SIAM J. CONTROL OPTIM.
Vol. 48, No. 5, pp. 3318–3353

LARGE-POPULATION LQG GAMES INVOLVING A MAJOR PLAYER: THE NASH CERTAINTY EQUIVALENCE PRINCIPLE*

MINYI HUANG†

Abstract. We consider linear-quadratic-Gaussian (LQG) games with a major player and a large number of minor players. The major player has a significant influence on others. The minor players individually have negligible impact, but they collectively contribute mean field coupling terms in the individual dynamics and costs. To overcome the dimensionality difficulty and obtain decentralized strategies, the so-called Nash certainty equivalence methodology is applied. The control synthesis is preceded by a state space augmentation via a set of aggregate quantities giving the mean field approximation. Subsequently, within the population limit the LQG game is decomposed into a family of limiting two-player games as each is locally seen by a representative minor player. Next, when solving these limiting two-player games, we impose certain interaction consistency conditions such that the aggregate quantities initially assumed coincide with the ones replicated by the closed loop of a large number of minor players. This procedure leads to a set of decentralized strategies for the original LQG game, which is an ε-Nash equilibrium.

Key words. mean field models, stochastic differential games, LQG control, decentralized control, Nash equilibria

AMS subject classifications. 91A10, 91A23, 91A25, 93E03, 93E20

DOI. 10.1137/080735370

1. Introduction.

In the 1940s, John von Neumann and Oskar Morgenstern envisioned the perspective of games with a large number of players [34, pp. 12–15], and since then there has accumulated a vast literature on such large population game theoretic models. Within the context of noncooperative game theory, large population models have been well studied in economics, social science, biological science, and engineering (see [5, 6, 10, 13, 19, 25, 30] and the survey [24]). In these areas, of particular interest is the class of games in which each player interacts with the average effect of many others and individually has negligible effect on the overall population. Such an interaction pattern may be modeled by mean field coupling, and it has naturally arisen in economics, engineering, and public health research [25, 10, 19, 5, 6]. While these models enjoy conceptual simplicity and capture key features of interaction of agents, they also impose a great challenge on analysis due to the complexity in optimizing the strategies of agents, especially in dynamic models.

For large population dynamic games, it is unrealistic for a player to collect detailed state information about all other players, and a central issue is the development of low complexity solutions so that each player may implement a strategy based on local information. For models with mean field coupling, recent advances have been made in effectively addressing the complexity issue. In [19, 21, 22, 20], the so-called Nash certainty equivalence (NCE) methodology has been developed, where the key idea is to break the large population game into localized optimal control problems via specifying a consistency relationship between the individual strategies and the aggregate population effect. A very appealing feature of the resulting solution, as

*Received by the editors September 15, 2008; accepted for publication (in revised form) October 12, 2009; published electronically January 29, 2010. This work was partially supported by a Discovery Grant of the Natural Sciences and Engineering Research Council (NSERC) of Canada.
http://www.siam.org/journals/sicon/48-5/73537.html

†School of Mathematics and Statistics, Carleton University, Ottawa, ON K1S 5B6, Canada (mhuang@math.carleton.ca).

an asymptotic Nash equilibrium, is that each agent's strategy depends only on its own state and some deterministic quantities which may be calculated off-line. This optimization methodology has inherent connections with statistical physics when one is studying a large number of interacting particles, and a more detailed discussion is given in [22]. A closely related approach for mean field games has been independently developed by Lasry and Lions [26, 27, 28]. For models with many firm dynamics, Weintraub, Benkard, and Von Roy considered decentralized strategies and proposed the notion of oblivious equilibria via a mean field approximation [35, 36].

In contrast to [19, 21, 20], where all agents are comparably small and may be regarded as peers, this paper investigates a stochastic dynamic game model with a different interaction pattern. The class of games considered here involves a major player and many minor players. The major player has a significant role in affecting others, but each minor player possesses only a weak influence. This kind of interaction modeling is motivated by many socio-economic problems. A typical situation is the interaction within one or more large corporations and many, much smaller, competitors (see, e.g., [1]). Traditionally, models differentiating the strength of players have been well studied in cooperative game theory, and they are customarily called mixed games, with the players accordingly called mixed players; the reader is referred to [31, 11, 15, 18, 32, 17] for historical background. For noncooperative games with a large player and many small players, the work [14] examined the notion of negligibility of small players. In this paper, we will adopt a population of mixed players for the development of noncooperative stochastic dynamic games.

1.1. The mean field stochastic dynamic game model. We consider the linear-quadratic-Gaussian (LQG) game with a major player $\mathcal{A}_0$ and a population of minor players $\{\mathcal{A}_i, 1 \le i \le N\}$. In our exposition, the terms "player" and "agent" will be used interchangeably. At time $t \ge 0$, the states of $\mathcal{A}_0$ and $\mathcal{A}_i$ are, respectively, denoted by $x_0(t)$ and $x_i(t)$, $1 \le i \le N$. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, t \ge 0, P)$ be the underlying filtration. The dynamics of the $N+1$ agents are given by a system of linear stochastic differential equations (SDEs) with mean field coupling:

(1.1)
$$dx_0(t) = [A_0 x_0(t) + B_0 u_0(t) + F_0 x^{(N)}(t)]dt + D_0 dW_0(t), \qquad t \ge 0,$$

(1.2)
$$dx_i(t) = \big[A(\theta_i)x_i(t) + Bu_i(t) + Fx^{(N)}(t) + Gx_0(t)\big]dt + DdW_i(t), \qquad 1 \le i \le N,$$

where $x^{(N)} = (1/N)\sum_{i=1}^N x_i$ is the average state of the minor players. The initial states are measurable on $\mathcal{F}_0$ and are given, respectively, by $x_0(0)$ and $x_i(0)$, $1 \le i \le N$, each with a finite second moment. A simple choice of $\mathcal{F}_t$ is to take it as the σ-algebra $\mathcal{F}_t^{x_\cdot(0),W_\cdot} \triangleq \sigma(x_j(0), W_j(\tau), 0 \le j \le N, \tau \le t)$. At the right-hand side of (1.2), the state x_0 has a constant coefficient G, while each x_j as a component in $x^{(N)}$, $0 < j \ne i$, is associated with a factor $1/N$. This modeling feature indicates that $\mathcal{A}_0$ has a significant influence on others while, in contrast, each minor player has only a negligible impact on others for large N. Our modeling may be generalized to the case of multiple major players, but for simplicity of analysis we will focus on the model with only one major player.

The states x_0, x_i and controls u_0, u_i are, respectively, n and n_1 dimensional vectors. The noise processes W_0, W_i are n_2 dimensional independent standard Brownian motions adapted to $\mathcal{F}_t$, which are also independent of the initial states $(x_j(0), 0 \le j \le N)$. The deterministic constant matrices A_0, B_0, F_0, D_0, $A(\cdot)$, B, F, G, and D

all have compatible dimensions. The number θ_i is a dynamic parameter associated with agent $\mathcal{A}_i$. The variability of the parameter θ_i is used to model a heterogeneous population of minor players. Notice that in (1.2) we take only $A(\cdot)$ to be dependent on θ_i for the purpose of notational simplicity. When other matrix parameters for $\mathcal{A}_i$ also depend on θ_i, the analysis is similar and will not be given in detail. We assume θ_i takes values from the finite set $\Theta = \{1, \dots, K\}$ so that there are K types of minor players. If $\theta_i = k$, $\mathcal{A}_i$ is called a k-type minor player. From now on, for notational brevity the time argument for a process (x_0, x_i, etc.) will often be suppressed when the value of that process at time t is used.

For $0 \le j \le N$, denote $u_{-j} = (u_0, \dots, u_{j-1}, u_{j+1}, \dots, u_N)$. The cost function for $\mathcal{A}_0$ is given by

$$\begin{aligned} &J_0(u_0(\cdot), u_{-0}(\cdot)) \\ &= E \int_0^\infty e^{-\rho t} \left\{ \left[x_0 - \Phi(x^{(N)})\right]^T Q_0 \left[x_0 - \Phi(x^{(N)})\right] + u_0^T R_0 u_0 \right\} dt, \end{aligned} \tag{1.3}$$

where $\Phi(x^{(N)}) = H_0 x^{(N)} + \eta_0$ and the constant $\rho > 0$ is a discount factor. The cost function for $\mathcal{A}_i$, $1 \le i \le N$, is given by

$$\begin{aligned} &J_i(u_i(\cdot), u_{-i}(\cdot)) \\ &= E \int_0^\infty e^{-\rho t} \left\{ \left[x_i - \Psi(x_0, x^{(N)})\right]^T Q \left[x_i - \Psi(x_0, x^{(N)})\right] + u_i^T R u_i \right\} dt, \end{aligned} \tag{1.4}$$

where $\Psi(x_0, x^{(N)}) = H x_0 + \hat{H} x^{(N)} + \eta$. In (1.3)–(1.4), all the deterministic constant matrices or vectors H_0, H, $\hat{H}$, $Q_0 \ge 0$, $Q \ge 0$, $R_0 > 0$, $R > 0$, η_0, and η have compatible dimensions. Parallel to (1.2), the cost (1.4) contains the term Hx_0 to capture the strong influence of the major player.

In this paper, we are interested in the asymptotic analysis when N increases towards infinity. This is essentially to consider a family of games with an increasing number of minor players. For the large population modeling of the minor players, a natural way for modeling the sequence of dynamic parameters $\theta_1, \dots, \theta_N$ is to view it as being truncated from an infinite sequence $\{\theta_i,\ i \ge 1\}$ which exhibits certain statistical properties; this is made precise by assumption (A1) introduced below. For a given N, define

$$\mathcal{I}_k = \{i | \theta_i = k, 1 \le i \le N\}, \qquad N_k = |\mathcal{I}_k|, \tag{1.5}$$

where $|\mathcal{I}_k|$ is the cardinality of the index set $\mathcal{I}_k$, $1 \le k \le K$. Let $\pi_k^{(N)} = N_k/N$. Then $\pi^{(N)} = (\pi_1^{(N)}, \dots, \pi_K^{(N)})$ is a probability vector which gives the empirical distribution of $\theta_1, \dots, \theta_N$.

We introduce the following assumptions.

(**A1**) There exists a probability vector π such that $\lim_{N\to\infty} \pi^{(N)} = \pi$, where $\pi = (\pi_1, \dots, \pi_K)$ and $\min_{1\le k\le K} \pi_k > 0$.

(**A2**) The initial states $x_j(0)$, $0 \le j \le N$, are independent; $Ex_i(0) = 0$ for each $i \ge 1$; and there exists $c_0 < \infty$ independent of N such that $\sup_{j\ge 0} E|x_j(0)|^2 \le c_0$.

It is implied by (A1) that when $N \to \infty$, the proportion of k-type minor players becomes stable for each k and that the number of each type of minor players tends to infinity. Throughout the paper we make the convention that N is suitably large such that $\min_{1\le k\le K} N_k \ge 1$.

For simplicity, in (A2) it is assumed that all minor players have zero initial mean. It is possible to generalize our analysis to deal with different initial means as long as $\{Ex_i(0), i \geq 1\}$ has a limiting empirical distribution; see related discussions in [20].

It is worthwhile to note that LQ and LQG games have been a fruitful research area for the analysis and computation of equilibrium strategies (see, e.g., [2, 3, 4, 7, 9, 12, 29, 33, 38]). The class of LQG games with mixed players and mean field coupling will reveal novel features concerning the interaction of the agents, which do not emerge in traditional LQG games with a small number of players.

1.2. Objective and information pattern. Within the noncooperative game setup, the primary objective of this paper is to develop decentralized control synthesis under large population conditions such that each player's strategy uses only limited information. In particular, we assume that the state x_0 of the major player is available to all players, while the state of each minor player is always known to itself.

1.3. Novelty, contributions, and organization. Due to the presence of a large number of individually insignificant minor players, one might conjecture an asymptotic Nash equilibrium solution of the following form: the major player $\mathcal{A}_0$'s strategy is a function of only $(t, x_0(t))$, and each minor player $\mathcal{A}_i$'s strategy is a function of only $(t, x_i(t), x_0(t))$. However, as is less anticipated, this objective is in general *not* achievable. Concerning the inadequacy of the state information in the above conjectured solution, the reason may be roughly explained as follows. In this model, due to responding to the same major player, the behavior of the minor players will have inherent correlation. Subsequently, their aggregate effect as a random process is directly impacted by the major player, and from the point of view of the major player, when determining its equilibrium strategy, it cannot "freeze" the aggregate effect by treating it as an exogenous signal. So, for the decision making of an individual player (especially the major player), it is crucial to identify how the aggregate effect statistically evolves. For this reason, we need to introduce additional system states based on available information for the control synthesis of the players. The above treatment is in remarkable contrast to the analysis in [20], where the solution begins by first isolating the aggregate effect of all comparably small agents as an exogenous signal and next identifying an individual-mass interaction consistency relationship. In the end, each agent's strategy depends on only its own state [20]. Thus the inclusion of a major player can dramatically alter the nature of the game.

To tackle the model with mixed players, our fundamental approach is to determine the aggregate effect, represented by a set of aggregate quantities, of the minor players such that within the population limit the major player optimally responds to that aggregate effect while each minor player optimally responds to the major player and that aggregate effect combined, and such that all the minor players also collectively produce the same aggregate effect initially presumed. Under reasonable conditions, we show the existence of an aggregate effect possessing the above fixed point property and prove that the resulting set of decentralized individual strategies has an asymptotic Nash equilibrium property; a solution with such a property is designated as the Nash certainty equivalence (NCE) principle. The main contributions of the paper are summarized as follows:

1. A state space augmentation approach is developed for the approximation of the minor players' aggregate effect.
2. The NCE methodology is applied to design decentralized strategies for the $N+1$ players by use of a limiting two-player game. Conditions for solvability of the NCE equation system are obtained.

3. Weighted stability of the closed-loop system for the $N+1$ players is proved, and the set of NCE-based strategies is shown to be a decentralized ε-Nash equilibrium for the LQG game.

To facilitate the presentation, throughout the paper we use C, C_0, C_1, etc. to denote a generic constant, which may vary from place to place. The organization of the paper is as follows. In section 2, as a motivating analysis we give some heuristics on the mean field approximation. Section 3 examines a limiting two-player game. Section 4 is devoted to the construction and analysis of the NCE equation system. The weighted closed-loop stability and approximation results are developed in section 5. The asymptotic equilibrium analysis is presented in section 6. Section 7 develops numerical solutions, and section 8 concludes the paper.

2. Heuristics for the mean field approximation. For obtaining decentralized individual strategies, a crucial step is to analyze the average state of the minor players: $x^{(N)} = (1/N)\sum_{i=1}^N x_i$. First we note that in a population solely consisting of comparably small players, each having state y_i and interacting with the average state $y^{(N)} = (1/N)\sum_{i=1}^N y_i$ (see [20, 21]), it has been shown that subject to self-optimizing behavior, such an averaged quantity $y^{(N)}$ will collapse into a deterministic process to be regarded as an exogenous signal by a given agent when the number of agents increases to infinity. However, as is briefly explained in section 1.3, within the model containing a major player $\mathcal{A}_0$, the aggregate effect of the minor players in general appears as a random process and cannot be treated as an exogenous process. Here we elaborate a little more on this aspect. When optimizing their own costs, the minor players are subject to a significant influence from $\mathcal{A}_0$, and consequently, the resulting controlled state processes x_i, $1 \le i \le N$, will each have significant correlation with the state process x_0 of $\mathcal{A}_0$. Accordingly, $x^{(N)}$ is under the direct influence of the behavior of $\mathcal{A}_0$, and the processes x_i, $1 \le i \le N$, will have significant correlation, which makes it unlikely to achieve a good approximation of $x^{(N)}$ by a deterministic function.

Now, the natural question is, Before the control design of all players can be carried out, how can we give a characterization of the process $x^{(N)}$? It needs to be mentioned that the actual situation is made even more complicated by the additional issue that one needs to consider what are the associated individual controls even before starting the analysis of $x^{(N)}$. Thus, the two issues of control design and the characterization of $x^{(N)}$ are coupled, which makes the study of such models challenging.

To deal with the difficulties in approximating $x^{(N)}$, we begin by a heuristic examination of the structural properties of $x^{(N)}$ under self-optimizing behavior. Our plan is the following: by starting with a full information-based game, we examine the asymptotic property of the closed-loop system. For a fixed N, if each agent has full state information of all agents, we may view the problem (1.1)–(1.4) as a standard LQG (Nash) game and use a set of coupled dynamic programming equations to calculate the individual strategies (assuming their existence) in a feedback form; see [3, 7, 33] for related analysis. We write the feedback Nash equilibrium strategy of $\mathcal{A}_i$, $1 \le i \le N$, in the general form

$$u_i(t) = M_1^{(\theta_i)} x_i(t) + M_2^{(\theta_i)} x_0(t) + \sum_{j\ne i, j=1}^{N} M_3^{\theta_i,\theta_j} x_j(t) + m^{(\theta_i)}(t), \tag{2.1}$$

where the coefficient matrices and the function $m^{(\theta_i)}$ depend on N. As indicated by the superscripts in (2.1), we assume that two agents $\mathcal{A}_i$ and $\mathcal{A}_{i'}$ of the same type (i.e., $\theta_i = \theta_{i'}$) share the same structure in their control laws, which means $M_1^{(\theta_i)} = M_1^{(\theta_{i'})}$,

$M_3^{\theta_i,\theta_k} = M_3^{\theta_{i'},\theta_k}$, and $m^{(\theta_i)} = m^{(\theta_{i'})}$, etc. In (2.1), the coefficients for the states x_i, x_0, x_j are restricted to constant matrices. The motivating reason is that for the infinite horizon game problem, the individual control gain coefficients may be formally solved from a set of coupled algebraic Riccati equations. Hence, we do not specify these coefficients as being time-varying. However, the last term $m^{(\theta_i)}(t)$ is given as a function of time, and this is due to the terms η_0 and η in the costs; in this case the control off-set term $m^{(\theta_i)}(t)$ is described by a linear ordinary differential equation (ODE) when the optimization time horizon increases to infinity, and the interested reader may study a similar phenomenon in a standard stochastic optimal tracking problem; see Appendix A or [20]. Some additional justification of assuming time-invariant coefficients for the state variables in (2.1) will be clear when we develop the NCE-based equation system in section 4.

Substituting (2.1) into (1.2), we obtain

$$dx_i = \left[\widetilde{A}^{(\theta_i)} x_i + F x^{(N)} + \widetilde{G}^{(\theta_i)} x_0 + \widetilde{m}^{(\theta_i)}\right] dt + \sum_{j\neq i, j=1}^{N} \widetilde{M}_3^{\theta_i,\theta_j} x_j dt + D dW_i, \tag{2.2}$$

where $\widetilde{A}^{(\theta_i)} = A(\theta_i) + BM_1^{(\theta_i)}$, $\widetilde{G}^{(\theta_i)} = G + BM_2^{(\theta_i)}$, $\widetilde{M}_3^{\theta_i,\theta_j} = BM_3^{\theta_i,\theta_j}$, and $\widetilde{m}^{(\theta_i)} = Bm^{(\theta_i)}$.

It is obvious that the dynamics (2.2) for x_i, $1 \le i \le N$, differ according to the specific value of $\theta_i \in \Theta = \{1,\ldots,K\}$. A useful step is to classify the states x_i, $1 \le i \le N$, into K groups. Let $\mathcal{I}_k$ and N_k be given by (1.5). Define

$$z_k = (1/N_k) \sum_{i\in\mathcal{I}_k} x_i, \qquad 1 \le k \le K,$$

which is the average state of the same type of agents. For each fixed k, we add up both sides of (2.2) with respect to all $i \in \mathcal{I}_k$ to obtain

$$\begin{aligned} N_k dz_k = {} & N_k \left[\widetilde{A}^{(k)} z_k + F x^{(N)} + \widetilde{G}^{(k)} x_0 + \widetilde{m}^{(k)}\right] dt \\ & + \sum_{i\in\mathcal{I}_k} \sum_{j\neq i, j=1}^{N} \widetilde{M}_3^{\theta_i,\theta_j} x_j dt + \sum_{i\in\mathcal{I}_k} D dW_i. \end{aligned} \tag{2.3}$$

Since $\widetilde{M}_3^{\theta_i,\theta_j}$ depends on types of agents rather than individual agents, we have the simple relation

$$\begin{aligned} \xi_k &\triangleq \sum_{i\in\mathcal{I}_k} \sum_{j\neq i, j=1}^{N} \widetilde{M}_3^{\theta_i,\theta_j} x_j \\ &= \sum_{i\in\mathcal{I}_k} \sum_{j: j\neq i, \theta_j = k} \widetilde{M}_3^{\theta_i,k} x_j + \sum_{i\in\mathcal{I}_k} \sum_{k'\in\Theta\setminus\{k\}} \sum_{j:\theta_j = k'} \widetilde{M}_3^{\theta_i,k'} x_j \\ &= \sum_{i\in\mathcal{I}_k} \sum_{j: j\neq i, \theta_j = k} \widetilde{M}_3^{k,k} x_j + \sum_{i\in\mathcal{I}_k} \sum_{k'\in\Theta\setminus\{k\}} \sum_{j:\theta_j = k'} \widetilde{M}_3^{k,k'} x_j \\ &= N_k(N_k - 1)\widetilde{M}_3^{k,k} z_k + N_k \sum_{k'\in\Theta\setminus\{k\}} N_{k'} \widetilde{M}_3^{k,k'} z_{k'}. \end{aligned} \tag{2.4}$$

Now it follows from (2.3) and (2.4) that

$$
\begin{aligned}
dz_k = & \left[\widetilde{A}^{(k)} z_k + F x^{(N)} + \widetilde{G}^{(k)} x_0 + \widetilde{m}^{(k)}\right] dt \\
& + (N_k - 1)\widetilde{M}_3^{k,k} z_k dt + \sum_{j\in\Theta\backslash\{k\}} N_j \widetilde{M}_3^{k,j} z_j dt + (1/N_k)\sum_{i\in\mathcal{I}_k} D dW_i.
\end{aligned} \tag{2.5}
$$

Furthermore we notice the relation

$$
x^{(N)} = (1/N)\sum_{j=1}^{K} N_j z_j = \sum_{j=1}^{K} \pi_j^{(N)} z_j. \tag{2.6}
$$

Due to the coupling coefficient $1/N$ between two minor players, it is plausible to assume that $\widetilde{M}_3^{k,k}$ and $\widetilde{M}_3^{k,k'}$, $k' \neq k$, both have a magnitude of $1/N$ such that $(N_k - 1)\widetilde{M}_3^{k,k}$ and $N_{k'}\widetilde{M}_3^{k,k'}$ converge in the limit to two matrices, as $N \to \infty$. In addition, when N is large, the diffusion term $(1/N_k)\sum_{i\in\mathcal{I}_k} DdW_i(t)$ becomes negligible since $\lim_{N\to\infty}(N_k/N) = \pi_k > 0$. Also, when $N \to \infty$, we assume that $\widetilde{A}^{(k)}$, $\widetilde{G}^{(k)}$, and $\widetilde{m}^{(k)}(t)$, respectively, converge to their limits $\overline{A}^{(k)}$, $\overline{G}_k$, $\overline{m}_k(t)$ for $k = 1, \ldots, K$.

Now we introduce the equation system

$$
dz_k = \sum_{j=1}^{K} \overline{A}_{k,j} z_j dt + \overline{G}_k x_0 dt + \overline{m}_k dt, \qquad 1 \le k \le K, \tag{2.7}
$$

as the limiting form of (2.5), where each matrix $\overline{A}_{k,j}$ is determined as the limit of the coefficient associated with z_j in (2.5) after the substitution of $x^{(N)}$ by (2.6). Meanwhile, in the dynamics (1.1) of x_0, the term $x^{(N)}$ is approximated by $\sum_{k=1}^{K} \pi_k z_k$.

It must be noted that in this section we use only heuristic arguments for identifying the limiting dynamics of the aggregate quantities $z_1, \ldots, z_K$, and various hypotheses used in the derivation are not fully justified. Yet, the coefficient matrices and the function $\overline{m}_k$ in (2.7) are still undetermined. However, the procedure used in this section is very informative. The introduction of the aggregate quantities and the associated linear equation (2.7) will motivate the state space augmentation idea subsequently to be used in the rigorous development of the NCE approach.

3. The limiting two-player model. In this section we formalize the auxiliary two-player game within the population limit via the approximation of the average state $x^{(N)}$. Since $\pi_k^{(N)} \approx \pi_k$ for large N and

$$
x^{(N)} = (1/N)\sum_{k=1}^{K}\sum_{i\in\mathcal{I}_k} x_i = \sum_{k=1}^{K} \pi_k^{(N)} (1/N_k) \sum_{i\in\mathcal{I}_k} x_i,
$$

we may approximate $x^{(N)}$ by $\sum_{k=1}^{K} \pi_k \bar{z}_k$, where $\bar{z}_k \in \mathbb{R}^n$ is used to approximate $(1/N_k)\sum_{i\in\mathcal{I}_k} x_i$. Denote $\bar{z} = [\bar{z}_1^T, \ldots, \bar{z}_K^T]^T$, which is to be called the set of aggregate quantities. The process $\bar{z}$ is described by the equation

$$
d\bar{z}(t) = \overline{A}\bar{z}(t)dt + \overline{G}\bar{x}_0(t)dt + \overline{m}(t)dt, \tag{3.1}
$$

where $\bar{z}(0) = 0$, $\overline{A} \in \mathbb{R}^{nK\times nK}$, and $\overline{G} \in \mathbb{R}^{nK\times n}$ are constant matrices, and $\overline{m}(t)$ is a continuous function on $[0, \infty)$. The zero initial condition for (3.1) is due to the zero initial mean for the minor players as specified in (A2). Some additional specification for $\overline{m}(t)$ will be introduced later. We note that the introduction of (3.1) is essentially motivated by (2.7). For $\bar{x}_0$ appearing in (3.1), we characterize it as being generated by the limiting SDE below.

After replacing $x^{(N)}$ appearing in (1.1)–(1.2) by $\sum_{k=1}^K \pi_k \bar{z}_k$, the dynamics of the limiting two-player game are given by

$$d\bar{x}_0 = \left[A_0\bar{x}_0 + B_0u_0 + F_0\sum_{k=1}^K \pi_k\bar{z}_k\right]dt + D_0dW_0, \qquad t \geq 0, \tag{3.2}$$

$$d\bar{x}_i = \left[A(\kappa)\bar{x}_i + Bu_i + F\sum_{k=1}^K \pi_k\bar{z}_k + G\bar{x}_0\right]dt + DdW_i, \tag{3.3}$$

where $\bar{x}_0(0) = x_0(0)$, $\bar{x}_i(0) = x_i(0)$, and we suppose the representative minor player has its dynamic parameter $\theta_i = \kappa$ so that $A(\theta_i) = A(\kappa)$. To distinguish from the original model with $N+1$ players, we use the new state variables $\bar{x}_0$ and $\bar{x}_i$. But we still use the same set of variables u_0, u_i, W_0, and W_i in this population limit model, and such a reuse of notation should cause no risk of confusion. Let $\bar{\mathcal{A}}_0$ and $\bar{\mathcal{A}}_i$ stand for the two players described by (3.1)–(3.3), which will still be called the major player and the minor player, respectively.

The cost functions for $\bar{\mathcal{A}}_0$ and $\bar{\mathcal{A}}_i$, respectively, are given by

$$\bar{J}_0(u_0(\cdot)) = E\int_0^\infty e^{-\rho t}\left\{(\bar{x}_0 - \overline{\Phi})^T Q_0(\bar{x}_0 - \overline{\Phi}) + u_0^T R_0 u_0\right\}dt, \tag{3.4}$$

$$\bar{J}_i(u_i(\cdot), u_0(\cdot)) = E\int_0^\infty e^{-\rho t}\left\{(\bar{x}_i - \overline{\Psi})^T Q(\bar{x}_i - \overline{\Psi}) + u_i^T R u_i\right\}dt, \tag{3.5}$$

where $\overline{\Phi} = H_0\sum_{k=1}^K \pi_k\bar{z}_k + \eta_0$ and $\overline{\Psi} = H\bar{x}_0 + \hat{H}\sum_{k=1}^K \pi_k\bar{z}_k + \eta$.

If the coefficients in (3.1) have been known, we may treat the model (3.1)–(3.3) with the associated costs (3.4)–(3.5) as a standard Nash stochastic differential game with two players $\bar{\mathcal{A}}_0$ and $\bar{\mathcal{A}}_i$. We may notice a very appealing decoupling feature; i.e., $(\bar{x}_i, u_i)$ arises in neither the dynamics nor the cost of $\bar{\mathcal{A}}_0$ so that $\bar{J}_0$ depends only on u_0. Thus, the equilibrium strategy $\hat{u}_0$ of $\bar{\mathcal{A}}_0$ may be solved solely as an optimal control problem described by (3.1), (3.2), and (3.4). After obtaining $\hat{u}_0$, the equilibrium strategy $\hat{u}_i$ of $\bar{\mathcal{A}}_i$, again, is solved as an optimal control problem with the dynamics given by (3.1), (3.3), and the closed-loop form of (3.2) under $\hat{u}_0$. However, one must be reminded that in reality, the triple $(\overline{A}, \overline{G}, \overline{m}(t))$ for approximation in the original LQG game is not known in advance and instead needs to be properly determined. This indeterminacy difficulty is to be tackled by the so-called NCE methodology.

4. The NCE-based control synthesis.

Let $k \geq 1$ be an integer. Define the function class

$$\begin{aligned} &C_{\rho/2}([0,\infty), \mathbb{R}^k) \\ &\quad = \{f \mid f \in C([0,\infty), \mathbb{R}^k),\ \ \sup\nolimits_{t\geq 0}|f(t)|e^{-(\rho'/2)t} < \infty \text{ for some } \rho' \in [0,\rho)\}. \end{aligned} \tag{4.1}$$

Notice that ρ' may vary with each f within the above set.

In the procedure described below, we assume $(\overline{A}, \overline{G}, \overline{m})$ has been given and $\overline{m} \in C_{\rho/2}([0,\infty), \mathbb{R}^{nK})$. By first assuming solvability, we proceed with the construction of the equilibrium strategies of the two players while the rigorous analysis of the associated equations will be developed later.

4.1. Optimal control of the major player. Let "$\otimes$" denote the Kronecker product of two matrices [16]. Denote $F_0^\pi = \pi \otimes F_0$ and $H_0^\pi = \pi \otimes H_0$. By (3.1) and (3.2), the dynamics of $\bar{\mathcal{A}}_0$ may be written in the form

$$\begin{bmatrix} d\bar{x}_0 \\ d\bar{z} \end{bmatrix} = \begin{bmatrix} A_0 & F_0^\pi \\ \overline{G} & \overline{A} \end{bmatrix} \begin{bmatrix} \bar{x}_0 \\ \bar{z} \end{bmatrix} dt + \begin{bmatrix} B_0 \\ 0_{nK\times n_1} \end{bmatrix} u_0 dt + \begin{bmatrix} 0_{n\times 1} \\ \overline{m} \end{bmatrix} dt + \begin{bmatrix} D_0 dW_0 \\ 0_{nK\times 1} \end{bmatrix},$$

where $\bar{x}_0(0) = x_0(0)$ and $\bar{z}(0) = 0$. Define

$$\mathbb{A}_0 = \begin{bmatrix} A_0 & F_0^\pi \\ \overline{G} & \overline{A} \end{bmatrix}, \quad \mathbb{B}_0 = \begin{bmatrix} B_0 \\ 0_{nK\times n_1} \end{bmatrix}, \quad \mathbb{M}_0 = \begin{bmatrix} 0_{n\times 1} \\ \overline{m} \end{bmatrix},$$
$$Q_0^\pi = [I, -H_0^\pi]^T Q_0 [I, -H_0^\pi],$$

and $\bar{\eta}_0 = [I, -H_0^\pi]^T Q_0 \eta_0$. Notice that $\mathbb{M}_0 \in C_{\rho/2}([0,\infty), \mathbb{R}^{n(K+1)})$.

We introduce the algebraic Riccati equation (ARE)

$$\rho P_0 = P_0 \mathbb{A}_0 + \mathbb{A}_0^T P_0 - P_0 \mathbb{B}_0 R_0^{-1} \mathbb{B}_0^T P_0 + Q_0^\pi, \tag{4.2}$$

and the ODE

$$\rho s_0 = \frac{ds_0}{dt} + (\mathbb{A}_0^T - P_0 \mathbb{B}_0 R_0^{-1} \mathbb{B}_0^T) s_0 + P_0 \mathbb{M}_0 - \bar{\eta}_0, \tag{4.3}$$

where s_0 is to be sought within the set $C_{\rho/2}([0,\infty), \mathbb{R}^{n(K+1)})$. If the corresponding conditions in Lemma A.2 are satisfied, the optimal control law for $\bar{\mathcal{A}}_0$ is given as

$$\hat{u}_0 = -R_0^{-1} \mathbb{B}_0^T \left[P_0 (\bar{x}_0^T, \bar{z}^T)^T + s_0 \right].$$

The closed-loop equation for $\bar{\mathcal{A}}_0$ is given as

$$\begin{bmatrix} d\bar{x}_0 \\ d\bar{z} \end{bmatrix} = \left(\mathbb{A}_0 - \mathbb{B}_0 R_0^{-1} \mathbb{B}_0^T P_0\right) \begin{bmatrix} \bar{x}_0 \\ \bar{z} \end{bmatrix} dt + \left(\mathbb{M}_0 - \mathbb{B}_0 R_0^{-1} \mathbb{B}_0^T s_0\right) dt + \begin{bmatrix} D_0 dW_0 \\ 0_{nK\times 1} \end{bmatrix}, \tag{4.4}$$

where $\bar{x}_0(0) = x_0(0)$ and $\bar{z}(0) = 0$.

4.2. Optimal control of the minor player. Denote $F^\pi = \pi \otimes F$ and $\hat{H}^\pi = \pi \otimes \hat{H}$. Assume $\theta_i = \kappa$. For determining the strategy of $\bar{\mathcal{A}}_i$, we combine the closed-loop equation (4.4) of $\bar{\mathcal{A}}_0$ with (3.3) to obtain the SDE

$$\begin{aligned} d \begin{bmatrix} \bar{x}_i \\ \bar{x}_0 \\ \bar{z} \end{bmatrix} &= \begin{bmatrix} A(\kappa) & [G \quad F^\pi] \\ 0 & \mathbb{A}_0 - \mathbb{B}_0 R_0^{-1} \mathbb{B}_0^T P_0 \end{bmatrix} \begin{bmatrix} \bar{x}_i \\ \bar{x}_0 \\ \bar{z} \end{bmatrix} dt + \begin{bmatrix} B \\ 0_{n(K+1)\times n_1} \end{bmatrix} u_i dt \\ &\quad + \begin{bmatrix} 0_{n\times 1} \\ \mathbb{M}_0 - \mathbb{B}_0 R_0^{-1} \mathbb{B}_0^T s_0 \end{bmatrix} dt + \begin{bmatrix} D dW_i \\ D_0 dW_0 \\ 0_{nK\times 1} \end{bmatrix}, \end{aligned} \tag{4.5}$$

where $\bar{x}_i(0) = x_i(0)$, $\bar{x}_0(0) = x_0(0)$, $\bar{z}(0) = 0$ and P_0 is determined from (4.2). For the optimal control problem associated with (4.5) and (3.5), we introduce the notation

$$\begin{aligned} &\mathbb{A}_\kappa = \begin{bmatrix} A(\kappa) & [G \quad F^\pi] \\ 0 & \mathbb{A}_0 - \mathbb{B}_0 R_0^{-1} \mathbb{B}_0^T P_0 \end{bmatrix}, \quad \mathbb{B} = \begin{bmatrix} B \\ 0_{n(K+1)\times n_1} \end{bmatrix}, \\ &\mathbb{M} = \begin{bmatrix} 0_{n\times 1} \\ \mathbb{M}_0 - \mathbb{B}_0 R_0^{-1} \mathbb{B}_0^T s_0 \end{bmatrix}, \\ &Q^\pi = [I, -H, -\hat{H}^\pi]^T Q [I, -H, -\hat{H}^\pi], \quad \bar{\eta} = [I, -H, -\hat{H}^\pi]^T Q \eta. \end{aligned} \tag{4.6}$$

We introduce the ARE

$$\rho P_\kappa = P_\kappa \mathbb{A}_\kappa + \mathbb{A}_\kappa^T P_\kappa - P_\kappa \mathbb{B} R^{-1} \mathbb{B}^T P_\kappa + Q^\pi \tag{4.7}$$

and the ODE

$$\rho s_\kappa = \frac{ds_\kappa}{dt} + \left(\mathbb{A}_\kappa^T - P_\kappa \mathbb{B} R^{-1} \mathbb{B}^T\right) s_\kappa + P_\kappa \mathbb{M} - \bar{\eta}, \tag{4.8}$$

where s_κ is to be sought within the set $C_{\rho/2}([0,\infty), \mathbb{R}^{n(K+2)})$. Parallel to the control law of $\bar{\mathcal{A}}_0$ in section 4.1, if the conditions in Lemma A.2 are satisfied, the optimal control law for $\bar{\mathcal{A}}_i$ is given by

$$\hat{u}_i = -R^{-1}\mathbb{B}^T \left[P_\kappa(\bar{x}_i^T, \bar{x}_0^T, \bar{z}^T)^T + s_\kappa\right]. \tag{4.9}$$

Finally, substituting (4.9) into (3.3) gives

$$\begin{aligned} d\bar{x}_i = {} & A(\kappa)\bar{x}_i dt + G\bar{x}_0 dt + F^\pi \bar{z} dt - BR^{-1}\mathbb{B}^T P_\kappa (\bar{x}_i^T, \bar{x}_0^T, \bar{z}^T)^T dt \\ & - BR^{-1}\mathbb{B}^T s_\kappa dt + D dW_i, \end{aligned} \tag{4.10}$$

where $\bar{x}_i(0) = x_i(0)$.

4.3. The consistency condition. For the matrices P_κ, $\kappa = 1, \ldots, K$, and P_0, we introduce the partition

$$P_\kappa = \begin{bmatrix} P_{\kappa,11} & P_{\kappa,12} & P_{\kappa,13} \\ P_{\kappa,21} & P_{\kappa,22} & P_{\kappa,23} \\ P_{\kappa,31} & P_{\kappa,32} & P_{\kappa,33} \end{bmatrix}, \qquad P_0 = \begin{bmatrix} P_{0,11} & P_{0,12} \\ P_{0,21} & P_{0,22} \end{bmatrix}, \tag{4.11}$$

where $P_{\kappa,11}$, $P_{\kappa,22} \in \mathbb{R}^{n\times n}$, $P_{\kappa,33} \in \mathbb{R}^{nK\times nK}$, and $P_{0,11} \in \mathbb{R}^{n\times n}$, $P_{0,22} \in \mathbb{R}^{nK\times nK}$. The matrices $\overline{A}$, $\overline{G}$ and function $\overline{m}(t)$ are represented in the form

$$\overline{A} = \begin{bmatrix} \overline{A}_1 \\ \vdots \\ \overline{A}_K \end{bmatrix}, \qquad \overline{G} = \begin{bmatrix} \overline{G}_1 \\ \vdots \\ \overline{G}_K \end{bmatrix}, \qquad \overline{m}(t) = \begin{bmatrix} \overline{m}_1(t) \\ \vdots \\ \overline{m}_K(t) \end{bmatrix}, \tag{4.12}$$

where $\overline{A}_k \in \mathbb{R}^{n\times nK}$, $\overline{G}_k \in \mathbb{R}^{n\times n}$, and $\overline{m}_k(t) \in \mathbb{R}^n$ for $1 \le k \le K$. Denote the $n \times nK$ matrix

$$\mathbf{e}_k = [0_{n\times n}, \ldots, 0_{n\times n}, I_n, 0_{n\times n}, \ldots, 0_{n\times n}], \tag{4.13}$$

where the $n\times n$ identity matrix I_n is at the kth block, $1 \le k \le K$. Now we consider the average state $(1/N_\kappa)\sum_{i\in\mathcal{I}_\kappa} x_i$ of N_κ κ-type minor players with closed-loop dynamics of the form (4.10). When $N \to \infty$ so that $N_\kappa \to \infty$, we obtain an equation for the aggregate quantity $\bar{z}_\kappa$ (for the approximation of $(1/N_\kappa)\sum_{i\in\mathcal{I}_\kappa} x_i$) as

$$\begin{aligned} d\bar{z}_\kappa = {} & \left\{\left[A(\kappa) - BR^{-1}B^T P_{\kappa,11}\right]\mathbf{e}_\kappa + F^\pi - BR^{-1}B^T P_{\kappa,13}\right\}\bar{z} dt \\ & + (G - BR^{-1}B^T P_{\kappa,12})\bar{x}_0 dt - BR^{-1}\mathbb{B}^T s_\kappa dt, \end{aligned} \tag{4.14}$$

where $\bar{z} = [\bar{z}_1^T, \ldots, \bar{z}_K^T]^T$ and $\bar{z}(0) = 0$. Compared with (4.10), the diffusion term has been averaged out in (4.14), leading to a stochastic ODE involving the driving random process $\bar{x}_0$. Now, under the NCE methodology, the resulting equation system (4.14) for the aggregate quantities should coincide with (3.1), which had been presumed in the first place, and we call this requirement the *consistency condition*.

By the consistency condition we compare the coefficients involved in both (3.1) and (4.14) to obtain the equality relations

$$\overline{A}_\kappa = [A(\kappa) - BR^{-1}B^T P_{\kappa,11}]\mathbf{e}_\kappa + F^\pi - BR^{-1}B^T P_{\kappa,13}, \tag{4.15}$$

$$\overline{G}_\kappa = G - BR^{-1}B^T P_{\kappa,12}, \tag{4.16}$$

$$\overline{m}_\kappa = -BR^{-1}\mathbb{B}^T s_\kappa \tag{4.17}$$

for $\kappa = 1, \dots, K$.

4.4. The NCE equation system. Combining (4.2), (4.7), (4.15), and (4.16), we introduce the algebraic equation system

$$\begin{cases} \rho P_0 = P_0\mathbb{A}_0 + \mathbb{A}_0^T P_0 - P_0\mathbb{B}_0 R_0^{-1}\mathbb{B}_0^T P_0 + Q_0^\pi, \\ \rho P_\kappa = P_\kappa\mathbb{A}_\kappa + \mathbb{A}_\kappa^T P_\kappa - P_\kappa\mathbb{B}R^{-1}\mathbb{B}^T P_\kappa + Q^\pi, \qquad \kappa = 1, \dots, K, \\ \overline{A}_\kappa = [A(\kappa) - BR^{-1}B^T P_{\kappa,11}]\mathbf{e}_\kappa + F^\pi - BR^{-1}B^T P_{\kappa,13} \qquad \forall\kappa, \\ \overline{G}_\kappa = G - BR^{-1}B^T P_{\kappa,12} \qquad \forall\kappa, \end{cases} \tag{4.18}$$

which will be called the consistency-constrained AREs.

Combining (4.3), (4.8), and (4.17), we introduce the ODE system

$$\begin{cases} \rho s_0 = \frac{ds_0}{dt} + (\mathbb{A}_0^T - P_0\mathbb{B}_0 R_0^{-1}\mathbb{B}_0^T)s_0 + P_0\mathbb{M}_0 - \bar{\eta}_0, \\ \rho s_\kappa = \frac{ds_\kappa}{dt} + (\mathbb{A}_\kappa^T - P_\kappa\mathbb{B}R^{-1}\mathbb{B}^T)s_\kappa + P_\kappa\mathbb{M} - \bar{\eta}, \qquad \kappa = 1, \dots, K, \\ \overline{m}_\kappa = -BR^{-1}\mathbb{B}^T s_\kappa \qquad \forall\kappa, \end{cases} \tag{4.19}$$

which will be called the consistency-constrained ODEs. Recall that $\overline{m}$ has been used in defining $\mathbb{M}_0$ and $\mathbb{M}$. The equation systems (4.18)–(4.19) combined will be called the NCE equation system.

Let P_κ, $1 \le \kappa \le K$, be partitioned as in (4.11). Denote

$$\begin{aligned} M_1 &= \begin{bmatrix} A(1) - BR^{-1}B^T P_{1,11} & & \\ & \ddots & \\ & & A(K) - BR^{-1}B^T P_{K,11} \end{bmatrix}, \\ M_2 &= \begin{bmatrix} BR^{-1}B^T P_{1,13} \\ \vdots \\ BR^{-1}B^T P_{K,13} \end{bmatrix}, \end{aligned} \tag{4.20}$$

where M_1 and M_2 are each an $nK \times nK$ matrix. Note that the third equality in (4.18) may be written in an equivalent compact form,

$$\overline{A} = M_1 + \mathbf{1}_K \otimes F^\pi - M_2, \tag{4.21}$$

where $\mathbf{1}_K \in \mathbb{R}^K$ is the column vector with all K entries equal to 1. Also, the last equation in (4.18) gives

$$\overline{G} = \begin{bmatrix} G - BR^{-1}B^T P_{1,12} \\ \vdots \\ G - BR^{-1}B^T P_{K,12} \end{bmatrix}. \tag{4.22}$$

DEFINITION 1. *The set of constant matrices* $(P_0, \overline{A}, \overline{G}, P_\kappa, \kappa = 1, \dots, K)$ *is said to be a consistent solution to* (4.18) *if*

$$P_0 \geq 0, \qquad P_\kappa \geq 0 \ \forall \kappa,$$

$$\mathbb{A}_\kappa - \mathbb{B}R^{-1}\mathbb{B}^T P_\kappa - (\rho/2)I \text{ is Hurwitz } \forall \kappa, \tag{4.23}$$

and (4.18) *is satisfied (a square real-valued matrix is Hurwitz if all its eigenvalues have negative real parts). If, furthermore,*

$$\mathbb{A}_\kappa - \mathbb{B}R^{-1}\mathbb{B}^T P_\kappa \text{ is Hurwitz } \forall \kappa, \tag{4.24}$$

we say $(P_0, \overline{A}, \overline{G}, P_\kappa, \kappa = 1, \dots, K)$ *is a stabilizing consistent solution to* (4.18).

PROPOSITION 2. *The condition* (4.23) *(resp.,* (4.24)*) implies that* $\mathbb{A}_0 - \mathbb{B}_0 R_0^{-1}\mathbb{B}_0^T P_0 - (\rho/2)I$ *and* $A(\kappa) - BR^{-1}B^T P_{\kappa,11} - (\rho/2)I$ *(resp.,* $\mathbb{A}_0 - \mathbb{B}_0 R_0^{-1}\mathbb{B}_0^T P_0$ *and* $A(\kappa) - BR^{-1}B^T P_{\kappa,11}$*) are Hurwitz,* $\kappa = 1, \dots, K$.

Proof. For any fixed κ, let P_κ be partitioned into the form

$$P_\kappa = \begin{bmatrix} P_{\kappa,11} & Q_{\kappa,12} \\ Q_{\kappa,21} & Q_{\kappa,22} \end{bmatrix},$$

where $P_{\kappa,11} \in \mathbb{R}^{n\times n}$ and $Q_{\kappa,22} \in \mathbb{R}^{n(K+1)\times n(K+1)}$. By use of (4.6), it is straightforward to show that

$$\mathbb{A}_\kappa - \mathbb{B}R^{-1}\mathbb{B}^T P_\kappa = \begin{bmatrix} A(\kappa) - BR^{-1}BP_{\kappa,11} & * \\ 0 & \mathbb{A}_0 - \mathbb{B}_0 R_0^{-1}\mathbb{B}_0^T P_0 \end{bmatrix},$$

where the upper right block of the right-hand side is not displayed. The proposition follows. ∎

DEFINITION 3. *Suppose* $(P_0, \overline{A}, \overline{G}, P_\kappa, \kappa = 1, \dots, K)$ *is a consistent solution to* (4.18) *and the matrices* $(P_0, P_\kappa, \kappa = 1, \dots, K)$ *are further used to define the equation system* (4.19). *The set of* $2K + 1$ *vector functions* $(s_0, s_\kappa, \overline{m}_\kappa, \kappa = 1, \dots, K)$ *is called a consistent solution to* (4.19) *if the following two conditions hold:*

(i) $s_0 \in C_{\rho/2}([0, \infty), \mathbb{R}^{n(K+1)})$, *and both* s_κ *and* $\overline{m}_\kappa$ *belong to* $C_{\rho/2}([0, \infty), \mathbb{R}^{n(K+2)})$ *for each* κ*;*

(ii) (4.19) *is satisfied.*

DEFINITION 4. *If* $(P_0, \overline{A}, \overline{G}, P_\kappa, \kappa = 1, \dots, K)$ *and* $(s_0, s_\kappa, \overline{m}_\kappa, \kappa = 1, \dots, K)$ *are, respectively, a consistent solution to* (4.18) *and* (4.19), *we call* $(P_0, \overline{A}, \overline{G}, P_\kappa, s_0, s_\kappa, \overline{m}_\kappa, \kappa = 1, \dots, K)$ *a solution to the NCE equation system* (4.18)–(4.19).

Suppose $(P_0, P_\kappa, \kappa = 1, \dots, K)$ has been obtained from a consistent solution to (4.18). Denote the $nK \times nK(K+2)$ matrix

$$\Lambda = I_K \otimes (BR^{-1}\mathbb{B}^T),$$

which is blockwise diagonal with K identical diagonal blocks of $BR^{-1}\mathbb{B}^T$. Denote $P_0 = [P_{0,1}, P_{0,2}]$, where $P_{0,1} \in \mathbb{R}^{n(K+1)\times n}$ and $P_{0,2} \in \mathbb{R}^{n(K+1)\times nK}$. Denote $P_\kappa = [P_{\kappa,1}, P_{\kappa,2}]$, where $P_{\kappa,1} \in \mathbb{R}^{n(K+2)\times n}$ and $P_{\kappa,2} \in \mathbb{R}^{n(K+2)\times n(K+1)}$. Furthermore, let $P_{\kappa,2a}$ be the last nK columns of P_κ, which implies that $P_{\kappa,2a}$ is also a submatrix of $P_{\kappa,2}$. Denote

$$\widehat{\mathbb{A}}_0 = \mathbb{A}_0 - \mathbb{B}_0 R_0^{-1}\mathbb{B}_0^T P_0 - \rho I, \qquad \widehat{\mathbb{A}}_\kappa = \mathbb{A}_\kappa - \mathbb{B}R^{-1}\mathbb{B}^T P_\kappa - \rho I,$$

$$\Gamma_1 = -\begin{bmatrix} \widehat{\mathbb{A}}_0^T & & & \\ & \widehat{\mathbb{A}}_1^T & & \\ & & \ddots & \\ & & & \widehat{\mathbb{A}}_K^T \end{bmatrix}, \qquad \Gamma_2 = \begin{bmatrix} 0_{n(K+1)\times n(K+1)} & P_{0,2}\Lambda \\ P_{1,2}\mathbb{B}_0 R_0^{-1}\mathbb{B}_0^T & P_{1,2a}\Lambda \\ \vdots & \vdots \\ P_{K,2}\mathbb{B}_0 R_0^{-1}\mathbb{B}_0^T & P_{K,2a}\Lambda \end{bmatrix},$$

and $\Gamma = \Gamma_1 + \Gamma_2$.

Define

$$s_* = [s_0^T, s_1^T, \dots, s_K^T]^T, \qquad \eta_* = [\bar{\eta}_0^T, \mathbf{1}_K^T \otimes \bar{\eta}^T]^T.$$

After expressing $\mathbb{M}_0$ and $\mathbb{M}$ in terms of s_*, the equation system (4.19) leads to the ODE

$$\frac{ds_*}{dt} = \Gamma s_* + \eta_*, \qquad t \geq 0, \tag{4.25}$$

where the initial condition $s_*(0)$ is undetermined.

Proposition 5. *Suppose* $(P_0, \overline{A}, \overline{G}, P_\kappa, \kappa = 1, \dots, K)$ *is a consistent solution to* (4.18)*, and denote* $d_s = n(K+1) + nK(K+2)$*. Then we have the following:*

(i) *Equation* (4.25) *always has at least one solution in the class* $C_{\rho/2}([0,\infty), \mathbb{R}^{d_s})$.

(ii) *If, furthermore, the real part of each eigenvalue of* Γ *is at least* $\rho/2$*, then there is a unique initial condition* $s_*(0)$ *such that the solution* s_* *is in the class* $C_{\rho/2}([0,\infty), \mathbb{R}^{d_s})$*, and in this case* s_* *is in fact bounded.*

Proof. We may assume that Γ is blockwise diagonal,

$$\Gamma = \begin{bmatrix} \Gamma_{11} & \\ & \Gamma_{22} \end{bmatrix}, \tag{4.26}$$

where the real part of all eigenvalues of Γ_{11} (resp., Γ_{22}) is less than (resp., at least) $\rho/2$. The general case may be treated by first applying a nonsingular linear transformation to (4.25) so that Γ is converted into the form (4.26).

(i) We split s_* into two parts s_*^-, s_*^+ and split η_* into two parts η_*^-, η_*^+, so that (4.25) gives

$$\begin{aligned} \frac{ds_*^-}{dt} &= \Gamma_{11} s_*^- + \eta_*^-, \\ \frac{ds_*^+}{dt} &= \Gamma_{22} s_*^+ + \eta_*^+. \end{aligned} \tag{4.27}$$

We may express

$$s_*^+(t) = e^{\Gamma_{22} t}\left[s_*^+(0) + \int_0^t e^{-\Gamma_{22}\tau} \eta_*^+ d\tau \right], \qquad t \geq 0.$$

If the initial condition for s_*^+ is given as

$$s_*^+(0) = -\int_0^\infty e^{-\Gamma_{22}\tau} \eta_*^+ d\tau, \tag{4.28}$$

it may be checked that the resulting function $s_*^+(t)$ is bounded on $[0,\infty)$. For any other initial condition different from (4.28), the solution $s_*^+(t)$ has a growth rate of at least $e^{(\rho/2)t}$. Subsequently, we may take any $s_*^-(0)$ for (4.27) so that $s_*^-(t)$ is in $C_{\rho/2}([0,\infty), \mathbb{R}^{d^-})$, where d^- is the dimension of s_*^-, and $s_* = [(s_*^-)^T, (s_*^+)^T]^T$ gives a solution in $C_{\rho/2}([0,\infty), \mathbb{R}^{d_s})$ as desired.

(ii) For this case there exists a unique $s_*(0) = -\int_0^\infty e^{-\Gamma\tau} \eta_* d\tau$ to give a bounded solution $s(t)$, and any other initial condition generates a solution with a growth rate of at least $e^{(\rho/2)t}$. ∎

Remark. Notice that when $P_0, P_\kappa, \kappa = 1, \dots, K$, are obtained from a consistent solution of (4.18), all eigenvalues of Γ_1 have a real part greater than $\rho/2$ by Proposition 2. Thus, to fulfill the assumption in part (ii) of Proposition 5 the perturbing term Γ_2 in $\Gamma = \Gamma_1 + \Gamma_2$ should act in a favorable manner so that the real part of each eigenvalue of Γ is at least $\rho/2$.

Proposition 5 has interesting implications. It suggests that the solvability of the NCE equation system (4.18)–(4.19) is essentially reduced to the solvability of (4.18). In particular, once a consistent solution to (4.18) is obtained, a consistent solution to (4.19) is guaranteed; in addition, if the real part of each eigenvalue of Γ in (4.25) is at least $\rho/2$, then (4.25) is uniquely solvable within $C_{\rho/2}([0,\infty), \mathbb{R}^{d_s})$, which yields a corresponding solution to (4.19). From this point of view, (4.18) is of central importance in the NCE-based control synthesis.

5. Closed-loop behavior of the agents. Suppose $(P_0, \overline{A}, \overline{G}, P_\kappa, s_0, s_\kappa, \overline{m}_\kappa, \kappa = 1, \dots, K)$ is a solution to the NCE equation system (4.18)–(4.19). We introduce the following assumption.

(**A3**) The matrix $M_1 - (\rho/2)I + \mathbf{1}_K \otimes F^\pi$ is Hurwitz, where $F^\pi = \pi \otimes F$ and M_1 is given by (4.20).

Remark. Intuitively, if the perturbing term $\mathbf{1}_K \otimes F^\pi$ is relatively small, one may expect $M_1 - (\rho/2)I + \mathbf{1}_K \otimes F^\pi$ to be Hurwitz since $M_1 - (\rho/2)I$ is Hurwitz by Proposition 2. But it should be noted that M_1 implicitly depends on F^π.

Now we examine the closed-loop behavior of the $N+1$ agents when the NCE-based strategies are applied. Consider the stochastic system described by (1.1)–(1.2) together with

$$dz = \overline{A}zdt + \overline{G}x_0dt + \overline{m}dt, \tag{5.1}$$

where $z = [z_1^T, \dots, z_K^T]^T$ and $z(0) = 0$. Let the control laws of $\mathcal{A}_0$ and $\mathcal{A}_i$, $1 \le i \le N$, be given by

$$\hat{u}_0 = -R_0^{-1}\mathbb{B}_0^T[P_0(x_0^T, z^T)^T + s_0], \tag{5.2}$$

$$\hat{u}_i = -R^{-1}\mathbb{B}^T\left[P_{\theta_i}(x_i^T, x_0^T, z^T)^T + s_{\theta_i}\right], \qquad 1 \le i \le N, \tag{5.3}$$

where θ_i is the dynamic parameter of $\mathcal{A}_i$. Concerning decentralized implementation of (5.2)–(5.3), the key observation here is that the evolution of z is driven by the major player's state which is available to all players.

After the control laws (5.2)–(5.3) are applied, the closed-loop dynamics of $\mathcal{A}_0$ and $\mathcal{A}_i$, $1 \le i \le N$, may be written in the form

$$dx_0 = \Big\{A_0x_0 - B_0R_0^{-1}\mathbb{B}_0^T[P_0(x_0^T, z^T)^T + s_0] + F_0x^{(N)}\Big\}dt + D_0dW_0, \tag{5.4}$$

$$\begin{aligned} dx_i = {} & \Big\{A(\theta_i)x_i - BR^{-1}\mathbb{B}^T[P_{\theta_i}(x_i^T, x_0^T, z^T)^T + s_{\theta_i}] + Fx^{(N)} + Gx_0\Big\}dt \\ & + DdW_i, \qquad 1 \le i \le N, \end{aligned} \tag{5.5}$$

where z is given by (5.1). In contrast to the limiting equation system (3.1)–(3.3) in section 3 for the two players $\bar{A}_0$ and $\bar{\mathcal{A}}_i$, the evolution of z in (5.1) is now driven by the actual state x_0 of the major player $\mathcal{A}_0$ instead of its large population limit version $\bar{x}_0$.

Let $\mathcal{I}_\kappa$ and N_κ be defined by (1.5). Denote $\check{z}_\kappa = (1/N_\kappa)\sum_{i\in\mathcal{I}_\kappa} x_i$ as the average state of κ-type minor players, and $\check{z} = [\check{z}_1^T, \dots, \check{z}_K^T]^T$. By use of (5.5), the dynamics for $\check{z}_\kappa$ are given as follows:

$$d\check{z}_\kappa = \Big\{A(\kappa)\check{z}_\kappa - BR^{-1}\mathbb{B}^T[P_\kappa(\check{z}_\kappa^T, x_0^T, z^T)^T + s_\kappa] + Fx^{(N)} + Gx_0\Big\}\,dt + (1/N_\kappa)D\sum_{i\in\mathcal{I}_\kappa} dW_i, \tag{5.6}$$

where $\check{z}_\kappa(0) = (1/N_\kappa)\sum_{i\in\mathcal{I}_\kappa} x_i(0)$.

We recall that by Proposition 2, $M_1 - (\rho/2)I$ and $\mathbb{A}_0 - \mathbb{B}_0\mathbb{R}_0^{-1}\mathbb{B}_0^T P_0 - (\rho/2)I$ are Hurwitz if $(P_0, \overline{A}, \overline{G}, P_\kappa, \kappa = 1, \dots, K)$ is a consistent solution to (4.18). Under (A3), let the fixed number $\hat{\rho} \in [0, \rho)$ be chosen such that

$$M_1 - (\hat{\rho}/2)I + \mathbf{1}_K \otimes F^\pi,\ M_1 - (\hat{\rho}/2)I,\ \mathbb{A}_0 - \mathbb{B}_0\mathbb{R}_0^{-1}\mathbb{B}_0^T P_0 - (\hat{\rho}/2)I \text{ are Hurwitz,} \tag{5.7}$$

$$\sup_{t\ge 0, 1\le\kappa\le K} e^{-(\hat{\rho}/2)t}(|s_0| + |s_\kappa| + |\overline{m}_\kappa|) < \infty. \tag{5.8}$$

Now we are in a position to state the weighted stability and approximation results.

THEOREM 6. *Assume* (A1)–(A3). *Suppose* $(P_0, \overline{A}, \overline{G}, P_\kappa, s_0, s_\kappa, \overline{m}_\kappa, \kappa = 1, \dots, K)$ *is a solution to the NCE equation system* (4.18)–(4.19). *For the system of* $N+1$ *agents, there exists* $N^{(0)} > 0$ *such that the closed-loop system* (5.1), (5.4)–(5.5) *satisfies*

$$\sup_{t\ge 0, 0\le j\le N} e^{-\hat{\rho}t}\left\{E|x_j(t)|^2 + E|z(t)|^2 + E|\check{z}(t)|^2\right\} \le C_0$$

for some constant C_0 *independent of* $N \ge N^{(0)}$. *In addition, the approximation*

$$\sup_{t\ge 0} e^{-\hat{\rho}t}E|\check{z}(t) - z(t)|^2 \le C_1(1/N + \epsilon_N^2) \tag{5.9}$$

holds, where C_1 *is independent of* $N \ge N^{(0)}$ *and* $\epsilon_N = \sup_{1\le k\le K} |\pi_k^{(N)} - \pi_k|$.

Proof. See Appendix B. ∎

Remark. Theorem 6 holds if $\hat{\rho}$ is replaced by ρ, but this gives a less tight estimate of the solution growth.

By inspecting the proof of Theorem 6, we obtain the following corollary.

COROLLARY 7. *Assume* (A1) *and* (A2) *hold,* $M_1 + \mathbf{1}_K \otimes F^\pi$ *is Hurwitz,* $(P_0, \overline{A}, \overline{G}, P_\kappa, \kappa = 1, \dots, K)$ *is a stabilizing consistent solution to* (4.18), *and* $(s_0, s_\kappa, \overline{m}_\kappa, \kappa = 1, \dots, K)$ *is a bounded consistent solution to* (4.19). *Then the conclusions of Theorem* 6 *hold with* $\hat{\rho} = 0$.

Corresponding to (5.1), (5.4)–(5.5), we now construct the limiting equation system

$$d\bar{x}_0 = \left\{A_0\bar{x}_0 - B_0R_0^{-1}\mathbb{B}_0^T[P_0(\bar{x}_0^T, \bar{z}^T)^T + s_0] + F_0^\pi\bar{z}\right\}dt + D_0dW_0, \tag{5.10}$$

$$d\bar{x}_i = \left\{A(\theta_i)\bar{x}_i - BR^{-1}\mathbb{B}^T[P_{\theta_i}(\bar{x}_i^T, \bar{x}_0^T, \bar{z}^T)^T + s_{\theta_i}] + F^\pi\bar{z} + G\bar{x}_0\right\}dt + DdW_i, \tag{5.11}$$

where $1 \le i \le N$, $\bar{x}_j(0) = x_j(0)$ for $j = 0, \dots, N$, and

$$d\bar{z} = \overline{A}\bar{z}dt + \overline{G}\bar{x}_0dt + \overline{m}dt, \tag{5.12}$$

with the initial condition $\bar{z}(0) = 0$.

The following comparison result shows the approximation error between the mean field model (5.1), (5.4)–(5.5) and the limiting model (5.10)–(5.12). The result is useful for cost estimation when proving Theorem 10 in section 6.

PROPOSITION 8. *Assume* (A1)–(A3). *Let* $(P_0, \overline{A}, \overline{G}, P_\kappa, s_0, s_\kappa, \overline{m}_\kappa, \kappa = 1, \dots, K)$ *be a solution to the NCE equation system* (4.18)–(4.19). *Let* $(x_0, x_1, \dots, x_N, z)$ *be the solution to the closed-loop system* (5.1), (5.4)–(5.5), *and let* $(\bar{x}_0, \bar{x}_1, \dots, \bar{x}_N, \bar{z})$ *be given by* (5.10)–(5.12). *Then we have*

$$\sup_{t\geq 0, 0\leq j\leq N} e^{-\hat{\rho}t}\left\{E|z(t)-\bar{z}(t)|^2 + E|x_j(t)-\bar{x}_j(t)|^2\right\} \leq C(1/N+\epsilon_N^2),$$

where $C > 0$ *is a constant independent of* $N \geq N^{(0)}$, *and* $\hat{\rho}$, ϵ_N, $N^{(0)}$ *are the same as in Theorem* 6.

Proof. Denote $\tilde{x}_j = x_j - \bar{x}_j$, $j = 0, \dots, N$, and $\tilde{z} = z - \bar{z}$. Then we have

$$\begin{aligned}
d\tilde{x}_0 &= \left\{ A_0\tilde{x}_0 - B_0R_0^{-1}\mathbb{B}_0^T P_0(\tilde{x}_0^T, \tilde{z}^T)^T + F_0\sum_{k=1}^K \pi_k^{(N)}\check{z}_k - F_0^\pi \bar{z} \right\} dt,\\
d\tilde{z} &= \overline{A}\tilde{z}dt + \overline{G}\tilde{x}_0 dt,\\
d\tilde{x}_i &= \left\{ A(\theta_i)\tilde{x}_i - BR^{-1}\mathbb{B}^T P_{\theta_i}(\tilde{x}_i^T, \tilde{x}_0^T, \tilde{z}^T)^T + F\sum_{k=1}^K \pi_k^{(N)}\check{z}_k - F^\pi\bar{z} + G\tilde{x}_0 \right\} dt,
\end{aligned}$$

where $\tilde{z}(0) = 0$ and $\tilde{x}_j(0) = 0$, $j = 0, \dots, N$. Analogously to (B.1)–(B.2) in Appendix B, we define $\tilde{x}_{j,\hat{\rho}} = e^{-(\hat{\rho}/2)t}\tilde{x}_j$, $j = 0, \dots, N$ and $\tilde{z}_{\hat{\rho}} = e^{-(\hat{\rho}/2)t}\tilde{z}$. We further have

$$\begin{aligned}
d\tilde{x}_{0,\hat{\rho}} &= \Big\{[A_0 - (\hat{\rho}/2)I]\tilde{x}_{0,\hat{\rho}} - B_0R_0^{-1}\mathbb{B}_0^T P_0(\tilde{x}_{0,\hat{\rho}}^T, \tilde{z}_{\hat{\rho}}^T)^T + F_0^\pi \tilde{z}_{\hat{\rho}}\\
&\quad + F_0\sum_{k=1}^K \pi_k(\check{z}_{k,\hat{\rho}} - z_{k,\hat{\rho}}) + F_0\sum_{k=1}^K (\pi_k^{(N)} - \pi_k)\check{z}_{k,\hat{\rho}}\Big\}dt, && (5.13)\\
d\tilde{z}_{\hat{\rho}} &= \Big[\overline{A} - (\hat{\rho}/2)I\Big]\tilde{z}_{\hat{\rho}}dt + \overline{G}\tilde{x}_{0,\hat{\rho}}dt, && (5.14)\\
d\tilde{x}_{i,\hat{\rho}} &= \Big\{[A(\theta_i) - (\hat{\rho}/2)I]\tilde{x}_{i,\hat{\rho}} - BR^{-1}\mathbb{B}^T P_{\theta_i}(\tilde{x}_{i,\hat{\rho}}^T, \tilde{x}_{0,\hat{\rho}}^T, \tilde{z}_{\hat{\rho}}^T)^T + G\tilde{x}_{0,\hat{\rho}} + F^\pi\tilde{z}_{\hat{\rho}}\\
&\quad + F\sum_{k=1}^K \pi_k(\check{z}_{k,\hat{\rho}} - z_{k,\hat{\rho}}) + F\sum_{k=1}^K (\pi_k^{(N)} - \pi_k)\check{z}_{k,\hat{\rho}}\Big\}dt, \qquad 1 \leq i \leq N. && (5.15)
\end{aligned}$$

By (5.13)–(5.14), the stability of $\mathbb{A}_0 - \mathbb{B}_0R_0^{-1}\mathbb{B}_0^T P_0 - (\hat{\rho}/2)I$, and Theorem 6, for $N \geq N^{(0)}$ we obtain

$$\sup_{t\geq 0}\left\{E|\tilde{x}_{0,\hat{\rho}}(t)|^2 + E|\tilde{z}_{\hat{\rho}}(t)|^2\right\} \leq C(1/N + \epsilon_N^2),$$

which, combined with (5.15), leads to (notice that $A(\theta_i) - BR^{-1}B^T P_{\theta_i,11} - (\hat{\rho}/2)I$ is Hurwitz by (5.7))

$$\sup_{t\geq 0, 1\leq i\leq N} E|\tilde{x}_{i,\hat{\rho}}(t)|^2 \leq C(1/N + \epsilon_N^2),$$

where C is independent of $N \geq N^{(0)}$. This completes the proof. ∎

6. Asymptotic equilibrium analysis.

Define

$$M_3 = \begin{bmatrix} A_0 & 0 & F_0^\pi \\ \overline{G} & \overline{A} & 0 \\ \overline{G} & -M_2 & M_1 + \mathbf{1}_K \otimes F^\pi \end{bmatrix}, \qquad L_{0,H} = Q_0^{1/2}[I, 0, -H_0^\pi], \tag{6.1}$$

where M_1 and M_2 are as given by (4.20). Similarly, we define $M_3^{(N)}$ (resp., $L_{0,H}^{(N)}$) when π is replaced by $\pi^{(N)}$ in the definition of M_3 (resp., $L_{0,H}$).

We introduce the following stability, observability, and detectability conditions:

(**A3′**) The matrix $M_1 + \mathbf{1}_K \otimes F^\pi$ is Hurwitz.

(**A4**) The pair $(L_{0,H}, M_3)$ is observable.

(**A5**) The pair $(L_a, \mathbb{A}_0 - (\rho/2)I)$ is detectable, and for each $\kappa = 1, \ldots, K$, the pair $(L_b, \mathbb{A}_\kappa - (\rho/2)I)$ is detectable, where $L_a = Q_0^{1/2}[I, -H_0^\pi]$ and $L_b = Q^{1/2}[I, -H, -\hat{H}^\pi]$.

Obviously, (A3′) is stronger than (A3), and it will be used in establishing the asymptotic equilibrium results for the model with mean field coupling in both the dynamics and costs. The above observability condition will be useful for obtaining prior integral estimates of the state process of the major player. From the point of view of the major player, the system dynamics may be specified in terms of the state vector $(x_0, z, \check{z})$, and M_3 essentially appears as the coefficient matrix of $(x_0, z, \check{z})$ in the limiting dynamics when $N \to \infty$.

Consider the system of $N + 1$ agents described by (1.1)–(1.2) and (5.1). For any $0 \le j \le N$, the admissible control set $\mathcal{U}_j$ of agent $\mathcal{A}_j$ consists of all Lipschitz feedback controls u_j with respect to $(x_0, x_1, \ldots, x_N, z)$ (i.e., u_j is a continuous function of $(t, x_0, x_1, \ldots, x_N, z)$ and is Lipschitz continuous with respect to $(x_0, x_1, \ldots, x_N, z)$) such that a unique strong solution to the closed-loop system of the $N+1$ agents exists on $[0, \infty)$. Note that $\mathcal{U}_j$ itself is not required to be decentralized. For $j = 0, \ldots, N$, denote $u_{-j} = (u_0, \ldots, u_{j-1}, u_{j+1}, \ldots, u_N)$, and let $J_j(u_j, u_{-j})$ be defined by (1.3)–(1.4).

DEFINITION 9. *A set of controls $u_k \in \mathcal{U}_k, 0 \le k \le N$, for the $N + 1$ players is called an ε-Nash equilibrium with respect to the costs $J_k, 0 \le k \le N$, where $\varepsilon \ge 0$, if for any i, $0 \le i \le N$, we have*

$$J_i(u_i, u_{-i}) \le J_i(u_i', u_{-i}) + \varepsilon,$$

when any alternative $u_i' \in \mathcal{U}_i$ is applied by player $\mathcal{A}_i$.

Below we state the main result on the asymptotic Nash equilibrium property of the decentralized strategies specified by (5.2)–(5.3).

THEOREM 10. *Assume that* (A1), (A2), (A3′), (A4), *and* (A5) *hold and that*

$$(P_0, \overline{A}, \overline{G}, P_\kappa, s_0, s_\kappa, \overline{m}_\kappa,\ \kappa = 1, \ldots, K)$$

is a stabilizing solution to the NCE equation system (4.18)–(4.19). *In addition, Q is nonsingular. Then the set of NCE-based strategies* (5.2)–(5.3) *is an ε-Nash equilibrium, where $\varepsilon = O(1/\sqrt{N} + \epsilon_N) \to 0$ as $N \to \infty$ and ϵ_N is given as in Theorem* 6.

Proof. See Appendix C. ∎

6.1. The model with mean field coupling only in costs.

For the model with only cost coupling, the dynamics (1.1)–(1.2) reduce to the simpler form

$$dx_0 = [A_0 x_0 + B_0 u_0]\, dt + D_0 dW_0, \qquad t \ge 0, \tag{6.2}$$

$$dx_i = [A(\theta_i) x_i + B u_i + G x_0]\, dt + D dW_i, \qquad 1 \le i \le N, \tag{6.3}$$

and the costs are still given by (1.3)–(1.4). We state the main result, where the solution to the NCE equation system is not required to be stabilizing. Note that when $F = 0$ as in (6.3), a solution to the NCE equation system (4.18)–(4.19) always satisfies (A3).

THEOREM 11. *Suppose that* (A1), (A2), (A4), *and* (A5) *hold and that*

$$(P_0, \overline{A}, \overline{G}, P_\kappa, s_0, s_\kappa, \overline{m}_\kappa,\ \kappa = 1, \dots, K)$$

is a solution to the NCE equation system (4.18)–(4.19), *and* $(Q^{1/2}, A(\kappa) - (\rho/2)I)$ *is detectable for* $\kappa = 1, \dots, K$. *Then the set of NCE-based strategies* (5.2)–(5.3) *is an* ε*-Nash equilibrium, where* $\varepsilon = O(1/\sqrt{N} + \epsilon_N) \to 0$ *as* $N \to \infty$ *and* ϵ_N *is given as in Theorem* 6.

Proof. Step 1 in the proof of Theorem 10 is still valid for this theorem. In particular, (C.12) still holds. Next, for a given minor player $\mathcal{A}_{i_0}$, we restrict our attention to control u_{i_0} satisfying (C.18) in which x_0 and $x^{(N)} - x_{i_0}/N$ may be separated from x_{i_0} and estimated; this gives

$$E \int_0^\infty e^{-\rho t} \left\{ x_{i_0}^T (I - \hat{H}^T/N) Q (I - \hat{H}/N) x_{i_0} + u_{i_0}^T R u_{i_0} \right\} dt \le C \tag{6.4}$$

for some C independent of $N \ge N^{(0)}$, where $N^{(0)}$ is the same as for (C.18). By (6.4), we use detectability of $(Q^{1/2}, A(\theta_{i_0}) - (\rho/2)I)$ and follow the argument in proving (A.12) to obtain $E \int_0^\infty e^{-\rho t} |x_{i_0}(t)|^2 dt \le C_1$ for C_1 independent of $N \ge N^{(0)}$. Subsequently, (C.28) and (C.29) may be established for u_{i_0} satisfying (C.18). Finally, similarly to Step 2 in the proof of Theorem 10, we may show that when $\mathcal{A}_{i_0}$ applies a strategy $u'_{i_0} \in \mathcal{U}_{i_0}$ other than $\hat{u}_{i_0}$, it can reduce its cost by at most $O(1/\sqrt{N} + \epsilon_N)$. ∎

7. Numerical solutions. In the numerical examples, we follow the notation used in section 1.1.

7.1. A model with homogeneous minor players. The dynamics of the major and minor players are given by

$$\begin{aligned} dx_0 &= 2x_0 dt + u_0 dt + 0.2x^{(N)} dt + dW_0, \\ dx_i &= 3x_i dt + x_0 dt + u_i dt + 0.3x^{(N)} dt + dW_i, \end{aligned}$$

where $1 \le i \le N$, and the parameters in the costs (1.3)–(1.4) are given by

$$[Q_0, R_0, H_0, \eta_0] = [1, 1, 0.3, 1.5], \qquad [Q, R, H, \hat{H}, \eta] = [1, 3, 0.4, 0.3, 1]. \tag{7.1}$$

The discount factor $\rho = 1$. The following heuristic algorithm is used for finding the numerical solution of $(P_0, \overline{A}, \overline{G}, P_1)$, where P_1 denotes the solution to the ARE (see (4.18)) for the minor players.

ALGORITHM A.

Step 1. Take the initial guess $\overline{A}^{(0)} = 0,\ \overline{G}^{(0)} = 0$ (for $\overline{A}, \overline{G}$).

Step 2. Given $(\overline{A}^{(k)}, \overline{G}^{(k)})$, $k \ge 0$, use this pair of parameters for (4.2) to obtain $P_0^{(k+1)}$.

Step 3. Use $P_0^{(k+1)}$ to determine coefficients in (4.7), which is then solved to give $P_1^{(k+1)}$.

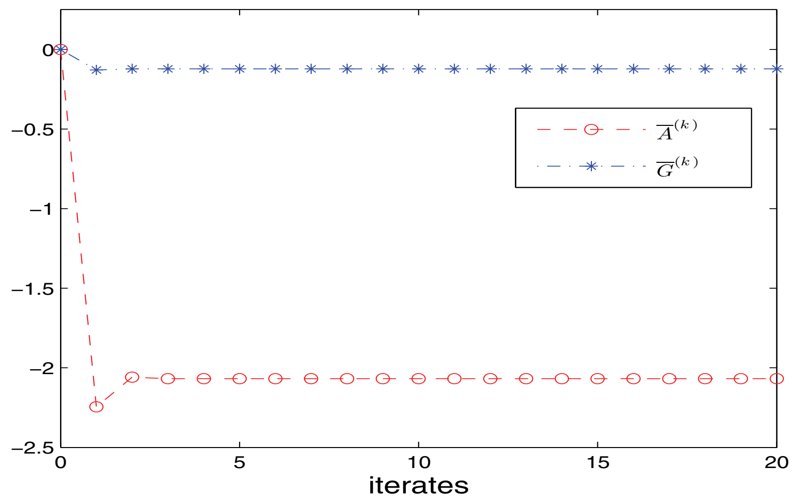

FIG. 1. *Iteration of* $(\overline{A}^{(k)}, \overline{G}^{(k)})$ *by Algorithm A.*

Step 4. Use (4.15)–(4.16) to give updated value $(\overline{A}^{(k+1)}, \overline{G}^{(k+1)})$. Go to Step 2 and continue until sufficient accuracy is obtained in solving $(P_0, \overline{A}, \overline{G}, P_1)$.

Remark. In Step 2, (4.2) has a well-defined solution $P_0^{(k+1)} \geq 0$ provided that the ARE (4.2) parametrized by $(\overline{A}^{(k)}, \overline{G}^{(k)})$ satisfies the standard stabilizability and detectability conditions [37, p. 276]. For this example our initialization in Step 1 satisfies these conditions.

By 20 iterates of Algorithm A, we obtain

$$\overline{A} = -2.06819117030469, \quad \overline{G} = -0.12205345839681, \tag{7.2}$$

$$P_0 = \begin{bmatrix} 3.29723523856799 & 0.08170803806570 \\ 0.08170803806570 & 0.02258535366897 \end{bmatrix}, \tag{7.3}$$

$$P_1 = \begin{bmatrix} 15.19740215917039 & 3.36616037519042 & 0.90717135174368 \\ 3.36616037519042 & 0.84943268921156 & 0.25418426077389 \\ 0.90717135174368 & 0.25418426077389 & 0.08179273643943 \end{bmatrix}. \tag{7.4}$$

The iteration of $(\overline{A}^{(k)}, \overline{G}^{(k)})$ is illustrated in Figure 1. The proof of convergence is beyond the scope of this paper. It may be checked that with $(P_0, \overline{A}, \overline{G}, P_1)$ given by (7.2)–(7.4), (4.18) is satisfied with an error by the order of 10^{-13}. For the calculation below, we display only eight digits after the decimal point. We may further show that $\mathbb{A}_1 - \mathbb{B}R^{-1}\mathbb{B}^T P_1$ has three eigenvalues $(-2.06580072, -1.31644101, -2.04898540)$ implying a stabilizing consistent solution. We can show that $M_1 + \mathbf{1}_K \otimes F^\pi$ in (A3$'$) is now equal to $-1.76580072 < 0$.

We form the three observability matrices

$$\mathbb{O} = \begin{bmatrix} L_{0,H} \\ L_{0,H}M_3 \\ L_{0,H}M_3^2 \end{bmatrix}, \quad \mathbb{O}_0 = \begin{bmatrix} L_a \\ L_a\mathbb{A}_0 \end{bmatrix}, \quad \mathbb{O}_1 = \begin{bmatrix} L_b \\ L_b\mathbb{A}_1 \\ L_b\mathbb{A}_1^2 \end{bmatrix}.$$

Let $\mathbf{i}$ denote the imaginary unit. We calculate the eigenvalues:

$$\begin{array}{ll} \text{For } \mathbb{O}: & 0.64178536,\ -0.21616041 \pm 0.92202101\mathbf{i}. \\ \text{For } \mathbb{O}_0: & 0.91022868 \pm 0.77648305\mathbf{i}. \\ \text{For } \mathbb{O}_1: & -0.58367366,\ 1.20868326 \pm 1.67225690\mathbf{i}. \end{array}$$

So each of $(L_{0,H}, M_3)$, $(L_a, \mathbb{A}_0)$, and $(L_b, \mathbb{A}_1)$ is an observable pair. This verifies the detectability and observability conditions in (A4) and (A5).

Finally, we calculate the eigenvalues of Γ given in (4.25) as

$$2.27099915 \pm 0.02306827\mathbf{i},\ 3.46108834,\ 3.04500020,\ 3.05095714,$$

which are all greater than $\rho/2 = 0.5$ so that condition (ii) in Proposition 5 holds. So we may obtain a unique bounded solution s_* for (4.25).

7.2. A model with two types of minor players. We consider a model with a major player and two types of minor players. The dynamics of each player are described by a one dimensional SDE, and for $\mathcal{A}_0$, the associated parameters are given by

$$[A_0,\ B_0,\ F_0,\ D_0] = [2,\ 1,\ 0.1,\ 1], \qquad [Q_0,\ R_0,\ H_0,\ \eta_0] = [1,\ 1,\ 0.3,\ 1.5].$$

For the minor players of type 1, the parameters are given as

$$[A(1),\ B,\ F,\ G,\ D] = [3,\ 1,\ 0.3,\ 0.3,\ 1], \tag{7.5}$$

$$[Q,\ R,\ H,\ \hat{H},\ \eta] = [1,\ 3,\ 0.2,\ 0.2,\ 0.8]. \tag{7.6}$$

For the minor players of type 2, the parameters are given by (7.5)–(7.6), with the exception that $A(2) = 1$. The discount factor is $\rho = 2$, and the empirical distribution in (A1) is $\pi = (\pi_1, \pi_2) = (0.4, 0.6)$.

For reasons of space, instead of examining all matrices involved in assumptions (A3), (A3′), (A4), and (A5), we focus on numerically solving the equation system (4.18) since it is of particular importance. For the above system, by 20 iterates of Algorithm A we obtain

$$\overline{A} = \begin{bmatrix} -1.08348395963536 & -0.09869992104338 \\ 0.10050060055886 & 0.53858981314775 \end{bmatrix},$$
$$\overline{G} = \begin{bmatrix} -0.04879686018588 \\ 0.23921892627785 \end{bmatrix}.$$

The solution to the major player's algebraic Riccati equation is given as

$$P_0 = \begin{bmatrix} 2.41194083638386 & -0.00658100734677 & -0.01476751788071 \\ -0.00658100734677 & 0.00375991606902 & 0.00914044231566 \\ -0.01476751788071 & 0.00914044231566 & 0.03099790296720 \end{bmatrix}.$$

For reasons of space, the solutions P_1 and P_2 associated with the minor players are not displayed. We may check that (4.18) is satisfied with an error by the order of 10^{-14}. Let

$$\mathbb{A}_1^c = \mathbb{A}_1 - \mathbb{B}R^{-1}\mathbb{B}^T P_1, \quad \mathbb{A}_2^c = \mathbb{A}_2 - \mathbb{B}R^{-1}\mathbb{B}^T P_2.$$

We can check that $\mathbb{A}_1^c$ has eigenvalues

$$-1.08166600,\ 0.55015848,\ -0.43185289,\ -1.07514057,$$

and $\mathbb{A}_2^c$ has eigenvalues

$$0.42264973,\ 0.55015848,\ -0.43185289,\ -1.07514057.$$

Clearly, neither $\mathbb{A}_1^c$ nor $\mathbb{A}_2^c$ is Hurwitz, but $\mathbb{A}_1^c-(\rho/2)I$ and $\mathbb{A}_2^c-(\rho/2)I$ are Hurwitz. So we numerically obtain a consistent solution, but not a stabilizing consistent solution, to (4.18).

8. Conclusion. This paper considers decentralized control for large population LQG games involving a major player and a large number of minor players. Our approach is to use a mean field approximation such that the game problem in the population limit is decomposed into a family of localized limiting two-player games, where the aggregate effect of all minor players is characterized by a linear stochastic ODE driven by the state of the major player. After introducing the so-called consistency condition for the aggregate effect and the individual strategies, the Nash certainty equivalence (NCE) approach is developed, which leads to decentralized strategies for all players. Asymptotic Nash equilibrium properties for the obtained strategies are established. For future work, we aim to extend the modeling and analysis to hierarchical games where the major player is endowed with a certain leadership. The associated decentralized strategy synthesis is of great interest.

Appendix A. Preliminaries on optimal tracking. Let $(\Omega,\mathcal{F},\mathcal{F}_t,t\geq 0,P)$ be an underlying filtration. Consider the n dimensional controlled SDE

$$dx(t)=Ax(t)dt+Bu(t)dt+f(t)dt+DdW(t),\qquad t\geq 0, \tag{A.1}$$

where $x(t)\in\mathbb{R}^n$, $u(t)\in\mathbb{R}^{n_1}$ is the control, $f\in C([0,\infty),\mathbb{R}^n)$, and $W(t)$ is an n_2 dimensional standard Brownian motion adapted to $\mathcal{F}_t$. The initial condition $x(0)$ is independent of $W(t)$ and $E|x(0)|^2<\infty$. All the constant matrices have compatible dimensions. A control $u(\cdot)$ is admissible if it is adapted to $\mathcal{F}_t$ and $E\int_0^\infty e^{-\rho t}|u(t)|^2dt<\infty$. Denote all such admissible controls by the set $\mathcal{U}$. We note that the notation or variables used in this appendix are not required to be identical to those appearing in section 1.1. For a given admissible control $u(\cdot)$, let the cost function be given by

$$J(u(\cdot))=E\int_0^\infty e^{-\rho t}\left\{[Hx(t)-g(t)]^T[Hx(t)-g(t)]+u^T(t)Ru(t)\right\}dt, \tag{A.2}$$

where $\rho>0$, $H\in\mathbb{R}^{n\times n}$, $R>0$, and $g\in C([0,\infty),\mathbb{R}^n)$.

Denote the algebraic Riccati equation (ARE)

$$\rho\Pi=\Pi A+A^T\Pi-\Pi BR^{-1}B^T\Pi+H^TH \tag{A.3}$$

and the ordinary differential equation (ODE)

$$\rho s(t)=\frac{ds(t)}{dt}+(A^T-\Pi BR^{-1}B^T)s(t)+\Pi f(t)-H^Tg(t),\qquad t\geq 0, \tag{A.4}$$

where the solution s is to be sought within the class $C_{\rho/2}([0,\infty),\mathbb{R}^n)$ as defined by (4.1). Notice that the initial condition $s(0)$ is not prespecified at this stage.

Similarly to [8, pp. 23–25], we use an algebraic approach to derive the optimal control law. But some special care needs to be taken to deal with (A.4) which has no boundary condition.

LEMMA A.1. *Suppose $y(t)$ is given by the n dimensional stochastic differential equation (SDE)*

$$dy(t) = \bar{A}y(t)dt + \bar{B}u(t)dt + \bar{f}(t)dt + \bar{D}dW(t),$$

where the constant coefficient matrices have compatible dimensions, $\bar{A}-(\rho/2)I$ is Hurwitz, $y(0)$ is independent of $W(\cdot)$ and satisfies $E|y(0)|^2 < \infty$, $\bar{f} \in C_{\rho/2}([0,\infty), \mathbb{R}^n)$, and $u(\cdot) \in \mathcal{U}$ satisfies $E\int_0^\infty e^{-\rho t}|u(t)|^2 dt \le c_1$ for some constant c_1. Then there exists a constant c_2 such that $E\int_0^\infty e^{-\rho t}|y(t)|^2 dt \le c_2$, where c_2 may be determined only in terms of $(c_1, \rho, \bar{A}, \bar{B}, \bar{f}, \bar{D}, E|y(0)|^2)$.

Proof. Denote $y_\rho = e^{-(\rho/2)t}y$ and $u_\rho = e^{-(\rho/2)t}u$. We have

$$dy_\rho(t) = \left[\bar{A} - (\rho/2)I\right] y_\rho(t)dt + \bar{B}u_\rho(t)dt + e^{-(\rho/2)t}\bar{f}(t)dt + e^{-(\rho/2)t}\bar{D}dW(t).$$

By expressing $y_\rho(t)$ in terms of $y_\rho(0)$ and integration with u_ρ, $\bar{f}$, and W, we obtain

$$E\int_0^\infty |y_\rho(t)|^2 dt \le C + E\int_0^\infty \left|\int_0^t e^{(\bar{A}-(\rho/2)I)(t-\tau)}Bu_\rho(\tau)d\tau\right|^2 dt. \tag{A.5}$$

Now we may find C (which may change in different places) and a fixed $\delta > 0$ such that

$$\begin{aligned}
E\int_0^\infty |y_\rho(t)|^2 dt &\le C + CE\int_0^\infty \left|\int_0^t e^{-\delta(t-\tau)}|u_\rho(\tau)|d\tau\right|^2 dt \\
&\le C + CE\int_1^\infty \left|\int_0^t e^{-\delta(t-\tau)}|u_\rho(\tau)|d\tau\right|^2 dt \\
&= C + CE\int_1^\infty e^{-2\delta t}\left|\int_0^t |u_\rho(\tau)|\delta e^{\delta\tau}/(e^{\delta t}-1)d\tau\right|^2 (e^{\delta t}-1)^2\delta^{-2}dt \\
&\le C + CE\int_1^\infty e^{-2\delta t}\int_0^t |u_\rho(\tau)|^2\delta e^{\delta\tau}/(e^{\delta t}-1)d\tau(e^{\delta t}-1)^2\delta^{-2}dt \qquad \text{(A.6)} \\
&\le C + CE\int_0^\infty e^{-\delta t}\int_0^t e^{\delta\tau}|u_\rho(\tau)|^2 d\tau dt \\
&= C + CE\int_0^\infty |u_\rho(\tau)|^2 d\tau \triangleq c_2, \qquad \text{(A.7)}
\end{aligned}$$

where (A.6) is obtained by Jensen's inequality and (A.7) results from an exchange of order of integration. It is clear that c_2 may be determined only in terms of $(c_1, \rho, \bar{A}, \bar{B}, \bar{f}, \bar{D}, E|y(0)|^2)$. ∎

LEMMA A.2. *For the optimal control problem* (A.1)–(A.2)*, assume* (i) *the pair $(H, A-(\rho/2)I)$ is detectable and* (A.3) *has a positive semidefinite solution Π such that $A - BR^{-1}B^T\Pi - (\rho/2)I$ is Hurwitz, and* (ii) *both f and g are in the class $C_{\rho/2}([0,\infty), \mathbb{R}^n)$. Then we have that*

(a) *there exists a unique solution s in the class $C_{\rho/2}([0,\infty), \mathbb{R}^n)$ for* (A.4)*;*

(b) *the optimal control law is given by*

$$\hat{u}(t) = -R^{-1}B^T[\Pi x(t) + s(t)]. \tag{A.8}$$

Proof. (a) Denote $\widehat{A} = A - BR^{-1}B^T\Pi - \rho I$. We have $\frac{ds}{dt} = -\widehat{A}^T s + H^T g - \Pi f$. Given an initial condition $s(0)$, the solution to (A.4) is

$$s(t) = e^{-\widehat{A}^T t}\left\{s(0) + \int_0^t e^{\widehat{A}^T\tau}[H^T g(\tau) - \Pi f(\tau)]d\tau\right\}. \tag{A.9}$$

It is straightforward to verify that $s(0) = -\int_0^\infty e^{\widehat{A}^T\tau}[H^T g(\tau) - \Pi f(\tau)]d\tau$ is finite and is the only initial condition which gives a solution $s \in C_{\rho/2}([0,\infty), \mathbb{R}^n)$, and that any other initial condition leads to a solution growth rate faster than $e^{(\rho/2)t}$.

(b) *Step* 1. Denote the auxiliary equation

$$dy(t) = (A - BR^{-1}B^T\Pi)y(t)dt - BR^{-1}B^T s(t)dt + f(t)dt + DdW(t),$$

where $y(0) = x(0)$. Take any $u(\cdot) \in \mathcal{U}$ such that $J(u(\cdot)) < \infty$ (such a control exists since it may be checked that $\hat{u}(\cdot)$ has this property), and let x be the associated solution to (A.1). This further implies

$$E\int_0^\infty e^{-\rho t}\left[x^T(t)H^T Hx(t) + u(t)^T Ru(t)\right]dt < \infty. \tag{A.10}$$

Denote $\tilde{x}(t) = x(t) - y(t)$ and $\tilde{u}(t) = u(t) + R^{-1}B^T[\Pi x(t) + s(t)]$. It may be checked that

$$d\tilde{x}(t) = (A - BR^{-1}B^T\Pi)\tilde{x}(t)dt + B\tilde{u}(t)dt, \tag{A.11}$$

where $\tilde{x}(0) = 0$. By (A.10), we may show

$$E\int_0^\infty e^{-\rho t}|x(t)|^2 dt < \infty. \tag{A.12}$$

The proof of (A.12) is postponed to Step 3. Now by (A.12) and the definition of $\tilde{u}$, it follows that $E\int_0^\infty e^{-\rho t}|\tilde{u}(t)|^2 dt < \infty$, and therefore $\tilde{u} \in \mathcal{U}$.

Step 2. The cost (A.2) may be written in the form $J = E\int_0^\infty e^{-\rho t}\xi(t)dt$, where

$$\begin{aligned}\xi(t) &= (Hy - g)^T(Hy - g) + (\Pi y + s)^T BR^{-1}B^T(\Pi y + s) \quad (\triangleq \xi_1)\\ &\quad + \tilde{x}^T H^T H\tilde{x} + (\tilde{u} - R^{-1}B^T\Pi\tilde{x})^T R(\tilde{u} - R^{-1}B^T\Pi\tilde{x}) \quad (\triangleq \xi_2)\\ &\quad + 2(H\tilde{x})^T(Hy - g) - 2(\tilde{u} - R^{-1}B^T\Pi\tilde{x})^T B^T(\Pi y + s) \quad (\triangleq \xi_3).\end{aligned}$$

Applying Ito's formula to $e^{-\rho t}\tilde{x}^T(t)[\Pi y(t) + s(t)]$ and taking expectation, after some elementary calculations we obtain

$$e^{-\rho T}E\{\tilde{x}^T(T)[\Pi y(T) + s(T)]\} = E\int_0^T e^{-\rho t}\zeta(t)dt, \tag{A.13}$$

where $0 < T < \infty$ and

$$\zeta(t) = \tilde{x}^T(H^T g - \Pi BR^{-1}B^T s) - \tilde{x}^T(H^T H + \Pi BR^{-1}B^T\Pi)y + \tilde{u}^T B^T(\Pi y + s).$$

By (A.11), we may show $E|\tilde{x}(t)|^2 = O(e^{\rho t})$. Furthermore, we may show that $E|y(t)|^2 = O(e^{\rho' t})$ for some $\rho' \in (0,\rho)$. Hence $E\int_0^\infty e^{-\rho t}|\zeta(t)|dt < \infty$. Then it is easy to show that $E\int_0^\infty e^{-\rho t}\zeta(t)dt = 0$ since the left-hand side of (A.13) tends to zero when $T \to \infty$. Notice that $\xi_3 = -2\zeta$. Hence

$$J(u(\cdot)) = E\int_0^\infty e^{-\rho t}\xi(t)dt = \int_0^\infty e^{-\rho t}(\xi_1 + \xi_2)dt$$

and optimality of the control law $\hat{u}(t)$ readily follows.

Step 3. We now complete the proof of (A.12). If necessary, a linear transformation may be used to decompose A into two diagonal blocks A_{11} and A_{22}, where the real part of all eigenvalues of A_{11} is at least $\rho/2$, and A_{22} has all eigenvalues with a real part less than $\rho/2$; accordingly, the matrices B, Π, H, D and function f, etc. will be subject to an associated linear transformation. Without introducing additional notation, we simply assume that A takes the form

$$A = \begin{bmatrix} A_{11} & \\ & A_{22} \end{bmatrix}.$$

We write the dynamics (A.1) by two components

$$dx_1(t) = A_{11}x_1(t)dt + B_1u(t)dt + f_1(t)dt + D_1dW(t),$$
$$dx_2(t) = A_{22}x_2(t)dt + B_2u(t)dt + f_2(t)dt + D_2dW(t).$$

Denote $H = [H_1, H_2]$, where the number of columns in H_1 is equal to the dimension of x_1. By the detectability of $(H, A - (\rho/2)I)$, it is straightforward to show that the pair $(H_1, A_{11} - (\rho/2)I)$ and hence (H_1, A_{11}) are observable (see [23, 37]).

Applying Lemma A.1, we obtain

$$E\int_0^\infty e^{-\rho t}|x_2(t)|^2dt < \infty. \tag{A.14}$$

Next, for any $r \in [0, 1]$ we have

$$\begin{aligned} x_1(t) = e^{A_{11}r}x_1(t-r) &+ \int_{t-r}^t e^{A_{11}(t-\tau)}[B_1u(\tau) + f_1(\tau)]d\tau \\ &+ \int_{t-r}^t e^{A_{11}(t-\tau)}D_1dW(\tau) \end{aligned} \tag{A.15}$$

for $t \geq r$. By $E\int_0^\infty e^{-\rho t}|u(t)|^2dt < \infty$ and elementary estimates, we may find a fixed $C < \infty$ (depending on $u(\cdot)$) such that $E\int_r^\infty e^{-\rho t}|\int_{t-r}^t e^{A_{11}(t-\tau)}B_1u(\tau)d\tau|^2dt \leq C$ holds for all $r \in [0, 1]$. By $E\int_0^\infty e^{-\rho t}x^T(t)H^THx(t)dt < \infty$ (see (A.10)) and (A.14), it follows that $E\int_r^\infty e^{-\rho t}x_1^T(t)H_1^TH_1x_1(t)dt < \infty$, which combined with (A.15) gives $E\int_r^\infty e^{-\rho t}x_1^T(t-r)(H_1e^{A_{11}r})^TH_1e^{A_{11}r}x_1(t-r)dt < \infty$. By the above estimates we see that there in fact exists a fixed $C > 0$ such that for all $r \in [0, 1]$,

$$E\int_0^\infty e^{-\rho t}x_1^T(t)(H_1e^{A_{11}r})^TH_1e^{A_{11}r}x_1(t)dt \leq C. \tag{A.16}$$

By taking integration with respect to $r \in [0, 1]$, we obtain

$$E\int_0^\infty e^{-\rho t}x_1^T(t)\left[\int_0^1(H_1e^{A_{11}r})^TH_1e^{A_{11}r}dr\right]x_1(t)dt \leq C.$$

On the other hand, by the observability of the pair (H_1, A_{11}), it follows that the matrix $\int_0^1 (H_1 e^{A_{11}r})^T H_1 e^{A_{11}r} dr$ is positive definite. Hence $E\int_0^\infty e^{-\rho t}|x_1(t)|^2 dt < \infty$, which combined with (A.14) proves (A.12). ∎

We continue to give the following lemma on a lower bound estimate of the cost for a perturbed version of the control problem (A.1)–(A.2). Let the dynamics and cost be given by

$$dx(t) = Ax(t)dt + Bu(t)dt + f(t)dt + \xi_a(t)dt + DdW(t) \tag{A.17}$$

and

$$J_\xi(u(\cdot)) = E\int_0^\infty e^{-\rho t}\left\{[Hx(t) - g(t) + \xi_b(t)]^T[Hx(t) - g(t) + \xi_b(t)] + u^T(t)Ru(t)\right\} dt,$$

where ξ_a and ξ_b are random processes with

$$\epsilon_a = \left\{\int_0^\infty e^{-\rho t} E|\xi_a(t)|^2 dt\right\}^{1/2} < \infty, \qquad \epsilon_b = \left\{\int_0^\infty e^{-\rho t} E|\xi_b(t)|^2 dt\right\}^{1/2} < \infty.$$

It is assumed that $W(t)$, $u(t)$, $\xi_a(t)$, and $\xi_b(t)$ are adapted to an underlying filtration $(\Omega, \mathcal{F}, \mathcal{F}_t, t \geq 0, P)$.

Lemma A.3. *Under the assumptions of Lemma* A.2*, for any $u(\cdot)$ adapted to $\mathcal{F}_t$, we have*

$$J_\xi(u(\cdot)) \geq J(\hat{u}(\cdot)) - O(\epsilon_a + \epsilon_b),$$

where J is given by (A.2) *and $\hat{u}$ is given by* (A.8)*, provided that for a fixed constant C_0, the pair (x, u) in* (A.17) *has the prior upper bound estimate*

$$E\int_0^\infty e^{-\rho t}(|x|^2 + |u|^2)dt \leq C_0. \tag{A.18}$$

Proof. We first write (A.17) in the form

$$dx = (A - BR^{-1}B^T\Pi)xdt + Bu'dt - BR^{-1}B^T sdt + fdt + \xi_a dt + DdW(t), \tag{A.19}$$

where $u'(t) = u(t) + R^{-1}B^T[\Pi x(t) + s(t)]$. By (A.18), it follows that

$$E\int_0^\infty e^{-\rho t}|u'(t)|^2 dt \leq C_1,$$

where $C_1 < \infty$ depends on C_0. For given $x(0)$ and u satisfying (A.18), u' is a well-defined random process adapted to $\mathcal{F}_t$. We construct the control problem with dynamics and cost:

(A.20)
$$\begin{aligned} d\bar{x} &= (A - BR^{-1}B^T\Pi)\bar{x}dt + Bu^\dagger dt - BR^{-1}B^T sdt + fdt + DdW(t),\\ \tilde{J}(u^\dagger(\cdot)) &= \int_0^\infty e^{-\rho t}\Big\{[H\bar{x} - g]^T[H\bar{x} - g]\\ &\qquad + [u^\dagger - R^{-1}B^T(\Pi\bar{x} + s)]^T R[u^\dagger - R^{-1}B^T(\Pi\bar{x} + s)]\Big\}dt, \end{aligned}$$

where $\bar{x}(0) = x(0)$ and $u^\dagger(t)$ is adapted to $\mathcal{F}_t$. By Lemma A.2, we see that $\tilde{J}(u^\dagger(\cdot))$ attains its minimum when $u^\dagger \equiv 0$, and the resulting optimal cost is equal to $J(\hat{u}(\cdot))$. Hence, we have the lower bound

$$\tilde{J}(u'(\cdot)) \geq \tilde{J}(0) = J(\hat{u}(\cdot)). \tag{A.21}$$

Take $u^\dagger = u'$ in (A.20) and let $\bar{x}'$ be the associated solution. We obtain the estimate $\int_0^\infty e^{-\rho t} E|\bar{x}'(t)|^2 dt \leq C_2$ by Lemma A.1, where C_2 depends on C_1 but not on (ξ_a, ξ_b). By (A.19)–(A.20), we further obtain

$$\int_0^\infty e^{-\rho t} E|x(t) - \bar{x}'(t)|^2 dt \leq C_3 \epsilon_a^2 \tag{A.22}$$

for some C_3 independent of ξ_a. By the relation

$$\begin{aligned} &[Hx(t) - g(t) + \xi_b(t)]^T [Hx(t) - g(t) + \xi_b(t)]^T + u^T(t) R u(t) \\ &\quad = [H\bar{x}' - g + H(x - \bar{x}') + \xi_b]^T [H\bar{x}' - g + H(x - \bar{x}') + \xi_b]^T + \Delta_u^T R \Delta_u, \end{aligned}$$

where $\Delta_u = u' - R^{-1}B^T(\Pi\bar{x}' + s) + R^{-1}B^T\Pi(\bar{x}' - x)$, we use (A.22) and Schwarz inequality to show

$$J_\xi(u(\cdot)) \geq \tilde{J}(u'(\cdot)) - C(\epsilon_a + \epsilon_b) \geq J(\hat{u}(\cdot)) - C(\epsilon_a + \epsilon_b),$$

where the second inequality is due to (A.21), and C depends on C_1, C_2, C_3, but not on (ξ_a, ξ_b). $\square$

Appendix B. Proof of Theorem 6. Denote the weighted processes

$$x_{j,\hat{\rho}} = e^{-(\hat{\rho}/2)t} x_j, \quad z_{\kappa,\hat{\rho}} = e^{-(\hat{\rho}/2)t} z_\kappa, \tag{B.1}$$

$$z_{\hat{\rho}} = e^{-(\hat{\rho}/2)t} z, \quad \check{z}_{\kappa,\hat{\rho}} = e^{-(\hat{\rho}/2)t} \check{z}_\kappa, \quad \check{z}_{\hat{\rho}} = e^{-(\hat{\rho}/2)t} \check{z}, \tag{B.2}$$

where $j = 0, \ldots, N$ and $\kappa = 1, \ldots, K$. Also denote $s_{0,\hat{\rho}} = e^{-(\hat{\rho}/2)t} s_0$, $s_{\kappa,\hat{\rho}} = e^{-(\hat{\rho}/2)t} s_\kappa$, and $\overline{m}_{\hat{\rho}} = e^{-(\hat{\rho}/2)t}\overline{m}$. Our method of estimation is to first obtain an equation system in terms of the states $(x_{0,\hat{\rho}}, z_{\hat{\rho}}, \check{z}_{\hat{\rho}})$, and then perform SDE estimates.

By (5.4) and Ito's formula, it follows that

$$\begin{aligned} dx_{0,\hat{\rho}} &= \left\{ [A_0 - (\hat{\rho}/2)I] x_{0,\hat{\rho}} - B_0 R_0^{-1} \mathbb{B}_0^T [P_0 (x_{0,\hat{\rho}}^T, z_{\hat{\rho}}^T)^T + s_{0,\hat{\rho}}] + F_0 \sum_{i=1}^K \pi_i^{(N)} \check{z}_{i,\hat{\rho}} \right\} dt \\ &\qquad + e^{-(\hat{\rho}/2)t} D_0 dW_0 \\ &= \left\{ [A_0 - (\hat{\rho}/2)I] x_{0,\hat{\rho}} - B_0 R_0^{-1} \mathbb{B}_0^T [P_0 (x_{0,\hat{\rho}}^T, z_{\hat{\rho}}^T)^T + s_{0,\hat{\rho}}] + F_0 \sum_{i=1}^K \pi_i^{(N)} z_{i,\hat{\rho}} \right\} dt \\ &\qquad + F_0 \sum_{i=1}^K \pi_i^{(N)} (\check{z}_{i,\hat{\rho}} - z_{i,\hat{\rho}}) dt + e^{-(\hat{\rho}/2)t} D_0 dW_0. \end{aligned} \tag{B.3}$$

For $z_{\hat{\rho}}$, we have the equation

$$dz_{\hat{\rho}} = [\overline{A} - (\hat{\rho}/2)I] z_{\hat{\rho}} dt + \overline{G} x_{0,\hat{\rho}} dt + \overline{m}_{\hat{\rho}}(t) dt. \tag{B.4}$$

Next, by (5.6) and Ito's formula, it follows that

$$
\begin{aligned}
\text{(B.5)}\qquad d\check z_{\kappa,\hat\rho} = \Big\{&[A(\kappa)-(\hat\rho/2)I]\check z_{\kappa,\hat\rho} - BR^{-1}\mathbb{B}^T[P_\kappa(\check z^T_{\kappa,\hat\rho}, x^T_{0,\hat\rho}, z^T_{\hat\rho})^T + s_{\kappa,\hat\rho}]\\
&+ F\sum_{i=1}^K \pi_i^{(N)} z_{i,\hat\rho} + Gx_{0,\hat\rho}\Big\}dt + F\sum_{i=1}^K \pi_i^{(N)}(\check z_{i,\hat\rho} - z_{i,\hat\rho})dt\\
&+ (1/N_\kappa)e^{-(\hat\rho/2)t} D\sum_{i\in\mathcal{I}_\kappa} dW_i, \qquad \kappa = 1,\dots,K.
\end{aligned}
$$

Let each P_κ, $\kappa=1,\dots,K$, be partitioned as in (4.11). Denote

$$
\text{(B.6)}\qquad F^{\pi^{(N)}} = \pi^{(N)}\otimes F, \qquad \Phi_1^{(N)} = \left[\overline{G}, \mathbf{1}_K\otimes F^{\pi^{(N)}} - M_2\right],
$$

where M_2 is defined as in (4.20). Recall that $\overline{G}$ may be represented in the form (4.22). Now (B.5) yields

$$
\begin{aligned}
\text{(B.7)}\qquad d\check z_{\hat\rho} = {}& [M_1 - (\hat\rho/2)I]\check z_{\hat\rho}dt + \Phi_1^{(N)}\begin{bmatrix} x_{0,\hat\rho}\\ z_{\hat\rho}\end{bmatrix}dt + \mathbf{1}_K\otimes F^{\pi^{(N)}}(\check z_{\hat\rho} - z_{\hat\rho})dt\\
&- e^{-(\hat\rho/2)t}\zeta_1 dt + e^{-(\hat\rho/2)t}d\zeta_2,
\end{aligned}
$$

where M_1 is defined as in (4.20). In the above,

$$
\text{(B.8)}\qquad \zeta_1 = \begin{bmatrix} BR^{-1}\mathbb{B}^T s_1\\ \vdots\\ BR^{-1}\mathbb{B}^T s_K\end{bmatrix}, \qquad \zeta_2 = \begin{bmatrix} (1/N_1)D\sum_{i\in\mathcal{I}_1} W_i\\ \vdots \\ (1/N_K)D\sum_{i\in\mathcal{I}_K} W_i\end{bmatrix}.
$$

Denote

$$
F_0^{\pi^{(N)}} = \pi^{(N)}\otimes F_0, \qquad \mathbb{A}_0^{(N)} = \begin{bmatrix} A_0 & F_0^{\pi^{(N)}}\\ \overline{G} & \overline{A}\end{bmatrix}.
$$

We have the equation

$$
\begin{aligned}
\begin{bmatrix} dx_{0,\hat\rho}\\ dz_{\hat\rho}\\ d\check z_{\hat\rho}\end{bmatrix} = {}& \begin{bmatrix} \mathbb{A}_0^{(N)} - \mathbb{B}_0R_0^{-1}\mathbb{B}_0^TP_0 - (\hat\rho/2)I & 0\\ \Phi_1^{(N)} & M_1 - (\hat\rho/2)I\end{bmatrix}\begin{bmatrix} x_{0,\hat\rho}\\ z_{\hat\rho}\\ \check z_{\hat\rho}\end{bmatrix}dt\\
\text{(B.9)}\qquad & + \begin{bmatrix} F_0^{\pi^{(N)}}\\ 0_{nK\times nK}\\ \mathbf{1}_K\otimes F^{\pi^{(N)}}\end{bmatrix}(\check z_{\hat\rho} - z_{\hat\rho})dt - e^{-(\hat\rho/2)t}\begin{bmatrix} B_0R_0^{-1}\mathbb{B}_0^Ts_0\\ -\overline{m}\\ \zeta_1\end{bmatrix}dt\\
&+ e^{-(\hat\rho/2)t}\begin{bmatrix} D_0dW_0\\ 0_{nK\times 1}\\ d\zeta_2\end{bmatrix},
\end{aligned}
$$

where ζ_1 and ζ_2 are given by (B.8). Notice that $\lim_{N\to\infty}\Phi_1^{(N)} = [\overline{G}, \mathbf{1}_K\otimes F^\pi - M_2]$, $\lim_{N\to\infty} F_0^{\pi^{(N)}} = F_0^\pi$, $\lim_{N\to\infty} F^{\pi^{(N)}} = F^\pi$, and $\lim_{N\to\infty}\mathbb{A}_0^{(N)} = \mathbb{A}_0$. Denote

$$
\text{(B.10)}\qquad \xi = (x^T_{0,\hat\rho}, z^T_{\hat\rho}, \check z^T_{\hat\rho})^T.
$$

Recalling (5.7)–(5.8), we have Hurwitz matrices $M_1 - (\hat\rho/2)I$, $\mathbb{A}_0^{(N)} - \mathbb{B}_0R_0^{-1}\mathbb{B}_0^TP_0 - (\hat\rho/2)I$ for all sufficiently large N, and $\sup_{t\ge 0} e^{-(\hat\rho/2)t}(|s_0| + |\overline{m}| + |\zeta_1|) < \infty$. By (B.9) and elementary linear SDE estimates, we may find constants $a_1 > 0$ and C, both independent of t, N such that for all $N \geq N'$ with some sufficiently large N', we have

$$
\begin{aligned}
E|\xi(t)|^2 &\leq C + CE\left|\int_0^t e^{-a_1(t-\tau)}|\check{z}_{\hat{\rho}}(\tau) - z_{\hat{\rho}}(\tau)|d\tau\right|^2 \\
&\leq C + C\int_0^t e^{-a_1(t-\tau)}d\tau \int_0^t e^{-a_1(t-\tau)}E|\check{z}_{\hat{\rho}}(\tau) - z_{\hat{\rho}}(\tau)|^2 d\tau \qquad \text{(B.11)}\\
&\leq C + C\int_0^t e^{-a_1(t-\tau)}E|\check{z}_{\hat{\rho}}(\tau) - z_{\hat{\rho}}(\tau)|^2 d\tau, \qquad \text{(B.12)}
\end{aligned}
$$

where (B.11) is obtained by Schwarz inequality.

We continue to estimate the integrand $E|\check{z}_{\hat{\rho}}(t) - z_{\hat{\rho}}(t)|^2$ in (B.12). First, by (B.4)–(B.5) and the relations (4.15)–(4.16), it follows that

$$
\begin{aligned}
\text{(B.13)}\quad & d(\check{z}_{\kappa,\hat{\rho}} - z_{\kappa,\hat{\rho}}) \\
&= \left\{[A(\kappa) - BR^{-1}B^TP_{\kappa,11} - (\hat{\rho}/2)I](\check{z}_{\kappa,\hat{\rho}} - z_{\kappa,\hat{\rho}}) + F\sum_{i=1}^K \pi_i^{(N)}(\check{z}_{i,\hat{\rho}} - z_{i,\hat{\rho}})\right\}dt \\
&\quad + F\sum_{i=1}^K(\pi_i^{(N)} - \pi_i)z_{i,\hat{\rho}}dt + (1/N_\kappa)e^{-(\hat{\rho}/2)t}D\sum_{i\in\mathcal{I}_\kappa} dW_i, \quad \kappa = 1, \dots, K.
\end{aligned}
$$

This equation may be written in the compact form

$$
\begin{aligned}
d(\check{z}_{\hat{\rho}} - z_{\hat{\rho}}) = {} & [M_1 - (\hat{\rho}/2)I + \mathbf{1}_K \otimes F^{\pi^{(N)}}](\check{z}_{\hat{\rho}} - z_{\hat{\rho}})dt \\
& + [\mathbf{1}_K \otimes (\pi^{(N)} - \pi) \otimes F]z_{\hat{\rho}}dt + e^{-(\hat{\rho}/2)t}d\zeta_2,
\end{aligned}
$$

where ζ_2 is given as in (B.8). Again, by use of the fact $\lim_{N\to\infty} \mathbf{1}_K \otimes F^{\pi^{(N)}} = \mathbf{1}_K \otimes F^\pi$ and (A3), we apply SDE estimates to obtain

$$
\begin{aligned}
E|\check{z}_{\hat{\rho}}(t) - z_{\hat{\rho}}(t)|^2 &\leq C\left[1/N + \int_0^t e^{-a_2(t-\tau)}\epsilon_N^2 E|z_{\hat{\rho}}(\tau)|^2 d\tau\right] \\
&\leq C\left[1/N + \int_0^t e^{-a_2(t-\tau)}\epsilon_N^2 E|\xi_\tau|^2 d\tau\right], \qquad \text{(B.14)}
\end{aligned}
$$

where $a_2 > 0$ is a fixed constant independent of $N \geq N''$ for a sufficiently large $N'' > 0$, and C is independent of t, N. Let $a = \min\{a_1, a_2\}$ and $\Delta_t = E|\xi(t)|^2$ for ξ defined in (B.10). Now it follows from (B.12) and (B.14) that

$$
\begin{aligned}
\Delta_t &\leq C + C\int_0^t e^{-a(t-\tau)}\int_0^\tau \epsilon_N^2 e^{-a(\tau-h)}\Delta_h dh d\tau \\
&= C + C\epsilon_N^2\int_0^t (t-h)e^{-a(t-h)}\Delta_h dh \\
&\leq C + \left(\max_{0\leq\tau\leq t}\Delta_\tau\right)\epsilon_N^2 C\int_0^t (t-h)e^{-a(t-h)}dh \leq C + \left(\max_{0\leq\tau\leq t}\Delta_\tau\right)\epsilon_N^2(C/a^2)
\end{aligned}
$$

for $N \geq \max\{N', N''\}$, which implies

$$
\max_{0\leq\tau\leq t}\Delta_\tau \leq C + \left(\max_{0\leq\tau\leq t}\Delta_\tau\right)\epsilon_N^2(C/a^2).
$$

We pick $N^{(0)} \geq \max\{N', N''\}$ such that $\epsilon_N^2(C/a^2) < 1/2$ for all $N \geq N^{(0)}$. Hence, for all $N \geq N^{(0)}$, we have $\max_{0\leq\tau\leq t} \Delta_\tau \leq C_0$ for some C_0 independent of t, N, so that

$$\sup_{t\geq 0} E|\xi(t)|^2 \leq C_0. \tag{B.15}$$

Based on (B.15), we may further use (5.5) to obtain $\sup_{1\leq i\leq N} \sup_{t\geq 0} E|x_{i,\hat\rho}(t)|^2 \leq C_0$ since each matrix $A(\theta_i) - BR^{-1}B^T P_{\theta_i,11} - (\hat\rho/2)I$ is Hurwitz. Now we combine (B.14) with (B.15) to obtain

$$\sup_{t\geq 0} E|\check z_{\hat\rho}(t) - z_{\hat\rho}(t)|^2 \leq C_1(1/N + \epsilon_N^2),$$

where C_1 is independent of $N \geq N^{(0)}$. This completes the proof. ∎

Appendix C. Proof of Theorem 10.

LEMMA C.1. *Let $y(t)$ be a nonnegative scalar integrable function of t on $[0, T]$ and $\sigma > 0$. Then for all $\delta \in [0, \sigma]$,*

$$\int_0^T e^{-\sigma t}\left[y(t) - \delta \int_0^t y(\tau)d\tau\right] dt \geq 0.$$

Proof. Since

$$\begin{aligned}\int_0^T e^{-\sigma t}\delta \int_0^t y(\tau)d\tau dt &= \int_0^T \delta y(\tau) \int_\tau^T e^{-\sigma t} dt d\tau \\ &\leq \int_0^T (\delta/\sigma) y(\tau) e^{-\sigma\tau} d\tau \leq \int_0^T y(\tau)e^{-\sigma\tau} d\tau,\end{aligned}$$

the lemma follows. ∎

LEMMA C.2. *Suppose $(P_0, \overline{A}, \overline{G}, P_\kappa, \kappa = 1, \ldots, K)$ is a stabilizing consistent solution to (4.18) and (A3′) holds. Define the matrix*

$$\Gamma = \begin{bmatrix} A_0 - B_0R_0^{-1}B_0^T P_{0,11} & -B_0R_0^{-1}B_0^T P_{0,12} & F_0^\pi \\ \overline{G} & \overline{A} & 0 \\ \overline{G} & -M_2 & M_1 + \mathbf{1}_K \otimes F^\pi \end{bmatrix}, \tag{C.1}$$

where $P_{0,11}$, $P_{0,12}$ are given by (4.11), and M_1, M_2 are defined by (4.20). Then Γ is Hurwitz.

Proof. Let $x_0^\dagger$, $z^\dagger$, and $\check z^\dagger$ each be in $\mathbb{R}^n$. We show that Γ is Hurwitz with the aid of the ODE

$$\frac{d}{dt}\begin{bmatrix} x_0^\dagger \\ z^\dagger \\ \check z^\dagger \end{bmatrix} = \Gamma \begin{bmatrix} x_0^\dagger \\ z^\dagger \\ \check z^\dagger \end{bmatrix}. \tag{C.2}$$

First, by the relation (4.21) we can check that

$$\frac{d}{dt}(z^\dagger - \check z^\dagger) = (M_1 + \mathbf{1}_K \otimes F^\pi)(z^\dagger - \check z^\dagger).$$

Since $M_1 + \mathbf{1}_K \otimes F^\pi$ is Hurwitz, given any initial condition $(x_0^\dagger(0), z^\dagger(0), \check{z}^\dagger(0))$, we have

$$\lim_{t\to\infty} |z^\dagger(t) - \check{z}^\dagger(t)| = 0. \tag{C.3}$$

Subsequently, we may write (C.2) in the equivalent form

$$\frac{d}{dt}\begin{bmatrix} x_0^\dagger \\ z^\dagger \\ \check{z}^\dagger \end{bmatrix} = \hat{\Gamma}\begin{bmatrix} x_0^\dagger \\ z^\dagger \\ \check{z}^\dagger \end{bmatrix} + \begin{bmatrix} F_0^\pi(\check{z}^\dagger - z^\dagger) \\ 0_{nK\times 1} \\ \mathbf{1}_K \otimes F^\pi(\check{z}^\dagger - z^\dagger) \end{bmatrix},$$

where

$$\hat{\Gamma} = \begin{bmatrix} A_0 - B_0R_0^{-1}B_0^TP_{0,11} & F_0^\pi - B_0R_0^{-1}B_0^TP_{0,12} & 0 \\ \overline{G} & \overline{A} & 0 \\ \overline{G} & \mathbf{1}_K \otimes F - M_2 & M_1 \end{bmatrix}$$

is Hurwitz by Proposition 2 and the fact that $(P_0, \overline{A}, \overline{G}, P_\kappa, \kappa = 1, \dots, K)$ is a stabilizing consistent solution to (4.18). Hence, by (C.3) and the stability of $\hat{\Gamma}$, given any initial condition $(x_0^\dagger(0), z^\dagger(0), \check{z}^\dagger(0))$, we may show $\lim_{t\to\infty}(|x_0^\dagger(t)| + |z^\dagger(t)| + |\check{z}^\dagger(t)|) = 0$, which implies that Γ is Hurwitz. ∎

Proof of Theorem 10. The proof is given in two steps.

Step 1. *The case for the major player $\mathcal{A}_0$ to use an alternative strategy.* Let the control of $\mathcal{A}_0$ be denoted by u_0, and let the minor players take the control law (5.3). The closed-loop system leads to the equations

$$dx_0 = \left\{A_0x_0 + B_0u_0 + F_0x^{(N)}\right\}dt + D_0dW_0(t), \tag{C.4}$$

$$dz = \overline{A}zdt + \overline{G}x_0dt + \overline{m}dt, \tag{C.5}$$

$$\begin{aligned} d\check{z}_\kappa = &\left\{A(\kappa)\check{z}_\kappa - BR^{-1}\mathbb{B}^T[P_\kappa(\check{z}_\kappa^T, x_0^T, z^T)^T + s_\kappa] + Fx^{(N)} + Gx_0\right\}dt \\ &+ (1/N_\kappa)D\sum_{i\in\mathcal{I}_\kappa} dW_i, \qquad \kappa = 1, \dots, K. \end{aligned} \tag{C.6}$$

We may write (C.4)–(C.6) in the compact form

$$d\begin{bmatrix} x_0 \\ z \\ \check{z} \end{bmatrix} = M_3^{(N)}\begin{bmatrix} x_0 \\ z \\ \check{z} \end{bmatrix}dt + \begin{bmatrix} B_0u_0 \\ 0_{nK\times 1} \\ 0_{nK\times 1} \end{bmatrix}dt + \begin{bmatrix} 0_{n\times 1} \\ \overline{m} \\ -\zeta_1 \end{bmatrix}dt + \begin{bmatrix} D_0dW_0 \\ 0_{nK\times 1} \\ d\zeta_2 \end{bmatrix},$$

where ζ_1 and ζ_2 are given by (B.8), and $M_3^{(N)}$ is specified as shown below (6.1).

For $0 \le j \le N$ and the control laws $\hat{u}_j$ given by (5.2)–(5.3), denote $\hat{u}_{-j} = (\hat{u}_0, \dots, \hat{u}_{j-1}, \hat{u}_{j+1}, \dots, \hat{u}_N)$. If u_0 is simply taken as $\hat{u}_0$, we may apply Theorem 6 to show that the associated cost $J_0(\hat{u}_0, \hat{u}_{-0})$ is upper bounded by a constant C_0 independent of $N \ge N^{(0)}$, where $N^{(0)}$ is specified as in Theorem 6. For Step 1 we assume $N \ge N^{(0)}$. By our convention, each of the constants C, C_0, etc. used below may vary from place to place. It suffices to restrict attention to all u_0 such that

$$\begin{aligned} &J_0(u_0, \hat{u}_{-0}) \\ &= E\int_0^\infty e^{-\rho t}\left\{[x_0 - H_0x^{(N)} - \eta_0]^TQ_0[x_0 - H_0x^{(N)} - \eta_0] + u_0^TR_0u_0\right\}dt \le C_0, \end{aligned} \tag{C.7}$$

which implies that for each $r \in [0,1]$,

$$E\int_r^\infty e^{-\rho t}\left\{[x_0 - H_0x^{(N)}]^TQ_0[x_0 - H_0x^{(N)}] + u_0^TR_0u_0\right\}dt$$

(C.8)
$$= E\int_r^\infty e^{-\rho t}\left[(x_0^T, z^T, \check z^T)(L_{0,H}^{(N)})^TL_{0,H}^{(N)}(x_0^T, z^T, \check z^T)^T + u_0^TR_0u_0\right]dt \le C_0,$$

where $L_{0,H}^{(N)}$ is specified as shown below (6.1).

Similarly to the derivation of (A.16) in Appendix A, we may show from (C.8) that

$$E\int_r^\infty e^{-\rho t}\left[(x_0^T, z^T, \check z^T)(e^{M_3^{(N)}r})^T(L_{0,H}^{(N)})^TL_{0,H}^{(N)}e^{M_3^{(N)}r}(x_0^T, z^T, \check z^T)^T\right]\Big|_{(t-r)}dt \le C_0$$

for each $r \in [0,1]$, where C_0 does not depend on (r, N). Hence

(C.9)
$$E\int_0^\infty e^{-\rho t}\left[(x_0^T, z^T, \check z^T)(e^{M_3^{(N)}r})^T(L_{0,H}^{(N)})^TL_{0,H}^{(N)}e^{M_3^{(N)}r}(x_0^T, z^T, \check z^T)^T\right]\Big|_t dt \le C_0.$$

By the observability of the pair $(L_{0,H}, M_3)$ and the convergence relation

$$\lim_{N\to\infty} M_3^{(N)} = M_3, \qquad \lim_{N\to\infty} L_{0,H}^{(N)} = L_{0,H},$$

there exists a sufficiently large $N^{(1)} \ge N^{(0)}$ such that for all $N \ge N^{(1)}$ we have

$$c_1I \le \int_0^1 \left(e^{M_3^{(N)}r}\right)^T\left(L_{0,H}^{(N)}\right)^T L_{0,H}^{(N)}e^{M_3^{(N)}r}dr \le c_2I, \tag{C.10}$$

where the two constants $0 < c_1 \le c_2 < \infty$ are independent of N. In proving (C.10), we have used the continuous dependence of observability on parameters of a linear system [37, p. 44]. By taking integration with respect to r on $[0,1]$ in (C.9), it follows from (C.10) and (C.7) that

$$E\int_0^\infty e^{-\rho t}\left(|x_0|^2 + |z|^2 + |\check z|^2 + |u_0|^2\right)dt \le C_0. \tag{C.11}$$

Letting ζ_2 be defined by (B.8), by (C.5)–(C.6) we have

$$d(\check z - z) = (M_1 + \mathbf{1}_K \otimes F^\pi)(\check z - z)dt + (\pi^{(N)} - \pi)\otimes \mathbf{1}_K \otimes F\check z dt + d\zeta_2.$$

Since (A3′) holds and (C.11) gives $E\int_0^\infty e^{-\rho t}|\check z(t)|^2dt \le C_0$, we may use the method in proving Lemma A.1 to show

$$\int_0^\infty e^{-\rho t}E|\check z(t) - z(t)|^2dt \le C(1/N + \epsilon_N^2) \tag{C.12}$$

for $N \ge N^{(1)}$, where the component $1/N$ results from the term $d\zeta_2$ and the initial condition $(\check z - z)(0)$.

We continue to give a refined upper bound for $J_0(\hat u_0, \hat u_{-0})$. In fact, if $u_0 = \hat u_0$, we may use (5.9), Proposition 8, and elementary estimates to show that

$$J_0(\hat u_0, \hat u_{-0}) \le \bar J_0(\hat u_0^\dagger) + C(1/\sqrt{N} + \epsilon_N), \tag{C.13}$$

where $\bar{J}_0(\hat{u}_0^\dagger)$ is the optimal cost associated with the control $\hat{u}_0^\dagger$ in the control problem with dynamics

$$d\bar{x}_0(t) = \Big\{A_0\bar{x}_0(t) + B_0 u_0^\dagger + F_0^\pi \bar{z}(t)\Big\}dt + D_0 dW_0(t),$$
$$d\bar{z}(t) = \overline{A}\bar{z}(t)dt + \overline{G}\bar{x}_0(t)dt + \overline{m}(t)dt,$$

with initial conditions $\bar{x}(0) = x(0)$, $\bar{z}(0) = 0$ and cost

$$\bar{J}_0(u_0^\dagger(\cdot)) = E\int_0^\infty e^{-\rho t}\Big\{[\bar{x}_0 - H_0^\pi \bar{z} - \eta_0]^T Q_0[\bar{x}_0 - H_0^\pi \bar{z} - \eta_0] + (u_0^\dagger)^T R_0 u_0^\dagger\Big\}dt.$$

To obtain a lower bound for $J_0(u_0, \hat{u}_{-0})$ subject to (C.11), in (C.4) we write $F_0 x^{(N)} = F_0^\pi z + F_0^\pi(\check{z} - z) + (F_0^{\pi^{(N)}} - F_0^\pi)\check{z} \triangleq F_0^\pi z + \xi_a$ and in $J_0(u_0, \hat{u}_{-0})$ write $H_0 x^{(N)} = H_0^\pi z + H_0^\pi(\check{z} - z) + (H_0^{\pi^{(N)}} - H_0^\pi)\check{z} \triangleq H_0^\pi z + \xi_b$, where $H_0^\pi = \pi \otimes H_0$ and $H_0^{\pi^{(N)}} = \pi^{(N)} \otimes H_0$. By (C.11)–(C.12), we may view (C.4)–(C.5) as a perturbed control model with state (x_0, z) and perturbation (ξ_a, ξ_b) and apply Lemma A.3 to show

$$J_0(u_0, \hat{u}_{-0}) \geq \bar{J}_0(\hat{u}_0^\dagger) - O(1/\sqrt{N} + \epsilon_N). \tag{C.14}$$

It follows from (C.13) and (C.14) that $J_0(u_0, \hat{u}_{-0}) \geq J_0(\hat{u}_0, \hat{u}_{-0}) - O(1/\sqrt{N} + \epsilon_N)$.

Step 2. *The case for any given minor player to use an alternative strategy.* Without loss of generality, we consider alternative strategies of $\mathcal{A}_{i_0}$, which is a k-type minor player. After all agents, except $\mathcal{A}_{i_0}$, apply the control laws (5.2) and (5.3), the closed-loop dynamics of $\mathcal{A}_0$ and $\mathcal{A}_i$, $i \neq i_0$, may be written in the form

$$dx_0 = \Big\{A_0 x_0 - B_0 R_0^{-1}\mathbb{B}_0^T[P_0(x_0^T, z^T)^T + s_0] + F_0 x^{(N)}\Big\}dt + D_0 dW_0, \tag{C.15}$$

$$\begin{aligned} dx_i = &\Big\{A(\theta_i)x_i - BR^{-1}\mathbb{B}^T[P_{\theta_i}(x_i^T, x_0^T, z^T)^T + s_{\theta_i}] + Fx^{(N)} + Gx_0\Big\}dt \\ &+ DdW_i, \qquad 1 \leq i \leq N,\ i \neq i_0, \end{aligned} \tag{C.16}$$

where

$$dz = \overline{A}z dt + \overline{G}x_0 dt + \overline{m}dt, \tag{C.17}$$

with the initial condition $z(0) = 0$. We also write the dynamics of $\mathcal{A}_{i_0}$ as follows:

$$dx_{i_0} = \Big\{A(\theta_{i_0})x_{i_0} + Bu_{i_0} + Fx^{(N)} + Gx_0\Big\}dt + DdW_{i_0}.$$

Similarly to (C.7), we restrict attention to $N \geq N^{(0)}$ and control u_{i_0} such that

$$\begin{aligned} E\int_0^\infty e^{-\rho t}\Big\{&[x_{i_0} - Hx_0 - (\hat{H}x^{(N)} + \eta)]^T Q[x_{i_0} - Hx_0 - (\hat{H}x^{(N)} + \eta)] \\ &+ u_{i_0}^T R u_{i_0}\Big\}dt \leq C, \end{aligned} \tag{C.18}$$

where C is independent of N. Since Q is nonsingular, this implies

$$E\int_0^\infty e^{-\rho t}\Big|x_{i_0} - Hx_0 - \hat{H}x^{(N)}\Big|^2 dt \leq C. \tag{C.19}$$

Below we aim to establish a prior upper bound for $E\int_0^\infty e^{-\rho t}|x_{i_0}(t)|^2 dt$ when (C.18) holds.

Corresponding to (C.16), for $1 \le j \le K$ and $j \ne k$, define $\check z_j' = (1/N_j)\sum_{i\in\mathcal{I}_j} x_i$,

$$\check z_k' = (1/(N_k-1)) \sum_{i\in\mathcal{I}_k\setminus\{i_0\}} x_i,$$

and $\check z' = [(\check z_1')^T, \ldots, (\check z_K')^T]^T$. Without loss of generality, we restrict attention to large N such that $N_k > 1$. The SDE for $(x_0, z, \check z')$ may be expressed in the form

$$\begin{bmatrix} dx_0 \\ dz \\ d\check z' \end{bmatrix} = \Gamma^{(N)} \begin{bmatrix} x_0 \\ z \\ \check z' \end{bmatrix} dt + D_1^{(N)}(x_{i_0}/N)dt + \zeta_3 dt + d\zeta_4, \tag{C.20}$$

where $\Gamma^{(N)}$ and $D_1^{(N)}$ may be determined in a straightforward manner. In fact

$$\lim_{N\to\infty} \Gamma^{(N)} = \Gamma, \tag{C.21}$$

where Γ is defined by (C.1) and is Hurwitz by Lemma C.2, and $D_1^{(N)}$ also converges to a finite limit when $N \to \infty$. The term ζ_3 is a linear combination of $s_0, \ldots, s_K, \overline{m}$; ζ_4 is determined from the Brownian motions W_j, $0 \le j \le N$, $j \ne i_0$. By solving the linear SDE (C.20) with x_{i_0} treated as an exogenous term, we may express $(x_0, z, \check z')$ in the form

$$\begin{bmatrix} x_0(t) \\ z(t) \\ \check z'(t) \end{bmatrix} = e^{\Gamma^{(N)}t} \begin{bmatrix} x_0(0) \\ z(0) \\ \check z'(0) \end{bmatrix} + \int_0^t e^{\Gamma^{(N)}(t-\tau)}(1/N)D_1^{(N)}x_{i_0}(\tau)d\tau + \zeta_5, \tag{C.22}$$

where ζ_5 depends on $s_0, \ldots, s_K, \overline{m}$ and the Brownian motions. Recall that Γ is Hurwitz and (C.21) holds. Taking a sufficiently large $N^{(2)} \ge N^{(0)}$, for $N \ge N^{(2)}$ we combine (C.19) with (C.22) to obtain

(C.23)
$$E\int_0^\infty e^{-\rho t}\left|[I-(1/N)\hat H]x_{i_0}(t) - D_2^{(N)}\int_0^t e^{\Gamma^{(N)}(t-\tau)}(1/N)D_1^{(N)}x_{i_0}(\tau)d\tau\right|^2 dt \le C,$$

where $(1/N)\hat H x_{i_0}$ is contained in $\hat H x^{(N)}$, and $D_2^{(N)}$ is determined from $(H, \hat H, \pi^{(N)})$ and has a finite limit when $N \to \infty$. Now, by (C.23) and the inequality $(\alpha-\beta)^2 \ge \alpha^2/2 - \beta^2$, for $N \ge N^{(2)}$ we have

$$\begin{aligned} E\int_0^\infty e^{-\rho t}\Big\{(1/2)\left|[I-(1/N)\hat H]x_{i_0}(t)\right|^2 \\ - \left|D_2^{(N)}\int_0^t e^{\Gamma^{(N)}(t-\tau)}(1/N)D_1^{(N)}x_{i_0}(\tau)d\tau\right|^2\Big\}dt \le C. \end{aligned} \tag{C.24}$$

Without loss of generality, we may assume for all $N \ge N^{(2)}$,

$$(1/2)[I-(1/N)\hat H]^T[I-(1/N)\hat H] \ge (1/3)I.$$

Then it follows from (C.24) that for all $N \geq N^{(2)}$,

(C.25)

$$E \int_0^\infty e^{-\rho t} \left\{ (1/3)|x_{i_0}(t)|^2 - \left| D_2^{(N)} \int_0^t e^{\Gamma^{(N)}(t-\tau)} (1/N) D_1^{(N)} x_{i_0}(\tau) d\tau \right|^2 \right\} dt \leq C.$$

Next we show that

(C.26)

$$E \int_0^\infty e^{-\rho t} \left\{ (1/4)|x_{i_0}(t)|^2 - \left| D_2^{(N)} \int_0^t e^{\Gamma^{(N)}(t-\tau)} (1/N) D_1^{(N)} x_{i_0}(\tau) d\tau \right|^2 \right\} dt \geq 0$$

for all sufficiently large N. By (C.21) we may find fixed constants $c_1 > 0$ and $c_2 > 0$, both independent of t, N such that

$$\begin{aligned} \left| D_2^{(N)} \int_0^t e^{\Gamma^{(N)}(t-\tau)} (1/N) D_1^{(N)} x_{i_0}(\tau) d\tau \right|^2 &\leq \frac{c_1}{N^2} \int_0^t |e^{\Gamma^{(N)}(t-\tau)}|^2 d\tau \int_0^t |x_{i_0}(\tau)|^2 d\tau \\ &\leq \frac{c_2}{N^2} \int_0^t |x_{i_0}(\tau)|^2 d\tau \end{aligned}$$

for all sufficiently large N. Now it follows from Lemma C.1 that

$$\begin{aligned} &\int_0^\infty e^{-\rho t} \left\{ (1/4)|x_{i_0}(t)|^2 - \left| D_2^{(N)} \int_0^t e^{\Gamma^{(N)}(t-\tau)} (1/N) D_1^{(N)} x_{i_0}(\tau) d\tau \right|^2 \right\} dt \\ &\qquad \geq \int_0^\infty e^{-\rho t} \left\{ (1/4)|x_{i_0}(t)|^2 - \frac{c_2}{N^2} \int_0^t |x_{i_0}(\tau)|^2 d\tau \right\} dt \geq 0 \end{aligned}$$

when N is sufficiently large, so that (C.26) holds. Hence it follows from (C.25) and (C.26) that

$$E \int_0^\infty e^{-\rho t} |x_{i_0}(t)|^2 dt \leq C \tag{C.27}$$

for all sufficiently large N, where C is independent of N. By combining (C.18), (C.22), and the prior estimate (C.27), it may be further shown that there exists a fixed $C > 0$ such that

$$E \int_0^\infty e^{-\rho t} \left(|x_{i_0}|^2 + |u_{i_0}|^2 + |x_0|^2 + |z|^2 + |\check{z}'|^2 \right) dt \leq C \tag{C.28}$$

for all sufficiently large N. By use of (C.28) and parallel to (C.12), we may take a sufficiently large $N^{(3)} \geq N^{(2)}$ and apply elementary SDE estimates to show that for all $N \geq N^{(3)}$,

$$\int_0^\infty e^{-\rho t} E \left| x^{(N)} - \sum_{k=1}^K \pi_k z_k \right|^2 dt \leq C(1/N + \epsilon_N^2). \tag{C.29}$$

Similarly to the treatment in Step 1, we apply Lemma A.3 to show that if the control of $\mathcal{A}_{i_0}$ changes from $\hat{u}_{i_0}$ to another one, it can reduce its cost by at most $O(1/\sqrt{N} + \epsilon_N)$. This completes the proof. ∎

REFERENCES

[1] R. J. Aumann, *Game theory*, in The New Palgrave: A Dictionary of Economics, J. Eatwell, M. Milgate, and P. Newman, eds., Vol. 2, Macmillan, London, 1987, pp. 460–482.

[2] H. Abou-Kandil and P. Bertrand, *Analytic solution for a class of linear quadratic open-loop Nash games*, Int. J. Control, 43 (1986), pp. 997–1002.

[3] T. Başar and G. J. Olsder, *Dynamic Noncooperative Game Theory*, 2nd ed., Academic Press, London, UK, 1995.

[4] T. Başar and S. Li, *Distributed computation of Nash equilibria in linear-quadratic stochastic differential games*, SIAM J. Control Optim., 27 (1989), pp. 563–578.

[5] C. T. Bauch and D. J. D. Earn, *Vaccination and the theory of games*, Proc. Natl. Acad. Sci. U.S.A., 101 (2004), pp. 13391–13394.

[6] R. Breban, R. Vardavas, and S. Blower, *Mean-field analysis of an inductive reasoning game: Application to influenza vaccination*, Phys. Rev. E, 76 (2007), 031127.

[7] A. Bensoussan, *Points de Nash dans le cas de fonctionelles quadratiques et jeux differentials lineaires a N personnes*, SIAM J. Control, 12 (1974), pp. 460–499.

[8] A. Bensoussan, *Stochastic Control of Partially Observable Systems*, Cambridge University Press, Cambridge, UK, 1992.

[9] M. C. Delfour, *Linear quadratic differential games: Saddle point and Riccati differential equation*, SIAM J. Control Optim., 46 (2007), pp. 750–774.

[10] G. M. Erickson, *Differential game models of advertising competition*, European J. Oper. Res., 83 (1995), pp. 431–438.

[11] F. Fogelman and M. Quinzii, *Asymptotic values of mixed games*, Math. Oper. Res., 5 (1980), pp. 86–93.

[12] G. Freiling, G. Jank, and S. R. Lee, *Existence and uniqueness of open-loop Stackelberg equilibria in linear-quadratic differential games*, J. Optim. Theory Appl., 110 (2001), pp. 515–544.

[13] D. Fudenberg and D. K. Levine, *The Theory of Learning in Games*, MIT Press, Cambridge, MA, 1998.

[14] D. Fudenberg, D. Levine, and W. Pesendorfer, *When are nonanonymous players negligible*? J. Econ. Theory, 79 (1998), pp. 46–71.

[15] Z. Galil, *The nucleolus in games with major and minor players*, Internat. J. Game Theory, 3 (1974), pp. 129–140.

[16] A. Graham, *Kronecker Products and Matrix Calculus: with Applications*, Ellis Horwood, Chichester, UK, 1981.

[17] O. Haimanko, *Nonsymmetric values of nonatomic and mixed games*, Math. Oper. Res., 25 (2000), pp. 591–605.

[18] S. Hart, *Values of mixed games*, Internat. J. Game Theory, 2 (1973), pp. 69–86.

[19] M. Huang, P. E. Caines, and R. P. Malhamé, *Individual and mass behaviour in large population stochastic wireless power control problems: Centralized and Nash equilibrium solutions*, in Proceedings of the 42nd IEEE Conference on Decision and Control, Maui, HI, 2003, pp. 98–103

[20] M. Huang, P. E. Caines, and R. P. Malhamé, *Large-population cost-coupled LQG problems with nonuniform agents: Individual-mass behavior and decentralized ε-Nash equilibria*, IEEE Trans. Automat. Control, 52 (2007), pp. 1560–1571.

[21] M. Huang, R. P. Malhamé, and P. E. Caines, *Nash equilibria for large-population linear stochastic systems of weakly coupled agents*, in Analysis, Control and Optimization of Complex Dynamic Systems, E. K. Boukas and R. P. Malhamé, eds., Springer, New York, 2005, pp. 215–252.

[22] M. Huang, R. P. Malhamé, and P. E. Caines, *Large population stochastic dynamic games: Closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle*, Comm. Inform. Systems, 6 (2006), pp. 221–252.

[23] T. Kailath, *Linear Systems*, Prentice-Hall, Englewood Cliffs, NJ, 1980.

[24] M. A. Khan and Y. Sun, *Non-cooperative games with many players*, in Handbook of Game Theory with Economic Applications, Vol. 3, R. J. Aumann and S. Hart, eds., Elsevier Science, North-Holland, Amsterdam, 2002, pp. 1761–1808.

[25] V. E. Lambson, *Self-enforcing collusion in large dynamic markets*, J. Econ. Theory, 34 (1984), pp. 282–291.

[26] J.-M. Lasry and P.-L. Lions, *Jeux à champ moyen.* I. *Le cas stationnaire*, C. R. Math. Acad. Sci. Paris, 343 (2006), pp. 619–625.

[27] J.-M. Lasry and P.-L. Lions, *Jeux à champ moyen.* II. *Horizon fini et contrôle optimal*, C. R. Math. Acad. Sci. Paris, 343 (2006), pp. 679–684.

[28] J.-M. Lasry and P.-L. Lions, *Mean field games*, Japan. J. Math., 2 (2007), pp. 229–260.
[29] D. L. Lukes, *Equilibrium feedback control in linear games with quadratic costs*, SIAM J. Control, 9 (1971), pp. 234–252.
[30] J. Maynard Smith, *Evolution and the Theory of Games*, Cambridge University Press, Cambridge, UK, 1982.
[31] J. W. Milnor and L. S. Shapley, *Values of large games* II: *Oceanic games*, Math. Oper. Res., 3 (1978), pp. 290–307.
[32] A. Neyman, *Weighted majority games have an asymptotic value*, Math. Oper. Res., 13 (1988), pp. 556–580.
[33] G. P. Papavassilopoulos, J. V. Medanic, and J. B. Cruz, Jr., *On the existence of Nash strategies and solutions to coupled Riccati equations in linear-quadratic Nash games*, J. Optim. Theory Appl., 28 (1979), pp. 49–76.
[34] J. von Neumann and O. Morgenstern, *Theory of Games and Economic Behavior*, 3rd ed., Princeton University Press, Princeton, NJ, 1953.
[35] G. Y. Weintraub, C. L. Benkard, and B. Van Roy, *Oblivious equilibrium: A mean field approximation for large-scale dynamic games*, in Advances in Neural Information Processing Systems, MIT Press, Cambridge, MA, 2005. Available online at http://books.nips.cc.
[36] G. Y. Weintraub, C. L. Benkard, and B. Van Roy, *Markov perfect industry dynamics with many firms*, Econometrica, 76 (2008), pp. 1375–1411.
[37] W. M. Wonham, *Linear Multivariable Control: A Geometric Approach*, 2nd ed., Springer-Verlag, New York, 1979.
[38] J. Yong, *A leader-follower stochastic linear quadratic differential game*, SIAM J. Control Optim., 41 (2002), pp. 1015–1041.