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On a Robustness of Quantile Hedging: Complete Market's Case

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Abstract. Recently, Föllmer and Leukert have introduced the notion of quantile-hedging. In their paper [3], three types of quantile-hedging-problems in particular have been formulated: i.e.,

1. problem of maximizing the probability of success,
2. problem of minimizing the cost for a given probability of success,
3. problem of minimizing 'a downside-risk',

and reduced to certain simple statistical-tests.

In this article, under an elementary complete-market with unknown (to the investor) constant drift of the risky-asset setting, we will measure a certain robustness of quantile-hedging against the uncertainty of the drift. We will discriminate the robustness by whether the associated statistical-test has uniformly the most powerful test function against *alternatives*. We claim that the solution of 3 is robust if the sign of the drift is known, the solution of 2, not robust, and the solution of 1, robust to some extent, which is affected by the shape of the contingent claim.

Key words: quantile-hedging, complete market, robustness, unknown drift, Neyman–Pearson's Lemma.

1. Setup and Results

Consider a complete financial market that consists of the nonrisky asset price process: $B_t = 1$, $(t \in [0, T])$ and the risky asset price process: $\mathrm{d}X_t = X_t\,(\sigma \mathrm{d}w_t + \mu \mathrm{d}t)$, $(t \in [0, T])$ on a probability space $(\Omega, \mathcal{F}, P)$ with a 1-dimensional Brownian motion $w = (w_t)_{t\in[0,T]}$ $(w_0 = 0)$ on it and the Brownian filtration $(\mathcal{F}_t)_{t\in[0,T]}$. We assume that σ is a positive known constant, μ, an unknown (to an investor) constant and $\mathcal{F} = \mathcal{F}_T$. In this market, the investor can perfectly replicate every contingent claim by admissible[1] self-financing strategy of the assets, i.e., for any nonnegative $H (\in L^1(P))$, there exist a unique equivalent martingale measure P^* defined by

$$\frac{\mathrm{d}P^*}{\mathrm{d}P} = \mathcal{E}\left(-\frac{\mu}{\sigma} w_T\right),$$

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$H_0 \in \mathbb{R}_+$, and predictable $(\xi_t^H)_{t\in[0,T]}$ which satisfy

$$E^*[H|\mathcal{F}_t] = H_0 + \int_0^t \xi_t^H \mathrm{d}X_t, \quad (t \in [0,T]),$$

where E^* denotes the expectation with respect to P^*. The right-hand-side of the above is interpreted as the replicated-portfolio-value at time t of the investor, and the initial cost $H_0 = E^*[H]$ of the portfolio is called the fair-price of H. For instance, if we take

$$H = (X_T - K)^+, \quad (K > 0), \tag{1}$$

i.e., the European-call-option with maturity T and strike-price K, then,

$$H_0 = X_0\Phi(d_+) - K\Phi(d_-)$$

where

$$\Phi(d) := \int_{-\infty}^{d} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2}\mathrm{d}x\,,$$

$$d_\pm := \frac{1}{\sigma\sqrt{T}}\left\{\log\left(\frac{X_0}{K}\right) \pm \frac{\sigma^2 T}{2}\right\},$$

the Black–Scholes-formula follows.

In [3], Föllmer and Leukert have suggested the question:

'But what if the investor is unwilling or unable to put up the initial cost H_0?'

and proposed the notion of quantile-hedging as an answer. It can be interpreted as a dynamic version of the Value at Risk concept; starting with a lower initial cost $\tilde{V}_0$ than the fair price H_0, they have especially formulated the following three types of quantile-hedging-problems:

Problem 1. (Maximizing the probability of success) Fix $\tilde{V}_0 \le H_0$. *Among admissible strategies, solve the following optimization-problem* :

$$\max P\left[H \le V_0 + \int_0^T \xi\,\mathrm{d}X\right] \quad \text{subject to} \quad V_0 \le \tilde{V}_0.$$

Problem 2. (Minimizing the cost for a given probability of success) Fix $\tilde{V}_0 \le H_0$ *and* $0 < \epsilon < 1$. *Among admissible strategies, solve the following optimization-problem* :

$$\min_{V_0 \le \tilde{V}_0} V_0 \quad \text{subject to} \quad P\left[H \le V_0 + \int_0^T \xi\,\mathrm{d}X\right] \ge 1 - \epsilon.$$

Problem 3. (Minimizing 'a downside-risk') Fix $\tilde{V}_0 \le H_0$. *Among admissible strategies, solve the following optimization-problem* :

$$\min E\left[\left(H - V_0 - \int_0^T \xi\,\mathrm{d}X\right)^+\right] \quad \text{subject to} \quad V_0 \le \tilde{V}_0.$$

Their idea to solve the problems above is to reduce them to the following 'payoff-modification-problems':

Problem 1′. Fix $\tilde{V}_0 \le H_0$. Solve the following optimization-problem:

$$\max_{A \in \mathcal{F}} P(A) \quad \text{subject to} \quad E^*[H1_A] \le \tilde{V}_0,$$

Problem 2′. Fix $\tilde{V}_0 \le H_0$ and $0 < \epsilon < 1$. Solve the following optimization-problem:

$$\min_{A \in \mathcal{F}} E^*[H1_A] \quad \text{subject to} \quad P(A) \ge 1 - \epsilon,$$

Problem 3′. Fix $\tilde{V}_0 \le H_0$. Solve the following optimization-problem:

$$\max_{\phi \in \mathcal{R}} E[H\phi] \quad \text{subject to} \quad E^*[H\phi] \le \tilde{V}_0,$$

where

$$\mathcal{R} := \{\phi : \Omega \to [0,1], \text{ measurable}\},$$

respectively. With the help of the solutions of Problems 1′–3′, they obtain

PROPOSITION 1. *(Föllmer–Leukert; Proposition (2.8) in [3])*

1. *Let $\tilde{A}$ be a solution of Problem 1′. The replicating strategy $(\tilde{\xi}_t)_{t \in [0,T]}$ of 'the modified claim' $H1_{\tilde{A}}$ defined by*
$$E^*\left[H1_{\tilde{A}} | \mathcal{F}_t\right] = E^*[H1_{\tilde{A}}] + \int_0^t \tilde{\xi}_u \mathrm{d}X_u,$$
is a solution of Problem 1.
2. *Let $\tilde{A}$ be a solution of Problem 2′. The expectation $E^*\left[H1_{\tilde{A}}\right]$ is a solution of Problem 2, and it is attained by the strategy $(\tilde{\xi}_t)_{t \in [0,T]}$ which is defined by*
$$E^*\left[H1_{\tilde{A}} | \mathcal{F}_t\right] = E^*[H1_{\tilde{A}}] + \int_0^t \tilde{\xi}_u \mathrm{d}X_u.$$
3. *Let $\tilde{\phi}$ be a solution of Problem 3′. The replicating strategy $(\tilde{\xi}_t)_{t \in [0,T]}$ of 'the modified claim' $H\tilde{\phi}$ defined by*
$$E^*\left[H\tilde{\phi} | \mathcal{F}_t\right] = E^*\left[H\tilde{\phi}\right] + \int_0^t \tilde{\xi}_u \mathrm{d}X_u,$$
is a solution of Problem 3.

Let us recall that Problems 1′–3′ are reduced again to the Neyman–Pearson's Lemma:

LEMMA 1. *(Neyman–Pearson) Let P_1, P_2 be probability-measures on $(\Omega, \mathcal{F})$, both dominated by another probability-measure P_3. If the relation*

$$P_2(A(c)) = \alpha \quad A(c) := \left\{\frac{\mathrm{d}P_1}{\mathrm{d}P_3} > c\frac{\mathrm{d}P_2}{\mathrm{d}P_3}\right\}$$

holds for some α $(\in (0,1))$ *and* $c(>0)$*, then,* $1_{A(c)}$ *is a most powerful test (abbrev. MP-test) function of the test:*

$$\max_{\phi\in\mathcal{R}} E_1[\phi] \quad \text{subject to} \quad E_2[\phi] \le \alpha$$

where $E_i(i = 1,2)$ *denotes the expectation with respect to* $P_i(i = 1,2)$*, respectively.*

Note that the critical region $A(c)$ of MP-test depends on the choice of alternative-hypothesis P_1. If an MP-test $1_{A(c)}$ against *any alternatives* happens to exist, it is called uniformly the most powerful test (abbrev. UMP-test). In our situation, UMP-property among *alternatives*, i.e., diffusions with unknown-drifts, assures the invariance of strategy (or initial-cost) of quantile-hedging.

Now, we give the following definition.

DEFINITION 1. *The solutions of Problems 1–3 are called robust if the associated Problems 1′–3′ have UMP-test-functions against ‘alternatives’,*

and observe the following:

THEOREM 1. *(1) For any contingent claim* H*, the solution of Problem 3 is robust on the parameter set* $\{\mu;\mu > 0\}$ *and* $\{\mu;\mu < 0\}$*, respectively. (2) Let* $H = F_T(X_T)$*, a European contingent claim with some measurable and integrable* $F_T : \mathbb{R} \mapsto \mathbb{R}_+$*. Then, the solution of Problem 2 is not robust.*

Proof. (1) Problem 3′ is rewritten as

$$\max_{A\in\mathcal{F}} Q\,(A) \quad \text{subject to} \quad Q^*(A) \le \tilde{V}_0/H_0\,,$$

where ‘the test hypothesis’

$$\frac{\mathrm{d}Q^*}{\mathrm{d}P^*} := \frac{H}{H_0},$$

and ‘alternative hypothesises’

$$\frac{\mathrm{d}Q}{\mathrm{d}P^*} = \frac{\mathrm{d}Q}{\mathrm{d}P}\frac{\mathrm{d}P}{\mathrm{d}P^*} := \frac{H}{E[H]}\mathcal{E}\left(\frac{\mu}{\sigma}w_T^*\right) = \text{const.} \times H(X_T)^{\mu/\sigma^2},$$

where w^* : Brownian-motion under P^*.

Therefore, ‘the critical region of MP-test’:

$$A(c) := \left\{\frac{\mathrm{d}Q}{\mathrm{d}P^*} > c\frac{\mathrm{d}Q^*}{\mathrm{d}P^*}\right\} = \left\{(X_T)^{\mu/\sigma^2} > \text{const.}\right\}$$

such that $Q^*(A(c)) = \tilde{V}_0/H_0$ is in the form of

$$A(c) = \begin{cases} \{X_T > d_1\} & \text{if } \mu > 0, \\ \{X_T < d_2\} & \text{if } \mu < 0, \end{cases}$$

with some $d_1, d_2 > 0$, independent of the value μ, since X_T is an exponential martingale with known volatility σ under P^*. Hence, $1_{A(c)}$ is UMP among the alternatives $\{\mu; \mu > 0\}$ and $\{\mu; \mu < 0\}$, respectively.

(2) The associated 'statistical-test' of Problem 2′ is

$$\max_{A^c \in \mathcal{F}} Q^* \left(A^c\right) \quad \text{subject to} \quad P(A^c) \leq \epsilon,$$

so 'the critical region of MP-test':

$$A^c(c) := \left\{\frac{\mathrm{d}Q^*}{\mathrm{d}P} > c\right\} = \left\{H(X_T)^{-\mu/\sigma^2} > \text{const.}\right\}$$

such that $P(A^c(c)) = \epsilon$ is in the form of $\{X_T \in B\}$ with some $B \in \mathcal{F}$, *depends* on the value μ, since X_T is a lognormal-diffusion with unknown drift μ under P. Therefore, $1_{A^c(c)}$ is not UMP. □

The solution of Problem 1 has somewhat robustness, which depends on the shape of H. e.g.,

THEOREM 2. *(1) Let H be the T-maturity European call option with strike price K, which is defined by the formula (1). Then, the solution of Problem 1 is robust on the parameter set $\{\mu; \mu \leq \sigma^2\}$. (2) Let $H = (K - X_T)^+$, the T-maturity European put option with strike price K. Then, the solution of Problem 1 is robust on the parameter set $\{\mu; \mu \geq 0\}$.*

Proof. The associated 'statistical-test' of Problem 1′ is

$$\max_{A \in \mathcal{F}} P(A) \quad \text{subject to} \quad Q^*(A) \leq \tilde{V}_0 / H_0,$$

so 'the critical region of MP-test':

$$A(c) := \left\{\frac{\mathrm{d}P}{\mathrm{d}P^*} > c\frac{\mathrm{d}Q^*}{\mathrm{d}P^*}\right\} = \left\{(X_T)^{\mu/\sigma^2} > \text{const.} H\right\}$$

such that $Q^*(A(c)) = \tilde{V}_0 / H_0$ is in the form of

$$A(c) = \begin{cases} \{X_T < d\} \text{ if } \mu \leq \sigma^2, \\ \{X_T < d_1\} \cup \{X_T > d_2\} \text{ if } \mu > \sigma^2, \end{cases}$$

in the case of call option,

$$A(c) = \begin{cases} \{X_T > d\} & \text{if} \quad \mu \geq 0, \\ \{X_T < d_1\} \cup \{X_T > d_2\} & \text{if} \quad \mu < 0, \end{cases}$$

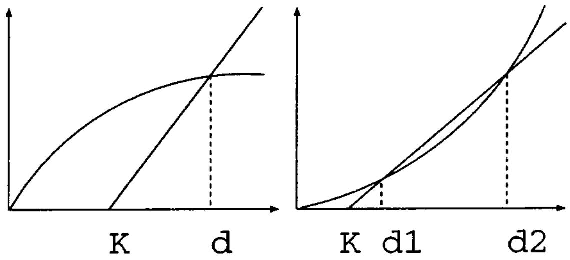

Figure 1. Call's case (left: $\mu \leq \sigma^2$, right: $\mu > \sigma^2$).

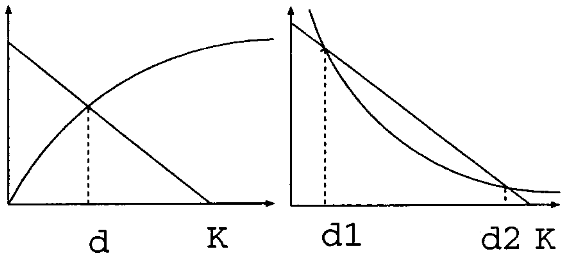

Figure 2. Put's case (left: $\mu \geq 0$, right: $\mu < 0$).

in the case of put option, respectively, with some $d, d_1, (<)d_2$. (cf., Section 3 of [3], and Figures 1, 2 above.) The constant d *does not* depend on the value μ, since X_T is an exponential martingale with known volatility σ under P^*, while the constants d_1, d_2 *depend* on the value μ, since they are determined as the solutions: (d_1, d_2, c) $(0 < d_1 < d_2)$ of

$$(d_i)^{\mu/\sigma^2} \pm c(d_i - K) = 0, \quad (i = 1, 2, +(-)\text{: put(call) case}),$$
$$P^*(\{X_T \leq d_1\} \cup \{X_T \geq d_2\}) = \tilde{V}_0/H_0 .$$

□

Let us state some remarks of our result and close this article.

(1) We have measured a certain robustness against the uncertainty of drift μ. The robustness of Problem 1 and 3 over some parameter space $\{\mu; \mu \in M\}$ $(M \subset \mathbb{R})$ means that the strategy of quantile-hedging is invariant under any choice of subjective probability measure: $P \circ (X^\mu)^{-1}$ ($\mu \in M$, here we denote $\mathrm{d}X_t^\mu = X_t^\mu(\sigma \mathrm{d}w_t + \mu \mathrm{d}t)$, $t \in [0, T]$).

(2) Robustness against richer uncertainty of drift μ (e.g., predictable process with certain integrability condition), robustness against uncertainty of volatility (cf., [1],[2]), and robustness under incomplete market setting still all remain as our challenging research in the future. This article is based on our casual consideration at the beginning.

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