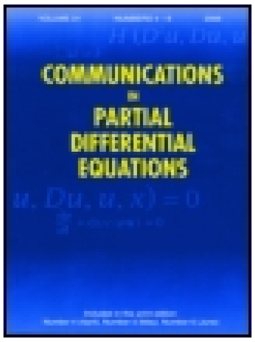

Taylor & Francis
Taylor & Francis Group

Communications in Partial Differential Equations

ISSN: 0360-5302 (Print) 1532-4133 (Online) Journal homepage: http://www.tandfonline.com/loi/lpde20

A strong comparison result for the bellman equation arising in stochastic exit time control problems and its applications

G. Barles & E. Rouy

COMMUN. IN PARTIAL DIFFERENTIAL EQUATIONS, 23(11&12), 1995–2033 (1998)

A STRONG COMPARISON RESULT FOR THE BELLMAN EQUATION ARISING IN STOCHASTIC EXIT TIME CONTROL PROBLEMS AND ITS APPLICATIONS *

G. Barles & E. Rouy
Université de Tours
Faculté des Sciences et Techniques
Parc de Grandmont
37200 Tours. France

1 Introduction

In this article, we consider the Dirichlet problem for the fully nonlinear, possibly degenerate, Hamilton-Jacobi-Bellman equation arising in stochastic optimal control with exit time. Our main contribution is to provide for this problem a rather general "Strong Comparison Result" i.e. a maximum principle type result for discontinuous viscosity solutions. The motivation for proving such a result is that it is a key argument when one wants to establish that

*This work was partially supported by the TMR Programme "Viscosity Solutions and their Applications"

the value function of a stochastic exit time control problem is continuous and the unique viscosity solution of the associated Bellman-Dirichlet problem. It is also a key result in the so-called "half-relaxed limits method" for proving the convergence of approximation schemes.

In order to be more specific, we first describe stochastic exit time control problems. We are given a system whose state is the solution $(X_t)_t$ of the controlled stochastic differential equation

$$dX_t = b(X_t, \alpha_t)dt + \sigma(X_t, \alpha_t)dW_t \ , \quad X_0 = x \in \Omega \ ,$$

where Ω is some smooth bounded domain in $I\!\!R^N$ and $(W_t)_t$ is a p-dimensional Brownian motion for some $p \in I\!\!N$. The process $(\alpha_t)_t$, the control, is some progressively measurable process with respect to the filtration associated to the Brownian motion with values in a compact metric space $\mathcal{A}$. The drift b and the diffusion matrix σ are continuous functions defined on $\overline{\Omega} \times \mathcal{A}$ taking values respectively in $I\!\!R^N$ and in the space of $N \times p$ matrices.

Then we define the value-function of the exit time control problem by

$$\mathbf{U}(x) = \inf_{(\alpha_s)_s} I\!\!E_x \left[\int_0^\tau f(X_t, \alpha_t)e^{-\lambda t}dt + \varphi(X_\tau)e^{-\lambda\tau} \right] \ , \tag{1}$$

where $I\!\!E_x$ denotes the conditional expectation with respect to the event $\{X_0 = x\}$, τ is the first exit time of the trajectory $(X_t)_t$ from Ω, $\lambda > 0$ and f, φ are continuous, real-valued functions defined respectively on $\overline{\Omega} \times \mathcal{A}$ and on $\partial\Omega$. Precise assumptions on the data will be given later on.

We refer the reader interested in stochastic control problems to A. Bensoussan[8], A. Bensoussan and J.L Lions[9, 10] where classical PDE approaches are described and to N. El Karoui[12], N.V Krylov[19] and E.D Sontag[25] where these problems are considered from a probabilistic point of view. The more recent approach by viscosity solutions was first introduced in P.L Lions[20, 21, 22] and is presented in the book of W.H Fleming and H.M Soner[13].

According to optimal control theory, it is natural to think $\mathbf{U}$ as being a solution (in fact, the "right solution") of the Hamilton-Jacobi-Bellman equation

$$H(x, u, Du, D^2u) = 0 \quad \text{in } \Omega \, , \tag{2}$$

together with the Dirichlet boundary condition

$$u = \varphi \quad \text{on } \partial\Omega \, , \tag{3}$$

where H, the so-called Hamiltonian, is given by

$$H(x, t, p, M) = \sup_{\alpha \in \mathcal{A}} \left\{ -\frac{1}{2}\mathrm{Tr}[a(x, \alpha)M] - b(x, \alpha) \cdot p + \lambda t - f(x, \alpha) \right\} \tag{4}$$

for any $x \in \overline{\Omega}$, $t \in I\!R$, $p \in I\!R^N$ and $M \in \mathcal{S}^N$ (the space of $N \times N$ symmetric matrices). Here, $a(x,\alpha) = \sigma(x,\alpha)\sigma^T(x,\alpha)$ for any $x \in \overline{\Omega}$ and $\alpha \in \mathcal{A}$ (cf. P.L Lions[20, 21, 22] or the book of W.H Fleming and H.M Soner[13]).

The first standard difficulty which arises in such control problems is that the value function $\mathbf{U}$ is not smooth enough to satisfy the above equation in the classical sense; in fact it is only expected to be continuous and may even present discontinuities. This difficulty was solved by the introduction of the notion of viscosity solutions which allows to give a sense for the value function $\mathbf{U}$ to be a solution of (2), even in the case when $\mathbf{U}$ is discontinuous (cf. again [20, 21, 22] and [13]).

But in such exit time control problems, the main difficulty comes from the treatment of the boundary condition (3). Indeed this Dirichlet boundary condition – if considered in a strong sense – is not well-adapted to the control problem : it is rather simple to build examples in which the value function is continuous in Ω and can be extended continuously to $\overline{\Omega}$ but where its extension does not satisfy (3). Note that these losses of boundary conditions may only occur when the diffusion σ degenerates along the normal direction to the boundary.

This is why it is necessary to relax the Dirichlet boundary condition which has to be read as

$$\min\Big(H(x,u,Du,D^2u), u - \varphi\Big) \leq 0 \quad \text{on } \partial\Omega\ , \tag{5}$$

and

$$\max\Big(H(x,u,Du,D^2u), u - \varphi\Big) \geq 0 \quad \text{on } \partial\Omega\ . \tag{6}$$

These inequalities have to be understood in the viscosity sense and we refer the reader to the User's Guide of M.G. Crandall, H. Ishii and P.L. Lions[11] for the presentation of these generalized boundary conditions. Roughly speaking, one may say that the equation has to hold up to the boundary wherever the solution does not assume the boundary data φ. This type of boundary conditions "in the viscosity sense" were first considered by P.L Lions[23] in the case of Neumann boundary conditions then by H.M Soner[24] for State Constraint problems and by H. Ishii[16] and B. Perthame and the first author[4, 5, 6] for exit time control problems. All these definitions can be seen as particular cases of the general definition of viscosity solutions for equations with discontinuous Hamiltonians due to H. Ishii[14, 15].

If one knows a priori that the value function $\mathbf{U}$ is continuous (and more precisely if $\mathbf{U}$ is continuous in Ω and continuously extendable to $\overline{\Omega}$), it is now standard in the theory of viscosity solutions to prove that $\mathbf{U}$ is actually the unique continuous (viscosity) solution of the generalized Dirichlet problem by combining the Dynamic Programming Principle type arguments of P.L Lions[20, 21, 22] (cf. also [13]) and the uniqueness argument of H.M

Soner[24]. It is worth mentioning that such results may handle cases when $\mathbf{U}$ is not equal to φ on the boundary.

But, in general, it is not known how to prove such a continuity property for $\mathbf{U}$ except in few cases: in the deterministic case, H.M Soner[24] proved this continuity in the State Constraint case (the case when $\varphi \equiv +\infty$) and H. Ishii[16] extended the method to treat any exit time problem. Recently, in the stochastic case, M. Katsoulakis[18] obtained a nice but still not completely general result.

These proofs of continuity are tedious and difficult to extend to the general cases we want to handle. To avoid them, B. Perthame and the first author[4, 5, 6] used the notion of semicontinuous viscosity sub and supersolutions. The strategy was the following: first, one proves that the upper semicontinuous envelope of $\mathbf{U}$, denoted by $\mathbf{U}^*$, is a viscosity subsolution of (2)-(5) and that the lower semicontinuous envelope of $\mathbf{U}$, denoted by $\mathbf{U}_*$ [(1)], is a viscosity supersolution of (2)-(6). The next step consists in using a so-called "Strong Comparison Result" i.e. a Maximum Principle type result which allows to compare <u>discontinuous</u> viscosity sub and supersolutions. Such a result implies in particular

$$\mathbf{U}^* \leq \mathbf{U}_* \quad \text{in } \Omega ,$$

and since we obviously have $\mathbf{U}_* \leq \mathbf{U}^*$ in Ω, it turns out that $\mathbf{U} = \mathbf{U}^* = \mathbf{U}_*$ is continuous in Ω and is the unique solution of (2)-(5)-(6). The Strong Comparison Result is the key point of this approach; the terminology "Strong" comes from the fact that comparing <u>continuous</u> viscosity sub and supersolutions is in general far easier although not completely trivial. We refer to M.G Crandall, H. Ishii and P.L Lions[11] for a review on the existing Strong (and not Strong) Comparison Results for any type of problems and boundary conditions and to the work of J. Burdeau and the first author[2] where such approach was first carried out in the case of stochastic control but with no control on the diffusion matrix σ.

In addition to these properties of the value functions, the Strong Comparison Result allows to prove the convergence of any suitable approximation scheme, and in particular of numerical schemes, following the work of P.E Souganidis and the first author[7].

Of course, in order to be true, such a Strong Comparison Result requires conditions on the control problems and especially near the boundary of Ω. These conditions are described in Section 2 and are admittedly rather complicated. We do not know whether they are far from being optimal or not. We only recall here that it was proved in [6] in the deterministic control case (i.e when $\sigma \equiv 0$) that non-uniqueness features for the associated problem (2)-(5)-(6) come from the role played by the trajectories, or equivalently in this case

[(1)] $\mathbf{U}^*(x) = \limsup_{y \to x} \mathbf{U}(y)$ and $\mathbf{U}_*(x) = \liminf_{y \to x} \mathbf{U}(y)$

by the vector fields b, which are tangent to the boundary $\partial\Omega$. To deal with this difficulty, the Hamiltonian was assumed in [6] to satisfy a "non-degeneracy condition" which can be explained in the determistic control case, i.e for H given by (4) but with $a \equiv 0$, in the following way: if, at $x \in \partial\Omega$, there exists a vector field $b(x,\alpha)$ which is tangent to the boundary then there exists a vector field $b(x,\alpha_1)$ pointing strictly inside Ω and a vector field $b(x,\alpha_2)$ pointing strictly outside Ω.

We use here similar control ideas properly adapted to the stochastic case; they lead to rather natural conditions on b and σ but it is worth pointing out that we have here no idea how to read these conditions directly on the function H.

The paper is organized as follows : in section 2, we describe the Strong Comparison Result, and its consequences on the control problem and for the convergence of approximation schemes are presented in section 3. Then in section 4 we provide several results connecting the boundary conditions (and in particular the losses of boundary conditions) with the behavior of the controlled dynamic in a neighborhood of $\partial\Omega$. These results can be considered as preliminaries to the (long) proof of the Strong Comparison Result which is given in Section 5 and we conclude this article by some comments on the strategy of our proof and by describing some extensions of our approach (cf. Section 6).

2 Presentation of the Strong Comparison Result

As we already mentioned it in the introduction, our Strong Comparison Result holds only for the Bellman Equation associated to a stochastic exit time control problem. Therefore our assumptions are formulated on the data of the control problem and not, as it is usually the case, on the Hamiltonian H.

We first make the following classical assumptions:

(H1) The functions σ, b and f are continuous on $\overline{\Omega} \times \mathcal{A}$. For any $\alpha \in \mathcal{A}$, $\sigma(.,\alpha)$ and $b(.,\alpha)$ are Lipschitz continuous functions on $\overline{\Omega}$; moreover

$$\sup_{\alpha\in\mathcal{A}} \|\phi(.,\alpha)\|_{C^{0,1}(\overline{\Omega})} < \infty,$$

for $\phi = \sigma_{ij}$, b_i $(1 \leq i \leq N,\ 1 \leq j \leq p)$.

(H2) $\lambda > 0$.

(H3) $\varphi \in C(\partial\Omega)$.

(H4) Ω is a smooth, bounded domain of $\mathbb{R}^N$ with a $W^{3,\infty}$-boundary $\partial\Omega$.

Assumptions **(H1)**-**(H3)** are classical: in particular, **(H1)** ensures existence and uniqueness of the dynamic $(X_t)_t$ and **(H2)** together with the boundedness of f and φ ensures that $\mathbf{U}$ is well-defined.

In addition to these classical assumptions, we need "non-degeneracy" type assumptions on (or in a neighborhood of) the boundary $\partial\Omega$. We are going to use the distance d to the boundary : by assumption **(H4)**, d is a $W^{3,\infty}$-function in a neighborhood of $\partial\Omega$; therefore d is C^2 in this neighborhood and we will denote $n(x) = -Dd(x)$ even if x is not on $\partial\Omega$; we recall that for $x \in \partial\Omega$, $n(x)$ is the outward unit normal to $\partial\Omega$ at x.

For $x \in \partial\Omega$, we set

$$\mathcal{A}_{in}(x) = \{\alpha \in \mathcal{A} /\ \sigma^T(x,\alpha)n(x) = 0 \text{ and } \frac{1}{2}\mathrm{Tr}[a(x,\alpha)D^2d(x)] + b(x,\alpha)\cdot Dd(x) \geq 0\}.$$

Roughly speaking $\mathcal{A}_{in}(x)$ is the subset, possibly empty, of controls for which the trajectory starting from $x \in \partial\Omega$ is expected to stay in $\overline{\Omega}$ for small time.

Then we denote by Γ_{in} the set of all $x \in \partial\Omega$ such that $\mathcal{A}_{in}(x) = \mathcal{A}$ and Γ_{out} the set of all $x \in \partial\Omega$ such that $\mathcal{A}_{in}(x) = \emptyset$.

Our first "non-degeneracy" type assumption is the following:

(H5) The sets Γ_{in} and Γ_{out} are both unions of connected components of $\partial\Omega$.

Then, taking **(H4)** and **(H5)** into account, $\Gamma = \partial\Omega\backslash(\Gamma_{in} \cup \Gamma_{out})$ is also a union of connected components of $\partial\Omega$ and if $x \in \Gamma$, then $\mathcal{A}_{in}(x) \neq \emptyset$ and $\mathcal{A}_{in}(x) \neq \mathcal{A}$. The "non-degeneracy" conditions on Γ reads

(H6) For every $x \in \Gamma$, one of the two following assumptions holds

(i) there exist $\eta = \eta(x) > 0$, $\mathcal{V} = \mathcal{V}(x)$ an $\overline{\Omega}$-neighborhood of x and three subsets $\mathcal{A}_1(x)$, $\mathcal{A}_2(x)$, $\mathcal{A}_3(x)$ of $\mathcal{A}$ such that, $\mathcal{A} = \mathcal{A}_1(x) \cup \mathcal{A}_2(x) \cup \mathcal{A}_3(x)$ and

a) for any $\alpha \in \mathcal{A}_1(x)$,

$$|\sigma^T(x,\alpha)n(x)| \geq \eta,$$

b) for any $\alpha \in \mathcal{A}_2(x) \cup \mathcal{A}_3(x)$,

$$\begin{cases} \sigma^T(y,\alpha)n(y) \equiv 0 \text{ on } \mathcal{V} \cap \Gamma \\ \phi_\alpha :\ y \mapsto \sigma^T(y,\alpha)n(y) \text{ is of class } W^{2,\infty} \text{ on } \mathcal{V} \end{cases}$$

and

$$\sup_{\alpha \in \mathcal{A}_2(x)\cup\mathcal{A}_3(x)} ||\phi_\alpha||_{W^{2,\infty}(\mathcal{V})} < +\infty.$$

c) for any $\alpha \in \mathcal{A}_2(x)$,

$$\frac{1}{2}\mathrm{Tr}\left[a(x,\alpha)D^2d(x)\right] + b(x,\alpha)\cdot Dd(x) \leq -\eta,$$

d) for any $\alpha \in \mathcal{A}_3(x)$ and any $y \in \mathcal{V}$, there exists $\alpha' = \alpha'(y) \in \mathcal{A}_3(x)$ such that

$$\begin{cases} a(y,\alpha') = a(y,\alpha) \\ \frac{1}{2}\mathrm{Tr}\left[a(y,\alpha')D^2d(y)\right] + b(y,\alpha')\cdot Dd(y) \geq \eta. \end{cases}$$

(ii) there exist $\mathcal{V} = \mathcal{V}(x)$ an $\overline{\Omega}$–neighborhood of x and, for any $y \in \mathcal{V}$, a control $\alpha(y) \in \mathcal{A}$ depending continuously of y such that

$$\begin{cases} \sigma^T(y,\alpha(y))n(y) \equiv 0 \text{ on } \mathcal{V}\cap\Gamma \\ y \mapsto \sigma^T(y,\alpha(y)) \text{ is of class } W^{1,\infty} \text{ on } \mathcal{V} \end{cases}$$

and

$$\begin{cases} \sigma(x,\alpha(x)) = 0 \\ b(x,\alpha(x))\cdot Dd(x) > 0\ . \end{cases}$$

In view of **(H5)**, it is clear that if $x \in \Gamma$ and **(H6)-(i)** holds then $\mathcal{A}_3(x) \neq \emptyset$ since $\mathcal{A}_{in}(x) \neq \emptyset$.

Our result is the

Theorem 2.1 : (Strong Comparison Result)
Assume that **(H1)-(H6)** *hold. If u (resp. v) is a bounded usc subsolution (resp. lsc supersolution) of (2)-(5)-(6), then*

$$u \leq v \quad in\,\Omega\ .$$

Moreover if we define $\tilde{u}$ and $\tilde{v}$ on $\overline{\Omega}$ by setting

$$\tilde{u}(x) = \begin{cases} \limsup\limits_{\substack{y\to x\\ y\in\Omega}} u(y) & on\ \partial\Omega\ , \\ u(x) & otherwise\ , \end{cases}$$

and

$$\tilde{v}(x) = \begin{cases} \liminf\limits_{\substack{y\to x\\ y\in\Omega}} v(y) & on\ \Gamma_{in}\cup\Gamma_{out}\ , \\ v(x) & otherwise\ , \end{cases}$$

then $\tilde{u}$ and $\tilde{v}$ are still respectively a usc subsolution and a lsc supersolution of (2)-(5)-(6) and

$$\tilde{u} \leq \tilde{v} \quad on\,\overline{\Omega}\ .$$

Remark 2.1 *It is worth noticing that $\tilde{u}$ and $\tilde{v}$ defined in Theorem 2.1 satisfy $\tilde{u} = u$ and $\tilde{v} = v$ in Ω while $\tilde{u} \le u$ and $\tilde{v} \ge v$ on $\partial\Omega$.*

Before examining the consequences of Theorem 2.1 and providing its proof, it is worth coming back to the admittedly strange "non-degeneracy" assumptions **(H5)**-**(H6)**.

First the subset Γ_{in} of $\partial\Omega$ is the set of points on the boundary through which the trajectories $(X_t)_t$ cannot exit Ω, and this for any choice of the control α. Therefore, on Γ_{in}, all these trajectories are staying inside $\overline{\Omega}$ and we will see (cf. Proposition 4.2) that the equation holds up to Γ_{in}. Therefore this boundary does not create any difficulty since its points behave like interior points.

The situation is completely different on Γ_{out} in a neighborhood of which all the controlled trajectories are pushed outside Ω; in this case the Dirichlet boundary condition holds in the classical sense as shown in Remark 4.1 below and this boundary does not create any difficulty either.

The fact that Γ_{in} and Γ_{out} must be *unions of connected components* of $\partial\Omega$ was already proved to be necessary in the case of deterministic control for Theorem 2.1 to hold (cf. [6] or [1]).

Through Γ, the controlled trajectories may or may not exit depending on the choice of the control; therefore this is on this boundary that the more interesting phenomena and difficulties occur. On this subset of the boundary, if $x \in \Gamma$ satisfies **(H6)-(i)**, this means that in the neighborhood $\mathcal{V}(x)$, H may be rewritten as $H = \max(H_1, H_2, H_3)$ where the H_j $(j = 1, 2, 3)$ are defined in the same way as H but by taking the supremum respectively on $\mathcal{A}_j(x)$.

It is worth noticing that, for those points, the diffusion is either uniformly non-degenerate in the normal direction, namely $|\sigma^T(y,\alpha)n(y)| \ge \eta > 0$ for $y \in \mathcal{V} \cap \Gamma$ if $\alpha \in \mathcal{A}_1(x)$ or degenerate, namely $\sigma^T(y,\alpha)n(y) = 0$ for $y \in \mathcal{V} \cap \Gamma$ if $\alpha \in \mathcal{A}_2(x) \cup \mathcal{A}_3(x)$. This is admittedly a rather strong assumption (and a key assumption in our approach) but it may be slightly improved by the following remark: if $H = \sup\limits_{\alpha \in \mathcal{A}'} (\cdots)$ where $\mathcal{A}' \subset \mathcal{A}$ then this assumption has to be checked only for $\mathcal{A}'$. To give a very rough idea of what we have in mind, this means, for example, that this assumption has to be satisfied by the extremal points of a convex set and not by all points of this convex set.

On another hand, we point out that $\mathcal{A}_1(x)$ may really depend on x because of points of Γ for which **(H6)-(ii)** holds: if there is no such points then $\mathcal{A}_1$ is a constant subset of $\mathcal{A}$ on each connected component of Γ.

The second main remark on **(H6)-(i)** concerns the $\mathcal{A}_3(x)$ part which contains the strangest requirement. It can be understood in the following way : the Hamiltonian H_3 defined above may be rewritten for $y \in \mathcal{V}(x)$ as

$$H_3(y,t,p,M) = \sup_{\alpha' \in \tilde{\mathcal{A}}_3(x,y)} \left\{ -\frac{1}{2}\mathrm{Tr}\,(a(y,\alpha')M) + F_{\alpha'}(y,t,p) \right\} ,$$

with

$$F_{\alpha'}(x,t,p) = \sup_{\substack{\alpha\in\mathcal{A}_3(x)\\ a(y,\alpha)=a(y,\alpha')}} \{-b(y,\alpha)\cdot p + \lambda t - f(y,\alpha)\} \ ,$$

for some subset $\tilde{\mathcal{A}}_3(x,y)$ of $\mathcal{A}_3(x)$ and for $t \in I\!R$, $p \in I\!R^N$ and $M \in \mathcal{S}^N$, where the semilinear Hamiltonian

$$-\frac{1}{2}\mathrm{Tr}[a(y,\alpha')M] + F_{\alpha'}(y,t,p) \ ,$$

satisfies, uniformly with respect to $\alpha' \in \tilde{\mathcal{A}}_3(x,y)$, the assumptions used in [2] for obtaining a Strong Comparison Result for semilinear equations.

The assumption **(H6)-(ii)** seems to be very restrictive but it clearly allows to handle both the case of first-order equations and what we may call the "controllable case", a simple example of which being the case when $\mathcal{A} = \{\alpha = (\alpha_1,\alpha_2);\ \alpha_1 \in \mathcal{B}(0,1) \text{ and } \alpha_2 \in B(0,1)\}$ where, for some norms, $\mathcal{B}(0,1)$ is the unit ball in the space of $N\times p$ matrices and $B(0,1)$ is the unit ball of $I\!R^N$ and when $\sigma(x,\alpha) = \alpha_1$, $b(x,\alpha) = \alpha_2$. We also want to point out here that the points $x \in \Gamma$ where this assumption holds are far easier to handle in the proof that the points where we have **(H6)-(i)**.

To conclude these comments on Theorem 2.1, we have to compare it with the other results in the literature and this is not so easy since Theorem 2.1 is only concerned with equations associated to stochastic control. In the optimal control context, Theorem 2.1 clearly extends the results which are known for first-order equations (cf. [6]) and essentially the result of [2] for semilinear equations (i.e. the case when σ does not depend on α) despite the assumption **(H6)-(i)** b) is a bit stronger here.

The comparison with the result of M. Katsoulakis[18] is more difficult since his result is not of the same type. Indeed he proves that such a comparison result holds for the state constraint problem if the subsolution (here $\tilde{u}$) satisfies a "cone condition" that we recall at the beginning of Section 4.2. Then he shows that this condition holds for the value function of the control problem under the following hypothesis : for any $x \in \partial\Omega$, there exists a control $\alpha(x)$ with a Lipschitz continuous dependence in x such that

$$\sigma^T(x,\alpha(x))n(x) = 0 \ ,$$

$$\frac{1}{2}\mathrm{Tr}\left[a(x,\alpha(x))D^2d(x)\right] + b(x,\alpha(x))\cdot Dd(x) \geq \eta > 0 \ ,$$

and that, if $(f_1,\cdots,f_{N-1})$ is an orthonormal basis of $T(x)$, the tangent hyperplane to $\partial\Omega$ at x, there is at most two j such that $\sigma^T(x,\alpha(x))f_j \neq 0$.

Under this hypothesis, the value function is proved to be continuous and can be compared with any possibly discontinuous sub and supersolution (thus providing that way the desired Strong Comparison property). This hypothesis is clearly more general than **(H6)-(ii)** but the comparison with **(H6)-(i)** is impossible, each set of hypotheses allowing to treat cases where the other fails.

3 Applications to Stochastic Exit Time Control Problems

As announced in the Introduction, our first result is the

Theorem 3.1 : *Under the assumptions of Theorem 2.1, the value function* $\mathbf{U}$ *given by* (1) *of the exit time control problem is continuous in* Ω *and can be extended continuously up to* $\partial\Omega$. *Moreover it is the unique viscosity solution of the Bellman-Dirichlet problem (2)-(5)-(6).*

We skip here the proof of this result which follows the strategy that we recalled in the Introduction and which is detailed in [2]. Of course one has to use Theorem 2.1 instead of the Strong Comparison Result of [2].

The next result is an example concerning the convergence of approximation schemes.

Theorem 3.2 : *Under the assumptions of Theorem 2.1, there exists a unique viscosity solution* $u_\varepsilon \in C(\overline{\Omega})$ *of*

$$-\frac{\varepsilon^2}{2}\Delta u_\varepsilon + H(x, u_\varepsilon, Du_\varepsilon, D^2 u_\varepsilon) = 0 \quad \text{in } \Omega\,,$$

where H *is given by* (4), *which satisfies in the classical sense*

$$u_\varepsilon = \varphi \quad \text{on } \partial\Omega\,.$$

Moreover when ε *tends to* 0, $u_\varepsilon \to \mathbf{U}$ *in* $C(\Omega)$ [(2)] *where* $\mathbf{U}$ *is given by (1).*

Proof of Theorem 3.2 : the classical difficulty to obtain such results comes from the a priori boundary layers which may occur on the boundary since u_ε agrees to the boundary data φ but $\mathbf{U}$ may not assume it continuously. As a consequence the uniform convergence of u_ε to $\mathbf{U}$ on $\overline{\Omega}$ is hopeless.

We only give few indications on the (standard) proof. We first remark that u_ε is the value function of an exit time control problem defined in the same way as $\mathbf{U}$ except that the dynamic is now

$$dX_t^\varepsilon = b(X_t^\varepsilon, \alpha_t)dt + \sigma(X_t^\varepsilon, \alpha_t)dW_t + \varepsilon d\tilde{W}_t\ ,\ X_0^\varepsilon = x \in \Omega\ ,$$

where $(\tilde{W}_t)_t$ is another Brownian motion in $\mathbb{R}^N$ which is independent of $(W_t)_t$.

The fact that u_ε is continuous up to $\overline{\Omega}$ is standard and may be seen as a consequence of the non-degeneracy of the diffusion (See Proposition 4.1).

[(2)] We recall that the convergence in $C(\Omega)$ is the uniform convergence on all compact subsets of Ω.

Moreover, from the definition of u_ε and since f is bounded, it is easy to see that the u_ε are uniformly bounded.

To prove the convergence, we use the so-called "half-relaxed limits" method. To do so, we introduce the notations

$$\underline{u}(x) = \liminf_{\substack{y\to x\\ \varepsilon\to 0}} u_\varepsilon(y)\ ,$$

and

$$\overline{u}(x) = \limsup_{\substack{y\to x\\ \varepsilon\to 0}} u_\varepsilon(y)\ ,$$

for $x \in \overline{\Omega}$.

The first step consists in applying the stability result of B. Perthame and the first author[4] : this result implies that the functions $\overline{u}$ and $\underline{u}$ are respectively viscosity sub and supersolution of (2)-(5) and of (2)-(6).

The advantage of such stability result is that it only requires an L^∞ estimate on the u_ε. But it is really useful only if we can connect $\underline{u}$ and $\overline{u}$. This is where the Strong Comparison Result is used. Indeed, since $\overline{u}$ and $\underline{u}$ are respectively sub and supersolution of (2)-(5)-(6), Theorem 2.1 implies

$$\overline{u} \leq \underline{u} \quad \text{in}\ \Omega\,. \tag{7}$$

But, by their very definition, $\underline{u} \leq \overline{u}$ in Ω, and therefore (7) immediately implies $\overline{u} = \underline{u}$ in Ω. Finally it is a simple exercice to show that this equality yields the local uniform convergence in Ω of u_ε to the continuous function $w = \overline{u} = \underline{u}$ (3), which turns out to be the unique bounded solution of (2)-(5)-(6). Moreover, by using again Theorem 2.1, we obtain $w = \mathbf{U}$ since $\mathbf{U}$ is a bounded solution of this problem. ◻

Similar arguments allow to prove the convergence in $C(\Omega)$ of any suitable approximation either of the control problem or of the Dirichlet problem. This gives for example the convergence of any stable, consistent and monotone numerical scheme following P.E Souganidis and the first author[7].

4 Dynamic and Boundary Conditions

4.1 When does the Dirichlet boundary condition hold in the classical sense and when do losses of boundary data occur?

In this section, we study the connections between the behavior of the dynamic in the neighborhood of the boundary and the way in which the boundary

(3) Notice that $\overline{u}$ is usc and $\underline{u}$ is lsc.

condition is satisfied. For related results in the deterministic case we refer the reader to [6] (See also [1]) and to [11], [2] for second-order equations and in particular for the case of stochastic control.

We consider the functions u, v, $\tilde{u}$ and $\tilde{v}$ defined as in the statement of Theorem 2.1. We first address ourselves to the problem of determining the points of $\partial\Omega$ where the Dirichlet boundary condition is satisfied in the classical sense and those where losses of this boundary data occur.

This problem is solved by a result which can be found in [2].

Proposition 4.1 *Under assumptions* **(H1)-(H4)**, *for any $x_0 \in \partial\Omega$, we have*

- *If $u(x_0) > \varphi(x_0)$ then $\mathcal{A}_{in}(x_0) = \mathcal{A}$.*
- *If $v(x_0) < \varphi(x_0)$ then $\mathcal{A}_{in}(x_0) \neq \emptyset$.*

From a control point of view, i.e. if we have in mind that $u = v = \mathbf{U}$, this result is very natural : indeed the fact that $\mathbf{U}(x_0) > \varphi(x_0)$ is clearly related to the fact that, in a neighborhood of x_0, there is no control which allows to exit Ω systematically while the inequality $\mathbf{U}(x_0) < \varphi(x_0)$ means that the optimal policy is to stay inside Ω for a while and that this objective can be achieved by the choice of some control.

Remark 4.1 *In particular, Proposition 4.1 implies that*

$$\tilde{u} \le u \le \varphi \le v \le \tilde{v} \text{ on } \Gamma_{out} ,$$

(i.e. on Γ_{out} the Dirichlet boundary condition is satisfied in the classical sense) and

$$\tilde{u} \le u \le \varphi \text{ on } \Gamma .$$

Now we turn to the Γ_{in}- part of the boundary. The result is the

Proposition 4.2 *Under the assumptions* **(H1)-(H4)**, *the function $\tilde{u}$ (resp. $\tilde{v}$) is a viscosity subsolution (resp. supersolution) of*

$$H(x, w, Dw, D^2w) = 0 \qquad \text{on } \Omega \cup \mathrm{Int}(\Gamma_{in}) .$$

Of course, in this Proposition, "$\mathrm{Int}(\Gamma_{in})$" means the interior of Γ_{in} relatively to $\partial\Omega$ and therefore if **(H5)** holds then $\mathrm{Int}(\Gamma_{in}) = \Gamma_{in}$.

This result is very natural from a control point of view : the points of Γ_{in} behave like interior points because all the controlled trajectories starting from points of Γ_{in} are staying in $\overline{\Omega}$. It is a consequence of the following Lemma which deals with linear equations.

Lemma 4.1 : *Let w be a bounded usc (resp. lsc) real-valued function defined on $\overline{\Omega} \cap \overline{B}(\bar{x}, r)$ with $\bar{x} \in \partial\Omega$ and $r > 0$, which is a subsolution (resp. supersolution) of the linear equation*

$$-\frac{1}{2}\mathrm{Tr}[\tilde{a}(x)D^2w] - \tilde{b}(x) \cdot Dw + \lambda w - \tilde{f}(x) = 0 \tag{8}$$

on $\Omega \cap B(\bar{x}, r)$ and which satisfies

$$w(x) = \limsup_{\substack{y\to x \\ y\in\Omega}} w(y) \quad (\text{ resp. } w(x) = \liminf_{\substack{y\to x \\ y\in\Omega}} w(y)) \quad \text{for all } x \in \partial\Omega \cap B(\bar{x}, r)\ .$$

We assume that **(H4)** *holds and that $\lambda > 0$, $\tilde{a} = \tilde{\sigma}\tilde{\sigma}^T$ and $\tilde{\sigma}$, $\tilde{b}$, $\tilde{f}$ are continuous functions defined on $\overline{\Omega} \cap B(\bar{x}, r)$. Finally, we assume that, for all $x \in \partial\Omega \cap B(\bar{x}, r)$,*

$$\tilde{\sigma}^T(x)n(x) = 0 \quad \text{and} \quad \frac{1}{2}\mathrm{Tr}[\tilde{a}(x)D^2d(x)] + \tilde{b}(x) \cdot Dd(x) > 0 \tag{9}$$

and that $x \to \tilde{\sigma}^T(x)n(x)$ is Lipschitz continuous on $\overline{\Omega} \cap B(\bar{x}, r)$.

Then w is a subsolution (resp. supersolution) of (8) *on $\overline{\Omega} \cap B(\bar{x}, r)$.*

Proof of Lemma 4.1. We only give the proof for the subsolution case, since the proof for the supersolution case is similar. Let $\phi \in C^2(\overline{\Omega} \cap B(\bar{x}, r))$ and let $x_0 \in \partial\Omega \cap B(\bar{x}, r)$ be a strict global maximum point of $w - \phi$ on $\overline{\Omega} \cap B(\bar{x}, r)$.

For $\varepsilon > 0$, we consider the function $x \mapsto w - \phi - \varepsilon/d$ on $\Omega \cap \overline{B}(\bar{x}, r)$. By classical arguments, this function achieves its maximum at $x_\varepsilon \in \Omega \cap \overline{B}(\bar{x}, r)$ and one can prove that $x_\varepsilon \to x_0$ as $\varepsilon \to 0$ with

$$\lim_{\varepsilon\to 0} w(x_\varepsilon) = w(x_0) \quad \text{and} \quad \lim_{\varepsilon\to 0} \frac{\varepsilon}{d(x_\varepsilon)} = 0.$$

Because of **(H4)**, for ε small enough, the projection x'_ε of x_ε on $\partial\Omega$ exists and is unique and we have, by (9),

$$|\tilde{\sigma}^T(x_\varepsilon)Dd(x_\varepsilon)| = |\tilde{\sigma}^T(x_\varepsilon)Dd(x_\varepsilon) - \tilde{\sigma}^T(x'_\varepsilon)Dd(x'_\varepsilon)| \le Cd(x_\varepsilon) \tag{10}$$

for some constant $C > 0$.

Moreover, since $x_\varepsilon \to x_0$, $x_\varepsilon \notin \partial B(\bar{x}, r)$ for ε small enough, so that x_ε is a local maximum point of $w - \phi - \varepsilon/d$ in $\Omega \cap B(\bar{x}, r)$. Hence the viscosity subsolution inequality holds

$$-\frac{1}{2}\mathrm{Tr}[\tilde{a}(x_\varepsilon)D^2\phi(x_\varepsilon)] - \tilde{b}(x_\varepsilon) \cdot D\phi(x_\varepsilon) + \lambda w(x_\varepsilon) - \tilde{f}(x_\varepsilon)$$

$$\le \frac{\varepsilon}{d(x_\varepsilon)^3}|\tilde{\sigma}^T(x_\varepsilon)Dd(x_\varepsilon)|^2 - \frac{\varepsilon}{d(x_\varepsilon)^2}\left(\frac{1}{2}\mathrm{Tr}[\tilde{a}(x_\varepsilon)D^2d(x_\varepsilon)] + \tilde{b}(x_\varepsilon) \cdot Dd(x_\varepsilon)\right).$$

By using (9) and (10), we get

$$-\frac{1}{2}\mathrm{Tr}[\tilde{a}(x_\varepsilon)D^2\phi(x_\varepsilon)] - \tilde{b}(x_\varepsilon)\cdot D\phi(x_\varepsilon) + \lambda w(x_\varepsilon) - \tilde{f}(x_\varepsilon) \le C^2\frac{\varepsilon}{d(x_\varepsilon)} = o(1)\ ,$$

and we conclude by letting ε tend to 0. □

Since on Γ_{in}, $\mathcal{A} = \mathcal{A}_{in}$, the proof of Proposition 4.2 is a direct application of Lemma 4.1 : we choose $\tilde{\psi}(x) = \psi(x, \alpha)$ for $\psi = \sigma$, b, f and successively for all $\alpha \in \mathcal{A}$, and then take the supremum over all $\alpha \in \mathcal{A}$.

We conclude this section by a result which will play a central role in the proof of Theorem 2.1.

Proposition 4.3 : *Assume that* **(H1)-(H6)** *hold. The function $\tilde{u}$ (resp. $\tilde{v}$) defined in the statement of Theorem 2.1 is a bounded usc subsolution (resp. a bounded lsc supersolution) of (2)-(5)-(6).*

Proof : We first prove the result for $\tilde{u}$. Of course, the only difficulty is on $\partial\Omega$ since $\tilde{u} = u$ in Ω.

If $x \in \partial\Omega$, two cases may happen : the first one is $u(x) \le \varphi(x)$. In this case, we have also $\tilde{u}(x) \le \varphi(x)$ since $\tilde{u}(x) \le u(x)$ and the conclusion follows. The second possibility is $u(x) > \varphi(x)$. In this case, Proposition 4.1 implies that $\mathcal{A}_{in}(x) = \mathcal{A}$ and the conclusion follows from Proposition 4.2 (recall that because of **(H5)**, $\mathrm{Int}(\Gamma_{in}) = \Gamma_{in}$).

The proof for $\tilde{v}$ is a little bit different. We first notice that $\tilde{v}$ is actually lsc because of **(H5)**. In order to prove that $\tilde{v}$ is a viscosity supersolution of (2)-(5)-(6), the only difficulty is again on $\partial\Omega$ since $\tilde{v} = v$ in Ω.

Let x be a point on $\partial\Omega$. As for $\tilde{u}$, the conclusion follows easily either if $v(x) \ge \varphi(x)$ or if $x \in \Gamma_{in}$. Therefore we have only to consider the case when $v(x) < \varphi(x)$ and $x \notin \Gamma_{in}$. Applying Proposition 4.1 yields $\mathcal{A}_{in}(x) \neq \emptyset$ and therefore $x \notin \Gamma_{out}$. Hence $x \in \Gamma$. But, by definition, $\tilde{v} = v$ on Γ and, by **(H5)**, Γ is a union of connected components of $\partial\Omega$; therefore $\tilde{v} = v$ in a neighborhood of x and the conclusion follows. □

4.2 On the behaviour of the subsolutions near the - boundary Γ

In the proof of a Strong Comparison Result for a Dirichlet problem, the behavior of the subsolutions and/or the supersolutions near the boundary plays an important role : this fact is pointed out in [18] where it is stated that such a comparison result holds under a "cone condition" on the subsolution. More specifically, the subsolution $\tilde{u}$ has to satisfy, for any $x \in \partial\Omega$ and for some $0 < \beta < 1$,

$$\tilde{u}(x) = \limsup_{\substack{y\to x\\ y\in K(x,\beta)}} \tilde{u}(y)\ ,$$

where $K(x,\beta) = \{y \in \Omega;\ (y-x)\cdot n(x) \le -\beta|y-x|\}$.

We investigate in this section similar properties. Though, it is worth pointing out that, in the case of **(H6)-(i)**, we will not be able to prove so strong a result and this will create additional difficulties in the proof of Theorem 2.1.

Again, we first provide a result in the case of a linear equation.

Theorem 4.1 : *Let w be a bounded usc real-valued function defined on $\overline{\Omega} \cap B(\bar{x}, r)$ with $\bar{x} \in \partial\Omega$ and $r > 0$, which is a subsolution of the linear equation*

$$-\frac{1}{2}\mathrm{Tr}[\tilde{a}(x)D^2 w] - \tilde{b}(x)\cdot Dw + \lambda w - \tilde{f}(x) \le 0 \ \text{ in } \overline{\Omega} \cap B(\bar{x}, r)\ ,$$

and which satisfies

$$w(x) = \limsup_{\substack{y \to x \\ y \in \Omega}} w(y) \quad \text{for all } x \in \partial\Omega \cap B(\bar{x}, r).$$

We assume that **(H4)** *holds and that $\lambda > 0$, $\tilde{a} = \tilde{\sigma}\tilde{\sigma}^T$ on $\overline{\Omega} \cap B(\bar{x}, r)$, $\tilde{b}$, $\tilde{f}$ are continuous and $\tilde{\sigma}$ is Lipschitz continuous on $\overline{\Omega} \cap B(\bar{x}, r)$. Then*

(i) if, for all $x \in \partial\Omega \cap B(\bar{x}, r)$, one has

$$\begin{cases} \tilde{\sigma}^T(x)n(x) = 0 \\ \frac{1}{2}\mathrm{Tr}[\tilde{a}(x)D^2 d(x)] + \tilde{b}(x)\cdot Dd(x) > 0, \end{cases}$$

there exists a sequence $(x_k)_k$ of points of Ω which converges to $\bar{x}$ such that

$$\begin{cases} w(x_k) \to w(\bar{x}) \\ |x_k - \bar{x}|^2 = o(1)d(x_k) \end{cases}$$

where $o(1) \to 0$ as $k \to +\infty$;

(ii) besides, if additionally $\tilde{\sigma}(\bar{x}) = 0$, the sequence $(x_k)_k$ also satisfies

$$|x_k - \bar{x}| = O(1)d(x_k)$$

where $O(1)$ is bounded when $k \to +\infty$.

From this result, we immediately deduce the following corollary which plays a key role in the proof of Theorem 2.1.

Corollary 4.1 : *Under the assumptions of Theorem 2.1, one has, for any $\bar{x} \in \Gamma$,*

(i) if $\bar{x}$ satisfies **(H6)-(i)**, *the conclusion of Theorem 4.1-(i) holds for $w = \tilde{u}$;*

(ii) if $\bar{x}$ satisfies **(H6)-(ii)**, *the conclusion of Theorem 4.1-(ii) holds for $w = \tilde{u}$.*

The proof of this corollary is immediate since, for $r > 0$ small enough, $\tilde{u}$ is a subsolution of

$$-\frac{1}{2}\mathrm{Tr}[a(x,\alpha(x))D^2w] - b(x,\alpha(x))\cdot Dw + \lambda w - f(x,\alpha(x)) \leq 0 \text{ in } \Omega \cap B(\bar{x},r) \ ,$$

where $\alpha(x)$ is either the constant control $\alpha'(\bar{x})$ given by d) in the case of assumption **(H6)-(i)** (recall that $\mathcal{A}_3(x) \neq \emptyset$), or given by **(H6)-(ii)**. Then, by applying Lemma 4.1 with $\tilde{\psi}(x) = \psi(x,\alpha(x))$ for $\psi = \sigma$, b, f, this equation eventually holds on $\overline{\Omega} \cap B(\bar{x},r)$ for r small enough and we apply Theorem 4.1.

Now we turn to the

Proof of Theorem 4.1 : The proof consists in several steps.

Step 1 : Reduction to the case when the boundary is flat. Because of **(H4)** and changing if necessary r by taking it small enough, there exists a C^2-diffeomorphism $\psi = (\psi_1, \ldots, \psi_N)$ from $B(\bar{x},r)$ onto an open set $D \subset \mathbb{R}^N$ such that

(i) $\psi(\Omega \cap B(\bar{x},r)) \subset \{y \in \mathbb{R}^N /\ y_N > 0\}$

(ii) $\psi(\partial\Omega \cap B(\bar{x},r)) \subset \{y \in \mathbb{R}^N /\ y_N = 0\}$

(iii) $\psi(\bar{x}) = 0$ and $\psi_N(x) = d(x)$.

We now introduce the function χ defined on $D \cap \{y \in \mathbb{R}^N /\ y_N \geq 0\}$ by

$$\chi(y) = w(\psi^{-1}(y)).$$

Tedious but straightforward computations imply that χ is a usc bounded subsolution of

$$-\frac{1}{2}\mathrm{Tr}[\bar{a}(x)D^2\chi] - \bar{b}(x)\cdot D\chi + \lambda\chi - \bar{f}(x) \leq 0 \text{ in } D \cap \{y \in \mathbb{R}^N /\ y_N \geq 0\}$$

where, by using the convention $y = \psi(x)$,

$$\bar{f}(y) = \tilde{f}(x)$$

$$\bar{b}_k(y) = \sum_{i=1}^{N} \frac{\partial \psi_k}{\partial x_i}(x)\tilde{b}_i(x) + \frac{1}{2}\sum_{i,j=1}^{N} \frac{\partial^2 \psi_k}{\partial x_i \partial x_j}(x)\tilde{a}_{ij}(x) \text{ for } 1 \leq k \leq N$$

$$\bar{a}_{kl}(y) = \sum_{i,j=1}^{N} \frac{\partial \psi_k}{\partial x_i}(x)\frac{\partial \psi_l}{\partial x_j}(x)\tilde{a}_{ij}(x) \text{ for } 1 \leq k,l \leq N$$

$$\bar{\sigma}_{km}(y) = \sum_{i=1}^{N} \frac{\partial \psi_k}{\partial x_i}(x)\tilde{\sigma}_{im}(x) \text{ for } 1 \leq k \leq N \text{ and } 1 \leq m \leq p.$$

Now, if $(e_1, \ldots, e_N)$ and $(\tilde{e}_1, \ldots, \tilde{e}_p)$ denote the standard orthonormal bases of $I\!R^N$ and $I\!R^p$ respectively, since $\psi_N(x) = d(x)$, we have, for all $m \in \{1, \ldots, p\}$,

$$\bar{\sigma}_{Nm}(y) = \sum_{i=1}^{N} \frac{\partial d}{\partial x_i}(x)\tilde{\sigma}_{im}(x) = \tilde{\sigma}^T(x)Dd(x) \cdot \tilde{e}_m$$

and therefore, if $x \in \partial\Omega$ (i.e. $y_N = 0$), then

$$\bar{\sigma}^T(y)e_N = \tilde{\sigma}^T(x)Dd(x) = 0,$$

and similarly

$$\bar{b}_N(y) = \tilde{b}(x) \cdot Dd(x) + \frac{1}{2}\mathrm{Tr}[\tilde{a}(x)D^2d(x)] > 0.$$

If we assume that Theorem 4.1 is true for χ and let $(y_k)_k$ be the sequence which is defined therein, then by setting $x_k = \psi^{-1}(y_k)$, the sequence $(x_k)_k$ converges to $\bar{x}$ and satisfies

$$w(x_k) = \chi(y_k) \to \chi(0) = w(\bar{x}) \text{ as } k \to +\infty$$

and

$$|x_k - \bar{x}|^2 \le C|y_k - 0|^2 = C\ o(1)y_N^k = C\ o(1)d(x_k) \text{ for all } k$$

in case (i) or

$$|x_k - \bar{x}| \le C|y_k - 0| = C\ O(1)y_N^k = C\ O(1)d(x_k) \text{ for all } k$$

in case (ii) and therefore the result in Ω is a consequence of the one for the flat boundary case.

Step 2 : The flat boundary case. We assume that $\partial\Omega \cap B(\bar{x}, r)$ is flat and we first consider case (i). We introduce for all $\varepsilon > 0$, the functions w_ε defined by

$$w_\varepsilon(y) = w(\varepsilon y', \varepsilon^2 y_N)$$

for all $y = (y', y_N) \in \tilde{\mathcal{O}}_\varepsilon = \{z \in I\!R^N /\ (\varepsilon z', \varepsilon^2 z_N) \in \overline{\Omega} \cap B(\bar{x}, r)\}$.

Then w_ε is a subsolution of

$$-\frac{1}{2}\mathrm{Tr}[\tilde{a}(y_\varepsilon)D^2_{y'y'}w_\varepsilon] - \frac{1}{2\varepsilon}\mathrm{Tr}[\tilde{a}(y_\varepsilon)D^2_{y'y_N}w_\varepsilon] - \frac{1}{2\varepsilon^2}\mathrm{Tr}[\tilde{a}(y_\varepsilon)D^2_{y_Ny_N}w_\varepsilon]$$

$$-\tilde{b}(y_\varepsilon) \cdot (\varepsilon D_{y'}w_\varepsilon + D_{y_N}w_\varepsilon) + \varepsilon^2\lambda w_\varepsilon - \varepsilon^2\tilde{f}(y_\varepsilon) \le 0 \quad \text{in } \overline{\mathcal{O}}_\varepsilon$$

where

$$D_{y_N}w_\varepsilon = \frac{\partial w_\varepsilon}{\partial y_N}e_N, \quad D_{y'}w_\varepsilon = Dw_\varepsilon - D_{y_N}w_\varepsilon\ ,$$

$$D^2_{y_N y_N} w_\varepsilon = \frac{\partial^2 w_\varepsilon}{\partial y_N \partial y_N} e_N \otimes e_N, \quad D^2_{y' y_N} w_\varepsilon = \sum_{i=1}^{N-1} \frac{\partial^2 w_\varepsilon}{\partial y_i \partial y_N} (e_i \otimes e_N + e_N \otimes e_i),$$
$$D^2_{y'y'} w_\varepsilon = D^2 w_\varepsilon - D^2_{y' y_N} w_\varepsilon - D^2_{y_N y_N} w_\varepsilon.$$

By using the boundedness and Lipschitz continuity of $\tilde{\sigma}$ and the fact that $\tilde{\sigma}^T(y', 0)e_N = 0$, we have for all $X \in \mathcal{S}^N$,

$$\begin{aligned}\frac{1}{2\varepsilon} \mathrm{Tr}\left[\tilde{a}(y_\varepsilon) \sum_{i=1}^{N-1} X_{iN}(e_i \otimes e_N + e_N \otimes e_i)\right] &= \frac{1}{\varepsilon} \sum_{i=1}^{N-1} X_{iN} \tilde{\sigma}^T(y_\varepsilon) e_i \cdot \tilde{\sigma}^T(y_\varepsilon) e_N \\ &= O(\varepsilon)\end{aligned}$$

and
$$\frac{1}{2\varepsilon^2} \mathrm{Tr}[\tilde{a}(y_\varepsilon) X_{NN} e_N \otimes e_N] = \frac{1}{2\varepsilon^2} X_{NN} |\tilde{\sigma}^T(y_\varepsilon) e_N|^2 = O(\varepsilon^2).$$

Besides, it is clear that the functions w_ε are uniformly bounded. Hence, if we set $\bar{w} = \limsup^* w_\varepsilon$, i.e.

$$\bar{w}(y) = \limsup_{\substack{z \to y \\ \varepsilon \to 0}} w_\varepsilon(z)$$

then (see the half-relaxed limits method in [11]) $\bar{w}$ is a subsolution of

$$-\tilde{b}_N(0) \frac{\partial \bar{w}}{\partial y_N} - \frac{1}{2} \mathrm{Tr}[\tilde{a}(0) D^2_{y'y'} \bar{w}] \leq 0 \quad \text{in } \mathbb{R}^{N-1} \times [0, +\infty). \tag{11}$$

We consider (11) as being a backward parabolic equation set, for example, in the strip $\mathbb{R}^{N-1} \times [0, 1)$, y_N playing the role usually devoted to the time variable. Here, we associate this equation to the final condition $\bar{w}(\cdot, 1)$. Then $\bar{w}$ is a subsolution of this problem whose maximal subsolution (and solution) is given by

$$\mathbb{E}[\bar{w}(Y^{y'}_{y_N}, 1)] \text{ where } Y^{y'}_{y_N} = y' + \frac{\tilde{\sigma}(0)}{\sqrt{\tilde{b}_N(0)}} (\widehat{W}_1 - \widehat{W}_{y_N})$$

where $(\widehat{W}_t)_t$ denotes a standard Brownian motion in $\mathbb{R}^{N-1}$.

From this fact, we deduce that

$$\bar{w}(0, 0) \leq \mathbb{E}[\bar{w}(Y^0_0, 1)]\,. \tag{12}$$

But, by the definition of $\bar{w}$ and since w is usc, we know that

$$\bar{w}(y', y_N) \leq w(0, 0) \quad \text{in } \mathbb{R}^{N-1} \times [0, 1] \quad \text{and} \quad \bar{w}(0, 0) = w(0, 0).$$

Therefore (12) implies

$$\bar{w}(Y^0_0, 1) = w(0, 0) \;\text{ a.s. },$$

or, in other words,

$$\bar{w}\left(\frac{\tilde{\sigma}(0)}{\sqrt{\tilde{b}_N(0)}}(\widehat{W}_1 - \widehat{W}_0), 1\right) = w(0,0) \text{ a.s. .}$$

Since the distribution of the random variable $\widehat{W}_1 - \widehat{W}_0$ is a Gaussian law $\mathcal{N}(0, I_{R^{N-1}})$, this last equality implies

$$\bar{w}\left(\frac{\tilde{\sigma}(0)}{\sqrt{\tilde{b}_N(0)}}y', 1\right) = w(0,0) \text{ a.e. in } I\!R^{N-1}. \tag{13}$$

Finally, since $\bar{w}$ is usc and $\bar{w}(y',1) \leq w(0,0)$ in $I\!R^{N-1}$, the equality (13) holds for all $y' \in I\!R^{N-1}$ and, in particular $\bar{w}(0,1) = w(0,0)$.

It just remains to interpret this property in terms of w_ε: there exist sequences $(\varepsilon_k)_k$ and $(y_k)_k$ converging respectively to 0 and $(0,1)$ such that, if $y_k = (y'_k, y^k_N)$, one has

$$w_{\varepsilon_k}(y'_k, y^k_N) = w(\varepsilon_k y'_k, \varepsilon_k^2 y^k_N) \to w(0,0) \text{ as } k \to +\infty.$$

Let $x_k = (\varepsilon_k y'_k, \varepsilon_k^2 y^k_N)$. Then

$$x_k \to 0, \quad w(x_k) \to w(0) \text{ as } k \to +\infty \ ,$$

and

$$d(x_k) = \varepsilon_k^2 y^k_N \ ,$$

while

$$|x_k - 0|^2 = \varepsilon_k^2 |y'_k|^2 + \varepsilon_k^4 |y^k_N|^2 = d(x_k)\left(\frac{|y'_k|^2}{y^k_N} + \varepsilon_k^2 y^k_N\right) = o(1) d(x_k) \ ,$$

and the proof of case (i) is complete.

For (ii), the strategy is similar but a little simplerfor the scaling argument is different. Here we set

$$w_\varepsilon(y) = w(\varepsilon y) \ ,$$

for $y \in \tilde{\mathcal{O}}_\varepsilon := \{z \in I\!R^N : \ \varepsilon z \in \overline{\Omega} \cap B(\bar{x}, r)\}$. The function $\bar{w}$, defined in the same way as above, is this time a subsolution of the backward transport equation

$$-\tilde{b}_N(0)\frac{\partial \bar{w}}{\partial y_N} - \tilde{b}'(0) \cdot D_{y'}\bar{w} \leq 0 \text{ in } I\!R^{N-1} \times [0, +\infty),$$

where $\tilde{b}'(0) = \tilde{b}(0) - \tilde{b}_N(0)e_N$. The maximal solution of this equation with the terminal data $\bar{w}(y',1)$ is given at (y', y_N) by

$$\bar{w}(y' - \frac{\tilde{b}'}{\tilde{b}_N(0)}(y_N - 1), 1) \ ,$$

and following the same argument as in the case (i), we obtain

$$\bar{w}(\frac{\tilde{b}'}{\tilde{b}_N(0)}, 1) = w(0,0) ,$$

and this property gives the result as in the case (i). ◻

Remark 4.2 : *In [18], M. Katsoulakis proves, by using probabilistic arguments, that the conclusion (ii) of Corollary 4.1 holds for the value function of the state constraint problem under the assumptions we recall at the end of Section 2. On the contrary, Corollary 4.1 is valid for any subsolution and is a key result in the proof of our Strong Comparison Result.*

5 Proof of Theorem 2.1

Because of Proposition 4.3, $\tilde{u}$ and $\tilde{v}$ are respectively a usc subsolution and a lsc supersolution of (2)-(5) -(6). The proof of Theorem 2.1 consists in comparing the subsolution $\tilde{u}$ and the supersolution $\tilde{v}$ (and not in arguing directly on u and v). However, for reasons which will be clear below, we are not going to consider the equation under its present form but we are going to make a change of variable which preserves the viscosity sub- and supersolutions.

More precisely, we know that, since $\tilde{u}$ (resp. $\tilde{v}$) is a viscosity subsolution (resp. supersolution) of (2)-(5) (resp. (2)-(6)) then, for any increasing C^2-function ω, the function $\bar{u} = \omega^{-1}(\tilde{u})$ (resp. $\bar{v} = \omega^{-1}(\tilde{v})$) is a viscosity subsolution (resp. supersolution) of

$$\overline{H}(x, w, Dw, D^2 w) = 0 \text{ in } \Omega \tag{14}$$

and

$$\min(\overline{H}(x, w, Dw, D^2 w), w - \overline{\varphi}) \leq 0 \text{ on } \partial\Omega ,$$

(resp. (14) and

$$\max(\overline{H}(x, w, Dw, D^2 w), w - \overline{\varphi}) \geq 0 \text{ on } \partial\Omega ,)$$

where $\overline{H}$ is defined on $\overline{\Omega} \times I\!R \times I\!R^N \times \mathcal{S}^N$ by

$$\overline{H}(x, u, p, X) = \frac{1}{\omega'(u)} H(x, \omega(u), \omega'(u)p, \omega'(u)X + \omega''(u) p \otimes p)$$

and where $\overline{\varphi}$ is given on $\partial\Omega$ by $\overline{\varphi} = \omega^{-1}(\varphi)$.

Thus, for

$$H(x, u, p, X) = \sup_{\alpha \in A} \left\{ -\frac{1}{2}\mathrm{Tr}[a(x,\alpha)X] - b(x,\alpha) \cdot p + \lambda u - f(x,\alpha) \right\} ,$$

we get

$$\overline{H}(x,u,p,X) = \sup_{\alpha\in A}\{-\frac{1}{2}\mathrm{Tr}[a(x,\alpha)X] - \frac{1}{2}\frac{\omega''(u)}{\omega'(u)}|\sigma^T(x,\alpha)p|^2 - b(x,\alpha)\cdot p + \lambda\frac{\omega(u)}{\omega'(u)} - \frac{f(x,\alpha)}{\omega'(u)}\}.$$

The proof of the inequality $\tilde{u} \leq \tilde{v}$ on $\overline{\Omega}$ is reduced in that way to the proof of the inequality $\bar{u} \leq \bar{v}$ on $\overline{\Omega}$. The choice of the change of variable ω will be given later on.

Let M be the supremum of the bounded function $\bar{u} - \bar{v}$ on $\overline{\Omega}$. Since $\bar{u} - \bar{v}$ is usc, there exists $x_0 \in \overline{\Omega}$ such that $\bar{u}(x_0) - \bar{v}(x_0) = M$. We want to prove that $M \leq 0$. To do so, we assume by contradiction that $M > 0$.

By Remark 4.1, it is clear that such a maximum point cannot be on Γ_{out}. In fact, we are led to investigate two cases, namely the case when there is a maximum point of $\bar{u} - \bar{v}$ on $\Omega \cup \Gamma_{in}$ and the case when all the maximum points are on Γ.

In both cases, the proof uses similar arguments and to avoid repeating them, we are going to summarize now their common features. The proof consists in considering, for some small parameter $\varepsilon > 0$, the global maximum $(x_\varepsilon, y_\varepsilon)$ of the function

$$(x,y) \mapsto \bar{u}(x) - \bar{v}(y) - \phi_\varepsilon(x,y)\ ,$$

where ϕ_ε is a suitable C^2 – test-function on $\overline{\Omega} \times \overline{\Omega}$.

Then we can apply the following lemma (see Theorem 3.2 of [11]).

Lemma 5.1 *For any $\eta > 0$, there exist X_ε, $Y_\varepsilon \in \mathcal{S}^N$ such that*

$$\begin{pmatrix} X_\varepsilon & 0 \\ 0 & -Y_\varepsilon \end{pmatrix} \leq D^2\phi_\varepsilon(x_\varepsilon, y_\varepsilon) + \eta\left(D^2\phi_\varepsilon(x_\varepsilon, y_\varepsilon)\right)^2\ , \tag{15}$$

and such that, if $x_\varepsilon \in \Omega$ or if $x_\varepsilon \in \partial\Omega$ with $\bar{u}(x_\varepsilon) > \overline{\varphi}(x_\varepsilon)$, one has

$$\overline{H}(x_\varepsilon, \bar{u}(x_\varepsilon), D_x\phi_\varepsilon(x_\varepsilon, y_\varepsilon), X_\varepsilon) \leq 0\ , \tag{16}$$

and if $y_\varepsilon \in \Omega$ or if $y_\varepsilon \in \partial\Omega$ with $\bar{v}(y_\varepsilon) < \overline{\varphi}(y_\varepsilon)$, one has

$$\overline{H}(y_\varepsilon, \bar{v}(y_\varepsilon), -D_y\phi_\varepsilon(x_\varepsilon, y_\varepsilon), Y_\varepsilon) \geq 0. \tag{17}$$

In the above lemma, we have dropped the dependence of X_ε and Y_ε with respect to η for the sake of simplicity of notations. In the same way, in order to simplify the exposure and to emphasize the main new arguments of our proof, we proceed in the following as if η were equal to 0. A complete correct proof would consist in arguing with $\eta > 0$ and then in letting η tend to 0.

For all $\alpha \in \mathcal{A}$, we set

$$G_\alpha(x,u,p) = -\frac{1}{2}\frac{\omega''(u)}{\omega'(u)}|\sigma^T(x,\alpha)p|^2 - b(x,\alpha)\cdot p + \lambda\frac{\omega(u)}{\omega'(u)} - \frac{f(x,\alpha)}{\omega'(u)} .$$

By **(H1)** and the compactness of $\mathcal{A}$, for any (x,u,p,M), there exists $\alpha \in \mathcal{A}$ such that

$$\overline{H}(x,u,p,X) = -\frac{1}{2}\mathrm{Tr}[a(x,\alpha)X] + G_\alpha(x,u,p).$$

If (17) holds, this leads to the existence of $\alpha_\varepsilon \in \mathcal{A}$ such that

$$-\frac{1}{2}\mathrm{Tr}[a(y_\varepsilon,\alpha_\varepsilon)Y_\varepsilon] + G_{\alpha_\varepsilon}(y_\varepsilon,\bar{v}(y_\varepsilon),-D_y\phi_\varepsilon(x_\varepsilon,y_\varepsilon)) \geq 0 . \tag{18}$$

If in addition (16) holds, by the form of $\overline{H}$, one also has

$$-\frac{1}{2}\mathrm{Tr}[a(x_\varepsilon,\alpha_\varepsilon)X_\varepsilon] + G_{\alpha_\varepsilon}(x_\varepsilon,\bar{u}(x_\varepsilon),D_x\phi_\varepsilon(x_\varepsilon,y_\varepsilon)) \leq 0 . \tag{19}$$

In order to use these inequalities, we have to investigate the properties of the functions G_{α_ε} and in particular to choose the change of variable ω. This is the aim of the following lemma.

Lemma 5.2 *Let $K = \max(\|u\|_\infty, \|v\|_\infty)$. There exists $A > 0$ depending only on $\|f\|_\infty$, K and λ such that if*

$$\omega'(s) = A - e^{\omega(s)}$$

then

$$\frac{\partial G_\alpha}{\partial s}(x,s,p) \geq \frac{\lambda}{2}(1 + |\sigma^T(x,\alpha)p|^2) ,$$

for any $\alpha \in \mathcal{A}$ and for all $x \in \overline{\Omega}$, $s \in [\omega^{-1}(-K), \omega^{-1}(K)]$ and $p \in I\!R^N$.

This result is strongly inspired by G. Barles and F. Murat [3] to which we refer for the proof.

We now come back to the matrices inequality (15) and point out that, with the convention $\eta = 0$, it may be rewritten as

$$X_\varepsilon p\cdot p - Y_\varepsilon q\cdot q \leq D^2\phi_\varepsilon(x_\varepsilon,y_\varepsilon)(p,q)\cdot(p,q) \tag{20}$$

for all p, q in $I\!R^N$.

In order to compute the traces arising in the inequalities (18) and (19), it is convenient to choose properly two orthonormal bases $(e_1,\ldots,e_p)$ and $(f_1,\ldots,f_p)$ of $I\!R^p$ and then to set, for all $k \in \{1,\ldots,p\}$,

$$p_k = \mu\sigma(x_\varepsilon,\alpha_\varepsilon)e_k \quad \text{and} \quad q_k = \nu\sigma(y_\varepsilon,\alpha_\varepsilon)f_k \tag{21}$$

where μ and ν are positive parameters which may depend on ε. By letting $p = p_k$ and $q = q_k$ in (20) and by summing over k, we get

$$\mu^2 \mathrm{Tr}[a(x_\varepsilon, \alpha_\varepsilon)X_\varepsilon] - \nu^2 \mathrm{Tr}[a(y_\varepsilon, \alpha_\varepsilon)Y_\varepsilon] \leq \sum_{k=1}^{p} D^2\phi_\varepsilon(x_\varepsilon, y_\varepsilon)(p_k, q_k)\cdot(p_k, q_k) \ . \quad (22)$$

Then we multiply respectively (18) and (19) by ν^2 and μ^2 and we substract them. By taking into account (22), this yields

$$\begin{aligned} \mu^2 G_{\alpha_\varepsilon}(x_\varepsilon, \bar{u}(x_\varepsilon), D_x\phi_\varepsilon(x_\varepsilon, y_\varepsilon)) - \nu^2 G_{\alpha_\varepsilon}(y_\varepsilon, \bar{v}(y_\varepsilon), -D_y\phi_\varepsilon(x_\varepsilon, y_\varepsilon)) \\ \leq \frac{1}{2}\sum_{k=1}^{p} D^2\phi_\varepsilon(x_\varepsilon, y_\varepsilon)(p_k, q_k)\cdot(p_k, q_k) \ . \end{aligned} \quad (23)$$

It remains to estimate the G_{α_ε}-terms. We set for $x \in \overline{\Omega}$, $u \in I\!R$ and $p \in I\!R^N$

$$\tilde{G}_\alpha(x, u, p) = G_\alpha(x, u, p) + b(x, \alpha)\cdot p \ .$$

The reason to do so will be clear later but we may say here that the "$b(x,\alpha)\cdot p$"-term plays a particular role in the equation (cf. **(H6)**) and this is the reason for dropping it out of G_α and for considering it apart from the other terms.

In order to simplify the statement of the next result, we set for $\alpha \in \mathcal{A}$, $x, y \in \overline{\Omega}$ and $p, q \in I\!R^N$

$$K_\alpha(x, y, p, q) = 1 + |\sigma^T(x,\alpha)p| + |\sigma^T(y,\alpha)q| \ .$$

For $\tilde{G}_\alpha$ we have the

Lemma 5.3 *Assume that* **(H1)**-**(H2)** *hold and* μ, ν *satisfy*

$$1/\overline{C} \leq \mu, \nu \leq \overline{C} \quad \text{and} \quad |\mu - \nu| \leq \overline{C}\varepsilon \ ,$$

for some constant $\overline{C} > 0$*, then, there exists a constant* $C > 0$ *such that, for any* $\alpha \in \mathcal{A}$*, for any* $x, y \in \overline{\Omega}$*,* $\omega^{-1}(-K) \leq v < u \leq \omega^{-1}(K)$ *and* $p \in I\!R^N$*, we have*

$$\begin{aligned} \mu^2\tilde{G}_\alpha(x,u,p) - \nu^2\tilde{G}_\alpha(y,v,q) \ \geq\ & \frac{1}{C}(u - v - C^2\varepsilon)\,[K_\alpha(x,y,p,q)]^2 \\ & -C|\sigma^T(x,\alpha)p - \sigma^T(y,\alpha)q|K_\alpha(x,y,p,q) \\ & -Cm_f(|x-y|) \ , \end{aligned}$$

where m_f *is the modulus of continuity of* f *in* x *which is uniform in* $\alpha \in \mathcal{A}$.

The proof of this lemma is easy and left to the reader.

The test-function ϕ_ε will be choosen of the form

$$\phi_\varepsilon(x,y) = \frac{|x-y|^4}{\varepsilon^4} + \overline{\phi}_\varepsilon(x,y)$$

where $\overline{\phi}_\varepsilon$ is a C^2 – function which will differ from case to case.

In the sequel we use the notations

$$p^x_\varepsilon = D_x\overline{\phi}_\varepsilon(x_\varepsilon, y_\varepsilon) \quad , \quad p^y_\varepsilon = -D_y\overline{\phi}_\varepsilon(x_\varepsilon, y_\varepsilon) \ ,$$

$$p_\varepsilon = \frac{4}{\varepsilon^4}(x_\varepsilon - y_\varepsilon)|x_\varepsilon - y_\varepsilon|^2 \quad \text{and} \quad Z_\varepsilon = \frac{4}{\varepsilon^4}(|x_\varepsilon - y_\varepsilon|^2 Id + 2(x_\varepsilon - y_\varepsilon)\otimes(x_\varepsilon - y_\varepsilon)) \ ,$$

and we choose the real numbers μ, ν and the orthonormal bases such that $1/C \leq \mu, \nu \leq C$ and

$$|\mu - \nu| \leq C\varepsilon \quad , \quad |e_k - f_k| \leq C|x_\varepsilon - y_\varepsilon| \quad \text{for all } k, \tag{24}$$

where, here and below, C denotes some constant which may vary from line to line but which depends only on the data of the problem (σ, b, λ, f) and on u and v.

By classical (and easy) computations and by taking **(H1)** into account, (22) yields

$$\sum_{k=1}^{p} D^2\phi(x_\varepsilon, y_\varepsilon)(p_k, q_k)\cdot(p_k, q_k) \leq C\frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon^4}(|\mu-\nu|^2 + |x_\varepsilon - y_\varepsilon|^2)$$
$$+\sum_{k=1}^{p} D^2\overline{\phi}_\varepsilon(x_\varepsilon, y_\varepsilon)(p_k, q_k)\cdot(p_k, q_k) \ .$$

Thus, (23) yields

$$\mu^2\tilde{G}_{\alpha_\varepsilon}(x_\varepsilon, \bar{u}(x_\varepsilon), p_\varepsilon + p^x_\varepsilon) - \nu^2\tilde{G}_{\alpha_\varepsilon}(y_\varepsilon, \bar{v}(y_\varepsilon), p_\varepsilon + p^y_\varepsilon)$$
$$\leq \mu^2 b(x_\varepsilon, \alpha_\varepsilon)\cdot(p_\varepsilon + p^x_\varepsilon) - \nu^2 b(y_\varepsilon, \alpha_\varepsilon)\cdot(p_\varepsilon + p^y_\varepsilon)$$
$$+C\frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon^4}(|\mu-\nu|^2 + |x_\varepsilon - y_\varepsilon|^2) + \sum_{k=1}^{p} D^2\overline{\phi}_\varepsilon(x_\varepsilon, y_\varepsilon)(p_k, q_k)\cdot(p_k, q_k).$$

For all the choices of test-functions $\overline{\phi}_\varepsilon$ we will make below, we will be able to prove that

$$\bar{u}(x_\varepsilon) - \bar{v}(y_\varepsilon) \to M > 0 \quad \text{and} \quad \frac{|x_\varepsilon - y_\varepsilon|}{\varepsilon} \to 0 \ . \tag{25}$$

Therefore $\bar{u}(x_\varepsilon) > \bar{v}(y_\varepsilon)$ for ε small enough and then by using these properties together with Lemma 5.3, **(H1)** and (24), we get the final estimate

$$\begin{aligned}
&\frac{1}{C}(\bar{u}(x_\varepsilon) - \bar{v}(y_\varepsilon) - C^2\varepsilon)\,[K_{\alpha_\varepsilon}(x_\varepsilon, y_\varepsilon, p_\varepsilon + p_\varepsilon^x, p_\varepsilon + p_\varepsilon^y)]^2 \\
&\le \mu^2 b(x_\varepsilon, \alpha_\varepsilon)\cdot p_\varepsilon^x - \nu^2 b(y_\varepsilon, \alpha_\varepsilon)\cdot p_\varepsilon^y + o(1)K_{\alpha_\varepsilon}(x_\varepsilon, y_\varepsilon, p_\varepsilon + p_\varepsilon^x, p_\varepsilon + p_\varepsilon^y) \\
&\quad + C|\sigma^T(x_\varepsilon, \alpha_\varepsilon)p_\varepsilon^x - \sigma^T(y_\varepsilon, \alpha_\varepsilon)p_\varepsilon^y| K_{\alpha_\varepsilon}(x_\varepsilon, y_\varepsilon, p_\varepsilon + p_\varepsilon^x, p_\varepsilon + p_\varepsilon^y) \\
&\qquad \frac{1}{2}\sum_{k=1}^{p} D^2\overline{\phi}_\varepsilon(x_\varepsilon, y_\varepsilon)(p_k, q_k)\cdot(p_k, q_k)\,,
\end{aligned} \tag{26}$$

where $o(1) \to 0$ as $\varepsilon \to 0$.

We now turn to the study of the two different cases we mentionned above.

5.1 The Case when the Maximum is achieved on $\Omega \cup \Gamma_{in}$

In this section, we consider the case when $x_0 \in \Omega \cup \Gamma_{in}$. For all $\varepsilon > 0$, we choose

$$\overline{\phi}_\varepsilon(x, y) = |x - x_0|^4.$$

We have the following classical result (cf. for example [11] or [1])

Lemma 5.4 *As $\varepsilon \to 0$, we have*

(i) $x_\varepsilon,\ y_\varepsilon \to x_0$,

(ii) $\bar{u}(x_\varepsilon) \to \bar{u}(x_0)$, $\bar{v}(y_\varepsilon) \to \bar{v}(x_0)$,

(iii) $\dfrac{|x_\varepsilon - y_\varepsilon|^4}{\varepsilon^4} \to 0$.

Since $\Omega \cup \Gamma_{in}$ is relatively open in $\overline{\Omega}$ because Ω is a smooth open subset of $I\!R^N$, $x_\varepsilon,\ y_\varepsilon \in \Omega \cup \Gamma_{in}$ for ε small enough. By Proposition 4.2, the viscosity inequalities (16) and (17) hold for $\bar{u}$ and $\bar{v}$ and Lemma 5.1 applies.

In this case, we choose $\mu = \nu = 1$ and the orthonormal basis $(e_1, \ldots, e_p)$ can be chosen arbitrarily but with $f_k = e_k$ for all k. If we set

$$p_0 = 4(x_\varepsilon - x_0)|x_\varepsilon - x_0|^2 \quad \text{and} \quad Z_0 = 4(|x_\varepsilon - x_0|^2 Id + 2(x_\varepsilon - x_0)\otimes(x_\varepsilon - x_0))$$

(where we dropped again the dependance in ε for the sake of simplicity of the notations), we have $p_\varepsilon^x = p_0$ and $p_\varepsilon^y = 0$.

By Lemma 5.4, (25) holds and thus we can turn to (26). We get

$$\begin{aligned}
\frac{1}{C}(M - o(1))\,[K_{\alpha_\varepsilon}(x_\varepsilon, y_\varepsilon, p_\varepsilon + p_0, p_\varepsilon)]^2 &\le b(x_\varepsilon, \alpha_\varepsilon)\cdot p_0 \\
&\quad + o(1)K_{\alpha_\varepsilon}(x_\varepsilon, y_\varepsilon, p_\varepsilon + p_0, p_\varepsilon) \\
&\quad + C|p_0| K_{\alpha_\varepsilon}(x_\varepsilon, y_\varepsilon, p_\varepsilon + p_0, p_\varepsilon) + \frac{1}{2}\sum_{k=1}^{p} Z_0 p_k \cdot p_k\,.
\end{aligned}$$

But by the estimates coming from Lemma 5.4, this inequality yields

$$\frac{1}{C}(M - o(1))\,[K_{\alpha_\varepsilon}(x_\varepsilon, y_\varepsilon, p_\varepsilon + p_0, p_\varepsilon)]^2 \leq o(1) K_{\alpha_\varepsilon}(x_\varepsilon, y_\varepsilon, p_\varepsilon + p_0, p_\varepsilon)\,.$$

And, by dividing by $[K_{\alpha_\varepsilon}(x_\varepsilon, y_\varepsilon, p_\varepsilon + p_0, p_\varepsilon)]^2$, this inequality leads to a contradiction for ε small enough. Therefore $\bar{u} - \bar{v}$ cannot have a maximum point on $\Omega \cup \Gamma_{in}$.

5.2 The Case when the Maximum is achieved on Γ

We have proved that, if $M > 0$, $\bar{u} - \bar{v}$ cannot have a maximum point on $\Omega \cup \Gamma_{in} \cup \Gamma_{out}$. This means that x_0 is on Γ. Because of the form of assumption **(H6)**, we have to examine two cases, namely the case when for x_0 the condition **(H6)-(i)** holds and the case when for x_0 the condition **(H6)-(ii)** holds.

5.2.1 The case when (H6)-(i) holds

By Corollary 4.1, there exists a sequence $(x_k)_k$ of points of Ω such that

$$x_k \to x_0 \text{ as } k \to +\infty$$
$$\bar{u}(x_k) = \omega^{-1}(\tilde{u}(x_k)) \to \omega^{-1}(\tilde{u}(x_0)) = \bar{u}(x_0) \text{ as } k \to +\infty\,,$$

with

$$|x_k - x_0|^2 = o(1) d(x_k) \text{ for all } k\,.$$

We set $\varepsilon_k^2 = d(x_k)$ and

$$\psi_k(x, y) = \bar{u}(x) - \bar{v}(y) - \phi_{\varepsilon_k}(x, y)$$

where

$$\overline{\phi}_{\varepsilon_k}(x, y) = \psi\left(\frac{d(x) - d(y)}{\varepsilon_k(\varepsilon_k + d(x) + d(y))}\right) + |x - x_0|^4$$

with

$$\psi(\tau) = [(\tau - 1)^-]^4.$$

Finally we denote by $(\overline{x}, \overline{y})$ a global maximum point of ψ_k on $\overline{\Omega} \times \overline{\Omega}$ (we drop again the dependence in k for the sake of simplicity of notations) and we set $M_k = \psi_k(\overline{x}, \overline{y})$.

We also use the notations

$$\Delta_k = \varepsilon_k + d(\overline{x}) + d(\overline{y}) \text{ and } \tau_k = \frac{d(\overline{x}) - d(\overline{y})}{\varepsilon_k \Delta_k}.$$

We then have the

Lemma 5.5 *As k goes to infinity, we have*

(i) $M_k \to M$,

(ii) $\bar{u}(\overline{x}) - \bar{v}(\overline{y}) \to M$,

(iii) $\overline{x},\ \overline{y} \to x_0,\ \dfrac{|\overline{x}-\overline{y}|^4}{\varepsilon_k^4} \to 0$ *and* $\liminf_{k\to+\infty} \tau_k \geq 1$.

Proof. Since $(\overline{x},\overline{y})$ is a global maximum point of ψ_k, we have, for all k,

$$\psi_k(x_k, x_0) \leq \psi_k(\overline{x},\overline{y}).$$

Therefore, by taking into account the properties of x_k and the choice of ε_k, we have

$$\psi_k(x_k, x_0) = M + o(1) \leq M_k \leq \bar{u}(\overline{x}) - \bar{v}(\overline{y}) - \phi_{\varepsilon_k}(\overline{x},\overline{y}) \leq \bar{u}(\overline{x}) - \bar{v}(\overline{y}). \quad (27)$$

From this inequality, we first deduce

$$\frac{|\overline{x}-\overline{y}|^4}{\varepsilon_k^4} + [(\tau_k - 1)^-]^4 + |\overline{x} - x_0|^4 \leq 2(\|\bar{u}\|_\infty + \|\bar{v}\|_\infty)\ ,$$

and thus $|\overline{x}-\overline{y}|$ goes to 0 as k goes to infinity.

Then by taking the lim sup in (27) and by using the upper semicontinuity of $\bar{u}$ and the lower semicontinuity of $\bar{v}$, we obtain

$$M \leq \limsup_{k\to+\infty} M_k \leq \limsup_{k\to+\infty} (\bar{u}(\overline{x}) - \bar{v}(\overline{y})) \leq M.$$

But it is also clear that

$$M \leq \liminf_{k\to+\infty} M_k \leq \liminf_{k\to+\infty} (\bar{u}(\overline{x}) - \bar{v}(\overline{y})) \leq \limsup_{k\to+\infty} (\bar{u}(\overline{x}) - \bar{v}(\overline{y})) \leq M,$$

and therefore *(i)* and *(ii)* are proven. But then

$$\phi_{\varepsilon_k}(\overline{x},\overline{y}) = \bar{u}(\overline{x}) - \bar{v}(\overline{y}) - M_k \to 0\ ;$$

this property clearly yields *(iii)* and the proof is complete. □

By Remark 4.1, we know that $\bar{u} \leq \overline{\varphi}$ on Γ and therefore $\bar{u}(x_0) \leq \overline{\varphi}(x_0)$. We claim that, if $\overline{y} \in \Gamma$ for k large enough, then $\bar{v}(\overline{y}) < \overline{\varphi}(\overline{y})$. Indeed, otherwise we would have

$$M = \lim_{k\to+\infty} (\bar{u}(\overline{x}) - \bar{v}(\overline{y})) \leq \limsup_{k\to+\infty} (\bar{u}(\overline{x}) - \overline{\varphi}(\overline{y})) \leq \bar{u}(x_0) - \overline{\varphi}(x_0) \leq 0$$

and this inequality cannot hold since $M > 0$. As a consequence the inequality (17) holds for k large enough.

But we do not know a priori if we have the inequality (16) since we may have $\overline{x} \in \Gamma$. It is easy to see that this "bad case" cannot occur for k large enough because of our choice of the test-function, in particular the $\psi(\tau_k)$ – term. Indeed, by Lemma 5.5-*(iii)*, $\liminf_{k\to+\infty} \tau_k \geq 1$, and this clearly implies that $d(\overline{x}) - d(\overline{y}) > 0$, and therefore $d(\overline{x}) > 0$, for k large enough .

Since $\overline{x} \in \Omega$ and either $\overline{y} \in \Omega$ or $\overline{v}(\overline{y}) < \overline{\varphi}(\overline{y})$ and since $\overline{u}(\overline{x}) > \overline{v}(\overline{y})$ for k large enough, we are led to (26). As we already mentioned it above, the remainder of the proof consists in choosing properly μ, ν and the orthonormal bases $(e_1...e_p)$ and $(f_1...f_p)$ and in estimating the different terms of this inequality to get a contradiction.

At this point it is also convenient to consider two cases : in view of Lemma 5.5, either there exists a subsequence $(\tau_{k'})_{k'}$ such that $\tau_{k'} \geq 1$ for all k', or for k large enough, $\tau_k < 1$. In the first case, $\psi'(\tau_{k'}) = \psi''(\tau_{k'}) = 0$ and we are bound to a situation similar to the case when $x_0 \in \Omega \cup \Gamma_{in}$. Therefore, in the sequel, we only consider the second case, namely the one when, for k large enough, $\tau_k < 1$ and $(\tau_k)_k$ converges to 1 as k goes to infinity.

To do so, we set

$$\Lambda_k = \frac{\psi'(\tau_k)}{\varepsilon_k \Delta_k} \quad \text{and} \quad \Theta_k = \frac{\psi''(\tau_k)}{\varepsilon_k^2 \Delta_k^2} \ .$$

Remark that Λ_k is a priori of order $o(1)\varepsilon_k^{-2}$ while Θ_k is of order $o(1)\varepsilon_k^{-4}$ since Δ_k may be of order ε_k and this behavior is unusual for quantities which appear respectively in the first and second order derivatives of the test-function. Indeed these terms are a priori "too large" to be handled by classical arguments. The main point of the proof (and one of the reasons why we need the "non-degeneracy" assumptions **(H5)**-**(H6)**) consists in taking care of this difficulty.

We set $\alpha_k = \alpha_{\varepsilon_k}$ where α_{ε_k} appears in (18) and (19). We have to investigate three cases which lead to different estimates. We recall that $\overline{x}$ and $\overline{y}$ are in a neighbourhood of x_0 for k large enough. For these k, we have to examine the cases when $\alpha_k \in \mathcal{A}_1(x_0)$, when $\alpha_k \in \mathcal{A}_2(x_0)$ and when $\alpha_k \in \mathcal{A}_3(x_0)$.

Case 1 : $\alpha_k \in \mathcal{A}_1(x_0)$.

We first consider the last term which appears in (26).

For $p,\ q \in I\!R^N$, we have

$$\begin{aligned} D^2\overline{\phi}_{\varepsilon_k}(\overline{x},\overline{y})(p,q)\cdot(p,q) =& \Lambda_k[(1-\varepsilon_k\tau_k)D^2d(\overline{x})p\cdot p - (1+\varepsilon_k\tau_k)D^2d(\overline{y})q\cdot q] \\ & -\frac{2\Lambda_k}{\Delta_k}[(1-\varepsilon_k\tau_k)Dd(\overline{x})\cdot p - (1+\varepsilon_k\tau_k)Dd(\overline{y})\cdot q][Dd(\overline{x})\cdot p + Dd(\overline{y})\cdot q] \\ & +\Theta_k[(1-\varepsilon_k\tau_k)Dd(\overline{x})\cdot p - (1+\varepsilon_k\tau_k)Dd(\overline{y})\cdot q]^2 + Z_0 p\cdot p \ . \qquad (28) \end{aligned}$$

In order to get rid of the two last terms of this expression, we first remark that since $\alpha_k \in \mathcal{A}_1(x_0)$ and because of **(H1)**, we have for k large enough

$$|\sigma^T(\overline{x}, \alpha_k) Dd(\overline{x})|, |\sigma^T(\overline{y}, \alpha_k) Dd(\overline{y})| \geq \eta/2 \,. \tag{29}$$

We set

$$e_1 = \frac{\sigma^T(\overline{x}, \alpha_k) Dd(\overline{x})}{|\sigma^T(\overline{x}, \alpha_k) Dd(\overline{x})|} \quad \text{and} \quad f_1 = \frac{\sigma^T(\overline{y}, \alpha_k) Dd(\overline{y})}{|\sigma^T(\overline{y}, \alpha_k) Dd(\overline{y})|}$$

and we choose, for $j \in \{2, \dots, p\}$, e_j and f_j such that the orthonormal bases $(e_1 \dots e_p)$ and $(f_1 \dots f_p)$ satisfy

$$|e_j - f_j| \leq C|\overline{x} - \overline{y}| \quad \text{for } j = 2, \cdots, p.$$

This is possible since $|e_1 - f_1| \leq C|\overline{x} - \overline{y}|$ and because of **(H1)** and (29).

Then we set

$$\mu = \frac{1}{(1 - \varepsilon_k \tau_k)|\sigma^T(\overline{x}, \alpha_k) Dd(\overline{x})|} \text{ and } \nu = \frac{1}{(1 + \varepsilon_k \tau_k)|\sigma^T(\overline{y}, \alpha_k) Dd(\overline{y})|} \,.$$

Such μ and ν are well-defined for k large enough since we are in the case when $0 < \tau_k < 1$ and since (29) holds. Moreover, by using **(H1)** and again (29), we have $|\mu - \nu| = O(\varepsilon_k + |\overline{x} - \overline{y}|)$ and by Lemma 5.5 we have $|\mu - \nu| \leq \overline{C}\varepsilon_k$ for some constant $\overline{C}$.

With these choices, we have, for all j

$$(1 - \varepsilon_k \tau_k) Dd(\overline{x}) \cdot p_j - (1 + \varepsilon_k \tau_k) Dd(\overline{y}) \cdot q_j = 0$$

where p_j, q_j are given by (21), since both terms are equal to 1 for $j = 1$ and to 0 for $j \geq 2$ and therefore

$$\sum_{j=0}^{p} D^2 \overline{\phi}_k(\overline{x}, \overline{y})(p_j, q_j) \cdot (p_j, q_j) = \mu^2 \mathrm{Tr}[a(\overline{x}, \alpha_k) Z_0]$$

$$+\Lambda_k \left[(1 - \varepsilon_k \tau_k)\mu^2 \mathrm{Tr}(a(\overline{x}, \alpha_k) D^2 d(\overline{x})) - (1 + \varepsilon_k \tau_k)\nu^2 \mathrm{Tr}(a(\overline{y}, \alpha_k) D^2 d(\overline{y}))\right] \,.$$

Now we set, for x close to Γ and $\alpha \in \mathcal{A}$,

$$Q_\alpha(x) = \frac{1}{2} \mathrm{Tr}\left[a(x, \alpha) D^2 d(x)\right] + b(x, \alpha) \cdot Dd(x) \,,$$

and we recall that here

$$p_{\varepsilon_k}^{\overline{x}} = p_0 + \Lambda_k (1 - \varepsilon_k \tau_k) Dd(\overline{x}) \quad \text{and} \quad p_{\varepsilon_k}^{\overline{y}} = \Lambda_k (1 + \varepsilon_k \tau_k) Dd(\overline{y}) \,.$$

By using this together with the above estimate and Lemma 5.5, (26) can be rewritten as

$$\frac{1}{C}(\bar{u}(\overline{x}) - \bar{v}(\overline{y}) - C^2 \varepsilon_k) \left[K_{\alpha_k}(\overline{x}, \overline{y}, p_{\varepsilon_k} + p_{\varepsilon_k}^{\overline{x}}, p_{\varepsilon_k} + p_{\varepsilon_k}^{\overline{y}})\right]^2 \leq$$

$$\mu^2\Lambda_k(1-\varepsilon_k\tau_k)Q_{\alpha_k}(\overline{x}) - \nu^2\Lambda_k(1+\varepsilon_k\tau_k)Q_{\alpha_k}(\overline{y})$$
$$+C(o(1)+|\overline{x}-\overline{y}|\Lambda_k)K_{\alpha_k}(\overline{x},\overline{y},p_{\varepsilon_k}+p^{\overline{x}}_{\varepsilon_k},p_{\varepsilon_k}+p^{\overline{y}}_{\varepsilon_k})+o(1)\ ,$$

where the $o(1)$ tends to 0 as $k\to\infty$.

Then, by taking **(H1)** and **(H4)** into account, the Q_{α_k}-function is Lipschitz continuous in a neighborhood of x_0 and this inequality reduces to

$$\frac{1}{C}(\bar{u}(\overline{x})-\bar{v}(\overline{y})-C^2\varepsilon_k)\left[K_{\alpha_k}(\overline{x},\overline{y},p_{\varepsilon_k}+p^{\overline{x}}_{\varepsilon_k},p_{\varepsilon_k}+p^{\overline{y}}_{\varepsilon_k})\right]^2\le$$

$$C(o(1)+\varepsilon_k\Lambda_k)K_{\alpha_k}(\overline{x},\overline{y},p_{\varepsilon_k}+p^{\overline{x}}_{\varepsilon_k},p_{\varepsilon_k}+p^{\overline{y}}_{\varepsilon_k})+o(1)\ ,$$

since $|\overline{x}-\overline{y}|\le C\varepsilon_k$ by Lemma 5.5.

The only remaining difficulty is to prove that Λ_k-term of the right-hand side is controlled by the term

$$\Xi_k := K_{\alpha_k}(\overline{x},\overline{y},p_{\varepsilon_k}+p^{\overline{x}}_{\varepsilon_k},p_{\varepsilon_k}+p^{\overline{y}}_{\varepsilon_k})$$

of the left-hand side. But since $\alpha_k\in\mathcal{A}_1(x_0)$, one has

$$|\sigma^T(\overline{x},\alpha_k)(p_{\varepsilon_k}+p^{\overline{x}}_{\varepsilon_k})|\ge\frac{\eta}{2}|\Lambda_k|(1-\varepsilon_k\tau_k)-C|p_{\varepsilon_k}|\ ,$$

and the same inequality holds for $|\sigma^T(\overline{y},\alpha_k)(p_{\varepsilon_k}+p^{\overline{y}}_{\varepsilon_k})|$ when replacing $1-\varepsilon_k\tau_k$ by $1+\varepsilon_k\tau_k$.

Therefore, either Λ_k satisfies along a subsequence

$$\frac{\eta}{2}|\Lambda_k(1-\varepsilon_k\tau_k)|\ge 2C|p_{\varepsilon_k}|\ ,$$

and we have

$$\Xi_k\ge 1+\frac{\eta}{4}\left(\Lambda_k(1-\varepsilon_k\tau_k)+\Lambda_k(1+\varepsilon_k\tau_k)\right)\ge\frac{1}{C}(1+\Lambda_k)\ ,$$

and by dividing by $[\Xi_k]^2$, one easily gets after tedious but straightforward computations

$$\frac{1}{C}(\bar{u}(\overline{x})-\bar{v}(\overline{y})-C^2\varepsilon_k)\le o(1)\ , \tag{30}$$

and this inequality is the desired contradiction if k is large enough.

Or, for k large enough,

$$\frac{\eta}{2}|\Lambda_k(1-\varepsilon_k\tau_k)|\le 2C|p_{\varepsilon_k}|\ ,$$

but then easy computations leads to $\varepsilon_k\Lambda_k\le o(1)$ and (30) also holds.

Remark 5.1 *The above argument is the justification of the change of variable we made : without this change of variable, we would be unable to control the dependence in Λ_k in the right-hand side and in particular in the Q_{α_k}-term. This change of variable deals with this difficulty essentially because of Lemma 5.2.*

Case 2 : $\alpha_k \in \mathcal{A}_2(x_0)$.

In this case, the choice of the two orthonormal bases is not relevant : we may choose any orthonormal basis $(e_1, \cdots, e_p)$ and then we take the same one for $(f_1, \cdots, f_p)$. On the contrary, the choice of μ and ν is the key point here and in the next section. We choose them in order to have

$$\mu^2\Lambda_k(1-\varepsilon_k\tau_k)Q_{\alpha_k}(\overline{x}) - \nu^2\Lambda_k(1+\varepsilon_k\tau_k)Q_{\alpha_k}(\overline{y}) = 0 \ ,$$

i.e. for example

$$\mu^2 = -\left[(1-\varepsilon_k\tau_k)Q_{\alpha_k}(\overline{x})\right]^{-1} > 0 \quad \text{and} \quad \nu^2 = -\left[(1+\varepsilon_k\tau_k)Q_{\alpha_k}(\overline{y})\right]^{-1} > 0 \ .$$

Because of assumption **(H6)**, $Q_{\alpha_k}(\overline{x}), Q_{\alpha_k}(\overline{y}) \leq -\eta/4 < 0$ and therefore μ, ν are well-defined and uniformly bounded. Moreover by **(H1)**,**(H4)** and Lemma 5.5, they also satisfy $|\mu-\nu| \leq \overline{C}\varepsilon_k$.

We have now to estimate the right-hand side of (26) and in particular the terms $D^2\overline{\phi}_{\varepsilon_k}(\overline{x},\overline{y})$ and $\sigma^T(\overline{x},\alpha_k)p^{\overline{x}}_{\varepsilon_k} - \sigma^T(\overline{y},\alpha_k)p^{\overline{y}}_{\varepsilon_k}$.

In order to do it, we first give the

Lemma 5.6 : *Assume that* **(H1)**, **(H4)** *and* **(H6)** *hold. If $\alpha \in \mathcal{A}_2(x_0) \cup \mathcal{A}_3(x_0)$, we have for any x, y in a neighborhood of x_0,*

$$|\sigma^T(x,\alpha)n(x)| \leq Cd(x) \ , \tag{31}$$

and

$$|\sigma^T(x,\alpha)Dd(x) - \sigma^T(y,\alpha)Dd(y)| \leq C(d(x)+d(y))|x-y| + C|d(x)-d(y)| \ . \tag{32}$$

The proof of Lemma 5.6 is easy and relies on the fact that $\sigma^T(x,\alpha)n(x) \equiv 0$ on $\mathcal{V}(x_0) \cap \partial\Omega$ if $\alpha \in \mathcal{A}_2(x_0) \cup \mathcal{A}_3(x_0)$. Indeed (31) is just a consequence of this property and of the Lipschitz continuity of $\sigma^T(\cdot,\alpha)Dd(\cdot)$ which is uniform with respect to α (we already used this in section 4). The inequality (32) is a consequence of the regularity assumption in **(H6)** and of the fact that, on $\mathcal{V}(x_0) \cap \partial\Omega$, the tangential derivative of the function $x \mapsto \sigma^T(x,\alpha)Dd(x)$ is 0. We leave the details of the proof to the reader.

We now examine the term

$$L_k := |(1-\varepsilon_k\tau_k)\sigma^T(\overline{x},\alpha_k)Dd(\overline{x}) - (1+\varepsilon_k\tau_k)\sigma^T(\overline{y},\alpha_k)Dd(\overline{y})| \ .$$

From (31) ,(32) and since $0 < \tau_k < 1$, we get

$$L_k \leq C\varepsilon_k(d(\overline{x}) + d(\overline{y})) + C(d(\overline{x}) + d(\overline{y}))|\overline{x} - \overline{y}| + C|d(\overline{x}) - d(\overline{y})| .$$

Then since $d(\overline{x})+d(\overline{y}) \leq \Delta_k$ and $|\overline{x}-\overline{y}| \leq \varepsilon_k$ for k large enough by Lemma 5.5, we finally have

$$L_k \leq C\varepsilon_k\Delta_k + C|d(\overline{x}) - d(\overline{y})| .$$

By using this inequality and Lemma 5.5, we first deduce

$$\begin{aligned} |\sigma^T(\overline{x},\alpha_k)p_{\varepsilon_k}^{\overline{x}} - \sigma^T(\overline{y},\alpha_k)p_{\varepsilon_k}^{\overline{y}}| &\leq \Lambda_k L_k + C|p_0| \\ &\leq C\psi'(\tau_k)\left(1 + \frac{|\overline{x}-\overline{y}|}{\varepsilon_k}\right) + o(1) \\ &\leq o(1). \end{aligned}$$

For the remaining terms of the $D^2\overline{\phi}_{\varepsilon_k}(\overline{x},\overline{y})$-term, we obtain in the same way

$$\begin{aligned} &\frac{\Lambda_k}{\Delta_k}\sum_{j=1}^{p}\Big[(1-\varepsilon_k\tau_k)\mu\sigma^T(\overline{x},\alpha_k)Dd(\overline{x})\cdot e_j - (1+\varepsilon_k\tau_k)\nu\sigma^T(\overline{y},\alpha_k)Dd(\overline{y})\cdot e_j\Big] \\ &\qquad\qquad \Big[\mu\sigma^T(\overline{x},\alpha_k)Dd(\overline{x})\cdot e_j + \nu\sigma^T(\overline{y},\alpha_k)Dd(\overline{y})\cdot e_j\Big] \\ &\qquad\qquad \leq C\frac{\Lambda_k}{\Delta_k}\left[|\mu-\nu|(d(\overline{x})+d(\overline{y})) + L_k\right](d(\overline{x})+d(\overline{y})) \\ &\qquad\qquad \leq C\Lambda_k(\varepsilon_k\Delta_k + L_k) \leq C\psi'(\tau_k) = o(1) , \end{aligned}$$

and

$$\begin{aligned} &\Theta_k\sum_{j=1}^{p}\Big[(1-\varepsilon_k\tau_k)\mu\sigma^T(\overline{x},\alpha_k)Dd(\overline{x})\cdot e_j - (1+\varepsilon_k\tau_k)\nu\sigma^T(\overline{y},\alpha_k)Dd(\overline{y})\cdot e_j\Big]^2 \\ &\qquad = \Big|(1-\varepsilon_k\tau_k)\mu\sigma^T(\overline{x},\alpha_k)Dd(\overline{x}) - (1+\varepsilon_k\tau_k)\nu\sigma^T(\overline{y},\alpha_k)Dd(\overline{y})\Big|^2 \\ &\qquad\qquad \leq \frac{C\psi''(\tau_k)}{\varepsilon_k^2\Delta_k^2}(\varepsilon_k\Delta_k + L_k)^2 \leq C\psi''(\tau_k) = o(1) . \end{aligned}$$

After plugging these estimates in (26), we are led to (30).

Remark 5.2 *In the study of case 1, the change of variable was playing the central role. In this second case and in the third one, this central role is played by the regularity assumption of* **(H6)** *and more particularly by the estimates of Lemma 5.6. The above computations may be seen as a justification of both assumption* **(H6)** *and of the curious form of the test-function in which we introduce a dependence in* $\dfrac{d(x)-d(y)}{\varepsilon^2+\varepsilon d(x)+\varepsilon d(y)}$ *instead, for example, in* $\dfrac{d(x)-d(y)}{\varepsilon^2}$.

Case 3 : $\alpha_k \in \mathcal{A}_3(x_0)$

The proof is similar to the one for the case when $\alpha_k \in \mathcal{A}_2(x_0)$ with an important difference : in case 2, we knew a priori that the quantities $Q_{\alpha_k}(\overline{x}), Q_{\alpha_k}(\overline{y})$ were bounded away from 0 and this was a key point in the choice of μ and ν since this allowed us to deduce several key properties for these parameters. Here we have to prove that this property holds, and this the aim of the following lemma which is the corner stone of Case 3.

Lemma 5.7 *Under the assumptions of Theorem 2.1, there exists a constant $\tilde{C} > 0$ such that, if $|\Lambda_k| \geq \tilde{C}(1 + |p_{\varepsilon_k}|)$ then*

$$Q_{\alpha_k}(\overline{x}), Q_{\alpha_k}(\overline{y}) \geq \eta/8 ,$$

where η is the constant which appears in **(H6)**.

We assume for the moment that this lemma is true and we conclude the proof of Theorem 2.1.

If $|\Lambda_k| \geq \tilde{C}(1 + |p_{\varepsilon_k}|)$, we follow exactly the arguments of Case 2 but with the choice

$$\mu^2 = [(1 - \varepsilon_k \tau_k) Q_{\alpha_k}(\overline{x})]^{-1} \quad \text{and} \quad \nu^2 = [(1 + \varepsilon_k \tau_k) Q_{\alpha_k}(\overline{y})]^{-1} .$$

Notice that the signs of the Q_{α_k} –terms have changed. Because of Lemma 5.7, μ, ν are well-defined and they satisfy the right estimates and this allows us to conclude by using the same computations as in the previous case.

If $|\Lambda_k| \leq \tilde{C}(1+|p_{\varepsilon_k}|)$, then the estimation of the difference of the Q_{α_k}–terms is not a problem anymore. We can choose $\mu = \nu = 1$ and follow the arguments of Case 2 to estimate the $D^2\overline{\phi}_{\varepsilon_k}(\overline{x}, \overline{y})$–term. We are led in the same way to (30) and the proof is complete. □

Now we turn to the

Proof of Lemma 5.7. By the definition of α_k (cf. (18)) we have for any $\alpha \in \mathcal{A}$

$$\begin{aligned} &-\frac{1}{2}\mathrm{Tr}[a(y_k, \alpha_k) Y_{\varepsilon_k}] + G_{\alpha_k}(\overline{y}, \overline{v}(\overline{y}), -D_y \phi_\varepsilon(\overline{x}, \overline{y})) \\ &\geq -\frac{1}{2}\mathrm{Tr}[a(\overline{y}, \alpha) Y_{\varepsilon_k}] + G_\alpha(\overline{y}, \overline{v}(\overline{y}), -D_y \phi_\varepsilon(\overline{x}, \overline{y})) . \end{aligned}$$

We test this inequality with the choice $\alpha = \alpha'_k = \alpha'_k(\overline{y})$ where $\alpha'_k(\overline{y})$ is the control associated to α_k by **(H6)**; by taking into account the properties of $\alpha'_k(\overline{y})$, the obtained inequality reduces to

$$-b(\overline{y}, \alpha_k) \cdot (p_{\varepsilon_k} + p^{\overline{y}}_{\varepsilon_k}) - f(\overline{y}, \alpha_k) \geq -b(\overline{y}, \alpha'_k) \cdot (p_{\varepsilon_k} + p^{\overline{y}}_{\varepsilon_k}) - f(\overline{y}, \alpha'_k) .$$

By substracting

$$(1+\varepsilon_k\tau_k)\Lambda_k \mathrm{Tr}[a(\overline{y},\alpha_k)D^2d(\overline{y}))] = (1+\varepsilon_k\tau_k)\Lambda_k \mathrm{Tr}[a(\overline{y},\alpha'_k)D^2d(\overline{y})]$$

on both side, and by taking into account the fact that $\Lambda_k \le 0$ since $\psi' \le 0$, one gets with few easy computations

$$|\Lambda_k|\left(Q_{\alpha_k}(\overline{y}) - Q_{\alpha'_k}(\overline{y})\right) \ge -C(1+|p_{\varepsilon_k}|)\ .$$

Finally, by recalling that $Q_{\alpha'_k}(\overline{y}) \ge \eta/4$ for k large enough, it is clear that if $|\Lambda_k| \ge \tilde{C}(1+|p_{\varepsilon_k}|)$ for $\tilde{C} > 0$ large enough then $Q_{\alpha_k}(\overline{y}) \ge \eta/8$ and the proof is complete. □

Remark 5.3 *The key point in this last case is Lemma 5.7 which relies on* **(H6)-(i)**-*d); without this result, we would be unable to control the dependence in Λ_k in the right-hand side of* (26) *and in particular in the Q_{α_k}-term. This may be seen as a (technical) justification of this assumption.*

5.2.2 The case when (H6)-(ii) holds

By Corollary 4.1, there exists a sequence $(x_k)_k$ of points of Ω such that

$$x_k \to x_0 \text{ as } k \to +\infty$$
$$\overline{u}(x_k) = \omega^{-1}(\tilde{u}(x_k)) \to \omega^{-1}(\tilde{u}(x_0)) = \overline{u}(x_0) \text{ as } k \to +\infty\ ,$$

with

$$|x_k - x_0| = O(1)d(x_k) \text{ for all } k.$$

In this case, the proof follows essentially the ideas of the case when **(H6)-(i)** holds but is far easier since the test-function has a classical form.

More specifically, for all k, we set $\varepsilon_k = d(x_k)$ and for $x,\ y \in \overline{\Omega}$, we choose the test-function

$$\psi_k(x,y) = \overline{u}(x) - \overline{v}(y) - |x-x_0|^4 - \psi\left(c_0 - \frac{|x-y|^2}{\varepsilon_k^2}\right) + \psi\left(\frac{d(x)-d(y)}{\varepsilon_k} - 1\right)$$

where

$$\psi(\tau) = (\tau^-)^4$$

and

$$c_0 \ge \limsup_{k\to+\infty} \frac{|x_k - x_0|^2}{d(x_k)^2}.$$

The rest of the proof follows the arguments of the beginning of section 5.2.1 (in particular the analogue of Lemma 5.5 holds) but with the easy computations of section 5.1. We leave the details to the reader.

Remark 5.4 : *It is worth mentioning that, in this section, we are in the framework of the Strong Comparison Result of M. Katsoulakis[18] because of the existence of the sequence $(x_k)_k$ which implies that the "cone condition" is satisfied.*

6 Concluding Remarks and Extensions

6.1 Remark on the strategy of the proof of Theorem 2.1

The classical difficulty one faces when one wants to prove a Strong Comparison Result like Theorem 2.1 is the lack of boundary condition for the subsolution $\tilde{u}$ on Γ : indeed the boundary condition $\tilde{u} \leq \varphi$ on $\partial\Omega$ is completely useless; in fact, we are in a state constraint case on Γ.

In the literature there are two main ways to turn around this difficulty. Our strategy of proof follows rather closely the classical proof of [24] (See also [11] or [13]) which consists in building the test-function in order that the maximum point associated to the subsolution (cf. $\overline{x}$) cannot be on the boundary by "pushing it inside Ω". Such strategy is also the one of [6] and [18].

The only difference – but this is part of the main difficulty here – is that the usual term which appears in such type of proof, namely either

$$\left(\frac{d(x)-d(y)}{\varepsilon}-1\right)^{-} \text{ or } \left(\frac{(x-y)\cdot n(x_0)}{\varepsilon}-1\right)^{-}$$

is replaced here, in the case when **(H6)-(i)** holds, by the term

$$\left(\frac{d(x)-d(y)}{\varepsilon(\varepsilon+d(x)+d(y))}-1\right)^{-}$$

i.e. by a term with a different scaling in ε.

The second way to attack such problems is to prove that the subsolution satisfies a boundary condition. Such strategy is used by H. Ishii and S. Koike[17] for first-order state constraints problems and in [2] for second-order semilinear equations.

It is worth pointing out that, on one hand, this approach leads here to a Ventsell type boundary condition and this type of boundary condition is more difficult to handle than the Neumann type boundary conditions of [2] and [17] and, on another hand, the mixing of conditions **(H6)-(i)** and **(H6)-(ii)** on Γ also creates additional difficulties. Because of that, we were unable to perform the proof of Theorem 2.1 with this approach despite we would be able to do it if one of the assumptions **(H6)-(i)** or **(H6)-(ii)** were satisfied on the whole boundary Γ.

In order to be a little more precise we sketch very briefly this other proof in the case when**(H6)-(i)** holds on Γ. First of all, the Ventsell type boundary condition is a consequence of Lemma 4.1 : indeed, it implies that, for $x \in \partial\Omega$, if we set

$$\tilde{\mathcal{A}}(x)=\bigcup_{\nu>0}\ \bigcap_{y\in B(x,\nu)\cap\partial\Omega}\mathcal{A}_{in}(x),$$

and if **(H1)**-**(H4)** hold, $\tilde{u}$ is a viscosity subsolution of

$$\sup_{\alpha\in\tilde{A}(x)}\left\{-\frac{1}{2}\mathrm{Tr}[a(x,\alpha)D^2\tilde{u}]-b(x,\alpha)\cdot D\tilde{u}+\lambda\tilde{u}-f(x,\alpha)\right\}=0 \text{ on } \partial\Omega. \quad (33)$$

Then, when all the maximum points of $\tilde{u}-\tilde{v}$ are on Γ, instead of using the test-function of section 5 to get $\overline{x}\in\Omega$ with the help of Theorem 4.1, we use the following test-function:

$$\tilde{u}(x)-\tilde{v}(y)-\frac{|x-y|^4}{\varepsilon^4}-\delta\psi\left(\frac{d(x)-d(y)}{\varepsilon^2+\varepsilon d(x)+\varepsilon d(y)}\right),$$

where δ is a small parameter and where the function $\psi\in C^2(I\!\!R)$ satisfies

$$\begin{array}{ll}\psi(\tau)=-\tau & \text{for } \tau\le 0\,,\\ \psi'(\tau)<0,\ \psi''(\tau)>0 & \text{for } 0<\tau<1\,,\\ \psi(\tau)=-1 & \text{for } \tau\ge 1.\end{array}$$

The main remarks on this test-function are the following ones : if (x_n,y_n) are maximum points of this test-function associated to a subsequence ε_n, then

1. We do not know a priori that $\dfrac{|x_n-y_n|^4}{\varepsilon_n^4}\to 0$ but only that
$$\frac{|x_n-y_n|^4}{\varepsilon_n^4}\le C\delta\,,$$

2. If we have $d(x_n)\le d(y_n)$ for any n then
$$\frac{|x_n-y_n|^4}{\varepsilon_n^4}\to 0\,, \quad (34)$$
this last property coming from the remark that, in this case, the ψ-term is positive.

The key point in the proof is that there is no subsequence satisfying the item 2. above with $x_n\in\partial\Omega$ for all n; otherwise (34) together with the properties of ψ and the Ventsell type boundary condition (33) yields to a contradiction. Therefore $x_n\in\Omega$ for n large enough and the computations follow along the lines of the proof of Theorem 2.1 above. The only difference is that these computations lead to $M\le\rho(\delta)$ where $\rho(\delta)\to 0$ as $\delta\to 0$. This provides the desired contradiction if δ is chosen small enough.

This other method for proving Theorem 2.1 may be summarized as "small term-large derivatives" since the ψ-term is anytime less than δ (despite, in general, it is not converging to 0 as $\varepsilon\to 0$) but has large derivatives, and in particular large first-order normal derivative. This is a key point in order to prove that the boundary condition (33) cannot hold and we think that this type of arguments may be useful in other situations.

[15] H. Ishii: *Perron's method for Hamilton-Jacobi Equations.* Duke Math. J. **55** (1987), 369-384.

[16] H. Ishii: *A boundary value problem of the Dirichlet type for Hamilton-Jacobi Equations.* Ann. Scuola. Norm. Pisa Cl. Sci. (4) 16, pp 105-135.

[17] H. Ishii and S. Koike: *A new formulation of state constraints problems for first-order PDEs.* SIAM J. on Control and Optimization, **34**, No 2, pp 554-571, (1996).

[18] M. Katsoulakis: *Viscosity solutions of second-order fully nonlinear elliptic equations with state constraints.* Indiana Univ. Math. J., **43**, No 2, pp 493-519.

[19] Krylov N.V: CONTROLLED DIFFUSION PROCESSES. Springer-Verlag, 1980.

[20] Lions P.L.: *Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations, Part I: The dynamic programming principle and applications,* Comm. P.D.E. **8** (1983).

[21] Lions P.L.: *Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations, Part II: Viscosity solutions and uniqueness,* Comm.P.D.E. **8** (1983).

[22] Lions P.L.: *Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations, Part III,* in Nonlinear PDE and Appl., Séminaire du Collège de France, vol V, Pitman (1985).

[23] P.L Lions: *Neumann type boundary conditions for Hamilton-Jacobi Equations.* Duke Math. J. 52 (1985) p.793-820.

[24] M.H Soner: *Optimal control problems with state-space constraints.* SIAM J. on Control and Optimisation **24** (1986), Part I: 552-562, Part II: 1110-1122.

[25] E.D Sontag: MATHEMATICAL CONTROL THEORY. Springer Verlag, New-York, 1990.

Received: September 1997

Revised: June 1998