \qquad \|b_n(s, x)\|\geqq N_0',$$

then

$$\left\|\frac{b_n(s, x)}{\langle b_n(s, x), \nabla\phi(x)\rangle}-\gamma(t, y)\right\|<\varepsilon' .$$

We assume that the neighborhood $U_0'$ is so small that for $(s, x)\in U_0'$ we have

$$|f(t, y)-f(s, x)|\leqq\varepsilon' ,$$

$$|\nabla f(t, y)-\nabla f(s, x)|\leqq\varepsilon' .$$

From this we conclude that

$$\frac{\langle b_n(s, x), \nabla f(s, x)\rangle}{\langle b_n(s, x), \nabla\phi(x)\rangle}\geqq -C\varepsilon' ,$$

and therefore that

$$\langle b_n(s, x), \nabla f(s, x)\rangle\geqq -C\,\|\nabla\phi(x)\|\,\varepsilon'\,\|b_n(s, x)\| .$$

Since $\varepsilon'$ was arbitrary, we can achieve that

$$C\,\|\nabla\phi(x)\|\,\varepsilon'<\varepsilon .$$

If we consider the covering $\{U_0'\}$ as $(t, y)$ varies over $B_0$, we can extract a finite subcovering for $B_0$. The union of these open sets will be denoted by $V_1$. Then $V_1 \supset B_0$ and there exist $N_0$ and $n_0$ such that, if

$$(s, x) \in V_1, \qquad n \geqq n_0 \qquad \text{and} \qquad \|b_n(s, x)\| \geqq N_0,$$

then

$$\langle b_n(s, x), \nabla f(s, x)\rangle \geqq -\varepsilon \|b_n(s, x)\|.$$

We can assume that $N_0 > L$ without any loss of generality. Also for each $(t, y) \in B_0$, there is a neighborhood $U_0''$ of $(t, y)$, a $\delta_0'' > 0$ and an $n_0''$ such that if

$$n \geqq n_0'', \qquad (s, x) \in U_0'', \qquad \text{and} \qquad \langle \nabla\phi(x), a_n(s, x)\nabla\phi(x)\rangle < \delta_0'',$$

then

$$\|b_n(s, x)\| \geqq N_0.$$

We can again extract a finite subcovering. The union of these open sets will be denoted by $V_2$. Clearly, $U_0 = V_1 \cap V_2$ and the natural choices of $n_0$ and $\delta_0$ work.

LEMMA 6.6. *Let $U_0$ be as in Lemma 6.5 and $B_1 = B \cap U_0^c$. Then $B_1 \subset J_1$ is compact. Given any $\varepsilon > 0$, there exist a neighborhood $U_1$ of $B_1$, a $\delta_0 > 0$ and an $n_0 < \infty$ such that if*

$$(s, x) \in U_1, \quad n \geqq n_0 \quad \text{and} \quad \langle \nabla\phi(x), a_n(s, x)\nabla\phi(x)\rangle < \delta_0,$$

*then*

$$f_t(s, x) + \langle b_n(s, x), \nabla f(s, x)\rangle \geqq -\varepsilon$$

*and*

$$\|a_n(s, x)\| \leqq \varepsilon.$$

*Moreover, there exists a constant $M$, independent of $\varepsilon$, such that the choice of $U_1$ can always be made so that, for $(s, x) \in U_1$ and for all $n$,*

$$\|b_n(s, x)\| \leqq M.$$

Proof: The proof is exactly like that of Lemma 6.5 in that it uses our hypotheses locally and the compactness of $B_1$ to get a global bound.

LEMMA 6.7.

$$\liminf_{U \downarrow B} \liminf_{n\to\infty} \int_{U \cap I} k_n(s, x)\, d\mu_n \geqq 0,$$

*where*

$$I = \{(s, x) : t_1 \leqq s \leqq t_2\}.$$



Proof: Let $\varepsilon > 0$ be arbitrary. We choose $U_0$, $U_1$, $\delta_0$ and $n_0$ according to Lemmas 6.5 and 6.6. Set $U = U_0 \cup U_1$. Let $n \geqq n_0$ and

$$U_{1,1}^{(n)} = U_1 \cap \{(s, x) : \langle \nabla\phi(x), a_n(s, x)\nabla\phi(x)\rangle < \delta_0\},$$
$$U_{1,2}^{(n)} = U_1 \cap \{(s, x) : \langle \nabla\phi(x), a_n(s, x)\nabla\phi(x)\rangle \geqq \delta_0\},$$
$$U_{0,1}^{(n)} = U_0 \cap U_1^c \cap \{(s, x) : \|b_n(s, x)\| \geqq N_0\},$$
$$U_{0,2}^{(n)} = U_0 \cap U_1^c \cap \{(s, x) : \|b_n(s, x)\| < N_0\}.$$

Then, for each $n$,

$$U = U_{1,1}^{(n)} \cup U_{1,2}^{(n)} \cup U_{0,1}^{(n)} \cup U_{0,2}^{(n)}.$$

We consider the behavior of $k_n$ on each of these sets separately.

*Case* 1. By Lemma 6.6,

$$\int_{U_{1,1}^{(n)} \cap I} k_n(s, x)\, d\mu_n(s, x) \geqq -C\varepsilon, \tag{6.11}$$

where $C$ depends on the bounds on $D^2 f$.

*Case* 2. Again by Lemma 6.6,

$$\int_{U_{1,2}^{(n)} \cap I} k_n(s, x)\, d\mu_n(s, x) \geqq -M'\mu_n(U_{1,2}^{(n)}), \tag{6.12}$$

where $M'$ depends on the bound on the $k_n$.

*Case* 3. By Lemma 6.5,

$$\int_{U_{0,1}^{(n)} \cap I} k_n(s, x)\, d\mu_n(s, x) \geqq -\varepsilon \int_{U_{0,1}^{(n)}} \|b_n(s, x)\|\, d\mu_n - C\mu_n(U_{0,1}^{(n)}), \tag{6.13}$$

where $C$ depends on the bound on $k_n - \langle b_n, \nabla f\rangle$.

*Case* 4. Finally, using the bound on $k_n$ restricted to $U_{0,2}^{(n)}$, we have

$$\int_{U_{0,2}^{(n)} \cap I} k_n(s, x)\, d\mu_n(s, x) \geqq -(A_1 + A_2 N_0)\mu_n(U_{0,2}^{(n)}). \tag{6.14}$$

Combining (6.11) through (6.14), one obtains

$$\int_{U \cap I} k_n(s, x)\, d\mu_n \geqq -C\varepsilon - M'\mu_n\{U_{1,2}^{(n)}\} - \varepsilon \int_{[0,T]\times G} \|b_n(s, x)\|\, d\mu_n - C\mu_n[U_{0,1}^{(n)}] - (A_1 + A_2 N_0]\mu_n\{U_{0,2}^{(n)}\}. \tag{6.15}$$

Now let $U \downarrow B$. Since $U_{0,2}^{(n)} \subset \{s, x : \langle \nabla\phi(x) a_n(s, x) \nabla\phi \rangle \geqq \delta\}$, we have, by Lemma 6.4,

$$\limsup_{U \downarrow B} \limsup_{n\to\infty} \mu_n[U_{0,2}^{(n)} \cup U_{1,2}^{(n)}] = 0 . \tag{6.16}$$

By Lemma 6.3,

$$\int_{[0,T]\times\bar{G}} \|b_n(s, x)\| \, d\mu_n \leqq C ,$$

and since, in $U_{0,1}^{(n)}$, $\|b_n(s, x)\| \geqq N_0 > L$ we have

$$\mu_n[U_{0,1}^{(n)}] \leqq \frac{C}{L} . \tag{6.17}$$

Therefore, using (6.16) and (6.17) in (6.15), we see that

$$\liminf_{\downarrow B} \liminf_{n\to\infty} \int_{U\cap J} k_n(s, x) \, d\mu_n \geqq -C'\varepsilon - C'\varepsilon - \frac{C'}{L} .$$

Since $\varepsilon$ and $L$ are arbitrary, we have

$$\liminf_{U\downarrow B} \liminf_{n\to\infty} \int_{U\cap J} k_n(s, x) \, d\mu_n \geqq 0 .$$

LEMMA 6.8.

$$\liminf_{n\to\infty} \int_{[t_1 \leqq s \leqq t_2]\times\bar{G}} k_n(s, x) \, d\mu_n \geqq \int_{[t_1 \leqq s \leqq t_2]\times\bar{G}} k(s, x) \, d\mu .$$

Proof: Since in the $t$-component, $\mu$ is Lebesgue measure, $[t_1, t_2] \times \bar{G}$ is a $\mu$-continuity set. We can, therefore, restrict our attention to $[t_1, t_2] \times \bar{G}$. Let $B$ be the support of $f$ and its derivatives on $[t_1, t_2] \times \partial G$. Choose $U$ to be an open neighborhood of $B$ so that $\mu(\partial U) = 0$. Since $k_n \to k$ uniformly on $U^c$,

$$\lim_{n\to\infty} \int_{U^c} k_n(s, x) \, d\mu_n = \int_{U^c} k(s, x) \, d\mu .$$

By Lemma 6.7, for any given $\varepsilon > 0$, we can choose $U$ so that

$$\liminf_{n\to\infty} \int_U k_n(s, x) \, d\mu_n \geqq -\varepsilon .$$



Therefore,

$$\liminf_{n\to\infty} \int_{[t_1,t_2]\times\bar{G}} k_n(s, x) \, d\mu_n \geqq \int_{U^c} k(s, x) \, d\mu - \varepsilon .$$

Letting $U \downarrow B$ and letting $\varepsilon \to 0$, we have the lemma.

Proof of Theorem 6.3: Set

$$\begin{aligned} Y(t) &= \int_0^t (f_s + L_s f)(s, x(s)) \chi_G(x(s)) \, ds \\ &= \int_0^t k(s, x(s)) \chi_G(x(s)) \, ds , \end{aligned}$$

$$Y_n(t) = h_n \sum_{0 \leqq jh_n < t} k_n(jh_n, x(jh_n)) ,$$

and

$$Z_n(t) = \sum_{0 \leqq jh_n < t} \int [f((j+1)h_n, y) - f(jh_n, x(jh_n))] \Pi_j^{h_n}(x(jh_n), dy) .$$

Relative to $\mathscr{M}_{kh_n}$ and $P_n$,

$$f(kh_n, x(kh_n)) - Z_n(kh_n)$$

is a martingale. Let us choose $t_0$ such that $0 \leqq t_0 < T$. Let $\Phi$ be a non-negative bounded continuous function on $D([0, T], \bar{G})$ which is $\mathscr{M}_{t_0}$ measurable. Let $t$ be an arbitrary element of $(t_0, T)$. Find $k_n$ and $k_n'$ such that $k_n h_n \to t_0$ from the right and $k_n' h_n \to t$. Then we have, for each $n$,

$$\int [f(k_n h_n, x(k_n h_n)) - Z_n(k_n h_n) - f(k_n' h_n, x(k_n' h_n)) + Z_n(k_n' h_n)] \Phi \, dP_n = 0 .$$

From Theorem 6.2 we know that

$$E^{P_n} \sup_{0 \leqq t \leqq T} |Z_n(t) - Y_n(t)| \to 0 \quad \text{as} \quad n \to \infty .$$

Also,

$$f(k_n h_n, x(k_n h_n)) \to f(t, x(t))$$

and

$$f(k_n' h_n, x(k_n' h_n)) \to f(t_0, x(t_0)) .$$

Hence,

$$E^P[\{f(t, x(t)) - f(t_0, x(t_0))\}\Phi]$$

$$\begin{aligned}
&= \lim_{n\to\infty} \int [Z_n(t) - Z_n(t_0)]\Phi \, dP_n \\
&= \lim_{n\to\infty} \int [Z_n(t) - Z_n(t_0)] \, dQ_n \\
&= \lim_{n\to\infty} \int_{t_0}^t \int_{\bar{G}} k_n(s, x) \, d\mu_n(s, x) \\
&\geqq \int_{t_0}^t \int_{\bar{G}} k(s, x) \, d\mu(s, x) \\
&= E^Q\left[\int_{t_0}^t (f_s + L_s f)(s, x(s))\chi_G(x(s)) \, ds\right] \\
&= E^P\left[\Phi \int_{t_0}^t (f_s + L_s f)(s, x(s))\chi_G(x(s)) \, ds\right].
\end{aligned}$$

Therefore,

$$E^P[f(t, x(t)) - f(t_0, x(t_0)) - \int_{t_0}^t \chi_G(f_s + L_s f)(s, x(s)) \, ds \mid \mathscr{M}_{t_0}] \geqq 0.$$

This proves property (ii) of the theorem. Property (i) is trivial.

*Remark.* If the uniqueness theorem is valid for the limiting coefficients $a$, $b$, $\rho$, and $\gamma$, then

$$\lim_{\substack{h\to 0\\ x\to z}} P_x^h = P_z ,$$

where $P_z$ is the unique solution to the submartingale problem for the limiting coefficients.

## 7. Appendix

Let $D = \{x \in R^d : x_1 > 0\}$. Given $u \in C_0^\infty([0, T] \times \bar{D})$, define

$$\|u\|_{p,T}^{(2)} = \|u\|_{p,T} + \|D_t u\|_{p,T} + \sum_{|\alpha|\leqq 2} \|D_x^\alpha u\|_{p,T} ,$$

where $\|\cdot\|_{p,T}$ is the ordinary $L_p$-norm on $[0, T] \times \bar{D}$. Denote by $W_p^{(2)}([0, T) \times \bar{G})$ the completion of $C_0^\infty([0, T) \times \bar{G})$ with respect to $\|\cdot\|_{p,T}^{(2)}$. If $T = \infty$, we drop the subscript $T$. Given a function $f$ on $[0, T) \times \bar{D}$, let $f^*$ denote the anti-asymmetric extension of $f$ across $x_1 = 0$. We define,



for $f \in C_0^\infty([0, T) \times \bar{D})$,

$$G_0 f(s, x) = \int_{-\infty}^\infty dt \int_{R^d} dy \, p(t - s, x - y) f^*(t, y) , \tag{7.1}$$

where

$$p(t, y) = \chi_{t\geqq 0} \frac{1}{(2\pi t)^{d/2}} e^{-|y|^2/2t} . \tag{7.2}$$

LEMMA 7.1. *If $f \in L_p([0, T) \times \bar{D})$, then for each $T < \infty$ and $1 \leqq p \leqq \infty$ there is a constant $C_{p,T}$ independent of $f$ such that*

$$\|G_0 f\|_{p,T} \leqq C_{p,T} \|f\|_{p,T} , \tag{7.3}$$

$$\sum_{|\alpha|=1} \|D_x^\alpha G_0 f\|_{p,T} \leqq C_{p,T} \|f\|_{p,T} . \tag{7.4}$$

*Moreover, for each $1 < p < \infty$ there is a constant $C_p$ independent of $T$ and $f$ such that*

$$\sum_{|\alpha|=2} \|D_x^\alpha G_0 f\|_p \leqq C_p \|f\|_p . \tag{7.5}$$

*In particular, for each finite $T$ the operator $G_0$ maps $L_p([0, T) \times \bar{D})$ boundedly into $W_p^{(2)}([0, \infty) \times D)$.*

Proof: The only nontrivial assertion is (7.5). However, by the estimate of B. Frank Jones (cf. [3]), for $|\alpha| = 2$ the $L_p([0, \infty) \times R^d)$-norm of $D_x^\alpha G_0 f$ is bounded by the $L_p([0, \infty) \times R^d)$-norm of $f^*$, and (7.5) follows immediately from this.

The estimates (7.3), (7.4) can be proved by checking that the $L_1([0, T) \times R^d)$-norms of $p(t, x)$ and $D_{x_i} p(t, x)$ are finite for each finite $T$.

Next let

$$L = \tfrac{1}{2} \sum_{i,j=1}^d a_{ij}(t, x) \frac{\partial^2}{\partial x_i \partial x_j}$$

be a second order elliptic operator. Define $\varepsilon_{ij}(t, x) = a_{ij}(t, x) - \delta_{ij}$ and let

$$T_\varepsilon f(t, x) = \tfrac{1}{2} \sum_{i,j=1}^d \varepsilon_{ij}(t, x) \frac{\partial^2 G_0 f}{\partial x_i \partial x_j}(t, x)$$

for $f \in C_0^\infty([0, \infty) \times \bar{D})$. By the estimate (7.4), we see that for each $1 < p < \infty$

and $\delta > 0$ there is an $\varepsilon_{p,\delta} < 0$ such that

$$\|T_\varepsilon f\|_p \leqq \delta \, \|f\|_p \tag{7.6}$$

if

$$\sup_{\substack{t \geqq 0 \\ x \in \bar{D}}} \sum_{i,j} |\varepsilon_{ij}(t, x)| \leqq \varepsilon_{p,\delta} \, .$$

In particular, if

$$\sup_{\substack{t \geqq 0 \\ x \in \bar{D}}} \sum_{i,j} |\varepsilon_{ij}(t, x)| \leqq \varepsilon_{p,\delta} \, ,$$

where $\delta < 1$, then $(I - T_\varepsilon)^{-1}$ is bounded from $L_p([0, \infty) \times \bar{D})$ into $L_p([0, \infty) \times \bar{D})$, and $\|(I - T_\varepsilon)^{-1}\|_p^p \leqq 1/(1 - \delta)$. Hence we can define

$$G_L = G_0(I - T_\varepsilon)^{-1} \, , \tag{7.7}$$

and we can assert the existence of constants $C_{p,T,L}$ such that

$$\|G_L f\|_p \leqq C_{p,T,L} \, \|f\|_{p,T} \, , \qquad 1 \leqq p \leqq \infty \, , \quad 0 \leqq T < \infty \, , \tag{7.8}$$

$$\|D_x^\alpha f\|_p \leqq C_{p,T,L} \, \|f\|_{p,T} \, , \qquad 1 \leqq p \leqq \infty \, , \quad 0 \leqq T < \infty \, , \quad |\alpha| = 1 \, , \tag{7.9}$$

for $f \in L_p([0, T) \times \bar{D})$, and constants $C_{p,L}$ such that

$$\|D_x^\alpha f\|_p \leqq C_{p,L} \, \|f\|_p \, , \qquad 1 < p < \infty \, , \quad |\alpha| = 2 \, , \tag{7.10}$$

for $f \in L_p([0, \infty) \times \bar{D})$. Of course, the dependence of $C_{p,T,L}$ and $C_{p,L}$ on $L$ is through the size of the $\varepsilon_{ij}$.

We next introduce certain $L_p$-type norms on $[0, \infty) \times \partial D$. In order to avoid confusion, we use $\|\cdot\|_{\sim p}$ to denote the ordinary $L_p$-norm on $[0, \infty) \times \partial D$. Given $f \in C_0^\infty([0, \infty) \times \partial D)$, define

$$\langle\langle f \rangle\rangle_{\sim p,x} = \left[ \int_0^\infty dt \int_{\partial D} \int_{\partial D} \frac{|f(t, x) - f(t, y)|^p}{|x - y|^{d+p-2}} \, dx \, dy \right]^{1/p} ,$$

$$\langle\langle f \rangle\rangle_{\sim p,t} = \left[ \int_{\partial D} dy \int_0^\infty \int_0^\infty \frac{|f(t, y) - f(s, y)|^p}{|t - s|^{(p+1)/2}} \, ds \, dt \right]^{1/p} ,$$

$$|||f|||_{\sim p} = \|f\|_{\sim p} + \langle\langle f \rangle\rangle_{\sim p,x} + \langle\langle f \rangle\rangle_{\sim p,t},$$

$$|||f|||_{\sim p}^{(0,1)} = |||f|||_{\sim p} + \sum_2^d |||D_{x_j} f|||_{\sim p} \, ,$$

$$|||f|||_{\sim p}^{(1,1)} = |||f|||_{\sim p}^{(0,1)} + |||D_t f|||_{\sim p} \, .$$



Associated with these norms we introduce the spaces $W_p([0, \infty) \times \partial D)$, $W_p^{(0,1)}([0, \infty) \times \partial D)$, and $W_p^{(1,1)}([0, \infty) \times \partial D)$. Before putting these spaces to work, we state the following simple facts.

Lemma 7.2. *Let $T$ be a bounded, linear, translation invariant map of $L_p([0, \infty) \times \partial D)$ into itself. Then*

$$\langle\langle Tf \rangle\rangle_{\sim p,x} \leqq \|T\|_{\sim p}^p \langle\langle f \rangle\rangle_{\sim p,x} \qquad \textit{and} \qquad \langle\langle Tf \rangle\rangle_{\sim p,t} \leqq \|T\|_{\sim p}^p \langle\langle f \rangle\rangle_{\sim p,t} \, .$$

*In particular, $T$ maps each of the spaces $W_p([0, \infty) \times \partial D)$, $W_p^{(0,1)}([0, \infty) \times \partial D)$, and $W_p^{(1,1)}([0, \infty) \times \partial D)$ into itself with norm $\|T\|_{\sim p}^p$.*

The basic reason for dealing with these spaces is the following. Define

$$h_{x_1}^{(0)}(t, \tilde{y}) = \chi_{t \geqq 0} \frac{x_1}{t(2\pi t)^{d/2}} \exp\{-(x_1^2 + |\tilde{y}|^2)/2t\} \, , \qquad x_1 > 0 \, , \quad y \in D \, ,$$

where $\tilde{y}$ is the projection of $y$ on $\partial D$. For $f \in C_0^\infty([0, \infty) \times \partial D)$, let

$$H_0 f(s, x) = H_{x_1}^{(0)} f(s, \tilde{x}) = \int_s^\infty dt \int_D d\tilde{y} \, h_{x_1}^{(0)}(t - s, x - \tilde{y}) f(t, \tilde{y}) \, . \tag{7.11}$$

It is easily checked that $H_0 f$ is the time-space harmonic extension for $(\partial/\partial s + \frac{1}{2}\Delta)$ of $f$ in $[0, \infty) \times \bar{D}$. The crucial fact which we need is that, for each $1 < p < \infty$, there is a constant $C_p$ such that

$$\sum_{|\alpha|=2} \|D_x^\alpha f\|_p \leqq C_p \, |||f|||_{\sim p}^{(0,1)} \, . \tag{7.12}$$

This estimate is proved on p. 294 of [5].

Theorem 7.1. *Let $L = \frac{1}{2} \sum_{i,j} a_{ij}(t, x) \, \partial^2/\partial x_i \, \partial x_j$ be an elliptic operator and let $\varepsilon_{ij}(t, x) = a_{ij}(t, x) - \delta_{ij}$. Define*

$$D_\varepsilon = \tfrac{1}{2} \sum \varepsilon_{ij}(t, x) \, \partial^2/\partial x_i \, \partial x_j \, .$$

*Given $f \in C_0^\infty([0, \infty) \times \partial D)$, let $H_L f$ denote the time-space harmonic extension of $f$ to $[0, \infty) \times \bar{D}$. If for some $T < \infty$ the $\varepsilon_{ij}$ vanish when $t \geqq T$, then for each $1 < p < \infty$ and $\delta > 0$ there is an $\varepsilon_{p,\delta,T} > 0$ such that*

$$\sup_{\substack{x \geqq 0 \\ x \in \bar{D}}} \sum_{i,j} \varepsilon_{ij}(t, x) \leqq \varepsilon_{p,\delta,T}$$

*implies*

$$\|(H_L - H_0)f\|_p^{(2)} \leqq \delta\, \|\|f\|\|_p^{(0,1)}\,. \tag{7.13}$$

Proof: Assume that the $\varepsilon_{ij}$ are small enough for (7.8), (7.9) and (7.10) to hold. A simple computation yields

$$H_L - H_0 = G_L\, D_\varepsilon\, H_0\,.$$

From (7.12) we see that $D_\varepsilon H_0 f \in L_p([0, T) \times \bar{D})$ and that

$$\|D_\varepsilon H_0 f\|_{p,T} \leqq C_p\, \varepsilon\, \|\|f\|\|_{\sim p}^{(0,1)},$$

where

$$\varepsilon = \sup_{\substack{t \geqq 0 \\ x \in D}} \sum_{i,j} |\varepsilon_{ij}(t, x)|\,.$$

Thus, by (7.8), (7.9) and (7.10),

$$\|(H_L - H_0)f\|_p^{(2)} \leqq C\varepsilon\, \|\|f\|\|_{\sim p}^{(0,1)}\,.$$

We want to study next the operators $\{H_{x_1}^{(0)}\}_{x_1>0}$ as transformations from $C_0^\infty([0, \infty) \times \partial D)$ into $C_b([0, \infty) \times \partial D)$. First note that

$$\widehat{h_{x_1}^{(0)}}(\mu, \tilde{\theta}) = \exp\{-x_1(2i\mu + |\tilde{\theta}|^2)^{1/2}\}\,. \tag{7.14}$$

This shows that $\{H_{x_1}^{(0)}\}_{x_1>0}$ is a semigroup. Let $K_0$ denote its generator. Then

$$\widehat{K_0 f}(\mu, \tilde{\theta}) = -(2i\mu + |\tilde{\theta}|^2)^{1/2} \hat{f}(\mu, \tilde{\theta})\,. \tag{7.15}$$

Denoting by $R_\lambda^{(0)}$ the resolvent operators $(\lambda - K_0)^{-1}$ associated with $\{H_{x_1}^{(0)}\}_{x_1>0}$, we have

$$\widehat{R_\lambda^{(0)} f}(\mu, \tilde{\theta}) = \frac{1}{\lambda + (2i\mu + |\tilde{\theta}|^2)^{1/2}}\, \hat{f}(\mu, \tilde{\theta})\,. \tag{7.16}$$

Using the representation

$$R_\lambda^{(0)} = \int_0^\infty e^{-\lambda x_1} H_{x_1}^{(0)}\, dx_1\,,$$

we see that

$$\|R_\lambda^{(0)} f\|_{\sim p} \leqq \frac{1}{\lambda}\, \|f\|_{\sim p}\,, \qquad 1 \leqq p \leqq \infty \tag{7.17}$$



Since $\{H_{x_1}^{(0)}\}_{x_1>0}$ is a convolution semigroup, in order to find the semigroup $\{H_{x_1}^{(\rho,\tilde{\gamma})}\}_{x_1>0}$ generated by $K_{\rho,\tilde{\gamma}} = K_0 + \langle \tilde{\gamma}, \tilde{\nabla}\rangle + \rho\, \partial/\partial t$, where $\rho \geqq 0$ and $\tilde{\gamma} \in R^{d-1}$ are constants, we simply have to take

$$h_{x_1}^{(\rho,\tilde{\gamma})}(t, \tilde{x}) = h_{x_1}^{(0)}(t - \rho x, \tilde{x} - x_1 \tilde{\gamma}) \tag{7.18}$$

and set

$$H_{x_1}^{(\rho,\tilde{\gamma})} f(s, \tilde{x}) = \int_s^\infty dt \int_{\partial G} d\tilde{y}\, h_{x_1}^{(\rho,\tilde{\gamma})}(t - s, \tilde{x} - \tilde{y}) f(t, \tilde{y})\,. \tag{7.19}$$

We can now read off the following facts about $\{H_{x_1}^{(\rho,\tilde{\gamma})}\}_{x_1>0}$, $K_{\rho,\tilde{\gamma}}$, and $R_\lambda^{(\rho,\tilde{\gamma})}$ from the corresponding statements about $\{H_{x_1}^{(0)}\}_{x_1>0}$, $K_0$, and $R_\lambda^{(0)}$:

$$K_{\rho,\tilde{\gamma}} f(\mu, \tilde{\theta}) = -((2i\mu + |\tilde{\theta}|^2)^{1/2} + i\langle \tilde{\gamma}, \tilde{\theta}\rangle + i\mu) \hat{f}(\mu, \tilde{\theta})\,, \tag{7.20}$$

$$R_\lambda^{(\rho,\tilde{\gamma})} f(\mu, \tilde{\theta}) = \frac{1}{\lambda + (2i\mu + |\tilde{\theta}|^2)^{1/2} + i\langle \tilde{\gamma}, \tilde{\theta}\rangle + i\rho\mu} \hat{f}(\mu, \tilde{\theta})\,, \tag{7.21}$$

$$\|R_\lambda^{(\rho,\tilde{\gamma})} f\|_p \leqq \frac{1}{\lambda}\, \|f\|_p\,, \qquad 1 \leqq p \leqq \infty\,. \tag{7.22}$$

THEOREM 7.2. *For each $1 < p < \infty$, $\rho \geqq 0$, $\tilde{\gamma} \in R^{d-1}$, there is a $C_{p,\rho,\tilde{\gamma}}$ such that*

$$\sum_{j=2}^d \|\|D_{x_j} R_\lambda^{(\rho,\tilde{\gamma})} f\|\|_{\sim p} \leqq C_{p,\rho,\tilde{\gamma}}\, \|\|f\|\|_{\sim p}\,, \qquad \lambda \geqq 0\,. \tag{7.23}$$

*In particular, if $\lambda > 0$, then*

$$\|\|R_\lambda^{(\rho,\tilde{\gamma})} f\|\|_{\sim p}^{(0,1)} \leqq C_{p,\rho,\tilde{\gamma},\lambda}\, \|\|f\|\|_{\sim p}\,. \tag{7.24}$$

*Moreover, if $\rho < 0$, then*

$$\|\|D_t R_\lambda^{(\rho,\tilde{\gamma})} f\|\|_{\sim p} \leqq C_{p,\rho,\tilde{\gamma}}\, \|\|f\|\|_{\sim p}\,, \tag{7.25}$$

*and therefore, if $\lambda > 0$, then*

$$\|\|R_\lambda^{(\rho,\tilde{\gamma})} f\|\|_{\sim p}^{(1,1)} \leqq C_{p,\rho,\tilde{\gamma},\lambda}\, \|\|f\|\|_{\sim p}\,. \tag{7.26}$$

Proof: Let $\rho = 0$. In order to prove (7.23) for $\lambda = 0$, it suffices to know that $\theta_j/[2i\mu + |\tilde{\theta}|^2)^{1/2} + i\langle \tilde{\gamma}, \tilde{\theta}\rangle]$ is an $L_p$-multiplier (cf. [3] or [6]), $1 < p < \infty$. That this is the case follows from the work of Fabes and Riviere [3] on multipliers of mixed homogeneity. To prove (7.23) when $\lambda > 0$, we use the resolvent equation

$$R_\lambda^{(0,\tilde{\gamma})} = R_0^{(0,\tilde{\gamma})} - \lambda R_0^{(0,\tilde{\gamma})} R_\lambda^{(0,\tilde{\gamma})}.$$

From this we see that

$$\|D_{x_j} R_\lambda^{(0,\tilde{\gamma})}\|_p \leqq 2\, \|D_{x_j} R_0^{(0,\tilde{\gamma})} f\|_p\,,$$

and so (7.23) follows immediately.

We next turn to the case when $\rho > 0$. Now we must know that

$$\frac{\mu}{[2i\mu + \tilde{\theta}^2)^{1/2} + i\langle\tilde{\gamma}, \tilde{\theta}\rangle + i\rho\mu]}$$

is an $L_p$-multiplier. This multiplier no longer fits into the category treated in [3]. However, it does satisfy the conditions of Theorem 4.5 in [6], and is therefore an $L_p$-multiplier for $1 < p < \infty$. Arguing as in the preceding paragraph, we now have (7.25) for all $\rho > 0$.

In order to prove (7.23) for $\rho > 0$, we must show that

$$\frac{\theta_j}{(2i\mu + |\tilde{\theta}|^2)^{1/2} + i\langle\tilde{\gamma}, \tilde{\theta}\rangle + i\rho\mu}$$

is an $L_p$-multiplier. But, according to the preceding paragraph,

$$\frac{\mu}{(2i\mu + |\tilde{\theta}|^2)^{1/2} + i\langle\tilde{\gamma}, \tilde{\theta}\rangle + i\rho\mu}$$

and therefore also

$$\frac{(2i\mu + |\tilde{\theta}|^2)^{1/2} + i\langle\tilde{\gamma}, \tilde{\theta}\rangle}{(2i\mu + |\tilde{\theta}|^2)^{1/2} + i\langle\tilde{\gamma}, \tilde{\theta}\rangle + i\rho\mu} = 1 - \frac{i\rho\mu}{(2i\mu + |\tilde{\theta}|^2)^{1/2} + i\langle\tilde{\gamma}, \tilde{\theta}\rangle + i\rho\mu}$$

are $L_p$-multipliers. By the Fabes-Riviere result, we know that

$$\frac{\theta_j}{(2i\mu + |\tilde{\theta}|^2)^{1/2} + i\langle\tilde{\gamma}, \tilde{\theta}\rangle}$$

is an $L_p$-multiplier. Combining this with the above and remembering that the product of two $L_p$-multipliers is again an $L_p$-multiplier, we see that

$$\frac{\theta_j}{(2i\mu + |\tilde{\theta}|^2)^{1/2} + i\langle\tilde{\gamma}, \tilde{\theta}\rangle + i\rho\mu}$$

is an $L_p$-multiplier.

The last result that we prove in this section concerns the possibility of perturbing in $W_p([0, \infty) \times \partial D)$.



THEOREM 7.3. *Let $\alpha$ be a function on $[0, \infty) \times \partial D$ such that*

$$\sup_{\substack{t \geqq 0 \\ x \in \partial G}} |\alpha(t, x)| \leqq \varepsilon \leqq 1$$

*and*

$$|\alpha(s, x) - \alpha(t, y)| \leqq A(|t - s| + |x - y|)\,.$$

*Then there is a constant $C_q$ for each $1 < p < \infty$ such that*

$$|||\alpha f|||_{\sim p} \leqq C_p\, \varepsilon^{1/(p+1)}\, |||f|||_{\sim p}\,.$$

Proof: Clearly, $\|\alpha f\|_{\sim p} \leqq \varepsilon\, \|f\|_{\sim p}$. We shall show that

$$\langle\langle \alpha f\rangle\rangle_{\sim p, x} \leqq C_p\, \varepsilon^{1/(p+1)}\, |||f|||_{\sim p}\,.$$

The proof that

$$\langle\langle \alpha f\rangle\rangle_{\sim p, t} \leqq C_p\, \varepsilon^{1/(p+1)}\, |||f|||_{\sim p}$$

is similar. We have

$$\begin{aligned}(\langle\langle \alpha f\rangle\rangle_{\sim p,x})^p &= \int_0^\infty dt \int_{\partial D}\int_{\partial D} \frac{|\alpha f(t, x) - \alpha f(t, y)|^p}{|x - y|^{d+p-2}}\, dx\, dy \\ &= \int_0^\infty dt \iint\limits_{|x-y| \geqq 1} \frac{|\alpha f(t, x) - \alpha f(t, y)|^p}{|x - y|^{d+p-2}}\, dx\, dy \\ &\quad + \int_0^\infty dt \iint\limits_{|x-y| < 1} \frac{|\alpha f(t, x) - \alpha f(t, y)|^p}{|x - y|^{d+p-2}}\, dx\, dy \\ &= I_1 + I_2.\end{aligned}$$

Note that

$$I_1 \leqq 2\varepsilon^p \int_0^\infty dt \int_{\partial D} |f(t, x)|^p\, dx \int\limits_{|y-x| \geqq 1} \frac{1}{|y - x|^{d+p-2}}\, dy \leqq C_p\, \varepsilon^p\, \|f\|_{\sim p}^p\,.$$

We treat $I_2$ next:

$$\begin{aligned}I_2 &\leqq C_p \int_0^\infty dt \int\limits_{|x-y| < 1} \frac{|\alpha(t, x)|^p\, |f(t, x) - f(t, y)|^p}{|x - y|^{d+p-2}}\, dx\, dy \\ &\quad + C_p \int_0^\infty dt \int\limits_{|x-y| < 1} \frac{|f(t, y)|^p\, |\alpha(t, x) - \alpha(t, y)|^p}{|x - y|^{d+p-2}}\, dx\, dy \\ &= C_p(I_3 + I_4)\,.\end{aligned}$$

Clearly,

$$I_3 \leqq \varepsilon^p (\langle\langle f \rangle\rangle_{p,x})^p .$$

Observe now that

$$\begin{aligned}
I_4 &= \int_0^\infty dt \iint_{|x-y|<\varepsilon^\mu} \frac{|f(t,y)|^p |\alpha(t,x) - \alpha(t,y)|^p}{|x-y|^{d+p-2}}\, dx\, dy \\
&\quad + \int_0^\infty dt \iint_{\varepsilon^\mu \leqq |x-y| \leqq 1} \frac{|\ (t,y)|^p |\alpha(t,x) - \alpha(t,y)|^p}{|x-y|^{d+p-2}}\, dx\, dy \\
&= I_5 + I_6 ,
\end{aligned}$$

$$I_5 \leqq A \int_0^\infty dt \int_{\partial D} f(t,y)^p\, dy \int_{|x-y| \leqq \varepsilon^\mu} \frac{1}{|x-y|^{d-2}}\, dx \leqq C\varepsilon^\mu \, \|f\|^p_{\sim p} ,$$

$$\begin{aligned}
I_6 &= \int_0^\infty dt \int_{\partial D} dy\, f(t,y)^p \int_{\varepsilon^\mu \leqq |x-y| \leqq 1} \frac{|\alpha(t,x) - \alpha(t,y)|}{|x-y|^{d+p-2}}\, dx \\
&\leqq C \frac{\varepsilon^p}{\varepsilon^{p\mu}} \int_0^\infty dt \int_{\partial D} |f(t,y)|^p \int_{|x-y| \leqq 1} \frac{1}{|x-y|^{d\ \ 2}}\, dx \leqq C\varepsilon^{p(1-\mu)} \, \|f\|^p_{\sim p} .
\end{aligned}$$

Combining these and taking $\mu = p/(p+1)$, we get our result.

*Remark* 1. Suppose $L$ is as in Theorem 7.1. Given $h \in C_0^\infty([0,\infty) \times \partial D)$, we write

$$H_L f = H_0 h + G_L D_\varepsilon H_0 h .$$

Then $H_0 h \in C_b^{1,2}([0,\infty) \times D)$ and $G_L D_\varepsilon H_0 h \in W_p^{(2)}([0,\infty) \times \bar{D})$. Thus if $Q_{s,x}$ is a solution to the martingale problem for $a$ starting from $x \in G$ at times $s$, then for $p > d + 2$

$$H_L\, h(t, x(t)) - \int_s^t \left(\frac{\partial}{\partial u} + L_u\right) H_L\, h(u, x(u))\, du$$

is a local $Q_{s,x}$-martingale in $D$. But $((\partial/\partial u) + L_u) H_L f = 0$ on $D$, and so $H_L f(t, x(t))$ is a local $Q_{s,x}$-martingale in $D$. In particular,

$$H_L\, h(s,x) = E^{Q_{s,x}}[h(\tau_s, x(\tau_s))] , \tag{7.27}$$



where $\tau_s = \inf\{t \geqq s : x(t) \notin D\}$.

*Remark* 2. Let $L$ be as in Remark 1, and $p > d + 2$. If $g \in C_0^\infty([0,\infty) \times \bar{D})$,

$$G_L\, g \in W_p^{(2)}([0,\infty) \times \bar{D})$$

and

$$\left(\frac{\partial}{\partial t} + L\right) G_L\, g = -g .$$

Hence,

$$G_L\, g(t, x(t)) + \int_s^t g(u, x(u))\, du$$

is a local $Q_{s,x}$-martingale in $D$. Combining this with the fact that $G_L\, g \to 0$ at $[0,\infty) \times \partial D$, we have

$$G_L\, g(s,x) = E^{Q_{s,x}}\left[\int_s^{\tau_s} g(u, x(u))\, du\right] . \tag{7.28}$$

*Remark* 3. Let $L$ be as before, $p > d + 2$, and let $h$ and $g$ be chosen as in Remarks 1 and 2, respectively. Define $u(s,x) = -G_L\, g(s,x) + H_L(s,x)$. Then,

$$u(s,x) = -E^{Q_{s,x}}\left[\int_s^{\tau_s} g(u, x(u))\, du\right] + E^{Q_{s,x}}[h(\tau_s, x(\tau_s))] .$$

Moreover, $u \in W_p^{(2)}([0,\infty) \times \bar{D})$. In particular, $u$ has a bounded continuous gradient in all directions on the boundary, and so Theorem 2.5 applies to $u$.

*Remark* 4. For $p > d + 2$, one can prove a Sobolev type theorem bounding the maximum of $|f|$ by $|||f|||_p$.

Conditions for Compactness in $D([0,T], R^d)$.

Theorem 7.4. *Let $P_n$ be a sequence of measures on $D([0,T], R^d)$ such that*

(i) *$\{P_n\}$ is uniformly stochastically continuous in the interval $[0,T]$,*

(ii) *there exists a constant $A$ independent of $n$ such that*

$$E^{P_n}[|x(t_1) - x(t_2)|^2 \, |x(t_3) - x(t_2)|^2] \leqq A\, |t_1 - t_3|^2$$

*for $0 \leqq t_1 < t_2 < t_3 \leqq T$. Then the sequence $\{P_n\}$ is relatively compact.*

This theorem is found in Chentsov [1].

Let $h > 0$ be arbitrary. For each $h$ let a stochastic process $P_h$ be defined on $D([0,T], R^d)$ in the following manner. $P_h$ is concentrated on step functions which are constant on intervals of the form $jh \leqq t < (j+1)h$. There is never a jump at $T$. The chains $x(jh)$ constitute a Markov chain in $j$ for each $h > 0$. For each $\delta > 0$, there is a constant $C_\delta$ such that

$$E^{P_h}[\|x(kh) - x(rh)\| \mid \mathscr{M}_{rh}] \leqq \delta + C_\delta |(k-r)h|^{1/2}$$

for $k \geqq r$. Moreover,

$$P_h[x(0) = z_h] = 1$$

and $z_h$ varies over a compact set as $h \to 0$. Then we can state

THEOREM 7.5. *The family* $\{P_h\}$ *is relatively compact in* $D([0, T], R^d)$ *as* $h \to 0$.

Proof: From the hypothesis it follows that

$$E^{P_h}[\|x(t)\|] \leqq A \quad \text{for} \quad 0 \leqq t \leqq T \quad \text{and} \quad h \to 0 .$$

Therefore, if $\{t_j\}$ is a dense set in $[0, T]$, for given $\eta > 0$, we can find constants $C_j$ such that

$$(7.29) \qquad P_h[\|x(t_j)\| \leqq C_j \text{ for all } j] \geqq 1 - \tfrac{1}{3}\eta \qquad \text{for} \qquad 0 < h \leqq 1 .$$

Moreover, we can find $\varepsilon_j \downarrow 0$, $\alpha_j \downarrow 0$ and $\beta_j \downarrow T$ such that

$$(7.30) \qquad P_h[\|x(\alpha_j) - x(0)\| \leqq \varepsilon_j \text{ for all } j] \geqq 1 - \tfrac{1}{6}\eta \qquad \text{for} \qquad 0 < h \leqq 1 ,$$

$$(7.31) \qquad P_h[\|x(\beta_j) - x(T)\| \leqq \varepsilon_j \text{ for all } j] \geqq 1 - \tfrac{1}{6}\eta \qquad \text{for} \qquad 0 < h \leqq 1 .$$

We shall now prove that

$$\lim_{\rho\to 0} \limsup_{h \to 0} P_h[\omega_x^*(\rho) \geqq \varepsilon] = 0 ,$$

where

$$\omega_x^*(\rho) = \sup_{\substack{0 \leqq t < u < s \leqq T \\ |t-s| < \rho}} [\inf\{|x(t) - x(u)|, |x(u) - x(s)|\}] .$$

Then the theorem will be proved in view of (7.29), (7.30), (7.31) and Theorem 7.2 of Parthasarathy [7].

Let us take $h > 0$ and set $x_j = x(jh)$, for convenience. Then $x_0$, $x_1$, $x_2, \cdots$ form a Markov chain. Let us define

$$\tau_1 = \inf[j : j > 0 \, |x_j - x_0| > \varepsilon] ,$$

$$\tau_2 = \inf[j : j > 0 \, |x_{\tau_1} - x_{\tau_1 + j}| > \varepsilon] ,$$

$$\vdots$$

$$\tau_k = \inf[j : j > 0 \, |x_{\tau_1 + \cdots + \tau_{k-1}} - x_{\tau_1 + \cdots + \tau_{k-1} + j}| > \varepsilon] .$$



Of course, after a certain stage, $\tau_k$ does not exist. In fact, since $\tau_{k+1} \geqq \tau_k + 1$, the process ends after $T/h$ steps. We wish actually to estimate

$$P_h[\min(\tau_1, \cdots, \tau_{k_0}) \leqq \rho h^{-1}] = \phi_h(\rho) ,$$

where $k_0$ is the last $j$ for which $\tau_j$ exists:

$$\begin{aligned} P_h[\min(\tau_1, \cdots, \tau_k) \leqq \rho h^{-1}] &\leqq P_h[\tau_1 \leqq \rho h^{-1}, \tau_1 h \leqq T] + \cdots \\ &\quad + P_h[\tau_k \leqq \rho h^{-1}, (\tau_1 + \cdots + \tau_k)h \leqq T] \\ &\quad + P_h[(\tau_1 + \cdots + \tau_{k+1})h \leqq T] \\ &\leqq P_h[\tau_1 \leqq \rho h^{-1}] + \cdots + P_h[\tau_k \leqq \rho h^{-1}] + P_h[(\tau_1 + \cdots + \tau_{k+1})h \leqq T] \\ &= P_h[h\tau_1 \leqq \rho] + \cdots + P_h[h\tau_k \leqq \rho] + P_h[h(\tau_1 + \cdots + \tau_{k+1}) \leqq T] . \end{aligned}$$

Here $k$ is an arbitrary integer. We note that

$$P_h[h\tau_1 \leqq \rho] = P_h\left[\sup_{0 \leqq t \leqq 0} |x(t) - x(0)| > \varepsilon\right] .$$

If $x_0, x_1, \cdots, x_n$ is a Markov chain and if

$$f_h(n, \varepsilon) = \sup_{\substack{x \\ 0 \leqq r < s < n}} P_h[|x_r - x_s| \geqq \tfrac{1}{2}\varepsilon \, | x_r = x] ,$$

we see, by a standard estimate, that

$$P_h\left[\sup_{0 \leqq j \leqq n} |x_0 - x_j| \geqq \varepsilon\right] \leqq \frac{f_h(n, \varepsilon)}{1 - f_h(n, \varepsilon)} .$$

Therefore,

$$P_h[h\tau_1 \leqq \rho] \leqq \frac{f_h(n, \varepsilon)}{1 - f_h(n, \varepsilon)} ,$$

where $n = [\rho h^{-1}]$. To estimate $f_h(n, \varepsilon)$ we note that, by hypothesis, for any $\delta > 0$

$$E^{P_h}[\|x_r - x_s\| \mid x_s = x] \leqq \delta + C_\delta |(r-s)h|^{1/2} .$$

Therefore,

$$\begin{aligned} f_h(n, \varepsilon) &\leqq \frac{2}{\varepsilon}[\delta + C_\delta (nh)^{1/2}] \\ &\leqq \frac{2}{\varepsilon}[\delta + C_\delta \rho^{1/2}] . \end{aligned}$$

Hence, for each $\varepsilon > 0$,

$$\sup_h P_h[h\tau_1 \leqq \rho] = f(\rho, \varepsilon) ,$$

where $f(\rho, \varepsilon) \to 0$ as $\rho \to 0$, for each $\varepsilon > 0$.

In view of the fact that we have the same estimate in terms of $f(\rho, \varepsilon)$ for all conditional distributions involved and because the strong Markov property is valid for any chain, we have

$$P_h[h\tau_j \leqq \rho] \leqq f(\rho, \varepsilon) ,$$

since all the $\tau_j$ are stopping times for the chain. Moreover,

$$P_h[h\tau_j \leqq \rho \mid \mathscr{M}_{h\tau_{j-1}}] \leqq f(\rho, \varepsilon) .$$

This means that

$$P_h[h(\tau_1 + \cdots + \tau_k) \leqq T] \leqq \psi_k(\varepsilon) ,$$

where $\psi_k(\varepsilon)$ is estimated in terms of $f(\rho, \varepsilon)$ and

$$\psi_k(\varepsilon) \to 0 \quad \text{as} \quad k \to \infty \quad \text{for} \quad \text{each} \quad \varepsilon > 0 .$$

Therefore,

$$\phi_k(\rho) \leqq kf(\rho, \varepsilon) + \psi_{k+1}(\varepsilon)$$

or

$$\sup_h \phi_h(\rho) = \phi(\rho) \to 0 \quad \text{as} \quad \rho \to 0 \quad \text{for} \quad \text{each} \quad \varepsilon > 0 .$$

It is easily verified that if $\min(\tau_1, \cdots, \tau_{k_0}) \leqq \rho h^{-1}$, then

$$\omega_x^*(\rho) \leqq 2\varepsilon .$$

Consequently,

$$\sup_h P_h[\omega_x^*(\rho) \geqq 2\varepsilon] \leqq \phi(\rho) ,$$

and this completes the proof of the theorem.

## Bibliography

[1] Chentsov, N. N, *Weak convergence of stochastic processes whose trajectories have no discontinuities of the second kind and the "heuristic" approach to the Kolmogorov-Smirnov tests*, Theor. Prob. Appl., Vol. 1, 1956, pp. 140–144.

[2] Doob, J. L., *Stochastic Processes*, John Wiley and Sons, New York, 1953.

[3] Fabes, E., and Rivière, N. M., *Singular integrals with mixed homogeneity*, Studia Math., Vol. 27, 1966, pp. 19–38.



[4] Kunita, H., and Watanabe, S., *On square integrable martingales*, Nagoya Math., J., Vol. 30, 1967, pp. 209–245.

[5] Ladyzhenskaya, O. A., Solonnikov, V. A., and Ural'tseva, N. N., *Linear and Quasi-linear Equations of Parabolic Type*, Amer. Math. Soc. Translations of Mathematical Monographs, No. 23, Amer. Math. Soc., Providence, R.I., 1968.

[6] Littman, W., McCarthy, C., and Rivière, N., *$L^p$ multiplier theorems*, Studia Math., Vol. 30, 1968, pp. 193–217.

[7] Parthasanathy, K. R., *Probability Measures on Metric Spaces*, Academic Press, New York, 1967.

[8] Stroock, D. W., and Varadhan, S. R. S., *Diffusion processes with continuous coefficients, I*, Comm. Pure Appl. Math., Vol. 22, 1969, pp. 345–400.

[9] Stroock, D. W., and Varadhan, S. R. S., *Diffusion processes with continuous coefficients, II*, Comm. Pure Appl. Math., Vol. 22, 1969, pp. 479–530.

[10] Meyer, P. A., *Probability and Potentials*, Blaisdell, Waltham, Toronto-London, 1966.

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