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PROBABILISTIC INTERPRETATION FOR SYSTEMS OF QUASILINEAR PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS*

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A probabilistic interpetation for systems of second order quasilinear parabolic partial differential equation is obtained by introducing a kind of backward stochastic differential equation. Similar applications to systems of elliptic partial differential equations and to optimal control problems are also briefly discussed.

KEY WORDS Parabolic partial differential equation, probability, stochastic differential equation, Feynman-Kac formula, optimal control.

1 INTRODUCTION

It is known that a solution of a linear second order parabolic (or elliptic) equation can be formulated as a functional of a solution of some stochastic differential equation (see [5] for detailed references). This kind of interpretation has found important applications both in the theory of partial differential equations and that of stochastic differential equations, such as large deviation [5], [6], optimal control theory [2], [11], [12], martingale problem [4], [13], [19], variational and quasi variational inequality, [2] etc.

A natural and interesting problem is: can one obtain a similar interpretation for a system of quasilinear parabolic (or elliptic) partial differential equations? This is the objective of this paper. A recent result in backward stochastic differential equation (see Pardoux and Peng [15]) produces new insights to this direction. In this paper, we will interpret a system of second order quasilinear parabolic partial differential equations as a solution of a backward stochastic differential equation. This backward equation is associated with some classical Itô forward stochastic differential equations. More precisely, we introduce a classical SDE of Itô's type defined on an interval $[t, T]$

$$\begin{cases} dy(s) = b(y(s), s)\, ds + \sigma(y(s), s)\, dW_s, \\ y(t) = x \in \mathbb{R}^n, \end{cases}$$

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where $(W_s; s \geq 0)$ is a Brownian motion. Then we consider a pair of adapted processes $(p., q.)$ which solves uniquely the following type of m-dimensional backward SDE

$$p_s = \Psi(y(T)) + \int_s^T f(p_r, q_r, y(r), r)\, dr - \int_t^T q_r\, dW_r, \qquad t \leq s \leq T.$$

We define a $\mathbb{R}^m$-valued function

$$\mathbf{u}(x, t) = p_t = Ep_t.$$

Then $\mathbf{u}(x, t)$ solves the following system of quasilinear parabolic PDE

$$\begin{cases} \partial_t \mathbf{u} + \mathscr{L}_{(x,t)}\mathbf{u}(x, t) + f(\mathbf{u}(x, t), \mathbf{u}_x(x, t)\sigma(x, t), x, t) = 0, \\ \mathbf{u}(x, T) = \Psi(x), \end{cases}$$

where $\mathscr{L}_{(x,t)}$ is the infinitesimal operator generated by the diffusion process $y(\cdot)$.

A particular case is that of linear one dimensional backward equation where f does not contain q:

$$f = f_0(x, t) + c(x, t)p.$$

In this case the corresponding system of equation becomes a linear parabolic PDE. So the classical probabilistic interpretation, known as Feynman-Kac formula, can be regarded as a special case of our formulation.

The paper is organized as follows: In Section 2, we present a result [14] concerning backward stochastic differential equations and adapt it to our situation. In Section 3 we apply the result of Section 2 to obtain stochastic interpretation for a system of quasilinear parabolic partial differential equations. We conclude this paper with a brief discussion of some possible applications of our idea.

2 BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS

In this section we will extend results obtained in [14] to more general case. Throughout this section K will be a given positive constant. Let $\tau = \tau(\omega)$ be an $\mathscr{F}_t$-stopping time with values in $[0, \infty]$.

Definition We denote by $M^2(0, \tau; \mathbb{R}^m)$ the set of all R^m valued $\mathscr{F}_t$-adapted processes such that

$$E\int_0^\tau |v(s)|^2\, ds < \infty.$$

$v(\cdot)$ will be said in $M^{2,K}(0, \tau, \mathbb{R}^m)$ if $v(s)\exp\{Ks/2\}$, $s \geq 0$ is in $M^2(0, \tau; \mathbb{R}^m)$. Some times we note only $M^{2,K}(0, \tau)$.

Let

$$X \in L^2(\Omega, \mathscr{F}_\tau, P; \mathbb{R}^m),$$

be given such that, there exists a constant C_b such that

$$E|X|^2 e^{K(T\wedge\tau)} < C_b \qquad \forall T > 0, \tag{2.1}$$

For a given $\varphi(\cdot) \in M^{2,K}(0, \tau; \mathbb{R}^m)$ we first consider the following problem: to find a pair of adapted processes (p, q) that solves

$$p_t = X + \int_{t\wedge\tau}^{\tau} \varphi(s)\, ds - \int_{t\wedge\tau}^{\tau} q_s\, dW_s, \qquad t \geq 0. \tag{2.2}$$

We have the following lemma.

LEMMA 2.1 *We suppose* (2.1). *Then there exists a unique pair of adapted processes*

$$(p_\cdot, q_\cdot) \in M^{2,K}(0, \tau; \mathbb{R}^m) \times M^{2,K}(0, \tau; \mathbb{R}^{m\times d})$$

which solves (2.2).

Proof Let

$$M(t) = E^{\mathscr{F}_t}\left[X + \int_0^\tau \varphi(s)\, ds\right]$$

Observe that $E|E^{\mathscr{F}_t}X|^2 \leq E|X|^2$ and

$$E\left|E^{\mathscr{F}_{\tau\wedge t}}\int_t^\tau \varphi(s)\, ds\right|^2 \leq E\left[\left(E^{\mathscr{F}_{\tau\wedge t}}\int_0^\tau e^{-Ks}\, ds\right)E^{\mathscr{F}_{\tau\wedge t}}\int_t^\tau |\varphi(s)|^2 e^{Ks}\, ds\right] < \infty,$$

thus $M(\cdot)$ is a square integrable martingale. From the well-known martingale representation theorem, there exists a unique process $q_\cdot \in M^2(O, T)$, $\forall T > 0$, such that

$$M(t) = M(0) + \int_0^t q_s\, dW_s = M(0) + \int_0^{t\wedge\tau} q_s\, dW_s.$$

Particularly

$$M(0) + \int_0^\tau q_s\, dW_s = M(\tau) = X + \int_0^\tau \varphi(s)\, ds,$$

or

$$M(0) + \int_0^{t\wedge\tau} q_s\, dW_s = X + \int_0^\tau \varphi(s)\, ds - \int_{t\wedge\tau}^\tau q_s\, dW_s. \tag{2.3}$$

We set

$$p_t = E^{\mathscr{F}_t}\left(X + \int_{t \wedge \tau}^{\tau} \varphi(s)\, ds\right).$$

It turns out that

$$\begin{aligned} p_t &= E^{\mathscr{F}_t}\left[X + \int_0^{\tau} \varphi(s)\, ds\right] - E^{\mathscr{F}_t} \int_0^{t \wedge \tau} \varphi(s)\, ds \\ &= M(0) + \int_0^{t \wedge \tau} q_s\, dW_s - \int_0^{t \wedge \tau} \varphi(s)\, ds \end{aligned}$$

This with (2.3) implies that $(p., q.)$ solves (2.2).

We now prove that $(p., q.) \in M^{2,K}(0, T)$. We apply Itô's formula to $e^{Ks}|p_s|^2$

$$E(e^{K(T \wedge \tau)}|p_\tau)|^2 - e^{K(t \wedge \tau)}|p_t|^2) = E \int_{t \wedge \tau}^{T \wedge \tau} (K|p_s|^2 - 2 < p_s, \varphi(s) > + |q_s|^2)e^{Ks}\, ds.$$

Thus

$$\begin{aligned} E(e^{K(t \wedge \tau)}|p_t|^2) + E \int_{t \wedge \tau}^{T \wedge \tau} \left(\frac{1}{2} K|p_s|^2 + |q_s|^2\right) e^{Ks}\, ds \\ \leq E(e^{K(T \wedge \tau)}|p_\tau|^2) + cE \int_0^{T \wedge \tau} |\varphi(s)|^2 e^{Ks}\, ds. \end{aligned}$$

It is easily seen that

$$\lim_{T \to \infty} E(|p(T \wedge \tau)|^2 e^{K(T \wedge \tau)}) = E(|X|^2 e^{K\tau}),$$

and

$$\lim_{T \to \infty} E|p_{T \wedge \tau} - X|^2 = 0.$$

Passing limit as $T \to \infty$, we have

$$\begin{aligned} E(e^{K(t \wedge \tau)}|p_t|^2 + E \int_{t \wedge \tau}^{\tau} \left(\frac{1}{2} K|p_s|^2 + |q_s|^2\right) e^{Ks}\, ds \\ \leq E(e^{K\tau}|p_\tau|^2) + E \int_0^{\tau} |\varphi(s)|^2 e^{Ks}\, ds < \infty. \end{aligned}$$

This completes the proof. □

Now we can consider a more general case: to find a pair of $\mathscr{F}_t$-adapted processes $(p_\cdot, q_\cdot)$ which solves

$$\begin{cases} -dp_t = f(p_t, q_t, t)\,dt - q_t\,dW_t, & t \in [0, \tau], \\ p_\tau = X, \end{cases} \tag{2.4}$$

or

$$p_t = X + \int_{t\wedge\tau}^{\tau} f(p_s, q_s, s)\,ds - \int_{t\wedge\tau}^{\tau} q_s\,dW_s. \tag{2.5}$$

where, for each $(p, q) \in \mathbb{R}^m \times \mathbb{R}^{m\times d}$, $f(p, q, \cdot)$ is a m-dimensional $\mathscr{F}_t$-adapted process. We assume that there exist constants $C_0 > 0$ and μ (μ can be negative) satisfing

$$d_0 = K + 2\mu - 2C_0 > 0$$

such that

$$\begin{cases} |f(p_1, q_1, s) - f(p_2, q_2, s)| \le C|p_1 - p_2| + C_0|q_1 - q_2|, & \forall p_1, p_2, q_1, q_2, s, \\ \langle p_1, f(p + p_1, q, s) - f(p, q, s)\rangle \le -\mu|p_1|^2, & \forall p, p_1, q, s, \\ E(e^{K(T\wedge\tau)}|X|^2) + E\int_0^\tau |f(0, 0, s)|^2 e^{Ks}\,ds \le C_1. & \forall T > 0 \end{cases} \tag{2.6}$$

We can claim the main result of this section.

THEOREM 2.2 *We assume* (2.6). *Then there exists a unique pair*

$$(p_\cdot, q_\cdot) \in M^{2,K}(0, \tau; \mathbb{R}^m) \times M^{2,K}(0, \tau; \mathbb{R}^{m\times d})$$

which solves the backward stochastic equation (2.4).

Remark The result of Theorem 2.2 contains the following three interesting cases:
Case (i) $\tau \le T < \infty$. In this case, $M^{2,K}(0, T) = M^2(O, T)$, $\forall K > 0$. Thus μ can be any positive or negative number. This assumption is equivalent to

$$\begin{cases} |f(p_1, q_1, s) - f(p_2, q_2, s)| \le C(|p_1 - p_2| + |q_1 - q_2|), & \forall p_1, p_2, q_1, q_2, s, \\ E\left[|X|^2 + \int_0^T |f(0, 0, s)|^2\,ds\right] < \infty. \end{cases}$$

Thus the result contains those in [14] as a special case.
Case (ii) $Ee^{K\tau} < \infty$ for some positive constant K.
Case (iii) $\tau = \infty$, $X = 0$. It is the so called infinite horizon case.

We shall use a kind of apriori method to prove the existence part of the proof for Theorem 2.2. We define, for any given $\alpha \in \mathbb{R}$,

$$f_\alpha(p, q, s) = \alpha f(p, q, s) - (1 - \alpha)\mu p.$$

Consider now to solve the following problem

$$p_t = X + \int_{t \wedge \tau}^{\tau} [f_\alpha(p_s, q_s, s) + F_0(s)]\, ds - \int_{t \wedge \tau}^{\tau} q_s\, dW_s \tag{2.7}$$

where $F_0(\cdot)$ is a given process in $M^{2,K}(0, \tau; \mathbb{R}^m)$. We have the following apriori lemma.

LEMMA 2.3 *We assume that, for a given* $\alpha_0 \in [0, 1]$ *and for any* $F_0(\cdot) \in M^{2,K}(0, \tau; \mathbb{R}^m)$, *equation* (2.7) *has a solution in* $M^{2,K}(0, \tau)$. *Then, there exists a* $\delta_0 = \delta_0(K, \mu, C_0)$, *such that for all* $\alpha \in [\alpha_0, \alpha_0 + \delta_0]$ *and for any* $F_0(\cdot) \in M^{2,K}(0, \tau; \mathbb{R}^m)$, *equation* (2.7) *has a solution in* $M^{2,K}(0, \tau; \mathbb{R}^{m \times d})$.

Proof Observe that

$$f_{\alpha_0+\delta}(p, q, s) = f_{\alpha_0}(p, q, s) + \delta(\mu p + f(p, q, s)).$$

We can set $(p^0, q^0) = 0$, and solve iteratively the following equations

$$p_t^{i+1} = X + \int_{t \wedge \tau}^{\tau} [f_{\alpha_0}(p_s^{i+1}, q_s^{i+1}, s) + \delta(\mu p_s^i + f(p_s^i, q_s^i, s))]\, ds - \int_{t \wedge \tau}^{\tau} q^{i+1}(s)\, dW_s, \tag{2.8}$$

such that $p^i, q^i \in M^{2,K}(0, \tau)$. We set

$$\hat{p}^{i+1} = p^{i+1} - p^i, \; \hat{q}^{i+1} = q^{i+1} - q^i.$$

Using Itô's formula to $|p_s^{i+1}|^2 e^{Ks}$, it follows

$$\begin{aligned}
E|\hat{p}_t^{i+1}|^2 e^{K(t \wedge \tau)} &+ E\int_{t \wedge \tau}^{\tau} (K|\hat{p}_s^{i+1}|^2 + |\hat{q}_s^{i+1}|^2) e^{Ks}\, ds \\
&\leq -2E\int_{t \wedge \tau}^{\tau} \langle \hat{p}_s^{i+1}, f(p_s^{i+1}, q_s^{i+1}, s) - f(p_s^i, q_s^i, s)\rangle e^{Ks}\, ds \\
&\quad - 2E\int_{t \wedge \tau}^{\tau} \langle \hat{p}_s^{i+1}, \delta(-\mu\hat{p}_s^i + f(p_s^i, q_s^i, s) - f(p_s^{i-1}, q_s^{i-1}, s))\rangle e^{Ks}\, ds \\
&\leq E\int_{t \wedge \tau}^{\tau} (-2\mu|\hat{p}_s^{i+1}|^2 + 2C_0|\hat{p}_s^{i+1}||\hat{q}_s^{i+1}|) e^{Ks}\, ds \\
&\quad + 2\delta E\int_{t \wedge \tau}^{\tau} (|\hat{p}_s^{i+1}|[(\mu + C_0|)|\hat{p}_s^i| + C_0|\hat{q}_s^i|] e^{Ks}\, ds
\end{aligned}$$

From which we can derive easily

$$E|\hat{p}^{i+1}(t)|^2 e^{K(t\wedge\tau)} + E\int_{t\wedge\tau}^{\tau} [(K + 2\mu - 2C_0^2 - 2\delta)|\hat{p}_s^{i+1}|^2 + \frac{1}{2}|\hat{q}_s^{i+1}|^2)]e^{Ks}\, ds$$

$$\leq E\int_{t\wedge\tau}^{\tau} [\delta(\mu + C_0)|\hat{p}_s^i|^2 + \delta C_0^2|\hat{q}_s^i|^2]e^{Ks}\, ds$$

Now it is seen that there exists a $\delta_0 = \delta_0(K, \mu, C_0) > 0$, such that, when $0 < \delta \leq \delta_0$,

$$E|\hat{p}^{i+1}(t)|^2 e^{K(t\wedge\tau)} + \frac{1}{2}E\int_{t\wedge\tau}^{\tau} [d_0|\hat{p}_s^{i+1}|^2 + |\hat{q}_s^{i+1}|^2)]e^{Ks}\, ds$$

$$\leq \frac{1}{4}E\int_{t\wedge\tau}^{\tau} [d_0|\hat{p}_s^i|^2 + |\hat{q}_s^i|^2]e^{Ks}\, ds$$

where we denote $d_0 = K + 2\mu - 2C_0$. It turns out that $\{(p^i, q^i)\}$ is a Cauchy sequence in $M^{2,K}(0, \tau)$. We denote its limit by (p^δ, q^δ). Passing limit in (2.8) as $i \to \infty$, we see that, when $0 < \delta \leq \delta(K, \mu, C_0)$, (p^δ, q^δ) solves the equation (2.7) for $\alpha = \alpha_0 + \delta$. The proof is complete. □

Now we can give

Proof of Theorem 2.2 *Uniqueness* Let (p^1, q^1) and (p^2, q^2) both solve (2.5). We set

$$\hat{p} = p^1 - p^2, \qquad \hat{q} = q^1 - q^2$$

Using Itô's formula to $|\hat{p}_s|^2 e^{Ks}$, we have

$$E|\hat{p}_t|^2 e^{K(t\wedge\tau)} + E\int_{t\wedge\tau}^{\tau} (K|\hat{p}_s|^2 + |\hat{q}_s|^2)e^{Ks}\, ds$$

$$\leq 2E\int_{t\wedge\tau}^{\tau} \langle \hat{p}_s, f(p_s^1, q_s^1, s) - f(p_s^2, q_s^2, s)\rangle e^{Ks}\, ds$$

From which we can derive easily

$$E|\hat{p}_t|^2 e^{K(t\wedge\tau)} + E\int_{t\wedge\tau}^{\tau}\left[(K + 2\mu - 2C_0^2)|\hat{p}_s|^2 + \frac{1}{2}|\hat{q}_s|^2)\right]e^{Ks}\, ds \leq 0$$

Thus $(p_s^1, q_s^1) = (p_s^2, q_s^2)$, $0 \leq s \leq \tau$.

Existence It is easy to see that, when $\alpha = 0$, equation (2.7) has a solution in $M^{2,K}(0, \tau)$. Indeed, since $F_0(\cdot) \in M^{2,K}(O, \tau)$, so $F_0(\cdot)e^{-\mu\cdot} \in M^{2,K+2\mu}(O, \tau)$. It follows from Lemma 2.1 that the following equation

$$p_t^\mu = e^{-\mu\tau}X + \int_{t\wedge\tau}^{\tau} e^{-\mu s}F_0(s)\, ds - \int_{t\wedge\tau}^{\tau} q_s^\mu\, ds$$

has a unique solution (p^μ, q^μ) in $M^{2,K+2\mu}(O, \tau)$. It is seen that the pair

$$(p_s, q_s) = (p_s^\mu e^{\mu s}, q_s^\mu e^{\mu s}) \in M^{2,K}(0, T)$$

solves equation (1) when $\alpha = 0$.

Now, according to Lemma 2.3, we can solve (2.7) succesively for the case $\alpha \in [0, \delta_0]$, $\alpha \in [\delta_0, 2\delta_0], \ldots$. It turns out that, when $\alpha = 1$, the solution of (1) exists. □

3 PROBABILISTIC FORMULATION FOR A SYSTEM OF QUASILINEAR PARABOLIC PDE

In this section, we will formulate a system of quasilinear parabolic PDE as a solution of a certain backward stochastic differential equation, of type (2.2), associated with some forward (classical) stochastic differential equations. We first introduce the forward equation.

Let

$$b(x, t)\colon \mathbb{R}^m \times [0, T] \to \mathbb{R}^m,$$

$$\sigma(x, t)\colon \mathbb{R}^m \times [0, T] \to \mathscr{L}(\mathbb{R}^d; \mathbb{R}^m).$$

Let G be a bounded domain in $\mathbb{R}^m$ with boundary $\partial G = S$. We denote $Q = G \times (0, T)$. Assume

$$\begin{cases} \text{(i) } \sigma(x, t) \text{ is of class } C^{2,1}(\bar{Q}), \\ \quad\; b(x, t) \text{ is of class } C^{1,1}(\bar{Q}); \\ \text{(ii) } S \text{ is a manifold of class } C^3; \\ \text{(iii) } \sum a_{ij}(x, t)\xi_i\xi_j \geq \beta|\xi|^2, \qquad \forall (x, t) \in \bar{Q}, \end{cases} \tag{H3.1}$$

where $\beta > 0$ is a constant and $a_{ij} = \frac{1}{2}[\sigma\sigma^*]_{ij}$. For any given $(x, t) \in \bar{Q}$, consider the following forward equation defined on $[t, T]$

$$\begin{cases} dy(s) = b(y(s), s)\, ds + \sigma(y(s), s)\, dW_s, \\ \;y(t) = x. \end{cases} \tag{3.1}$$

We define the following stopping time

$$\tau = \tau_x = \inf\{s \geq t;\ y(s) \notin G\}.$$

From (H3.1), the diffusion process $y(\cdot)$ and related stopping time τ are well defined and there exists a positive constant K such that

$$Ee^{K\tau} < \infty.$$

We then consider the associated backward stochastic equation

$$p_{s\wedge\tau} = \Psi(y(\tau)) + \int_{s\wedge\tau}^{\tau} f(p_r, q_r, y(r), r)\, dr - \int_{s\wedge\tau}^{\tau} q_r\, dW_r, \tag{3.2}$$

where

$$f\colon \mathbb{R}^n \times \mathscr{L}(\mathbb{R}^d; \mathbb{R}^n) \times \mathbb{R}^m \times [0, T] \to \mathbb{R}^n,$$

$$\Psi\colon \mathbb{R}^m \to \mathbb{R}^n.$$

We assume

$$\begin{cases} \text{(i) } f \text{ is continuously differentiable on } \mathbb{R}^n \times \mathscr{L}(\mathbb{R}^d; \mathbb{R}^n) \times \bar{Q}, \\ \quad\ \text{the derivatives are bounded;} \\ \text{(ii) } \Psi \text{ is of class } C^3. \end{cases} \tag{H3.2}$$

According to Lemma 2.2, (p_r, q_r) is $\mathscr{F}_r^t$-adapted.

We can now define $\mathbf{u}\colon \bar{Q} \to \mathbb{R}^n$ by

$$\mathbf{u}(x, t) = E^{x,t} p_t = p_t.$$

We will show that the function $\mathbf{u}(x, t)$ solves the following system of parabolic PDE

$$\begin{cases} \partial_t \mathbf{u} + \mathscr{L}_{(x,t)}\mathbf{u}(x, t) + f(\mathbf{u}(x, t), \mathbf{u}_x(x, t)\sigma(x, t), x, t) = 0, & (x, t) \in Q, \\ \mathbf{u}(x, T) = \Psi(x), \qquad \forall x \in G; & \\ \mathbf{u}(x, t) = \Psi(x), \qquad \forall (x, t) \in S \times [0, T]. & \end{cases} \tag{3.3}$$

where we denote, for $\mathbf{u}(x, t) = (u^1(x, t), \ldots, u^n(x, t))^T$,

$$u_t = \partial_t u, \qquad \mathbf{u}_t = \begin{pmatrix} u_t^n \\ \vdots \\ u_t^n \end{pmatrix}, \qquad u_x = \partial_x u, \qquad \mathbf{u}_x = \begin{pmatrix} u_{x_1}^1 & \cdots & u_{x_m}^1 \\ \vdots & \ddots & \vdots \\ u_{x_1}^n & \cdots & u_{x_m}^n \end{pmatrix},$$

$$\mathscr{L}_{(x,t)} u = \sum_{i,j} a_{ij}(x, t)\partial_{x_i x_j} u + \sum_i b_i(x, t)\partial_{x_i} u, \qquad \mathscr{L}\mathbf{u} = \begin{pmatrix} \mathscr{L}u^1 \\ \vdots \\ \mathscr{L}u^n \end{pmatrix}.$$

We need the following lemma.

LEMMA 3.1 *We assume* ($H3.1$) *and* ($H3.2$), *we assume also the following compatibility condition*

$$[\mathscr{L}_{(x,T)}\Psi(x) + f(\Psi(x), \Psi_x(x)\sigma(x, T), x, T)]_{x\in S} = 0.$$

Then, (3.3) *has an unique solution in* $C^{2,1}$.

The proof of this lemma can be found in [9], (Th .7.1, Ch. VII).
With this lemma, we can assert

THEOREM 3.2 *We assume the same conditions as in Lemma* 3.1. *Then, for any given* (x, t), *the solution of the system of parabolic Equation* (3.3) *has the following interpretation*

$$U(x, t) = E^{x,t}p_t = p_t,$$

where p_t *is determined uniquely by* (3.1), (3.2).

Proof We can apply Itô's formula to the solution $\mathbf{u}$ of (3.3),

$$\begin{aligned}\mathbf{u}(y(\tau), \tau) - \mathbf{u}(y(s \wedge \tau), s \wedge \tau) &= \int_{s\wedge\tau}^{\tau} [\mathbf{u}_t(y(r), r) + \mathscr{L}_{(y(r),r)}\mathbf{u}(y(r), r)]\, dr \\ &\quad + \int_{s\wedge\tau}^{\tau} \mathbf{u}_x(y(r), r)\sigma(y(r), r)\, dW_r.\end{aligned}$$

Thus, since $\mathbf{u}$ solves (3.3),

$$\begin{aligned}\mathbf{u}(y(s \wedge \tau), s \wedge \tau) &= \Psi(y(\tau)) + \int_{s\wedge\tau}^{\tau} f(\mathbf{u}(y(r), r), \mathbf{u}_x(y(r), r)\sigma(y(r), r), y(r), r)\, dr \\ &\quad - \int_{s\wedge\tau}^{\tau} \mathbf{u}_x(y(r), r)\sigma(y(r), r)\, dW_r.\end{aligned}$$

From Lemma 2.2, we see that $((\mathbf{u}(y(s), s), \mathbf{u}_x\sigma(y(s), s))$ coincides with the unique solution of (3.2), (p_s, q_s) in $s \in [t, \tau]$. It follows that

$$u(x, t) = p_t.$$

The proof is complete. □

4 THE CASE FOR QUASILINEAR ELLIPTIC PDE

In this section, we briefly discuss how to formulate a quasilinear elliptic partial differential equation as a solution of a certain backward stochastic differential equation, of type (2.4), associated with some forward (classical) stochastic differential equations. Let σ, b, f, Ψ and G be given as in Section 3, with $n = 1$, and let (H3.1) and (3.2) be held. We assume that all those coefficients are time-invariant. For any given $x \in \bar{G}$, consider the following forward equation

$$\begin{cases} dy(s) = b(y(s))\,ds + \sigma(y(s))\,dW_s, \\ y(0) = x. \end{cases} \tag{4.1}$$

We define the following stopping time

$$\tau = \tau_x = \inf\{s \geq 0;\ y(s) \notin G\}.$$

From (H3.1), the diffusion process $y(\cdot)$ and related stopping time τ are well defined and there exists a positive constant K such that $Ee^{K\tau}$ is bounded. We make the following assumption: there exists a constant μ such that $K + 2\mu - 2C_0 > 0$, and

$$\langle p_1, f(x, p + p_1, q) - f(x, p, q)\rangle < \mu |p_1|^2,$$

where C_0 is the Lipschitz constant of f with respect to (p, q). Then, according to, Theorem 2.2, we can solve uniquely the following adapted backward stochastic equation

$$p_{s\wedge\tau} = \Psi(y(\tau)) + \int_{s\wedge\tau}^{\tau} f(p_r, q_r\, y(r))\,dr - \int_{s\wedge\tau}^{\tau} q_r\,dW_r. \tag{4.2}$$

We can now define $u\colon \bar{G} \to \mathbb{R}$ by

$$u(x) = E^x p_0 = p_0.$$

We will show that the function $u(x)$ solves the following elliptic PDE

$$\begin{cases} \mathscr{L}_{(x)} u(x) + f(u(x), u_x(x)\sigma(x), x) = 0, & x \in Q, \\ u(x) = \Psi(x), \qquad \forall x \in G; \end{cases} \tag{4.3}$$

Indeed, we can assert

THEOREM 4.1 *The solution of the elliptic equation* (4.3) *has a unique equation* $u \in C^2$. *Furthermore,* $u(x)$ *has the following interpretation*

$$u(x) = E^x p_0 = p_0.$$

where p_t *is determined uniquely by* (4.1) *and* (4.2).

Proof The existence result of the elliptic equation (4.3) can be found in [10]. Then similarly as in the proof of Theorem 4.2, we can use Itô's formula to $u(y(s))$,

$0 \leq s \leq \tau$ to check that the pair

$$(p_s, q_s) \equiv (u(y(s)), Du(y(s)\sigma(y(s)))$$

is the unique solution of the backward equation (4.2). □

5 OTHER RELATED PROBLEMS

We discuss briefly some topics that can be treated with the same methology. Detailed treatment will appear elsewhere.

5.1 *Systems of Quasilinear Elliptic Partial Differential Equations*

Let σ, b, f are time invariant. Let $\mathbf{u}(x)$: $G \to \mathbb{R}^n$ solve the following system of elliptic PDE

$$\mathscr{L}_{(x)}\mathbf{u}(x) + f(\mathbf{u}(x), \mathbf{u}_x(x)\sigma(x), x) = 0, \quad x \in Q; \quad \mathbf{u}(x) = \Psi(x), \quad x \in S, \tag{5.1}$$

where

$$\mathscr{L}_{(x)}u = \sum_{i,j} a_{ij}(x)\partial_{x_i x_j}u + \sum_i b_i(x)\partial_{x_i}u.$$

We can also interpret this system of PDE as follows: Define

$$\tau = \tau_x = \inf\{s;\ y(s) \notin G\},$$

where $y(\cdot)$ solves

$$\begin{cases} dy(s) = b(y(s))\,ds + \sigma(y(s))\,dW_s, \\ y(0) = x. \end{cases}$$

Then, we consider the following associated backward stochastic equation defined on $s \in [0, \tau]$

$$p_{s\wedge\tau} = \Psi(y(\tau)) + \int_{s\wedge\tau}^{\tau} f(p_r, q_r, y(r))\,dr - \int_{s\wedge\tau}^{\tau} q_r\,dW_r.$$

Then, under some suitable condition,

$$\mathbf{u}(x) = E^x p_0 = p_0.$$

5.2 *Generalized Hamilton–Jacobi–Bellman Equation*

Using the above method, we can generalize a Hamilton–Jacobi–Bellman equation (see [12], [11], [2]) in several directions. We take only a very simple one. Let

U be a subset of $\mathbb{R}^k$ (called control constraint set). An admissible control is a U valued adapted process. A trajectory corresponding to an admissible control $v(\cdot)$ is the solution of the following forward stochastic differential equation

$$\begin{cases} dy(s) = b(y(s), v(s))\, ds + \sigma(y(s), v(s))\, dW_s, \\ y(0) = x. \end{cases}$$

where

$$b(x, v)\colon \mathbb{R}^m \times U \to \mathbb{R}^m,$$

$$\sigma(x, v)\colon \mathbb{R}^m \times U \to \mathscr{L}(\mathbb{R}^d; \mathbb{R}^m).$$

Define

$$\tau = \tau_x = \inf\{s;\ y(s) \notin G\}.$$

Then one can consider the following backward stochastic equation

$$p_{s\wedge\tau} = \Psi(y(\tau)) + \int_{s\wedge\tau}^{\tau} f(p_r, q_r, y(r), v(r))\, dr - \int_{s\wedge\tau}^{\tau} q_r\, dW_r.$$

Where f, Ψ are both $\mathbb{R}$-valued

$$f(p, q, x, v)\colon \mathbb{R}^n \times \mathscr{L}(\mathbb{R}^d; \mathbb{R}^n) \times \mathbb{R}^m \times U \to \mathbb{R}, \qquad \Psi(x)\mathbb{R}^m \to \mathbb{R}.$$

We can introduce the so called cost function

$$J_x(v(\cdot)) = E^x p_0 = p_0.$$

Define the so called value function by minimizing $J_x(v(\cdot))$ over the set of admissible controls:

$$u(x) = \inf_{v(\cdot)} J_x(v(\cdot)).$$

Under some suitable conditions, $u(x)$ solves the following generalized Hamilton–Jacobi–Bellman equation

$$\inf_{v\in U}\{\mathscr{L}_{(x,v)}u(x) + f(u(x), u_x(x)\sigma(x), x, v)\} = 0, \quad x \in Q, \quad u(x)|_{x\in S} = \Psi(x),$$

where

$$\mathscr{L}_{(x,v)}u = \sum_{i,j} a_{ij}(x, v)\partial_{x_i x_j}u + \sum_i b_i(x, v)\partial_{x_i}u.$$

Remark The stochastic maximum principle for such kinds of control problems can be found in [16].

Remark Some results of this work were announced in [18].

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