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Mean field games*

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Abstract. We survey here some recent studies concerning what we call mean-field models by analogy with Statistical Mechanics and Physics. More precisely, we present three examples of our mean-field approach to modelling in Economics and Finance (or other related subjects...). Roughly speaking, we are concerned with situations that involve a very large number of "rational players" with a limited information (or visibility) on the "game". Each player chooses his optimal strategy in view of the global (or macroscopic) informations that are available to him and that result from the actions of all players. In the three examples we mention here, we derive a mean-field problem which consists in nonlinear differential equations. These equations are of a new type and our main goal here is to study them and establish their links with various fields of Analysis. We show in particular that these nonlinear problems are essentially well-posed problems i.e., have unique solutions. In addition, we give various limiting cases, examples and possible extensions. And we mention many open problems.

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Contents

1. Introduction

1.1. General introduction

We present here some recent modelling issues arising in Economics and Finance which lead to new classes of nonlinear equations that we also briefly analyse here.

In a recent series of papers (J.-M. Lasry and P.-L. Lions [16,18,19,17,20, 21]), we introduce a general mathematical modelling approach to situations which involve a great number of "agents". Roughly speaking, we derive these models from a "continuum limit" (in other words letting the number of agents go to infinity) which is somewhat reminiscent of the classical mean field approaches in Statistical Mechanics and Physics (as for instance, the derivation of Boltzmann or Vlasov equations in the kinetic theory of gases) or in Quantum Mechanics and Quantum Chemistry (density functional models, Hartree or Hartree–Fock type models…). This general approach leads in various situations to new nonlinear equations which contain as particular examples many classical problems and are linked to several research fields of Analysis. We describe rapidly these equations in the next section. And we conclude this introduction by a brief overview of the economic and/or financial issues that we address through our "mean-field" approach.

We consider here three different illustrations of such an approach that are treated in sections 2–4. In section 2, we consider stochastic differential games and N players Nash points. Then, we derive rigorously the mean field limit equations as N goes to infinity (in a stationary setting). And we analyse mathematically the limit equations. We also consider time-dependent problems and deterministic limits. We next give an interpretation of such systems of equations in terms of the optimal control of (some) partial differential equations. And we indicate various directions that can be (or need to be) investigated together with several open problems.

Section 3 is devoted to the second example which leads to a new class of free boundary problems. In the one-dimensional case, we state and solve this problem showing the existence and uniqueness of a smooth solution. We next discuss briefly and explicitly some stationary problems. And we mention various directions of interest.

The third example concerns the formation of volatility in financial markets. In this context, our mean field approach leads to a nonlinear differential equation in infinite dimensions. And we show that i) its local solvability is induced by a striking property of solutions of parabolic partial differential equations, ii) the model possesses remarkable invariance properties, iii) which allow us to solve the equation globally in a semi-explicit way.

1.2. Mathematical models

We discuss here the mathematical structure of the three examples mentioned in the previous section. We begin with the example which will be mostly detailed here namely "**mean-field differential games**". A typical example of the models we derive is given by the following system of equations

$$\begin{cases} -\nu\Delta u + H(x,\nabla u) + \lambda = V(x,m) \\ -\nu\Delta m - \operatorname{div}\Big(\dfrac{\partial H}{\partial p}(x,\nabla u)m\Big) = 0 \\ m > 0, \displaystyle\int m\,dx = 1 \end{cases} \tag{1}$$

where $\nu > 0$, u is a scalar function, $H(x,p)$ is a given function (or nonlinearity) which is assumed to be convex in p, $\lambda \in \mathbf{R}$ is unknown, and $V(x,r)$ is another given function (or nonlinearity). Precise assumptions on H and V are given in section 2 below. Various types of boundary conditions are possible and we mention here the simplest possible case where the equations in (1) are set in $\mathbf{R}^d, H(x,p), V(x,r)$ and the unknowns u and m are required to be periodic in x_i (of a given period $T_i > 0$) for each $1 \le i \le d (x \in \mathbf{R}^d)$, and $\int m\,dx$ obviously means $\int_Q m\,dx$ where $Q = \prod_{i=1}^{d}(0,T_i)$.

The above problem corresponds to stationary situations. The time-dependent analogue is typically of the following form (where $t \in [0,T]$)

$$(2) \qquad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + H(x, \nabla u) = V(x,m), u|_{t=0} = V_0(x, m(x,0)) \\ \dfrac{\partial m}{\partial t} + \Delta m + \operatorname{div}\Big(\dfrac{\partial H}{\partial p}(x, \nabla u) m\Big) = 0, m|_{t=T} = m_0 \\ m > 0, \displaystyle\int m\, dx = 1 \text{ for all } t \in [0,T]. \end{cases}$$

The nonlinearities H and V are as above, m_0 is a given "initial" condition while $V_0(x,r)$ is a given function (or nonlinearity).

If m were not present in the first equations of (1) and (2) (assume for instance that V and u_0 only depend on x), these equations would simply be a general class of Hamilton–Jacobi–Bellman equations arising in stochastic control theory (see for instance W.H. Fleming and H.M. Soner [12], M. Bardi and I. Capuzzo–Dolcetta [5] and the references therein ...). And the equations would then be seen as the linearized problem, backward in time in the case of (2). In general, the coupling between the first and the second equation in both systems (with the additional feature in the case of (2) of the equation in u written forward in time while the equation for m is backward in time) make these systems novel ones for which no existing theory or approach seems to be applicable directly. In addition, (1) and (2) contain several particular cases of interest such as the Hartree equations in Quantum Mechanics, or the compressible Euler equations (in the barotropic or isentropic regime) when we let ν go to 0_+.

For these novel systems, we prove general existence results together with uniqueness results at least when V (and V_0) are nondecreasing in r (for all x). We also investigate the limit as ν goes to 0_+ (deterministic limit) and show the links with optimal control problems (in which case u— or m— may be interpreted as the primal state while m—or u— is then the dual state).

The second example concerns the formation of prices. Again, we indicate here a typical example of the models we derive namely

$$(3) \qquad \begin{cases} \dfrac{\partial f}{\partial t} - \dfrac{\sigma^2}{2}\dfrac{\partial^2 f}{\partial x^2} = -\dfrac{\sigma^2}{2}\dfrac{\partial f}{\partial x}(p(t),t)\Big\{\delta(x-p(t)+a) - \delta(x-p(t)-a)\Big\} \\ f(x,t) > 0 \text{ if } x < p(t), t \geq 0; f(x,t) < 0 \text{ if } x > p(t), t \geq 0; \\ f|_{t=0} = f_0 \text{ on } \mathbf{R}, p(0) = p_0 \end{cases}$$

where $\sigma > 0, f_0$ is a given smooth function with fast decay at infinity, $p_0 \in \mathbf{R}$ and the following compatibility condition holds

$$(4) \qquad f_0(x) > 0 \text{ if } x < p_0, f_0(x) < 0 \text{ if } x > p_0.$$

Finally, $a > 0$ is given and δ is a "delta-like" function i.e., a smooth non negative function compactly supported in $(-a,+a)$ such that $\int \delta = 1$. The problem

(3) is clearly a free boundary problem (note that $f(p(t),t) = 0$ for all $t \geq 0$) which appears to be new. Our main results state the existence and uniqueness of a smooth solution (u,p) (with fast decay at infinity).

The final example we consider concerns the formation of volatility in financial markets. Postulating a simple linear elastic law for the impact of trading on stock prices, our mean field approach leads to the following nonlinear differential equation in an infinite dimensional space (which can be taken to be, for example, $C_b^{2,\alpha}(\mathbf{R})$ i.e., the space of bounded C^2 functions with bounded, Hölder continuous of exponent $\alpha \in (0,1)$, first and second derivatives): we look for a mapping σ from $C_b^{2,\alpha}(\mathbf{R})$ into $C_b^{0,\alpha}(\mathbf{R} \times [0,T])$ —the space of bounded, Hölder continuous of exponent $\alpha \in (0,1)$, functions on $\mathbf{R} \times [0,T]$, where $T > 0$ is fixed— which satisfies

$$\sigma'(\Phi) = k\sigma(\Phi)\,.\,\Gamma \text{ for all } \Phi \in C_b^{2,\alpha}(\mathbf{R}) \tag{5}$$

where $k > 0, \Gamma$ is the operator defined on $C_b^{2,\alpha}$ by $\Gamma\Psi = \frac{\partial^2 u}{\partial x^2}$ and u solves the following parabolic equation (written backward in time).

$$\frac{\partial u}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} = 0 \text{ on } \mathbf{R} \times (0,T), u|_{t=T} = \Psi\,. \tag{6}$$

Of course, σ in (6) stands for the function of x and t given by $\sigma(\Phi)$. And in (5) "$\sigma(\Phi)$." simply corresponds to the multiplication operator.

Such nonlinear differential equations as (5) cannot be solved in general unless some very specific compatibility condition (related to the symmetry of second differentials) is satisfied. And we show that this "symmetry" condition is indeed satisfied in our model thanks to a remarkable property of solutions of parabolic equations. This insures the fact that (5) is locally well-posed (in a maximal neighbourhood of any Φ_0, provided we specify $\sigma(\Phi_0)$). We also show a general invariance property which allows us to prove that (5) is globally well-posed and to construct its solutions in a semi-explicit way (via the solution of some nonlinear parabolic equation...).

1.3. The economic and financial context

In many economic and financial situations, it is natural to consider a very large number of rational agents which have limited information. Here, the word rational is taken from the theory of rational anticipations and, roughly speaking, means that each agent tries to maximize his strategy (utility maximization). In section 4 below, we consider a model for the formation of volatility in financial markets in an attempt to reconcile the classical Black–Scholes theory (see [8] and R. Merton [24]) with financial practice where the (implicit) volatility used for option pricing and hedging differs from the historical volatility (of, say, a stock for example). We do so by postulating (as in [15], [3]) an impact of trading on the price dynamics and considering, in a self-consistent (or mean-field) way, an infinite (continuum) number of traders.

Section 3 is devoted to a toy model for price formation in an idealized situation where two populations of, respectively, vendors and buyers of a single good typically agree to a certain price at which some transactions take place. Once more, we do not consider a single vendor or buyer but continua of them through their densities and the price is then determined by a dynamic equilibrium.

Finally, in section 2, we consider a general class of stochastic differential games for a large number N of players. The limit behaviour, as N goes to infinity, is intimately connected to the modelling of economic equilibrium with rational anticipations. One indeed postulates in such a context that each agent is rational (and assumes the other agents to be rational as well ...) but also has a "tiny" (infinitesimal) influence on the equilibrium. A fundamental contribution to this issue has been given by R. Aumann [1], and, since then, many works have investigated it (see the recent work by G. Carmona [9] and the references therein). We propose a different approach based upon stochastic control. Roughly speaking, each player maximizes the expectation of a criterion by choosing a strategy on the parameters of a stochastic evolution. This criterion depends on one hand on individual parameters as is customary in stochastic control and on the other hand on the (spatial) density of the other players. The consistency of this equilibrium, that we call a "mean field equilibrium", in the sense of rational anticipations is insured by the fact that the dynamics of the density of players results from the individual optimal strategies. Let us mention that the stochastic set-up we use for this dynamical equilibrium allows us to circumvent the use of approximations proposed in the abstract static set-up of the works mentioned above. We also emphasize the fact that the deduction of these mean-field equilibria is justified rigorously from the limit, as N goes to infinity, of Nash equilibria. And, as it is to be expected, we observe a significant simplification of the complexity of such N players equilibria thanks to that limit.

The proposed terminology of mean-field games is an explicit reference to statistical Mechanics and Physics and to the study of systems composed of a very large number of particles (where the dynamics of each particle is determined by a mean field created by the density of particles) and we shall see later on that this is more than a simple analogy although the main difference certainly lies in the possibility for each "player/particle" to choose its best strategy in our case. One could as well talk of a "micro-macro" approach of equilibrium, in which each ("microscopic") player behaves rationally with respect to his preferences and to global data or informations (of a "macroscopic" nature).

2. Mean field games

2.1. Stationary problems: Nash points or N players

We consider here the simplest possible case and we shall mention later on variants and extensions. We consider N players ($N \geq 1$) whose dynamics are given respectively by

$$dX_t^i = \sigma^i dW_t^i - \alpha^i dt, X_0^i = x^i \in \mathbf{R}^d, 1 \leq i \leq N \tag{7}$$

where $d \geq 1, \sigma^i > 0$ for all $i, (W_t^1, \dots, W_t^N)$ are N independent Brownian motions in $\mathbf{R}^d$ and α^i corresponds to the strategy (or the control) of player i that we take to be a bounded process for $t \geq 0$ and adapted to W_t^i. We discuss below this assumption which is natural when dealing with a large number of players but is rather restrictive from a game theoretical viewpoint.

In order to simplify the presentation, we assume that $X_t^i \in \mathbf{T}^d$ (identified to $Q = [0,1]^d$ with periodicity) and that all the functions given below are periodic in $x_j^i (1 \leq i \leq N, 1 \leq j \leq d)$ of period 1 (for instance). And we introduce a cost function for each $X = (x^1, \dots, x^N) \in Q^N$

$$J^i(\alpha^1, \dots, \alpha^N) = \liminf_{T \to +\infty} \frac{1}{T} E\left[\int_0^T L^i(X_t^i, \alpha_t^i) + F^i(X_t^1, \dots, X_t^N)\, dt\right] \tag{8}$$

where, for each i, F^i is Lipschitz on Q^N, L^i is Lipschitz in $x^i \in Q$ uniformly in α^i bounded and

$$\inf_{x^i} L^i(x^i, \alpha^i)/|\alpha^i| \to +\infty \text{ if } |\alpha^i| \to +\infty. \tag{9}$$

We then recall the definition of a Nash point for $X \in Q^N$ fixed : $(\bar{\alpha}^1, \dots, \bar{\alpha}^N)$ is a Nash point if

$$J^i(\bar{\alpha}^1, \dots, \bar{\alpha}^{i-1}, \alpha^i, \bar{\alpha}^{i+1}, \dots, \bar{\alpha}^N) \geq J^i(\bar{\alpha}^1, \dots, \bar{\alpha}^N), \forall \alpha^i, \forall i. \tag{10}$$

Finally, we denote by $H^i(x,p) = \sup_{\alpha \in \mathbf{R}^d} (p.\alpha - L^i(x,\alpha))$ for $x \in \mathbf{Q}, p \in \mathbf{R}^d$ and $\nu^i = \frac{1}{2}(\sigma^i)^2 (\forall 1 \leq i \leq d)$. And we shall assume, for the sake of simplicity, that H^i is of class C^1 in p (for all i,x).

The existence of Nash points (or Nash equilibria) may be shown under very general conditions on the data. We refer the reader to A. Bensoussan and J. Frehse [6,7] for the general connections existing between Nash points and partial differential equations. We only present one sample of possible existence results where we assume that the Hamiltonians H^i satisfy (for all $1 \leq i \leq N$)

$$\exists \theta \in (0,1), \inf_x \Big(\frac{\partial H^i}{\partial x} \cdot p + \frac{\theta}{d} \nu^i (H^i)^2 \Big) > 0 \text{ for } |p| \text{ large.} \tag{11}$$

Theorem 2.1. *Under the above conditions*
i) There exist $\lambda_1,\dots,\lambda_N \in \mathbf{R}, v_1,\dots,v_N \in C^2(Q), m_1,\dots,m_N \in W^{1,p}(Q)$ *(for all* $1 \le p < \infty$*) such that* ($\forall 1 \le i \le N$)
(12)
$$-\nu^i \Delta v_i + H^i(x,\nabla v_i) + \lambda_i = \int_{Q^{N-1}} F^i(x^1,\dots,x^{i-1},x,x^{i+1},\dots,x^N) \prod_{j\neq i} m_j(x_j)\,dx^j,$$

$$\int_Q v_i\,dx = 0, \int_Q m_i\,dx = 1, m_i > 0 \text{ on } Q, \tag{13}$$

$$-\nu^i \Delta m_i - \text{ div } \Big(\frac{\partial H^i}{\partial p}(x,\nabla v_i) m_i\Big) = 0. \tag{14}$$

ii) For any solution $(\lambda_1,\dots,\lambda_N), (v_1,\dots,v_N), (m_1,\dots,m_N)$ *of the preceding system,* $\bar{\alpha}^i = \frac{\partial H^i}{\partial p}(x,\nabla v_i)$ *defines a feedback which is a Nash point for all* $X \in Q^N$ *and, in addition, we have for each* $X \in Q^N$

$$\lambda_i = J^i(\bar{\alpha}^1,\dots,\bar{\alpha}^N) = \lim_{T\to+\infty} \frac{1}{T} E\left[\int_0^T L^i(\bar{X}_t^i, \bar{\alpha}^i(\bar{X}_t^i)) + F^i(\bar{X}_t^1,\dots,\bar{X}_t)\right] \tag{15}$$

where $\bar{X}_t^i$ *is the solution of (7) corresponding to the feedback* $\bar{\alpha}^i$.

We do not detail the proof of this result here. Let us only indicate that the existence follows from a priori estimates on λ_i namely

$$|\lambda_i| \le \sup |F^i| + \sup |H^i(x,0)| \tag{16}$$

(an easy consequence of the maximum principle, choosing maximum and minimum points of v^i …), and more importantly on ∇v_i. The latter estimate is obtained by Bernstein's method i.e., deducing a bound on $|\nabla v^i(x)|^2$ using the maximum principle in the equation satisfied by it.

The part ii) of the above result is shown by a simple verification using Itôs formula and observing that the ergodicity of $\bar{X}_t^j$ insures that we have for each X

$$\lim_{T\to+\infty} \frac{1}{T} \Big\{ E\left[\int_0^T F^i(\bar{X}_t^1,\dots,\bar{X}_t^{i-1}, X_t^i, \bar{X}_t^{i+1},\dots,\bar{X}_t^N)\,dt\right] + \\ - E\left[\int_0^T dt \int_{Q^{N-1}} F^i(x^1,\dots,x^{i-1},X_t^i,x^{i+1},\dots,x^N) \prod_{j\neq i} m_j(x^j)\,dx^j\right]\Big\} = 0.$$

Remarks. i) The above result carries over (with appropriate assumptions on data …) to the more complex case where the cost $L^i(X_t^i,\alpha_t^i) + F^i(X_t^1,\dots,X_t^N)$

is replaced by $L^i(X_t^1,\dots,X_t^N,\alpha_t)$ (or even $L^i(X_t^1,\dots,X_t^N,\alpha_t^1,\dots,\alpha_t^N)$ if we restrict ourselves to feedback strategies ...) in which case H^i in (14) and $H^i - \int_{Q^{N-1}} F^i \bigotimes_{j\neq i} dm^j$ is replaced by

$$H^i = \sup_{\alpha\in\mathbf{R}^d}\Big(p\,.\,\alpha - \int_{Q^{N-1}} L^i(x^1,\dots,x^{i-1},x,x^{i+1},\dots,x^N,\alpha)\Big)\prod_{j\neq i} m_j(x^j)\,dx^j).$$

ii) The periodicity set-up is the simplest one in which we can discuss ergodic (a long-term) control problems. We could as well consider situations with reflecting boundary conditions on a domain in $\mathbf{R}^d$, or control problems with constraints (in which the processes X_t^i are required to stay inside some domain), or even problems set in $\mathbf{R}^N$ with cost functions going to infinity at infinity ...

iii) The condition (11) is not needed when $d=1$. Indeed, one obtains a bound on $\frac{dv_i}{dx}$ noticing that $\frac{d^2v_i}{dx^2}$ is bounded from below and has zero mean on $(0,1)$ (hence is bounded in L^1).

iv) Another possible extension consists in replacing $\sigma^i dW_t^i$ in (7) by $\sigma^i(X_t^i)dW_t^i$ (or even $\sigma^i(X_t^i,\alpha_t^i)dW_t^i$) and $-\alpha^i dt$ by $b^i(X_t^i,\alpha_t^i)dt$, provided one assumes (in order to avoid considerable technical difficulties) that σ^i is non degenerate i.e., $\sigma^i(\sigma^i)^T$ is positive definite (σ^i maybe a $d\times m_i$ matrix in which case W_t^i is a m_i-dimensional Brownian motion).

v) If we assume that the data are smooth, then one can check easily that the solutions are smooth.

vi) A much more delicate situation concerns the case when we allow α_t^i to be adapted to all Brownian motions in which case the partial differential equations (or system of equations) which yields a Nash point is now given by $(\forall 1\le i\le N)$

$$(17)\qquad \begin{cases} -\sum_{j=1}^N \nu^j \Delta_{x^j} v_i + \sum_{j\neq i} \frac{\partial H^j}{\partial p}(x^j,\nabla_{x^j}v_j)\,.\,\nabla_{x^j}v_j + H^i(x^i,\nabla_{x^i}v_i) \\ = F^i(x^1,\dots,x^N) - \lambda^i \ \text{ in } Q^N, \int_{Q^N} v_i = 0\,. \end{cases}$$

The system (12)–(14) is then recovered upon integrating (17) with respect to the measure $\bigotimes_{j\neq i} m_j(x^j)$ and assuming that each v_i only depends on the variable x^i. Once again, it is possible to prove existence results in such a general context.

vii) **Example:** In the very particular case where $L^i(x,\alpha) = \frac{\mu^i}{2}|\alpha|^2$ with $\mu^i > 0$ $(\forall i)$, one may check, denoting by $\varphi_i = e^{-v_i/(2\nu^i\mu^i)}(\int_Q e^{-v_i/(\nu^i\mu^i)}\,dx)^{1/2}$, that we have $m_i=\varphi_i^2, \int_Q \varphi_i^2\,dx = 1$ and the system (12)–(14) then reduce to

(18)

$$-(2(\nu^i)^2\mu^i)\Delta\varphi_i + \Big(\int_{Q^{N-1}} F^i(x^1,\dots,x^{i-1},x,x^{i+1},\dots,x^N)\prod_{j\neq i}\varphi_j^2(x^j)\,dx^j\Big)\varphi_i$$
$$= \lambda_i\varphi_i, \varphi_i > 0.$$

In particular, if $\nu^i = \frac{1}{2}, \mu^i = 1$ and $F^i = \frac{1}{2}\sum_{j\neq i} V(x_i - x_j) + V_0(x_i)$, (18) is the so-called Hartree equation (or system) in Quantum Mechanics

$$-\frac{1}{2}\Delta\varphi_i + \frac{1}{2}\sum_{j\neq i}(V \star \varphi_j^2)\varphi_i + V_0\varphi_i = \lambda_i\varphi_i, \varphi_i > 0. \tag{19}$$

And, as is well-known, the uniqueness of solutions of such systems is in general false.

viii) The previous example also shows that even in a totally symmetric situation corresponding to undistinguishable players, there is non-uniqueness of Nash equilibria.

In a totally symmetric situation, one can find symmetrical solutions as shown in

Theorem 2.2. *If we assume in addition that $\nu^i = \nu, H^i = H$ for all $1 \leq i \leq N$ and $F^i(x^1,\ldots,x^N) = F^j(x^1,\ldots,x^{i-1},x^j,x^{i+1},\ldots,x^{j-1},x^i,x^{j+1},\ldots,x^N)$ for all $1 \leq i < j \leq N$, then there exists a solution of the system (12)–(14) satisfying $\lambda_1 = \cdots = \lambda_N, v_1 = \cdots = v_N, m_1 = \cdots = m_N$.*

Remark. Even such symmetrical Nash equilibria are not unique. A simple example is given by the one introduced in Remark vii) above (assuming that the 2-body potential V is even, as is natural from a Physics viewpoint) which yields the Hartree equation

$$-\frac{1}{2}\Delta\varphi + (W \star \varphi^2)\varphi + V_0\varphi = \lambda\varphi, \int_Q \varphi^2 = 1, \varphi > 0 \tag{20}$$

(with $W = \frac{N-1}{2}V$). And, in general, there is no uniqueness of solutions of (20).

In order to simplify the presentation, we shall call, in the sections below, a Nash equilibrium any solution $(\lambda_1,\ldots,\lambda_N),(v_1,\ldots,v_N),(m_1,\ldots,m_N)$ of the system (12)–(14).

2.2. Stationary problems: $N \to +\infty$

We now let N go to $+\infty$ assuming that all players are identical and thus $\nu^i = \nu, H^i = H$ for all $1 \leq i \leq N$. In addition, we assume that the criterion F^i only depends on x^i and on the empirical density of the other players namely $\frac{1}{N-1}\sum_{j\neq i}\delta_{x^j}$ (we might as well use $\frac{1}{N}\sum_j \delta_{x^j}\ldots$). The latter dependence is expressed through an operator V from the space of probability measures on Q into a bounded set of Lipschitz functions on Q i.e., $F^i(x^1,\ldots,x^N) = V[\frac{1}{N-1}\sum_{j\neq i}\delta_{x^j}](x^i)$. A typical example is given by $V[m](x) = F(K \star m(x), x)$ where K is a Lipschitz function on $\mathbf{R}^d \times Q, K \star m(x) = \int_Q K(x,y)m(y)\,dy$ and F is locally Lipschitz on $\mathbf{R} \times Q$. Let us observe also that we may sum and multiply such operators $(V_1 \bullet V_2)[m](x) = V_1[m](x)V_2[m](x)$ thus forming an algebra ...

We shall need in our proof the following continuity assumption on the operator V.

(21) $V[m_n]$ converges uniformly on Q to $V[m]$ if m_n converges weakly to m.

Observe that $V[m_n]$ is by construction always relatively compact in $C(Q)$ and thus (21) is a somewhat natural assumption which is obviously satisfied for the specific class of operators mentioned above.

Theorem 2.3. *Under the above conditions, any Nash equilibria* $(\lambda_1^N, \dots, \lambda_N^N)$, $(v_1^N, \dots, v_N^N)$, $(m_1^N, \dots, m_N^N)$ *satisfy the following properties*

i) $(\lambda_i^N)_{i,N}$ *is bounded in* $\mathbf{R}$, $(v_i^N)_{i,N}$ *is relatively compact in* $C^2(Q)$, $(m_i^N)_{i,N}$ *is relatively compact in* $W^{1,p}(Q)$ *(for any* $1 \le p < \infty$*),*

ii) $\sup\limits_{i,j}(|\lambda_i^N - \lambda_j^N| + \|v_i^N - v_j^N\|_\infty + \|m_i^N - m_j^N\|_\infty) \to 0$ *as* $N \to \infty$,

iii) Any converging subsequence $(\lambda_1^{N'}, v_1^{N'}, m_1^{N'})_{N'}$ *in* $\mathbf{R} \times C^2 \times W^{1,p}(\forall 1 \le p < \infty)$ *converges to* (λ, v, m) *which satisfies*

$$-\nu\Delta v + H(x, \nabla v) + \lambda = V[m] \tag{22}$$

$$\int_Q v\,dx = 0, \int_Q m\,dx = 1, m > 0 \tag{23}$$

$$-\nu\Delta m - \operatorname{div}\left(\frac{\partial H}{\partial p}(x, \nabla v)m\right) = 0. \tag{24}$$

Remarks. i) All the remarks made after Theorem 2.2 may be adapted to the preceding result.

ii) It is also possible to consider situations where the data (in the dynamics and in the cost functions) depend on $\frac{1}{N-1}\sum\limits_{j\neq i}\delta_{x^j}$ and even on $\frac{1}{N-1}\sum\limits_{j\neq i}\delta_{\alpha^j}$ thus introducing operators on measures over the feedbacks. This, at least formally and rigorously under appropriate conditions then leads to systems like (22)–(24) which are, however, much more coupled and nonlinear. They may contain nonlinear terms such as the ones to be found in Vlasov type equations for kinetic models ... !

iii) By a simple regularization procedure (replacing m by $m \star \rho_\varepsilon$ where ρ_ε is a regularizing kernel in the operator V and letting ε go to 0_+), we may consider systems (22)–(24) contains as "particular cases" (i.e., after the limit procedure indicated above) systems where

$$V[m](x) = F(m(x), x) \tag{25}$$

and F is a function on $\mathbf{R} \times Q$. We may even consider functions of m and its derivatives like $V[m] = -\gamma\Delta m + F(m)$ (and $\gamma \geq 0$) ...

iv) As we shall see later on, one may consider several large groups of identical players and recover systems involving several (v, m).

v) We shall present below an interpretation of such systems in terms of optimal control of partial differential equations (which by the way allows us to have a notion of Pareto's optimality for such equilibria).

vi) If we look again at the particular case when $H(x,p) = \frac{1}{2}|p|^2 - F_0(x)$, the system (22)–(24) reduces to the following generalized Hartree equation

$$-\Delta\varphi + (F_0 + V[\varphi^2])\varphi = \lambda\varphi, \varphi > 0, \int_Q \varphi^2\, dx = 1. \tag{26}$$

vii) Uniqueness of solutions of (22)–(24) is not true in general. Indeed, take $F_0 = 0, V[\varphi^2] = -c\varphi^2$ with $c > 0$ in the example leading to (26). Then, we need to solve

$$-\Delta\varphi = \lambda\varphi + c\varphi^3, \varphi > 0, \int_Q \varphi^2\, dx = 1. \tag{27}$$

A particular solution is given by $\varphi \equiv 1, \lambda = -c$. On the other hand, another solution may be found if $d = 1$ for c large enough by solving

$$\min_{\int\varphi^2=1} \frac{1}{2}\int(\varphi')^2 - \frac{c}{4}\int\varphi^4;$$

Indeed, if we let $\varphi = (1+\varepsilon v)(1+\varepsilon^2\int v^2)^{-1/2}$ where $\int v = 0$, we obtain

$$\frac{1}{2}\int(\varphi')^2 - \frac{c}{4}\int\varphi^4 = -\frac{c}{4} + \varepsilon^2\left\{\frac{1}{2}\int(v')^2 - \frac{3c}{2}\int v^2\right\} + o(\varepsilon^2) < -\frac{c}{4}$$

as soon as $c > \frac{4}{3}\pi^2$ if we choose $v = \cos(2\pi x)$.

On the other hand, we shall prove in the following section that uniqueness of solutions of (22)–(24) holds in general situations.

viii) The above result and the general uniqueness result presented in the following section (while non-uniqueness always holds for a finite number of players ...) illustrate clearly the claim we made in the Introduction namely the simplification which occurs as N goes to infinity since all equilibria become asymptotically symmetric and uniqueness holds in general situations.

ix) Another advantage of dealing with a continuum of players is the possibility of modelling easily the "birth and death" of players (in which case the total number of players may vary ...) through "source" terms in the equation form. These extra source terms may also involve diffusion terms or may be localized in some regions ...

x) We conjecture that the above convergence result is still valid if we consider N players with full information. As we saw in the previous section, we then need to consider the system (17). If we denote by $(v_i^N)_{1\leq i\leq N}$ a solution of that system and if we introduce the global invariant measure m^N on Q^N i.e., the solution of

$$-\nu\sum_{i=1}^{N}\Delta_{x^i}m^N-\sum_{i=1}^{N}\operatorname{div}_{x^i}\Big(\frac{\partial H}{\partial p}(x,\nabla_{x^i}v_i^N)m^N\Big)=0 \tag{28}$$

with $m^N>0$ on Q^N and $\int_{Q^N}m^N\,dx=1$, we expect at least formally that each v_i^N will behave asymptotically as a function $v(x^i)$ (note that at least formally $\nabla_{x^j}v_i^N$ should be of order $1/N$ for $j\neq i$). Then, m^N should behave asymptotically like $\prod_{i=1}^{N}m(x^i)$ and m,v satisfy (23)–(24). Finally, in order to recover (22), we integrate (17) with respect to the measure $\prod_{j\neq i}m(x^j)$ recalling that $v_i^N\approx v(x^i)$. We have been able to carry out such a program in very particular cases and we emphasize the fact that the formal considerations we just made are, even in such particular cases, a caricature of the delicate rigorous arguments that need to be made.

2.3. Stationary problems: mathematical analysis

We have in fact already proven in the preceding section by the limit N going to infinity, a general existence result at least when V satisfies the conditions introduced in that section. This is why we begin an analysis with a general uniqueness that we state, for the sake of simplicity, for smooth solutions of (22)–(24). It is easy from the argument made below to adapt it to situations with a limited regularity information.

Theorem 2.4. *Let us assume that either V is monotone in L^2 i.e.,*

$$\int_Q(V[m_1]-V[m_2])(m_1-m_2)\,dx\geq 0,\forall m_1,m_2 \tag{29}$$

and H is strictly convex namely for all $(x,p)\in Q\times\mathbf{R}^d$

$$H(x,p+q)-H(x,p)-\frac{\partial H}{\partial p}(x,p)\,.\,q=0\Rightarrow q=0, \tag{30}$$

or that V is strictly monotone i.e.,

$$\int_Q(V[m_1]-V[m_2])(m_1-m_2)\,dx\leq 0\Rightarrow m_1\equiv m_2. \tag{29$'$}$$

Then, the uniqueness of solutions of (22)–(24) holds.

Remark. In fact, it is possible to replace (30) by a weaker condition namely

$$(30') \quad H(x,p+q)-H(x,p)-\frac{\partial H}{\partial p}(x,p)\ .\ q=0 \Rightarrow \frac{\partial H}{\partial p}(x,p+q)=\frac{\partial H}{\partial p}(x,p)$$

(a condition which is satisfied by $H=|p|$ while (30) does not hold in that example).

Proof. Let $(\lambda_1,v_1,m_1),(\lambda_2,v_2,m_2)$ be two solutions of (22)–(24). We multiply (22) by (m_1-m_2) and (24) by (v_1-v_2), subtract the resulting identities, use (23) and find finally

(31)

$$\begin{cases} \int_Q (V[m_1]-V[m_2])(m_1-m_2)\,dx+\int_Q m_1(H(x,\nabla v_2)-H(x,\nabla v_1)-\frac{\partial H}{\partial p}(x,\nabla v_2). \\ .\ \nabla(v_2-v_1)\,dx+\int_Q m_2(H(x,\nabla v_1)-H(x,\nabla v_2)-\frac{\partial H}{\partial p}(x,\nabla v_2)\ .\ \nabla(v_1-v_2)\,dx=0. \end{cases}$$

Then, we observe that in both cases covered by the above result V is at least monotone and H is at least convex. Therefore, each of the three terms in (31) is non-negative and thus must vanish. Then, if H is strictly convex, we deduce that $\nabla v_1 \equiv \nabla v_2$ and thus $v_1 \equiv v_2$ in view of (23). We then conclude easily that $m_1 \equiv m_2$ because of (24). On the other hand, if V is strictly monotone then $V[m_1]=V[m_2]$. In that case, we use the classical uniqueness results for ergodic Hamilton–Jacobi–Bellman equations to deduce that $v_1 \equiv v_2$. And we conclude easily. □

It is also possible to revisit the existence issue in particular if we wish to allow for more general operators $V[m]$. Several types of existence results (weak solutions, smooth solutions ...) are possible depending upon the type of conditions one is willing to make upon H and V. In order to be more specific, let us mention the simple example where $H(x,p)=\mu|p|^\alpha(\alpha \geq 1,\mu>0)$ and $V[m]=cm^\beta+f(x)(c\in\mathbf{R},\beta>0)$. Then, the type of existence results we know depends upon α,β and the sign of c. In order to restrict the length of this article which only aims at a survey of the problems, we shall not pursue here the discussion in such a technical direction which however is very interesting from a purely mathematical stand-point. Let us however mention that in strong anti-monotone situations, i.e., $c<0,\beta$ large in the above example, existence may not hold for arbitrary data. This reflects the fact that solutions may blow up in the asymptotic regularization limit mentioned in the previous section ($V[m\star\rho_\varepsilon]$ going to $V[m]$ ...).

2.4. Dynamical problems: mathematical analysis

We now present the analogue of the mean-field games equations in the context of finite horizon control problems. We keep the same notation as before and still work in the periodic case for the sake of simplicity although we could set up the models in the whole space (with minor restrictions on the growth of data and solutions at infinity) or in a domain with arbitrary boundary conditions (Dirichlet, Neumann, state constraints ...). With these conventions, we consider some fixed horizon $T > 0$ and introduce the following mean-field system of equations

$$\frac{\partial v}{\partial t} - \nu \Delta v + H(x, \nabla v) = V[m] \quad \text{in } Q \times (0,T) \tag{32}$$

$$\frac{\partial m}{\partial t} + \nu \Delta m + \operatorname{div}\left(\frac{\partial H}{\partial p}(x, \nabla v) m\right) = 0 \quad \text{in } Q \times (0,T) \tag{33}$$

with the time boundary conditions

$$v\,|_{t=0} = V_0[m(0)] \text{ on } Q, m\,|_{t=T} = m^0 \text{ on } Q. \tag{34}$$

The assumptions on ν and H are the same as before. The operators $V[m]$ and $V[m(0)]$ (V_0 only acts on the value at $t = 0$ of $m(x,t)$) are, exactly as in the previous section, quite arbitrary. Typical examples include nonlinear non-local smoothing operators (see the preceding section) or local ones given by

$$V[m] = F(m(x,t), x, t), V_0[m] = F_0(m(x), x) \tag{35}$$

where F, F_0 are functions on $\mathbf{R} \times Q \times [0,T], \mathbf{R} \times Q$ respectively. Finally, m^0 is a given, say, smooth and positive function on Q such that $\int_Q m^0\, dx = 1$.

Exactly as in the stationary case, the above system can be deduced, at least formally, from Nash equilibria for N players in which case we have to solve at the level of a finite number N of players the following system of equations

$$\begin{cases} \dfrac{\partial v_i^N}{\partial t} - \nu \displaystyle\sum_{j=1}^{N} \Delta_{x^j} v_i^N + \sum_{j \neq i} \frac{\partial H}{\partial p}(x, \nabla_{x^j} v_j^N) \,.\, \nabla_{x^j} v_i^N + H(x, \nabla_{x^i} v_i^N) \\ = V\left[\dfrac{1}{N-1} \displaystyle\sum_{j \neq i} \delta_{x^j}\right](x^i) \quad \text{in } Q^N \times (0,T) \\ \text{and } v_i^N\,|_{t=0} = V_0\left[\dfrac{1}{N-1} \displaystyle\sum_{j \neq i} \delta_{x^j}\right](x^i) \quad \text{on } Q. \end{cases} \tag{36}$$

Of, course, there is no simple connection between (32)–(34) and (36) since m^0 does not even appear in (36). The interpretation of m^0 is the probability distribution of each player in Q at $t = 0$: more precisely, we consider N players whose initial positions are random, independent with the same probability distribution m^0. Then, if each player follows its optimal (in the sense of Nash equilibria) strategy, the density of players m^N (on Q^N) obeys the following Fokker–Planck equation on $Q^N \times (0,T)$

$$
(37) \qquad \begin{cases} \dfrac{\partial m^N}{\partial t} + \nu \displaystyle\sum_{i=1}^{N} \Delta_{x^i} m^N + \sum_{i=1}^{N} \operatorname{div}_{x^i}\Big(\dfrac{\partial H}{\partial p}(x^i, \nabla_{x^i} v_i^N) m^N\Big) = 0 \\ \text{with } m^N\,|_{t=T} = \displaystyle\prod_{i=1}^{N} m^0(x^i) \text{ on } Q^N. \end{cases}
$$

(Note that we reversed time so that $m^N(t)$ stands for the density of players at time $T-t$).

We next explain formally how to deduce (32)–(34) from the behaviour of m^N and of each v_i^N (that could be interpreted as a random process on Q^N equipped with the probability measure $\prod_{i=1}^{N} m^0(x^i)$, in which case the behaviour of v_i^N described below is naturally interpreted in probability— instead of everywhere— with respect to that probability). We first expect v_i^N to have very little dependence upon x^j (for $j \neq i$) and to become asymptotically symmetric in view of the righthand sides. More precisely we expect $\nabla_{x^j} v_i^N$ to be of order $\frac{1}{N}$ for $j \neq i$. We then should deduce that m^N is asymptotically factorized i.e., $m^N \approx \prod_{i=1}^{N} m(x^i,t)$ (the so-called propagation of chaos in Statistical Mechanics and Physics which essentially means that all players become independent ...). Then, we consider $\bar{v}_i^N(x^i,t) = \int_{Q^{N-1}} v_i^N(x^1,\dots,x^N,t) \prod_{j\neq i} m(x^j,t) dx^j$ which by symmetry should be essentially (asymptotically) independent of i. And we recover (32)–(34) upon integrating (36) with respect to $\prod_{j\neq i} m(x^j,t)$ once we notice that, as N goes to $+\infty$, we have

$$
\int_{Q^{N-1}} V\left[\frac{1}{N-1}\sum_{j\neq i} \delta_{x^j}\right](x^i,t) \prod_{j\neq i} m(x^j,t)\, dx^j \to V[m](x^i,t)
$$

by the law of large numbers.

This is obviously a rough sketch of what could be a rigorous derivation of (32)–(34). We have been able to make this type of arguments rigorous only in very particular situations. And it is obviously an outstanding open problem (a conjecture) to make it rigorous with some generality at least. Let us finally mention that, if we look at steady states of (32)–(34), we indeed recover the stationary problems introduced, rigorously justified and studied in the previous section.

Exactly as in the case of stationary problems, the variety of operators V and V_0 precludes the possibility of very general existence results. And, in some cases, existence should not be correct if the operators are too singular (and anti-monotone) as we shall explain later on. Similarly, uniqueness cannot be true without conditions. This is only we shall first present a very general uniqueness result and then give some samples of existence results of various types (smooth or weak solutions). We thus begin with the following uniqueness result in which we assume all solutions to be smooth in order to simplify the presentation.

Theorem 2.5. *We assume that either V and V_0 are monotone (respectively in $L^2(Q\times(0,T)), L^2(Q)$) and H is strictly convex (i.e., satisfies (30), or V and V_0 are strictly monotone (i.e., satisfy (29')) in $Q\times(0,T), Q$ respectively). Then, the uniqueness of solutions of (32)–(34) holds.*

The proof of this result is a trivial adaptation of the one we made in the previous section. Indeed, it suffices to multiply the equations for v_i by (m_1-m_2) and for m_i by (v_1-v_2) $(i=1,2)$ where $(v_1,m_1),(v_2,m_2)$ are two solutions of (32–34), to subtract the identities and to integrate over $Q\times(0,T)\ldots$

Remarks. i) It is possible to give examples of non uniqueness with $V\equiv 0, V_0$ is a local operator $F_0(m(x))$ and F_0 is decreasing.

ii) It is also possible to show some uniqueness results in the "small" like for instance uniqueness results when T is "small" enough ...

We now turn to existence results. As we said above, there are many possible directions in which one may obtain some existence results and we shall only mention two that correspond to very natural choices of the operators V and V_0. The first one concerns the case of operators V and V_0 which are "regularizing operators". And we assume (for instance) that V, V_0 satisfy

(38)
$$\begin{cases} V(\text{resp.} V_0) \text{ maps the subset } X \text{ of } C([0,T];L^1(Q)) \text{ (resp. } L^1(Q))\text{defined by} \\ m\geq 0, \displaystyle\int_Q m\,dx\equiv 1 \text{ into a bounded set of } L^\infty(0,T;W^{1,\infty}(Q)) \text{ (resp. } W^{1,\infty}(Q)) \end{cases}$$

(39) $$V \text{ is continuous from } X \text{ into } C(Q\times[0,T])$$

(40) $$V, V_0 \text{ are bounded maps from } C^{k,\alpha} \text{ into } C^{k+1,\alpha}(Vk\geq 0, \forall\alpha\in(0,1)).$$

We also assume that H is smooth on $Q\times\mathbf{R}^d$ and satisfies for some $C\geq 0$ either

(41) $$\left|\frac{\partial H}{\partial p}\right|\leq C(1+|p|), \forall(x,p)\in Q\times\mathbf{R}^d$$

or

(42) $$\left|\frac{\partial H}{\partial x}\right|\leq C(1+|p|), \forall(x,p)\in Q\times\mathbf{R}^d.$$

Theorem 2.6. *Under the above conditions, there exists at least a smooth solution of (32)–(34).*

Another natural case concerns local operators V and V_0 i.e., operators given by (35). For such operators, once again many results are possible depending upon the growths of F and F_0 leading to smooth solutions or weak ones. Since the above result states the existence of a smooth solution, we pick now a sample of existence results of weak solutions. We assume that F, F_0, H are continuous and satisfy for all their arguments the following conditions for some contents $a > 1, b > 1, q > 1, \delta > 0, C \geq 0$

$$F(x,t,\lambda)\lambda \geq \delta |F(x,t,\lambda)|^a - C, \tag{43}$$

$$F_0(x,\lambda)\lambda \geq \delta |F_0(x,\lambda)|^b - C, \tag{44}$$

$$\begin{cases} \delta |p|^q - C \leq H(x,p) \leq C|p|^q + C \\ \dfrac{\partial H}{\partial p}(x,p).p \geq qH - C, \left|\dfrac{\partial H}{\partial p}(x,p)\right| \leq C|p|^{q-1} + C. \end{cases} \tag{45}$$

Theorem 2.7. *Under the above conditions, there exists a solution of (32)–(34) such that* $v \in L^q(0,T;W^{1,q}(Q))$, *$v$ is bounded from below,* $m|\nabla v|^q \in L^1(0,T;L^1(Q)), v \in C([0,T];L^r(Q))$ *where* $r = \min(b, \frac{ad}{d-2(a-1)})$ *if* $a < 1 + \frac{d}{2}, r = b$ *if* $a \geq 1 + \frac{d}{2}$ *and* $m \in C([0,T];L^1(Q))$.

Remarks. i) All the remarks made in the previous section on stationary can be adapted to (32)–(34). Let us also point out that some extensions are explicitly mentioned in section 2.7 below.

ii) We saw above that uniqueness holds if F and F_0 are non-decreasing functions. In the above existence result, we see that the assumptions made upon F and F_0 imply that F and F_0 cannot be "too decreasing". A further indication that existence and/or uniqueness are lost when we consider general F and F_0 is given by the following linearization argument. Let us take for instance $V_0 \equiv 0$ and $F(\lambda) = -c\lambda$ with $c > 0$ so that F is strictly decreasing. We also choose $H = \frac{1}{2}|p|^2$. Then, $m \equiv 1, v \equiv -ct$ is an explicit solution of (32)–(34) corresponding to $m^0 \equiv 1$. And the linearized system around that solution takes the following form

$$\frac{\partial u}{\partial t} - \nu \Delta u = -cf, \frac{\partial f}{\partial t} + \nu \Delta f + \Delta u = 0, u\,|_{t=0} = 0, f\,|_{t=T} = f_0. \tag{46}$$

And one easily checks by a straightforward computation using Fourier series that this system is well-posed if and only if T is small enough $(T < \frac{\nu}{c})$.

iii) A further evidence of the role of monotonicity is given in the following section on the deterministic limit $(\nu \to 0_+)$.

2.5. Deterministic limits

We now let ν go to 0_+. This amounts to let the "noise" disappear from the player's dynamics. And we begin with the stationary problem. We only consider one example of the possible issues to be studied (and the richness of the theme ...). We then assume that $V[m] = F(m) + f_0(x)$ where f_0 is Lipschitz over Q, F is locally Lipschitz on $\mathbf{R}$ and $\inf\limits_{\mathbf{R}}\text{ess}\, F' > 0$ and that H satisfies: $H(x,p) \geq H(x,0) = 0, \forall p \in \mathbf{R}^d, \forall x \in Q$. As we saw in the previous sections, we know that, for $\nu > 0$, there exists a unique solution $(\lambda_\nu, v_\nu, m_\nu)$ of (22)–(24). And we have the

Theorem 2.8. *As ν goes to 0_+, (λ_ν, m_ν) converges in $\mathbf{R} \times L^2(Q)$ towards (λ, m) which is determined by*

$$m(x) = (F^{-1}(\lambda - f_0(x)))_+ \text{ on } Q, \int_Q m\,dx = 1. \tag{47}$$

Remark. If we recall that a local operator $F(m)$ is deduced by a limit, as ε goes to 0_+, of $F(m \star \rho_\varepsilon)$ where ρ_ε is a smoothing kernel, one sees that the solution determined in the above result corresponds to the successive limits $N \to +\infty$, then ε goes to 0_+ and finally ν goes to 0_+. In general, those limits do not commute. Indeed, let us consider, for example, the case when $H^i(x,p) = \frac{1}{2}|p|^2, \nu^i = \nu$, one first observes that, if we first let ν go to 0_+, any Nash point $(\bar{x}^1,\dots,\bar{x}^N)$ of $(F^1,\dots,F^N)$ leads to a Nash equilibrium with $m_i = \delta_{\bar{x}^i}, \lambda_i = \inf\limits_{x\in Q} F^i(\bar{x}^1,\dots,\bar{x}^{i-1},x,\bar{x}^{i+1},\dots,\bar{x}^N)$ and v_i solves in Q (in the sense of viscosity solutions)

$$\frac{1}{2}|\nabla v^i|^2 = F^i(\bar{x}^1,\dots,\bar{x}^{i-1},x,\bar{x}^{i+1},\dots,\bar{x}^N) - \lambda_i, \int_Q v_i\,dx = 0. \tag{48}$$

Next, if $F^i(x^1,\dots,x^N) = F(\frac{1}{N-1}\sum\limits_{j\neq i}\rho_\varepsilon(x^i - x^j)) + f_0(x^i)$ with $F(0) = 0$ (F as above ...), $f_0 \geq 0, f_0 \not\equiv 0$, meas $\{f_0 = 0\} > 0, \rho_\varepsilon = \frac{1}{\varepsilon^d}\rho(\frac{\cdot}{\varepsilon}), \rho \in C_0^\infty(\mathbf{R}^d)$, $\int_{\mathbf{R}^d}\rho\,dx = 1$, Supp $\rho \subset B_1$, then, for all $N \geq 1$, one may find, for $\varepsilon > 0$ small enough, points x^i inside the set $\{f_0 = 0\}$ such that $|\bar{x}^i - \bar{x}^j| > \varepsilon$ if $i \neq j$. Then, $(\bar{x}^1,\dots,\bar{x}^N)$ is clearly a Nash point which yields: $\lambda_i = 0 (\forall i)$. And we conclude that not only the "limit" value λ, as N goes to $+\infty$, vanishes but also the measures m_i may not converge as N goes to $+\infty$. in particular, the limit $\nu \to 0_+$, then $\varepsilon \to 0_+$ and finally $N \to +\infty$ yields $\lambda = 0$ while λ determined from (47) is clearly positive in that case, proving thus our claim about the absence of commutativity of the limits.

We now discuss briefly the case of time-dependent problems. First of all, we argue formally and let ν go to 0_+ in the system (32)–(34). We then recover the following system of equations

$$\frac{\partial v}{\partial t} + H(x, \nabla v) = V[m] \text{ in } Q \times (0,T), \tag{49}$$

$$\frac{\partial m}{\partial t} + \mathrm{div}\left(\frac{\partial H}{\partial p}(x, \nabla v) m\right) = 0 \text{ in } Q \times (0,T), \tag{50}$$

$$v\,|_{t=0} = V_0[m] \text{ on } Q, m\,|_{t=T} = m^0 \text{ on } Q. \tag{51}$$

And we first consider the case of operators V, V_0 that are smoothing operators namely map non-negative L^1 functions such that $\int_Q m\,dx = 1$ into the set of Lipschitz functions in x such that $D^2 v(x) \leq CI_d$ for some fixed $C \geq 0$ (semi-concave functions). And we assume that H satisfies (42) (for example). In that case, one can prove the existence of a solution (which is unique under the conditions of Theorem 2.5) (v,m) such that v is Lipschitz in $(x,t), D^2 v(x,t) \leq CI_d$ on $Q \times (0,T), m \in L^\infty(Q \times (0,T))$ and (49) holds in viscosity sense while (50) holds in the sense of distributions. In addition, we can justify the above formal limit as ν goes to 0_+.

Next, we discuss the case when V and V_0 are given by local operators (see (35)) with F, F_0 smooth in (x, λ). And we first choose $H(x,p) = \frac{1}{2}|p|^2, F(x,\lambda) = F(\lambda), F_0(x,\lambda) = F_0(\lambda)$. In that case, if we denote by $U = \nabla v$, we find

$$\begin{cases} \dfrac{\partial U}{\partial t} + U_j \dfrac{\partial}{\partial x_j} U = \dfrac{\partial}{\partial x_i} F(m) \\ \dfrac{\partial m}{\partial t} + \text{ div } (Um) = 0. \end{cases}$$

or

$$\begin{cases} \dfrac{\partial m}{\partial t} + \text{ div } (Um) = 0 \\ \dfrac{\partial (mU)}{\partial t} + \text{div } (mU \otimes U) + \nabla \pi(m) = 0 \end{cases} \tag{52}$$

where $\pi'(\lambda) = -F'(\lambda)\lambda$.

In other words, we recover the classical compressible Euler equations of Fluid Mechanics in the so-called barotropic and potential regime (see P.-L. Lions [23] for more details ...). In this interpretation, π is the pressure law. And one knows that π should be non-decreasing (as a consequence of the second law of thermodynamics) which corresponds precisely to requiring that F is non-increasing (i.e., the "bad" case for the systems we are studying). This is consistent with the results obtained in the preceding section since, if π is increasing, we know that (52) is a nonlinear hyperbolic system and solutions of (52) develop discontinuities in finite time and there is thus almost no hope to solve (51) in that case ...! On the other hand, if π is decreasing i.e., F increasing, the system (49)–(50) is formally a nonlinear elliptic system. Indeed, if we write, using (49), $m = G(\frac{\partial v}{\partial t} + H(x, \nabla v))$ where $G = F^{-1}$ is increasing, then (50) becomes

$$G'(F(m))\Big(\frac{\partial^2 v}{\partial t^2} + \frac{\partial H}{\partial p} \,.\, \nabla_x \frac{\partial v}{\partial t}\Big) + m \frac{\partial^2 H}{\partial p^2} \,.\, D_x^2 v$$
$$+ G'(F(m)) \frac{\partial H}{\partial p} \,.\, \nabla_x \Big(\frac{\partial v}{\partial t} + H(\nabla_x v)\Big) = 0$$

i.e.,

$$(53) \qquad \frac{\partial^2 V}{\partial t^2} + 2\frac{\partial H}{\partial p} \,.\, \nabla_x \frac{\partial v}{\partial t} + \frac{\partial H}{\partial p} \,.\, D^2 v \,.\, \frac{\partial H}{\partial p} + \frac{m}{G'(F(m))} \frac{\partial^2 H}{\partial p^2} \,.\, D_x^2 v = 0,$$

which is clearly an elliptic equation! In addition, the boundary conditions at $t = 0$ and $t = T$ may be written as

$$(54) \qquad \frac{\partial v}{\partial t} + H(\nabla_x v) \,|_{t=T} = G(m^0) \text{ on } Q,$$

which is a nonlinear Neumann boundary condition, and denoting by $h_0 = F_0 \circ G^{-1}$

$$(55) \qquad v - h_0\Big(\frac{\partial v}{\partial t} + H(\nabla_x v)\Big)\Big|_{t=0} = 0 \text{ on } Q,$$

another nonlinear boundary condition which requires h_0 (i.e., F_0) to be increasing in order to insure the maximum principle.

Once more, we see that the conditions on the monotonicity of F and F_0 are natural ones. And, at least if $d = 1$, these observations can be used in order to solve this system of equations. It turns out that when $d = 1$ a particular case of such systems has been studied by A. Guionnet [13] and A. Guionnet and O. Zeitouni [14] in the context of large deviations theory (and their applications to Physics).

2.6. Links with optimal control

We now explain in this section how the system (32)–(34) is connected to optimal control problems at least in the particular case where we assume that the operators V and V_0 are gradient operators in $L^2(Q \times (0,T))$ (resp. $L^2(Q)$) of some functionals Φ and Ψ respectively.

The first interpretation in terms of optimal control corresponds to the optimal control of the following (backward) Fokker–Planck equation

$$(56) \qquad \frac{\partial m}{\partial t} + \nu \Delta m + \operatorname{div}(\alpha m) = 0 \text{ in } Q \times (0,T), m \,|_{t=T} = m^0 \text{ in } Q$$

where $\alpha = \alpha(x,t)$ is a (distributed) control. Then, we introduce the following optimal control problem

$$\inf_{\alpha}\left\{\Phi(m)+\int_0^T dt\int_Q dxL(x,\alpha)m+\Psi(m(0))\right\}. \tag{57}$$

If α_0 is an optimal control i.e., minimizes over all possible α the preceding expression and m denotes the corresponding state i.e., solution of (5), then we build the dual state v which is determined by $\alpha_0 = \frac{\partial H}{\partial p}(x,\nabla v)$ (assuming that H is strictly convex ...). And we deduce from the usual necessary conditions for optimality that (v,m) solves (32)–(34) with $V = \Phi'$ and $V_0 = \Psi'$.

And it is important to observe that (57) is a convex optimization problem as soon as L is convex in α, Φ and Ψ are convex. And the latter condition is precisely equivalent to the fact that V and V_0 are monotone operators! Furthermore, if it is the case, the dual problem has an interpretation as a control problem for Hamilton–Jacobi–Bellman equations namely

$$\frac{\partial v}{\partial t} - \nu\Delta v + H(x,\nabla v) = \beta \text{ in } Q\times(0,T), v\,|_{t=0} = \gamma \text{ in } Q. \tag{58}$$

And one looks at the following minimization problem

$$\inf_{\beta,\gamma}\left\{\Phi^{\star}(\beta)+\Psi^{*}(\gamma)-\int_Q m^0 v(T)\right\}, \tag{59}$$

where $\Phi^{\star}, \Psi^{\star}$ denote respectively the dual convex functions of Φ, Ψ.

We also wish to mention, without any further explanation, that, in the convex case, these optimal control interpretations allow to give a notion of Pareto's optimality to the mean-field equilibria we introduced and studied.

Example. In the case when $H(x,p) = \frac{1}{2}|p|^2 - f_0(x)$, if we set (as we did many times above) $\text{ø} = e^{-v/(2\nu)}$, (59) is an optimal control problem for

$$\frac{\partial \text{ø}}{\partial t} - \nu\Delta\text{ø} + (f_0+\beta)\text{ø} = 0 \;\text{ in } Q\times[0,T]$$

where the potential β is the control.

It is also possible to adapt the optimal control problems above to the stationary mean-field equations. And in the example mentioned above, the optimal control problem takes the following form

$$\inf_{\beta}\{\Phi(\beta)-\lambda\}$$

where λ is the first eigenvalue of the Schrödinger operator $-\nu\Delta + (f_0+\beta)$.

2.7. Variants and extensions

We already mentioned in the previous sections many possible variants and extensions corresponding to more general dynamics, more complex interactions between the players and more complex criteria to minimize for each player. We also mentioned the possibility of incorporating in the equations source terms or even additional differential operators that may correspond to the "death and birth" of players, or to drift-diffusion phenomena for the density of players. One can also consider situations where the equation for v is an obstacle problem in which case, at least formally, the equation for m is naturally set on the zone where the solution does not coincide with the obstacle.

Even in the cases we mentioned and studied, much remains to be done as far as existence, uniqueness and regularity are concerned. And there are fundamental open problems in the derivation of these mean-field equations (i.e., the rigorous treatment of the limit as N goes to $+\infty$). It is even possible to consider the issue of an asymptotic expansion in N of the solutions (at least in the stationary case for symmetric Nash equilibria ...).

One general class of problems we thus introduced (although most of the extensions mentioned above are not even contained in that class) is given by

$$\frac{\partial v}{\partial t} - F(x,t,v,D_x v, D_x^2 v; m) = 0 \text{ in } Q \times (0,T), v\,|_{t=0} = V_0[m], \tag{60}$$

$$\frac{\partial m}{\partial t} + D_x^2\Big(\frac{\partial F}{\partial A} m\Big) + D_x\Big(\frac{\partial F}{\partial p}\Big) = 0 \text{ in } Q \times (0,T), m\,|_{t=T} = m^0, \tag{61}$$

where $F = F(x,t,\lambda,p,A)$ is non decreasing in A for the partial ordering of symmetric matrices (an ellipticity condition) and the dependence upon m is a functional one.

Finally, an important direction for future work corresponds to the case of several populations, each of which consists of a large number of identical players but the characteristics of the players vary from one population to the other. For instance, if we only consider two populations, the mean-field equilibria are then characterized by the solutions of $(i = 1,2)$

$$\frac{\partial v^i}{\partial t} - \nu^i \Delta v^i + H^i(x, \nabla v^i) = V^i[m_1, m_2], v^i\,|_{t=0} = V_0^i[m_1, m_2] \tag{62}$$

$$\frac{\partial m^i}{\partial t} + \nu^i \Delta m^i + \operatorname{div}\Big(\frac{\partial H^i}{\partial p}(x, \nabla v^i) m^i\Big) = 0, m_i\,|_{t=T} = m_0^i \tag{63}$$

(at least in the most elementary case of our general approach ...).

3. Price formation and dynamic equilibria

3.1. The model

We introduce a simple mean-field model for the dynamical formation of a price. We consider an idealized population of players (which however somehow reflects the nature or microstructure of financial markets) consisting of two groups namely one group of buyers of a certain good and one group of vendors of the same good. Postulating some exogenous randomness in price preferences, we describe this population by two densities f_B, f_V i.e., non-negative functions of (x,t) where t stands for time and x stands for a possible value of the price (roughly speaking $f_B(x,t)$ represents the number of potential buyers at a price x at time t). We denote by $p(t)$ the price resulting from a dynamical equilibrium and we assume that there is some friction measured by a positive parameter a (one could think of $2a$ to be the bid-ask spread). And we obtain the following system of mean-field equations

$$\begin{cases} \dfrac{\partial f_B}{\partial t} - \dfrac{\sigma^2}{2}\dfrac{\partial^2 f_B}{\partial x^2} = \lambda\delta(x-p(t)+a), \text{ if } x < p(t), t > 0 \\ f_B \geq 0, f_B(x,t) = 0 \text{ if } x \geq p(t), t \geq 0, \end{cases} \tag{64}$$

$$\begin{cases} \dfrac{\partial f_V}{\partial t} - \dfrac{\sigma^2}{2}\dfrac{\partial^2 f_V}{\partial t} = -\lambda\delta(x-p(t)+a), \ \text{ if } \ x > p(t), t > 0 \\ f_V \geq 0, f_V(x,t) = 0 \ \text{ if } \ x \leq p(t), t \geq 0, \end{cases} \tag{65}$$

$$\lambda = -\frac{\sigma^2}{2}\frac{\partial f_B}{\partial x}(p(t),t) = +\frac{\sigma^2}{2}\frac{\partial f_V}{\partial x}(p(t),t). \tag{66}$$

The multiplier λ measures the number of transactions at time t (i.e., the flux of buyers which must be equal to the flux of vendors). The parameter $\sigma > 0$ measures the randomness. And δ denotes either the usual delta function δ_0, or a smoothed version of it (when we wish to ignore some technicalities ...) that is a smooth non-negative function with compact support in $(-a,+a)$ and such that $\int \delta = 1$. In the next section, we use a smooth δ while in the last section we use the usual one in order to simplify the presentation as much as possible. Of course, (64)–(66) is to be completed by an initial condition

$$f_B\,|_{t=0} = f_B^0, f_V\,|_{t=0} = f_V^0 \tag{67}$$

where (to make matters as simple as possible) we assume that

$$f_B^0(x) > 0 \text{ if } x < p_0, f_B^0(x) = 0 \text{ if } x \geq p_0$$

$$f_V^0(x) > 0 \text{ if } x > p_0, f_V^0(x) = 0 \text{ if } x \leq p_0$$

for some $p_0 \in \mathbf{R}$.

A very natural invariance property of the above system is given by the invariance in t of the total number of goods and of the total number of players. Indeed, if we consider

$$\frac{d}{dt}\int_{\mathbf{R}} f_B(x,t)\,dx = \frac{d}{dt}\int_{-\infty}^{p(t)} f_B(x,t)\,dx = \int_{-\infty}^{p(t)} \frac{\partial f_B}{\partial t}(x,t)\,dx$$

since f_B vanishes at $p(t)$. Thus, we have

$$\begin{aligned}\frac{d}{dt}\int_{\mathbf{R}} f_B(x,t)\,dx &= \int_{\infty}^{p(t)} \frac{\sigma^2}{2}\frac{\partial^2 f_B}{\partial x^2}\,dx + \lambda(t)\\ &= \frac{\sigma^2}{2}\frac{\partial f_B}{\partial x}(p(t),t) + \lambda(t) = 0.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\frac{d}{dt}\int_{\mathbf{R}} f_V(x,t)\,dx &= \frac{d}{dt}\int_{p(t)}^{+\infty} f_V(x,t)\,dx = \int_{p(t)}^{+\infty}\frac{\partial f_V}{\partial t}(x,t)\,dx\\ &= \int_{p(t)}^{+\infty}\frac{\sigma^2}{2}\frac{\partial^2 f_V}{\partial x^2}dx - \lambda(t)\\ &= -\frac{\sigma^2}{2}\frac{\partial f_V}{\partial x}(p(t),t) - \lambda(t) = 0.\end{aligned}$$

In the next two sections, we shall review some known results on the system (64)–(66). Let us mention that there are again many possible (and relevant) directions of research concerning the derivation of this model from Nash points and utility maximization, extensions to more general dynamics, to situations with several possible goods or where transactions may involve more than a unit quantity of the good... Although we have some very preliminary results in those directions, it is quite clear that much remains to be done both from a modelling and from a mathematical standpoint.

3.2. Main results

We first perform a reduction of the system (64)–(66) which is not really necessary for our argument but allows to simplify the presentation. We introduce

$$f(x,t) = f_B(x,t) \;\text{ if }\; x \le p(t) = -f_V(x,t) \;\text{ if }\; x > p(t)$$

and we observe that (64)–(66) simply reduces to

(3)

$$\begin{cases}\dfrac{\partial f}{\partial t} - \dfrac{\sigma^2}{2}\dfrac{\partial^2 f}{\partial x^2} = -\dfrac{\sigma^2}{2}\dfrac{\partial f}{\partial x}(p(t),t)\Big\{\delta(x-p(t)+a) - \delta(x-p(t)-a)\Big\} \text{ on } \mathbf{R}\times(0,\infty)\\ f(x,t) > 0 \;\text{ if }\; x < p(t), t \ge 0; f(x,t) < 0 \;\text{ if }\; x > p(t), t \ge 0,\\ f\,|_{t=0} = f_0 \;\text{ on }\; \mathbf{R}, p(0) = p_0.\end{cases}$$

And we assume that f_0 is a smooth function on $\mathbf{R}$ with fast decay at infinity (in all that follows, fast decay means that, for example, one can bound the function by $\frac{C}{1+|x|^2}$ for some positive constant C). Furthermore, we assume that (4) holds. Let us point out that these reduction and assumption require p_0 to be an equilibrium price at $t=0$ i.e., $f'(p_{0_+}) = f'(p_{0_-})$ or $(f_B^0)'(p_0) = -(f_V^0)'(p_0)$.

This is not strictly necessary for our analysis but if we do not make this assumption, f^0 cannot be smooth at p^0 and quite a few technicalities that we wish to avoid in this survey have to be incorporated.

Then, our main result is the following

Theorem 3.1. *Under the above conditions, there exists a unique smooth solution (f,p) of (3) such that f has fast decay for all $t \geq 0$.*

Both from a theoretical viewpoint and from a numerical approximation viewpoint, we also investigate the time-implicit discretization of (3) which takes the following form

$$\begin{cases} \lambda^2 f - \dfrac{d^2 f}{dx^2} = g - \dfrac{df}{dx}(p)(\delta_{p-a} - \delta_{p+a}) \\ f(x) > 0 \ \text{ if } \ x < p, f(x) < 0 \ \text{ if } \ x > p, \end{cases} \tag{68}$$

where $\lambda > 0$ (in a discretization, $\lambda^2 = \frac{2}{\sigma^2 \Delta t}$ where Δt is the time step ...).

Here, we take the classical Dirac mass δ to make the explicit computations below as simple as possible. And g is a given smooth function (with fast decay) such that we have for some p

$$g > 0 \ \text{ if } \ x < p_0, g < 0 \ \text{ if } \ x > p_0. \tag{69}$$

We now use the fact that $\frac{1}{2\lambda} e^{-\lambda |x|}$ is the Green's function of the operator $\lambda^2 - \frac{\sigma^2}{2} \frac{d^2}{dx^2}$ and we can thus recast (68) as

$$f = G - \frac{1}{2\lambda} \frac{df}{dx}(p) \left\{ e^{-\lambda|x-p+a|} - e^{-\lambda|x-p-a|} \right\}, \tag{70}$$

where $G = \frac{1}{2\lambda} e^{-\lambda|x|} \star g$. Hence, $f(p) = G(p)$. Therefore, p will be determined provided we show that G has a unique zero. Furthermore, in order to determine $\frac{df}{dx}(p)$, we observe that, if we differentiate (69) and choose $x = p$, we deduce from (70)

$$\frac{df}{dx}(p) = \frac{dG}{dx}(p) + e^{-\lambda a} \frac{df}{dx}(p).$$

And we only have to prove that G "crosses zero" at a single point. In order to do so, we find observe that without loss of generality, we may assume that $p_0 = 0$ (making a translation if necessary ...). And we write

$2\lambda G(x)$

$$= \int e^{-\lambda|x-g|} g(y)\, dy = \int_{-\infty}^{0} e^{-\lambda|x-y|} g_+(y)\, dy - \int_0^{+\infty} e^{-\lambda|x-y|} g_-(y)\, dy$$

$$= \begin{cases} e^{\lambda x} \left\{ e^{-2\lambda x} \int_{-\infty}^{x} e^{\lambda y} g_+(y)\, dy + \int_x^0 e^{-\lambda y} g_+(y)\, dy - \int_0^{+\infty} e^{-\lambda y} g_-(y)\, dy \right\} \\ \hfill \text{for } x > 0, \\ e^{-\lambda x} \left\{ \int_{-\infty}^{0} e^{\lambda y} g_+(y)\, dy - \int_0^x e^{\lambda y} g_-(y)\, dy - e^{2\lambda x} \int_x^{+\infty} e^{-\lambda y} g_-(y)\, dy \right\} \\ \hfill \text{for } x > 0. \end{cases}$$

We next observe that $S_-(x) = e^{-2\lambda x} \int_{-\infty}^{x} g_+(y) e^{\lambda y}\, dy + \int_x^0 e^{-\lambda y} g_+(y)\, dy - \int_0^{\infty} e^{-\lambda y} g_-(y)\, dy$ is decreasing for $x < 0$ since $S'_- = -2\lambda e^{-2\lambda x} \int_{\infty}^{x} g_+(y) e^{\lambda y}\, dy$, while $S_+(x) = \int_{-\infty}^{0} e^{\lambda y} g_+(y)\, dy - \int_0^x e^{\lambda y} g_-(y)\, dy - e^{2\lambda x} \int_x^{+\infty} e^{-\lambda y} g_-(y)\, dy$ is also decreasing for $x > 0$ since $S'_+ = -2\lambda e^{2\lambda x} \int_x^{+\infty} e^{-\lambda y}\, g_-(y)\, dy$. In addition, $S_-(0) = \int_{-\infty}^{0} g_+(y)\, dy - \int_{-\infty}^{0} g_-(y)\, dy = S_+(0)$. Therefore, G vanishes at most at one point and it does so if and only if $\lim\limits_{x \to -\infty} S_-(x) > 0 > \lim\limits_{x \to +\infty} S_+(x)$. And this last condition is easily seen to be equivalent to

(71)
$$\int_{-\infty}^{0} e^{-\lambda y} g_+(y)\, dy - \int_0^{+\infty} e^{-\lambda y} g_-(y)\, dy > 0 > \int_{-\infty}^{0} e^{-\lambda y} g_+(y)\, dy - \int_0^{+\infty} e^{\lambda y} g_-(y)\, dy.$$

This allows to prove the

Proposition 3.2. *There exists a solution (f, p) of (68) if and only if (71) holds. If it is the case, the solution is unique.*

3.3. Stationary problems

We consider in this section an example of the stationary version of (3). We do so in the simplest possible case where explicit solutions can be easily determined. In a fixed interval $(0, A)$ (with $A > 2a$), we consider the following stationary problem

$$(72) \quad \begin{cases} -\dfrac{\sigma^2}{2} \dfrac{d^2 f}{dx^2} = -\dfrac{\sigma^2}{2} \dfrac{df}{dx}(p)(\delta_{p-a} - \delta_{p+a}) \ \text{ in } (0, A) \\ \dfrac{df}{dx}(0) = \dfrac{df}{dx}(A) = 0, f > 0 \ \text{ if } \ x < p, f < 0 \ \text{ if } \ x > p, a < p < A - a. \end{cases}$$

where we prescribe the total number of agents and the total number of goods or equivalently $N_1 = \int_0^p f\, dx$ (the number of buyers) and $N_2 = \int_p^A (-f)\, dx$ (the number of vendors). Observing that we have

$$\begin{cases} \dfrac{df}{dx} = 0 \ \text{ if } \ x < p - a \ \text{ or if } \ x > p + a \\ \dfrac{df}{dx} = \dfrac{df}{dx}(p) \ \text{ if } \ p - a < x < p + a, \end{cases}$$

we deduce easily the following algebraic equations for p and $\Theta = -\frac{df}{dx}(p) > 0$.

$$N_1 = \Theta ap - \Theta\frac{a^2}{2}, N_2 = \Theta a(A-p) - \Theta\frac{a^2}{2}$$

hence $\frac{N_1-N_2}{N_1+N_2} = \frac{2p-A}{A-a}$ and a solution exists if and only if $\frac{|N_1-N_2|}{N_1+N_2} < 1 - \frac{a}{A-a}$, a restriction which corresponds to the restriction $p \in (a, A-a)$ ...Let us also observe that p is an increasing function of the ratio $\frac{N_1}{N_2}$, which is a very natural property from an economic viewpoint: indeed, if the (relative) number of buyers grows, one should indeed expect the price to go up!

4. Formation of volatility

4.1. The model

For a detailed presentation of the financial background, we refer the reader to J.-M. Lasry and P.-L. Lions [18, 19]. Let us only mention here that if one postulates that an agent impacts on the dynamics of an asset price by trading (either investing or hedging options), and that this impact is given by a simple linear (elastic) law as originally proposed by A.S. Kyle [15] (see also [3, 4, 2, 12, 11, 22] ...), one can show using stochastic control theory (and an extension of it that we developed in [16] because of this problem) that the volatility of the asset is modified by the gamma of the option (i.e., the second derivative of the option price). Next, if one assumes that the volatility in fact depends upon a "macroscopic" pay-off (some kind of cumulative pay-off), and that there exist a large number of players which all have a small impact on the volatility as soon as they trade an option, one is led to a mean field nonlinear differential equation for the volatility seen as a functional on a space of pay-offs. More precisely, let us consider for example pay-off functions in $X = C_b^{2,\alpha}(\mathbf{R})$ for some $\alpha \in (0,1)$ fixed (where $C_b^{2,\alpha}$ means the space of bounded $C^{2,\alpha}$ functions with bounded first and second derivatives). We look, as described in rough terms above, for a volatility mapping σ from X into $C_b^{0,\alpha}(\mathbf{R}\times[0,T]) = Y$ (bounded $C^{0,\alpha}$ functions). Then, as shown in [18], one obtains the following mean-field or self-consistent equation

$$\sigma'(\Phi) = k\sigma(\Phi)\,.\,\Gamma \quad \text{on } X \tag{5}$$

where $k > 0, \sigma(\Phi)$ denotes the multiplication operator by the function $\sigma(\Phi)$ and Γ is the operator defined on $C_b^{2,\alpha}$ (at least as long as $\sigma(\Phi)$ or $\sigma^2(\Phi)$ is bounded from below on $(0,\infty)$) by $\Gamma\Psi = \frac{\partial^2 u}{\partial x^2}$ and u solves the following parabolic equation (written backward in time)

$$\frac{\partial u}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} = 0 \text{ on } \mathbf{R}\times(0,T), u\,|_{t=T} = \Psi \text{ on } \mathbf{R}. \tag{6}$$

In (6), σ stands for the function of x and t given by $\sigma(\Phi)$.

In fact, the equation (5) is not complete since we should add the restriction that $\sigma(\Phi)$ lies in the open set O defined by $\{\sigma \in Y / \inf_{\mathbf{R}} \sigma > 0\}$. One can also formulate an equivalent problem in terms of $a = \sigma^2$ in which case (5) becomes

$$a'(\Phi) = 2ka(\Phi) \,.\, \Gamma \text{ on } X, a(\Phi) \in O. \tag{73}$$

And we wish to solve locally or globally (73) given an "initial" condition

$$a(\Phi_0) = a_0 \tag{74}$$

where Φ_0 is given arbitrarily in X and a_0 is given arbitrarily in O.

4.2. Local well-posedness

In order to make sure that (73), (74) is possibly a well-posed problem locally near any point Φ_0 in X, a compatibility condition is required. This condition is easily explained as a consequence of the symmetry of the second derivative operator. Indeed, if we apply (73) to a test function $\Psi_1 \in X$ and differentiate the resulting identity with respect to Φ in the direction of another test function $\Psi_2 \in X$, one obtains formally

$$\begin{aligned} a''(\Phi)(\Psi_1, \Psi_2) &= 2k(a'(\Phi) \,.\, \Psi_2)(\Gamma\Psi_1) + 2ka(\Phi). \\ &\quad .\left(\left\langle \frac{\partial \Gamma}{\partial a}, a'(\Phi) \,.\, \Psi_2 \right\rangle . \,\Psi_1\right) \\ &= 4k^2 a(\Phi)(\Gamma\Psi_2)(\Gamma\Psi_1) + 4k^2 a(\Phi) \,.\, \left(\left\langle \frac{\partial \Gamma}{\partial a}, a\Gamma\Psi_2 \right\rangle . \,\Psi_1\right). \end{aligned}$$

And thus we have to check the symmetry in (Ψ_1, Ψ_2) of the quantity

$$Q = \left\langle \frac{\partial \Gamma}{\partial a}, a\Gamma\Psi_2 \right\rangle . \,\Psi_1$$

The fact that Q is indeed symmetric in (Ψ_1, Ψ_2) is far from being obvious a priori and is shown as follows. We first observe that

$$Q = \frac{\partial^2}{\partial x^2}\left\{\left\langle \frac{\partial u}{\partial a}, a\Gamma\Psi_2 \right\rangle\right\}$$

where u is the solution of (6) with Ψ replaced by Ψ_1 and we look at it as a functional of a. Next, we remark that $v = \langle \frac{\partial u}{\partial a}, a\chi \rangle$ solves

$$\frac{\partial v}{\partial t} + \frac{a}{2}\frac{\partial^2 v}{\partial x^2} + \frac{a\chi}{2}\frac{\partial^2 u}{\partial x^2} = 0 \text{ on } \mathbf{R} \times (0,T), v\,|_{t=T} = 0. \tag{75}$$

And if we insert in (75) $\chi = \Gamma\Psi_2$ and we recall that $\frac{\partial^2 u}{\partial x^2} = \Gamma\Psi_1$, we realize that $u = \langle \frac{\partial u}{\partial a}, a\Gamma\Psi_2\rangle$ is nothing but the solution of

$$\frac{\partial v}{\partial t} + \frac{a}{2}\frac{\partial^2 v}{\partial x^2} + \frac{a}{2}(\Gamma\Psi_2)(\Gamma\Psi_1) = 0 \text{ on } \mathbf{R}\times(0,T), v\,|_{t=T} = 0.$$

And the symmetry of v and thus of Q follows.

Once we have checked this fundamental symmetry property, it is possible (using various known facts on parabolic equations) to prove that (73)–(74) can be solved uniquely on a maximal path-connected open set I of X containing Φ_0 with the following additional information: if $\Phi \in \partial I$ and $\Phi_n \in I, \Phi_n \xrightarrow{n} \Phi$ in X , then either $\|a(\Phi_n)\|_Y \xrightarrow{n} +\infty$ or $\inf_{\mathbf{R}} a(\Phi_n) \xrightarrow{n} 0$.

4.3. Global solution

We now show in this section how to solve globally (73)–(74) (in other words $I = X$) and to construct "almost" explicitly the solution.

This is a consequence of remarkable invariance property enjoyed by solutions of (73) namely

$$\frac{\partial}{\partial t}(\log a) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(a) \text{ is independent of } \Phi \tag{76}$$

if a solves (73). Indeed, we have

$$(\log a)'(\Phi) = 2k\Gamma$$

hence

$$\Big(\frac{\partial}{\partial t}(\log a)\Big)'(\Phi) = 2k\frac{\partial}{\partial t}\Gamma = 2k\frac{\partial^2}{\partial x^2}\Big(\frac{\partial U}{\partial t}\Big)$$

where U is the operator defining by $U\Psi = u$ solution of (6).

Therefore, we deduce

$$\begin{aligned}\Big(\frac{\partial}{\partial t}(\log a)\Big)'(\Phi) &= 2k\frac{\partial^2}{\partial x^2}\Big(\frac{a}{2}\frac{\partial^2}{\partial x^2}U\Big) = k\frac{\partial^2}{\partial x^2}(a\Gamma)\\ &= \frac{1}{2}\frac{\partial^2}{\partial x^2}(a'(\Phi))\\ &= \Big(\frac{1}{2}\frac{\partial^2 a}{\partial x^2}\Big)'(\Phi).\end{aligned}$$

Next, if a solves (73), (74), we deduce from (76) that $a(\Phi)$ as a function of (x,t) satisfies for any $\Phi \in X$

$$\frac{\partial}{\partial t}(\log a) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(a) = D \text{ on } \mathbf{R}\times(0,T) \tag{77}$$

where $D = \frac{\partial}{\partial t}(\log a_0) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(a_0)$.

And the construction of a will be complete using the nonlinear parabolic equation (77) provided we can identify a or $\log a$ at $t = T$. This is indeed possible since $U\Psi \mid_{t=T}$ is, by definition, Ψ hence $\Gamma\Psi \mid_{t=T} = \frac{d^2}{dx^2}(\Psi)$. Therefore, (73) implies that we have

$$(\log a)'(\Phi) \mid_{t=T} = 2k\frac{d^2}{dx^2}$$

and we conclude

$$(\log a(\Phi) - \log a_0) \mid_{t=T} = 2k\frac{d^2}{dx^2}(\Phi - \Phi_0) \text{ on } \mathbf{R}. \tag{78}$$

Or, $a(\Phi) \mid_{t=T} = (a_0 \mid_{t=T}) \exp(2k\frac{d^2}{dx^2}(\Phi - \Phi_0))$.

This allows us to identify completely $a(\Phi)$ thanks to (77) and (78).

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