SIAM J. CONTROL OPTIM.
Vol. 54, No. 4, pp. 2202–2233

OPTIMAL SWITCHING IN FINITE HORIZON UNDER STATE CONSTRAINTS*

IDRIS KHARROUBI†

Abstract. We study an optimal switching problem with a state constraint: the controller is only allowed to choose strategies that keep the controlled diffusion in a closed domain. We prove that the value function associated with this problem is the limit of value functions associated with unconstrained switching problems with penalized coefficients, as the penalization parameter goes to infinity. This convergence allows one to set a dynamic programming principle for the constrained switching problem. We then prove that the value function is a solution to a system of variational inequalities (SVI) in the constrained viscosity sense. We finally prove that uniqueness for our SVI cannot hold, and we give a weaker characterization of the value function as the maximal solution to this SVI. All our results are obtained without any regularity assumption on the constraint domain.

Key words. optimal switching, state constraints, dynamic programming, variational inequalities, energy and resources management

AMS subject classifications. 60H10, 60H30, 91G80, 93E20

DOI. 10.1137/15M1012281

1. Introduction.

Optimal control of multiple switching regimes consists of looking for the value of an optimization problem where the allowed strategies are sequences of interventions. It naturally arises in many applied disciplines where it is not realistic to assume that the involved quantities can be continuously controlled. More precisely, the optimal switching problem supposes that the control strategies are sequences $\alpha = (\tau_k, \zeta_k)_k$, where the sequence $(\tau_k)_k$ represents the intervention times of the controller and ζ_k corresponds to the level of intervention of the agent at each time τ_k.

Such a class of strategies allows us to consider discrete actions for the controller which can be more relevant than continuous time controls. Therefore, the modelization with optimal switching problems has attracted a lot of interest during recent decades (see, e.g., Brennan and Schwartz [2] for resource extraction, Dixit [8] for production facility problems, Carmona and Ludkovski [4] for power plant management, or Ly Vath, Pham, and Villeneuve [14] for the dividend decision problem with reversible technology investment).

Another specificity to take into account in the modelization with optimal switching is the limitation of the quantities involved in the control problem. Indeed, in most management problems the controlled system is subject to a constraint on the possible states that it can take. For example, a solvency condition is usually imposed on the investors in a financial market, and the energy producer has to take into account the limited storage capacities. This leads us to impose a state constraint on the controlled diffusion X of the form

$$X_s \in \mathcal{D} \quad \text{for all } s,$$

*Received by the editors March 12, 2015; accepted for publication (in revised form) June 3, 2016; published electronically August 30, 2016.
http://www.siam.org/journals/sicon/54-4/M101228.html
Funding: This research supported by the French ANR research grant LIQUIRISK (ANR-11-JS01-0007).
†Université Paris Dauphine, PSL Research University, CNRS, UMR 7534, CEREMADE, Place du Maréchal De Lattre De Tassigny, F-75775 PARIS CEDEX 16 (kharroubi@ceremade.dauphine.fr).

where $\mathcal{D}$ is a closed set. We therefore need to restrict our control problem to the set $\mathcal{A}^{\mathcal{D}}_{t,x}$ of strategies that keep the controlled diffusion starting from (t,x) in the constraint domain $\mathcal{D}$. Unfortunately, such a constraint leads to strong difficulties due, in particular, to the complicated structure of the set-valued function $(t,x) \mapsto \mathcal{A}^{\mathcal{D}}_{t,x}$. To the best of our knowledge, no rigorous study of the optimal switching problem in the constrained case has been done before, and our aim is to fill this gap.

In the continuous time control case, Soner [15] gives a first study of the constrained problem in a deterministic framework, introducing the notion of constrained viscosity solutions. To characterize the value function, his approach relies on a continuity argument under an assumption on the boundary of the constraint domain $\partial\mathcal{D}$. He then extends this result to the case of piecewise deterministic processes in [16]. The continuous time stochastic control case is studied by Katsoulakis in [12]. His approach also relies on continuity, and he imposes regularity conditions on the constraint domain $\mathcal{D}$. In our case, such an approach is not possible since the value function may be discontinuous even for a smooth domain $\mathcal{D}$ as the counterexample presented in subsection 2.2 will show.

Let us also mention the recent approach of Goreac, Ivanescu, and Serea presented in [10]. They formulate the initial problem as a linear problem which concerns the occupation measures induced by the controlled diffusion processes. Under convexity assumptions, the authors characterize (see Theorem 11 in [10]) the value function associated to the weak formulation of the continuous time stochastic control problem under state constraints. (The weak formulation means that the controller is allowed to choose the probability space in addition to the control strategy.) Unfortunately, such an approach cannot be applied to the optimal switching under state constraints since the the set of values taken by the controls is not convex.

In this work, we present an original approach which allows us to deal with the lack of regularity of the associated value function. Moreover, our method does not need any regularity or convexity assumption. In particular, we only need to assume that the constraint domain $\mathcal{D}$ is closed.

To be more precise, our approach relies on the simple structure of switching controls. Indeed, they can be seen as random variables taking values in $([0,T]\times\mathcal{I})^{\mathbb{N}}$, where $\mathcal{I}$ is a finite set and $T>0$ is a given constant. From the Tychonov theorem we get the compactness of this space, which allows us to prove the tightness of a sequence $(\alpha^n)_n$ of switching strategies and hence the convergence in law up to a subsequence. Then, applying the Skorokhod representation theorem, we are able to provide a probability space and a sequence $(\tilde{\alpha}^n)_n$ that converges almost surely to some $\tilde{\alpha}$ and such that $\tilde{\alpha}^n$ is equal in law to α^n for all n.

We use this sequential compactness property in the following way. We first introduce a sequence $(v_n)_n$ of unconstrained switching problems with n-penalized terminal and running reward coefficients out of the constraint domain $\mathcal{D}$. For each penalized switching problem v_n, we take α^n as a $\frac{1}{n}$-almost optimal strategy for v_n, and we make $\tilde{\alpha}^n$ converge to $\tilde{\alpha}$ as described previously. Then we construct a switching strategy α^* which is equal in law to $\tilde{\alpha}$. To this end we prove stability results for measurability and convergence properties for a sequence of diffusion driven by converging Brownian motions. These results, which have their own interest, are presented separately in the appendix.

The strong convergence of $\tilde{\alpha}^n$ to $\tilde{\alpha}$ allows us to prove that α^* is optimal for the switching problem under constraint. As a byproduct, we get the convergence of the unconstrained penalized switching problems to the constrained one. Using existing results on classical optimal switching problems, this convergence allows us to set a dynamic programming principle for the constrained switching problem.

We then focus on the PDE characterization of the value function. Using the dynamic programming principle proved before, we show that the value function is a constrained viscosity solution to a system of variational inequalities (SVI) defined on the constraint domain $\mathcal{D}$. We then investigate the uniqueness of a solution to this SVI. The usual approach to getting uniqueness of a viscosity solution consists of proving a comparison theorem for the PDE. As a consequence of such a comparison theorem, the unique solution has to be continuous. Unfortunately, the continuity of the value functions is not true in general, as shown by the counterexample given in subsection 2.2. Therefore, we cannot hope to state such a uniqueness result for the SVI on $\mathcal{D}$. Instead, we characterize our value function as the maximal viscosity solution of the SVI under an additional growth assumption. This maximality property is also obtained from the convergence of the penalized unconstrained problems to the constrained one.

The organization of the paper is as follows. In section 2 we lay out in detail the formulation of the optimal switching problem under state constraints, and we provide a simple example to stress the possible lack of regularity for the value function. We then give in section 3 some examples of applications. In section 4, we provide an approximation of our constrained problem by unconstrained problems with penalized coefficients. We prove the convergence of the penalized problems to the constrained one as the penalization parameter goes to infinity. In section 5, we state a dynamic programming principle, and we prove that the value function is a constrained viscosity solution to an SVI. Finally, in section 6 we focus on uniqueness. Since we cannot prove uniqueness of a solution for the SVI, we characterize the value function as the maximal constrained viscosity solution to the SVI under an additional growth assumption. Some examples where this additional growth condition is satisfied are then given.

2. Problem formulation.

2.1. Optimal switching under state constraints. We fix a complete probability space $\big(\Omega, \mathcal{G}, \mathbb{P}\big)$ which is endowed with a Brownian motion $W = (W_t)_{t\geq 0}$ valued in $\mathbb{R}^d$. We denote by $\mathbb{F}$ the complete and right continuous filtration generated by W. We also consider a terminal time given by a constant $T > 0$.

Controls. We then define the set $\mathcal{A}_t$ of admissible switching controls at time $t \in [0, T]$ as the set of double sequences $\alpha = (\tau_k, \zeta_k)_{k\geq 0}$, where

- $(\tau_k)_{k\geq 0}$ is a nondecreasing sequence of $\mathbb{F}$-stopping times with $\tau_0 = t$ and $\lim_{k\to\infty} \tau_k > T$,
- ζ_k is an $\mathcal{F}_{\tau_k}$-measurable random variables valued in the set $\mathcal{I}$ defined by $\mathcal{I} = \{1, \dots, m\}$.

With a strategy $\alpha = (\tau_k, \zeta_k)_{k\geq 0} \in \mathcal{A}_t$ we associate the process $(\alpha_s)_{s\geq t}$ defined by

$$\alpha_s = \sum_{k\geq 0} \zeta_k \mathbb{1}_{[\tau_k, \tau_{k+1})}(s), \quad s \geq t.$$

Controlled diffusion. We are given two functions, $\mu :\ \mathbb{R}^d \times \mathcal{I} \to \mathbb{R}^d$ and $\sigma :\ \mathbb{R}^d \times \mathcal{I} \to \mathbb{R}^{d\times d}$. We make the following assumption:

(H1) There exists a constant L such that

$$|\mu(x, i) - \mu(x', i)| + |\sigma(x, i) - \sigma(x', i)| \leq L|x - x'|$$

for all $(x, x', i) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{I}$.

For $(t,x) \in [0,T] \times \mathbb{R}^d$ and $\alpha \in \mathcal{A}_t$ we consider the controlled diffusion $X^{t,x,\alpha}$ defined by the following SDE:

$$X_s^{t,x,\alpha} = x + \int_t^s \mu\big(X_r^{t,x,\alpha}, \alpha_r\big)dr + \int_t^s \sigma\big(X_r^{t,x,\alpha}, \alpha_r\big)dW_r, \quad s \geq t. \tag{2.1}$$

Under **(H1)**, we have the existence and uniqueness of an $\mathbb{F}$-adapted solution $X^{t,x,\alpha}$ to (2.1) for any initial condition $(t,x) \in [0,T] \times \mathbb{R}^d$ and any switching control $\alpha \in \mathcal{A}_t$.

We also have the following classical estimate (see, e.g., [13, Chapter 2, section 5, Corollary 12]): for any $q \geq 1$ there exists a constant C_q such that

$$\sup_{\alpha \in \mathcal{A}_t} \mathbb{E}\left[\sup_{s \in [t,T]} \big|X_s^{t,x,\alpha}\big|^q\right] \leq C_q\big(1 + |x|^q\big) \tag{2.2}$$

for all $(t,x) \in [0,T] \times \mathbb{R}^d$.

Expected payoff. We consider terminal and running reward functions $g:\ \mathbb{R}^d \times \mathcal{I} \to \mathbb{R}$ and $f:\ \mathbb{R}^d \times \mathcal{I} \to \mathbb{R}$ and a cost function $c:\ \mathbb{R}^d \times \mathcal{I} \times \mathcal{I} \to \mathbb{R}$ on which we impose the following assumption:

(H2)

(i) The functions f, g, and c are locally Lipschitz: for any $R > 0$ there exists a constant L_R such that

$$|g(x,i) - g(x',i)| + |f(x,i) - f(x',i)| + |c(x,i,j) - c(x',i,j)| \leq L_R|x - x'|$$

for all $i,j \in \mathcal{I}$ and $x, x' \in \mathbb{R}^d$ such that $|x| \leq R$ and $|x'| \leq R$.

(ii) There exist a constant C and an integer q such that

$$|g(x,i)| + |f(x,i)| + |c(x,i,j)| \leq C\big(1 + |x|^q\big)$$

for all $x \in \mathbb{R}^d$ and $i,j \in \mathcal{I}$.

(iii) There exists a constant $\bar{c} > 0$ such that

$$c(x,i,j) \geq \bar{c}$$

for all $x \in \mathbb{R}^d$ and $i,j \in \mathcal{I}$.

We then define the functional payoff J up to time T by

$$J\big(t,x,\alpha\big) = \mathbb{E}\left[g\big(X_T^{t,x,\alpha}, \alpha_T\big) + \int_t^T f\big(X_s^{t,x,\alpha}, \alpha_s\big)ds - \sum_{k \geq 1} c\big(X_{\tau_k}^{t,x,\alpha}, \zeta_{k-1}, \zeta_k\big)\mathbb{1}_{\tau_k \leq T}\right]$$

for all $(t,x) \in [0,T] \times \mathbb{R}^d$ and $\alpha \in \mathcal{A}_t$.

Under **(H1)** and **(H2)** we get from (2.2) that $J\big(t,x,\alpha\big)$ is well defined for any initial condition $(t,x) \in [0,T] \times \mathbb{R}^d$ and any control $\alpha \in \mathcal{A}_t$.

State constraint. Let $\mathcal{D}$ be a nonempty closed subset of $\mathbb{R}^d$. For $(t,x,i) \in [0,T] \times \mathcal{D} \times \mathcal{I}$ we denote by $\mathcal{A}_{t,x,i}^{\mathcal{D}}$ the set of strategies $\alpha \in \mathcal{A}_t$ such that $\zeta_0 = i$ and

$$\mathbb{P}\Big(X_s^{t,x,\alpha} \in \mathcal{D} \text{ for all } s \in [t,T]\Big) = 1.$$

Value function. We then define the value function v associated with the switching problem under state constraints by

$$v(t,x,i) = \sup_{\alpha \in \mathcal{A}^{\mathcal{D}}_{t,x,i}} J\big(t,x,\alpha\big) \tag{2.3}$$

for all $(t,x,i) \in [0,T] \times \mathcal{D} \times \mathcal{I}$, with the convention $v(t,x,i) = -\infty$ if $\mathcal{A}^{\mathcal{D}}_{t,x,i} = \emptyset$. Our aim is to give an analytic characterization of the function v.

2.2. Lack of smoothness for the value function. In general control theory, we expect to get a continuous value function, as we assume that the parameters are continuous. In the framework of optimal switching under constraints, such a property fails to be true. Indeed, the following simple example provides a discontinuous value function.

Fix $d = 2$, and consider the case where $\mathcal{D}$ is the smooth domain $\mathbb{R} \times \mathbb{R}_+$. Take $\mathcal{I} = \{1,2\}$, and define the diffusion coefficients μ and σ by

$$\mu(x,1) \;=\; \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad \mu(x,2) \;=\; \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{and} \quad \sigma(x,1) \;=\; \sigma(x,2) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

for all $x \in \mathbb{R}^2$. Define the gain coefficients g and $f$and the cost functions $c(.,1,2)$ and $c(.,2,1)$ by

$$g(x,1) = g(x,2) = 0, \quad f(x,1) = f(x,2) = 1, \quad \text{and} \quad c(x,1,2) = c(x,2,1) = c > 0$$

for all $x \in \mathbb{R}^2$. Since the reward coefficients f and g do not depend on the state position x, we only need to focus on the constraint. In particular a strategy is optimal if it minimizes the number of switching orders and satisfies the state constraint.

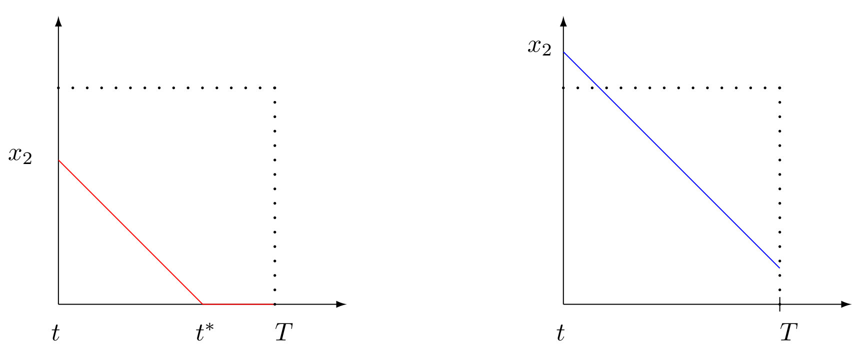

FIG. 1. *Second component of optimal trajectories in the cases $x_2 < T - t$, $i = 1$ (left panel, red curve) and $x_2 \geq T - t$, $i = 1$ (right panel, blue curve).*

As shown by Figure 1, in the case $x_2 < T - t$ and $i = 1$, the agent has to act at time t^* to keep the second component nonnegative (see the red curve). In contrast, in the case $x_2 \geq T - t$ and $i = 1$, the blue curve shows that the system will satisfy the constraint until terminal time T, and there is no need to switch. We therefore get the following expression for the value function:

$$v(t,x,1) \;=\; \begin{cases} T-t & \text{if } x_2 \geq T-t, \qquad T-t \qquad \text{if } x_2 \geq T-t, \\ T-t-c & \text{if } x_2 < T-t, \qquad T-t-c \quad \text{if } x_2 < T-t \end{cases} \tag{2.4}$$

for all $x = \binom{cx_1}{x_2} \in \mathcal{D}$ and all $t \in [0, T]$.

In particular, the function $v(.,1)$ is discontinuous at each point $(t,(x_1, T-t))$ for all $t \in [0,T]$ and all $x_1 \in \mathbb{R}$. Hence the function v is discontinuous even on the interior $Int(\mathcal{D})$ of the constraint domain. These discontinuities are induced by the state constraints that force the operator to act so as to keep the diffusion in $\mathcal{D}$, even if this action is suboptimal.

3. Examples of applications.

We present in this section some models involving an optimal switching problem under state constraint.

3.1. Hydroelectric pumped storage model.

The following simplified hydroelectric pumped storage model is inspired by [4]. Pumped storage (currently, the dominant type of electricity storage) consists of a large reservoir of water held by a hydroelectric dam at a higher elevation. When desired, the dam can be opened, which activates the turbines and moves the water to another, lower reservoir. The generated electricity is sold to a power grid. As the water flows, the upper reservoir is depleted. Conversely, in times of low electricity demand, the water can be pumped back into the reservoir with required energy purchased from grid. A strategy α consists of a sequence of $\mathbb{F}$-stopping times $(\tau_k)_k$, representing the intervention times, and a sequence of $\mathcal{F}_{\tau_k}$-measurable random variables $(\zeta_k)_k$, representing the changes of regime. There are three possible regimes:

(i) $\zeta_k = 1$: pump; in this case we set $\mu_1(x,1) = 1$ and $\sigma_1(x,1) = 0$.
(ii) $\zeta_k = 2$: store; in this case we set $\mu_1(x,1) = 0$ and $\sigma_1(x,1) = 0$.
(iii) $\zeta_k = 3$: generate; in this case we set $\mu_1(x,1) = -1$ and $\sigma_1(x,1) = 0$.

For a given strategy $\alpha = (\tau_k, \zeta_k)_k$, we denote by L_t^α the controlled water level in the upper reservoir. It satisfies the equation

$$L_t^\alpha = L_0 + \int_0^t \mu_1(L_s^\alpha, \alpha_s)ds + \int_0^t \sigma_1(L_s^\alpha, \alpha_s)dW_s, \quad t \geq 0.$$

Denote by P the electricity price process, and suppose that it is a diffusion defined on $(\Omega, \mathcal{G}, \mathbb{P})$ by

$$P_t = P_0 + \int_0^t \mu_2(P_s)ds + \int_0^t \sigma_2(P_s)dW_s, \quad t \geq 0.$$

Let X^α be the controlled process defined by $X^\alpha = \binom{L^\alpha}{P}$. Then it satisfies the SDE

$$X_t^\alpha = X_0 + \int_0^t \mu(X_s^\alpha, \alpha_s)ds + \int_0^t \sigma(X_s^\alpha, \alpha_s)dW_s, \quad t \geq 0,$$

with $\mu = \binom{\mu_1}{\mu_2}$ and $\sigma = \binom{\sigma_1}{\sigma_2}$. Suppose also that the cost of changing the regime from i to j is given by a constant $c(i,j)$. The expected payoff for a strategy α is then given by

$$\begin{aligned} J(0, X_0, \alpha) &= \mathbb{E}\left[\int_0^T -P_t dL_t^\alpha - \sum_{\tau_k \leq T} c(\zeta_{k-1}, \zeta_k)\right] \\ &= \mathbb{E}\left[\int_0^T f(X_t^\alpha, \alpha_t)dt - \sum_{\tau_k \leq T} c(\zeta_{k-1}, \zeta_k)\right], \end{aligned}$$

where f is defined by $f(p, \ell, i) = -p \times \mu_1(\ell, i)$ for all $(p, \ell, i) \in \mathbb{R} \times \mathbb{R} \times \{1, 2, 3\}$.

Since the reservoir capacity is not infinite, the strategy α has to satisfy the constraint $0 \leq L_t^\alpha \leq \ell_{max}$ for all $t \in [0,T]$. This corresponds to the general constraint $X_t^\alpha \in \mathcal{D}$, where $\mathcal{D} = \mathbb{R} \times [0, \ell_{max}]$. The goal of the energy producer is to maximize $J(0, X_0, \alpha)$ over the strategies α satisfying the constraint on the water level L^α.

3.2. Valuation of natural resources. The following model comes from [2]. We consider an agent that holds a mine that produces a single homogeneous commodity. We suppose that the commodity price S is given by

$$S_t = S_0 + \int_0^t \mu_1(S_u)du + \int_0^t \sigma_1(S_u)dW_u, \quad t \geq 0.$$

The agent can choose to extract or not the commodity from the mine. Thus, the strategy α consists of a sequence of $\mathbb{F}$-stopping times $(\tau_k)_k$, representing the intervention times, and a sequence of $\mathcal{F}_{\tau_k}$-measurable random variables $(\zeta_k)_k$, representing the changes of regime. There are two possible regimes:

(i) $\zeta_k = 1$: extraction; in this case we set $\mu_2(x, 1) = -1$ and $\sigma_2(x, 1) = 0$.

(ii) $\zeta_k = 0$: no extraction; in this case we set $\mu_2(x, 2) = 0$ and $\sigma_2(x, 2) = 0$.

For a strategy $\alpha = (\tau_k, \zeta_k)_k$, we denote by Q_t^α the physical inventory of the mine at time t. Therefore, it satisfies the equation

$$Q_t^\alpha = Q_0 + \int_0^t \mu_2(Q_s^\alpha, \alpha_s)ds + \int_0^t \sigma_2(Q_s^\alpha, \alpha_s)dW_s, \qquad t \geq 0.$$

Denote by X^α the controlled process defined by $X^\alpha = \binom{S}{Q^\alpha}$. Then it satisfies the SDE

$$X_t^\alpha = X_0 + \int_0^t \mu(X_s^\alpha, \alpha_s)ds + \int_0^t \sigma(X_s^\alpha, \alpha_s)dW_s, \qquad t \geq 0,$$

with $\mu = \binom{\mu_1}{\mu_2}$ and $\sigma = \binom{\sigma_1}{\sigma_2}$. Suppose also that the cost of changing the regime from i to j is given by a constant $c(i, j)$. The expected payoff for a strategy α is then given by

$$\begin{aligned} J(0, X_0, \alpha) &= \mathbb{E}\left[\int_0^T S_t dQ_t^\alpha - \sum_{\tau_k \leq T} c(\zeta_{k-1}, \zeta_k)\right] \\ &= \mathbb{E}\left[\int_0^T f(X_t^\alpha, \alpha_t)dt - \sum_{\tau_k \leq T} c(\zeta_{k-1}, \zeta_k)\right], \end{aligned}$$

where f is defined by $f(s, q, i) = -s \times \mu_2(q, i)$ for all $(s, q, i) \in \mathbb{R} \times \mathbb{R} \times \{0, 1\}$.

Since the physical inventory is nonnegative, the strategy α has to satisfy the constraint $Q_t^\alpha \geq 0$ for all $t \in [0,T]$. This corresponds to the general constraint $X_t^\alpha \in \mathcal{D}$, where $\mathcal{D} = \mathbb{R} \times \mathbb{R}_+$. Thus, the aim of the agent is to maximize $J(0, X_0, \alpha)$ over the strategies α satisfying the constraint on the inventory Q^α.

3.3. Reversible technology investment. We present a simplified version of the model studied in [14]. We consider a firm whose activities generate a cash process by using some technology. The firm has at any time the possibility of choosing between two technologies: a modern one and an old one. Therefore, its strategy α consists of a sequence of $\mathbb{F}$-stopping times $(\tau_k)_k$, representing the times of change of technology, and a sequence of $\mathcal{F}_{\tau_k}$-measurable random variables $(\zeta_k)_k$, representing the chosen technology at each time τ_k. Thus, there are two possible regimes:

(i) $\zeta_k = 1$: old technology; in this case we set $\mu(x,1) = \delta_1 x$ and $\sigma(x,1) = \gamma_1 x$.

(ii) $\zeta_k = 2$: modern technology; in this case we set $\mu(x,2) = \delta_2 x$ and $\sigma(x,2) = \gamma_2 x$.

Here γ_1, γ_2, δ_1, and δ_2 are four constants with $\delta_1 < \delta_2$ and $\gamma_1 < \gamma_2$. (The modern technology has a better rate but a worse uncertainty than the old technology.) For a strategy $\alpha = (\tau_k, \zeta_k)_k$, we denote by X_t^α the cash reserve at time t of the firm. We suppose that it satisfies the equation

$$X_t^\alpha = X_0 + \int_0^t \mu(X_s^\alpha, \alpha_s)ds + \int_0^t \sigma(X_s^\alpha, \alpha_s)dW_s, \qquad t \geq 0.$$

We also suppose that the cost of changing the technology from i to j is given by a constant $c(i,j)$. Then the expected payoff at terminal time T for a strategy α is given by

$$J(0, X_0, \alpha) = \mathbb{E}\left[X_T^\alpha - \sum_{\tau_k \leq T} c(\zeta_{k-1}, \zeta_k)\right].$$

We suppose that the firm has to satisfy the following solvency constraint: $X_t^\alpha \geq 0$ for all $t \in [0,T]$. This corresponds to the constraint domain $\mathcal{D} = \mathbb{R}_+$. Thus, the goal of the firm is to maximize $J(0, X_0, \alpha)$ over the strategies α satisfying the constraint on the cash reserve R^α.

4. Unconstrained penalized switching problem.

4.1. An unconstrained penalized approximating problem. We now introduce an approximation of our initial constrained problem. This approximation consists of a penalization of the coefficients f and g out of the domain $\mathcal{D}$ where the controlled underlying diffusion is constrained to stay.

Consider, for $n \geq 1$, the functions $f_n : \mathbb{R}^d \times \mathcal{I} \to \mathbb{R}$ and $g_n : \mathbb{R}^d \times \mathcal{I} \to \mathbb{R}$ defined by

$$f_n(x,i) = f(x,i) - n\Theta_n(x), \tag{4.1}$$

$$g_n(x,i) = g(x,i) - n\Theta_n(x) \tag{4.2}$$

for all $(x,i) \in \mathbb{R}^d \times \mathcal{I}$, where the function $\Theta_n : \mathbb{R}^d \to [0,1]$ is given by

$$\Theta_n(x) = n\left(d\big(x, \mathcal{D}\big) \wedge \frac{1}{n}\right) \;=\; nd(x,\mathcal{D}) \wedge 1, \tag{4.3}$$

with $d(x,\mathcal{D}) = \inf_{x' \in \mathcal{D}} |x - x'|$ for all $x \in \mathbb{R}^d$.

Given an initial condition (t,x) and a switching control $\alpha = (\tau_k, \zeta_k)_{k \geq 0} \in \mathcal{A}_t$, we consider the total penalized profit starting from $(t,x,i) \in [0,T] \times \mathbb{R}^d \times \mathcal{I}$ at horizon T, defined by

$$J_n(t,x,\alpha) = \mathbb{E}\left[g_n\big(X_T^{t,x,\alpha}, \alpha_T\big) + \int_t^T f_n\big(X_s^{t,x,\alpha}, \alpha_s\big)ds - \sum_{k \geq 1} c\big(X_{\tau_k}^{t,x,\alpha}, \zeta_{k-1}, \zeta_k\big)\mathbb{1}_{\tau_k \leq T}\right].$$

We can then define the penalized unconstrained value function $v_n : [0,T] \times \mathbb{R}^d \times \mathcal{I} \to \mathbb{R}$ by

$$v_n(t,x,i) = \sup_{\alpha \in \mathcal{A}_{t,i}} J_n(t,x,\alpha) \tag{4.4}$$

for all $n \geq 1$ and all $(t,x,i) \in [0,T] \times \mathbb{R}^d \times \mathcal{I}$, where $\mathcal{A}_{t,i}$ is the set of strategies $\alpha = (\tau_k, \zeta_k)_{k \geq 0} \in \mathcal{A}_t$ such that $\zeta_0 = i$.

4.2. Convergence of the penalized unconstrained problems. We now state the main result of this section, which concerns the convergence of the functions v_n to v. The main line of the proof is to take a sequence of almost optimal strategies for the functions v_n and to make it converge to a strategy that we expect to be optimal. To do this, we need to prove measurability and convergence results for diffusion driven by a converging sequence of Brownian motions. These results are presented in detail in Appendix A.1.

THEOREM 4.1. *Under* **(H1)** *and* **(H2)**, *the sequence* $(v_n)_{n\geq 1}$ *is nonincreasing and converges on* $[0,T]\times\mathcal{D}\times\mathcal{I}$ *to the function* v*:*

$$v_n(t,x,i)\downarrow v(t,x,i) \qquad \text{as } n\uparrow +\infty \tag{4.5}$$

for all $(t,x,i)\in[0,T]\times\mathcal{D}\times\mathcal{I}$. *Moreover, for any* $(t,x,i)\in[0,T]\times\mathcal{D}\times\mathcal{I}$ *there exists a strategy* $\alpha^*\in\mathcal{A}^{\mathcal{D}}_{t,x,i}$ *such that*

$$v(t,x,i)=J(t,x,\alpha^*).$$

Proof. Fix $(t,x,i)\in[0,T]\times\mathcal{D}\times\mathcal{I}$. Since $f_{n+1}\leq f_n$ and $g_{n+1}\leq g_n$ we get

$$J_{n+1}(t,x,\alpha)\leq J_n(t,x,\alpha)$$

for all $n\geq 1$ and $\alpha\in\mathcal{A}_t$. From this last inequality we deduce that

$$v_{n+1}(t,x,i)\leq v_n(t,x,i),\quad n\geq 1.$$

We now prove that $(v_n)_n$ converges to v. We first notice that

$$J_n(t,x,\alpha)=J(t,x,\alpha)$$

for any $n\geq 1$, any initial condition $(t,x,i)\in[0,T]\times\mathcal{D}\times\mathcal{I}$, and any switching strategy $\alpha\in\mathcal{A}^{\mathcal{D}}_{t,x,i}$. Therefore, we get $v_n\geq v$ for all $n\geq 1$. Denote by $\bar{v}$ the pointwise limit of $(v_n)_n$:

$$\bar{v}(t,x,i)=\lim_{n\to\infty}v_n(t,x,i),\qquad (t,x,i)\in[0,T]\times\mathcal{D}\times\mathcal{I}.$$

Then we have $\bar{v}(t,x,i)\geq v(t,x,i)$. If $\bar{v}(t,x,i)=-\infty$, we obviously get $\bar{v}(t,x,i)=v(t,x,i)$.

We now suppose that $\bar{v}(t,x,i)>-\infty$ and prove that $\bar{v}(t,x,i)\leq v(t,x,i)$. We proceed in three steps.

Step 1. Convergence of a sequence of almost optimal strategies for the unconstrained problems.

Substep 1.1. Bounded sequence of almost optimal strategies. For $n\geq 1$, let $\alpha^n=(\tau^n_k,\zeta^n_k)_{k\geq 0}\in\mathcal{A}_{t,i}$, a switching strategy such that

$$J_n(t,x,\alpha^n)\geq v_n(t,x,i)-\frac{1}{n}.$$

We can suppose without loss of generality that

$$\tau^n_k\in[0,T]\cup\{T+1\},\qquad \mathbb{P}\text{-a.s.} \tag{4.6}$$

for all $n\geq 1$ and all $k\geq 0$. Indeed, fix $n\geq 1$, and consider the strategy $\hat{\alpha}^n=(\hat{\tau}^n_k,\hat{\zeta}^n_k)_{k\geq 0}\in\mathcal{A}_{t,i}$ defined by

$$\begin{aligned}\hat{\tau}^n_k&=\tau^n_k\mathbb{1}_{\tau^n_k\leq T}+(T+1)\mathbb{1}_{\tau^n_k>T},\\ \hat{\zeta}^n_k&=\zeta^n_k\mathbb{1}_{\tau^n_k\leq T}+i\mathbb{1}_{\tau^n_k>T}.\end{aligned}$$

Then we have $J_n(t,x,\alpha^n) = J_n(t,x,\hat{\alpha}^n)$, and we can replace α^n by $\hat{\alpha}^n$, which satisfies (4.6).

Substep 1.2. Tightness and convergence of $(W,\alpha^n)_n$. We now prove that the sequence of $C([0,T],\mathbb{R}^d)\times(\mathbb{R}_+\times\mathcal{I})^{\mathbb{N}}$-valued random variables $(W,\alpha^n)_{n\geq 1}$ is tight. Fix a sequence $(\delta_\ell)_\ell$ of positive numbers such that

$$\delta_\ell \xrightarrow[\ell\to\infty]{} 0 \quad \text{and} \quad 2^\ell\delta_\ell \ln\left(\frac{2T}{\delta_\ell}\right) \xrightarrow[\ell\to\infty]{} 0. \tag{4.7}$$

We define for $\eta>0$ and $C>0$ the subset $\mathcal{K}^C_\eta$ of $C([0,T],\mathbb{R}^d)$ by

$$\mathcal{K}^C_\eta = \bigcap_{\ell\geq 1}\mathcal{K}^C_{\eta,\ell},$$

where

$$\mathcal{K}^C_{\eta,\ell} = \left\{h\in C([0,T],\mathbb{R}^d)\ :\ h(0)=0 \ \text{ and } \ \mathrm{mc}_{\delta_\ell}(h) \ \leq \ C\frac{2^\ell\delta_\ell\ln\left(\frac{2T}{\delta_\ell}\right)}{\eta}\right\}$$

and mc denotes the modulus of continuity defined by

$$\mathrm{mc}_\delta(h) = \sup_{\substack{s,\,t\,\in\,[0,\,T]\\ |s\,-\,t|\,\leq\,\delta}} \big|h(s)-h(t)\big|$$

for any $h\in C([0,T],\mathbb{R}^d)$ and any $\delta>0$. Using the Arzéla–Ascoli theorem, we get from (4.7) that $\mathcal{K}^C_\eta$ is a compact subset of $C([0,T],\mathbb{R}^d)$. We now define the subset $\mathbf{K}^C_\eta$ of $C([0,T],\mathbb{R}^d)\times(\mathbb{R}_+\times\mathcal{I})^{\mathbb{N}}$ by

$$\mathbf{K}^C_\eta = \mathcal{K}^C_\eta \times \big([0,T+1]\times\mathcal{I}\big)^{\mathbb{N}}.$$

From the Tychonov theorem and since $\mathcal{K}^C_\eta$ is compact, we get that $\mathbf{K}^C_\eta$ is a compact subset of $C([0,T],\mathbb{R}^d)\times(\mathbb{R}_+\times\mathcal{I})^{\mathbb{N}}$ endowed with the norm $\|\cdot\|$ defined by

$$\big\|\big(h,(t_k,z_k)_{k\geq 0}\big)\big\| = \sup_{t\in[0,T]}|h(t)| + \sum_{k\geq 0}\frac{(|t_k|+|z_k|)\wedge 1}{2^k}$$

for all $h\in C([0,T],\mathbb{R}^d)$ and $(t_k,z_k)_{k\geq 0}\in(\mathbb{R}_+\times\mathcal{I})^{\mathbb{N}}$. We then have from (4.6)

$$\mathbb{P}\Big((W,\alpha^n)\in\mathbf{K}^C_\eta\Big) = \mathbb{P}\Big(W\in\mathcal{K}^C_\eta\Big)$$

for all $\eta>0$, $C>0$, and $n\geq 1$. Using the Markov inequality, we get

$$\begin{aligned}
\mathbb{P}\Big(W\in\mathcal{K}^C_\eta\Big) &= 1-\mathbb{P}\Big(W\notin\mathcal{K}^C_\eta\Big)\\
&\geq 1-\sum_{\ell\geq 1}\mathbb{P}\Big(W\notin\mathcal{K}^C_{\eta,\ell}\Big)\\
&\geq 1-\sum_{\ell\geq 1}\frac{\mathbb{E}[\mathrm{mc}_{\delta_\ell}(W)]}{C\frac{2^\ell\delta_\ell\ln\left(\frac{2T}{\delta_\ell}\right)}{\eta}}\,. \qquad (4.8)
\end{aligned}$$

From Theorem 1 in [9], there exists a constant C^* such that

$$\mathbb{E}\Big[\mathrm{mc}_\delta(W)\Big] \leq C^*\delta\ln\left(\frac{2T}{\delta}\right) \tag{4.9}$$

for all $\delta > 0$. Therefore, we get from (4.8) and (4.9) that

$$\mathbb{P}\Big((W,\alpha^n) \in \mathbf{K}_\eta^{C^*}\Big) \geq 1-\eta$$

for all $\eta \in (0,1)$, and the sequence $(W,\alpha^n)_n$ is tight.

We deduce from the Prokhorov theorem that, up to a subsequence,

$$\mathbb{P}\circ(W,\alpha^n)^{-1} \xrightarrow[n\to\infty]{} \mathcal{L},$$

with $\mathcal{L}$ a probability measure on $\big(C([0,T],\mathbb{R}^d)\times(\mathbb{R}\times\mathcal{I})^{\mathbb{N}}, \|\cdot\|\big)$.

Step 2. Change of probability space. Since $\big(C([0,T],\mathbb{R}^d)\times(\mathbb{R}\times\mathcal{I})^{\mathbb{N}}, \|\cdot\|\big)$ is separable, we get from the Skorokhod representation theorem that there exists a probability space $(\tilde\Omega,\tilde{\mathcal{G}},\tilde{\mathbb{P}})$ on which are defined Brownian motions $\tilde W^n$, $n\geq 1$, and $\tilde W$, and random variables $\tilde\alpha^n = (\tilde\tau_k^n,\tilde\zeta_k^n)_{k\geq 0}$, $n\geq 1$, and $\tilde\alpha = (\tilde\tau_k,\tilde\zeta_k)_{k\geq 0}$ such that

$$\tilde{\mathbb{P}}\circ(\tilde W^n,\tilde\alpha^n)^{-1} = \mathbb{P}\circ(W,\alpha^n)^{-1} \tag{4.10}$$

for all $n\geq 1$ and

$$\Big\|\big(\tilde W^n,\tilde\alpha^n\big) - \big(\tilde W,\tilde\alpha\big)\Big\| \xrightarrow[n\to\infty]{\tilde{\mathbb{P}}\text{-a.s.}} 0. \tag{4.11}$$

In particular we get

$$\mathcal{L} = \tilde{\mathbb{P}}\circ(\tilde W,\tilde\alpha)^{-1}.$$

Substep 2.1. Measurability properties for $\tilde\alpha^n$ and $\tilde\alpha$. We now prove that each $\tilde\tau_k$ is an $\tilde{\mathbb{F}}$-stopping time and ζ_k is $\tilde{\mathcal{F}}_{\tilde\tau_k}$-measurable, where $\tilde{\mathbb{F}} = (\tilde{\mathcal{F}}_t)_{t\geq 0}$ is the complete right-continuous filtration generated by $\tilde W$.

For $n\geq 1$, denote by $\tilde{\mathbb{F}}^n = (\tilde{\mathcal{F}}_t^n)_{t\geq 0}$ the complete right-continuous filtration generated by $\tilde W^n$. Using Proposition A.2, we get from (4.10) that $\tilde\tau_k^n$ is an $\tilde{\mathbb{F}}^n$-stopping time and that $\tilde\zeta_k^n$ is $\tilde{\mathcal{F}}^n_{\tilde\tau_k^n}$-measurable for all $n\geq 1$ and $k\geq 0$. Then, using Proposition A.3, we get from (4.11) that $\tilde\tau_k$ is an $\tilde{\mathbb{F}}$-stopping time and that $\tilde\zeta_k$ is $\tilde{\mathcal{F}}_{\tilde\tau_k}$-measurable for all $k\geq 0$.

Substep 2.2. Equality of the penalized gains and convergence of the associated controlled diffusions. From the previous substep, we can define the diffusions $\tilde X^{t,x,\tilde\alpha^n}$ and $\tilde X^{t,x,\tilde\alpha}$ on $(\tilde\Omega,\tilde{\mathcal{G}},\tilde{\mathbb{P}})$ by

$$\tilde X_s^{t,x,\tilde\alpha^n} = x + \int_t^s b(\tilde X_r^{t,x,\tilde\alpha^n},\tilde\alpha_r^n)dr + \int_t^s \sigma(\tilde X_r^{t,x,\tilde\alpha^n},\tilde\alpha_r^n)d\tilde W_r^n, \quad s\geq t,$$

and

$$\tilde X_s^{t,x,\tilde\alpha} = x + \int_t^s b(\tilde X_r^{t,x,\tilde\alpha},\tilde\alpha_r)dr + \int_t^s \sigma(\tilde X_r^{t,x,\tilde\alpha},\tilde\alpha_r)d\tilde W_r, \quad s\geq t,$$

and the associated gains $J_n(t,x,\tilde\alpha^n)$ and $J(t,x,\tilde\alpha)$ by

$$\begin{aligned}&\tilde J_n(t,x,\tilde\alpha^n)\\ &\quad = \mathbb{E}^{\tilde{\mathbb{P}}}\left[g_n\big(\tilde X_T^{t,x,\tilde\alpha^n},\tilde\alpha_T^n\big) + \int_t^T f_n\big(\tilde X_s^{t,x,\tilde\alpha^n},\tilde\alpha_s^n\big)ds - \sum_{k\geq 1} c\big(\tilde X_{\tilde\tau_k^n}^{t,x,\tilde\alpha^n},\tilde\zeta_{k-1}^n,\tilde\zeta_k^n\big)\mathbb{1}_{\tilde\tau_k^n<T}\right]\end{aligned}$$

and

$$\tilde{J}(t,x,\tilde{\alpha}) = \mathbb{E}^{\tilde{\mathbb{P}}}\left[g\big(\tilde{X}_T^{t,x,\tilde{\alpha}},\tilde{\alpha}_T\big) + \int_t^T f\big(\tilde{X}_s^{t,x,\tilde{\alpha}},\tilde{\alpha}_s\big)ds - \sum_{k\geq 1} c\big(\tilde{X}_{\tilde{\tau}_k}^{t,x,\tilde{\alpha}},\tilde{\zeta}_{k-1},\tilde{\zeta}_k\big)\mathbb{1}_{\tilde{\tau}_k<T}\right].$$

Since (W,α^n) and $(\tilde{W}^n,\tilde{\alpha}^n)$ have the same law, we deduce from **(H1)** and **(H2)** that

$$J_n(t,x,\alpha^n) = \tilde{J}_n(t,x,\tilde{\alpha}^n) \geq v_n(t,x,i) - \frac{1}{n}, \quad n \geq 1. \tag{4.12}$$

We now prove that, up to a subsequence,

$$\limsup_{n\to\infty} \tilde{J}_n(t,x,\tilde{\alpha}^n) \leq \tilde{J}(t,x,\tilde{\alpha}). \tag{4.13}$$

We first notice that $\limsup_{n\to\infty}\tilde{J}_n(t,x,\tilde{\alpha}^n) \leq \limsup_{n\to\infty}\tilde{J}(t,x,\tilde{\alpha}^n)$. From Proposition A.4 and (4.11) we have

$$\mathbb{E}^{\tilde{\mathbb{P}}}\left[\sup_{s\in[t,T]}\big|\tilde{X}_s^{t,x,\tilde{\alpha}} - \tilde{X}_s^{t,x,\tilde{\alpha}^n}\big|^2\right] \xrightarrow[n\to\infty]{} 0. \tag{4.14}$$

We therefore get, up to a subsequence,

$$\sup_{s\in[t,T]}\left|\tilde{X}_s^{t,x,\tilde{\alpha}^n} - \tilde{X}_s^{t,x,\tilde{\alpha}}\right| \xrightarrow[n\to\infty]{\tilde{\mathbb{P}}\text{-a.s.}} 0. \tag{4.15}$$

This implies, with **(H2)**(i) and (ii) and (4.11),

$$g\big(\tilde{X}_T^{t,x,\tilde{\alpha}^n},\tilde{\alpha}_T^n\big) + \int_t^T f\big(\tilde{X}_s^{t,x,\tilde{\alpha}^n},\tilde{\alpha}_s^n\big)ds \xrightarrow[n\to\infty]{\tilde{\mathbb{P}}\text{-a.s.}} g\big(\tilde{X}_T^{t,x,\tilde{\alpha}},\tilde{\alpha}_T\big) + \int_t^T f\big(\tilde{X}_s^{t,x,\tilde{\alpha}},\tilde{\alpha}_s\big)ds.$$

Moreover, since $\bar{v}(t,x,i) > -\infty$ we have from **(H2)**(ii) that

$$\sup_{n\geq 1}\#\big\{k\geq 1 \ : \ \tilde{\tau}_k^n \leq T\big\} < +\infty\,, \quad \tilde{\mathbb{P}}\text{-a.s.}$$

This last estimate, (4.6), (4.11), and (4.15) imply

$$\liminf_{n\to\infty}\sum_{k\geq 1} c\big(\tilde{X}_{\tilde{\tau}_k}^{t,x,\tilde{\alpha}^n},\tilde{\zeta}_{k-1}^n,\tilde{\zeta}_k^n\big)\mathbb{1}_{\tilde{\tau}_k^n\leq T} \geq \sum_{k\geq 1} c\big(\tilde{X}_{\tilde{\tau}_k}^{t,x,\tilde{\alpha}},\tilde{\zeta}_{k-1},\tilde{\zeta}_k\big)\mathbb{1}_{\tilde{\tau}_k\leq T}, \quad \tilde{\mathbb{P}}\text{-a.s.}$$

We finally conclude by using Fatou's lemma.

Substep 2.3. The process $\tilde{X}^{t,x,\tilde{\alpha}}$ satisfies the constraint $\tilde{X}_s^{t,x,\tilde{\alpha}} \in \mathcal{D}$ for all $s \in [t,T]$. For $\varepsilon > 0$, we define the set $\mathcal{D}_\varepsilon$ by

$$\mathcal{D}_\varepsilon = \Big\{x' \in \mathbb{R}^d \ : \ d(x',\mathcal{D}) < \varepsilon\Big\}.$$

Suppose that there exists some $\varepsilon > 0$ such that

$$\mathbb{E}^{\tilde{\mathbb{P}}}\left[\int_t^T \mathbb{1}_{\mathcal{D}_\varepsilon^c}(\tilde{X}_s^{t,x,\tilde{\alpha}})ds\right] > 0.$$

From (4.15) and the dominated convergence theorem we can find $\eta > 0$ and $n_\eta \geq 1$ such that, up to a subsequence,

$$\mathbb{E}^{\tilde{\mathbb{P}}}\left[\int_t^T \mathbb{1}_{\mathcal{D}_\varepsilon^c}(\tilde{X}_s^{t,x,\tilde{\alpha}^n})ds\right] \geq \eta$$

for all $n \geq n_\eta$. From the definition of f_n and g_n and the previous inequality, there exists a constant C such that

$$\tilde{J}(t,x,\tilde{\alpha}^n) \leq C\mathbb{E}^{\tilde{\mathbb{P}}}\left[\sup_{s\in[t,T]}\left|\tilde{X}_s^{t,x,\tilde{\alpha}^n}\right|\right] - n\eta$$

for any $n \geq \frac{1}{\varepsilon} \vee n_\eta$. Sending n to infinity, we get from (4.12) and (2.2) applied on $(\tilde{\Omega}, \tilde{\mathcal{G}}, \tilde{\mathbb{P}})$ that

$$\bar{v}(t,x,i) = \lim_{n\to\infty} \tilde{J}_n(t,x,\tilde{\alpha}^n) = -\infty,$$

which contradicts $\bar{v}(t,x,i) > -\infty$. We therefore obtain

$$\mathbb{E}^{\tilde{\mathbb{P}}}\left[\int_t^T \mathbb{1}_{\mathcal{D}_\varepsilon^c}(\tilde{X}_s^{t,x,\tilde{\alpha}})ds\right] = 0$$

for all $\varepsilon > 0$ and $\mathbb{E}^{\tilde{\mathbb{P}}}\big[\int_t^T \mathbb{1}_{\{\tilde{X}_s^{t,x,\tilde{\alpha}}\notin\mathcal{D}\}}ds\big] = 0$. Since $\tilde{X}^{t,x,\tilde{\alpha}}$ is continuous, we get

$$\tilde{\mathbb{P}}\Big(\tilde{X}^{t,x,\tilde{\alpha}} \in \mathcal{D} \text{ for all } s \in [t,T]\Big) = 1.$$

Step 3. Back to $(\Omega, \mathcal{G}, \mathbb{P})$ and conclusion. We construct $\alpha^* \in \mathcal{A}_{t,i}$ such that (W, α^*) has the same law as $(\tilde{W}, \tilde{\alpha})$. Using Proposition A.1, we can find Borel functions ψ_k and ϕ_k, $k \geq 1$, such that

$$\tilde{\tau}_k = \psi_k\big((\tilde{W}_s)_{s\in[0,T]}\big) \quad \text{and} \quad \tilde{\zeta}_k = \phi_k\big((\tilde{W}_s)_{s\in[0,T+1]}\big), \quad \tilde{\mathbb{P}}\text{-a.s.}$$

for all $k \geq 0$. Define the strategy $\alpha^* = (\tau_k^*, \zeta_k^*)_{k\geq 0}$ by

$$\tau_k^* = \psi_k\big((W_s)_{s\in[0,T]}\big) \quad \text{and} \quad \zeta_k^* = \phi_k\big((W_s)_{s\in[0,T+1]}\big)$$

for all $k \geq 0$. Obviously (W, α^*) has the same law as $(\tilde{W}, \tilde{\alpha})$. Moreover, from Proposition A.2, each τ_k^* is an $\mathbb{F}$-stopping time and each ζ_k^* is $\mathcal{F}_{\tau_k^*}$-measurable. We deduce that $\alpha^* \in \mathcal{A}_{t,i}$. Using substep 2.3, we also get $\alpha^* \in \mathcal{A}_{t,x,i}^{\mathcal{D}}$. From (4.12) and (4.13) we get, up to a subsequence,

$$\tilde{J}(t,x,\tilde{\alpha}) \geq \limsup_{n\to\infty} \tilde{J}_n(t,x,\tilde{\alpha}^n) = \limsup_{n\to\infty} J_n(t,x,\alpha^n) \geq \bar{v}(t,x,i).$$

Since (W, α^*) and $(\tilde{W}, \tilde{\alpha})$ have the same law and $\alpha^* \in \mathcal{A}_{t,x,i}^{\mathcal{D}}$, we get

$$v(t,x,i) \geq J(t,x,\alpha^*) = \tilde{J}(t,x,\tilde{\alpha}) \geq \bar{v}(t,x,i). \qquad \square$$

In general, proving a regularity result on the value function of a constrained optimization problem is very technical (see, e.g., [15] or [12]). In our case, Theorem 4.1 gives a semiregularity result for v.

COROLLARY 4.1. *Under* **(H1)** *and* **(H2)**, *the function $v(.,i)$ is upper semicontinuous on $[0,T) \times \mathcal{D}$ for all $i \in \mathcal{I}$.*

Proof. Fix $i \in \mathcal{I}$. From **(H1)** and **(H2)** the value function $v_n(.,i)$ associated to the penalized optimal switching problem is continuous on $[0,T) \times \mathbb{R}^d$ (see, e.g., [1] or [17]). From Theorem 4.1, the function $v(.,i)$ is upper semicontinuous on $[0,T) \times \mathcal{D}$ as an infimum of continuous functions. $\square$

5. Dynamic programming and variational inequalities.

5.1. The dynamic programming principle. In this section we state the dynamic programming principle. We first need the following lemmata. We postpone their proofs to Appendix A.2 to focus on the dynamic programming principle and its proof.

LEMMA 5.1. *Under* **(H2)***, the functions f_n and g_n are locally Lipschitz continuous and have polynomial growth:*

- *for any $n \geq 1$ and any $R > 0$ there exists a constant $L_{R,n}$ such that*

$$|g_n(x,i) - g_n(x',i)| + |f_n(x,i) - f_n(x',i)| \leq L_{R,n}|x - x'|$$

for all $x, x' \in \mathbb{R}^d$ such that $|x| \leq R$ and $|x'| \leq R$, and all $i \in \mathcal{I}$;

- *for any $n \geq 1$ there exists a constant C_n such that*

$$|g_n(x,i)| + |f_n(x,i)| \leq C_n\big(1 + |x|^q\big)$$

for all $x \in \mathbb{R}^d$ and all $i \in \mathcal{I}$.

LEMMA 5.2. *Under* **(H1)** *and* **(H2)***, there exists a constant C such that*

$$v_n(t,x,i) \leq C\big(1 + |x|^q\big) \tag{5.1}$$

for all $n \geq 1$ and all $(t,x,i) \in [0,T] \times \mathcal{D} \times \mathcal{I}$.

We are now able to state the dynamic programming principle.

THEOREM 5.1. *Under* **(H1)** *and* **(H2)***, the value function v satisfies the following dynamic programming equality:*

$$\begin{aligned} v(t,x,i) = \sup_{\alpha=(\tau_k,\zeta_k)_k \in \mathcal{A}^{\mathcal{D}}_{t,x,i}} \mathbb{E}\Bigg[\int_t^\nu f(X_s^{t,x,\alpha}, \alpha_s)ds - \sum_{t \leq \tau_k \leq \nu} c(X_{\tau_k}^{t,x,\alpha}, \zeta_{k-1}, \zeta_k) \\ + v\Big(\nu, X_\nu^{t,x,\alpha}, \alpha_\nu\Big)\Bigg] \end{aligned} \tag{5.2}$$

for any $(t,x,i) \in [0,T] \times \mathcal{D} \times \mathcal{I}$ and any stopping time ν valued in $[t,T]$.

Proof. We first notice that the left-hand side of (5.2) is well defined. Indeed, for a given stopping time ν valued in $[t,T]$ and a strategy $\alpha \in \mathcal{A}^{\mathcal{D}}_{t,x,i}$, we get from the regularity of v given by Corollary 4.1 that the random quantity $v\big(\nu, X_\nu^{t,x,\alpha}, \alpha_\nu\big)$ is measurable. Moreover, from Lemma 5.2, (2.2), and the inequality $v \leq v_n$, we get that its expectation is well defined.

Fix $(t,x,i) \in [0,T] \times \mathcal{D} \times \mathcal{I}$. If $\mathcal{A}^{\mathcal{D}}_{t,x,i} = \emptyset$, then the two sides of (5.2) are equal to $-\infty$, so the equality holds.

Suppose now that $\mathcal{A}^{\mathcal{D}}_{t,x,i} \neq \emptyset$, and let $\alpha = (\tau_k, \zeta_k)_k \in \mathcal{A}^{\mathcal{D}}_{t,x,i}$ and ν be a stopping time valued in $[t,T]$. From Lipschitz properties of f_n and g_n given by Lemma 5.1, we have by Lemma 4.4 in [1]

$$v_n(t,x,i) \geq \mathbb{E}\left[\int_t^\nu f_n\big(X_s^{t,x,\alpha}, \alpha_s\big)ds - \sum_{t \leq \tau_k \leq \nu} c\big(X_{\tau_k}^{t,x,\alpha}, \zeta_{k-1}, \zeta_k\big) + v_n\big(\nu, X_\nu^{t,x,\alpha}, \alpha_\nu\big)\right]$$

for all $n \geq 1$. Since $\alpha \in \mathcal{A}^{\mathcal{D}}_{t,x,i}$ we have from the definition of f_n that

$$f_n(X_s^{t,x,\alpha}, \alpha_s) = f(X_s^{t,x,\alpha}, \alpha_s)$$

for $d\mathbb{P} \otimes ds$-almost all $(s,\omega) \in [t,T] \times \Omega$. From Theorem 4.1, Lemma 5.2, (2.2), and the monotone convergence theorem, by sending n to infinity we get

$$v(t,x,i) \geq \mathbb{E}\left[\int_t^\nu f\big(X_s^{t,x,\alpha},\alpha_s\big)ds - \sum_{t\leq\tau_k\leq\nu} c\big(X_{\tau_k}^{t,x,\alpha},\zeta_{k-1},\zeta_k\big) + v\big(\nu, X_\nu^{t,x,\alpha},\alpha_\nu\big)\right].$$

We now prove the reverse inequality. From the definitions of the performance criterion and the value functions, the law of iterated conditional expectations, and the Markov property of our model, we get the successive relations

$$\begin{aligned}
&J(t,\, x\,, \alpha) \\
&\quad = \mathbb{E}\Bigg[\int_t^\nu f(s, X_s^{t,x,\alpha},\alpha_s)ds - \sum_{t\leq\tau_k\leq\nu} c(X_{\tau_k}^{t,x,\alpha},\zeta_{k-1},\zeta_k) \\
&\qquad\quad + \mathbb{E}\left[g(X_T^{t,x,\alpha}) + \int_\nu^T f(X_s^{t,x,\alpha},\alpha_s)ds - \sum_{\nu<\tau_k\leq T} c(X_{\tau_k}^{t,x,\alpha},\zeta_{k-1},\zeta_k)\middle|\mathcal{F}_\nu\right]\Bigg] \\
&\quad = \mathbb{E}\left[\int_t^\nu f\big(X_s^{t,x,\alpha},\alpha_s\big)ds - \sum_{t\leq\tau_k\leq\nu} c\big(X_{\tau_k}^{t,x,\alpha},\zeta_{k-1},\zeta_k\big) + J\big(\nu, X_\nu^{t,x,\alpha},\alpha\big)\right] \\
&\quad \leq \mathbb{E}\left[\int_t^\nu f\big(X_s^{t,x,\alpha},\alpha_s\big)ds - \sum_{t\leq\tau_k\leq\nu} c\big(X_{\tau_k}^{t,x,\alpha},\zeta_{k-1},\zeta_k\big) + v\big(\nu, X_\nu^{t,x,\alpha},\alpha_\nu\big)\right].
\end{aligned}$$

Since ν and α are arbitrary, we obtain the required inequality. ☐

5.2. Viscosity properties. We prove in this section that the function v is a solution to a system of variational inequalities. More precisely we consider the following PDE:

$$\min\left[-\frac{\partial v}{\partial t} - \mathcal{L}v - f, v - \mathcal{H}v\right] = 0 \quad \text{on } [0,T) \times \mathcal{D} \times \mathcal{I}, \tag{5.3}$$

$$\min\Big[v - g, v - \mathcal{H}v\Big] = 0 \quad \text{on } \{T\} \times \mathcal{D} \times \mathcal{I}, \tag{5.4}$$

where $\mathcal{L}$ is the second order local operator defined by

$$\mathcal{L}v(t,x,i) = \left(\mu^\intercal Dv + \frac{1}{2}\text{tr}[\sigma\sigma^\intercal D^2 v)]\right)(t,x,i)$$

and $\mathcal{H}$ is the nonlocal operator defined by

$$\mathcal{H}v(t,x,i) = \max_{\substack{j\,\in\,\mathcal{I}\\ j\,\neq\, i}} \big[v(t,x,j) - c(x,i,j)\big]$$

for all $(t,x,i) \in [0,T] \times \mathcal{D} \times \mathcal{I}$. As usual, the value functions need not be smooth, nor even known to be continuous a priori. So, we shall work with the notion of (discontinuous) viscosity solutions (see [6]). Generally, for PDEs arising in optimal control problems involving state constraints, we need the notion of a constrained viscosity solution introduced by [15] for first order equations to take into account the boundary conditions induced by the state constraints.

For a locally bounded function u on $[0,T] \times \mathcal{D} \times \mathcal{I}$, we define its lower semicontinuous (lsc) envelope u_*, and upper semicontinuous (usc) envelope u^* by

$$u_*(t,x,i) = \liminf_{\substack{(t',x') \to (t,x),\\ (t',x') \in [0,T) \times \mathcal{D}}} u(t',x',i), \qquad u^*(t,x,i) = \limsup_{\substack{(t',x') \to (t,x),\\ (t',x') \in [0,T) \times \mathcal{D}}} u(t',x',i)$$

for all $(t,x,i) \in [0,T] \times \mathcal{D} \times \mathcal{I}$.

Remark 1. From Corollary 4.1 and the definition of the usc envelope, we have $v = v^*$ on $[0,T) \times \mathcal{D} \times \mathcal{I}$. However, this equality may not to be true on $\{T\} \times \mathcal{D} \times \mathcal{I}$.

We now give the definition of a constrained viscosity solutions to (5.3) and (5.4).

DEFINITION 5.1 (constrained viscosity solutions to (5.3)–(5.4)).

(i) *A function u, lsc (resp., usc) on $[0,T) \times \mathcal{D} \times \mathcal{I}$, is called a viscosity supersolution on $[0,T) \times Int(\mathcal{D}) \times \mathcal{I}$ (resp., subsolution on $[0,T) \times \mathcal{D} \times \mathcal{I}$) to* (5.3)–(5.4) *if we have*

$$\min\left[-\frac{\partial\varphi}{\partial t}(t,x,i) - \mathcal{L}\varphi(t,x,i) - f(x,i)\ ,\ u(t,x,i) - \mathcal{H}u(t,x,i)\right] \geq (resp.,\ \leq)\, 0$$

for any $(t,x,i) \in [0,T) \times Int(\mathcal{D}) \times \mathcal{I}$ (resp., $(t,x,i) \in [0,T) \times \mathcal{D} \times \mathcal{I}$) and any $\varphi \in C^{1,2}([0,T] \times \mathbb{R}^d, \mathbb{R})$ such that

$$\varphi(t,x) - u(t,x,i) = \max_{[0,T]\times\mathcal{D}} (\varphi - u(.,i)) \quad \left(resp.,\ \min_{[0,T]\times\mathcal{D}} (\varphi - u(.,i))\right)$$

and

$$\min\Big[u(T,x,i) - g(x,i)\ ,\ u(T,x,i) - \mathcal{H}u(T,x,i)\Big] \geq (resp.,\ \leq)\, 0$$

for any $x \in Int(\mathcal{D})$ (resp., $x \in \mathcal{D}$).

(ii) *A locally bounded function u on $[0,T] \times \mathcal{D} \times \mathcal{I}$ is called a constrained viscosity solution to* (5.3)–(5.4) *if its lsc envelope u_* is a viscosity supersolution to* (5.3)–(5.4) *on $[0,T] \times Int(\mathcal{D}) \times \mathcal{I}$ and its usc envelope u^* is a viscosity subsolution on $[0,T] \times \mathcal{D} \times \mathcal{I}$ to* (5.3)–(5.4).

We can now state the viscosity property of v.

THEOREM 5.2. *Suppose that the function v is locally bounded. Under* **(H1)** *and* **(H2)**, *v is a constrained viscosity solution to* (5.3)–(5.4).

Proof of the supersolution property on $[0,T) \times Int(\mathcal{D}) \times \mathcal{I}$. First, for any $(t,x,i) \in [0,T) \times \mathcal{D} \times \mathcal{I}$, we see, as a consequence of (5.2) applied to $\nu = t$, and by choosing any admissible control $\alpha \in \mathcal{A}^{\mathcal{D}}_{t,x,i}$ with immediate switch j at t, that

$$v(t,x,i) \geq \mathcal{H}v(t,x,i). \tag{5.5}$$

Now, let $(\bar{t},\bar{x},i) \in [0,T) \times Int(\mathcal{D}) \times \mathcal{I}$ and $\varphi \in C^{1,2}([0,T] \times \mathbb{R}^d, \mathbb{R})$ s.t.

$$\varphi(\bar{t},\bar{x}) - v_*(\bar{t},\bar{x},i) = \max_{[0,T]\times\mathcal{D}} (\varphi - v_*(.,i)). \tag{5.6}$$

Since $v \geq \mathcal{H}v$ on $[0,T) \times Int(\mathcal{D}) \times \mathcal{I}$, we get from the definition of the operator $\mathcal{H}$ and **(H2)**(i) that

$$v_*(\bar{t},\bar{x},j) \geq v_*(\bar{t},\bar{x},j) - c(\bar{x},i,j)$$

for all $j \in \mathcal{I}$. Therefore we obtain

$$v_*(\bar{t}, \bar{x}, i) \geq \mathcal{H}v_*(\bar{t}, \bar{x}, i).$$

So it remains to show that

$$-\frac{\partial \varphi}{\partial t}(\bar{t}, \bar{x}, i) - \mathcal{L}\varphi(\bar{t}, \bar{x}, i) - f(\bar{x}, i) \geq 0. \tag{5.7}$$

From the definition of v_* there exists a sequence $(t_m, x_m)_m$ valued in $[0,T) \times Int(\mathcal{D})$ such that

$$(t_m, x_m, v(t_m, x_m, i)) \xrightarrow[m\to\infty]{} (\bar{t}, \bar{x}, v_*(\bar{t}, \bar{x}, i)).$$

By continuity of φ, $\gamma_m := v(t_m, x_m, i) - \varphi(t_m, x_m) - v_*(\bar{t}, \bar{x}, i) + \varphi(\bar{t}, \bar{x})$ converges to 0 as m goes to infinity. Since $(\bar{t}, \bar{x}) \in [0,T) \times Int(\mathcal{D})$, there exists $\eta > 0$ such that, for m large enough, $t_m < T$ and

$$\begin{aligned}\left(\left(t_m - \frac{\eta}{2}\right) \wedge 0, t_m + \frac{\eta}{2}\right) \times B\left(x_m, \frac{\eta}{2}\right) \\ \subset ((t-\eta) \wedge 0, t+\eta) \times B(x, \eta) \subset [0,T) \times Int(\mathcal{D}).\end{aligned}$$

Let us consider an admissible control α^m in $\mathcal{A}^{\mathcal{D}}_{t_m,x_m,i}$ with no switch until the first exit time τ_m before T of the associated process $(s, X^m_s) := (s, X^{t_m,x_m,\alpha^m}_s)$ from $(t_m - \frac{\eta}{2}, t_m + \frac{\eta}{2}) \times B(x_m, \frac{\eta}{2})$:

$$\tau_m := \inf\left\{s \geq t_m \ : \ (s - t_m) \vee |X^m_s - x_m| \geq \frac{\eta}{2}\right\}.$$

Consider also a strictly positive sequence $(h_m)_m$ such that h_m and γ_m/h_m converge to 0 as m goes to infinity. By using the dynamic programming principle (5.2) for $v(t_m, x_m, i)$ and $\nu = \hat{\tau}_m := \inf\{s \geq t_m \ : \ (s-t_m) \vee |X^m_s - x_m| \geq \frac{\eta}{4}\} \wedge (t_m + h_m)$, we get

$$\begin{aligned}v(t_m, x_m, i) &= \gamma_m + v_*(\bar{t}, \bar{x}, i) - \varphi(\bar{t}, \bar{x}, i) + \varphi(t_m, x_m, i) \\ &\geq \mathbb{E}\left[\int_{t_m}^{\hat{\tau}_m} f(X^m_s, i)ds + v\big(\hat{\tau}_m, X^m_{\hat{\tau}_m}, i\big)\right].\end{aligned}$$

Using (5.6), we obtain

$$v(t_m, x_m, i) \geq \mathbb{E}\left[\int_{t_m}^{\hat{\tau}_m} f(X^m_s, i)ds + \varphi\big(\hat{\tau}_m, X^m_{\hat{\tau}_m}\big)\right].$$

Applying Itô's formula to $\varphi(s, X^m_s)$ between t_m and $\hat{\tau}_m$ and since $\sigma(X^m_s, i)D\varphi(s, X^m_s)$ is bounded for $s \in [t_m, \hat{\tau}_m]$, we obtain

$$\frac{\gamma_m}{h_m} + \mathbb{E}\left[\frac{1}{h_m}\int_{t_m}^{\hat{\tau}_m}\left(-\frac{\partial \varphi}{\partial t} - \mathcal{L}\varphi - f\right)(s, X^m_s, i)ds\right] \geq 0 \tag{5.8}$$

for all $m \geq 1$. From the continuity of the process X^m, we have

$$\mathbb{P}\Big(\exists m, \ \forall m' \geq m \ : \ \hat{\tau}_{m'} = t_{m'} + h_{m'}\Big) = 1.$$

Hence, by the mean-value theorem, the random variable inside the expectation in (5.8) converges a.s. to $(-\frac{\partial \varphi}{\partial t} - \mathcal{L}\varphi - f)(\bar{t}, \bar{x}, i)$ as m goes to infinity. We conclude by the dominated convergence theorem and get (5.7). ∎

Proof of the subsolution property on $[0,T)\times\mathcal{D}\times\mathcal{I}$. We first recall that $v^* = v$ on $[0,T)\times\mathcal{D}\times\mathcal{I}$ from Remark 1. Let $(\bar t,\bar x,i)\in[0,T)\times\mathcal{D}\times\mathcal{I}$ and $\varphi\in C^{1,2}([0,T]\times\mathbb{R}^d,\mathbb{R})$ such that

$$\varphi(\bar t,\bar x)-v(\bar t,\bar x,i)=\min_{[0,T]\times\mathcal{D}}(\varphi-v(.,i)). \tag{5.9}$$

If $v(\bar t,\bar x,i)\le\mathcal{H}v(\bar t,\bar x,i)$, then the subsolution property trivially holds. Consider now the case $v(\bar t,\bar x,i)>\mathcal{H}v(\bar t,\bar x,i)$ and argue from contradiction by assuming on the contrary that

$$\eta:=-\frac{\partial\varphi}{\partial t}(\bar t,\bar x)-\mathcal{L}\varphi(\bar t,\bar x,i)-f(\bar x,i)>0.$$

By continuity of φ and its derivatives, there exists some $\delta>0$ such that $\bar t+\delta<T$ and

$$\Big(-\frac{\partial\varphi}{\partial t}-\mathcal{L}\varphi-f\Big)(t,x,i)\ge\frac{\eta}{2} \tag{5.10}$$

for all $(t,x)\in\mathcal{V}:=((\bar t-\delta,\bar t+\delta)\cap[0,T))\times B(\bar x,\delta)$. By the dynamic programming principle (5.2), given $m\ge 1$, there exists $\hat\alpha^m=(\hat\tau_n^m,\hat\zeta_n^m)_n\in\mathcal{A}^{\mathcal{D}}_{\bar t,\bar x,i}$ such that, for any stopping time τ valued in $[\bar t,T]$, we have

$$v(\bar t,\bar x,i)\le\mathbb{E}\left[\int_{\bar t}^{\tau}f(\hat X_s^m,i)-\sum_{\bar t\le\hat\tau_n^m\le\tau}c(\hat X^m_{\hat\tau_n^m},\hat\zeta_n^m,\hat\zeta_n^m)+v(\tau,\hat X_\tau^m,i)\right]+\frac{1}{m},$$

where $\hat X^m:=X^{\bar t,\bar x,\hat\alpha^m}$. By choosing $\tau=\bar\tau_m:=\hat\tau_1^m\wedge\nu^m$, where

$$\nu^m:=\inf\{s\ge\bar t\;:\;(s,\hat X_s^m)\notin\mathcal{V}\}$$

is the first exit time of $(s,\hat X_s^m)$ from $\mathcal{V}$, we then get

$$\begin{aligned}v(\bar t,\bar x,i)&\le\mathbb{E}\left[\int_{\bar t}^{\bar\tau^m}f(\hat X_s^m,i)ds\right]+\mathbb{E}\left[v(\bar\tau^m,\hat X^m_{\bar\tau^m},i)\mathbb{1}_{\nu^m<\hat\tau_1^m}\right]\\&\quad+\mathbb{E}\left[[v(\bar\tau^m,\hat X^m_{\bar\tau^m},\hat\zeta_1^m)-c(\hat X^m_{\bar\tau^m},i,\hat\zeta_1^m)]\mathbb{1}_{\nu^m\ge\hat\tau_1^m}\right]+\frac{1}{m}\\&\le\mathbb{E}\left[\int_{\bar t}^{\bar\tau^m}f(\hat X_s^m,i)ds\right]+\mathbb{E}\left[v(\bar\tau^m,\hat X^m_{\bar\tau^m},i)\mathbb{1}_{\nu^m<\hat\tau_1^m}\right]\\&\quad+\mathbb{E}\left[\mathcal{H}v(\bar\tau^m,\hat X^m_{\bar\tau^m},i)\mathbb{1}_{\nu^m\ge\hat\tau_1^m}\right]+\frac{1}{m}.\end{aligned} \tag{5.11}$$

Now, since $v\ge\mathcal{H}v$ on $[0,T]\times\mathcal{D}\times\mathcal{I}$ and $\hat\alpha^m\in\mathcal{A}^{\mathcal{D}}_{\bar t,\bar x,i}$, we obtain from (5.9) that

$$\varphi(\bar t,\bar x,i)\le\mathbb{E}\left[\int_{\bar t}^{\bar\tau_m}f(\hat X_s^m,i)ds+\varphi(\bar\tau^m,\hat X^m_{\bar\tau^m})\right]+\frac{1}{m}.$$

Applying Itô's formula to $\varphi(s,\hat X_s^m)$ between t_m and $\bar\tau^m$, we get

$$0\le\mathbb{E}\left[\int_{t_m}^{\bar\tau^m}\left(\frac{\partial\varphi}{\partial t}+\mathcal{L}\varphi+f\right)(s,\hat X_s^m,i)\right]+\frac{1}{m}\le-\frac{\eta}{2}\mathbb{E}\left[\bar\tau^m-\bar t\right]+\frac{1}{m}.$$

This implies

$$\lim_{m\to+\infty} \mathbb{E}[\bar{\tau}^m] = \bar{t}. \tag{5.12}$$

From the definition of ν^m and (5.12) we have, up to a subsequence,

$$\mathbb{P}\big(\nu^m \geq \hat{\tau}_1^m\big) \xrightarrow[m\to\infty]{} 1. \tag{5.13}$$

On the other hand, we get from (5.11)

$$\begin{aligned} v(\bar{t},\bar{x},i) \leq \mathbb{E}\left[\int_{\bar{t}}^{\bar{\tau}_m} f(\hat{X}_s^m,i)ds\right] + \mathbb{P}\big(\nu^m < \hat{\tau}_1^m\big) \sup_{(t',x')\in Adh(\mathcal{V})} v(t',x',i) \\ + \mathbb{P}\big(\nu^m \geq \hat{\tau}_1^m\big) \sup_{(t',x')\in Adh(\mathcal{V})} \mathcal{H}v(t',x',i) + \frac{1}{m}. \end{aligned}$$

From Lemma 5.2, (5.12), and (5.13) we get, by sending m to ∞,

$$v(\bar{t},\bar{x},i) \leq \sup_{(t',x')\in Adh(\mathcal{V})} \mathcal{H}v(t',x',i).$$

Since $v = v^*$, we get, by sending m to infinity and δ to zero,

$$v(\bar{t},\bar{x},i) \leq (\mathcal{H}v)^*(\bar{t},\bar{x},i) \ \leq \ \mathcal{H}v(\bar{t},\bar{x},i),$$

which is the required contradiction. ☐

Proof of the viscosity supersolution property on $\{T\}\times Int(\mathcal{D})\times\mathcal{I}$. Fix some $(\bar{x},i) \in Int(\mathcal{D})\times\mathcal{I}$ and consider a sequence $(t_m,x_m)_{m\geq1}$ valued in $[0,T)\times Int(\mathcal{D})$ such that

$$\big(t_m,x_m,v(t_m,x_m,i)\big) \xrightarrow[m\to\infty]{} \big(T,\bar{x},v_*(T,\bar{x},i)\big).$$

Let $\delta > 0$ such that $B(\bar{x},\delta) \in Int(\mathcal{D})$. We first can suppose w.l.o.g. that

$$B\left(x_m,\frac{\delta}{2}\right) \subset B(\bar{x},\delta) \tag{5.14}$$

for all $m \geq 1$. By taking a strategy $\alpha^m = (\tau_k^m,\zeta_k^m)_k \in \mathcal{A}_{t_m,x_m,i}^{\mathcal{D}}$ with no switch before $\nu_m := \inf\{s \geq t_m,\ X_s^m \notin B(x_m,\frac{\delta}{2})\} \wedge T$ with $X^m := X^{t_m,x_m,\alpha^m}$, we have from (5.2) applied to $\tau_m := \inf\{s \geq t_m,\ X_s^m \notin B(x_m,\frac{\delta}{4})\} \wedge T$ and α_m that

$$v(t_m,x_m,i) \geq \mathbb{E}\left[\int_{t_m}^{\tau^m} f(X_s^m,i)ds\right] + \mathbb{E}\big[v(\tau^m,X_{\tau^m}^m,i)\big].$$

Since $v(T,.) = g$ we obtain from (5.14) that

$$\begin{aligned} v(t_m,x_m,i) &\geq \mathbb{E}\left[\int_{t_m}^{\tau^m} f(X_s^m,i)ds\right] + \mathbb{E}\Big[v(\tau^m,X_{\tau^m}^m,i)\mathbb{1}_{\tau^m<T}\Big] + \mathbb{E}\Big[g(X_{\tau^m}^m,i)\mathbb{1}_{\tau^m=T}\Big] \\ &\geq \mathbb{E}\left[\int_{t_m}^{\tau^m} f(X_s^m,i)ds\right] + \mathbb{P}\big(\tau^m < T\big) \inf_{\substack{t\,<\,T \\ x\,\in\,Adh(B(\bar{x},\,\delta))}} v(t,x,i) \\ &\quad + \mathbb{P}\big(\tau^m = T\big) \inf_{x\in Adh(B(\bar{x},\delta))} g(x). \end{aligned} \tag{5.15}$$

Since $\mathbb{E}[\sup_{s\in[t_m,T]}|X_s^m - x_m|]$ converges to zero (see, e.g., [13, Chapter 2, section 5, Corollary 12]), we have, up to a subsequence,

$$\sup_{s\in[t_m,T]}|X_s^m - x_m| \xrightarrow[m\to\infty]{\mathbb{P}\text{-a.s.}} 0.$$

From the convergence of $(x_m)_m$ to $x \in Int(\mathcal{D})$, we deduce that

$$\mathbb{P}\big(\tau^m = T\big) \xrightarrow[m\to\infty]{} 1.$$

Sending m to infinity and δ to 0 in (5.15), we get

$$v_*(T,\bar{x},i) \geq g(\bar{x},i). \tag{5.16}$$

On the other hand, we know from (5.5) that $v \geq \mathcal{H}v$ on $[0,T)\times Int(\mathcal{D})$, and thus

$$v(t_m,x_m,i) \geq \mathcal{H}v(t_m,x_m,i) \geq \mathcal{H}v_*(t_m,x_m,i)$$

for all $m \geq 1$. Recalling that $\mathcal{H}v_*$ is lsc, we obtain, by sending m to infinity,

$$v_*(T,\bar{x},i) \geq \mathcal{H}v_*(T,\bar{x},i).$$

Together with (5.16), this proves the required viscosity supersolution property of (5.4). ∎

Proof of the viscosity subsolution property on $\{T\}\times\mathcal{D}\times\mathcal{I}$. We argue by contradiction by assuming that there exists $(\bar{x},i) \in \mathcal{D}\times\mathcal{I}$ such that

$$\min\big[v^*(T,\bar{x},i) - g(\bar{x},i), \mathcal{H}v^*(T,\bar{x},i)\big] := 2\varepsilon > 0. \tag{5.17}$$

One can find a sequence of smooth functions $(\varphi^n)_{n\geq 0}$ on $[0,T]\times\mathbb{R}^d$ such that φ^n converges pointwise to $v^*(.,i)$ on $[0,T]\times\mathcal{D}\times\mathcal{I}$ as $n\to\infty$. Moreover, by (5.17) and the upper semicontinuity of v^*, we may assume that the inequality

$$\min\left[\varphi^n - g(.,i),\ \varphi^n - \max_{j\in\mathcal{I}}\{v^*(.,j) + c(.,i,j)\}\right] \geq \varepsilon \tag{5.18}$$

holds on some bounded neighborhood B^n of $(T,\bar{x})$ in $[0,T]\times\mathcal{D}$, for n large enough. Let $(t_m,x_m)_{m\geq 1}$ be a sequence in $[0,T)\times\mathcal{D}$ such that

$$\big(t_m,x_m,v(t_m,x_m,i)\big) \xrightarrow[m\to\infty]{} \big(T,\bar{x},v^*(T,\bar{x},i)\big).$$

Then there exists $\delta^n > 0$ such that $B_m^n := [t_m,T]\times B(x_m,\delta^n) \subset B^n$ for m large enough, so that (5.18) holds on B_m^n. Since v is locally bounded, there exists some $\eta > 0$ such that $|v^*| \leq \eta$ on B^n. We can then assume that $\varphi^n \geq -2\eta$ on B^n. Let us define the smooth function $\tilde{\varphi}_m^n$ by

$$\tilde{\varphi}_m^n(t,x) := \varphi^n(t,x) + \left(4\eta\frac{|x-x_m|^2}{|\delta^n|^2} + \sqrt{T-t}\right)$$

for $(t,x) \in [0,T)\times Int(\mathcal{D})$ and observe that

$$(v^* - \tilde{\varphi}_m^n)(t,x,i) \leq -\eta \tag{5.19}$$

for $(t,x) \in [t_m, T] \times \partial B(x_m, \delta^n)$. Since $\frac{\partial\sqrt{T-t}}{\partial t} \longrightarrow -\infty$ as $t \to T$, we have for m large enough that

$$-\frac{\partial \tilde{\varphi}_m^n}{\partial t} - \mathcal{L}\tilde{\varphi}_m^n(.,i) \geq 0 \quad \text{on } B_m^n. \tag{5.20}$$

Let $\alpha^m = (\tau_j^m, \zeta_j^m)_j$ be a $\frac{1}{m}$-optimal control for $v(t_m, x_m, i)$ with corresponding state process $X^m = X^{t_m, x_m, \alpha^m}$, and define $\theta_n^m = \inf\{s \geq t_m \ : \ (s, X_s^m) \notin B_m^n\} \wedge \tau_1^m \wedge T$. From (5.2) we have

$$\begin{aligned} v(t_m, x_m, i) - \frac{1}{m} \leq \mathbb{E}\left[\int_{t_m}^{\theta_n^m} f(X_s^m, i) ds\right] &+ \mathbb{E}\Big[\mathbb{1}_{\theta_n^m < \tau_1^m \wedge T}\ v(\theta_n^m, X_{\theta_n^m}^m, i)\Big] \\ &+ \mathbb{E}\Big[\mathbb{1}_{\theta_n^m = T < \tau_1^m}\ g(X_{\theta_n^m}^m, i)\Big] \\ &+ \mathbb{E}\Big[\mathbb{1}_{\tau_1^m = \theta_n^m \leq T}\ \Big(v\big(\tau_1^m, X_{\tau_1^m}^m, \zeta_1^m\big) + c(X_{\tau_1^m}^m, i, \zeta_1^m)\Big)\Big]. \end{aligned} \tag{5.21}$$

Now, by applying Itô's lemma to $\tilde{\varphi}_n^m(s, X_s^m)$ between t_m and θ_n^m, we get from (5.18), (5.19), and (5.20) that

$$\begin{aligned} \tilde{\varphi}_m^n(t_m, x_m) &\geq \mathbb{E}\Big[\mathbb{1}_{\theta_n^m < \tau_1^m}\ \tilde{\varphi}_m^n(\theta_n^m, X_{\theta_n^m}^m)\Big] + \mathbb{E}\Big[\mathbb{1}_{\tau_1^m \leq \theta_n^m} \tilde{\varphi}_m^n\big(\tau_1^m, X_{\tau_1^m}^m\big)\Big] \\ &\geq \mathbb{E}\Big[\mathbb{1}_{\theta_n^m < \tau_1^m \wedge T}\Big(v^*(\theta_n^m, X_{\theta_n^m}^m, i) + \eta\Big)\Big] + \mathbb{E}\Big[\mathbb{1}_{\theta_n^m = T < \tau_1^m}\Big(g(X_{\theta_n^m}^m, i) + \varepsilon\Big)\Big] \\ &\quad + \mathbb{E}\Big[\mathbb{1}_{\tau_1^m = \theta_n^m \leq T}\Big(v^*\big(\tau_1^m, X_{\tau_1^m}^m, \zeta_1^m\big) + c(X_{\tau_1^m}^k, i, \zeta_1^m) + \varepsilon\Big)\Big]. \end{aligned}$$

Together with (5.21), this implies

$$\tilde{\varphi}_m^n(t_m, x_m) \geq v(t_m, x_m, i) - \mathbb{E}\left[\int_{t_m}^{\theta_n^m} f(X_s^m, i) ds\right] - \frac{1}{m} + \varepsilon \wedge \eta.$$

Sending m and then n to infinity, we get the required contradiction: $v^*(T, \bar{x}, i) \geq v^*(T, \bar{x}, i) + \varepsilon \wedge \eta$. ∎

6. Uniqueness result.

6.1. Maximality of the value function as a solution to the SVI. In general, the uniqueness of a viscosity solution to some PDE is given by a comparison theorem. Such a result says that, for u a *usc* supersolution and and w an *lsc* subsolution, we have $u \geq w$. Applying this result to $u = v_*$ the *lsc* envelope of v and $w = v^*$ the *usc* envelope of v, we would get that $v_* = v^*$ and v would be continuous. As the counterexample presented in subsection 2.2 shows, such a property cannot hold for SVI (5.3)–(5.4).

We therefore provide a weaker characterization of v. To this end, we introduce, for $n \geq 1$, the SVI with penalized coefficients defined on the whole space $[0,T] \times \mathbb{R}^d \times \mathcal{I}$:

$$\min\left[-\frac{\partial v}{\partial t} - \mathcal{L}v - f_n, v - \mathcal{H}v\right] = 0 \quad \text{on } [0,T) \times \mathbb{R}^d \times \mathcal{I}, \tag{6.1}$$

$$\min\big[v - g_n, v - \mathcal{H}v\big] = 0 \quad \text{on } \{T\} \times \mathbb{R}^d \times \mathcal{I}. \tag{6.2}$$

Under assumptions **(H1)** and **(H2)**, we can use Lemma 5.1 to apply Proposition 5.1 in [1], and we get from Proposition 4.12 in [1] the following comparison result for this PDE.

THEOREM 6.1. *Suppose that* **(H1)** *and* **(H2)** *hold. Let* u *and* w *be respectively a subsolution and a supersolution to* (6.1)–(6.2)*. Suppose that there exist two constants* $C_u > 0$ *and* $C_w > 0$ *and an integer* $\gamma \geq 1$ *such that*

$$u(t,x,i) \leq C_u\big(1+|x|^\gamma\big),$$
$$w(t,x,i) \geq -C_w\big(1+|x|^\gamma\big)$$

for all $(t,x,i) \in [0,T] \times \mathbb{R}^d \times \mathcal{I}$*. Then we have* $u \leq w$ *on* $[0,T] \times \mathbb{R}^d \times \mathcal{I}$*.*

We now introduce the following additional assumption on the function v.

(H3) There exist a constant $C > 0$ and an integer $q \geq 1$ such that

$$v(t,x,i) \geq -C\big(1+|x|^q\big) \tag{6.3}$$

for all $(t,x,i) \in [0,T] \times \mathcal{D} \times \mathcal{I}$.

We give in the next subsection some examples where **(H3)** is satisfied. We can state our maximality result as follows.

THEOREM 6.2. *Under* **(H1)**, **(H2)**, *and* **(H3)** *the function* v *is the maximal constrained viscosity solution to* (5.3)–(5.4) *satisfying* (6.3): *for any function* $w : [0,T] \times \mathcal{D} \times \mathcal{I} \to \mathbb{R}$ *such that*

- w *is a constrained viscosity solution to* (5.3)–(5.4)*, and*
- *there exist a constant* C *and an integer* $\eta \geq 1$ *such that*

$$w(t,x,i) \geq -C\big(1+|x|^\eta\big) \tag{6.4}$$

  for all $(t,x,i) \in [0,T] \times \mathcal{D} \times \mathcal{I}$,

we have $v \geq w$ *on* $[0,T] \times \mathcal{D} \times \mathcal{I}$*.*

Proof. Let $w : [0,T]\times\mathcal{D}\times\mathcal{I} \to \mathbb{R}$ be a constrained viscosity solution to (5.3)–(5.4) satisfying (6.4). We proceed in four steps to prove that $w \leq v$.

Step 1. Extension of the definition of w to $[0,T] \times \mathbb{R}^d \times \mathcal{I}$. For $n \geq 1$, we define the function $\tilde{w}_n$ on $[0,T] \times \mathbb{R}^d \times \mathcal{I}$ by

$$\tilde{w}_n(t,x,i) = \begin{cases} w(t,x,i) & \text{for } (t,x,i) \in [0,T] \times \mathcal{D} \times \mathcal{I}, \\ -C_n e^{-\rho_n t}\big(1+|x|^{2\eta}\big) & \text{for } (t,x,i) \in [0,T] \times (\mathbb{R}^d \setminus \mathcal{D}) \times \mathcal{I}, \end{cases} \tag{6.5}$$

where ρ_n and C_n are two positive constants. From **(H1)**, **(H2)**, Lemma 5.1, and (6.4), we can find ρ_n and C_n (large enough) such that

$$-\frac{\partial \tilde{w}_n}{\partial t} - \mathcal{L}\tilde{w}_n - f_n \leq 0 \quad \text{on } [0,T) \times (\mathbb{R}^d \setminus \mathcal{D}) \times \mathcal{I}, \tag{6.6}$$

$$\tilde{w}_n - g_n \leq 0 \quad \text{on } \{T\} \times \mathbb{R}^d \times \mathcal{I}, \tag{6.7}$$

and

$$\tilde{w}_n(t,x,i) \geq -C_n e^{-\rho_n t}\big(1+|x|^{2\eta}\big) \quad \text{for } (t,x,i) \in [0,T) \times \mathbb{R}^d \times \mathcal{I}. \tag{6.8}$$

Step 2. Viscosity property of $\tilde{w}_n$. For C_n and ρ_n such that (6.6), (6.7), and (6.8) hold, we obtain that $\tilde{w}_n$ is a viscosity subsolution to (6.1)–(6.2). Indeed, let $\varphi \in C^{1,2}([0,T] \times \mathbb{R}^d, \mathbb{R})$ and $(t,x,i) \in [0,T] \times \mathbb{R}^d \times \mathcal{I}$ such that

$$(\tilde{w}_n^* - \varphi)(t,x,i) = \max_{[0,T]\times\mathbb{R}^d\times\mathcal{I}} (\tilde{w}_n^* - \varphi). \tag{6.9}$$

We first notice from (6.8) that the usc envelope $\tilde{w}_n^*$ of $\tilde{w}_n$ is given by

$$\tilde{w}_n^*(t,x,i) = \begin{cases} w^*(t,x,i) & \text{for } (t,x,i) \in [0,T] \times \mathcal{D} \times \mathcal{I}, \\ -C_n e^{-\rho_n t}\big(1+|x|^{2\eta}\big) & \text{for } (t,x,i) \in [0,T] \times (\mathbb{R}^d \setminus \mathcal{D}) \times \mathcal{I}. \end{cases} \tag{6.10}$$

We now prove that $\tilde{w}_n$ is a subsolution to (6.1)–(6.2). Using (6.7), (6.10), and the viscosity subsolution property of w, we get

$$\tilde{w}_n^* \leq g_n \qquad \text{on } \{T\} \times \mathbb{R}^d \times \mathcal{I}.$$

For the viscosity property on $[0,T) \times \mathbb{R}^d \times \mathcal{I}$, we distinguish two cases:

- Case 1: $(t,x,i) \in [0,T) \times \mathcal{D} \times \mathcal{I}$. From (6.9) and (6.10), we have

$$(\tilde{w}_n^* - \varphi)(t,x,i) = \max_{[0,T]\times\mathcal{D}\times\mathcal{I}} (\tilde{w}_n^* - \varphi).$$

  Since w is a constrained viscosity solution to (5.3)–(5.4) and $f = f_n$ on $\mathcal{D}$, we get

$$\min\left[-\frac{\partial\varphi}{\partial t}(t,x,i) - \mathcal{L}\varphi(t,x,i) - f_n(t,x,i), \varphi(t,x,i) - \mathcal{H}\tilde{w}_n^*(t,x,i)\right] \leq 0.$$

- Case 2: $(t,x,i) \in [0,T) \times (\mathbb{R}^d \setminus \mathcal{D}) \times \mathcal{I}$. From (6.6) and (6.10), we also get

$$\min\left[-\frac{\partial\varphi}{\partial t}(t,x,i) - \mathcal{L}\varphi(t,x,i) - f_n(t,x,i), \varphi(t,x,i) - \mathcal{H}\tilde{w}_n^*(t,x,i)\right] \leq 0.$$

Therefore, $\tilde{w}_n$ is a viscosity subsolution to (6.1)–(6.2).

Step 3. Growth condition on v_n. We prove that for each $n \geq 1$ there exists a constant $C_n > 0$ such that

$$v_n(t,x,i) \geq -C_n\big(1+|x|^{2q}\big), \qquad (t,x,i) \in [0,T] \times \mathbb{R}^d \times \mathcal{I}.$$

Fix $(t,x,i) \in [0,T] \times \mathbb{R}^d \times \mathcal{I}$, and denote by ${}^0\alpha = ({}^0\tau_k, {}^0\zeta_k)_k$ the trivial strategy of $\mathcal{A}_{t,i}$, i.e., ${}^0\tau_0 = t$, ${}^0\zeta_0 = i$, and ${}^0\tau_k > T$ for $k \geq 1$. Then we have

$$v_n(t,x,i) \geq J_n(t,x,{}^0\alpha).$$

From the definition of J_n, (2.2), and Lemma 5.1 there exists a constant $\tilde{C}_n > 0$ such that

$$v_n(t,x,i) \geq -\tilde{C}_n\big(1+|x|^q\big)$$

for all $(t,x,i) \in [0,T] \times \mathbb{R}^d \times \mathcal{I}$.

Step 4. Comparison on $[0,T] \times \mathbb{R}^d \times \mathcal{I}$. From Proposition 4.2 in [1], we know that v_n is a viscosity solution to (6.1)–(6.2). Using the results of Steps 2 and 3, we can apply Theorem 6.1 to $\tilde{w}_n$ and v_n with $\gamma = 2\eta + q$, and we get

$$\tilde{w}_n(t,x,i) \leq \tilde{w}_n^*(t,x,i) \leq v_n(t,x,i)$$

for all $(t,x,i) \in [0,T] \times \mathbb{R}^d \times \mathcal{I}$. Sending n to infinity and using Theorem 4.1 and (6.5), we get $w \leq v$ on $[0,T] \times \mathcal{D} \times \mathcal{I}$. ∎

Remark 2. We notice that the counterexample given in subsection 2.2 also satisfies assumption **(H3)**. In particular, this gives an example where classical uniqueness does not hold and where our maximality result is valid.

6.2. Sufficient conditions for (H3). We end this section by providing explicit examples where **(H3)** is satisfied. The idea consists of constructing switching strategies with finite number of switches and satisfying the constraint imposed on the controlled diffusion. This allows us to get a lower bound for the value function. Thanks to the estimate of Lemma 5.2, this proves the polynomial growth of the value function.

The first example deals with the case where there exists a regime that stops the controlled diffusion. By switching immediately on it, we keep the controlled diffusion in $\mathcal{D}$. The second example considers the case where for any initial condition there exists an associated regime that keeps the associated diffusion in $\mathcal{D}$. By switching on such a regime at the first time the diffusion meets the boundary $\partial\mathcal{D}$ of $\mathcal{D}$, we get a strategy satisfying the constraint. Finally, the last example concerns the case of a convex domain $\mathcal{D}$. Using a viability condition involving the normal cone, we also ensure the existence of a regime keeping the diffusion in $\mathcal{D}$. We notice that all the presented conditions are satisfied by the examples presented in section 3.

PROPOSITION 6.1. (i) *Suppose that for any $x \in \partial\mathcal{D}$ there exists $i_x \in \mathcal{I}$ such that $\mu(x, i_x) = 0$ and $\sigma(x, i_x) = 0$. Then* **(H3)** *holds.*

(ii) *Suppose that for each $(t, x) \in [0, T] \times \mathcal{D}$ there exists $i_{t,x} \in \mathcal{I}$ such that the process $X^{t,x}$ defined by*

$$X_s^{t,x} = x + \int_t^s \mu(X_r^{t,x}, i_{t,x})dr + \int_t^s \sigma(X_r^{t,x}, i_{t,x})dW_r, \quad s \geq t,$$

satisfies

$$\mathbb{P}\big(X_s^{t,x} \in \mathcal{D} \ \forall s \in [t, T]\big) = 1. \tag{6.11}$$

Then **(H3)** *is satisfied.*

(iii) *Suppose that $\mathcal{D}$ is convex and there exists $i^* \in \mathcal{I}$ such that*

$$p^\intercal \mu(x, i^*) + \frac{1}{2} tr[\sigma(x, i^*)\sigma(x, i^*)^\intercal A)] \leq 0$$

for all $x \in \partial\mathcal{D}$ and all $(p, A) \in \mathcal{N}^2_{\mathcal{D}}(x)$, where $\mathcal{N}^2_{\mathcal{D}}(x)$ is the second order normal cone to $\mathcal{D}$ at x defined by

$$\mathcal{N}_{\mathcal{D}}(x) = \Big\{(p, A) \in \mathbb{R}^d \times \mathbb{S}^d \ : \ p^\intercal(y - x) + \frac{1}{2}(y - x)^\intercal A(y - x) \ \leq \ o(|y - x|^2) \quad \text{as } y \to x \text{ and } y \in \mathcal{D}\Big\}$$

and $\mathbb{S}^d$ is the set of $d \times d$ symmetric matrices. Then **(H3)** *holds.*

Proof. (i) Fix an initial condition $(t, x, i) \in [0, T]\times\mathcal{D}\times\mathcal{I}$. Let $X^{t,x}$ be the diffusion defined by

$$X_s^{t,x} = x + \int_t^s \mu(X_r^{t,x}, i)dr + \int_t^s \sigma(X_r^{t,x}, i)dW_r, \qquad s \geq t.$$

Consider the strategy $\alpha : (\tau_k, \zeta_k)_k$ defined by $(\tau_0, \zeta_0) = (t, i)$,

$$\tau_1 = \inf\big\{s \geq 0 \ : \ X_s \in \partial\mathcal{D}\big\},$$
$$\zeta_1 = i_{X_{\tau_1}},$$

and $\tau_k > T$ and $\zeta_k = \zeta_1$ for $k \geq 2$. We then have $\mu(X_s^{t,x,\alpha}, \alpha_s) = 0$ and $\sigma(X_s^{t,x,\alpha}, \alpha_s) = 0$ for $s \in [\tau_1, T]$. Therefore, we get $\alpha \in \mathcal{A}_{t,x,i}^{\mathcal{D}}$ and

$$v(t, x, i) \geq J(t, x, \alpha).$$

From (2.2) and **(H2)**(ii) there exists a constant $C > 0$ such that

$$v(t, x, i) \geq -C(1 + |x|^q).$$

By combining this inequality with Lemma 5.2, we get **(H3)**.

(ii) Fix $(t, x, i) \in [0, T] \times \mathcal{D} \times \mathcal{I}$. Consider the strategy $\alpha = (\tau_k, \zeta_k)_k$ defined by $(\tau_0, \zeta_0) = (t, i)$, $(\tau_1, \zeta_1) = (t, i_{t,x})$, and $\tau_k > T$ for $k \geq 2$. From (6.11) we get $\alpha \in \mathcal{A}_{t,x,i}^{\mathcal{D}}$. We then have

$$v(t, x, i) \geq J(t, x, \alpha).$$

From (2.2) and **(H2)**(ii) there exists a constant $C > 0$ such that

$$v(t, x, i) \geq -C(1 + |x|^q).$$

This inequality with Lemma 5.2 gives **(H3)**.

(iii) From Proposition 8 and Remark 9 in [10], we get that for any initial condition $(t, x, i) \in [0, T] \times \mathcal{D} \times \mathcal{I}$, the control $\alpha = (\tau_k, \zeta_k)_k$ defined by

$$(\tau_0, \zeta_0) = (t, i),$$
$$(\tau_1, \zeta_1) = (t, i^*),$$

and $\tau_k > T$ for $k \geq 2$ satisfies $\alpha \in \mathcal{A}_{t,x,i}^{\mathcal{D}}$. We then have

$$v(t, x, i) \geq J(t, x, \alpha).$$

From (2.2) and **(H2)**(ii), there exists a constant $C > 0$ such that

$$v(t, x, i) \geq -C(1 + |x|^q).$$

Using Lemma 5.2, we get **(H3)** from this last inequality. ∎

Appendix A. Additional results and proofs.

A.1. Additional results on convergence and measurability.

We first present two results about stopping times and measurability.

PROPOSITION A.1. *Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a complete probability space endowed with a Brownian motion B. Let $\mathbb{H} = (\mathcal{H})_{t\geq 0}$ be the complete right-continuous filtration generated by B, τ an $\mathbb{H}$-stopping time, and ζ an $\mathcal{H}_\tau$-measurable random variable. Suppose that there exists a constant M such that $\mathbb{P}(\tau \leq M) = 1$. Then there exist two Borel functions ψ and ϕ such that*

$$\tau = \psi\big((B_s)_{s\in[0,M]}\big) \quad \textit{and} \quad \zeta = \phi\big((B_s)_{s\in[0,M+1]}\big), \quad \mathbb{P}\text{-a.s.}$$

Proof. Using $\tau \leq M$ $\mathbb{P}$-a.s., we can write

$$\tau = \int_0^M \mathbb{1}_{\tau > s} ds = \lim_{n\to\infty} \frac{M}{n} \sum_{k=0}^{n-1} \mathbb{1}_{\tau > \frac{k}{n}M}, \quad \mathbb{P}\text{-a.s.} \tag{A.1}$$

Since τ is a $\mathbb{H}$-stopping time and $\mathbb{H}$ is the complete right-continuous extension of the natural filtration of B, from [7, Chapter 2, Remark 32] we can write

$$\underline{\psi}_n^k\big((B_s)_{s\in[0,M]}\big) \leq \mathbb{1}_{\tau>\frac{k}{n}M} \leq \bar{\psi}_n^k\big((B_s)_{s\in[0,M]}\big) \tag{A.2}$$

and

$$\mathbb{P}\Big(\underline{\psi}_n^k\big((B_s)_{s\in[0,M]}\big) \neq \bar{\psi}_n^k\big((B_s)_{s\in[0,M]}\big)\Big) = 0, \tag{A.3}$$

where $\underline{\psi}_n^k$ and $\bar{\psi}_n^k$ are two Borel functions for any $n \geq 1$ and any $k \in \{0,\dots,n-1\}$. Define the Borel functions $\bar{\psi}_n$ and $\underline{\psi}_n$ by

$$\bar{\psi}_n = \frac{M}{n}\sum_{k=0}^{n-1}\bar{\psi}_n^k \quad \text{and} \quad \underline{\psi}_n = \frac{M}{n}\sum_{k=0}^{n-1}\underline{\psi}_n^k\,.$$

We then get from (A.1), (A.2), and (A.3) that

$$\limsup_{n\to\infty}\underline{\psi}_n\big((B_s)_{s\in[0,M]}\big) \leq \tau \leq \limsup_{n\to\infty}\bar{\psi}_n\big((B_s)_{s\in[0,M]}\big), \quad \mathbb{P}\text{-a.s.},$$

and

$$\mathbb{P}\left(\limsup_{n\to\infty}\underline{\psi}_n\big((B_s)_{s\in[0,M]}\big) \neq \limsup_{n\to\infty}\bar{\psi}_n^k\big((B_s)_{s\in[0,M]}\big)\right) = 0.$$

Taking $\psi = \limsup_{n\to\infty}\bar{\psi}_n$, we get $\tau = \psi\big((B_s)_{s\in[0,M]}\big)$ $\mathbb{P}$-a.s.

We now turn to ζ. Since ζ is $\mathcal{H}_\tau$-measurable, $\zeta\mathbb{1}_{\tau\leq t}$ is $\mathcal{H}_t$-measurable for all $t \geq 0$. Using $\tau \leq M$ $\mathbb{P}$-a.s., we get that ζ is $\mathcal{H}_M$-measurable. Using [7, Chapter 2, Remark 32], as previously, we get a Borel function ϕ such that $\zeta = \phi\big((B_s)_{s\in[0,M+1]}\big)$ $\mathbb{P}$-a.s. $\square$

Proposition A.2. *Let $(\Omega^i, \mathcal{G}^i, \mathbb{P}^i)$, $i = 1, 2$, be two compete probability spaces. Suppose that each $(\Omega^i, \mathcal{G}^i, \mathbb{P}^i)$ is endowed with a Brownian motion W^i and denote by $\mathbb{F}^i = (\mathcal{F}_t^i)_t$ the filtration satisfying usual conditions generated by W^i.*

Fix (τ^i, ζ^i), a couple of random variables defined on $(\Omega^i, \mathcal{G}^i, \mathbb{P}^i)$ for $i = 1, 2$, and suppose that
- *τ^1 is an $\mathbb{F}^1$-stopping time,*
- *ζ^1 is $\mathcal{F}^1_{\tau^1}$-measurable,*
- *(W^2, τ^2, ζ^2) has the same law as (W^1, τ^1, ζ^1).*

Then τ^2 is an $\mathbb{F}^2$-stopping time and ζ^2 is $\mathcal{F}^2_{\tau^2}$-measurable.

Proof. Since τ^1 is an $\mathbb{F}^1$-stopping time and $\mathbb{F}^1$ is the complete right-continuous filtration of $(W_s^1)_{s\geq 0}$, we can write from [7, Chapter 2, Remark 32] that, for any $r \geq 0$ and any $\varepsilon > 0$,

$$\underline{\psi}\big((W_s^1)_{s\in[0,r+\varepsilon]}\big) \;\leq\; \mathbb{1}_{\tau^1\leq r} \;\leq\; \bar{\psi}\big((W_s^1)_{s\in[0,r+\varepsilon]}\big)$$

and

$$\mathbb{P}^1\big(\underline{\psi}\big((W_s^1)_{s\in[0,r+\varepsilon]}\big) \neq \bar{\psi}\big((W_s^1)_{s\in[0,r+\varepsilon]}\big)\big) = 0,$$

where $\underline{\psi}$ and $\bar{\psi}$ are two Borel functions. Since (W^1, τ^1) and (W^2, τ^2) have the same law, we get

$$\mathbb{P}^2\Big(\underline{\psi}\big((W_s^2)_{s\in[0,r+\varepsilon]}\big) \;\leq\; \mathbb{1}_{\tau^2\leq r} \;\leq\; \bar{\psi}\big((W_s^2)_{s\in[0,r+\varepsilon]}\big)\Big) = 1$$

and

$$\mathbb{P}^2\Big(\underline{\psi}\big((W^2_s)_{s\in[0,r+\varepsilon]}\big)\neq\bar{\psi}\big((W^2_s)_{s\in[0,r+\varepsilon]}\big)\Big)=0.$$

Since $\mathbb{F}^2$ is complete, this implies that $\mathbb{1}_{\tau^2\leq r}$ is $\mathcal{F}^2_{r+\varepsilon}$-measurable. Using the right-continuity of $\mathbb{F}^2$, we deduce that $\mathbb{1}_{\tau^2\leq r}$ is $\mathcal{F}^2_r$-measurable and τ^2 is an $\mathbb{F}^2$-stopping time.

By the same argument, we get that the random variable $\zeta^2\mathbb{1}_{\tau^2\leq r}$ is $\mathcal{F}^2_r$-measurable for all $r\geq 0$, which is equivalent to the $\mathcal{F}^2_{\tau^2}$-measurability of ζ^2. ∎

We now provide two results on measurability and convergence for a sequence of processes defined on the same space but with different filtrations.

In what follows, we fix a complete probability space $(\Omega,\mathcal{G},\mathbb{P})$ on which is defined a sequence of Brownian motions $(B^n)_{n\geq 0}$. For $n\geq 0$, we denote by $\mathbb{F}^n=(\mathcal{F}^n_t)_{t\geq 0}$ the complete right-continuous filtration generated by B^n.

PROPOSITION A.3. *For $n\geq 1$ let τ^n be an $\mathbb{F}^n$-stopping time and ζ^n be an $\mathcal{F}^n_{\tau^n}$-measurable random variable. We suppose that the following hold:*

(i) *B^n converges to B^0:*

$$\sup_{t\in[0,T]}|B^n_t-B^0_t|\xrightarrow[n\to\infty]{\mathbb{P}\text{-}a.s.}0.$$

(ii) *The sequences $(\tau^n)_{n\geq 1}$ and $(\zeta^n)_{n\geq 1}$ are uniformly bounded.*

(iii) *There exist random variables τ^0 and ζ^0 such that*

$$(\tau^n,\zeta^n)\xrightarrow[n\to\infty]{\mathbb{P}\text{-}a.s.}(\tau^0,\zeta^0).$$

Then, τ^0 is an $\mathbb{F}^0$-stopping time and ζ^0 is $\mathcal{F}^0_\tau$-measurable.

Proof. We first prove that τ^0 is an $\mathbb{F}^0$-stopping time. Fix $t>0$ and define for $p\geq 1$ the bounded and continuous functions Φ_p by

$$\Phi_p(x)=\mathbb{1}_{x\leq t-\frac{1}{p}}+p\mathbb{1}_{t-\frac{1}{p}<x\leq t}(t-x),\quad x\in\mathbb{R}_+.$$

From (iii) and Theorem 3.1 in [3], we get

$$\mathbb{E}\big[\Phi_p(\tau^n)|\mathcal{F}^n_t\big]\xrightarrow[n\to\infty]{\mathbb{P}}\mathbb{E}\big[\Phi_p(\tau^0)|\mathcal{F}^0_t\big].$$

Since τ^n is an $\mathbb{F}^n$-stopping time, we have $\mathbb{E}[\Phi_p(\tau^n)|\mathcal{F}^n_t]=\Phi_p(\tau^n)$. Indeed, we can write $\Phi_p=\lim_{k\infty}\Phi^k_p$, where Φ^k_p is defined by

$$\Phi^k_p(x)=\mathbb{1}_{x\leq t-\frac{1}{p}}+\sum_{j=1}^{k}\frac{j}{kp}\mathbb{1}_{t-\frac{j}{kp}<x\leq t-\frac{j-1}{kp}},\quad x\in\mathbb{R}_+.$$

Then since τ^n is an $\mathbb{F}^n$ stopping time, the random variable $\Phi^k_p(\tau^n)$ is $\mathcal{F}^n_t$-measurable. Sending k to infinity, we get that $\Phi_p(\tau^n)$ is $\mathcal{F}^n_t$-measurable.

Since Φ_p is continuous, we get from (iii) that

$$\Phi_p(\tau_n)\xrightarrow[n\to\infty]{\mathbb{P}\text{-a.s.}}\Phi_p(\tau^0).$$

Therefore $\Phi_p(\tau^0)=\mathbb{E}[\Phi_p(\tau^0)|\mathcal{F}^0_t]$. Sending p to infinity, we get $\mathbb{1}_{\tau^0\leq t}=\mathbb{E}[\mathbb{1}_{\tau^0\leq t}|\mathcal{F}^0_t]$, and τ^0 is an $\mathbb{F}^0$-stopping time since $\mathbb{F}^0$ is complete.

To prove that ζ^0 is $\mathcal{F}^0_{\tau^0}$-measurable, we proceed in the same way and consider $\zeta^n\Phi_p(\tau^n)$ instead of $\Phi_p(\tau^n)$ for $n\geq 0$. ∎

We now turn to stability of diffusions. For $n \geq 0$, we fix random functions $b_n : [0,T] \times \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ and $a_n : [0,T] \times \Omega \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$. We suppose that the following holds.

(HA)

(i) For each $n \geq 0$, b_n and a_n are $\mathbb{F}^n$-progressive$\otimes\mathcal{B}(\mathbb{R}^d)$-measurable.

(ii) There exists $\delta > 0$ such that

$$\mathbb{E}\left[\int_0^T \Big(|b^n(t,0)|^{2+\delta} + |a^n(t,0)|^{2+\delta}\Big)dt\right] < +\infty, \qquad n \geq 0.$$

(iii) There exists a constant L such that

$$|b^n(t,x) - b^n(t,x')| + |a^n(t,x) - a^n(t,x')| \leq L|x-x'|, \quad x, x' \in \mathbb{R}^d,\ n \geq 0.$$

Then, for a given deterministic initial condition X_0, we can define for each $n \geq 0$ the solution X^n to the SDE

$$X_t^n = X_0 + \int_0^t b^n(s, X_s^n)ds + \int_0^t a^n(s, X_s^n)dB_s^n, \qquad t \geq 0.$$

Proposition A.4. *Suppose that*

$$\sup_{t\in[0,T]} |B_t^n - B_t^0| \xrightarrow[n\to\infty]{\mathbb{P}\text{-}a.s.} 0 \tag{A.4}$$

and

$$\mathbb{E}\left[\int_0^T \left|a^n(s,x) - a^0(s,x)\right|^2 ds\right] + \mathbb{E}\left[\int_0^T |b^n(s,x) - b^0(s,x)|^2 ds\right] \xrightarrow[n\to+\infty]{} 0 \tag{A.5}$$

for all $x \in \mathbb{R}^d$. Then, under **(HA)**, *we have*

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left|X_t^n - X_t^0\right|^2\right] \xrightarrow[n\to\infty]{} 0. \tag{A.6}$$

To prove this result we cannot use classical estimates on diffusion processes since the driving Brownian motion evolves with n. In particular, the stochastic integrals $\int a^n dB^0$ are not defined. We therefore need to use approximations by step processes, as done in the construction of the Itô integral.

Proof. We proceed in two steps.

Step 1. We first consider the case where the b^n and a^n do not depend on the variable x. For $p \geq 1$, let H^p be an $\mathbb{F}$-adapted piecewise constant process of the form

$$H_t^p = \sum_{k=0}^{N_p} \tilde{H}_k^p \mathbb{1}_{[t_k^p, t_{k+1}^p)}(t), \quad t \in [0,T],$$

where $\tilde{H}_k^p \in \mathbf{L}^{2+\delta}(\Omega, \mathcal{F}_{t_k^p}, \mathbb{P})$ for $0 \leq k \leq N_p$ such that

$$\mathbb{E}\left[\int_0^T |H_s^p - a_s|^2 ds\right] \leq \frac{1}{p}. \tag{A.7}$$

We then have

$$\mathbb{E}\left[\left|\int_0^T a^n dB^n - \int_0^T a dB^0\right|^2\right] \leq 2\left(E\left[\left|\int_0^T a^n dB^n - \int_0^T H^p dB^0\right|^2\right] + \frac{1}{p}\right). \tag{A.8}$$

We then define the process $H^{p,n}$ by

$$H_t^{p,n} = \sum_{k=0}^{N_p} \mathbb{E}\Big[\tilde{H}_k^p \big| \mathcal{F}_{t_k^p}^n\Big] \mathbb{1}_{[t_k^p, t_{k+1}^p)}(t), \qquad t \in [0,T].$$

We can write the following decomposition:

$$\begin{aligned}\mathbb{E}\left[\left|\int_0^T a^n dB^n - \int_0^T H^p dB^0\right|^2\right] \leq 2 &\left(E\left[\left|\int_0^T a^n dB^n - \int_0^T H^{p,n} dB^n\right|^2\right]\right.\\ &\left.+ E\left[\left|\int_0^T H^{p,n} dB^n - \int_0^T H^p dB^0\right|^2\right]\right).\end{aligned} \tag{A.9}$$

From (A.4), we can apply Proposition 2 in [5], and we get

$$\mathbb{E}\Big[\tilde{H}_k^p \big| \mathcal{F}_{t_k^p}^n\Big] \xrightarrow[n\to+\infty]{\mathbb{P}} \tilde{H}_k^p, \qquad 0 \leq k \leq N. \tag{A.10}$$

In particular, we get from (A.4) and (A.10) that

$$\mathbb{E}\left[\left|\int_0^T H^{p,n} dB^n - \int_0^T H^p dB^0\right|^2\right] \xrightarrow[n\to+\infty]{} 0. \tag{A.11}$$

Moreover, from Itô isometry and (A.7), we have

$$\begin{aligned}E\left[\left|\int_0^T a^n dB^n - \int_0^T H^{p,n} dB^n\right|^2\right] &= E\left[\int_0^T \left|a_s^n - H_s^{p,n}\right|^2 ds\right]\\ &\leq 3\left(E\left[\int_0^T \left|a_s^n - a_s^0\right|^2 ds\right] + \frac{1}{p}\right.\\ &\qquad \left. + E\left[\int_0^T \left|H_s^p - H_s^{p,n}\right|^2 ds\right]\right).\end{aligned} \tag{A.12}$$

Then, using (A.10), we also get

$$\mathbb{E}\left[\int_0^T \left|H_s^p - H_s^{p,n}\right|^2 ds\right] \xrightarrow[n\to+\infty]{} 0. \tag{A.13}$$

Therefore, we get from (A.5), (A.12), and (A.13) that

$$\limsup_{n\to\infty} \mathbb{E}\left[\left|\int_0^T a^n dB^n - \int_0^T H^{p,n} dB^n\right|^2\right] \leq \frac{1}{p}.$$

From this last inequality, (A.8), (A.9), and (A.11) we get

$$\limsup_{n\to\infty}\mathbb{E}\left[\left|\int_0^T a^n dB^n - \int_0^T a^0 dB^0\right|^2\right] \leq \frac{4}{p}, \qquad p \geq 1.$$

Therefore, we get

$$\lim_{n\to\infty}\mathbb{E}\left[\left|\int_0^T a^n dB^n - \int_0^T a^0 dB^0\right|^2\right] = 0.$$

From Theorem 3.1 in [3], we deduce that

$$\lim_{n\to\infty}\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t a^n dB^n - \int_0^t a^0 dB^0\right|^2\right] = 0.$$

From this last equality and (A.5), we get (A.6).

Step 2. We now consider the general case. For $n \geq 0$, we denote by $(X^{n,p})_{p\geq 0}$ the sequence of processes defined by

$$X_t^{n,0} = X_0, \quad t \geq 0,$$

and

$$X_t^{n,p+1} = X_0 + \int_0^t b^n(s, X_s^{n,p})ds + \int_0^t a^n(s, X_s^{n,p})dB_s^n, \quad t \geq 0,$$

for $p \geq 0$. From **(HA)**(ii) and since X_0 is deterministic, we get by induction on p that

$$\mathbb{E}\left[\sup_{t\in[0,T]}|X_t^{n,p}|^{2+\delta}\right] < \infty$$

for all $n, p \geq 1$. Still using an induction, we get from Step 1 that

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|X_t^{n,p} - X_t^{0,p}\right|^2\right] \xrightarrow[n\to\infty]{} 0 \tag{A.14}$$

for all $p \geq 0$. From the argument on diffusion processes, we have (see, e.g., the proof of [11, Chapter 5, Theorem 2.9])

$$\sup_{n\geq 0}\mathbb{E}\left[\sup_{t\in[0,T]}|X_t^{n,p} - X_t^n|^2\right] \leq \psi(p),$$

where $\psi(p) \to 0$ as $p \to +\infty$. We then get

$$\limsup_{n\to+\infty}\mathbb{E}\left[\sup_{t\in[0,T]}\left|X_t^n - X_t^0\right|^2\right] \leq 2\psi(p) + \lim_{n\to+\infty}\mathbb{E}\left[\sup_{t\in[0,T]}\left|X_t^{n,p} - X_t^{0,p}\right|^2\right] \leq 2\psi(p).$$

Sending p to ∞, we get the result. ∎

A.2. Proofs of Lemmata 5.1 and 5.2.

Proof of Lemma 5.1. Fix $n \geq 1$, $R > 0$, and $i \in \mathcal{I}$. From the definition of f_n we have

$$|f_n(x,i) - f_n(x',i)| \leq n\big|\Theta_n(x) - \Theta_n(x')\big| + \big|f(x,i) - f(x',i)\big|$$

for all $x, x' \in \mathbb{R}^d$ and $i \in \mathcal{I}$. Since $d(.,\mathcal{D})$ is Lipschitz continuous, we get from the definition of Θ_n and **(H2)**(i) the existence of a constant $L_{R,n}$ such that

$$|f_n(x,i) - f_n(x',i)| \leq L_{R,n}|x - x'|$$

for all $x, x' \in \mathbb{R}^d$.

We turn to the growth property. From the definition of f_n we have

$$|f_n(x,i)| \leq n\big|\Theta_n(x)\big| + \big|f(x,i)\big|$$

for all $x \in \mathbb{R}^d$ and $i \in \mathcal{I}$. Since $d(.,\mathcal{D})$ is Lipschitz continuous, it has a linear growth, and we get from the definition of Θ_n and **(H2)**(ii) that there exists a constant C_n such that

$$|f_n(x,i)| \leq C_n\big(1 + |x|^q\big).$$

The proof is the same for the function g_n. ◻

Proof of Lemma 5.2. Fix $n \geq 1$ and $(t,x,i) \in [0,T] \times \mathcal{D} \times \mathcal{I}$. Using the definition of f_n and g_n, we have

$$J_n(t,x,\alpha) \leq J_1(t,x,\alpha) \tag{A.15}$$

for any $\alpha \in \mathcal{A}_{t,i}$. From (2.2) and **(H2)** there exists a constant C such that

$$J_1(t,x,\alpha) \leq C\big(1 + |x|^q\big)$$

for any $\alpha \in \mathcal{A}_{t,i}$. From (A.15) and the definition of $v_n(t,x,i)$, we get (5.1). ◻

REFERENCES

[1] B. Bouchard, *A stochastic target formulation for optimal switching problems in finite horizon*, Stochastics, 81 (2009), pp. 171–197.
[2] M. Brennan and E. Schwartz, *Evaluating natural resource extraction*, J. Business, 58 (1985), pp. 135–137.
[3] P. Briand, B. Delyon, and J. Mémin, *Donsker-type theorem for BSDEs*, Electron. Comm. Probab., 6 (2001), pp. 1–14.
[4] R. Carmona and M. Ludkovski, *Valuation of energy storage: An optimal switching approach*, Quant. Finance, 10 (2009), pp. 359–374.
[5] F. Coquet, J. Mémin, and L. Slominski, *On weak convergence of filtrations*, in Séminaire de Probabilités XXXV, Lecture Notes in Math. 1755, Springer, New York, 2001, pp. 306–328.
[6] M. Crandall, H. Ishii, and P. L. Lions, *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc., 27 (1992), pp. 1–67.
[7] C. Dellacherie and P. A. Meyer, *Probabilités et Potentiel*, Hermann, Paris, 1975.
[8] A. Dixit, *Entry and exit decisions under uncertainty*, J. Political Econ., 97 (1989), pp. 620–638.
[9] M. Fischer and G. Nappo, *On the moments of the modulus of continuity of Itô processes*, Stoch. Anal. Appl., 28 (2009), pp. 103–122.
[10] D. Goreac, I. Ivanescu, and O. Serea, *LP Approach to dynamic programming principles for stochastic control problems with state constraints*, Nonlinear Anal. Ser. A, 77 (2013), pp. 59–73.

[11] I. Karatzas and S. E. Shreve, *Brownian Motion and Stochastic Calculus*, 2nd ed., Grad. Texts in Math., Springer-Verlag, Berlin, 1991.

[12] M. A. Katsoulakis, *Viscosity solutions of second order fully nonlinear elliptic equations with state constraints*, Indiana Univ. Math. J., 43 (1994), pp. 493–519.

[13] N. V. Krylov, *Controlled Diffusion Processes*, Stochast. Model. App. Probab., Springer-Verlag, New York, 1980.

[14] V. Ly Vath, H. Pham, and S. Villeneuve, *A mixed singular/switching control problem for a dividend policy with reversible technology investment*, Ann. Appl. Probab., 18 (2008), pp. 1164–1200.

[15] H. M. Soner, *Optimal control with state-space constraint* I, SIAM J. Control Optim., 24 (1986), pp. 552–561, doi:10.1137/0324032.

[16] H. M. Soner, *Optimal control with state-space constraint* II, SIAM J. Control Optim., 24 (1986), pp. 1110–1122, doi:10.1137/0324067.

[17] S. Tang and J. Yong, *Finite horizon stochastic optimal switching and impulse controls with a viscosity solution approach*, Stochastics Stochastics Rep., 45 (1993), pp. 145–176.