

# $\epsilon$ -NASH MEAN FIELD GAME THEORY FOR NONLINEAR STOCHASTIC DYNAMICAL SYSTEMS WITH MAJOR AND MINOR AGENTS\*

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**Abstract.** This paper studies large population dynamic games involving nonlinear stochastic dynamical systems with agents of the following mixed types: (i) a major agent and (ii) a population of  $N$  minor agents where  $N$  is very large. The major and minor agents are coupled via both (i) their individual nonlinear stochastic dynamics and (ii) their individual finite time horizon nonlinear cost functions. This problem is analyzed by the so-called  $\epsilon$ -Nash mean field game theory. A distinct feature of the mixed agent mean field game problem is that even asymptotically (as the population size  $N$  approaches infinity) the noise process of the major agent causes random fluctuation of the mean field behavior of the minor agents. To deal with this, the overall asymptotic ( $N \rightarrow \infty$ ) mean field game problem is decomposed into (i) two nonstandard stochastic optimal control problems with random coefficient processes which yield forward adapted stochastic best response control processes determined from the solution of (backward in time) stochastic Hamilton–Jacobi–Bellman (SHJB) equations and (ii) two stochastic coefficient McKean–Vlasov (SMV) equations which characterize the state of the major agent and the measure determining the mean field behavior of the minor agents. This yields a stochastic mean field game (SMFG) system which is in contrast to the deterministic mean field game systems of standard MFG problems with only minor agents. Existence and uniqueness of the solutions to SMFG systems (SHJB and SMV equations) is established by a fixed point argument in the Wasserstein space of random probability measures. In the case where minor agents are coupled to the major agent only through their cost functions, the  $\epsilon_N$ -Nash equilibrium property of the SMFG best responses is shown for a finite  $N$  population system where  $\epsilon_N = O(1/\sqrt{N})$ .

**Key words.** mean field games, mixed agents, stochastic dynamic games, stochastic optimal control, decentralized control, stochastic Hamilton–Jacobi–Bellman equation, stochastic McKean–Vlasov equation, Nash equilibria

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**1. Introduction.** An important class of games is that of dynamic games with a very large number of minor agents in which each agent interacts with the average (or so-called mean field) effect of other agents via couplings in their individual dynamics and individual cost functions. A minor agent is an agent which, asymptotically as the population size goes to infinity, has a negligible influence on the overall system while the overall population’s effect on it is significant. Stochastic dynamic games with mean field couplings arise in fields such as wireless power control [14], consensus dynamics [33], flocking [34], charging control of plug-in electric vehicles [28], synchronization of coupled nonlinear oscillators [45], crowd dynamics [8], and economics [44, 11].

For large population stochastic dynamic games with mean field couplings and no major agent,  $\epsilon$ -Nash mean field game ( $\epsilon$ -NMFG) (or Nash certainty equivalence)

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theory was originally developed as a decentralized methodology in a series of papers by Huang together with Caines and Malhamé; see [14, 16] for the  $\epsilon$ -NMFG linear-quadratic-Gaussian (LQG) framework and [17, 15, 7] for a general formulation of nonlinear McKean–Vlasov (MV) type  $\epsilon$ -NMFG problems. For this class of game problems a closely related approach was independently developed by Lasry and Lions [25, 24, 26, 11], where the term mean field games (MFGs) was introduced. For models of the dynamics of industries with many firms, Weintraub, Benkard, and Van Roy proposed the notion of oblivious equilibrium by use of mean field approximations [43]. The  $\epsilon$ -NMFG framework for LQG systems is extended to systems of agents with ergodic (long time average) costs in [27], while Kolokoltsov, Li, and Wang extended the  $\epsilon$ -NMFG theory to general nonlinear Markov systems [22]. The extension of the  $\epsilon$ -NMFG framework so as to model the collective system dynamics which include large population of leaders and followers, and an unknown (to the followers) reference trajectory for the leaders, is studied in [32]. Adaptive  $\epsilon$ -NMFG algorithms for LQG systems with ergodic cost functions are developed in [20], where the agents estimate their own dynamical parameters and the population's dynamical and cost function distribution parameters. See the survey paper [4] for some of the research on MFG theory up to 2011.

The central idea of the  $\epsilon$ -NMFG theory is to establish a certain equilibrium relationship between the individual strategies and the mass effect (i.e., the overall effect of the population on a given agent) as the population size goes to infinity [16]. Specifically, in the equilibrium, (i) the individual strategy of each agent is a best response to the infinite population mass effect in the sense of a so-called  $\epsilon$ -Nash equilibrium, and (ii) the set of strategies collectively replicates the mass effect, this being a dynamic game theoretic fixed point property. The defining property of the  $\epsilon$ -NMFG equilibrium with individual strategies  $\{u_i^o : 1 \leq i \leq N\}$  requires that for any given  $\epsilon > 0$ , there exists  $N(\epsilon)$  such that for any population size  $N(\epsilon) \leq N$ , when any agent  $j$ ,  $1 \leq j \leq N$ , distinct from  $i$  employs  $u_j^o$ , then agent  $i$  can benefit by at most  $\epsilon$  by unilaterally deviating from his strategy  $u_i^o$ , and this holds for all  $1 \leq i \leq N$ . The estimates in [14, 17, 16] show  $\epsilon = O(1/\sqrt{N})$ , while distinct estimates are obtained in the framework of [22].

A stochastic maximum principle for control problems of mean field type is studied in [1], where the state process is governed by a stochastic differential equation (SDE) in which the coefficients depend on the law of the SDE. We refer the reader to [5, 6] for the analysis of forward-backward stochastic differential equations (FBSDEs) of mean field type and their related partial differential equations.

Recently, Huang [18] introduced a large population LQG dynamic game model with mean field couplings which involves not only a large number of multiclass minor agents but also a major agent with a significant influence on minor agents. (See [13, 12, 29] for static cooperative games of agents with different influences or so-called mixed agents.) Since all minor agents respond to the same major agent, the mean field behavior of minor agents in each class is directly impacted by the major agent and hence is a random process [18]. This is in contrast to the situation in the standard MFG models with only minor agents. A state-space augmentation approach for the approximation of the mean field behavior of the minor agents is taken in order to Markovianize the problem and hence to obtain  $\epsilon$ -NMFG equilibrium strategies [18]. An extension of the model in [18] to the systems of agents with Markov jump parameters in their dynamics and random parameters in their cost functions is studied in [42] in a discrete-time setting. See also [19] for the extension of the model in [18] to the case of systems with egoistic and altruistic agents.

The model of [18] with finite classes of minor agents is extended in [30] to the case of minor agents parameterized by an infinite set of dynamical parameters where the state augmentation trick cannot be applied to obtain a finite-dimensional Markov model. Exploiting the LQ structure of the problem, an appropriate representation for the mean field behavior of the minor agents as a random process is assumed which depends linearly on the random initial state and Brownian motion of the major agent. Appropriate approximation of the model by LQG control problems with random parameters in the dynamics and costs yields non-Markovian forward adapted  $\epsilon$ -NMFG strategies resulting from backward stochastic differential equations (BSDEs) obtained by a stochastic maximum principle [30].

In this paper we extend the LQG model for major and minor (MM) agents [18] to the case of a nonlinear stochastic dynamic games formulation of controlled MV type [17]. Specifically, we consider large population dynamic games involving nonlinear stochastic dynamical systems with agents of the following mixed types: (i) a major agent and (ii) a population of  $N$  minor agents where  $N$  is very large. The MM agents are coupled via both (i) their individual nonlinear stochastic dynamics and (ii) their individual finite time horizon nonlinear cost functions.

Applications of the MM formulation may be found in charging control of plug-in electric vehicles [46, 28], economic and social opinion models with an influential leader (e.g., [9]), and, as an extension of [21], power markets involving large consumers and large utilities together with many domestic consumers represented by smart meter agents and possibly large numbers of renewable energy based generators.

A distinctive feature of the mixed agent MFG problem is that even asymptotically (as the population size  $N$  approaches infinity) the noise process of the major agent causes random fluctuation of the mean field behavior of the minor agents [18, 30].

The main contributions of this paper are as follows:

- The overall asymptotic ( $N \rightarrow \infty$ ) MFG problem is decomposed into (i) two nonstandard stochastic optimal control problems (SOCs) with random coefficient processes which yield forward adapted stochastic best response control processes determined from the solution of (backward in time) stochastic Hamilton–Jacobi–Bellman (SHJB) equations and (ii) two stochastic coefficient McKean–Vlasov (SMV) equations which characterize the state of the major agent and the measure determining the mean field behavior of the minor agents. This yields a stochastic mean field game (SMFG) system which are in contrast to the deterministic MFG systems of standard MFG problems with only minor agents.
- Existence and uniqueness of the solutions to SMFG systems (SHJB and SMV equations) is established by a fixed point argument in the Wasserstein space of random probability measures.
- In the case where minor agents are coupled to the major agent only through their cost functions, the  $\epsilon_N$ -Nash equilibrium property of the SMFG best responses is shown for a finite  $N$  population system, where  $\epsilon_N = O(1/\sqrt{N})$ .
- As a special but important case, the results of Nguyen and Huang [30] for MM agent MFG LQG systems with a homogeneous population are retrieved in Appendix G in [36].
- Finally, the results of this paper are illustrated with an MM agent version of a game model of the synchronization of coupled nonlinear oscillators [45] (see Appendix H in [36]).

It is to be emphasized that the nonstandard nature of the SOCs in (i) above, which consists of the coupling through the SMV equations in (ii), arises from a

distinct feature of the problem formulation. The source of this nonstandard nature is the game structure whereby the minor agents are (through the principle of optimality) optimizing with respect to the future stochastic evolution of the major agent's state which is partly a result of that agent's future best response control actions. This feature vanishes in the non-game theoretic setting of one controller with one cost function with respect to the trajectories of all the system components (the classical SOCPs); moreover it also vanishes in the infinite population limit of the standard  $\epsilon$ -NMFG models with no major agent. This is true for both completely and partially observed SOCPs. The nonstandard feature of the SOCPs here gives rise to the analysis of systems with (not necessarily Markovian) stochastic parameters. Here, as in [30, 48], the theory of BSDEs (see in particular [2, 38, 39, 40]) is used in the resulting stochastic dynamic game theory. More specifically, we utilize techniques from [39] (see e.g. [10] for an application to portfolio optimization, which applies the principle of optimality to a stochastic nonlinear control problem with random coefficients; this leads to a formulation of an SHJB equation by use of (i) a semimartingale representation for the corresponding stochastic value function and (ii) the Itô–Kunita formula.

The organization of the paper is as follows. Section 2 is dedicated to the problem formulation. An MV approximation for an MM agent system is studied in section 3. Section 4 presents a preliminary nonlinear SOCP with random parameters. The SMFG system of equations of the MM agents is given in section 5, and the existence and uniqueness of its solution is established in section 6. The  $\epsilon$ -Nash equilibrium property of the resulting SMFG control laws is studied in section 7. Finally, section 8 concludes the paper.

**1.1. Notation and terminology.** The following notation will be used throughout the paper. Let  $\mathbb{R}^n$  denote the  $n$ -dimensional real Euclidean space with the standard Euclidean norm  $|\cdot|$  and the standard Euclidean inner product  $\langle \cdot, \cdot \rangle$ . The transpose of a vector (or matrix)  $x$  is denoted by  $x^T$ .  $\text{tr}(A)$  denotes the trace of a square matrix  $A$ . Let  $\mathbb{R}^{n \times m}$  be the Hilbert space consisting of all  $(n \times m)$ -matrices with the inner product  $\langle A, B \rangle := \text{tr}(AB^T)$  and the norm  $|A| := \langle A, A \rangle^{1/2}$ . The set of nonnegative real numbers is denoted by  $\mathbb{R}_+$ .  $T \in [0, \infty)$  is reserved to denote the terminal time. The integer  $N$  is reserved to designate the population size of the minor agents. The superscript  $N$  for a process (such as state, control, or cost function) is used to indicate the dependence on the population size  $N$ . We use the subscript 0 for the major agent  $\mathcal{A}_0$  and an integer valued subscript for an individual minor agent  $\{\mathcal{A}_i : 1 \leq i \leq N\}$ . At time  $t \geq 0$ , (i) the states of agents  $\mathcal{A}_0$  and  $\mathcal{A}_i$  are respectively denoted by  $z_0^N(t)$  and  $z_i^N(t)$ ,  $1 \leq i \leq N$ , and (ii) for the system configuration of minor agents  $(z_1^N(t), \dots, z_N^N(t))$  the empirical distribution  $\delta_t^N$  is defined as the normalized sum of Dirac's masses, i.e.,  $\delta_t^N := (1/N) \sum_{i=1}^N \delta_{z_i^N(t)}$ , where  $\delta_{(\cdot)}$  is the Dirac measure.  $C(S)$  is the set of continuous functions and  $C^k(S)$  the set of  $k$ -times continuously differentiable functions on  $S$ . The symbol  $\partial_t$  denotes the partial derivative with respect to the variable  $t$ . We denote  $D_x$  and  $D_{xx}^2$  as the gradient and Hessian operators with respect to the variable  $x$ . Let  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$  be a complete filtered probability space.  $\mathbb{E}$  denotes expectation. The conditional expectation with respect to the  $\sigma$ -field  $\mathcal{V}$  is denoted by  $\mathbb{E}_{\mathcal{V}}$ . For a Euclidean space  $H$  we denote by  $L_G^2([0, T]; H)$  the space of all  $\{\mathcal{G}_t\}_{t \geq 0}$ -adapted  $H$ -valued processes  $f(t, \omega)$  such that  $\mathbb{E} \int_0^T |f(t, \omega)|^2 dt < \infty$ . We use the notation  $(\mathbb{E}_{\omega} h)(z) := \int h(z, \omega) \mathbb{P}_{\omega}(d\omega)$  for any function  $h(z, \omega)$  and sample point  $\omega \in \Omega$ . Finally, note that we may not display the dependence of random variables or stochastic processes on the sample point  $\omega \in \Omega$ .

**2. Problem formulation.** We consider a dynamic game involving (i) a major agent  $\mathcal{A}_0$  and (ii) a population of  $N$  minor agents  $\{\mathcal{A}_i : 1 \leq i \leq N\}$ , where  $N$  is very large. We assume homogeneous minor agents, although the modeling may be generalized to the case of multiclass heterogeneous minor agents [17, 18] (see [37]).

The dynamics of the agents are given by the following controlled Itô stochastic differential equations on  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ :

$$(2.1) \quad dz_0^N(t) = \frac{1}{N} \sum_{j=1}^N f_0[t, z_0^N(t), u_0^N(t), z_j^N(t)] dt \\ + \frac{1}{N} \sum_{j=1}^N \sigma_0[t, z_0^N(t), z_j^N(t)] dw_0(t), \quad z_0^N(0) = z_0(0), \quad 0 \leq t \leq T,$$

$$(2.2) \quad dz_i^N(t) = \frac{1}{N} \sum_{j=1}^N f[t, z_i^N(t), u_i^N(t), z_0^N(t), z_j^N(t)] dt \\ + \frac{1}{N} \sum_{j=1}^N \sigma[t, z_i^N(t), z_0^N(t), z_j^N(t)] dw_i(t), \quad z_i^N(0) = z_i(0), \quad 1 \leq i \leq N,$$

with terminal time  $T \in (0, \infty)$ , where (i)  $z_0^N : [0, T] \rightarrow \mathbb{R}^n$  is the state of the major agent  $\mathcal{A}_0$  and  $z_i^N : [0, T] \rightarrow \mathbb{R}^n$  is the state of the minor agent  $\mathcal{A}_i$ ; (ii)  $u_0^N : [0, T] \rightarrow U_0$  and  $u_i^N : [0, T] \rightarrow U$  are respectively the control inputs of  $\mathcal{A}_0$  and  $\mathcal{A}_i$ ; (iii)  $f_0 : [0, T] \times \mathbb{R}^n \times U_0 \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $\sigma_0 : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$ ,  $f : [0, T] \times \mathbb{R}^n \times U \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  and  $\sigma : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$ ; (iv) the set of initial states is given by  $\{z_j^N(0) = z_j(0) : 0 \leq j \leq N\}$ ; and (v) the sequence  $\{(w_j(t))_{t \geq 0} : 0 \leq j \leq N\}$  denotes  $N+1$  mutually independent standard Brownian motions in  $\mathbb{R}^m$ . We denote the filtration  $\mathcal{F}_t$  as the  $\sigma$ -field generated by the initial states and the Brownian motions up to time  $t$ , i.e.,  $\mathcal{F}_t := \sigma\{z_j(0), w_j(s) : 0 \leq j \leq N, 0 \leq s \leq t\}$ . We also set  $\mathcal{F}_t^{w_0} = \sigma\{z_0(0), w_0(s) : 0 \leq s \leq t\}$ . These filtrations are augmented by all the  $\mathbb{P}$ -null sets in  $\mathcal{F}$ .

For  $0 \leq j \leq N$  denote  $u_{-j}^N := \{u_0^N, \dots, u_{j-1}^N, u_{j+1}^N, \dots, u_N^N\}$ . The objective of each agent is to minimize its finite time horizon nonlinear cost function given by

$$(2.3) \quad J_0^N(u_0^N; u_{-0}^N) := \mathbb{E} \int_0^T \left( \frac{1}{N} \sum_{j=1}^N L_0[t, z_0^N(t), u_0^N(t), z_j^N(t)] \right) dt,$$

$$(2.4) \quad J_i^N(u_i^N; u_{-i}^N) := \mathbb{E} \int_0^T \left( \frac{1}{N} \sum_{j=1}^N L[t, z_i^N(t), u_i^N(t), z_0^N(t), z_j^N(t)] \right) dt$$

for  $1 \leq i \leq N$ , where  $L_0 : [0, T] \times \mathbb{R}^n \times U_0 \times \mathbb{R}^n \rightarrow \mathbb{R}_+$  and  $L(z_i, u_i, z_0, x) : [0, T] \times \mathbb{R}^n \times U \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}_+$  are the nonlinear cost-coupling functions of the MM agents. For  $0 \leq j \leq N$ , we indicate the dependence of  $J_j$  on  $u_j^N$ ,  $u_{-j}^N$  and the population size  $N$  by  $J_j^N(u_j^N; u_{-j}^N)$ .

We note that in the system of equations (2.1)–(2.4) the major agent  $\mathcal{A}_0$  has a significant influence on minor agents, while each minor agent has an asymptotically negligible impact on other agents in a large  $N$  population system. The MM agents are coupled via both (i) their individual nonlinear stochastic dynamics (2.1)–(2.2) and (ii) their individual finite time horizon nonlinear cost functions (2.3)–(2.4).

We further note that the coupling terms may be written as functionals of the empirical distribution  $\delta_{(\cdot)}^N$  by the formula  $\int_{\mathbb{R}^n} \phi(x) \delta_t^N(dx) = (1/N) \sum_{i=1}^N \phi(x_i(t))$  for a bounded continuous function  $\phi$  in  $\mathbb{R}^n$ .

*Remark 2.1.* Under suitable conditions, the results of this paper may be adapted to deal with cost-couplings of the form

$$L_0 \left[ t, z_0^N(t), u_0^N(t), z_j^N(t), \frac{1}{N} \sum_{j=1}^N z_j^N(t) \right], \quad L \left[ t, z_i^N(t), u_i^N(t), z_0^N(t), z_j^N(t), \frac{1}{N} \sum_{j=1}^N z_j^N(t) \right]$$

in (2.3)–(2.4).

**2.1. Assumptions.** Let the empirical distribution of  $N$  minor agents' initial states be defined by  $F_N(x) = (1/N) \sum_{i=1}^N 1_{\{\mathbb{E}z_i(0) < x\}}$ , where  $1_{\{\mathbb{E}z_i(0) < x\}} = 1$  if  $\mathbb{E}z_i(0) < x$  and  $1_{\{\mathbb{E}z_i(0) < x\}} = 0$  otherwise. We enunciate the following assumptions:

- (A1) The initial states  $\{z_j(0) : 0 \leq j \leq N\}$  are  $\mathcal{F}_0$ -adapted random variables mutually independent and independent of all Brownian motions  $\{(w_j(t))_{t \geq 0} : 0 \leq j \leq N\}$ , and there exists a constant  $k$  independent of  $N$  such that  $\sup_{0 \leq j \leq N} \mathbb{E}|z_j(0)|^2 \leq k < \infty$ .
- (A2)  $\{F_N : N \geq 1\}$  converges to a probability distribution  $F$  weakly, i.e., for any bounded and continuous function  $\phi$  on  $\mathbb{R}^n$  we have  $\lim_{N \rightarrow \infty} \int_{\mathbb{R}^n} \phi(x) dF_N(x) = \int_{\mathbb{R}^n} \phi(x) dF(x)$ .
- (A3)  $U_0$  and  $U$  are compact metric spaces.
- (A4) The functions  $f_0[t, x, u, y]$ ,  $\sigma_0[t, x, y]$ ,  $f[t, x, u, y, z]$ , and  $\sigma[t, x, y, z]$  are continuous and bounded with respect to all their parameters and Lipschitz continuous in  $(x, y, z)$ . In addition, their first order derivatives (w.r.t.  $x$ ) are all uniformly continuous and bounded with respect to all their parameters and Lipschitz continuous in  $(y, z)$ .
- (A5)  $f_0[t, x, u, y]$  and  $f[t, x, u, y, z]$  are Lipschitz continuous in  $u$ .
- (A6)  $L_0[t, x, u, y]$  and  $L[t, x, u, y, z]$  are continuous and bounded with respect to all their parameters and Lipschitz continuous in  $(x, y, z)$ . In addition, their first order derivatives (w.r.t.  $x$ ) are all uniformly continuous and bounded with respect to all their parameters and Lipschitz continuous in  $(y, z)$ .
- (A7) (Nondegeneracy assumption) There exists a positive constant  $\alpha$  such that

$$\sigma_0[t, x, y] \sigma_0^T[t, x, y] \geq \alpha I, \quad \sigma[t, x, y, z] \sigma^T(t, x, y, z) \geq \alpha I \quad \forall (t, x, y, z),$$

where  $\sigma_0$  and  $\sigma$  are given in (2.1) and (2.2).

**3. MV approximation for MFG analysis.** Motivated by the analysis in section I.1 of [41] and in section 8.1 of [17], we take a probabilistic approach to establish the following asymptotic properties: (i) The influence of any minor agent  $\mathcal{A}_i$  on any other minor agent  $\mathcal{A}_j$  is asymptotically negligible as the population size  $N$  goes to infinity and (ii) in the limit, the effect of the mass of agents on a given minor agent  $\mathcal{A}_i$  is that of the behavior of a mass of predictable generic agents. This is in the form of a single mean field function in the LQG case [14, 16] or a predictable state probability distribution in the nonlinear case [17, 15, 26].

Let  $\varphi_0(\omega, t, x) : \Omega \times [0, T] \times \mathbb{R} \rightarrow U_0$  and  $\varphi(\omega, t, x) : \Omega \times [0, T] \times \mathbb{R} \rightarrow U$  be two arbitrary  $\mathcal{F}_t^{w_0}$ -measurable stochastic processes for which we introduce the following assumption:

- (H4)  $\varphi_0(\omega, t, x)$  and  $\varphi(\omega, t, x)$  are Lipschitz continuous in  $x$ , and  $\varphi_0(\omega, t, 0) \in L_{\mathcal{F}_t^{w_0}}^2([0, T]; U_0)$  and  $\varphi(\omega, t, 0) \in L_{\mathcal{F}_t^{w_0}}^2([0, T]; U)$ .

We assume that  $\varphi_0(t, x) := \varphi_0(\omega, t, x)$  and  $\varphi(t, x) := \varphi(\omega, t, x)$  are respectively used by the MM agents as their control laws in (2.1) and (2.2) (i.e.,  $u_0 = \varphi_0$  and  $u_i = \varphi$  for  $1 \leq i \leq N$ ). Then we have the following closed-loop equations with random coefficients:

$$\begin{aligned} d\hat{z}_0^N(t) &= \frac{1}{N} \sum_{j=1}^N f_0[t, \hat{z}_0^N(t), \varphi_0(t, \hat{z}_0^N(t)), \hat{z}_j^N(t)]dt \\ &\quad + \frac{1}{N} \sum_{j=1}^N \sigma_0[t, \hat{z}_0^N(t), \hat{z}_j^N(t)]dw_0(t), \quad \hat{z}_0^N(0) = z_0(0), \quad 0 \leq t \leq T, \\ d\hat{z}_i^N(t) &= \frac{1}{N} \sum_{j=1}^N f[t, \hat{z}_i^N(t), \varphi(t, \hat{z}_i^N(t)), \hat{z}_0^N(t), \hat{z}_j^N(t)]dt \\ &\quad + \frac{1}{N} \sum_{j=1}^N \sigma[t, \hat{z}_i^N(t), \hat{z}_0^N(t), \hat{z}_j^N(t)]dw_i(t), \quad \hat{z}_i^N(0) = z_i(0), \quad 1 \leq i \leq N. \end{aligned}$$

Under (A4)–(A5) and (H4) there exists a unique solution  $(z_0^N(\cdot), \dots, z_N^N(\cdot))$  to the above system (see Theorem 6.16, Chapter 1 of [47, p. 49]).

We now introduce the MV system

$$\begin{aligned} d\bar{z}_0(t) &= f_0[t, \bar{z}_0(t), \varphi_0(t, \bar{z}_0(t)), \mu_t]dt + \sigma_0[t, \bar{z}_0(t), \mu_t]dw_0(t), \quad 0 \leq t \leq T, \\ d\bar{z}(t) &= f[t, \bar{z}(t), \varphi(t, \bar{z}(t)), \bar{z}_0(t), \mu_t]dt + \sigma[t, \bar{z}(t), \bar{z}_0(t), \mu_t]dw(t) \end{aligned}$$

with initial condition  $(\bar{z}_0(0), \bar{z}(0))$ , where for an arbitrary function  $g \in C(\mathbb{R}^s)$  for appropriate  $s$  and probability distribution  $\mu_t$  in  $\mathbb{R}^n$  we set

$$g[t, z, \varphi, z_0, \mu_t] = \int_{\mathbb{R}^n} g[t, z, \phi, z_0, x] \mu_t(dx),$$

when the indicated integral converges. In using the MV system it is assumed that the infinite population of minor agents can be modeled by the collection of sample paths of individual agents subject to their individual initial conditions and their individual Brownian sample paths.

In the above MV system  $(\bar{z}_0(\cdot), \bar{z}(\cdot), \mu_{(\cdot)})$  is a “consistent solution” if  $(\bar{z}_0(\cdot), \bar{z}(\cdot))$  is a solution to the above MV system, and  $\mu_t$ ,  $0 \leq t \leq T$ , is the conditional law of  $\bar{z}(t)$  given  $\mathcal{F}_t^{w_0}$  (i.e.,  $\mu_t := \mathcal{L}(\bar{z}(t) | \mathcal{F}_t^{w_0})$ ).

Under (A4)–(A5) and (H4) it can be shown by a fixed point argument that there exists a unique solution  $(\bar{z}_0(\cdot), \bar{z}(\cdot), \mu_{(\cdot)})$  to the above system (see Theorem 1.1 in [41] or Theorem 6.7 below).

We also introduce the equations

$$\begin{aligned} d\bar{z}_0(t) &= f_0[t, \bar{z}_0(t), \varphi_0(t, \bar{z}_0(t)), \mu_t]dt + \sigma_0[t, \bar{z}_0(t), \mu_t]dw_0(t), \quad 0 \leq t \leq T, \\ d\bar{z}_i(t) &= f[t, \bar{z}_i(t), \varphi(t, \bar{z}_i(t)), \bar{z}_0(t), \mu_t]dt + \sigma[t, \bar{z}_i(t), \bar{z}_0(t), \mu_t]dw_i(t), \quad 1 \leq i \leq N, \end{aligned}$$

with initial conditions  $\bar{z}_j(0) = z_j(0)$ ,  $0 \leq j \leq N$ , which can be viewed as  $N$  independent samples of the MV system above. We develop a decoupling result below such that each  $\hat{z}_i^N$ ,  $1 \leq i \leq N$ , has the natural limit  $\bar{z}_i$  in the infinite population limit (see Theorem 12 in [17]).

The proof of the following theorem, which is based on the Cauchy–Schwarz inequality, Gronwall’s lemma, and the conditional independence of minor agents given  $\mathcal{F}_t^{w_0}$ , is given in Appendix A of [36].

THEOREM 3.1 (MV convergence result). Assume (A1), (A3)–(A5), and (H4) hold. Then we have

$$(3.1) \quad \sup_{0 \leq j \leq N} \sup_{0 \leq t \leq T} \mathbb{E} |\hat{z}_j^N(t) - \bar{z}_j(t)| = O(1/\sqrt{N}),$$

where the right-hand side may depend upon the terminal time  $T$ .

**4. A preliminary nonlinear SOCP with random coefficients.** Let  $(W(t))_{t \geq 0}$  and  $(B(t))_{t \geq 0}$  be mutually independent standard Brownian motions in  $\mathbb{R}^m$  with  $\mathcal{F}_t^{W,B} := \sigma\{W(s), B(s) : s \leq t\}$  and  $\mathcal{F}_t^W := \sigma\{W(s) : s \leq t\}$ , where both are augmented by all the  $\mathbb{P}$ -null sets in  $\mathcal{F}$ .

We now consider the following single agent nonlinear SOCP on  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ :

$$(4.1) \quad dz(t, \omega) = f[t, \omega, z, u]dt + \sigma[t, \omega, z]dW(t) + \varsigma[t, \omega, z]dB(t), \quad 0 \leq t \leq T,$$

$$(4.2) \quad \inf_{u \in \mathcal{U}} J(u) := \inf_{u \in \mathcal{U}} \mathbb{E} \left[ \int_0^T L[t, \omega, z(t), u(t)]dt \right],$$

where the coefficients  $f, \sigma, \varsigma$ , and  $L$  are random depending on  $\omega \in \Omega$  explicitly. In (4.1)–(4.2), (i)  $z : [0, T] \times \Omega \rightarrow \mathbb{R}^n$  is the state of the agent with  $\mathcal{F}_0^{W,B}$ -adapted random initial state  $z(0)$  such that  $\mathbb{E}|z(0)|^2 < \infty$ ; (ii)  $u : [0, T] \times \Omega \rightarrow U$  is the control input where  $U$  is a compact metric space; (iii) the functions  $f : [0, T] \times \Omega \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^n$ ,  $\sigma, \varsigma : [0, T] \times \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$  are  $\mathcal{F}_t^W$ -adapted stochastic processes; and (iv) the admissible control set  $\mathcal{U}$  is taken as  $\mathcal{U} := \{u(\cdot) \in U : u(t) \text{ is adapted to } \mathcal{F}_t^{W,B} \text{ and } \mathbb{E} \int_0^T |u(t)|^2 dt < \infty\}$ . We introduce the following assumptions (see [39]):

- (H1)  $f[t, x, u]$  and  $L[t, x, u]$  are a.s. continuous in  $(x, u)$  for each  $t$  and a.s. continuous in  $t$  for each  $(x, u)$ ,  $f[t, 0, 0] \in L_{\mathcal{F}_t}^2([0, T]; \mathbb{R}^n)$ , and  $L[t, 0, 0] \in L_{\mathcal{F}_t}^2([0, T]; \mathbb{R}_+)$ . In addition, they and all their first derivatives (w.r.t.  $x$ ) are a.s. continuous and bounded.
- (H2)  $\sigma[t, x]$  and  $\varsigma[t, x]$  are a.s. continuous in  $x$  for each  $t$  and a.s. continuous in  $t$  for each  $x$  and  $\sigma[t, 0]$ , and  $\varsigma[t, 0] \in L_{\mathcal{F}_t}^2([0, T]; \mathbb{R}^{n \times m})$ . In addition, they and all their first derivatives (w.r.t.  $x$ ) are a.s. continuous and bounded.
- (H3) (Nondegeneracy assumption) There exist nonnegative constants  $\alpha_1$  and  $\alpha_2$  such that

$$\sigma[t, \omega, x]\sigma^T[t, \omega, x] \geq \alpha_1 I, \quad \varsigma[t, \omega, x]\varsigma^T(t, \omega, x) \geq \alpha_2 I \quad \text{a.s.} \quad \forall (t, \omega, x),$$

where  $\alpha_1$  or  $\alpha_2$  (but not both) can be zero.

The value function for the SOCP (4.1)–(4.2) is defined by (see [39])

$$(4.3) \quad \phi(t, x(t)) = \inf_{u \in \mathcal{U}} \mathbb{E}_{\mathcal{F}_t^W} \int_t^T L[s, \omega, z(s), u(s)]ds,$$

where  $x(t)$  is the initial condition for the process  $z(\cdot)$ . We note that  $\phi(t, x(t))$  is an  $\mathcal{F}_t^W$ -adapted process which is sample path continuous a.s. under the assumptions (H1)–(H2). We assume that there exists an optimal control law  $u^o \in \mathcal{U}$  such that

$$\phi(t, x(t)) = \mathbb{E}_{\mathcal{F}_t^W} \int_t^T L[s, \omega, x(s), u^o(s, \omega, x(s))]ds,$$



where  $x(\cdot)$  is the closed-loop solution when the control law  $u^o$  is applied. By the principle of optimality, it can be shown that the process

$$(4.4) \quad \zeta(t) := \phi(t, x(t)) + \int_0^t L[s, \omega, x(s), u^o(s, x(s))] ds$$

is an  $\{\mathcal{F}_t^W\}_{0 \leq t \leq T}$ -martingale (see [3]). Next, by the martingale representation theorem (see Theorem 5.7, Chapter 1 in [47]) along the optimal solution  $x(\cdot)$  there exists an  $\mathcal{F}_t^W$ -adapted process  $\psi(\cdot, x(\cdot))$  such that

$$(4.5) \quad \zeta(t) = \phi(0, x(0)) + \int_0^t \psi^T(s, x(s)) dW(s), \quad t \in [0, T].$$

From (4.4)–(4.5) and the fact that  $\phi(T, x(T)) = 0$ , it follows that

$$\zeta(T) = \int_0^T L[s, \omega, x(s), u^o(s, x(s))] ds = \phi(0, x(0)) + \int_0^T \psi^T(s, x(s)) dW(s),$$

which gives

$$(4.6) \quad \phi(0, x(0)) = \int_0^T L[s, \omega, x(s), u^o(s, x(s))] ds - \int_0^T \psi^T(s, x(s)) dW(s).$$

Hence, combining (4.4)–(4.6) yields

$$(4.7) \quad \begin{aligned} \phi(t, x(t)) &= \int_t^T L[s, \omega, x(s), u^o(s, x(s))] ds - \int_t^T \psi^T(s, x(s)) dW(s) \\ &=: \int_t^T \Gamma(s, x(s)) ds - \int_t^T \psi^T(s, x(s)) dW(s), \quad t \in [0, T], \end{aligned}$$

where  $\phi(s, x(s))$ ,  $\Gamma(s, x(s))$ , and  $\psi(s, x(s))$  are  $\mathcal{F}_s^W$ -adapted stochastic processes. (See the assumed semimartingale representation form (3.5) in [39].)

Using the extended Itô–Kunita formula (see Appendix B in [36]) and the principle of optimality, Peng [39] showed that since  $\phi(t, x)$  can be expressed in the semimartingale form (4.7), and if  $\phi(t, x)$ ,  $\psi(t, x)$ ,  $D_x \phi(t, x)$ ,  $D_{xx}^2 \phi(t, x)$ , and  $D_x \psi(t, x)$  are a.s. continuous in  $(t, x)$ , then the pair  $(\phi(s, x), \psi(s, x))$  satisfies the following backward in time SHJB equation:

$$(4.8) \quad \begin{aligned} -d\phi(t, \omega, x) &= \left[ H[t, \omega, x, D_x \phi(t, \omega, x)] + \langle \sigma[t, \omega, x], D_x \psi(t, \omega, x) \rangle \right. \\ &\quad \left. + \frac{1}{2} \text{tr}(a[t, \omega, x] D_{xx}^2 \phi(t, \omega, x)) \right] dt - \psi^T(t, \omega, x) dW(t, \omega), \quad \phi(T, x) = 0, \end{aligned}$$

where  $(t, x) \in [0, T] \times \mathbb{R}^n$ ,  $a[t, \omega, x] := \sigma[t, \omega, x] \sigma^T[t, \omega, x] + \varsigma[t, \omega, x] \varsigma^T[t, \omega, x]$ , and the stochastic Hamiltonian  $H : [0, T] \times \Omega \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  is given by

$$H[t, \omega, x, p] := \inf_{u \in \mathcal{U}} \{ \langle f[t, \omega, x, u], p \rangle + L[t, \omega, x, u] \}.$$

We note that the appearance of the term  $\langle \sigma[t, \omega, x], D_x \psi(t, \omega, x) \rangle$  in (4.8) corresponds to the Brownian motion  $W(\cdot)$  in the extended Itô–Kunita formula (see (B.1) in

Appendix B of [36]) for the composition of  $\mathcal{F}_t^W$ -adapted stochastic processes  $\phi(t, \omega, x)$  and  $z(t, \omega)$  given in (4.7) and (4.1), respectively.

The solution to the backward in time SHJB equation (4.8) is a unique forward in time  $\mathcal{F}_t^W$ -adapted pair  $(\phi, \psi)(t, x) \equiv (\phi(t, \omega, x), \psi(t, \omega, x))$  (see [39, 47]). We omit the proof of the following theorem, which closely resembles that of Theorem 4.1 in [39].

**THEOREM 4.1.** *Assume (H1)–(H3) hold. Then the SHJB equation (4.8) has a unique solution  $(\phi(t, x), \psi(t, x))$  in  $(L^2_{\mathcal{F}_t}([0, T]; \mathbb{R}), L^2_{\mathcal{F}_t}([0, T]; \mathbb{R}^m))$ .*

The forward in time  $\mathcal{F}_t^W$ -adapted optimal control process of the SOCP (4.1)–(4.2) is given by (see [39])

$$(4.9) \quad \begin{aligned} u^o(t, \omega, x) &:= \arg \inf_{u \in U} H^u[t, \omega, x, D_x \phi(t, \omega, x), u] \\ &= \arg \inf_{u \in U} \{ \langle f[t, \omega, x, u], D_x \phi(t, \omega, x) \rangle + L[t, \omega, x, u] \}. \end{aligned}$$

By a verification theorem approach, Peng [39] showed that if the unique solution  $(\phi, \psi)(t, x)$  of the SHJB equation (4.8) satisfies

- (i) for each  $t$ ,  $(\phi, \psi)(t, \cdot)$  is a  $C^2(\mathbb{R}^n)$  map from  $\mathbb{R}^n$  into  $\mathbb{R} \times \mathbb{R}^m$ ,
- (ii) for each  $x$ ,  $(\phi, \psi)(t, x)$  and  $(D_x \phi, D_{xx}^2 \phi, D_x \psi)(t, x)$  are continuous  $F_t^W$ -adapted stochastic processes,

then  $\phi(x, t)$  coincides with the value function (4.3) of the SOCP (4.1)–(4.2).

**5. The MM agent SMFG system.** In the formulation (2.1)–(2.4) all minor agents are reacting to the same major agent and hence the major agent has non-negligible influence on the mean field behavior of the minor agents. In other words, the noise process of the major agent  $w_0$  causes random fluctuation of the mean-field behavior of the minor agents and makes it stochastic. (See the discussion in section 2 of [18] for the MM agent MFG LQG model.)

In this section, we first construct two auxiliary SOCPs with random coefficients for the major and a generic minor agent in sections 5.1 and 5.2, respectively. Then, we present the stochastic mean field system for the MM agents game formulation (2.1)–(2.4) via the MFG consistency condition in section 5.3.

**5.1. SOCP of the major agent.** By the MV convergence result in Theorem 3.1 which indicates that a single minor agent's statistical properties can effectively approximate the empirical distribution produced by all minor agents, we may approximate the empirical distribution of minor agents  $\delta_{(\cdot)}^N$  with a stochastic probability measure  $\mu_{(\cdot)}$  which depends on the noise process of the major agent  $w_0$ .

In this section, let  $\mu_t(\omega)$ ,  $0 \leq t \leq T$ , be an exogenous nominal minor agent stochastic measure process such that  $\mu_0(dx) := dF(x)$ , where  $F$  is defined in (A.2). Note that in section 5.3  $\mu_t(\omega)$  will be characterized via the MFG consistency condition as the random measure of minor agents' mean field behavior.

We define the following SOCP (4.1)–(4.2) with  $\mathcal{F}_t^{w_0}$ -adapted random coefficients from the major agent's model (2.1) and (2.3) in the infinite population limit:

$$(5.1) \quad dz_0(t) = f_0[t, z_0(t), u_0(t), \mu_t(\omega)]dt + \sigma_0[t, z_0(t), \mu_t(\omega)]dw_0(t, \omega), \quad z_0(0),$$

$$(5.2) \quad \inf_{u_0 \in \mathcal{U}_0} J_0(u_0) := \inf_{u_0 \in \mathcal{U}_0} \mathbb{E} \left[ \int_0^T L_0[t, z_0(t), u_0(t), \mu_t(\omega)]dt \right],$$

where we explicitly indicate the dependence of the random measure  $\mu_{(\cdot)}$  on the sample point  $\omega \in \Omega$ .

*Step I* (major agent's SHJB equation). The value function of the major agent's SOCP (5.1)–(5.2) is defined by

$$(5.3) \quad \phi_0(t, x(t)) = \inf_{u_0 \in \mathcal{U}_0} \mathbb{E}_{\mathcal{F}_t^{w_0}} \int_t^T L_0[s, z_0(s), u_0(s), \mu_s(\omega)] ds,$$

where  $x(t)$  is the initial condition for the process  $z_0(s)$  (see (4.3)). As in section 4,  $\phi_0(t, x(t))$  has the form (see (4.7))

$$\phi_0(t, x(t)) = \int_t^T \Gamma_0(s, x(s)) ds - \int_t^T \psi_0^T(s, x(s)) dw_0(s), \quad t \in [0, T],$$

where  $\phi_0(s, x(s))$ ,  $\Gamma_0(s, x(s))$ , and  $\psi_0(s, x(s))$  are  $\mathcal{F}_s^{w_0}$ -adapted stochastic processes. If  $\phi_0(t, x)$ ,  $\psi_0(t, x)$ ,  $D_x \phi_0(t, x)$ ,  $D_{xx}^2 \phi_0(t, x)$ , and  $D_x \psi_0(t, x)$  are a.s. continuous in  $(x, t)$ , then the pair  $(\phi_0(s, x), \psi_0(s, x))$  satisfies the following SHJB equation:

$$(5.4) \quad -d\phi_0(t, \omega, x) = \left[ H_0[t, \omega, x, D_x \phi_0(t, \omega, x)] + \langle \sigma_0[t, x, \mu_t(\omega)], D_x \psi_0(t, \omega, x) \rangle \right. \\ \left. + \frac{1}{2} \text{tr}(a_0[t, \omega, x] D_{xx}^2 \phi_0(t, \omega, x)) \right] dt \\ - \psi_0^T(t, \omega, x) dw_0(t, \omega), \quad \phi_0(T, x) = 0,$$

where  $(t, x) \in [0, T] \times \mathbb{R}^n$ ,  $a_0[t, \omega, x] := \sigma_0[t, x, \mu_t(\omega)] \sigma_0^T[t, x, \mu_t(\omega)]$ , and the stochastic Hamiltonian  $H_0 : [0, T] \times \Omega \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  is given by

$$H_0[t, \omega, x, p] := \inf_{u \in \mathcal{U}_0} \{ \langle f_0[t, x, u, \mu_t(\omega)], p \rangle + L_0[t, x, u, \mu_t(\omega)] \}.$$

The solution to the backward in time SHJB equation (5.4) is a forward in time  $\mathcal{F}_t^{w_0}$ -adapted pair  $(\phi_0(t, x), \psi_0(t, x)) \equiv (\phi_0(t, \omega, x), \psi_0(t, \omega, x))$  (see [39]).

We note that the appearance of the term  $\langle \sigma_0[t, x, \mu_t(\omega)], D_x \psi_0(t, \omega, x) \rangle$  in (5.4) corresponds to the major agent's Brownian motion  $w_0(\cdot)$  in the extended Itô–Kunita formula (see (B.1) in Appendix B of [36]) for the composition of  $\mathcal{F}_t^{w_0}$ -adapted processes  $\phi_0(t, \omega, x)$  and  $z_0(t, \omega)$  in (5.1).

The best response process of the major agent's SOCP (5.1)–(5.2) is given by

$$(5.5) \quad u_0^o(t, \omega, x) \equiv u_0^o(t, x, \{\mu_s(\omega)\}_{0 \leq s \leq T}) := \arg \inf_{u_0 \in U_0} H_0^{u_0}[t, \omega, x, u_0, D_x \phi_0(t, \omega, x)] \\ \equiv \arg \inf_{u_0 \in U_0} \{ \langle f_0[t, x, u_0, \mu_t(\omega)], D_x \phi_0(t, \omega, x) \rangle + L_0[t, x, u_0, \mu_t(\omega)] \},$$

where the infimum exists a.s. here and in all analogous infimizations in the chapter due to the continuity of all functions appearing in  $H_0^{u_0}$  and the compactness of  $U_0$ . It should be noted that the stochastic best response control  $u_0^o$  is a forward in time  $\mathcal{F}_t^{w_0}$ -adapted process which depends on the Brownian motion  $w_0$  via the stochastic measure  $\mu_t(\omega)$ ,  $0 \leq t \leq T$ . The notation in (5.5) indicates that  $u_0^o$  at time  $t$  depends upon the stochastic measure  $\mu_s(\omega)$  on the whole interval  $0 \leq s \leq T$ .

*Step II* (major agent's SMV equation). By substituting the best response control process  $u_0^o$  (5.5) into the major agent's dynamics (5.1) we get the following SMV dynamics with random coefficients:

$$(5.6) \quad dz_0^o(t, \omega) = f_0[t, z_0^o, u_0^o(t, \omega, z_0^o), \mu_t(\omega)] dt + \sigma_0[t, z_0^o, \mu_t(\omega)] dw_0(t, \omega)$$

with  $z_0^o(0) = z_0(0)$ , where  $f_0$  and  $\sigma_0$  are random processes via the stochastic measure  $\mu$  and  $u_0^o$ .

**5.2. SOCP of the generic minor agent.** As in section 5.1 let  $\mu_t$ ,  $0 \leq t \leq T$ , denote the exogenous nominal minor agent stochastic measure process approximating the empirical distribution produced by all minor agents in the infinite population limit such that  $\mu_0(dx) = dF(x)$ , where  $F$  is defined in (A.2). We let  $z_0^o(\cdot)$  be the solution to the major agent's SMV equation (5.6).

We define the following SOCP (4.1)–(4.2) with  $\mathcal{F}_t^{w_0}$ -adapted random coefficients from the  $i$ th generic minor agent's model (2.2), (2.4) in the infinite population limit:

$$(5.7) \quad dz_i(t) = f[t, z_i(t), u_i(t), z_0^o(t, \omega), \mu_t(\omega)]dt + \sigma[t, z_i(t), z_0^o(t, \omega), \mu_t(\omega)]dw_i(t),$$

$$(5.8) \quad \inf_{u_i \in \mathcal{U}} J_i(u_i) := \inf_{u_i \in \mathcal{U}} \mathbb{E} \left[ \int_0^T L[t, z_i(t), u_i(t), z_0^o(t, \omega), \mu_t(\omega)]dt \right], \quad z_i(0),$$

where we explicitly indicate the dependence of the solution to the major agent's SMV equation  $z_0^o(\cdot)$  and the nominal minor agent's random measure  $\mu(\cdot)$  on the sample point  $\omega \in \Omega$ .

*Step I* (generic minor agent's SHJB equation). The value function of the generic minor agent's SOCP (5.7)–(5.8) is defined by

$$(5.9) \quad \phi_i(t, x(t)) = \inf_{u_i \in \mathcal{U}_0} \mathbb{E}_{\mathcal{F}_t^{w_0}} \int_t^T L[s, z_i(s), u_i(s), z_0^o(s, \omega), \mu_s(\omega)]ds,$$

where  $x(t)$  is the initial condition for the process  $z_i(\cdot)$ . As in section 4,  $\phi_i(t, x(t))$  has the form (see (4.7))

$$\phi_i(t, x(t)) = \int_t^T \Gamma_i(s, x(s))ds - \int_t^T \psi_i^T(s, x(s))dw_0(s), \quad t \in [0, T],$$

where  $\phi_i(s, x(s))$ ,  $\Gamma_i(s, x(s))$ , and  $\psi_i(s, x(s))$  are  $\mathcal{F}_s^{w_0}$ -adapted stochastic processes. If  $\phi_i(t, x)$ ,  $\psi_i(t, x)$ ,  $D_x \phi_i(t, x)$ , and  $D_{xx}^2 \phi_i(t, x)$  are a.s. continuous in  $(x, t)$ , then the pair  $(\phi_i(s, x), \psi_i(s, x))$  satisfies the following backward in time SHJB equation (see (4.8)):

$$(5.10) \quad -d\phi_i(t, \omega, x) = \left[ H[t, \omega, x, D_x \phi_i(t, \omega, x)] + \frac{1}{2} \text{tr}(a[t, \omega, x] D_{xx}^2 \phi_i(t, \omega, x)) \right] dt - \psi_i^T(t, \omega, x)dw_0(t, \omega), \quad \phi_i(T, x) = 0,$$

where  $(t, x) \in [0, T] \times \mathbb{R}^n$ ,  $a[t, \omega, x] := \sigma[t, x, z_0^o(t, \omega), \mu_t(\omega)]\sigma^T[t, x, z_0^o(t, \omega), \mu_t(\omega)]$ , and the stochastic Hamiltonian  $H : [0, T] \times \Omega \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  is given by

$$H[t, \omega, x, p] := \inf_{u \in \mathcal{U}} \{ \langle f[t, x, u, z_0^o(t, \omega), \mu_t(\omega)], p \rangle + L[t, x, u, z_0^o(t, \omega), \mu_t(\omega)] \}.$$

The solution to the backward in time SHJB equation (5.10) is a forward in time  $\mathcal{F}_t^{w_0}$ -adapted pair  $(\phi_i(t, x), \psi_i(t, x)) \equiv (\phi_i(t, \omega, x), \psi_i(t, \omega, x))$  (see [39]). We note that since the coefficients of the SOCP (5.7)–(5.8) are  $\mathcal{F}_t^{w_0}$ -adapted random processes we have the major agent's Brownian motion  $w_0$  in (5.10), which allows us to seek for a forward in time adapted solution to the backward in time SHJB equation (5.10).

It is important to note that in (5.10), unlike the major agent's SHJB equation (5.4) we do not have the term  $\langle \sigma[t, x, z_0^o(t, \omega), \mu_t(\omega)] D_x \psi_i(t, \omega, x) \rangle$  since the coefficients in the minor agent's model (5.7)–(5.8) are  $\mathcal{F}_t^{w_0}$ -adapted random processes depending upon the major agent's Brownian motion ( $w_0$ ), which is independent of the minor agent's Brownian motion ( $w_i$ ). (See the extended Itô–Kunita formula in Appendix B of [36].)

As in section 5.1, the stochastic best response process of the minor agent's SOCP (5.7)–(5.8) is

$$(5.11) \quad \begin{aligned} u_i^o(t, \omega, x) &\equiv u_i^o(t, x | \{z_0^o(s, \omega), \mu_s(\omega)\}_{0 \leq s \leq T}) := \arg \inf_{u \in U} H^u[t, \omega, x, u, D_x \phi_i(t, \omega, x)] \\ &\equiv \arg \inf_{u \in U} \{ \langle f[t, x, u, z_0^o(t, \omega), \mu_t(\omega)], D_x \phi_i(t, \omega, x) \rangle \\ &\quad + L[t, x, u, z_0^o(t, \omega), \mu_t(\omega)] \}, \end{aligned}$$

where the infimum exists a.s. here and in all analogous infimizations in the chapter due to the continuity of all functions appearing in  $H^u$  and the compactness of  $U$ . It should be noted that the stochastic best response process of the generic minor agent  $u_i^o$  is a forward in time  $\mathcal{F}_t^{w_0}$ -adapted random process which depends on the Brownian motion  $w_0$  via the major agent's state  $z_0^o(t, \omega)$  and the stochastic measures  $\mu_t(\omega)$ ,  $0 \leq t \leq T$ . The notation in (5.11) indicates that  $u_i^o$  at time  $t$  depends upon  $z_0^o(s, \omega)$  and  $\mu_s(\omega)$  on the whole interval  $0 \leq s \leq T$ .

*Step II* (minor agent's SMV and stochastic coefficient Fokker–Planck–Kolmogorov (SFPK) equations). By substituting the best response control process  $u_i^o$  (5.11) into the minor agent's dynamics (5.7) we get the following SMV dynamics with random coefficients:

$$(5.12) \quad \begin{aligned} dz_i^o(t, \omega, \omega') &= f[t, z_i^o, u_i^o(t, \omega, z_i), z_0^o(t, \omega), \mu_t(\omega)] dt \\ &\quad + \sigma[t, z_i^o, z_0^o(t, \omega), \mu_t(\omega)] dw_i(t, \omega'), \quad z_i^o(0) = z_i(0), \end{aligned}$$

where  $f$  and  $\sigma$  are random processes via  $z_0^o$ ,  $\mu$ , and the best response control process  $u_i^o$  which all depend on the Brownian motion of the major agent ( $w_0$ ).

Based on the MV approximation in section 3, the generic agent's statistical properties can effectively approximate the empirical distribution produced by all minor agents in a large population system. Hence, we obtain a new stochastic measure  $\hat{\mu}_t(\omega)$  for the mean field behavior of minor agents as the conditional law of the generic minor agent's process  $z_i^o(t, \omega)$  given  $\mathcal{F}_t^{w_0}$ . We characterize  $\hat{\mu}_t(\omega)$ ,  $0 \leq t \leq T$ , by  $P(z_i^o(t, \omega) \leq \alpha | \mathcal{F}_t^{w_0}) = \int_{-\infty}^{\alpha} \hat{\mu}(t, \omega, dx)$  a.s. for all  $\alpha \in \mathbb{R}^n$  and  $0 \leq t \leq T$  with  $\hat{\mu}_0(dx) = \mu_0(dx) = dF(x)$ , where  $F$  is defined in (A.2).

An equivalent method for characterizing the SMV of the generic minor agent is to express (5.12) in the form of an SFPK equation with random coefficients:

$$(5.13) \quad \begin{aligned} d\hat{p}(t, \omega, x) &= \left( - \langle D_x, f[t, x, u_i^o(t, \omega, x), z_0^o(t, \omega), \mu_t(\omega)] \hat{p}(t, \omega, x) \rangle \right. \\ &\quad \left. + \frac{1}{2} \text{tr} \langle D_{xx}^2, a[t, \omega, x] \hat{p}(t, \omega, x) \rangle \right) dt, \quad \hat{p}(0, x) = p_0(x), \end{aligned}$$

in  $[0, T] \times \mathbb{R}^n$  where  $p(t, \omega, x)$  is the conditional probability density of  $z_i^o(t, \omega)$  given  $\mathcal{F}_t^{w_0}$ . By the MV approximation (see section 3) it is possible to characterize the mean field behavior of minor agents in terms of a generic agent's density function  $\hat{p}(t, \omega, x)$ . The reason that the generic minor agent's FPK equation (5.13) does not include the Itô integral term with respect to  $w_i$  is that  $p(t, \omega, x)$  is the conditional probability density given  $\mathcal{F}_t^{w_0}$ , and the Brownian motions  $w_0$  and  $w_i$ ,  $1 \leq i \leq N$ , are independent.

The density function  $\hat{p}(t, \omega, x)$  generates the random measure of the minor agent's mean field behavior  $\hat{\mu}_t(\omega)$  such that  $\hat{\mu}(t, \omega, dx) = \hat{p}(t, \omega, x) dx$  (a.s.),  $0 \leq t \leq T$ .

We note that the major agent's SOCP (5.1)–(5.2) and minor agent's SOCP (5.7)–(5.8) may be written with respect to the random density  $p(t, \omega, x)$  of the stochastic measure  $\mu(t, \omega, dx)$  by  $\mu(t, \omega, dx) = p(t, \omega, x)dx$  (a.s.),  $0 \leq t \leq T$ .

**5.3. The MFG consistency condition.** Based on the MFG or Nash certainty equivalence consistency (see [17] and [26]), we close the “measure and control” mapping loop by setting  $\hat{\mu}_t(\omega) = \mu_t(\omega)$  a.s.,  $0 \leq t \leq T$ , or  $\hat{p}(t, \omega, x) = p(t, \omega, x)$  a.s. for  $(t, x) \in [0, T] \times \mathbb{R}^n$ . The MFG consistency is demonstrated in (i) the major agent's SMFG system

$$(5.14) \quad \begin{aligned} [\text{MFG-SHJB}] \quad -d\phi_0(t, \omega, x) = & \left[ H_0[t, \omega, x, D_x\phi_0(t, \omega, x)] \right. \\ & \left. + \langle \sigma_0[t, x, \mu_t(\omega)], D_x\psi_0(t, \omega, x) \rangle + \frac{1}{2} \text{tr}(a_0[t, \omega, x] D_{xx}^2\phi_0(t, \omega, x)) \right] dt \\ & - \psi_0^T(t, \omega, x) dw_0(t, \omega), \quad \phi_0(T, x) = 0, \end{aligned}$$

$$(5.15) \quad \begin{aligned} u_0^o(t, \omega, x) \equiv u_0^o(t, x | \{\mu_s(\omega)\}_{0 \leq s \leq T}) \\ := \arg \inf_{u_0 \in U_0} \{ \langle f_0[t, x, u_0, \mu_t(\omega)], D_x\phi_0(t, \omega, x) \rangle + L_0[t, x, u_0, \mu_t(\omega)] \}, \end{aligned}$$

$$(5.16) \quad \begin{aligned} [\text{MFG-SMV}] \quad dz_0^o(t, \omega) = & f_0[t, z_0^o, u_0^o(t, \omega, z_0^o), \mu_t(\omega)] dt \\ & + \sigma_0[t, z_0^o, \mu_t(\omega)] dw_0(t, \omega), \quad z_0^o(0) = z_0(0), \end{aligned}$$

together with (ii) the minor agents' SMF system

$$(5.17) \quad \begin{aligned} [\text{MFG-SHJB}] \quad -d\phi(t, \omega, x) = & \left[ H[t, \omega, x, D_x\phi(t, \omega, x)] \right. \\ & \left. + \frac{1}{2} \text{tr}(a[t, \omega, x] D_{xx}^2\phi(t, \omega, x)) \right] dt - \psi^T(t, \omega, x) dw_0(t, \omega), \quad \phi(T, x) = 0, \end{aligned}$$

$$(5.18) \quad \begin{aligned} u^o(t, \omega, x) \equiv u^o(t, x | \{z_0^o(s, \omega), \mu_s(\omega)\}_{0 \leq s \leq T}) \\ \equiv \arg \inf_{u \in U} \{ \langle f[t, x, u, z_0^o(t, \omega), \mu_t(\omega)], D_x\phi(t, \omega, x) \rangle + L[t, x, u, z_0^o(t, \omega), \mu_t(\omega)] \}, \end{aligned}$$

$$(5.19) \quad \begin{aligned} [\text{MFG-SMV}] \quad dz^o(t, \omega, \omega') = & f[t, z^o, u^o(t, \omega, z^o), z_0^o(t, \omega), \mu_t(\omega)] dt \\ & + \sigma[t, z^o, z_0^o(t, \omega), \mu_t(\omega)] dw(t, \omega'), \end{aligned}$$

where  $(t, x) \in [0, T] \times \mathbb{R}^n$ , and  $z^o(0)$  has the measure  $\mu_0(dx) = dF(x)$ , where  $F$  is defined in (A.2). We note that in the minor agents' SMFG system (5.17)–(5.19) we dropped index  $i$  from the generic minor agent's equations (5.7)–(5.12). The MM agent SMFG system is given by (5.14)–(5.16) and (5.17)–(5.19).

The solution of the MM-SMFG system consists of 8-tuple  $\mathcal{F}_t^{w_0}$ -adapted random processes

$$(\phi_0(t, \omega, x), \psi_0(t, \omega, x), u_0^o(t, \omega, x), z_0^o(t, \omega), \phi(t, \omega, x), \psi(t, \omega, x), u^o(t, \omega, x), z^o(t, \omega)),$$

where  $z^o(t, \omega)$  generates the conditional random law  $\mu_t(\omega)$ , i.e.,  $P(z^o(t, \omega) \leq \alpha | \mathcal{F}_t^{w_0}) = \int_{-\infty}^{\alpha} \mu_t(\omega, dx)$  for all  $\alpha \in \mathbb{R}^n$  and  $0 \leq t \leq T$ . Note that the MM-SMFG equations (5.14)–(5.16) and (5.17)–(5.19) are coupled together through  $z_0^o(\cdot)$  and  $\mu_t(\cdot)$ .

We observe that the solution to the MM-SMFG system is a “stochastic mean field” in contrast to the deterministic mean field of the standard MFG problems with only minor agents considered in [17, 15, 25, 24, 26]. If the noise process of the major agent vanishes then the MM-SMFG system reduces to a deterministic MFG system (see (6)–(9) in [15]).

In the analysis of the next section, the fact that  $\mu_t^0(\omega)$ ,  $0 \leq t \leq T$ , is a unit mass random measure concentrated at  $z_0^o(t, \omega)$  permit us to replace  $\mu_t^0(\omega)$  by the Dirac delta symbol  $\delta_{z_0^o(t, \omega)}$ .

## 6. Existence and uniqueness of solutions to the MM SMFG system.

In this section we establish existence and uniqueness for the solution of the joint MM agents’ SMFG system (5.14)–(5.16) and (5.17)–(5.19). The analysis is based on providing sufficient conditions for a map that goes from the random measure of minor agents  $\mu_{(\cdot)}(\omega)$  back to itself, through (5.14)–(5.16) and (5.17)–(5.19), to be a contraction operator on the space of random probability measures:

$$\begin{array}{ccccc} \mu_{(\cdot)}(\omega) & \xrightarrow{(5.14)} & (\phi_0(\cdot, \omega, x), \psi_0(\cdot, \omega, x)) & \xrightarrow{(5.15)} & u_0^o(\cdot, \omega, x) \\ \uparrow (5.19) & & & & \downarrow (5.16) \\ u^o(\cdot, \omega, x) & \xleftarrow{(5.18)} & (\phi(\cdot, \omega, x), \psi(\cdot, \omega, x)) & \xleftarrow{(5.17)} & \mu_{(\cdot)}^0(\omega) \equiv \delta_{z_0^o(t, \omega)} \end{array}$$

In this section we first introduce some preliminary material about the Wasserstein space of probability measures. Second, we analyze the SHJB and SMV equations of the major agent and minor agents in sections 6.1 and 6.2, respectively. Third, the analysis of the joint MM agents’ SMFG system is carried out in section 6.3, where the main result is given in Theorem 6.12, which provides sufficient conditions for a contraction operator map that goes from the random measure of minor agents  $\mu_{(\cdot)}(\omega)$  back to itself.

On the Banach space  $C([0, T]; \mathbb{R}^n)$  we define the metric  $\rho_T(x, y) = \sup_{0 \leq t \leq T} |x(t) - y(t)|^2 \wedge 1$ , where  $\wedge$  denotes minimum. It can be shown that  $C_\rho := (C([0, T]; \mathbb{R}^n), \rho_T)$  forms a separable complete metric space (i.e., a Polish space). Let  $\mathcal{M}(C_\rho)$  be the space of all Borel probability measures  $\mu$  on  $C([0, T]; \mathbb{R}^n)$  such that  $\int |x|^2 d\mu(x) < \infty$ . We also denote  $\mathcal{M}(C_\rho \times C_\rho)$  as the space of probability measures on the product space  $C([0, T]; \mathbb{R}^n) \times C([0, T]; \mathbb{R}^n)$ . As in [17] the process  $x$  is defined to be a generic random process with the sample space  $C([0, T]; \mathbb{R}^n)$ , i.e.,  $x(t, \omega) = \omega(t)$  for  $\omega \in C([0, T]; \mathbb{R}^n)$ .

Based on the metric  $\rho_T$ , we introduce the Wasserstein metric on  $\mathcal{M}(C_\rho)$ :

$$D_T^\rho(\mu, \nu) = \inf_{\gamma \in \Pi(\mu, \nu)} \left[ \int_{C_\rho \times C_\rho} \rho_T(x(\omega_1), x(\omega_2)) d\gamma(\omega_1, \omega_2) \right]^{1/2},$$

where  $\Pi(\mu, \nu) \subset \mathcal{M}(C_\rho \times C_\rho)$  is the set of Borel probability measures  $\gamma$  such that  $\gamma(A \times C([0, T]; \mathbb{R}^n)) = \mu(A)$  and  $\gamma(C([0, T]; \mathbb{R}^n) \times A) = \nu(A)$  for any Borel set  $A \in C([0, T]; \mathbb{R}^n)$ . The metric space  $\mathcal{M}_\rho := (\mathcal{M}(C_\rho), D_T^\rho)$  is a Polish space since  $C_\rho \equiv (C([0, T]; \mathbb{R}^n), \rho_T)$  is a Polish space.

We also introduce the class  $\mathcal{M}_\rho^\beta$  of stochastic measures in the space  $\mathcal{M}_\rho$  with a.s. Hölder continuity of exponent  $\beta$ ,  $0 < \beta < 1$ . (See Definition 3 in [17] for the nonstochastic case.)

**DEFINITION 6.1.** A stochastic probability measure  $\mu_t(\omega)$ ,  $0 \leq t \leq T$ , in the space  $\mathcal{M}_\rho$  is in  $\mathcal{M}_\rho^\beta$  if  $\mu$  is a.s. uniformly Hölder continuous with exponent  $0 < \beta < 1$ , i.e., there exists  $\beta \in (0, 1)$  and constant  $c$  such that for any bounded and Lipschitz continuous function  $\phi$  on  $\mathbb{R}^n$ ,

$$\left| \int_{\mathbb{R}^n} \phi(x) \mu_t(\omega, dx) - \int_{\mathbb{R}^n} \phi(x) \mu_s(\omega, dx) \right| \leq c(\omega) |t - s|^\beta \quad a.s.$$

for all  $0 \leq s < t \leq T$ , where  $c$  may depend upon the Lipschitz constant of  $\phi$  and the sample point  $\omega \in \Omega$ .

As in [17], we may take  $\mu_t$ ,  $0 \leq t \leq T$ , to be a Dirac measure at any constant  $x \in \mathbb{R}^n$  to show that the set  $\mathcal{M}_\rho^\beta$  is nonempty. We introduce the following assumption:

(A8) For any  $p \in \mathbb{R}^n$  and  $\mu, \mu^0 := \delta_{z_0} \in \mathcal{M}_\rho^\beta$ , the sets

$$S_0(t, \omega, x, p) := \arg \inf_{u_0 \in U_0} H_0^{u_0}[t, \omega, x, u_0, p],$$

$$S(t, \omega, x, p) := \arg \inf_{u \in U} H^u[t, \omega, x, u, p],$$

where  $H_0^{u_0}$  and  $H^u$  are respectively defined in (5.5) and (5.11), are singletons and the resulting  $u$  and  $u_0$  as functions of  $[t, \omega, x, p]$  are a.s. continuous in  $t$ , Lipschitz continuous in  $(x, p)$ , uniformly with respect to  $t$ , and  $\mu, \mu^0 \in \mathcal{M}_\rho^\beta$ .

In addition,  $u_0[t, \omega, 0, 0]$  and  $u[t, \omega, 0, 0]$  are in the space  $L_{\mathcal{F}_t}^2([0, T]; \mathbb{R}^n)$ .

The first part of (A8) may be satisfied under suitable convexity conditions with respect to  $u_0$  and  $u$  (see [17]).

**6.1. Analysis of the major agent's SMFG system.** Let  $\mu_t(\omega)$ ,  $0 \leq t \leq T$ , be a fixed stochastic measure in the set  $\mathcal{M}_\rho^\beta$  with  $0 < \beta < 1$  such that  $\mu_0(dx) := dF(x)$ , where  $F$  is defined in (A2). Then, the functionals of  $\mu_{(\cdot)}(\omega)$  in (5.1)–(5.2) become random functions which we write as

$$(6.1) \quad \begin{aligned} f_0^*[t, \omega, z_0, u_0] &:= f_0[t, z_0, u_0, \mu_t(\omega)], & \sigma_0^*[t, \omega, z_0] &:= \sigma_0[t, z_0, \mu_t(\omega)], \\ L_0^*[t, \omega, z_0, u_0] &:= L_0[t, z_0, u_0, \mu_t(\omega)]. \end{aligned}$$

We have the following result, which broadly follows Proposition 4 in [17].

**PROPOSITION 6.2.** Assume (A3) holds for  $U_0$ . Let  $\mu_t(\omega)$ ,  $0 \leq t \leq T$ , be a fixed stochastic measure in the set  $\mathcal{M}_\rho^\beta$  with  $0 < \beta < 1$ . For  $f_0^*$ ,  $\sigma_0^*$  and  $L_0^*$  defined in (6.1) the following hold:

- (i) Under (A4) for  $f_0$  and  $\sigma_0$ , the functions  $f_0^*[t, \omega, z_0, u_0]$  and  $\sigma_0^*[t, \omega, z_0]$  and their first order derivatives (w.r.t.  $z_0$ ) are a.s. continuous and bounded on  $[0, T] \times \mathbb{R}^n \times U_0$  and  $[0, T] \times \mathbb{R}^n$ .  $f_0^*[t, \omega, z_0, u_0]$  and  $\sigma_0^*[t, \omega, z_0]$  are a.s. Lipschitz continuous in  $z_0$ . In addition,  $f_0^*[t, \omega, 0, 0]$  is in the space  $L_{\mathcal{F}_t}^2([0, T]; \mathbb{R}^n)$  and  $\sigma_0^*[t, \omega, 0]$  is in the space  $L_{\mathcal{F}_t}^2([0, T]; \mathbb{R}^{n \times m})$ .
- (ii) Under (A5) for  $f_0$ , the function  $f_0^*[t, \omega, z_0, u_0]$  is a.s. Lipschitz continuous in  $u_0 \in U_0$ , i.e., there exist a constant  $c > 0$  such that

$$\sup_{t \in [0, T], z_0 \in \mathbb{R}^n} |f_0^*[t, \omega, z_0, u_0] - f_0^*[t, \omega, z_0, u'_0]| \leq c(\omega) |u_0 - u'_0| \quad (a.s.).$$

- (iii) Under (A6) for  $L_0$ , the function  $L_0^*[t, \omega, z_0, u_0]$  and its first order derivative (w.r.t.  $z_0$ ) is a.s. continuous and bounded on  $[0, T] \times \mathbb{R}^n \times U_0$ .  $L_0^*[t, \omega, z_0, u_0]$  is a.s. Lipschitz continuous in  $z_0$ . In addition,  $L_0^*[t, \omega, 0, 0]$  is in the space  $L_{\mathcal{F}_t}^2([0, T]; \mathbb{R}_+)$ .
- (iv) Under (A8) for  $H_0^{u_0}$ , the set of minimizers

$$\arg \inf_{u_0 \in U_0} \{ \langle f_0^*[t, \omega, z_0, u_0], p \rangle + L_0^*[t, \omega, z_0, u_0] \}$$

is a singleton for any  $p \in \mathbb{R}^n$ , and the resulting  $u_0$  as a function of  $[t, \omega, z_0, p]$  is a.s. continuous in  $t$ , a.s. Lipschitz continuous in  $(z_0, p)$ , uniformly with respect to  $t$ . In addition,  $u_0[t, \omega, 0, 0]$  is in the space  $L_{\mathcal{F}_t}^2([0, T]; \mathbb{R}^n)$ .



*Proof.*

- (i) We show only the results for  $f_0^*$ ; the analysis for  $\sigma_0^*$  is similar. For  $\omega \in \Omega$ , we take  $(t, z, u)$  and  $(s, z', u')$  both from  $[0, T] \times \mathbb{R}^n \times U_0$ . We have

$$\begin{aligned} |f_0^*[t, \omega, z, u] - f_0^*[s, \omega, z', u']| &\equiv |f_0[t, z, u, \mu_t(\omega)] - f_0[s, z', u', \mu_s(\omega)]| \\ &\leq |f_0[t, z, u, \mu_t(\omega)] - f_0[s, z', u', \mu_t(\omega)]| \\ &\quad + |f_0[s, z', u', \mu_t(\omega)] - f_0[s, z', u', \mu_s(\omega)]| \\ &\leq |f_0[t, z, u, \mu_t(\omega)] - f_0[s, z, u, \mu_t(\omega)]| \\ &\quad + |f_0[s, z, u, \mu_t(\omega)] - f_0[s, z', u', \mu_t(\omega)]| \\ &\quad + |f_0[s, z', u', \mu_t(\omega)] - f_0[s, z', u', \mu_s(\omega)]|. \end{aligned}$$

By (A4),  $f_0[t, \omega, z, u]$  is continuous with respect to  $(t, z, u)$  and therefore

$$\begin{aligned} &|f_0[t, z, u, \mu_t(\omega)] - f_0[s, z, u, \mu_t(\omega)]| \\ &\quad + |f_0[s, z, u, \mu_t(\omega)] - f_0[s, z', u', \mu_t(\omega)]| \rightarrow 0, \end{aligned}$$

as  $|t - s| + |z - z'| + |u - u'| \rightarrow 0$ . Since  $\mu_{(\cdot)}(\omega)$  is in the set  $\mathcal{M}_\rho^\beta$ ,  $0 < \beta < 1$ , and by (A4) there exists a constant  $k > 0$  independent of  $(s, z, u)$  such that

$$|f_0[s, z, u, y] - f_0[s, z, u, y']| \leq k|y - y'|,$$

we get  $|f_0[s, z', u', \mu_t(\omega)] - f_0[s, z', u', \mu_s(\omega)]| \rightarrow 0$  as  $|t - s| \rightarrow 0$ . This concludes the a.s. continuity of  $f_0^*[t, \omega, z_0, u_0]$  on  $[0, T] \times \mathbb{R}^n \times U_0$ .

Using the Leibniz rule we have

$$D_{z_0} f_0^*[t, \omega, z_0, u_0] = \int D_{z_0} f_0[t, z_0, u_0, x] \mu_t(\omega)(dx), \quad \text{a.s.,}$$

where the partial derivative exists due to the boundedness of the first order derivative (w.r.t.  $z_0$ ) of  $f_0$  by (A4). The a.s. continuity of  $D_{z_0} f_0^*$  on  $[0, T] \times \mathbb{R}^n \times U_0$  may be proved by a similar argument above for  $f_0^*$ . Other results of the proposition follow directly from (A4).

- (ii) This is a direct result of (A5).  
 (iii) The proofs are similar to the proofs for  $f_0^*$  in part (i).  
 (iv) This is a direct result of (A8) for  $S_0$  using the measure  $\mu_{(\cdot)}(\omega) \in \mathcal{M}_\rho^\beta$ ,  $0 < \beta < 1$ .  $\square$

Employing the results of section 4, we analyze the SHJB equation (5.14), where the probability measure  $\mu_{(\cdot)}(\omega)$  is in the set  $\mathcal{M}_\rho^\beta$ ,  $0 < \beta < 1$ .

**THEOREM 6.3.** *Assume (A3)–(A7) for  $U_0$ ,  $f_0$ ,  $\sigma_0$ , and  $L_0$  hold, and the probability measure  $\mu_{(\cdot)}(\omega)$  is in the set  $\mathcal{M}_\rho^\beta$ ,  $0 < \beta < 1$ . Then the SHJB equation for the major agent (5.14) has a unique solution  $(\phi_0(t, x), \psi_0(t, x))$  in  $(L^2_{\mathcal{F}_t}([0, T]; \mathbb{R}), L^2_{\mathcal{F}_t}([0, T]; \mathbb{R}^m))$ .*

*Proof.* Proposition 6.2 indicates that the SOCP of the major agent (5.1)–(5.2) satisfies Assumptions (H1)–(H3) of section 4 with  $\varsigma[t, x] = 0$ . The result follows directly from Theorem 4.1.  $\square$

Let  $\mu_{(\cdot)}(\omega) \in \mathcal{M}_\rho^\beta$ ,  $0 < \beta < 1$ , be given. We assume that the unique solution  $(\phi_0, \psi_0)(t, x)$  to the SHJB equation (5.14) satisfies the regularity properties: (i) for each  $t$ ,  $(\phi_0, \psi_0)(t, x)$  is a  $C^2(\mathbb{R}^n)$  map from  $\mathbb{R}^n$  into  $\mathbb{R} \times \mathbb{R}^m$ , (ii) for each  $x$ ,  $(\phi_0, \psi_0)$  and  $(D_x \phi_0, D^2_{xx} \phi_0, D_x \psi_0)$  are continuous  $F_t^W$ -adapted stochastic processes. Then,

$\phi_0(x, t)$  coincides with the value function (5.3) [39], and under (A8) for  $H_0^{u_0}$  we get the best response control process (5.5):

$$(6.2) \quad u_0^o(t, \omega, x) \equiv u_0^o(t, x | \{\mu_s(\omega)\}_{0 \leq s \leq T}) := \arg \inf_{u_0 \in U_0} H_0^{u_0}[t, \omega, x, u_0, D_x \phi_0(t, \omega, x)],$$

where  $(t, x) \in [0, T] \times \mathbb{R}^n$ .

We introduce the following assumption (see (H6) in [17]):

(A9) For any  $\mu_{(\cdot)}(\omega) \in \mathcal{M}_\rho^\beta$ ,  $0 < \beta < 1$ , the best response control  $u_0^o(t, \omega, x)$  is a.s. continuous in  $(t, x)$  and a.s. Lipschitz continuous in  $x$ .

We denote  $C_{\text{Lip}(x)}([0, T] \times \Omega \times \mathbb{R}^n; H)$  be the class of a.s. continuous functions from  $[0, T] \times \Omega \times \mathbb{R}^n$  to  $H$ , which are a.s. Lipschitz continuous in  $x$  [17]. We introduce the following well-defined map:

$$(6.3) \quad \begin{aligned} \Upsilon_0^{\text{SHJB}} : \mathcal{M}_\rho^\beta &\longrightarrow C_{\text{Lip}(x)}([0, T] \times \Omega \times \mathbb{R}^n; U_0), \quad 0 < \beta < 1, \\ \Upsilon_0^{\text{SHJB}}(\mu_{(\cdot)}(\omega)) &= u_0^o(t, \omega, x) \equiv u_0^o(t, x | \{\mu_s(\omega)\}_{0 \leq s \leq T}). \end{aligned}$$

We now analyze the major agent's SMV equation (5.16) with  $\mu_{(\cdot)}(\omega) \in \mathcal{M}_\rho^\beta$ , where  $0 < \beta < 1$ , and  $u_0^o(t, \omega, x) \in C_{\text{Lip}(x)}([0, T] \times \Omega \times \mathbb{R}^n; U_0)$  are given in (6.2).

**THEOREM 6.4.** *Assume (A3)–(A7), for  $U_0$ ,  $f_0$ , and  $\sigma_0$ , and (A9) hold. Let  $\mu_{(\cdot)}(\omega) \in \mathcal{M}_\rho^\beta$ , where  $0 < \beta < 1$ , and let  $u_0^o(t, \omega, x)$  be given in (6.2). Then, there exists a unique solution  $z_0^o$  on  $[0, T] \times \Omega$  to the major agent's SMV equation (5.16).*

*Proof.* Proposition 6.2 indicates that the major agent's SMV equation (5.16) satisfies Assumption (RC) in [47, p. 49]. The result follows directly from Theorem 6.16, Chapter 1 of [47, p. 49].  $\square$

**THEOREM 6.5.** *Assume (A3)–(A7), for  $U_0$ ,  $f_0$ , and  $\sigma_0$ , and (A9) hold. Let  $\mu_{(\cdot)}(\omega) \in \mathcal{M}_\rho^\beta$ , where  $0 < \beta < 1$ , and let  $u_0^o(t, \omega, x)$  be given in (6.2). Then, the probability measure  $\mu_{(\cdot)}^0(\omega)$  as the unit mass measure concentrated at  $z_0^o(t, \omega)$  (i.e.,  $\mu_t^0(\omega) = \delta_{z_0^o(t, \omega)}$ ) which is obtained from the major agent's SMV equation (5.16) is in the class  $\mathcal{M}_\rho^\gamma$ , where  $0 < \gamma < 1/2$ .*

*Proof.* We take  $0 \leq s < t \leq T$ . Since  $\mu_t^0(\omega) = \delta_{z_0^o(t, \omega)}$ , for any bounded and Lipschitz continuous function  $\phi$  on  $\mathbb{R}^n$  with a Lipschitz constant  $K > 0$ , we have

$$\begin{aligned} \mathbb{E} \left| \int_{\mathbb{R}^n} \phi(x) \mu_t^0(\omega, dx) - \int_{\mathbb{R}^n} \phi(x) \mu_s^0(\omega, dx) \right| &= \mathbb{E} |\phi(z_0^o(t, \omega)) - \phi(z_0^o(s, \omega))| \\ &\leq K \mathbb{E} |z_0^o(t, \omega) - z_0^o(s, \omega)|. \end{aligned}$$

On the other hand, Theorem 6.4 indicates that there exists a unique solution to the SMV equation (5.16) such that

$$z_0^o(t, \omega) - z_0^o(s, \omega) = \int_s^t f_0[\tau, z_0^o, u_0^o, \mu_\tau(\omega)] d\tau + \int_s^t \sigma_0[\tau, z_0^o, \mu_\tau(\omega)] dw_0(\tau).$$

Boundedness of  $f_0$  and  $\sigma_0$  (see (A4)), the Cauchy–Schwarz inequality and the property of Itô integral yield

$$\mathbb{E} |z_0^o(t, \omega) - z_0^o(s, \omega)|^2 \leq 2C_1^2 |t - s|^2 + 2C_2^2 |t - s|,$$

where  $C_1$  and  $C_2$  are upper bounds for  $f_0$  and  $\sigma_0$ , respectively. Hence,

$$\begin{aligned} \mathbb{E} \left| \int_{\mathbb{R}^n} \phi(x) \mu_t^0(\omega, dx) - \int_{\mathbb{R}^n} \phi(x) \mu_s^0(\omega, dx) \right| &\leq \sqrt{2}K (C_1 |t - s| + C_2 |t - s|^{1/2}) \\ &\leq \sqrt{2}K (C_1 \sqrt{T} + C_2) |t - s|^{1/2}. \end{aligned}$$

By Kolmogorov's theorem (Theorem 18.19, p. 266, in [23]), for each  $0 < \gamma < 1/2$ ,  $T > 0$ , and almost every  $\omega \in \Omega$ , there exists a constant  $c(\omega, \gamma, K, T)$  such that

$$\left| \int_{\mathbb{R}^n} \phi(x) \mu_t^0(\omega, dx) - \int_{\mathbb{R}^n} \phi(x) \mu_s^0(\omega, dx) \right| \leq c(\omega, \gamma, K, T) |t - s|^\gamma$$

for all  $0 \leq s < t \leq T$ . Hence,  $\mu_{(\cdot)}^0(\omega)$  is in the class  $\mathcal{M}_\rho^\gamma$ , where  $0 < \gamma < 1/2$ .  $\square$

By Theorems 6.4 and 6.5 we may now introduce the following well-defined map:

$$(6.4) \quad \begin{aligned} \Upsilon_0^{\text{SMV}} : M_\rho^\beta \times C_{\text{Lip}(x)}([0, T] \times \Omega \times \mathbb{R}^n; U_0) &\longrightarrow M_\rho^\gamma, \quad 0 < \beta < 1, \quad 0 < \gamma < 1/2, \\ \Upsilon_0^{\text{SMV}}(\mu_{(\cdot)}(\omega), u_0^o(t, \omega, x)) &= \mu_{(\cdot)}^0(\omega) \equiv \delta_{z_0^o(t, \omega)}. \end{aligned}$$

**6.2. Analysis of the minor agents' SMFG system.** Let  $\mu_{(\cdot)}(\omega) \in \mathcal{M}_\rho^\beta$ ,  $0 < \beta < 1$ , be the fixed stochastic measure assumed in section 6.1. In this section we assume that  $\mu_{(\cdot)}^0(\omega) \in \mathcal{M}_\rho^\gamma$ ,  $0 < \gamma < 1/2$ , is the unit mass random measure concentrated at  $z_0^o(\cdot, \omega)$  (i.e.,  $\mu_t^0(\omega) = \delta_{z_0^o(t, \omega)}$ ) obtained from the composite map:

$$(6.5) \quad \begin{aligned} \Upsilon_0 : M_\rho^\beta &\longrightarrow M_\rho^\gamma, \quad 0 < \beta < 1, \quad 0 < \gamma < 1/2, \\ \Upsilon_0(\mu_{(\cdot)}(\omega)) &:= \Upsilon_0^{\text{SMV}}\left(\mu_{(\cdot)}(\omega), \Upsilon_0^{\text{SHJB}}(\mu_{(\cdot)}(\omega))\right) = \mu_{(\cdot)}^0(\omega) \equiv \delta_{z_0^o(t, \omega)}, \end{aligned}$$

where  $\Upsilon_0^{\text{SHJB}}$  and  $\Upsilon_0^{\text{SMV}}$  are given in (6.3) and (6.4), respectively.

Following arguments exactly parallel to those used in section 6.1, we analyze the SHJB equation (5.17), where the probability measures  $\mu_{(\cdot)}(\omega) \in \mathcal{M}_\rho^\beta$ ,  $0 < \beta < 1$ , and  $\mu_{(\cdot)}^0(\omega) \in \mathcal{M}_\rho^\gamma$ ,  $0 < \gamma < 1/2$ .

**THEOREM 6.6.** Assume (A3)–(A7) for  $U$ ,  $f$ ,  $\sigma$ , and  $L$  hold, and  $\mu_{(\cdot)}(\omega) \in \mathcal{M}_\rho^\beta$ ,  $0 < \beta < 1$ , and  $\mu_{(\cdot)}^0(\omega)$  is in the set  $\mathcal{M}_\rho^\gamma$ ,  $0 < \gamma < 1/2$ . Then the SHJB equation for the generic minor agent (5.10) has a unique solution  $(\phi_i(t, x), \psi_i(t, x))$  in  $(L_{\mathcal{F}_t}^2([0, T]; \mathbb{R}), L_{\mathcal{F}_t}^2([0, T]; \mathbb{R}^m))$ .

*Proof.* An argument similar to Proposition 6.2 for the generic minor agent (see Proposition C.1 in Appendix C of [36]) indicates that the SOCP of the generic minor agent (5.7)–(5.8) satisfies Assumptions (H1)–(H3) of section 4 with  $\sigma[t, x] = 0$ . The result follows directly from Theorem 4.1.  $\square$

For the probability measure  $\mu_{(\cdot)}(\omega) \in \mathcal{M}_\rho^\beta$ ,  $0 < \beta < 1$ , and  $\mu_{(\cdot)}^0(\omega) \in \mathcal{M}_\rho^\gamma$ ,  $0 < \gamma < 1/2$ , we assume that the unique solution  $(\phi_i, \psi_i)(t, x)$  to the SHJB equation (5.10) satisfies the regularity properties: (i) for each  $t$ ,  $(\phi_i, \psi_i)(t, x)$  is a  $C^2(\mathbb{R}^n)$  map from  $\mathbb{R}^n$  into  $\mathbb{R} \times \mathbb{R}^m$ , (ii) for each  $x$ ,  $(\phi_i, \psi_i)$  and  $(D_x \phi_i, D_{xx}^2 \phi_i, D_x \psi_i)$  are continuous  $F_t^W$ -adapted stochastic processes. Then,  $\phi_i(x, t)$  coincides with the value function (5.9) [39], and under (A8) for  $H^u$  we get the best response control process (5.11):

$$(6.6) \quad \begin{aligned} u_i^o(t, \omega, x) &\equiv u_i^o(t, x | \{\mu_s^0(\omega), \mu_s(\omega)\}_{0 \leq s \leq T}) \\ &:= \arg \inf_{u_i \in U} H^u[t, \omega, x, u_i, D_x \phi_i(t, \omega, x)], \end{aligned}$$

where  $(t, x) \in [0, T] \times \mathbb{R}^n$ .

We introduce the following assumption (see (A9) or (H6) in [17]):

(A10) For any  $\mu_{(\cdot)}(\omega) \in \mathcal{M}_\rho^\beta$ ,  $0 < \beta < 1$ , and  $\mu_{(\cdot)}^0(\omega) \in \mathcal{M}_\rho^\gamma$ ,  $0 < \gamma < 1/2$ , the best response control process  $u_i^o(t, \omega, x)$  is a.s. continuous in  $(t, x)$  and a.s. Lipschitz continuous in  $x$ .

We introduce the following well-defined map for the generic minor agent  $i$ :

$$(6.7) \quad \Upsilon_i^{\text{SHJB}}: M_\rho^\beta \times M_\rho^\gamma \longrightarrow C_{\text{Lip}(x)}([0, T] \times \Omega \times \mathbb{R}^n; U), \quad 0 < \beta < 1, \quad 0 < \gamma < 1/2,$$

$$\Upsilon_i^{\text{SHJB}}(\mu_{(\cdot)}(\omega), \mu_{(\cdot)}^0(\omega)) = u_i^o(t, \omega, x) \equiv u_i^o(t, x | \{\mu_s^0(\omega), \mu_s(\omega)\}_{0 \leq s \leq T}).$$

For given probability measure  $\mu_{(\cdot)}^0(\omega) \in \mathcal{M}_\rho^\gamma$ ,  $0 < \gamma < 1/2$ , we analyze the generic minor agent's SMV equation (5.12):

$$(6.8) \quad dz_i^o(t, \omega, \omega') = f[t, z_i^o, u_i^o(t, \omega, z_i^o), \mu_t^0(\omega), \mu_t(\omega)]dt$$

$$+ \sigma[t, z_i^o, \mu_t^0(\omega), \mu_t(\omega)]dw_i(t, \omega'), \quad z_i^o(0) = z_i(0),$$

where  $u_i^o(t, \omega, x) \in C_{\text{Lip}(x)}([0, T] \times \Omega \times \mathbb{R}^n; U)$  is given in (6.6). We call the pair  $(z_i^o(\cdot, \omega, \omega'), \mu_{(\cdot)}(\omega))$  a consistent solution of the generic minor agent's SMV equation (6.8) if  $(z_i^o(\cdot, \omega, \omega'), \mu_{(\cdot)}(\omega))$  solves (6.8) and  $\mu_{(\cdot)}(\omega)$  is the law of the process  $z_i^o(\cdot, \omega, \omega')$ , i.e.,  $\mu_{(\cdot)} = \mathcal{L}(z_i^o(\cdot, \omega, \omega'))$ . We define  $\Lambda$  as the map which associates to  $\mu_{(\cdot)}(\omega) \in \mathcal{M}_\rho^\beta$ ,  $0 < \beta < 1/2$ , the law of the process  $z_i^o(\cdot, \omega, \omega')$  in (6.8):

$$(6.9) \quad z_i^o(t, \omega, \omega') = z_i^o(0) + \int_0^t \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f[s, z_i^o, u_i^o, y, z] d\mu_s^0(\omega)(y) d\mu_s(\omega)(z) \right) ds$$

$$+ \int_0^t \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sigma[s, z_i^o, y, z] d\mu_s^0(\omega)(y) d\mu_s(\omega)(z) \right) dw_i(s, \omega'),$$

where we observe that the law  $\Lambda$  depends on the sample point  $\omega \in \Omega$ .

We now show that there exists a unique  $\mu_{(\cdot)}(\omega) \in \mathcal{M}_\rho^\beta$ ,  $0 < \beta < 1$ , such that  $\mu(\omega) = \Lambda(\mu(\omega))$ . The proof of the following theorem, which is given in Appendix D of [36], is based upon a fixed point argument with random parameters. (See Theorem 6 in [17] and Theorem 1.1 in [41] for the standard fixed point argument.)

**THEOREM 6.7.** Assume (A3)–(A7), for  $U$ ,  $f$ , and  $\sigma$ , and (A10) hold. Let  $\mu_{(\cdot)}^0(\omega)$  be in the set  $\mathcal{M}_\rho^\gamma$ , where  $0 < \gamma < 1/2$ , and  $u_i^o(t, \omega, x)$  be given in (6.6). Then, there exists a unique consistent solution pair  $(z_i^o(\cdot, \omega, \omega'), \mu_{(\cdot)}(\omega))$  to the generic minor agent's SMV equation (6.8) where  $\mu_{(\cdot)}(\omega) = \mathcal{L}(z_i^o(\cdot, \omega, \omega'))$ .

**THEOREM 6.8.** Assume (A3)–(A7), for  $U$ ,  $f$ , and  $\sigma$ , and (A10) hold. Let  $\mu_{(\cdot)}^0(\omega)$  be in the set  $\mathcal{M}_\rho^\gamma$ , where  $0 < \gamma < 1/2$ . For given  $u_i^o(t, \omega, x)$  in (6.6), let  $(z_i^o(\cdot, \omega, \omega'), \mu_{(\cdot)}(\omega))$  be the consistent solution pair of the SMV equation (6.8). Then, the probability measure  $\mu_{(\cdot)}(\omega)$  is in the class  $\mathcal{M}_\rho^\beta$ , where  $0 < \beta < 1$ .

*Proof.* We take  $0 \leq s < t \leq T$ . For any bounded and Lipschitz continuous function  $\phi$  on  $\mathbb{R}^n$  with a Lipschitz constant  $K > 0$ , we have

$$\mathbb{E} \left| \int_{\mathbb{R}^n} \phi(x) \mu_t(\omega, dx) - \int_{\mathbb{R}^n} \phi(x) \mu_s(\omega, dx) \right| = \mathbb{E} |\mathbb{E}_\omega(\phi(z_i^o(t, \omega, \omega')) - \phi(z_i^o(s, \omega, \omega')))|$$

$$\leq K \mathbb{E} |\mathbb{E}_\omega(z_i^o(t, \omega, \omega') - z_i^o(s, \omega, \omega'))|.$$

On the other hand, Theorem 6.7 indicates that there exists a unique solution to the SMV equation (6.8) such that

$$\mathbb{E}_\omega(z_i^o(t, \omega, \omega') - z_i^o(s, \omega, \omega')) = \int_s^t f[\tau, z_i^o, u_i^o, \mu_\tau^0(\omega), \mu_\tau(\omega)] d\tau,$$

where we note that  $\mathbb{E}_\omega \int_0^t \sigma[\tau, z_i^o, \mu_\tau^0(\omega), \mu_\tau(\omega)] dw_i(\tau, \omega') = 0$  for  $0 \leq t \leq T$ . Boundedness of  $f$  (see (A4)) yields

$$\mathbb{E} |\mathbb{E}_\omega (z_i^o(t, \omega, \omega') - z_i^o(s, \omega, \omega'))| \leq C_1 |t - s|,$$

where  $C_1$  is the upper bound for  $f$ .

By Kolmogorov's theorem (Theorem 18.19 in [23, p. 266]), for each  $0 < \gamma < 1$ ,  $T > 0$ , and almost every  $\omega \in \Omega$ , there exists a constant  $c(\omega, \gamma, K, T)$  such that

$$\left| \int_{\mathbb{R}^n} \phi(x) \mu_t(\omega, dx) - \int_{\mathbb{R}^n} \phi(x) \mu_s(\omega, dx) \right| \leq c(\omega, \gamma, K, T) |t - s|^\gamma$$

for all  $0 \leq s < t \leq T$ . Hence,  $\mu_{(\cdot)}(\omega)$  is in the class  $\mathcal{M}_\rho^\beta$ , where  $0 < \beta < 1$ .  $\square$

By Theorems 6.7 and 6.8 we may now introduce the following well-defined map:

$$(6.10) \quad \Upsilon_i^{\text{SMV}} : M_\rho^\beta \times M_\rho^\gamma \times C_{\text{Lip}(x)}([0, T] \times \Omega \times \mathbb{R}^n; U_0) \longrightarrow M_\rho^\beta, \quad 0 < \beta < 1, \quad 0 < \gamma < 1/2, \\ \Upsilon_i^{\text{SMV}}(\mu_{(\cdot)}(\omega), \mu_{(\cdot)}^0(\omega), u_i^o(t, \omega, x)) = \mu_{(\cdot)}(\omega).$$

**6.3. Analysis of the joint MM agents' SMFG system.** Based on the analysis of sections 6.1 and 6.2 we obtain the following well-defined map:

$$(6.11) \quad \Upsilon : M_\rho^\beta \longrightarrow M_\rho^\beta, \quad 0 < \beta < 1, \\ \Upsilon(\mu_{(\cdot)}(\omega)) = \Upsilon_i^{\text{SMV}}(\mu_{(\cdot)}(\omega), \Upsilon_0(\mu_{(\cdot)}(\omega)), \Upsilon_i^{\text{SHJB}}(\mu_{(\cdot)}(\omega), \Upsilon_0(\mu_{(\cdot)}(\omega)))) = \mu_{(\cdot)}(\omega),$$

which is the composition of the maps  $\Upsilon_0$ ,  $\Upsilon_i^{\text{SHJB}}$ , and  $\Upsilon_i^{\text{SMV}}$  introduced in (6.5), (6.7), and (6.10), respectively. Subsequently, the problem of existence and uniqueness of solutions to the MM SMV system (5.14)–(5.16) and (5.17)–(5.19) is translated into a fixed point problem with random parameters for the map  $\Upsilon$  on the Polish space  $\mathcal{M}_\rho^\beta$ ,  $0 < \beta < 1$ .

We introduce the following assumption without which one needs to work with the “expectation” of the Wasserstein metric  $D_{(\cdot)}^\rho$  of stochastic measures:

(A11) We assume that the diffusion coefficient of the major agent  $\sigma_0$  in (2.1) does not depend on its own state  $z_0^N$  and the states of the minor agents  $z_i^N$ ,  $1 \leq i \leq N$ .

The proof of the following lemma is given in Appendix E of [36].

LEMMA 6.9.

(i) Assume (A3)–(A7), for  $U_0$ ,  $f_0$ , and  $\sigma_0$ , and (A11) hold. Let  $\mu_{(\cdot)}(\omega)$  be in the set  $\mathcal{M}_\rho^\beta$  where  $0 < \beta < 1$ . Then, for given  $u_0, u'_0 \in C_{\text{Lip}(x)}([0, T] \times \Omega \times \mathbb{R}^n; U_0)$  there exists a constant  $c_0$  such that

$$(6.12) \quad \left( D_T^\rho(\mu^0(\omega), \nu^0(\omega)) \right)^2 \leq c_0 \sup_{(t, x) \in [0, T] \times \mathbb{R}^n} |u_0(t, \omega, x) - u'_0(t, \omega, x)|^2 \quad a.s.,$$

where  $\mu^0(\omega), \nu^0(\omega) \in \mathcal{M}_\rho^\gamma$ ,  $0 < \gamma < 1/2$ , are induced by the map  $\Upsilon_0^{\text{SMV}}$  in (6.4) using the two control processes  $u_0$  and  $u'_0$ , respectively.

(ii) Assume (A3)–(A7), for  $U_0$ ,  $f_0$ , and  $\sigma_0$ , and (A11) hold. Let  $u_0^o$  be in the space  $C_{\text{Lip}(x)}([0, T] \times \Omega \times \mathbb{R}^n; U_0)$ . Then, for given  $\mu(\omega), \nu(\omega) \in \mathcal{M}_\rho^\beta$ ,  $0 < \beta < 1$ , there exists a constant  $c_1$  such that

$$(6.13) \quad \left( D_T^\rho(\mu^0(\omega), \nu^0(\omega)) \right)^2 \leq c_1 \left( D_T^\rho(\mu(\omega), \nu(\omega)) \right)^2 \quad a.s.,$$

where  $\mu^0(\omega), \nu^0(\omega) \in \mathcal{M}_\rho^\gamma$ ,  $0 < \gamma < 1/2$ , are induced by the map  $\Upsilon_0^{SMV}$  in (6.4) using the stochastic measures  $\mu(\omega)$  and  $\nu(\omega)$ , respectively.

- (iii) Assume (A3)–(A7) for  $U$ ,  $f$ , and  $\sigma$  hold. Let  $\mu_{(\cdot)}^0(\omega)$  be in the set  $\mathcal{M}_\rho^\gamma$ , where  $0 < \gamma < 1/2$ . Then, for given  $u, u' \in C_{Lip(x)}([0, T] \times \Omega \times \mathbb{R}^n; U)$  there exists a constant  $c_2$  such that

$$(6.14) \quad \left( D_T^\rho(\mu(\omega), \nu(\omega)) \right)^2 \leq c_2 \sup_{(t,x) \in [0,T] \times \mathbb{R}^n} |u(t, \omega, x) - u'(t, \omega, x)|^2 \quad a.s.,$$

where  $\mu(\omega), \nu(\omega) \in \mathcal{M}_\rho^\beta$ ,  $0 < \beta < 1$ , are induced by the map  $\Upsilon_i^{SMV}$  in (6.10) using the two control processes  $u$  and  $u'$ , respectively.

- (iv) Assume (A3)–(A7) for  $U$ ,  $f$ , and  $\sigma$  hold. Let  $u_i^\alpha$  be in the space  $C_{Lip(x)}([0, T] \times \Omega \times \mathbb{R}^n; U)$ . Then, for given  $\mu^0(\omega), \nu^0(\omega) \in \mathcal{M}_\rho^\gamma$ ,  $0 < \gamma < 1/2$ , there exists a constant  $c_3$  such that

$$(6.15) \quad \left( D_T^\rho(\mu(\omega), \nu(\omega)) \right)^2 \leq c_3 \left( D_T^\rho(\mu^0(\omega), \nu^0(\omega)) \right)^2 \quad a.s.,$$

where  $\mu(\omega), \nu(\omega) \in \mathcal{M}_\rho^\beta$ ,  $0 < \beta < 1$ , are induced by the map  $\Upsilon_i^{SMV}$  in (6.10) using the stochastic measures  $\mu^0(\omega)$  and  $\nu^0(\omega)$ , respectively.

We define the Gâteaux derivative of the function  $F(t, x, \mu)$  with respect to the measure  $\mu(y)$  as [22]

$$\partial_{\mu(y)} F(t, x, \mu) = \lim_{\epsilon \rightarrow 0} \frac{F(t, x, \mu + \epsilon \delta(y)) - F(t, x, \mu)}{\epsilon},$$

where  $\delta$  is the Dirac delta function. We introduce the following assumptions:

- (A12) (i) In (5.1)–(5.2) the Gâteaux derivative of  $f_0$ ,  $\sigma_0$ , and  $L_0$  with respect to  $\mu$  exist and are  $C^\infty(\mathbb{R}^n)$  and a.s. uniformly bounded. (ii) In (5.7)–(5.8) the partial derivatives of  $f$ ,  $\sigma$ , and  $L$  with respect to  $\mu^0$  and  $\mu$  exist and are  $C^\infty(\mathbb{R}^n)$  and a.s. uniformly bounded.

The proof of the following lemma is based on the sensitivity analysis of the SHJB equations (5.14) and (5.17) to the stochastic measures  $\mu_{(\cdot)}(\omega)$  and  $\mu_{(\cdot)}^0(\omega)$  developed in Appendix F of [36] (see also section 6 in [22]).

LEMMA 6.10.

- (i) Assume (A3)–(A7), for  $U_0$ ,  $f_0$ ,  $\sigma_0$ ,  $L_0$ , and (A12)(i) hold. Let  $(\phi_0(t, x), \psi_0(t, x))$  be the unique solution pair to (5.14) which is  $C^\infty(\mathbb{R}^n)$  and is a.s. uniformly bounded. In addition, we assume (A8) holds for  $S_0$  and the resulting  $u_0$  is also a.s. Lipschitz continuous in  $\mu$ . Then, for  $\mu_{(\cdot)}(\omega)$  and  $\nu_{(\cdot)}(\omega)$  in the set  $\mathcal{M}_\rho^\beta$ ,  $0 < \beta < 1$ , there exists a constant  $c_4$  such that

$$(6.16) \quad \sup_{(t,x) \in [0,T] \times \mathbb{R}^n} |u_0(t, \omega, x) - u'_0(t, \omega, x)|^2 \leq c_4 \left( D_T^\rho(\mu(\omega), \nu(\omega)) \right)^2 \quad a.s.,$$

where  $u_0, u'_0 \in C_{Lip(x)}([0, T] \times \Omega \times \mathbb{R}^n; U_0)$  are induced by the map  $\Upsilon_0^{SHJB}$  in (6.3) using two stochastic measures  $\mu_{(\cdot)}(\omega)$  and  $\nu_{(\cdot)}(\omega)$ , respectively.

- (ii) Assume (A3)–(A7), for  $U$ ,  $f$ ,  $\sigma$ ,  $L$ , and (A12)(ii) hold. Let  $(\phi(t, x), \psi(t, x))$  be the unique solution pair to (5.17) which is  $C^\infty(\mathbb{R}^n)$  and is a.s. uniformly bounded. In addition, we assume (A8) holds for  $S$  and the resulting  $u$  is also

a.s. Lipschitz continuous in  $\mu$ . Then, for  $\mu_{(\cdot)}^0(\omega) \in \mathcal{M}_\rho^\gamma$ ,  $0 < \gamma < 1/2$ , and  $\mu_{(\cdot)}(\omega)$  and  $\nu_{(\cdot)}(\omega)$  in the set  $\mathcal{M}_\rho^\beta$ ,  $0 < \beta < 1$ , there exists a constant  $c_5$  such that

(6.17)

$$\sup_{(t,x) \in [0,T] \times \mathbb{R}^n} |u(t, \omega, x) - u'(t, \omega, x)|^2 \leq c_5 \left( D_T^\rho(\mu(\omega), \nu(\omega)) \right)^2 \quad a.s.,$$

where  $u, u' \in C_{Lip(x)}([0, T] \times \Omega \times \mathbb{R}^n; U)$  are induced by the map  $\Upsilon_i^{SHJB}$  in (6.7) using two stochastic measures  $\mu_{(\cdot)}(\omega)$  and  $\nu_{(\cdot)}(\omega)$ , respectively.

- (iii) Assume (A3)–(A7), for  $U$ ,  $f$ ,  $\sigma$ ,  $L$ , and (A12)(ii) hold. Let  $(\phi(t, x), \psi(t, x))$  be the unique solution pair to (5.17) which is  $C^\infty(\mathbb{R}^n)$  and is a.s. uniformly bounded. In addition, we assume (A8) holds for  $S$  and the resulting  $u$  is also a.s. Lipschitz continuous in  $\mu^0$ . Then, for  $\mu_{(\cdot)}(\omega) \in \mathcal{M}_\rho^\beta$ ,  $0 < \beta < 1$ , and  $\mu_{(\cdot)}^0(\omega)$  and  $\nu_{(\cdot)}^0(\omega)$  in the set  $\mathcal{M}_\rho^\gamma$ ,  $0 < \gamma < 1/2$ , there exists a constant  $c_6$  such that

(6.18)

$$\sup_{(t,x) \in [0,T] \times \mathbb{R}^n} |u(t, \omega, x) - u'(t, \omega, x)|^2 \leq c_6 \left( D_T^\rho(\mu^0(\omega), \nu^0(\omega)) \right)^2 \quad a.s.,$$

where  $u, u' \in C_{Lip(x)}([0, T] \times \Omega \times \mathbb{R}^n; U)$  are induced by the map  $\Upsilon_i^{SHJB}$  in (6.7) using the two stochastic measures  $\mu_{(\cdot)}^0(\omega)$  and  $\nu_{(\cdot)}^0(\omega)$ , respectively.

*Proof.* (i) Assumption (A8) for  $S_0$  together with the fact that the resulting  $u_0$  in (A8) is also a.s. Lipschitz continuous in  $\mu$  yields

$$(6.19) \quad |u_0(t, \omega, x) - u'_0(t, \omega, x)| \leq k_1 D_t^\rho(\mu(\omega), \nu(\omega)) + k_2 |D_x \phi_0^\mu(t, \omega, x) - D_x \phi_0^\nu(t, \omega, x)|$$

with positive constants  $k_1, k_2$ , where we indicate the dependence of  $\phi_0$  on measures  $\mu$  and  $\nu$  by  $\phi_0^\mu$  and  $\phi_0^\nu$ , respectively.

We consider the Gâteaux derivative of  $\phi_0$  with respect to the measure  $\mu$ . The assumptions of the theorem imply that the conditions for Proposition F.1 in Appendix F of [36] hold. Therefore, Proposition F.1 in [36] concludes that the Gâteaux derivative of  $D_x \phi_0$  with respect to measure  $\mu$  is a.s. uniformly bounded. This together with the mean value theorem yields

$$(6.20) \quad |D_x \phi_0^\mu(t, \omega, x) - D_x \phi_0^\nu(t, \omega, x)| \leq k_3 D_t^\rho(\mu(\omega), \nu(\omega))$$

with positive constant  $k_3$ . Equations (6.19) and (6.20) give

$$|u_0(t, \omega, x) - u'_0(t, \omega, x)| \leq k D_t^\rho(\mu(\omega), \nu(\omega))$$

with  $k := k_1 + k_2 k_3$ , which yields the result.  $\square$

*Remark 6.11.* In the standard MFG model of [17] a similar condition to (6.16)–(6.18) is taken as an assumption. (See the feedback regularity condition (37) in [17].) Following the argument in section 7.1 of [17], one can show that the inequalities (6.16)–(6.18) hold in the LQG model with Lipschitz continuous nonlinear couplings.

We recall the map  $\Upsilon$  given in (6.11) which is the composition of the maps  $\Upsilon_0$ ,  $\Upsilon_i^{SHJB}$ , and  $\Upsilon_i^{SMV}$  introduced in (6.5), (6.7), and (6.10), respectively:

$$\begin{array}{ccc}
 \mu_{(\cdot)}(\omega) & \xrightarrow{\Upsilon_0^{\text{SHJB}}} & u_0^o(\cdot, \omega, x) \\
 \uparrow \Upsilon_i^{\text{SMV}} & & \downarrow \Upsilon_0^{\text{SMV}} \\
 u^o(\cdot, \omega, x) & \xleftarrow{\Upsilon_i^{\text{SHJB}}} & \mu_{(\cdot)}^0(\omega) \equiv \delta_{z_0^o(t, \omega)}
 \end{array}$$

**THEOREM 6.12 (main result).** *Let the assumptions of both Lemma 6.9 and Lemma 6.10 hold. If the constants  $\{c_i : 0 \leq i \leq 6\}$  for (6.12)–(6.15) and (6.16)–(6.18) satisfy the gain condition*

$$\max \{c_2 c_5, c_2 c_6 c_0, c_2 c_6 c_1, c_3 c_1, c_3 c_0 c_4\} < 1,$$

*then there exists a unique solution for the map  $\Upsilon$  and hence a unique solution to the MM-SMFG system (5.14)–(5.16) and (5.17)–(5.19).*

*Proof.* The result follows from the Banach fixed point theorem for the map  $\Upsilon$  given in (6.11) on the Polish space  $\mathcal{M}_\rho^\beta$ ,  $0 < \beta < 1$ , since the gain condition ensures that  $\Upsilon$  is a contraction.  $\square$

As in the classical FBSDEs, the gain condition in Theorem 6.12 is expected to hold for short time horizon  $T$ . Another approach to the solution existence of the MM-SMFG system (5.14)–(5.16) and (5.17)–(5.19) is Schauder's fixed point argument which is the topic of future work.

#### 7. $\epsilon$ -Nash equilibrium property of the SMFG control laws.

We let  $(\phi_0(t, \omega, x), \psi_0(t, \omega, x), u_0^o(t, \omega, x), z_0^o(t, \omega), \phi(t, \omega, x), \psi(t, \omega, x), u^o(t, \omega, x), z^o(t, \omega))$

be the unique solution of the MM-SMFG system (5.14)–(5.16) and (5.17)–(5.19) such that SMFG best response  $u_0^o(t, \omega, x)$  and  $u^o(t, \omega, x)$  are a.s. continuous in  $(t, x)$  and a.s. Lipschitz continuous in  $x$ .

We now apply the SMFG best responses  $u_0^o(t, \omega, x)$  and  $u^o(t, \omega, x)$  into a finite  $N + 1$  MM population (2.1)–(2.2). This yields the following closed loop individual dynamics:

(7.1)

$$\begin{aligned}
 dz_0^{o,N}(t) &= \frac{1}{N} \sum_{j=1}^N f_0[t, z_0^{o,N}(t), u_0^o(t, z_0^{o,N}(t)), z_j^{o,N}(t)] dt \\
 &\quad + \frac{1}{N} \sum_{j=1}^N \sigma_0[t, z_0^{o,N}(t), z_j^{o,N}(t)] dw_0(t), \quad z_0^{o,N}(0) = z_0(0), \quad 0 \leq t \leq T,
 \end{aligned}$$

(7.2)

$$\begin{aligned}
 dz_i^{o,N}(t) &= \frac{1}{N} \sum_{j=1}^N f[t, z_i^{o,N}(t), u^o(t, z_i^{o,N}(t)), z_0^{o,N}(t), z_j^{o,N}(t)] dt \\
 &\quad + \frac{1}{N} \sum_{j=1}^N \sigma[t, z_i^{o,N}(t), z_0^{o,N}(t), z_j^{o,N}(t)] dw_i(t), \quad z_i^{o,N}(0) = z_i(0), \quad 1 \leq i \leq N.
 \end{aligned}$$

We set the admissible control set of agent  $\mathcal{A}_j$ ,  $0 \leq j \leq N$ , to be

$$\begin{aligned}
 \mathcal{U}_j &= \left\{ u_j(\cdot, \omega) := u_j(\cdot, \omega, z_0(\cdot, \omega), \dots, z_N(\cdot, \omega)) \in C_{\text{Lip}(z_0, \dots, z_N)} : u_j(t, \omega) \text{ is a } \right. \\
 &\quad \mathcal{F}_t^{w_0}\text{-measurable process adapted to sigma-field} \\
 &\quad \left. \sigma\{z_i(\tau, \omega) : 0 \leq i \leq N, 0 \leq \tau \leq t\} \text{ such that } \mathbb{E} \int_0^T |u_j(t, \omega)|^2 dt < \infty \right\}.
 \end{aligned}$$



We note that  $\mathcal{U}_j$ ,  $0 \leq j \leq N$ , are the full information admissible control which are not restricted to be decentralized.

**DEFINITION 7.1.** *Given  $\epsilon > 0$ , the admissible control laws  $(u_0^o, \dots, u_N^o)$  for  $N+1$  agents generate an  $\epsilon$ -Nash equilibrium with respect to the costs  $J_j^N$ ,  $0 \leq j \leq N$ , if  $J_j^N(u_j^o; u_{-j}^o) - \epsilon \leq \inf_{u_j \in \mathcal{U}_j} J_j^N(u_j; u_{-j}^o) \leq J_j^N(u_j^o; u_{-j}^o)$  for any  $0 \leq j \leq N$ .*

We now show that the SMFG best responses for a finite  $N+1$  MM population system (7.1)–(7.2) is an  $\epsilon$ -Nash equilibrium with respect to the cost functions (2.3)–(2.4) in the case where minor agents are coupled to the major agent only through their cost functions. (See the MM-MFG LQG model in [30].)

(A13) Assume the functions  $f$  and  $\sigma$  in (2.2) (and hence in (7.2)) do not contain the state of major agent  $z_0^N$ .

Note that in the case of assumption (A13) the major agent  $\mathcal{A}_0$  has non-negligible influence on the minor agents through their cost functions (2.4). An analysis based on the anticipative variational calculations used in the MM-MFG LQG case [31] may establish the  $\epsilon$ -Nash equilibrium property of the SMFG best responses in the general case. This is currently under investigation and will be reported in future work.

**THEOREM 7.2.** *Assume (A1)–(A6) and (A13) hold and there exists a unique solution to the MM-SMFG system (5.14)–(5.16) and (5.17)–(5.19) such that the SMFG best response control processes  $z_0^o(t, \omega, x)$  and  $u^o(t, \omega, x)$  are a.s. continuous in  $(t, x)$  and a.s. Lipschitz continuous in  $x$ . Then  $(u_0^o, u_1^o, \dots, u_N^o)$  where  $u_i^o \equiv u^o$ ,  $1 \leq i \leq N$ , generates an  $O(\epsilon_N + 1/\sqrt{N})$ -Nash equilibrium with respect to the cost functions (2.3)–(2.4) such that  $\lim_{N \rightarrow \infty} \epsilon_N = 0$ .*

*Proof:* Under (A13) we have the the following closed loop individual dynamics under the SMFG best response control processes:

$$\begin{aligned} dz_0^{o,N}(t) &= \frac{1}{N} \sum_{j=1}^N f_0[t, z_0^{o,N}(t), u_0^o(t, z_0^{o,N}(t)), z_j^{o,N}(t)] dt \\ &\quad + \frac{1}{N} \sum_{j=1}^N \sigma_0[t, z_0^{o,N}(t), z_j^{o,N}(t)] dw_0(t), \quad z_0^{o,N}(0) = z_0(0), \quad 0 \leq t \leq T, \\ dz_i^{o,N}(t) &= \frac{1}{N} \sum_{j=1}^N f[t, z_i^{o,N}(t), u^o(t, z_i^{o,N}(t)), z_j^{o,N}(t)] dt \\ &\quad + \frac{1}{N} \sum_{j=1}^N \sigma[t, z_i^{o,N}(t), z_j^{o,N}(t)] dw_i(t), \quad z_i^{o,N}(0) = z_i(0), \quad 1 \leq i \leq N. \end{aligned}$$

We also introduce the associated MV system

$$(7.3) \quad \begin{aligned} dz_0^o(t) &= f_0[t, z_0^o(t), u_0^o(t, z_0^o), \mu_t] dt + \sigma_0[t, z_0^o(t), \mu_t] dw_0(t), \\ dz_i^o(t) &= f[t, z_i^o(t), u^o(t, z_i^o), \mu_t] dt + \sigma[t, z_i^o, \mu_t] dw_i(t) \end{aligned}$$

with the initial condition  $z_j^o(0) = z_j(0)$ ,  $0 \leq j \leq N$ . In the above MV equation  $\mu_t$ ,  $0 \leq t \leq T$ , is the conditional law of  $z_i^o(t)$ ,  $1 \leq i \leq N$ , given  $\mathcal{F}_t^{w_0}$  (i.e.,  $\mu_t := \mathcal{L}(z_i^o(t) | \mathcal{F}_t^{w_0})$ ,  $1 \leq i \leq N$ ). Theorem 3.1 implies that

$$(7.4) \quad \sup_{0 \leq j \leq N} \sup_{0 \leq t \leq T} \mathbb{E} |z_j^{o,N}(t) - z_j^o(t)| = O(1/\sqrt{N}),$$

where the right-hand side may depend upon the terminal time  $T$ .

Let  $z(0) = \int_{\mathbb{R}^n} x dF(x)$  be the mean value of the minor agents' initial states (see (A2)). We denote

$$(\epsilon_N)^2 = \left| \int_{\mathbb{R}^N} x^T x dF_N(x) - 2z^T(0) \int_{\mathbb{R}^N} x dF_N(x) + z^T(0)z(0) \right|.$$

It is evident from (A2) that  $\lim_{N \rightarrow \infty} \epsilon_N = 0$ . To prove the  $\epsilon$ -Nash equilibrium property we consider two cases as follows.

*Case I* (strategy change for the major agent  $\mathcal{A}_0$ ). While the minor agents are using the SMFG best response control law  $u^0(t, \omega, x)$ , a strategy change from  $u_0^0(t, \omega, x)$  to the  $\mathcal{F}_t^{w_0}$ -adapted process  $u_0(t, \omega, x, z_{-0}^{o,N}(t, \omega)) \in \mathcal{U}_0$  for the major agent yields

$$\begin{aligned} dz_0^N(t) &= \frac{1}{N} \sum_{j=1}^N f_0[t, z_0^N(t), u_0(t, z_0^N(t), z_{-0}^{o,N}(t)), z_j^{o,N}(t)] dt \\ &\quad + \frac{1}{N} \sum_{j=1}^N \sigma_0[t, z_0^N(t), z_j^{o,N}(t)] dw_0(t), \quad z_0^N(0) = z_0(0), \quad 0 \leq t \leq T, \end{aligned}$$

where  $z_{-0}^{o,N} \equiv (z_1^{o,N}, \dots, z_N^{o,N})$ . Since minor agents are coupled to the major agent only through their cost functions (see (A13)) the strategy change of the major agent does not affect the minor agents' states  $z_i^{o,N}$  and  $z_i^o$ ,  $1 \leq i \leq N$ , above.

Let  $\hat{z}_0^N(\cdot)$  be the solution to

$$\begin{aligned} d\hat{z}_0^N(t) &= \frac{1}{N} \sum_{j=1}^N f_0[t, \hat{z}_0^N(t), u_0(t, \hat{z}_0^N(t), z_{-0}^o(t)), z_j^o(t)] dt \\ &\quad + \frac{1}{N} \sum_{j=1}^N \sigma_0[t, \hat{z}_0^N(t), z_j^o(t)] dw_0(t), \quad \hat{z}_0^N(0) = z_0(0), \quad 0 \leq t \leq T, \end{aligned}$$

where  $z_{-0}^o \equiv (z_1^o, \dots, z_N^o)$  is given by the MV system above. Theorem 3.1 and the Gronwall's lemma imply that

$$(7.5) \quad \sup_{0 \leq t \leq T} \mathbb{E}|z_0^N(t) - \hat{z}_0^N(t)| = O(1/\sqrt{N}).$$

We also introduce

$$d\hat{z}_0(t) = f_0[t, \hat{z}_0(t), u_0(t, \hat{z}_0(t), z_{-0}^o(t)), \mu_t] dt + \sigma_0[t, \hat{z}_0(t), \mu_t] dw_0(t)$$

with initial condition  $\hat{z}_0(0) = z_0(0)$ , where  $\mu_{(\cdot)}$  is the minor agents' measure given by the MV system above. Again, by Theorem 3.1 and the Gronwall's lemma it can be shown that

$$(7.6) \quad \sup_{0 \leq t \leq T} \mathbb{E}|\hat{z}_0^N(t) - \hat{z}_0(t)| = O(1/\sqrt{N}).$$

Assumptions (A3) and (A6), (7.4)–(7.6), and Theorem 3.1 yield

$$\begin{aligned}
(7.7) \quad J_0^N(u_0; u_{-0}^o) & \equiv \mathbb{E} \int_0^T \left( (1/N) \sum_{j=1}^N L_0[t, z_0^N(t), u_0(t, z_0^N, z_{-0}^{o,N}), z_j^{o,N}(t)] \right) dt \\
& \stackrel{(7.4)}{\geq} \mathbb{E} \int_0^T \left( (1/N) \sum_{j=1}^N L_0[t, z_0^N(t), u_0(t, z_0^N, z_{-0}^o), z_j^o(t)] \right) dt - O(\epsilon_N + 1/\sqrt{N}) \\
& \stackrel{(7.5)}{\geq} \mathbb{E} \int_0^T \left( (1/N) \sum_{j=1}^N L_0[t, \hat{z}_0^N(t), u_0(t, \hat{z}_0^N, z_{-0}^o), z_j^o(t)] \right) dt - O(\epsilon_N + 1/\sqrt{N}) \\
& \stackrel{(7.6)}{\geq} \mathbb{E} \int_0^T \left( (1/N) \sum_{j=1}^N L_0[t, \hat{z}_0(t), u_0(t, \hat{z}_0, z_{-0}^o), z_j^o(t)] \right) dt - O(\epsilon_N + 1/\sqrt{N}) \\
& \stackrel{(3.1)}{\geq} \mathbb{E} \int_0^T L_0[t, \hat{z}_0(t), u_0(t, \hat{z}_0, z_{-0}^o), \mu_t] dt - O(\epsilon_N + 1/\sqrt{N}),
\end{aligned}$$

where the appearance of the  $\epsilon_N$  term in the first inequality of (7.7) is due to the fact that here the sequence of minor agents' initials  $\{z_j^o(0) : 1 \leq j \leq N\}$  in the SMV system (7.3) is generated by independent randomized observations on the distribution  $F$  given in (A2).

Furthermore, by the construction of the major agent's SMFG system (5.14)–(5.16) (see the major agent's SOCP (5.1)–(5.2)) we have

$$(7.8) \quad \mathbb{E} \int_0^T L_0[t, \hat{z}_0(t), u_0(t, \hat{z}_0, z_{-0}^o), \mu_t] dt \geq \mathbb{E} \int_0^T L_0[t, z_0^o(t), u_0^o(t, z_0^o), \mu_t] dt.$$

But, Theorem 3.1 and (7.4) imply

$$\begin{aligned}
(7.9) \quad & \mathbb{E} \int_0^T L_0[t, z_0^o(t), u_0^o(t, z_0^o), \mu_t] dt \\
& \stackrel{(3.1)}{\geq} \mathbb{E} \int_0^T \left( (1/N) \sum_{j=1}^N L_0[t, z_0^o(t), u_0(t, z_0^o), z_j^o(t)] \right) dt - O(\epsilon_N + 1/\sqrt{N}) \\
& \stackrel{(7.4)}{\geq} \mathbb{E} \int_0^T \left( (1/N) \sum_{j=1}^N L_0[t, z_0^{o,N}(t), u_0(t, z_0^{o,N}), z_j^{o,N}(t)] \right) dt - O(\epsilon_N + 1/\sqrt{N}) \\
& \equiv J_0^N(u_0^o; u_{-0}^o) - O(\epsilon_N + 1/\sqrt{N}).
\end{aligned}$$

It follows from (7.7)–(7.9) that  $J_0^N(u_0^o; u_{-0}^o) - O(\epsilon_N + 1/\sqrt{N}) \leq \inf_{u_0 \in \mathcal{U}_0} J_0^N(u_0; u_{-0}^o)$ .

*Case II* (strategy change for the minor agents). Without loss of generality, we assume that the first minor agent changes its MF best response control strategy  $u^o(t, \omega, x)$  to  $u_1(t, \omega, x, z_{-1}(t, \omega)) \in \mathcal{U}_1$ . This leads to

$$\begin{aligned}
dz_0^N(t) &= \frac{1}{N} \sum_{j=1}^N f_0[t, z_0^N, u_0^o(t, z_0^N), z_j^N] dt + \frac{1}{N} \sum_{j=1}^N \sigma_0[t, z_0^N, z_j^N] dw_0(t), \\
dz_1^N(t) &= \frac{1}{N} \sum_{j=1}^N f[t, z_1^N, u_1(t, z_1^N, z_{-1}^N), z_j^N] dt + \frac{1}{N} \sum_{j=1}^N \sigma[t, z_1^N, z_j^N] dw_1(t),
\end{aligned}$$

$$\begin{aligned} dz_2^N(t) &= \frac{1}{N} \sum_{j=1}^N f[t, z_2^N, u^o(t, z_2^N), z_j^N] dt + \frac{1}{N} \sum_{j=1}^N \sigma[t, z_2^N, z_j^N] dw_2(t), \\ &\vdots \\ dz_N^N(t) &= \frac{1}{N} \sum_{j=1}^N f[t, z_N^N, u^o(t, z_N^N), z_j^N] dt + \frac{1}{N} \sum_{j=1}^N \sigma[t, z_N^N, z_j^N] dw_N(t). \end{aligned}$$

By the same argument as in proving Theorem 3.1 (see Appendix A in [36]) it can be shown that

$$\begin{aligned} \sup_{j=0,2,\dots,N} \sup_{0 \leq t \leq T} \mathbb{E}|z_j^{o,N}(t) - z_j^N(t)| &= O(1/\sqrt{N}), \\ \sup_{j=0,2,\dots,N} \sup_{0 \leq t \leq T} \mathbb{E}|z_j^o(t) - z_j^N(t)| &= O(1/\sqrt{N}). \end{aligned}$$

Let  $\hat{z}_1^N(\cdot)$  be the solution to

$$\begin{aligned} d\hat{z}_1^N(t) &= \frac{1}{N} \sum_{j=1}^N f[t, \hat{z}_1^N(t), u_1(t, \hat{z}_1^N(t), z_{-1}^o(t)), z_j^o(t)] dt \\ &\quad + \frac{1}{N} \sum_{j=1}^N \sigma[t, \hat{z}_1^N(t), z_j^o(t)] dw_1(t), \quad \hat{z}_1^N(0) = z_1(0), \quad 0 \leq t \leq T, \end{aligned}$$

where  $z_{-1}^o \equiv (z_1^o, \dots, z_N^o)$  is given by the MV system above. Theorem 3.1 and the Gronwall's lemma imply that

$$(7.10) \quad \sup_{0 \leq t \leq T} \mathbb{E}|z_1^N(t) - \hat{z}_1^N(t)| = O(1/\sqrt{N}).$$

We also introduce

$$d\hat{z}_1(t) = f[t, \hat{z}_1(t), u_1(t, \hat{z}_1(t), z_{-1}^o(t)), \mu_t] dt + \sigma[t, \hat{z}_1(t), \mu_t] dw_1(t)$$

with initial condition  $\hat{z}_1(0) = z_1(0)$ , where  $\mu_{(\cdot)}$  is the minor agents' measure given by the MV system above. Again, by Theorem 3.1 and the Gronwall's lemma it can be shown that

$$(7.11) \quad \sup_{0 \leq t \leq T} \mathbb{E}|\hat{z}_1^N(t) - \hat{z}_1(t)| = O(1/\sqrt{N}).$$

Using (7.4) and (7.10)–(7.11), and by the same argument as in (7.7)–(7.9), we can show that  $J_1^N(u_1^o; u_{-1}^o) - O(\epsilon_N + 1/\sqrt{N}) \leq \inf_{u \in \mathcal{U}_1} J_1^N(u_1; u_{-1}^o)$ .  $\square$

**8. Conclusion.** This paper studies a stochastic mean field game system for a class of dynamic games involving nonlinear stochastic dynamical systems with major and minor agents. The SMFG system consists of coupled (i) backward in time stochastic Hamilton-Jacobi-Bellman equations and (ii) forward in time stochastic McKean-Vlasov or stochastic Fokker-Planck-Kolmogorov equations. Existence and uniqueness of the solution to the MM-SMFG system is established by a fixed point argument in the Wasserstein space of random probability measures. In the case where minor agents

are coupled to the major agent only through their cost functions, the  $\epsilon_N$ -Nash equilibrium property of the SMFG best responses is shown for a finite  $N$  population system, where  $\epsilon_N = O(1/\sqrt{N})$ . As a special but important case, the results of Nguyen and Huang [30] for MM-SMFG linear-quadratic-Gaussian systems with a homogeneous population can be retrieved and, in addition, the results of this paper are illustrated with a major and minor agent version of a game model of the synchronization of coupled nonlinear oscillators (see Appendices G and H in [36]).

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