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SMOOTH FIT PRINCIPLE FOR IMPULSE CONTROL OF MULTIDIMENSIONAL DIFFUSION PROCESSES*

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Abstract. Value functions of impulse control problems are known to satisfy quasi-variational inequalities (QVIs) [A. Bensoussan and J.-L. Lions, *Impulse Control and Quasivariational Inequalities*, Heyden & Son, Philadelphia, 1984; translation of *Contrôle Impulsionnel et Inéquations Quasi Variationnelles*, Gauthier-Villars, Paris, 1982]. This paper proves the smooth-fit C^1 property of the value function for multidimensional controlled diffusions, using a viscosity solution approach. We show by examples how to exploit this regularity property to derive explicitly optimal policy and value functions.

Key words. stochastic impulse control, viscosity solution, quasi-variational inequality, smooth fit, controlled diffusion

AMS subject classifications. 49J20, 49N25, 49N60

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1. Introduction.

This paper considers the following impulse control problem for an n-dimensional diffusion process $X(t)$. In the absence of control, $X(t)$ is governed by an Itô's stochastic differential equation,

$$dX(t) = \mu(X(t))dt + \sigma(X(t))dW(t), \quad X(0) = x, \tag{1.1}$$

where W is a standard Brownian motion in a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$. If a control policy $V = (\tau_1, \xi_1; \tau_2, \xi_2; \dots)$ is adopted, then $X(t)$ evolves as

$$dX(t) = \mu(X(t-))dt + \sigma(X(t-))dW(t) + \sum_i \delta(t - \tau_i)\xi_i, \tag{1.2}$$

where $\delta(\cdot)$ denotes the Dirac delta function. Here the control $V = (\tau_1, \xi_1; \tau_2, \xi_2; \dots)$ is of an impulse type such that $\tau_1, \tau_2, \dots$ is an increasing sequence of stopping times with respect to $\mathcal{F}_t$ (the natural filtration generated by W), and ξ_i is an $\mathbb{R}^n$-valued, $\mathcal{F}_{\tau_i}$-measurable random variable.

The problem is to choose an appropriate impulse control $(\tau_1, \xi_1; \tau_2, \xi_2; \dots)$ so that the following objective function is minimized:

$$\mathbb{E}_x \left(\int_0^\infty e^{-rt} f(X(t))dt + \sum_{i=1}^\infty e^{-r\tau_i} B(\xi_i) \right). \tag{1.3}$$

Here f is a running cost function, B is a transaction cost function, and $r > 0$ is a discount factor.

This multidimensional control problem has been proposed and studied in various forms in different contexts of risk management, including optimal cash management [7] and inventory controls [16, 15, 39, 38]. More recent papers in the literature

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of mathematical finance include those on transaction cost in portfolio management [2, 21, 22, 10, 29, 32], insurance models [18, 5], liquidity risk [25, 4], optimal control of exchange rates [19, 30, 6], and finally, real options [40, 27].

Compared to regular controls, impulse control provides a more natural mathematical framework when the state space is discontinuous. It is a more general version of singular control allowing for nonzero fixed cost [15] and therefore harder to analyze. Indeed, in contrast to the singular/regular control theory, which enjoys a vast literature in financial engineering (see, for instance, Merton [28] and Karatzas and Shreve [20] among others), impulse control is less well understood, especially in terms of the structure of the optimal policy and regularity properties of the value function. In fact, regarding optimal policy, the best known work is perhaps still due to [7], which characterized the (u, U, d, D) form of the optimal policy for an inventory system. Although there have been various extensions of this structural result [16, 15, 39, 38, 34], most were derived through the verification theorem approach and by assuming a priori the smooth-fit property through the action/continuation regions. In the end, this approach usually amounts to solving complex algebraic equations that are hard to verify without a priori knowledge of the regularity property; thus the correctness of the "solution" is dubious. Indeed, there are (see, e.g., [41] and [3]) a few examples of singular control and stopping problems with explicit solutions, where value functions are not C^2, or even C^1, and only recently (see [26, 14], and [35]) the smooth-fit principle for one-dimensional singular control and the closely related switching control problems were established. In [1] value functions were shown to be the solutions of quasi-variational inequalities (QVIs) and the regularity properties were established for the case when the control is strictly positive and the state space is in a bounded region. However, to the best of our knowledge, regularity properties for value functions involving all-direction controls have not been fully established. This is an important omission in light of the wide range of applications mentioned earlier.

Our work. This paper studies regularity properties of the impulse control problem (1.3) on multidimensional diffusions in (2.1) subject to our conditions (A1)–(A4). Unlike the approach in [1], where the regularity was established through studying the corresponding QVIs, we first prove the value function to be the unique viscosity solution to the corresponding Hamilton–Jacobi–Bellman (HJB) equation. The main difficulties in proving the uniqueness of the viscosity solution are the unusual nonlocal property of the associated operator in the HJB equation and the unboundedness of the state space. We overcome these by exploiting and clarifying the definition of viscosity solutions in a local sense and by relating the problem to an optimal stopping problem (see also Remark 1). Next, we establish the regularity property of the value function, and in particular, the smooth-fit C^1 property through the boundaries between action and continuation regions. The existing technique in [1] does not apply here as it relies on a certain smoothness assumption that fails in our case (see also Remark 2). Finally, we show how to exploit this smooth-fit property to explicitly derive the form of optimal policy and the action/continuation regions for special cases that were first studied and analyzed in [7].

2. Formulation and assumptions.

2.1. Model formulation. Let us first define precisely the family of admissible controls. An *admissible impulse control* V consists of a sequence of stopping times $\tau_1, \tau_2, \ldots$ with respect to $\mathcal{F}_t$ (the natural filtration generated by W) and a corresponding sequence of $\mathbb{R}^n$-valued random variables $\xi_1, \xi_2, \ldots$ satisfying the conditions

$$\begin{cases} 0 \le \tau_1 \le \tau_2 \le \cdots \le \tau_i \le \cdots, \\ \tau_i \to \infty \text{ almost surely as } i \to \infty, \end{cases}$$

and $\xi_i \in \mathcal{F}_{\tau_i}$ $\forall i \ge 1$.

As explained in the introduction, given an initial state $x \in \mathbb{R}^n$ and an admissible control $V = (\tau_1, \xi_1; \tau_2, \xi_2; \ldots)$, the underlying process $X(t)$ is governed by the stochastic differential equation

$$\begin{cases} dX(t) = \mu(X(t-))dt + \sigma(X(t-))dW(t) + \displaystyle\sum_i \delta(t - \tau_i)\xi_i, \\ X(0) = x, \end{cases} \tag{2.1}$$

where $\delta(\cdot)$ denotes the Dirac delta function. Here the coefficients $\mu(\cdot)$ and $\sigma(\cdot)$ satisfy the Lipschitz conditions to ensure the existence and uniqueness of (1.1) (see, for instance, [37, Chapter V, Theorem 11.2]). Equation (2.1) is interpreted in a piecewise sense as in [1].

The associated total expected cost (objective function) is given by

$$J_x[V] := \mathbb{E}_x \left(\int_0^\infty e^{-rt} f(X(t))dt + \sum_{i=1}^\infty e^{-r\tau_i} B(\xi_i) \right), \tag{2.2}$$

where f is the "running cost," B is the "transaction cost," and $r > 0$ is the discount factor. We will specify the conditions on f and B in section 2.2 below.

The goal is to find the admissible $\widetilde{V}$ and the associated control sequence (τ_i, ξ_i) to minimize the total cost, i.e.,

$$J_x[\widetilde{V}] \le J_x[V] \quad \text{for any admissible } V.$$

We define the value function

$$u(x) = \inf_V J_x[V], \tag{2.3}$$

where the infimum is taken over all admissible control policies.

2.2. Assumptions and notations. Throughout this paper, we shall impose the following standing assumptions:

(A1) Lipschitz conditions on $\mu, \sigma : \mathbb{R}^n \to \mathbb{R}$: there exist constants $C_\mu, C_\sigma > 0$ such that

$$\begin{cases} |\mu(x) - \mu(y)| \le C_\mu |x - y| \\ |\sigma(x) - \sigma(y)| \le C_\sigma |x - y| \end{cases} \qquad \forall x, y \in \mathbb{R}^n. \tag{2.4}$$

(A2) Lipschitz condition on the running cost $f \ge 0$: there exists a constant $C_f > 0$ such that

$$|f(x) - f(y)| \le C_f |x - y| \quad \forall x, y \in \mathbb{R}^n. \tag{2.5}$$

(A3) Conditions on the transaction cost function $B : \mathbb{R}^n \to \mathbb{R}$:

$$\begin{cases} \displaystyle\inf_{\xi \in \mathbb{R}^n} B(\xi) = K > 0, \\ B \in C(\mathbb{R}^n \backslash \{0\}), \\ |B(\xi)| \to \infty \text{ as } |\xi| \to \infty, \text{ and} \\ B(\xi_1) + B(\xi_2) \ge B(\xi_1 + \xi_2) + K \quad \forall \xi_1, \xi_2 \in \mathbb{R}^n. \end{cases} \tag{2.6}$$

(A4) $r > 2C_\mu + C_\sigma^2$.

We will also use the following notations for the operators:

$$\mathcal{M}\varphi(x) = \inf_{\xi \in \mathbb{R}^n} (\varphi(x+\xi) + B(\xi)), \tag{2.7}$$

$$\mathcal{L}\varphi(x) = -\operatorname{tr}\left[A \cdot D^2\varphi(x)\right] - \mu(x) \cdot D\varphi(x) + r\varphi(x), \tag{2.8}$$

where the matrix $A = (a_{ij})_{n\times n} = \frac{1}{2}\sigma(x)\sigma(x)^{\mathsf{T}}$.

Denote by $\Xi(x)$ the set of all the points ξ for which $\mathcal{M}u$ achieves the minimum value, where u is the value function, i.e.,

$$\Xi(x) := \{\xi \in \mathbb{R}^n : \mathcal{M}u(x) = u(x+\xi) + B(\xi)\}. \tag{2.9}$$

We also adopt the following standard notations for function spaces:

$$\begin{aligned} UC(\mathbb{R}^n) &= \text{space of all uniformly continuous functions on } \mathbb{R}^n, \\ UC_{bb}(\mathbb{R}^n) &= \{f \in UC(\mathbb{R}^n) : f \text{ is bounded below}\}, \\ W^{k,p}(U) &= \text{space of all } L^p \text{ functions with } \beta\text{th weak partial} \\ &\quad\ \text{derivatives belonging to } L^p\ \forall |\beta| \le k, \\ C_c^\infty(U) &= \{f \in C^\infty(U) : f \text{ has compact support in } U\}, \\ C^{k,\alpha}(D) &= \left\{ f \in C^k(D) : \sup_{x,y\in D} \left\{ \frac{|D^\beta f(x) - D^\beta f(y)|}{|x-y|^\alpha} \right\} < \infty \forall |\beta| \le k \right\}. \end{aligned}$$

2.3. Preliminary results. We first establish some preliminary results about the value function, as well as the operator $\mathcal{M}$, under the standing assumptions.

LEMMA 2.1. *The value function $u(x)$ defined by* (2.3) *is Lipschitz.*

The proof of this lemma is a standard argument using Itô's formula and Gronwall's inequality. (See [23] and Theorem 10.1 in [12] for similar results and techniques.)

LEMMA 2.2 (basic properties of $\mathcal{M}$).

(1) *$\mathcal{M}$ is concave: for any $\varphi_1, \varphi_2 \in C(\mathbb{R}^n)$ and $0 \le \lambda \le 1$,*

$$\mathcal{M}(\lambda\varphi_1 + (1-\lambda)\varphi_2) \ge \lambda\mathcal{M}\varphi_1 + (1-\lambda)\mathcal{M}\varphi_2.$$

(2) *$\mathcal{M}$ is increasing: for any $\varphi_1 \le \varphi_2$ everywhere,*

$$\mathcal{M}\varphi_1 \le \mathcal{M}\varphi_2.$$

(3) *The operator $\mathcal{M}$ maps $C(\mathbb{R}^n)$ into $C(\mathbb{R}^n)$. In particular, $\mathcal{M}u(\cdot)$ is continuous. Moreover, $\mathcal{M}$ maps $UC(\mathbb{R}^n)$ into $UC(\mathbb{R}^n)$ and maps a Lipschitz function to a Lipschitz function.*

Proof. (1) and (2) are obvious.

(3) Suppose $\varphi \in C(\mathbb{R}^n)$. Then for any $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n$, $\varepsilon > 0$,

$$-\varepsilon < \varphi(x+\xi+y) - \varphi(x+\xi) < \varepsilon,$$

provided that $|y| < \delta$ sufficiently small. Hence

$$\varphi(x+\xi) + B(\xi) - \varepsilon < \varphi(x+\xi+y) + B(\xi) < \varphi(x+\xi) + B(\xi) + \varepsilon.$$

This holds for arbitrary ξ, so by taking the infimum we get

$$\mathcal{M}\varphi(x) - \varepsilon \le \mathcal{M}\varphi(x+y) \le \mathcal{M}\varphi(x) + \varepsilon,$$

provided $|y| < \delta$ is small enough.

The last statement regarding Lipschitz functions can be proved similarly. ∎

LEMMA 2.3. *u and $\mathcal{M}u$ defined as above satisfy $u(x) \le \mathcal{M}u(x)\, \forall\, x \in \mathbb{R}^n$.*

Proof. Suppose $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n$, and $V = (\tau_1, \xi_1; \tau_2, \xi_2; \ldots)$ is an admissible control policy. Then $V' = (0, \xi; \tau_1, \xi_1; \tau_2, \xi_2; \ldots)$ is also admissible. Moreover,

$$u(x) \le J_x[V'] = J_{x+\xi}[V] + B(\xi).$$

Taking the infimum over V and then the infimum over $\xi \in \mathbb{R}^n$, we get $u(x) \le \mathcal{M}u(x)$. ∎

Now define the *continuation region* $\mathcal{C}$ and the *action region* $\mathcal{A}$ as follows:

$$\mathcal{C} := \{x \in \mathbb{R}^n : u(x) < \mathcal{M}u(x)\}, \tag{2.10}$$

$$\mathcal{A} := \{x \in \mathbb{R}^n : u(x) = \mathcal{M}u(x)\}. \tag{2.11}$$

Then, since u and $\mathcal{M}u$ are continuous, we have the following.

PROPOSITION 1. *$\mathcal{C}$ is open.*

PROPOSITION 2. *Suppose $x \in \mathcal{A}$; then*

(1) *the set*

$$\Xi(x) := \{\xi \in \mathbb{R}^n : \mathcal{M}u(x) = u(x+\xi) + B(\xi)\}$$

is nonempty; i.e., the infimum is in fact a minimum;

(2) *moreover, for any $\xi(x) \in \Xi(x)$, we have*

$$u(x+\xi(x)) \le \mathcal{M}u(x+\xi(x)) - K,$$

in particular,

$$x + \xi(x) \in \mathcal{C}.$$

Proof. (1) Given $x \in \mathcal{A}$, take sequence $\{\xi_n\}$ such that

$$\mathcal{M}u(x) \le u(x+\xi_n) + B(\xi_n) \le \mathcal{M}u(x) + \frac{1}{n}.$$

Then $\{\xi_n\}$ is bounded since $|B(\xi)| \to \infty$ as $|\xi| \to \infty$. Extract a convergent subsequence $\{\xi_{n_k}\}$ that converges to ξ^*.

Claim. $\xi^* \ne 0$ and $\mathcal{M}u(x) = u(x+\xi^*) + B(\xi^*)$. Suppose $\xi^* = 0$. Since

$$u(x+\xi_{n_k}) + K \le u(x+\xi_{n_k}) + B(\xi_{n_k}) \le \mathcal{M}u(x) + \frac{1}{n_k},$$

by sending $k \to \infty$, we deduce that $u(x) + K \le \mathcal{M}u(x) = u(x)$. This is a contradiction. Now that $\xi^* \ne 0$, clearly $\xi^* \in \Xi(x)$ since $B(\cdot)$ is also continuous at $\xi^* \ne 0$.

(2) Recall that $B(\xi_1) + B(\xi_2) \ge K + B(\xi_1 + \xi_2)$; hence

$$\begin{aligned}
\mathcal{M}u(x) &= \inf_{\xi \in \mathbb{R}} (u(x+\xi) + B(\xi)) \\
&= \inf_{\eta \in \mathbb{R}} (u(x+\xi(x)+\eta) + B(\xi(x)+\eta)) \\
&\le \inf_{\eta \in \mathbb{R}} [u(x+\xi(x)+\eta) + B(\eta)] + B(\xi(x)) - K \\
&= \mathcal{M}u(x+\xi(x)) + B(\xi(x)) - K.
\end{aligned}$$

On the other hand, $\mathcal{M}u(x) = u(x+\xi(x)) + B(\xi(x))$. We get the desired result. ∎

Here we need $K > 0$ (assumption (A3)) to deduce $u(x+\xi(x)) < \mathcal{M}u(x+\xi(x))$.

3. Value function as viscosity solution. We show in this section that under certain conditions, the value function of the impulse control problem is the unique viscosity solution of the corresponding HJB equation

$$\max(\mathcal{L}u - f, u - \mathcal{M}u) = 0. \tag{HJB}$$

3.1. Definition of viscosity solutions. First, recall (see [24]) the following definition of viscosity subsolutions (supersolutions, resp.):

(a) If $\varphi \in C^2(\mathbb{R}^n)$, $u - \varphi$ has a *global* maximum (minimum, resp.) at x_0 and $u(x_0) = \varphi(x_0)$, then

$$\max(\mathcal{L}\varphi(x_0) - f(x_0), \varphi(x_0) - \mathcal{M}\varphi(x_0)) \leq 0 \quad (\geq 0 \text{ resp.}). \tag{3.1}$$

However, note that the operator $\mathcal{M}$ is nonlocal; i.e., $\mathcal{M}\varphi(x_0)$ is not determined by values of φ in a neighborhood of x_0, and $\mathcal{M}\varphi(x_0)$ might be very small if φ is small away from x_0. Therefore, one has no control over $\mathcal{M}\varphi(x_0)$ by simply requiring that $u-\varphi$ have a local maximum (minimum, resp.) at x_0. In light of this, one can modify the definition of viscosity subsolutions (supersolutions, resp.) as follows: Suppose $u \in UC(\mathbb{R}^n)$.

(b) If $\varphi \in C^2(\mathbb{R}^n)$, $u - \varphi$ has a local maximum (minimum, resp.) at x_0 and $u(x_0) = \varphi(x_0)$, then

$$\max(\mathcal{L}\varphi(x_0) - f(x_0), u(x_0) - \mathcal{M}u(x_0)) \leq 0 \quad (\geq 0 \text{ resp.}). \tag{3.2}$$

In fact, one can show the following.

THEOREM 3.1. *The above two definitions of viscosity subsolutions (supersolutions, resp.) are equivalent.*

Proof. We will prove only the equivalence of subsolutions.

(b) $\Rightarrow$ (a). Suppose $\varphi \in C^2(\mathbb{R}^n)$, $u - \varphi$ has a global maximum at x_0, and $u(x_0) = \varphi(x_0)$. Then $u \leq \varphi$ globally, and by Lemma 2.2,

$$\varphi(x_0) = u(x_0) \leq \mathcal{M}u(x_0) \leq \mathcal{M}\varphi(x_0).$$

(a) $\Rightarrow$ (b). Suppose $\varphi \in C^2(\mathbb{R}^n)$, $u - \varphi$ has a local maximum at x_0, and $u(x_0) = \varphi(x_0)$. For any $\varepsilon > 0$, take $r > 0$ so small that

$$u \leq \varphi \leq u + \varepsilon \text{ in } \bar{B}_{2r}(x_0) := \{x \in \mathbb{R}^n : |x - x_0| \leq 2r\}.$$

There also exists a function $\tilde{\varphi} \in C^\infty(\mathbb{R}^n)$ such that

$$u \leq \tilde{\varphi} \leq u + \varepsilon \text{ in } \mathbb{R}^n.$$

(For instance, the usual mollification $\tilde{\varphi} = u * \eta^\delta + \varepsilon$, with $\delta > 0$ small enough.) Take a cutoff function $\zeta(x)$ such that

$$0 \leq \zeta(x) \leq 1;\ \zeta \equiv 1 \text{ on } \bar{B}_r(x_0);\ \zeta \equiv 0 \text{ off } \bar{B}_{2r}(x_0).$$

Now define

$$\psi(x) = \zeta(x)\varphi(x) + (1 - \zeta(x))\tilde{\varphi}(x).$$

Then clearly by construction,

$$u(x) \leq \psi(x) \leq u(x) + \varepsilon,$$

and ψ attains a global maximum at x_0. Thus

$$\mathcal{L}\psi(x_0) - f(x_0) \le 0, \quad \psi(x_0) \le \mathcal{M}\psi(x_0).$$

Note that by $\psi(x_0) = \varphi(x_0) = u(x_0)$, $D\psi(x_0) = D\varphi(x_0)$, $D^2\psi(x_0) = D^2\varphi(x_0)$, we have

$$\mathcal{L}\varphi(x_0) - f(x_0) \le 0, \quad u(x_0) \le \mathcal{M}\psi(x_0) \le \mathcal{M}u(x_0) + \varepsilon.$$

Finally, since $\varepsilon > 0$ is arbitrary, by sending it to 0, we have (3.2). ∎

In light of Theorem 3.1, throughout the paper we shall adopt the following definition of viscosity solution.

Definition 1. *The function u is called a viscosity solution of* (HJB) *if the following hold:*

(1) *(subsolution property.) For any $\varphi \in C^2(\mathbb{R}^n)$, if $u - \varphi$ has a local maximum at x_0 and $u(x_0) = \varphi(x_0)$, then we have*

$$\max(\mathcal{L}\varphi(x_0) - f(x_0), u(x_0) - \mathcal{M}u(x_0)) \le 0. \tag{3.3}$$

(2) *(supersolution property.) For any $\varphi \in C^2(\mathbb{R}^n)$, if $u - \varphi$ has a local minimum at x_0 and $u(x_0) = \varphi(x_0)$, then we have*

$$\max(\mathcal{L}\varphi(x_0) - f(x_0), u(x_0) - \mathcal{M}u(x_0)) \ge 0. \tag{3.4}$$

Then we have the following known result [33]. (For the reader's convenience, we provide the proof in Appendix A.)

Theorem 3.2. *The value function defined by* (2.3) *is a viscosity solution of the HJB equation*

$$\max\{\mathcal{L}u - f, u - \mathcal{M}u\} = 0. \tag{HJB}$$

3.2. Uniqueness of viscosity solution. In this section we shall show that the viscosity solution for (HJB) is unique in $UC_{bb}(\mathbb{R}^n)$.

The key idea is to relate the impulse control problem to an optimal stopping problem via the following operator $\mathcal{T}$, as in Bensoussan and Lions [1] and Ramaswamy and Dharmatti [36]. More precisely, given $\phi \in UC(\mathbb{R}^n)$, consider the following optimal stopping time problem:

$$\mathcal{T}\phi(x) := \inf_{\tau} \mathbb{E}\left(\int_0^{\tau} e^{-rt} f(X(t))dt + e^{-r\tau}\mathcal{M}\phi(x(\tau))\right) \tag{3.5}$$

subject to (1.1) and with the infimum taken over all $\mathcal{F}_t$ stopping times.

We shall first prove the uniqueness of the viscosity solution to the HJB equation (3.7) associated with this optimal stopping problem (3.5). We then exploit the properties of the operator $\mathcal{T}$ to establish the uniqueness of the viscosity solution to (HJB) for the impulse control problem.

3.2.1. Related optimal stopping problems. Now given (1.1), consider the following more generic optimal stopping problem with a terminal (nonnegative) cost $g(\cdot)$:

$$v(x) = \inf_{\tau} \mathbb{E}\left(\int_0^{\tau} e^{-rt} f(X(t))dt + e^{-r\tau} g(x(\tau))\right), \tag{3.6}$$

where f is the same as before, and the infimum is taken over all $\mathcal{F}_t$ stopping times.

First, the following result is well known [31].

PROPOSITION 3. *Assume that $g \in C(\mathbb{R}^n)$. Then the value function $v(x)$ defined by* (3.6) *is a continuous viscosity solution of the HJB equation*

$$\max\{\mathcal{L}w - f, w - g\} = 0 \text{ in } \mathbb{R}^n, \tag{3.7}$$

where $\mathcal{L}$ is defined in (2.8)*.*

Next, we show the following.

THEOREM 3.3 (unique viscosity solution for optimal stopping). *Suppose $g \in UC(\mathbb{R}^n)$ and suppose there are some constants $C, \Lambda > 0$ such that*

$$\begin{cases} |\mu(x)| \leq C & \forall x \in \mathbb{R}^n, \\ a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 & \forall x, \xi \in \mathbb{R}^n, \end{cases} \tag{3.8}$$

where $(a_{ij}(x))_{n\times n} = \frac{1}{2}\sigma(x)\sigma(x)^{\mathsf{T}}$. Then (3.7) *has only one viscosity solution in $UC(\mathbb{R}^n)$.*

To prove Theorem 3.3, the following observation is useful.

LEMMA 3.4. *w is a viscosity solution of $\max\{\mathcal{L}w - f,\, w - g\} = 0$ if and only if it is a viscosity solution of*

$$F(x, w(x), Dw(x), D^2w(x)) = 0,[1] \tag{3.9}$$

where $F : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times S^n \to \mathbb{R}$ is defined by

$$F(x, t, p, X) = rt + \max\{G(x, p, X), -rg(x)\}, \tag{3.10}$$

$$G(x, p, X) = -\operatorname{tr}[AX] - \mu(x)\cdot p - f(x). \tag{3.11}$$

The key to proving Theorem 3.3 is the following comparison result.

LEMMA 3.5. *Let $D \subset \mathbb{R}^n$ be a bounded open set $f, g \in UC(\mathbb{R}^n)$, let $v_* \in C(\overline{D})$ be a subsolution, and let $v^* \in C(\overline{D})$ be a supersolution of*

$$\max\{\mathcal{L}w - f, w - g\} = 0 \text{ in } D.$$

Assume also that $v_* \leq v^*$ on ∂D. Then $v_* \leq v^*$ in $\overline{D}$.

The proof of Lemma 3.5 is based on the classical comparison theorem of second order degenerate elliptic differential equations in bounded domains (cf. [8]), and we defer it to Appendix B.

The following lemma (see Theorem 1 and the remarks on the fully nonlinear case in [9]) extends the above comparison result from bounded domains to an unbounded domain.

LEMMA 3.6. *Suppose $F(x, t, p, X)$ is elliptic in X such that there exist constants $\alpha, C, \Lambda > 0$ satisfying*

$$F(x, t + s, p + q, X + Y) \geq F(x, t, p, X) + \alpha s - C|q| - \Lambda \operatorname{tr}(Y) \tag{3.12}$$

$\forall x, p, q \in \mathbb{R}^n$, $t \in \mathbb{R}$, $s \geq 0$, $X, Y \in S^n$, $Y \geq 0$. Suppose also that we have a comparison result for the equation $F = 0$ in bounded domains. If v_* and v^* are the

[1] When there is no risk of confusion, we will also abbreviate (3.9) as $F = 0$.

continuous sub- and supersolution, respectively, of $F(x, w(x), Dw(x), D^2w(x)) = 0$ *in* $\mathbb{R}^n$ *with at most polynomial growth, then*

$$v_* \le v^* \text{ in } \mathbb{R}^n.$$

Now we can return to the following.

Proof of Theorem 3.3. By Lemmas 3.5 and 3.6, and noticing also

$$w \in UC(\mathbb{R}^n) \Rightarrow \sup_{x\in\mathbb{R}^n} \frac{|w(x)|}{1+|x|} < \infty,$$

it remains to show that F, defined by (3.10), satisfies (3.12) as follows:

$$\begin{aligned} &F(x, t+s, p+q, X+Y) - F(x, t, p, X) \\ &\quad = rs + \max\{G(x, p+q, X+Y), -rg(x)\} - \max\{G(x, p, X), -rg(x)\} \\ &\quad \ge \begin{cases} rs \text{ if } G(x, p, X) \le -rg(x), \\ rs + G(x, p+q, X+Y) - G(x, p, X) \text{ otherwise.} \end{cases} \end{aligned}$$

If $Y \ge 0$, using (3.8), we have

$$\begin{aligned} G(x, p+q, X+Y) - G(x, p, X) &= -\operatorname{tr}(AY) + \mu(x)q \\ &\ge -C|q| - \Lambda \operatorname{tr}(Y), \end{aligned}$$

since $\operatorname{tr}(AY) = \operatorname{tr}(S^\mathsf{T}AS) \le C \operatorname{tr}(S^\mathsf{T}S) = \Lambda \operatorname{tr}(Y)$, where $A = \frac{1}{2}\sigma\sigma^\mathsf{T}$, $Y = SS^\mathsf{T}$. This completes the proof. □

3.2.2. Uniqueness for impulse control problems. Now we are ready to prove the following.

THEOREM 3.7 (unique viscosity solution for impulse controls). *Assume that there are some constants* $C, \Lambda > 0$*, such that*

$$\text{(3.8)} \qquad \begin{cases} |\mu(x)| \le C & \forall x \in \mathbb{R}^n, \\ a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2 & \forall x, \xi \in \mathbb{R}^n, \end{cases}$$

and $r > C_f$*. Then the value function for the impulse control problem defined by* (2.3) *is a unique viscosity solution in* $UC_{bb}(\mathbb{R}^n)$ *for the* HJB *equation.*

The proof relies on the following properties of the operator $\mathcal{T}$.

LEMMA 3.8.

(1) $\mathcal{T} : UC(\mathbb{R}^n) \to UC(\mathbb{R}^n)$.

(2) *If* $w \le v$ *in* $\mathbb{R}^n$*, then* $\mathcal{T}w \le \mathcal{T}v$ *in* $\mathbb{R}^n$.

(3) $\mathcal{T}$ *is concave on* $UC(\mathbb{R}^n)$.

Lemma 3.8 is immediate by the monotone and concave properties of $\mathcal{M}$ in Lemma 2.2, and by a direct application of Itô's formula and Gronwall's inequality.

Proof of Theorem 3.7. Suppose $w, v \in UC_{bb}(\mathbb{R}^n)$ are two solutions of (HJB). Without loss of generality, we can assume $w, v \ge 0$. Otherwise if $-C$ is a lower bound, then $w + C$, $v + C$ are nonnegative solutions to (HJB) of the same structure with f replaced by $f + rC$.

First, by definition of $\mathcal{T}$ together with Proposition 3 and Theorem 3.3, $\mathcal{T}w$ is the unique viscosity solution of

$$\max\{\mathcal{L}(\mathcal{T}w) - f, \mathcal{T}w - \mathcal{M}w\} = 0 \text{ in } \mathbb{R}^n.$$

On the other hand, w is also a viscosity solution of

$$\max\{\mathcal{L}w - f, w - \mathcal{M}w\} = 0 \text{ in } \mathbb{R}^n.$$

By uniqueness, $w = \mathcal{T}w$. Similarly, $v = \mathcal{T}v$.

Next, it suffices to show that if w, v satisfies $w - v \leq \gamma w$ for some $\gamma \in [0,1]$, then there exists some constant $\nu \in (0,1)$ such that

$$w - v \leq \gamma(1-\nu)w. \tag{3.13}$$

Indeed, if this claim holds, then note that $w - v \leq w$ since $v \geq 0$, and we have $w - v \leq (1-\nu)w$. Repeating the argument, we have

$$w - v \leq (1-\nu)^n w.$$

Sending $n \to \infty$, we get $w \leq v$. Switching the roles of w and v, we get $w = v$ as desired.

Now it remains to check the claim (3.13). First, by concavity of $\mathcal{T}$, we have

$$\mathcal{T}v \geq \mathcal{T}[(1-\gamma)w] \geq (1-\gamma)\mathcal{T}w + \gamma\mathcal{T}0.$$

Since f is at most with a linear growth, we deduce that

$$\begin{aligned} w = \mathcal{T}w &= \inf_{\tau} \mathbb{E}\left(\int_0^{\tau} e^{-rt} f(X(t))dt + e^{-r\tau}\mathcal{M}w(x(\tau))\right) \\ &\leq \mathbb{E}\int_0^{\infty} e^{-rt} f(X(t))dt \leq C\int_0^{\infty} e^{-rt}(1 + \mathbb{E}|X(t)|)dt \\ &\leq C(1+|x|)\int_0^{\infty} e^{-rt}e^{Ct}dt =: w_0 < \infty. \end{aligned}$$

Here, the last line follows from Gronwall's inequality.

Thus, $\mathcal{M}w(x) = \inf(w(x+\xi) + B(\xi)) \leq w_0 + K = K/\nu$, where

$$\nu := \frac{K}{K + w_0} \in (0,1).$$

Note that since $\mathcal{M}0 = K$, we obtain

$$\begin{aligned} \mathcal{T}0 &= \inf_{\tau} \mathbb{E}\left(\int_0^{\tau} e^{-rt} f(X(t))dt + e^{-r\tau}K\right) \\ &\geq \nu \inf_{\tau} \mathbb{E}\left(\int_0^{\tau} e^{-rt} f(X(t))dt + e^{-r\tau}\mathcal{M}w(X(t))\right) = \nu\mathcal{T}w. \end{aligned}$$

Therefore,

$$\mathcal{T}v \geq (1-\gamma)\mathcal{T}w + \gamma\mathcal{T}0 \geq (1-\gamma)\mathcal{T}w + \gamma\nu\mathcal{T}w.$$

The claim (3.13) is now clear by plugging in $\mathcal{T}w = w$, $\mathcal{T}v = v$. ∎

REMARK 1. *It is worth noting that in* [4] *the uniqueness of the viscosity solution was characterized for the value function of a finite horizon impulse control problem with execution delay involving the liquidity risk. However, the technique of* [4] *relies on the particular setup of a positive delay parameter and cannot be reduced to our case.*

4. Regularity of value function. We shall show in this section that the value function u is in the Sobolev space $W^{2,p}(\mathcal{O})$ for any open bounded region $\mathcal{O}$ and for any $p < \infty$, and in particular, $u \in C^1(\mathbb{R}^n)$. Throughout this section, we assume the operator $\mathcal{L}$ to be strictly elliptic: there exists a contant $c > 0$, such that $a_{ij}(x)\xi_i\xi_j \geq c|\xi|^2 \; \forall x, \xi \in R^n$.

Recall

$$\begin{aligned} \mathcal{C} &:= \{x \in \mathbb{R}^n : u(x) < \mathcal{M}u(x)\}, \\ \mathcal{A} &:= \{x \in \mathbb{R}^n : u(x) = \mathcal{M}u(x)\}. \end{aligned}$$

First, $u(\cdot)$ is clearly C^2 in the continuation region $\mathcal{C}$.

LEMMA 4.1 (C^2-regularity in $\mathcal{C}$). (1) *The value function $u(\cdot) \in C^2(\mathcal{C})$, and it satisfies the following differential equation in the classical sense:*

$$\mathcal{L}u(x) - f(x) = 0, \quad x \in \mathcal{C}. \tag{4.1}$$

(2) *For any set $D \Subset \mathcal{C}$,*[2]

$$u \in C^{2,\alpha}(\overline{D})$$

for some $\alpha > 0$.

Proof. Recall that from Theorem 3.2, u satisfies (4.1) in $\mathcal{C}$ in a viscosity sense. Now for any open ball $B \subset \mathcal{C}$, consider the Dirichlet problem

$$\begin{cases} \mathcal{L}w - f = 0 & \text{in } B, \\ w = u & \text{on } \partial B. \end{cases}$$

Classical Schauder estimates (cf. [13, Theorem 6.13]) ensure that such a solution w exists and belongs to $C^{2,\alpha}(B)$ because $f \in C^{0,\alpha}(B)$ for some $\alpha > 0$. Thus w satisfies the differential equation in a viscosity sense, whence $w = u$ in B by classical uniqueness results of the viscosity solution of a linear PDE in a bounded domain (cf. [8]). Hence,

$$u \in C^{2,\alpha}(B).$$

Finally, if $D \Subset \mathcal{C}$, then $\overline{D}$ can be covered by finitely many open balls contained in $\mathcal{C}$, and hence

$$u \in C^{2,\alpha}(\overline{D}). \qquad \square$$

Now we are are ready to establish the main regularity theorem.

THEOREM 4.2 ($W^{2,p}_{\text{loc}}$-regularity). *Assume that*

$$\sigma \in C^{1,1}(D) \textit{ for any compact set } D \subset \mathbb{R}^n. \tag{4.2}$$

Then for any bounded open set $\mathcal{O} \subset \mathbb{R}^n$, and $p < \infty$, we have

$$u \in W^{2,p}(\mathcal{O}).$$

In particular, $u \in C^1(\mathbb{R}^n)$, by the Sobolev embedding.

[2] $D \Subset \mathcal{C}$ means that D is *compactly contained* in $\mathcal{C}$; i.e., there exists a compact set F such that $D \subset F \subset \mathcal{C}$.

Proof. Given any bounded open set $\mathcal{O}$, we denote by $\mathcal{C}'$ ($\mathcal{A}'$, resp.) the restriction of the continuation (action, resp.) region within $\mathcal{O}$.

Our approach is to prove, for some constant C depending on $\mathcal{O}$,

$$-C \le \mathcal{L}u \le C, \tag{4.3}$$

in the sense of distribution. That is, for any smooth test function $\varphi \in C_c^\infty(\mathcal{O})$ with $\varphi \ge 0$, we have

$$-C\int_{\mathcal{O}} \varphi dx \le \int_{\mathcal{O}} \left(a_{ij}u_{x_i}\varphi_{x_j} + b_i u_{x_i}\varphi + ru\varphi\right) dx \le C\int_{\mathcal{O}} \varphi dx. \tag{4.4}$$

First, by (4.2), we can write the differential operator $\mathcal{L}$ in divergence form

$$\mathcal{L}u = -(a_{ij}u_{x_i})_{x_j} - b_i u_{x_i} + ru, \tag{4.5}$$

with $a_{ij} \in C^{1,1}(\overline{\mathcal{O}})$ and b_i Lipschitz. Note also that the first weak derivatives u_{x_i} are well defined since u is Lipschitz ($u \in W^{1,\infty}(\mathcal{O})$).

Observe that u is a viscosity subsolution of $\mathcal{L}u = f$ in $\mathcal{O}$ since it is a viscosity solution of (HJB). Thus u is also a distribution subsolution according to Ishii [17, Theorem 1]. Hence we have

$$\mathcal{L}u \le C$$

in the sense of distribution, with $C = \sup_{\overline{\mathcal{O}}} |f|$.

Next, we show that

$$\mathcal{L}u^\varepsilon(x_0) \ge -C \quad \forall x_0 \in \mathcal{O}, \tag{4.6}$$

where $u^\varepsilon = u * \eta^\varepsilon \in C^\infty$ is the mollification of u, $\eta^\varepsilon = \eta(x/\varepsilon)/\varepsilon^n$, and $\eta(\cdot)$ is the standard mollifier. This is proved according to three different cases.

Case 1. $x_0 \in \mathcal{C}'$. Then by Lemma 4.1, $\mathcal{L}u(x_0) = f(x_0)$ in the classical sense. Hence

$$|\mathcal{L}u^\varepsilon(x_0)| = |f^\varepsilon(x_0)| \le C,$$

where $C = \sup_{\overline{\mathcal{O}}} |f|$, which does not depend on x_0.

Case 2. $x_0 \in \partial\mathcal{A}'$. Then there exists a sequence $\{x_n\} \subset \mathcal{C}'$ converging to x_0. Since $|\mathcal{L}u^\varepsilon(x_n)| \le C \;\forall n$ by the proof in Case 1, we obtain (4.6) by taking the limit as $n \to \infty$.

Case 3. $x_0 \in \operatorname{Int}\mathcal{A}'$, the interior of $\mathcal{A}'$. Since $\mathcal{A}' \subset \mathcal{O}$ is bounded, and $|B(\xi)| \to \infty$ as $|\xi| \to \infty$, we can find an open ball $\mathcal{O}' \supset \mathcal{O}$ so that $\xi(y) \in \mathcal{O}' \;\forall y \in \mathcal{A}'$ and $\xi(y) \in \Xi(y)$, because

$$B(\xi(y)) = \mathcal{M}u(y) - u(y + \xi(y)) \le \mathcal{M}u(y) \le \sup_{\overline{\mathcal{O}'}} \mathcal{M}u.$$

Now we define the set

$$D := \left\{ y \in \mathcal{O}' : u(y) < \mathcal{M}u(y) - \frac{K}{2} \right\}. \tag{4.7}$$

Clearly, $D \Subset \mathcal{C}$. From Lemma 4.1,

$$u \in C^{2,\alpha}(\overline{D}).$$

For any $x \in \operatorname{Int}\mathcal{A}'$, suppose $B_{\rho_1}(x) \subset \operatorname{Int}\mathcal{A}'$. Let us take $\xi(x) \in \Xi(x)$; then $y := x + \xi(x)$ satisfies

$$u(y) - \mathcal{M}u(y) \le K$$

by Proposition 2. Hence $y \in D$.

On the other hand, since $u - \mathcal{M}u$ is uniformly continuous on $\overline{\mathcal{O}'}$, there exists $\rho_2 > 0$ such that

$$|y - y'| \le \rho_2 \Rightarrow |u(y') - \mathcal{M}u(y') - (u(y) - \mathcal{M}u(y))| \le \frac{K}{4}.$$

Therefore, for any constant $\lambda \in [-1, 1]$ and any unit vector $\chi \in \mathbb{R}^n$, $y' = y + \lambda\rho_2\chi$ satisfies

$$u(y') - \mathcal{M}u(y') \le u(y) - \mathcal{M}u(y) + \frac{K}{4} < -\frac{K}{2}.$$

Hence, if we let $0 < \rho \le \rho_1 \wedge \rho_2$, then $\forall\, \lambda \in [-1, 1]$, $\chi \in \mathbb{R}^n$, with $|\chi| = 1$, we have

$$\begin{aligned} &y = x + \xi(x) \in D, \quad y' = y + \lambda\rho\chi \in D,\\ &x \in \operatorname{Int}\mathcal{A}', \quad x + \lambda\rho\chi \in \operatorname{Int}\mathcal{A}'. \end{aligned}$$

By definition, we obtain

$$\begin{aligned} u(x) = \mathcal{M}u(x) &= u(x + \xi(x)) + B(\xi(x)),\\ u(x \pm \rho\chi) = \mathcal{M}u(x \pm \rho\chi) &\le u(x \pm \rho\chi + \xi(x)) + B(\xi(x)), \end{aligned}$$

and hence the second difference quotient at x is

$$\begin{aligned} &\frac{1}{\rho^2}[u(x + \rho\chi) + u(x - \rho\chi) - 2u(x)]\\ &\quad \le \frac{1}{\rho^2}[u(y + \rho\chi) + u(y - \rho\chi) - 2u(y)]\\ &\quad = \frac{1}{|\rho|}\int_0^1 [Du(y + \lambda\rho\chi) - Du(y - \lambda\rho\chi)] \cdot \chi d\lambda\\ &\quad \le C_D, \end{aligned}$$

where $C_D = \sup_{x\in\overline{D}} |D^2u(x)| \le \|u\|_{C^{2,\alpha}(\overline{D})}$.

Now, with $x_0 \in \operatorname{Int}\mathcal{A}'$ given, suppose $B_\theta(x_0) \subset \operatorname{Int}\mathcal{A}'$; then for any $0 < \varepsilon < \frac{\theta}{2}$, $\rho_1 = \frac{\theta}{2}$, $z \in B_\varepsilon(0)$, we have $B_{\rho_1}(x_0 - z) \subset \operatorname{Int}\mathcal{A}'$. Therefore, for $0 < \rho \le \rho_1 \wedge \rho_2$ and unit vector $\chi \in \mathbb{R}^n$,

$$\begin{aligned} &\frac{1}{\rho^2}\left[u^\varepsilon(x_0 + \rho\chi) + u^\varepsilon(x_0 - \rho\chi) - 2u^\varepsilon(x_0)\right]\\ &= \frac{1}{\rho^2}\int_{B_\varepsilon(0)} [u(x_0 - z + \rho\chi) + u(x_0 - z - \rho\chi) - 2u(x_0 - z)]\eta^\varepsilon(z)\, dz\\ &\le C_D \int_{B_\varepsilon(0)} \eta^\varepsilon(z)\, dz = C_D. \end{aligned}$$

Sending $\rho \to 0$ we get

$$\chi^{\mathsf{T}} D^2 u^\varepsilon(x_0)\chi \le C_D.$$

Hence,

$$\begin{aligned}\operatorname{tr}(\sigma(x_0)\sigma(x_0)^\mathsf{T} D^2u^\varepsilon(x_0)) &= \operatorname{tr}(\sigma^\mathsf{T}(x_0)D^2u^\varepsilon(x_0)\sigma(x_0))\\ &= \sum_k \sigma_k^\mathsf{T} D^2u^\varepsilon\sigma_k\\ &\le C_D\sum_{i,j}|\sigma_{ij}(x_0)|^2\\ &\le C,\end{aligned}$$

where σ_k is the kth column of the matrix σ, σ_{ij} is the (i,j)th element of σ, and the last inequality is due to continuity of σ.

Note that $|u^\varepsilon(x_0)| + |Du^\varepsilon(x_0)| \le \|u\|_{W^{1,\infty}(\overline{\mathcal{O}})}$ and $\mu(x)$ bounded; we deduce

$$\mathcal{L}u^\varepsilon(x_0) = -\frac{1}{2}\operatorname{tr}\left(\sigma(x_0)\sigma(x_0)^\mathsf{T} D^2u^\varepsilon(x_0)\right) - \mu(x_0)\cdot Du^\varepsilon(x_0) + ru^\varepsilon(x_0) \ge -C,$$

where C is independent of x_0.

Finally, (4.6) implies that for any smooth test function $\varphi \in C_c^\infty(\mathcal{O})$, $\varphi \ge 0$,

$$\int_{\mathcal{O}}\left(a_{ij}u^\varepsilon_{x_i}\varphi_{x_j} + b_iu^\varepsilon_{x_i}\varphi + ru^\varepsilon\varphi\right)dx \ge -C\int_{\mathcal{O}}\varphi dx. \tag{4.8}$$

Since $u \in W^{1,2}(\mathcal{O})$, $u^\varepsilon \to u$ in $L^2(\mathcal{O})$, and $u^\varepsilon_{x_i} \to u_{x_i}$ in $L^2(\mathcal{O})$. Sending $\varepsilon \to 0$ in the above inequality, we obtain (4.3). Therefore,

$$\mathcal{L}u \in L^\infty(\mathcal{O}).$$

By the Calderón–Zygmund estimate (see, e.g., [13]),

$$u \in W^{2,p}(\mathcal{O}) \quad \forall p < \infty. \qquad \square$$

REMARK 2. *Compared to the regularity results of Bensoussan and Lions* [1], *we deal with a control which is unbounded and not necessarily positive. Moreover, we prove the regularity property for the value function as a viscosity solution of the HJB equation as opposed to their weak solutions of QVIs with $u \in H_0^1$ satisfying*

$$\begin{cases} a(u, v-u) \ge \langle f, v-u\rangle & \forall v \in H_0^1,\ v \le \mathcal{M}u,\\ 0 \le u \le \mathcal{M}u, \end{cases}$$

where $a(\phi,\psi) = \langle \mathcal{L}\phi, \psi\rangle$ and $\langle\cdot,\cdot\rangle$ is the paring between the Hilbert space H_0^1 and its dual space. (See Lemmas 2.3–2.4 *and Theorem* 2.2 *of Chapter* 4 *in* [1]*). In addition, their key lemma, Lemma* 2.3, *requires, in our notation, $\gamma(x) := \inf_{x+\xi\in\partial O} B(\xi) \in W^{2,\infty}(O)$, and therefore C^1. However, this condition fails in our case; see our example in section* 5, *where the corresponding γ has a corner.*

5. Structure of the value function. Having obtained the regularity results for the value function, in this section we shall characterize the structure of the value function as well as the continuation/action regions for the following special case: $n = 1$ and the cost functions f and B are given by

$$f(x) = \begin{cases} hx & \text{if } x \ge 0,\\ -px & \text{if } x \le 0, \end{cases} \tag{5.1}$$

$$B(\xi) = \begin{cases} K^+ + k^+\xi & \text{if } \xi \ge 0,\\ K^- - k^-\xi & \text{if } \xi < 0, \end{cases} \tag{5.2}$$

where h, p, k^+, k^-, K^+, K^- are positive constants. Moreover, we assume that μ and σ are all constant:

$$\sigma(x) \equiv \sigma, \quad \mu(x) \equiv \mu. \tag{5.3}$$

In addition, we will impose the following condition to rule out triviality:

$$h - rk^- > 0, \quad p - rk^+ > 0. \tag{5.4}$$

This case was first characterized in [7] with a verification-type argument. Here we provide an alternative derivation by exploiting the regularity property established earlier. We shall show directly the following.

THEOREM 5.1 (characterization of the solution structure). *Assuming* (5.1), (5.2), (5.3), *and* (5.4),

(1) *there exist constants* $-\infty < q < s < \infty$ *such that*

$$\mathcal{C} := \{x \in \mathbb{R} : u(x) < \mathcal{M}u(x)\} = (q, s),$$
$$\mathcal{A} := \{x \in \mathbb{R} : u(x) = \mathcal{M}u(x)\} = (-\infty, q] \cup [s, +\infty);$$

(2) *the value function* u *defined in* (2.3) *satisfies*

$$\begin{cases} \mathcal{L}u(x) = f(x), & q < x < s, \\ u(x) = u(s) + k^-(x - s), & x \geq s, \\ u(x) = u(q) + (q - x)k^+, & x \leq q; \end{cases}$$

(3) *there are points* $Q, S \in (q, s)$ *such that*

$$u'(q) = u'(Q) = -k^+, \qquad u(q) = u(Q) + K^+ + k^+(Q - q),$$
$$u'(s) = u'(S) = k^-, \qquad u(s) = u(S) + K^- - k^-(S - s),$$

as shown in Figure 5.1.

The proof of Theorem 5.1 is based on a series of lemmas.

LEMMA 5.2. *Under assumptions* (5.1) *and* (5.2), *for any* $x_0 \in \mathcal{A}$ *and*

$$\xi_0 \in \Xi(x_0) := \{\xi \in \mathbb{R} : \mathcal{M}u(x_0) = u(x_0 + \xi) + B(\xi)\},$$

we have

$$u'(x_0) = u'(x_0 + \xi_0) = \begin{cases} -k^+, & \xi_0 > 0, \\ k^-, & \xi_0 < 0. \end{cases} \tag{5.5}$$

Proof. First, such a ξ_0 exists and $\xi_0 \neq 0$ by Proposition 2. By definition, $u(x_0) = \mathcal{M}u(x_0) = u(x_0 + \xi_0) + B(\xi_0)$, which means ξ_0 is a global minimum of the function $u(x_0 + \cdot) + B(\cdot)$. Also, $\xi_0 \neq 0$ implies that B is also differentiable at ξ_0, and hence

$$u'(x_0 + \xi_0) = -B'(\xi_0) = \begin{cases} -k^+, & \xi_0 > 0, \\ k^-, & \xi_0 < 0. \end{cases}$$

Now, for any $\delta \neq 0$, we have

$$u(x_0 + \delta) \leq \mathcal{M}u(x_0 + \delta) \leq u(x_0 + \delta + \xi_0) + B(\xi_0).$$

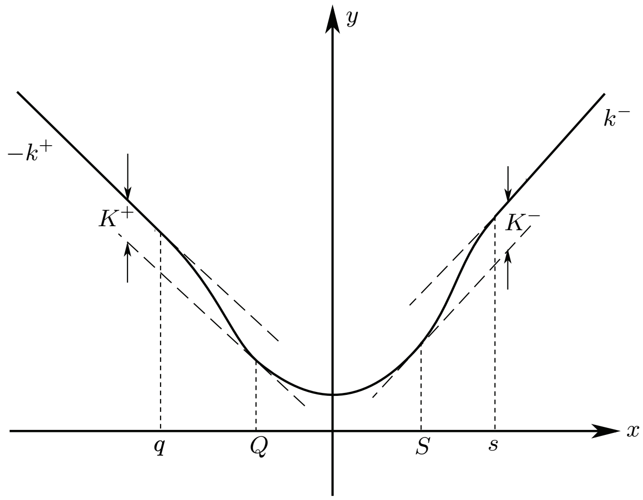

FIG. 5.1. *The value function u.*

Thus,

$$\frac{u(x_0+\delta)-u(x_0)}{\delta} \le \frac{u(x_0+\xi_0+\delta)-u(x_0+\xi_0)}{\delta}, \quad \delta > 0,$$
$$\frac{u(x_0+\delta)-u(x_0)}{\delta} \ge \frac{u(x_0+\xi_0+\delta)-u(x_0+\xi_0)}{\delta}, \quad \delta < 0.$$

Taking the limit as $\delta \to 0^+$ ($\delta \to 0^-$, resp.), we conclude that

$$u'(x_0) = u'(x_0+\xi_0). \qquad \square$$

LEMMA 5.3. *Assume* (5.1), (5.2), *and* (5.3). *For any $x_0 \in \mathcal{A}$ and $\xi_0 \in \Xi(x_0)$,*
(1) *if $x_0 > 0$, then $\xi_0 < 0$ and $u'(x_0) = k^-$;*
(2) *if $x_0 < 0$, then $\xi_0 > 0$ and $u'(x_0) = -k^+$.*

Proof. (1) Suppose not; then there exists a $\xi_0 \in \Xi(x_0)$ with $\xi_0 > 0$ and

$$u(x_0) = \mathcal{M}u(x_0) = u(x_0+\xi_0) + B(\xi_0).$$

First, take an ε-optimal strategy $V = (\tau_1, \xi_1; \tau_2, \xi_2; \ldots)$ for the initial level $x_0 + \xi_0$, i.e.,

$$J_{x_0+\xi_0}[V] \le u(x_0+\xi_0) + \varepsilon,$$

where $\varepsilon > 0$ is arbitrarily small, to be chosen later.

Construct a strategy for x_0,

$$V_1 = (0, \xi_0; \tau_1, \xi_1; \tau_2, \xi_2; \ldots).$$

Then by definition,

$$J_{x_0}[V_1] = J_{x_0+\xi_0}[V] + B(\xi_0) \le u(x_0+\xi_0) + \varepsilon + B(\xi_0) = u(x_0) + \varepsilon. \tag{5.6}$$

On the other hand, we can construct another strategy for x_0,

$$V_2 = (\tau, \xi_0; \tau_1, \xi_1; \tau_2, \xi_2; \ldots),$$

where

$$\tau = \inf\{t : x(t; x_0, V) < 0\}.$$

Here, we use $x(t; x_0, V)$ to denote the solution of (2.1) with initial value x_0 and strategy V.

Since the system is linear by (5.3), we have

$$x(t; x_0, V_2) = \begin{cases} x(t; x_0, V_1) - \xi_0 > 0, & t < \tau; \\ x(t; x_0, V_1), & t \geq \tau. \end{cases}$$

It follows that $f(x(t; x_0, V_2)) \leq f(x(t; x_0, V_1)) \, \forall \, t$. So

$$\begin{aligned} &J_{x_0}[V_2] - J_{x_0}[V_1] \\ &\quad = \mathbb{E}\left(\int_0^\infty [f(x(t; x_0, V_2)) - f(x(t; x_0, V_1))] e^{-rt} dt + e^{-r\tau} B(\xi_0) - B(\xi_0) \right) \\ &\quad \leq B(\xi_0)(\mathbb{E} e^{-r\tau} - 1) =: -\nu. \end{aligned} \tag{5.7}$$

It remains to show that $\nu > 0$.

Claim. $\tau > 0$ a.s. if ε is sufficiently small.

Clearly, $\tau \geq \tau_1 \wedge \inf\{t : x_0 + \mu t + \sigma W(t) < 0\}$ and obviously $\inf\{t : x_0 + \mu t + \sigma W(t) < 0\} > 0$ a.s., since $x_0 > 0$. Now we need to prove $\tau > 0$ a.s. Suppose not; then

$$\begin{aligned} J_{x_0+\xi_0}[V] &= \mathbb{E}\left\{ J_{x_0+\xi_0+\xi_1}[V \setminus (\tau_1, \xi_1)] + B(\xi_1) \right\} \\ &\geq \mathbb{E}\{u(x_0 + \xi_0 + \xi_1) + B(\xi_1)\} \\ &\geq \mathcal{M}u(x_0 + \xi_0). \end{aligned}$$

However, since $x_0 + \xi_0 \in \mathcal{C}$ by Proposition 2, if we take $0 < \varepsilon \leq (\mathcal{M}u(x_0 + \xi_0) - u(x_0 + \xi_0))/2$, we have

$$J_{x_0+\xi_0}[V] \leq u(x_0 + \xi_0) + \varepsilon = \mathcal{M}u(x_0 + \xi_0) - \varepsilon.$$

This is a contradiction. Thus we proved the claim, and it follows that $\nu = -B(\xi_0)(\mathbb{E} e^{-r\tau} - 1) > 0$.

Combining (5.6) and (5.7) and taking $\varepsilon < \nu/2$,

$$J_{x_0}[V_2] \leq J_{x_0}[V_1] - \nu \leq u(x_0) + \varepsilon - \nu < u(x_0) - \nu/2.$$

This is a contradiction, and we have $\xi_0 > 0$. It follows from Lemma 5.2 that $u'(x_0) = k^-$.

The proof of (2) is exactly the same. $\square$

Recall $\mathcal{C}$ is an open set, i.e., a union of open intervals. The following lemma rules out the possibility that $\mathcal{C}$ contains unbounded intervals.

LEMMA 5.4. *Under assumption* (5.4), *$\mathcal{C}$ does not contain any of the intervals $(a, +\infty)$ or $(-\infty, b)$, with $a \geq -\infty$, $b \leq +\infty$.*

Proof. Suppose $\mathcal{C} \supset (a, +\infty)$. Then we have, for $c > \max\{a, 0\}$,

$$-\frac{1}{2}\sigma^2 u'' - \mu u' + ru = hx, \quad x \in (c, +\infty).$$

The ODE has a general solution

$$u(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} + \frac{hx}{r} + \frac{\mu h}{r^2},$$

where $\lambda_1 = \frac{-\mu - \sqrt{\mu^2 + 2\sigma^2 r}}{\sigma^2} < 0$, $\lambda_2 = \frac{-\mu + \sqrt{\mu^2 + 2\sigma^2 r}}{\sigma^2} > 0$. Note that $C_2 = 0$; otherwise $u'(x) = C_1 \lambda_1 e^{\lambda_1 x} + C_2 \lambda_2 e^{\lambda_2 x} + \frac{h}{r}$ is unbounded, approaching $+\infty$ or $-\infty$ as $x \to +\infty$, which contradicts the fact that u is Lipschitz.

Now, for any $x > c > 0$,

$$C_1 e^{\lambda_1 x} + \frac{hx}{r} + \frac{\mu h}{r^2} = u(x) < \mathcal{M}u(x) \le u(c) + K^- - k^-(c - x)$$

or

$$\left(\frac{h}{r} - k^-\right) x + C_1 e^{\lambda_1 x} < u(c) + K^- - k^- c - \frac{\mu h}{r^2}.$$

As $x \to +\infty$, we get a contradiction, noticing that $h - rk^- > 0$.

$p - rk^+ > 0$ will ensure that $\mathcal{C}$ cannot contain intervals of $(-\infty, b)$ type. Therefore, we prove the lemma. ∎

Finally, we see the following.

LEMMA 5.5. *Assume* (5.1), (5.2), (5.3), *and* (5.4). *Then $\mathcal{C}$ is connected.*

Proof. Suppose not. We prove by contradiction through the following steps.

Step 1. By assumption, there are points $y_1 < y_2 < y_3$ so that $y_1, y_3 \in \mathcal{C}$ while $y_2 \in \mathcal{A}$. Define

$$x_1 := \inf\{x \in \mathcal{A} : x \le y_2, [x, y_2] \subset \mathcal{A}\},$$
$$x_2 := \sup\{x \in \mathcal{A} : x \ge y_2, [y_2, x] \subset \mathcal{A}\}.$$

Clearly, x_1, x_2 exist and are finite, with $[x_1, x_2] \subset \mathcal{A}$. (We do not rule out the possibility that $x_1 = x_2$ here.)

By Lemma 5.2, $u'(x) = k^-$ or $-k^+$, for any $x \in \mathcal{A}$. Since $u \in C^1(\mathbb{R})$, u' is a constant on $[x_1, x_2]$. Assume $u'(x) = k^- \, \forall\, x \in [x_1, x_2]$, and consider u at the point x_2. (The other case $u' = -k^+$ is similar. In that case we consider the point x_1 instead.)

Step 2. We show that

$$u(x) \le u(x_2) + k^-(x - x_2) \quad \forall x \ge x_2, \tag{5.8}$$

and the inequality is strict if $x > x_2$ and $x \in \mathcal{C}$.

Let $\xi_2 \in \Xi(x_2)$. Then $\xi_2 < 0$ by Lemma 5.3, and hence $B(\xi_2) = K^- - k^- \xi_2 = B(\xi_2 - y) - k^- y$ for $y = x - x_2 \ge 0$. Therefore,

$$\begin{aligned} u(x) &\le (\text{or} < \text{if } x \in \mathcal{C})\ \mathcal{M}u(x) \le u(x_2 + \xi_2) + B(x_2 + \xi_2 - x) \\ &= u(x_2 + \xi_2) + B(\xi_2) + k^-(x - x_2) = u(x_2) + k^-(x - x_2). \end{aligned}$$

Step 3. We show that

$$-\mu k^- + ru(x_2) \le hx_2. \tag{5.9}$$

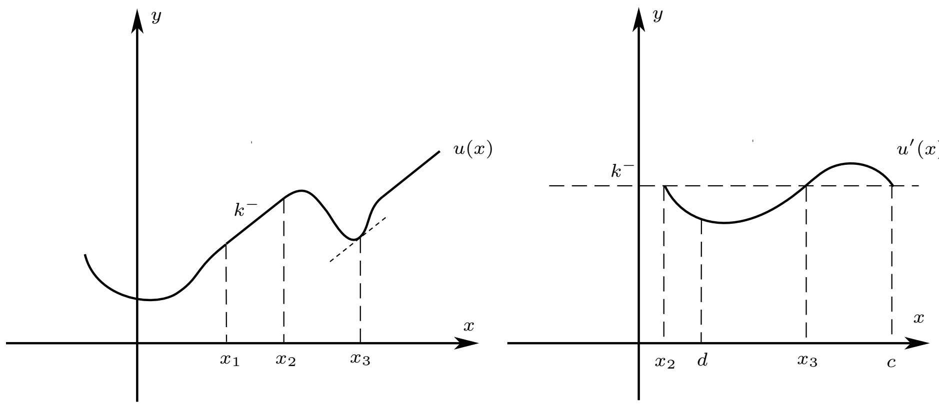

FIG. 5.2. *Proof of Lemma* 5.5.

Since $x_2 \in \mathcal{A}$, $u'(x_2) = k^-$, Lemma 5.3 implies that $x_2 \geq 0$. However, (5.8) implies that x_2 is a local maximum of $u - \phi$, where $\phi(x) = u(x_2) + k^-(x - x_2)$ is linear. By the viscosity subsolution property, we have $\mathcal{L}\phi(x_2) \leq f(x_2) = hx_2$, which is (5.9).

Step 4. There exists a point $x_3 > x_2$ such that

$$-\mu k^- + ru(x_3) \geq hx_3. \tag{5.10}$$

Suppose (x_2, c) is an open interval component of $\mathcal{C}$. Then by Lemma 5.4, $c < \infty$ and thus $c \in \mathcal{A}$. Lemma 5.3 implies that $u'(c) = k^-$ since $c > x_2 \geq 0$. Take $d \in (x_2, c)$ such that $u'(d) < k^-$. Such a d exists since $u(x) < u(x_2) + k^-(x - x_2)$ for $x \in (x_2, c)$. (See Figure 5.2.) Let

$$x_3 = \inf\{d \leq x \leq c : u'(x) = k^-\},$$

which is well defined, since c is in this set. Clearly $x_3 > d > x_2$. Moreover, $u'(x) < k^- = u'(x_3)$ for $d \leq x < x_3$ by definition. So

$$u''(x_3) \geq 0.$$

Thus $hx_3 = -\frac{1}{2}\sigma^2 u''(x_3) - \mu u'(x_3) + ru(x_3) \leq -\mu k^- + ru(x_3)$.

Step 5. From (5.9) and (5.10), it follows that

$$u(x_3) - u(x_2) \geq h/r(x_3 - x_2) > k^-(x_3 - x_2),$$

by (5.4). This is a contradiction to (5.8). ∎

Proof of Theorem 5.1. (1) Since $\mathcal{C}$ is connected, by Lemma 5.4, $\mathcal{C} = (q, s)$ for some $-\infty < q < s < \infty$.

(2) Suppose $x \geq s$ and $\xi \in \Xi(x)$. Because $x + \xi \in \mathcal{C} = (q, s)$, we have $\xi < 0$, whence $u'(x) = k^-$ by Lemma 5.2. Thus, $u(x) = u(s) + k^-(x - s)$ for $x \geq s$. A similar argument shows that $u(x) = u(q) + (q - x)k^+$, for $x \leq q$.

(3) Let $\xi \in \Xi(s)$ and $S = s + \xi$. Then $S \in (q, s)$, $u'(s) = u'(S) = k^-$, and

$$u(s) = \mathcal{M}u(s) = u(S) + B(S - s) = u(S) + K^- - k^-(S - s).$$

The remaining statement is similar. ∎

Appendix.

Appendix A. Proof of Theorem 3.2.

Proof. (1) (subsolution property.) Suppose $\varphi \in C^2(\mathbb{R}^n), u-\varphi$ has a local maximum at x_0, and $u(x_0) = \varphi(x_0)$. By Lemma 2.3, it suffices to prove

$$\mathcal{L}\varphi(x_0) - f(x_0) \leq 0.$$

For any admissible control $V = \{\tau_1, \xi_1; \tau_2, \xi_2; \ldots\}$ and $\tau > 0$, the control $V' = \{\tau_1 + \tau, \xi_1; \tau_2 + \tau, \xi_2; \ldots\}$ is also admissible, and thus

$$u(x_0) \leq J_{x_0}[V'] = \mathbb{E}\left(\int_0^\tau f(X(t))e^{-rt}dt + e^{-r\tau}J_{x(\tau)}[V]\right),$$

which implies

$$u(x_0) \leq \mathbb{E}\left(\int_0^\tau f(X(t))e^{-rt}dt + e^{-r\tau}u(x(\tau))\right), \tag{A.1}$$

where $X(t)$ is the solution of

$$\begin{cases} dX(t) = \mu(X(t))dt + \sigma(X(t))dW(t), \\ X(0) = x_0. \end{cases} \tag{A.2}$$

Meanwhile, Dynkin's formula gives

$$\mathbb{E}\left(e^{-r\tau}\varphi(x(\tau))\right) - \varphi(x_0) = -\mathbb{E}\left(\int_0^\tau e^{-rt}\mathcal{L}\varphi(X(t))dt\right). \tag{A.3}$$

Noting that $u \leq \varphi$ near x_0 and $u(x_0) = \varphi(x_0)$, combining (A.1) and (A.3), and sending $\tau \to 0^+$, we have $\mathcal{L}\varphi - f \leq 0$ at x_0.

(2) (supersolution property.) Suppose $\varphi \in C^2(\mathbb{R}^n)$, $u - \varphi$ has a local minimum at x_0, and $u(x_0) = \varphi(x_0)$. If $u(x_0) = \mathcal{M}u(x_0)$, then (3.4) is trivially true. Thus we assume $u(x_0) < \mathcal{M}u(x_0)$. By continuity of $\mathcal{M}$, there exist constants $\delta > 0, \rho > 0$ such that

$$\varphi(x) \leq u(x), u(x) - \mathcal{M}u(x) < -\delta \text{ whenever } |x - x_0| < \rho. \tag{A.4}$$

Define

$$\tau_\rho := \inf\{t > 0 : |X(t) - x_0| \geq \rho\}.$$

For any $\varepsilon > 0$, choose an ε-optimal control $V = (\tau_1, \xi_1; \tau_2, \xi_2; \ldots)$, i.e.,

$$J_{x_0}[V] \leq u(x_0) + \varepsilon.$$

Then for any stopping time $\tau \leq \tau_1$ a.s.,

$$u(x_0) + \varepsilon \geq J_{x_0}[V] = \mathbb{E}\left(\int_0^\tau f(X(t))e^{-rt}dt + e^{-r\tau}J_{x(\tau^-)}[V']\right), \tag{A.5}$$

where $V' = (\tau_1 - \tau, \xi_1; \tau_2 - \tau, \xi_2; \ldots)$ is admissible.

Fix $R > 0$ and let $\bar{\tau} = \tau_\rho \wedge R$.

Claim. $\mathbb{P}\{\tau_1 < \bar{\tau}\} \to 0$ as $\varepsilon \to 0$.

Consider (A.5) with $\tau = \tau_1$. On the set $\{\tau_1 < \bar{\tau}\}$,

$$\begin{aligned} J_{x(\tau_1^-)}[V'] &= \mathbb{E}\left(J_{x(\tau_1^-)+\xi_1}[\tilde{V}] + B(\xi_1)\right) \\ &\geq \mathbb{E}\left(u(x(\tau_1^-)+\xi_1) + B(\xi_1)\right) \geq \mathcal{M}u(x(\tau_1^-)) \\ &\geq u(x(\tau_1^-)) + \delta, \end{aligned}$$

because of (A.4). Otherwise, we still have $J_{x(\tau_1^-)}[V'] \geq u(x(\tau_1^-))$. Thus,

$$\begin{aligned} u(x_0) + \varepsilon &\geq \mathbb{E}\left(\int_0^{\tau_1} f(X(t))e^{-rt}dt + e^{-r\tau_1}J_{x(\tau_1^-)}[V'']\right) \\ &\geq \mathbb{E}\left(\int_0^{\tau_1} f(X(t))e^{-rt}dt + e^{-r\tau_1}u(x(\tau_1^-))\right) + e^{-rR}\delta \cdot \mathbb{P}\{\tau_1 < \bar{\tau}\} \\ &\geq u(x_0) + e^{-rR}\delta \cdot \mathbb{P}\{\tau_1 < \bar{\tau}\}. \end{aligned}$$

This proves the claim.

Take $\tau = \bar{\tau} \wedge \tau_1$; by sending $\varepsilon \to 0$ in (A.5), we get

$$u(x_0) \geq \mathbb{E}\left(\int_0^{\bar{\tau}} f(X(t))e^{-rt}dt + e^{-r\bar{\tau}}u(x(\bar{\tau}^-))\right).$$

Note that $\varphi(x(\bar{\tau}^-)) \leq u(x(\bar{\tau}^-))$ and $\varphi(x_0) = u(x_0)$, and that the above inequality together with Dynkin's formula (A.3) gives

$$\mathbb{E}\left(\int_0^{\bar{\tau}} e^{-rt}(\mathcal{L}\varphi - f)(X(t))dt\right) \geq 0.$$

Dividing by $\mathbb{E}(\bar{\tau})$ and sending $\rho \to 0$, we obtain the desired result (3.4). ◻

Appendix B. Proof of Lemma 3.5. To prove this lemma, we first recall a well-known comparison theorem on elliptic PDEs.

THEOREM B.1 (Theorem 3.3 in [8]). *Let U be a bounded open subset of $\mathbb{R}^n$, and let $F \in C(U \times \mathbb{R} \times \mathbb{R}^n \times S^n)$ satisfy the following:*

(1) *$F(x,t,p,X) \leq F(x,s,p,Y)$ whenever $t \leq s$, $Y \leq X$.*

(2) *There exists some $\gamma > 0$ such that, for $r \geq s$ and $(x,p,X) \in \bar{U} \times \mathbb{R}^n \times S^n$,*

$$\gamma(r-s) \leq F(x,r,p,X) - F(x,s,p,X).$$

(3) *There is a function $\omega : [0,\infty] \to [0,\infty]$ with $\omega(0+) = 0$ such that*

$$F(y,r,\alpha(x-y),Y) - F(x,r,\alpha(x-y),X) \leq \omega(\alpha|x-y|^2 + |x-y|)$$

whenever $x, y \in U$, $r \in \mathbb{R}$, $X, Y \in S^n$, and

$$-3\alpha \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

Let $u \in C(\bar{U})$ be a viscosity subsolution, and let $v \in C(\bar{U})$ be a viscosity supersolution of $F = 0$ in U with $u \leq v$ on ∂U. Then $u \leq v$ in $\bar{U}$.

Here S^n is the collection of $n \times n$ real symmetric matrices equipped with the usual ordering and I is the identity matrix.

Proof of Lemma 3.5. In view of Lemmas 3.4 and 3.5, it suffices to verify that F, defined by (3.10), satisfies the conditions of Theorem B.1. Clearly, F is continuous, and

$$F(x,t,p,X) \le F(x,s,p,Y) \text{ whenever } t \le s, Y \le X.$$

$$F(x,t,p,X) - F(x,s,p,X) = r(t-s).$$

Finally, we are to prove that there exists a function $\omega : [0,\infty] \to [0,\infty]$ satisfying $\omega(0+) = 0$ such that if $x, y \in D$, $t \in \mathbb{R}$, and X, Y are symmetric and satisfying for some $\alpha > 0$,

$$-3\alpha \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix},$$

then

$$F(y,t,\alpha(x-y),Y) - F(x,t,\alpha(x-y),X) \le \omega(\alpha|x-y|^2 + |x-y|). \tag{B.1}$$

It is easy to check that $G(x,p,X)$ and $-rg(x)$ satisfy (B.1), since $f, g \in UC(\mathbb{R}^n)$ (cf. Example 3.6 in [8]). Hence

$$\begin{aligned} &F(y,t,\alpha(x-y),Y) - F(x,t,\alpha(x-y),X) \\ &= \max\{G(y,\alpha(x-y),Y), -rg(y)\} - \max\{G(x,\alpha(x-y),X), -rg(x)\} \\ &\le \begin{cases} -r(g(y)-g(x)) & \text{if } G(y,\alpha(x-y),Y) \le -rg(y), \\ G(y,\alpha(x-y),Y) - G(x,\alpha(x-y),X) & \text{otherwise} \end{cases} \\ &\le \omega(\alpha|x-y|^2 + |x-y|). \qquad \square \end{aligned}$$

Appendix C. Sobolev embedding. We summarize here some relevant results concerning embeddings of various Sobolev spaces (cf. [11]).

THEOREM C.1 (general Sobolev inequalities). *Let U be a bounded open subset of $\mathbb{R}^n$, with C^1 boundary. Assume $u \in W^{k,p}(U)$.*

(1) *If*

$$k < \frac{n}{p},$$

then $u \in L^q(U)$, where

$$\frac{1}{q} = \frac{1}{p} - \frac{k}{n}.$$

In addition,

$$\|u\|_{L^q(U)} \le C\|u\|_{W^{k,p}(U)}.$$

Here the constant C depends only on k, p, n, and U.

(2) *If*

$$k > \frac{n}{p},$$

then $u \in C^{k-[\frac{n}{p}]-1\gamma}(\bar{U})$, *where*

$$\gamma = \begin{cases} [\frac{n}{p}] + 1 - \frac{n}{p} & \text{if } \frac{n}{p} \text{ is not an integer,} \\ \text{any positive number} < 1 & \text{if } \frac{n}{p} \text{ is an integer.} \end{cases}$$

In addition,

$$\|u\|_{C^{k-[\frac{n}{p}]-1,\gamma}(\bar{U})} \leq C\|u\|_{W^{k,p}(U)},$$

where the constant C *depends only on* k, p, n, γ, *and* U.

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