JOURNAL OF FUNCTIONAL ANALYSIS 62, 379–396 (1985)

Hamilton–Jacobi Equations in Infinite Dimensions I. Uniqueness of Viscosity Solutions

MICHAEL G. CRANDALL

Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706

AND

PIERRE-LOUIS LIONS

Université de Paris-IX, Dauphine, place de Lattre-de-Tassigny, 75775 Paris Cedex 16, France

Communicated by the Editors

Received October 22, 1984

INTRODUCTION

This paper is the first of a series devoted to the study of Hamilton–Jacobi equations in infinite-dimensional spaces. To pose a typical problem, we consider a (real) Banach space V, its dual space V^*, and solutions of an equation

$$H(x, u, Du) = 0 \qquad \text{in } \Omega \tag{HJ}$$

set in a subset Ω of V. In (HJ), $H: V \times \mathbf{R} \times V^* \to \mathbf{R}$ is continuous and Du stands for the Fréchet derivative of u. Thus, a classical solution u of (HJ) in Ω is a continuously (Fréchet) differentiable function $u: \Omega \to \mathbf{R}$ such that $H(x, u(x), Du(x)) = 0$ for $x \in \Omega$. In particular, we will prove that under appropriate conditions classical solutions of (HJ) are uniquely determined by their boundary values. However, global classical solutions of fully nonlinear first-order partial differential equations are rare even in finite-dimensional spaces, and we introduce an appropriate weakened notion below for which the uniqueness results are still valid.

There are various reasons for studying (HJ). First of all, this is the form of the basic partial differential equations satisfied by value functions arising in deterministic control problems, deterministic differential games, and the calculus of variations. A simple example is $u(x) = |x|$ (the norm of x in V), which is nothing but the length of the shortest path from x to 0 and which is a classical solution of $|Du| = 1$ in $V \backslash \{0\}$ provided the norm of V is dif-

0022-1236/85 $3.00

ferentiable on $V\setminus\{0\}$. For more complex control problems the reader may consult Barbu and Da Prato [1], while relations between control problems, the calculus of variations, and Hamilton–Jacobi equations in finite dimensions are recalled in P. L. Lions [14].

A second impetus for the current work lies in the simple desire to contribute to the understanding of nonlinear partial differential equations in infinite-dimensional spaces. At stake (eventually) are not only the various dynamic programming equations (also called Bellman, Hamilton–Jacobi–Bellman, and Isaacs equations, depending on the problem considered), but also the equations associated with filtering or control of finite-dimensional stochastic systems under partial observation.

The remark above concerning the differentiability of $u(x)=|x|$ brings into focus the fact that geometrical properties of V will play a role in the theory. In particular, questions related to the existence of an equivalent norm on V which is differentiable on $V\setminus\{0\}$ are relevant to the theory in infinite dimensions. However, in what follows, we partly obscure this fact by including various assumptions of this sort that we need in the assumptions concerning the Hamiltonian H. The outstanding explicit geometrical assumption made on V in most of the presentation is that V has the Radon–Nikodym property (i.e., "V is RN"). For example, reflexive spaces and separable dual spaces are RN. The Radon–Nikodym property is important for us because if V is RN, φ is a bounded and lower semicontinuous real-valued function on a closed ball B in V and $\varepsilon>0$, then there is an element x^* of V^* of norm at most ε such that $\varphi+x^*$ attains its minimum on B. This fact is proved in Ekeland and Lebourg [10] under more severe restrictions on V (which are probably met in most applications of our results) and in full generality in Stegall [16]. See also Bourgain [2].

Our main goal here will be to use this fact to show that the naive extension of the notion of viscosity solutions to Banach spaces succeeds in spaces with the Radon–Nikodym property. That is, the basic comparison (and therefore uniqueness) theorems remain valid. Other papers in this series will concern relations with control theory and differential games, existence theorems, and uniqueness of other classes of unbounded viscosity solutions. Indeed, existence in Hilbert (and more general) spaces is established in M. G. Crandall and P. L. Lions [6] by use of the relationship between differential games and viscosity solutions.

In the next section we briefly give a definition of viscosity solutions of (HJ) and prove some uniqueness results. The comparison results in infinite dimensions will be given in a natural generality which is new even in finite dimensions. In particular, we give the first complete formulation and proof under assumptions which are invariant under nice changes of the independent variable. This generality and the basic outline of proof has been evolving in the papers by Crandall and Lions [4, 5], Crandall, Evans, and Lions

[3], and Ishii [11, 12]. However, the proofs must be modified since bounded continuous functions on closed balls in infinite-dimensional spaces do not have maxima in general. This difficulty is oversome here by use of the variational principle mentioned above. We would like to thank N. Ghoussoub for bringing this result to our attention. This allowed us to simplify a preliminary version of this paper which was based on a more complex notion of viscosity solution than that given here and on Ekeland's principle [9]. However, not every Banach space is RN, so this more complex notion may prove significant in later developments. It is therefore presented in an appendix. One can equally well prove uniqueness results using it. However, we have chosen not to do so here because the theory (which is already getting technical) becomes less attractive and it is not yet clear if there will be either an acompanying existence theory or applications sufficient to justify this degradation of the presentation.

Finally, let us recall that the basic theory of Hamilton–Jacobi equations in finite-dimensional spaces is now fairly well understood via the notion of viscosity solutions (recalled below). This notion is given various equivalent forms in Crandall and Lions [4], where the fundamental uniqueness theorems were first proved. The uniqueness proofs below correspond to the modified arguments given in Crandall, Evans, and Lions [3] as sharpened in the various papers mentioned above, and the relevance to control theory was exhibited by Lions in [14] using the dynamic programming principle. See Crandall and Souganidis [8] for a more extensive resume and bibliography of recent work in finite dimensions.

The only previous work concerning viscosity solutions of Hamilton–Jacobi equations in infinite dimensions of which we are aware is by R. Jensen (verbal communication). Jensen, working in Hilbert spaces, uses the notion of a viscosity solution on a closed set and compactness assumptions to obtain the existence of extrema.

I. Viscosity Solutions and Comparison Theorems

Let Ω be an open subset of the (real) Banach space V. We will denote the value of $p \in V^*$ at $x \in V$ by (p, x). If $v\colon \Omega \to \mathbf{R}$ is continuous (i.e., $v \in C(\Omega)$) and $x \in \Omega$, we define the superdifferential $D^+v(x)$ and the subdifferential $D^-v(x)$ of v at x just as in [3]:

$$
\begin{aligned}
D^+v(x) &= \{p \in V^*\colon \limsup_{\substack{y \in \Omega \\ y \to x}} (v(y) - v(x) - (p, y-x))/|y-x| \leqslant 0\}, \\
D^-v(x) &= \{p \in V^*\colon \liminf_{\substack{y \in \Omega \\ y \to x}} (v(y) - v(x) - (p, y-x))/|y-x| \geqslant 0\}.
\end{aligned} \tag{1.1}
$$

We now define the notion of viscosity solutions.

DEFINITION 1. Let $u \in C(\Omega)$. Then u is a viscosity subsolution of (HJ) on Ω if

$$H(x, u(x), p) \leqslant 0 \qquad \text{for every } x \in \Omega \text{ and } p \in D^{+}u(x). \tag{1.2}$$

Similarly, u is a viscosity supersolution of (HJ) on Ω if

$$H(x, u(x), p) \geqslant 0 \qquad \text{for every } x \in \Omega \text{ and } p \in D^{-}v(x). \tag{1.3}$$

Finally, u is a viscosity solution Ω if it is both a viscosity subsolution and a viscosity supersolution.

Since we are assuming that Ω is open, the restriction $y \in \Omega$ in (1.1) is superfluous. However, (1.1) as it stands can be used whether or not Ω is open, and the above definition generalizes at once. We will not use this generality here, but see Jensen [13]. We will use (for example) the phrases "solution of $H \leqslant 0$" and "subsolution of $H = 0$" interchangeably. The above definition is one of several possible alternatives. A more convenient form for analytical work is contained in the following obvious proposition.

PROPOSITION 1. *Let $u \in C(\Omega)$. Then u is a viscosity subsolution of* (HJ) *in Ω if and only if for every $\varphi \in C(\Omega)$*

$$H(y, u(y), D\varphi(y)) \leqslant 0 \textit{ at each local maximum } y \in \Omega \textit{ of } u - \varphi \textit{ at which } \varphi \textit{ is differentiable.} \tag{1.4}$$

Similarly, u is a viscosity supersolution of (HJ) *when*

$$H(y, u(y), D\varphi(y)) \geqslant 0 \textit{ at each local minimum } y \in \Omega \textit{ of } u - \varphi \textit{ at which } \varphi \textit{ is differentiable.} \tag{1.5}$$

Remarks 1. The corresponding result in finite dimensions states that the proposition remains true if $\varphi \in C(\Omega)$ is replaced by $\varphi \in C^1(\Omega)$ and that one may replace local extrema in the statement by strict local extrema. Here, for example, when we say $y \in \Omega$ is a strict local maximum of a function v, we mean "very strict"—that is, there is a number $a > 0$ and a positive nondecreasing function $g:(0, a] \to (0, \infty)$ such that $v(x) \leqslant v(y) - g(|x-y|)$ for $|x-y| \leqslant a$. Of course, we may work with strict extrema in the general case. Moreover, if the space V admits a function $\zeta: V \to [0, \infty)$ such that $\zeta(x)/|x|$ is bounded above and below by positive constants on $V\setminus\{0\}$ and ζ is boundedly differentiable on $V\setminus\{0\}$, then the proposition remains correct with everywhere differentiable $\varphi \in C(\Omega)$ and $D\varphi$ continuous at y in (1.4) and (1.5). This may be established in a manner similar to (e.g.) [3, Proposition 1.1].

Before we formulate some hypotheses on H, we need to make our strategy—which has already been implemented above—more explicit. We will state and prove one principal result concerning comparison and uniqueness of solutions of (HJ) in all details. This proof will clearly illustrate how one may account for the infinite-dimensional difficulties. We will then state further results and, in particular, the corresponding result concerning comparison of solutions of the related time-dependent initial-value problem, without proof. The proofs will not be given because, by this time, a knowledgeable reader will be able to construct them in a straightforward way using the methods which have already been clearly illustrated here and in the literature. Thus it is not appropriate for us to do this here. The same is true for many other results. Hence we feel justified (here and later) to simply state when results "remain true in infinite dimensions" provided that we have verified the assertions for ourselves. For example, the results of Crandall and Lions [5] and Ishii [12] concerning moduli of continuity remain true in Hilbert spaces. By contrast, assertions concerning existence, especially when the finite-dimensional proofs employ compactness arguments, cannot usually be verified without considerable work ([6]). We will return to the question of moduli of continuity in [6], where it plays an essential role.

We turn to formulating the conditions on H that we will use. First of all, one does not expect boundary problems for (HJ) to have unique solutions unless $H(x, u, p)$ is monotone in u, and it is convenient to emphasize this monotonicity by considering problems of the form

$$u + H(x, u, Du) = 0. \tag{HJ$'$}$$

Of course, (HJ) may be transformed into a problem of the form (HJ$'$) (with a "new H") in a variety of ways. We will be imposing conditions on H in (HJ$'$).

These conditions will involve two auxiliary functions d: $V \times V \to [0, \infty)$ and v: $V \to [0, \infty)$. These functions are to satisfy a collection of conditions we will simply call (C). In the statement of these conditions and below, $| \ |$ is used to denote the norm of V as well as the corresponding dual norm on V^* and the absolute value on $\mathbf{R}$. If $x, y \in V$, then $L(x, y)$ denotes the line segment joining them. It may be useful to the reader to keep in mind the case in which V is Hilbert, $d(x, y) = |x - y|$ and $v(x) = |x|$ while reading further. The conditions (C) are:

(C) Let $y \in V$. The nonnegative function $x \to d(x, y)$ is Fréchet differentiable at every point except y and the derivative is denoted by $d_x(x, y)$. Similarly, $y \to d(x, y)$ is differentiable except at x and its derivative is $d_y(x, y)$. The function v is bounded on bounded sets. Moveover, there are constants $K, k > 0$ such that the nonnegative function v is differentiable on $\{x \in V: v(x) > K\}$,

$$|d_x(x, y)|, \qquad |d_y(x, y)|, \qquad |Dv(x)| \leqslant K \tag{1.6}$$

whenever the quantities on the left are defined,

$$\liminf_{|x| \to \infty} \frac{v(x)}{|x|} \geqslant k, \tag{1.7}$$

and

$$k\,|x-y| \leqslant d(x, y) \leqslant K\,|x-y| \qquad \text{for} \quad x, y \in V. \tag{1.8}$$

We continue. A function m: $[0, \infty) \to [0, \infty)$ will be called a *modulus* if it is continuous, nondecreasing, nonnegative, and subadditive and satisfies $m(0) = 0$. We will use m, m_H, etc., to denote such functions. We will also say a function σ: $[0, \infty) \times [0, \infty) \to [0, \infty)$ is a *local modulus* if $r \to \sigma(r, R)$ is a modulus for each $R \geqslant 0$ and $\sigma(r, R)$ is continuous and nondecreasing in both variables. (The words indicate that $\sigma(r, R)$ is a modulus in r when something else is "local," i.e., bounded by R.) $B_R(x)$ denotes the closed ball of center x and radius R in V and int $B_R(x)$ is its interior. Assuming that conditions (C) hold, the properties of H: $V \times \mathbf{R} \times V^* \to \mathbf{R}$ that we will employ are:

There is a local modulus σ such that

$$|H(x, r, p) - H(x, r, q)| \leqslant \sigma(|p-q|, R) \tag{H0}$$

for $x \in V$, $p, q \in V^*$, and $r \in \mathbf{R}$ satisfying $|x|$, $|p|$, $|q| \leqslant R$.

For each $(x, p) \in V \times V^*$, $r \to H(x, r, p)$ is nondecreasing. (H1)

There is a local modulus σ_H such that

$$H(x, r, p) - H(x, r, p + \lambda Dv(x)) \leqslant \sigma_H(\lambda, |p| + \lambda) \tag{H2}$$

whenever $0 \leqslant \lambda$, $(x, p) \in \Omega \times V^*$, and $v(x) > K$.

and

There is a modulus m_H such that

$$H(y, r, -\lambda d_y(x, y)) - H(x, r, \lambda d_x(x, y)) \leqslant m_H(\lambda d(x, y) + d(x, y)) \tag{H3}$$

for all x, $y \in \Omega$ with $x \neq y$ and $L(x, y) \subset \Omega$, $r \in R$, and $\lambda \geqslant 0$.

We formulate the following comparison result for (HJ′) in such a way as to exhibit an appropriate continuity of the solution in the equation. This is useful for existence proofs [6], a fact which justifies the added complexity of the statement. In the theorem, $\bar{\Omega}$ is the closure of Ω and $\partial\Omega$ is its boundary. We remind the reader that *everywhere below K, k are the constants of conditions* (C).

THEOREM 1. *Let H, $\hat{H} \in C(V \times \mathbf{R} \times V^*)$ and the conditions* (C) *hold. Let each of H and $\hat{H}$ satisfy* (H0) *and H satisfy* (H1), (H2), *and* (H3). *Let $u, v \in C(\bar{\Omega})$ be viscosity solutions of*

$$u + H(x, u, Du) \leqslant 0 \qquad \text{and} \qquad 0 \leqslant v + \hat{H}(x, v, Dv) \qquad \text{in } \Omega. \quad (1.9)$$

Let there be a modulus m such that

$$|u(x) - u(y)| + |v(x) - v(y)| \leqslant m(|x - y|) \qquad \text{if} \quad L(x, y) \subset \Omega. \quad (1.10)$$

If $\Omega \neq V$, then for ε, $a > 0$ satisfying

$$\varepsilon \leqslant (ka)^2/(m(a) + 1)$$

we have

$$\begin{aligned} u(x) - v(x) \leqslant & \sup_{\partial\Omega} (u - v)^+ + 2m(a) + m_H(2m(a) + (\varepsilon m(a))^{1/2}) \\ & + \sup\{(\hat{H}(z, r, p) - H(z, r, p))^+ : (z, r, p) \in \Omega \times R \times V^* \text{ and} \\ & |p| \leqslant 2K(m(a)/\varepsilon)^{1/2}\} \end{aligned} \quad (1.11)$$

for $x \in \Omega$. If $\Omega = V$, then (1.11) *holds with the terms* $\sup_{\partial\Omega} (u - v)^+$ *and* $2m(a)$ *replaced by* 0. *In either case, if $\hat{H} = H$ and $u \leqslant v$ on $\partial\Omega$, then $u \leqslant v$ in Ω.*

Remarks 2. We pause and attempt to illuminate this result a bit, as it is packed with interesting technical subtleties in addition to the infinite-dimensional formulation. The uniqueness asserted in the theorem was proved in the case $V = \mathbf{R}^N$, $d(x, y) = |x - y|$, and u and v uniformly continuous in Crandall and Lions [5], who also assumed that (H1) was replaced by the stronger condition of uniform continuity of $H(x, r, p)$ in (x, p) for bounded p. It was remarked in [5] that a formulation using something like "d" of condition (C) would yield a class of problems invariant under nice changes of the independent variables. Subsequently, Ishii [12] improved this result by coupling the case $v(x) = |x - x_0|$ for some x_0 with $d(x, y) = |x - y|$, eliminating the restrictive uniform continuity assumption on H mentioned above. Ishii also chose some comparison functions which improved the presentation, and we use analogues here.

An obviously interesting test class with respect to the generality of the hypotheses is the linear problem in which $H(x, p) = (p, b(x))$ where $b: V \to V$. If b is bounded on bounded sets, then (H0) is satisfied. The requirement (H1) is clearly satisfied. If V is Hilbert and $d(x, y) = |x-y|$, the requirement (H3) amounts to asking that there be a constant c such that $x \to b(x) + cx$ is "monotone" in the sense of Minty, Browder, etc. Further specializing to $V = \mathbf{R}$, $v(x) = |x - x_0|$, (H3) amounts to asking that $b(x)$ be bounded below on $x > 0$ and above on $x < 0$. Let $a_0 = 0$, $a_i = a_{i-1} + i$, and $a_{-i} = a_i$ for $i > 0$. It is easy to construct an even function v satisfying the requirements of (C) with Dv supported on I_i for i odd and an odd function b with support in I_i for i even with b' bounded below but b unbounded below on $[0, \infty)$. Then H satisfies (H2) with this v and $d(x, y) = |x-y|$, but it does not satisfy Ishii's condition. The situation is rather subtle.

Let us subject a problem $u + H(x, u, Du) = 0$, where v, d satisfy the conditions (C) and H satisfies (H0)–(H3), to a change of independent variable $x = G(z)$ where $G: V \to V$ and its inverse are everywhere defined, Lipschitz continuous, and continuously differentiable diffeomorphisms. Denote the resulting equation for $v(z) = u(G(z))$ by $v(z) + F(z, v(z), D_z v(z)) = 0$. We will not write the formulas, but the reader can verify that the new Hamiltonian satisfies the conditions of the theorem with the "transformed" d and v (let's call them d_G and v_G) given by $d_G(z, w) = d(G(z), G(w))$ and $v_G(z) = v(G(z))$. In particular, if $d(x, y) = |x-y|$, then $d_G(z, w) = |G(z) - G(w)|$. This provides a wide class of examples.

Finally, we remark that explicit error estimates in the spirit of Theorem 1 (but in finite dimensions and a different technical setting) appeared in Souganidis [15], and it was Ishii [12] who pointed out that one only needed uniform continuity of u and v on line segments in Ω in the sense of (1.10) rather than on Ω itself in the arguments.

Proof of Theorem 1. Let us first observe that the final assertion of the theorem follows immediately upon letting $\varepsilon \to 0$ and then $a \to 0$ in (1.11). We give the proof of (1.11) for the case $\Omega \neq V$ (the alternative being the simpler case). Without loss of generality, we assume that $\sup_{\partial\Omega} (u-v)^+ < \infty$. Let

$$\rho(x) = \text{distance}(x, \partial\Omega). \tag{1.12}$$

One easily deduces from (1.10) that if $L(x, y) \subset \Omega$, then

$$u(x) - v(y) \leqslant \sup_{\partial\Omega} (u-v)^+ + m(\min(\rho(x), \rho(y))) + m(|x-y|) \tag{1.13}$$

and, in particular,

$$u(x) - v(x) \leqslant \sup_{\partial\Omega} (u-v)^+ + m(\rho(x))$$

for $x \in \Omega$. Since m (being a modulus) and ρ have at most linear growth, we conclude that there are constants A, B for which

$$u(x) - v(x) \leqslant A + B\,|x| \qquad \text{for} \quad x \in \Omega. \tag{1.14}$$

We will use an auxiliary function $\zeta \in C^1(\mathbf{R})$ satisfying

$$\zeta(r) = 0 \quad \text{for} \quad r \leqslant 1, \qquad \zeta(r) = r - 2 \quad \text{for} \quad r \geqslant 3, \qquad \text{and} \quad 0 \leqslant \zeta' \leqslant 1. \tag{1.15}$$

Let a, ε, β, $R > 0$ and consider the function

$$\Phi(x, y) = u(x) - v(y) - \left(\frac{d(x, y)^2}{\varepsilon} + \beta \zeta(v(x) - R) \right) \tag{1.16}$$

on the set

$$\Delta(a) = \{(x, y) \in \Omega \times \Omega : \rho(x), \rho(y) > a \text{ and } |x - y| < a\}. \tag{1.17}$$

Roughly speaking, the result will be obtained by considering Φ near its maximum. We claim that if

$$\beta k > B, \qquad \varepsilon \leqslant (ka)^2/(m(a) + 1) \qquad \text{and} \qquad R > K \tag{1.18}$$

where B is from (1.14), then

$$\begin{aligned} \Phi(x, y) \leqslant{} & \sup_{\partial\Omega} (u - v)^+ + 2m(a) + \sigma_H(2m(a) + (\varepsilon m(a))^{1/2}) \\ & + \sigma_H(\beta, 2Km(a) + \beta) \\ & + \sup\{(\hat{H}(z, r, p) - H(z, r, p))^+ : (z, r, p) \in \Omega \times \mathbf{R} \times V^* \\ & \quad \text{and } |p| \leqslant 2K(m(a)/\varepsilon)^{1/2}\} \end{aligned} \tag{1.19}$$

on $\Delta(a)$. Let us show that the claim implies the theorem and then prove the claim. Since $u(x) - v(x) = \Phi(x, x)$ if $v(x) < R$, we may let $R \to \infty$ to see that a bound on Φ on $\Delta(a)$ which is independent of large R is also a bound on $u(x) - v(x)$ for $\rho(x) > a$. Since we also have (1.13), $u - v$ is therefore bounded if Φ is bounded. But then we are free to choose β as small as desired and the estimate on $u - v$ arising from letting $\beta \to 0$ in the bound on Φ together with (1.13) yields the theorem.

It remains to prove the claim. It clearly suffices to show that if

$$\sup_{\Delta(a)} \Phi > \sup_{\partial\Omega} (u - v)^+ + 2m(a) \tag{1.20}$$

then (1.19) holds. To this end, choose a sequence $(x_n, y_n) \in \Delta(a)$, $n = 1, 2, \ldots$, such that

$$\Phi(x_n, y_n) \text{ increases to } \sup_{\Delta(a)} \Phi \text{ and } \Phi(x_n, y_n) \geqslant \Phi(x_n, x_n). \tag{1.21}$$

It follows from (1.7), (1.8), (1.14), the inequality $u(x) - v(y) \leqslant u(x) - v(x) + m(a)$ on $\Delta(a)$, and $\beta k > C$ that $\Phi(x, y) \leqslant -1$ if $|x| + |y|$ is large, and we conclude that

$$(x_n, y_n) \text{ is bounded.} \tag{1.22}$$

Moreover, it follows from (1.13) and (1.16) that

$$\Phi(x, y) \leqslant \sup_{\partial\Omega} (u - v)^+ + m(\min(\rho(x), \rho(y))) + m(a),$$

so, from (1.20), (1.21) we conclude that there is a $\gamma > 0$ such that

$$\rho(x_n), \rho(y_n) \geqslant a + \gamma \qquad \text{for large } n. \tag{1.23}$$

Next, since

$$\Phi(x, y) - \Phi(x, x) = v(x) - v(y) - \frac{d(x, y)^2}{\varepsilon} \leqslant m(|x - y|) - \frac{d(x, y)^2}{\varepsilon}$$

on $\Delta(a)$ it follows from (1.21) that

$$d(x_n, y_n)^2 \leqslant \varepsilon m(|x_n - y_n|) \leqslant \varepsilon m(a). \tag{1.24}$$

Using (1.8) we see also that (1.24) implies $|x_n - y_n| \leqslant (\varepsilon m(a))^{1/2}/k$ and so, using (1.18),

$$|x_n - y_n| \leqslant a(m(a)/(m(a) + 1))^{1/2} < a. \tag{1.25}$$

The upshot of these considerations is that we can assume that there is a $\gamma > 0$ such that

$$S_n = \{(x, y) \in V \times V^*: |x - x_n|^2 + |y - y_n|^2 \leqslant \gamma^2\} \subset \Delta(a) \tag{1.26}$$

for all n. Put

$$\delta_n = \sup_{\Delta(a)} \Phi - \Phi(x_n, y_n) \tag{1.27}$$

and consider

$$\Psi(x, y) = \Phi(x, y) - (3\delta_n/(k\gamma)^2)(d(x, x_n)^2 + d(y, y_n)^2) \tag{1.28}$$

on S_n. We assume that $\delta_n > 0$ for all n, the other possibility being considerably simpler. Using (1.8), (1.27), (1.28), we see that

$$\Psi(x, y) \leqslant \Psi(x_n, y_n) - 2\delta_n \tag{1.29}$$

on ∂S_n. It follows that if $P\colon S_n \to \mathbf{R}$ varies by less than $2\delta_n$ over S_n and $\Psi + P$ has a maximum point with respect to S_n, then this point must be interior to S_n. According to Stegall [16], there are elements p_n, $q_n \in V^*$ satisfying

$$(|p_n|^2 + |q_n|^2)^{1/2} \leqslant \delta_n/\gamma \tag{1.30}$$

such that $\Psi(x, y) - (p_n, x) - (q_n, y)$ attains its maximum over S_n at some point $(\hat{x}, \hat{y})$, which must be an interior point by the above considerations. Now, according to the fact that u and v satisfy (1.9) in the viscosity sense, Proposition 1, $\hat{x}$ is a local maximum of $x \to \Psi(x, \hat{y})$, $\hat{y}$ is a local minimum of $y \to \Psi(\hat{x}, y)$, and the various assumptions, we conclude that

$$\begin{aligned} u(\hat{x}) + H(\hat{x}, u(\hat{x}), p_{1\varepsilon} + \beta q + \theta_{1n}) &\leqslant 0, \\ v(\hat{y}) + \hat{H}(\hat{y}, v(\hat{y}), p_{2\varepsilon} + \theta_{2n}) &\geqslant 0, \end{aligned} \tag{1.31}$$

where

$$\begin{gathered} p_{1\varepsilon} = 2d(\hat{x}, \hat{y})\, d_x(\hat{x}, \hat{y})/\varepsilon, \qquad p_{2\varepsilon} = -2d(\hat{x}, \hat{y})\, d_y(\hat{x}, \hat{y})/\varepsilon, \\ q = \zeta'(v(\hat{x}) - R)\, Dv(\hat{x}), \\ \theta_{1n} = p_n + K_n d(\hat{x}, x_n)\, d_x(\hat{x}, x_n), \qquad \theta_{2n} = -q_n - K_n d(\hat{y}, y_n)\, d_y(\hat{y}, y_n), \\ K_n = 6\delta_n/(k\gamma)^2, \end{gathered} \tag{1.32}$$

where the indexing is chosen to show only key dependencies for what follows. The reader will notice that we have written expressions above which are not always defined, e.g., $d_x(\hat{x}, \hat{y})$ and $Dv(\hat{x})$. However, in the event they may not be defined, e.g., if $\hat{x} = \hat{y}$ or $v(\hat{x}) < K$, they have coefficients which vanish, e.g., $d(\hat{x}, \hat{y})$ and $\zeta'(v(\hat{x}) - R)$. These products are defined to be 0.

We next write several chains of inequalities and then explain how each arises. We have:

$$\begin{aligned} \Phi(x_n, y_n) = \Psi(x_n, y_n) &\leqslant \Psi(\hat{x}, \hat{y}) \leqslant u(\hat{x}) - v(\hat{y}) \\ &\leqslant \hat{H}(\hat{y}, v(\hat{y}), p_{2\varepsilon}) - H(\hat{x}, u(\hat{x}), p_{1\varepsilon} + \beta q) + \varepsilon_n, \end{aligned} \tag{1.33}$$

where

$$\varepsilon_n \to 0 \qquad \text{as} \quad n \to \infty. \tag{1.34}$$

Moreover

$$\begin{aligned}
&\hat{H}(\hat{y}, v(\hat{y}), p_{2\varepsilon}) - H(\hat{x}, u(\hat{x}), p_{1\varepsilon} + \beta q) \\
&\quad \leqslant \hat{H}(\hat{y}, v(\hat{y}), p_{2\varepsilon}) - H(\hat{y}, v(\hat{y}), p_{2\varepsilon}) \\
&\qquad + H(\hat{y}, u(\hat{x}), p_{2\varepsilon}) - H(\hat{x}, u(\hat{x}), p_{1\varepsilon}) \\
&\qquad + H(\hat{x}, u(\hat{x}), p_{1\varepsilon}) - H(\hat{x}, u(\hat{x}), p_{1\varepsilon} + \beta q) \\
&\quad \leqslant \sup\{(\hat{H}(x, r, p) - H(x, r, p))^+ : (x, r, p) \in B \times \mathbf{R} \times V^*, |p| \leqslant |p_{2\varepsilon}|\} \\
&\qquad + m_H(2d(\hat{x}, \hat{y})^2/\varepsilon + d(\hat{x}, \hat{y})) + \sigma_H(\beta, 2Kd(\hat{x}, \hat{y})/\varepsilon + \beta),
\end{aligned} \tag{1.35}$$

where B is a sufficiently large ball in V and

$$d(\hat{x}, \hat{y}) \leqslant d(x_n, y_n) + 2K\gamma \leqslant (\varepsilon m(a))^{1/2} + 2K\gamma. \tag{1.36}$$

All of (1.33) but the final inequality follows at once from the definitions and the nonnegativity of the various functions. The last inequality in (1.33) with the relation (1.34) comes from (1.31) the assumption that H and $\hat{H}$ satisfy (H0) (and so are uniformly continuous in p when x and p are bounded), the fact that $(\hat{x}, \hat{y})$ lies in a bounded set (independent of n) by (1.22), $p_{i\varepsilon}$, $i = 1, 2$, are bounded for fixed ε by (1.32) and (1.6), while $|\theta_{in}| \to 0$ as $n \to \infty$ by $\delta_n \to 0$ and (1.30) and (1.32). The first inequality in (1.35) is valid because of the monotonicity (H2) and $u(\hat{x}) - v(\hat{x}) \geqslant 0$ (by (1.33)), which imply that $H(\hat{y}, v(\hat{y}), p) \leqslant H(\hat{y}, u(\hat{x}), p)$ for all p. The second inequality arises in the obvious way from (H2) and (H3) together with (1.32), (1.6), and (1.15). Finally, (1.36) arises from $(\hat{x}, \hat{y}) \in S_n$ and the Lipschitz continuity of d implied by (1.6).

From (1.32), (1.36), and (1.6) we further deduce that

$$|p_{i\varepsilon}| \leqslant 2K((\varepsilon m(a))^{1/2} + K\gamma)/\varepsilon. \tag{1.37}$$

Now we use (1.33)–(1.37) in an obvious way and let $n \to \infty$ and then $\gamma \to 0$ (as we may do) to conclude that

$$\begin{aligned}
\lim_{n \to \infty} \Phi(x_n, y_n) \leqslant \sup\{&(\hat{H}(x, r, p) - H(x, r, p))^+ : (x, r, p) \in \Omega \times \mathbf{R} \times V^*, \\
&|p| \leqslant 2K(m(a)/\varepsilon)^{1/2}\} \\
&+ m_H(2m(a) + (\varepsilon m(a))^{1/2}) + \sigma_H(\beta, 2Km(a) + \beta)
\end{aligned}$$

and this proves the claim.

Remark. A key ingredient in the above estimates was (1.24), which allowed us to estimate $d(x_n, y_n)^2$ by $\varepsilon m(a)$. In fact, this is far from sharp. Using (1.8) we found $|x_n - y_n| \leqslant (\varepsilon m(a))^{1/2}/k$, which may in turn be used in

(1.24) to find $d(x_n, y_n)^2 \leqslant \varepsilon m((\varepsilon m(a))^{1/2}/k$ and then the process can be iterated arbitrarily often. Ishii [12] uses one iteration in his proof of uniqueness. From the point of view of uniqueness, the question is not serious. From the point of view of error estimates, one might be interested in more precision. For example, if $m(a) = a^\alpha$ where $0 < \alpha < 1$, the best estimate is of the form $d(x_n, y_n)^2 \leqslant c\varepsilon^{(1/(2-\alpha))}$, and this allows one to sharpen the result above.

We next formulate a typical result for a Cauchy problem. Thus we consider two inequations

$$u_t + H(x, t, u, Du) \leqslant 0 \quad \text{in} \quad \Omega \times (0, T), \tag{1.38}$$

and

$$v_t + \hat{H}(x, t, v, Dv) \geqslant 0 \quad \text{in} \quad \Omega \times (0, T), \tag{1.39}$$

where $T > 0$. Of course, the notion of a viscosity solution of (1.38), (1.39) is contained in the notion for (HJ)—one just regards them as equations of the form (HJ) in the subset $\Omega \times (0, T)$ of the space $V \times \mathbf{R}$. The conditions we will impose on the Hamiltonians are quite analogous to those in the stationary case. Namely, we ask for conditions (C) and

There is a local modulus σ such that

$$|H(x, t, r, p) - H(x, t, r, q)| \leqslant \sigma(|p - q|, R) \tag{H0*}$$

for $(x, t, r) \in V \times [0, T] \times \mathbf{R}$, $p, q \in V^*$ satisfying $|x|, |p|, |q| \leqslant R$.

There is a $\lambda > 0$ such that for $(x, t, p) \in V \times [0, T] \times V^*$, $r \to H(x, r, p) + \lambda r$ is nondecreasing. (H1)*

There is a local modulus σ_H such that

$$H(x, t, r, p) - H(x, t, r, p + \lambda Dv(x)) \leqslant \sigma_H(\lambda, |p| + \lambda) \tag{H2*}$$

whenever $0 \leqslant \lambda$, $(x, t, r, p) \in V \times [0, T] \times \mathbf{R} \times V^*$, and $v(x) > K$.

and

There is a modulus m_H such that

$$H(y, t, r, -\lambda d_y(x, y)) - H(x, t, r, \lambda d_x(x, y)) \leqslant m_H(\lambda d(x, y) + d(x, y)) \tag{H3*}$$

for all $x, y \in V$ with $x \neq y$, $(t, r) \in [0, T] \times \mathbf{R}$, and $\lambda \geqslant 0$.

THEOREM 2. *Let* $u, v \in C(\bar{\Omega} \times [0, T])$, H, $\hat{H} \in C(V \times [0, T] \times \mathbf{R} \times V^*)$, *and* (1.38), (1.39) *hold in the viscosity sense. Assume that* H *and* $\hat{H}$ *satisfy*

(H0)*, *while* H *satisfies* (H1)*–(H3)*. *Let* m *be a modulus such that* $|u(x,t)-u(y,t)|+|v(x,t)-v(y,t)|\leqslant m(|x-y|)$ (1.40) *if* $L(x,y)\subset\Omega$ *and* $0\leqslant t\leqslant T$, *and also*

$$\lim_{t\downarrow 0} u(x,t)-v(x,t)=u(x,0)-v(x,0)$$

uniformly for x *in bounded subsets of* $\bar{\Omega}$.

Then, there is a constant C *depending only on* λ, k, K, *and* T *such that*

$$\begin{aligned} u(x,t)-v(x,t)\leqslant C(&\sup\{(u(x,t)-v(x,t))^+ : (x,t)\in\partial\Omega\times[0,T]\\ &\cup\,\Omega\times\{0\}\}\\ &+2m(a)+m_H(C(m(a)+(\varepsilon m(a))^{1/2})\\ &+\sup\{(\hat{H}(x,t,r,p)-H(x,t,r,p))^+ : (x,t,r,p)\in\Omega\\ &\times[0,T]\times\mathbf{R}\times V^* \text{ and } |p|\leqslant C(m(a)/\varepsilon)^{1/2}\}) \end{aligned}$$

for $0<\varepsilon<a^2/C(m(a)+1)$.

Remarks on the Proof. The reader will be able to construct the proof using existing ingredients—in particular, the proofs of Theorem 1 and Ishii [12, Theorem 1(i)] together with the lemma:

LEMMA 1. *Let conditions* (C) *hold and* H *and* $\hat{H}$ *be continuous. Let* u *and* v *be viscosity solutions of* (1.38) *and* (1.39) *on* $\Omega\times[0,T]$. *Then* $z(x,y,t)=u(x,t)-v(y,t)$ *is a viscosity solution of*

$$z_t+H(x,t,u(x,t),D_xz)-\hat{H}(y,t,v(y,t),-D_yz)\leqslant 0$$

on $\Omega\times\Omega\times(0,T)$.

The lemma may be proved as in [5, Lemma 2]. To this end, recall Remarks 1 and observe that $\zeta(x)=d(x,0)$ has the properties required in these remarks.

Remark on the Statement. Let us call the sup on x, r, p in the term involving $(\hat{H}-H)$ in the estimate $g(t)$ (with ε and a fixed). Formally applying Theorem 2 to u and $v+\int_0^t g(s)\,ds$, which satisfies a suitable inequation, an estimate arises which amounts to replacing g by $\int_0^t g(s)\,ds$ in the assertion. Some technical considerations concerning regularity in t need to be disposed of (by hypotheses or argument) to make this precise.

The original uniqueness results in finite dimenions [4] were formulated so as to display a trade-off between assumptions on the Hamiltonian H and regularity properties of the solutions u and remarking that the same results are valid in infinite dimensions. In particular, we consider the following strong form of (H3):

There is a local modulus $\hat{\sigma}$ such that

$$H(y, r, -\lambda d_y(x, y)) - H(x, r, \lambda d_x(x, y)) \leqslant \hat{\sigma}(d(x, y), \lambda d) \qquad (\mathrm{H3})_s$$

for all $x, y \in H$ with $L(x, y) \subset \Omega$ and $x \neq y$, $r \in \mathbf{R}$, and $\lambda \geqslant 0$.

as well as the weak form

There is a local modulus $\hat{\sigma}$ such that

$$H(y, r, -\lambda d_y(x, y)) - H(x, r, \lambda d_x(x, y)) \leqslant \hat{\sigma}(d(x, y), \lambda) \qquad (\mathrm{H3})_w$$

for all $x, y \in H$ with $L(x, y) \subset \Omega$ and $x \neq y$, $r \in \mathbf{R}$, and $\lambda \geqslant 0$,

If H satisfies the conditions of Theorem 1 with $(\mathrm{H3})_s$ instead of (H3), then all continuous and bounded viscosity solutions u and v of

$$u + H(x, u, Du) \leqslant 0 \qquad \text{and} \qquad 0 \leqslant v + H(x, v, Dv) \qquad \text{on } V \quad (1.40)$$

satisfy $u \leqslant v$. On the other hand, if H merely satisfies the conditions of Theorem 1 with $(\mathrm{H3})_w$ in place of (H3) and u and v are Lipschitz continuous viscosity solutions of (1.40), then $u \leqslant v$. Analogous remarks hold for the Cauchy problem.

Let us remark that the Cauchy problem is distinguished from the pure boundary problem in two respects—the linearity of the equation in u_t, which allows a more general dependence of H on t, and the fact that in the Cauchy problem the estimate on $u - v$ does not involve the part $t = T$ of the boundary of $\bar{\Omega} \times [0, T]$. Of course, while we did not do so here, one can identify irrelevant parts of the boundary in general—see, e.g., Crandall and Newcomb [7].

Our final observation concerns uniqueness results for bounded uniformly continuous functions. In this case, an easy examination of the proofs above shows that (H0), (H3) need only to be satisfied for bounded t and, more importantly, that the auxiliary function v which appears in (H2) need not satisfy (1.7) in (C). Indeed, it suffices to have

$$v(x) \to +\infty \qquad \text{as} \quad |x| \to \infty. \qquad (1.7')$$

Precise results and examples are given in [6].

APPENDIX: VISCOSITY SOLUTIONS WITHOUT THE RADON–NIKODYM PROPERTY

In this appendix we will define a notion of viscosity solution of (HJ) which is useful for studying equations in spaces which are not RN. Since the defintion appears to be more restrictive in the case in which V is RN, we will call this type of solution a "strict" viscosity solution.

To begin, we generalize the notion of sub- and superdifferentials to the notion of ε-sub- and superdifferentials (see Ekeland and Lebourg [10] and Ekeland [9]). Let $\Omega \subset V$ be open, $v \in C(\Omega)$, $x \in \Omega$, $\varepsilon > 0$ and set

$$\begin{aligned} D_\varepsilon^+ v(x) &= \left\{ p \in V^*: \limsup_{\substack{y \to x \\ y \in \Omega}} \frac{v(y) - v(x) - (p, y-x)}{|y-x|} \leqslant \varepsilon \right\}, \\ D_\varepsilon^- v(x) &= \left\{ p \in V^*: \liminf_{\substack{y \to x \\ y \in \Omega}} \frac{v(y) - v(x) - (p, y-x)}{|y-x|} \geqslant -\varepsilon \right\}. \end{aligned} \tag{A.1}$$

$D_\varepsilon^+ v(x)$, $D_\varepsilon^- v(x)$ are closed and convex (possibly empty) sets. It is clear that v is differentiable at x if and only if both $D^+ v(x)$ and $D^- v(x)$ are nonempty, and then $\{Dv(x)\} = D^+ v(x) = D^- v(x)$. The analogous statement here is that v is differentiable at x if and only if $D_\varepsilon^+ v(x)$ and $D_\varepsilon^- v(x)$ are nonempty for every $\varepsilon > 0$. Finally, by Ekeland's principle, for every $\varepsilon > 0$, $D_\varepsilon^+ v(x)$ (respectively, $D_\varepsilon^- v(x)$) is nonempty for a dense set of x.

We again consider the Hamilton–Jacobi equation

$$H(x, u, Du) = 0 \qquad \text{in } \Omega. \tag{HJ}$$

DEFINITION 2. A continuous function $u \in C(\Omega)$ is a strict viscosity subsolution of (HJ) in Ω if for each $\varepsilon > 0$, $x \in \Omega$, and $p \in D_\varepsilon^+ u(x)$

$$\inf\{ H(x, u(x), p+q): |q| \leqslant \varepsilon \} \leqslant 0. \tag{A.2}$$

Similarly, u is a strict viscosity supersolution if

$$\sup\{ H(x, u(x), p+q): |q| \leqslant \varepsilon \} \geqslant 0 \tag{A.3}$$

for all $x \in \Omega$ and $p \in D_\varepsilon^- u(x)$.

It is easy to see that $p \in D_\varepsilon^+ u(y)$ exactly when there is a continuous function φ which is differentiable at y and such that $D\varphi(y) = p$ and $x \to u(x) - \varphi(x) - \varepsilon\,|x-y|$ has a local maximum at y. Thus we make contact with Ekeland's principle [9], which may be used to replace Stegall's theorem in proofs of uniqueness.

It is clear that $D^+ u(x) = \bigcap \{D_\varepsilon^+ u(x): \varepsilon > 0\}$, etc., and therefore that strict viscosity solutions are viscosity solutions. The converse is almost certainly false in general, although it will be true with restraints of H and V. It is true in finite dimensions.

PROPOSITION. *Let H be continuous and $V=\mathbf{R}^N$ with the Euclidean norm. Then u is a viscosity solution of $H\leqslant 0$ $(H\geqslant 0)$ if and only if it is a strict viscosity solution of $H\leqslant 0$ (respectively, $H\geqslant 0$).*

Proof. One direction is trivial as remarked above. We show that if u is a viscosity solution of $H(x, u, Du)\leqslant 0$ then it is also a strict viscosity solution. To this end, let $\varepsilon>0$ and $p\in D_\varepsilon^+ u(y)$. Then there is a function φ differentiable at y such that $D\varphi(y)=p$ and $u(x)-\varphi(x)-\varepsilon\,|y-x|$ has a minimum at y. Because of the special choice of V we may assume that in fact φ is continuously differentiable and the maximum is strict. Then the function $u(x)-\varphi(x)-\varepsilon(|x-y|^2+\delta)^{1/2}$ will have a maximum at some point x_δ which tends to y as $\delta\downarrow 0$. Since u is a viscosity subsolution we also have

$$H(x_\delta, u(x_\delta), D\varphi(x_\delta)+q_\delta)\leqslant 0$$

where $q_\delta=\varepsilon(x_\delta-y)/(|x_\delta-y|^2+\delta)^{1/2}$ has norm less than ε. Since $D\varphi(x_\delta)\to p$, it follows that $\inf\{H(y, u(y), p+q)\colon |q|\leqslant\varepsilon\}\leqslant 0$, and the result is proved.

The above proof can be adapted to the case in which V is RN, the norm of V is continuously differentiable on $V\backslash\{0\}$, and H has the property that if x_n, $x\in V$, $r_n, r\in R$, and $q_n, p_n, p\in V^*$ satisfy $x_n\to x$, $p_n\to p$, $r_n\to r$, and $|q_n|\leqslant\varepsilon$, then

$$\liminf_{n\to\infty} H(x_n, r_n, p_n+q_n)\geqslant \inf_{|q|\leqslant\varepsilon} H(x, r, p+q)$$

and

$$\limsup_{n\to\infty} H(x_n, r_n, p_n+q_n)\leqslant \sup_{|q|\leqslant\varepsilon} H(x, r, p+q). \tag{A.4}$$

It is clear that many perturbations of the notion of a strict solution are possible. For example, rather than require that (A.2) and (A.3) hold for all $\varepsilon>0$, one could require it for small ε or make the range depend on x, etc. We will not pursue this issue here.

REFERENCES

1. V. BARBU AND G. DA PRATO, "Hamilton–Jacobi Equations in Hilbert Spaces," Pitman, London, 1983.
2. J. BOURGAIN, La propriété de Radon–Nikodym, Cours de 3e cycle polycopié no. 36, Université P. et M. Curie, Paris, 1979.
3. M. G. CRANDALL, L. C. EVANS, AND P. L. LIONS, Some properties of viscosity solutions of Hamilton–Jacobi equations, *Trans. Amer. Math. Soc.* **282** (1984), 487–502.

4. M. G. CRANDALL AND P. L. LIONS, Viscosity solutions of Hamilton–Jacobi equations, *Trans. Amer. Math. Soc.* **277** (1983), 1–42.
5. M. G. CRANDALL AND P. L. LIONS, On existence and uniqueness of solutions of Hamilton–Jacobi equations, *Nonlinear Anal. T. M. A.*, in press.
6. M. G. CRANDALL, AND P. L. LIONS, Hamilton–Jacobi equations in infinite dimensions. II. Existence of bounded viscosity solutions, in preparation.
7. M. G. CRANDALL AND R. NEWCOMB, Viscosity solutions of Hamilton–Jacobi equations at the boundary, *Proc. Amer. Math. Soc.*, in press.
8. M. G. CRANDALL AND P. E. SOUGANIDIS, Developments in the theory of nonlinear first order partial differential equations, *in* "Differential Equations, I" (W. Knowles and R. T. Lewis, Eds.), pp. 131–143, North-Holland Mathematics Studies 92, North-Holland, Amsterdam, 1984.
9. I. EKELAND, Nonconvex minimization problems, *Bull. Amer. Math. Soc.* **1** (1979), 443–474.
10. I. EKELAND AND G. LEBOURG, Generic Fréchet differentiability and perturbed optimization in Banach spaces, *Trans. Amer. Math. Soc.* **224** (1976), 193–216.
11. H. ISHII, Uniqueness of unbounded solutions of Hamilton–Jacobi equations, *Indiana Univ. Math. J.* **33** (1984), 721–748.
12. H. ISHII, Existence and uniqueness of solutions of Hamilton–Jacobi equations, preprint.
13. R. JENSEN, private communication and a work in preparation.
14. P. L. LIONS, "Generalized Solutions of Hamilton–Jacobi Equations," Pitman, London, 1982.
15. P. E. SOUGANIDIS, Existence of viscosity solutions of the Hamilton–Jacobi equation, *J. Differential Equations*, in press.
16. C. STEGALL, Optimization of functions on certain subsets of Banach spaces, *Math. Ann.* **236** (1978), 171–176.