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Local martingales, bubbles and option prices

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Abstract. In this article we are interested in option pricing in markets with bubbles. A bubble is defined to be a price process which, when discounted, is a local martingale under the risk-neutral measure but not a martingale. We give examples of bubbles both where volatility increases with the price level, and where the bubble is the result of a feedback mechanism. In a market with a bubble many standard results from the folklore become false. Put-call parity fails, the price of an American call exceeds that of a European call and call prices are no longer increasing in maturity (for a fixed strike). We show how these results must be modified in the presence of a bubble. It turns out that the option value depends critically on the definition of admissible strategy, and that the standard mathematical definition may not be consistent with the definitions used for trading.

Key words: Bubbles, feedback, local martingales, derivative pricing, put-call parity

JEL Classification: G13, D84

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1 Introduction

The last few decades have seen a spectacular growth in the trading of financial securities and derivatives, and a spectacular growth in the sophistication of the mathematical tools used by traders. They have also been associated with a series

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of bubbles and crashes where systematic over-pricing was followed by market corrections, often with serious adverse consequences for investors. The purpose of this article is to consider a class of models of a pricing bubble, and more importantly to consider the implications for option pricing within that class.

In economic terms a bubble is a deviation between the trading price of an asset and its underlying value. The problem with this definition, and the reason why bubbles are difficult to spot until they have burst, is that it depends on the elusive concept of underlying value. In his classic text 'A random walk down Wall Street' Malkiel [20] outlines two basic theories which describe how financial assets are assigned value. The first, called the 'firm-foundation' theory, attempts to define an intrinsic value for a share via an analysis of the balance sheet, the expected future dividends and the growth prospects of a company. The second, sometimes called the 'castle-in-the-air' or 'greater-fool' theory, says that an asset is worth whatever another investor will pay for it.

Within this framework there have been numerous attempts to explain why bubbles may arise. Some explanations fall under the general heading of 'Irrational exuberance' (Shiller [21]), and attempt to explain the mispricing in terms of investor psychology and herd behaviour. Indeed Shiller [22] also takes issue with the description of such behaviour as 'irrational', preferring to place the emphasis on human shortcomings rather than human error. One explanation he gives for speculative bubbles is that price increases in a stock lead to increased enthusiasm for, and attention upon, that stock, which in turn leads to increased demand. Further price increases follow in a feedback mechanism. Alternatively the price bubble can be seen as desirable for risk-loving (compared to the rest of the market) investors since there is the potential for much larger gains to be made.

In this work we are interested in the consequences of bubbles for derivative pricing. We want to remain within the mathematical paradigm of no-arbitrage, and for this reason we will assume the existence of an equivalent local martingale measure. The key fact is that we will assume that under the pricing measure the asset price is a strict local martingale. This is the sense in which our price process satisfies a bubble – the current price exceeds the expected discounted future price (under the risk-neutral measure). To a certain extent this work is expository in nature, and the new mathematics is confined to Theorem 4.3 and Lemma A.2. Instead the emphasis is on interpretation and re-interpretation of existing models and results.

Let S denote the discounted trading price of an asset. Suppose that under the pricing measure $\mathbb{Q}$, the price process is a local martingale. Since S is non-negative it is a supermartingale. We are interested in the consequences of assuming that S is not a martingale. In particular, we want to consider whether some basic properties of derivative prices remain valid: does put-call parity hold, are American and European prices the same for convex payoffs, do call prices tend to zero as strike increases to infinity, and, for diffusion models, are call prices convex in the underlying? The counter-intuitive answer to each of the above questions turns out to be no.

Some of the above questions were asked in the book by Lewis [19] and a paper of Heston et al. [15]. Lewis makes an extensive study of the option pricing PDE when the underlying is a local martingale, and makes the key observation that this PDE may have multiple solutions. Heston et al. [15] place an economic interpretation on these results and our definition of a bubble is taken from that paper. The main distinction between this paper and [19] and [15] is that we define the option price (in the standard mathematical fashion) as the cheapest initial fortune with which it is possible to super-replicate the option. In contrast, Lewis asserts that the price is given by a different formula, whereas Heston et al. propose a range of prices at which the option might trade. A further difference is that Lewis and Heston et al. use a PDE approach in a Markovian setting, whereas we express results in terms of local martingales and stochastics.

In Sect. 2 we give some examples of bubbles, both from financial history and in mathematical terms via strict local martingales. One of the examples has a direct interpretation in terms of a bubble via a feedback mechanism.

In Sects. 3, 4 and 5 we review some of the implications of our definitions and give several characterisations of the presence of a bubble. In Sect. 3 we show that many standard results from the folklore (such as put-call parity) fail to hold when there is a bubble. We then discuss some of the interesting conclusions which can be derived. In Sect. 4 we show that although call prices are not strictly increasing in maturity (for fixed strike) it is possible to derive static no-arbitrage restrictions which relate prices for calls with maturity T to calls with maturity T'.

One of the key ideas is the notion of an admissible strategy, and in Sect. 5 we discuss the ways in which the price of a derivative depends on the rules for option trading imposed by internal or external regulation. Some of these issues also arise in martingale models, including the Black-Scholes model, but they are much more apparent in a strict local martingale model where the prices of American and European options differ. For this reason the study of models with bubbles is very useful for the light it throws upon the issues of admissibility and arbitrage.

2 Examples

2.1 Examples of bubbles from history

There are many famous episodes in financial history of bubbles in prices. One of the earliest examples was the Dutch tulip mania in the 17th century. Speculative trading in tulip bulbs led first to astronomical increases in prices (there was a twenty-fold increase in prices in January 1637) and then, in February 1637, to a massive crash. See Malkiel [20] for more details on this and other examples.

In the 20th century financial markets (and other markets such as property) were subject to a series of bubbles and crashes. The Roaring Twenties, the sixties technology bubble, the roaring eighties, the Internet bubble of the 1990s – in each case (over)-optimism about a new era of progress and wealth generation led to a speculative growth in stock prices and a subsequent market correction. Recent instances of pricing bubbles leading to violations of, for example, put-call parity include Palm and RJR Nabisco. We refer the reader to Heston et al. [15] for a further discussion.

2.2 Examples of bubbles from mathematics

Let S be the discounted price of a financial security. We assume throughout that S is continuous, though it is clear that many of the arguments can be extended to the jump case. No-arbitrage theory tells us that S is a local martingale under the pricing measure. More precisely, for locally bounded semi-martingale price processes, there is no-free-lunch-with-vanishing-risk if and only if there exists an equivalent local martingale measure (Delbaen and Schachermayer [10]). Note that we specify models under the pricing measure rather than the physical measure. In this way we sidestep the issue of whether an equivalent local martingale measure exists.

For most examples (such as the Black-Scholes model) S is a martingale. However, for some models the price process is a local martingale, but not a true martingale. We will use the term strict local martingale to refer to the fact that S is a local martingale, but not a martingale. Note that, since S is non-negative we must have that S is a supermartingale.

Definition 2.1 *The price process S has a bubble if S is a strict local martingale under the risk-neutral measure $\mathbb{Q}$.*

The key fact is that simply because S is a strict local martingale does not mean that there are arbitrage opportunities. In particular, the strategy of selling the asset short may not be admissible, since the liability is unbounded.

2.2.1 Time inhomogeneous models

Throughout we suppose that t is the current time. Suppose $S_t = s > 0$ and

$$dS_u = \frac{S_u}{\sqrt{T-u}} dB_u.$$

Then $S_u = s\exp(\tilde{B}_{A_u} - A_u/2)$ where $\tilde{B}$ is a Brownian motion with $\tilde{B}_t = 0$, and $A_u = -\ln(1-(u-t)/T)$. The process is a true martingale over $[t,u]$ for each $u < T$, but $S_T = 0$, almost surely.

2.2.2 CEV models

Let S solve $S_t = s > 0$, $dS_u = S_u^\alpha dB_u$ for $\alpha > 1$. As Lewis [19] observes, in this case S is a strict local martingale.

If $\alpha = 2$ then it is possible to write down simple expressions for the transition density of the process. (When $\alpha = 2$, we have that S is the reciprocal of the radial part of 3-dimensional Brownian motion. This is the classical example, due to Johnson and Helms, of an L^2-bounded strict local martingale, and some of the implications for no-arbitrage pricing theory have already been investigated by Delbaen and Schachermayer [9].) We have

$$\begin{aligned}\mathbb{P}(S_T \in dz) = {} & \frac{s}{z^3} \frac{dz}{\sqrt{2\pi(T-t)}} \\ & \times \left\{ \exp\left(-\frac{(1/z - 1/s)^2}{2(T-t)} \right) - \exp\left(-\frac{(1/z + 1/s)^2}{2(T-t)} \right) \right\}.\end{aligned}$$

It follows that

$$\mathbb{E}[S_T|S_u] = \Delta(S_u, u) = S_u\left(1 - 2\Phi\left(-\frac{1}{S_u\sqrt{T-u}}\right)\right).$$

Note that $\Delta(S_u, u)$ is a true martingale, and moreover

$$d\Delta(S_u, u) = \left(1 - 2\Phi\left(-\frac{1}{S_u\sqrt{T-u}}\right)\right) dS_u = \Delta(S_u, u)S_u dB_u.$$

2.2.3 Stochastic volatility models

Consider the following stochastic volatility model, first proposed by Hull and White [18], under which the non-negative volatility process follows an exponential Brownian motion:

$$\begin{aligned} dS_u &= S_u V_u dB_u^S, \\ dV_u &= \eta V_u dB_u^V + \mu V_u du. \end{aligned}$$

Here the dynamics are specified under a risk-neutral measure $\mathbb{Q}$, B^S and B^V are Brownian motions with correlation ρ, and the initial values S_t and V_t are known positive constants. (The original Hull-White model took $\rho = 0$, but the extension to constant ρ is straight-forward.) Write $B^V = \rho B^S + \rho^\perp B^\perp$ where $\rho^\perp = +\sqrt{1-\rho^2}$ and $B^\perp$ is a Brownian motion orthogonal to B^S.

This model is incomplete, except when $|\rho| = 1$, and to that extent it falls outside the general philosophy of this article, but it still gives interesting examples of strict local martingales when ρ is positive, as we now show.

The following argument is essentially due to Sin [23], and can be applied to other stochastic volatility models. We want to decide whether S is a true martingale or a strict local martingale. Suppose S is a true martingale. Then we can define a probability measure $\mathbb{P}$, equivalent to $\mathbb{Q}$, via $d\mathbb{P}/d\mathbb{Q} = S_T/S_t$. By Girsanov's theorem, under $\mathbb{P}$, the process $B^{S,\mathbb{P}}$ defined via $dB_u^{S,\mathbb{P}} = dB_u^S - V_u du$ is a Brownian motion. Further, under $\mathbb{P}$, V solves

$$dV_u = \eta V_u(\rho dB_u^{S,\mathbb{P}} + \rho^\perp dB_u^\perp) + (\mu V_u + \eta\rho V_u^2)du.$$

If $\eta\rho > 0$ then this process explodes to plus infinity in finite time. Since V does not explode under $\mathbb{Q}$, this contradicts the assumption that $\mathbb{P}$ and $\mathbb{Q}$ are equivalent. Hence, when $\eta\rho > 0$, S is a strict local martingale.

2.2.4 Complete-market models of stochastic volatility

Consider the case of a complete stochastic volatility model (Hobson and Rogers [17], Davis [8]) where the underlying and the volatility are driven by the same Brownian motion.

As a first and simple example, suppose that under the pricing measure $\mathbb{Q}$

$$dS_u = S_u V_u dB_u, \qquad dV_u = V_u^2 dB_u$$

and $S_t = V_t = s$. Then (one solution is) $S_u = V_u$, for all $u \geq t$, and since V is a strict local martingale, so is S (this is the CEV model of Sect. 2.2.2).

The next example is adapted from [17] and is a model of a speculative feedback bubble. Suppose that $dS_u = S_u V_u dB_u$ where $V_u = \eta(\Sigma_u)$ and Σ is given by

$$\Sigma_u = \int_0^\infty \lambda e^{-\lambda r} \ln(S_u/S_{u-r})dr = S_u - \int_0^\infty \lambda e^{-\lambda r} \ln(S_{u-r})dr.$$

The idea is that Σ is a measure of the displacement between the current price level and its past average, and that the volatility of future stock price movements are a function of this displacement. (Here Σ is a proxy for the amount of attention the stock receives.) Then,

$$d\Sigma_u = \eta(\Sigma_u)dB_u - \left(\lambda\Sigma_u + \frac{1}{2}\eta(\Sigma_u)^2\right) du.$$

If S is a true martingale under $\mathbb{Q}$ then S_T/S_t is the density with respect to $\mathbb{Q}$ of an equivalent measure $\mathbb{P}$, and then under $\mathbb{P}$,

$$d\Sigma_u = \eta(\Sigma_u)dB_u^{\mathbb{P}} - \left(\lambda\Sigma_u - \frac{1}{2}\eta(\Sigma_u)^2\right) du.$$

If $\eta(\Sigma)^2 = (1 + \Sigma^2)$ the behaviour of Σ is different under the two sets of dynamics: in one case it explodes to minus infinity, and in the other it explodes to plus infinity. Hence $\mathbb{Q}$ and $\mathbb{P}$ are not equivalent, S is not a true martingale, and the price process S has a speculative feedback bubble.

2.2.5 Bubbles in the Black-Scholes world

Suppose that S is an exponential Brownian motion with unit volatility.

For real x and positive r define

$$v(x,r) = \sum_{n\geq 0} \frac{x^{2n}}{(2n)!} h^{(n)}(r).$$

Here $h^{(n)}$ denotes the n^{th} derivative of h, and h is any non-negative (and not identically zero) function for which h and and all its derivatives at 0 are zero. The canonical example is $h(r) = e^{-1/r}$. Note that v as defined solves the heat equation $v_{xx} = v_r$ with zero initial condition $v(x,0) = 0$. Now set

$$Y_u = \Lambda(S_u, u) = S_u^{1/2} e^{-(T-u)/8} v\left(\ln S_u, \frac{T-u}{2}\right), \tag{1}$$

so that $dY_u = \Lambda'(S_u,u)dS_u$ where Λ' denotes the space derivative. The process Y is a local martingale for which $Y_T = 0$, almost surely.

We can think of Y as a degenerate price for the derivative security with payoff zero at expiry. Further it is possible to add Y to the price process of any derivative to create a new price process with bubble-like characteristics. We return to a discussion of this example at the end of Sect. 5.

2.3 Alternative definitions of bubbles

Although Definition 2.1 captures many of the essential features of bubbles, it is not the only definition used in the literature. For example, Andersen and Sornette [1] propose a model with exponential growth followed by a downward jump (or crash) at an unpredictable stopping time. Much closer in spirit to this work is the paper by Cassese [6]. Recall that we work under the risk neutral measure $\mathbb{Q}$, which we assume to exist and to be equivalent to the real-world measure, here denoted $\mathbb{R}$. Cassese works under $\mathbb{R}$, and considers the situation where Z is a strict local $\mathbb{R}$-martingale, where Z is the candidate change of measure density. In particular, there is no equivalent local martingale measure $\mathbb{Q}$: instead Z can be used to define a finitely additive measure. In that sense the paper of Cassese might be described as a discussion of the no-arbitrage condition rather than bubbles (see Sect. 5 of [6]). However, many of the conclusions which we derive have parallels in situations considered by Cassese. For example, Theorem 3.4(iii) should be compared with Example 9.2 of Fernholz et al. [13]. In their model of a weakly diverse market Fernholz et al. show that put-call parity fails. However, this is a consequence of the fact that the candidate Girsanov change of measure martingale is a strict local martingale, and that the model contains a free-lunch-with-bounded-risk, rather than a bubble in the sense of Definition 2.1.

3 European and American options

We work on a probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q})$, where T_∞ is the terminal date of the economy and $\mathcal{F}_{T_\infty} = \mathcal{F}$. Defined on this space there is a complete market model of a traded asset with continuous (discounted) price process S, so that by assumption any $\mathbb{Q}$-local martingale can be written as a stochastic integral of a predictable process θ with respect to S.

Consider an option on the traded asset with non-negative payoff $H \equiv H(S_T)$, and expiry $T < T_\infty$. We are interested in the problem of pricing the derivative H from the point of view of a financial institution which is considering the sale of the derivative.

We begin by defining the set of admissible strategies available to this institution.

Definition 3.1 *An admissible wealth process is a self-financing process W of the form $W_u = W_t + (\theta \cdot S)_u$, where θ is a predictable S-integrable process, such that*

$$\lim_k k\mathbb{Q}\left(\inf_{u\in[t,T]} W_u < -k\right) = 0.$$

The set $\mathcal{W}$ of admissible wealth processes contains as subsets the sets

$$\begin{aligned}
\mathcal{W}_+ &= \{W_u : W_u \geq 0, \forall u \in [t,T], a.s.\},\\
\mathcal{W}_f &= \bigcup_a \{W_u : W_u \geq -a, \forall u \in [t,T], a.s.\},\\
\mathcal{W}_{mg} &= \{W_u : W_u \geq M_u, \forall u \in [t,T], \text{for some martingale } M\}.
\end{aligned}$$

Each of these sets has been proposed as an appropriate definition of admissibility (Harrison and Pliska [14], Dybvig and Huang [11], Ansel and Stricker [2]). The need for some restriction to rule out doubling strategies was first observed in [14]. Our formulation is due to Strasser [24]. The key point is that by any of these definitions an admissible wealth process is a $\mathbb{Q}$-supermartingale.

Given a definition of admissible strategies, it is now possible to define (in the standard mathematical fashion) the fair price for a derivative. We recall this definition here, since later we will discuss alternative definitions.

Definition 3.2 *The fair price of a derivative security is the smallest initial fortune required to finance an admissible super-replicating wealth process.*

Theorem 3.3 *The time-t fair price $V_t^E(H)$ of a European option with payoff H is given by*

$$V_t^E(H) = \mathbb{E}_t[H(S_T)]$$

and the fair price $V_t^A(H)$ of an American option is given by

$$V_t^A(H) = \sup_{t \le \tau \le T} \mathbb{E}_t[H(S_\tau)].$$

Proof The proof is based on the fact that $V_u^E(H) = \mathbb{E}_u[H(S_T)]$ is a (super)-martingale which super-replicates the option payout, and can be shown to be the smallest such process. Since H is non-negative it is admissible by the complete market assumption. The American option can be treated similarly. □

When $H(x) = (x - K)^+$ for some $K \ge 0$ we write $C_t^E(K)$ and $C_t^A(K)$ for the European and American call prices. Similarly, we denote the prices of put options by $P_t^E(K)$ and $P_t^A(K)$.

Suppose, temporarily, that S is a true martingale. It follows from the identity

$$(S_T - K)^+ - (K - S_T)^+ = S_T - K \tag{2}$$

that put-call parity holds:

$$C_t^E(K) - P_t^E(K) = S_t - K. \tag{3}$$

Now consider options with a general non-negative convex payoff H. Jensen's inequality tells us that it is never optimal to exercise the American option before maturity, and that

$$V_t^E(H) = V_t^A(H) \qquad \forall H \ge 0, \quad H \text{ convex}. \tag{4}$$

If S is a strict local martingale, then both of the statements (3) and (4) are false. Instead we have the following result:

Theorem 3.4 *The local martingale S has a bubble if and only if any of the following conditions holds:*

(i) *S is a strict supermartingale;*
(ii) $\mathbb{E}_t[S_T] < S_t$, *so that the forward price is below the current price;*
(iii) $C_t^E(K) - P_t^E(K) < S_t - K$ *for some K, so that put-call parity fails;*
(iv) $\lim_{K\uparrow\infty} P_t^E(K) - K + S_t > 0$*;*
(v) $C_t^E(K) < C_t^A(K)$ *for some K, so that American calls are more expensive than their European counterparts;*
(vi) $\lim_{K\uparrow\infty} C_t^A(K) > 0$.

In all these cases $C_t^A(K) - C_t^E(K) = S_t - \mathbb{E}_t(S_T) > 0$ which is independent of K, so that if (iii) or (v) holds for some K then it holds for all K.

Another necessary and sufficient condition for a bubble is that

$$\limsup_k k\mathbb{Q}\left(\sup_{u\in[t,T]} S_u > k\right) > 0, \tag{5}$$

Proof By definition S has a bubble if (i) holds. Further, if S is a non-negative local martingale then S is a supermartingale and (ii) is a necessary and sufficient condition for S to be a strict supermartingale. The equivalence of (ii) and (iii) follows on taking expectations in (2).

Note that $P_t^E(K) - K + S_t = S_t - \mathbb{E}_t[S_T] + C_t^E(K)$. By definition $C_t^E(K) = \mathbb{E}_t[(S_T - K)^+]$ which tends to zero as $K \uparrow \infty$, so that (iv) if and only if (ii).

The final condition is taken from Azéma et al. [3].

To complete the proof, and to prove (v) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (vi) it is sufficient to apply Theorem A.1 with $G(x) = (x - K)^+$. $\square$

Example 3.5 Consider the example given in Sect. 2.2.1. Then $S_T = 0$ almost surely, and $C_t^E(K) = 0$, $P_t^E(K) = K$. Hence put-call parity fails. Also, by comparison with holding the stock we immediately see that $C_t^A(K) \le s$; equality follows from considering the first-hitting times of levels n.

Example 3.6 Suppose $dS_t = S_t^2 dB_t$. Then

$$\begin{aligned}\mathbb{E}[(S_T - K)^+] = S_t\,&\{\Phi(\kappa - \delta) - \Phi(-\delta) + \Phi(\delta) - \Phi(\delta + \kappa)\}\\ &- K\left\{\Phi(\kappa + \delta) - \Phi(\delta - \kappa) + \delta^{-1}\left[\Phi'(\kappa + \delta) - \Phi'(\kappa - \delta)\right]\right\}\end{aligned}$$

where $\delta^{-1} = S_t\sqrt{T-t}$, $\kappa^{-1} = K\sqrt{T-t}$ and Φ and Φ' are the cumulative normal distribution and density respectively.

Note that

$$\lim_{s\uparrow\infty} \mathbb{E}[(S_T - K)^+] = \frac{2}{\sqrt{2\pi(T-t)}} - K\left[2\Phi\left(\frac{1}{K\sqrt{T-t}} - 1\right)\right].$$

so that the price of a call option is bounded when considered as a function of S_t.

Example 3.6 can also be used as a counter-example to another standard result from the folklore of option pricing. Suppose S is a diffusion process. There are several proofs (Bergman et al. [4] via PDEs, El Karoui et al. [12] via stochastic flows, Hobson [16] via coupling) that subject to certain regularity conditions, the price of an option with a convex payoff is convex in S. Example 3.6 shows that this can fail when S is a strict local martingale; the European price of an option with convex payoff $H(S_T) = S_T$ is not convex in s.

4 Prices at intermediate times

Given time-t call prices with maturity T, what are the possible call prices at time T' consistent with no arbitrage? Rather than specify a model and finding a model-specific solution, we will consider the problem from the point of view of a financial institution which observes traded call prices. The only assumption we make is that these observed prices are consistent with some model and are defined according to the formulas in Theorem 3.3.

Since call prices are now also a function of the maturity of the option we write $C_t^E = C_t^E(K,T)$. For a fixed time T, if call prices are given by Definition 3.2, then $C_t^E(K,T)$ is decreasing and convex in K. If these conditions fail then there is a simple static strategy (e.g. a spread or a butterfly spread) which can be used to generate arbitrage. Conversely, if there is a model under which the asset price is a local martingale and under which the expected call payoffs are given by $C_t^E(K,T)$, then there is no static portfolio of investments in the calls which can be used to generate arbitrage.

Definition 4.1 *There is no static arbitrage in the family of call prices $\{C_t^E(K_\alpha, T_\alpha); \alpha \in \mathcal{A}\}$ if there is a model for which S is a non-negative local martingale, and under which call prices with strikes K_α and maturities T_α are given by $C_t^E(K_\alpha, T_\alpha)$.*

If S is a true martingale then call prices are increasing in T. It is well known that if $t < T' < T$ then a necessary and sufficient condition for there to be no static arbitrage in the pair of call price functions

$$\{C_t^E(K',T'), C_t^E(K,T); K' \geq 0, K \geq 0\}$$

is that $C_t^E(K'',T'')$ – for T'' equal to each of T, T' – is a decreasing, convex function of K'' with $C_t^E(0,T'') = S_t$ and $\lim_{K''\uparrow\infty} C_t^E(K'',T'') = 0$, and together they satisfy $(S_t - K)^+ \leq C_t^E(0,T') \leq C_t^E(0,T)$. However, it is easy to see from the example of a call with strike zero that this is no longer true when S is a strict supermartingale.

Instead the following result is true:

Theorem 4.2 *Fix $t < T' < T$. There is no static arbitrage provided $C_t^E(K,T')$ and $C_t^E(K,T)$ are decreasing convex functions of K with zero limit at strike infinity, $0 \leq C_t^E(0,T) \leq C_t^E(0,T') \leq S_t$ and*

$$(S_t - K)^+ - S_t + C_t^E(0,T') \leq C_t^E(K,T') \leq C_t^E(K,T) - C_t^E(0,T) + C_t^E(0,T'),$$

or equivalently

$$(S_t - K)^+ - S_t \leq C_t^E(K,T') - \mathbb{E}_t(S_{T'}) \leq C_t^E(K,T) - \mathbb{E}_t(S_T).$$

Given the relationships between puts and calls, the theorem will follow immediately from the following theorem.

Theorem 4.3 *Fix $t < T' < T$. There is no static arbitrage provided $P_t^E(K,T')$ and $P_t^E(K,T)$ are non-negative, increasing, convex functions of K with*

$$(K - S_t)^+ \le P_t^E(K,T') \le P_t^E(K,T) \le K. \tag{6}$$

Proof We need to show that (6) is both necessary, and, in combination with the other conditions, sufficient. Necessity follows from the following argument using monotonicity and Jensen:

$$P_t^E(K,T') = \mathbb{E}_t[(K - S_{T'})^+] \le \mathbb{E}_t[(K - \mathbb{E}_{T'}[S_T])^+] \le \mathbb{E}_t[(K - S_T)^+],$$

together with an equally trivial argument for the lower bound.

Now we need to show sufficiency. To do this we give a model which is consistent with the given put price functions. We are going to do this over the interval $[T', T]$, but it is clear we could repeat the construction over $[t, T']$ and by concatenation we have a model which prices correctly at t, T' and T.

By the results of Breeden and Litzenberger [5], given $P_t^E(K,T) = \mathbb{E}_t[(K - S_T)^+]$ it is possible to infer the time-T law of S from the relationship $P_t^E(K,T) = \mathbb{E}_t[(K - S_T)^+]$. Denote the laws of $S_{T'}$ and S_T by μ_0 and μ_1 respectively. Let m_i denote the mean of μ_i, so that $m_1 \le m_0$. Define $P_i(k) = \int_0^\infty (k - x)^+ \mu_i(dx)$, so that $P_0(k) = P_t^E(k, T')$ for example. We are going to construct a local martingale process Y_u defined on $[0,1]$ such that $Y_0 \sim \mu_0$ and $Y_1 \sim \mu_1$.

If $m_0 = m_1$ then $P_0(k) \le P_1(k)$, uniformly in k, and this is exactly the necessary and sufficient condition for there to exist a martingale with the desired initial and terminal laws. Indeed there are many solutions to the Skorokhod embedding problem (for an example see Chacon and Walsh [7]) which describe how to construct a martingale (specifically a time-change of Brownian motion) for which $Y_0 \sim \mu_0$ and $Y_1 \sim \mu_1$.

So suppose $m_1 < m_0$. Let z be the unique positive root of F where $F(k) = k - m_1 - P_0(k)$. Fix $\epsilon \in (0,1)$ and define the increasing convex function P_ϵ by

$$P_\epsilon(k) = \begin{cases} P_0(k) & k < z \\ k - m_1 & k \ge z \end{cases}$$

Let μ_ϵ be the probability measure associated with P_ϵ in the sense of Breeden and Litzenberger.

We plan to construct Y such that Y is a non-negative local martingale, $Y_0 \sim \mu_0$, $Y_\epsilon \sim \mu_\epsilon$ and $Y_1 \sim \mu_1$.

Let B be a Brownian motion with $B_0 \sim \mu_0$ and define $\tau = \inf\{r : B_r \le z\}$. Then $B_\tau \sim \mu_\epsilon$, and, for $u \le \epsilon$ we set $Y_u = B_{\tau \wedge (\epsilon - u)^{-1}}$.

For the final part, note that $m_\epsilon = m_1$ and $P_\epsilon(k) \le P_1(k)$. Hence, we can use any martingale solution to the Skorokhod embedding problem to generate a suitable continuation Y_u. □

5 European calls with collateral requirements

When the underlying price process is a local martingale, but not a martingale, the price process is necessarily unbounded above. Indeed, it must be the case that $\limsup x\mathbb{Q}(\sup_{0\leq t\leq T} S_t > x) > 0$, recall (5). In these cases Heston et al. [15] argue that the standard mathematical definition of the European option price of Definition 3.2 is not the economically meaningful one.

Consider a call option in one of the models described in Sect. 2.2. In a strict local martingale model there is a probability of order $1/n$ that the value of the underlying will rise to n, at which point the intrinsic value of the option will be large. Most exchanges require traders who are short options to post collateral in such circumstances. Even though the trader may be following a trading strategy corresponding to a super-replicating wealth process, it may not generate sufficient wealth to cover these collateral requirements. Thus, for a wealth process W to describe a super-replicating strategy, in practice it must satisfy both the terminal condition $W_T \geq H(S_T)$ and a collateral condition at intermediate times.

A related phenomenon was applicable during the Internet bubble. European call options on stocks cannot be exercised before maturity, but the terms and conditions of options on Internet stocks often included the proviso that if the firm was subject to a take-over at time $T' < T$, then the option paid $(S_{T'} - K)^+$. In order to super-replicate the call option it is necessary to have a wealth process which satisfies both a condition at maturity and this condition at intermediate times. (Note that if S is a true martingale then a super-replicating wealth process for a call automatically satisfies $W_u \geq \mathbb{E}_u[(S_T - K)^+] \geq (S_u - K)^+$, but if the underlying is a local martingale this need not be the case.)

Definition 5.1 *Consider a contingent claim with maturity T. The fair price of an option with payoff H and collateral requirement G is the smallest initial fortune which is required to construct a self-financing wealth process W satisfying the super-replication condition $W_T \geq H(S_T)$ and the collateral condition $W_u \geq G(S_u)$.*

Essentially the above definition involves redefining the set of admissible strategies to be the set of self-financing wealth processes which satisfy $W_u \geq G(S_u)$.

It is clear that any super-replicating wealth process W for $H(S_T)$ which also satisfies $W_u \geq G(S_u)$ is also a super-replicating wealth process for an American-style claim which pays $H(S_T)$ if exercised at time T, and $G(S_u)$ if exercised at time $u < T$. The converse also holds in that a super-replicating wealth process for the American claim is also a super-replicating wealth process satisfying a collateral requirement. Hence there is an equivalence between the price of a contingent claim with collateral requirement, and the price of American options where the payoff depends on the time of exercise.

Let $V_t(G, H)$ denote the smallest fortune with which it is possible to super-replicate the claim. Note that $V_t(0, H) = V_t^E(H)$ and $V_t(H, H) = V_t^A(H)$. The following result is proved in the appendix.

Theorem 5.2 *Suppose G is a positive convex function, with $\limsup G(x)/x = \beta$, and that $H \geq G$ is arbitrary. Then $V_t(G, H) = \mathbb{E}_t[H(S_T)] + \beta(S_t - \mathbb{E}_t[S_T])$.*

Corollary 5.3 *Consider a European call option and suppose that the exchange requires collateral of* $\alpha(S_u - K)^+$*, with* $\alpha \in (0,1]$*, to be posted when the option is in the money. Then the fair price of the option is* $(1-\alpha)C_t^E(K) + \alpha C_t^A(K) = C_t^E(K) + \alpha(S_t - \mathbb{E}_t[S_T])$.

Consider now the actual price (rather than the fair price) of the option. Lewis [19, p.295] asserts that this price should be $C_t^A(K)$, but does not give a convincing reason for his choice. In contrast, Heston et al. [15] argue either C_t^E or C_t^A could be the traded price, amongst others, and that without some additional conditions it is not possible to determine the true price. We have shown that the correct additional condition is related to the notion of admissible strategy. Given a zero collateral requirement the unique, arbitrage-free fair price of a European call option is $C_t^E(K)$, and with a collateral requirement of $(S_u - K)^+$ the fair price is $C_t^A(K)$.

Now consider a similar question, but from a different perspective. Suppose that the option trades for price C_t^{Tr}. Is it possible to make arbitrage profits?

Again, the correct answer depends on the notion of admissibility, and this phenomenon is not a byproduct of a strict local martingale model, but rather is to be found in almost any model. Consider the Black-Scholes model, and consider a claim with zero payoff. Suppose the time-t traded price of this claim is given by $V_t^{Tr} = \Lambda(S_t, t)$, where Λ is as given by (1), and suppose moreover that the traded price process of the claim is $V_u^{Tr} = \Lambda(S_u, u)$. Consider an agent who sells such a claim, for price V_t^{Tr}, and who makes no effort to hedge the claim. At maturity the agent has a fortune V_t^{Tr}, and an arbitrage.

Now suppose the agent is required to keep his net wealth positive, where net wealth includes the obligation resulting from the claim sold short. In particular the agent must have $\tilde{W}_u \geq 0$ where

$$\tilde{W}_u = V_t^{Tr} + (\theta \cdot S)_u - V_u^{Tr}.$$

Since this net wealth process is a non-negative local martingale, started at zero, it must be identically zero, and the agent cannot generate an arbitrage. This argument extends easily to net wealth processes satisfying $W_T \geq 0$ (super-replication) and an admissibility condition on net wealth of the form

$$\lim_k k\mathbb{Q}\left(\inf_{u\in[t,T]} \tilde{W}_u < -k\right) = 0.$$

In summary, even though there is an apparent arbitrage which results from the claim with zero payoff trading at a positive price, it may not be possible to make profits from this situation. It is observations such as this which underpin the importance of Definitions 3.2 and 5.1.

6 Conclusion

Most examples of stock prices are such that the stock is a martingale under the risk-neutral measure. In such a model many simple properties such as put-call parity follow trivially. However, the general no-arbitrage pricing theory does not require the stock price to be a true martingale; instead it merely presupposes that the stock price is a local martingale.

Following Heston et al. [15], we give an economic interpretation to this strict local martingale property in terms of a bubble. There are several examples from the literature where, wittingly or otherwise, the proposed dynamics correspond to a strict local martingale; similarly there are repeated examples from economic history of pricing bubbles.

Great care is needed when pricing options under such a model as many intuitively obvious statements turn out to be false – the presence of a bubble requires us to be careful about the definitions of admissible strategies. It may be that the standard mathematical definition is not the appropriate financial definition if collateral restrictions are imposed.

Appendix: Optimisation for local martingales

Suppose $M_0 = 1$ and that $(M_t)_{0 \le t \le 1}$ is a continuous non-negative $(\mathbb{Q}, \mathcal{F}_t)$-local martingale. Let $m_1 = \mathbb{E}[M_1]$; then $m_1 \le 1$ with equality if M is a true martingale, otherwise M is a strict supermartingale.

Theorem A.1 *Let $G : \mathbb{R}^+ \to \mathbb{R}$ be a convex function such that $\limsup_{x\uparrow\infty} G(x)/x = \beta \in (-\infty, \infty]$. Then, for τ a stopping time,*

$$\sup_{\tau \le 1} \mathbb{E}[G(M_\tau)] = \mathbb{E}[G(M_1)] + \beta^+(1 - m_1) \tag{7}$$

In the true martingale case where $m_1 = 1$ the result follows easily from an application of a conditional Jensen's inequality. Hence it is sufficient to consider the case where $m_1 < 1$ and the local martingale M is a strict supermartingale.

For $x > 0$ define $H_x = \inf\{u : M_u \ge x\}$ where the infimum of the empty set is taken to be 1. To cover the case where we start at time t, let Θ_t denote the shift operator, so that $H_x \circ \Theta_t = \inf\{u > t : M_u \ge x\} \wedge 1$. Then $(M_{u \wedge (H_x \circ \Theta_t)})_{t \le u \le 1}$ is a bounded local martingale and hence a a true martingale. Hence, for $t < 1$,

$$\begin{aligned} M_t &= \mathbb{E}_t[M_{H_x \circ \Theta_t}] \\ &= M_t I_{(M_t > x)} + x\mathbb{Q}_t(t < H_x \circ \Theta_t < 1) + \mathbb{E}_t[M_1 I_{(H_x \circ \Theta_t = 1)}]. \end{aligned}$$

Taking limits and rearranging we find

$$M_t - \mathbb{E}_t[M_1] = \lim_{x\uparrow\infty} x\mathbb{Q}_t(H_x \circ \Theta_t < 1) = \lim_{x\uparrow\infty} x\mathbb{Q}_t\left(\sup_{u\in[t,1]} M_u \ge x\right).$$

Write $\gamma_t = M_t - \mathbb{E}_t[M_1]$. Then $\gamma_t = 0$ if M is a true martingale, whereas in the case we are considering $\gamma_t \ge 0$.

Theorem A.1 follows on taking $t = 0$ and $F = G$ in the following lemma, however we need this more general form in Sect. 5.

Lemma A.2 *Suppose that G is convex and $\limsup_{x\uparrow\infty} G(x)/x = \beta$. Let F be any function with $F \geq G$ and let $J(x,t)$ be defined via*

$$J(x,t) = G(x)I_{(t<1)} + F(x)I_{(t=1)}.$$

Then

$$\sup_{t\leq\tau\leq 1} \mathbb{E}_t[J(M_\tau,\tau)] = \mathbb{E}_t[F(M_1)] + \beta^+\gamma_t. \tag{8}$$

Proof Suppose $\beta \leq 0$ whence G is a decreasing convex function. The result then follows easily using Jensen's inequality, and the fact that M is a supermartingale.

So suppose $\beta > 0$. Fix $\epsilon > 0$ and let $x_n \uparrow \infty$ be a sequence such that $G(x_n)/x_n > \beta - \epsilon$. Write H_n as shorthand for $H_{x_n} \circ \Theta_t$. Then, for n large enough so that $M_t < x_n$,

$$\begin{aligned}\sup_{t\leq\tau\leq 1} \mathbb{E}_t[J(M_\tau,\tau)] &\geq \mathbb{E}_t[J(M_{H_n}, H_n)]\\ &= \mathbb{E}_t[F(M_1)I_{(H_n=1)}] + G(x_n)\mathbb{Q}_t(H_n < 1).\end{aligned}$$

Taking limits, and since ϵ is arbitrary, we conclude that

$$\sup_{t\leq\tau\leq 1} \mathbb{E}_t[J(M_\tau,\tau)] \geq \mathbb{E}_t[F(M_1)] + \beta\gamma_t.$$

Now we wish to prove the reverse inequality. (Note that there is nothing to prove if $\beta = \infty$.) Let Y^F, Y^G be the martingales given by $Y^F_u = \mathbb{E}_u[F(M_1)]$ and $Y^G_u = \mathbb{E}_u[G(M_1)]$. Suppose we can show that

$$J(M_u, u) \leq Y^F_u + \beta\gamma_u \tag{9}$$

for all u including $u = 1$. The right-hand-side of this expression is a supermartingale and if (9) holds for all u then we have $J(M_\tau,\tau) \leq Y^F_\tau + \beta\gamma_\tau$ for all stopping times τ. It follows that

$$\mathbb{E}_t[J(M_\tau,\tau)] \leq \mathbb{E}_t[Y^F_\tau + \beta\gamma_\tau] \leq Y^F_t + \beta\gamma_t,$$

and we are done.

Clearly (9) holds when $u = 1$. For $u < 1$ we show $G(M_u) \leq Y^G_u + \beta\gamma_u$ from which (9) follows instantly.

Define $G^\beta(x) = \beta x - G(x)$. Then G^β is an increasing concave function with $\liminf_{x\uparrow\infty} G^\beta(x)/x = 0$. By Jensen's inequality

$$\mathbb{E}_u[G^\beta(M_1)] \leq G^\beta(\mathbb{E}_u[M_1]) \leq G^\beta(M_u).$$

Rewriting this in terms of G we find

$$G(M_u) \leq \beta(M_u - \mathbb{E}_u[M_1]) + \mathbb{E}_u[G(M_1)] = \beta\gamma_u + Y^G_u,$$

and the result is proved. $\square$

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