An Optimal Execution Problem in Finance Targeting the Market Trading Speed: an MFG Formulation

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***Abstract*— The stock market can be modeled as a large population non-cooperative game where (following standard financial models) each trader has stochastic linear dynamics with quadratic cost function. We consider the case where there exists one major trader with significant influence on market movements together with a large number of minor traders, each with individually asymptotically negligible effect on the market. The traders are coupled in their dynamics and cost functions by the market's average trading rate (a component of the system mean field). In this work the theory of partially observed mean field games is extended to cover indefinite LQG problems and then employed to obtain ϵ-Nash equilibria for the market, together with the best response trading strategies where each agent attempts to (i) maximize its wealth, and then also (ii) track a fraction of market's average trading rate, and (iii) avoid large execution prices and large trading accelerations. Illustrative simulations are presented.**

I. INTRODUCTION

Partially observed Mean Field Game (PO MFG) theory for systems with a major agent was introduced and developed in [1]–[6] where it is assumed the major agent's state is partially observed by each minor agent, and the major agent completely observes its own state. This theory for partially observed major minor LQG mean field game systems (PO MM LQG MFG) was further extended in [7] to major-minor LQG systems in which both the major agent and the minor agents partially observe the major agent's state. The existence of ϵ-Nash equilibria, together with the individual agents' control laws yielding the equilibria, was established wherein each minor agent recursively generates (i) an estimate of the major agent's state, and (ii) an estimate of the major agent's estimate of its own state (in order to estimate the major agent's control feedback), and hence generates a version of the system's mean field. It is to be noted that the case where each agent has only partial observations on its own state was addressed in the LQG case in [8] and in the nonlinear case in [9], [10].

Optimal execution problems have been addressed in the literature (see e.g. [11]–[14]) where an agent must liquidate or acquire a certain amount of shares over a pre-specified time horizon at a trading speed to balance the price impact (from trading quickly) and the price uncertainty (from trading slowly), while it maximizes its final wealth. Further, in [15] the partially observed setting where the market liquidity variable is not observed was studied. This problem with the linear models in [11] was formulated as for the nonlinear major minor (MM) MFG model in [16]. The PO MM LQG MFG theory was first applied to an optimal execution problem with linear models of [11] in [17] where an institutional investor, interpreted as a major agent, aims to liquidate a specific amount of shares and it has only partial observations of its own inventories. Furthermore, there is a large population of high frequency traders (HFTs), interpreted as minor agents, who wish to liquidate their shares, and each of them has partial observations of its own inventories and the major agent's inventories. This work was improved in the modeling of market dynamics, and also was extended to consider two populations of HFTs with liquidation or acquisition objectives who wish to, respectively, liquidate or acquire a certain number of shares within a specific duration of time in [18]. In order to minimize the impact of its trading on market movements, it is not only important for each trader to slow down the speed of trading but also to take into consideration its speed relative to the market's average speed. To address this problem the PO LQG MFG theory is extended in the current work to cover indefinite LQ problems. The theory is then utilized to establish the existence of ϵ-Nash equilibria together with the best response trading strategies such that each agent targets a percentage of market's average trading rate while it attempts to maximize its own wealth and avoid the occurrence of large execution prices, and large trading accelerations which are appropriately weighted in the agent's performance function.

We note that the terms major trader (respectively, minor trader), and institutional trader (respectively, HFT) are used interchangeably in this paper.

II. TRADING DYNAMICS OF AGENTS IN THE MARKET

As stated in the Introduction, the institutional investor is considered as a major agent in the mean field model of the market which liquidates its shares and the HFTs are considered as minor agents, where two types of them are considered: liquidators and acquirers. Employing the trading model in [11], the trading dynamics of the major agent and any generic minor agent in the market are described by the linear time evolution of the (i) inventories, (ii) trading rates and (iii) prices while the bilinear cash process appears in the quadratic performance function for each agent.

A. Inventory Dynamics

It is assumed that the institutional investor liquidates its inventory of shares, $Q_0(t)$, by trading at a rate $\nu_0(t)$ during the trading period $[0, T]$. Hence the major agent's inventory

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dynamics is given by

$$dQ_0(t) = \nu_0(t)dt + \sigma_0^Q dw_0^Q, \quad 0 \leq t \leq T,$$

where w_0^Q is a Wiener process modeling the noise in the inventory information that the institutional trader collects from its branches in different locations; σ_0^Q is a positive scalar and we assume that $Q_0(0) \gg 1$. The same dynamical model is adopted for the trading dynamics of a generic HFT

$$dQ_i(t) = \nu_i(t)dt + \sigma_i^Q dw_i^Q, \quad 1 \leq i \leq N_A + N_L,\ 0 \leq t \leq T$$

where N_a and N_l are, respectively, liquidators' and acquirers' population among N minor traders, i.e. $N = N_a + N_l$, w_i^Q is a Wiener process that models the HFT's information noise, σ_i^Q is a positive scalar, $\nu_i(t)$ is the agent's rate of trading which can be positive or negative depending on whether the agent is acquirer or liquidator, respectively; $Q_i(t)$ is the minor liquidator agent's remaining shares at time t, or the shares the minor acquirer agent has bought until time t. However, the initial share stock of the HFTs, $\{Q_i(0), 1 \leq i \leq N_a + N_l\}$, are not considered to be large, furthermore they are not motivated to retain shares and are assumed to trade them quickly.
We assume that the trading rate of the major agent is controlled via $u_0(t)$ as

$$d\nu_0(t) = u_0(t)dt, \quad 0 \leq t \leq T,$$

where the trading strategy $u_0(t)$ can be seen to be the trading acceleration of the major trader. Correspondingly, $u_i(t)$ controls the trading rate of minor agent, $\mathcal{A}_i$, by

$$d\nu_i(t) = u_i(t)dt, \quad 1 \leq i \leq N_a + N_l,\ 0 \leq t \leq T,$$

where $u_i(t)$ is the trading acceleration of the minor trader.

B. Price Dynamics

The trading rate of the major agent and the average trading rate of the minor agents give rise to the fundamental asset price which models the permanent effect of agents' trading rates on the market price. Further, each agent has a temporary effect on the asset price which only persists during the action of the trade and which determines the execution price, that is to say the price at which each agent can trade.

1) Fundamental Asset Price: We model the dynamics of the fundamental asset price, as seen from the major agent's viewpoint, by

$$dF_0(t) = \big(\lambda_0\nu_0(t) + \lambda\nu^N(t)\big)dt + \sigma dw_0^F(t),\ 0 \leq t \leq T,$$

where $\nu^N(t) = \frac{1}{N}\sum_{i=1}^N \nu_i(t)$ is the average of the minor agents' rates of trading, the Wiener process $w_0^F(t)$ models the aggregate effect of all traders in the market which - unlike the major and minor agents $\mathcal{A}_0$, $\mathcal{A}_i$, - have no complete or partial observations on any of the state variables appearing in the dynamical market model (these are termed uninformed traders). Further, σ denotes the intensity of the market volatility and $\lambda_0, \lambda \geq 0$ denote the strength of the linear permanent impact of the major and minor agents' trading on the fundamental asset price, respectively. Similarly, we model the fundamental asset price dynamics, as seen by a minor agent $\mathcal{A}_i$, by

$$dF_i(t) = \big(\lambda_0\nu_0(t) + \lambda\nu^N(t)\big)dt + \sigma dw_i^F(t),\ 0 \leq t \leq T,$$

where $1 \leq i \leq N_a + N_l$, $\nu^N(t) = \frac{1}{N}\sum_{i=1}^N \nu_i(t)$ is again the average trading rate of the minor traders, and the Wiener process, $w_i^F(t)$, represents the mass effect of all uninformed traders in the market. The time differences between agents in getting data from fast changing limit order book make the Wiener processes, $w_i^F,\ 0 \leq i \leq N_a + N_l$ independent.

2) Execution Price: The major agent's execution price $S_0(t)$ evolution is assumed to be given by

$$dS_0(t) = dF_0(t) + a_0 d\nu_0(t), \quad 0 \leq t \leq T, \tag{1}$$

where $a_0 \geq 0$ is the temporary impact strength of the major agent on fundamental asset price. Likewise, a minor agent's execution price, $S_i(t)$, is assumed to evolve by

$$dS_i(t) = dF_i(t) + a d\nu_i(t),\ 1 \leq i \leq N_a + N_l\ 0 \leq t \leq T, \tag{2}$$

where a models the temporary impact of a minor agent's trading on its execution price.

C. Cash Process

The cash processes for the major agent and a generic minor agent, $Z_0(t)$, $Z_i(t)$, respectively, are given by

$$dZ_0(t) = -S_0(t)dQ_0(t), \quad 0 \leq t \leq T, \tag{3}$$
$$dZ_i(t) = -S_i(t)dQ_i(t), \quad 1 \leq i \leq N_a + N_l,\ 0 \leq t \leq T, \tag{4}$$

where $Z_0(t)$, and $Z_i(t), 1 \leq i \leq N_l$ are the cash obtained through liquidation of shares, and $Z_i(t), 0 \leq i \leq N_a$ is the cash paid for acquisition of shares up to time t. We note that the value of $dQ_0(t)$ in a stock sale (respectively, buy) is negative (respectively, positive) and hence for positive $S_0(t)$, $Z_0(t)$ increases (respectively, decreases).

D. Cost Function

1) Major Liquidator Trader: The objective for the major trader is to liquidate $\mathcal{N}_0$ shares and maximize the cash it holds at the end of the trading horizon, i.e. maximize $Z_0(T)$, and if the remaining inventory at the final time T is $Q_0(T)$, it can liquidate it at a lower price than the market asset price reflected at cost function by $Q_0(T)(F_0(T) - \alpha Q_0(T))$. Further, the major trader's utility in minimizing the inventory over the period $[0, T]$ is modeled by including the penalty $\phi\int_0^T Q_0^2(s)ds$ in its objective function, and the utility of avoiding very high execution prices, and large trading accelerations by including the terms $\epsilon S_0^2(T)$, $\int_0^T \delta S_0^2(s)ds$, and $\int_0^T R_0 u_0^2(s)ds$ in the objective function. Additionally, to minimize the movements of the market by the major trader, a percentage of the market's average speed of trading is targeted by incorporating the terms $\beta\big(\nu_0(T) - \rho\nu^N(T)\big)^2$ and $\int_0^T \theta\big(\nu_0(s) - \rho\nu^N(s)\big)^2 ds$. Therefore, its cost function to be minimized is given by

$$J_0(u_0,u_{-0})=\mathbb{E}\Big[-rZ_0(T)-pQ_0(T)\big(F_0(T)-\alpha Q_0(T)\big)$$
$$+\epsilon S_0^2(T)+\beta\big(\nu_0(T)-\rho\nu^N(T)\big)^2+\int_0^T\big(\phi Q_0^2(s)+\delta S_0^2(s)$$
$$+\theta\big(\nu_0(s)-\rho\nu^N(s)\big)^2+R_0u_0^2(s)\big)ds\Big],\quad (5)$$

where r, p, α, ϵ, β, ϕ, δ, θ, and R_0 are positive scalars, $0\le\rho\le 1$, and $u_{-0}:=(u_1,...,u_{N_a+N_l})$ are trading strategies of the minor traders. Note that for larger values of ϕ the trader attempts to liquidate its inventory more quickly.

2) Minor Liquidator Trader: In a similar way, the objective function to be minimized for a liquidator HFT who wants to liquidate $\mathcal{N}_l$ shares during the interval $[0,T]$ is given by

$$J_i(u_i,u_{-i})=\mathbb{E}\Big[-r_lZ_i(T)-p_lQ_i(T)\big(F_i(T)-\psi_lQ_i(T)\big)$$
$$+\xi_lS_i^2(T)+\mu_l\big(\nu_i(T)-\rho_l\nu^{N_l}(T)\big)^2$$
$$+\int_0^T\Big(\kappa_lQ_i^2(s)+\gamma_lS_i^2(s)+\varrho_l\big(\nu_i^2(s)-\rho_l\nu^{N_l}(s)\big)^2$$
$$+R_lu_i^2(s)\Big)ds\Big],\ 1\le i\le N_l,\quad (6)$$

where r_l, p_l, ψ_l, ξ_l, μ_l, κ_l, γ_l, ϱ_l and R_l are positive scalars, $0\le\rho_l\le 1$, and $u_{-i}:=(u_0,u_1,...,u_{i-1},u_{i+1},...,u_{N_a+N_l})$. Note that $\mathcal{N}_l\ll\mathcal{N}_0$. Note that the cost function (6) is defined in a way that a generic minor liquidator trader tracks a fraction of the market's average selling rate $\nu^{N_l}=\frac{1}{N}\sum_{i=1}^{N_l}\nu_i$ by including the terms $\mu_l\big(\nu_i(T)-\rho_l\nu^{N_l}(T)\big)^2$ and $\int_0^T\varrho_l\big(\nu_i^2(s)-\rho_l\nu^{N_l}(s)\big)^2ds$.

3) Minor Acquirer Trader: The objective for a minor acquirer trader is to buy $\mathcal{N}$ shares over the trading horizon $[0,T]$, while it minimizes the execution cost including the cash $Z_i(T)$ paid up to time T, and the cash must be paid at time T to buy the remaining shares at once at a higher price than the market's asset price, i.e. $(\mathcal{N}-Q_i(T))\big(F_i(T)+\psi_a(\mathcal{N}-Q_i(T))\big)$. It is also intended to (i) target a fraction of the acquirers' average trading rate $\nu^{N_a}=\frac{1}{N_a}\sum_{i=1}^{N_a}\nu_i$ by incorporating the terms $\mu_a\big(\nu_i(T)-\rho_a\nu^{N_a}(T)\big)^2+\int_0^T\varrho_a\big(\nu_i(s)-\rho_a\nu^{N_a}(s)\big)^2ds$, and (ii) to avoid high execution prices, and large trading accelerations modeled by including $\xi_aS_i^2(T)+\int_0^T\big(\gamma_aS_i^2(s)+R_au_i^2(s)\big)ds$ in its objective function

$$J_i(u_i,u_{-i})=\mathbb{E}\Big[p_a(\mathcal{N}-Q_i(T))\big(F_i(T)+\psi_a(\mathcal{N}-Q_i(T))\big)$$
$$+r_aZ_i(T)+\xi_aS_i^2(T)+\mu_a\big(\nu_i(T)-\rho_a\nu^{N_a}(T)\big)^2+$$
$$\int_0^T\Big(\kappa_a(\mathcal{N}-Q_i(s))^2+\gamma_aS_i^2(s)+\varrho_a\big(\nu_i(s)-\rho_a\nu^{N_a}(s)\big)^2+$$
$$R_au_i^2(s)\Big)ds\Big],\ 1\le i\le N_a,\quad (7)$$

where $\int_0^T\kappa_a(\mathcal{N}-Q_i(s))^2ds$ is to penalize the agent for the remaining shares to be bought up to T and to expedite the acquisition. The parameters p_a, ψ_a, r_a, ξ_a, μ_a, κ_a, γ_a, ϱ_a, and R_a are positive scalars, $0\le\rho_a\le 1$, and $u_{-i}:=(u_0,u_1,...,u_{i-1},u_{i+1},...,u_{N_a+N_l})$.

III. MFG Formulation of the Optimal Execution Problem

In this section we formulate the optimal execution problem in the MM LQG MFG framework.

A. Finite Populations

1) Major Agent: The dynamics of the major trader in the market can be modeled as

$$d\nu_0(t)=u_0(t)dt,$$
$$dQ_0(t)=\nu_0(t)dt+\sigma_0^Q dw_0^Q,$$
$$dS_0(t)=\big(\lambda_0\nu_0(t)+\lambda\nu^N(t)\big)dt+a_0u_0(t)dt+\sigma dw_0^F(t).$$

Let the major agent's state be denoted by $x_0=[\nu_0,Q_0,S_0]^T$, then its dynamics can be expressed as

$$dx_0=A_0x_0dt+B_0u_0dt+E_0x^Ndt+D_0dw_0\quad (8)$$

where

$$A_0=\begin{bmatrix}0&0&0\\1&0&0\\\lambda_0&0&0\end{bmatrix},\ B_0=\begin{bmatrix}1\\0\\a_0\end{bmatrix},\ w_0=\begin{bmatrix}w_0^Q\\w_0^F\end{bmatrix}$$
$$E_0=\begin{bmatrix}0&0&0\\0&0&0\\\lambda&0&0\end{bmatrix},\ D_0=\begin{bmatrix}0&0\\\sigma_0^Q&0\\0&\sigma\end{bmatrix}.$$

The major trader's cost function (5) can also be described in terms of its states with replacing the final cash process by $\mathbb{E}[Z_0(T)]=-\mathbb{E}[\int_0^T S_0(s)\nu_0(s)ds]$, and the fundamental asset price $F_0(T)$ using (1). The equation (8) together with the cost function (5) form the stochastic LQG problem for the major trader. Note that the major trader is involved with the market's average trading rate in its dynamics while involved with the market's average selling rate in its cost function.

2) Minor Liquidator Agents: Similarly, the stochastic optimal control problem for a minor liquidator trader $\mathcal{A}_i,\ 1\le i\le N_l$, is given by the set of dynamical equations

$$d\nu_i(t)=u_i(t)dt,$$
$$dQ_i(t)=\nu_i(t)dt+\sigma_i^Q dw_i^Q,$$
$$dS_i(t)=\big(\lambda_0\nu_0(t)+\lambda\nu^N(t)\big)dt+au_i(t)dt+\sigma dw_i^F.$$

Similar to the major trader, we define a generic minor trader's state vector as $x_i=[\nu_i,Q_i,S_i]^T$, and its dynamics can be written as

$$dx_i=A_lx_idt+B_lu_idt+E_lx^Ndt+D_ldw_{l_i}\quad (9)$$

with

$$A_l=\begin{bmatrix}0&0&0\\1&0&0\\0&0&0\end{bmatrix},\quad E_l=\begin{bmatrix}0&0&0\\0&0&0\\\lambda&0&0\end{bmatrix},\quad B_l=\begin{bmatrix}1\\0\\a\end{bmatrix}$$
$$G_l=\begin{bmatrix}0&0&0\\0&0&0\\\lambda_0&0&0\end{bmatrix},\ D_l=\begin{bmatrix}0&0\\\sigma_i^Q&0\\0&\sigma\end{bmatrix},\ w_{li}=\begin{bmatrix}w_i^Q\\w_i^F\end{bmatrix}.$$

The quadratic cost function (6) can also be expressed in terms of the minor agent's state when the final cash process in (6) is replaced by $\mathbb{E}[Z_i(T)]=-\mathbb{E}[\int_0^T S_i(s)\nu_i(s)ds]$ using (4), and the fundamental asset price $F_i(T)$ is replaced using (2). The equations (9) and (6) form the stochastic LQG problem for a generic minor liquidator. Additionally, they show that a minor liquidator agent is coupled with the major agent's trading rate and the market's average trading rate in its dynamics while coupled with the market's average selling rate in its cost function.

3) Minor Acquirer Agents: The stochastic optimal control problem for a minor acquirer trader $\mathcal{A}_i,\ 1 \leq i \leq N_a$, is given by the set of dynamical equations

$$d\nu_i(t) = u_i(t)dt, \tag{10}$$

$$dY_i(t) = -\nu_i(t)dt + \sigma_i^Q dw_i^Q, \tag{11}$$

$$dS_i(t) = \big(\lambda_0\nu_0(t) + \lambda\nu^N(t)\big)dt + au_i(t)dt + \sigma dw_i^F, \tag{12}$$

where $Y_i(t) = \mathcal{N}_a - Q_i(t)$ is the remaining shares at t to be acquired until the end of trading horizon. We define a generic minor acquirer trader's state vector as $x_i = [\nu_i, Y_i, S_i]$, hence its dynamics in compact form would be

$$dx_i = A_a x_i dt + B_a u_i dt + E_a x^N dt + D_a dw_{a_i}, \tag{13}$$

where

$$A_a = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ E_a = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \lambda & 0 & 0 \end{bmatrix},\ B_a = \begin{bmatrix} 1 \\ 0 \\ a \end{bmatrix}$$

$$G_a = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \lambda_0 & 0 & 0 \end{bmatrix},\ D_a = \begin{bmatrix} 0 & 0 \\ \sigma_i^Q & 0 \\ 0 & \sigma \end{bmatrix},\ w_{ai} = \begin{bmatrix} w_i^Q \\ w_i^F \end{bmatrix}.$$

Accordingly, the cost function for acquisition is given by

$$\begin{aligned} J_i(u_i, u_{-i}) = \mathbb{E}\Big[& p_a Y_i(T)\big(S_i(T) - a\nu_i(T) + \psi_a Y_i(T)\big) \\ & + \xi_a S_i^2(T) + \mu_a\big(\nu_i(T) - \rho_a\bar{\nu}(T)\big)^2 + \\ & \int_0^T \big(\kappa_a Y_i(s)^2 + \gamma_a S_i^2(s) + \varrho_a\big(\nu_i(s) - \rho_a\bar{\nu}(s)\big)^2 \\ & - r_a S_i(s)\nu_i(s) + R_a u_i^2(s)\big)ds\Big],\ 1 \leq i \leq N_a. \end{aligned} \tag{14}$$

The set of equations (13)-(14) constitute the standard stochastic LQG problem for a minor acquirer trader. It can be seen that a generic minor acquirer trader interacts with the major agent's trading rate as well as the market's average trading rate through it dynamics, and with the market's average buying rate through its cost function.

B. Mean Field Evolution

Following the LQG MFG methodology [19], the mean field, $\bar{x}$, is defined as the L^2 limit, when it exists, of the average of minor agents' states when population size goes to infinity

$$\bar{x}(t) = \lim_{N\to\infty} x^N(t) = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^N x_i(t),\ a.s.$$

Now, if the control strategy for each minor agent is considered to have the general feedback form

$$u_i = L_1 x_i + L_2 x_0 + \sum_{j\neq i, j=1}^N L_4 x_j + L_3, \quad 1 \leq i \leq N, \tag{15}$$

then the mean field dynamics can be obtained by substituting (15) in the minor liquidator (respectively, acquirer) agents' dynamics (9) (respectively, (13)), and taking the average and then its L^2 limit as $N \to \infty$.

The set of mean field equations for the optimal execution problem can be written as

$$d\bar{x} = \bar{A}\bar{x}dt + \bar{G}x_0 dt + \bar{m}dt, \tag{16}$$

where $\bar{x} = [\bar{x}_a^T, \bar{x}_l^T]^T$ consists of the mean field $\bar{x}_l$ of the liquidator population, and the mean field $\bar{x}_a$ of the acquirer population. The matrices in (16) are defined as

$$\bar{A} = \begin{bmatrix} \bar{A}_a & \bar{A}_{al} \\ \bar{A}_{la} & \bar{A}_l \end{bmatrix}, \bar{G} = \begin{bmatrix} \bar{G}_a \\ \bar{G}_l \end{bmatrix}, \bar{m} = \begin{bmatrix} \bar{m}_a \\ \bar{m}_l \end{bmatrix}, \tag{17}$$

which shall be determined from consistency equations (30) in section IV-C.

C. Infinite Populations

Following the mean field game methodology with a major agent [20], the optimal execution problem is first solved in the infinite population case where the average term in the finite population dynamics and cost function of each agent is replaced with its infinite population limit, i.e. the mean field. Then specializing to MFG linear systems [19], the major agent's state is extended with the mean field, while the minor agent's state is extended with the mean field and the major agent's state; this yields LQG problems for each trader linked only through the mean field and the major agent's state. Finally the infinite population best response strategies are applied to the finite population system which yields an ϵ-Nash equilibria (see Theorem 4.1).

In this paper we address the optimal execution problem in the MFG framework when the traders have, first, complete observations and, second, partial observations of their state and the major trader's state in Sections IV and V, respectively.

IV. Completely Observed Optimal Execution Problems

In the completely observed (CO) optimal execution problem it is assumed that the major trader completely observes its own state, and each generic minor trader completely observes its own state and the major trader's state. The best response MFG trading strategies which are obtained later in this section yield an ϵ-Nash equilibria for the market by the following theorem.

Theorem 4.1 (ϵ-Nash Equilibria for CO MM-MF Systems): Subject to reasonable technical assupmtions (see [19]), the system equations (8), (9), (13) together with the mean field equations (30) generate the set of control laws $\mathcal{U}_{MF}^N \triangleq \{u_i^\circ; 0 \leq i \leq N\}$, $1 \leq N < \infty$, given by (25) and (28) such that

(i) All agent systems $\mathcal{A}_i,\ 0 \leq i \leq N$, are second order stable.

(ii) $\{\mathcal{U}_{MF}^N; 1 \leq N < \infty\}$ yields an ϵ-Nash equilibrium for all ϵ, i.e. for all $\epsilon > 0$, there exists $N(\epsilon)$ such that for all $N \geq N(\epsilon)$;

$$J_i^{s,N}(u_i^\circ, u_{-i}^\circ) - \epsilon \leq \inf_{u_i \in \mathcal{U}_{i,y}^N} J_i^{s,N}(u_i, u_{-i}^\circ) \leq J_i^{s,N}(u_i^\circ, u_{-i}^\circ).$$

After applying the mean field methodology to decouple the agents, the problem of obtaining the best response trading strategy is transformed to a stochastic indefinite LQ problem that is solved for using the following theorem which is a restriction to the constant matrix parameter case of the general result in [21].

Theorem 4.2 (Stochastic Indefinite LQ Problem): Let $T > 0$ be given. For any $(s, y) \in [0, T) \times \mathbb{R}^n$, consider the following linear system

$$dx = \big[Ax + Bu + b\big]dt + \big[Cx + Du + \sigma\big]dw, \quad (18)$$

where $t \in [s, T],\ x(s) = y$ and A, B, C, D, b, σ are matrix valued functions of suitable sizes, and $w(.)$ is a standard Wiener process. In addition, a quadratic cost function is given

$$J(s, y, u(.)) = \mathbb{E}\{\frac{1}{2}\int_0^T \big[\langle Px(t), x(t)\rangle + \langle Nx(t), u(t)\rangle + \langle Ru(t), u(t)\rangle\big]dt + \frac{1}{2}\langle \bar{P}x(T), x(T)\rangle\} \quad (19)$$

where P, N and R are $\mathcal{S}^n$, $\mathbb{R}^{m\times n}$ and $\mathcal{S}^m$-valued functions, respectively, and $G \in \mathcal{S}^n$. Let $\Pi(.) \in C([s, T]; \mathcal{S}^n)$ be the solution of the Riccati equation

$$\dot{\Pi} + \Pi A + A^T\Pi + C^TPC + P - (B^T\Pi + N + D^T\Pi C)^T \times (R + D^T\Pi D)^{-1}(B^T\Pi + N + D^T\Pi C) = 0,\ a.e.t \in [s, t]$$
$$\Pi(T) = \bar{P}, \quad (20)$$

where $R + D^TPD > 0,\ a.e.t \in [s, T]$, and $s(.) \in C([s, T]; \mathbb{R}^n)$ be the solution of the offset equation

$$\dot{s} + [A - B(R + D^T\Pi D)^{-1}(B^TP + s + D^TPC)]^Ts + [C - D(R + D^T\Pi D)^{-1}(B^T\Pi + N + D^T\Pi C)]^T\Pi\sigma + \Pi b = 0,\ a.e.t \in [s, T], \quad s(T) = 0. \quad (21)$$

Let's define $\Psi \triangleq (R+D^T\Pi D)^{-1}[B^T\Pi+N+D^T\Pi C]$, and $\psi \triangleq (R+D^T\Pi D)^{-1}[B^Ts+D^T\Pi\sigma]$. Then the stochastic LQ problem (18)-(19) is solvable at s with the optimal control $u^\circ(.)$ being of a state feedback form

$$u^\circ(t) = -\Psi(t)x(t) - \psi(t),\ t \in [s, T]. \quad (22)$$

□

Henceforth we discuss the stochastic optimal control problem for the major trader, and a generic minor trader.

A. Major Liquidator Agent

The dynamics for the major agent's extended state $x_0^{ex} = [x_0^T, \bar{x}^T]^T$ in the infinite population is given by

$$\begin{bmatrix} dx_0 \\ d\bar{x} \end{bmatrix} = \begin{bmatrix} A_0 & [E_0, E_0] \\ \bar{G} & \bar{A} \end{bmatrix}\begin{bmatrix} x_0 \\ \bar{x} \end{bmatrix}dt + \begin{bmatrix} 0_{3\times 1} \\ \bar{m} \end{bmatrix}dt + \begin{bmatrix} B_0 \\ 0_{6\times 1} \end{bmatrix}u_0(t)dt + \begin{bmatrix} D_0 & 0_{3\times 6} \\ 0_{6\times 3} & 0_{6\times 6} \end{bmatrix}\begin{bmatrix} dw_0 \\ 0 \end{bmatrix} \quad (23)$$

where the average state in dynamics (8) was replaced by its L^2 limit, i.e. the mean field. Accordingly, the following matrices are defined

$$\mathbb{A}_0 = \begin{bmatrix} A_0 & [E_0, E_0] \\ \bar{G} & \bar{A} \end{bmatrix}, \quad \mathbb{M}_0 = \begin{bmatrix} 0_{3\times 1} \\ \bar{m} \end{bmatrix},$$
$$\mathbb{B}_0 = \begin{bmatrix} B_0 \\ 0_{6\times 1} \end{bmatrix}, \quad \mathbb{D}_0 = \begin{bmatrix} D_0 & 0_{3\times 6} \\ 0_{6\times 3} & 0_{6\times 6} \end{bmatrix}.$$

The performance function of the major trader (5) in terms of its extended state x_0^{ex} is given as

$$J_0 = \mathbb{E}\Big[\|x_0^{ex}(T)\|^2_{\bar{\mathbb{P}}_0} + \int_0^T\big(\|x_0^{ex}(s)\|^2_{\mathbb{P}_0} + \|u_0(s)\|^2_{R_0}\big)ds\Big]. \quad (24)$$

But specializing to the specific form (5) gives the weight matrices

$$\mathbb{P}_0 = \begin{bmatrix} P_0 & H_0 \\ H_0^T & H_0^TH_0 \end{bmatrix}, \ \bar{\mathbb{P}}_0 = \begin{bmatrix} \bar{P}_0 & \bar{H}_0 \\ \bar{H}_0^T & \bar{H}_0^T\bar{H}_0 \end{bmatrix}, \ R_0 > 0,$$

where P_0, $\bar{P}_0$ are matrix coefficients associated with the major agent's state in the running and final costs, respectively,

$$\bar{P}_0 = \begin{bmatrix} \beta & \frac{1}{2}pa_0 & 0 \\ \frac{1}{2}pa_0 & p\alpha & -\frac{1}{2}p \\ 0 & -\frac{1}{2}p & \epsilon \end{bmatrix}, P_0 = \begin{bmatrix} \theta & 0 & \frac{1}{2}r \\ 0 & \phi & 0 \\ \frac{1}{2}r & 0 & \delta \end{bmatrix},$$

and H_0, $\bar{H}_0$ are those associated with the mean field,

$$\bar{H}_0 = \begin{bmatrix} -\rho\beta & 0_{1\times 2} & -\rho\beta & 0_{1\times 2} \\ 0_{2\times 1} & 0_{2\times 2} & 0_{2\times 1} & 0_{2\times 2} \end{bmatrix},$$
$$H_0 = \begin{bmatrix} -\rho\theta & 0_{1\times 2} & -\rho\theta & 0_{1\times 2} \\ 0_{2\times 1} & 0_{2\times 2} & 0_{2\times 1} & 0_{2\times 2} \end{bmatrix}.$$

Consequently, using *Theorem 4.2*, the infinite population best response control is given by

$$u_0^\circ(t) = -R_0^{-1}\mathbb{B}_0^T\big[\Pi_0\big(x_0^T, \bar{x}^T\big)^T + s_0\big], \quad (25)$$

where Π_0 and s_0 are calculated via

$$-\dot{\Pi}_0 = \Pi_0\mathbb{A}_0 + \mathbb{A}_0^T\Pi_0 - \Pi_0\mathbb{B}_0R_0^{-1}\mathbb{B}_0^T\Pi_0 + \mathbb{P}_0,\ \Pi_0(T) = \bar{\mathbb{P}}_0$$
$$-\dot{s}_0 = (\mathbb{A}_0 - \mathbb{B}_0R_0^{-1}\mathbb{B}_0^T\Pi_0)^Ts_0 + \Pi_0\mathbb{M}_0,\ s_0(T) = 0.$$

B. Minor Acquirer/Liquidator Agents

For brevity, the notation $(.)_{a/l}$ is used in the rest of this paper to denote the matrices and parameters corresponding to a generic acquirer or a liquidator agent, respectively. Accordingly, a generic minor (acquirer/liquidator) agent $\mathcal{A}_i$'s extended dynamics with the extended state $x_i^{ex} = [x_i^T, x_0^T, \bar{x}^T]^T$ is

$$\begin{bmatrix} dx_i \\ dx_0 \\ d\bar{x} \end{bmatrix} = \begin{bmatrix} A_{a/l} & [G_{a/l}, E_{a/l}, E_{a/l}] \\ 0_{9\times 3} & \mathbb{A}_0 \end{bmatrix}\begin{bmatrix} x_i \\ x_0 \\ \bar{x} \end{bmatrix}dt + \begin{bmatrix} 0_{3\times 1} \\ \mathbb{M}_0 \end{bmatrix}dt + \begin{bmatrix} 0_{3\times 1} \\ \mathbb{B}_0 \end{bmatrix}u_0(t)dt + \begin{bmatrix} B_{a/l} \\ 0_{6\times 1} \end{bmatrix}u_i(t)dt + \begin{bmatrix} D_{a/l} & 0_{3\times 9} \\ 0_{9\times 3} & \mathbb{D}_0 \end{bmatrix}\begin{bmatrix} dw_i \\ dw_0 \\ 0 \end{bmatrix}. \quad (26)$$

Substituting the major agent's control action (25) into (26), we define

$$\mathbb{A}_{a/l} = \begin{bmatrix} A_{a/l} & [G_{a/l}, E_{a/l}, E_{a/l}] \\ 0_{9\times 3} & \mathbb{A}_0 - \mathbb{B}_0R_0^{-1}\mathbb{B}_0^T\Pi_0 \end{bmatrix}, \ \mathbb{M}_{a/l} = \begin{bmatrix} 0_{3\times 1}, \\ \mathbb{M}_0 \end{bmatrix},$$
$$\mathbb{B}_{a/l} = \begin{bmatrix} B_{a/l} \\ 0_{9\times 1} \end{bmatrix}, \quad \mathbb{D}_{a/l} = \begin{bmatrix} D_{a/l} & 0_{3\times 9} \\ 0_{9\times 3} & \mathbb{D}_0 \end{bmatrix}.$$

The cost function ((6) or (7)) for a generic (liquidator or acquirer) minor trader in terms of its extended state x_i^{ex} is

$$J_i = \mathbb{E}\Big[\|x_i^{ex}(T)\|^2_{\bar{\mathbb{P}}_{a/l}} + \int_0^T \big(\|x_i^{ex}(s)\|^2_{\mathbb{P}_{a/l}} + \|u_i(s)\|^2_{R_{a/l}}\big) ds\Big]. \tag{27}$$

Again specializing to the specific forms (6) and (7) yields

$$\mathbb{P}_{a/l} = \begin{bmatrix} P_{a/l} & 0_{3\times3} & H_{a/l} \\ 0_{3\times3} & 0_{3\times3} & 0_{3\times6} \\ H_{a/l}^T & 0_{6\times3} & H_{a/l}^T H_{a/l} \end{bmatrix},$$

$$\bar{\mathbb{P}}_{a/l} = \begin{bmatrix} \bar{P}_{a/l} & 0_{3\times3} & \bar{H}_{a/l} \\ 0_{3\times3} & 0_{3\times3} & 0_{3\times6} \\ \bar{H}_{a/l}^T & 0_{6\times3} & \bar{H}_{a/l}^T \bar{H}_{a/l} \end{bmatrix}, \ R_{a/l} > 0,$$

where for a minor acquirer trader the matrix coefficients corresponding to its states $\bar{P}_a, P_a$ and the mean field $\bar{H}_a, H_a$ are, respectively,

$$\bar{P}_a = \begin{bmatrix} \mu_a & -\frac{1}{2}p_a a & 0 \\ -\frac{1}{2}p_a a & p_a\psi_a & \frac{1}{2}p_a \\ 0 & \frac{1}{2}p_a & \xi_a \end{bmatrix}, P_a = \begin{bmatrix} \varrho_a & 0 & -\frac{1}{2}r_a \\ 0 & \kappa_a & 0 \\ -\frac{1}{2}r_a & 0 & \gamma_a \end{bmatrix},$$

$$\bar{H}_a = \begin{bmatrix} \begin{bmatrix} -\rho_a\mu_a & 0_{1\times2} \\ 0_{2\times1} & 0_{2\times2} \end{bmatrix} & \begin{bmatrix} 0_{3\times3} \end{bmatrix} \end{bmatrix},$$

$$H_a = \begin{bmatrix} \begin{bmatrix} -\rho_a\varrho_a & 0_{1\times2} \\ 0_{2\times1} & 0_{2\times2} \end{bmatrix} & \begin{bmatrix} 0_{3\times3} \end{bmatrix} \end{bmatrix},$$

and similarly those matrices for a minor liquidator trader are

$$\bar{P}_l = \begin{bmatrix} \mu_l & \frac{1}{2}p_l a & 0 \\ \frac{1}{2}p_l a & p_l\psi_l & -\frac{1}{2}p_l \\ 0 & -\frac{1}{2}p_l & \xi_l \end{bmatrix}, P_l = \begin{bmatrix} \varrho_l & 0 & \frac{1}{2}r_l \\ 0 & \kappa_l & 0 \\ \frac{1}{2}r_l & 0 & \gamma_l \end{bmatrix},$$

$$\bar{H}_l = \begin{bmatrix} \begin{bmatrix} 0_{3\times3} \end{bmatrix} & \begin{bmatrix} -\rho_l\mu_l & 0_{1\times2} \\ 0_{2\times1} & 0_{2\times2} \end{bmatrix} \end{bmatrix},$$

$$H_l = \begin{bmatrix} \begin{bmatrix} 0_{3\times3} \end{bmatrix} & \begin{bmatrix} -\rho_l\varrho_l & 0_{1\times2} \\ 0_{2\times1} & 0_{2\times2} \end{bmatrix} \end{bmatrix}.$$

We utilize *Theorem 4.2* again to obtain the best response control for a generic minor agent as

$$u_i^\circ(t) = -R_{a/l}^{-1}\mathbb{B}_{a/l}^T\big[\Pi_{a/l}\big(x_i^T, x_0^T, \bar{x}^T\big)^T + s_{a/l}\big], \tag{28}$$

where $\Pi_{a/l}$ and $s_{a/l}$ is given by

$$-\dot{\Pi}_{a/l} = \Pi_L\mathbb{A}_{a/l} + \mathbb{A}_{a/l}^T\Pi_{a/l} - \Pi_{a/l}\mathbb{B}_{a/l}R_{a/l}^{-1}\mathbb{B}_{a/l}^T\Pi_{a/l} + \mathbb{P}_{a/l},$$
$$-\dot{s}_{a/l} = (\mathbb{A}_{a/l} - \mathbb{B}_{a/l}R_{a/l}^{-1}\mathbb{B}_{a/l}^T\Pi_{a/l})^T s_{a/l} + \Pi_{a/l}\mathbb{M}_{a/l},$$

with $\Pi_{a/l}(T) = \bar{\mathbb{P}}_{a/l}$ and $s_{a/l}(T) = 0$.

C. Consistency Conditions

The closed loop trading dynamics of a generic minor agent $\mathcal{A}_i,\ 1 \le i \le N_{a/l}$ applying (28) is consequently

$$d\nu_i = -R_{a/l}^{-1}\mathbb{B}_{a/l}^T\Pi_{a/l}\big(x_i^T, x_0^T, \bar{x}^T\big)^T dt - R_{a/l}^{-1}\mathbb{B}_{a/l}^T s_{a/l}(t)dt,$$

then the average of closed loop trading dynamics over acquirer or liquidator population is obtained as

$$\frac{1}{N_{a/l}}\sum_{i=1}^{N_{a/l}} d\nu_i = -\frac{1}{N_{a/l}}\sum_{i=1}^{N_{a/l}} R_{a/l}^{-1}\mathbb{B}_{a/l}^T\Pi_{a/l}\big(x_i^T, x_0^T, \bar{x}^T\big)^T dt - \frac{1}{N_{a/l}}\sum_{i=1}^{N_{a/l}} R_{a/l}^{-1}\mathbb{B}_{a/l}^T s_{a/l}(t)dt, \tag{29}$$

where $\bar{x} = [\bar{x}_a^T, \bar{x}_l^T]^T$. Then taking the L^2 limit of (29) as the population size $N_{a/l}$ goes to infinity yields the trading rate mean field dynamics

$$d\bar{\nu}_{a/l} = \lim_{N_{a/l}\to\infty} d\nu^{N_{a/l}} = -R_{a/l}^{-1}\mathbb{B}_{a/l}^T\Pi_{a/l} \times \lim_{N_{a/l}\to\infty}\big((x^{N_{a/l}})^T, x_0^T, \bar{x}_a^T, \bar{x}_l^T\big)^T dt - R_{a/l}^{-1}\mathbb{B}_{a/l}^T s_{a/l}dt,$$

and hence the consistency equations become

$$\begin{aligned}
\bar{A}_{a,11} &= -R_a^{-1}(\Pi_{a,11} + \Pi_{a,17}) - aR_a^{-1}(\Pi_{a,31} + \Pi_{a,37}),\\
\bar{A}_{a,12} &= -R_a^{-1}(\Pi_{a,12} + \Pi_{a,18}) - aR_a^{-1}(\Pi_{a,32} + \Pi_{a,38}),\\
\bar{A}_{a,13} &= -R_a^{-1}(\Pi_{a,13} + \Pi_{a,19}) - aR_a^{-1}(\Pi_{a,33} + \Pi_{a,39}),\\
\bar{A}_{al,11} &= -R_a^{-1}(\Pi_{a,110} + a\Pi_{a,310}),\\
\bar{A}_{al,12} &= -R_a^{-1}(\Pi_{a,111} + a\Pi_{a,311})\\
\bar{A}_{al,13} &= -R_a^{-1}\Pi_{a,112} - aR_a^{-1}\Pi_{a,312}\\
\bar{G}_{a/l,11} &= -R_{a/l}^{-1}(\Pi_{a/l,14} + a\Pi_{a/l,34}),\\
\bar{G}_{a/l,12} &= -R_{a/l}^{-1}(\Pi_{a/l,15} + a\Pi_{a/l,35}),\\
\bar{G}_{a/l,13} &= -R_{a/l}^{-1}(\Pi_{a/l,16} + a\Pi_{a/l,36}),\\
\bar{m}_{a/l,1} &= -R_{a/l}^{-1}\mathbb{B}_{a/l}^T s_{a/l},
\end{aligned} \tag{30}$$

where $\Pi_{a/l,ij} = \Pi_{a/l}(i,j)$ for $i = \{1,3\}$, $j = \{1,2,3,...,12\}$. Hence the matrices in (17) are given as

$$\bar{A}_{a/l} = \begin{bmatrix} \bar{A}_{a/l,11} & \bar{A}_{a/l,12} & \bar{A}_{a/l,13} \\ 1 & 0 & 0 \\ (\lambda + a\bar{A}_{a/l,11}) & a\bar{A}_{a/l,12} & a\bar{A}_{a/l,13} \end{bmatrix},$$

$$\bar{A}_{al} = \begin{bmatrix} \bar{A}_{al,11} & \bar{A}_{al,12} & \bar{A}_{al,13} \\ 0 & 0 & 0 \\ a\bar{A}_{al,11} & a\bar{A}_{al,12} & a\bar{A}_{al,13} \end{bmatrix}, \ \bar{m} = \begin{bmatrix} \bar{m}_{a/l,1} \\ 0 \\ a\bar{m}_{a/l,1} \end{bmatrix}$$

$$\bar{A}_{la} = \begin{bmatrix} \bar{A}_{la,11} & \bar{A}_{la,12} & \bar{A}_{la,13} \\ 0 & 0 & 0 \\ a\bar{A}_{la,11} & a\bar{A}_{la,12} & a\bar{A}_{la,13} \end{bmatrix},$$

$$\bar{G}_{a/l} = \begin{bmatrix} \bar{G}_{a/l,12} & \bar{G}_{a/l,22} & \bar{G}_{a/l,23} \\ 0 & 0 & 0 \\ (\lambda_0 + a\bar{G}_{a/l,21}) & a\bar{G}_{a/l,22} & a\bar{G}_{a/l,23} \end{bmatrix}.$$

where the equations for the entries of matrices $\bar{A}_{la}, \bar{A}_l$ due to space limitation are skipped (see [22]).

V. Partially Observed Optimal Execution Problems

We now follow the general development in [7] for PO MM LQG MFG systems where the major agent has only partial observations on its own states.

A. Major Liquidator Agent

Let the major agent's observation process be

$$dy_0 = \mathbb{H}_0[x_0^T, \bar{x}^T]^T dt + \sigma_{v_0} dv_0 \tag{31}$$

where $\mathbb{H}_0$ is a constant matrix with appropriate dimension. Then the corresponding Kalman filter equation to generate the estimates of the major agent's states is given by

$$d\hat{X}_{0|\mathcal{F}_0^y} = \mathbb{A}_0\hat{X}_{0|\mathcal{F}_0^y}dt+\mathbb{M}_0dt + \mathbb{B}_0\hat{u}_{0|\mathcal{F}_0^y}dt + K_0(t)[dy_0 - \mathbb{H}_0\hat{\mathcal{X}}_{0|\mathcal{F}_0^y}dt] \quad (32)$$

with the filter gain given by

$$K_0(t) = V_0(t)\mathbb{H}_0^T R_{v_0}^{-1}, \quad (33)$$

where $R_{v_0} = \sigma_{v_0}\sigma_{v_0}^T$. The associated Riccatti equation is

$$\dot{V}_0(t)=\mathbb{A}_0V_0(t)+V_0(t)\mathbb{A}_0^T-K_0(t)R_{v_0}K_0(t)^T+Q_{w_0}. \quad (34)$$

Assumption: $[\mathbb{A}_0, Q_{w_0}]$ is controllable and $[\mathbb{H}_0, \mathbb{A}_0]$ is observable.
Following the methodology in [7] the cost function (24) can be decomposed

$$J_0 = \mathbb{E}\Big[\|\hat{x}_{0|\mathcal{F}_0^y}(T)\|^2_{\bar{P}_0} + \int_0^T\!\big(\|\hat{x}_{0|\mathcal{F}_0^y}(s)\|^2_{P_0} + \|u_0(s)\|^2_{R_0}\big)ds + \|x_0(T)-\hat{x}_{0|\mathcal{F}_i^y}(T)\|^2_{\bar{P}_0}+\int_0^T \big(\|x_0(s)-\hat{x}_{0|\mathcal{F}_0^y}(s)\|^2_{P_0}\big)ds\Big],$$

and thence employing the Separation Principle of LQG stochastic control the corresponding infinite population best response control action is given by

$$\hat{u}_0^\circ = -R_0^{-1}\mathbb{B}_0^T\big[\Pi_0\big(\hat{x}^T_{0|\mathcal{F}_0^y}, \hat{\bar{x}}^T_{|\mathcal{F}_0^y}\big)^T + s_0\big] \quad (35)$$

B. Minor (Acquirer/Liquidator) Agent

The extended state shall be denoted by

$$X_i = [x_i^T, x_0^T, \bar{x}^T, \hat{x}^T_{0|\mathcal{F}_0^y}, \hat{\bar{x}}^T_{|\mathcal{F}_0^y}]^T. \quad (36)$$

Let the minor agent's observation process be given by

$$dy_i(t) = \mathbb{H}_{a/l}[x_i^T, x_0^T, \bar{x}^T, \hat{x}^T_{0|\mathcal{F}_0^y}, \hat{\bar{x}}^T_{|\mathcal{F}_0^y}]^T dt + \sigma_{v_i}dv_i \quad (37)$$

with $\mathbb{H}_{a/l}$ constant matrix. Then the extended dynamics of minor agent is given by

$$\begin{bmatrix} dx_i \\ dx_0 \\ d\bar{x} \\ d\hat{x}_{0|\mathcal{F}_0^y} \\ d\hat{\bar{x}}_{|\mathcal{F}_0^y}\end{bmatrix} = \begin{bmatrix} A_{a/l} & [G_{a/l}, E_{a/l}, E_{a/l}] & 0_{3\times 9} \\ 0_{9\times 3} & \mathbb{A}_0 & -\mathbb{B}_0R_0^{-1}\mathbb{B}_0^T\Pi_0 \\ 0_{9\times 3} & K_0\mathbb{H}_0 & \mathbb{A}_0 - \mathbb{B}_0R_0^{-1}\mathbb{B}_0^T\Pi_0 - K_0\mathbb{H}_0 \end{bmatrix} \times \begin{bmatrix} x_i \\ x_0 \\ \bar{x} \\ \hat{x}_{0|\mathcal{F}_0^y} \\ \hat{\bar{x}}_{|\mathcal{F}_0^y}\end{bmatrix} dt + \begin{bmatrix} 0_{3\times 1} \\ \mathbb{M}_0 \\ \mathbb{M}_0 \end{bmatrix} dt + \begin{bmatrix} \mathbb{B}_{a/l} \\ 0_{6\times 1}\end{bmatrix} u_i dt + \begin{bmatrix} \mathbb{D}_{a/l} & 0_{12\times r_v} \\ 0_{9\times 12} & K_0 \end{bmatrix}\begin{bmatrix} dw_i \\ dw_0 \\ 0 \\ dv_0\end{bmatrix}$$

or equivalently

$$dX_i=(\mathcal{A}_{a/l}X_i+\mathcal{M}_{a/l}+\mathcal{B}_{a/l}u_i)dt+\Sigma_{a/l}\big[dw_i^T, dw_0^T, 0, dv_0^T\big]^T$$

Then the Kalman filter which generates the estimates of the minor (liquidator/acquirer) agent's states is

$$d\hat{X}_{i|\mathcal{F}_i^y} = \mathcal{A}_{a/l}\hat{X}_{i|\mathcal{F}_i^y}dt + \mathcal{M}_{a/l}dt + \mathcal{B}_{a/l}\hat{u}_{i|\mathcal{F}_i^y}dt + K_{a/l}(t)\big[dy_i - \mathbb{H}_{a/l}\hat{X}_{i|\mathcal{F}_i^y}dt\big] \quad (38)$$

where the filter gain is given as

$$K_{a/l}(t) = V_{a/l}(t)\mathbb{H}^T_{a/l}R_{v_i}^{-1}, \quad (39)$$

with $R_{v_i} = \sigma_{v_i}\sigma_{v_i}^T$.
Assumption: $[\mathbb{A}_{a/l}, Q_w]$ is controllable (respectively stabilizable) and $[\mathbb{H}_{a/l}, \mathbb{A}]$ is observable (respectively detectable). The corresponding Riccati equation is

$$\dot{V}_{a/l}(t)=\mathcal{A}_{a/l}V_{a/l}(t)+V_{a/l}(t)\mathcal{A}^T_{a/l}-K_{a/l}(t)R_vK_{a/l}(t)^T+Q_w. \quad (40)$$

The same procedure as in [7] can be used to decompose the cost function (27), employing the Separation Principle, as

$$J_i = \mathbb{E}\Big[\|\hat{x}_{i|\mathcal{F}_i^y}(T)\|^2_{\bar{P}_{a/l}}+\int_0^T\!\big(\|\hat{x}_{i|\mathcal{F}_i^y}(s)\|^2_{P_{a/l}}+\|u_i(s)\|^2_{R_{a/l}}\big)ds + \|x_i(T)-\hat{x}_{i|\mathcal{F}_i^y(T)}\|^2_{\bar{P}_{a/l}}+\int_0^T \|x_i(s)-\hat{x}_{i|\mathcal{F}_i^y}(s)\|^2_{P_{a/l}}ds\Big].$$

So employing the Separation Principle the corresponding infinite population best response control for a generic minor trader is seen to be

$$\hat{u}_i^\circ = -R_{a/l}^{-1}\mathbb{B}_{a/l}^T\big[\Pi_{a/l}\big(\hat{x}^T_{i|\mathcal{F}_i^y}, \hat{x}^T_{0|\mathcal{F}_i^y}, \hat{\bar{x}}^T_{|\mathcal{F}_i^y}\big)^T + s_{a/l}\big]. \quad (41)$$

Theorem 5.1 (ϵ-Nash Equilibria for PO MM-MF Systems): [7] Subject to reasonable technical assumptions (see [7]), the KF-MF state estimation scheme (32)-(34) and (38)-(40) together with the MM-MFG equation scheme (30) generate the set of control laws $\hat{\mathcal{U}}^N_{MF} \triangleq \{\hat{u}_i^\circ; 0 \le i \le N\}$, $1 \le N < \infty$, given by (35) and (41) such that

(i) All agent systems $\mathcal{A}_i$, $0 \le i \le N$, are second order stable.
(ii) $\{\hat{\mathcal{U}}^N_{MF}; 1 \le N < \infty\}$ yields an ϵ-Nash equilibrium for all ϵ, i.e. for all $\epsilon > 0$, there exists $N(\epsilon)$ such that for all $N \ge N(\epsilon)$;

$$J_i^{s,N}(\hat{u}_i^\circ, \hat{u}_{-i}^\circ)-\epsilon \le \inf_{u_i\in\mathcal{U}^N_{i,y}} J_i^{s,N}(u_i, \hat{u}_{-i}^\circ) \le J_i^{s,N}(\hat{u}_i^\circ, \hat{u}_{-i}^\circ).$$

□

VI. Simulations

In the numerical experiments it is assumed that the trading action takes place within $T = 100$ seconds. The temporary impact strength of the major agent's trading and a generic minor agent's trading on the market are $a_0 = a = 5.43\times 10^{-6}$, while their permanent impact strengths are taken to be $\lambda_0 = \lambda = 2\times 10^{-8}$. The diffusion coefficients in trading dynamics are selected as $\sigma_0^Q = 0.05$, and $\sigma_i^Q = 0.02$. The weights in the cost function for the major trader are: $\alpha = 5a_0 \times 10^5$, $\phi = 10^{-6}a_0$, $\delta = 1/(2a0)$, $\epsilon = 1/(2\alpha)$, $\theta = 1/(2\delta)$, $\beta = 10$; and those of a generic minor (liquidator/acquirer) trader are: $\psi_L = \psi_A = 5a \times 10^5$, $\kappa_L = \kappa_A = 10^{-1}a$, $\xi_L = \xi_A = 1/(2\psi)$, $\gamma_L = \gamma_A = 1/(2a)$, $\varrho_L = \varrho_A = 1/(2\gamma)$, $\mu = 10$. Furthermore, the market volatility is $\sigma = 0.6565$, the initial asset price is taken to be $F_0(0) = F_i(0) = \$35$, and the initial inventory stock of the major trader to be liquidated is set to $Q_0(0) = 5 \times 10^6$ while the minor liquidator HFT aims to sell $Q_i(0) = 5000$ shares and the acquirer HFT

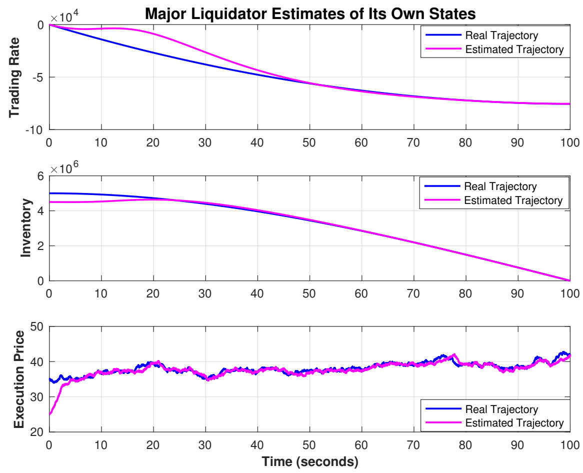

Fig. 1. Major Agent's State Trajectories and Major Agent's Estimates of its Own States in the Partial Observation Case.

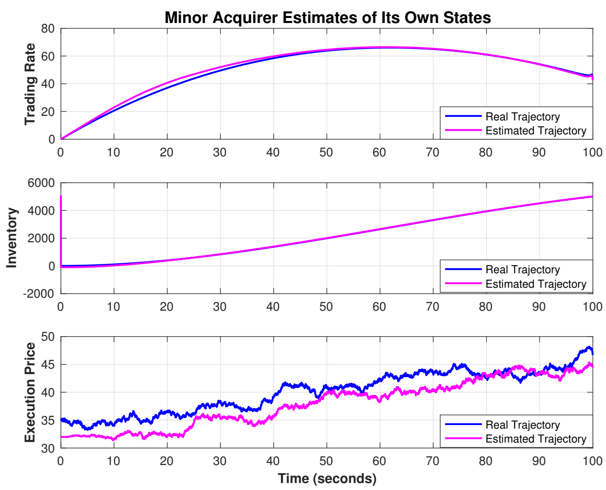

Fig. 2. A Generic Minor Acquirer Agent's State Trajectories and Minor Agent's Estimates of its Own States in the Partial Observation Case

wants to buy $Q_i(0) = 5000$ shares. In the estimation part, the measurement noise standard deviation for the major trader is $\sigma_0 = 0.05$, and for the HFT is $\sigma = 0.5$.

The resulting ϵ-Nash equilibrium trajectories of the major agent and generic liquidator and acquirer HFTs for the partial observation case together with those of complete observation case are depicted in Fig. 1-3.

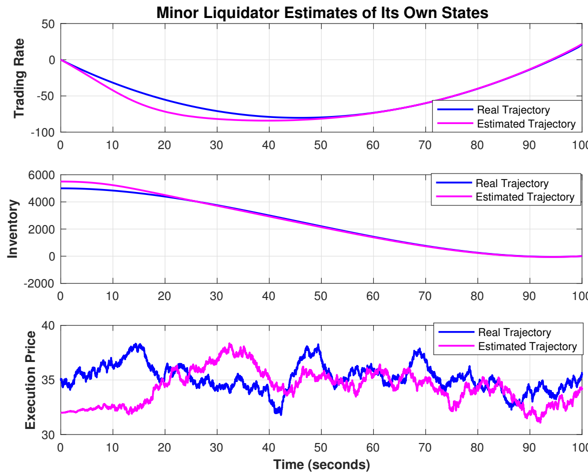

Fig. 3. A Generic Minor Liquidator Agent's State Trajectories and Minor Agent's Estimates of its Own States in the Partial Observation Case

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