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Real Options Games in Complete and Incomplete Markets with Several Decision Makers*

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Abstract. We consider optimal investment policies for irreversible capital investment projects under uncertainty in a monopoly situation and in a Stackelberg leader-follower game. We consider two types of payoffs: lump-sum and cash flows. The decisions are the times to enter into the market. The problems belong to the class of optimal stopping times, for which the right approach is that of variational inequalities (V.I.s). In the case of complete markets, payoffs are expected values with respect to the risk-neutral probability. In the case of incomplete markets, the risk-neutral probability is not defined. We consider an investor maximizing his/her utility function, and we consider the investment in the project as an additional decision, besides portfolio investment and consumption decisions. This decision remains a stopping time, conversely to the portfolio investment and consumption decisions (continuous controls). The game problem raises new difficulties. The leader's V.I. has a nondifferentiable obstacle. The weak formulation of the V.I. handles this difficulty. In some cases, the solution of the V.I. may be continuously differentiable although the obstacle is not. An additional difficulty occurs for lump-sum payoffs in the case of incomplete markets. We cannot compare gains and losses at different times. We propose an alternative approach, using equivalence (indifference) considerations. In the case of payoffs characterized by cash flows, this difficulty does not exist, but an intermediary problem arises which has a nice interpretation as a differential game. The solutions thus obtained for the Stackelberg game are not intuitive. Therefore, competition has important consequences on investment decisions.

Key words. variational inequality, utility-based pricing, optimal stopping, real options

AMS subject classifications. 60G40, 62L15

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1. Introduction.

Advances in the development and implementation of options models to value and manage capital investment projects have progressed at a prodigious rate following Stewart Myers's 1977 assertion that a firm's growth opportunities may be viewed as "real options." Early milestones in real options development included Brennan and Schwartz [5], McDonald and Siegel [34], and Paddock, Siegel, and Smith [44]. Brennan and Schwartz [5] applied option pricing to valuation and operation of a copper mine. The owner of an active mine holds a put option to suspend operation should copper prices fall below a certain threshold, while the owner of a suspended mine has a call option to reopen should prices rise above a

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different, higher threshold. Fixed suspension and resumption costs drive a wedge between the thresholds, creating a "hysteresis" effect. Free boundary conditions determine threshold prices.

Classic real options models did not address the possibility that the investment strategy exercised by one firm could have an impact on the optimal investment policies of a competitor. If a firm fears pre-emption, the delay option value is less significant, and project value approaches the traditional net present value. Notable contributions in this area include papers by Grenadier [15], [16], [17]. Grenadier [15] applied option game theory to the real estate market to explain development cascades and overbuilding. Extensions to the analysis (see Grenadier [16], [17]) consider equilibrium strategies with option exercise games.[1] While pre-emption fears erode delay option value, Dixit and Pindyck [10, p. 314] note that the situation is different if the roles of leader and follower are exogenously determined as in the Stackelberg game. In this case, the leader has the ability to wait, recognizing the option value. They discuss the possible option values and investment strategies in narrative terms without rigorous proof. We address the problem rigorously and obtain the optimal strategies for the leader and the follower in both complete and incomplete market cases. The latter case is particularly important for real options for the reasons discussed below.

Capital investment projects are typically nontraded. That is, claims to these assets are not traded in well-developed secondary markets. The result is that real options markets are incomplete. Nevertheless, many articles simply assume markets are complete; i.e., all risk can be hedged completely. In some models the static NPV proxies for the nontraded underlying asset, a strategy that Copeland and Antikarov [7] call the "marketed asset disclaimer." Brennan and Schwartz [3], [4] and Dixit and Pindyck [10] use a capital asset pricing model, and then they estimate a risk-adjusted project return rate and solve the option value as a threshold optimization. Financial mathematicians address the issue by employing strategies such as (1) minimizing tracking error (see, for example, Duffie and Richardson [12]); (2) selecting a martingale measure for pricing using minimal martingale or minimal entropy methods (see Follmer and Sonderman [13], Schweizer [45], Delbaen and Schachermayer [9], and Frittelli [14]); and (3) employing utility functions to estimate an indifference price (see Henderson [20], [21], Henderson and Hobson [23], Musiela and Zariphopoulou [39], [41], Oberman and Zariphopoulou [43], and Hodges and Neuberger [26]). For incomplete markets, we adopt in this paper the utility-based approach, considering utility maximization with joint decisions of stopping times, portfolio investment strategies, and/or consumption rules.

We consider a duopoly model of Stackelberg leader-follower type in an irreversible capital investment under uncertainty. The decision can be made over an infinite horizon with the constraint that the follower be forbidden to invest until the leader has already done so. Leader and follower roles are predetermined. The Stackelberg game has been applied in many areas, including electricity pricing (see Ho, Luh, and Muralidharan [25]), NOx allowances in the electric power industry (see Chen et al. [6]), supply chain management and marketing channels (see He et al. [19]), server-proxy-user systems in computer science (see Han and Xia [18]), and others.

Our framework assumes that uncertainties embedded in the irreversible capital investment

[1] See, for example, Dixit and Pindyck [10, Chap. 9], Grenadier [15], [16], [17], Smets [46], Lambrecht and Perraudin [32], Huisman [28], Smit and Trigeorgis [48], Smit and Ankum [47], etc.

project arise from the uncertainties of investment payoffs. We consider two types of investment payoffs: lump-sum investment payoffs and investment payoffs as a series of cash flows.

The primary advantage to modeling uncertainty as a lump-sum investment payoff lies in its simplicity. As noted by Dixit and Pindyck [10], modeling uncertainty in terms of project value in many cases leads to the same optimal policies as modeling uncertainty in cash flows.[2] On the other hand, focusing on cash flows permits comparison between results of our model and more conventional decision making paradigms. We model the uncertainty of lump-sum investment payoffs by assuming that the investment project value is governed by an externally determined geometric Brownian motion. To capture the uncertainty from cash flows, we consider project cash flows to evolve according to an arithmetic Brownian motion. The possibility of negative cash flows deals explicitly with the fact that the project's operations may generate losses. The option we are interested in is the time to invest (when to enter into the market) for both the follower and the leader. These problems are optimal stopping time problems. From Bensoussan and Lions [1], it is known that the right tool to solve these problems is variational inequalities (V.I.s), introduced by Stampacchia in 1964 and Lions and Stampacchia [33] in the context of physical and mechanical applications. The definition of payoffs is easy in the complete market case. There exists a unique risk-neutral probability. So the payoffs are mathematical expectations with respect to this probability. For the follower, as in the case of a single decision maker, the V.I. has an obstacle function (the outcome received at the stopping time to be decided) which is C^1 (continuously differentiable). There is then a *strong* formulation for the V.I., and the solution can be obtained by a threshold strategy. This is not the case for the leader's problem. The obstacle function is C^0, not C^1. We need to use a *weak* formulation for the V.I. We prove the existence and uniqueness of the solutions in the weak form. We then study the regularity of the solution and find that it is indeed C^1 and piecewise C^2. So the solution of the V.I. is more regular than the obstacle. However, the optimal strategy is not given by a single threshold. This is related to the lack of smoothness of the obstacle. The optimal strategy is obtained by two intervals. For the cash flow model characterized by an arithmetic Brownian motion process, we encounter the additional difficulty of an unbounded obstacle function.

In the incomplete market case, we use a utility-based approach for the valuation. The manager's (investor's) risk preferences are modeled through an exponential utility function. We consider the manager's (investor's) utility maximization with joint decisions of stopping times, portfolio investment strategies, and/or consumption rules. For the case of lump-sum investment payoffs, we do not allow intermediate consumption in order to simplify exposition and to avoid the penalization arising from the case of negative consumption. The manager (investor) maximizes his/her expected utility of wealth taking portfolio investment strategies and stopping times into consideration. The follower's and the single decision maker's value functions are solutions of V.I.s, which can be written in a strong formulation, and the optimal strategies are obtained through a threshold approach. The situation is much more complex for the leader's problem since we are unable to compare gains and losses at different times as in the complete market case. We circumvent the problem by employing equivalence (indifference)

[2]While this statement holds in complete markets, the relationship between the two models is in general an open question.

considerations as a surrogate of the present value used as a comparison benchmark in the complete market situation. In addition, we encounter the same mathematical difficulty as in the complete market case, namely, that the leader's obstacle function is nonsmooth. We thus formulate and solve V.I.s in the weak sense. However, we have neither the situation of a more regular solution nor of a two-interval strategy. We comment on this problem, the full solution of which is open.

For the case of investment payoffs as a series of cash flows, the manager (investor) faces a utility maximization problem in which he/she chooses a portfolio investment strategy, a consumption rate, and a stopping time for market entry (capital investment). The problem can be split into two parts. We first assume that the investment project has already been undertaken. This leads to a control problem of portfolio selection and consumption rate, augmented by the stochastic income stream from the capital project now in place. We then consider the problem of choosing the optimal stopping time, portfolio selection, and consumption rules before the optimal stopping. We get a V.I. with a nonlinear differential operator. The obstacle is defined by the first part. The solution to this V.I. has an interesting interpretation, that of the stochastic differential game, different from the Stackelberg game. For the follower and the single decision maker, we prove the existence and uniqueness of the solution of the V.I., which is C^1 and piecewise C^2. For the leader's problem, we first characterize his/her payoff at the optimal stopping time. The leader must take into account the effect of the follower's anticipated entry on his/her portfolio investment and consumption decisions. Again, we encounter a nonsmooth obstacle function, and we must interpret the V.I. in the weak sense. We study the regularity of the solution, employing penalization, and here the challenge is to find suitable a priori estimates. We prove the existence and uniqueness of the solutions in their strong forms. The leader's optimal stopping rule is given by a two-interval strategy.

The remainder of this paper is organized as follows. In section 2, we briefly describe general features of problems and models studied. In sections 3 and 4, we present models and generate results for the case of lump-sum investment payoffs, respectively, for the complete market and the incomplete market cases. In sections 5 and 6, we present models and results for the case when the investment payoff is characterized as a series of cash flows. We present concluding remarks in section 7.

2. General features of problems and models.

2.1. Asset valuation in complete and incomplete markets. The issue of valuation is key in finance. We have to assign a "fair" value to random outcomes, often called "contingent claims." The expected value is not a fair value because risk is not taken into consideration. The essential property of complete markets is that there exists one and only one fair value. It is the expected value of the random outcome with respect to a new probability measure, called the risk-neutral probability. This risk-neutral probability is uniquely defined. So in the case of complete markets we will measure the value of losses and gains by taking simply the expected value with respect to the risk-neutral probability. For our later purpose, the framework is as follows. Consider a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ with a Wiener process $W(t)$. We assume that the market is characterized by a single asset, $S(t)$, whose evolution is governed by

$$dS(t) = rS(t)dt + \sigma S(t)(\lambda dt + dW(t)), \tag{2.1}$$

where r is a risk-free interest rate, σ is the volatility, and λ is the Sharpe ratio; they are all constants. The market is complete, and there exists a unique risk-neutral probability. It is obtained as follows: we define

$$\widehat{W}(t) = W(t) + \lambda t$$

and the process $Z(t)$ by

$$dZ(t) = -\lambda Z(t) dW(t), \quad Z(0) = 1.$$

The risk-neutral probability measure $\widehat{\mathbb{Q}}$ is then

$$\frac{d\widehat{\mathbb{Q}}}{d\mathbb{Q}}|_{\mathcal{F}_t} = Z(t)$$

with $\mathcal{F}_t = \sigma(W(s), s \leq t)$. Under $\widehat{\mathbb{Q}}$, $\widehat{W}(t)$ is a standard Wiener process. The fair value of a contingent claim is simply the expected value of the random outcome with respect to the risk-neutral probability.

Let us now consider the situation of incomplete markets. The risk-neutral probability is not unique. The classical valuation approach is inappropriate, and alternatives must be considered. One proposed solution is to introduce a rational utility-maximizing investor who evaluates unhedgeable risk based on the investor's risk preferences. Utility-based valuation in stochastic dynamic market environments derives from the famous work of Merton [35], who developed the original dynamic, stochastic model of expected utility maximization. We adopt the utility-based valuation in our incomplete market model. We assume an exponential utility function to describe the investor's risk preferences. As noted in Musiela and Zariphopoulou [41], there are advantages working with an exponential utility function. For example, the asset value is given in terms of a nonlinear pricing rule that has certainty equivalent characteristics. However, this nonlinear pricing functional is not the static analogue of the certainty equivalent corresponding to the exponential preferences. It is distorted, with a magnitude that depends only on the correlation between the traded and the nontraded assets. Another advantage is that the measure under which the indifference price is computed is a measure under which the traded asset price is a martingale and which has the minimal entropy with respect to the historical one.

2.2. Structures of the market. In addition to the consideration of market completeness, we consider two different types of market structures: one market player or two market players. Market players' decisions are the time to enter into the market. In the case of one market player, the entrepreneur may operate without consideration of any potential competitor's decisions (i.e., a monopolist in the product market). By paying an investment cost K at the market entry time τ, the firm expects to receive the whole operation income.

In the case of two market players, we consider a Stackelberg leader-follower game. The roles of leader and follower have been predetermined, for example, by regulations, by competitive advantages of market powers, etc. The follower is forbidden to invest until the leader has already done so. The leader enters the market at time θ and the follower at time $\tau \geq \theta$. Each player pays an investment cost K upon entry. The leader receives the whole operation income prior to the follower's entry. Upon the follower's entry, depending on the competitiveness or the regulation, the leader may share the market with the follower evenly or the leader may maintain larger portions of the operation income, leaving smaller portions to the follower.

2.3. Lump-sum and cash flow models. We consider two types of operation incomes from the capital investment undertaken. One is the operation income characterized by a lump-sum, which means that the investment project yields a one time payoff at the time of investment for a given cost K. We model this situation in terms of the investment project value, governed by a geometric Brownian motion process. In this model, the leader has to incorporate into his/her objective function his/her surrender project value to the follower upon the follower's optimal entry, so one has to compare values of events taking place at different times. In the case of complete markets, the payoff is the expected value with respect to the unique risk-neutral probability, so gains/losses at different times are comparable (by expected present values with respect to the risk-neutral probability). In the case of incomplete markets, we encounter a major hurdle for the leader's value function. Due to the absence of the unique martingale measure, we choose to work with the investor's utility, and there is no identity relationship for values at different times. We circumvent the problem by employing equivalence (indifference) considerations. However, we then encounter another mathematical difficulty. The "obstacle" function is nonsmooth and, in this case, we are unable to get the result that the solution of the V.I. is C^1 (i.e., smoother than the obstacle) as in the case of complete markets. By exploring with an explicit calculation for a two-interval solution, we arrive at only a partial answer.

In the other model, the operation income is characterized by cash flows, which means that by paying a given cost K at the investment time the entrepreneur receives a series of cash flows thereafter. The cash flow evolves as an arithmetic Brownian motion process. Unlike the lump-sum model, the problem of comparing gains and losses at different times does not arise in this model, in either the complete or the incomplete case, because it is a cash flow valuation.

We present a detailed problem formulation, models, and valuation procedures for the case of lump-sum investment payoffs in sections 3 and 4, and for the case of the investment payoff characterized as a series of cash flows in sections 5 and 6.

3. Lump-sum payoffs and complete market assumption.

3.1. Single player. We assume that the investment opportunity is driven by a stochastic value process:

$$dV(s) = rV(s)ds + \eta V(s)(\xi ds + dW(s)) \tag{3.1}$$

$$= (r + \eta(\xi - \lambda))V(s)ds + \eta V(s)d\widehat{W}(s); \quad s \geq t; \quad V(t) = v, \tag{3.2}$$

where η and ξ are constants. We denote the solution of (3.2) by $V_{v,t}(s)$, $s \geq t$. Beginning at t, the firm undertakes the capital investment project at time $\tau > t$, at which point the firm invests at a cost K in return for the project value $(1-a)V_{v,t}(\tau)$, where $a \in [0, 1)$.[3] The firm's problem is to find an optimal stopping time to maximize the expected discounted project value:

$$F(v,t) = \sup_{\tau \geq t} \widehat{E}\big[e^{-r(\tau-t)}\big((1-a)V_{v,t}(\tau) - K\big)\mathbb{1}_{\tau<\infty}\big], \tag{3.3}$$

[3]This specification is for preparing the situation of two players. a represents market share by other active participants. For the single decision maker case, a is zero. In the Stackelberg games that follow, a represents the leader's market share after the follower's entry into the market. In the context of the leader-follower model, we assume $a \geq \frac{1}{2}$. Otherwise, the leader will surrender a majority interest in the project, providing a strong disincentive to enter at all.

where $\widehat{E}[\cdot]$ is the expectation with respect to the risk-neutral probability measure $\widehat{\mathbb{Q}}$.

This is a stationary problem; hence the value function (3.3) becomes

$$F(v) = \sup_{\tau \geq 0} \widehat{E}\big[e^{-r\tau}\big((1-a)V_v(\tau) - K\big)\mathbb{1}_{\tau<\infty}\big], \qquad \text{where} \quad V_v(t) = V_{v,0}(t). \tag{3.4}$$

Clearly, since $\tau = \infty$ is possible (i.e., the firm never invests), $F(v) \geq 0$. Assuming that the function $F(v)$ is sufficiently smooth, $F(v)$ solves the following V.I. as a consequence of dynamic programming:

$$\begin{cases} F(v) \geq ((1-a)v - K), \\ F'(v)v\big(r + \eta(\xi - \lambda)\big) + \frac{1}{2}F''(v)v^2\eta^2 - rF(v) \leq 0, \\ \big[F(v) - ((1-a)v - K)\big]\big[F'(v)v\big(r + \eta(\xi - \lambda)\big) + \frac{1}{2}F''(v)v^2\eta^2 - rF(v)\big] = 0, \\ F(0) = 0;\ F(v) \geq 0;\ F(v) \text{ has linear growth at infinity.} \end{cases} \tag{3.5}$$

From Chapter 5 in Dixit and Pindyck [10], we have the following theorem.

Theorem 3.1. *Assume $\xi < \lambda$:*

$$F(v) = \begin{cases} \dfrac{K}{\beta - 1}\Big(\dfrac{v}{\hat{v}}\Big)^{\beta}, & v \leq \hat{v}, \\ (1-a)v - K, & v \geq \hat{v}, \end{cases} \tag{3.6}$$

where $\beta = \frac{1}{2} - \frac{r+\eta(\xi-\lambda)}{\eta^2} + \sqrt{\big(\frac{1}{2} - \frac{r+\eta(\xi-\lambda)}{\eta^2}\big)^2 + \frac{2r}{\eta^2}} > 1$, *and* $\hat{v} = \frac{\beta K}{(1-a)(\beta-1)}$.

A proof is needed to check that $F(v)$ in (3.6) is the solution of the V.I. since a threshold is not necessarily a solution of the V.I. We omit the proof.

The optimal stopping rule which achieves the supremum in (3.4) is $\hat{\tau}(v) = \inf\{t | V_v(t) \geq \hat{v}\}$. Using $V_v(\hat{\tau}(v)) = \hat{v}$ if $\hat{\tau}(v) < \infty$ and $v < \hat{v}$, we have the probabilistic representation

$$F(v) = \big((1-a)\hat{v} - K\big)\widehat{E}[\exp\{-r\hat{\tau}(v)\}] \quad \text{if } v < \hat{v}. \tag{3.7}$$

3.2. Two players: A Stackelberg game. We now consider a two firm (i.e., two player) Stackelberg game. The roles of leader and follower have been predetermined. Each firm may invest in the capital project, but the actions, actual and/or anticipated, of one player affect the other's decision. Both players face an optimal stopping problem similar to the monopolist's, albeit with the complications inherent in the leader-follower framework. The follower is forbidden to invest until the leader has already done so. The leader enters the market at time θ and the follower at time $\tau \geq \theta$, both stopping times of the filtration $\mathcal{F}_t$. Again, each player pays an investment cost K upon entry. The leader knows that the follower, acting rationally, will enter at time $\hat{\tau}_\theta$, at which time he/she must surrender a portion of the project to the follower.

3.2.1. Statement of the leader's problem. We first notice that

$$\hat{\tau}_\theta = \theta + \hat{\tau}(V_v(\theta)) \tag{3.8}$$

with a slight abuse of notation. We consider the function $\hat{\tau}(v)$, in which v is deterministic. It is an entry time for the process $V_v(t)$. So it depends on v and on all the values of the process

$V_v(t)$. It can be considered as a functional of v and $\widehat{W}(.)$. If we replace the initial time 0 by a random time θ which is a stopping time with respect to $\mathcal{F}_t$, then we mean the same functional on arguments $V_v(\theta)$ and $\widehat{W}(.+\theta)-\widehat{W}(\theta)$. It is a stopping time with respect to $\mathcal{F}_{t+\theta}$. It is important to notice that $\widehat{W}(.+\theta)-\widehat{W}(\theta)$ is independent of $V_v(\theta)$. Therefore we have the following formula for any test function $\Psi(x,s)$:

$$\widehat{E}[\Psi(V_v(\hat{\tau}_\theta),\hat{\tau}_\theta)|\mathcal{F}_\theta]=\Psi(V_v(\theta),\theta)\mathbb{1}_{V_v(\theta)\geq\hat{v}}+\mathbb{1}_{V_v(\theta)<\hat{v}}\widehat{E}[\Psi(\hat{v},t+\hat{\tau}(v))]|_{v=V_v(\theta),t=\theta}. \tag{3.9}$$

From (3.9) and (3.7), with $\Psi(x,s)=\exp\{-r(s-t)\}$, $s\geq t$, we can write explicitly the Laplace transform of the conditional density as

$$\widehat{E}[\exp\{-r(\hat{\tau}_\theta-\theta)\}|\mathcal{F}_\theta]=\mathbb{1}_{V_v(\theta)\geq\hat{v}}+\mathbb{1}_{V_v(\theta)<\hat{v}}\frac{F(V_v(\theta))}{(1-a)\hat{v}-K}. \tag{3.10}$$

When the leader enters at time $\theta<\infty$, by paying cost K, he/she receives $V_v(\theta)$ minus the expected discounted value that he/she will surrender to the follower at the time of the follower's entry. Assuming the follower is rational, the leader knows that the follower will enter at time $\hat{\tau}_\theta$. At time $\hat{\tau}_\theta$, the leader surrenders $(1-a)V_v(\hat{\tau}_\theta)$. So at time θ, if $\theta<\infty$, the leader receives

$$\begin{aligned}&V_v(\theta)-K-(1-a)\widehat{E}\big[e^{-r(\hat{\tau}_\theta-\theta)}V_v(\hat{\tau}_\theta)\mathbb{1}_{\hat{\tau}_\theta<\infty}|\mathcal{F}_\theta\big]\\&=aV_v(\theta)\mathbb{1}_{V_v(\theta)\geq\hat{v}}+(V_v(\theta)-\beta F(V_v(\theta))\mathbb{1}_{V_v(\theta)<\hat{v}}-K,\end{aligned}$$

where we use the fact that $\frac{(1-a)\hat{v}}{(1-a)\hat{v}-K}=\beta$ and $F(v)$ is defined in (3.6). To facilitate the presentation, we define

$$\Psi(v)=av\mathbb{1}_{v\geq\hat{v}}+(v-\beta F(v))\mathbb{1}_{v<\hat{v}}-K. \tag{3.11}$$

The leader's problem can now be expressed as

$$L(v)=\sup_{\theta\geq 0}\widehat{E}\big[e^{-r\theta}\Psi(V_v(\theta))\mathbb{1}_{\theta<\infty}\big]. \tag{3.12}$$

The leader's value function $L(v)$ must satisfy

$$L(v)\geq 0;\quad L(v)\geq\Psi(v). \tag{3.13}$$

The obstacle $\Psi(v)$ presents a problem because it is only continuous, not $C^1(0,\infty)$. There is, however, only one point of nondifferentiability, $\hat{v}$, which we observe from the following:

$$\Psi'(v)=\begin{cases}a, & v>\hat{v},\\ 1-\beta(1-a)\left(\frac{v}{\hat{v}}\right)^{\beta-1}, & v<\hat{v},\end{cases} \tag{3.14}$$

with $\Psi'(\hat{v}-0)=1-\beta(1-a)<\Psi'(\hat{v}+0)=a$.

We next find bounds associated with the obstacle function, $\Psi(v)$, and the value function, $L(v)$. We first proceed to the bound of $\Psi(v)$. By (3.11), $\Psi(v)$ can be expressed as

$$\Psi(v)=av-K+\widetilde{\Psi}(v)\mathbb{1}_{v<\hat{v}},$$

where $\tilde{\Psi}(v) = (1-a)v - \beta F(v)$ satisfies

$$\tilde{\Psi}'(v) = (1-a) - \beta F'(v) = \begin{cases} 1 - a\left[1 - \beta\left(\dfrac{v}{\hat{v}}\right)^{\beta-1}\right], & v \le \hat{v}, \\ (1-a)(1-\beta), & v \ge \hat{v}. \end{cases}$$

Hence, $\tilde{\Psi}'(v) \ge 0$ if $0 < v < v^* = \hat{v}\left(\frac{1}{\beta}\right)^{\frac{1}{\beta-1}} < \hat{v}$ and $\tilde{\Psi}'(v) < 0$ if $v > v^*$.

Moreover, $\tilde{\Psi}(0) = \tilde{\Psi}(\hat{v}) = 0$. Therefore,

$$av - K \le \Psi(v) \le av. \tag{3.15}$$

We now proceed to bounds for $L(v)$. By (3.12) and (3.15), we have

$$L(v) \le a \sup_{\theta \ge 0} \widehat{E}\left[e^{-r\theta} V_v(\theta) \mathbb{1}_{\theta<\infty}\right]. \tag{3.16}$$

From (3.2), we have

$$\begin{aligned} dV_v(t)e^{-rt} &= \eta(\xi - \lambda)V_v(t)e^{-rt}dt + \eta V_v(t)e^{-rt}d\widehat{W}(t) \\ &\le \eta V_v(t)e^{-rt}d\widehat{W}(t), \end{aligned}$$

which shows the supermartingale property for the discounted value process. By the optional sampling theorem, this implies

$$\widehat{E}\left[V_v(\theta \wedge T)e^{-r(\theta\wedge T)}\right] \le v, \text{ from which it follows that } \widehat{E}\left[V_v(\theta)e^{-r\theta}\mathbb{1}_{\theta<\infty}\right] \le v.$$

Therefore, (3.16) implies $0 \le L(v) \le av$. From (3.12) and (3.15), we have $L(v) \ge \Psi(v) \ge av - K$; hence we arrive at the bound of $L(v)$:

$$(av - K)^+ \le L(v) \le av. \tag{3.17}$$

We now proceed to formulate the leader's problem in a different way. By (3.12), we need to evaluate $\widehat{E}\left[e^{-r\theta}V_v(\theta)\mathbb{1}_{\theta<\infty}\right]$. Using

$$\widehat{E}\left[V_v(\theta \wedge T)e^{-r(\theta\wedge T)}\right] = v + \eta(\xi - \lambda)\widehat{E}\left[\int_0^{\theta\wedge T} e^{-rs}V_v(s)ds\right],$$

letting $T \to \infty$, and using

$$\widehat{E}\left[V_v(T)e^{-rT}\right] = ve^{\eta(\xi-\lambda)T} \to 0 \quad \text{as } T \to \infty,$$

we obtain

$$\widehat{E}\left[V_v(\theta)e^{-r\theta}\mathbb{1}_{\theta<\infty}\right] = v + \eta(\xi - \lambda)\widehat{E}\left[\int_0^{\theta} e^{-rs}V_v(s)ds\right]. \tag{3.18}$$

Rewriting (3.12) by making use of (3.18) with some algebraic manipulation and setting $U(v) = L(v) - av + K$, we have[4]

$$U(V) = \sup_{\theta \geq 0} \widehat{E}\Big\{ e^{-r\theta} \Big(\Psi\big(V_v(\theta)\big) - aV_v(\theta) + K \Big) \mathbb{1}_{\theta<\infty} + \int_0^\theta e^{-rs} \Big(a\eta(\xi-\lambda)V_v(s) + rK \Big) ds \Big\}$$

$$= \sup_{\theta \geq 0} \widehat{E}\Big\{ e^{-r\theta} \chi\big(V_v(\theta)\big) \mathbb{1}_{\theta<\infty} + \int_0^\theta e^{-rs} f\big(V_v(s)\big) ds \Big\}. \tag{3.19}$$

This formulation leads to an optimal stopping time problem with an obstacle $\chi(v) = \Psi(v) - av + K$ and a running profit $f(v) = a\eta(\xi-\lambda)v + rK$.

We have

$$0 \leq \chi(v) \leq K \quad \text{and} \quad 0 \leq U(v) \leq K. \tag{3.20}$$

3.2.2. The leader's problem V.I. We now formulate the analytic problem corresponding to (3.19) by dynamic programming. We cannot proceed as in (3.5) because we cannot guarantee that $U(v)$ is $C^1(0,\infty)$ since $\chi(v)$ is not $C^1(0,\infty)$. We must establish the variational formulation of (3.19) in the weak sense.

We introduce the following useful functional spaces with a weight function $w(x) = \frac{1}{(1+x^2)^\varrho}$:

$$L^2_\varrho(\Omega) = \Big\{ \Phi(x) | \int_\Omega w(x)\Phi^2(x)dx < \infty \Big\}, \tag{3.21}$$

$$H^1_\varrho(\Omega) = \Big\{ \Phi \in L^2_\varrho(\Omega) | x\Phi'(x) \in L^2_\varrho(\Omega) \text{ if } \Omega \subseteq \mathbb{R}^+;\ \Phi'(x) \in L^2_\varrho(\Omega) \text{ if } \Omega \subseteq \mathbb{R} \Big\}, \tag{3.22}$$

where $L^2_\varrho(\Omega)$ is a Hilbert space with the weighted scalar product

$$(\Phi, \tilde{\Phi})_\varrho = \int_\Omega \Phi(x)\tilde{\Phi}(x)w(x)dx \tag{3.23}$$

and $H^1_\varrho(\Omega)$ is a Sobolev space with the scalar product

$$((\Phi, \tilde{\Phi}))_\varrho = (\Phi, \tilde{\Phi})_\varrho + \int_\Omega x^2 w(x)\Phi'(x)\tilde{\Phi}'(x)dx. \tag{3.24}$$

We work on $L^2_\varrho(0,\infty)$ and $H^1_\varrho(0,\infty)$.[5] We define on $H^1_\varrho(0,\infty)$ the bilinear form

$$b(\Phi, \tilde{\Phi}) = \int_0^\infty v\Phi'(v) \Big[-\big(r + \eta(\xi-\lambda)\big) + \eta^2 \frac{1 - v^2(\varrho-1)}{1+v^2} \Big] \tilde{\Phi}(v)w(v)dv + \frac{1}{2}\int_0^\infty \Phi'(v)\tilde{\Phi}'(v)v^2\eta^2 w(v)dv + \int_0^\infty r\Phi(v)\tilde{\Phi}(v)w(v)dv. \tag{3.25}$$

[4]It is straightforward that the uniqueness and existence of the solution, $U(v)$, guarantees the uniqueness and existence of the solution, $L(v) = U(v) + av - K$.

[5]$\varrho > \frac{3}{2}$.

Clearly, $b(\Phi,\Phi)$[6] is continuous on $H^1_\varrho(0,\infty)$. For $\alpha \geq \alpha_0$, we have the coercivity property

$$b(\Phi,\Phi) + \alpha(\Phi,\Phi)_\varrho \geq c_0\|\Phi\|^2_\varrho,\ c_0 > 0. \tag{3.26}$$

We consider the convex subset of $H^1_\varrho(0,\infty)$:

$$\mathcal{K} = \left\{\Phi \in H^1_\varrho(0,\infty) | \Phi(v) \geq \chi(v)\ \forall\ v,\ \Phi(0) = K\right\}. \tag{3.27}$$

Note that $\mathcal{K}$ is not empty because the constant K belongs to $\mathcal{K}$.

The V.I. corresponding to (3.19) is

$$b(U,\tilde{U} - U) \geq (f,\tilde{U} - U)_\varrho \qquad \forall\ \tilde{U} \in \mathcal{K},\ U \in \mathcal{K}. \tag{3.28}$$

To study the existence and uniqueness of the solution to (3.28), we introduce the modified problem

$$b(U,\tilde{U} - U) + \alpha(U,\tilde{U} - U)_\varrho \geq (f,\tilde{U} - U)_\varrho + \alpha(G,\tilde{U} - U)_\varrho \qquad \forall\ \tilde{U} \in \mathcal{K},\ U \in \mathcal{K}. \tag{3.29}$$

In (3.29), G is given with

$$0 \leq G(v) \leq K. \tag{3.30}$$

The coercivity property (3.26) guarantees the problem (3.29) has a unique solution that we denote by $\Gamma_\alpha(G)$ to emphasize the dependence on G.

We first check that the unique solution, $\Gamma_\alpha(G)$, to (3.29) is in the interval $[0,K]$, the bound of $U(v)$.

Lemma 3.2. *We have the property*

$$0 \leq \Gamma_\alpha(G) \leq K. \tag{3.31}$$

Proof. See Appendix A. ■

We next check the contraction mapping property.

Lemma 3.3. *We have*

$$\|\Gamma_\alpha(G_1) - \Gamma_\alpha(G_2)\|_{L^\infty} \leq \frac{\alpha}{\alpha + r}\|G_1 - G_2\|_{L^\infty}. \tag{3.32}$$

Proof. See Appendix B. ■

Theorem 3.4. *Assume $\xi - \lambda < 0$. The V.I.* (3.28) *has a unique solution U such that*

$$0 \leq U \leq K. \tag{3.33}$$

Proof. It is a consequence of the fact that a solution to (3.28) satisfying (3.33) is a fixed point of Γ_α, defined as a map from the interval (3.33) into itself. ■

Theorem 3.5. *The solution U of the V.I.* (3.28) *coincides with* (3.19).

Proof. See Appendix C. ■

[6]Replace $\tilde{\Phi}$ with Φ in (3.25).

3.2.3. Smoothness of the solution. It turns out that the solution $U(v)$ will be smoother than the obstacle. This is due to the fact that the obstacle is not continuously differentiable in a single point. So the V.I. will have a strong formulation, which is given by the following relations:

$$\begin{cases} -\frac{1}{2}U''(v)\eta^2 v^2 - (r + \eta(\xi - \lambda))vU'(v) + rU(v) \geq a\eta(\xi - \lambda)v + rK, \\ U(v) \geq \chi(v), \\ [U(v) - \chi(v)][-\frac{1}{2}U''(v)\eta^2 v^2 - (r + \eta(\xi - \lambda))vU'(v) \\ \qquad\qquad + rU(v) - a\eta(\xi - \lambda)v - rK] = 0, \\ U(0) = K \end{cases} \tag{3.34}$$

with $\chi(v) = \big((1-a)v - \beta F(v)\big)\mathbb{1}_{v<\hat{v}}$ satisfying

$$\begin{cases} -\frac{1}{2}\chi''(v)\eta^2 v^2 - (r + \eta(\xi - \lambda))v\chi'(v) + r\chi(v) = -(1-a)\eta(\xi - \lambda)v & \text{if } v < \hat{v}, \\ \chi(0) = \chi(\hat{v}) = (1-a)\hat{v} - \frac{\beta K}{\beta - 1} = 0, \end{cases} \tag{3.35}$$

where $\hat{v} = \frac{\beta K}{(1-a)(\beta-1)}$.

We look for a solution to (3.34) in an interval $0 \leq U(v) \leq \overline{U}(v)$, where $\overline{U}(v)$ will be a ceiling function which is C^1 and vanishes for v sufficiently large. We consider

$$\overline{U}(v) = -a\left[v - \bar{v} - \frac{\bar{v}}{\beta}\left(\left(\frac{v}{\bar{v}}\right)^{\beta} - 1\right)\right], \quad v < \bar{v}. \tag{3.36}$$

This function and its derivative vanish at $\bar{v}$, i.e., $\overline{U}(\bar{v}) = \overline{U}'(\bar{v}) = 0$. It is extended by 0 for $v > \bar{v}$.

We first need to check that $\overline{U}(v)$ defined above is an appropriate ceiling function. Using (3.36), we see that

$$\begin{aligned} -\frac{1}{2}v^2\eta^2\overline{U}'' - v\overline{U}'\big(r + \eta(\xi - \lambda)\big) + r\overline{U} &= av\eta(\xi - \lambda) + ar\bar{v}\left(1 - \frac{1}{\beta}\right) \\ &\geq av\eta(\xi - \lambda) + rK \end{aligned} \tag{3.37}$$

for $\bar{v}$ sufficiently large.

We also need to check the property that $\overline{U}(v) > \chi(v)$. Since $\chi(v) = 0$ for $v > \hat{v}$, it is sufficient to prove $\overline{U}(v) > \chi(v)$ for $v < \hat{v}$. From (3.37), we have

$$av\eta(\xi - \lambda) + ar\bar{v}\left(1 - \frac{1}{\beta}\right) > -(1-a)v\eta(\xi - \lambda),$$

provided $ar\bar{v}(1 - \frac{1}{\beta}) > -v\eta(\xi - \lambda)$, which occurs if $ar\bar{v}(1 - \frac{1}{\beta}) > -\hat{v}\eta(\xi - \lambda)$, i.e., if $\bar{v}$ is sufficiently large. This inequality together with (3.35) implies $\overline{U}(v) \geq \chi(v)$. Combining with the first inequality implies

$$U(v) \leq \overline{U}(v). \tag{3.38}$$

Thus, $\overline{U}(v)$ is an appropriate ceiling function.

We now apply penalty approximation techniques to (3.34). We approximate (3.34) by a smoother problem, a penalized problem. That is, we look for U_ϵ such that

$$\begin{cases} -\frac{1}{2}U_\epsilon''(v)\eta^2v^2 - (r+\eta(\xi-\lambda))vU_\epsilon'(v) + rU_\epsilon(v) = a\eta(\xi-\lambda)v + rK + \frac{1}{\epsilon}(\chi - U_\epsilon(v))^+, \\ U_\epsilon(0) = K; \quad U_\epsilon(\bar{v}) = 0. \end{cases} \tag{3.39}$$

We also have

$$U_\epsilon(v) \le \overline{U}(v). \tag{3.40}$$

We can check the property of (3.40). Indeed, setting $\widetilde{U}_\epsilon = U_\epsilon - \overline{U}$, we have

$$\begin{cases} -\frac{1}{2}\widetilde{U}_\epsilon''(v)\eta^2v^2 - (r+\eta(\xi-\lambda))v\widetilde{U}_\epsilon'(v) + r\widetilde{U}_\epsilon(v) \le \frac{1}{\epsilon}\big(\widetilde{U}_\epsilon(v)\big)^-, \\ \widetilde{U}_\epsilon(0) = 0; \quad \widetilde{U}_\epsilon(\bar{v}) = 0. \end{cases}$$

This implies $\widetilde{U}_\epsilon \le 0$ by the maximum principle, and (3.40) is satisfied.

The key point is to check the estimate

$$\frac{(\chi - U_\epsilon)^+}{\epsilon} \le c_0. \tag{3.41}$$

Using the property of $\chi(v)$, we have

$$\begin{aligned} -\frac{1}{2}U_\epsilon''(v)\eta^2v^2 - (r+\eta(\xi-\lambda))vU_\epsilon'(v) + \left(r + \frac{1}{\epsilon}\right)U_\epsilon(v) &\ge a\eta(\xi-\lambda)v + rK \\ &\ge a\eta(\xi-\lambda)\bar{v} + rK \quad \text{for } v < \bar{v}; \end{aligned}$$

hence

$$U_\epsilon(v) \ge \epsilon\big(a\eta(\xi-\lambda)\bar{v} + rK\big). \tag{3.42}$$

Consider next $(0, \hat{v})$ and set

$$U_\epsilon^*(v) = U_\epsilon(v) - \chi(v).$$

$U_\epsilon^*(v)$ satisfies the following equations:

$$\begin{cases} -\frac{1}{2}U_\epsilon^{*\prime\prime}(v)\eta^2v^2 - (r+\eta(\xi-\lambda))vU_\epsilon^{*\prime}(v) + rU_\epsilon'(v) = a\eta(\xi-\lambda)v + rk + \frac{1}{\epsilon}U_\epsilon^*(v)^-, \\ U_\epsilon^*(0) = K; \quad U_\epsilon^*(\hat{v}) \ge \epsilon\big(a\eta(\xi-\lambda)\bar{v} + rK\big). \end{cases} \tag{3.43}$$

This implies

$$U_\epsilon^*(v) \ge \min\big[\epsilon\big(\eta(\xi-\lambda)\hat{v} + rK\big), \epsilon\big(a\eta(\xi-\lambda)\bar{v} + rK\big)\big].$$

Therefore, we obtain (3.41) with

$$c_0 = -rK - \eta(\xi-\lambda)\max(\hat{v}, a\bar{v}) > 0. \tag{3.44}$$

Referring back to (3.37), we have

$$\int_0^{\bar{v}} v^2 |U_\epsilon''| dv \le C. \tag{3.45}$$

Therefore, we can extract from $U_\epsilon(v)$ a subsequence, still denoted $U_\epsilon(v)$, such that

$$\begin{cases} U_\epsilon(v) \to U(v) \text{ in } L^\infty(0,\bar{v}) \text{ weak star}, \\ vU_\epsilon'(v) \to vU'(v) \text{ in } L^2(0,\bar{v}) \text{ weakly}, \\ v^2U_\epsilon''(v) \to v^2U''(v) \text{ in } L^\infty(0,\bar{v}) \text{ weak star}. \end{cases} \tag{3.46}$$

In particular, we can take

$$U(\bar{v}) = 0; \qquad U(0) = K.$$

Passing to the limit in (3.39), we obtain

$$\begin{cases} -\frac{1}{2}U''(v)\eta^2v^2 - (r + \eta(\xi - \lambda))vU'(v) + rU(v) = a\eta(\xi - \lambda)v + rK + \varphi(v), \\ U(v) \ge \chi(v), \\ (U(v) - \chi(v))\varphi(v) = 0, \\ U(0) = K, \\ U(\bar{v}) = 0, \\ U(v) \in C^1(0,\bar{v}),\ U''(v) \in L^\infty(0,\bar{v}). \end{cases} \tag{3.47}$$

The function $U(v)$ is the solution of (3.34). Let us check that

$$U'(\bar{v}) = 0. \tag{3.48}$$

Indeed, $U'(\bar{v}) \le 0$ since $U(v) > 0$ for $v < \bar{v}$. But, from (3.40), we have $U(v) \le \overline{U}(v)$, yielding

$$\frac{U(v) - U(\bar{v})}{v - \bar{v}} \ge \frac{\overline{U}(v) - \overline{U}(\bar{v})}{v - \bar{v}} \quad \text{for } v < \bar{v},$$

and thus $U'(\bar{v}) \ge 0$. We see that (3.48) holds. It follows that the function $U(v)$, the solution of (3.47), extended by 0 is the solution of (3.34).

The next interesting result is that we can get an explicit solution for $U(v)$. It is not a threshold solution, which is due to the lack of smoothness of the obstacle $\chi(v)$ in $\hat{v}$. The solution will be referred as a *two-interval solution*, as explained in the following.

Theorem 3.6. *There exist three points $0 < v_1 < v_2 < \hat{v} < v_3$ such that*

$$\begin{cases} -\frac{1}{2}U''(v)\eta^2v^2 - (r + \eta(\xi - \lambda))vU'(v) + rU(v) = a\eta(\xi - \lambda)v + rK, \\ \qquad\qquad 0 < v < v_1 \textit{ and } v_2 < v < v_3, \\ U(v) = \chi(v) \quad \textit{for } v_1 \le v \le v_2, \\ U(v) = 0 \quad \textit{for } v > v_3 \end{cases} \tag{3.49}$$

with matching conditions $U'(v_1) = \chi'(v_1)$, $U'(v_2) = \chi'(v_2)$, and $U'(v_3) = 0$.

Proof. See Appendix D. ■

So the solution coincides with the obstacle on two intervals. Knowing this fact, it is possible to make a direct calculation taking into account

$$\begin{cases} U(v) = -av + K + Av^{\beta} \quad \text{if } v < v_1, \\ U(v) = -av + K + Cv^{\beta} + Dv^{\beta_1} \quad \text{if } v_2 < v < v_3, \end{cases} \tag{3.50}$$

where β and β_1 are, respectively, the positive and negative roots of the characteristic equation $\frac{1}{2}\eta^2\beta^2 + \left(r + \eta(\xi - \lambda) - \frac{1}{2}\eta^2\right) - r = 0$, and A, C, and D are constants to be defined. We leave it to the reader to perform these calculations.

We can define the leader's optimal stopping rule as

$$\hat{\theta}(v) = \begin{cases} \inf\{t|V_v(t) \geq v_1\} \quad \text{if } 0 \leq v < v_1, \\ 0 \quad \text{if } v_1 \leq v \leq v_2, \\ \inf\{t|V_v(t) \leq v_2 \text{ or } V_v(t) \geq v_3\} \quad \text{if } v_2 < v < v_3, \\ 0 \quad \text{if } v \geq v_3. \end{cases} \tag{3.51}$$

3.2.4. Optimal rules in the case of complete markets. We summarize the leader's and the follower's optimal investment rules as follows:

1. If $v < v_1$, the leader waits to enter until $v \geq v_1$, and the follower enters when $v \geq \hat{v}$.
2. If $v_1 < v < v_2$, the leader enters immediately, and the follower enters when $v \geq \hat{v}$.
3. If $v_2 < v < v_3$, the leader waits to enter until v moves outside the interval bounded by v_2 and v_3. The follower's optimal policy is more complicated, and there are two possibilities. One possibility is $v_2 < v < \hat{v} < v_3$. If the leader enters because $v \leq v_2$, the follower will make his/her move when $v \geq \hat{v}$. If the leader enters because $v \geq v_3$, the follower enters immediately since $v \geq v_3 > \hat{v}$. The other possibility is $v_2 < \hat{v} < v < v_3$. Although $v \geq \hat{v}$, indicating that the follower should enter, the follower is blocked from doing so until the leader enters. If the leader enters because $v \leq v_2$, the follower will make his/her move when $v \geq \hat{v}$ again. If the leader enters because $v \geq v_3$, the follower enters immediately since $v \geq v_3 > \hat{v}$.
4. If $v \geq v_3$, the leader invests immediately, and the follower then makes his/her move.

Note that both the leader's and the follower's investment rules deviate from the monopolist's case. Although the follower's optimal investment trigger, $\hat{v}$, is the same as the monopolist's, he/she is forbidden to enter the market until the leader has already done so. The leader's optimal investment trigger, characterized by two intervals, which are $[v_1, v_2]$ and $[v_3, \infty)$, differs from the monopolist's, $\hat{v}$, due to the consideration of strategic interactions from the follower's action. By Theorem 3.6, the unique triple for the leader's optimal stopping is such that $0 < v_1 < v_2 < \hat{v} < v_3$; the leader will invest suboptimally if he/she ignores the follower's action.

4. Lump-sum payoffs and incomplete market assumption.

4.1. Utility-based pricing model. The asset S representing the market still evolves as (2.1). The project value process V of (3.1), generated by the capital investment project, now evolves as

$$dV(t) = rV(t)dt + \eta V(t)\left(\xi dt + \rho dW(t) + \sqrt{1-\rho^2}dW^0(t)\right), \tag{4.1}$$

where $W(t)$ and $W^0(t)$ are independent Wiener processes. The market asset S depends on only one of the Wiener processes, but the project value depends on both. The parameter $0 < |\rho| < 1$ is the correlation coefficient between market uncertainty and project value process uncertainty. The market asset S can span only a portion of the project value risk driven by the Wiener process $W(t)$. This leaves the remaining risk driven by $W^0(t)$ unhedgeable. Therefore, the market is incomplete, and there is more than one risk-neutral pricing measure. Alternatives to arbitrage-free pricing must be developed in order to correctly value assets and define the related risk management.

We adopt the utility-based pricing approach where the risk-averse investor's preferences are characterized by an exponential utility function given by

$$U(x) = -\frac{1}{\gamma}e^{-\gamma x}, \tag{4.2}$$

where the argument x is the investor's wealth ($x > 0$), and γ is his/her risk aversion parameter, with $\gamma > 0$.

The rational investor maximizes his/her expected utility of wealth. To simplify exposition, we ignore consumption. The evolution of wealth is uniquely driven by the investment portfolio strategy. Given initial wealth, x, the risk-averse investor optimizes his/her portfolio by dynamically choosing allocations in the market asset S and the riskless bond. The investor's wealth, X, evolves as

$$dX(t) = rX(t)dt + X(t)\pi(t)\sigma\big(\lambda dt + dW(t)\big), \tag{4.3}$$

where $\pi(t)$ is the portion of wealth invested in asset S. We use the discounted values $\widetilde{V}(t) = V(t)e^{-rt}$ and $\widetilde{X}(t) = X(t)e^{-rt}$, where

$$d\widetilde{V}(t) = \widetilde{V}(t)\eta\big(\xi dt + \rho dW(t) + \sqrt{1-\rho^2}dW^0(t)\big), \tag{4.4}$$

$$d\widetilde{X}(t) = \widetilde{X}(t)\pi(t)\sigma\big(\lambda dt + dW(t)\big). \tag{4.5}$$

Processes $\widetilde{V}(t)$ and $\widetilde{X}(t)$ are positive. Let $\mathcal{F}_t = \sigma(W(s), W^0(s); s \le t)$. We consider stopping times τ with respect to $\mathcal{F}_t$ starting with initial values $\widetilde{V}(0) = v$, $\widetilde{X}(0) = x$. At τ, the individual invests and receives $\widetilde{X}_x(\tau) + (\widetilde{V}_v(\tau) - K)^+$.[7] With a pair $(\pi(\cdot), \tau)$ we associate the objective function

$$J_{x,v}(\pi(\cdot), \tau) = E\left[e^{\frac{\lambda^2}{2}\tau}U\left(\widetilde{X}_x(\tau) + \big(\widetilde{V}_v(\tau) - K\big)^+\right)\right]. \tag{4.6}$$

We assume τ is finite a.s. The function (4.6) is well defined, but the value $-\infty$ is possible. When $|\rho| = 1$, we expect the criterion to be equivalent to $\widehat{E}\big(\widetilde{V}_v(\tau) - K\big)^+$, which differs from the risk-neutral pricing, $\widehat{E}[e^{-r\tau}(V_v(\tau) - K)^+]$. These changes are motivated by the quest for

[7] This formulation is similar to the specification that the investment cost grows at the risk-free rate.

an analytical solution. The positive exponential is used because U defined in (4.2) is negative. The coefficient $\frac{\lambda^2}{2}$ permits an essential simplification.[8]

The investor's problem is to maximize his/her expected utility taking portfolio investment strategies and stopping times into consideration. We define the associated value function as

$$F(x, v) = \sup_{\pi(\cdot),\tau} J_{x,v}(\pi(\cdot), \tau). \tag{4.7}$$

4.2. Single player. In this section, we want to solve the single player problem. We prepare for the sharing of the resource by using a coefficient a. At time τ, the individual invests and receives $\widetilde{X}_x(\tau) + (1-a)\widetilde{V}_v(\tau) - K$.

With a pair $(\pi(\cdot), \tau)$ we associate the objective function

$$J_{x,v}(\pi(\cdot), \tau) = E\left[e^{\frac{\lambda^2}{2}\tau} U\left(\widetilde{X}_x(\tau) + \left((1-a)\widetilde{V}_v(\tau) - K\right)^+\right)\right]. \tag{4.8}$$

We again assume τ is finite a.s. The associated value function is defined as

$$F(x, v) = \sup_{\pi(\cdot),\tau} J_{x,v}(\pi(\cdot), \tau). \tag{4.9}$$

We solve the problem by V.I. By Henderson [22], we have the following theorem.

Theorem 4.1. *We assume $\xi - \rho\lambda < 0$ and set*

$$\beta = 1 - \frac{2(\xi - \rho\lambda)}{\eta} > 1. \tag{4.10}$$

Define $\hat{v}$ by

$$(1-a)\hat{v} = K + \frac{\varpi}{\gamma(1-\rho^2)} \tag{4.11}$$

with ϖ the unique positive solution of

$$e^{\varpi} - \frac{\varpi}{\beta} = 1 + \frac{K\gamma(1-\rho^2)}{\beta}. \tag{4.12}$$

We look for a solution $F(x, v) = U(x)\Phi(v)$ and have

$$F(x, v) = \begin{cases} U(x)\big(\chi(v)\big)^{\frac{1}{1-\rho^2}} & \text{if } 0 < v < \hat{v}, \\ U(x)e^{-\gamma\left((1-a)v-K\right)} & \text{if } v \geq \hat{v}, \end{cases} \tag{4.13}$$

[8]Henderson [22] formulated the same investment problem, recognizing that this is a nonstandard situation. Wealth must be evaluated at a finite intermediate time. She also showed how this formulation eliminates bias that might influence the manager's choice of exercise/investment time in the infinite horizon setting. She also noted that the choice of $\frac{\lambda^2}{2}$ is a modeling choice, and it is not essential to solve the model in closed form. It is essential for unbiased investment timing.

where $\chi(v) = 1-(1-e^{-\varpi})\left(\frac{v}{\hat{v}}\right)^{\beta}$. *Note that if* $|\rho|$ *is close to one or* γ *is close to zero (i.e., risk neutrality), then* $\varpi \sim \frac{K\gamma(1-\rho^2)}{\beta-1}$ *and* $\hat{v} = \frac{K\beta}{(\beta-1)(1-a)}$. *It is the value identified in section 3.1, except for a different value of* β.[9]

Differing from Henderson's verification through a probabilistic argument, we prove that $F(x,v)$ in (4.13) is the solution of the V.I. (which is the equivalent of the Bellman equation for stopping times). We omit the proof. The optimal strategy for the investor is

$$\hat{\tau}(x,v) = \inf\{t \geq 0 | \widetilde{V}_v(t) \text{ is outside } (0,\hat{v})\} = \hat{\tau}(v) \tag{4.14}$$

and $\hat{\tau}(v) < \infty$ a.s. This is the optimal stopping time and does not depend on the initial wealth x.

4.3. Two players: A Stackelberg game. As in section 3.2, we consider a Stackelberg leader and follower game. The leader enters the market at time θ and the follower at time $\tau \geq \theta$, both stopping times of the filtration $\mathcal{F}_t$. The leader knows that the follower, acting rationally, will enter at time $\hat{\tau}_\theta$, at which time he/she must surrender a portion of the project to the follower.

The follower's investment problem is given by solving the one player problem. After the leader's entry, he/she makes the optimal stopping decision. By paying an investment cost K at τ, the follower receives $\widetilde{X}_x(\tau) + (1-a)\widetilde{V}(\tau)$. The follower's problem is to maximize the expected discounted utility of wealth by choosing stopping time τ and investment strategy π. Thus, the follower's strategy is identical to that described in section 4.2. We have the follower's value function as defined by (4.13), where $\hat{v}$ is given by (4.11) with ϖ the unique value of (4.12). Thus, the optimal stopping strategy for the follower, $\hat{\tau}(v)$, is the same as defined in (4.14).

The stopping time $\hat{\tau}(v)$ is the optimal entry if the follower can enter the market at time zero. Since the follower enters after the leader (who starts at θ), for finite θ, the follower will enter at time

$$\hat{\tau}_\theta = \theta + \hat{\tau}\big(\widetilde{V}_v(\theta)\big) \tag{4.15}$$

with the same considerations as for (3.9).

4.3.1. The leader's problem. The leader must take into account the fact that the follower will enter according to $\hat{\tau}(v)$. The evolution of the leader's discounted wealth, and of the discounted value process, are as in (4.5) and (4.4), respectively.

Beginning with initial wealth x, the leader rebalances his/her portfolio holdings by dynamically choosing the investment allocations, $\pi(\cdot)$, in the asset S and riskless bond. The leader chooses a stopping time θ (with respect to $\mathcal{F}_t$), at which time he/she invests. At stopping time θ, the leader receives $\widetilde{X}_x(\theta) + (\widetilde{V}_v(\theta) - K)^+$. However, he/she must anticipate the follower's optimal entry and the corresponding sharing of the market.

[9]By assuming investment cost grows at the risk-free rate in the complete market, we can see directly that when $|\rho| \to 1$ or $\gamma \to 0$ the optimal investment rule converges to the complete market/risk-neutral valuation case.

The follower may enter the market after θ, and he/she will decide to do so according to the rule $\hat{\tau}_\theta$ described in (4.15). The leader anticipates the follower's actions. So we have the following rules:

1. If $\widetilde{V}(\theta) > \hat{v}$, the follower enters immediately and the leader gets $\widetilde{X}(\theta) + a\widetilde{V}(\theta) - K$, where a is the leader's market share after entry by the follower. This payoff is larger than $\widetilde{X}(\theta)$ since, by assumption, $a > \frac{1}{2}$.
2. If $\widetilde{V}(\theta) < \hat{v}$, then $\hat{\tau}\big(\widetilde{V}(\theta)\big) > 0$, and the leader gets $\widetilde{X}(\theta) + (\widetilde{V}(\theta) - K)$ immediately. However, he/she surrenders, at time $\theta + \hat{\tau}\big(\widetilde{V}(\theta)\big)$, a percentage $(1-a)$ of project value to the follower.

We encounter a hurdle in the latter scenario. Unlike the solution in the complete market, we cannot compare gains and losses occurring at different times directly. At time θ, the leader must determine the value surrendered to the follower, taking into account the follower's optimal entry at $\hat{\tau}_\theta$. In the complete market case, this is straightforward because the valuation is in the monetary unit. In the incomplete market case, we are unable to evaluate cash flows generated (or outflows incurred) at different points in time. We will circumvent this problem converting the surrender value by an equivalence (indifference) consideration. We describe this amount in reference to the follower because this surrender value must be equivalent to an inflow that he/she will receive in his/her portfolio optimization problem after the time of entrance $\hat{\tau}_\theta$. The leader has no choice but to take into account a surrender value which is acceptable to the follower.

We proceed by first obtaining the surrender project value with equivalence (indifference) consideration. We can start at the origin, which will later be the time of entrance of the leader. Let $\widetilde{X}(t)$, $\widetilde{V}(t)$ be the wealth of the follower and the value process governed by (4.5) and (4.4) with $\widetilde{X}(0) = x$, $\widetilde{V}(0) = v$. If $v > \hat{v}$, then everything takes place at time 0. The follower receives $(1-a)v$. He/she values this operation by $U\big(x+(1-a)v\big)$ since his/her wealth is $x + (1-a)v$.

If $0 < v < \hat{v}$, then the follower receives, at time $\hat{\tau}(v)$, $(1-a)\widetilde{V}\big(\hat{\tau}(v)\big)$, and the corresponding value is

$$e^{\frac{\lambda^2}{2}\hat{\tau}(v)}U\big(\widetilde{X}_x\big(\hat{\tau}(v)\big) + (1-a)\widetilde{V}_v\big(\hat{\tau}(v)\big)\big).$$

Note that K does not enter this calculation because we are looking at the equivalent only at the initial time (later the entrance time of the leader) of what the follower will receive from the leader at time $\hat{\tau}(v)$. The amount K plays a role in calculating $\hat{\tau}(v)$.

Since he/she can manage his/her portfolio on the interval of time $\big(0, \hat{\tau}(v)\big)$, the value is in fact

$$H(x,v) = \max_{\pi(\cdot)} E\big[e^{\frac{\lambda^2}{2}\hat{\tau}(v)}U\big(\widetilde{X}_x\big(\hat{\tau}(v)\big) + (1-a)\widetilde{V}_v\big(\hat{\tau}(v)\big)\big)\big] \tag{4.16}$$

and we have the boundary condition

$$H(x,\hat{v}) = U\big(x + (1-a)\hat{v}\big). \tag{4.17}$$

If $v = 0$, then $\hat{\tau}(v) = 0$; therefore

$$H(x,0) = U(x). \tag{4.18}$$

To simplify exposition, we define the nonlinear differential operator

$$\Gamma H(x,v) = \frac{\partial H}{\partial v} v\eta\xi + \frac{1}{2}\frac{\partial^2 H}{\partial v^2} v^2\eta^2 + \frac{1}{2}\lambda^2 H - \frac{1}{2}\frac{\left(\lambda\frac{\partial H}{\partial x} + \frac{\partial^2 H}{\partial x \partial v}\rho\eta v\right)^2}{\frac{\partial^2 H}{\partial x^2}},$$

and $H(x,v)$ must satisfy

$$\begin{cases} \Gamma H(x,v) = 0, & x \in \mathbb{R},\ v < \hat{v}, \\ H(x,0) = U(x); & H(x,\hat{v}) = U(x + (1-a)\hat{v}). \end{cases} \tag{4.19}$$

We look for a solution

$$H(x,v) = U(x)\Lambda(v) \tag{4.20}$$

and find that

$$\Lambda(v) = \left(1 + Bv^\beta\right)^{\frac{1}{1-\rho^2}}, \tag{4.21}$$

where $B = -\frac{1 - e^{-\left(\varpi + K\gamma(1-\rho^2)\right)}}{\hat{v}^\beta}$, β is defined in (4.10), and ϖ is the unique solution of (4.12). Comparing to the function $\Phi(v)$ defined in Theorem 4.1, we have

$$\Phi(v)e^{-\gamma K} \leq \Lambda(v) \leq \Phi(v). \tag{4.22}$$

We consider that the follower is indifferent between receiving $H^e(v)$ at time 0 and losing his/her right to enter into the market at time $\hat{\tau}(v)$, or to exert his/her right to enter into the market. The leader will use this value to express an equivalent to the commitment he/she has to share the market with the follower in due time, at the time when he/she enters into the market. We define now the indifference value $H^e(v)$ by

$$\begin{aligned} H(x + H^e(v), 0) &= H(x,v) \\ &= U\left(x + H^e(v)\right) \\ &= U(x)e^{-\gamma H^e(v)}; \end{aligned} \tag{4.23}$$

then

$$e^{-\gamma H^e(v)} = \Lambda(v), \qquad 0 < v < \hat{v}. \tag{4.24}$$

We have the extension

$$H^e(v) = (1-a)v, \qquad v \geq \hat{v}, \tag{4.25}$$

since the follower jumps into the market immediately and the leader surrenders $(1-a)v$. For $v < \hat{v}$, we have $\hat{\tau}(v) > 0$ and it is necessary that $H^e(v) < (1-a)v$ since $H^e(v)$ is the time 0 value which makes him/her indifferent by having this amount at time 0 and not having $(1-a)v$ at time $\hat{\tau}(v)$. Therefore, we need to have the property

$$H^e(v) \leq (1-a)v \qquad \text{if } v < \hat{v}, \tag{4.26}$$

which is satisfied since this amounts to

$$
\begin{aligned}
G(v) &= \Lambda(v) - e^{-\gamma(1-a)v} \\
&= 1 + Bv^{\beta} - e^{-\gamma(1-a)(1-\rho^2)v} \\
&\geq 0 \qquad \text{if } v < \hat{v}.
\end{aligned}
\tag{4.27}
$$

Equation (4.27) is true because $G''(v) \leq 0$, $G'(0) = \gamma(1-a)(1-\rho^2) > 0$, and

$$
G'(\hat{v}) = \frac{-\gamma(1-a)(1-\rho^2)(1-e^{-\gamma K(1-\rho^2)})}{e^{\varpi}-1} < 0.
$$

The leader deducts the amount $K + H^e(v)$ from the value v that he/she receives at time 0 if he/she decides to invest at time 0. We can now formulate the leader's problem completely.

For each pair $(\pi(\cdot), \theta)$, the leader's objective function is

$$
J_{x,v}(\pi(\cdot), \theta) = E\left[e^{\frac{\lambda^2}{2}\theta} U\left(\widetilde{X}(\theta) + \left(\widetilde{V}(\theta) - K - H^e(\widetilde{V}(\theta))\right)^+\right)\right]
\tag{4.28}
$$

and the leader's problem is to maximize (4.28) with respect to portfolio investment strategies, π, and stopping times, θ. Thus, we define the leader's value function:

$$
L(x,v) = \sup_{\pi(\cdot),\theta} J_{x,v}\big(\pi(\cdot), \theta\big).
\tag{4.29}
$$

We impose $\theta < \infty$ a.s.

4.3.2. The leader's problem V.I. We first note from the definition (4.29)

$$
L(x,0) \geq \sup_{\theta} E e^{\frac{\lambda^2}{2}\theta} U(x) = U(x).
$$

Next, we may write the V.I. in the strong sense that $L(x,v)$ in (4.29) must satisfy as a consequence of dynamic programming, assuming sufficient smoothness,

$$
\begin{cases}
\Gamma L(x,v) \leq 0, \\
L(x,v) \geq U\big(x + \big(v - K - H^e(v)\big)^+\big), \\
\Gamma L(x,v)\big[L(x,v) - U\big(x + \big(v - K - H^e(v)\big)^+\big)\big] = 0.
\end{cases}
\tag{4.30}
$$

We can see that for $v = 0$, $U(x)$ satisfies all the conditions. Therefore we may add the boundary condition

$$
L(x,0) = U(x).
\tag{4.31}
$$

We look for a solution of the form

$$
L(x,v) = U(x)L(v)
\tag{4.32}
$$

and obtain

$$\begin{cases} L'(\xi-\lambda\rho)+\frac{1}{2}v\eta(L''-\rho^2\frac{L'^2}{L})\geq 0, \\ L(v)\leq e^{-\gamma\left(v-K-H^e(v)\right)^+}, \\ \left[L'(\xi-\lambda\rho)+\frac{1}{2}v\eta(L''-\rho^2\frac{L'^2}{L})\right]\left[L(v)-e^{-\gamma\left(v-K-H^e(v)\right)^+}\right]=0, \\ L(0)=1. \end{cases} \tag{4.33}$$

We want a solution such that

$$0\leq L(v)\leq 1. \tag{4.34}$$

Setting $L(v)=\Sigma(v)^{\frac{1}{1-\rho^2}}$, we obtain the V.I.

$$\begin{cases} \Sigma'(\xi-\lambda\rho)+\frac{1}{2}v\eta\Sigma''\geq 0, \\ \Sigma(v)\leq e^{-\gamma(1-\rho^2)\left(v-K-H^e(v)\right)^+}, \\ \left[\Sigma'(\xi-\lambda\rho)+\frac{1}{2}v\eta\Sigma''\right]\left[\Sigma(v)-e^{-\gamma(1-\rho^2)\left(v-K-H^e(v)\right)^+}\right]=0, \\ 0\leq\Sigma(v)\leq 1;\quad \Sigma(0)=1. \end{cases} \tag{4.35}$$

We encounter the same mathematical difficulty as we have for the leader's problem in the complete market. The obstacle

$$\psi(v)=e^{-\gamma(1-\rho^2)\left(v-K-H^e(v)\right)^+} \tag{4.36}$$

is continuous but not C^1. At point $\hat{v}$, we have $\psi(\hat{v})=e^{-\gamma(1-\rho^2)(a\hat{v}-K)}$. Moreover, we note the following property.

Lemma 4.2. *The function $v-K-H^e(v)$ vanishes at a single point v^o such that $K<v^o<\hat{v}$.*

Proof. See Appendix E. ∎

The function $\left(v-K-H^e(v)\right)^+$ is continuous but not C^1. The derivative is discontinuous at v^o and $\hat{v}$. The obstacle $\psi(v)$ has the same property. Since the obstacle is not in C^1, we must consider (4.35) in a weak sense. Nevertheless the function $\Sigma(v)$ will be the value function of an optimal stopping problem (no continuous control). Namely, we have the state equation[10]

$$dV(t)=V(t)\eta\big((\xi-\lambda\rho)dt+dW(t)\big),\quad V(0)=v, \tag{4.37}$$

and the value function

$$\Sigma(v)=\inf_{\tau_\Sigma}J_v(\tau_\Sigma) \tag{4.38}$$

with

$$J_v(\tau_\Sigma)=E\left[\psi\big(V_v(\tau_\Sigma)\big)\right], \tag{4.39}$$

[10]Note that this is a different $V(t)$ process from the original model. This $V(t)$ process arises from our redefined control problem $\Sigma(v)$. This is the process for which we define the optimal stopping rule for $\hat{\tau}_\Sigma$, not the original $V(t)$ process.

and τ_Σ must be a stopping time which is a.s. finite. To formulate (4.38) in a weak form sense, we introduce the Sobolev space $H^1_\varrho(0,\infty)$ with the scalar product (3.24). We define the bilinear form

$$
\begin{aligned}
b(\phi,\tilde{\phi}) = \int_0^\infty \phi'(v)\left\{-\eta(\xi-\lambda\rho)+\eta^2\frac{1-v^2(\varrho-1)}{1-v^2}\right\}\frac{\tilde{\phi}(v)}{(1+v^2)^\varrho}dv \\
+\frac{1}{2}\int_0^\infty \frac{\phi'(v)\tilde{\phi}'(v)v^2\eta^2}{(1+v^2)^\varrho}dv.
\end{aligned}
\tag{4.40}
$$

Consider next the convex set

$$
\mathcal{K} = \{\phi \in H^1_\varrho(0,\infty) | 0 \le \phi(v) \le \psi(v) \ \forall\ v\},
\tag{4.41}
$$

which contains $\psi(v)$; i.e., it is not empty.

The V.I. corresponding to (4.38) is

$$
b(\Sigma, \widetilde{\Sigma}-\Sigma) \ge 0 \qquad \forall\ \widetilde{\Sigma} \in \mathcal{K},\ \Sigma \in \mathcal{K}.
\tag{4.42}
$$

We observe from (4.33), (4.34), and (4.35) that τ_Σ, the optimal stopping we obtain in (4.38), and θ, the optimal stopping we obtain in (4.29), coincide. To simplify the notation and to make use of the optimal stopping notation we defined for the leader, we slightly abuse the notation by replacing τ_Σ with θ.

We turn now to prove the existence and uniqueness of the solution of (4.42), whose solution coincides with that of the value function (4.38). We proceed by first establishing the property that the set of solutions of (4.42) has a maximum and a minimum element within a convenient interval. In Theorem 4.3, we prove that the minimum and the maximum solutions coincide, equal to the value function (4.38).

Theorem 4.3. *Assume $\xi - \lambda\rho < 0$. Then there exists one and only one solution of (4.42). It coincides with the value function (4.38).*

Proof. See Appendix F. ■

4.3.3. Obtaining a two-interval solution. We can investigate whether the solution $\Sigma(v)$ of the V.I. (4.42) is smoother than the obstacle, as is the case in the situation of complete markets. Unfortunately, here, because of the presence of the term $H^e(v)$, we cannot have a general result. Nevertheless, we can explore directly the possibility that a C^1 solution exists by looking for a two-interval strategy for which we will give only a partial account. Namely, we seek three values v_1, v_2, v_3 such that

$$
v_0 < v_1 < v_2 < \hat{v} < v_3
\tag{4.43}
$$

and

$$
\begin{cases}
\Sigma'(\xi-\lambda\rho)+\frac{1}{2}v\eta\Sigma''=0, & 0<v<v_1,\\
\Sigma(v)=\psi(v), & v_1<v<v_2,\\
\Sigma'(\xi-\lambda\rho)+\frac{1}{2}v\eta\Sigma''=0, & v_2<v<v_3,\\
\Sigma(v)=\psi(v), & v=v_3,
\end{cases}
\tag{4.44}
$$

with the value matching conditions

$$\begin{cases} \Sigma(v_1) = \psi(v_1); & \Sigma'(v_1) = \psi'(v_1), \\ \Sigma(v_2) = \psi(v_2); & \Sigma'(v_2) = \psi'(v_2), \\ \Sigma(v_3) = \psi(v_3); & \Sigma'(v_3) = \psi'(v_3). \end{cases} \tag{4.45}$$

We consider an explicit calculation. We begin with v_1. We have $\Sigma(v) = 1+Dv^\beta$ for $0 < v < v_1$ since $\Sigma(0) = 1$. It follows that we have the conditions (by (4.45))

$$\begin{cases} 1 + Dv_1^\beta = e^{-\gamma(1-\rho^2)\left(v_1-K-H^e(v_1)\right)}, \\ \beta D v_1^{\beta-1} = -\gamma(1-\rho^2)\left(1-(H^e)'(v_1)\right)e^{-\gamma(1-\rho^2)\left(v_1-K-H^e(v_1)\right)}; \end{cases}$$

hence

$$e^{\gamma(1-\rho^2)\left(v_1-K-H^e(v_1)\right)} - \frac{\gamma(1-\rho^2)v_1}{\beta}\left(1-(H^e)'(v_1)\right) = 1. \tag{4.46}$$

On the other hand, by (4.24), we obtain

$$-\frac{\gamma(1-\rho^2)v_1}{\beta}\left(H^e(v)\right)' = 1 - e^{\gamma(1-\rho^2)H^e(v_1)},$$

and using this in (4.46) we obtain the equation for v_1:

$$e^{\gamma(1-\rho^2)\left(v_1-K-H^e(v_1)\right)} + e^{\gamma(1-\rho^2)H^e(v_1)} = 2 + \frac{\gamma v_1(1-\rho^2)}{\beta}. \tag{4.47}$$

Proposition 4.4. *Assume*

$$(1-a)\frac{e^{\varpi+\gamma K(1-\rho^2)}-1}{e^{\varpi}-1} < 1; \tag{4.48}$$

then the solution of (4.47)*, which exists, belongs to* $(v_0, \hat{v})$*. Moreover, with any solution* v_1*, one can associate a negative constant* D *such that the first conditions in* (4.45) *are satisfied.*

Proof. See Appendix G. ■

Having obtained v_1, we next look for the values of v_2 and v_3. For $v_2 < v < v_3$, $\Sigma(v) = A + Cv^\beta$, where A and C are constants. We write the conditions as follows (by (4.45)):

$$\begin{cases} A + Cv_2^\beta = e^{-\gamma(1-\rho^2)\left(v_2-K-H^e(v_2)\right)}, \\ C\beta v_2^{\beta-1} = -\gamma(1-\rho^2)\left(1-(H^e)'(v_2)\right)e^{-\gamma(1-\rho^2)\left(v_2-K-H^e(v_2)\right)}, \\ A + Cv_3^\beta = e^{-\gamma(1-\rho^2)(av_3-K)}, \\ C\beta v_3^{\beta-1} = -a\gamma(1-\rho^2)e^{-\gamma(1-\rho^2)(av_3-K)}. \end{cases} \tag{4.49}$$

We eliminate the constants A and C to obtain a system of two equations with two unknowns, v_2 and v_3:

$$\begin{cases} \left(1+\frac{1}{\beta}v_3(1-\rho^2)a\right)e^{\gamma(1-\rho^2)\left(v_2-K-H^e(v_2)\right)} = e^{\gamma(1-\rho^2)(av_3-K)} \\ \qquad\qquad \left[2+\frac{\gamma}{\beta}v_2(1-\rho^2) - e^{\gamma(1-\rho^2)H^e(v_2)}\right], \\ \frac{\gamma}{\beta}\frac{a(1-\rho^2)e^{\gamma(1-\rho^2)\left(v_2-K-H^e(v_2)\right)}}{v_3^{\beta-1}} = \frac{e^{\gamma(1-\rho^2)(av_3-K)}}{v_2^{\beta-1}}\left[\frac{\gamma}{\beta}(1-\rho^2) + \frac{1-e^{\gamma(1-\rho^2)H^e(v_2)}}{v_2}\right]. \end{cases} \tag{4.50}$$

Unfortunately this nonlinear system of algebraic equations is not easy to study analytically. Even if we can solve it, it will remain to prove that the two-interval solution is indeed a solution of the V.I.

4.3.4. Optimal rules in the case of incomplete markets. In the incomplete market situation, we need a weak formulation for the leader's problem V.I. The optimal stopping is the first time when the solution and the obstacle coincide. However, we cannot state that the optimal strategy is characterized by two intervals, as is the case in the situation of complete markets. Once the leader has entered into the market, the follower will enter according to the threshold strategy $\hat{v}$ defined by (4.11). The single decision maker's optimal entry time is the same as in the complete market case if the correlation between market risk and investment risk approaches unity (i.e., $|\rho| \to 1$) or γ approaches zero (i.e., a risk-neutral investor). We conclude that market completeness and risk aversion are important inputs into the optimal investment policy.

5. Cash flow payoffs and complete market assumption.

We now turn to consider the second type of investment operation incomes characterized by a series of cash flows (cf. section 2.3).

5.1. Single player.

We assume that the cash flow process $Y(t)$ from investment operation evolves as

$$dY(t) = \alpha dt + \varsigma dW(t) \tag{5.1}$$

$$= (\alpha - \lambda\varsigma)dt + \varsigma d\widehat{W}(t), \quad Y(0) = y, \tag{5.2}$$

where α and ς are constants. The reason for modeling the stochastic investment cash flow stream as an arithmetic Brownian motion is the recognition of the possibility of loss from operations. Negative values from the arithmetic Brownian motion may be interpreted as a loss generated from operations.

If the firm exploits the investment opportunity by paying cost K, it will obtain a continuous cash flow $\delta Y(t)$ per unit time. At time t, the project value, $V(t)$, from the operation is the expected discounted cash flow stream under the risk-neutral probability measure given as

$$V(t) = \delta \widehat{E}\left[\int_t^{\infty} e^{-r(s-t)} Y_y(s) ds | \mathcal{F}_t\right] = \delta\left(\frac{Y_y(t)}{r} + \frac{\alpha - \lambda\varsigma}{r^2}\right). \tag{5.3}$$

From (5.3), the expected discounted payoff from the capital investment project undertaken at time τ is

$$J_y(\tau) = \widehat{E}\left[e^{-r\tau}\left(\delta\left(\frac{Y_y(\tau)}{r} + \frac{\alpha - \lambda\varsigma}{r^2}\right) - K\right)\mathbb{1}_{\tau<\infty}\right]. \tag{5.4}$$

The firm's objective is to find an optimal stopping time to maximize the expected discounted payoff. That is,

$$F(y) = \sup_{\tau \geq 0} J_y(\tau). \tag{5.5}$$

Assuming $F(y)$ is sufficiently smooth, we can write the V.I. in the strong sense that $F(y)$ must satisfy as a consequence of dynamic programming

$$\text{(5.6)}\quad \begin{cases} (\alpha-\lambda\varsigma)F'(y)+\frac{1}{2}\varsigma^2F''(y)-rF(y)\le 0,\\ F(y)\ge\left(\delta\left(\frac{y}{r}+\frac{\alpha-\lambda\varsigma}{r^2}\right)-K\right),\\ \left[F(y)-\left(\delta\left(\frac{y}{r}+\frac{\alpha-\lambda\varsigma}{r^2}\right)-K\right)\right]\left[(\alpha-\lambda\varsigma)F'(y)+\frac{1}{2}\varsigma^2F''(y)-rF(y)\right]=0,\\ F(y)\ge 0;\quad F \text{ has linear growth at } \infty. \end{cases}$$

The solution is $C^1(-\infty,\infty)$ and piecewise C^2. This suffices to give a meaning to (5.6) for almost all y. Such a solution, $F(y)$, if it exists, will be unique since, by a classical verification argument, it will coincide with (5.5).

Theorem 5.1. *Let*

$$\text{(5.7)}\quad \beta=-\frac{\alpha-\lambda\varsigma}{\varsigma^2}+\sqrt{\left(\frac{\alpha-\lambda\varsigma}{\varsigma^2}\right)^2+\frac{2r}{\varsigma^2}}>0$$

and

$$\text{(5.8)}\quad \hat{y}=\frac{1}{\beta}+\frac{r}{\delta}K-\frac{\alpha-\lambda\varsigma}{r}>0;$$

then there exists a unique solution F of (5.6), which is C^1, given by

$$\text{(5.9)}\quad F(y)=\begin{cases} \frac{\delta}{r\beta}\exp\{-\beta(\hat{y}-y)\} & \text{if } y\le\hat{y},\\ \delta\left(\frac{y}{r}+\frac{\alpha-\lambda\varsigma}{r^2}\right)-K & \text{if } y\ge\hat{y}. \end{cases}$$

Proof. See Appendix H. ■

We next define the optimal stopping rule $\hat{\tau}(y)$ that achieves the supremum in (5.5). From general results on V.I., we have

$$\hat{\tau}(y)=\inf\left\{t|F(Y_y(t))=\delta\left(\frac{Y_y(t)}{r}+\frac{\alpha-\lambda\varsigma}{r^2}\right)-K\right\}=\inf\{t|G(Y_y(t))=0\},$$

where $G(y)=F(y)-\delta(\frac{y}{r}+\frac{\alpha-\lambda\varsigma}{r^2})+K$. From the proof of Theorem 5.1, we see that $G(y)$ decreases from K to zero on the interval $(0,\hat{y})$ and remains zero for $y>\hat{y}$. Thus, $Y_y(t)=\hat{y}$ is the solution to the above equation. Therefore, we can write

$$\text{(5.10)}\quad \hat{\tau}(y)=\inf\{t|Y_y(t)\ge\hat{y}\}.$$

By (5.10), the manager undertakes the capital investment as soon as the cash flow process reaches the threshold $\hat{y}$ from below. Using $Y_y(\hat{\tau}(y))=\hat{y}$ if $\hat{\tau}(y)<\infty$ and $y<\hat{y}$, we have the probabilistic representation of value function $F(y)$:

$$\text{(5.11)}\quad F(y)=\frac{\delta}{r\beta}\widehat{E}\left[\exp\left(-r\hat{\tau}(y)\right)\right]\quad \text{if } y<\hat{y}.$$

5.2. Two players: A Stackelberg game.

We now consider the situation of section 3.2 but with the investment payoff as a series of cash flows. Upon entry, the leader receives a continuous cash flow $\delta_1 Y(t)$ per unit time prior to the follower's entry. Once both have entered, each gets a continuous cash flow $\delta_2 Y(t)$ per unit time, with $\delta_2 < \delta_1$.[11]

5.2.1. Statement of the leader's problem.

As in section 3.2.1, the follower's optimal stopping is

$$\hat{\tau}_\theta = \theta + \hat{\tau}\big(Y_y(\theta)\big) \tag{5.12}$$

with considerations similar to those for (3.9) and we can write explicitly the Laplace transform of the conditional density as

$$\widehat{E}[e^{-r(\hat{\tau}_\theta - \theta)}|\mathcal{F}_\theta] = \mathbb{1}_{Y_y(\theta)\geq \hat{y}} + \mathbb{1}_{Y_y(\theta)<\hat{y}} e^{-\beta\left(\hat{y}-Y_y(\theta)\right)}. \tag{5.13}$$

When the leader enters at time $\theta < \infty$, by paying cost K, he/she receives a continuous cash flow $\delta_1 Y_y(t)$ per unit time prior to the follower's entry and gets a continuous cash flow $\delta_2 Y_y(t)$ per unit time after the follower's entry. By anticipating the follower's optimal entry at time $\hat{\tau}_\theta$, entering at time $\theta < \infty$, the leader's expected discounted payoff is

$$\begin{aligned}
&-K + \delta_1 \widehat{E}\left[\int_\theta^{\hat{\tau}_\theta} e^{-r(s-\theta)} Y(s)ds|\mathcal{F}_\theta\right] + \delta_2 \widehat{E}\left[\mathbb{1}_{\hat{\tau}_\theta<\infty}\int_{\hat{\tau}_\theta}^{\infty} e^{-r(s-\theta)} Y(s)ds|\mathcal{F}_\theta\right]\\
&= -K + \delta_2\left(\frac{Y_y(\theta)}{r} + \frac{\alpha-\lambda\varsigma}{r^2}\right)\\
&\qquad + (\delta_1-\delta_2)\mathbb{1}_{Y_y(\theta)<\hat{y}}\left[\frac{Y_y(\theta)}{r} + \frac{\alpha-\lambda\varsigma}{r^2} - \left(\frac{1}{r\beta}+\frac{K}{\delta_2}\right)e^{-\beta(\hat{y}-Y_y(\theta))}\right],
\end{aligned}$$

where we use the fact that $\frac{\hat{y}}{r} + \frac{\alpha-\lambda\varsigma}{r^2} = \frac{1}{r\beta} + \frac{K}{\delta_2}$. To facilitate the presentation, we define

$$\Psi(y) = -K + \delta_2\left(\frac{y}{r} + \frac{\alpha-\lambda\varsigma}{r^2}\right) + (\delta_1-\delta_2)\mathbb{1}_{y<\hat{y}}\left[\frac{y}{r} + \frac{\alpha-\lambda\varsigma}{r^2} - \left(\frac{1}{r\beta}+\frac{K}{\delta_2}\right)e^{-\beta(\hat{y}-y)}\right]. \tag{5.14}$$

The leader's objective is to find a stopping time, θ, to maximize the expected discounted payoff:

$$L(y) = \sup_{\theta\geq 0} \widehat{E}\left[e^{-r\theta}\Psi\big(Y_y(\theta)\big)\mathbb{1}_{\theta<\infty}\right]. \tag{5.15}$$

The leader's value function $L(y)$ must satisfy

$$L(y) \geq 0; \quad L(y) \geq \Psi(y). \tag{5.16}$$

The obstacle, $\Psi(y)$, presents a challenge because it is $C^0(-\infty,\infty)$, not $C^1(-\infty,\infty)$. The only point of nondifferentiability is $\hat{y}$. In addition, $\Psi(y)$ is unbounded as $y \to \pm\infty$, which we observe from the following. By (5.14), we may express $\Psi(y)$ as

$$\Psi(y) = \frac{\delta_2}{r}\left[\frac{1}{\beta} + y - \hat{y} + \frac{\delta_1-\delta_2}{\delta_2}\mathbb{1}_{y<\hat{y}}\left(y - \hat{y} + \mu\big(1-e^{-\beta(\hat{y}-y)}\big)\right)\right], \tag{5.17}$$

where $\mu = \hat{y} + \frac{\alpha-\lambda\varsigma}{r} = \frac{\delta_2 + r\beta K}{\beta\delta_2}$.

[11] The condition of $\delta_2 < \delta_1$ expresses the idea of the first-mover advantages; otherwise, the leader would have no desire to enter into the market prior to the follower.

5.2.2. The leader's problem V.I. As in section 3.2.2, the nondifferentiability point of the obstacle function requires us to write the variational formulation of (5.15) in the weak sense. We introduce the useful functional spaces, the Hilbert space $L^2_\varrho(-\infty,\infty)$ defined by (3.21), and the Sobolev space $H^1_\varrho(-\infty,\infty)$ defined by (3.22), with the corresponding scalar products defined by (3.23) and (3.24), respectively. We define on $H^1_\varrho(-\infty,\infty)$ the bilinear form

$$b(\Phi,\tilde{\Phi}) = -\int_{-\infty}^{\infty}\left(\alpha-\lambda\varsigma+\frac{\varrho y\varsigma^2}{1+y^2}\right)\Phi'(y)\tilde{\Phi}'(y)w(y)dy+\frac{1}{2}\int_{-\infty}^{\infty}\varsigma^2\Phi'(y)\tilde{\Phi}(y)w(y)dy$$
$$+r\int_{-\infty}^{\infty}\Phi(y)\tilde{\Phi}(y)w(y)dy. \tag{5.18}$$

We consider the convex subset

$$\mathcal{K}=\{\Phi\in H^1_\varrho(-\infty,\infty)|\Phi(y)\geq\Psi(y)\ \forall y\}, \tag{5.19}$$

where $\mathcal{K}$ is not empty since it contains Ψ.

The V.I. corresponding to (5.15) is

$$b(L,\tilde{L}-L)\geq 0 \qquad \forall\tilde{L}\in\mathcal{K},\ L\in\mathcal{K}. \tag{5.20}$$

Theorem 5.2. *The value function* (5.15) *is the unique solution of the V.I.* (5.20).

Proof. See Appendix I. ■

5.2.3. Smoothness of the solution. Because the obstacle is nondifferentiable only at a single point, $\hat{y}$, it turns out that the solution $L(y)$ will be smoother than the obstacle. Therefore the V.I. will have a strong formulation:

$$\begin{cases}-\frac{1}{2}\varsigma^2L''(y)-(\alpha-\varsigma\lambda)L'(y)+rL(y)\geq 0,\\ L(y)\geq\Psi(y),\\ (L(y)-\Psi(y))[-\frac{1}{2}\varsigma^2L''(y)-(\alpha-\varsigma\lambda)L'(y)+rL(y)]=0.\end{cases} \tag{5.21}$$

Note that $\Psi(y)$ satisfies

$$\begin{cases}-\frac{1}{2}\varsigma^2\Psi''(y)-(\alpha-\varsigma\lambda)\Psi'(y)+r\Psi=\delta_1y-rK, \quad y<\hat{y},\\ \Psi(\hat{y})=\frac{\delta_2}{r}\hat{y}+\frac{\delta_2^2}{r^2}(\alpha-\varsigma\lambda)-K=\frac{\delta_2}{r\beta}.\end{cases} \tag{5.22}$$

We would like $L\geq 0$.

Consider

$$u(y)=L(y)-\left(\frac{\delta_2}{r}y+\frac{\delta_2}{r^2}(\alpha-\varsigma\lambda)\right)+K,$$
$$m(y)=\Psi(y)-\left(\frac{\delta_2}{r}y+\frac{\delta_2}{r^2}(\alpha-\varsigma\lambda)\right)+K,$$

giving us

$$\begin{cases}-\frac{1}{2}\varsigma^2u''(y)-(\alpha-\varsigma\lambda)u'(y)+ru\geq-\delta_2y+rK,\\ u(y)\geq m(y),\\ (u(y)-m(y))[-\frac{1}{2}\varsigma^2u''(y)-(\alpha-\varsigma\lambda)u'(y)+ru+\delta_2y-rK]=0,\\ u(y)\geq-(\frac{\delta_2}{r}y+\frac{\delta_2}{r^2}(\alpha-\varsigma\lambda))+K.\end{cases} \tag{5.23}$$

The function m satisfies

$$-\frac{1}{2}\varsigma^2 m''(y) - (\alpha - \varsigma\lambda)m(y) + rm(y) = (\delta_1 - \delta_2)y, \quad y < \hat{y}. \tag{5.24}$$

Proposition 5.3. *There exists a unique $u \in C^1(-\infty, \infty)$, piecewise C^2 solution of (5.23). This function vanishes for y sufficiently large.*

Proof. This is a special case of the more complex problem presented in Theorem 6.5, and thus we refer the reader to the proof of Theorem 6.5. ∎

Next, we are able to arrive at a two-interval solution stated in the proposition that follows.

Proposition 5.4. *There exists a triple $y_1 < y_2 < \hat{y} < y_3$ such that*

$$\begin{cases} -\frac{1}{2}\varsigma^2 u''(y) - (\alpha - \varsigma\lambda)u'(y) + ru = -\delta_2 y + rK \ \ \textit{for} \ \ y < y_1 \ \ \textit{and} \ \ y_2 < y < y_3, \\ u(y) \geq m(y), \\ u(y) = m(y) \ \textit{for} \ y_1 \leq y \leq y_2, \\ u(y) = 0 \ \textit{for} \ y \geq y_3, \\ u'(y_1) = m'(y_1); \ u'(y_2) = m'(y_2); \ u'(y_3) = 0. \end{cases}$$

Proof. The proof is similar to the proof of Theorem 6.6, a more complex problem. ∎

We can define the leader's optimal stopping rule as

$$\hat{\theta}(y) = \begin{cases} \inf\{t | Y_y(t) \geq y_1\} & \text{if } y < y_1, \\ 0 & \text{if } y_1 \leq y \leq y_2, \\ \inf\{t | Y_y(t) \leq y_2 \text{ or } Y_y(t) \geq y_3\} & \text{if } y_2 < y < y_3, \\ 0 & \text{if } y \geq y_3. \end{cases} \tag{5.25}$$

5.2.4. Optimal rules in the case of complete markets.

As in section 3.2.4, the leader's optimal stopping rule (i.e., optimal investment strategy) is characterized as a two-interval solution, which is $[y_1, y_2]$ and $[y_3, \infty)$ with the relation such that $y_1 < y_2 < \hat{y} < y_3$, where $\hat{y}$ is the follower's optimal investment trigger (same as the monopolist's). It is observed that the optimal stopping for both parties differs from the monopolist's. Although the follower's optimal investment trigger coincides with the monopolist's, he/she does not act the same as the monopolist due to the concern of strategic interactions from the leader. The leader will invest nonoptimally if he/she ignores the strategic effect of competition.

6. Cash flow payoffs and incomplete market assumption.

6.1. Utility-based pricing model.

We consider the model of section 4.1 with a cash flow process rather than a project value process. The asset S representing the market still evolves as (2.1). The cash flow process Y of (5.1), generated by the capital investment project, evolves as

$$dY(t) = \alpha dt + \varsigma\big(\rho dW(t) + \sqrt{1-\rho^2}dW^0(t)\big), \tag{6.1}$$

where $W(t)$ and $W^0(t)$ are independent Wiener processes.

The rational utility-maximizing individual investor's exponential (i.e., constant absolute risk aversion (CARA)) utility function is given by

$$U(C) = -\frac{1}{\gamma}e^{-\gamma C}, \tag{6.2}$$

where the argument C is the investor's consumption.

Remark 6.1.1. We allow for negative consumption. For $C \in \mathbb{R}$, U increases from $-\infty$ to 0. As $C \to -\infty$, it leads to huge negative values. We interpret this effect as a penalty to the utility maximization investor. We could, of course, impose the constraint of nonnegative consumption. However, imposing nonnegativity on the consumption would rule out the analytical solutions for further developments, a property we would like to retain for the full analysis. Therefore, we choose to accept negative consumption which could lead to huge negative utility values (big penalties for our utility maximization investor) instead of imposing the nonnegativity constraint on the consumption. We also note that the negative consumption occurs when x or y becomes very negative and we cannot avoid this situation since $x, y \in \mathbb{R}$.

The rational investor maximizes his/her expected utility of consumption. Given the initial wealth, x, the risk-averse investor optimizes his/her portfolio by dynamically choosing allocations in the market asset S, the riskless bond, and the consumption rate C. The investor's wealth, X, evolves as

$$\begin{cases} dX(t) = \pi(t)X(t)\sigma(\lambda dt + dW(t)) + rX(t)dt - C(t)dt, \ t < \tau, \\ X(\tau) = X(\tau - 0) - K, \\ dX(t) = \pi(t)X(t)\sigma(\lambda dt + dW(t)) + rX(t)dt - C(t)dt + \delta Y(t)dt, \ t > \tau, \\ dY(t) = \alpha dt + \varsigma\big(\rho dW(t) + \sqrt{1-\rho^2}dW^0(t)\big), \\ X(0) = x, \ Y(0) = y, \ \text{with } x, y \in \mathbb{R}, \end{cases} \tag{6.3}$$

where $\pi(t)$ is the ratio of wealth invested in asset S, $C(t)$ is the consumption rate, and τ is a stopping time, chosen optimally by the investor. At τ, he/she invests in the project by paying an investment cost K. Thus, there is a jump in the wealth process, making it discontinuous at τ. After τ, the wealth process evolves differently since the investor receives an additional income stream, i.e., a cash flow stream at a rate $\delta Y(t)$ per unit time. Prior to τ, allocations in the market asset are for portfolio investment purposes and provide partial hedges against the uncertain capital investment payoff. After τ, the investor solves the optimal investment/consumption portfolio problem to invest his/her total wealth.

We see from (6.3) that the wealth process has two possible evolution regimes. To facilitate further representation, we introduce the processes, regime 0, X^0 and regime 1, X^1:

$$dX^0(t) = \pi(t)X^0(t)\sigma(\lambda dt + dW(t)) + rX^0(t)dt - C(t)dt, \quad X^0(0) = x, \tag{6.4}$$

$$dX^1(t) = \pi(t)X^1(t)\sigma(\lambda dt + dW(t)) + rX^1(t)dt - C(t)dt + \delta Y(t)dt. \tag{6.5}$$

The wealth process X corresponds to X^0 before the stopping time τ and X^1 after τ.

After τ, the investor solves a control problem of portfolio selection and consumption rate, augmented by the stochastic income $\delta Y(t)$. With a pair of $(C(\cdot), \pi(\cdot))$ we associate the objective function

$$J_{x,y}\big(C(\cdot)\big) = E\left[\int_0^\infty e^{-\mu t} U\big(C(t)\big)\,dt\right], \tag{6.6}$$

where μ is the discount rate. This function may take the value $-\infty$. We define the utility maximization control problem as

$$F^1(x,y) = \sup_{\{\pi(\cdot),C(\cdot)\}\in\mathcal{U}^1_{x,y}} J\big(C(\cdot)\big), \tag{6.7}$$

where $\mathcal{U}^1_{x,y}$ is the set of admissible controls to be defined later (cf. section 6.2.1).

At time τ, the investor stops, receiving $F^1(X^0(\tau)-K, Y(\tau))$. With a triple of $(C(\cdot),\pi(\cdot),\tau)$ we associate the objective function

$$J_{x,y}\big(C(\cdot),\pi(\cdot),\tau\big) = E\left[\int_0^\tau e^{-\mu t} U\big(C(t)\big)\,dt + F^1\big(X^0(\tau)-K, Y(\tau)\big)e^{-\mu\tau}\right]. \tag{6.8}$$

We assume τ is finite a.s. The investor's problem is to maximize the expected discounted utility from consumption by choosing stopping time τ, consumption rate C, and portfolio investment strategy π. We define the associated value function as

$$F(x,y) = \sup_{\{\pi(\cdot),C(\cdot),\tau\}\in\mathcal{U}^0_{xy}} J_{x,y}\big(C(\cdot),\pi(\cdot),\tau\big) \tag{6.9}$$

with $\mathcal{U}^0_{xy}$ a set of admissible controls to be defined later (cf. section 6.2.2).

6.2. Single player. In this section, we solve a single player's problem, and, as such, the manager/investor has no need to consider the actions of any other firm. By paying cost K, the firm expects to receive a continuous cash flow $\delta Y(t)$ per unit time. The investor's wealth evolves as (6.3), for which we introduce regimes (6.4) and (6.5) to facilitate exposition.

The investor's problem is to maximize the expected discounted utility from consumption with respect to investment time τ, consumption rate C, and investment strategy π. As described in section 6.1, it is necessary to solve the problem in two steps: post- and preinvestment utility maximization, i.e., after the stopping time τ and prior to τ. The obstacle used in solving preinvestment utility maximization is defined by the solution of the postinvestment utility maximization. We proceed in detail in the following sections.

6.2.1. Postinvestment utility maximization. Before considering the optimal stopping problem, we begin with a control problem relative to the process $X^1(t)$. That is, we assume the capital investment project has been undertaken, and we solve the investor's utility maximization as a control problem of portfolio selections and consumption rules, augmented by a stochastic income stream, $\delta Y(t)$ per unit time.

To facilitate the notation, for $\mathcal{F}_t$-adapted processes $\pi(t)$ and $C(t)$, we introduce the local integrability condition I^i, $i = 0, 1$:

$$I^i = \begin{cases} E\int_0^T \big(\pi(t)X^i(t)\big)^2 dt < \infty \quad \forall\, T, \\ E\int_0^T \big(C(t)\big)^2 dt < \infty \quad \forall\, T, \end{cases} \tag{6.10}$$

and we also define

$$\tau_N^i = \inf\{t | \mathbb{1}_{i=1}\big(rX^1(t) + \delta Y(t)\big) + \mathbb{1}_{i=0}X^0(t) \le -N\}, \quad i = 0, 1.$$

To a pair of $(C(\cdot), \pi(\cdot))$ we introduce the objective function

$$J_{x,y}\big(C(\cdot)\big) = E\bigg[\int_0^\infty e^{-\mu t} U\big(C(t)\big)dt\bigg]. \tag{6.11}$$

This function may take the value $-\infty$. The investor's problem is to maximize his/her expected discounted utility from consumption. We define the associated value function as

$$F^1(x, y) = \sup_{\{\pi(\cdot), C(\cdot)\} \in \mathcal{U}^1_{x,y}} J_{x,y}\big(C(\cdot)\big), \tag{6.12}$$

where $\mathcal{U}^1_{x,y} = \{(\pi, C) : I^1;\ \tau_N^1 \uparrow \infty$ a.s. as $N \uparrow \infty;\ e^{-\mu T} E[e^{-\gamma\left(rX^1(T) + \delta Y(T)\right)}] \to 0$ as $T \to \infty\}$. We associate the value function $F^1(x, y)$ with the Bellman equation

$$\begin{aligned} &-\mu F^1 + \frac{\partial F^1}{\partial x}(rx + \delta y) + \frac{\partial F^1}{\partial y}\alpha + \frac{1}{2}\frac{\partial^2 F^1}{\partial y^2}\varsigma^2 + \sup_C\bigg\{U(C) - C\frac{\partial F^1}{\partial x}\bigg\} \\ &\quad + \sup_\pi\bigg\{\pi x\sigma\bigg(\lambda\frac{\partial F^1}{\partial x} + \varsigma\rho\frac{\partial^2 F^1}{\partial xy}\bigg) + \frac{1}{2}\frac{\partial^2 F^1}{\partial x^2}\pi^2 x^2 \sigma^2\bigg\} = 0. \end{aligned} \tag{6.13}$$

Define the feedbacks

$$\begin{cases} \widehat{C}(x, y) = -\dfrac{1}{\gamma}\ln\dfrac{\partial F^1}{\partial x}, \\ \widehat{\pi}(x, y)\sigma x = -\dfrac{\lambda + \frac{\partial F^1}{\partial x} + \varsigma\rho\frac{\partial^2 F^1}{\partial x \partial y}}{\frac{\partial^2 F^1}{\partial x^2}}, \end{cases} \tag{6.14}$$

and (6.13) may be rewritten as

$$\begin{aligned} -uF^1 + \frac{\partial F^1}{\partial x}\bigg(rx + \delta y - \frac{1}{\gamma} + \frac{1}{\gamma}\ln\frac{\partial F^1}{\partial x}\bigg) + \frac{\partial F^1}{\partial y}\alpha + \frac{1}{2}\frac{\partial F^1}{\partial y^2}\varsigma^2 \\ - \frac{1}{2}\frac{\left(\lambda\frac{\partial F^1}{\partial x} + \varsigma\rho\frac{\partial^2 F^1}{\partial x \partial y}\right)^2}{\frac{\partial^2 F^1}{\partial x^2}} = 0. \end{aligned} \tag{6.15}$$

We look for a solution of (6.15) as

(6.16)

$$F^1(x, y) = -\frac{1}{r\gamma}\exp\bigg\{-r\gamma\bigg(x + \frac{\delta}{r}y\bigg) + 1 - \frac{\mu + \frac{\lambda^2}{2}}{r} - \frac{\delta\gamma}{r}\bigg(\alpha - \lambda\varsigma\rho - \frac{1}{2}\varsigma^2\delta\gamma(1 - \rho^2)\bigg)\bigg\}.$$

Theorem 6.1. *The function $F^1(x, y)$ defined by (6.16) coincides with the value function (6.12).*

Proof. See Appendix J. ∎

6.2.2. Preinvestment utility maximization. After obtaining the solution to postinvestment utility maximization in section 6.2.1, we solve the full problem by making use of the value function $F^1(x,y)$. That is, we now turn to the problem of optimal stopping. Before the stopping time τ, the wealth process is governed by (6.4), and the cash flow evolves as (6.1). At time τ, the investor stops, receiving $F^1(X^0(\tau)-K,Y(\tau))$. Therefore, the objective function is

$$J_{x,y}\big(C(\cdot),\pi(\cdot),\tau\big)=E\left[\int_0^\tau e^{-\mu t}U\big(C(t)\big)dt+F^1\big(X^0(\tau)-K,Y(\tau)\big)e^{-\mu\tau}\right] \tag{6.17}$$

and we define the value function

$$F(x,y)=\sup_{\{\pi(\cdot),C(\cdot),\tau\}\in\mathcal{U}^0_{xy}}J_{x,y}\big(C(\cdot),\pi(\cdot),\tau\big), \tag{6.18}$$

where $\mathcal{U}^0_{xy}=\{(\pi,C,\tau):I^0;\ \tau<\infty\text{ a.s.};\ \tau^*=\lim\uparrow\tau^0_N\geq\tau\text{ a.s.}\}$.

As a consequence of dynamic programming, assuming sufficient smoothness of $F(x,y)$, we write the V.I. in the strong sense that $F(x,y)$ must satisfy

$$\begin{cases}-\mu F+\frac{\partial F}{\partial x}rx+\frac{\partial F}{\partial y}\alpha+\frac{1}{2}\frac{\partial^2F}{\partial y^2}\varsigma^2+\sup_C\big(U(C)-C\frac{\partial F}{\partial x}\big)+\sup_\pi\big[\pi x\sigma\big(\lambda\frac{\partial F}{\partial x}+\varsigma\rho\frac{\partial^2F}{\partial x\partial y}\big)\\ \qquad\qquad+\frac{1}{2}\pi^2x^2\sigma^2\frac{\partial^2F}{\partial x^2}\big]\leq0,\\ F(x,y)\geq F^1(x-K,y),\\ \Big(F(x,y)-F^1(x-K,y)\Big)\Big[-\mu F+\frac{\partial F}{\partial x}rx+\frac{\partial F}{\partial y}\alpha+\frac{1}{2}\frac{\partial^2F}{\partial y^2}\varsigma^2+\sup_C\big(U(C)-C\frac{\partial F}{\partial x}\big)\\ \qquad\qquad+\sup_\pi\big[\pi x\sigma\big(\lambda\frac{\partial F}{\partial x}+\varsigma\rho\frac{\partial^2F}{\partial x\partial y}\big)+\frac{1}{2}\pi^2x^2\sigma^2\frac{\partial^2F}{\partial x^2}\big]\Big]=0,\end{cases} \tag{6.19}$$

where in the curly brace we find the nonlinear 2nd order differential operator appearing at the left-hand side of the first inequality. Since $x,y\in\mathbb{R}$, there are no boundary conditions. We look for a solution:

$$F(x,y)=-\frac{1}{r\gamma}\exp\left[-r\gamma\big(x+g(y)\big)+1-\frac{\mu+\frac{\lambda^2}{2}}{r}\right]. \tag{6.20}$$

Going back to the expression of $F^1(x,y)$ in (6.16), it will be convenient to introduce

$$f(y)=\frac{\delta y}{r}+\frac{\delta}{r^2}\left(\alpha-\lambda\varsigma\rho-\frac{1}{2}\varsigma^2\rho\gamma(1-\rho^2)\right) \tag{6.21}$$

so that

$$F^1(x,y)=-\frac{1}{r\gamma}\exp\left[-r\gamma\big(x+f(y)\big)+1-\frac{\mu+\frac{\lambda^2}{2}}{r}\right]. \tag{6.22}$$

Define the feedbacks

$$\begin{cases}\widehat{C}(x,y)=r\big(x+g(y)\big)-\frac{1}{\gamma}\big(1-\frac{\mu+\frac{\lambda^2}{2}}{r}\big),\\ \widehat{\pi}(x,y)=\frac{\lambda}{r\gamma}-\varsigma\rho g'(y).\end{cases} \tag{6.23}$$

Substituting (6.23) and the functions $F(x,y)$ and $F^1(x,y)$, defined in (6.20) and (6.22), respectively, into (6.19), the V.I. (6.19) that $F(x,y)$ must satisfy reduces to the following V.I. expressed in terms of $g(y)$:

$$\begin{cases} \frac{1}{2}\varsigma^2 g'' + g'(\alpha - \lambda\varsigma\rho) - \frac{1}{2}\varsigma^2 r\gamma(1-\rho^2)g'^2 - rg \le 0, \\ g(y) \ge f(y) - K, \\ \big(g(y) - f(y) + K\big)\big[\frac{1}{2}\varsigma^2 g'' + g'(\alpha - \lambda\varsigma\rho) - \frac{1}{2}\varsigma^2 r\gamma(1-\rho^2)g'^2 - rg\big] = 0. \end{cases} \tag{6.24}$$

This V.I. cannot be interpreted simply as a control problem. Indeed, the nonlinear operator is connected to a minimization problem and the inequalities are connected to a maximization problem for a stopping time. Therefore, $g(y)$ is more appropriately the value function of a differential game than of a control problem.

Considering $u(y) = g(y) - f(y) + K$, the V.I. (6.24) becomes

$$\begin{cases} -\frac{1}{2}\varsigma^2 u'' - u'\big(\alpha - \lambda\varsigma\rho - \varsigma^2\delta\gamma(1-\rho^2)\big) + \frac{1}{2}\varsigma^2 r\gamma(1-\rho^2)u'^2 + ru \ge -\delta y + rK, \\ u \ge 0, \\ u\big[-\frac{1}{2}\varsigma^2 u'' - u'\big(\alpha - \lambda\varsigma\rho - \varsigma^2\delta\gamma(1-\rho^2)\big) + \frac{1}{2}\varsigma^2 r\gamma(1-\rho^2)u'^2 + ru + \delta y - rK\big] = 0. \end{cases} \tag{6.25}$$

We study (6.25) by the threshold approach. For $\hat{y}$ fixed, we solve the Dirichlet problem

$$\begin{cases} -\frac{1}{2}\varsigma^2 u'' - u'\big(\alpha - \lambda\varsigma\rho - \varsigma^2\delta\gamma(1-\rho^2)\big) + \frac{1}{2}\varsigma^2 r\gamma(1-\rho^2)u'^2 + ru = -\delta y + rK, \quad y < \hat{y}, \\ u(\hat{y}) = 0 \end{cases} \tag{6.26}$$

and we require linear growth for $y \to -\infty$.

Theorem 6.2. *For each $\hat{y}$, there exists a unique solution of (6.26) with the estimate*

$$-\frac{(-\delta\hat{y} + rK)^-}{r} \le u(y) \le -\frac{\delta}{r}(y - \hat{y}) + \frac{1}{r}\left(-\delta\hat{y} + rK - \frac{\delta}{r}\big(\alpha - \lambda\varsigma\rho - \gamma\delta\varsigma^2(1-\rho^2)\big)\right)^+ \quad \textit{for } y < \hat{y}. \tag{6.27}$$

The solution is C^2 in $(-\infty, \hat{y})$.

There exists a unique $\hat{y}$ such that

$$u'(\hat{y}) = 0, \qquad \hat{y} \ge \frac{rK}{\delta}. \tag{6.28}$$

The corresponding solution of (6.26) extended by 0 beyond $\hat{y}$ is the unique solution of the V.I. (6.25). It is C^1 and piecewise C^2.

Proof. See Appendix K. ■

Returning to (6.24), from Theorem 6.2, we have proven that there exists a unique $g(y) \in C^1$ and piecewise C^2. Moreover, there exists a unique $\hat{y}$ such that

$$\begin{cases} -\frac{1}{2}\varsigma^2 g'' - g'(\alpha - \lambda\varsigma\rho) + \frac{1}{2}\varsigma^2 r\gamma(1-\rho^2)g'^2 + rg = 0, \quad y < \hat{y}, \\ g(y) = \frac{\delta}{r}y - K + \frac{\delta}{r^2}\big(\alpha - \lambda\varsigma\rho - \frac{1}{2}\varsigma^2\delta\gamma(1-\rho^2)\big), \quad y \ge \hat{y}, \\ g'(\hat{y}) = \frac{\delta}{r}. \end{cases} \tag{6.29}$$

The function $g(y)$ is the value function of the stochastic control problem:

$$\begin{aligned} g(y) = \inf_{v(\cdot)} E\Bigg[\int_0^{\theta_y(v(\cdot))} & e^{-rt}\frac{1}{2}v^2(t)dt \\ & + e^{-r\theta_y(v(\cdot))}\left[\frac{\delta}{r}y - K + \frac{\delta}{r^2}\left(\alpha - \lambda\varsigma\rho - \frac{1}{2}\varsigma^2\delta\gamma(1-\rho^2)\right)\right]\Bigg] \end{aligned} \tag{6.30}$$

with the state equation

$$dY(t) = (\alpha - \lambda\varsigma\rho - \varsigma^2\delta\gamma(1-\rho^2) + \varsigma\sqrt{r\gamma(1-\rho^2)}v(t))dt + \varsigma dW(t), \quad Y(0) = y < \hat{y}, \tag{6.31}$$

where $v(\cdot)$ is adapted and locally square integrable, and $\theta_y(v(\cdot))$ is the first time the process (6.31) reaches $\hat{y}$.

As $y \to -\infty$, $\theta_y(v(\cdot)) \to \infty$ for any control $v(\cdot)$; hence,

$$g(y) \to 0 \text{ as } y \to -\infty; \quad g(y) \geq 0. \tag{6.32}$$

The positivity follows from the property $u(y) \geq -f(y) + K$. Indeed, $u(y) = -f(y) + K$ satisfies the first inequality in (6.25) as an equality.

Theorem 6.3. *The function $F(x, y)$ given by* (6.20) *coincides with the value function* (6.18).

Proof. See Appendix L. ■

We next define the optimal stopping rule, which achieves supremum in (6.18), as

$$\hat{\tau}(y) = \inf\{t | Y_y(t) \geq \hat{y}\},$$

where $\hat{y}$ is the unique value defined by the V.I. (6.26) and (6.28).

Remark 6.2.1. When $\rho^2 \to 1$, the investor's investment problem converges to the solution of the optimal stopping problem

$$g(y) = \sup_{\tau} \widehat{E}[e^{-r\tau}(V(\tau) - K)\mathbb{1}_{\tau<\infty}].$$

Proof. When $\rho^2 \to 1$, the solution of $g(y)$ in (6.24) coincides with the solution of $F(y)$ in (5.6). ■

Remark 6.2.2. When $\gamma \to 0$, the investor's investment problem converges to the solution of the optimal stopping problem under the minimal martingale measure:[12]

$$g(y) = \sup_{\tau} \widehat{E}[e^{-r\tau}(V(\tau) - K)\mathbb{1}_{\tau<\infty}].$$

The justification is similar to Remark 6.2.1.

Remark 6.2.3. In the incomplete market, the investor's option value to invest, $g(y)$, decreases with respect to the risk aversion parameter.

Proof. By (6.24), it is obvious that $g(y)$ is decreasing with respect to γ. ■

[12]The unique measure makes the market asset price a martingale while preserving the conditional of the nontraded factor.

6.3. Two players: A Stackelberg game. We consider the situation of section 5.2 in the case of incomplete markets. The follower's investment problem is similar to that of the single player. After the leader's entry, he/she makes the optimal stopping decision. By paying an investment cost K at time τ, the follower receives a continuous cash flow stream, $\delta_2 Y(t)$ per unit time. Like the single player, the follower's problem is to maximize the expected discounted utility from consumption by choosing stopping time τ, consumption rate C, and investment strategy π. Thus, the follower's strategy is identical to that described in section 6.2 with δ becoming δ_2. We have the follower's value function as defined by (6.20) with $g(y)$ satisfying (6.24), where $\hat{y}$ is the unique value defined by the V.I. (6.25) and (6.28). We take $\delta = \delta_2$.

The optimal stopping strategy for the follower is

$$\hat{\tau}(y) = \inf\{t | Y_y(t) \geq \hat{y}\},$$

where $\hat{y}$ is the unique value defined by the V.I. (6.25) and (6.28). Note again that we must take $\delta = \delta_2$ and thus $\hat{y} \geq \frac{rK}{\delta_2}$. The stopping time $\hat{\tau}(y)$ is the optimal entry if the follower can enter the market at time zero. Since the follower enters after the leader (who starts at θ), for finite θ, the follower will enter at time

$$\hat{\tau}_\theta = \theta + \hat{\tau}\big(Y_y(\theta)\big) \tag{6.33}$$

with considerations similar to those for (3.9).

6.3.1. The leader's problem. By paying cost K, the leader expects to receive a continuous cash flow $\delta_1 Y(t)$ per unit time prior to the follower's entry and $\delta_2 Y(t)$ per unit time after the follower's entry. The leader's wealth evolution is similar to (6.3), but it is complicated by the fact that the follower will enter according to the optimal stopping rule $\hat{\tau}_\theta$. Thus, the wealth evolves according to three regimes, X^0 (cf. (6.4)), X^1 (cf. (6.34)), and X^2 (cf. (6.34)), which correspond to (1) before the leader's stopping, θ, (2) after the leader's stopping but prior to the follower's optimal entry, and (3) after the follower's optimal entry, respectively.

The leader's problem is to maximize the expected discounted utility from consumption by choosing stopping time θ, consumption rate C, and investment strategy π. Again, as described in section 6.1, we solve the utility maximization problem in two steps: post- and preinvestment utility maximization. The leader's postinvestment utility maximization is complicated by the fact that, upon the follower's optimal entry, the leader's stochastic income stream will be changed from $\delta_1 Y(t)$ per unit time to $\delta_2 Y(t)$ per unit time. Consequently, the leader must solve the postinvestment utility maximization problem with respect to consumption rules and investment strategies under two stochastic income streams. As in the single player's preinvestment utility maximization, the obstacle for the preinvestment utility maximization is obtained from the solution of postinvestment utility maximization.

6.3.2. Leader's postinvestment utility maximization. As in section 6.2.1, we begin with a control problem assuming the capital investment project has been undertaken. We solve the investor's utility maximization problem of portfolio strategy with stochastic incomes.

A key aspect of the analysis is to define carefully what the leader receives at time θ when he/she decides to exploit the cash flow $Y(t)$. Suppose $\theta = 0$, his/her wealth is x, and the cash flow $y > 0$. The wealth becomes $x - K$ immediately since he/she has to pay the fixed cost of entry. The follower will enter at $\hat{\tau}(y)$. We then have the following evolution of wealth:

(6.34)
$$\begin{cases} dX^1(t) = \pi(t)X^1(t)\sigma\big(\lambda dt + dW(t)\big) + rX^1(t)dt + \delta_1 Y(t)dt - C(t)dt,\ 0 < t < \hat{\tau}(y),\\ X^1(0) = x,\\ dX^2(t) = \pi(t)X^2(t)\sigma\big(\lambda dt + dW(t)\big) + rX^2(t)dt + \delta_2 Y(t)dt - C(t)dt,\ t > \hat{\tau}(y),\\ X^2\big(\hat{\tau}(y)\big) = X^1\big(\hat{\tau}(y)\big),\\ dY(t) = \alpha dt + \varsigma\big(\rho dW(t) + \sqrt{1-\rho^2}dW^0(t)\big), \quad Y(0) = y. \end{cases}$$

If $\theta = 0$ and $y \geq \hat{y}$, the follower enters at time 0 and the leader's problem is exactly the same as the follower's, namely problem (6.12) with $\delta = \delta_2$. So we consider the function

$$L^2(x,y) = -\frac{1}{r\gamma}e^{-r\gamma\left(x+f(y)\right)+1-\frac{\mu+\frac{\lambda^2}{2}}{r}} \tag{6.35}$$

with

$$f(y) = \frac{\delta_2 y}{r} + \frac{\delta_2}{r^2}\left(\alpha - \lambda\varsigma\rho - \frac{1}{2}\varsigma^2\delta_2\gamma(1-\rho^2)\right). \tag{6.36}$$

If $\theta = 0$ and $y < \hat{y}$, the leader's problem is described as follows. The wealth process and cash flow process are governed by $X^1(t)$ and $Y(t)$ defined in (6.34).

As in the single player case, to facilitate the notation, for $\mathcal{F}_t$-adapted processes $\pi(t)$ and $C(t)$, we introduce the local integrability condition I^i, $i = 0, 1$, which is defined in (6.10) and define

$$\tau_N^i = \inf\{t | \mathbb{1}_{i=1}X^1(t) + \mathbb{1}_{i=0}X^0(t) \leq -N\}, \quad i = 0, 1.$$

At the follower's entry time, $\hat{\tau}(y)$, the leader gets $L^2\big(X^1\big(\hat{\tau}(y)\big), Y\big(\hat{\tau}(y)\big)\big)$ (cf. (6.35)) as shown in the above case where both market players are in the market. With a pair $(C(\cdot), \pi(\cdot))$ we associate the objective function

$$J_{x,y}\big(C(\cdot), \pi(\cdot)\big) = E\left[\int_0^{\hat{\tau}(y)} e^{-\mu t}C(t)d(t) + L^2\big(X^1\big(\hat{\tau}(y)\big), Y\big(\hat{\tau}(y)\big)\big)e^{-\mu\hat{\tau}(y)}\right], \tag{6.37}$$

where we recall $\hat{\tau}(y) < \infty$ a.s. We consider the value function

$$L^1(x,y) = \sup_{\{\pi(\cdot), C(\cdot)\} \in \mathcal{U}^1_{x,y}} J_{x,y}\big(C(\cdot), \pi(\cdot)\big), \tag{6.38}$$

where $\mathcal{U}^1_{x,y} = \{(\pi, C) : I^1;\ \tau^* = \lim \uparrow \tau_N^1 \geq \hat{\tau}(y) \text{ a.s.}\}$. We associate the value function with the Bellman equation

$$\begin{cases} -\mu L^1 + \frac{\partial L^1}{\partial x}(rx + \delta_1 y) + \frac{\partial L^1}{\partial y}\alpha + \frac{1}{2}\frac{\partial^2 L^1}{\partial y^2}\varsigma^2 + \sup_C\big(U(C) - C\frac{\partial L^1}{\partial x}\big)\\ \qquad + \sup_\pi\big[\pi x\sigma\big(\lambda\frac{\partial L^1}{\partial x} + \varsigma\rho\frac{\partial^2 L^1}{\partial x \partial y}\big) + \frac{1}{2}\pi^2 x^2\sigma^2\frac{\partial^2 L^1}{\partial x^2}\big] = 0,\\ L^1(x,\hat{y}) = L^2(x,\hat{y}), \qquad y < \hat{y}. \end{cases} \tag{6.39}$$

Define the feedbacks

$$\begin{cases} \widehat{C}(x,y) = -\dfrac{1}{\gamma}\ln\dfrac{\partial L^1}{\partial x}, \\ \widehat{\pi}(x,y)\sigma x = -\dfrac{\lambda + \frac{\partial L^1}{\partial x} + \varsigma\rho\frac{\partial^2 F^1}{\partial x\partial y}}{\frac{\partial^2 L^1}{\partial x^2}}. \end{cases} \tag{6.40}$$

Substituting (6.40) into (6.39), we can rewrite (6.39) as

$$\begin{aligned} -\mu L^1 + \frac{\partial L^1}{\partial x}\left(rx + \delta_1 y - \frac{1}{\gamma} + \frac{1}{\gamma}\ln\frac{\partial L^1}{\partial x}\right) + \frac{\partial L^1}{\partial y}\alpha + \frac{1}{2}\frac{\partial^1 L^1}{\partial y^2}\varsigma^2 \\ - \frac{1}{2}\frac{\left(\lambda\frac{\partial L^1}{\partial x} + \varsigma\rho\frac{\partial^2 L^1}{\partial x\partial y}\right)^1}{\frac{\partial^2 L^1}{\partial x^2}} = 0, \qquad y < \hat{y}, \end{aligned} \tag{6.41}$$

and we look for a solution

$$L^1(x,y) = -\frac{1}{r\gamma}e^{-r\gamma\left(x+g(y)\right)+1-\frac{\mu+\frac{\lambda^2}{2}}{r}} \tag{6.42}$$

with g the solution of

$$\begin{cases} -\frac{1}{2}\varsigma^2 g'' - (\alpha - \varsigma\rho)g' + \frac{1}{2}r^2\gamma^2\varsigma^2(1-\rho^2)g'^2 + rg = \delta_1 y, \qquad y < \hat{y}, \\ g(\hat{y}) = f(\hat{y}); \qquad\qquad g \text{ has linear growth at } -\infty. \end{cases} \tag{6.43}$$

Considering the difference $m = g - f$, we rewrite (6.43) as

$$\begin{cases} -\frac{1}{2}\varsigma^2 m'' - m'\left(\alpha - \lambda\varsigma\rho - \varsigma^2\delta_2\gamma(1-\rho^2)\right) + \frac{1}{2}\varsigma^2 r\gamma(1-\rho^2)m'^2 + rm = (\delta_1 - \delta_2)y, \\ m(\hat{y}) = 0; \quad m \text{ has linear growth at } y \to -\infty. \end{cases} \tag{6.44}$$

We look for a solution of (6.44) in the interval

$$\frac{\delta_1 - \delta_2}{r}(y - y^*) \le m(y) \le \frac{\delta_1 - \delta_2}{r}\left[y - y_0 + (y_0 - \hat{y})e^{\beta(y-\hat{y})}\right] \quad \text{for } y < \hat{y}, \tag{6.45}$$

where $\beta > 0$ is the solution of

$$-\frac{1}{2}\varsigma^2\beta^2 - \beta\left(\alpha - \lambda\varsigma\rho - \gamma\delta\varsigma^2(1-\rho^2)\right) + r = 0,$$

and $y_0 = -\frac{\alpha - \lambda\varsigma\rho - \delta_2\gamma\varsigma^2(1-\rho^2)}{r}$ with $f(y_0 + \frac{\delta_2^2\gamma\varsigma^2(1-\rho^2)}{r^2}) = 0$, and we take

$$y^* = \max\left(\hat{y}, y_0 + \frac{\gamma\varsigma^2(1-\rho^2)(\delta_1 - \delta_2)}{2r}\right). \tag{6.46}$$

Similar to Theorem 6.2, there exists one and only one solution of (6.44) in the interval (6.45), which is C^1 and piecewise C^2. We thus have proven there exists a unique $g(y) \in C^1$ and piecewise C^2. As a result, we prove the existence and uniqueness solution of the Bellman equation (6.39). It remains to prove the form of solution defined by (6.42) is indeed the value function defined by (6.38).

Theorem 6.4. *The function $L^1(x,y)$ defined by (6.42) coincides with the value function given in (6.38).*

Proof. See Appendix M. ■

6.3.3. Leader's preinvestment utility maximization. After obtaining the solution to the leader's postinvestment utility maximization in section 6.3.2, we can solve the full problem by making use of the value function $L^1(x,y)$. We turn to the leader's optimal stopping problem (i.e., choice of θ). Before the stopping time θ, wealth and cash flow evolve as (6.4) and (6.1).

At time θ, the leader stops and receives $L^1(X^0(\theta)-K, Y(\theta))$; the objective of the leader is to maximize

$$J_{x,y}\big(C(\cdot),\pi(\cdot),\theta\big) = E\left[\int_0^\theta U\big(C(t)\big)e^{-\mu t}dt + L^1\big(X^0(\theta)-K, Y(\theta)\big)e^{-\mu\theta}\right] \tag{6.47}$$

and we define the value function

$$L(x,y) = \sup_{\{\pi(\cdot),C(\cdot),\theta\}\in\mathcal{U}^0_{x,y}} J_{x,y}\big(C(\cdot),\pi(\cdot),\theta\big), \tag{6.48}$$

where $\mathcal{U}^0_{x,y} = \{(\pi, C, \theta) : I^0;\, \theta < \infty \text{ a.s.};\, \tau^* = \lim \uparrow \tau^0_N \geq \theta \text{ a.s.}\}$. As a consequence of dynamic programming, we write the V.I. in the strong sense that the value function $L(x,y)$ must satisfy

$$\begin{cases} -\mu L + rx\frac{\partial L}{\partial x} + \alpha\frac{\partial L}{\partial y} + \frac{1}{2}\varsigma^2\frac{\partial^2 L}{\partial y^2} + \sup_C\big(U(C) - C\frac{\partial L}{\partial x}\big) \\ \qquad\qquad + \sup_\pi\big[\pi\sigma x\big(\lambda\frac{\partial L}{\partial x} + \varsigma\rho\frac{\partial^2 L}{\partial x\partial y}\big) + \frac{1}{2}\pi^2x^2\sigma^2\frac{\partial^2 L}{\partial x^2}\big] \leq 0, \\ L(x,y) \geq L^1(x-K,y), \\ \big(L(x,y) - L^1(x-K,y)\big)\Big[-\mu L + rx\frac{\partial L}{\partial x} + \alpha\frac{\partial L}{\partial y} + \frac{1}{2}\varsigma^2\frac{\partial^2 L}{\partial y^2} + \sup_c\big(U(C) - C\frac{\partial L}{\partial x}\big) \\ \qquad\qquad + \sup_\pi\big[\pi\sigma x\big(\lambda\frac{\partial L}{\partial x} + \varsigma\rho\frac{\partial^2 L}{\partial x\partial y}\big) + \frac{1}{2}\pi^2x^2\sigma^2\frac{\partial^2 L}{\partial x^2}\big]\Big] = 0. \end{cases} \tag{6.49}$$

We look for a solution

$$L(x,y) = -\frac{1}{r\gamma}e^{-r\gamma\left(x+h(y)\right)+1-\frac{\mu+\frac{\lambda^2}{2}}{r}} \tag{6.50}$$

and obtain that $h(y)$ must satisfy the V.I.

$$\begin{cases} \frac{1}{2}\varsigma^2h'' + (\alpha - \lambda\varsigma\rho)h' - \frac{1}{2}\varsigma^2r\gamma(1-\rho^2)h'^2 - rh \leq 0, \\ h(y) \geq g(y) - K, \\ \big(h(y) - g(y) + K\big)\big[\frac{1}{2}\varsigma^2h'' + (\alpha - \lambda\varsigma\rho)h' - \frac{1}{2}\varsigma^2r\gamma(1-\rho^2)h'^2 - rh\big] = 0. \end{cases} \tag{6.51}$$

We meet with the classical difficulty that the obstacle $g(y) - K$ is C^0 but not C^1, so that the V.I. (6.51) must be interpreted in a weak sense. We cannot as in (6.25) consider $u(y) = h(y) - g(y) + K$ since $g(y)$ is not sufficiently smooth. We will nonetheless consider the function

$$u(y) = h(y) - f(y) + K,$$

which satisfies

$$\begin{cases} -\frac{1}{2}\varsigma^2u'' - \big(\alpha - \lambda\varsigma\rho - \varsigma^2\delta_2\gamma(1-\rho^2)\big)u' + \frac{1}{2}\varsigma^2r\gamma(1-\rho^2)u'^2 + ru \geq -\delta_2y + rK, \\ u \geq m, \\ (u-m)\big[-\frac{1}{2}\varsigma^2u'' - \big(\alpha - \lambda\varsigma\rho - \varsigma^2\delta_2\gamma(1-\rho^2)\big)u' + \frac{1}{2}\varsigma^2r\gamma(1-\rho^2)u'^2 \\ \qquad\qquad + ru + \delta_2y - rK\big] = 0. \end{cases} \tag{6.52}$$

The function $m(y) = g(y) - f(y)$ is defined by (6.44), which can be interpreted as, after the leader has entered, the difference between the leader's project value in anticipation of the follower's (optimal) entry and the leader's project value after the follower's entry. We see at once that if $m \le 0$, then u coincides with the solution of (6.26). The leader and the follower have the same strategy since the leader has no advantage in anticipation. So we will consider the case

$$m \text{ is not always negative.} \tag{6.53}$$

In that case, m is positive near $\hat{y}$. Indeed, otherwise, we consider the first point $y^* < \hat{y}$ such that $m(y^*) = 0$. Necessarily, $y^* < 0$. Otherwise, m would have a negative local minimum in the interval $(0, \hat{y})$. This is impossible from the maximum principle since the right-hand side of (6.44) is positive if $y > 0$. So we may assume

$$m'(\hat{y} - 0) < 0. \tag{6.54}$$

Since $u(y) = -f(y) + K$ satisfies the first inequality in (6.52) as an equality, we have

$$u(y) \ge -f(y) + K, \tag{6.55}$$

which is the same constraint as for the follower; thus $h(y) \ge 0$.

We look for a solution of (6.52) in an interval $0 \le u(y) \le \bar{u}(y)$, where $\bar{u}(y)$ will be a ceiling function which is C^1 and vanishes for y sufficiently large. We consider the ceiling function

$$\bar{u}(y) = \frac{\delta_2}{r}\left[-(y - k\hat{y}) + \frac{e^{\beta(y-k\hat{y})} - 1}{\beta}\right], \tag{6.56}$$

where k is sufficiently large. This function is such that $\bar{u}(k\hat{y}) = \bar{u}'(k\hat{y}) = 0$. We extend it by 0 for $y > k\hat{y}$. Also, $\bar{u}(y) > 0$ for $y < k\hat{y}$.

Theorem 6.5. *Assume* (6.53). *There exists a unique $u \in C^1(-\infty, \infty)$, piecewise C^2 solution of* (6.52). *This function vanishes for y sufficiently large. It is the value function*

$$u(y) = \inf_{v(\cdot)} \sup_{\theta} J_y\big(v(\cdot), \theta\big) = \sup_{\theta} \inf_{v(\cdot)} J_y\big(v(\cdot), \theta\big) = J_y\big(\hat{v}(\cdot), \hat{\theta}\big), \tag{6.57}$$

where

$$\begin{cases} J_y\big(v(\cdot), \theta\big) = E\big[\int_0^\theta \big(-\delta_2 Y_y(t) + rK + \frac{1}{2}v^2(t)\big)e^{-rt}dt + m\big(Y_y(\theta)\big)e^{-r\theta}\mathbb{1}_{\theta<\infty}\big], \\ v(\cdot) \in \mathcal{U}_y = \big\{\limsup_{T\to\infty}\big(-EY_y(T)e^{-rT}\big) = 0\big\}. \end{cases}$$

Moreover, $u(y) + \frac{\delta_2}{r}y$ is bounded for $y \to -\infty$, and $u \ge 0$.

Proof. See Appendix N. ∎

We now want to show that the solution to (6.52) is characterized by two intervals. The difficulty is that one cannot interpret $u(y)$ simply as the value function of a control problem. We approach the problem directly.

We must find three points y_1, y_2, y_3 such that

$$y_1 < y_2 < \hat{y} < y_3 \tag{6.58}$$

and

$$
\begin{cases} -\frac{1}{2}\varsigma^2 u''(y) - \big(\alpha - \lambda\varsigma\rho - \varsigma^2\delta_2\gamma(1-\rho^2)\big)u'(y) + \frac{1}{2}\varsigma^2 r\gamma(1-\rho^2)u'^2(y) + ru(y) \\ \qquad\qquad\qquad\qquad = -\delta_2 y + rK \qquad \text{for } y < y_1 \text{ and } y_2 < y < y_3, \\ u(y) = m(y) \quad \text{for } y_1 \le y \le y_2, \\ u(y) = 0 \quad \text{for } y \ge y_3, \\ u'(y_1) = m'(y_1); \quad u'(y_2) = m'(y_2); \quad u'(y_3) = 0. \end{cases} \tag{6.59}
$$

We can take $y_3 = \bar{y} = k\hat{y}$ with k sufficiently large. We show the existence of a unique two-interval solution in the following theorem.

Theorem 6.6. *Assume* (6.53). *Then the solution u of* (6.52) *is of the form* (6.59). *There exists a unique triple y_1, y_2, y_3 with $y_1 < y_2 < \hat{y} < y_3$ such that* (6.59) *holds.*

Proof. See Appendix O. ■

By Theorem 6.6, for the process $Y(t)$ given in (6.1), we can define the optimal stopping rule as

$$
\hat{\theta}(y) = \begin{cases} \inf\{t | Y_y(t) \ge y_1\} \quad \text{if } y < y_1, \\ 0 \quad \text{if } y_1 \le y \le y_2, \\ \inf\{t | Y_y(t) \le y_2 \text{ or } Y_y(t) \ge y_3\} \quad \text{if } y_2 < y < y_3, \\ 0 \quad \text{if } y \ge y_3. \end{cases} \tag{6.60}
$$

It remains to show that the function $L(x, y)$ defined by (6.50) is the value function (6.48).

Theorem 6.7. *Assume* (6.53). *Then the function $L(x, y)$ defined by* (6.50) *is the value function* (6.48).

Proof. The feedbacks

$$
\begin{cases} \widehat{C}(x, y) = r\big(x + h(y)\big) - \frac{1}{\gamma}\Big(1 - \frac{\mu + \frac{\lambda^2}{2}}{r}\Big), \\ \widehat{\pi}(x, y)\sigma x = \frac{\lambda}{r\gamma} - \varsigma\rho h'(y) \end{cases} \tag{6.61}
$$

and the stopping time (6.60) provide the optimal control. The proof is similar to that of Theorem 6.3. ■

6.3.4. Optimal rules in the case of incomplete markets. We prove that the single player and the follower have a unique optimal stopping strategy characterized by a threshold. The leader's optimal stopping strategy is characterized by a two-interval solution. The optimal stopping rules for the leader and the follower can be characterized as those described in the complete market section. In the single decision maker model, we are able to recover complete market results by allowing $|\rho|$ (the correlation between the traded and the nontraded assets) $\to 1$. And when γ (the risk aversion coefficient) $\to 0$, the problem becomes one of optimal stopping under a minimal martingale measure. The option value to invest is decreasing in γ. Market completeness and risk aversion are essential factors for evaluating optimal investment policies. For managers, we caution that naively assuming market completeness may lead to nonoptimal investment decisions.

7. Conclusion. We study optimal investment policies in an irreversible capital investment under uncertainty in a monopoly situation and in a Stackelberg leader-follower game. We consider two types of investment payoffs: lump-sum and cash flows. The decisions are the times to enter into the market. We employ V.I.s for solving these optimal stopping time problems.

In the case of complete markets, we work with the risk-neutral probability measure for the valuation. In the case of incomplete markets, we adopt the utility-based valuation where the investor solves the utility maximization with joint decisions of stopping times, portfolio investment, and/or consumption rules.

For the case of lump-sum payoffs, we proved that the single player and consequently the follower have a unique optimal stopping strategy characterized by a threshold, both for the complete and incomplete capital markets. In the complete market, the leader's optimal investment rule is characterized by a two-interval strategy. In the incomplete market, we need the weak formulation of the leader's problem and can characterize the optimal stopping only as the first time the solution touches the obstacle. For the case of flow payoffs, we prove that the single player and the follower have a unique optimal stopping strategy characterized by a threshold, and that the leader's optimal stopping strategy can be characterized by the two-interval solution in both the complete and the incomplete markets.

In both payoff situations, we find that differences in the investment stopping time depend on the degree of completeness in the market and conclude that market completeness and risk aversion are important inputs into the optimal investment policy. In addition, optimal investment policies for both the leader and the follower deviate from those of the single decision maker. Strategic considerations are shown to be important in the formulation of capital investment project decisions. Use of classical real option rules for a single decision maker will lead to nonoptimal investment.

Appendix A. Proof of Lemma 3.2. We take $\tilde{U} = K - (U-K)^-$ as a test function in (3.29). Indeed, $\tilde{U} \in \mathcal{K}$ so that

$$-b(U,(U-K)^+) - \alpha(U,(U-K)^+)_\varrho \geq -(f,(U-K)^+)_\varrho - \alpha(G,(U-K)^+)_\varrho.$$

Hence,

$$\begin{aligned}-b((U-K)^+,(U-K)^+) - \alpha|(U-K)^+|_\varrho^2 &\geq -(f-rK,(U-K)^+)_\varrho \\ &\quad - \alpha(G-K,(U-K)^+)_\varrho \\ &\geq 0,\end{aligned}$$

which implies $(U-K)^+ = 0$. That is, $\Gamma_\alpha(G) \leq K$.

The property $\Gamma_\alpha(G) \geq 0$ is obvious since $\Gamma_\alpha(G) \in \mathcal{K}$; hence $\Gamma_\alpha(G) \geq \chi \geq 0$.

Appendix B. Proof of Lemma 3.3. To simplify notation, let $U_1 = \Gamma_\alpha(G_1)$, and let $U_2 = \Gamma_\alpha(G_2)$. By definition,

$$\text{(B.1)}\qquad \begin{cases} b(U_1,\tilde{U}-U_1) + \alpha(U_1,\tilde{U}-U_1)_\varrho \geq (f,\tilde{U}-U_1)_\varrho + \alpha(G_1,\tilde{U}-U_1)_\varrho, \\ b(U_2,\tilde{U}-U_2) + \alpha(U_2,\tilde{U}-U_2)_\varrho \geq (f,\tilde{U}-U_2)_\varrho + \alpha(G_2,\tilde{U}-U_2)_\varrho. \end{cases}$$

Set $M = \frac{\alpha\|G_1 - G_2\|_{L^\infty}}{\alpha + r}$.

We take $\tilde{U} = U_1 - (U_1 - U_2 - M)^+$ in the first relation (B.1) and $\tilde{U} = U_2 + (U_1 - U_2 - M)^+$ in the second relation (B.1). Adding, we deduce

$$\begin{aligned} -b(U_1 - U_2, (U_1 - U_2 - M)^+) - \alpha(U_1 - U_2, (U_1 - U_2 - M)^+)_\varrho \\ \geq -\alpha(G_1 - G_2, (U_1 - U_2 - M)^+)_\varrho; \end{aligned}$$

hence,

$$\begin{aligned} -b(U_1 - U_2, (U_1 - U_2 - M)^+) - \alpha(U_1 - U_2, (U_1 - U_2 - M)^+)_\varrho \\ - (r + \alpha)(M, (U_1 - U_2 - M)^+) \\ \geq -\alpha(G_1 - G_2, (U_1 - U_2 - M)^+)_\varrho. \end{aligned}$$

Since $(r + \alpha)M \geq \alpha(G_1, G_2)$, we have

$$b((U_1 - U_2 - M)^+, (U_1 - U_2 - M)^+) + \alpha|(U_1 - U_2 - M)^+|^2 \leq 0.$$

Therefore, $U_1 - U_2 \leq M$. Similarly, $U_2 - U_1 \leq M$, and we conclude that (3.32) holds.

Appendix C. Proof of Theorem 3.5. We can approximate $\chi(v)$ by a sequence of smooth functions $\chi^\epsilon(v)$ such that

$$\sup_v |\chi^\epsilon(v) - \chi(v)| \to 0 \quad \text{as } \epsilon \to 0.$$

In the present context, such an approximation is easily obtained because $\chi(v)$ is smooth except in $\hat{v}$. As a result, the solution U is more regular than $H^1_\varrho(0, \infty)$. The solution is locally H^2, and the V.I. can be written as in (3.5), calling U^ϵ the solution of the V.I. (3.28) with χ^ϵ instead of χ; we have

$$\begin{cases} v(r + \eta(\xi - \lambda))U^{\epsilon\prime}(v) + \frac{1}{2}U^{\epsilon\prime\prime}(v)\eta^2 v^2 - rU^\epsilon(v) + f(v) \leq 0, \\ U^\epsilon(v) \geq \chi^\epsilon(v), \\ \left[U^\epsilon(v) - \chi^\epsilon(v)\right]\left[v(r + \eta(\xi - \lambda))U^{\epsilon\prime}(v) + \frac{1}{2}U^{\epsilon\prime\prime}(v)\eta^2 v^2 - rU^\epsilon(v) + f(v)\right] = 0, \\ U^\epsilon(0) = K. \end{cases}$$

It is standard to check that

$$U^\epsilon(v) = \sup_{\theta \geq 0} \widehat{E}\left[e^{-r\theta}\chi^\epsilon\big(V_v(\theta)\big)\mathbb{1}_{\theta<\infty} + \int_0^\theta e^{-rs} f\big(V_v(s)\big)ds\right]$$

and, clearly,

$$\sup_v |U^\epsilon(v) - U(v)| \leq \sup_v |\chi^\epsilon(v) - \chi(v)|.$$

On the other hand, U^ϵ remains in a bounded subset of $H^1_\varrho(0, \infty)$, and we can extract a subsequence converging to $U(v)$, a solution of (3.28), in $H^1_\varrho(0, \infty)$ weakly. From the uniqueness of the solution of (3.28), the sequence converges, and thus the solution of (3.28) coincides with (3.19).

Appendix D. Proof of Theorem 3.6. Since $U(0) = K > \chi(0) = 0$ and $U(\bar v) = \chi(\bar v) = 0$, there exists a first point $v_1 \le \bar v$ such that $U(v_1) = \chi(v_1)$. We must have $v_1 < \hat v$. Otherwise, $v_1 = \bar v$. But, from (3.5), we have

$$\begin{cases} F'(v)v\big(r+\eta(\xi-\lambda)\big) + \frac{1}{2}F''(v)v^2\eta^2 - rF(v) \le 0, \\ F(v) \ge (1-a)v - K, \\ \big[F(v) - (1-a)v + K\big]\big[F'(v)v\big(r+\eta(\xi-\lambda)\big) + \frac{1}{2}F''(v)v^2\eta^2 - rF(v)\big] = 0, \\ F(0) = 0, \end{cases}$$

which, by setting $U(v) = F(v) - (1-a)v + K$, may be re-expressed as

(D.1)
$$\begin{cases} -\frac{1}{2}U''(v)v^2\eta^2 - U'(v)v\big(r+\eta(\xi-\lambda)\big) + rU(v) \ge (1-a)v\eta(\xi-\lambda) + rK, \\ U(v) \ge 0, \\ U(v)\big[-\frac{1}{2}U''(v)v^2\eta^2 - U'(v)v\big(r+\eta(\xi-\lambda)\big) + rU(v) - (1-a)v\eta(\xi-\lambda) - rK\big] = 0, \\ U(0) = K. \end{cases}$$

We then observe that $U(v)$ coincides with the solution of (D.1) (with $(1-a)$ replaced by a), i.e., the same system of (3.34) with $\chi(v) = 0$. But then $\bar v = \hat v$; hence $v_1 = \hat v$. In this case $\widetilde U(v) = U(v) - \chi(v)$ satisfies

$$\text{(D.2)} \qquad \begin{cases} -\frac{1}{2}\widetilde U''(v)v^2\eta^2 - \widetilde U'(v)v\big(r+\eta(\xi-\lambda)\big) + r\widetilde U(v) = v\eta(\xi-\lambda) + rK, \\ \widetilde U(0) = K; \quad \widetilde U(\hat v) = 0. \end{cases}$$

Since $U'(\hat v) = 0$, it follows that $\widetilde U'(\hat v - 0) = -\chi'(\hat v - 0)$. This in turn implies $\widetilde U'(\hat v - 0) > 0$ because $\chi'(\hat v - 0) < 0$. It then follows that $\widetilde U(v) < 0$ for v close to $\hat v$, which is impossible. Therefore $v_1 \le \hat v$. Indeed, setting $\widetilde U(v) = U(v) - \chi(v)$, it satisfies

$$-\frac{1}{2}\widetilde U''(v)v^2\eta^2 - \widetilde U'(v)v\big(r+\eta(\xi-\lambda)\big) + r\widetilde U(v) = v\eta(\xi-\lambda) + rK$$

with $\widetilde U(0) = K$, $\widetilde U(v_1) = 0$, $\widetilde U'(v_1) = 0$. The matching of the derivative comes from the fact that $\widetilde U(v)$ is C^1 and $\widetilde U(v) \ge 0$, $\widetilde U(v_1) = 0$. So v_1 is the local minimum; hence $\widetilde U'(v_1) = 0$, and we have $\widetilde U'(v_1) = \chi'(v_1)$.

Since $U(\hat v) > \chi(\hat v) = 0$, there exists an interval in which $\hat v$ is contained such that the equation holds in this interval, denoted as (v_2, v_3) with $v_3 = \bar v$ and $v_2 < \hat v$. Necessarily, we have $v_2 \ge v_1$; otherwise, $U(v)$ will be the solution of the equation on $(0, v_3)$, which is impossible. So we have $U(v_2) = \chi(v_2)$ and necessarily $U'(v_2) = \chi'(v_2)$.

On the other hand, on the interval (v_1, v_2), $\chi(v)$ satisfies all conditions (3.35) and the right-hand side $-(1-a)\eta(\xi-\lambda)v > a\eta(\xi-\lambda)v + rK$; therefore, $\chi(v)$ satisfies all conditions (3.34) on (v_1, v_2). Therefore $U(v) = \chi(v)$ on (v_1, v_2).

By the uniqueness of $U(v)$, the triple v_1, v_2, v_3 is necessarily unique.

Appendix E. Proof of Lemma 4.2. From the relation $-\gamma(1-\rho^2)H^e(v) = \ln(1+Bv^\beta)$, we obtain for $v < \hat{v}$

$$\big(H^e(v)\big)' = \frac{-\beta B v^{\beta-1}}{\gamma(1-\rho^2)(1+Bv^\beta)},$$
$$\big(H^e(v)\big)'' = \frac{-\beta B}{\gamma(1-\rho^2)} \frac{(\beta-1)Bv^{\beta-2} - Bv^{2\beta-2}}{(1+Bv^\beta)^2} > 0.$$

So $\big(H^e(v)\big)'$ increases from 0 to

$$\big(H^e\big)'(\hat{v}-0) = (1-a)\frac{e^{\varpi+K\gamma(1-\rho^2)}-1}{e^{\varpi}-1} > (1-a) = \big(H^e\big)'(\hat{v}+0).$$

It follows that $v - K - H^e(v)$ may either be increasing from 0 to $\hat{v}$ or may pass by a positive maximum. At any rate, there is one and only one 0.

Appendix F. Proof of Theorem 4.3. We first prove that (4.42) has a set of solutions with a maximum and a minimum element. For that we consider a monotone increasing sequence and monotone decreasing sequence Σ_n and Σ^n defined as follows:

$$b(\Sigma^{n+1}, \widetilde{\Sigma} - \Sigma^{n+1}) + \alpha(\Sigma^{n+1}, \widetilde{\Sigma} - \Sigma^{n+1})_\varrho \ge \alpha(\Sigma^n, \widetilde{\Sigma} - \Sigma^{n+1})_\varrho, \tag{F.1}$$

$$b(\Sigma_{n+1}, \widetilde{\Sigma} - \Sigma_{n+1}) + \alpha(\Sigma_{n+1}, \widetilde{\Sigma} - \Sigma_{n+1})_\varrho \ge \alpha(\Sigma_n, \widetilde{\Sigma} - \Sigma_{n+1})_\varrho. \tag{F.2}$$

We start with

$$\Sigma^0 = \psi; \qquad \Sigma_0 = 0. \tag{F.3}$$

In (F.1) and (F.2) α is a sufficiently large number, which ensures that the bilinear form $b(\Sigma, \widetilde{\Sigma}) + \alpha(\Sigma, \widetilde{\Sigma})_\varrho$ is coercive. It follows that the sequences Σ^n and Σ_n are well defined. Clearly $\Sigma^1 \le \Sigma^0$. Let us check that if $\Sigma^n \le \Sigma^{n-1}$, then $\Sigma^{n+1} \le \Sigma^n$. Consider (F.1) with $n-1$ replacing n; hence

$$b(\Sigma^n, \widetilde{\Sigma} - \Sigma^n) + \alpha(\Sigma^n, \widetilde{\Sigma} - \Sigma^n)_\varrho \ge \alpha(\Sigma^{n-1}, \widetilde{\Sigma} - \Sigma^n)_\varrho. \tag{F.4}$$

We take $\widetilde{\Sigma} = \Sigma^{n+1} - (\Sigma^{n+1} - \Sigma^n)^+ = \min(\Sigma^n, \Sigma^{n+1})$ in (F.1) and $\widetilde{\Sigma} = \Sigma^n + (\Sigma^{n+1} - \Sigma^n)^+ = \max(\Sigma^n, \Sigma^{n+1})$ in (F.4). We get

$$-b\big(\Sigma^{n+1}, (\Sigma^{n+1} - \Sigma^n)^+\big) - \alpha\big(\Sigma^{n+1}, (\Sigma^{n+1} - \Sigma^n)^+\big)_\varrho \ge -\alpha\big(\Sigma^n, (\Sigma^{n+1} - \Sigma^n)^+\big)_\varrho,$$
$$b\big(\Sigma^n, (\Sigma^{n+1} - \Sigma^n)^+\big) + \alpha\big(\Sigma^n, (\Sigma^{n+1} - \Sigma^n)^+\big)_\varrho \ge \alpha\big(\Sigma^{n-1}, (\Sigma^{n+1} - \Sigma^n)^+\big)_\varrho.$$

Adding, we easily deduce that $(\Sigma^{n+1} - \Sigma^n)^+ = 0$. As $n \to \infty$, we obtain that $\Sigma^n \downarrow \overline{\Sigma}$, the maximum solution of (4.42). Indeed, if Σ is a solution such that $\Sigma \le \Sigma^0$, then $\Sigma \le \Sigma^1$. It follows that $\Sigma \le \Sigma^n$; hence $\Sigma \le \overline{\Sigma}$. Similarly $\Sigma_n \uparrow \underline{\Sigma}$, the minimum solution of (4.42).

We next prove that the maximum and the minimum solutions coincide. We first verify that the optimal stopping time $\hat{\theta}$ is finite a.s. and proceed with convergence of solutions from the smooth function construction. Suppose now $\psi \in C^1$. Then we rely on the classical result on V.I.s that the minimum and the maximum solutions coincide with the value function defined in (4.38) and (4.39). The optimal stopping time $\hat{\theta}(v)$ is defined by

$$\hat{\theta}(v) = \inf\{t|\Sigma\big(V_v(t)\big) = \psi\big(V_v(t)\big)\}. \tag{F.5}$$

We have $\hat{\theta}(v) < \infty$ a.s. Indeed, we first note that Σ is also C^1 and there exists a number $M > \hat{v}$ such that

$$\Sigma(v) = \psi(v), \quad v \geq M. \tag{F.6}$$

Otherwise, $\Sigma(v) < \psi(v)\ \forall\ v$ and the differential equation $\Sigma'(\xi - \lambda\rho) + \frac{1}{2}v\eta\Sigma'' = 0$ holds for any v. Therefore, we have $\Sigma(v) = 1 + DV^\beta$ with D a negative constant. This is not possible since $\Sigma(v) \geq 0\ \forall\ v$.

If

$$\Sigma(M) = e^{-\gamma(1-\rho^2)(aM-K)},$$
$$\Sigma'(M) = -a\gamma(1-\rho^2)e^{-\gamma(1-\rho^2)(aM-K)},$$

then for $v \geq M$, $\Sigma(v) = e^{-\gamma(1-\rho^2)(av-K)}$ satisfies conditions (4.35) since $\Sigma'(\xi-\lambda\rho)+\frac{1}{2}v\eta\Sigma'' \geq 0$ thanks to assumption $\xi - \lambda\rho < 0$. Therefore, we have (F.6). Take $v < M$ and define

$$\theta_M(v) = \inf\{t|V_v(t) = 0 \text{ or } V_v(t) \geq M\};$$

then $\theta_M(v) < \infty$ a.s. But $\hat{\theta}(v) < \theta_M(v)$ since $\Sigma\big(V_v\big(\theta_M(v)\big)\big) = \psi\big(V_v\big(\theta_M(v)\big)\big)$, and we have $\hat{\theta} < \infty$ a.s.

We next construct smooth functions $\psi_\epsilon(v)$ which approximate $\psi(v)$ from below or from above. We consider a sequence of C^1 approximations $\psi_\epsilon(v) \geq 0$ of the obstacle $\psi(v)$. Define a convex subset

$$\mathcal{K}_\epsilon = \{\phi|0 \leq \phi \leq \psi_\epsilon\}$$

and let Σ_ϵ be the unique solution of the V.I.

$$b(\Sigma_\epsilon, \widetilde{\Sigma} - \Sigma_\epsilon) \geq 0 \qquad \forall\ \widetilde{\Sigma} \in \mathcal{K}_\epsilon,\ \Sigma_\epsilon \in \mathcal{K}_\epsilon, \tag{F.7}$$

which, thanks to the regularity of ψ_ϵ, can be written as

$$\begin{cases} \Sigma_\epsilon'(\xi - \lambda\rho) + \frac{1}{2}v\eta\Sigma_\epsilon'' \geq 0, \\ \Sigma_\epsilon(v) \leq \psi_\epsilon(v), \\ \big[\Sigma_\epsilon'(\xi - \lambda\rho) + \frac{1}{2}v\eta\Sigma_\epsilon''\big]\big[\Sigma_\epsilon(v) - \psi_\epsilon(v)\big] = 0, \\ 0 \leq \Sigma_\epsilon(v) \leq 1, \\ \Sigma_\epsilon(0) = 1, \end{cases} \tag{F.8}$$

and $\Sigma_\epsilon(v)$ is the value function:

$$\Sigma_\epsilon(v) = \inf_\theta E\big[\psi_\epsilon\big(V_v(\theta)\big)\big]. \tag{F.9}$$

We begin an approximation such that

$$\psi_\epsilon(v) \uparrow \psi(v) \quad \text{as } \epsilon \downarrow 0. \tag{F.10}$$

We can define the increasing processes $\Sigma_{\epsilon,n}(v)$ corresponding to $\psi_\epsilon(v)$ and $\Sigma_{\epsilon,n}(v) \uparrow \Sigma_\epsilon(v)$ as $n \uparrow \infty$. However, if we compare $\Sigma_{\epsilon,n}(v)$ and $\Sigma_n(v)$, it is easy to check that

$$\Sigma_{\epsilon,n}(v) \leq \Sigma_n(v) \leq \underline{\Sigma}(v); \tag{F.11}$$

therefore, we obtain

$$\Sigma_\epsilon(v) \leq \underline{\Sigma}(v). \tag{F.12}$$

But from formula (F.9), $\Sigma_\epsilon(v)$ converges toward the value function. We obtain that the value function is smaller than the minimum solution $\underline{\Sigma}(v)$ of (4.42).

Next consider an approximation

$$\psi_\epsilon(v) \downarrow \psi(v) \quad \text{as } \epsilon \downarrow 0 \tag{F.13}$$

and the monotone decreasing process Σ_ϵ^n. We have $\Sigma_\epsilon^n \downarrow \Sigma_\epsilon$; therefore, this time $\Sigma_\epsilon^n \geq \Sigma_\epsilon \geq \overline{\Sigma}$, and $\Sigma_\epsilon \geq \overline{\Sigma}$. But Σ_ϵ converges again towards the value function. Therefore the minimum and maximum solutions of (4.42) coincide and are equal to the value function. This completes the proof.

Appendix G. Proof of Proposition 4.4. To prove that solutions of (4.47) exist, we consider the function

$$Z(v) = e^{\gamma(1-\rho^2)\left(v-K-H^e(v)\right)} + e^{\gamma(1-\rho^2)H^e(v)} - \left(2 + \frac{\gamma v(1-\rho^2)}{\beta}\right)$$

and note that $Z(0) = e^{-\gamma(1-\rho^2)K} - 1 < 0$.

On the other hand,

$$Z(\hat{v}) = e^{\varpi + \frac{2a-1}{1-a}\left(\varpi + \gamma K(1-\rho^2)\right)} + e^{\varpi + \gamma K(1-\rho^2)} - 2 - \frac{1}{1-a}(e^\varpi - 1).$$

Setting $u = \frac{2a-1}{1-a}$, $u \in (0,\infty)$, and considering the function

$$h(u;\varpi) = e^{\varpi + u\left(\varpi + \gamma K(1-\rho^2)\right)} + e^{\varpi + \gamma K(1-\rho^2)} - 2 - (u+2)(e^\varpi - 1),$$

one can easily check that $h(u;\varpi) > 0$. Hence, $Z(\hat{v}) > 0$. Therefore, there are values v_1 such that $Z(v_1) = 0$. It remains to verify that $v_1 > v_0$. By (4.46), we can rewrite (4.47) as

$$e^{\gamma(1-\rho^2)\left(v_1-K-H^e(v_1)\right)} = 1 + \frac{\gamma v_1(1-\rho^2)}{\beta}\left(1 - \left(H^e(v_1)\right)'\right).$$

But from the assumption and Lemma 4.2, we have $1-\left(H^e(v_1)\right)' > 0$; hence $v_1 - K - H^e(v_1) > 0$, which implies $v_1 > v_0$. It remains to define D by

$$Dv_1^\beta = e^{-\gamma(1-\rho^2)\left(v_1-K-H^e(v_1)\right)} - 1$$

to satisfy the first conditions (4.45). This completes the proof.

Appendix H. Proof of Theorem 5.1. The solution (5.9) is $C^1(-\infty,\infty)$ and piecewise C^2. It satisfies $F(y) \geq 0$ and the condition that $F(y)$ has linear growth at ∞. By construction, it satisfies the complimentary slackness condition (i.e., the product condition of (5.6)). To complete the proof, we must verify the inequalities

$$F(y) \geq \delta\left(\frac{y}{r} + \frac{\alpha - \lambda\varsigma}{r^2}\right) - K \quad \text{if } y \leq \hat{y} \tag{H.1}$$

and

$$(\alpha - \lambda\varsigma)F'(y) + \frac{1}{2}\varsigma^2 F''(y) - rF(y) \leq 0 \quad \text{if } y \geq \hat{y}. \tag{H.2}$$

To prove (H.1), consider $G(y) = F(y) - \delta(\frac{y}{r} + \frac{\alpha-\lambda\varsigma}{r^2}) + K$. We have, if $y < \hat{y}$,

$$G'(y) = \frac{\delta}{r}\left(e^{-\beta(\hat{y}-y)} - 1\right) < 0 \quad \text{and} \quad G''(y) = \frac{\delta\beta}{r}e^{-\beta(\hat{y}-y)} > 0.$$

Hence $G'(y)$ increases up to zero on $(-\infty,\hat{y})$. Therefore, $G'(y) \leq 0$ on $(-\infty,\hat{y})$ and $G(y)$ is decreasing on $(-\infty,\hat{y})$. Since $G(\hat{y}) = 0$, $G(y) \geq 0$ on $(-\infty,\hat{v})$, we have shown that (H.1) is true.

The inequality (H.2) is equivalent to $y \geq \frac{rK}{\delta}$ for $y \geq \hat{y}$, which implies $\frac{1}{\beta} > \frac{\alpha-\lambda\varsigma}{r}$. This is true because $r - \beta(\alpha - \lambda\varsigma) = \frac{1}{2}\varsigma^2\beta^2$.

Appendix I. Proof of Theorem 5.2. Before proceeding to prove the value function (5.15) is the unique solution of the V.I. (5.20), we introduce the approximation function, $\Psi_M(y)$, to overcome the problem of the unbounded obstacle, $\Psi(y)$, and define the corresponding approximated value function, $L_M(y)$. Note that

$$\Psi(y) = \frac{\delta_2}{r\beta} + \frac{\delta_2}{r}(y-\hat{y})^+ - \frac{\delta_1}{r}(y-\hat{y})^- + \mu\frac{\delta_1-\delta_2}{r}\mathbb{1}_{y<\hat{y}}\left(1 - e^{-\beta(\hat{y}-y)}\right). \tag{I.1}$$

We consider the following expression:

$$\Psi_M(y) = \frac{\delta_2}{r\beta} + \frac{\delta_2}{r}\frac{M(y-\hat{y})^+}{M+(y-\hat{y})^+} - \frac{\delta_1}{r}(y-\hat{y})^- + \mu\frac{\delta_1-\delta_2}{r}\mathbb{1}_{y<\hat{y}}\left(1 - e^{-\beta(\hat{y}-y)}\right), \tag{I.2}$$

with

$$\Psi_M(y) \uparrow \Psi(y) \quad \text{as } M \uparrow \infty. \tag{I.3}$$

In correspondence with $\Psi_M(y)$, we set

$$L_M(y) = \sup_{\theta\geq 0} \widehat{E}\left[e^{-r\theta}\Psi_M\left(Y_y(\theta)\right)\mathbb{1}_{\theta<\infty}\right]. \tag{I.4}$$

It is easy to check that

$$L_M(y) \uparrow L(y) \quad \text{as } M \uparrow \infty. \tag{I.5}$$

From (I.2), we have $\Psi_M(y) \le \frac{\delta_2}{r\beta} + \frac{\delta_2}{r}M + \frac{\mu(\delta_1-\delta_2)}{r} = \mathcal{V}_M$, and therefore we have $0 \le L_M(y) \le \mathcal{V}_M$.

We consider the convex subset of $H^1_\varrho(-\infty,\infty)$:

$$\mathcal{K}_M = \{\Phi \in H^1_\varrho(-\infty,\infty) | \Phi(y) \ge \Psi_M(y)\ \forall y\}, \tag{I.6}$$

which is not empty since it contains Ψ_M.

Lemma I.1. *There exists a unique L_M in $\mathcal{K}_M$ such that*

$$b(L_M, \tilde{L} - L_M) \ge 0 \qquad \forall \tilde{L} \in \mathcal{K}_M \tag{I.7}$$

and

$$0 \le L_M(y) \le \mathcal{V}_M. \tag{I.8}$$

Lemma I.2. *The unique solution of (I.7) and (I.8) coincides with the value function (I.4).*

Lemma I.3. *The function $L(y)$ defined by (5.15) is the solution of the V.I. (5.20) with*

$$0 \le L(y) \le \frac{\delta_2}{r}(y - \hat{y})^+ + C, \tag{I.9}$$

where C is a convenient constant.

Lemma I.4. *The set of solutions of (5.20) such that*

$$0 \le L(y) \le L^0(y) \tag{I.10}$$

has a maximum and a minimum element, where

$$L^0(y) = \frac{1}{2}y^2 + \frac{\alpha - \lambda\varsigma}{r}y + C^0, \tag{I.11}$$

with

$$C^0 > \frac{1}{2r^2}(\alpha - \lambda\varsigma)^2 + \max\left(\frac{\delta_1}{r\beta} + \frac{\delta_1 - \delta_2}{\delta_2}K, \frac{\delta_2^2}{2r^2}\right). \tag{I.12}$$

We are now ready to prove Theorem 5.2. Since the discontinuity of Ψ' occurs at one point only, we may construct smooth functions $\Psi_\epsilon(y)$ which approximate $\Psi(y)$ from below or above and which, in addition, have linear growth. Since $\Psi_\epsilon(y)$ is smooth, the corresponding solution of the V.I. (5.20) is smoother than $H^1_\varrho(-\infty,\infty)$. In fact, $\Psi''_\epsilon(y) \in L^2_\varrho(-\infty,\infty)$ and the V.I. can be written in the classical form

$$\begin{cases} (\alpha - \lambda\varsigma)L'_\epsilon(y) + \frac{1}{2}L''_\epsilon(y) - rL_\epsilon(y) \le 0, \\ L_\epsilon(y) \ge \Psi_\epsilon(y), \\ \big(L_\epsilon(y) - \Psi_\epsilon(y)\big)\left[(\alpha - \lambda\varsigma)L'_\epsilon(y) + \frac{1}{2}L''_\epsilon(y) - rL_\epsilon(y)\right] = 0, \\ L_\epsilon(y) \ge 0;\ L_\epsilon(y) \text{ has linear growth.} \end{cases}$$

However, it is classical that smooth solutions are unique and coincide with the value function

$$L_\epsilon(y) = \sup_{\theta \ge 0} \widehat{E}[e^{-r\theta}\Psi_\epsilon(Y_y(\theta))\mathbb{1}_{\theta<\infty}]. \tag{I.13}$$

Consider now the case when

$$\Psi_\epsilon(y) \uparrow \Psi(y) \quad \text{as } \epsilon \downarrow 0. \tag{I.14}$$

We may define the increasing process $L_{\epsilon,n}(y)$ as defined in Lemma I.4, and naturally $L_{\epsilon,n}(y) \uparrow L_\epsilon(y)$ as $n \uparrow \infty$. However, if we compare $L_{\epsilon,n}(y)$ and $L_n(y)$, one may verify that

$$L_{\epsilon,n}(y) \le L_n(y) \le \underline{L}(y). \tag{I.15}$$

We therefore obtain

$$L_\epsilon(y) \le \underline{L}(y). \tag{I.16}$$

As $\epsilon \downarrow 0$ $L_\epsilon(y) \uparrow$ toward a solution of (5.20) and from (I.16) it is necessarily the smallest solution. From formula (I.13), one may verify that $L_\epsilon(y) \uparrow$ toward the value function (5.15). So the value function coincides with the minimum solution. Consider now an approximation such that

$$\Psi_\epsilon(y) \downarrow \Psi(y) \quad \text{as } \epsilon \downarrow 0. \tag{I.17}$$

We prove in a similar manner that $L_\epsilon(y) \downarrow \overline{L}(y)$, the maximum solution. Therefore, the minimum and maximum solutions coincide. We have shown that the value function (5.15) is the unique solution of (5.20). The property (I.9) is a consequence of the properties of the value function.

Appendix J. Proof of Theorem 6.1. We compute the feedbacks

$$\begin{cases} \widehat{C}(x,y) = rx + \delta y - \dfrac{1}{\gamma}\left(1 - \dfrac{\mu + \frac{\lambda^2}{2}}{r}\right) + \dfrac{\delta}{r}\left(\alpha - \lambda\varsigma\rho - \dfrac{1}{2}\varsigma^2\delta\gamma(1-\rho^2)\right), \\ \widehat{\pi}(x,y)x\sigma = \dfrac{\lambda}{\gamma r} - \dfrac{\varsigma\rho\delta}{r} \end{cases} \tag{J.1}$$

and note that $U\big(\widehat{C}(x,y)\big) = rF^1(x,y)$. We also have

$$\widehat{\pi}(x,y)x\sigma\lambda + rx - \widehat{C}(x,y) + \delta y = \frac{1}{\gamma}\left(1 - \frac{\mu - \frac{\lambda^2}{2}}{r}\right) - \frac{\delta}{r}\left(\alpha - \frac{1}{2}\varsigma^2\delta\gamma(1-\rho^2)\right).$$

So the corresponding wealth is given by

$$d\widehat{X}^1 = \left[\frac{1}{\gamma}\left(1 - \frac{\mu - \frac{\lambda^2}{2}}{r}\right) - \frac{\delta}{r}\left(\alpha - \frac{1}{2}\varsigma^2\delta\gamma(1-\rho^2)\right)\right]dt + \left(\frac{\lambda}{r\gamma} - \frac{\varsigma\rho\delta}{r}\right)dW. \tag{J.2}$$

We also note

$$\begin{cases} \dfrac{\partial F^1}{\partial x}\widehat{\pi}(x,y)x\sigma + \dfrac{\partial F^1}{\partial y}\varsigma\rho = -\lambda F^1(x,y), \\ \dfrac{\partial F^1}{\partial y}\varsigma\sqrt{1-\rho^2} = -\gamma\delta\varsigma\sqrt{1-\rho^2}F^1(x,y). \end{cases} \tag{J.3}$$

Using the Itô differential together with the above calculation, we have

$$d\Big(F^1\big(\widehat{X}^1(t),Y(t)\big)e^{-\mu t}\Big) = -re^{-\mu t}F^1\big(\widehat{X}^1(t),Y(t)\big)dt - e^{-\mu t}F^1\big(\widehat{X}^1(t),Y(t)\big)\big[\lambda dW(t)+\gamma\delta\varsigma\sqrt{1-\rho^2}dW^0(t)\big].$$

Integrating between 0 and $T\wedge\hat{\tau}_N$, we can write

$$\text{(J.4)}\qquad E\big[F^1\big(\widehat{X}^1(T\wedge\hat{\tau}_N),Y(T\wedge\hat{\tau}_N)\big)e^{-\mu T\wedge\hat{\tau}_N}\big] = F^1(x,y) - rE\bigg[\int_0^{T\wedge\hat{\tau}_N} e^{-\mu t}F^1\big(\widehat{X}^1(t),Y(t)\big)dt\bigg],$$

where $\hat{\tau}_N = \inf\{t|r\widehat{X}^1(t)+\delta Y(t)\le -N\}$, and

$$\exp\big(-\mu T\big)E\big\{\exp\big[-\gamma\big(r\widehat{X}^1(T\wedge\hat{\tau}_N)+\delta Y(T\wedge\hat{\tau}_N)\big)\big]\big\}\le\exp\big(-\gamma(rx+\delta y)\big).$$

Hence, $\exp(-\mu T)\exp(\gamma N)\,\mathbb{P}(\hat{\tau}_N<T)\le\exp\big(-\gamma(rx+\delta y)\big)$.

Since $\hat{\tau}_N\uparrow\hat{\tau}^*$ a.s. and $\mathbb{P}(\hat{\tau}^*<T)\le\mathbb{P}(\hat{\tau}_N<T)$, we get $\mathbb{P}(\hat{\tau}^*<T)=0\ \forall\ T$; hence, $\hat{\tau}^*=\infty$ a.s.

Passing to the limit as $N\to\infty$ in (J.4), we obtain

$$E\big[F^1\big(\widehat{X}^1(T),Y(T)\big)e^{-\mu T}\big] = F^1(x,y) - rE\bigg[\int_0^{T} e^{-\mu t}F^1\big(\widehat{X}^1(t),Y(t)\big)dt\bigg];$$

therefore, $E\big[F^1\big(\widehat{X}^1(T),Y(T)\big)e^{-\mu T}\big]=F^1(x,y)e^{-rT}$ and the control

$$\begin{cases}\widehat{C}(t)=\widehat{C}\big(\widehat{X}^1(t),Y(t)\big),\\ \widehat{\pi}(t)=\widehat{\pi}\big(\widehat{X}^1(t),Y(t)\big)\end{cases}$$

is admissible. Clearly,

$$\text{(J.5)}\qquad J\big(\widehat{C}(\cdot)\big)=F^1(x,y).$$

On the other hand, we easily check that for any admissible control

$$J\big(C(\cdot)\big)\le F^1(x,y);$$

hence the desired result has been obtained.

Appendix K. Proof of Theorem 6.2. For each $\hat{y}$ fixed, the solution of (6.26) is the value function of a stochastic control problem. The state equation is

$$\text{(K.1)}\quad dY(t)=\big(\alpha-\lambda\varsigma\rho-\gamma\delta\varsigma^2(1-\rho^2)+\varsigma\sqrt{r\gamma(1-\rho^2)}v(t)\big)dt+\varsigma dW(t),\quad Y(0)=y<\hat{y},$$

where $v(\cdot)$ is adapted and locally square integrable. Let $\theta_y\big(v(\cdot)\big)$ be the first time the process reaches $\hat{y}$; it can be ∞. We then have

$$\text{(K.2)}\qquad u(y)=\inf_{v(\cdot)}E\bigg[\int_0^{\theta_y(v(\cdot))}e^{-rt}\Big(-\delta Y_y(t)+rK+\frac{1}{2}v^2(t)\Big)dt\bigg].$$

For $0 < t < \theta_y\big(v(\cdot)\big)$, we have $Y_y(t) < \hat{y}$, and thus

$$E\bigg[\int_0^{\theta_y(v(\cdot))} e^{-rt}\Big(-\delta Y_y(t) + rK + \frac{1}{2}v^2(t)\Big)dt\bigg] \geq \frac{-\delta\hat{y} + rK}{r}\big(1 - Ee^{-r\theta_y(v(\cdot))}\big)$$
$$\geq -\frac{(-\delta\hat{y} + rK)^-}{r};$$

hence we get the left inequality in (6.27). To prove the right inequality, we just notice that $u(y) \leq z(y)$, where $z(y)$ is the solution of

$$\begin{cases} -\frac{1}{2}\varsigma^2 z'' - z'\big(\alpha - \lambda\varsigma\rho - \varsigma^2\delta\gamma(1-\rho^2)\big) + rz = -\delta y + rK, \\ z(\hat{y}) = 0. \end{cases}$$

The function $z(y)$ is given explicitly by

$$z(y) = -\frac{\delta}{r}(y - \hat{y}) + (1 - e^{\beta(y-\hat{y})})\bigg[\frac{-\delta\hat{y} + rK}{r} - \frac{\delta}{r^2}\big(\alpha - \lambda\varsigma\rho - \gamma\delta\varsigma^2(1-\rho^2)\big)\bigg], \tag{K.3}$$

where $\beta > 0$ is the solution of

$$-\frac{1}{2}\varsigma^2\beta^2 - \beta\big(\alpha - \lambda\varsigma\rho - \gamma\delta\varsigma^2(1-\rho^2)\big) + r = 0,$$

which proves the right inequality in (6.27).

We now check the condition (6.28). We denote by $u(y;\hat{y})$ the unique solution of (6.26) and (6.27) for a given $\hat{y}$ and $y < \hat{y}$. From (6.27), we have $u(y;\hat{y}) \geq 0$ if $\hat{y} \leq \frac{rK}{\delta}$; hence $u'(\hat{y};\hat{y}) \leq 0$ for $\hat{y} \leq \frac{rK}{\delta}$.

From (K.3),

$$z'(y;\hat{y}) = -\frac{\delta}{r} - \beta e^{\beta(y-\hat{y})}\bigg[\frac{-\delta\hat{y} + rK}{r} - \frac{\delta}{r^2}\big(\alpha - \lambda\varsigma\rho - \gamma\delta\varsigma^2(1-\rho^2)\big)\bigg];$$

hence,

$$z'(\hat{y};\hat{y}) = -\frac{\delta}{r} - \beta\bigg[\frac{-\delta\hat{y} + rK}{r} - \frac{\delta}{r^2}\big(\alpha - \lambda\varsigma\rho - \gamma\delta\varsigma^2(1-\rho^2)\big)\bigg].$$

Therefore, $z'(\hat{y};\hat{y}) > 0$ if $\hat{y}$ is sufficiently large. It follows that $z(y;\hat{y}) < 0$ if $y < \hat{y}$ close to $\hat{y}$. Therefore, $u(y;\hat{y}) < 0$ if $y < \hat{y}$ close to $\hat{y}$. Hence $u'(\hat{y};\hat{y}) > 0$.

Therefore, there exists $\hat{y}$ such that (6.28) holds. We have $\hat{y} \geq \frac{rK}{\delta}$; otherwise $u''(\hat{y}) < 0$, which is impossible.

We then check that the function $u(y)$ is the solution of the V.I. (6.25). First, we check that $u(y) \geq 0$. Since $u''(\hat{y}) > 0$, we have $u(y) > 0$ near $\hat{y}$; hence $u'(y) < 0$ near $\hat{y}$. The function $u(y)$ cannot have a positive maximum on $(\frac{rK}{\delta}, \hat{y})$; hence $u(y)$ decreases on $(\frac{rK}{\delta}, \hat{y})$. Therefore, $u(\frac{rK}{\delta}) > 0$. Necessarily, $u(y) > 0$ on $(-\infty, \frac{rK}{\delta})$. Since for $y > \hat{y}$, $-\delta y + rK \leq -\delta\hat{y} + rK \leq 0$, the differential inequality is also verified for $y > \hat{y}$.

Finally, to complete the proof we prove the uniqueness of $\hat{y}$ and the fact that the V.I. (6.25) has a solution of the form (6.26) and (6.28). We first check that $\hat{y}$ is uniquely defined. For that we will rely on an interesting interpretation of the solution of (6.26) and (6.28). Again call $u(y;\hat{y})$ this solution, in which $u(y;\hat{y}) = 0$ for $y \geq \hat{y}$. If we define

$$\chi(y;\hat{y}) = \begin{cases} -\delta y + Kr, & y < \hat{y}, \\ 0, & y > \hat{y}, \end{cases}$$

which is an L^∞ function (not continuous in $\hat{y}$), then the function u appears as the solution of

$$\text{(K.4)} \qquad \begin{cases} -\frac{1}{2}\varsigma^2 u'' - u'(\alpha - \lambda\varsigma\rho - \varsigma^2\delta\gamma(1-\rho^2)) + \frac{1}{2}\varsigma^2 r\gamma(1-\rho^2)u'^2 + ru = \chi(y;\hat{y}), \\ u \text{ has linear growth as } y \to -\infty. \end{cases}$$

But then $u(y;\hat{y})$ has the probabilistic interpretation

$$\text{(K.5)} \qquad u(y;\hat{y}) = \inf_{v(\cdot)} E\left[\int_0^{\theta_y\left(v(\cdot)\right)} e^{-rt}\left(\chi\big(Y_y(t);\hat{y}\big) + \frac{1}{2}v^2(t)\right)dt\right],$$

where the state equation dY is as in (K.1), $v(\cdot)$ is adapted and locally square integrable, and $\theta_y\big(v(\cdot)\big)$ is the first time the process reaches $\hat{y}$.

The function $\chi(y;\hat{y})$ decreases with $\hat{y}$; hence $u(y;\hat{y})$ also decreases with $\hat{y}$. Hence, if $\hat{y}_1 < \hat{y}_2$, we have $u(y;\hat{y}_1) \geq u(y;\hat{y}_2)$. But then for $\hat{y}_1 < y < \hat{y}_2$, $u(y;\hat{y}_2) \leq 0$. Since it is also ≥ 0, $u(y,\hat{y}_2) = 0$ for $\hat{y}_1 < y < \hat{y}_2$. Necessarily, $u(y;\hat{y}_1) = u(y;\hat{y}_2)$ for $y < \hat{y}_1$ since they satisfy the same equation. Therefore, $u(y;\hat{y}_1) = u(y;\hat{y}_2)$. Therefore, $\hat{y}_1 = \hat{y}_2$ since we naturally interpret $\hat{y}$ in (6.28) as the first point for which $u(\hat{y}) = 0$, $u'(\hat{y}) = 0$.

To complete the proof, it remains to check that a solution of the V.I. (6.25) is necessarily of the form (6.26) and (6.28). Indeed, let us consider $\hat{y}^*$ to be the first point for which $u(\hat{y}^*) = 0$. We claim that $\hat{y}^* \neq \infty$. If $\hat{y}^* = \infty$, then u satisfies

$$\text{(K.6)} \qquad \begin{cases} -\frac{1}{2}\varsigma^2 u'' - u'\big(\alpha - \lambda\varsigma\rho - \varsigma^2\delta\gamma(1-\rho^2)\big) + \frac{1}{2}\varsigma^2 r\gamma(1-\rho^2)u'^2 + ru = -\delta y + rK, \\ u \geq 0; \quad u \text{ has linear growth as } y \to -\infty, \end{cases}$$

which is of the form (K.4) with $\chi(y;\infty)$. We know a function $u(y;\hat{y})$ for a certain $\hat{y}$ finite and $u(y;\hat{y}) \geq u(y;\infty)$. But then $u(y;\infty) = 0$ for $y > \hat{y}$, which is not possible from (K.6). So $\hat{y}^* < \infty$. We claim that $u'(\hat{y}^*) = 0$. We know that $u'(\hat{y}^*) \leq 0$. If $u'(\hat{y}^*) < 0$, then $u(y)$ becomes negative after $\hat{y}^*$, which is impossible. But then $u(y) = u(y;\hat{y}^*)$ for $y \leq \hat{y}^*$. Moreover $\hat{y}^* > \frac{rK}{\delta}$. We also have $u(y) = 0$ for $y > \hat{y}^*$. Otherwise $u(y)$ becomes positive after $\hat{y}^*$. But then it cannot always stay positive since we would have (K.6) holding for $y > \hat{y}^*$, with a right-hand side negative and $u(\hat{y}^*) = 0$. Necessarily, $u(y) \leq 0$, which is impossible. Since $u(y)$ must vanish after $\hat{y}^*$, it will have a positive maximum after $\hat{y}^*$, which is also not possible. Therefore, $u = 0$ for $y \geq \hat{y}^*$. The proof has been completed.

Appendix L. Proof of Theorem 6.3. Consider the feedbacks

$$\text{(L.1)} \qquad \begin{cases} \widehat{C}(x,y) = r\big(x + g(y)\big) - \frac{1}{\gamma}\left(1 - \frac{\mu + \frac{\lambda^2}{2}}{r}\right), \\ \widehat{\pi}(x,y)\sigma x = \frac{\lambda}{r\gamma} - \varsigma\rho g'(y), \end{cases}$$

and we have $U\big(\widehat{C}(x,y)\big) = rF(x,y)$. Let $\widehat{X}^0(t)$ be the optimal wealth trajectory given by (6.4), applying the feedbacks (L.1). We have

$$\widehat{\pi}(x,y)\sigma\lambda x + xr - \widehat{C}(x,y) = -\lambda\varsigma\rho g'(y) - rg(y) + \frac{1}{\gamma}\left(1 - \frac{\mu - \frac{\lambda^2}{2}}{r}\right);$$

hence

(L.2)
$$\begin{cases} d\widehat{X}^0(t) = \left[-rg(Y(t)) - \lambda\varsigma\rho g'(Y(t)) + \frac{1}{\gamma}\left(1 - \frac{\mu-\frac{\lambda^2}{2}}{r}\right)\right]dt + \left(\frac{\lambda}{r\gamma} - \varsigma\rho g'(Y(t))\right)dW(t),\\ \widehat{X}^0(0) = x,\\ dY(t) = \alpha dt + \varsigma\big(\rho dW(t) + \sqrt{1-\rho^2}dW^0(t)\big),\\ Y(0) = y. \end{cases}$$

Let $\hat{\tau}(y) = \inf\{t|Y_y(t) \geq \hat{y}\}$, which does not depend on x. We have $E[\hat{\tau}(y)] = \frac{\hat{y}-y}{\alpha}$ for $y < \hat{y}$; therefore, $\hat{\tau}(y) < \infty$ a.s. As in (J.4), we can write

$$E\big[F\big(\widehat{X}^0(\hat{\tau}\wedge\hat{\tau}_N), Y(\hat{\tau}\wedge\hat{\tau}_N)\big)e^{-\mu\hat{\tau}\wedge\hat{\tau}_N}\big] - F(x,y) = -rE\left[\int_0^{\hat{\tau}\wedge\hat{\tau}_N} e^{-\mu t}F\big(\widehat{X}^0(t), Y(t)\big)dt\right],$$

where $\hat{\tau}_N = \inf\{t|\widehat{X}^0(t) \leq -N\}$, and we deduce

$$E\big\{\exp\big[-r\gamma\big(\widehat{X}^0(\hat{\tau}\wedge\hat{\tau}_N) + g\big(Y(\hat{\tau}\wedge\hat{\tau}_N)\big)\big)\big]\exp\big(-\mu\hat{\tau}\wedge\hat{\tau}_N\big)\big\} \leq \exp\big(-\gamma r(x+g(y))\big).$$

Since $Y(\hat{\tau}\wedge\hat{\tau}_N) < \hat{y}$, $g\big(Y(\hat{\tau}\wedge\hat{\tau}_N)\big)$ is bounded; hence $\exp(r\gamma N)E\big[\exp(-\mu\hat{\tau})\mathbb{1}_{\hat{\tau}_N<\hat{\tau}}\big] \leq \exp\big(-\gamma r(x+g(y))\big)$. Therefore, if $\hat{\tau}^* = \lim\uparrow\hat{\tau}_N$, we obtain $\hat{\tau}^* \geq \hat{\tau}_N$ a.s. It follows that

$$\begin{cases} \widehat{C}(t) = \widehat{C}\big(\widehat{X}^0(t), Y(t)\big),\\ \widehat{\pi}(t) = \widehat{\pi}\big(\widehat{X}^0(t), Y(t)\big),\\ \hat{\tau}(y) \end{cases}$$

forms an admissible control. We obtain $F(x,y) = J_{x,y}\big(\widehat{C}(\cdot), \widehat{\pi}(\cdot), \hat{\tau}\big)$.

On the other hand, for any admissible control, we have

$$F(x,y) \geq E\left[\int_0^{\tau\wedge\tau_N} U\big(C(t)\big)e^{-\mu t}dt + F\big(X^0(\tau\wedge\tau_N), Y(\tau\wedge\tau_N)\big)\exp(\tau\wedge\tau_N)\right].$$

Letting N tend to ∞, we obtain $F(x,y) \geq J_{xy}\big(C(\cdot), \pi(\cdot), \tau\big)$, which completes the proof of the desired result.

Appendix M. Proof of Theorem 6.4. We consider the feedbacks

(M.1)
$$\begin{cases} \widehat{C}(x,y) = r\big(x+g(y)\big) - \frac{1}{\gamma}\left(1 - \frac{\mu+\frac{\lambda^2}{2}}{r}\right),\\ \widehat{\pi}(x,y)\sigma x = \frac{\lambda}{r\gamma} - \varsigma\rho g'(y). \end{cases}$$

We have $U\big(\widehat{C}(x,y)\big) = rL^1(x,y)$ if $y < \hat{y}$. The corresponding wealth process $\widehat{X}^1(t)$ has the Itô differential

$$d\widehat{X}^1 = \left[\delta_1 Y - rg(Y) - \lambda\varsigma\rho g'(Y) + \frac{1}{\gamma}\left(1 - \frac{\mu - \frac{\lambda^2}{2}}{r}\right)\right]dt + \left(\frac{\lambda}{r\gamma} - \varsigma\rho g'(Y)\right)dW. \tag{M.2}$$

Let $\hat{\tau}_N$ be the first time when $\widehat{X}^1(t)$ reaches $-N$. We get

$$\begin{aligned} &E\big[L^1\big(\widehat{X}^1\big(\hat{\tau}(y)\wedge\hat{\tau}_N\big), Y\big(\hat{\tau}(y)\wedge\hat{\tau}_N\big)\big)e^{-\mu\hat{\tau}(y)\wedge\hat{\tau}_N}\big] - L^1(x,y) \\ &= -rE\left[\int_0^{\hat{\tau}(y)\wedge\hat{\tau}_N} L^1\big(\widehat{X}^1(t), Y(t)\big)e^{-\mu t}dt\right]. \end{aligned}$$

We deduce

$$E\big[L^1\big(\widehat{X}^1\big(\hat{\tau}(y)\wedge\hat{\tau}_N\big), Y\big(\hat{\tau}(y)\wedge\hat{\tau}_N\big)\big)e^{-\mu\hat{\tau}(y)\wedge\hat{\tau}_N}\big] \leq e^{-\gamma r\left(x+g(y)\right)}.$$

Calling $\hat{\tau}^* = \lim\uparrow\hat{\tau}_N$, we obtain $\hat{\tau}^* \geq \hat{\tau}(y)$ a.s. Therefore, the control

$$\begin{cases} \widehat{C}(t) = \widehat{C}\big(\widehat{X}^1(t), Y(t)\big), \\ \widehat{\pi}(t) = \widehat{\pi}\big(\widehat{X}^1(t), Y(t)\big) \end{cases}$$

is admissible. Moreover,

$$\begin{aligned} L^1(x,y) = E&\left[\int_0^{\hat{\tau}(y)\wedge\hat{\tau}_N} U\big(\widehat{C}(t)\big)e^{-\mu t}dt\right] \\ &+ E\big[L^1\big(\widehat{X}^1\big(\hat{\tau}(y)\wedge\hat{\tau}_N\big), Y\big(\hat{\tau}(y)\wedge\hat{\tau}_N\big)\big)e^{-\mu\hat{\tau}(y)\wedge\hat{\tau}_N}\big] \end{aligned}$$

and, from the boundary condition that $L^1(x,y)$ satisfies at $y = \hat{y}$, we get

$$L^1(x,y) = J\big(\widehat{C}(\cdot), \widehat{\pi}(\cdot)\big).$$

On the other hand, for any admissible pair $\big(C(\cdot), \pi(\cdot)\big)$, we have

$$\begin{aligned} L^1(x,y) \geq E&\left[\int_0^{\hat{\tau}(y)\wedge\tau_N} U\big(C(t)\big)e^{-\mu t}dt\right. \\ &\left. + L^1\big(X^1\big(\hat{\tau}(y)\wedge\tau_N\big), Y\big(\hat{\tau}(y)\wedge\tau_N\big)\big)e^{-\mu\hat{\tau}(y)\wedge\tau_N}\right], \end{aligned}$$

and letting N tend to ∞, we obtain $L^1(x,y) \geq J\big(C(\cdot), \pi(\cdot)\big)$ and the proof is completed.

Appendix N. Proof of Theorem 6.5. We consider the ceiling function

$$\bar{u}(y) = \frac{\delta_2}{r}\left[-(y - k\hat{y}) + \frac{e^{\beta(y-k\hat{y})} - 1}{\beta}\right], \tag{N.1}$$

where k is sufficiently large. This function is such that $\bar{u}(k\hat{y}) = \bar{u}'(k\hat{y}) = 0$. We extend it by 0 for $y > k\hat{y}$. Also, $\bar{u}(y) > 0$ for $y < k\hat{y}$. We check

$$\bar{u} > m. \tag{N.2}$$

Since $m = 0$ for $y > \hat{y}$, it is sufficient to prove (N.2) for $y < \hat{y}$. However,

$$\begin{aligned}
&-\frac{1}{2}\varsigma^2\bar{u}'' - \big(\alpha - \lambda\varsigma\rho - \varsigma^2\delta_2\gamma(1-\rho^2)'\big)\bar{u}' + \frac{1}{2}\varsigma^2 r\gamma(1-\rho^2)\bar{u}'^2 + r\bar{u} \\
&\geq \frac{\delta_2}{r}\big(\alpha - \lambda\varsigma\rho - \varsigma^2\delta_2\gamma(1-\rho^2)\big) - \delta_2(y - k\hat{y}) - \frac{\delta_2}{\beta} \\
&\geq (\delta_1 - \delta_2)y \quad \text{for } y < \hat{y},
\end{aligned} \tag{N.3}$$

provided $\frac{\delta_2}{r}\big(\alpha - \lambda\varsigma\rho - \varsigma^2\delta_2\gamma(1-\rho^2)\big) + \delta_2 k\hat{y} - \frac{\delta_2}{\beta} > \delta_1\hat{y}$. This implies (N.2). We also have

$$\frac{\delta_2}{r}\big(\alpha - \lambda\varsigma\rho - \varsigma^2\delta_2\gamma(1-\rho^2)\big) - \delta_2(y - k\hat{y}) - \frac{\delta_2}{\beta} > \delta_2 y + rK,$$

provided $\frac{\delta_2}{r}\big(\alpha - \lambda\varsigma\rho - \varsigma^2\delta_2\gamma(1-\rho^2)\big) + \delta_2 k\hat{y} - \frac{\delta_2}{\beta} > rK$. Therefore, $\bar{u}$ satisfies the two inequalities (6.52), so it can be taken as a ceiling function.

Combining with (6.55), we must have

$$-f(y) + k \leq u(y) \leq \bar{u}(y). \tag{N.4}$$

Since $f(y) = \frac{\delta_2}{r}(y - \tilde{y}_0)$ with $\tilde{y}_0 = -\frac{\alpha - \lambda\varsigma\rho - \delta_2\gamma\varsigma^2(1-\rho^2)}{r} + \frac{\delta_2^2\gamma\varsigma^2(1-\rho^2)}{r^2}$, we have $u + \frac{\delta_2 y}{r}$ bounded as $y \to -\infty$.

We use the penalty technique to approximate (6.52). We approximate (6.52) by a smoother problem, a penalized problem. We introduce the problem with u_ϵ such that

$$\begin{cases}
-\frac{1}{2}\varsigma^2 u_\epsilon'' - \big(\alpha - \lambda\varsigma\rho - \varsigma^2\delta_2\gamma(1-\rho^2)\big)u_\epsilon' + \frac{1}{2}\varsigma^2 r\gamma(1-\rho^2)u_\epsilon'^2 + ru_\epsilon \\
\qquad\qquad = -\delta_2 y + rK + \frac{1}{\epsilon}(m - u_\epsilon)^+, \\
u_\epsilon(\bar{y}) = 0; \quad u_\epsilon(y) + f(y) \text{ is bounded as } y \to -\infty.
\end{cases} \tag{N.5}$$

In (N.5), $\bar{y} = k\hat{y}$, where k is sufficiently large.

We have the property

$$u_\epsilon(y) \leq \bar{u}(y). \tag{N.6}$$

Let us define $\tilde{u}_\epsilon(y) = u_\epsilon(y) - m(y)$. We have $\tilde{u}_\epsilon(\bar{y}) = 0$ and $\tilde{u}_\epsilon \to \infty$ as $y \to -\infty$. If $\tilde{u}_\epsilon(y) \geq 0$ on $(-\infty, \bar{y})$, then $(m - u_\epsilon)^+ = 0$.

Suppose that $\tilde{u}_\epsilon(y)$ becomes negative; then we can consider points of negative minimum. We check that $\hat{y}$ cannot be such a point. Indeed, we know that $m'(\hat{y} - 0) < 0$ from (6.54). If $\hat{y}$ were a minimum of $\tilde{u}_\epsilon$, we would have $\tilde{u}_\epsilon(\hat{y} - 0) \leq 0 \leq \tilde{u}_\epsilon(\hat{y} + 0)$ and since $u_\epsilon'(\hat{y} - 0) = u_\epsilon'(\hat{y} + 0)$, $m'(\hat{y} - 0) \geq 0$, which is a contradiction. Therefore, if y_ϵ^* is a negative minimum of $\tilde{u}_\epsilon(y)$, we have $y_\epsilon^* < \hat{y}$ or $y_\epsilon^* > \hat{y}$.

If $y_\epsilon^* > \hat{y}$, it is also a local minimum of u_ϵ. But then considering (N.5)

$$ru_\epsilon(y_\epsilon^*) \geq -\delta_2\bar{y} + rK - \frac{1}{\epsilon}u_\epsilon(y_\epsilon^*),$$

which implies $u_\epsilon(y_\epsilon^*) \geq \epsilon(-\delta_2\bar{y} + rK)$; hence $\tilde{u}_\epsilon(y_\epsilon^*) \geq \epsilon(-\delta_2\bar{y} + rK)$.

Suppose now $y_\epsilon^* < \hat{y}$. We note that on $(-\infty, \hat{y})$, $\tilde{u}_\epsilon$ satisfies

$$-\frac{1}{2}\varsigma^2\tilde{u}_\epsilon'' - \left(\alpha - \lambda\varsigma\rho - \left(\frac{\delta_2}{r} + m'\right)\varsigma^2 r\gamma(1-\rho^2)\right)\tilde{u}_\epsilon' + \frac{1}{2}\varsigma^2 r\gamma(1-\rho^2)\tilde{u}_\epsilon'^2 + r\tilde{u}_\epsilon$$
$$= -\delta_1 y + rK + \frac{1}{\epsilon}\tilde{u}_\epsilon^-,$$

but then $r\tilde{u}_\epsilon(y_\epsilon^*) \geq -\delta_1\hat{y} + rK - \frac{1}{\epsilon}\tilde{u}_\epsilon(y_\epsilon^*)$, and $\tilde{u}_\epsilon(y_\epsilon^*) \geq \epsilon(-\delta_1\hat{y} + rK)$. Therefore,

$$\tilde{u}_\epsilon(y) = u_\epsilon(y) - m(y) \geq \epsilon\big(rK - \max(\delta_2\bar{y}, \delta_1\hat{y})\big) \geq -\epsilon c_0$$

with $c_0 = \max(\delta_2\bar{y}, \delta_1\hat{y}) > 0$, and we get

$$\frac{(m - u_\epsilon)^+}{\epsilon} \leq c_0. \tag{N.7}$$

Consider next $\underline{u}(y)$ to be the solution of

(N.8)
$$\begin{cases} -\frac{1}{2}\varsigma^2\underline{u}'' - \big(\alpha - \lambda\varsigma\rho - \varsigma^2\delta_2\gamma(1-\rho^2)\big)\underline{u}' + \frac{1}{2}\varsigma^2 r\gamma(1-\rho^2)\underline{u}'^2 + r\underline{u} = -\delta_2 y + rK, \quad y < \underline{y}, \\ \underline{u}(\underline{y}) = 0. \end{cases}$$

Then we have the estimates

$$-f(y) + K \leq \underline{u}(y) \leq u_\epsilon(y) \leq \bar{u}(y). \tag{N.9}$$

Since $\underline{u}(\underline{y}) = \bar{u}(\underline{y}) = 0$, we have

$$\underline{u}'(\underline{y} - 0) \geq u_\epsilon'(\underline{y} - 0) \geq \bar{u}'(\underline{y}) = 0. \tag{N.10}$$

From the estimates (N.9), we have

$$\frac{|u_\epsilon(y)|}{(1+y^2)^\varrho} \leq C \quad \forall\, \varrho > \frac{1}{2}. \tag{N.11}$$

We now check

$$\frac{|u_\epsilon'(y)|}{(1+y^2)^\varrho} \leq C \quad \forall\, \varrho > \frac{1}{2},\ y \leq \underline{y}, \tag{N.12}$$

with $u_\epsilon'(\underline{y}) = u_\epsilon'(\underline{y} - 0)$.

From (N.10), we know that this is the time for $y = \underline{y}$.

Consider $z_\epsilon = \frac{u_\epsilon}{(1+y^2)^\varrho}$; then z_ϵ satisfies

$$-\frac{1}{2}\frac{\varsigma^2 z_\epsilon''}{(1+y^2)^\varrho} - z_\epsilon'\left\{-\frac{2\varrho r\gamma\varsigma^2(1-\rho^2)y}{1+y^2} + \frac{2\varrho y}{(1+y^2)^{\varrho+1}} + \frac{\alpha-\lambda\varsigma\rho-\gamma\delta_2\varsigma^2(1-\rho^2)}{(1+y^2)^\varrho}\right\} + \frac{1}{2}r\gamma\varsigma^2(1-\rho^2)z_\epsilon'^2 = \zeta_\epsilon(y), \quad y<\bar{y}, \tag{N.13}$$

and $|\zeta_\epsilon(y)| \le C$ if $\varrho > \frac{1}{2}$.

Consider a point where z_ϵ' is maximum positive or minimum negative. Let y_ϵ be such a point. If $y_\epsilon < \bar{y}$, then $z_\epsilon''(y_\epsilon) = 0$ and from (N.13) we necessarily have $|z_\epsilon'(y_\epsilon)| \le C$. Now, on $\underline{y}$, we have $z_\epsilon'(\bar{y}-0) = \frac{u_\epsilon'(y-0)}{(1+y^2)^\varrho}$, and thus $|z_\epsilon'(y)| \le C$. Since $\frac{u_\epsilon'(y)}{(1+y^2)^\varrho} = z_\epsilon'(y) + \frac{2\varrho y}{1+y^2}z_\epsilon(y)$, we obtain (N.12). From (N.5), we deduce

$$\frac{|u_\epsilon''(y)|}{(1+y^2)^{2\varrho}} \le C. \tag{N.14}$$

Therefore, we can extract from $u_\epsilon(y)$ a subsequence, still denoted $u_\epsilon(y)$, such that

$$u_\epsilon(y) \to u(y),\ u_\epsilon'(y) \to u(y) \text{ pointwise on } (-\infty,\bar{y}). \tag{N.15}$$

Calling $\chi^\epsilon = \frac{1}{\epsilon}(m-u_\epsilon)^+$, we also have

$$\begin{cases} \chi^\epsilon \to \chi \text{ in } L^\infty(-\infty,\bar{y}) \text{ weak star}, \\ \frac{u_\epsilon''(y)}{(1+y^2)^{2\varrho}} \to \frac{u''(y)}{(1+y^2)^{2\varrho}} \text{ in } L^\infty(-\infty,\bar{y}) \text{ weak star}. \end{cases} \tag{N.16}$$

The limit u satisfies

(N.17)
$$\begin{cases} -\frac{1}{2}\varsigma^2 u'' - \left(\alpha-\lambda\varsigma\rho-\varsigma^2\delta_2\gamma(1-\rho^2)\right)u' + \frac{1}{2}\varsigma^2 r\gamma(1-\rho^2)u'^2 + ru = -\delta_2 y + rK + \chi, \\ u \ge m, \\ (u-m)\chi = 0, \\ u(\underline{y}) = 0. \end{cases}$$

Then, necessarily,

$$u'(\underline{y}-0) = 0. \tag{N.18}$$

Indeed $u(y) \ge m(y)$ is positive for y close to $\underline{y}$; hence $u'(\underline{y}-0) \le 0$. Moreover, from (N.10), $u'(\underline{y}-0) \ge 0$. Hence, we have (N.18). Therefore, u extended by 0 beyond $\underline{y}$ remains C^1 and is the solution of the V.I. (6.52). We can take $\underline{y}$ to be the first point where $u(y)$ touches 0. Necessarily, $u(y) \ge 0$.

Consider again the first point $y^* < \hat{y}$ such that $m(y^*) = 0$. Then $y^* < 0$ and $m(y) < 0$ for $y < y^*$. Therefore, for $y < y^*$ the following differential equation holds:

$$-\frac{1}{2}\varsigma^2 u'' - \left(\alpha-\lambda\varsigma\rho-\varsigma^2\delta_2\gamma(1-\rho^2)\right)u' + \frac{1}{2}\varsigma^2 r\gamma(1-\rho^2)u'^2 + ru = -\delta_2 y + rK,\ y<y^*. \tag{N.19}$$

There must exist a point $y^{**} < y^*$ such that $u'(y^{**}) < 0$. Otherwise, $u'(y) \geq 0 \; \forall \; y < y^*$; hence $u(y)$ is bounded for $y < y^*$, which is impossible since $u(y) \to \infty$ as $y \to -\infty$. We claim

$$u'(y) < 0 \quad \text{if } y < y^{**}. \tag{N.20}$$

Otherwise, there would be a point of maximum. From (N.19) the value of this maximum would be negative, which is impossible. Hence, we have (N.20).

Define the feedback $\hat{v}(y) = -u'(y)\varsigma\sqrt{r\gamma(1-\rho^2)}$, and from (N.20)

$$\hat{v}(y) \geq \hat{v}(y)\mathbb{1}_{y^{**}<y<\bar{y}} \geq v_0. \tag{N.21}$$

We associate with (N.5) a stochastic differential game. The state equation is governed by

$$\begin{cases} dY(t) = \big(\alpha - \lambda\varsigma\rho - \varsigma^2\delta_2\gamma(1-\rho^2) + v(t)\varsigma\sqrt{r\gamma(1-\rho^2)}\big)dt + \varsigma dW(t), \\ Y(0) = y, \qquad y < \bar{y}, \end{cases} \tag{N.22}$$

where $v(t)$ is the control (adapted process). We define the cost function

$$J_y^\epsilon\big(v(\cdot),\vartheta(\cdot)\big) = E\left[\int_0^{\bar{\theta}} \left(-\delta_2 Y_y(t) + rK + \frac{1}{2}v^2(t) + \frac{1}{\epsilon}m\big(Y_y(t)\big)\vartheta(t)\right)e^{-\int_0^t \left(r+\frac{\vartheta(s)}{\epsilon}\right)ds}dt\right], \tag{N.23}$$

where $\bar{\theta} = \inf\{t | Y_y(t) = \bar{y}\}$. We then have

$$u_\epsilon(y) = \inf_{v(\cdot)}\sup_{\vartheta(\cdot)} J_y^\epsilon\big(v(\cdot),\vartheta(\cdot)\big) = \sup_{\vartheta(\cdot)}\inf_{v(\cdot)} J_y^\epsilon\big(v(\cdot),\vartheta(\cdot)\big). \tag{N.24}$$

Thus, there exists a saddle point. Optimal processes, $\hat{v}_\epsilon(t)$ and $\hat{\vartheta}_\epsilon$, are obtained from feedbacks

$$\begin{cases} \hat{v}_\epsilon(y) = -u_\epsilon'(y)\varsigma\sqrt{r\gamma(1-\rho^2)}, \\ \hat{\vartheta}_\epsilon(y) = \mathbb{1}_{m(y)>u_\epsilon(y)} \end{cases}$$

by the formula

$$\begin{cases} \hat{v}_\epsilon(t) = \hat{v}_\epsilon\big(Y_y(t)\big), \\ \hat{\vartheta}_\epsilon(t) = \hat{\vartheta}_\epsilon\big(Y_y(t)\big). \end{cases}$$

Consider the optimal trajectory $\widehat{Y}_y(t)$ given by (see (N.22))

$$d\hat{Y} = \big(\alpha - \lambda\varsigma\rho - \gamma\delta_2\varsigma^2(1-\rho^2) + \hat{v}(Y)\varsigma\sqrt{r\gamma(1-\rho^2)}\big)dt + \varsigma dW. \tag{N.25}$$

We have from (N.21) $d\widehat{Y} \geq -c_1 dt + \varsigma dW$; hence $E\widehat{Y}_y(t) \geq y - c_1 t$. Therefore, from the estimates (N.4) and (N.1),

$$Eu\big(\widehat{Y}_y(t)\big) \leq \frac{\delta_2}{r}\big[-E\widehat{Y}_y(t) + c_2\big] \leq \frac{\delta_2}{r}\big[-y + c_1 t + c_2\big]. \tag{N.26}$$

We next define optimal stopping:

$$\hat{\theta} = \hat{\theta}(y) = \inf\big\{t | u\big(\widehat{Y}_y(t)\big) = m\big(\widehat{Y}_y(t)\big)\big\}. \tag{N.27}$$

Using Itô's formula, we can write

$$u(y) = E\left[\int_0^{T\wedge\hat{\theta}} \left(-\delta_2 \widehat{Y}_y(t) + rK + \frac{1}{2}\hat{v}^2(t)\right)e^{-rt}dt + u\big(Y_y(T\wedge\hat{\theta})\big)e^{-rT\wedge\hat{\theta}}\right],$$

where $\hat{v}(t) = \hat{v}\big(\widehat{Y}_y(t)\big)$. Therefore,

$$u(y) = E\left[\int_0^{T\wedge\hat{\theta}} \left(-\delta_2 \widehat{Y}_y(t) + rK + \frac{1}{2}v^2(t)\right)e^{-rt}dt + m\big(Y_y(\hat{\theta})\big)e^{-rT}\mathbb{1}_{\hat{\theta}<T} \right.$$
$$\left. + u\big(Y_y(T)\big)e^{-rT}\mathbb{1}_{T\leq\hat{\theta}}\right].$$

Letting $T\uparrow\infty$, making use of the estimate (N.27), we obtain

$$u(y) = J_y\big(\hat{v}(\cdot),\hat{\theta}\big). \tag{N.28}$$

Define the feedback $\hat{\vartheta}(y) = \mathbb{1}_{u(y)=m(y)}$.

We define next the cost associated with any control $v(\cdot)$ and the feedback $\hat{\vartheta}(\cdot)$:

$$J_y(v(\cdot),\hat{\vartheta}(\cdot)) = E\left[\int_0^{\hat{\theta}} \left(-\delta_2 Y_y(t) + rK + \frac{1}{2}v^2(t)\right)e^{-rt}dt + m\big(Y_y(\hat{\theta})\big)e^{-r\hat{\theta}}\mathbb{1}_{\hat{\theta}<\infty}\right],$$

where $\hat{\theta} = \inf\big\{t|u\big(Y_y(t)\big) = m\big(Y_y(t)\big)\big\}$. We check

$$u(y) \leq E\left[\int_0^{T\wedge\hat{\theta}} \left(-\delta_2 Y_y(t) + rK + \frac{1}{2}v^2(t)\right)e^{-rt}dt + u\big(Y_y(T\wedge\hat{\theta})\big)e^{-rT\wedge\hat{\theta}}\right]$$
$$= E\left[\int_0^{T\wedge\hat{\theta}} \left(-\delta_2 Y_y(t) + rK + \frac{1}{2}v^2(t)\right)e^{-rt}dt + m\big(Y_y(\hat{\theta})\big)e^{-r\hat{\theta}}\mathbb{1}_{\hat{\theta}<T}\right.$$
$$\left. + u\big(Y_y(T)\big)e^{-rT}\right].$$

Again, $Eu\big(Y_y(T)\big)e^{-rT} \leq \frac{\delta_2}{r}\big[-EY_y(T)e^{-rT} + c_2e^{-rT}\big]$ and from the transversality condition $Eu\big(Y_y(T)\big)e^{-rT} \to 0$ as $T\to\infty$. Therefore,

$$u(y) \leq J_y\big(v(\cdot),\hat{\vartheta}(\cdot)\big);$$

thus $u(y) \leq \sup_\theta \inf_{v(\cdot)} J_y\big(v(\cdot),\theta\big)$.

Now, considering again $\hat{Y}_y(t)$, we have for any stopping time θ

$$u(y) \geq E\left[\int_0^{T\wedge\theta} \left(-\delta_2 \widehat{Y}_y(t) + rK + \frac{1}{2}v^2(t)\right)e^{-rt}dt + u\big(\widehat{Y}_y(T\wedge\theta)\big)e^{-rT\wedge\theta}\right]$$
$$\geq E\left[\int_0^{T\wedge\theta} \left(-\delta_2 \widehat{Y}_y(t) + rK + \frac{1}{2}v^2(t)\right)e^{-rt}dt + m\big(\widehat{Y}_y(\theta)\big)e^{-re\theta}\mathbb{1}_{\theta<T}\right]$$

and, letting $T \to \infty$,

$$u(y) \geq J_y\big(\hat{v}(\cdot), \theta\big).$$

Therefore, $u(y) \geq \inf_{v(\cdot)} \sup_\theta J_y\big(v(\cdot), \theta\big)$. This completes the proof.

Appendix O. Proof of Theorem 6.6. We take y_1 to be the first point such that $u(y_1) = m(y_1)$. We must have $y_1 < \hat{y}$; otherwise we are in the case $m \leq 0$, which is excluded. We also have

$$u'(y_1) = m'(y_1) \tag{O.1}$$

and $\delta_1 y_1 \geq rK$.

Indeed, set $\tilde{u}(y) = u(y) - m(y)$. Then $\tilde{u}(y)$ satisfies

$$\begin{aligned} -\frac{1}{2}\varsigma^2\tilde{u}'' - \left(\alpha - \lambda\varsigma\rho - \left(\frac{\delta_2}{r} - m'\right)\varsigma^2 r\gamma(1-\rho^2)\right)\tilde{u}' + \frac{1}{2}\varsigma^2 r\gamma(1-\rho^2)\tilde{u}'^2 + r\tilde{u} \\ = -\delta_1 y + rK \end{aligned} \tag{O.2}$$

with $\tilde{u}(y_1) = 0$, $\tilde{u}'(y_1) = 0$. The matching of the derivatives comes from the fact that $\tilde{u}(y)$ is C^1 and $\tilde{u}(y) \geq 0$, $\tilde{u}(y_1) = 0$. So y_1 is the local minimum; hence $\tilde{u}'(y_1) = 0$ and we have (O.1). Now suppose $\delta_1 y_1 \leq rK$; from (O.2), we see that $\tilde{u}''(y_1 - 0) < 0$; therefore, $\tilde{u}(y) < 0$ for $y < y_1$ close to y_1, which is impossible.

Call y_2 the left end of the interval (y_2, y_3) with $y_3 = \bar{y}$ and $y_2 < \hat{y}$, on which the equation holds. So we have $u(y_2) = m(y_2)$, and necessarily $u'(y_2) = m'(y_2)$.

On the other hand, on the interval (y_1, y_2), m satisfies all conditions (6.52). So $u = m$ on the interval (y_1, y_2). By the uniqueness of u, the triple y_1, y_2, y_3 is necessarily unique.

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