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Comparison Principle for Dirichlet-Type Hamilton–Jacobi Equations and Singular Perturbations of Degenerated Elliptic Equations

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Abstract. Under a nondegeneracy condition on the boundary, we prove a comparison principle for discontinuous viscosity sub- and supersolutions of the generalized Dirichlet boundary-value problem for a first-order Hamilton-Jacobi equation

$$\begin{cases} H(x, u, Du) = 0 & \text{in } \Omega, \\ \operatorname{Max}(H(x, u, Du); u - \varphi) \geq 0 & \text{on } \partial\Omega, \\ \operatorname{Min}(H(x, u, Du); u - \varphi) \leq 0 & \text{on } \partial\Omega. \end{cases}$$

For optimal control problems, we interpret this nondegeneracy as a condition on the controlled vector fields. Finally, we use this to extend classical singular perturbation results to degenerated elliptic equations.

1. Introduction

We are interested in comparison results between discontinuous viscosity sub- and supersolutions of the generalized Dirichlet problem for first-order Hamilton-Jacobi equations. Namely, let $\bar{u}$, an upper semicontinuous (u.s.c.) function, satisfy, in the viscosity sense,

$$H(x, \bar{u}, D\bar{u}) \leq 0 \qquad \text{in } \Omega, \tag{1}$$

$$\operatorname{Min}(H(x, \bar{u}, D\bar{u}), \bar{u} - \varphi) \leq 0 \qquad \text{on } \partial\Omega, \tag{1'}$$

and let $\underline{u}$, a lower semicontinuous (l.s.c.) function, satisfy, in the viscosity sense,

$$H(x, \underline{u}, D\underline{u}) \geq 0 \qquad \text{in } \Omega, \tag{2}$$

$$\operatorname{Max}(H(x, \underline{u}, D\underline{u}), \underline{u} - \varphi) \geq 0 \qquad \text{on } \partial\Omega. \tag{2'}$$

We intend to give conditions on H and $\bar{u}$, $\underline{u}$, such that it is possible to conclude that $\underline{u}$ is larger than $\bar{u}$. This kind of comparison result allows us to treat very simply singular perturbation problems arising in the context of large deviations and we give some applications to degenerated equations.

Our methods and results are based on the notion of viscosity solutions introduced by Crandall and Lions [8] (see also [6] and [18]); it gives sense to (1), (1′), (2), (2′) and we assume the reader is familiar with this notion. Although these references only consider pure Dirichlet boundary condition, this notion was extended to various boundary conditions such as Neumann conditions [19] and [22], state-constraint conditions [24], [5]. Conditions (1′) and (2′) were introduced more recently to characterize the cost functions of deterministic exit-time problems [16], [3], [4]. The only uniqueness result for discontinuous solutions of (1), (1′), (2), (2′) is the result in [4]. But it is also possible to compare $\bar{u}$ and $\underline{u}$ if one of them is continuous (see [4] and [16]). Moreover, a general comparison principle cannot exist since it would imply that the solution of (1), (1′), (2), (2′) is continuous, and this is wrong (consider, for example, exit-time control problems).

Our aim is to give some sufficient additional conditions on H, $\bar{u}$, $\underline{u}$, at the boundary ensuring a comparison result for discontinuous sub- and supersolutions of (1), (1′), (2), (2′); a result in that direction has been given in [16] for optimal control problems. We have two main motivations: the first concerns exit-time control problems (and differential games). Our result gives sufficient conditions on the controlled vector fields ensuring that the value function is continuous and is the unique viscosity solution to (1), (1′), (2), (2′). Our second motivation comes from the study of the vanishing viscosity method: we want to study the behavior, as ε goes to zero, of the solution u^ε of

$$\begin{cases} -\varepsilon \Delta u^\varepsilon + H(x, u^\varepsilon, Du^\varepsilon) = 0 & \text{in } \Omega, \\ u^\varepsilon = \varphi & \text{on } \partial\Omega. \end{cases} \tag{3}$$

Following [3] and [4], we define

$$\bar{u}(x) = \limsup_{\varepsilon \to 0, y \to x} u^\varepsilon(y), \qquad \underline{u}(x) = \liminf_{\varepsilon \to 0, y \to x} u_\varepsilon(y),$$

these functions are respectively a u.s.c. subsolution of (1), (1′) and an l.s.c. supersolution of (2), (2′). A comparison result implies that $\bar{u} \leq \underline{u}$, and thus that $\bar{u} = \underline{u}$, and this means that u^ε converges uniformly to the unique solution to (1), (1′), (2), (2′). We give examples of degenerated equations such as (3) where this argument is very powerful since no *a priori* bounds on u^ε are known which would imply its uniform convergence.

In Section 2 we state our general results and the two conditions on H, and $\bar{u}$, $\underline{u}$, that we need to prove a comparison principle. In Section 3 we are interested in control problems; we interpret our nondegeneracy condition on H in terms of the behavior of the controlled vector field at the boundary. Section 4 is devoted to some extensions to the Cauchy problem. We end the paper with some applications of our general results to some singular perturbation problems such as (3) arising in the context of large deviations. These are merely extensions to degenerated diffusions of the examples given by Evans and Ishii [9].

2. Comparison Principle

In this section we give a general comparison principle under a nondegeneracy condition on H at the boundary. We begin with a result involving some "regularity" condition of $\bar{u}$, $\underline{u}$ on $\partial\Omega$, and we show how this condition can be avoided.

We are given an open bounded domain Ω of $\mathbb{R}^N$ and a hamiltonian H satisfying:

(H1) H is continuous on $\bar{\Omega}\times[-R, R]\times\bar{B}_R$ $(\forall R<+\infty)$.

(H2) For all $R>0$, there exists $\gamma_R>0$ such that
$H(x, t, p)-H(x, s, p)\geq\gamma_R(t-s)$,
$\forall x\in\bar{\Omega}$, $-R\leq s\leq t\leq R$, $p\in\mathbb{R}^N$.

(H3) For all $R>0$, there exists a modulus of continuity m_R such that
$|H(x, t, p)-H(y, t, p)|\leq m_R\{|x-y|(1+|p|)\}$
for x, $y\in\bar{\Omega}$, $|t|\leq R$, $p\in\mathbb{R}^N$, and $H(x, t, p)$ is uniformly continuous in p, for $|t|\leq R$ and x in a neighborhood of $\partial\Omega$.

These three assumptions, which can be relaxed (see the remark below), are classical in the theory of Hamilton–Jacobi equations. Now we state our "nondegeneracy" conditions. Let $d(\cdot)$ denote the distance to $\partial\Omega$ and assume d is a $C^{1,1}$ function in a neighborhood of $\partial\Omega$ and set $n(x)=-\nabla d(x)$. We assume that for some subsets Γ_1, Γ_2 of $\partial\Omega$, which are specified later, the following conditions hold:

(H4) $\forall R>0$, $\forall x\in\Gamma_1$, $\exists C$ such that if $|t|\leq R$, $|x-y|\leq 1/C$, $y\in\bar{\Omega}$, and $\lambda\geq C(1+|p|)$, then

$$H(y, t, p+\lambda n(y))>0.$$

(H5) $\forall R>0$, $\forall x\in\Gamma_2$, $\exists C$ such that if $|t|\leq R$, $|x-y|\leq 1/C$, $y\in\bar{\Omega}$, and $\lambda\geq C(1+|p|)$, then

$$H(y, t, p-\lambda n(y))<0.$$

Our main result is

Theorem 1. *Let H satisfy* (H1)–(H3) *and let φ be continuous. Let u be a bounded u.s.c. subsolution to* (1), (1′) *and let v be a bounded l.s.c. supersolution to* (2), (2′). *Assume in addition that* (H4) *holds with* $\Gamma_1=\{x\in\partial\Omega,\ u(x)\leq\varphi(x),\ v(x)<\varphi(x)\}$ *and* (H5) *holds with* $\Gamma_2=\{x\in\partial\Omega,\ u(x)>\varphi(x),\ v(x)\geq\varphi(x)\}$. *Assume finally that*

$$\forall x\in\Gamma_1,\quad u(x)=\limsup_{\substack{y\to x\\ y\in\Omega}} u(y),\qquad \forall x\in\Gamma_2,\quad v(x)=\liminf_{\substack{y\to x\\ y\in\Omega}} v(y). \tag{4}$$

Then

$$u \le v \qquad \text{in } \bar{\Omega}.$$

Since we do not know Γ_1 and Γ_2 in general and assumption (4) is difficult to check (for example, in the vanishing viscosity method), we give some applications and variants of Theorem 1 and some precisions on Γ_1, Γ_2.

Proposition 2. *Let H satisfy* (H1), *let φ be continuous, and let u and v be as in Theorem* 1, *then*

$$\{v<\varphi\} \subset \Gamma_1', \qquad \{u>\varphi\} \subset \Gamma_2',$$

with

$$\Gamma_1' = \left\{ x_0 \in \partial\Omega;\ \limsup_{\varepsilon\to 0} \left\{ H(x, t, p+\lambda n(x));\ |x-x_0| \le \varepsilon, \frac{\lambda}{1+|p|} \ge \frac{1}{\varepsilon}, |t| \le R \right\} \right.$$
$$\left. \ge 0 \textit{ for all } R>0 \right\},$$

$$\Gamma_2' = \left\{ x_0 \in \partial\Omega;\ \limsup_{\varepsilon\to 0} \left\{ H(x, t, p-\lambda n(x));\ |x-x_0| \le \varepsilon, \frac{\lambda}{1+|p|} \ge \frac{1}{\varepsilon}, |t| \le R \right\} \right.$$
$$\left. \le 0 \textit{ for all } R>0 \right\}.$$

Corollary 3. *Assume that H satisfies* (H1)-(H3), *that φ is continuous, and that* (H4) *holds with $\Gamma_1 = \Gamma_1'$ and that* (H5) *holds with $\Gamma_2 = \Gamma_2'$. Let u and v be as in Theorem* 1, *then*

$$u \le v \qquad \text{in } \Omega.$$

Our last application concerns the state-constraint problem, i.e., roughly speaking the case $\varphi = +\infty$. Then (1), (1′), (2), (2′) reduces to

$$\begin{cases} H(x, u, Du) = 0 & \text{in } \Omega, \\ H(x, u, Du) \ge 0 & \text{on } \partial\Omega. \end{cases} \tag{5}$$

Corollary 4. *Assume* (H1)-(H3) *hold and that* (H4) *holds with $\Gamma_1 = \partial\Omega$. Then if u and v are respectively a bounded u.s.c. subsolution and a bounded l.s.c. supersolution of* (5), *then*

$$u \le v \qquad \text{in } \Omega.$$

Remark 1. All these results hold with (H3)-(H5) replaced by (H3′)-(H5′) below. Let us be given a function $d \in C^1(\Omega; \mathbb{R}^+)$ such that d is constant out of a neighborhood of $\partial\Omega$ and such that

$$d(x) = 0 \quad \Leftrightarrow \quad x \in \partial\Omega.$$

Set

$$\underline{n}(x) = -\nabla d(x).$$

Condition (H3) may be replaced, for any continuous function $\phi_R(t, s)$, by

$$\text{(H3}'\text{)}\quad |H(x, t, \lambda(x-y)+\mu\underline{n}(x)+p)-H(y, t, \lambda(x-y)+\mu\underline{n}(y))| \\ \le m_R(|x-y|+|p|+\lambda|x-y|^2) \\ \times \phi_R(H(x, t, \lambda(x-y)+\mu\underline{n}(x)+p), H(y, t, \lambda(x-y)+\mu\underline{n}(y)))$$

for $\lambda \ge 0$, $t \le R$, $|\mu| \le \lambda|x-y|$, $p \in \mathbb{R}^N$, and x, y in a neighborhood of $\partial\Omega$.

(H4′) Condition (H4) with $n(x)$ replaced by $\underline{n}(x)$.
(H5′) Condition (H5) with $n(x)$ replaced by $\underline{n}(x)$.

It is easy to build examples where $\underline{n}$ is tangent to $\partial\Omega$ and where uniqueness holds even though (H3)–(H5) are not satisfied.

Remark 2. Condition (H4) with $\Gamma_1 = \partial\Omega$ was used in [5] to obtain existence results for (5).

Remark 3. The introduction of the term $\phi_R(t, s)$ in (H3′) is due to Barles and Lions [2]. We refer to this paper for modification of the proofs below when (H3) is replaced by (H3′).

We now prove our results. We only consider the case of (H3)–(H5), the proof of the other results (for (H3′)–(H5′)) is obtained by replacing $d(x)$ by $\underline{d}(x)$.

Proof of Theorem 1. We are interested in

$$M = \operatorname*{Max}_{x\in\bar{\Omega}} (u-v)(x) = (u-v)(x_0) \qquad \text{for some} \quad x_0 \in \partial\Omega$$

(if this maximum is only achieved at interior points, we conclude as usual). Four cases may occur:

(i) $u(x_0) \le \varphi(x_0)$, $v(x_0) \ge \varphi(x_0)$,
(ii) $u(x_0) \le \varphi(x_0)$, $v(x_0) < \varphi(x_0)$,
(iii) $u(x_0) > \varphi(x_0)$, $v(x_0) \ge \varphi(x_0)$,
(iv) $u(x_0) > \varphi(x_0)$, $v(x_0) < \varphi(x_0)$.

In case (i), the proof is finished. Let us only consider case (ii), indeed case (iii) is similar and case (iv) is easier. We set

$$R = \max(\|u\|_\infty, \|v\|_\infty)$$

and we drop the dependence in R of the assumptions. Since $x_0 \in \Gamma_1$, (4) asserts that we may choose x_n such that

$$u(x_n) \to u(x_0) \qquad \text{as} \quad n \to \infty,$$

and let us introduce the function

$$\psi_n(x, y) = u(x) - v(y) - \frac{|x-y|^2}{\varepsilon_n^2} - \left[\left(\frac{d(x)-d(y)}{\alpha_n} - 1\right)^-\right]^2 - |x-x_0|^2,$$

where

$$\varepsilon_n = |x_n - x_0|, \qquad \alpha_n = d(x_n),$$

and set

$$M_n = \operatorname*{Max}_{x,y\in\bar\Omega} \psi_n(x, y).$$

Considering the particular points $x = x_n$, $y = x_0$, we get

$$\liminf_{n\to\infty} M_n \geqslant M. \tag{6}$$

Let us point out that the new feature here is to take two scales, namely $\alpha_n < \varepsilon_n$, in order to get (6). We denote $(\bar{x}_n, \bar{y}_n)$ a maximum point of ψ_n, then we easily check that, as $n \to \infty$,

$$v(\bar{y}_n) \to v(x_0),$$

$$u(\bar{x}_n) \to u(x_0),$$

$$\frac{|\bar{x}_n - \bar{y}_n|^2}{\varepsilon_n^2} \to 0,$$

$$|\bar{x}_n - x_0|^2 \to 0,$$

$$\left(\frac{d(\bar{x}_n) - d(\bar{y}_n)}{\alpha_n} - 1\right)^- \to 0,$$

since

$$\liminf_{n\to\infty} M_n + \frac{|\bar{x}_n - \bar{y}_n|^2}{\varepsilon_n^2} + \left[\left(\frac{d(\bar{x}_n) - d(\bar{y}_n)}{\alpha_n} - 1\right)^-\right]^2 + |\bar{x}_n - x_0|$$
$$\le \liminf_{n\to\infty} u(\bar{x}_n) - v(\bar{y}_n) \le u(x_0) - v(x_0) = M.$$

Therefore, $\bar{x}_n \in \Omega$ and either $\bar{y}_n \in \Omega$ or $\bar{y}_n \in \partial\Omega$ and $v(\bar{y}_n) < \varphi(\bar{y}_n)$ (for n large enough). Thus, we may use the equations on u and v, and we get

$$H\left(\bar{x}_n, u(\bar{x}_n), \frac{2(\bar{x}_n - \bar{y}_n)}{\varepsilon_n^2} + \lambda_n n(\bar{x}_n) + \rho_n\right) \le 0,$$

$$H\left(\bar{y}_n, v(\bar{y}_n), \frac{2(\bar{x}_n - \bar{y}_n)}{\varepsilon_n^2} + \lambda n(\bar{y}_n)\right) \ge 0,$$

where

$$\rho_n = 2(\bar{x}_n - x_0), \qquad \lambda_n = \frac{2}{\alpha_n}\left(\frac{d(\bar{x}_n) - d(\bar{y}_n)}{\alpha_n} - 1\right)^-.$$

Now (H4) asserts that, for some constant $C(x_0)$,

$$0 \le \lambda_n \le C\left(1 + \frac{|x_n - y_n|}{\varepsilon_n^2}\right), \tag{7}$$

and (H2), (H3) and the Lipschitz continuity of $n(x)$ (or (H3′)) concludes the proof of Theorem 1. □

Proof of Proposition 2. Let $x_0 \in \{x \in \partial\Omega;\ v(x) < \varphi(x)\}$ and let us prove that $x_0 \in \Gamma_1'$. Consider the function

$$\psi_\varepsilon(x) = v(x) + \frac{|x-x_0|^2}{\varepsilon^2} + \frac{d(x)}{\varepsilon^2}.$$

If x_ε is a minimum point of ψ_ε, then $x_\varepsilon \to x_0$, $v(x_\varepsilon) \to v(x_0)$ so that, for ε small enough, if $x_\varepsilon \in \partial\Omega$, then $v(x_\varepsilon) < \varphi(x_\varepsilon)$. Hence

$$H\left(x_\varepsilon, v(x_\varepsilon), \frac{-2(x_\varepsilon - x_0)}{\varepsilon^2} + \frac{n(x_\varepsilon)}{\varepsilon^2}\right) \geq 0,$$

and thus $x_0 \in \Gamma_1'$ which we want to prove. The same proof applies for Γ_2' and we do not give it here. □

Proof of Corollary 3. Let us modify v so that it satisfies (4) on the set $\{x \in \partial\Omega;\ v(x) \geq \varphi(x)\} \subset \Gamma_2'$. To do so, define $\tilde{v}$ by

$$\tilde{v}(x) = \begin{cases} v(x) & \text{if} \quad x \notin \Gamma_2', \\ \liminf\limits_{y \to x,\, y \in \Omega} v(y) & \text{if} \quad x \in \Gamma_2'. \end{cases}$$

Let us first remark that, by their definition in Corollary I.1, Γ_1' and Γ_2' are clearly closed but they are also open by assumptions (H4) and (H5). Hence, Γ_1' and Γ_2' are both unions of connex components of $\partial\Omega$. This remark shows that $\tilde{v}$ is still l.s.c.

We now prove that $\tilde{v}$ is still a viscosity supersolution of (1) in Γ_2'. We have only to check it at the points x where $\tilde{v}(x) < \varphi(x)$. We claim that at these points $\tilde{v}(x) = v(x)$. Indeed, assume, by contradiction, that $\tilde{v}(x) > v(x)$, and consider the function ψ defined by

$$\psi(y) = v(y) + \frac{|x-y|^2}{\varepsilon^2} - \frac{d(y)}{\varepsilon^2}.$$

Since $\tilde{v}(x) > v(x)$, it is clear that ψ has a local minimum point achieved at $y_\varepsilon \in \partial\Omega$. Moreover, $v(y_\varepsilon) \to v(x) < \varphi(x)$, therefore, for ε small enough, $v(y_\varepsilon) < \varphi(y_\varepsilon)$. But v is a viscosity supersolution of (1), hence

$$H\left(y_\varepsilon, v(y_\varepsilon), \frac{2(y_\varepsilon - x)}{\varepsilon^2} - \frac{1}{\varepsilon^2} n(y_\varepsilon)\right) \geq 0.$$

Since $y_\varepsilon \to x$, the above inequality contradicts the fact that $x \in \Gamma_2$ and (H5). Using this result, we easily prove that $\tilde{v}$ is a viscosity supersolution of (2′) on Γ_2' (because the points y, such that $\tilde{v}(y) \geq \varphi(y)$, do not play any role when we look at a minimum point of $\tilde{v} - \phi$, $\phi \in C^1$, achieved at x, $\tilde{v}(x) < \varphi(x)$).

Now, to conclude, we claim that the same properties hold for u (we define $\tilde{u}\ldots$ in the same way), we can compare $\tilde{u}$ and $\tilde{v}$ by Theorem 1 and therefore

$$\tilde{u} \leq \tilde{v} \qquad \text{in } \bar{\Omega},$$

giving the announced result in Ω. □

Proof of Corollary 4. In fact, the proof is exactly the same as the proof of Corollary I.1, even simpler since we can take $\Gamma_2=\varnothing$, $\Gamma_1=\partial\Omega$. We redefine u on $\partial\Omega$ by

$$u(x)=\limsup_{\substack{y\to x\\ y\in\Omega}} u(y).$$

u is still u.s.c. and is still a viscosity subsolution of (5) since there is no condition on $\partial\Omega$. We conclude by using Theorem 1 with $\varphi=\|u\|_\infty+\|v\|_\infty+1$. □

3. Exit-Time Control Problems

Here we briefly describe the problem to state our notations. For a complete presentation, in particular in connection with the boundary conditions (1′), (2′), we refer the reader to [3], [4], and [16].

We consider a system in which the state is given by the solution of the differential equation

$$\frac{dy_x(t)}{dt}+b(y_x(t),v(t))=0, \qquad y_x(0)=x\in\Omega.$$

To $v(\cdot)$, we associate the cost function

$$J(x,v(\cdot))=\int_0^\tau f(y_x(t),v(t))\,e^{-\lambda t}\,dt+\varphi(y_x(\tau))\,e^{-\lambda\tau},$$

where φ, f, b are Lipschitz continuous functions, λ is a positive real number, and $v(\cdot)$ is any measurable function with value in V, some compact metric space (the space of controls). Finally, τ $(\equiv\tau(x,v(\cdot)))$ denotes the first exit time of the trajectory $y_x(t)$ from the *open set* Ω.

Let us recall [4] that we could use different stopping times, for example, the first exit time from $\bar\Omega$, but τ plays a particular role. Indeed, if we consider the value function

$$u(x)=\operatorname{Inf}\{J(x,v(\cdot));\ v(\cdot)\in L^\infty(\mathbb{R}^+,V)\}, \tag{8}$$

then the following properties of u are as proved in [4].

If we set

$$v_*(x)=\liminf_{y\to x,\,y\in\bar\Omega} v(y), \qquad v^*(x)=\limsup_{y\to x,\,y\in\bar\Omega} v(y),$$

and

$$u_0(x)=\begin{cases} u_*(x) & \text{for } x\in\Omega,\\ \liminf\limits_{y\to x,\,y\in\Omega} & \text{for } x\in\partial\Omega,\end{cases} \tag{9}$$

and

$$H(x,t,p)=\sup_{v\in V}\{b(x,v)\cdot p+\lambda t-f(x,v)\}, \tag{10}$$

then we have

$$u_0 \text{ is a supersolution of (2), (2')}, \tag{11}$$

$$(u_0)^* \text{ is a subsolution of (1), (1')}, \tag{12}$$

$$((u_0)^*)_* = u_0 \qquad \text{in } \bar{\Omega}. \tag{13}$$

Therefore, u_0 has a particular property among sub- and supersolutions of (1), (1′), (2), (2′), namely (13) and this could be a uniqueness criteria.

As in Section 2 we can ask the question: when is it possible to compare the subsolution (u_0) and the supersolution u_0? If this is possible we obtain

$$(u_0)^* \leq u_0,$$

and thus

$$(u_0)^* = u_0 \in C(\bar{\Omega}).$$

To do so, it is sufficient to give conditions on b on the boundary which ensures that (H4) holds on Γ_1' and (H5) holds on Γ_2'. These conditions are:

(H6) $\forall x \in \partial\Omega$; if there exists $v \in V$ with $b(x, v) \cdot n(x) \geq 0$, then there exists $v' \in V$ with $b(x, v') \cdot n(x) > 0$.

(H7) $\forall x \in \partial\Omega$; if for all $v \in V$, $b(x, v) \cdot n(x) \geq 0$, then, for all $v \in V$, $b(x, v) \cdot n(x) > 0$.

It is clear that if $x_0 \in \Gamma_1'$, then, for some $v \in V$, $b(x_0) \cdot n(x_0) \geq 0$. Hence (H6) implies that (H4) holds for this point x_0. Thus (H6) implies that (H4) holds on Γ_1'. In the same way we obtain that (H7) implies that (H5) holds on Γ_2'. Therefore we have the following versions of Corollaries 3 and 4.

Proposition 3′. *Assume* $\lambda > 0$, f, $b_i \in W^{1,\infty}(\Omega)$, $\varphi \in C(\partial\Omega)$ *and* (H6) *and* (H7) *hold, then* u_0 *defined by* (8), (9) *is continuous in* $\bar{\Omega}$ *and is the unique viscosity solution of*

$$H(x, u, Du) = 0 \qquad \text{in } \Omega, \tag{14}$$

with the generalized Dirichlet boundary conditions (2), (2′).

Proposition 4′ (State Constraints). *Assume* $\lambda > 0, f, b_i \in W^{1,\infty}(\Omega)$, $\varphi = \|f\|_\infty + 1/\lambda$, *and, for some* η,

$$\forall x \in \partial\Omega, \quad v_x \in V, \qquad b(x, v_x) \cdot n(x) \geq \eta > 0, \tag{15}$$

then u_0 *defined by* (8), (9) *is continuous in* $\bar{\Omega}$ *and is the unique viscosity solution of* (14) *with the boundary condition* (2′).

Remark. Let us state that, in Propositions 3′ and 4′, the uniqueness in $\bar{\Omega}$ holds in the class of functions which are either continuous in $\bar{\Omega}$ or satisfy

$$u(x) = \liminf_{y \to x,\, y \in \Omega} u(y) \qquad \text{on } \partial\Omega.$$

If no assumption is made on the boundary, uniqueness holds in Ω (any u.s.c. subsolution is smaller than u_0 in Ω, any l.s.c. supersolution is larger than u_0).

Finally, let us mention that (15) was introduced by Soner [24] to prove the continuity of the value function for state-constraint optimal control problems. It is possible to build examples in which (15) fails at one single point and for which u_0 is not continuous. Therefore, (15) is nearly optimal (however, it could be modified in the same spirit as (H3′)-(H5′)). Condition (15) was extended to

$$\forall x \in \partial\Omega, \qquad \lim_{\lambda/(1+|p|)\to+\infty} H(x, t, p+\lambda n(x)) = +\infty, \qquad t \in \mathbb{R}$$

(which is equivalent to (H4) on $\partial\Omega$ as in Corollary 4, since b is Lipschitzian) by Capuzzo-Dolcetta and Lions [5] in order to treat general state-constraint boundary conditions for Hamilton–Jacobi equations. Our result in Proposition 4′ is therefore equivalent to the one in [5], except if we use the generalization of (H3′) and (H4′). We refer to [5] for detailed discussions concerning relations between the assumptions on H and b.

4. Extensions to the Cauchy Problem

We are now interested in the problem

$$\frac{\partial u}{\partial t} + H(x, t, u, Du) = 0 \qquad \text{in} \quad \Omega\times]0, T[, \tag{16}$$

with some boundary conditions. In this case, the boundary conditions are not so clear, in the sense that if we want to copy (1), φ is not defined in $\Omega\times\{T\}$. The first idea is to impose no condition on this set and to copy (1) for the other parts, i.e.,

$$\operatorname{Max}\left(\frac{\partial u}{\partial t} + H(x, t, u, Du), u - \varphi_*\right) \geq 0 \qquad \text{on} \quad \partial\Omega\times]0, T[, \tag{17}$$

$$\operatorname{Min}\left(\frac{\partial u}{\partial t} + H(x, t, u, Du), u - \varphi^*\right) \leq 0 \qquad \text{on} \quad \partial\Omega\times]0, T[, \tag{18}$$

$$\operatorname{Max}\left(\frac{\partial u}{\partial t} + H(x, t, u, Du), u - u_{0*}, u - \varphi_*\right) \geq 0 \qquad \text{in} \quad \bar\Omega\times\{0\}, \tag{19}$$

$$\operatorname{Min}\left(\frac{\partial u}{\partial t} + H(x, t, u, Du), u - u_0^*, u - \varphi^*\right) \leq 0 \qquad \text{in} \quad \bar\Omega\times\{0\} \tag{20}$$

(in order to simplify the equation we have set $\varphi_* = +\infty$ in Ω, $\varphi^* = -\infty$ in Ω), φ and u_0 are respectively the boundary condition and the initial condition. For the moment we do not assume that they are continuous. Our first aim is to define this problem. For $t = 0$ or T, some terms in the above equations are superfluous since we have

Proposition 5. *Assume that H is continuous, φ, u_0 are bounded. Let u (resp. v) be a bounded viscosity subsolution of* (16)-(20) (*resp. supersolution*), *then u* (*resp.*

v) satisfies

$$\frac{\partial u}{\partial t}+H(x,t,u,Du)\leq 0 \qquad \textit{in} \quad \Omega\times]0,T], \tag{16'}$$

$$\operatorname{Min}\left(\frac{\partial u}{\partial t}+H(x,t,u,Du),u-\varphi^*\right)\leq 0 \qquad \textit{on} \quad \partial\Omega\times]0,T] \tag{17'}$$

and

$$\begin{cases} u^*\leq (u_0)^* & \textit{in} \quad \Omega\times\{0\},\\ u^*\leq \operatorname{Max}(u_0^*,\varphi^*) & \textit{on} \quad \partial\Omega\times\{0\} \end{cases} \tag{20'}$$

(resp.

$$\frac{\partial v}{\partial t}+H(x,t,v,Dv)\geq 0 \qquad \textit{in} \quad \Omega\times]0,T] \tag{16''}$$

and

$$\operatorname{Max}\left(\frac{\partial v}{\partial t}+H(x,t,v,Dv),v-\varphi_*\right)\geq 0 \qquad \textit{on} \quad \partial\Omega\times]0,T], \tag{18'}$$

$$\begin{cases} v_*\geq (u_0)_* & \textit{in} \quad \Omega\times\{0\},\\ v_*\geq \operatorname{Min}((u_0)_*,\varphi_*) & \textit{on} \quad \partial\Omega\times\{0\}). \end{cases} \tag{19'}$$

Let us state that the u.s.c. or l.s.c. envelopes of u and v are extended to $\bar{\Omega}\times\{T\}$ by "regularity," i.e.,

$$u^*(x,T)=\limsup_{\substack{y\in\bar{\Omega}\\ y\to x\\ t\to T\\ t<T}} u^*(y,t), \qquad v_*(x,t)=\liminf_{\substack{y\in\bar{\Omega}\\ y\to x\\ t\to T\\ t<T}} v_*(y,t).$$

From here on we assume that u_0, φ are continuous and

$$u_0(x)=\varphi(x,0), \qquad \forall x\in\partial\Omega. \tag{21}$$

We are interested in comparison results for the problem

$$\begin{cases} \dfrac{\partial u}{\partial t}+H(x,t,u,Du)=0 & \text{in} \quad \Omega\times]0,T],\\ u(x,0)=u_0(x) & \text{in } \bar{\Omega},\\ \operatorname{Max}\left(\dfrac{\partial u}{\partial t}+H(x,t,u,Du),u-\varphi\right)\geq 0 & \text{on} \quad \partial\Omega\times]0,T],\\ \operatorname{Min}\left(\dfrac{\partial u}{\partial t}+H(x,t,u,Du),u-\varphi\right)\leq 0 & \text{on} \quad \partial\Omega\times]0,T]. \end{cases} \tag{22}$$

In order to avoid a long list of assumptions, we say that H satisfies (H1)-(H3) if the hamiltonian $F(y,u,p)$, defined by

$$F(y,u,p)=H(x,t,u,p),$$

satisfies (H1)-(H3) in the open subset $Q=\Omega\times]0,T[$, where $y=(x,t)$, and, in (H2), γ_R is not required to be nonnegative.

We recall that we consider that Ω is bounded and $C^{1,1}$. Our nondegeneracy conditions are now:

(H6) $\forall(x_0,t_0)\in\Gamma_1,\ \exists\eta>0,\ \exists C_1^R,\ C_2^R>0$ such that
$H(x,t,u,p+\lambda n(x))\geq C_1^R\lambda-C_2^R(1+|p|)$
for $(x,t)\in B((x_0+t_0),\eta)\cap(\bar{\Omega}\times[0,T])$, $|u|\leq R$, $p\in\mathbb{R}^n$, $\lambda\geq 0$.

(H7) $\forall(x_0,t_0)\in\Gamma_2,\ \exists\eta,\ \exists C_1^R,\ C_2^R>0$ such that
$H(x,t,u,p-\lambda n(x))\leq -C_1^R\lambda+C_2^R(1+|p|)$
for $(x,t)\in B((x_0,t_0),\eta)\cap(\bar{\Omega}\times[0,T])$, $|u|\leq R$, $p\in\mathbb{R}^n$, $\lambda\geq 0$.

Our result is

Proposition 6. *Assume that H satisfies* (H1)-(H3), *that u_0, φ are continuous, and that* (21) *holds. Assume, in addition, that* (H6) *and* (H7) *hold with $\Gamma_1=\Gamma_1''$ and $\Gamma_2=\Gamma_2''$ where Γ_1'' and Γ_2'' are defined by*

Γ_1'' is the set of points $(x_0,t_0)\in\partial\Omega\times]0,T]$ such that

$$\lim_{\varepsilon\to 0}\operatorname{Sup}\left\{p_0+H(x,t,u,p+\lambda n(x));\ |x-x_0|+|t-t_0|\leq\varepsilon,\ \frac{\lambda}{1+|p|+|p_0|}\geq\frac{1}{\varepsilon},\ |t|\leq R\right\}\geq 0\qquad\text{for all } R>0,$$

Γ_2'' is the set of points $(x_0,t_0)\in\partial\Omega\times]0,T]$ such that

$$\lim_{\varepsilon\to 0}\operatorname{Inf}\left\{p_0+H(x,t,u,p+\lambda n(x));\ |x-x_0|+|t-t_0|\leq\varepsilon,\ \frac{\lambda}{1+|p|+|p_0|}\geq\frac{1}{\varepsilon},\ |t|\leq R\right\}\leq 0.$$

Let u be a bounded u.s.c. viscosity subsolution of (22) *and let v be a bounded l.s.c. viscosity supersolution of* (22), *then*

$$u\leq v\qquad\text{in } \Omega.$$

We do not give the analogous result to Corollary 3 that the reader could easily write.

Remark. With our conventions, (H3) imposes some restrictions on the modulus of continuity of H in t, connected to p. This assumption is unusual and it is an open problem to know if we can drop it. In fact, we can already relax it by assuming that it holds only in a neighborhood of $\partial\Omega$.

Now we turn to the proofs.

Proof of Proposition 5. We only prove the claims for u, the proofs for v are totally similar. We first prove (16′). Let $x\in\Omega$ and let ϕ be a C^1 function such that $u^*(y,t)-\phi(y,t)$ achieves a local maximum point at (x,T). We may assume that it is a strict maximum point. Now let us consider the function

$$\psi(y,t)=u^*(y,t)-\phi(y,t)-\left|\left(\frac{T-t}{\varepsilon}-1\right)^-\right|^2.$$

For ε small enough, ψ has a local maximum point $(x_\varepsilon, t_\varepsilon)$ in $\Omega\times]0,T[$; moreover, $(x_\varepsilon,t_\varepsilon)\to(x,T)$ when $\varepsilon\to 0$. Using (16) we get

$$\phi_t(x_\varepsilon,t_\varepsilon)+\frac{2}{\varepsilon}\left(\frac{T-t_\varepsilon}{\varepsilon}-1\right)^- + H(x_\varepsilon,t_\varepsilon,u^*(x_\varepsilon,t_\varepsilon),D\phi(x_\varepsilon,t_\varepsilon))\le 0.$$

Since $u^*(x_\varepsilon,t_\varepsilon)\to_{\varepsilon\to 0} u^*(x,T)$ and $(2/3)((T-t_\varepsilon)/\varepsilon-1)^-\ge 0$, we conclude easily, using the continuity of H by letting $\varepsilon\to 0$.

Now we prove (20′). Let $x_0\in\bar\Omega$, we consider the function

$$\psi(x,t)=u^*(x,t)-\frac{|x-x_0|^2}{\varepsilon^2}-Ct.$$

When $(\varepsilon, C)\to(0,+\infty)$ the maximum point (x,t) of ψ (we drop the dependence of (x,t) in (ε,C) for the sake of clarity) converges to $(x_0,0)$ and $u^*(x,t)\to u^*(x_0,0)$. Now if (x,t) belongs to $\Omega\times]0,T[$, or if $u^*(x,t)>\varphi^*(x,t)$ or $u_0^*(x)$, we have

$$C+H\left(x,t,u^*(x,t),\frac{2(x-x_0)}{\varepsilon^2}\right)\le 0.$$

We choose

$$C=\operatorname{Max}\left\{H(y,s,r,p),\, y\in\bar\Omega,\, s\in[0,T],\, |r|\le\|u^*\|_\infty,\, |p|\le\frac{1}{\varepsilon^2}\right\}.$$

Hence, the above inequality cannot hold and, therefore, either $t=0$ and $u^*(x,0)\le u_0^*(x)$ or $t>0$, $x\in\partial\Omega$, and $u^*(x,t)\le\varphi^*(x,t)$. Letting $\varepsilon\to 0$, we conclude. □

Proof of Proposition 6. We only sketch the proof since it is basically a straightforward adaptation of the proofs of Theorem 1 and Corollary 3. We first make the change of unknown functions $\tilde u=e^{-\beta t}u$, $\tilde v=e^{-\beta t}v$, with $\beta=\bar\gamma+1$, $\gamma=\gamma_R$ given by (H2), and $R=\max(\|u\|_\infty,\|v\|_\infty)$, in order to get some growth in $\tilde u$ and $\tilde v$. The equations become

$$\tilde u_t+\beta\tilde u+e^{-\beta t}H(x,t,e^{\beta t}\tilde u, e^{\beta t}D\tilde u)=0.$$

If the maximum of $\tilde u-\tilde v$ is achieved at $(x_0,t_0)\in\Gamma_1''$, we introduce the function

$$\psi(x,y,t,s)=\tilde u(x,t)-\tilde v(y,s)-\frac{|x-y|^2}{2}-\frac{|t-s|^2}{2}-\cdots$$
$$-\left[\left(\frac{d(x)-d(y)}{\alpha}-1\right)^-\right]^2-|x-x_0|^2-|t-t_0|^2.$$

We choose (ε,α) as in the proof of Theorem 1 in such a way that, as $(\varepsilon,\alpha)\to(0,0)$,

$$\operatorname{Max}\psi(x,y,t,s)\to\operatorname{Max}[\tilde u(x,t)-\tilde v(x,t)].$$

We also get the same estimate

$$\lambda_\alpha = \frac{2}{\alpha}\left(\frac{d(x)-d(y)}{\alpha}-1\right)^{-} \leq C\left(1+\frac{|x-y|}{\varepsilon^2}+\frac{|t-s|}{\varepsilon^2}\right)$$

for the maximum points (x, y, t, s) of ψ. We write the equation for $\tilde{u}$ and $\tilde{v}$, we make the difference so that

$$\begin{aligned}\beta(\tilde{u}(x, t)-\tilde{v}(y, s))&+e^{-\beta t}[H(x, t, e^{+\beta t}\tilde{u}(x, t), p_\varepsilon)-H(x, t, e^{+\beta s}\tilde{v}(y, s), p_\varepsilon)]\\ &\leq e^{-\beta s}H(y, s, e^{\beta s}\tilde{v}(y, s), q_\varepsilon)-e^{-\beta t}H(x, t, e^{+\beta s}\tilde{v}(y, s), p_\varepsilon)-2(t-t_0),\end{aligned}$$

where

$$p_\varepsilon = \left(\frac{2(x-y)}{\varepsilon^2}+\lambda_\alpha n(x)+2(x-x_0)\right)e^{\beta t},$$

$$q_\varepsilon = \left(\frac{2(x-y)}{\varepsilon^2}+\lambda_\alpha n(y)\right)e^{\beta s}.$$

Using (H3), we can see that the right-hand side goes to zero when (ε, α) goes to zero, indeed, it may be bounded (denoting ρ_H the modulus of continuity of H in p) by

$$\begin{aligned}C|t-s|H(x, t, e^{\beta s}\tilde{v}(y, s), p_\varepsilon)&+C(|t-s|+|x-y|)|p_\varepsilon|+\rho_H(|p_\varepsilon-q_\varepsilon|)+2|t-t_0|\\ &\leq C\frac{|t-s|^2}{\varepsilon^2}+C\frac{|x-y|^2}{\varepsilon^2}+\rho_H\left(\frac{|x-y|^2}{\varepsilon^2}+\frac{|t-s|^2}{\varepsilon^2}+|x-x_0|\right)+2|t-t_0|.\end{aligned}$$

Concerning the left-hand side, if $\operatorname{Max}(\tilde{u}-\tilde{v})>0$, then $e^{-\beta t}\tilde{u}(x, t)\geq e^{-\beta s}\tilde{v}(y, s)$ for (ε, α) small enough, hence

$$\beta(\tilde{u}(x, t)-\tilde{v}(y, s))+e^{-\beta t}\gamma\cdot(e^{-\beta t}\tilde{u}(x, t)-e^{-\beta s}\tilde{u}(y, s))\leq \rho(\varepsilon, \alpha),$$

where $\rho(\varepsilon, \alpha)\to 0$ when $(\varepsilon, \alpha)\to 0$. Since $\beta=\bar{\gamma}+1$, this implies that

$$\tilde{u}(x, t)-\tilde{v}(y, s)\leq \tilde{\rho}(\varepsilon, \alpha),$$

where $\tilde{\rho}(\varepsilon, \alpha)\to 0$ when $(\varepsilon, \alpha)\to 0$ and the result follows. □

Let us conclude this part by some remarks on the case when condition (21) is violated. A new difficulty appears at the points of $\partial\Omega\times\{0\}$, and it is no longer clear that a comparison principle holds. We could write a general result but for the sake of simplicity we restrict ourselves to the state-constraint case (in the proof of Proposition 10 below we mix the ideas of Propositions 6 and 7).

Let us consider the problem

$$\begin{cases}\dfrac{\partial u}{\partial t}+H(x, t, u, Du)=0 & \text{in} \quad \Omega\times]0, T],\\[2mm] \dfrac{\partial u}{\partial t}+H(x, t, u, Du)\geq 0 & \text{on} \quad \partial\Omega\times]0, T],\\[2mm] u(x, 0)=u_0(x) & \text{in } \Omega,\\ u(x, 0)\geq u_0(x) & \text{on } \partial\Omega\end{cases} \tag{23}$$

(we formally take $\varphi=+\infty$ in (17)-(20) and thus we loosen the subsolution condition on $\partial\Omega\times\{0\}$). Again we can compare sub- and supersolutions:

Proposition 7. *Assume that H satisfies* (H1)–(H3) *and* (H6) *with $\Gamma_1 = \partial\Omega$ and that u_0 is continuous. Let u (resp. v) be a bounded u.s.c. (resp. l.s.c.) subsolution (resp. supersolution) of* (23), *then*

$$u \le v \qquad \text{in} \quad \Omega \times]0, T[.$$

Proof of Proposition 7. In order to avoid technical details let us admit that the sup below are attained and that γ in (H2) is positive; if not, an additional term $e^{-\beta t}$ is necessary as in the proof of Proposition 6. As in the proof of Corollary 4, we may also assume that

$$u(x, t) = \limsup_{\substack{(y,s)\to(x,t)\\ (y,s)\in\Omega\times[0,T]}} u(y, s), \qquad \forall (x, t) \in \partial(\Omega \times [0, T]). \tag{24}$$

It is enough to prove that

$$u \le u_0 \qquad \text{on} \quad \partial\Omega \times \{0\}, \tag{25}$$

since with (25), the proof of Proposition 6 applies, (25) replaces (21).

To prove (25), we build a sequence of supersolutions of (23) to show that we may compare them with u. Let $u_\eta \in C^1(\bar{\Omega})$, $u_\eta \to u_0$ as $\eta \to 0$, (H6) shows that for some C_η large enough the function ω_η defined by

$$\omega_\eta(x, t) = C_\eta t + u_\eta(x), \qquad x \in \bar{\Omega}, \quad t \in [0, T],$$

is a viscosity supersolution of (23).

To compare u and ω_η, the only new case is when

$$\operatorname*{Max}_{\bar{\Omega}\times[0,T]} (u - \omega_\eta) = (u - \omega_\eta)(x_0, 0), \qquad x_0 \in \partial\Omega.$$

Then we introduce the test function

$$-\psi_\eta(x, y, t, s) = u(x, t) - \omega_\eta(y, s) - \frac{|x-y|^2}{\varepsilon_n^2} - \left|\left(\frac{d(x)-d(y)}{\alpha_n} - 1\right)^-\right|^2 - \left(\frac{s-t}{\varepsilon_n} - 1\right)^2.$$

As in Theorem 1, ε_n and α_n are given by

$$\varepsilon_n = |x_n - x_0|,$$

$$\alpha_n = d(x_n),$$

where $(x_n, t_n) \in \Omega \times]0, T[$ is a sequence such that

$$u(x_n, t_n) \to u(x_0, 0), \qquad (x_n, t_n) \to (x_0, t_0),$$

following (25). Since ω_η is continuous, we know that

$$\operatorname*{Max}_{\bar{\Omega}\times[0,T]} (u - \omega_\eta) \le \liminf_{\eta\to 0} \operatorname*{Max}_{x,y,t,s} \psi_\eta(x, y, t, s) := M_\eta.$$

If the maximum of ψ_η is achieved for $t = 0$ we clearly conclude that $M_\eta \to 0$ since the corresponding x belongs to Ω and since ω_η is continuous. If it is achieved for $t>0$, then the corresponding s is positive and we are reduced to the case of Proposition 6. Therefore,

$$u \le C_\eta t + u_\eta \quad \text{in} \quad \bar{\Omega}\times[0,T],$$

hence

$$u(x,0) \le u_\eta(x), \qquad x\in\bar{\Omega},$$

as η goes to 0 we get

$$u(x,0) \le u_0(x), \qquad x\in\Omega.$$

This proves (25) and Proposition 7. □

5. Singular Perturbations for Degenerated Diffusions

In this section we apply the above ideas to some singular perturbation problems of Wentzell-Freidlin type for degenerated diffusions. Namely, we generalize the examples treated by Evans and Ishii [9] to degenerated hamiltonians; we refer to this paper for the probabilistic motivations.

As was pointed out in [4], the main interest of discontinuous viscosity solutions is that they allow us to pass to the limit in the vanishing viscosity method without any *a priori* estimates on the gradient of the solution. When considering degenerated diffusions, such estimates are not known and may be wrong, and the method introduced in [4] is therefore well adapted.

Moreover, we introduce two additional arguments which lead, for the results presented below, to the same, completely automatic and easy proofs. In order to study the large deviation behavior of u^ε, the first argument consists in replacing the change of variable introduced by Fleming [12],

$$v^\varepsilon = -\varepsilon \log u^\varepsilon,$$

by the change

$$\omega^\varepsilon = -(u^\varepsilon)^\varepsilon$$

obtained from the preceding by making Kruzkov's change of type [17]

$$\omega^\varepsilon = -e^{-v^\varepsilon}.$$

The point is that ω^ε is bounded when v^ε is bounded from below and generally L^∞-estimates for ω^ε are easy to deduce while v^ε may go to $+\infty$ at some points. The second argument is a uniqueness one for the limit equation. Due to the above change, the hamiltonian is now

$$(-\omega\cdot\omega_t) + a_{ij}^{(x)}\omega_i\cdot\omega_j + (b_i(x)\cdot\omega_i)\omega = 0;$$

since (a_{ij}) may be degenerated and since ω may be 0 at some points, this type of hamiltonian does not seem to be covered by the classical uniqueness theorems. Precise results are given in the Appendix.

Using these arguments, we give two kinds of results. The first consists in assuming that the diffusion matrix a_{ij}, although degenerated, has some properties which allow the construction of a continuous supersolution of the Hamilton–Jacobi equation of interest. Then we prove the large deviation behavior using only the argument of [4]; this is the case if a_{ij} is hypoelliptic, for example. The second kind of result we give only requires the nondegeneracy of (a_{ij}) along the normal vector to $\partial\Omega$ at each point of $\partial\Omega$ and is based on the ideas of Theorem 1 and the Appendix.

5.1. Asymptotic of the Exit Time

Let $(a^\varepsilon_{ij}(x))$, $x\in\Omega$ (a smooth open subset of $\mathbb{R}^n$), be a positive matrix satisfying

$$\begin{cases} a^\varepsilon_{ij} \xrightarrow{\varepsilon\to 0} a_{ij} \quad \text{in } C(\bar\Omega), & 1\le i,j\le N, \\ a(x)=\sigma^T(x)\sigma(x), & \sigma_{ij}\in W^{1,\infty}, \quad 1\le i\le m, \quad 1\le j\le N. \end{cases} \tag{26}$$

We consider the solution u^ε of

$$\begin{cases} -\varepsilon^2 a^\varepsilon_{ij}\dfrac{\partial^2 u^\varepsilon}{\partial x_i \partial x_j}+\lambda u^\varepsilon=0 & \text{in } \Omega, \\ u^\varepsilon=1 \qquad \text{on } \partial\Omega. \end{cases}$$

Our result is

Proposition 8. *Assume* (26) *holds and either*

There exists $W\in C(\bar\Omega)$ satisfying in the viscosity sense

$$a_{ij}W_iW_j\ge\lambda \quad \text{in } \Omega; \qquad W=0 \quad \text{on } \partial\Omega, \tag{27}$$

or

$$a_{ij}(x)n_i(x)n_j(x)>0 \qquad \text{on } \partial\Omega, \tag{28}$$

where $n(x)$ denotes the outward unit normal to $\partial\Omega$ at x. Then

$$-\varepsilon \operatorname{Log} u^\varepsilon(x) \underset{\varepsilon\to 0}{\longrightarrow} I(x) \qquad \text{in } \Omega,$$

where I is the value function of the control problem

$$\dot y(s)=2a(y(s))v(s), \qquad y(0)=x,$$

$$I(x)=\operatorname{Inf}\left\{\int_0^\tau [a(y(s))v(s)\cdot v(s)+\lambda]\,ds;\ v(\cdot)\in L^\infty(\mathbb{R}^+,\mathbb{R}^N)\right\}\le+\infty,$$

where τ is the first exit time of y from Ω. If (26) *holds, $I\in C(\bar\Omega)$ and the above convergence is uniform on $\bar\Omega$.*

Remark. The assumption $a=\sigma^T\sigma$ in (26), which is necessary in degenerated cases, has been introduced by Oleinik [20]; it is very natural from the probabilistic point of view (see [25]). It has been used to treat first-order Hamilton–Jacobi equations by Sayiah [23]. Here we need this assumption in order to prove that (H_3') holds (with $\underline{d}(x)=d(x)$) and to use the argument of [2].

Remark. A typical example when (27) holds is the case of matrices (a_{ij}) which are "hypoelliptic." Indeed, in that case we can prove the local controllability of the optimal control problem appearing in the statement of Proposition 7 (see [26] and [10]) and the function I which solves

$$a_{ij}(x)I_i \cdot I_j = \lambda \quad \text{in } \Omega, \qquad I = 0 \quad \text{on } \partial\Omega,$$

plays the role of W.

5.2. Asymptotic of a Diffusion with a Drift

Now we consider the solution u^ε of

$$\begin{cases} -\varepsilon a_{ij}^\varepsilon \dfrac{\partial^2 u^\varepsilon}{\partial x_i \partial x_j} - b_i(x) \dfrac{\partial u^\varepsilon}{\partial x_i} = 0 & \text{in } \Omega, \\ u^\varepsilon = 0 \quad \text{on } \tilde{\Gamma}_0, \qquad u^\varepsilon = 1 & \text{on } \tilde{\Gamma}_1, \end{cases}$$

where

$$b_i \in W^{1,\infty}(\Omega), \qquad 1 \le i \le N, \tag{29}$$

$$\tilde{\Gamma}_0 \text{ and } \tilde{\Gamma}_1 \text{ are closed disjoint subsets of } \partial\Omega, \qquad \tilde{\Gamma}_0 \cup \tilde{\Gamma}_1 = \partial\Omega. \tag{30}$$

We also assume

$$\begin{cases} \exists T < +\infty, \quad \forall y(\cdot), \qquad \dot{y} = b(y), \qquad y(0) \in \bar{\Omega}, \\ \exists s \le T \quad \text{with} \quad y(s) \notin \bar{\Omega}. \end{cases} \tag{31}$$

Our result is

Proposition 9. *Assume* (16), (28), (29), (30), *and* (31) *hold. Then*

$$-\varepsilon \operatorname{Log} u^\varepsilon(x) \xrightarrow[\varepsilon \to 0]{} J(x) \qquad \text{in } \Omega, \tag{32}$$

where J is the value function of the control problem

$$\dot{y}(s) = 2a(y(s))v(s) + b(y(s)), \qquad y(0) = x \in \Omega,$$

$$J(x) = \operatorname{Inf}\left\{ \int_0^\theta a(y(s))v(s) \cdot v(s)\, ds;\ v(\cdot) \in L^\infty(\mathbb{R}^+, \mathbb{R}^N), \right.$$

$$\left. y(\theta) \in \Gamma_1, y(s) \in \bar{\Omega},\ s \le \theta \right\} \le +\infty. \tag{33}$$

Remark. If $J \in C(\bar{\Omega})$, then, as in Proposition 7, we do not need (28) and (32) holds uniformly on compact subsets of $\partial \cup \Gamma_1$. Again this is true if a_{ij} is "hypoelliptic."

Remark. It is easy to see that condition (31) is necessary to get a large deviation behavior for u^ε (see [9], [12], and [21]). Condition (31) is in fact equivalent to

$$\exists \phi \in C^1(\bar{\Omega}), \quad \exists \gamma > 0, \qquad a_{ij}(x)\phi_i\phi_j - b_i\phi_i \le -\gamma \quad \text{in } \Omega. \tag{31'}$$

Indeed, if (31) holds in Ω, it also holds in

$$\Omega_\delta = \{x \in \mathbb{R}^n \mid d(x, \bar{\Omega}) \le \delta\}$$

for δ small enough. Then we refer to [21] for a proof that the cost function associated with the equation

$$d_{ij}u_i u_j - b_i u_i = -\gamma < 0 \qquad \text{in } \Omega_\delta$$

(where $d_{ij} \ge \nu(Id)$ for some $\nu > 0$) is bounded and thus Lipschitz continuous. Choosing

$$d = a + Id$$

we obtain ϕ by regularizing the obtained cost function by convolution in $\Omega_{\delta/2}$ (see [18]). Conversely, if we consider the optimal control problem associated with the equation

$$a_{ij}(x)u_i u_j - b_i(x)u_i = -\gamma \qquad \text{in } \Omega,$$

i.e.,

$$\dot{y}(s) = 2a(y(s))v(s) + b(y(s)), \qquad y(0) = x \in \Omega,$$

and V_α defined by

$$V_\alpha(x) = \operatorname{Inf}\left\{\int_0^\theta (a(y(s))v(s)\cdot v(s) - \alpha)\, ds,\ v(\cdot) \in L^2(\mathbb{R}^+, \mathbb{R}^N),\right.$$
$$\left. y(\theta) \in \partial\Omega,\ y(s) \in \bar{\Omega},\ s \le \theta\right\}$$

by (31'), V_α is bounded from below (see [21]), therefore choosing $v \equiv 0$ we obtain that θ is bounded for any trajectory of $\dot{y} = b(y)$ and thus (31) holds.

5.3. An Evolution Problem

We consider now the third example of [9]. Let u^ε solve

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} - \varepsilon a_{ij}^\varepsilon(x)\dfrac{\partial^2 u^\varepsilon}{\partial x_i\, \partial x_j} + b_i(x)\dfrac{\partial u^\varepsilon}{\partial x_i} = 0 & \text{in} \quad \Omega \times (0, T), \\ u^\varepsilon = 1 \quad \text{in } \Omega \times \{0\}, \qquad u^\varepsilon = 0 \quad \text{on } \partial\Omega \times (0, T), \end{cases}$$

where (a_{ij}^ε) and (b_i) satisfy respectively (23), (24), and (29).

Our result is

Proposition 10. *Assume* (26), (28), *and* (29) *hold. Then*

$$-\varepsilon \operatorname{Log} u^\varepsilon(x, t) \underset{\varepsilon\to 0}{\longrightarrow} K(x, t) \qquad \textit{in} \quad \Omega \times (0, T), \tag{34}$$

where K is the value function of the control problem

$$\dot{y}(s)=2a(y(s))v(s)+b(y(s)), \qquad y(0)=x\in\Omega,$$

$$K(x,t)=\operatorname{Inf}\left\{\int_0^t a(y(s))v(s)\cdot v(s)\,ds;\ v(\cdot)\in L^\infty(\mathbb{R}^+,\mathbb{R}^N),\right.$$
$$\left. y(s)\in\bar{\Omega},\ s\le t\right\}. \tag{35}$$

5.4. *Proof of Propositions 8–10*

We end this section by the proof of the above results.

Proof of Proposition 8. We make the change of variable

$$\omega^\varepsilon=1-(u^\varepsilon)^\varepsilon, \qquad 0\le\omega^\varepsilon\le 1. \tag{36}$$

Using the argument of [4], we define

$$\bar{\omega}(x)=\limsup_{\substack{y\to x\\ \varepsilon\to 0}}\omega^\varepsilon(y), \qquad \underline{\omega}(x)=\liminf_{\substack{y\to x\\ \varepsilon\to 0}}\omega^\varepsilon(y).$$

$\bar{\omega}$ and $\underline{\omega}$ are respectively sub- and supersolutions of the problem

$$\begin{cases} a_{ij}(x)\omega_i\omega_j+\lambda\omega(2-\omega)-\lambda=0 & \text{in } \bar{\Omega},\\ \operatorname{Min}(\omega,\ a_{ij}(x)\omega_i\omega_j+\lambda\omega(2-\omega)-\lambda)\le 0 & \text{on } \partial\Omega,\\ \operatorname{Max}(\omega,\ a_{ij}(x)\omega_i\omega_j+\lambda\omega(2-\omega)-\lambda)\ge 0 & \text{on } \partial\Omega. \end{cases} \tag{37}$$

If (27) holds, $1-e^{-W}$ is a supersolution with $W=0$ on $\partial\Omega$, therefore we may conclude as in [4], once we have noticed that (H3′) holds with $\underline{d}=d(x,\partial\Omega)$ for the hamiltonian in (37) because

$$\begin{aligned} &a_{ij}(x)p_ip_j-a_{ij}(y)q_iq_j\\ &\quad=|\sigma(x)\cdot p|^2-|\sigma(y)\cdot q|^2\\ &\quad\le(\|\nabla\sigma\|_\infty|x-y|\cdot|p|+\|\sigma\|_\infty|p-q|)(\sqrt{a_{ij}(x)p_ip_j}+\sqrt{a_{ij}(y)q_iq_j}). \end{aligned}$$

If (28) holds, we use Theorem 1. The hamiltonian in (37) also satisfies (H1) and (H2) except for $t=1$, but this is enough since in the proof we never consider points where $\bar{\omega}=\underline{\omega}=1$ (we assume by contradiction that $\operatorname{Max}(\bar{\omega}-\underline{\omega})>0$). Now we may apply Theorem 1. Since $\underline{z}=0$ on $\partial\Omega$, we only have to check (H5) on Γ_2' (following Proposition 2). But Γ_2' is clearly empty because of (27). Therefore we get $\bar{\omega}\le\underline{\omega}$. As usual this means that

$$\omega^\varepsilon\xrightarrow{\varepsilon\to 0}\underline{\omega}=\bar{\omega} \qquad \text{in } C(\bar{\Omega}),$$

and this is a much stronger result than the statement of Proposition 7 and representation formulae (see [18]) show that $I=\underline{\omega}$. □

Proof of Proposition 9. We make the change of variable

$$u^\varepsilon + \varepsilon^{-A/\varepsilon} = \exp(-\omega_A^\varepsilon/\varepsilon),$$

then, defining $\bar\omega_A$, $\underline{\omega}_A$ with lim sup and lim inf as before, we get

$$\bar\omega_A = \mathrm{Min}(A, \bar\omega), \qquad \underline{\omega}_A = \mathrm{Min}(A, \underline{\omega}), \tag{38}$$

and $\bar\omega_A$, $\underline{\omega}_A$ are sub- and supersolutions of

$$\begin{cases} a_{ij}\omega_i\omega_j + b_i\omega_i = 0 & \text{in } \Omega, \\ \mathrm{Min}(\omega - A,\ a_{ij}\omega_i\omega_j + b_i\omega_i) \le 0 & \text{on } \partial\Omega, \\ \mathrm{Max}(\omega - A,\ a_{ij}\omega_i\omega_j + b_i\omega_i) \ge 0 & \text{on } \partial\Omega. \end{cases} \tag{39}$$

In order to get (4) we define

$$\omega_A^*(x) = \limsup_{\substack{y\to x \\ y\in\Omega\cup\tilde\Gamma_1}} \bar\omega_A(x), \qquad \omega_{A*}(x) = \liminf_{\substack{y\to x \\ y\in\Omega\cup\tilde\Gamma_0}} \underline{\omega}_A(x),$$

and ω_A^*, ω_{*A} are still sub- and supersolutions of (39).

Now we may apply Theorem 1. From (28), (H4) clearly holds on Γ_1, and since $\omega_* \ge 0 = \varphi$ on $\tilde\Gamma_1$, Γ_1 is included in $\tilde\Gamma_0$; but from the definition of ω^*, (4) holds on $\tilde\Gamma_0$, therefore the assumptions of Theorem 1 concerning Γ_1 are true. Then we have to check (H5) on $\Gamma_2 \subset \Gamma_2'$ (by Proposition 2) and Γ_2' is clearly empty because of assumption (27).

It remains to consider (H2) and (H3); these are wrong for the hamiltonian in (39) and we now indicate how to keep a comparison principle. First, (H2) is not necessary since, using the argument of Ishii [15], it is enough to compare, for any $\alpha \in]0, 1[$, the functions $\underline{\omega}_A$ and ω^α with

$$\omega^\alpha = \alpha\bar\omega^A + (1-\alpha)\phi$$

(ϕ is defined in (31′)). Since the hamiltonian in (39) is convex, ω^α is a *strict* subsolution to (38): we have

$$\begin{cases} a_{ij}\omega_i^\alpha\omega_j^\alpha + b_i\omega_i^\alpha \le -\alpha\gamma & \text{in } \Omega, \\ \mathrm{Min}(a_{ij}\omega_i^\alpha\omega_j^\alpha + b_i\omega_i^\alpha + \alpha\gamma,\ \omega_i^\alpha - A - \alpha\|\phi\|_\infty) \le 0 & \text{on } \partial\Omega. \end{cases}$$

We refer the reader to the Appendix for a comparison principle between supersolutions and strict subsolutions for this problem (with a degenerated matrix a_{ij}).

This comparison principle shows that $\omega^\alpha \le \underline{\omega}_A + \alpha\|\phi\|_\infty$ and thus $\bar\omega^A \le \underline{\omega}_A$. Setting A to infinity, (38) shows that

$$\bar\omega = \underline{\omega} \qquad \text{on } \{x;\ \underline{\omega}(x) < +\infty\},$$

$$\omega^\varepsilon \xrightarrow[\varepsilon\to 0]{} \bar\omega = \underline{\omega} \qquad \text{uniformly on compact subsets of } \{x;\ \underline{\omega}(x) < +\infty\},$$

and Proposition 9 is proved. (Again $\bar\omega = J$ by classical representation formulae.) □

Proof of Proposition 10. Let us indicate only the main modifications of the classical proof required to treat this case. With the same change of variable as in the proof of Proposition 9, we obtain functions $\bar{\omega}_A$, $\underline{\omega}_A$ which are sub- and supersolutions to

$$\begin{cases} \omega_t + a_{ij}\omega_i\omega_j + b_i\omega_i = 0 & \text{in} \quad \Omega \times]0, T], \\ \operatorname{Max}(\omega_t + a_{ij}\omega_i\omega_j + b_i\omega_i, \omega - A) = 0 & \text{on} \quad \partial\Omega \times]0, T], \\ \operatorname{Min}(\omega_t + a_{ij}\omega_i\omega_j + b_i\omega_i, \omega - A) = 0 & \text{on} \quad \partial\Omega \times]0, T], \\ \omega(x, 0) = 0 & \text{in} \quad \Omega \times \{0\}. \end{cases} \tag{40}$$

By the nondegeneracy condition on $\partial\Omega$, the function Ct for C large enough is a continuous supersolution to the state-constraint problem

$$\begin{cases} \omega_t + a_{ij}\omega_i\omega_j + b_i\omega_i \geq 0 & \text{in} \quad \Omega \times]0, T], \\ \omega_t + a_{ij}\omega_i\omega_j + b_i\omega_i \geq 0 & \text{on} \quad \partial\Omega \times]0, T], \\ \omega(x, 0) = 0 & \text{in} \quad \bar{\Omega} \times \{0\}. \end{cases}$$

By Proposition 7 we get

$$0 \leq \underline{\omega}_A \leq \bar{\omega}_A \leq Ct \qquad \text{on} \quad \Omega \times]0, T] \tag{41}$$

(assumption (H3) is wrong here since a_{ij} is degenerated, but the argument of the Appendix also gives a comparison principle for evolution equations).

Then (41) replaces assumption (21) (we leave this point to the reader) and Proposition 6 applies and shows that

$$\underline{\omega}_A = \bar{\omega}^A \leq Ct,$$

i.e., setting A to infinity,

$$\omega^\varepsilon \xrightarrow[\varepsilon \to 0]{} \underline{\omega} = \bar{\omega} \qquad \text{uniformly in} \quad K \times [0, T]$$

for compact subsets K of Ω. □

Appendix. A Comparison Principle for the Hamiltonian $a_{ij}p_ip_j + b_ip_i$

In this appendix we give a uniqueness argument for a hamiltonian appearing in the context of large deviations. For the sake of simplicity we only give a particular result for the stationary case; a complete uniqueness proof being obtained by combining the idea of the proof below and the arguments given in [4] as in the preceding parts.

Theorem 11. *Let u be a u.s.c. subsolution to*

$$H(x, Du) \leq -\gamma < 0 \qquad \text{in } \Omega,$$

and let v be an l.s.c. supersolution to

$$H(x, Dv) \geq 0 \qquad \text{in } \Omega$$

with

$$H(x,p) = a_{ij}p_i p_j + b_i p_i.$$

If

$$a = \sigma^T \cdot \sigma, \qquad \sigma_{ij}, b_j \in W^{1,\infty}(\Omega), \qquad 1 \le i \le m, \quad 1 \le j \le N,$$

then

$$\operatorname*{Max}_{\bar\Omega}(u-v) = \operatorname*{Max}_{\partial\Omega}(u-v).$$

Remark. Variants of this result are possible. For instance, Theorem 1 holds with this hamiltonian if $a_{ij}n_i n_j$ is positive on $\partial\Omega$. The same results also hold for the Cauchy problem.

Proof of Theorem 11. We prove that, for $0 < \mu < 1$, we have

$$\operatorname*{Max}_{\bar\Omega}(\mu u - v) = \operatorname*{Max}_{\partial\Omega}(\mu u - v). \tag{42}$$

If (42) was wrong, we could find, for ε small enough, some $(\bar x, \bar y) \in \Omega \times \Omega$, a maximum point of

$$\mu u(x) - v(y) - \frac{|x-y|^2}{\varepsilon^2}. \tag{43}$$

Then set

$$p = 2\frac{(\bar x - \bar y)}{\varepsilon^2}, \qquad A = |\sigma(x)\cdot p|, \qquad \Delta_\varepsilon = (\sigma(\bar x) - \sigma(\bar y))\cdot p.$$

We have

$$\frac{A^2}{\mu} + b(\bar x)\cdot p \le -\gamma\mu, \qquad |A + \Delta_\varepsilon|^2 + b(\bar y)\cdot p \ge 0.$$

The difference gives

$$\frac{1-\mu}{\mu}A^2 + 2A\cdot\Delta_\varepsilon + \Delta_\varepsilon^2 + (b(\bar x) - b(\bar y))\cdot p \le -\gamma\mu. \tag{44}$$

But we have

$$|\Delta_\varepsilon|, |b(\bar x) - b(\bar y)|\cdot|p| \le \frac{C|\bar x - \bar y|^2}{\varepsilon^2} \xrightarrow[\varepsilon\to 0]{} 0,$$

therefore (44) is impossible and (42) holds. Setting μ to 1 we conclude. □

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