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STOCHASTIC HAMILTON-JACOBI-BELLMAN EQUATIONS*

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Abstract. This paper studies the following form of nonlinear stochastic partial differential equation:

$$-d\Phi_t = \inf_{v\in U}\left\{\frac{1}{2}\sum_{i,j}[\sigma\sigma^*]_{ij}(x,v,t)\,\partial_{x_ix_j}\Phi_t(x)+\sum_i b_i(x,v,t)\,\partial_{x_i}\Phi_t(x)+L(x,v,t)\right.$$
$$\left.+\sum_{i,j}\sigma_{ij}(x,v,t)\,\partial_{x_i}\Psi_{j,t}(x)\right\}dt-\Psi_t(x)\,dW_t,\qquad \Phi_T(x)=h(x),$$

where the coefficients σ_{ij}, b_i, L, and the final datum h may be random. The problem is to find an adapted pair $(\Phi,\Psi)(x,t)$ uniquely solving the equation. The classical Hamilton-Jacobi-Bellman (HJB) equation can be regarded as a special case of the above problem. An existence and uniqueness theorem is obtained for the case where σ does not contain the control variable v. An optimal control interpretation is given. The linear quadratic case is discussed as well.

Key words. stochastic control, dynamic programming, Riccati equation, backward stochastic differential equation, stochastic partial differential equation

AMS(MOS) subject classifications. 93E, 60H, 35K

1. Introduction. In this paper, we study a class of backward stochastic partial differential equations which can be derived from certain stochastic optimal control problems where the coefficients may be random variables. We call these kinds of equations stochastic Hamilton-Jacobi-Bellman equations, or HJB equations.

It is well known that the classical HJB equation is the following form of (deterministic) second-order, nonlinear, partial differential equation of parabolic type:

$$\begin{aligned}-\partial_t\Phi&=H_1(D^2\Phi,D\Phi,x,t),\\ \Phi(x,T)&=h(x),\end{aligned}\tag{1.1}$$

where $H_1:\mathbb{R}^{n\times n}\times\mathbb{R}^n\times\mathbb{R}^n\times[0,T]\to\mathbb{R}$ is given by

$$H_1(A,p,x,v,t)=\inf_{v\in U}\{\tfrac{1}{2}\operatorname{tr}[\sigma\sigma^*(x,v,t)A]+(b(x,v,t),v)+L(x,v,t)\}$$

with

$$\begin{aligned}&U\subset\mathbb{R}^k,\\ &\sigma(x,v,t):\mathbb{R}^n\times U\times[0,T]\to\mathbb{R}^n,\\ &b(x,v,t):\mathbb{R}^n\times U\times[0,T]\to\mathbb{R}^n,\\ &L(x,v,t):\mathbb{R}^n\times U\times[0,T]\to\mathbb{R}^n,\\ &h(x):\mathbb{R}^n\to\mathbb{R}.\end{aligned}$$

This equation has a stochastic control interpretation: Let $\{W_t,t\geqq 0\}$ be, for example, a one-dimensional standard Wiener process. For any given initial data $(x,t)\in\mathbb{R}^n\times[0,T]$, consider the following stochastic control system:

$$\begin{aligned}dz_s&=b(z_s,v_s,s)\,ds+\sigma(z_s,v_s,s)\,dW_s,\qquad t\leqq s\leqq T,\\ z_t&=x,\end{aligned}\tag{1.2}$$

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where v_s, $0 \leqq s \leqq T$ is a U-valued adapted stochastic process called admissible control. The optimal control problem is to minimize the following functional:

$$J_{x,t}(v.) = E\left\{\int_t^T L(z_s, v_s, s)\, ds + h(z_T)\right\} \tag{1.3}$$

over admissible controls. We can define the following value function:

$$\Phi(x, t) = \inf_{v.} J_{x,t}(v.),$$

which is defined on $\mathbb{R}^n \times [0, T]$ with values in $\mathbb{R}$. It can be proved that, under some reasonable assumptions, the value function is the unique solution to (1.1). We refer to Fleming and Rishel [6], Bensoussan and Lions [3], Krylov [11], [12], and Lions [14] for details.

In (1.1), the functions $\sigma(x, v, t)$, $b(x, v, t)$, $L(x, v, t)$, and $h(x)$ are all deterministic. The objective of this paper is to study the case when σ, b, L, h are possibly randomly perturbed. A typical case is when $h = h(x, \omega)$ is $\mathscr{F}_T$ measurable. Here $\mathscr{F}_t$ is the filtration generated by the Brownian motion W_t. In this case, the corresponding value function becomes random. In fact, it is a solution of the following type of backward stochastic partial differential equation:

$$\begin{aligned} -d\Phi_t &= H(D^2\Phi, D\Phi, D\Psi, x, t)\, dt - \Psi\, dW_t, \\ \Phi(x, T) &= h(x), \end{aligned} \tag{1.4}$$

with

(1.5)

$$H(A, p, x, B, t) = \inf_{v \in U} \{\tfrac{1}{2}\mathrm{tr}[\sigma\sigma^*(x, v, t)A] + (b(x, v, t), v) + (\sigma(x, v, t), B) + L(x, v, t)\}.$$

Here for any fixed $x \in \mathbb{R}^n$, $\Phi_t(x, \omega)$ and $\Psi_t(x, \omega)$ are $\mathscr{F}_t$-adapted processes. We call this equation a stochastic HJB equation.

The main result in this paper asserts the existence and uniqueness of an adapted pair (Φ, Ψ) with values in $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ satisfying (1.4). Due to the limitations of our approach, we can only deal with the case where σ does not contain the control variable v. Some nondegeneracy assumption is also needed.

The uniqueness part follows by an application of the backward Itô rule, while the existence is obtained by solving the equation iteratively and applying Itô's rule to estimate the errors between successive estimates.

The unusual feature here is the fact that Ψ is uniquely determined although no time derivative is specified. For understanding this point, let us recall the finite-dimensional situation (see Theorem 2.1 below) in which (1.4) is replaced by

$$X = x_t + \int_t^T b(x_s, m_s, s)\, ds + \int_t^T [\sigma(x_s) + m_s]\, dW_s, \qquad 0 \leqq t \leqq T, \tag{1.6}$$

where the terminal value X is $\mathscr{F}_T$-measurable. Applying Itô's rule to the difference of two solutions $\rho_t = E(|x_t - x_t'|^2)$ yields

$$\begin{aligned} \rho_t + E\int_t^T |\sigma(x) - \sigma(x') + m - m'|^2\, ds \\ = 2E\int_t^T < b(x, m) - b(x', m'), x - x' > ds, \end{aligned}$$

$$0 \leqq t \leqq T.$$

Simple estimation then yields

$$\rho_t + \tfrac{1}{2} E \int_t^T |m - m'|^2 \leqq C \int_t^T \rho_s \, ds.$$

Now ignoring the m-term and using Gronwall's inequality yields $x = x'$, and hence $m = m'$. The same method is used to establish convergence of Picard iteration of solution (x, m). With more work the same idea lies at the basis of the existence and uniqueness problem for (1.4).

In the paper, a connection of (1.4) with an optimal control problem is described. This connection is similar to that of (1.1) with the optimal control problems (1.2) and (1.3). It should be pointed out that the function H given in (1.5) is itself allowed to be an adapted process.

As an important special case, we study the linear quadratic optimal control problem with random coefficients. A kind of matrix-valued backward stochastic differential equation, called a stochastic Riccati equation, is investigated, and a result of Bismut [4] is nontrivially generalized.

The attention to such kind of backward stochastic differential equations (SDEs) determined by a pair of unknown adapted processes was originated in the study of the stochastic maximum principle for optimal control systems, where the adjoint processes are a pair of adapted processes. This adapted pair can be characterized by a linear backward stochastic differential equation called an adjoint equation (see Bensoussan [1], Bismut [5], Haussmann [7], Kushner [13], and Peng [17], for the finite-dimensional case, and Bensoussan [2] and Hu and Peng [9] for the infinite-dimensional case). The study of forward stochastic partial differential equations can be found in Pardoux [15]. Recently, Pardoux and Peng [16] obtained an existence and uniqueness result for nonlinear backward stochastic differential equations. The infinite-dimensional case (for semi linear evolution systems) can be found in Hu and Peng [8]. The last two results have been applied in this paper. Some results of this paper were briefly announced in Peng [18].

This paper is organized as follows. In the next section, we recall some preliminary results, mainly concerning existence and uniqueness theorems for the backward stochastic differential equations. Section 3 is devoted to the stochastic control interpretation of the stochastic HJB equations and a related verification theorem. The existence and uniqueness result is discussed in § 4. In § 5, we consider linear quadratic optimal control problems with random coefficients.

2. Preliminaries. We first recall some results concerning adapted backward stochastic differential equations and a generalized Itô formula that will be used in the following.

2.1. Finite-dimensional case. Let $(\Omega, \mathscr{F}, P)$ be a probability space equipped with filtration $\mathscr{F}_t^*$. Let $\{W_t, t \geqq 0\}$ be a d-dimensional standard Wiener process defined on it. We denote

$$\mathscr{F}_t = \sigma\{W_s ; 0 \leqq s \leqq t\}.$$

For any given Hilbert (or Euclidean) space H, we will denote by $\mathscr{M}^2(H)$ the space of all $\mathscr{F}_t$-adapted processes with values in H, such that

$$E \int_0^T |x_t|_H^2 \, dt < \infty, \qquad \forall x. \in \mathscr{M}^2(H).$$

Obviously, $\mathcal{M}^2(H)$ is a Hilbert space. We denote by $(\cdot,\cdot)$ (respectively, $|\cdot|$), the scalar product (respectively, norm) of a Euclidean space. Note that the space $\mathcal{L}(H;\mathbb{R}^n)$ is also a Hilbert (respectively, Euclidean) space under the scalar product

$$(m_1, m_2) = \operatorname{tr}(m_1^* m_2), \qquad \forall m_1, m_2 \in \mathcal{L}(\mathbb{R}^n; H),$$

where we denote by $\operatorname{tr}(A)$, the trace of an $n\times n$ matrix A. Let the following functions be given:

$$\begin{aligned} &f(x, m, t, \omega): \mathbb{R}^n \times \mathcal{L}(\mathbb{R}^d; \mathbb{R}^n) \times [0, T] \times \Omega \to \mathbb{R}^n,\\ &\sigma(x, m, t, \omega): \mathbb{R}^n \times \mathcal{L}(\mathbb{R}^d; \mathbb{R}^n) \times [0, T] \times \Omega \to \mathcal{L}(\mathbb{R}^d; \mathbb{R}^n),\\ &\quad X(\omega): \Omega \to \mathbb{R}^n. \end{aligned}$$

We assume

(2.1)
- (i) for each $(x, m) \in \mathbb{R}^n \times \mathcal{L}(\mathbb{R}^d; \mathbb{R}^n)$, $f(x, m, \cdot) \in \mathcal{M}^2(\mathbb{R}^n)$ $\sigma(x, m, \cdot) \in \mathcal{M}^2(\mathcal{L}(\mathbb{R}^d; \mathbb{R}^n))$;
- (ii) for each $(t, \omega) \in [0, T] \times \Omega$, $f(x, m, t, \omega)$, $\sigma(x, m, t, \omega)$ are differentiable with respect to (x, m), and the derivatives are all bounded;
- (iii) $X(\omega)$ is $\mathcal{F}_T$-measurable, and $E|X|^2 < \infty$.

Consider the following backward stochastic differential equation:

$$X = x_t + \int_t^T f(x_s, m_s, s)\, ds + \int_t^T [\sigma(x_s) + m_s]\, dW_s. \tag{2.2}$$

Our problem is to find a pair of adapted $\mathbb{R}^n \times \mathcal{L}(\mathbb{R}^d; \mathbb{R}^n)$-valued processes (x_s, m_s), which solves equation (2.2). We have the following result.

THEOREM 2.1. *Assume* (2.1) *holds. Then, there exists a unique pair* (x_s, m_s) *in* $\mathcal{M}^2(\mathbb{R}^n) \times \mathcal{M}^2(\mathcal{L}(\mathbb{R}^d; \mathbb{R}^n))$, *which solves* (2.2). *We have*

$$E \sup_{t\in[0,T]} |x_t|^2 < \infty. \tag{2.3}$$

Moreover, if

$$K = \sup_{\omega} \left\{ |X|^2 + \int_0^T |f(0, 0, s)|^2\, ds + \int_0^T |\sigma\sigma^*(0, s)|^2\, ds \right\} < \infty, \tag{2.4}$$

then

$$|x_t|^2 \leqq K\, e^{C(T-t)}, \tag{2.5}$$

where C is a positive constant depending only on the bound of $|f_x|$, $|\sigma_x|$.

Proof. The proof of existence and uniqueness for (2.2) and (2.3) can be found in [16]. We only prove (2.5). Applying Itô's formula to (2.2), we have, for $0 \leqq r \leqq t \leqq T$,

$$E^{\mathcal{F}_r}|x_t|^2 + E^{\mathcal{F}_r}\int_t^T |(\sigma(x_s, s) + m_s)|^2\, ds = E^{\mathcal{F}_r}|X|^2 - 2E^{\mathcal{F}_r}\int_t^T (x_s, f(x_s, m_s, s))\, ds. \tag{2.6}$$

Thus

$$\begin{aligned} E^{\mathcal{F}_r}|x_t|^2 + \tfrac{1}{2}E^{\mathcal{F}_r}\int_t^T |m_s|^2\, ds \leqq{}& E^{\mathcal{F}_r}|X|^2 + E^{\mathcal{F}_r}\int_t^T |\sigma(x_s, s)|^2\, ds\\ &+ 2E^{\mathcal{F}_r}\int_t^T |x_s||f(x_s, m_s, s)\, ds. \end{aligned}$$

Since the derivatives of f, σ with respect to (x, m) are bounded,

$$\begin{aligned} 2|x||f(x, m, s)| &\leqq 2|x||f(0, 0, s)| + 2C_1|x|(|x|+|m|) \\ &\leqq |f(0,0,s)|^2 + (4C_1^2 + 2C_1 + 1)|x|^2 + \tfrac{1}{4}|m|^2, \\ |\sigma(x, s)|^2 &\leqq |\sigma(0, s)|^2 + C_1|x|^2. \end{aligned}$$

It follows that

$$\begin{aligned} &E^{\mathscr{F}_r}|x_t|^2 + \tfrac{1}{4}E^{\mathscr{F}_r}\int_t^T |m_s|^2\,ds \\ &\leqq E^{\mathscr{F}_r}|X|^2 + E^{\mathscr{F}_r}\int_t^T (|f(0,0,s)|^2 + |\sigma(0,s)|^2 + (4C_1^2+3C_1+1)|x|^2)\,ds \\ &\leqq K + \int_t^T + (4C_1^2+3C_1+1)E^{\mathscr{F}_r}|x|^2)\,ds. \end{aligned}$$

Applying Gronwall's inequality, we obtain finally

$$E^{\mathscr{F}_r}|x_t|^2 \leqq K\,e^{C(T-t)}, \qquad \forall 0 \leqq r \leqq t \leqq T,$$

with $C = 4C_1^2 + 3C_1 + 1$. This implies (2.5). $\square$

2.2. Infinite-dimensional case. Let V, H be two separable Hilbert spaces such that V is densely embedded in H. We identify H with its dual space, and denote by V' the dual of V. We have then

$$V \subset H \subset V'.$$

We will denote by $\|\cdot\|$, $|\cdot|$, $\|\cdot\|_*$ the norms of V, H, and V', respectively; by $\langle\cdot,\cdot\rangle$ the duality product between V and V', and by $(\cdot,\cdot)$ the scalar product in H. For any $\varphi \in \mathscr{M}^2(\mathscr{L}(\mathbb{R}^n; H))$, we can define an H-valued random variable, called stochastic integral $\int_0^t \varphi_s\,dW_s$ by (see [15])

$$\left(h, \int_0^t \varphi_s\,dW_s\right) = \int_0^t (\varphi_s^* h)\,dW_s, \qquad \forall h \in H.$$

Let the following functions be given:

$$\begin{aligned} A(t,\omega) &: [0,T]\times\Omega \to \mathscr{L}(V; V'), \\ f(y,t,\omega) &: H\times[0,T]\times\Omega \to V', \\ g(y,z,t,\omega) &: V\times\mathscr{L}(\mathbb{R}^d; H)\times[0,T]\times\Omega \to H, \\ \eta(y,t,\omega) &: H\times[0,T]\times\Omega \to \mathscr{L}(\mathbb{R}^d; H), \\ Y(\omega) &: \Omega \to H. \end{aligned}$$

We assume the following:

$$\begin{aligned} &\text{For each } (y,z),\ A(t), f(y,t), g(y,z,t), \eta(y,t) \text{ are } \mathscr{F}_t\text{-adapted,} \\ &\text{such that, } f(0,t)\in\mathscr{M}^2(V'),\ \eta(0,t)\in\mathscr{M}^2(\mathscr{L}(\mathbb{R}^d; H)), \\ &g(0,0,t)\in\mathscr{M}^2(H),\ A(t)\in\mathscr{M}^2(\mathscr{L}(V,V'), \\ &\sup\nolimits_{t,\omega}\|A(t,\omega)\|_{\mathscr{L}(V,V')} \leqq C; \end{aligned} \tag{2.7}$$

$$\begin{aligned} &\exists\,\alpha > 0 \text{ and } \lambda, \text{ such that, } \forall y \in V, \\ &\langle A(t)y, y\rangle + \lambda|y|^2 \geqq \alpha\|y\|^2, \qquad \forall t; \end{aligned} \tag{2.8}$$

$$
(2.9)\quad
\begin{aligned}
&|g(y_1, z_1, t) - g(y_2, z_2, t)| \leqq C(\|y_1 - y_2\| + |z_1 - z_2|),\\
&\quad \forall (y_1, z_1), (y_2, z_2) \in (V \times \mathscr{L}(\mathbb{R}^d; H)),\\
&|\eta(y_1, t) - \eta(y_2, t)| \leqq C|y_1 - y_2|, \qquad \forall y_1, y_2 \in H,\\
&|\langle f(y_1, t) - f(y_2, t), y\rangle| \leqq C|y_1 - y_2| \cdot \|y\|, \qquad \forall (y_1, y_2) \in H, \quad y \in V.
\end{aligned}
$$

Consider the following semilinear backward stochastic equation:

$$
(2.10)\quad Y = y_t + \int_t^T (A(s)y_s + f(y_s, s) + g(y_s, z_s, s))\, ds + \int_t^T (\eta(y_s, s) + z_s)\, dW_s.
$$

The problem is to find a pair of adapted processes (y, z) satisfying the above equation. We have the following theorem (see [8] for proof).

THEOREM 2.2. *We assume* (2.7)-(2.9). *Then there exists a unique pair* $(y., z.) \in \mathscr{M}^2(V) \times \mathscr{M}^2(\mathscr{L}(\mathbb{R}^d; H))$, *satisfying the backward evolution equation* (2.10).

Remark. We can also obtain an estimate similar to (2.5).

We need also the following generalized Itô's formula due to Kunita [10].

THEOREM 2.3. *Let $F_t(x)$, $(x, t) \in \mathbb{R}^n \times [0, T]$ be a random field which is continuous in (x, t) almost surely, satisfying*

(i) *For each t, $F_t(\cdot)$ is a C^2-map from $\mathbb{R}^n$ into R almost surely*;

(ii) *For each x, $F_t(x)$ is a continuous semimartingale and it satisfies*

$$
F_t(x) = F_0(x) + \sum_{j=1}^m \int_0^t f_s^j(x)\, dY_s^j, \qquad \forall x \in \mathbb{R}^n,
$$

almost surely, where Y_s^j, $j = 1, \cdots, d$ are continuous semimartingales, $f_s^j(x)$, $x \in \mathbb{R}^n$, $s \in [0, T]$, are random fields that are continuous in (x, s) and satisfy

(a) *for each s, $f_s^j(x)$ is a C^2-map from $\mathbb{R}^n$ to R,*

(b) *for each x, $f_s^j(x)$ is an adapted process.*

Let $X_t = (X_t^1, \cdots, X_t^n)$ be continuous semimartingales. Then we have

$$
(2.11)\quad
\begin{aligned}
F_t(X_t) = F_0(X_0) &+ \sum_{j=1}^m \int_0^t f_s^j(X_s)\, dY_s^j + \sum_{i=1}^n \int_0^t \partial_{x_i} F_s(X_s)\, dX_s^i\\
&+ \sum_{j=1}^m \sum_{i=1}^n \int_0^t \partial_{x_i} f_s^j(X_s)\, d\langle Y^j, X^i\rangle_s\\
&+ \sum_{i,j=1}^n \int_0^t \partial_{x_i x_j} F_s(X_s)\, d\langle X^i, X^j\rangle_s,
\end{aligned}
$$

where $\langle \cdot, \cdot\rangle_s$ stands for the quadradic variation of semimartingales.

3. Optimal control formulation of the stochastic HJB equation. We begin by considering a stochastic optimal control system in which all coefficients may be also stochastic processes. This can lead us to formulate what we call stochastic HJB equations. We present a general form of such equations. The discussion is formal because, in general, we have no sufficient regularity properties of the value function. We will also discuss the so-called "verification theorem" approach in which the rigorous proof can be easily given.

3.1. Optimal control system. Let $\{W_t^1; t \geqq 0\}$ be a p-dimensional standard Wiener process in $(\Omega, \mathscr{F}, P)$ which is independent of $\{W_t; t \geqq 0\}$. Without loss of generality, we consider only the case where W_t is a one-dimensional Brownian motion, i.e., $d = 1$. We assume

$$
\mathscr{F}_t^* = \sigma\{W_s, W_s^1; 0 \leqq s \leqq t\}.
$$

Consider the following stochastic control system parametrized by the initial data $(x, t) \in \mathbb{R}^n \times [0, T]$:

(3.1) $\quad dy_s = b(y_s, v_s, s)\, ds + \sigma(y_s, v_s, s)\, dW_s + \pi(y_s, v_s, s)\, dW_s^1, \qquad t \leqq s \leqq T, \quad y_t = x,$

where

$$b(x, v, t) : \mathbb{R}^n \times \mathbb{R}^k \times [0, T] \to \mathbb{R}^n,$$
$$\sigma(x, v, t) : \mathbb{R}^n \times \mathbb{R}^k \times [0, T] \to \mathbb{R}^n,$$
$$\pi(x, v, t) : \mathbb{R}^n \times \mathbb{R}^k \times [0, T] \to \mathscr{L}(\mathbb{R}^p; \mathbb{R}^n).$$

An admissible control is an $\mathscr{F}_t^*$-adapted process v_t with values in U, such that

$$E \int_0^T |v_t|^2\, dt < \infty.$$

We denote the set of admissible controls by $\mathscr{U}$. By analogy with the classical optimal control problem, our problem is, for a given initial data $y_0 = x_0$ in (3.1), to find an admissible control v_t which minimizes the following cost function:

$$E\left\{\int_0^T L(y_s, v_s, s)\, ds + h(y_T)\right\}, \tag{3.2}$$

where we denote

$$\begin{aligned} L(x, v, t) &: \mathbb{R}^n \times \mathbb{R}^k \times [0, T] \to R, \\ h(x, \omega) &: \mathbb{R}^n \times \Omega \to R. \end{aligned} \tag{3.3}$$

The main difference between the classical optimal control problem and the above one is that h is a random variable. We assume that $h(x)$ is $\mathscr{F}_T$-measurable.

Remark. We can also treat the case where σ, π, b, and L are $\mathscr{F}_t$-adapted random processes. The approach and the conclusion are the same.

Following the idea of dynamic programming, for any fixed initial data (x, t) in (3.1), we minimize the following cost function over admissible controls:

$$J_{x,t}(v.) = E^{\mathscr{F}_t} \int_t^T L(y_s, v_s, s)\, ds + h(y_T). \tag{3.4}$$

The value function is defined as follows:

$$\Phi_t(x) = \inf_{v. \in \mathscr{U}} J_x, t(v.).$$

Observe that, for any fixed x, Φ_t is an $\mathscr{F}_t$-adapted process with values in $\mathbb{R}$. In general, Φ_t is not a bounded variation function with respect to t. In fact, we can only expect that Φ_t is a semimartingale

$$\Phi_t(x) = \int_t^T \Gamma_s(x)\, ds - \int_t^T \Psi_s(x)\, dW_s, \qquad x \in \mathbb{R}^n, \qquad 0 \leqq t \leqq T, \tag{3.5}$$

where, for each $x \in \mathbb{R}^n$ and $\Gamma_s(x)$, $\Psi_s(x)$ are $\mathscr{F}_s$-adapted real processes.

If $\Phi_t(x)$ can be expressed in the form (3.5), and if $\Phi_t(x), \Gamma_t(x), \Psi_t(x)$ are almost surely continuous in (x, t) and are smooth enough with respect to x, then we can prove that the pair $(\Phi_t(x), \Psi_t(x))$ satisfies the following stochastic partial differential equation:

$$-d\Phi_t = H(D^2\Phi, D\Phi, D\Psi, x, t)\, dt - \Psi\, dW_t, \qquad \Phi(x, T) = h(x), \tag{3.6}$$

or

$$\Phi_t = h(x) + \int_t^T H(D^2\Phi, D\Phi, D\Psi, x, s)\, ds - \int_t^T \Psi\, dW_s, \tag{3.7}$$

with

$$H(A, p, x, B, t) = \inf_{v\in U} \{\operatorname{tr}[a(x, v, t)A] + (b(x, v, t), p) + (\sigma(x, v, t), B) + L(x, v, t)\},$$

$$A \in \mathbb{R}^{n\times n}, \quad B \in \mathbb{R}^n,$$

where $D\Phi$ is the gradient of Φ, $D^2\Phi$ is the Hessian of Φ, and

$$a(x, v, t) = \tfrac{1}{2}[\sigma\sigma^*(x, v, t) + \pi\pi^*(x, v, t)].$$

Equation (3.6) or (3.7) is called a stochastic HJB equation. Comparing (3.6) with the classical HJB equation, which is a deterministic partial differential equation, an interesting feature of this equation is that its solution consists of a pair (Φ_t, Ψ_t).

The following procedure can verify that (Φ_t, Ψ_t) as in (3.5) satisfies (3.7). First, similar to the classical case, we introduce Bellman's optimality principle

$$\begin{aligned}\Phi_t(x) &= \inf_{v.\in\mathcal{U}} E^{\mathcal{F}_t}\left\{\int_t^{t+r} L(y_s, v_s, s)\, ds + E^{\mathcal{F}_{t+r}}\left(\int_{t+r}^T L(y_s, v_s, s)\, ds + h(y_T)\right)\right\}\\ &= \inf_{v.\in\mathcal{U}} E^{\mathcal{F}_t}\left\{\int_t^{t+r} L(y_s, v_s, s)\, ds + \Phi_{t+r}(y_{t+r})\right\}.\end{aligned} \tag{3.8}$$

Then, applying Itô-Kunita's formula (2.11) to the process $\Phi_s(y_s)$ yields

$$\begin{aligned}\Phi_{t+r}(y_{t+r}) = \Phi_t(x) + E^{\mathcal{F}_t}\int_t^{t+r} &[\operatorname{tr}(a(y_s, v_s, s)D^2\Phi_s(y_s)) + (b(y_s, v_s, s), D\Phi_s(y_s))\\ &+\operatorname{tr}(\sigma(y_s, v_s, s)D\Psi_s(y_s)) + \Gamma_s(y_s)]\, ds.\end{aligned}$$

This with (3.8) implies

$$\begin{aligned}r^{-1}\inf_{v.\in\mathcal{U}} E^{\mathcal{F}_t}\int_t^{t+r} &[L(y_s, v_s, s) + \operatorname{tr}(a(y_s, v_s, s)D^2\Phi_s(y_s)) + (b(y_s, v_s, s), D\Phi_s(y_s))\\ &+(\sigma(y_s, v_s, s), D\Psi_s(y_s)) + \Gamma_s(y_s)]\, ds = 0.\end{aligned}$$

Passing limit as $r \to 0$, we have

$$\Gamma_t(x) = -H(D^2\Phi_t(x), D\Phi_t(x), D\Psi_t(x), x, t).$$

Substituting it into (3.5) yields (3.7).

Unfortunately, as in the classical case, even when the coefficients b, σ, π, L, h are sufficiently regular with respect to (x, v), it is still difficult to verify the regularities of $(\Phi_t(x), \Psi_t(x))$.

3.2. Verification theorem approach. We now show that a sufficiently smooth solution of (3.7) coincides with the value function. In the classical case, this approach is called the verification theorem approach (see [6]).

Indeed, let a sufficiently smooth pair $(\varphi_t(x), \psi_t(x))$ be a solution of (3.7). Assume

(i) For each t, $(\varphi_t(x), \psi_t(x))$ is a C^2-map from $\mathbb{R}^n$ into $\mathbb{R}^1\times\mathbb{R}^d$.

(ii) For each x, $(\varphi_t(x), \psi_t(x))$, $(D\varphi_t(x), D^2\varphi_t(x), D\psi_t(x))$ are continuous $\mathcal{F}_t$-adapted processes.

Without loss of generality, let $u_t(x)$ be a random field that is $\mathscr{F}_1 \times \mathscr{B}(\mathbb{R}^n)/\mathscr{B}(\mathbb{R}^k)$-measurable with values in $\mathscr{U}$, such that

$$\operatorname{tr}[a(x, u_t, t)D^2\varphi_t] + (b(x, u_t, t), D\varphi_t) + (\sigma(x, u_t, t), D\psi_t) + L(x, u_t, t)$$
$$= H(D^2\varphi_t(x), D\varphi_t(x), D\psi_t(x), x, t).$$

Furthermore, suppose that $u_t(x)$ is regular enough such that the "feedback" control system

$$dy_s = b(y_s, u_s(y_s), s)\, ds + \sigma(y_s, u_s(y_s), s)\, dW_s + \pi(y_s, u_s(y_s), s)\, dW_s^1,$$
$$t \leqq s \leqq T, \qquad y_t = x,$$

is well posed. Finally, suppose that $H(D^2\varphi_t(x), D\varphi_t(x), D\psi_t(x), x, t)$ is smooth with respect to (x, t). Then $(\varphi_t(x), \psi_t(x))$ coincides with $(\Phi_t(x), \Psi_t(x))$. Furthermore, for any initial data $(x_0, 0)$, $u_s^* = u_s(y_s)$ minimize the cost function (3.2).

Indeed, let v_s be an admissible control, and let z_s be the corresponding solution

$$dz_s = b(z_s, v_s, s)\, ds + \sigma(z_s, v_s, s)\, dW_s + \pi(z_s, v_s, s)\, dW_s^1, \qquad t \leqq s \leqq T, \qquad z_t = x.$$

From Itô-Kunita's formula,

$$E^{\mathscr{F}_t}\varphi_T(z_T) = \varphi_t(x) + E^{\mathscr{F}_t}\int_t^T [\operatorname{tr}(a(z_s, v_s, s)D^2\varphi_s(z_s)) + (b(z_s, v_s, s), D\varphi_s(z_s))$$
$$+ (\sigma(z_s, v_s, s), D\psi_s(z_s)) - H(D^2\varphi_s(z_s), D\varphi_s(z_s), D\psi_s(z_s), z_s, s)]\, ds.$$

This yields

$$\varphi_t(x) \leqq E^{\mathscr{F}_t}\left\{\int_t^T L(z_s, v_s, s)\, ds + h(z_T)\right\}$$
$$= J_{x,t}(v.). \tag{3.9}$$

On the other hand, if we take the above $u_s^* = u_s(y_s)$ as a control, then, again from Itô-Kunita's formula,

$$E^{\mathscr{F}_t}\varphi_T(y_T) = \varphi_t(x) + E^{\mathscr{F}_t}\int_t^T [\operatorname{tr}(a(y_s, u_s^*, s)D^2\varphi_s(y_s)) + (b(y_s, u_s^*, s), D\varphi_s(y_s))$$
$$+ (\sigma(y_s, u_s^*, s), D\psi_s(y_s)) - H(D^2\varphi_s(y_s), D\varphi_s(y_s), D\psi_s(y_s), y_s, s)]\, ds.$$

It follows from the definition of u_s^* that

$$\varphi_t(x) = E^{\mathscr{F}_t}\left\{\int_t^T L(y_s, u_s^*, s)\, ds + h(y_T)\right\}$$
$$= J_{x,t}(u.^*).$$

This with (3.9) implies

$$\varphi_t(x) = \inf_{v. \in \mathscr{U}} J_{x,t}(v.) = \Phi_t(x).$$

Consequently, $\psi_t(x) = \Psi_t(x)$. The last assertion is easy to see since

$$E\left\{\int_0^T L(y_s, v_s, s)\, ds + h(y_T)\right\} = EE^{\mathscr{F}_0}\left\{\int_0^T L(y_s, v_s, s)\, ds + h(y_T)\right\}.$$

4. Existence and uniqueness. An existence and uniqueness theorem for the stochastic HJB equation (3.7), in terms of Sobolev space technique, is given in this section. We can only treat the case where no control variable appears in the diffusion coefficients σ and π. We also need a nondegeneracy assumption. The corresponding control system (3.1) becomes

$$dy_s = b(y_s, v_s, s)\,ds + \sigma(y_s, s)\,dW_s + \pi(y_s, s)\,dW_s^1, \qquad t \leqq s \leqq T, \qquad y_t = x, \tag{4.1}$$

with the following cost function to be minimized:

$$J_{x,t}(v.) = E^{\mathscr{F}_t}\left\{\int_t^T L(y_s, v_s, s)\,ds + h(y_T)\right\}. \tag{4.2}$$

We assume that

(4.3)
- (i) For each t, $b(x, v, t)$, $\sigma(x, t)$, $\pi(x, t)$, $L(x, v, t)$, $h(x)$ are continuous in (x, v);
- (ii) For each (v, t), they are continuously differentiable in x;
- (iii) They and all their derivatives in x are bounded;
- (iv) For each (x, v), they are continuous in t.

We also need the following nondegeneracy assumption on π:

$$\pi\pi^*(x, t) \geqq 2\alpha I, \qquad \forall (x, t), \tag{4.4}$$

where α is a positive constant.

Remark. Technically, the above nondegeneracy assumption will be used to satisfy (2.8). In general, if we have no such kind of condition, some kind of notion like "viscosity solution" seems necessary. A typical case is when h is deterministic and $\sigma = \pi = 0$. In this case the most suitable notation is the viscosity solution (see [14]).

Our problem is to find a unique pair $(\Phi_t(x), \Psi_t(x))$ satisfying (3.7), where the function H becomes

$$a(x, t) = \tfrac{1}{2}(\sigma\sigma^*(x, t) + \pi\pi^*(x, t)),$$

$$H(A, p, B, x, t) = \operatorname{tr}(a(x, t)A) + (\sigma(x, t), B) + \inf_{v \in U}\{(b(x, v, t), p) + L(x, v, t)\}$$

$$\forall x \in \mathbb{R}^n, \qquad t \in [0, T], \qquad A \in S^n, \qquad B \in \mathbb{R}^n.$$

We introduce the following Sobolev space:

$$H^1(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) : Du \in L^2(\mathbb{R}^n)\}.$$

We identity $L^2(\mathbb{R}^n)$ with its dual space, and denote the dual space of $H^1(\mathbb{R}^n)$ by $H^{-1}(\mathbb{R}^n)$. This gives us a triple as in § 2.2

$$(H^1(\mathbb{R}^n), L^2(\mathbb{R}^n), H^{-1}(\mathbb{R}^n)) = (V, H, V').$$

We can make the following assertion.

THEOREM 4.1. *Assume that* (4.3) *and* (4.4) *hold. Then there exists a unique pair* $(\Phi_t(x), \Psi_t(x))$ *in* $(\mathcal{M}^2(V), \mathcal{M}^2(H))$ *satisfying the stochastic* HJB *equation* (3.7).

To prove this theorem, we need a lemma that generalizes Theorem 2.2. Let $g(y, z, t)$, Y, $A(t)$ be as in Theorem 2.2. Let

$$A_1(t, \omega) : [0, T] \times \Omega \to \mathscr{L}(V; V'),$$

$$f(y, z, t, \omega) : H \times H \times [0, T] \times \Omega \to V'.$$

We assume that the following are true:

(4.5) For each (y, z), $A_1(t)$ and $f(y, z, t)$ are $\mathscr{F}_t$-adapted, $f(0, 0, t) \in \mathscr{M}^2(V')$;

$$\sup_{t,\omega} \|A_1(t, \omega)\|_{\mathscr{L}(V,V')} \leqq C, \qquad \langle A_1(t)y, y\rangle \geqq 0, \qquad \forall y \in V; \tag{4.6}$$

$$|\langle f(y_1, z_1, t) - f(y_2, z_2, t), y\rangle| \leqq C|y_1 - y_2|_H |y|_V + |z_1 - z_2|_H \langle 2A_1(t)y, y\rangle^{1/2}, \tag{4.7}$$
$$\text{for each } (y_1, z_1), (y_2, z_2) \in H \times H, \qquad \text{for each } y \in V.$$

Consider the following semilinear backward stochastic equation:

$$Y = y_t + \int_t^T (A(s)y_s + A_1(s)y_s + f(y_s, z_s, s) + g(y_s, z_s, s))\, ds + \int_t^T z_s\, dW_s. \tag{4.8}$$

We have the following lemma.

LEMMA 4.2. *Let $g(y, z, t)$, Y, $A(t)$ satisfy the assumptions of Theorem 2.2. Let (4.5)-(4.7) hold. Then, there exists a unique pair $(y_., z_.) \in \mathscr{M}^2(V) \times \mathscr{M}^2(H)$, satisfying the backward evolution equation (4.8).*

Proof. (i). *Uniqueness.* Let (y_s^1, z_s^1), (y_s^2, z_s^2) be two solutions of (4.8). From Itô's formula, we can derive

$$E|y_t^1 - y_t^2|^2 + 2E\int_t^T \langle (A(s) + A_1(s))(y_s^1 - y_s^2), y_s^1 - y_s^2\rangle + E\int_t^T |z_s^1 - z_s^2|^2\, ds$$
$$= -2E\int_t^T (g(y_s^1, z_s^1, s) - g(y_s^2, z_s^2, s), y_s^1 - y_s^2)\, ds$$
$$-2E\int_t^T \langle f(y_s^1, z_s^1, s) - f(y_s^2, z_s^2, s), y_s^1 - y_s^2\rangle\, ds.$$

Thus, by (2.8), (2.9), and (4.7), we obtain

$$E|y_t^1 - y_t^2|^2 + 2E\int_t^T \langle (A_1(s))(y_s^1 - y_s^2), y_s^1 - y_s^2\rangle\, ds + 2\alpha E\int_t^T \|y_s^1 - y_s^2\|^2\, ds$$
$$+ E\int_t^T |z_s^1 - z_s^2|^2\, ds$$
$$\leqq 2\lambda E\int_t^T |y_s^1 - y_s^2|^2\, ds + 2CE\int_t^T (|y_s^1 - y_s^2|^2 + |y_s^1 - y_s^2||z_s^1 - z_s^2|)\, ds$$
$$+2E\int_t^T [C|y_s^1 - y_s^2|\,\|y_s^1 - y_s^2\| + |z_s^1 - z_s^2|\langle (2A_1(s))(y_s^1 - y_s^2), y_s^1 - y_s^2\rangle^{1/2}]\, ds.$$

Since $\|A_1(s)\|_{\mathscr{L}(V,V')} \leqq C$, we can choose a small constant δ with $1 > 2\delta > 0$ such that

$$\frac{\delta}{1-2\delta}\langle 2A_1(s)y, y\rangle \leqq \frac{\alpha}{2}\|y\|^2, \qquad \forall y \in V.$$

Also, we have

$$2C|y^1 - y^2|\,\|y^1 - y^2\| < \frac{\alpha}{2}\|y^1 - y^2\|^2 + \frac{2}{\alpha}C^2|y^1 - y^2|^2$$

and

$$2|z^1-z^2|\langle 2A_1(s)(y^1-y^2), y^1-y^2\rangle^{1/2}$$
$$\leqq (1-2\delta)|z^1-z^2|^2+\frac{1}{1-2\delta}\langle 2A_1(s)(y^1-y^2), y^1-y^2\rangle,$$
$$2C|y^1-y^2||z^1-z^2|\leqq C\delta^{-1}|y^1-y^2|^2+\delta|z^1-z^2|^2.$$

It follows that

$$E|y_t^1-y_t^2|^2+\frac{\alpha}{2}E\int_t^T \|y_s^1-y_s^2\|^2\,ds+\delta E\int_t^T |z_s^1-z_s^2|^2\,ds$$
$$\leqq C_1E\int_t^T |y_s^1-y_s^2|^2\,ds, \qquad \forall t\in[0,T],$$

with $C_1=2\lambda+2C+4\alpha^{-1}C^2+\delta^{-1}C^2$. Thus we can apply Gronwall's inequality to obtain $y^1=y^2$, $z^1=z^2$.

(ii) *Existence.* Set $z_t^0=0$. According to Theorem 2.2, we can alternatively solve the following equation:

$$Y=y_t^j+\int_t^T (A(s)y_s^j+A_1(s)y_s^j+f(y_s^j, z_s^{j-1}, s)+g(y_s^j, z_s^j, s))\,ds$$
$$+\int_t^T z_s^j\,dW_s, \qquad j=1,2,\cdots. \tag{4.9}$$

By Itô's formula, we have

$$E|y_t^{j+1}-y_t^j|^2+2E\int_t^T \langle (A(s)+A_1(s))(y_s^{j+1}-y_s^j), y_s^{j+1}-y_s^j\rangle$$
$$+E\int_t^T |z_s^{j+1}-z_s^j|^2\,ds$$
$$=-2E\int_t^T (g(y_s^{j+1}, z_s^{j+1}, s)-g(y_s^j, z_s^j, s), y_s^{j+1}-y_s^j)\,ds$$
$$-2E\int_t^T \langle f(y_s^{j+1}, z_s^j, s)-f(y_s^j, z_s^{j-1}, s), y_s^{j+1}-y_s^j\rangle\,ds.$$

Again, from (2.8), (2.9), and (4.7), we have

$$E|y_t^{j+1}-y_t^j|^2+2E\int_t^T \langle A_1(s)(y_s^{j+1}-y_s^j), y_s^{j+1}-y_s^j\rangle\,ds$$
$$+2\alpha E\int_t^T \|y_t^{j+1}-y_t^j\|^2\,dt+E\int_t^T |z_s^{j+1}-z_s^j|^2\,ds$$
$$\leqq CE\int_t^T (2\|y_t^{j+1}-y_t^j\|+|z_s^{j+1}-z_s^j|)|y_s^{j+1}-y_s^j\,ds$$
$$+(2\lambda+2C)E\int_t^T |y_s^{j+1}-y_s^j|^2\,ds$$
$$+2E\int_t^T |z_s^j-z_s^{j-1}|\langle 2A_1(s)(y_s^{j+1}-y_s^j), y_s^{j+1}-y_s^j\rangle^{1/2}\,ds.$$

Taking δ as in the proof of the uniqueness and using the similar technique yield

$$
(4.10)\qquad \begin{aligned} &E|y_t^{j+1}-y_t^j|^2+\frac{\alpha}{2}E\int_t^T \|y_s^{j+1}-y_s^j\|^2\,dt+(1-\delta)E\int_t^T |z_s^{j+1}-z_s^j|^2\,ds\\ &\leqq C_2E\int_t^T |y_s^{j+1}-y_s^j|^2\,ds+(1-2\delta)E\int_t^T |z_s^j-z_s^{j-1}|^2\,ds,\end{aligned}
$$

where the constant C_2 is the same as in the proof of the uniqueness. Define

$$
Y_s^j=E\int_t^T |y_s^{j+1}-y_s^j|^2\,ds,\qquad j=1,2,\cdots,
$$

$$
Z_s^j=E\int_t^T |z_s^{j+1}-z_s^j|^2\,ds,\qquad j=0,1,2,\cdots.
$$

It follows from (4.10) that

$$
(4.11)\qquad -\frac{d}{dt}Y_t^j+(1-\delta)Z_t^j\leqq C_2Y_t^j+(1-2\delta)Z_t^{j-1}\qquad Y_0^j=0,\qquad j=1,2,\cdots,
$$

or

$$
-\frac{d}{dt}(Y_t^j\,e^{C_2t})+(1-\delta)\,e^{C_2t}Z_t^j\leqq(1-2\delta)\,e^{C_2t}Z_t^{j-1}\qquad Y_0^j=0,\qquad j=1,2,\cdots.
$$

Integrating from t to T yields

$$
(4.12)\qquad Y_t^j+(1-\delta)\int_t^T e^{C_2(s-t)}Z_s^j\,ds\leqq(1-2\delta)\int_t^T e^{C_2(s-t)}Z_s^{j-1}\,ds.
$$

It follows, in particular, that

$$
\int_0^T e^{C_2s}Z_s^j\,ds\leqq\left(\frac{1-2\delta}{1-\delta}\right)^jK,\qquad j=1,2,\cdots,
$$

with $K=E\int_0^T|z_s^1|^2\,ds$; also then

$$
(4.13)\qquad Y_t^j\leqq Y_0^j\leqq\left(\frac{1-2\delta)}{1-\delta}\right)^jK,\qquad j=1,2,\cdots.
$$

From (4.11) and the fact that $dY_t^j\,dt\leqq 0$, we derive that

$$
Z_0^j\leqq K_1\left(\frac{1-2\delta}{1-\delta}\right)^j+\frac{(1-2\delta)}{(1-\delta)}Z_0^{j-1},\qquad j=1,2,\cdots,
$$

with $K_1=C_2K(1-\delta)^{-1}$. This implies

$$
Z_0^j=E\int_0^T |z_s^{j+1}-z_s^j|^2\,ds\leqq\left(\frac{1-2\delta}{1-\delta}\right)^j(jK_1+Z_0^0),\qquad j=1,2,\cdots.
$$

It turns out that $\{(y^j,z^j)\}$ is a Cauchy sequence in $\mathcal{M}^2(H)\times\mathcal{M}^2(H)$. From (4.10), $\{y^j\}$ is also a Cauchy sequence in $\mathcal{M}^2(V)$ and $L^2(\Omega, C(0,T;H))$. Passing limit in (4.9) as $j\to\infty$, we obtain that the pair (y_t,z_t) defined by

$$
y=\lim_{j\to\infty}y^j\ \text{ in }\mathcal{M}^2(V),
$$

$$
z=\lim_{j\to\infty}z^j\ \text{ in }\mathcal{M}^2(H)
$$

solves (4.8). $\square$

We now proceed to the proof of Theorem 4.1.

Proof of Theorem 4.1. Set

$$\langle A_1(t)\varphi_1, \varphi\rangle = \frac{1}{2}\int_{R^n} (\sigma\sigma^*(x, t)D\varphi_1, D\varphi)\,dx, \qquad \forall \varphi_1, \varphi \in V,$$

$$\langle A(t)\varphi_1, \varphi\rangle = \frac{1}{2}\int_{R^n} (\pi\pi^*(x, t)D\varphi_1, D\varphi)\,dx, \qquad \forall \varphi_1, \varphi \in V,$$

$$\langle f(\psi, t), \varphi\rangle = \int_{R^n} (\psi(x), \sigma(x, t)D\varphi)\,dx, \qquad \forall \psi \in H, \varphi \in V,$$

$$\begin{aligned} g(\varphi, \psi, t) = {} & \frac{1}{2}\sum_{i,j=1}^{n} \partial_{x_i}(\sigma\sigma^*)_{ij}\,\partial_{x_j}\varphi \\ & + \frac{1}{2}\sum_{i,j=1}^{n} \partial_{x_i}(\pi\pi^*)_{ij}\,\partial_{x_j}\varphi - \psi\sum_{i=1}^{n}\partial_{x_i}\sigma_i(x, t) \\ & + \inf_{v\in U}\{(b(x, v, t), D\varphi(x)) + L(x, v, t)\}, \qquad \forall \varphi \in V, \end{aligned}$$

$$\eta = 0, \qquad Y(\omega) = h(x, \omega).$$

It is easy to see that

$$\begin{gathered} A_1(t), A_2(t) : [0, T] \to \mathscr{L}(V; V'), \\ f(\psi, t) : H \times [0, T] \to V', \\ g(\varphi, \psi, t) : V \times H \times [0, T] \to H, \\ Y(\omega) : \Omega \to H. \end{gathered}$$

Thus, for each $w, \varphi \in V, \psi \in H$, we have that

$$\langle (A(s) + A_1(s))\varphi + f(\psi, s) + g(\varphi, \psi, s), w\rangle = \int_{R^n} H(D^2\varphi, D\varphi, D\psi, x, t)w(x)\,dx.$$

With the above notations, we can write the stochastic HJB equation (3.7) in the form of (4.8).

It remains to check that the conditions of Lemma 4.2 are satisfied. We will only verify (4.6) and (4.7) because the other conditions are easy to check. First, for (4.6), we need to verify

$$\begin{aligned} \int_{R^n} \Big| & \inf_{v\in U}\{(b(x, v), D\varphi_1(x)) + L(x, v)\} \\ & - \inf_{v\in U}\{(b(x, v), D\varphi_2(x)) + L(x, v)\}\Big|^2\,dx\| \leqq C\|\varphi_2 - \varphi_1\|_V^2. \end{aligned} \tag{4.14}$$

For any $\varepsilon > 0$ and $\varphi_1, \varphi_2 \in V$, we can choose two measurable functions $v_l^\varepsilon(x) : \mathbb{R}^n \to U$, $l = 1, 2$, such that

$$\begin{aligned} & \int_{R^n} |(b(x, v_l^\varepsilon, t), D\varphi_l(x)) + L(x, v_l^\varepsilon, t) - \inf_{v\in U}\{(b(x, v, t), D\varphi_l(x)) + L(x, v, t)\}|^2 \\ & \quad = \int_{R^n} |\delta_l^\varepsilon(x)|^2\,dx \leqq \varepsilon, \qquad l = 1, 2. \end{aligned}$$

For each $x\in\mathbb{R}^n$, we have

$$\inf_{v\in U}\{(b(x,v,t),D\varphi_1(x))+L(x,v,t)\}-\inf_{v\in U}\{b(x,v,t)D\varphi_2(x)+L(x,v,t)\}$$
$$\geqq(b(x,v_1^\varepsilon,t),D\varphi_1(x))+L(x,v_1^\varepsilon,t)-(b(x,v_1^\varepsilon,t),D\varphi_2(x))-L(x,v_1^\varepsilon,t)-|\delta_1^\varepsilon(x)|$$
$$\geqq -C|D\varphi_1(x)-D\varphi_2(x)|-|\delta_1^\varepsilon(x)|.$$

Similarly,

$$\inf_{v\in U}\{(b(x,v,t),D\varphi_2(x))+L(x,v,t)\}-\inf_{v\in U}\{(b(x,v,t),D\varphi_1(x))+L(x,v,t)\}$$
$$\geqq -C|D\varphi_1(x)-D\varphi_2(x)|-|\delta_2^\varepsilon(x)|.$$

The above two inequalities imply

$$\int_{R^n}\left|\inf_{v\in U}\{(b(x,v,t),D\varphi_1(x))+L(x,v,t)\}-\inf_{v\in U}\{(b(x,v,t),D\varphi_2(x))+L(x,v,t)\}\right|^2 dx$$
$$\leqq C\|\varphi_1-\varphi_2\|_V^2+|\delta_1^\varepsilon|_H^2+|\delta_2^\varepsilon|_H^2$$
$$\leqq C\|\varphi_1-\varphi_2\|_V^2+2\varepsilon.$$

Let $\varepsilon\to 0$, we obtain (4.14).

Finally, the last Lipschitz condition in (2.9) can be derived as follows. For any $\psi_1,\psi_2\in H$, $\varphi\in V$, $t\in[0,T]$, we have

$$\begin{aligned}|\langle f(\psi_1,t)-f(\psi_2,t),\varphi\rangle| &= \left|\int_{R^n}(\psi_1-\psi_2,\sigma^*(x,t)D\varphi)\,dx\right|\\
&\leqq |\psi_1-\psi_2|_H|\sigma^*D\varphi|_H\\
&= |\psi_1-\psi_2|_H\left[\int_{R^n}(\sigma\sigma^*(x,t)D\varphi,D\varphi)\,dx\right]^{1/2}\\
&= |\psi_1-\psi_2|_H\langle A_1(t)\varphi,\varphi\rangle^{1/2}.\end{aligned}$$ □

5. Linear quadratic case: Stochastic Riccati equation.

5.1. Problem formulation. As a particular but important case, we consider a linear stochastic control system with quadratic cost function and random coefficients. For simplicity, we assume that σ, π, b, L are all time-invariant. We assume also that W_t^1 is a one-dimensional Brownian motion ($p=1$). In this case all data in (3.1), (3.2) can be written as follows:

$$b(x,v)=Ax+Bv,$$
$$\sigma(x,v)=Cx+Dv,$$
$$\pi(x,v)=Gx+Hv,$$
$$L(x,v)=(Rx,x)+(Nv,v),$$
$$h(x,\omega)=(Q(\omega)x,x),$$

where A, C, and G are $n\times n$ matrices; B, D, and H are $n\times k$ matrices; $R\in S^n$; $N\in S^k$; $Q(\omega):\Omega\to S^n$. Here S^n (respectively, S^k) denotes the space of all $n\times n$ (respectively, $k\times k$ symmetric matrices), with scalar product $\langle Q_1,Q_2\rangle=\mathrm{tr}\,(Q_1,Q_2)$.

We assume

$$Q(\omega) \text{ is } \mathscr{F}_T\text{-measurable, bounded, and nonnegative,} \tag{5.1}$$
$$R \text{ is nonnegative, } N \text{ is positive.}$$

The admissible controls now are valued in $\mathbb{R}^k$. Equation (2.1) now becomes

$$dy_s = (Ay_s + Bv_s)\, ds + (Cy_s + Dv_s)\, dW_s + (Gy_s + Hv_s)\, dW_s^1, \quad 0 \leqq s \leqq t \leqq T, \qquad y_t = x, \tag{5.2}$$

and the value function is

$$\Phi_t(x) = \inf_{v \in \mathscr{M}^2(R^k)} E^{\mathscr{F}_t}\left\{\int_t^T [(Ry_s, y_s) + (Ny_s, y_s)]\, ds + (Qy_T, y_T)\right\}. \tag{5.2}$$

By classical methods, we can prove that $\Phi_t(x)$ is of the quadratic form

$$\Phi_t(x) = (K_t x, x), \qquad \forall (x, t), \tag{5.3}$$

where K_t is an S^n-valued $\mathscr{F}_t$-adapted process. From § 3, we know that $\Phi_t(x)$ can be formally written as

$$\Phi_t(x) = (Qx, x) + \int_t^T \Gamma_s(x)\, ds - \int_t^T \Psi_s(x)\, dW_s, \tag{5.4}$$

where Γ_t and Ψ_t are $\mathscr{F}_t$-adapted real processes. Formally, we have

$$\Psi_t(x) = (M_t x, x), \tag{5.5}$$

where M_t is an $\mathscr{F}_t$-adapted process valued in S^n. If such (K_t, M_t) is bounded, we can use the dynamic programming principle to verify that the pair $(\Phi_t(x), \Psi_t(x))$ is a solution of the stochastic HJB equation (3.7).

We can formally derive the equation that characterizes (K_t, M_t). Substituting (5.4), (5.5), and all linear or quadratic data b, σ, π, L, h into (3.7), from Itô's formula we can obtain

$$\begin{aligned} -dK_t = [&A^*K_t + K_tA + C^*K_tC + G^*K_tG + M_tC + C^*M_t \\ &- \hat{B}(K_t, M_t)\hat{N}^{-1}(K_t)\hat{B}^*(K_t, M_t)]\, dt - M_t\, dW_t, \\ K_T = Q&, \end{aligned} \tag{5.6}$$

where for each $K, M \in S^n$, we denote

$$\begin{aligned} \hat{B} &= \hat{B}(K, M) = KB + MD + C^*KD + G^*KH, \\ \hat{N} &= \hat{N}(K) = N + D^*KD + H^*KH. \end{aligned} \tag{5.7}$$

Since $\hat{B}\hat{N}^{-1}\hat{B}$ are nonlinear with respect to K and M, (5.6) is a nonlinear backward stochastic equation. Equation (5.6) is called a stochastic matrix Riccati equation. When Q is deterministic, it becomes an ordinary (deterministic) Riccati equation.

5.2. Existence and uniqueness result. We can regard (5.6) as a nonlinear backward stochastic equation in the form (2.2), defined in the Euclidean space S^n. Theorem 2.1 cannot be applied directly, however, because $\hat{B}\hat{N}^{-1}\hat{B}$ does not satisfy the global Lipschitz condition. This kind of equation was first investigated by Bismut [4]. He obtained an existence theorem for the case where $C=0$, $D=0$ by a method based on the fixed point theorem. We will treat the case where only $D=0$. In this case the bounded variation part of K_t contains M_t. We must use a different method to overcome this difficulty. The case where D is nonzero is still an open problem.

When $D=0$, $\hat{B}$ and $\hat{N}$ become

$$\hat{B}(K, t)=KB+G^*KH, \qquad (5.8)$$
$$\hat{N}(K, t)=N+H^*KH.$$

We assert the following.

THEOREM 5.1. *We assume* (5.1) *and* $D=0$. *Then there exists a pair* (K_t, M_t) *in* $(\mathcal{M}^2(S^n))^2$ *satisfying the stochastic Riccati equation* (5.6) *such that* K_t *is nonnegative and bounded.*

To prove this theorem, the following lemma is in order.

LEMMA 5.2. *Let* $\hat{A}_t, \hat{C}_t, \hat{G}_t$, *be* $\mathbb{R}^{n\times n}$*-valued, and* $\hat{R}_t$ *be* S^n*-valued,* $\mathcal{F}_t$*-adapted processes. Assume that they are all bounded. Let* $\hat{Q}$ *be a bounded* $\mathcal{F}_T$*-measurable random variable with values in* S^n. *Then there exists a pair* $(\hat{K}_t, \hat{M}_t)$ *in* $(\mathcal{M}^2(S^n))^2$ *satisfying the following linear equation*:

$$-d\hat{K}_t=[\hat{A}_t^*\hat{K}_t+\hat{K}_t\hat{A}_t+C_t^*\hat{K}_tC_t+\hat{G}_t^*\hat{K}_t\hat{G}_t +(\hat{M}_t\hat{C}_t+\hat{C}_t^*\hat{M}_t)+\hat{R}_t], \qquad (5.9)$$

$$dt-\hat{M}_t dW_t,\ \hat{K}_T=\hat{Q}.$$

Moreover,

$$\sup_{t,\omega}|\hat{K}_t(\omega)|^2\leqq k_0, \qquad (5.10)$$

where the constant k_0 *only depends on*

$$\sup_{t,\omega}(|\hat{A}_t|+|\hat{C}_t|+|\hat{G}_t|),$$

and

$$\sup_{\omega}\left(|\hat{Q}|^2+\int_0^T|\hat{R}_t|^2\right)(\omega).$$

If $\hat{R}_t, \hat{Q}$ *are nonnegative, almost surely, then* $\hat{K}_t$ *is also nonnegative, almost surely.*

Proof. The above SDE can be regarded as an ordinary backward SDE in the Euclidean space S^n. According to Theorem 2.1, the existence and uniqueness as well as (5.10) hold. It remains to prove the nonnegativity of $\hat{K}$. For given (x, t) let y_s be the solution of

$$dy_s=\hat{A}_s y_s\,ds+\hat{C}_s y_s\,dW_s+\hat{G}_s y_s\,dW_s^1,$$
$$y_t=x, \qquad t\leqq s\leqq T.$$

Then, we can apply Itô's formula

$$d(\hat{K}_s y_s, y_s)=-(\hat{R}_s y_s, y_s)\,ds+(\hat{M}_s y_s, y_s)\,dW_s+2(\hat{K}_s y_s, \hat{C}_s y_s\,dW_s+\hat{G}_s y_s dW_s^1).$$

Thus

$$(\hat{K}_t x, x)=E^{\mathcal{F}_t}\left[\int_t^T(\hat{R}_s y_s, y_s)\,ds+(\hat{Q}y_T, y_T)\right].$$

It follows that K_t is nonnegative whenever $\hat{R}_s, \hat{Q}_s$ are nonnegative. □

We now proceed to prove Theorem 5.1.

Proof of Theorem 5.1. (i) *Existence.* We define $F(K, M, U)$: $S^n\times S^n\times \mathcal{L}(\mathbb{R}^n; \mathbb{R}^k)\to S^n$ by

$$F(K, M, U)=(A+BU)^*K+K(A+BU)+C^*KC +(G+HU)^*K(G+HU)+(MC+C^*M^*).$$

We also define $\hat{U}(K):(S^n)^+ \to \mathscr{L}(\mathbb{R}^n;\mathbb{R}^k)$ by

$$\hat{U}(K) = -\hat{N}^{-1}(K)\hat{B}(K),$$

where $(S^n)^+$ is the set of nonnegative elements of S^n. With these notations, we can rewrite (5.6) as

$$\begin{aligned} -dK_t &= [F(K_t, M_t, \hat{U}(K_t)) + \hat{U}^*(K_t)N\hat{U}(K_t)]\, dt - M_t\, dW_t, \\ K_T &= Q. \end{aligned} \tag{5.11}$$

It is seen that

$$F(K, M, \hat{U}(K)) + \hat{U}^*(K)N\hat{U}(K) \leqq F(K, M, U) + U^*NU, \qquad \forall U \in \mathscr{L}(\mathbb{R}^n;\mathbb{R}^k). \tag{5.12}$$

We now iteratively construct a sequence of approximating solutions. First, we define (K_1, M_1) by solving the following linear backward SDE:

$$\begin{aligned} -dK_{1,t} &= F(K_{1,t}, M_{1,t}, 0)\, dt - M_{1,t}\, dW_t, \\ K_{1,T} &= Q. \end{aligned}$$

By Lemma 5.2, we can easily check that the above solution exists, and that $K_{1,t}$ is bounded and nonnegative. Thus $\hat{U}(K_{1,t})$ has meaning and is bounded. Then, we define (K_2, M_2) by solving

$$\begin{aligned} -dK_{2,t} &= [F(K_{2,t}, M_{2,t}, \hat{U}(K_{1,t})) + \hat{U}^*(K_{1,t})N\hat{U}(K_{1,t})]\, dt - M_{2,t}\, dW_t, \\ K_{2,T} &= Q. \end{aligned}$$

Again from Lemma 5.2, there is a unique solution (K_2, M_2), which is bounded and nonnegative. Thus $\hat{U}(K_{2,t})$ is well defined and bounded. Inductively, we can define (K_{j+1}, M_{j+1}), which is the unique bounded and nonnegative solution of

$$\begin{aligned} -dK_{j+1,t} &= F(K_{j+1,t}, M_{j+1,t}, \hat{U}(K_{j,t}))\, dt + \hat{U}^*(K_{j,t})N\hat{U}(K_{j,t})]\, dt - M_{j+1,t}\, dW_t, \\ K_{j+1,T} &= Q, \qquad j = 1, 2, \cdots. \end{aligned} \tag{5.13}$$

We claim that the sequence $\{K_{j,t}\}$ is nonincreasing. Indeed, we have

$$\begin{aligned} -d(K_{j,t} - K_{j+1,t}) &= [F(K_{j,t}, M_{j,t}, \hat{U}(K_{j,t})) - F(K_{j+1,t}, M_{j+1,t}, \hat{U}(K_{j,t}))]\, dt \\ &\quad + [F(K_{j,t}, M_{j,t}, \hat{U}(K_{j-1,t})) + \hat{U}^*(K_{j-1,t})N\hat{U}(K_{j-1,t}) \\ &\quad - F(K_{j,t}, M_{j,t}, \hat{U}(K_{j,t})) - \hat{U}^*(K_{j,t})N\hat{U}(K_{j,t})]\, dt \\ &\quad - (M_{j,t} - M_{j+1,t})\, dW_t \\ &= F(K_{j,t} - K_{j+1,t}, M_{j,t} - M_{j+1,t}, \hat{U}(K_{j,t}))\, dt \\ &\quad + R_{j,t}\, dt - (M_{j,t} - M_{j+1,t})\, dW_t, \\ K_{j,T} - K_{j+1,T} &= 0, \end{aligned}$$

with

$$\begin{aligned} R_{j,t} = {} & F(K_{j,t}, M_{j,t}, \hat{U}(K_{j-1,t})) + \hat{U}^*(K_{j-1,t})N\hat{U}(K_{j-1,t}) \\ & - F(K_{j,t}, M_{j,t}, \hat{U}(K_{j,t})) - \hat{U}^*(K_{j,t})N\hat{U}(K_{j,t}). \end{aligned}$$

From (5.12), R_j is nonnegative. Thus, according to Lemma 5.2, $K_{j,t} - K_{j+1,t}$ is also nonnegative. This implies that $\{K_{j,t}\}$ is the nonincreasing sequence

$$CI \geqq K_{1,t} \geqq K_{2,t} \geqq \cdots \geqq K_{j,t} \geqq \cdots \geqq 0.$$

It follows that $\{K_{j,t}\}$ converges almost surely to a nonnegative, S^n-valued process K_t. According to Lebesgue's convergence theorem, we have

$$E\int_0^T |K_{j,t}-K_t|^q \to 0, \quad \text{as } j\to\infty, \qquad \forall q>0.$$

Thus $\{K_{j,t}\}$, and also then $\{U(K_{j,t})\}$ is a Cauchy sequence in the above sense. We have also almost everywhere

$$E|K_{j,t}-K_t|^q \to 0, \quad \text{as } j\to\infty, \qquad \forall q>0.$$

By definition (5.13), we can apply Itô's formula to $|K_{k,t}-K_{j,t}|^2$,

$$\begin{aligned} E|K_{k,0}-K_{j,0}|^2 &+ E\int_0^T |M_{k,t}-M_{j,t}|^2\,dt \\ &= 2E\int_0^T \operatorname{tr}[(K_{k,t}-K_{j,t})((M_{k,t}-M_{j,t})C+C^*(M_{k,t}-M_{j,t}))]\,dt + R(j,k), \\ &\leq \frac{1}{2}E\int_0^T |M_{k,t}-M_{j,t}|^2\,dt + CE\int_0^T |K_{k,t}-K_{j,t}|^2\,dt + R(j,k), \end{aligned}$$

where $R(j,k)\to 0$ as $\min(j,k)\to\infty$. Thus $\{M_{j,t}\}$ is a Cauchy sequence in $\mathcal{M}^2(\mathscr{L}(\mathbb{R}^d;S^n))$. Passing to the limit in (5.13), we obtain that (K_t, M_t) is a solution of (5.6), with

$$M_t = \lim_{j\to\infty} M_{j,t} \quad \text{in } \mathcal{M}^2(S^n).$$

(ii) *Uniqueness.* Let (K_t, M_t) and (K'_t, M'_t) be two pairs in $\mathcal{M}^2(S^n)\times\mathcal{M}^2(S^n)$ satisfying (5.6) (or (5.11)), such that K_t, K'_t are nonnegative and bounded. Then $\hat{U}(K_t)$ and $\hat{U}(K'_t)$ are well defined and bounded. We have

$$\begin{aligned} -d(K_t-K'_t) &= F(K_t-K'_t, M_t-M'_t, \hat{U}(K_t))\,dt \\ &\quad +[F(K'_t, M'_t, \hat{U}(K_t))+\hat{U}^*(K_t)N\hat{U}(K_t) \\ &\quad -F(K'_t, M'_t, \hat{U}(K'_t))+\hat{U}^*(K'_t)N\hat{U}(K'_t)]\,dt - M_t\,dW_t, \\ K_T-K'_T &= 0, \end{aligned}$$

or

$$\begin{aligned} -d(K_t-K'_t) &= F(K_t-K'_t, M_t-M'_t, \hat{U}(K_t))\,dt \\ &\quad +R'\,dt+(M_t-M'_t)\,dW_t, \\ K_T-K'_T &= 0, \end{aligned}$$

with

$$R' = F(K'_t, M'_t, \hat{U}(K_t))+\hat{U}^*(K_t)N\hat{U}(K_t)-F(K'_t, M'_t, \hat{U}(K'_t))+\hat{U}^*(K'_t)N\hat{U}(K'_t).$$

By (5.12), R' is nonnegative. It follows from Lemma 5.2 that $K_t-K'_t$ is nonnegative. Similarly, we can obtain that K'_t-K_t is nonnegative. This implies $K_t=K'_t$. Consequently (from the uniqueness part of Lemma 5.2), $M_t=M'_t$. □

As we mentioned in § 2 (verification theorem), once we obtain the solution of (5.6), which is regular enough, then the value function can be obtained by $\Phi_t=(K_t x, x)$. Moreover, the optimal feedback control can also be given. Specifically, we have the following corollary.

COROLLARY 5.3. *Let the assumptions of Theorem* 5.1 *hold. Then the value function is equal to* $(K_t x, x)$. *The optimal feedback control is*

$$u_t(x) = \hat{U}(K_t)x.$$

Proof. Since K_t is bounded and positive, $\hat{U}(K_t)$ is well posed and bounded. Let (x_0, t_0) be any given initial data. We want to minimize

$$J_{x_0,t_0}(v.) = E\int_{t_0}^{T} [(Rx_t, x_t) + (N_t v_t, v_t)]\, ds + E(Qx_T, x_T)$$

subject to

$$dx_t = (Ax_t + Bv_t)\, dt + Cx_t\, dW_t + (Gx_t + Hv_t)\, dW_t^1, \qquad x_{t_0} = x_0.$$

For any given admissible control v_t, we can apply Itô's formula to $(K_t x_t, x_t)$, and thus verify that

$$(K_{t_0} x_{t_0}, x_{t_0}) \leqq E^{\mathscr{F}_{t_0}}\left\{\left[(Qx_T, x_T) + \int_{t_0}^{T} (Rx_t, x_t) + (N_t v_t, v_t)\right] ds\right\}.$$

On the other hand, let y_t be a solution of

$$dy_t = (A + B\hat{U}(K_t))y_t\, dt + Cy_t\, dW_t + (G + H\hat{U}(K_t))y_t\, dW_t^1, \qquad y_{t_0} = x_0.$$

We can again apply Itô's formula to $(K_t y_t, y_t)$,

$$(K_{t_0} x_{t_0}, x_{t_0}) = E^{\mathscr{F}_{t_0}}\left\{\left[(Qy_T, y_T) + \int_{t_0}^{T} (Ry_t, y_t) + (N_t\hat{U}(K_t)y_t, \hat{U}(K_t)y_t)\right] ds\right\}.$$

It follows that, for almost surely (x_0, t_0) $(K_{t_0} x_0, x_0)$ is equal to the value function. Furthermore, if we set $u_t = \hat{U}(K_t)y_t$,

$$\begin{aligned}
J_{x_0,t_0}(v.) &= E\int_{t_0}^{T} [(Rx_t, x_t) + (N_t v_t, v_t)]\, ds + E(Qx_T, x_T) \\
&= EE^{\mathscr{F}_{t_0}}\left[\int_{t_0}^{T} [(Rx_t, x_t) + (N_t v_t, v_t)]\, ds + (Qx_T, x_T)\right] \\
&\geqq EE^{\mathscr{F}_{t_0}}\left[\int_{t_0}^{T} [(Ry_t, y_t) + (N_t u_t, u_t)]\, ds + (Qy_T, y_T)\right] \\
&= J_{x_0}, t_0(u.)
\end{aligned}$$

It follows that u_t is the optimal control and so $U(K_t)x$ is the optimal feedback. $\square$

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