Nonlinear Expectations, Nonlinear Evaluations and Risk Measures

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1 Introduction

1.1 Searching the Mechanism of Evaluations of Risky Assets

We are interested in the following problem: let $(X_t)_{0\leq t\leq T}$ be an $\mathbf{R}^d$–valued process, Y a random value depending on the trajectory of X. Assume that, at each fixed time $t \leq T$, the information available to an agent (an individual, a firm, or even a market) is the trajectory of X before t. Thus at time T, the random value $Y(\omega)$ will become known to this agent. The question is: how this agent evaluates Y at the time t? If this Y is traded in a financial market, it is called a derivative, i.e. a contract whose outcome depends on the evolution of the underlying process X. The output of this evaluation can be the maximum value the agent can accept to buy it or the minimum value to sell it. It may depend on his economic situation, his knowledge on the history of X, his risk aversion and utility function. In many situation this individual evaluation may be very different from the actual market price.

Examples of derivatives are futures and option contracts based on the underlying asset X, such as a commodity, a stock index, the interest rate, an exchange rate; or an individual stock; or a mortgage backed security. Here the term derivative is in general sense, i.e., it may be a positive or a negative number.

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K. Back et al.: LNM 1856, M. Frittelli and W. Runggaldier (Eds.), pp. 165–253, 2004.

The well–known Black & Scholes option pricing theory (1973) has made the most significant contribution, over the last 30 years, in modeling the evaluation of derivatives in financial markets.

One of the important limitations of Black–Scholes-Merton approach is that it is heavily based on the assumption that the statistic behavior of the stochastic process X is exogenously specified. The fact that the Black–Scholes pricing of Y is independent of the preference of the involved individuals is also frequently argued. On the other hand, in the situation where Y is not traded, the main arguments of BS model, i.e. the replication of a claim in an arbitrage–free market, are no longer viable, and the evaluation of Y is often preference–dependent.

In this lecture the evaluation of Y will be treated under a new viewpoint. We will introduce an evaluation operator $\mathcal{E}_{t,T}[Y]$ to define the value of Y evaluated by the agent at time t. This operator $\mathcal{E}_{t,T}[\cdot]$ assigns an $(X_s)_{0\leq s\leq T}$–dependent random variable Y to an $(X_s)_{0\leq s\leq t}$–dependent $\mathcal{E}_{t,T}[Y]$. Although this value $\mathcal{E}_{t,T}[Y]$ is very complicated and is different from one agent to anther, we can still find some axiomatic assumptions to describe the mathematical properties of this operator. The evaluation of Y is treated as a filtration consistent nonlinear expectation or, more general, a filtration consistent nonlinear evaluation. We will prove that this expectation or evaluation is completely determined by a simple function g.

1.2 Axiomatic Assumptions for Evaluations of Derivatives

General Situations: $\mathcal{F}_t^X$–Consistent Nonlinear Evaluations

Let us give a more specific formulation to the above evaluation problem. Let $X=(X_t)_{t\geq 0}$ be a d–dimensional process, it may be the prices of stocks in a financial market, the rates of exchanges, the rates of local and global inflations etc. We assume that at each time $t\geq 0$, the information of an agent (a firm, a group of people, a financial market) is the history of X during the time interval $[0,t]$. Namely, his actual filtration is

$$\mathcal{F}_t^X=\sigma\{X_s; s\leq t\}.$$

We denote the set of all real valued $\mathcal{F}_t^X$–measurable random variables by $m\mathcal{F}_t^X$. Under this notation an X–underlying derivative Y, with maturity $t\in[0,\infty)$, is an $\mathcal{F}_t^X$–measurable random variable, i.e., $Y\in m\mathcal{F}_t^X$. We will find the law of evaluation of Y at each time $s\in[0,t]$. We denote this evaluated value by $\mathcal{E}_{s,t}[Y]$. It is reasonable to assume that $\mathcal{E}_{s,t}[Y]$ is $\mathcal{F}_s^X$–measurable. We thus have the following system of evaluator: for each $Y\in m\mathcal{F}_t^X$

$$\mathcal{E}_{s,t}[Y]: m\mathcal{F}_t^X\longrightarrow m\mathcal{F}_s^X.$$

In particular

$$\mathcal{E}_{0,t}[Y]: m\mathcal{F}_t^X\longrightarrow \mathbf{R}.$$

We will make the following **Axiomatic Assumptions** for $(\mathcal{E}_{s,t}[\cdot])_{0\le s\le t<\infty}$:

(A1) Monotonicity: $\mathcal{E}_{s,t}[Y] \ge \mathcal{E}_{s,t}[Y']$, if $Y \ge Y'$.
(A2) $\mathcal{E}_{s,s}[Y] = Y$, if $Y \in m\mathcal{F}_s^X$, particularly $\mathcal{E}_{0,0}[c] = c$.
(A3) Time consistency: $\mathcal{E}_{s,t}[\mathcal{E}_{t,T}[Y]] = \mathcal{E}_{s,T}[Y]$, if $s \le t \le T$, $Y \in m\mathcal{F}_T^X$.
(A4) "Zero–one law": for each $s \le t$, $\mathcal{E}_{s,t}[1_A Y] = 1_A \mathcal{E}_{s,t}[Y]$, $\forall A \in \mathcal{F}_s^X$.

Remark 1.1. Conditions (A1) and (A2) are obvious. Condition (A3) means that at the time $t \le T$, $\mathcal{E}_{t,T}[Y]$ can be also treated as a derivative with the maturity t. At the time $s \le t$, the price $\mathcal{E}_{s,t}[\mathcal{E}_{t,T}[Y]]$ of this derivative is the same as the price of the original derivative Y with maturity T, i.e., $\mathcal{E}_{s,T}[Y]$.

Remark 1.2. The meaning of condition (A4) is: at time s, the agent knows whether $X_{\cdot\wedge s}$ is in A. If it is in A, then the value $\mathcal{E}_{s,t}[1_A Y]$ is the same as $\mathcal{E}_{s,t}[Y]$ $1_A Y = Y$. Otherwise $1_A Y$ is zero thus it costs nothing. A more generalization of (A4) is
(A4') For each $s \le t$,

$$1_A \mathcal{E}_{s,t}[1_A Y] = 1_A \mathcal{E}_{s,t}[Y], \ \ \forall A \in \mathcal{F}_s^X.$$

In this lecture we will not study this case (see Peng 2003 [Peng2003b]).

$\mathcal{F}_t^X$–Consistent Nonlinear Expectations

In many situations we assume furthermore, instead of **(A2)** , that

(A2') For each $0 \le s \le t$, $Y \in m\mathcal{F}_s^X$, $\mathcal{E}_{s,t}[Y] = Y$.

Remark 1.3. The meaning of condition (A2') is: the market has a zero–interesting rate, i.e., $r_t \equiv 0$. We observe that in many cases, even when $r_t \not\equiv 0$, we can still define the following discounted evaluation

$$\mathcal{E}_{t,T}^r[Y] := \mathcal{E}_{t,T}[Y \exp(-\int_t^T r_s ds)].$$

This $\mathcal{E}_{t,T}^r[\cdot]$ satisfies (A2').

Let us fix a sufficiently large $T < \infty$ and consider $\mathcal{E}_{s,t}[Y]$ for $0 \le s \le t \le T$ and $Y \in m\mathcal{F}_t^X$. By (A2')

$$\mathcal{E}_{s,t}[Y] = \mathcal{E}_{s,t}[\mathcal{E}_{t,T}[Y]] = \mathcal{E}_{s,T}[Y].$$

We then only need to treat $\mathcal{E}[Y|\mathcal{F}_s^X] := \mathcal{E}_{s,T}[Y]$:

$$\mathcal{E}[Y|\mathcal{F}_s^X] : m\mathcal{F}_T^X \to m\mathcal{F}_s^X,$$
$$\mathcal{E}[Y] = \mathcal{E}[Y|\mathcal{F}_0^X] : m\mathcal{F}_T^X \to \mathbf{R}.$$

By the Axiomatic assumptions, we have, for each Y, $Z \in m\mathcal{F}_T^X$ and $t \leq T$,

(A1) Monotonicity: $\mathcal{E}[Y|\mathcal{F}_t^X] \geq \mathcal{E}[Z|\mathcal{F}_t^X], \quad$ if $Y \geq Z$;
(A2') Constant–preserving: $\quad \mathcal{E}[Y|\mathcal{F}_t^X] = Y, \quad$ if $Y \in m\mathcal{F}_t^X$;
(A3) Time consistency: $\mathcal{E}[\mathcal{E}[Y|\mathcal{F}_t^X]|\mathcal{F}_s^X] = \mathcal{E}[Y|\mathcal{F}_s^X], \quad$ if $s \leq t \leq T$;
(A4) "Zero–one law": $\mathcal{E}[1_A Y|\mathcal{F}_t^X] = 1_A\mathcal{E}[Y|\mathcal{F}_t^X], \ \ \forall A \in \mathcal{F}_t^X$.

In particular, the functional $\mathcal{E}[\cdot]$ is a nonlinear expectation, i.e., it satisfies

(a1) Monotonicity: $\mathcal{E}[Y] \geq \mathcal{E}[Z]$, if $Y \geq Z$;
(a2) Constant–preserving: $\quad \mathcal{E}[c] = c$.

From (A3) and (A4) we have, each $0 \leq T < \infty$ and $Y \in m\mathcal{F}_T^X$,

$$\mathcal{E}[1_A\mathcal{E}[Y|\mathcal{F}_t^X]] = \mathcal{E}[1_A Y], \ \forall A \in \mathcal{F}_t^X. \tag{1}$$

We recall that this is just the classical definition of the conditional expectation given $\mathcal{F}_t^X$. In the next section we will prove that in nonlinear situations we can also derive all the Axiomatic assumptions (A1), (A2'), (A3) and (A4) by this definition (1) provided $\mathcal{E}$ is strictly monotone. In this case we call $\mathcal{E}[\cdot]$ an $\mathcal{F}_t^X$**–consistent nonlinear expectation**.

Remark 1.4. From the above reasoning it is clear that the Axiomatic assumptions (A1)–(A4) are also applied in many other situations to measuring a risky value Y in a dynamical situation. In fact, an advantage is that they are also workable in the situation where the risky value Y is not exchanged in markets. For example, a result of a decision is in general not exchangeable. For example, it is applicable to an individual or a group's evaluation of a derivative Y. In some situation an agent can not have all information $\mathcal{F}_t^X$, but this formulation can be also applied to the situation of partially observation, i.e., with a smaller filtration $\mathcal{G}_t \subset \mathcal{F}_t$, $t \geq 0$.

Remark 1.5. It is clear that for the formulation of an $\mathcal{F}_t^X$–consistent evaluation it is not needed to introduce an a priori probability space. But in this lecture we will be within the framework of Brownian Motion filtration $(\mathcal{F}_t)_{t\geq 0}$. For more general situation, see [Peng2002].

1.3 Organization of the Lecture

In the next section, we will give the formulations of filtration consistent evaluations and expectations under the filtration $\mathcal{F}_t$ generated by a Brownian Motion. Then in Section 3, we present BSDE theory and introduce a large sort of filtration consistent nonlinear evaluations and expectations, i.e., g–evaluations and g–expectations. This g–evaluation is entirely determined by a simple real function g. We also present a nonlinear decomposition theorem of Doob–Meyer's type, for the related g–supermartingale. This result plays a central role in Section 4, in which we will prove that the notion of g–expectations is large enough to represent all "regular" $\mathcal{F}_t$–consistent nonlinear expectations. This result permit us to find the simple mechanism, i.e., the function g, of the above apparently very abstract evaluations. We also provide a simple method to test and then find the function g. In Section 5, we present some basic method to solve numerically BSDE such as g–expectations and g–evaluations.

The nonlinear martingale theorem in self–content in this lecture, including the related upcrossing inequalities.

2 Brownian Filtration Consistent Evaluations and Expectations

2.1 Main Notations and Definitions

In this lecture, we will study the above evaluation problem within the following standard framework. Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$, $(B_t)_{t\geq 0}$ be a standard d-dimensional Brownian Motion defined on this space. We assume that $(\mathcal{F}_t)$ is the natural filtration of B:

$$\mathcal{F}_t = \sigma\{\sigma\{B_s;\, 0 \leq s \leq t\} \cup \mathcal{N}\}, \;\; \mathcal{F}^0_\infty := \bigcup_{t>0} \mathcal{F}_t.$$

where $\mathcal{N}$ is the collection of P–null sets in Ω. A vector valued stochastic process $X_t = X(\omega, t)$, $t \geq 0$, is said to be $\mathcal{F}_t$-adapted (or more specifically $(\mathcal{F}_t)_{0\leq t<\infty}$ -adapted), if for each $t \in [0, \infty)$, $(X_t(\cdot))$ is an $\mathcal{F}_t$–measurable random variable. $\mathcal{F}_t$ represents our information before time t. Thus the meaning that X is $\mathcal{F}_t$-adapted process is that at the current time t_0, we know all trajectories of X_t for $t \leq t_0$. All processes discussed in this lecture are assumed to be $\mathcal{F}_t$–adapted. We need the following notations. Let $p \geq 1$ and $\tau \leq T$ be a given $\mathcal{F}_t$–stopping time.

- The scalar product and norm of the Euclid space $\mathbf{R}^n$ are respectively denoted by $\langle \cdot, \cdot \rangle$ and $|\cdot|$.
- $L^p(\mathcal{F}_\tau; R^m)$:={the space of all real valued $\mathcal{F}_\tau$–measurable random variables such that $E[|\xi|^p] < \infty$};
- $L^p_{\mathcal{F}}(0, \tau; R^m)$:={R^m–valued and $\mathcal{F}_t$–adapted and stochastic processes such that $E\int_0^\tau |\phi_t|^p dt < \infty$};
- $D^p_{\mathcal{F}}(0, \tau; R^m)$:={all RCLL processes in $L^p_{\mathcal{F}}(0, \tau; R^m)$ such that $E[\sup_{0\leq t\leq \tau} |\phi_t|^p] < \infty$};
- $S^p_{\mathcal{F}}(0, \tau; R^m)$:={all continuous processes in $D^p_{\mathcal{F}}(0, \tau; R^m)$ };
- $\mathcal{S}_T$:={the collection of all $\mathcal{F}_t$–stopping times bounded by $\tau \leq T$};

- $\mathcal{S}_T^0 := \{\tau \in \mathcal{S}_T$ and $\cup_{i=1}^n \{\tau = t_i\} = \Omega$, with some deterministic $0 \leq t_1 < \cdots < t_N\}$.

In the case $m = 1$, we denote them by $L^p(\mathcal{F}_\tau)$, $L^p_{\mathcal{F}}(0,\tau)$, $D^p_{\mathcal{F}}(0,\tau)$ and $S^p_{\mathcal{F}}(0,\tau)$. We observe that all elements in $D^2_{\mathcal{F}}(0,T)$ are $\mathcal{F}_t$–predictable. When $p = 2$, the above L^p are separable Hilbert spaces.

We observe the following fact: for each $\phi \in L^p_{\mathcal{F}}(0,T)$ there exists a progressively measurable process $\bar{\phi}$ which is stochastically equivalent to ϕ, i.e.,

$$P(\omega : \phi_t(\omega) = \bar{\phi}_t(\omega)) = 1, \ \forall t \in [0,T].$$

In this lecture, we will not distinguish the two processes.

We now give a rigorous definition of $\mathcal{F}_t$–consistent evaluations and expectations:

Definition 2.1. *The system of operators*

$$\mathcal{E}_{s,t}[\cdot] : L^2(\mathcal{F}_t) \to L^2(\mathcal{F}_s), \ 0 \leq s \leq t \leq T \tag{2}$$

is called an $\mathcal{F}_t$–consistent nonlinear evaluation defined on $L^2(\mathcal{F}_T)$ if for each $0 \leq s \leq t < T$ and for each Y and $Y' \in L^2(\mathcal{F}_t)$, we have

(A1) Monotonicity: $\mathcal{E}_{t,T}[Y] \geq \mathcal{E}_{t,T}[Y']$, a.s., if $Y \geq Y'$, a.s.;
(A2) $\mathcal{E}_{t,t}[Y] = Y$, if $Y \in L^2(\mathcal{F}_t)$, a.s., particularly $\mathcal{E}_{0,0}[c] = c$;
(A3) Time consistency: $\mathcal{E}_{r,s}[\mathcal{E}_{s,t}[Y]] = \mathcal{E}_{r,t}[Y]$, a.s., if $r \leq s \leq t \leq T$;
(A4) "Zero–one law": for each $s \leq t$, $\mathcal{E}_{s,t}[1_A Y] = 1_A \mathcal{E}_{s,t}[Y]$, a.s., $\forall A \in \mathcal{F}_s$.

Remark 2.1. By (A4) it is easy to check that $\mathcal{E}_{s,t}[0] = 0$, a.s.. A condition weaker than (A4) is

(A4') For each $s \leq t$, $1_A \mathcal{E}_{s,t}[1_A Y] = 1_A \mathcal{E}_{s,t}[Y]$, a.s., $\forall A \in \mathcal{F}_s$.

As we discussed in the introduction, if (A2) is strengthen to

(A2') $\mathcal{E}_{s,t}[Y] = Y$, a.s., $\forall Y \in L^2(\mathcal{F}_s)$

then we have

Proposition 2.1. *We assume (A1), (A2'), (A3) and (A4). Then, with the definition*

$$\mathcal{E}[Y|\mathcal{F}_t] := \mathcal{E}_{t,T}[Y], \ a.s., \ Y \in L^2(\mathcal{F}_T) \tag{3}$$

We have

(A1) Monotonicity: $\mathcal{E}[Y|\mathcal{F}_t] \geq \mathcal{E}[Z|\mathcal{F}_t]$, a.s., if $Y \geq Z$, a.s.;
(A2') Constant–preserving: $\mathcal{E}[Y|\mathcal{F}_t] = Y$, a.s., if $Y \in L^2(\mathcal{F}_t)$;
(A3) Time consistency: $\mathcal{E}[\mathcal{E}[Y|\mathcal{F}_t]|\mathcal{F}_s] = \mathcal{E}[Y|\mathcal{F}_s]$, a.s., if $s \leq t \leq T$;
(A4) "Zero–one law": for each t, $\mathcal{E}[1_A Y|\mathcal{F}_t] = 1_A \mathcal{E}[Y|\mathcal{F}_t]$, a.s., $\forall A \in \mathcal{F}_t$.

Definition 2.2. *The system of operators*

$$\mathcal{E}[\cdot|\mathcal{F}_t] : L^2(\mathcal{F}_T) \to L^2(\mathcal{F}_t),\ 0 \le t < T \tag{4}$$

satisfying the above axiomatic assumptions (A1), (A2'), (A3) and (A4) is called an $\mathcal{F}_t$–consistent nonlinear expectation (or simply $\mathcal{F}$–expectation) defined on $L^2(\mathcal{F}_T)$.

2.2 $\mathcal{F}_t$–Consistent Nonlinear Expectations

The above $\mathcal{F}_t$–consistent nonlinear expectations can be also introduced in a classical way, beginning from the notion of nonlinear expectations:

Definition 2.3. *A nonlinear expectation defined on $L^2(\mathcal{F}_T)$ is a functional:*

$$\mathcal{E}[\cdot] : L^2(\mathcal{F}_T) \longmapsto \mathbf{R}$$

satisfying the following properties: (a1) Strict monotonicity:

$$\begin{gathered} \textit{if} \quad Y_1 \ge Y_2 \quad a.s., \ \textit{then} \quad \mathcal{E}[Y_1] \ge \mathcal{E}[Y_2]; \\ \textit{if} \quad Y_1 \ge Y_2 \quad a.s., \quad \mathcal{E}[Y_1] = \mathcal{E}[Y_2] \quad \iff \quad Y_1 = Y_2 \quad a.s. \end{gathered}$$

(a2) preserving of constants:

$$\mathcal{E}[c] = c, \quad \textit{for each constant } c.$$

Lemma 2.1. *Let $t \le T$ and $\eta_1, \eta_2 \in L^2(\mathcal{F}_t)$. If for each $A \in \mathcal{F}_t$,*

$$\mathcal{E}[\eta_1 1_A] = \mathcal{E}[\eta_2 1_A],$$

then we have

$$\eta_2 = \eta_1, \quad a.s. \tag{5}$$

Proof. We choose $A = \{\eta_1 \ge \eta_2\} \in \mathcal{F}_t$. Since $(\eta_1 - \eta_2)1_A \ge 0$ and $\mathcal{E}[\eta_1 1_A] = \mathcal{E}[\eta_2 1_A]$, it follows that $\eta_1 1_A = \eta_2 1_A$ a.s.. Thus $\eta_2 \ge \eta_1$ a.s. With the same argument we can prove that $\eta_1 \ge \eta_2$ a.s. It follows that (5) holds. The proof is complete. □

Definition 2.4. *A nonlinear expectation is called an $\mathcal{F}$-expectation defined on $L^2(\mathcal{F}_T)$ if for each $Y \in L^2(\mathcal{F}_T)$ and $t \in [0,T]$, there exists a random variable $\eta \in L^2(\mathcal{F}_t)$, such that*

$$\mathcal{E}[Y 1_A] = \mathcal{E}[\eta 1_A], \quad \forall A \in \mathcal{F}_t. \tag{6}$$

From Lemma 2.1, such an η is uniquely defined. We also denote it by $\eta = \mathcal{E}[Y|\mathcal{F}_t]$. $\mathcal{E}[Y|\mathcal{F}_t]$ is called the conditional $\mathcal{F}$-expectation of Y under $\mathcal{F}_t$. It is characterized by

$$\mathcal{E}[Y 1_A] = \mathcal{E}[\mathcal{E}[Y|\mathcal{F}_t] 1_A], \quad \forall A \in \mathcal{F}_t. \tag{7}$$

We will see that, in fact this definition of $\mathcal{E}[\cdot|\mathcal{F}_t]$ coincides with Definition 2.2. Indeed, we have the following lemmas. The first one checks (A3) and (A2'):

Lemma 2.2. *We have, for each* $0 \le s \le t \le T$ *and* $Y \in \mathcal{F}_T$,

$$\mathcal{E}[\mathcal{E}[Y|\mathcal{F}_t]|\mathcal{F}_s] = \mathcal{E}[Y|\mathcal{F}_s] \quad a.s. \ . \tag{8}$$

In particular,

$$\mathcal{E}[\mathcal{E}[Y|\mathcal{F}_t]] = \mathcal{E}[Y]. \tag{9}$$

If $Y \in L^2(\mathcal{F}_t)$, *we also have*

$$\mathcal{E}[Y|\mathcal{F}_t] = Y, \quad a.\ s..$$

Proof. Since $A \in \mathcal{F}_s \subset \mathcal{F}_t$, thus

$$\begin{aligned}\mathcal{E}[\mathcal{E}[\mathcal{E}[Y|\mathcal{F}_t]|\mathcal{F}_s]1_A] &= \mathcal{E}[\mathcal{E}[Y|\mathcal{F}_t]1_A] \\ &= \mathcal{E}[Y1_A] \\ &= \mathcal{E}[\mathcal{E}[Y|\mathcal{F}_s]1_A].\end{aligned}$$

It follows from Lemma 2.1 that (8) holds. (9) follows easily from the fact that $\mathcal{F}_0$ is the trivial σ-algebra (since $B_0 = 0$). Finally, if $Y \in L^2(\mathcal{F}_t)$, then the only $\eta \in L^2(\mathcal{F}_t)$ satisfying (6) is Y itself. Thus (A2') holds. $\square$

The second lemma checks (A4):

Lemma 2.3. *We have*

$$\mathcal{E}[Y1_A|\mathcal{F}_t] = \mathcal{E}[Y|\mathcal{F}_t]1_A, \quad \forall A \in \mathcal{F}_t, \quad a.s. \ . \tag{10}$$

Proof. For each $B \in \mathcal{F}_t$, we have

$$\begin{aligned}\mathcal{E}[\mathcal{E}[Y1_A|\mathcal{F}_t]1_B] &= \mathcal{E}[Y1_A1_B] \\ &= \mathcal{E}[\mathcal{E}[Y|\mathcal{F}_t]1_{A\cap B}] \\ &= \mathcal{E}[[\mathcal{E}[Y|\mathcal{F}_t]1_A]1_B].\end{aligned}$$

Thus (10) holds. $\square$

$\mathcal{E}[\cdot|\mathcal{F}_t]$ has also the monotonicity property:

Lemma 2.4. *For any* X, $Y \in L^2(\mathcal{F}_T)$, *if* $X \le Y$ *a.s., then we have for each* $t \in [0,T]$,

$$\mathcal{E}[X|\mathcal{F}_t] \le \mathcal{E}[Y|\mathcal{F}_t] \quad a.s.$$

In this case, if for some $t \in [0,T)$, *one has* $\mathcal{E}[X|\mathcal{F}_t] = \mathcal{E}[Y|\mathcal{F}_t]$, *a.s., then* $X = Y$, *a.s..*

Proof. Define $X_t = \mathcal{E}[X|\mathcal{F}_t]$ and $Y_t = \mathcal{E}[Y|\mathcal{F}_t]$, and let $A \in \mathcal{F}_t$. Because of the monotonicity of $\mathcal{E}$, we have

$$\mathcal{E}[X_t1_A] = \mathcal{E}[X1_A] \le \mathcal{E}[Y1_A] = \mathcal{E}[Y_t1_A].$$

Now, take $A = \{X_t > Y_t\}$. If $P(A) > 0$, the strict monotonicity of $\mathcal{E}$ implies that

$$\mathcal{E}[X_t 1_A] > \mathcal{E}[Y_t 1_A].$$

Comparing the two above inequalities, we conclude that $P(A) = 0$.

Now if for some $t \in [0, T)$, one has $\mathcal{E}[X|\mathcal{F}_t] = \mathcal{E}[Y|\mathcal{F}_t]$, then $\mathcal{E}[X] = \mathcal{E}[Y]$. It follows from the strict monotonicity of $\mathcal{E}[\cdot]$ that $X = Y$, a.s.. $\square$

We then can conclude

Proposition 2.2. *Let $\mathcal{E}[\cdot]$ be defined in Definition 2.3. If for each $0 \le t \le T < \infty$ and $Y \in L^2(\mathcal{F}_T)$, there exists a $\mathcal{E}[Y|\mathcal{F}_t] \in L^2(\Omega, \mathcal{F}_t, P)$ satisfying relation (7), then $(\mathcal{E}[Y|\mathcal{F}_t])_{0 \le t < T}$ is an $\mathcal{F}_t$–consistent nonlinear expectation defined on $L^2(\mathcal{F}_T)$.*

Proof. We have already (A1), (A3) and (A4). (A2') can be checked by a similar argument. $\square$

Lemma 2.5. *For any Y, $Y' \in L^2(\mathcal{F}_T)$ and for each $t \in [0, T]$ and $A \in \mathcal{F}_t$ we have*

$$\mathcal{E}[Y 1_A + Y' 1_{A^C}|\mathcal{F}_t] = \mathcal{E}[Y|\mathcal{F}_t] 1_A + \mathcal{E}[Y'|\mathcal{F}_t] 1_{A^C} \tag{11}$$

Proof. We have

$$\begin{aligned}
\mathcal{E}[Y 1_A + Y' 1_{A^C}|\mathcal{F}_t] &= \mathcal{E}[Y 1_A + Y' 1_{A^C}|\mathcal{F}_t] 1_A + \mathcal{E}[Y 1_A + Y' 1_{A^C}|\mathcal{F}_t] 1_{A^C} \\
&= \mathcal{E}[(Y 1_A + Y' 1_{A^C}) 1_A|\mathcal{F}_t] + \mathcal{E}[(Y 1_A + Y' 1_{A^C}) 1_{A^C}|\mathcal{F}_t] \\
&= \mathcal{E}[Y 1_A|\mathcal{F}_t] + \mathcal{E}[Y' 1_{A^C}|\mathcal{F}_t] \\
&= \mathcal{E}[Y|\mathcal{F}_t] 1_A + \mathcal{E}[Y'|\mathcal{F}_t] 1_{A^C}.
\end{aligned}$$

$\square$

Remark 2.2. (11) is equivalent to (A4'): $1_A \mathcal{E}[Y 1_A|\mathcal{F}_t] = 1_A \mathcal{E}[Y|\mathcal{F}_t]$.

2.3 $\mathcal{F}_t$-Consistent Nonlinear Evaluations

Just as in Subsection 2.2, we can also introduce $\mathcal{F}_t$-consistent nonlinear evaluations in the following way:

Definition 2.5. *An evaluation is a family of nonlinear functionals parameterized by $t \in [0, T]$*

$$\mathcal{E}_{0,t}[\cdot] : L^2(\mathcal{F}_t) \longmapsto \mathbf{R}$$

which satisfies the following strict monotonicity properties: for each $t \ge 0$ and $Y_1, Y_2 \in L^2(\mathcal{F}_t)$, we have

$$\begin{aligned}
\textit{if} \quad Y_1 \ge Y_2 \quad \textit{a.s., then} \quad & \mathcal{E}_{0,t}[Y_1] \ge \mathcal{E}_{0,t}[Y_2]; \\
\textit{if} \quad Y_1 \ge Y_2 \quad \textit{a.s., then } \mathcal{E}_{0,t}[Y_1] = \mathcal{E}_{0,t}[Y_2] \quad & \textit{iff} \quad Y_1 = Y_2 \quad \textit{a.s..}
\end{aligned}$$

Lemma 2.6. *For each $t \le T$ and $\eta_1, \eta_2 \in L^2(\mathcal{F}_t)$. If*

$$\mathcal{E}_{0,t}[\eta_1 1_A] \le \mathcal{E}_{0,t}[\eta_2 1_A], \quad \forall A \in \mathcal{F}_t,$$

then

$$\eta_1 \le \eta_2, \quad a.s.$$

If

$$\mathcal{E}_{0,t}[\eta_1 1_A] = \mathcal{E}_{0,t}[\eta_2 1_A], \quad \forall A \in \mathcal{F}_t,$$

then

$$\eta_2 = \eta_1, \quad a.s. \tag{12}$$

Proof. To prove the first assertion, we set $A = \{\eta_1 \ge \eta_2\} \in \mathcal{F}_t$. Since $(\eta_1 - \eta_2)1_A \ge 0$, thus the monotonicity yields $\mathcal{E}_{0,t}[\eta_1 1_A] \ge \mathcal{E}_{0,t}[\eta_2 1_A]$. With $\mathcal{E}_{0,t}[\eta_1 1_A] \le \mathcal{E}_{0,t}[\eta_2 1_A]$, it then follows from the strict monotonicity that $\eta_1 1_A = \eta_2 1_A$ a.s.. i.e., $\eta_1 \le \eta_2$ a.s. The second assertion is a simple consequence of the first one. □

We can also define $\mathcal{F}-$ evaluation operators

Definition 2.6. *A nonlinear evaluation $(\mathcal{E}_{0,t}[\cdot])_{t\in[0,T]}$ defined on $L^2(\mathcal{F}_T)$ is called $\mathcal{F}$-evaluation if for each $0 \le s \le t \le T$ and $Y \in L^2(\mathcal{F}_t)$ there exists a random variable $\eta \in L^2(\mathcal{F}_s)$, such that*

$$\mathcal{E}_{0,t}[Y 1_A] = \mathcal{E}_{0,s}[\eta 1_A], \quad \forall A \in \mathcal{F}_s.$$

From Lemma 2.6, such η is uniquely defined. We denote it by $\eta = \mathcal{E}_{s,t}[Y]$. $\mathcal{E}_{s,t}[\cdot]$ satisfies

$$\mathcal{E}_{0,t}[Y 1_A] = \mathcal{E}_{0,s}[\mathcal{E}_{s,t}[Y] 1_A], \quad \forall A \in \mathcal{F}_s. \tag{13}$$

We can prove that $(\mathcal{E}_{s,t}[\cdot])_{0\le s\le t\le T}$ is the $\mathcal{F}_t$–consistent nonlinear evaluation defined on $L^2(\mathcal{F}_T)$. We first check (A3):

Lemma 2.7. *For each $0 \le r \le s \le t \le T$ and $Y \in L^2(\mathcal{F}_t)$, we have*

$$\mathcal{E}_{r,s}[\mathcal{E}_{s,t}[Y]] = \mathcal{E}_{r,t}[Y] \quad a.s. \tag{14}$$

In particular,

$$\mathcal{E}_{0,s}[\mathcal{E}_{s,t}[Y]] = \mathcal{E}_{0,t}[Y] \quad a.s.. \tag{15}$$

Proof. Since $A \in \mathcal{F}_r \subset \mathcal{F}_s$. Thus by (13),

$$\begin{aligned}\mathcal{E}_{0,r}[\mathcal{E}_{r,s}[\mathcal{E}_{s,t}[Y]]1_A] &= \mathcal{E}_{0,s}[\mathcal{E}_{s,t}[Y]1_A] \\ &= \mathcal{E}_{0,t}[Y 1_A] \\ &= \mathcal{E}_{0,r}[\mathcal{E}_{r,t}[Y]1_A].\end{aligned}$$

It follows from Lemma 2.6 that (14) holds.

Let $r = 0$, (15) follows then easily from the fact that $\mathcal{F}_0$ is the trivial σ-algebra (since $B_0 = 0$). □

We then check (A4):

Lemma 2.8. *For each* $0 \le s \le t$, $Y \in L^2(\mathcal{F}_t)$ *and* $A \in \mathcal{F}_s$, *we have*

$$\mathcal{E}_{s,t}[Y1_A] = \mathcal{E}_{s,t}[Y]1_A, \quad a.s.. \tag{16}$$

Proof. For each $0 \le s \le t$ and $B \in \mathcal{F}_s$, we have, by (13),

$$\begin{aligned}\mathcal{E}_{0,s}[\mathcal{E}_{s,t}[Y1_A]1_B] &= \mathcal{E}_{0,t}[Y1_A1_B] \\ &= \mathcal{E}_{0,s}[\mathcal{E}_{s,t}[Y]1_{A\cap B}] \\ &= \mathcal{E}_{0,s}[[\mathcal{E}_{s,t}[Y]1_A]1_B].\end{aligned}$$

It follows from Lemma 2.6 that (16) holds. □

We also have (A2):

Lemma 2.9. *For each* $0 \le t < T$, *and* $\eta \in L^2(\mathcal{F}_t)$, *we have*

$$\mathcal{E}_{t,t}[\eta] = \eta, \ a.s..$$

Proof. By (13) we have

$$\mathcal{E}_{0,t}[\mathcal{E}_{t,t}[\eta]1_A] = \mathcal{E}_{0,t}[\eta 1_A], \ \ \forall A \in L^2(\mathcal{F}_t).$$

□

We then can conclude

Proposition 2.3. *Let* $(\mathcal{E}_{0,t}[\cdot])_{t\in[0,T]}$ *be a nonlinear evaluation defined on* $L^2(\mathcal{F}_T)$. *If for each* $0 \le s \le t \le T$ *and* $Y \in L^2(\mathcal{F}_t)$, *there exists an* $\mathcal{E}_{s,t}[Y] \in L^2(\mathcal{F}_s)$ *satisfying relation (13), then* $(\mathcal{E}_{s,t}[Y])_{0\le s\le t<T}$ *satisfies the Axiomatic assumptions (A1)–(A4) listed in Definition 2.1.*

Proof. The above three lemmas have proved (A2)–(A4). (A1) is a direct consequence of the first assertion of Lemma 2.6. □

Moreover, we can prove the following strict monotonicity $\mathcal{E}_{s,t}[\cdot]$.

Lemma 2.10. *For each* $0 \le s \le t \le T$ *and* X, $Y \in L^2(\mathcal{F}_t)$ *such that* $X \le Y$ *a.s., if* $\mathcal{E}_{s,t}[X] = \mathcal{E}_{s,t}[Y]$, *a.s, then* $X = Y$ *a.s..*

Proof. Since $\mathcal{E}_{s,t}[X] = \mathcal{E}_{s,t}[Y]$, thus

$$\begin{aligned}\mathcal{E}_{0,t}[X] &= \mathcal{E}_{0,s}[\mathcal{E}_{s,t}[X]] = \mathcal{E}_{0,s}[\mathcal{E}_{s,t}[Y]] \\ &= \mathcal{E}_{0,t}[Y].\end{aligned}$$

It follows from the strict monotonicity of $\mathcal{E}[\cdot]$ that $X = Y$, a.s..

□

We also have the following properties

Lemma 2.11. *For each* $0 \le s \le t \le T$, X, $Y \in L^2(\mathcal{F}_t)$ *and* $A \in \mathcal{F}_s$, *we have*

$$\mathcal{E}_{s,t}[X1_A + Y1_{A^C}] = \mathcal{E}_{s,t}[X]1_A + \mathcal{E}_{s,t}[Y]1_{A^C}.$$

Proof. According to Lemma 2.8,

$$\begin{aligned}\mathcal{E}_{s,t}[X1_A + Y1_{A^C}] &= \mathcal{E}_{s,t}[X1_A + Y1_{A^C}]1_A + \mathcal{E}_{s,t}[X1_A + Y1_{A^C}]1_{A^C} \\ &= \mathcal{E}_{s,t}[(X1_A + Y1_{A^C})1_A] + \mathcal{E}_{s,t}[(X1_A + Y1_{A^C})1_{A^C}] \\ &= \mathcal{E}_{s,t}[X1_A] + \mathcal{E}_{s,t}[Y1_{A^C}] \\ &= \mathcal{E}_{s,t}[X]1_A + \mathcal{E}_{s,t}[Y]1_{A^C}.\end{aligned}$$

□

3 Backward Stochastic Differential Equations: g–Evaluations and g–Expectations

3.1 BSDE: Existence, Uniqueness and Basic Estimates

BSDE Theory plays a central role in this lecture. A lot of $\mathcal{F}_t$–consistent nonlinear expectations and evaluations are derived by BSDEs. We first consider the following form of BSDE:

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s. \tag{17}$$

The setting of our problem is somewhat unusual: to find a pair of $\mathcal{F}_t$-adapted processes $(Y, Z) \in L^2_{\mathcal{F}}(0, T; \mathbf{R}^m \times \mathbf{R}^{m\times d})$ satisfying BSDE (17).

Remark 3.1. The solution Y is an ordinary Itô's process:

$$Y_t = Y_0 - \int_0^t g(s, Y_s, Z_s)ds + \int_0^t Z_s dB_s.$$

To prove the existence and uniqueness of BSDE (17) we first consider a very simple case: g is a real valued process that is independent of the variable (y, z). We have

Lemma 3.1. *For a fixed* $\xi \in L^2(\mathcal{F}_T)$ *and* $g_0(\cdot)$ *satisfying*

$$\mathbf{E}(\int_0^T |g_0(t)|dt)^2 < \infty,$$

there exists a unique pair of processes $(y_\cdot, z_\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbf{R}^{1+d})$, *satisfies the following BSDE*

$$y_t = \xi + \int_t^T g_0(s)ds - \int_t^T z_s dB_s. \tag{18}$$

If $g_0(\cdot) \in L^2_{\mathcal{F}}(0,T)$, then $(y_\cdot, z_\cdot) \in S^2_{\mathcal{F}}(0,T) \times L^2_{\mathcal{F}}(0,T;\mathbf{R}^d)$. We have the following basic estimate:

$$\begin{aligned} &|y_t|^2 + \mathbf{E}^{\mathcal{F}_t} \int_t^T [\frac{\beta}{2}|y_s|^2 + |z_s|^2] e^{\beta(s-t)} ds \\ &\le \mathbf{E}^{\mathcal{F}_t}|\xi|^2 e^{\beta(T-t)} + \frac{2}{\beta}\mathbf{E}^{\mathcal{F}_t} \int_t^T |g_0(s)|^2 e^{\beta(s-t)} ds \end{aligned} \tag{19}$$

In particular

$$\begin{aligned} &|y_0|^2 + \mathbf{E} \int_0^T [\frac{\beta}{2}|y_s|^2 + |z_s|^2] e^{\beta s} ds \\ &\le \mathbf{E}|\xi|^2 e^{\beta T} + \frac{2}{\beta}\mathbf{E} \int_0^T |g_0(s)|^2 e^{\beta s} ds, \end{aligned} \tag{20}$$

where β is an arbitrary constant. We also have

$$\mathbf{E}[\sup_{0\le t\le T} |y_t|^2] \le c\ \mathbf{E}[|\xi|^2 + \int_0^T |g_0(s)|^2 ds], \tag{21}$$

where the constant c depends only on T.

Proof. We define

$$M_t = \mathbf{E}^{\mathcal{F}_t}[\xi + \int_0^T g_0(s) ds].$$

M is a square integrable $(\mathcal{F}_t)$-martingale. By representation theorem of Brownian martingale (see Lemma 7.1), there exists a unique adapted process $(z_t) \in L^2_{\mathcal{F}}(0,T;\mathbf{R}^d)$ such that

$$M_t = M_0 + \int_0^t z_s dB_s.$$

Thus

$$M_t = M_T - \int_t^T z_s dB_s.$$

We denote

$$y_t = M_t - \int_0^t g_0(s) ds = M_T - \int_0^t g_0(s) ds - \int_t^T z_s dB_s.$$

Since $M_T = \xi + \int_0^T g_0(s) ds$, we have immediately (18).

The uniqueness is a simple consequence of the estimate (20). We only need to prove these two estimates. To prove (19), we first consider the case where ξ and $g_0(\cdot)$ are both bounded. Since

$$y_t = \mathbf{E}^{\mathcal{F}_t}\left[\xi + \int_t^T g_0(s) ds\right]$$

thus the process y is also bounded. We then apply Itô's formula to $|y_s|^2e^{\beta s}$ for $s\in[t,T]$:

$$\begin{aligned}&|y_t|^2e^{\beta t}+\int_t^T[\beta|y_s|^2+|z_s|^2]e^{\beta s}ds\\ &=|\xi|^2e^{\beta T}+\int_t^T 2y_sg_0(s)e^{\beta s}ds-\int_t^T e^{\beta s}2y_sz_sdB_s.\end{aligned}$$

We take conditional expectation under $\mathcal{F}_t$ on both sides of the above relation:

$$\begin{aligned}&|y_t|^2e^{\beta t}+\mathbf{E}^{\mathcal{F}_t}\int_t^T[\beta|y_s|^2+|z_s|^2]e^{\beta s}ds\\ &=\mathbf{E}^{\mathcal{F}_t}|\xi|^2e^{\beta T}+\mathbf{E}^{\mathcal{F}_t}\int_t^T 2y_sg_0(s)e^{\beta s}ds.\end{aligned}$$

Thus

$$\begin{aligned}&|y_t|^2+\mathbf{E}^{\mathcal{F}_t}\int_t^T[\beta|y_s|^2+|z_s|^2]e^{\beta(s-t)}ds\\ &=\mathbf{E}^{\mathcal{F}_t}|\xi|^2e^{\beta(T-t)}+\mathbf{E}^{\mathcal{F}_t}\int_t^T 2y_sg_0(s)e^{\beta(s-t)}ds\\ &\le\mathbf{E}^{\mathcal{F}_t}|\xi|^2e^{\beta(T-t)}+\mathbf{E}^{\mathcal{F}_t}\int_t^T[\frac{\beta}{2}|y_s|^2+\frac{2}{\beta}|g_0(s)|^2]e^{\beta(s-t)}ds.\end{aligned}$$

From this it follows (19) and (20).

We now consider the case where ξ and $g_0(\cdot)$ are possibly unbounded. We set

$$\xi^n:=(\xi\wedge n)\vee(-n),\qquad g_0^n(s):=(g_0(s)\wedge n)\vee(-n)$$

and

$$y_t^n:=\xi^n+\int_t^T g_0^n(s)ds-\int_t^T z_s^ndB_s. \tag{22}$$

We observe that, for each positive integers n and k, ξ^n, ξ^k, g_0^n as well as g_0^k are all bounded. We thus have

$$\begin{aligned}&|y_t^n|^2e^{\beta t}+\mathbf{E}^{\mathcal{F}_t}\int_t^T[\beta|y_s^n|^2+|z_s^n|^2]e^{\beta s}ds \qquad (23)\\ &=\mathbf{E}^{\mathcal{F}_t}|\xi^n|^2e^{\beta T}+\mathbf{E}^{\mathcal{F}_t}\int_t^T 2y_s^ng_0^n(s)e^{\beta s}ds\end{aligned}$$

and

$$\begin{aligned}&|y_t^n|^2+\mathbf{E}^{\mathcal{F}_t}\int_t^T[\frac{\beta}{2}|y_s^n|^2+|z_s^n|^2]e^{\beta(s-t)}ds \qquad (24)\\ &\le\mathbf{E}^{\mathcal{F}_t}|\xi^n|^2e^{\beta(T-t)}+\frac{2}{\beta}\mathbf{E}^{\mathcal{F}_t}\int_t^T|g_0^n(s)|^2e^{\beta(s-t)}ds\end{aligned}$$

as well as

$$\mathbf{E}\int_0^T [\frac{\beta}{2}|y_s^n - y_s^k|^2 + |z_s^n - z_s^k|^2]e^{\beta s}ds$$
$$\leq \mathbf{E}|\xi^n - \xi^k|^2 e^{\beta T} + \frac{2}{\beta}\mathbf{E}\int_0^T |g_0^n(s) - g_0^k(s)|^2 e^{\beta s}ds.$$

The last inequality implies that both $\{y^n\}$ and $\{z^n\}$ are Cauchy sequences in $L^2_{\mathcal{F}}(0,T)$. Thus (19) is proved by letting n tends to ∞ in (24).

It is clear that the solution y has continuous paths. (21) is a simple consequence of (20) together with B-D-G inequality applied to (18). Thus $y \in S^2_{\mathcal{F}}(0,T)$. □

Remark 3.2. By passing to the limit in both sides of (22) as $n \to \infty$, we also have the relation

$$|y_t|^2 e^{\beta t} + \mathbf{E}^{\mathcal{F}_t}\int_t^T [\beta|y_s|^2 + |z_s|^2]e^{\beta s}ds \tag{25}$$
$$= \mathbf{E}^{\mathcal{F}_t}|\xi|^2 e^{\beta T} + \mathbf{E}^{\mathcal{F}_t}\int_t^T 2y_s g_0(s)e^{\beta s}ds$$

and, in particular,

$$|y_0|^2 + \mathbf{E}\int_0^T [\beta|y_s|^2 + |z_s|^2]e^{\beta s}ds \tag{26}$$
$$= \mathbf{E}|\xi|^2 e^{\beta T} + \mathbf{E}\int_0^T 2y_s g_0(s)e^{\beta s}ds.$$

With the above basic estimates we can consider the general case of BSDE (17). We assume that

$$g = g(\omega, t, y, z) : \Omega \times [0,T] \times \mathbf{R}^m \times \mathbf{R}^{m\times d} \to \mathbf{R}^m$$

satisfies the following conditions: for each $(y,z) \in \mathbf{R}^m \times \mathbf{R}^{m\times d}$, $g(\cdot, y, z)$ is an $\mathbf{R}^m$–valued and $\mathcal{F}_t$–adapted process satisfying the Lipschitz condition in (y,z), i.e., for each $y, y' \in \mathbf{R}^m$ and $z, z' \in \mathbf{R}^{m\times d}$

$$|g(t,y,z) - g(t,y',z')| \leq C(|y-y'| + |z-z'|). \tag{27}$$

We also assume

$$g(\cdot, 0, 0) \in L^2_{\mathcal{F}}(0,T). \tag{28}$$

The following is the basic result of BSDE: the existence and uniqueness theorem.

Theorem 3.1. *Assume that g satisfies (28) and (27). Then for any given terminal condition $\xi \in L^2(\mathcal{F}_T; \mathbf{R}^m)$, the BSDE (17) has a unique solution, i.e., there exists a unique pair of $\mathcal{F}_t$-adapted processes $(Y,Z) \in S^2_{\mathcal{F}}(0,T;\mathbf{R}^m) \times L^2_{\mathcal{F}}(0,T;\mathbf{R}^{m\times d})$ satisfying (17).*

Proof. In the basic estimate (19) we fix $\beta = 8(1+C^2)$, where C is the Lipschitz constant of g in (y,z). To this β, we introduce a norm in the Hilbert space $L^2_{\mathcal{F}}(0,T;\mathbf{R}^n)$:

$$\|v(\cdot)\|_\beta \equiv \{\mathbf{E}\int_0^T |v_s|^2 e^{\beta s} ds\}^{\frac{1}{2}}.$$

Clearly this is equivalent to the original norm of $L^2_{\mathcal{F}}(0,T;\mathbf{R}^n)$. But this norm is more convenient to construct a contraction mapping in order to apply the fixed point theorem. We thus set

$$Y_t = \xi + \int_t^T g(s, y_s, z_s) ds - \int_t^T Z_s dB_s$$

We define a mapping

$$I[(y.,z.)] := (Y.,Z.) : L^2_{\mathcal{F}}(0,T;\mathbf{R}^m \times \mathbf{R}^{m\times d}) \to L^2_{\mathcal{F}}(0,T;\mathbf{R}^m \times \mathbf{R}^{m\times d}).$$

We need to prove that I is a contraction mapping under the norm $\|\cdot\|_\beta$. For any two elements (y,z) and (y',z') in $L^2_{\mathcal{F}}(0,T;\mathbf{R}^m \times \mathbf{R}^{m\times d})$ we set

$$(Y,Z) = I[(y,z)], \qquad (Y',Z') = I[(y',z')],$$

and denote their differences by $(\hat{y},\hat{z}) = (y-y', z-z')$, $(\hat{Y},\hat{Z}) = (Y-Y', Z-Z')$. By the basic estimate (20) we have

$$\mathbf{E}\int_0^T (\frac{\beta}{2}|\hat{Y}_s|^2 + |\hat{Z}_s|^2) e^{\beta s} ds \le \frac{2}{\beta}\mathbf{E}\int_0^T |g(s,y_s,z_s) - g(s,y'_s,z'_s)|^2 e^{\beta s} ds.$$

Since g satisfies Lipschitz condition, we have

$$\mathbf{E}\int_0^T [\frac{\beta}{2}|\hat{Y}_s|^2 + |\hat{Z}_s|^2] e^{\beta s} ds \le \frac{4C^2}{\beta}\mathbf{E}\int_0^T [|\hat{y}_s|^2 + |\hat{z}_s|^2] e^{\beta s} ds.$$

Since $\beta = 8(1+C^2)$, thus

$$\mathbf{E}\int_0^T [|\hat{Y}_s|^2 + |\hat{Z}_s|^2] e^{\beta s} ds \le \frac{1}{2}\mathbf{E}\int_0^T [|\hat{y}_s|^2 + |\hat{z}_s|^2] e^{\beta s} ds,$$

or

$$\|(\hat{Y},\hat{Z})\|_\beta \le \frac{1}{\sqrt{2}}\|(\hat{y},\hat{z})\|_\beta.$$

Thus I is a strict contraction mapping of $L^2_{\mathcal{F}}(0,T;\mathbf{R}^m \times \mathbf{R}^{m\times d})$. It follows by the fixed point theorem that BSDE (17) has a unique solution. $(Y,Z) \in L^2_{\mathcal{F}}(0,T;\mathbf{R}^m \times \mathbf{R}^{m\times d})$. It then follows from (28) and (27) that $g(\cdot,Y.,Z.) \in L^2_{\mathcal{F}}(0,T)$. Thus by Lemma 3.1 $Y \in S^2_{\mathcal{F}}(0,T)$. □

The basic estimates (19) and (20) can also be applied to prove the continuous dependence theorem of BSDE (17) with respect to parameters. Let (Y^1,Z^1) and (Y^2,Z^2) be respectively the solution of the following two BSDEs:

$$Y_t^1 = \xi^1 + \int_t^T [g(s, Y_s^1, Z_s^1) + \varphi^1{}_s]ds - \int_t^T Z_s^1 dB_s. \tag{29}$$

$$Y_t^2 = \xi^2 + \int_t^T [g(s, Y_s^2, Z_s^2) + \varphi^2{}_s]ds - \int_t^T Z_s^2 dB_s. \tag{30}$$

Here the terminal condition ξ^1 and ξ^2 are given elements in $L^2(\mathcal{F}_T; \mathbf{R}^m)$ and φ^1 and φ^2 are two given processes in $L^2_{\mathcal{F}}(0, T; \mathbf{R}^m)$. Let g be the same as in Theorem3.1. Analogue to the previous method, using Itô's formula applied to $|Y_s^1 - Y_s^2|^2 e^{\beta(s-t)}$ in the interval $[t, T]$, we can obtain the following estimate.

Theorem 3.2. *The difference of the solutions of BSDE (29) and (30)satisfies*

$$|Y_t^1 - Y_t^2|^2 + \frac{1}{2}\mathbf{E}^{\mathcal{F}_t} \int_t^T [|Y_s^1 - Y_s^2|^2 + |Z_s^1 - Z_s^2|^2] e^{\beta(s-t)} ds \tag{31}$$
$$\leq \mathbf{E}^{\mathcal{F}_t} |\xi^1 - \xi^2|^2 e^{\beta(T-t)} + \mathbf{E}^{\mathcal{F}_t} \int_t^T |\varphi_s^1 - \varphi_s^2| e^{\beta(s-t)} ds,$$

where $\beta = 16(1 + C^2)$. *We also have*

$$\mathbf{E}[\sup_{0 \leq t \leq T} |Y_t^1 - Y_t^2|^2] \leq c\mathbf{E}[|\xi^1 - \xi^2|^2] + c\mathbf{E} \int_0^T |\varphi_s^1 - \varphi_s^2|^2 ds. \tag{32}$$

In particular, when $\varphi_s^1 \equiv 0$, *(set* $\xi^2 = 0$, $\varphi_s^2 = -g(s, 0, 0)$*)*,

$$\mathbf{E}[\sup_{0 \leq t \leq T} |Y_t^1|^2] \leq c\mathbf{E}[|\xi^1|^2] + c\mathbf{E} \int_0^T |g(s, 0, 0)|^2 ds. \tag{33}$$

where the constant c *depends only on the Lipschitz constant of* g *and* T.

Proof. We apply estimate (19) to $(y_t, z_t) = (Y_t^1 - Y_t^2, Z_t^1 - Z_t^2)$:

$$|y_t|^2 + \mathbf{E}^{\mathcal{F}_t} \int_t^T [\frac{\beta}{2}|y_s|^2 + |z_s|^2] e^{\beta(s-t)} ds$$
$$\leq \mathbf{E}^{\mathcal{F}_t} |\xi|^2 e^{\beta(T-t)} + \frac{2}{\beta}\mathbf{E}^{\mathcal{F}_t} \int_t^T |\hat{g}(s)|^2 e^{\beta(s-t)} ds,$$

where $\hat{g}(s) := g(s, Y_s^1, Z_s^1) - g(s, Y_s^2, Z_s^2) + \varphi_s^1 - \varphi_s^2$. This with $|\hat{g}(s)| \leq C(|y_t| + |z_t|) + |\varphi_s^1 - \varphi_s^2|$, yields (32). This estimate with (21) yields (33). $\square$

For a fixed $t_0 \in [0, T]$, we denote

$$\mathcal{F}_t^{t_0} = \sigma\{\sigma(B_s - B_{t_0}; t_0 \leq s \leq t) \cup \mathcal{N}\}, \quad t \in [t_0, T].$$

The following is a simple corollary of the uniqueness of BSDE (17).

Proposition 3.1. *We still assume that g satisfies Assumptions (28) and (27). If moreover, for a fixed $t_0 \in [0,T]$ and for each $(y,z) \in \mathbf{R}^m \times \mathbf{R}^{m\times d}$, the process $g(\cdot,y,z)$ is $(\mathcal{F}_t^{t_0})$-adapted on the interval $[t_0,T]$ and $\xi \in L^2(\Omega,\mathcal{F}_T^{t_0},P;\mathbf{R}^m)$. Then the solution $(Y.,Z.)$ of BSDE (17) is also $(\mathcal{F}_t^{t_0})$-adapted on $[t_0,T]$. In particular, Y_{t_0} and Z_{t_0} are deterministic.*

Proof. Let (Y',Z') be the solution of $(\mathcal{F}_t^{t_0})$-adapted solution, on the interval $[t_0,T]$ of the BSDE

$$Y_t' = \xi + \int_t^T g(s,Y_s',Z_s')ds - \int_t^T Z_s' dB_s^0,$$

where we denote $B_t^0 \equiv B_t - B_{t_0}$. Observe that $(B_t^0)_{t_0\le s\le T}$ is an $(\mathcal{F}_t^{t_0})$ - Brownian Motion on $[t_0,T]$. On the other hand the same processes $(Y_t',Z_t')_{t_0\le t\le T}$ is also $\mathcal{F}_t$–adapted and

$$\int_t^T Z_s' dB_s = \int_t^T Z_s' dB_s^0, \quad t\in[t_0,T].$$

Thus from the uniqueness result of Theorem 3.1, The solution (Y,Z) of BSDE (17) coincides with (Y',Z') on $[t_0,T]$. Thus (Y,Z) is $(\mathcal{F}_t^{t_0})$-adapted. □

Remark 3.3. A special situation of BSDE (17) is when ξ is deterministic and $g(t,y,z)$ is a deterministic function of (t,y,z). In this case the solution of BSDE (17) is simply $(Y.,Z.) \equiv (Y_0(\cdot),0)$, where $Y_0(\cdot)$ is the solution of the following ordinary differential equation defined on $[0,T]$:

$$-\dot{Y}_0(t) = g(t,Y_0(t),0), \quad Y_0(T)=\xi.$$

3.2 1–Dimensional BSDE

We will see that each standard 1–dimensional BSDE on $[0,T]$ induces an $\mathcal{F}_t$–consistent evaluation, called g–evaluation, where $g=g(t,y,z)$ is the generator of the corresponding BSDE which is a simple real valued function. If (and only if) g satisfies $g(t,y,0)\equiv 0$, then the corresponding $\mathcal{F}_t$–consistent evaluation becomes an $\mathcal{F}_t$–consistent expectation. We also notice that the present state of art of mathematical finance corresponds mostly to $m=1$. It also covers many linear or nonlinear parabolic and elliptic PDEs. In fact $m>1$ corresponds to systems of PDEs.

The function g is defined as follows

$$g(\omega,t,y,z) : \Omega\times[0,T]\times\mathbf{R}\times\mathbf{R}^d \longmapsto \mathbf{R}.$$

We assume, for each $y,y'\in\mathbf{R}$, $z,z'\in\mathbf{R}^d$, $t\in[0,T]$, g satisfies

$$\begin{cases} \text{(i)} & g(\cdot,y,z)\in L^2_{\mathcal{F}}(0,T), \quad \text{for each } y\in\mathbf{R},\ z\in\mathbf{R}^d; \\ \text{(ii)} & |g(t,y_1,z_1)-g(t,y_2,z_2)|\le \nu|y_1-y_2|+\mu|z_1-z_2|; \\ & \text{and } \textbf{one} \text{ of the following three conditions} \\ \text{(iii)} & g(\cdot,y,z)|_{y=0,\ z=0}\equiv 0; \\ \text{(iii')} & g(\cdot,y,0)\equiv 0; \\ \text{(iii'')} & g \text{ is independent of } y \text{ and } g(\cdot,z)|_{z=0}\equiv 0. \end{cases} \tag{34}$$

where $\mu,\ \nu$ are given non negative constants. It is clear that (iii”) $\Rightarrow$ (iii’) $\Rightarrow$ (iii).

Comparison Theorem

We first present an important property: The **Comparison Theorem** of BSDE. We will present this theorem in the case where the solution Y is possibly a RCLL (right continuous with left limit) process i.e., P-almost all of its paths of $Y_{\cdot}(\omega)$ are right continuous with left limit. An RCLL process $(A_t(\omega))_{t\in[0,T]}$ is called an increasing process if P-almost all of its paths are non-decreasing with $A_0(\omega)=0$.
We first consider the following problem:
to find a solution $(Y,Z)\in L^2_{\mathcal{F}}(0,T;\mathbf{R}^{1+d})$ of the following BSDE

$$Y_t=\xi+\int_t^T g(s,Y_s,Z_s)ds+(V_T-V_t)-\int_t^T Z_s dB_s, \tag{35}$$

The following is a simple corollary of Theorem 3.1.

Proposition 3.2. *We assume (34)–(i), (ii). Then, for each $\xi\in L^2(\mathcal{F}_T)$ and $V\in D^2_{\mathcal{F}}(0,T)$, there exists a unique solution $(Y,Z)\in D^2_{\mathcal{F}}(0,T)\times L^2_{\mathcal{F}}(0,T;\mathbf{R}^d)$ of the BSDE (35). Moreover $Y+V\in S^2_{\mathcal{F}}(0,T)$.*

Proof. The case $V_t\equiv 0$ corresponds to Theorem 3.1. For the general situation we let $\bar{Y}_t := Y_t+V_t$. The existence and uniqueness of BSDE (35) is equivalent to the solution $(\bar{Y},Z)\in D^2_{\mathcal{F}}(0,T)\times L^2_{\mathcal{F}}(0,T;\mathbf{R}^d)$ of the following standard BSDE :

$$\bar{Y}_t=\xi+V_T+\int_t^T g(s,\bar{Y}_s-V_s,Z_s)ds-\int_t^T Z_s dB_s.$$

□

For a given random variable

$$\hat{\xi}\in L^2(\mathcal{F}_T),\ \hat{V}\in D^2_{\mathcal{F}}(0,T) \tag{36}$$

let $(\hat{Y},\hat{Z})\in L^2_{\mathcal{F}}(0,T;\mathbf{R}^{1+d})$ be the solution of the following BSDE

$$\hat{Y}_t=\hat{\xi}+\int_t^T g(s,\hat{Y}_s,\hat{Z}_s)ds+(\hat{V}_T-\hat{V}_t)-\int_t^T \hat{Z}_s dB_s. \tag{37}$$

It is easy to prove that the difference $(Y-\hat{Y},Z-\hat{Z})$ satisfies exactly the same estimate (31) given in Theorem 3.2. Using B–D–G inequality, we then derive the following estimate.

Proposition 3.3. *We assume (34)–(i), (ii). Then the difference of the solutions of BDSE (35) and (37) satisfies*

$$\mathbf{E}[\sup_{0\le t\le T}|Y_t+V_t-(\hat{Y}_t+\hat{V}_t)|^2]+\mathbf{E}\int_0^T|Z_s-\hat{Z}_s|^2ds\le cE|\xi+V_T-(\hat{\xi}+\hat{V}_T)|^2. \tag{38}$$

where the constant c depends only on T and the Lipschitz constant of g w.r.t. (y,z).

We now present

Theorem 3.3. *(Comparison Theorem of BSDE) We make the same assumption as in Proposition 3.2. Let (Y',Z') be the solution of the following simple BSDE*

$$Y'_t=\xi'+\int_t^T\bar{g}_s ds+V'_T-V'_t-\int_t^T Z'_s dB_s. \tag{39}$$

where $(\bar{g}_t)$, (V'_t) $\in L^2_{\mathcal{F}}(0,T;\mathbf{R})$ and $\xi'\in L^2(\mathcal{F}_T)$ are given such that

$$\xi\ge\xi',\quad g(Y'_t,Z'_t,t)\ge\bar{g}_t,\ a.s.,\ a.e., \tag{40}$$

and such that $\hat{V}=V-V'$ is an increasing process. We then have

$$Y_t\ge Y'_t,\quad a.e.,\ a.s.. \tag{41}$$

We also have Strict Comparison Theorem: under the above conditions

$$Y_0=Y'_0\quad\Longleftrightarrow\xi=\xi',\quad g(s,Y'_s,Z'_s)\equiv\bar{g}_s\quad \text{and } V_s\equiv V'_s. \tag{42}$$

Sketch of the Proof. We only consider the case $d=1$ (i.e., B is a 1-dimensional Brownian Motion) and prove the case $t=0$. The general situation is left to the reader as an exercise. We set $\hat{g}_s=g(s,Y'_s,Z'_s)-\bar{g}_s$ and

$$\hat{Y}=Y-Y',\ \hat{Z}=Z-Z',\ \hat{\xi}=\xi-\xi'.$$

The pair $(\hat{Y},\hat{Z})$ can be regarded as the solution of the following linear BSDE:

$$\begin{cases}-d\hat{Y}_s=(a_s\hat{Y}_s+b_s\hat{Z}_s+\hat{g}_s)ds+d\hat{V}_s-\hat{Z}_s dB_s,\\ \quad\hat{Y}_T=\hat{\xi},\end{cases}$$

where

$$a_s:=\begin{cases}\frac{g(s,Y_s,Z_s)-g(s,Y'_s,Z_s)}{Y_s-Y'_s}, & \text{if } Y_s\ne Y'_s,\\ 0, & \text{if } Y_s=Y'_s,\end{cases}$$
$$b_s:=\begin{cases}\frac{g(s,Y'_s,Z_s)-g(s,Y'_s,Z'_s)}{Z_s-Z'_s}, & \text{if } Z_s\ne Z'_s,\\ 0, & \text{if } Z_s=Z'_s.\end{cases}$$

Since g satisfies Lipschitz condition, thus $|a_s|\le C$ and $|b_s|\le C$. We set

$$Q_t := \exp\left[\int_0^t b_s dB_s - \frac{1}{2}\int_0^t |b_s|^2 ds + \int_0^t a_s ds\right].$$

We apply Itô's formula to $Q_t \hat{Y}_t$ on the interval $[0,T]$ and then take expectation:

$$\hat{Y}_0 = \mathbf{E}[\hat{Y}_T Q_T + \int_0^T Q_t \hat{g}_t dt + \int_0^T Q_t d\hat{V}_t] \geq 0.$$

From this we have $Y_0 \geq Y_0'$. This method also applies to prove $Y_t \geq Y_t'$ when $t > 0$.

By Girsanov Theorem,

$$\mathbf{E}[\hat{Y}_T Q_T + \int_0^T Q_t \hat{g}_t dt + \int_0^T Q_t d\hat{V}_t] = 0$$

if and only the following non negative quantities are zero: $\hat{Y}_t = 0$, $\hat{g}_t \equiv 0$ and $\hat{V}_T = 0$, a.s, a.e.. Thus we have the strict comparison. □

Remark 3.4. In many situations the Comparison Theorem is applied to compare the following type of two BSDEs:

$$Y_t^1 = \xi^1 + \int_t^T [g(s, Y_s^1, Z_s^1) + c_s^1] ds - \int_t^T Z_s^1 dB_s, \tag{43}$$

and

$$Y_t^2 = \xi^2 + \int_t^T [g(s, Y_s^2, Z_s^2) + c_s^2] ds - \int_t^T Z_s^2 dB_s, \tag{44}$$

where $c^1(\cdot), c^2(\cdot) \in L^2_{\mathcal{F}}(0,T)$. In this case if we have

$$c_s^1 \geq c_s^2, \text{ a.s., a.e., } \xi^1 \geq \xi^2, \text{ a.s..}$$

Then it is easy to apply Theorem 3.3 to derive $Y_t^1 \geq Y_t^2$, a.s., a.e..

Example 3.1. We consider a special case of BSDE (43) with $g(s,0,0) \equiv 0$.In this case if $c_s^2 \equiv 0$ and $\xi^2 = 0$, then the unique solution of BSDE (44) is $(Y_s^2, Z_s^2) \equiv 0$. It then follows from Remark 3.4 that if ξ^1 and $c^1(\cdot)$ are both non negative, then the solution Y^1 of (43) is also non negative. In this case we have also, by strict comparison,

$$Y_0^1 = 0 \Longleftrightarrow c_s^1 \equiv 0 \text{ and } \xi^1 = 0.$$

An interpretation in finance is: If an investor want to obtain an opportunity of non negative return, i.e., $\xi^1 \geq 0$, then he must invest at the present time some nonnegative value, i.e., $Y_0^1 \geq 0$. If $\xi \geq 0$, a.s. and $\mathbf{E}[\xi^1] > 0$, then his investment has to be positive: $Y_0^1 > 0$.

Example 3.2. We assume that $g(s,0,0) \equiv 0$ and $\xi \geq 0$ with $\mathbf{E}[\xi] > 0$. Consider the following BSDE parameterized by $\lambda \in (0,\infty)$:

$$Y_t^\lambda = \lambda\xi + \int_t^T g(s, Y_s^\lambda, Z_s^\lambda)ds - \int_t^T Z_s^\lambda dB_s.$$

We can prove that

$$\lim_{\lambda\uparrow\infty} Y_0^\lambda = +\infty.$$

In fact we compare its solution with the one of the following BSDE

$$\bar{Y}_t^\lambda = \lambda\xi + \int_t^T C(-|\bar{Y}_s^\lambda| - |\bar{Z}_s^\lambda|)ds - \int_t^T \bar{Z}_s^\lambda dB_s,$$

where $C > 0$ is the Lipschitz constant of g with respect to (y,z). By Comparison Theorem, we have

(i) $Y_0^\lambda \geq \bar{Y}_0^\lambda$, for each $\lambda > 0$;
(ii) $\bar{Y}_0^1 > 0$, when $\lambda = 1$

We also observe that for each $\lambda \geq 0$, we have $\bar{Y}_t^\lambda \equiv \lambda\bar{Y}_t^1$ and $\bar{Z}_t^\lambda \equiv \lambda\bar{Z}_t^1$. From this and (i), (ii) it follows that

$$Y_0^\lambda \geq \bar{Y}_0^\lambda = \lambda\bar{Y}_0^1 \uparrow \infty.$$

Exercise 3.1. Prove that Y_0^λ is also bounded by:

$$Y_0^\lambda \leq \lambda\hat{Y}_0,$$

where $\hat{Y}_0$ is a constant.

Backward Stochastic Monotone Semigroups and g–Evaluations

We now discuss the backward semigroup property of the solution Y of a BSDE. We introduce the following definition: Given $t \leq T$ and $Y \in L^2(\mathcal{F}_t)$. We consider the following BSDE defined on the interval $[0,t]$

$$y_s = Y + \int_s^t g(r, y_r, z_r)dr - \int_s^t z_r dB_r, \quad s \in [0,t]. \tag{45}$$

Definition 3.1. *We define, for each* $0 \leq s \leq t < \infty$ *and* $Y \in L^2(\mathcal{F}_t)$*,*

$$\mathcal{E}_{s,t}^g[Y] := y_s. \tag{46}$$

The system $\mathcal{E}_{s,t}^g[\cdot] : L^2(\mathcal{F}_t) \to L^2(\mathcal{F}_s)$, $0 \leq s \leq t \leq T$ *is called* g*–evaluation.*

Remark 3.5. s and t can be also two uniformly bounded $\mathcal{F}_t$–stopping times.

Theorem 3.4. *Let the function g satisfies (i)–(iii) of (34). Then the g–evaluation $\mathcal{E}^g_{s,t}[\cdot]$ defined in (46) satisfies the Axiomatic assumptions (A1)–(A4) listed in Definition 2.1: it is an $\mathcal{F}_t$–consistent nonlinear evaluation operator. Furthermore, we have:*
(A5) *For each Y_1, $Y_2 \in L^2(\mathcal{F}_t)$*

$$\mathcal{E}^{-g_{\mu v}}_{s,t}[Y_1 - Y_2] \leq \mathcal{E}^g_{s,t}[Y_1] - \mathcal{E}^g_{s,t}[Y_2] \leq \mathcal{E}^{g_{\mu v}}_{s,t}[Y_1 - Y_2]. \tag{47}$$

In particular, If g is independent of y, i.e., (iii") satisfies, then we have

$$\mathcal{E}^{-g_{\mu}}_{s,t}[Y_1 - Y_2] \leq \mathcal{E}^g_{s,t}[Y_1] - \mathcal{E}^g_{s,t}[Y_2] \leq \mathcal{E}^{g_{\mu}}_{s,t}[Y_1 - Y_2]. \tag{48}$$

Here $g_{\mu,\nu}(y,z) := \nu|y| + \mu|z|$, $g_\mu(z) := \mu|z|$,ν and μ are the Lipschitz constants of g w.r.t. y and z, respectively.

Proof. (A1) is directly from Comparison Theorem. (A2) is obvious. As for (A4), we multiply the BSDE (45) by 1_A, $A \in \mathcal{F}_s$ on the interval $[s,t]$. Since $g(r,0,0) \equiv 0$, we have, for $u \in [s,t]$,

$$\begin{aligned} y_u 1_A &= Y 1_A + \int_u^t 1_A g(r, y_r, z_r) dr - \int_u^t 1_A z_r dB_r \\ &= Y 1_A + \int_u^t g(r, 1_A y_r, 1_A z_r) dr - \int_u^t 1_A z_r dB_r. \end{aligned}$$

This implies that $(1_A y_r, 1_A z_r)_{r\in[s,t]}$ is the solution of the same backward equation with terminal condition $Y 1_A$. Thus

$$1_A \mathcal{E}^g_{s,t}[Y] = \mathcal{E}^g_{s,t}[1_A Y].$$

Thus we have (A4). (A3) simply follows from the uniqueness of BSDE, i.e., for each $s \leq u \leq t$, we have

$$\mathcal{E}^g_{s,t}[Y] = \mathcal{E}^g_{s,u}[y_u] = \mathcal{E}^g_{s,u}[\mathcal{E}^g_{u,t}[Y]]. \tag{49}$$

(A5) is the direct consequence of the following proposition. □

Proposition 3.4. *We assume that g_1 and g_2 satisfy (i)–(ii) of assumption (34). If g_1 is dominated by g_2 in the following sense*

$$g_1(t,y,z) - g_1(t,y',z') \leq g_2(t, y-y', z-z'), \ \ \forall y, y' \in \mathbf{R}, \ \forall z, z' \in \mathbf{R}^d, \tag{50}$$

then $\mathcal{E}^{g_1}[\cdot]$ is also dominated by $\mathcal{E}^{g_2}[\cdot]$ in the following sense: for each $t > 0$ and Y, $Y' \in L^2(\mathcal{F}_{\mathbf{t}})$, we have

$$\mathcal{E}^{g_1}_{u,t}[Y] - \mathcal{E}^{g_1}_{u,t}[Y'] \leq \mathcal{E}^{g_2}_{u,t}[Y - Y']. \tag{51}$$

If g is dominated by itself, then $\mathcal{E}_g[\cdot]$ is also dominated by itself.

Proof. We consider the following three BSDEs

$$\begin{aligned} -dy_r &= g_1(r, y_r, z_r)dr - z_r dB_r, \quad y_t = Y, \\ -dy'_r &= g_1(r, y'_r, z'_r)dr - z'_r dB_r, \quad y'_t = Y' \end{aligned}$$

and

$$-dY_r = g_2(r, Y_r, Z_r)dr - Z_r dB_r, \quad Y_t = Y - Y'.$$

We denote $(\hat{y}_r, \hat{z}_r) = (y_r - y'_r, z_r - z'_r)$ and $\hat{g}_r = g_1(r, y_r, z_r) - g_1(r, y'_r, z'_r)$

$$-d\hat{y}_r = \hat{g}_r dr - \hat{z}_r dB_r, \quad \hat{y}_t = Y - Y'.$$

Condition (50) implies $g_2(r, \hat{y}_r, \hat{z}_r) \geq \hat{g}_r$. It follows from Comparison Theorem that

$$\hat{y}_u \leq Y_u, \ \forall u \in [0, t], \text{ a.s.}$$

By the definition of $\mathcal{E}^g[\cdot]$ it follows that (51) holds. □

Example: Black–Scholes Evaluations

Consider a financial market consisting of $d+1$ assets: a bond and d stocks. We denote by $P_0(t)$ the price of the bond and by $P_i(t)$ the price of i-th stock at time t. We assume that $P_0(\cdot)$ is the solution of the ordinary differential equation

$$dP_0(t) = r(t)P_0(t)dt, \quad P_0(0) = 1,$$

$\{P_i(\cdot)\}_{i=1}^d$ is the solution of the following SDE

$$\begin{aligned} dP_i(t) &= P_i(t)[b_i(t)dt + \sum\nolimits_{j=1}^d \sigma_{ij}(t)dB_t^j], \\ P_i(0) &= p_i, \quad i = 1, \cdots, d. \end{aligned}$$

Here r is the interest rate of the bond; $\{b_i\}_{i=1}^d$ is the rate of the expected return, $\{\sigma_{ij}\}_{i,j=1}^d$ the volatility of the stocks. We assume that r, b, σ and σ^{-1} are all $\mathcal{F}_t$–adapted and uniformly bounded processes on $[0, \infty)$. The problem is how a market evaluates an European type of derivative $\xi \in L^2(\mathcal{F}_T)$ with maturity T? To solve this problem we consider an investor who has, at a time $t \leq T$, $n_0(t)$ bonds and $n_i(t)$ i-stocks, $i = 1, \cdots, d$, i.e., he invests $n_0(t)P_0(t)$ in bond and $\pi_i(t) = n_i(t)P_i(t)$ in the i-th stock. $\pi(t) = (\pi_1(t), \cdots, \pi_d(t))$, $0 \leq t \leq T$ is an $\mathbf{R}^d$ valued, square-integrable and adapted process. We define by $y(t)$ the investor's wealth invested in the market at time t:

$$y(t) = n_0(t)P_0(t) + \sum\nolimits_{i=1}^d \pi_i(t).$$

We make the so called self–financing assumption: in the period $[0, T]$, the investor does not withdraw his money from, or put some other person's money into his account y_t. Under this condition, his wealth y evolves according to

$$dy(t) = n_0(t)dP_0(t) + \sum_{i=1}^{d} n_i(t)dP_i(t).$$

or

$$dy(t) = [r(t)y(t) + \sum_{i=1}^{d}(b_i(t) - r(t))\pi_i(t)]dt + \sum_{i,j=1}^{d}\sigma_{ij}(t)\pi_i(t)dB_t^j.$$

We denote

$$g(t, y, z) = -r(t)y - \sum_{i,j=1}^{d}(b_i(t) - r(t))\sigma_{ji}^{-1}(t)z_j.$$

Then, by the change of variable $z_j(t) = \sum_{i=1}^{d}\sigma_{ij}(t)\pi_i(t)$, the above equation becomes

$$-dy(t) = g(t, y(t), z(t))dt - z(t)dB_t.$$

We observe that the function g satisfies (i) and (ii) of (34). It follows from the existence and uniqueness theorem of BSDE (Theorem 3.1) that for each derivative $\xi \in L^2(\mathcal{F}_T)$, there exists a unique solution $(y(\cdot), z(\cdot)) \in L^2_{\mathcal{F}}(0, T; \mathbf{R}^{1+d})$ with the terminal condition $y_T = \xi$. This meaning is significant: in order to replicate the derivative ξ, the investor needs and only needs to invest $y(t)$ at the present time t and then, during the time interval $[t, T]$, to perform the strategy $\pi_i(s) = \sigma_{ij}^{-1}(s)z_j(s)$. Furthermore, by Comparison Theorem of BSDE, if he wants to replicate a ξ' which is bigger than ξ, (i.e., $\xi' \geq \xi$, a.s., $P(\xi' \geq \xi) > 0$), then he must pay more. This means that no arbitrage–free strategy exists. This $y(t)$ is called the Black–Scholes price, or Black–Scholes evaluation, of ξ at the time t. We define, as in (46) $\mathcal{E}^g_{t,T}[\xi] = y_t$. We observe that the function g satisfies (i)–(iii) of condition (34). It follows from Theorem 3.4 that $\mathcal{E}^g_{t,T}[\cdot]$ satisfies (A1)–(A4) of $\mathcal{F}_t$–consistent evaluation.

g–Expectations

In this subsection we will consider a particularly interesting situation of the above stochastic semigroups: when g satisfies $g(s, y, z)|_{z=0} \equiv 0$, i.e., it satisfy (i), (ii) and (iii') in (34). In this situation $\mathcal{E}^g_{s,t}[Y]$ satisfies (A2'):

Proposition 3.5. *For each* $0 \leq s \leq t \leq T$*, and* $Y \in L^2(\mathcal{F}_s)$*, we have*

$$\mathcal{E}^g_{s,t}[Y] = Y. \tag{52}$$

Proof. We consider the solution (y, z) of (45) with the same terminal condition Y, but defined on $[s, t]$:

$$y_u = Y + \int_u^t g(r, y_r, z_r)dr - \int_u^t z_r dB_r, \quad u \in [s, t]. \tag{53}$$

We have $y_u = \mathcal{E}^g_{u,t}[Y]$. But by Assumption (34)–(iii'), it is easy to check $(y_u, z_u) \equiv (Y, 0)$. We thus have (52). □

Thus we can define $\mathcal{F}_t$–consistent nonlinear expectation $\mathcal{E}_g[Y|\mathcal{F}_t]$:

Definition 3.2. *We define*

$$\mathcal{E}_g[Y] := \mathcal{E}^g_{0,T}[Y],\ \mathcal{E}_g[Y|\mathcal{F}_t] := \mathcal{E}^g_{t,T}[Y],\ \ Y \in L^2(\mathcal{F}_T). \tag{54}$$

$\mathcal{E}_g[Y]$ is called g–expectation of Y. In particular, if $g = \mu|z|$ then we denote $\mathcal{E}_g[Y] = \mathcal{E}^\mu[Y]$.

g–expectations is nonlinear but it satisfies all other properties of a classical linear expectation.

Proposition 3.6. *We assume that g satisfies (i), (ii) and (iii') in (34). Then the g–expectation $\mathcal{E}_g[\cdot]$ defined in (54) is an $\mathcal{F}_t$–consistent nonlinear expectation defined on $L^2(\mathcal{F}_T)$. That is, it satisfies (A1), (A2'), (A3) and (A4) listed in Definition 2.2. Moreover, $\mathcal{E}_g[\cdot]$ is dominated by $\mathcal{E}^\mu[\cdot]$ and $\mathcal{E}^{g_{\mu v}}[\cdot]$ in the following sense:*

$$-\mathcal{E}^\mu[-Y|\mathcal{F}_t] \le \mathcal{E}_g[Y|\mathcal{F}_t] \le \mathcal{E}^\mu[Y|\mathcal{F}_t],\ \ \forall Y \in L^2(\mathcal{F}_T). \tag{55}$$

and

$$\begin{aligned} &\mathcal{E}^{-g_{\mu,\nu}}_{t,T}[Y_1 - Y_2] \le \mathcal{E}_g[Y_1|\mathcal{F}_t] - \mathcal{E}_g[Y_2|\mathcal{F}_t] \le \mathcal{E}^{g_{\mu,\nu}}_{t,T}[Y_1 - Y_2], \\ &\quad \forall Y_1, Y_2 \in L^2(\mathcal{F}_T). \end{aligned} \tag{56}$$

Proof. Since $\mathcal{E}^g_{s,t}[\cdot]$ satisfies (A1), (A2'), (A3) and (A4), by Proposition 2.1, $\mathcal{E}_g[\cdot|\mathcal{F}_t]$ defined in (54) satisfies (A1), (A2'), (A3) and (A4) of $\mathcal{F}_t$–expectations.

(56) is directly by (47). (55) is proved from the comparison theorem of BSDE since $\mathcal{E}^\mu[\cdot] = \mathcal{E}_{g_\mu}[\cdot]$, with $g_\mu(z) = \mu|z| \ge g(t, y, z)$. □

Definition 3.3. *Let $\tau \le T$ be a stopping time. We also define*

$$\mathcal{E}_g[Y|\mathcal{F}_\tau] = \mathcal{E}^g_{\tau,T}[Y].$$

Definition 3.4. *(g-**martingales**) A process $(Y_t)_{0\le t\le T}$ with $E[Y_t^2] < \infty$ for all t is called a g-martingale (resp. g-supermartingale, g-submartingale) if, for each $s \le t \le T$, we have*

$$\mathcal{E}_g[Y_t|\mathcal{F}_s] = Y_s,\quad (\mathit{resp.} \le Y_s, \ge Y_s).$$

The importance of this special setting follows from the following economically meaningful property.

Lemma 3.2. *Let the function g satisfies (i), (ii) and (iii") of (34). Then*

$$\mathcal{E}_g[Y + \eta|\mathcal{F}_t] = \mathcal{E}_g[Y|\mathcal{F}_t] + \eta,\quad \forall \eta \in L^2(\Omega, \mathcal{F}_t, P). \tag{57}$$

Proof. Consider the BSDE

$$\begin{aligned} -dy_s &= g(s, z_s)ds - z_s dB_s,\ t \le s \le T, \\ y_T &= Y. \end{aligned}$$

We have by the definition $\mathcal{E}_g[Y|\mathcal{F}_t] = y_t$. On the other hand, it is easy to check that $(y'_s, z'_s) := (y_s + \eta, z_s)$, $s \in [t, T]$ solve the above equation with the terminal condition $y'_T = Y + \eta$. It then follows that

$$\mathcal{E}_g[Y + \eta|\mathcal{F}_t] = y'_t = y_t + \eta = \mathcal{E}_g[Y|\mathcal{F}_t] + \eta.$$

□

Remark 3.6. Economically, (57) means that the nonlinearity of $\mathcal{E}_g[Y + \eta]$ is only due to the risky part of $Y + \eta$.

We will always write in the sequel $\mathcal{E}^\mu[Y] := \mathcal{E}_g[Y]$ for $g = \mu|z|$ and $\mathcal{E}^{-\mu}[Y] := \mathcal{E}_g[Y]$ for $g \equiv -\mu|z|$. Note that

$$\forall c > 0, \quad \mathcal{E}^\mu[cY|\mathcal{F}_t] = c\mathcal{E}^\mu[Y|\mathcal{F}_t] \tag{58}$$

and

$$\forall c < 0, \quad \mathcal{E}^\mu[cY|\mathcal{F}_t] = -c\mathcal{E}^\mu[-Y|\mathcal{F}_t].$$

An important feature of $\mathcal{E}^\mu[\cdot]$ is

Proposition 3.7. *Let g satisfy (i), (ii) and (iii') of Assumption (34), then $\mathcal{E}_g[\cdot]$ is dominated by $\mathcal{E}^\mu[\cdot]$ in the following sense, for each $t \geq 0$,*

$$\mathcal{E}_g[Y|\mathcal{F}_t] - \mathcal{E}_g[Y'|\mathcal{F}_t] \leq \mathcal{E}^{g_{\mu,v}}_{t,T}[Y - Y'],\ \forall Y, Y' \in L^2(\mathcal{F}_T). \tag{59}$$

If g is independent of y, i.e., (iii") satisfies, then we have

$$\mathcal{E}_g[Y|\mathcal{F}_t] - \mathcal{E}_g[Y'|\mathcal{F}_t] \leq \mathcal{E}^\mu[Y - Y'|\mathcal{F}_t],\ \forall Y, Y' \in L^2(\mathcal{F}_T). \tag{60}$$

In particular, $\mathcal{E}^\mu[\cdot]$ is dominated by itself:

$$\mathcal{E}^\mu[Y|\mathcal{F}_t] - \mathcal{E}^\mu[Y'|\mathcal{F}_t] \leq \mathcal{E}^\mu[Y - Y'|\mathcal{F}_t],\ \forall Y, Y' \in L^2(\mathcal{F}_T). \tag{61}$$

Proof. We observe that $\mathcal{E}^{g_{\mu,0}}_{t,T}[Y] = \mathcal{E}^\mu[Y|\mathcal{F}_t]$. Thus (59) as well as (61) are directly derived by (A5) of Theorem 3.4. □

The self–domination property (61) of $\mathcal{E}^\mu[\cdot]$ permit us to defined a norm

Definition 3.5. *We define*

$$\|Y\|_\mu := \mathcal{E}^\mu[|Y|],\ Y \in L^2(\mathcal{F}_T).$$

Proposition 3.8. $\|\cdot\|_\mu$ *forms a norm in $L^2(\mathcal{F}_T)$.*

Proof. The triangle inequality $\|Y\|_\mu + \|Z\|_\mu \leq \|Y + Z\|_\mu$ follows from (61) with $t = 0$. By (58) we also have $\|cY\|_\mu = c\,\|Y\|_\mu$, $c \geq 0$. □

Proposition 3.9. *Under $\|\cdot\|_\mu$, $\mathcal{E}_g[\cdot|\mathcal{F}_t]$ is a contraction mapping:*

$$\|\mathcal{E}_g[Y|\mathcal{F}_t] - \mathcal{E}_g[Y'|\mathcal{F}_t]\|_\mu \leq \|Y - Y'\|_\mu\,.$$

Proof. It is an easy consequence of (59). □

Proposition 3.10. *For each $\mu > 0$, and $T > 0$, there exist a constant $c_{\mu,T}$ such that*

$$\mathbf{E}[|Y|] \leq \mathcal{E}^\mu[|Y|] \leq c_{\mu,T}(\mathbf{E}[|Y|^2])^{1/2}. \tag{62}$$

Proof. By definition,

$$\begin{aligned}\mathcal{E}^\mu[|Y||\mathcal{F}_t] &= |Y| + \int_t^T \mu|Z_s|ds - \int_t^T Z_s dB_s \qquad (63)\\ &= |Y| + \int_t^T b_\mu(s) Z_s ds - \int_t^T Z_s dB_s,\end{aligned}$$

where $b_\mu(s) = \mu\frac{Z_s}{|Z_s|}1_{\{|Z_s|>0\}}$. Let $Q_\cdot^\mu$ be the solution of SDE

$$dQ_t^\mu = b_\mu(t) Q_t^\mu dB_t, \;\; Q_0^\mu = 1.$$

Using Itô's formula to $Q_t^\mu \mathcal{E}^\mu[|Y||\mathcal{F}_t]$, we have

$$\mathcal{E}^\mu[|Y|] = \mathcal{E}^\mu[|Y||\mathcal{F}_0] = E[Q_T^\mu |Y|] \leq \{E[(Q_T^\mu)^2]\}^{1/2} \cdot \{E[|Y|^2]\}^{1/2}.$$

But since $|b_\mu| \leq \mu$, there exists a constant $c_{\mu,T}$ depending only on μ and T, such that $E[(Q_T^\mu)^2]^{1/2} \leq c_{\mu,T}$. We thus have the second inequality of (62). The first inequality is derived by taking $t = 0$ on both sides of (63) and then taking expectation. □

We then have

Corollary 3.1. *Let T be fixed. Then the extension $L_\mu(\mathcal{F}_T)$ of $L^2(\mathcal{F}_T)$ under the norm $\|\cdot\|_\mu$ is a Banach space.*

Lemma 3.3. *We have for all $\mu > 0$ and $Y \in L^2(\mathcal{F}_T)$,*

$$\mathbf{E}[\mathcal{E}^\mu[Y|\mathcal{F}_t]^2] \leq e^{\mu^2(T-t)} E[Y^2].$$

Proof. By definition,

$$\mathcal{E}^\mu[Y|\mathcal{F}_t] = Y + \int_t^T \mu|Z_s|ds - \int_t^T Z_s dB_s.$$

Ito's formula gives

$$\mathcal{E}^\mu[Y|\mathcal{F}_t]^2 = Y^2 + \int_t^T 2\mu\mathcal{E}^\mu[Y|\mathcal{F}_s]|Z_s|ds - 2\int_t^T \mathcal{E}^\mu[Y|\mathcal{F}_s]Z_s dB_s - \int_t^T Z_s^2 ds.$$

Taking expectations, we deduce that

$$\begin{aligned}\mathbf{E}[\mathcal{E}^\mu[Y|\mathcal{F}_t]^2] &= \mathbf{E}[Y^2] + \int_t^T \mathbf{E}[2\mu\mathcal{E}^\mu[Y|\mathcal{F}_s]|Z_s|]ds - \int_t^T \mathbf{E}[Z_s^2]ds\\ &\leq \mathbf{E}[Y^2] + \mu^2 \int_t^T \mathbf{E}[\mathcal{E}^\mu[Y|\mathcal{F}_s]^2]ds\end{aligned}$$

(because of $2ab \leq a^2 + b^2$). The claim follows then immediately from Gronwall's inequality. □

Upcrossing Inequality of $\mathcal{E}^g$–Supermartingales and Optional Sampling Inequality

We begin with an easy upcrossing inequality which reveals the main idea to prove such kind of inequalities in nonlinear situation.

Proposition 3.11. *Let g satisfy (i), (ii), (iii') of (34) and let (Y_t) be a g - supermartin–gale on $[0,T]$. Let $0 = t_0 < t_1 < \cdots < t_n = T$, and $a < b$ be two constants. Then there exists a constant $c > 0$ such that the number $U_a^b[Y,n]$ of upcrossings of $[a,b]$ by $\{Y_{t_j}\}_{0\le j\le n}$ satisfies*

$$\mathcal{E}^{-\mu}[U_a^b[Y,n]] \le \frac{\mathcal{E}^{\mu}[(Y_T-a)^-]}{b-a}. \tag{64}$$

Sketch of Proof. We only prove the case $d=1$. For $j=1,2,\cdots n$, we consider the following BSDE

$$y_t^j = Y_{t_j} + \int_t^{t_j} g(s,y_s^j,z_s^j)ds - \int_t^{t_j} z_s^j dB_s,\ \ t\in[t_{j-1},t_j].$$

Then we define, for $s\in[t_{j-1},t_j]$,

$$a_s^j := \begin{cases} (z_s^j)^{-1}g(s,y_s^j,z_s^j), & \text{if } z_s^j \neq 0;\\ 0, & \text{otherwise.}\end{cases}$$

and then $a_s := \sum_{j=1}^n a_s^j 1_{(t_{j-1},t_j]}(s)$. Since g is Lipschitz in z and $g(t,y,0)\equiv 0$, it is clear that $|a_s|\le\mu$. We also have, for each j,

$$g(s,y_s^j,z_s^j) = a_s z_s^j,\ s\in(t_{j-1},t_j].$$

We set

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_T} := \exp\{\int_0^T a_s dB_s - \frac{1}{2}\int_0^T |a_s|^2 ds\}.$$

By Girsanov Theorem, Q is a probability measure and

$$\mathbf{E}_Q[Y_{t_j}|\mathcal{F}_{t_{j-1}}] = \mathcal{E}^g[Y_{t_j}|\mathcal{F}_{t_{j-1}}] \le Y_{t_{j-1}},\ j=1,2,\cdots n,$$

for Y is a g - supermartingale. This implies that $\{Y_{t_j}\}_{j=1}^n$ is a (discrete) Q - supermartingale. We then can apply the classical up crossing theorem ((see e.g., [HWY1992], Theorem 2.14 and 2.42))

$$\mathbf{E}_Q[U_a^b[Y,n]] \le \frac{\mathbf{E}_Q[(Y_T-a)^-]}{b-a}.$$

This with $|a_s|\le\mu$, we then can apply the comparison theorem to prove (64).
□

We now consider a more general situation. Let $(Y_t)_{t\in[0,T]}$ be an adapted process. For a given time sequence $t_0, t_1, t_2\cdots$ in $[0,T]$ with $0\le t_0 < t_1 < t_2\cdots$, we denote $\tau_{-1} := t_0$ and

$$\begin{aligned}
\tau_0 &:= \inf\{t_i \ge t_0; Y_{t_i} \le a\}\\
\tau_1 &:= \inf\{t_i \ge \tau_1; Y_{t_i} \ge b\}\\
&\cdots\cdots\\
\tau_{2i} &:= \inf\{t_i \ge \tau_{2i-1}; Y_{t_i} \le a\}\\
\tau_{2i+1} &:= \inf\{t_i \ge \tau_{2i+1}; Y_{t_i} \ge b\}\\
&\cdots\cdots
\end{aligned}$$

If $\tau_{2j-1} \le T$, sequence $(Y_{\tau_0}, \cdots Y_{\tau_{2i-1}})$ upcrosses the interval $[a,b]$ i times. We denote by $U_a^b(Y,k)$ the number of upcrossing $[a,b]$ of the sequence $(Y_{t_0},\cdots,Y_{t_k})$. It is clear that

$$\{U_a^b(Y,k) = i\} = \{\tau_{2i-1} \le t_k < \tau_{2i+1}\}$$

We now fix an integer n. We have the following upcrossing inequality

Theorem 3.5. *Let g satisfy (i) and (ii) of (34) and let $(Y_t)_{t\in[0,T]}$ be a g-supermartingale. Then we have*

$$[U_a^b(Y,n)] \le \frac{1}{b-a} e^{2\mu(t_n-t_0)}\{\mathcal{E}^\mu[(Y_{t_n}-a)^- + \mathcal{E}^\mu[\int_{t_0}^{t_n} e^{\mu s}|g_s^0|ds] + a\mu(t_n-t_0)\} \tag{65}$$

where $g_s^0 := g(s,0,0)$.

Proof. We set $\tau_i^n := \tau_i \wedge t_n$, for each $i = 0,1,\cdots$, and consider the following BSDE:

$$\begin{aligned}
-dy_t^i &= g(t,y_t^i,z_t^i)dt - z_t^i dB_t, \;\; t\in[0,\tau_{2i+1}^n],\\
y_{\tau_{2i+1}^n}^i &= Y_{\tau_{2i+1}^n}.
\end{aligned}$$

As in the proof of Comparison Theorem, we can write

$$g(t,y_t^i,z_t^i) = \alpha_t^i y_t^i + \beta_t^i\cdot z_t^i + g(t,0,0), \;\; t\in[0,\tau_{2i+1}^n],$$

with $|\alpha_s^i|\le\mu$, $|\beta_t^i|\le\mu$. For $t\in[0,T]$, we define

$$\begin{aligned}
\alpha_t &:= \sum_{i=0}^n 1_{(\tau_{2i}^n,\tau_{2i+1}^n)}(t)\alpha_t^i,\\
\beta_t &:= \sum_{i=0}^n 1_{(\tau_{2i}^n,\tau_{2i+1}^n)}(t)\beta_t^i.
\end{aligned}$$

We then introduce a new probability Q by

$$\frac{dQ}{dP}|\mathcal{F}_T := \exp[-\frac{1}{2}\int_0^T |\beta_s|^2 ds + \int_0^T \beta_s dB_s].$$

Since Y is an $\mathcal{E}^g$–supermartingale, we have, for each $i = 0, 1, \cdots$, by Lemma 7.8,

$$\begin{aligned} Y_{\tau^n_{2i}} &\geq \mathcal{E}^g_{\tau^n_{2i},\tau^n_{2i+1}}[Y_{\tau^n_{2i+1}}] \\ &= E_Q[Y_{\tau^n_{2i+1}} \exp(\int_{\tau^n_{2i}}^{\tau^n_{2i+1}} \alpha_s ds) + \int_{\tau^n_{2i}}^{\tau^n_{2i+1}} \exp(\int_{\tau^n_{2i}}^{s} \alpha_r dr) g^0_s ds)|\mathcal{F}_{\tau^n_{2i}}] \quad (66) \end{aligned}$$

We now estimate the term $u_i := E_Q[\exp(\int_0^{\tau^n_{2i+1}} \alpha_s ds) 1_{\{\tau_{2i+1}\leq t_n\}}]$. Since $(Y_{\tau^n_{2i+1}} - a) \geq b - a$ on $\{\tau_{2i+1} \leq t_n\}$ and $\{\tau_{2i} < t_n\} = \{\tau_{2i+1} \leq t_n\} + \{\tau_{2i} < t_n < \tau_{2i+1}\}$, we have

$$\begin{aligned} u_i &\leq \frac{1}{b-a} E_Q[(Y_{\tau^n_{2i+1}} - a)\exp(\int_0^{\tau^n_{2i+1}} \alpha_s ds) I_{\{\tau_{2i+1}\leq t_n\}}] \\ &\leq \frac{1}{b-a} E_Q[(Y_{\tau^n_{2i+1}} - a)\exp(\int_0^{\tau^n_{2i+1}} \alpha_s ds) I_{\{\tau_{2i}< t_n\}}] \\ &\quad + \frac{1}{b-a} E_Q[(Y_{t_n} - a)^- \exp(\int_0^{t_n} \alpha_s ds) I_{\{\tau_{2i}<t_n<\tau_{2i+1}\}}] \end{aligned}$$

With $\{\tau_{2i} < t_n\} \in \mathcal{F}_{\tau_{2i}}$, we apply (66) to the first term of the right side:

$$\begin{aligned} u_i \leq & \frac{1}{b-a} E_Q[\{(Y_{\tau^n_{2i}} - a) I_{\{\tau_{2i}<t_n\}} + \int_{\tau^n_{2i}}^{\tau^n_{2i+1}} e^{\mu s}|g^0_s| ds\} \exp(\int_0^{\tau^n_{2i}} \alpha_s ds)] \\ & + \frac{a}{b-a} E_Q[|\exp(\int_0^{\tau^n_{2i}} \alpha_s ds) - \exp(\int_0^{\tau^n_{2i+1}} \alpha_s ds)|] \\ & + \frac{1}{b-a} e^{\mu t_n} E_Q[(Y_{t_n} - a)^- I_{\{\tau_{2i}<t_n<\tau_{2i+1}\}}] \end{aligned}$$

Since $I_{\{\tau_{2i}<t_n\}}(Y_{\tau^n_{2i}} - a) = I_{\{\tau_{2i}<t_n\}}(Y_{\tau_{2i}} - a) \leq 0$, and

$$\begin{aligned} & |\exp(\int_0^{\tau^n_{2i}} \alpha_s ds) - \exp(\int_0^{\tau^n_{2i+1}} \alpha_s ds)| \\ = & |\int_{\tau^n_{2i}}^{\tau^n_{2i+1}} \alpha_s \exp(\int_{\tau^n_{2i}}^{s} \alpha_r dr) ds| \leq \mu e^{\mu(t_n - t_0)}(\tau^n_{2i+1} - \tau^n_{2i-1}), \end{aligned}$$

we thus have

$$\begin{aligned} u_i \leq \quad & \tfrac{1}{b-a} e^{\mu(t_n-t_0)} E_Q[a\mu(\tau^n_{2i+1} - \tau^n_{2i-1}) \\ + \quad & \textstyle\int_{\tau^n_{2i-1}}^{\tau^n_{2i+1}} e^{\mu s}|g^0_s| ds + (Y_{t_n} - a)^- I_{\{\tau_{2i}<t_n<\tau_{2i+1}\}}] \end{aligned}$$

We observe that $I_{\{\tau_{2i}<t_n<\tau_{2i+1}\}} \leq I_{\{U_a^b(Y,n)=i\}}$ and, in the expression of u_i, $\{\tau_{2i+1} \leq t_n\} = \{U_a^b(Y,n) > i\}$. Thus

$$e^{-\mu(t_n-t_0)}E_Q[I_{\{U_a^b(Y,n)>i\}}] \leq \frac{1}{b-a}e^{\mu(t_n-t_0)}\{E_Q[(Y_{t_n}-a)^- I_{\{U_a^b(Y,n)=i\}}] + E_Q[\int_{\tau_{2i-1}^n}^{\tau_{2i+1}^n} e^{\mu s}|g_s^0|ds] + a\mu E_Q[\tau_{2i+1}^n - \tau_{2i-1}^n]\}.$$

Summering both sides for all i yields

$$e^{-\mu(t_n-t_0)}E_Q[U_a^b(Y,n)] \leq \frac{1}{b-a}e^{\mu(t_n-t_0)}\{E_Q[(Y_{t_n}-a)^-] + \frac{1}{b-a}E_Q[\int_{t_0}^{t_n} e^{\mu s}|g_s^0|ds] + a\mu(t_n-t_0)\}.$$

This with $\mathcal{E}^{-\mu}[\cdot] \leq E_Q[\cdot] \leq \mathcal{E}^{\mu}[\cdot]$ derives the upcrossing inequality. □

Remark 3.7. Since, $\mathcal{E}^{-\mu}[\cdot]_{\mu=0} = \mathcal{E}^{\mu}[\cdot]_{\mu=0} = E[\cdot]$, thus in the case where $\mu = 0$ and $g_s^0 \equiv 0$, the about upcrossing inequality becomes a classical one:

$$(b-a)E[U_a^b(Y,n)] \leq E[(Y_{t_n}-a)^-].$$

To extend the above upcrossing inequality to denumerable sets, following (Peng, 1997 [Peng1997b]), we now extend the domain of $\mathcal{E}^g[\cdot]$ from $L^2(\mathcal{F}_T)$ to a larger space . We consider

$$L_2^0(\mathcal{F}_T) := \{X^+ \in L^0(\mathcal{F}_T),\ X^- \in L^2(\mathcal{F}_T)\}.$$

We need the following result:

Lemma 3.4. *Let $X \in L_2^0(\mathcal{F}_T)$ and let $\{X_i\}_{i=1}^{\infty}$ and $\{X_i'\}_{i=1}^{\infty}$ be two non decreasing sequences in $L^2(\mathcal{F}_T)$ such that $X_i \nearrow X$, a.s $X_i' \nearrow X$ a.s.. Then we have*

$$\lim_{i\to\infty} \mathcal{E}_g[X_i] = \lim_{i\to\infty} \mathcal{E}_g[X_i'].$$

Proof. We only need to consider the case where $X_i \geq X_i'$, a.s., for all $i = 1, 2, \cdots$. In this case

$$\lim_{i\to\infty} \mathcal{E}_g[X_i] \geq \lim_{i\to\infty} \mathcal{E}_g[X_i'].$$

On the other hand, for each fixed integer i_0, we have $X_{i_0} \wedge X_i' \nearrow X_{i_0}$ in $L^2(\mathcal{F}_T)$. It follows from the continuity of $\mathcal{E}_g[\cdot]$ in L^2 that $\lim_{i\to\infty} \mathcal{E}_g[X_i'] \geq \lim_{i\to\infty} \mathcal{E}_g[X_{i_0} \wedge X_i'] = \mathcal{E}_g[X_{i_0}]$. Thus $\lim_{i\to\infty} \mathcal{E}_g[X_i'] \geq \lim_{i\to\infty} \mathcal{E}_g[X_i]$. □

Definition 3.6. *For each $X \in L_2^0(\mathcal{F}_T)$, we define*

$$\mathcal{E}_g[X] = \lim_{i\to\infty} \mathcal{E}_g[X_i],$$

where $\{X_i\}_{i=1}^{\infty}$ is a non decreasing sequence in $L^2(\mathcal{F}_T)$ such that $X_i \nearrow X$, a.s.

From the above lemma, the functional $\mathcal{E}_g[\cdot] : L^0_2(\mathcal{F}_T) \to R \cup \{+\infty\}$ is clearly defined. We are interested in the situation where $g = g_{-\mu}(z) = -\mu|z|$.

Lemma 3.5. *For each nonnegative $X \in L^0_2(\mathcal{F}_T)$, if $\mathcal{E}^{-\mu}[X] = \mathcal{E}_{g_{-\mu}}[X] < +\infty$, then $X < +\infty$, dP–a.s.*

Proof. We set $A := \{\omega \in \Omega : X(\omega) = +\infty\}$. It is clear that $\lambda 1_A \le X$, a.s, for each $\lambda \in [0,\infty)$. Thus, by comparison theorem,

$$\mathcal{E}^{-\mu}[\lambda 1_A] \le \mathcal{E}^{-\mu}[X],\ \forall \lambda \in [0,\infty).$$

But we have $\mathcal{E}^{-\mu}[\lambda 1_A] = \lambda \mathcal{E}^{-\mu}[1_A]$ and, by strict comparison theorem, $\mathcal{E}^{-\mu}[1_A] > 0 \Leftrightarrow P(A) > 0$. It follows that A must be a P–zero subset. The proof is complete. □

Let $Y = (Y_t)_{t\in[0,T]}$ be an $\mathcal{F}_t$–adapted process, $u = \{t_1, t_2, \cdots, t_n\} \subset [0,T]$ with $t_1 < \cdots < t_n$. We denote by $U^b_a(Y,u)$ the upcrossing number of $\{Y_{t_1}, \cdots, Y_{t_n}\}$. For any subset D of $[0,T]$, define

$$U^b_a(Y,D) := \sup\{U^b_a(Y,u) \text{: } u \text{ is a finite subset of } D\}.$$

If D is a denumerable dense subset of $[0,T]$. Let $\{u_n\}^\infty_{n=1}$ be a sequence of finite subsets in D such that $u_n \subset u_{n+1}$for each n with $\cup_n u_n = D$. It is clear that

$$U^b_a(Y,D) = \lim_{n\to\infty} U^b_a(Y,u_n).$$

Theorem 3.6. *We assume that g satisfies (i) and (ii) of (34). Let $Y = (Y_t)_{t\in[0,T]}$ be a $\mathcal{E}^g$–supermartingale, D be a denumerable dense subset of $[0,T]$. Then for each $a, b \in R$, $r, s \in [0,T]$ such that $a < b$ and $r < s$, we have*

$$\mathcal{E}^{-\mu}[U^b_a(Y, D\cap[r,s]) \le \frac{e^{2\mu(s-r)}}{b-a}\{\mathcal{E}^\mu[(Y_s-a)^-]+\mathcal{E}^\mu[\int_r^s e^{\mu t}|g^0_t|dt]+a\mu(s-r)\}, \tag{67}$$

where μ is the Lipschitz constant of g and $g^0_s = g(s,0,0)$. In particular

$$\mathcal{E}^{-\mu}[U^b_a(Y,D)] \le \frac{e^{2\mu T}}{b-a}\{\mathcal{E}^\mu[(Y_T-a)^-] + \mathcal{E}^\mu[\int_0^T e^{\mu t}|g^0_t|dt]\ + a\mu T\}. \tag{68}$$

Moreover, $U^b_a(Y,D) < \infty$, a.s.

Proof. Let $u_n = \{t_0, t_1, t_2, \cdots, t_n\}$ be defined as the above with $t_0 = r$ and $t_n = s$. Since $\{U^b_a(Y,u_n)\}^\infty_{n=1}$ is an increasing and positive sequence such that $U^b_a(Y,u_n) \in L^2(\mathcal{F}_T)$ for each n, it follows that

$$\mathcal{E}^{-\mu}[U^b_a(Y, D\cap[r,s])] = \lim_{n\to\infty} \mathcal{E}^{-\mu}[U^b_a(Y, D\cap u_n)].$$

The sequence $\{\mathcal{E}^{-\mu}[U^b_a(Y, D\cap u_n)]\}^\infty_{n=1}$ is increasing and uniformly bounded by the left hand of (65). It follows that $\mathcal{E}^{-\mu}[U^b_a(Y, D\cap[r,s])]$ and $\mathcal{E}^{-\mu}[U^b_a(Y,D)]$ are well–defined and bounded. By Lemma 3.5, $U^b_a(Y,D) < \infty$, a.s. . □

Remark 3.8. From the above upcrossing inequality we can deduce a downcrossing inequality of a $\mathcal{E}^g$–submartingale Y. In fact, from the relation $D_a^b(Y,n) = U_{-b}^{-a}(-Y,n)$, one can directly obtain the downcrossing inequality of $D_a^b(Y,n)$ of a $\mathcal{E}^g$–submartingale Y from the corresponding upcrossing inequality of $U_{-b}^{-a}(-Y,n)$ of $\mathcal{E}^{\bar{g}}$–supermartingale $-Y$, where $\bar{g}(s,y,z) := -g(s,-y,-z)$.

From the above result, and combine the condition $E[\sup_{t\in[0,T]}|Y_t|^2] < \infty$, we have the following classical result.

Theorem 3.7. *We assume that g satisfies (i) and (ii) of (34). Let $Y = (Y_t)_{t\in[0,T]}$ be a $\mathcal{E}^g$–supermartingale, D be a denumerable dense subset of $[0,T]$. Then for almost all ω and for any $t \in [0,T]$, $\lim_{s\in D, s\searrow t} Y_s$ and $\lim_{s\in D, s\nearrow t} Y_s$ exist and are finite. Furthermore the process $(\bar{Y}_t)_{t\in[0,T]}$ defined by*

$$\bar{Y}_t := \lim_{s\in D, s\searrow t} Y_s,\ t \in [0,T)$$

is an $\mathcal{F}_t$–adapted process with $E[\sup_{0\le t\le T}|\bar{Y}_t|^2] < \infty$. If g also satisfies (iii) of (34), then $\bar{Y}$ is an $\mathcal{E}^g$–supermartingale.

Proof. We only need to prove that $\bar{Y}$ is an $\mathcal{E}^g$–supermartingale. The rest of the proofs can be find in, e.g., [HWY1992]. Let $s < t$, $s,t \in [0,T]$ and $s_n \in D$, $s_n < t$, $s_n \downarrow\downarrow s$, $t_n \in D$, $t_n \downarrow\downarrow t$ and $s_n \le t_n$. Then, for $m \ge n$,

$$\mathcal{E}^g_{s_m,t_n}[Y_{t_n}] \le Y_{s_m}.$$

We fix n and let $m \to \infty$. We have $Y_{s_m} \to \bar{Y}_s$ and, by $\mathcal{E}^g_{s,t_n}[Y_{t_n}])_{s\in[0,t_n]} \in S^2_{\mathcal{F}}(0,t_n)$, we also have $\mathcal{E}^g_{s_m,t_n}[Y_{t_n}] \to \mathcal{E}^g_{s,t_n}[Y_{t_n}]$, we derive

$$\mathcal{E}^g_{s,t_n}[Y_{t_n}] \le \bar{Y}_s,\ \text{a.s.}$$

Now let $n \to \infty$. We have $Y_{t_n} \to \bar{Y}_t$, in $L^2(\mathcal{F}_T)$. It follows that

$$|\mathcal{E}^g_{s,t_n}[Y_{t_n}] - \mathcal{E}^g_{s,t}[\bar{Y}_t]| \le |\mathcal{E}^g_{s,t_n}[Y_{t_n}] - \mathcal{E}^g_{s,t_n}[\bar{Y}_t]| + |\mathcal{E}^g_{s,t_n}[\bar{Y}_t] - \mathcal{E}^g_{s,t}[\bar{Y}_t]|.$$

We then can apply a technique used in the estimate of (141) to prove that $|\mathcal{E}^g_{s,t_n}[Y_{t_n}] - \mathcal{E}^g_{s,t}[\bar{Y}_t]| \to 0$. Thus

$$\mathcal{E}^g_{s,t}[\bar{Y}_t] \le \bar{Y}_s.$$

□

Remark 3.9. By this proposition we can prove that, in many typical cases a g - supermartingale Y admits a RCLL modification. More details on this topic will be given in Lemma 4.8, for a more general situation. We will always take its RCLL version.

Lemma 3.6. *Let Y be an RCLL g–supermartingale on $[0,T]$ and let σ and τ be two $\mathcal{F}_t$–stopping times. Then we have*

$$\mathcal{E}_g[Y_\tau|\mathcal{F}_\sigma] \le Y_{\tau\wedge\sigma}.$$

Proof. See Theorem 7.4.

3.3 A Monotonic Limit Theorem of BSDE

For a given stopping time $\tau \le T < \infty$, we consider a process (y_t) the solution of the following BSDE

$$y_t = \xi + \int_{t\wedge\tau}^{\tau} g(y_s, z_s, s)ds + (A_\tau - A_{t\wedge\tau}) - \int_{t\wedge\tau}^{\tau} z_s dB_s \tag{69}$$

where $\xi \in L^2(\mathcal{F}_\tau)$, A is a given RCLL increasing process with $\mathbf{E}[(A_\tau)^2] < \infty$. The following terms will be frequently used.

Definition 3.7. *If (y, z) is a solution of BSDE (69) then we call (y_t) a g-supersolution on $[0, \tau]$. If $A_t \equiv 0$ on $[0, \tau]$, then we call y a g-solution on $[0, \tau]$.*

We recall that a g-solution y on $[0, \tau]$ is uniquely determined if its terminal condition $y_\tau = \xi$ is given, a g-supersolution y on $[0, \tau]$ is uniquely determined if y_τ and $(A_t)_{0\le t\le\tau}$ are given. If y is a g-solution and y' is a g-supersolution on $[0, \tau]$ such that $y_\tau \le y'_\tau$ a.s., then for all stopping time $\sigma \le \tau$ we have also $y_\sigma \le y'_\sigma$.

Proposition 3.12. *Let y be a g-supersolution defined on an interval $[0, \tau]$. Then there is a unique $z \in L^2(0, \tau; \mathbf{R}^d)$ and a unique increasing RCLL process A on $[0, \tau]$ with $\mathbf{E}[(A_\tau)^2] < \infty$ such that the triple (y_t, z_t, A_t) satisfies (69).*

Proof. If both (y, z, A) and (y, z', A') satisfy (69), then we apply Itô's formula to $(y_t - y_t)^2 (\equiv 0)$ on $[0, \tau]$ and take expectation:

$$\mathbf{E}\int_0^\tau |z_s - z'_s|^2 ds + \mathbf{E}[\sum_{t\in(0,\tau]} (\Delta(A_t - A'_t))^2] = 0.$$

Thus $z_t \equiv z'_t$. From this it follows that $A_t \equiv A'_t$. □

Thus we can define

Definition 3.8. *Let y be a g–supersolution on $[0, \tau]$ and let (y, A, z) be the related unique triple in the sense of BSDE (69). Then we call (A, z) the (unique) decomposition of (y_t).*

Let us now consider the following sequence of g-supersolution $\{y^i\}_{i=1}^\infty$ on $[0, T]$, i.e.,

$$y_t^i = y_T^i + \int_t^T g(y_s^i, z_s^i, s)ds + (A_T^i - A_t^i) - \int_t^T z_s^i dB_s, \qquad i = 1, 2, \cdots . \tag{70}$$

Here A^i are RCLL increasing processes with $A_0^i = 0$ and $\mathbf{E}[(A_T^i)^2] < \infty$.

The following theorem shows that the limit of $\{y^i\}_{i=1}^\infty$ is still a g–supersolution.

Theorem 3.8. *We assume that g satisfies (i) and (ii) of Assumptions (34). For each $i = 1, 2, \cdots$, let A^i be a continuous and increasing processes with $A^i_0 = 0$ and $\mathbf{E}[(A^i_T)^2] < \infty$ and (y^i, z^i) be the solution of BSDE (70). If, as $i \to \infty$, $\{y^i\}_{i=1}^\infty$ converges monotonically up to a process y with $\mathbf{E}[\operatorname{esssup}_{0\le t\le T}|y_t|^2] < \infty$. Then this limit y is still a g-supersolution, i.e., there exists $z \in L^2_{\mathcal{F}}(0,T;\mathbf{R}^d)$ and an RCLL increasing process A with $\mathbf{E}[(A_T)^2] < \infty$ such that*

$$y_t = y_T + \int_t^T g(y_s, z_s, s)ds + (A_T - A_t) - \int_t^T z_s dB_s, \quad t \in [0,T]. \tag{71}$$

To prove this theorem, we need the following lemma. This lemma says that both $\{z^i\}$ and $\{(A^i_T)^2\}$ are uniformly bounded in L^2:

Lemma 3.7. *Under the assumptions of Theorem 3.8, there exists a constant C that is independent of i such that*

$$\begin{array}{l} (i)\ \mathbf{E}\int_0^T |z^i_s|^2 ds \le C, \\ (ii)\quad \mathbf{E}[(A^i_T)^2] \le C. \end{array} \tag{72}$$

Proof. From BSDE (70), we have

$$\begin{aligned} A^i_T &= y^i_0 - y^i_T - \int_0^T g(y^i_s, z^i_s, s)ds + \int_0^T z^i_s dB_s \\ &\le |y^i_0| + |y^i_T| + \int_0^T [\nu|y^i_s| + \mu|z^i_s| + |g(0,0,s)|]ds + |\int_0^T z^i_s dB_s|. \end{aligned}$$

We observe that $|y^i_t|$ is dominated by $|y^1_t| + |y_t|$. Thus there exists a constant, independent of i, such that

$$\mathbf{E}[\sup_{0\le t\le T} |y^i_t|^2] \le C. \tag{73}$$

It follows that, there exists a constant C_1, independent of i, such that

$$\mathbf{E}|A^i_T|^2 \le C_1 + 2(1+\mu^2 T)\mathbf{E}\int_0^T |z^i_s|^2 ds. \tag{74}$$

On the other hand, we use Itô's formula applied to $|y^i_t|^2$:

$$|y^i_0|^2 + \mathbf{E}\int_0^T |z^i_s|^2 ds = \mathbf{E}|y^i_T|^2 + 2\mathbf{E}\int_0^T y^i_s g(y^i_s, z^i_s, s)ds + 2\mathbf{E}\int_0^T y^i_s dA^i_s$$

The last two terms are bounded by

$$\begin{aligned} 2y^i_s g(y^i_s, z^i_s, s) &\le 2|y^i_s|(\nu|y^i_s| + \mu|z^i_s| + |g(0,0,s)|) \\ &\le 2(\nu+\mu^2)|y^i_s|^2 + \frac{1}{2}|z^i_s|^2 + |g(0,0,s)| \end{aligned}$$

and $2\mathbf{E}\int_0^T |y_s^i| dA_s^i \le 2[\mathbf{E}\sup_{0\le s\le T}|y_s^i|^2]^{1/2}[\mathbf{E}|A_T^i|^2]^{1/2}$. Thus

$$\begin{aligned}\mathbf{E}\int_0^T |z_s^i|^2 ds &\le C + 4[\mathbf{E}\sup_{0\le s\le T}|y_s^i|^2]^{1/2}[\mathbf{E}|A_T^i|^2]^{1/2}\\ &\le C + 16(1+\mu^2 T)\mathbf{E}[\sup_{0\le s\le T}|y_s^i|^2] + \frac{1}{4(1+\mu^2 T)}\mathbf{E}|A_T^i|^2\\ &= C_1 + \frac{1}{4(1+\mu^2 T)}\mathbf{E}|A_T^i|^2,\end{aligned}$$

where, from (73), the constants C and C_1 are all independent of i. This with (74) it follows that (72)–(i) and then (72)–(ii) holds true. The proof is complete. □

Combining this Lemma with Theorem 7.2 in Appendix, we can easily prove Theorem 3.8.

Proof of Theorem 3.8. In (70), we set $g_t^i := -g(y_t^i, z_t^i, t)$; Since $\{z^i\}$ is bounded in $L^2_{\mathcal{F}}(0,T;\mathbf{R}^d)$, thanks to the monotonic limit theorem of Itô processes (see Appendix: Theorem 7.2), there exists a $z \in L^2_{\mathcal{F}}(0,T;\mathbf{R}^d)$ such that, for each $p \in [0,2)$, $\{z^i\}_{i=1}^\infty$ strongly converges to z in $L^p_{\mathcal{F}}(0,T;\mathbf{R}^d)$.

As result, $\{g^i\} = \{-g(y^i, z^i, \cdot)\}$ also strongly converges in $L^p_{\mathcal{F}}(0,T;\mathbf{R}^d)$ to g^0 and

$$g^0(s) = -g(y_s, z_s, s), \quad \text{a.s., a.e.}$$

From this it follows immediately that (y,z) is the solution of the BSDE (71). The proof is complete. □

3.4 *g*–Martingales and (Nonlinear) *g*–Supermartingale Decomposition Theorem

More general than the martingales under g–expectations, we now introduce the notion of g–martingales under g–evaluations. Under this general framework, we will prove a general g–supermartingale decomposition theorem of Doob–Meyer's type.

Definition 3.9. *An $\mathcal{F}_t$-progressively measurable real-valued process Y with*

$$\mathbf{E}[ess\sup_{0\le t\le T}|Y_t|^2] < \infty,\ \forall T < \infty$$

is called a g–martingale (resp. g–supermartingale, g–submartingale) on $[0,T]$ if for each $0 \le s \le t \le T$,

$$\mathcal{E}^g_{s,t}[Y_t] = Y_s,\ \ \textit{(resp.}\ \le Y_s,\ \ \ge Y_s\textit{) a.s.}$$

In this subsection we will consider g–supermartingales. By Comparison Theorem of BSDE, it is easy to prove the following result

Proposition 3.13. *We assume that g satisfies (i) and (ii) of (34). Let $(A_t)_{0\le t<\infty}$ be an RCLL increasing (resp. decreasing) process with $\mathbf{E}[(A_T)^2]<\infty$ for each $T>0$. Let (y,z) be the solution of the following BSDE, for each $T>0$,*

$$y_t = y_T + \int_t^T g(y_s,z_s,s)ds + (A_T - A_t) - \int_t^T z_s dB_s, \quad t\in[0,T], \tag{75}$$

Then $(y_t)_{0\le t\le T}$ is a g–supermartingale (resp. g–submartingale).

In this section we are concerned with the inverse problem: can we say that a right-continuous $\mathcal{E}^g$-supermartingale is also a $\mathcal{E}^g$-supersolution? This problem is more difficult since it is in fact a nonlinear version of Doob-Meyer Decomposition Theorem. We claim

Theorem 3.9. *We assume that g satisfies (i) and (ii) of (34). Let (Y_t) be a right-continuous g–supermartingale on $[0,T]$. Then (Y_t) is an g-supersolution: there exists a unique RCLL increasing process (A_t) with $\mathbf{E}[(A_T)^2]<\infty$, for each $T>0$, such that (Y_t) coincides with the unique solution (y_t) of the BSDE. For each $T>0$,*

$$y_t = Y_T + \int_t^T g(y_s,z_s,s)ds + (A_T - A_t) - \int_t^T z_s dB_s, \quad t\in[0,T], \tag{76}$$

In order to prove this theorem, we consider the following family of BSDE parameterized by $i=1,2,\cdots$.

$$y_t^i = Y_T + \int_t^T g(y_s^i,z_s^i,s)ds + i\int_t^T (Y_s - y_s^i)ds - \int_t^T z_s^i dB_s. \tag{77}$$

An important observation is that, for each i, y_t^i is always bounded from above by Y_t. Thus y^i is a g-supersolution on $[0,T]$:

Lemma 3.8. *We have, for each $i=1,2,\cdots$,*

$$Y_t \ge y_t^i,\ \forall t\in[0,T],\ a.s..$$

Proof. For a $\delta>0$ and a given integer $i>0$, we define

$$\sigma^{i,\delta} := \inf\{t;\ y_t^i \ge Y_t+\delta\}\wedge T.$$

If $P(\sigma^{i,\delta}<T)=0$, for all i and δ, then the proof is done. If it is not the case, then there exist $\delta>0$ and a positive integer i such that $P(\sigma^{i,\delta}<T)>0$. We can then define the following stopping times

$$\tau := \inf\{t\ge\sigma^{i,\delta};\ y_t^i\le Y_t\}$$

It is clear that $\sigma^{i,\delta}\le\tau\le T$. Since $Y_\cdot - y_\cdot^i$ is RCLL, we have

$$y^i_\tau \le Y_\tau.$$

But since $(Y(s) - y^i(s)) \le 0$ on $[\sigma^{i,\delta}, \tau]$, by monotonicity of $\mathcal{E}^g[\cdot]$,

$$\begin{aligned} y^i_{\sigma^{i,\delta}} &\le \mathcal{E}^g_{\sigma^{i,\delta},\tau}[y^i_\tau | \mathcal{F}_{\sigma^{i,\delta}}] \\ &\le \mathcal{E}^g_{\sigma^{i,\delta},\tau}[Y_\tau | \mathcal{F}_{\sigma^{i,\delta}}] \\ &\le Y_{\sigma^{i,\delta}}.\ \text{a.s.} \end{aligned}$$

The last step is due to Theorem 7.3. But on the other hand, we have $P(\sigma^{i,\delta} < T) > 0$ and, by the definition of $\sigma^{i,\delta}$, $y^i_{\sigma^{i,\delta}} \ge Y_{\sigma^{i,\delta}} + \delta$ on $\{\sigma^{i,\delta} < T\}$. This induces a contradiction. The proof is complete. $\square$

Remark 3.10. From the above result, the term $i(Y_s - y^i_s)$ in (77) equals to $i(Y_s - y^i_s)^+$. By Comparison Theorem y^i_t are pushed up to be above the supermartingale Y_t. But in fact they can never surpass Y_t. We will see that this effect will force y^i to converge to the supermartingale Y itself. Thus, by Limit Theorem 3.8 Y itself is also a form of (76). Specifically, we have:

Proof of Theorem 3.9. The uniqueness is due to the uniqueness of g-supersolution i.e. Proposition 3.12. We now prove the existence. We rewrite BSDE (77) as

$$y^i_t = Y_T + \int_t^T g(y^i_s, z^i_s, s)ds + A^i_T - A^i_t - \int_t^T z^i_s dB_s,$$

where we denote

$$A^i_t := i\int_0^t (Y_s - y^i_s)ds.$$

From Lemma 3.8, $Y_t - y^i_t = |Y_t - y^i_t|$. It follows from the Comparison Theorem that $y^i_t \le y^{i+1}_t$. Thus $\{y^i\}$ is a sequence of continuous $\mathcal{E}^g$-supersolutions that is monotonically converges up to a process (y_t). Moreover (y_t) is bounded from above by Y_t. It is then easy to check that all conditions in Theorem 3.8 are satisfied. (y_t) is a $\mathcal{E}^g$-supersolution on $[0,T]$ of the following form.

$$y_t = Y_T + \int_t^T g(y_s, z_s, s)ds + (A_T - A_t) - \int_t^T z_s dB_s, \quad t \in [0,T],$$

where (A_t) is a RCLL increasing process. It then remains to prove that $y = Y$. From Lemma 3.7–(ii) we have

$$\mathbf{E}[|A^i_T|^2] = i^2 \mathbf{E}\left[\int_0^T |Y_t - y^i_t| dt\right]^2 \le C.$$

It then follows that $Y_t \equiv y_t$. The proof is complete $\square$.

4 Finding the Mechanism: Is an $\mathcal{F}$–Expectation a g–Expectation?

4.1 $\mathcal{E}^\mu$-Dominated $\mathcal{F}$-Expectations

Now we will study $\mathcal{F}$-expectations dominated by $\mathcal{E}^\mu = \mathcal{E}^{g_\mu}$, with $g_\mu(z) := \mu|z|$, for some large enough $\mu > 0$, according to the following

Definition 4.1. ***($\mathcal{E}^\mu$-domination)*** *Given $\mu > 0$, we say that an $\mathcal{F}$ - expectation $\mathcal{E}$ is dominated by $\mathcal{E}^\mu$ if*

$$\mathcal{E}[X+Y] - \mathcal{E}[X] \le \mathcal{E}^\mu[Y], \ \forall X, Y \in L^2(\mathcal{F}_T) \tag{78}$$

By Proposition 3.6, for any g satisfying (i), (ii) (iii") of (34), the associated g-expectation is dominated by $\mathcal{E}^\mu$, where μ is the Lipschitz constant in (34).

Lemma 4.1. *If $\mathcal{E}$ is dominated by $\mathcal{E}^\mu$ for some $\mu > 0$, then*

$$\mathcal{E}^{-\mu}[Y] \le \mathcal{E}[X+Y] - \mathcal{E}[X] \le \mathcal{E}^\mu[Y]. \tag{79}$$

Proof. It is a simple consequence of

$$\mathcal{E}^{-\mu}[Y|\mathcal{F}_t] = -\mathcal{E}^\mu[-Y|\mathcal{F}_t].$$

□

Lemma 4.2. *If $\mathcal{E}$ is dominated by $\mathcal{E}^\mu$ for some $\mu > 0$, then $\mathcal{E}[\cdot]$ is a continuous operator on $L^2(\mathcal{F}_T)$ in the following sense:*

$$\exists C > 0, \quad |\mathcal{E}[\xi_1] - \mathcal{E}[\xi_2]| \le C \|\xi_1 - \xi_2\|_{L^2}, \quad \forall \xi_1, \xi_2 \in L^2(\mathcal{F}_T). \tag{80}$$

Proof. The claim follows easily from Lemma 4.1 above and Lemma 3.3. □

From now on we will deal with $\mathcal{F}$-expectations $\mathcal{E}[\cdot]$ also satisfying the following condition:

$$\mathcal{E}[X+Y|\mathcal{F}_t] = \mathcal{E}[X|\mathcal{F}_t] + Y, \quad \forall X \in L^2(\mathcal{F}_T) \quad \text{and} \quad Y \in L^2(\mathcal{F}_t) \tag{81}$$

Recall that, when $\mathcal{E}[\cdot]$ is a g-expectation, (81) means that g satisfies (34)–(iii") (see (57)). We observe that an expectation $E_Q[\cdot]$ under a Girsanov transformation $\dfrac{dQ}{dP}$ satisfies this assumption.

We need to introduce a new notation: for a given $\zeta \in L^2(\mathcal{F}_T)$, we consider the mapping $\mathcal{E}_\zeta[\cdot]$ defined by

$$\mathcal{E}_\zeta[X] := \mathcal{E}[X+\zeta] - \mathcal{E}[\zeta] : L^2(\mathcal{F}_T) \longmapsto R. \tag{82}$$

Lemma 4.3. *If $\mathcal{E}[\cdot]$ is an $\mathcal{F}$-expectation satisfying (78) and (81), then the mapping $\mathcal{E}_\zeta[\cdot]$ is also an $\mathcal{F}$-expectation satisfying (78) and (81). Its conditional expectation under $\mathcal{F}_t$ is*

$$\mathcal{E}_\zeta[X|\mathcal{F}_t] = \mathcal{E}[X+\zeta|\mathcal{F}_t] - \mathcal{E}[\zeta|\mathcal{F}_t]. \tag{83}$$

Proof. It is easily seen that $\mathcal{E}_\zeta[\cdot]$ is a nonlinear expectation.

We now prove that the notion $\mathcal{E}_\zeta[X|\mathcal{F}_t]$ defined in (83) is actually the conditional expectation induced by $\mathcal{E}_\zeta[\cdot]$ under $\mathcal{F}_t$.

Indeed, put $G(X,\zeta,\mathcal{F}_t) = \mathcal{E}[X+\zeta|\mathcal{F}_t] - \mathcal{E}[\zeta|\mathcal{F}_t]$. We want to show that, for all $A \in \mathcal{F}_t$, $\mathcal{E}_\zeta(G(X,\zeta,\mathcal{F}_t)1_A) = \mathcal{E}_\zeta(X1_A)$. Computations give:

$$\begin{aligned}
\mathcal{E}_\zeta[G(X,\zeta,\mathcal{F}_t)] &= \mathcal{E}[\mathcal{E}[X+\zeta|\mathcal{F}_t] - \mathcal{E}[\zeta|\mathcal{F}_t] + \zeta] - \mathcal{E}[\zeta] \quad \text{(by (9))} \\
&= \mathcal{E}[\mathcal{E}[X+\zeta|\mathcal{F}_t] - \mathcal{E}[\zeta|\mathcal{F}_t] + \mathcal{E}[\zeta|\mathcal{F}_t]] - \mathcal{E}[\zeta] \quad \text{(by (81))} \\
&= \mathcal{E}[\mathcal{E}[X+\zeta|\mathcal{F}_t]] - \mathcal{E}[\zeta] \\
&= \mathcal{E}[X+\zeta] - \mathcal{E}[\zeta].
\end{aligned}$$

Thus we have

$$\mathcal{E}_\zeta[G(X,\zeta,\mathcal{F}_t)] = \mathcal{E}_\zeta[X], \quad \forall X. \tag{84}$$

Now for each $A \in \mathcal{F}_t$, we have,

$$\begin{aligned}
G(X1_A,\zeta,\mathcal{F}_t) &= \mathcal{E}[X1_A + \zeta 1_A + \zeta 1_{A^C}|\mathcal{F}_t] - \mathcal{E}[\zeta|\mathcal{F}_t] \\
&= \mathcal{E}[(X+\zeta)1_A + \zeta 1_{A^C}|\mathcal{F}_t] - \mathcal{E}[\zeta|\mathcal{F}_t] \\
&= \mathcal{E}[X+\zeta|\mathcal{F}_t]1_A + \mathcal{E}[\zeta|\mathcal{F}_t]1_{A^C} - \mathcal{E}[\zeta|\mathcal{F}_t] \\
&= (\mathcal{E}[X+\zeta|\mathcal{F}_t] - \mathcal{E}[\zeta|\mathcal{F}_t])1_A \\
&= G(X,\zeta,\mathcal{F}_t)1_A.
\end{aligned}$$

From this with (84) it follows that $\mathcal{E}_\zeta[X|\mathcal{F}_t]$ satisfies (7):

$$\mathcal{E}_\zeta[G(X,\zeta,\mathcal{F}_t)1_A] = \mathcal{E}_\zeta[G(X1_A,\zeta,\mathcal{F}_t)] = \mathcal{E}_\zeta[X1_A], \quad \forall A \in \mathcal{F}_t.$$

Thus $\mathcal{E}_\zeta[\cdot]$ is an $\mathcal{F}$-expectation with $\mathcal{E}_\zeta[\cdot|\mathcal{F}_t]$ given by (83).

We now check that (78) is satisfied. For each $X, Y \in L^2(\mathcal{F}_T)$,

$$\begin{aligned}
\mathcal{E}_\zeta[X+Y] - \mathcal{E}_\zeta[X] &= (\mathcal{E}[X+Y+\zeta] - \mathcal{E}[\zeta]) - (\mathcal{E}[X+\zeta] - \mathcal{E}[\zeta]) \\
&= \mathcal{E}[X+Y+\zeta] - \mathcal{E}[X+\zeta].
\end{aligned}$$

Since $\mathcal{E}[\cdot]$ satisfies (78), $\mathcal{E}_\zeta[\cdot]$ satisfies

$$\mathcal{E}_\zeta[X+Y] - \mathcal{E}_\zeta[X] \leq \mathcal{E}^\mu[Y].$$

Finally, let $Y \in L^2(\mathcal{F}_t)$; since $\mathcal{E}[\cdot]$ satisfies property (81), thus

$$\begin{aligned}
\mathcal{E}_\zeta[X+Y|\mathcal{F}_t] &= \mathcal{E}[X+\zeta|\mathcal{F}_t] - \mathcal{E}[\zeta|\mathcal{F}_t] + Y \\
&= \mathcal{E}_\zeta[X|\mathcal{F}_t] + Y.
\end{aligned}$$

Thus $\mathcal{E}_\zeta[\cdot]$ also satisfies property (81). The proof is complete. □

Lemma 4.4. *Let $\mathcal{E}[\cdot]$ be an $\mathcal{F}$-expectation satisfying (78) and (81). Then, for each $t \le T$, we have a.s.*

$$\mathcal{E}^{-\mu}[X|\mathcal{F}_t] \le \mathcal{E}_\zeta[X|\mathcal{F}_t] \le \mathcal{E}^{\mu}[X|\mathcal{F}_t],\ \forall X,\ \zeta \in L^2(\mathcal{F}_T).$$

This lemma is a simple consequence of the following one, whose proof is inspired by [BCHMP2000].

Lemma 4.5. *Let $\mathcal{E}_1[\cdot]$ and $\mathcal{E}_2[\cdot]$ be two $\mathcal{F}$-expectations satisfying (78) and (81). If*

$$\mathcal{E}_1[X] \le \mathcal{E}_2[X], \quad \forall X \in L^2(\mathcal{F}_T),$$

then a.s. and for all t,

$$\mathcal{E}_1[X|\mathcal{F}_t] \le \mathcal{E}_2[X|\mathcal{F}_t], \quad \forall X \in L^2(\mathcal{F}_T).$$

Proof. Indeed, for all $Y \in L^2(\mathcal{F}_T)$, we have by (81)

$$\begin{aligned}\mathcal{E}_1[Y - \mathcal{E}_1[Y|\mathcal{F}_t]] &= \mathcal{E}_1[\mathcal{E}_1[Y - \mathcal{E}_1[Y|\mathcal{F}_t]|\mathcal{F}_t]] \\ &= \mathcal{E}_1[\mathcal{E}_1[Y|\mathcal{F}_t] - \mathcal{E}_1[Y|\mathcal{F}_t]] \\ &= \mathcal{E}_1[0] = 0.\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathcal{E}_1[Y - \mathcal{E}_1[Y|\mathcal{F}_t]] &\le \mathcal{E}_2[Y - \mathcal{E}_1[Y|\mathcal{F}_t]] \\ &= \mathcal{E}_2[\mathcal{E}_2[Y - \mathcal{E}_1[Y|\mathcal{F}_t]|\mathcal{F}_t]].\end{aligned}$$

Thus

$$\mathcal{E}_2[\mathcal{E}_2[Y|\mathcal{F}_t] - \mathcal{E}_1[Y|\mathcal{F}_t]] \ge 0, \quad \forall Y \in L^2(\mathcal{F}_T).$$

Now, for a fixed $X \in L^2(\mathcal{F}_T)$, we set $\eta = \mathcal{E}_2[X|\mathcal{F}_t] - \mathcal{E}_1[X|\mathcal{F}_t]$. Since

$$\begin{aligned}\eta 1_{\{\eta<0\}} &= 1_{\{\eta<0\}}\mathcal{E}_2[X|\mathcal{F}_t] - 1_{\{\eta<0\}}\mathcal{E}_1[X|\mathcal{F}_t] \\ &= \mathcal{E}_2[X1_{\{\eta<0\}}|\mathcal{F}_t] - \mathcal{E}_1[X1_{\{\eta<0\}}|\mathcal{F}_t],\end{aligned}$$

we have then

$$\mathcal{E}_2[\eta 1_{\{\eta<0\}}] = 0.$$

But since $\eta 1_{\{\eta<0\}} \le 0$, it follows from the strict monotonicity of $\mathcal{E}_2[\cdot]$ that $\eta 1_{\{\eta<0\}} = 0$ a.s.. Thus

$$\mathcal{E}_2[X|\mathcal{F}_t] - \mathcal{E}_1[X|\mathcal{F}_t] \ge 0 \quad \text{a.s.}$$

The proof is complete. □

Lemma 4.6. *If $\mathcal{E}$ meets (78) and (81), there exists a positive constant C such that, for all X and Y in $L^2(\mathcal{F}_T)$, and for all $t \ge 0$,*

$$\mathcal{E}[\mathcal{E}[X+Y|\mathcal{F}_t] - \mathcal{E}[X|\mathcal{F}_t]] \le C\|Y\|_{L^2}.$$

Proof. Indeed, Lemmas 4.3 and 4.4 above imply that

$$\begin{aligned}\mathcal{E}[\mathcal{E}[X+Y|\mathcal{F}_t]-\mathcal{E}[X|\mathcal{F}_t]] &= \mathcal{E}[\mathcal{E}_X[Y|\mathcal{F}_t]] \\ &\leq \mathcal{E}[\mathcal{E}^\mu[Y|\mathcal{F}_t]] \\ &\leq \mathcal{E}^\mu[\mathcal{E}^\mu[Y|\mathcal{F}_t]] = \mathcal{E}^\mu[Y] \leq C\|Y\|_{L^2}.\end{aligned}$$

The last inequality is from Lemma 4.2.

□

4.2 $\mathcal{F}_t$-Consistent Martingales

In this subsection we assume that $\mathcal{E}$ is an $\mathcal{F}$-expectation satisfying (78) for some $\mu > 0$, and (81) as well.

Definition 4.2. *A process $(X_t)_{t\in[0,T]} \in L^2_{\mathcal{F}}(0,T)$ is called an $\mathcal{E}$-martingale (resp. $\mathcal{E}$-supermartingale, -submartingale) if for each $0 \leq s \leq t \leq T$*

$$X_s = \mathcal{E}[X_t|\mathcal{F}_s], \text{ (resp. } \geq \mathcal{E}[X_t|\mathcal{F}_s], \leq \mathcal{E}[X_t|\mathcal{F}_s]).$$

Lemma 4.7. *An $\mathcal{E}^\mu$-supermartingale (ξ_t) is both an $\mathcal{E}$ - supermartingale and $\mathcal{E}^{-\mu}$ - supermartingale. An $\mathcal{E}^{-\mu}$ - submartingale (ξ_t) is both an $\mathcal{E}$ - and $\mathcal{E}^\mu$ - submartingale. An $\mathcal{E}$ - martingale (ξ_t) is an $\mathcal{E}^{-\mu}$ - supermartingale and an $\mathcal{E}^\mu$-submartingale.*

Proof. It comes simply from the fact that, for each $0 \leq s \leq t \leq T$,

$$\mathcal{E}^{-\mu}[\xi_t|\mathcal{F}_s] \leq \mathcal{E}[\xi_t|\mathcal{F}_s] \leq \mathcal{E}^\mu[\xi_t|\mathcal{F}_s].$$

□

The next result is the first step in a procedure that will eventually prove that every $\mathcal{E}$-martingale admits continuous paths.

Lemma 4.8. *For each $X \in L^2(\mathcal{F}_T)$ the process $\mathcal{E}[X|\mathcal{F}_t]$, $t \in [0,T]$ admits a unique modification with a.s. RCLL paths.*

Proof. We can deduce from Lemma 4.7 that the process $\mathcal{E}[X|\mathcal{F}_t]$, $t \in [0,T]$, is an $\mathcal{E}^{-\mu}$-supermartingale. Hence we can apply the downcrossing inequality of Proposition 3.11.

This downcrossing inequality tells us that $\mathcal{E}[X|\mathcal{F}_t]$, $t \in [0,T]$ has P-a.s. finitely many downcrossings of every interval $[a,b]$ with rational $a < b$. By classical methods, this imply the almost sure existence of left and right limits for the paths of $\mathcal{E}[X|\mathcal{F}_\cdot]$.

We thus can define $Y_t = \lim_{\substack{s\searrow t \\ s\in\mathbf{Q}\cap[0,T]}} \mathcal{E}[X|\mathcal{F}_s]$. For each $A \in \mathcal{F}_t$, we have that

$$Y_t 1_A = \lim_{\substack{s \searrow t \\ s \in \mathbf{Q} \cap [0,T]}} \mathcal{E}[X|\mathcal{F}_s]1_A, \text{ in } L^2(\mathcal{F}_T).$$

From Lemma 4.2, it follows that

$$\mathcal{E}[Y_t 1_A] = \lim_{\substack{s \searrow t \\ s \in \mathbf{Q} \cap [0,T]}} \mathcal{E}[\mathcal{E}[X|\mathcal{F}_s]1_A].$$

But

$$\mathcal{E}[\mathcal{E}[X|\mathcal{F}_s]1_A] = \mathcal{E}[1_A \mathcal{E}[X|\mathcal{F}_t]].$$

It follows that a.s. $Y_t = \mathcal{E}[X|\mathcal{F}_t]$.

Now it's again classical to prove, using the existence of left and right limits, that the process Y defined above is a RCLL modification of $\mathcal{E}[X|\mathcal{F}_t]$, $t \in [0,T]$, and the lemma is proved. □

Henceforth, and without needing to recall it, we will always consider the RCLL modifications of the $\mathcal{E}$-martingales we have to deal with.

Lemma 4.8 has an immediate consequence as follows :

Lemma 4.9. *Let $\mathcal{E}[\cdot]$ be an $\mathcal{F}$-expectation satisfying (78) and (81). Then for each $X \in L^2(\mathcal{F}_T)$ and $g \in L^2_{\mathcal{F}}(0,T)$ the process $\mathcal{E}[X + \int_t^T g_s ds|\mathcal{F}_t]$, $t \in [0,T]$ is RCLL a.s.*

Proof. Indeed, we can write

$$\begin{aligned}\mathcal{E}[X + \int_t^T g_s ds|\mathcal{F}_t] &= \mathcal{E}[X + \int_0^T g_s ds - \int_0^t g_s ds|\mathcal{F}_t] \\ &= \mathcal{E}[X + \int_0^T g_s ds|\mathcal{F}_t] - \int_0^t g_s ds\end{aligned}$$

because of (81). The claim follows then easily from Lemma 4.8. □

Lemma 4.10. *For each $X \in L^2(\mathcal{F}_T)$, let*

$$y_t = \mathcal{E}[X|\mathcal{F}_t].$$

Then there exists a pair $(g(\cdot), z(\cdot)) \in L^2_{\mathcal{F}}(0,T; R \times R^d)$ with

$$|g_t| \leq \mu |z_t| \tag{85}$$

such that

$$y_t = X + \int_t^T g_s ds - \int_t^T z_s dB_s. \tag{86}$$

Furthermore, take $X' \in L^2(\mathcal{F}_T)$, put $y'_t = \mathcal{E}[X'|\mathcal{F}_t]$, and let $(g'(\cdot), z'(\cdot)) \in L^2_{\mathcal{F}}(0,T; R \times R^d)$ be the corresponding pair. Then we have

$$|g_t - g'_t| \leq \mu |z_t - z'_t| \tag{87}$$

Proof. Since

$$y_t = \mathcal{E}[X|\mathcal{F}_t], \quad 0 \le t \le T,$$

is an $\mathcal{E}$ - martingale, and since it is RCLL, it is a right-continuous $\mathcal{E}^\mu$-submartingale (resp. $\mathcal{E}^{-\mu}$ - supermartingale). By the domination $\mathcal{E}^{-\mu}[X|\mathcal{F}_t] \le \mathcal{E}[X|\mathcal{F}_t] \le \mathcal{E}^\mu[X|\mathcal{F}_t]$, we also have $\mathbf{E}[\sup_{t\in[0,T]}|y_t|^2] < \infty$. Thus, from the g - supermartingale decomposition theorem (Theorem 3.9) that there exist (z^μ, A^μ) and $(z^{-\mu}, A^{-\mu})$ in $L^2_{\mathcal{F}}([0,T]; R\times R^d)$ with A^μ and $A^{-\mu}$ RCLL and increasing such that $A^\mu(0) = 0$, $A^{-\mu}(0) = 0$ and such that

$$y_t = y_T + \int_t^T \mu|z_s^\mu|ds - A_T^\mu + A_t^\mu - \int_t^T z_s^\mu dB_s$$

and

$$y_t = y_T - \int_t^T \mu|z_s^{-\mu}|ds + A_T^{-\mu} - A_t^{-\mu} - \int_t^T z_s^{-\mu} dB_s.$$

Hence, the martingale parts and the bounded variation parts of the above two processes must coincide:

$$z_t^\mu \equiv z_t^{-\mu},$$
$$-\mu|z_t^\mu|dt + dA_t^\mu \equiv \mu|z_t^\mu|dt - dA_t^{-\mu},$$

whence

$$2\mu|z_t^\mu|dt \equiv dA_t^\mu + dA_t^{-\mu}.$$

It follows that A^μ and $A^{-\mu}$ are both absolutely continuous and we can write:

$$dA_t^\mu = a_t^\mu dt, \quad dA_t^{-\mu} = a_t^{-\mu} dt$$

with

$$0 \le a_t^\mu, \quad 0 \le a_t^{-\mu}.$$

We also have

$$a_t^\mu + a_t^{-\mu} \equiv 2\mu|z_t^\mu|,$$

so, if we define

$$z_t = z_t^\mu$$
$$g_t = \mu|z_t| - a_t^\mu,$$

we get (86) and (85).

Now, we prove (87). We have

$$\begin{aligned} y_t - y_t' &= \mathcal{E}[X|\mathcal{F}_t] - \mathcal{E}[X'|\mathcal{F}_t] \\ &= \mathcal{E}[X - X' + X'|\mathcal{F}_t] - \mathcal{E}[X'|\mathcal{F}_t] \\ &= \mathcal{E}_{X'}[X - X'|\mathcal{F}_t] \end{aligned}$$

Recall (Lemma 4.3) that $\mathcal{E}_{X'}[\cdot]$ is another $\mathcal{F}$-expectation satisfying (78) and (81). Thus there also exists a pair $(\tilde{g}(\cdot), \tilde{z}(\cdot)) \in L^2_{\mathcal{F}}(0,T; R\times R^d)$ with

$$|\tilde{g}_t| \le \mu|\tilde{z}_t| \tag{88}$$

such that the $\mathcal{E}_{X'}$-martingale $y_t - y'_t$ satisfies

$$y_t - y'_t = X - X' + \int_t^T \tilde{g}_s ds - \int_t^T \tilde{z}_s dB_s.$$

On the other hand, we have

$$y_t - y'_t = X - X' + \int_t^T [g_s - g'_s] ds - \int_t^T [z_s - z'_s] dB_s.$$

It follows then that

$$\tilde{g}_t \equiv g_t - g'_t, \quad \text{and} \quad \tilde{z}_t \equiv z_t - z'_t.$$

This with (88) yields (87). The proof is complete. □

Remark 4.1. From the above lemma, the result of Lemma 4.9 can be improved to: for each $X \in L^2(\mathcal{F}_T)$ and $g \in L^2_{\mathcal{F}}(0,T)$, the process $\mathcal{E}[X + \int_t^T g_s ds|\mathcal{F}_t]$, $t \in [0,T]$ is continuous a.s..

4.3 BSDE under $\mathcal{F}_t$–Consistent Nonlinear Expectations

Here again, $\mathcal{E}$ denotes an $\mathcal{F}$-expectation satisfying (78) for some $\mu > 0$, and (81) as well. Let a function f be given

$$f(\omega, t, y) : \Omega \times [0,T] \times R \longmapsto R$$

satisfying, for some constant $C_1 > 0$,

$$\begin{cases} \text{(i)} \;\; f(\cdot, y) \in L^2_{\mathcal{F}}(0,T), \quad \text{for each } y \in R; \\ \text{(ii)} \;\; |f(t, y_1) - f(t, y_2)| \le C_1 |y_1 - y_2|, \quad \forall y_1,\; y_2 \in R. \end{cases} \tag{89}$$

For a given terminal data $X \in L^2(\mathcal{F}_T)$, we consider the following type of equation:

$$Y_t = \mathcal{E}[X + \int_t^T f(s, Y_s) ds|\mathcal{F}_t] \tag{90}$$

Theorem 4.1. *We assume (89). Then there exists a unique process $Y(\cdot)$ solution of (90). Moreover, $Y(\cdot)$ admits continuous paths.*

Proof. Define a mapping $\Lambda(y(\cdot)) : L^2_{\mathcal{F}}(0,T) \longmapsto L^2_{\mathcal{F}}(0,T)$ by

$$\Lambda_t(y(\cdot)) := \mathcal{E}[X + \int_t^T f(s, y_s) ds|\mathcal{F}_t].$$

Using Lemma 78,

$$\Lambda_t(y_1(\cdot)) - \Lambda_t(y_2(\cdot)) \le \mathcal{E}^{\mu}[\int_t^T (f(s,y_1(s)) - f(s,y_2(s)))ds|\mathcal{F}_t].$$

Thus

$$\begin{aligned}|\Lambda_t(y_1(\cdot)) - \Lambda_t(y_2(\cdot))| &\le \mathcal{E}^{\mu}[\int_t^T |f(s,y_1(s)) - f(s,y_2(s))|ds|\mathcal{F}_t] \\ &\le C_1\mathcal{E}^{\mu}[\int_t^T |y_1(s)-y_2(s)|ds|\mathcal{F}_t], \text{ by (89).}\end{aligned}$$

Using Lemma 3.3, it follows that

$$\begin{aligned}E[|\Lambda_t(y_1(\cdot)) - \Lambda_t(y_2(\cdot))|^2] &\le C_1^2\mathbf{E}[\mathcal{E}^{\mu}[\int_t^T |y_1(s)-y_2(s)|ds|\mathcal{F}_t]^2] \\ &\le C_1^2 e^{\mu^2(T-t)}\mathbf{E}[\int_t^T |y_1(s)-y_2(s)|ds]^2 \\ &\le C_2\mathbf{E}[\int_t^T |y_1(s)-y_2(s)|^2 ds].\end{aligned}$$

where $C_2 := TC_1^2 e^{\mu^2 T}$.

We observe that, for any finite number β, the following two norms are equivalent in $L^2_{\mathcal{F}}(0,T)$

$$\mathbf{E}\int_0^T |\phi_s|^2 ds \sim \mathbf{E}\int_0^T |\phi_s|^2 e^{\beta s} ds.$$

Thus we multiply $e^{2C_2 t}$ on both sides of the above inequality and then integrate them on $[0,T]$. It follows that

$$\begin{aligned}\mathbf{E}\int_0^T |\Lambda_t(y.) - \Lambda_t(y'.)|^2 e^{2C_2 t}dt &\le C_2\mathbf{E}\int_0^T e^{2C_2 t}\int_t^T |y_s - y'_s|^2 ds dt \\ &= C_2\mathbf{E}\int_0^T\int_0^s e^{2C_2 t}dt|y_s - y'_s|^2 ds \\ &= (2C_2)^{-1}C_2\mathbf{E}\int_0^T (e^{2C_2 s}-1)|y_s - y'_s|^2 ds.\end{aligned}$$

We then have

$$\mathbf{E}\int_0^T |\Lambda_t(y.) - \Lambda_t(y'.)|^2 e^{2C_2 t}dt \le \frac{1}{2}\mathbf{E}\int_0^T |y_t - y'_t|^2 e^{2C_2 t}dt.$$

Namely, Λ is a contraction mapping on $L^2_{\mathcal{F}}(0,T)$. It follows that this mapping has a unique fixed point Y:

$$Y_t = \mathcal{E}[X + \int_t^T f(s,Y_s)ds|\mathcal{F}_t].$$

Finally, Lemma 4.9 and Remark 4.1 proves that the solution of (90) admits continuous paths, and the proof is complete. □

Theorem 4.2. ***(Comparison Theorem).*** *Let Y be the solution of (90) and let Y' be the solution of*

$$Y'_t = \mathcal{E}[X' + \int_t^T [f(s, Y'_s) + \phi_s] ds | \mathcal{F}_t]$$

where $X' \in L^2(\mathcal{F}_T)$ and $\phi \in L^2_{\mathcal{F}}(0, T)$. If

$$X' \geq X, \quad \phi_t \geq 0, \quad dP \times dt\text{-a.e.}, \tag{91}$$

then we have

$$Y'_t \geq Y_t, \quad dP \times dt\text{-a.e.} \tag{92}$$

(92) becomes equality if and only if (91) become equalities.

Proof. We begin with the case $\phi_t \equiv 0$. For each $\delta > 0$, we define

$$\tau_1^\delta = \inf\{t \geq 0; Y'_t \leq Y_t - \delta\} \wedge T.$$

It is clear that if, for all $\delta > 0$, $\tau_1^\delta = T$ a.s., then (92) holds. Now if for some $\delta > 0$ we have

$$P(A) > 0, \text{ with } A = \left\{\tau_1^\delta < T\right\} \in \mathcal{F}_{\tau_1^\delta}$$

we then can define

$$\tau_2 = \inf\{t \geq \tau_1^\delta; Y'_t \geq Y_t\}.$$

Since $Y'_T = X' \geq X = Y_T$, thus $\tau_2 \leq T$ and $1_A Y'(\tau_2) = 1_A Y(\tau_2)$. It follows that, for $\tau \in [\tau_1^\delta, \tau_2]$,

$$1_A Y_\tau = \mathcal{E}[1_A Y_{\tau_2} + \int_\tau^{\tau_2} 1_A f(s, 1_A Y_s) ds | \mathcal{F}_\tau],$$
$$1_A Y'_\tau = \mathcal{E}[1_A Y_{\tau_2} + \int_\tau^{\tau_2} 1_A f(s, 1_A Y'_s) ds | \mathcal{F}_\tau].$$

By the uniqueness result of Theorem 4.1, the solutions of the above two equations must coincide with each other. Thus $Y'_{\tau_1^\delta} 1_A = Y_{\tau_1^\delta} 1_A$. This contradicts $P(A) > 0$.

In order to prove the general case when $\phi_s \geq 0$, we define for $n = 1, 2, 3, \cdots$, $Y^n(\cdot)$ to be the solution of

$$Y_t^n = \mathcal{E}\left[[X' + \int_{\frac{iT}{n}}^T \phi_s ds] + \int_t^T f(s, Y_s^n) ds | \mathcal{F}_t\right],$$

$$\text{for } t \in [t_i^n, t_{i+1}^n), \; t_i^n := \frac{iT}{n}, \;\; i = 0, 1, \cdots, n-1..$$

This equation can be written, piece by piece, as

$$Y_t^n = \mathcal{E}\left[[Y^n_{t^n_{i+1}} + \int_{t^n_t}^{t^n_{i+1}} \phi_s ds] + \int_t^{t^n_{i+1}} f(s, Y^n_s) ds | \mathcal{F}_t \right],$$
$$t \in [t^n_i, t^n_{i+1}),\ Y^n_T = Y^n_{t^n_n} = X'.$$

From the first part of the proof. We have, for $i = n-1$, $Y^n_t \geq Y_t$, $t \in [t^n_{n-1}, T)$. In particular, $Y^n_{t^n_{n-1}} \geq Y_{t^n_{n-1}}$. An obvious iteration of this algorithm gives

$$Y^n_t \geq Y_t,\ t \in [t^n_i, t^n_{i+1}),\quad i = 0, \cdots, n-2.$$

Thus $Y^n_t \geq Y_t$, $t \in [0, T]$.

In order to prove that $Y'_t \geq Y_t$, It suffices to show the convergence of the sequence (Y^n) to Y'. A computation analogous to the proof of Theorem 4.1 shows that, for fixed $t \in [t^n_i, t^n_{i+1})$ and an appropriate constant C,

$$E[|Y^n_t - Y'_t|^2] \leq C\mathbf{E}[(\int_{\frac{iT}{n}}^t |\phi_s| ds + C_1 \int_t^T |Y^n_s - Y'_s| ds)^2]$$

Using Schwartz inequality, one has for all $t \in [0, T]$

$$\mathbf{E}[|Y^n_t - Y'_t|^2] \leq 2C\frac{T}{n}\mathbf{E}\int_0^T |\phi_s|^2 ds + 2CC_1^2 TE\int_t^T |Y^n_s - Y'_s|^2 ds. \tag{93}$$

Gronwall's Lemma applied to the above inequality shows that

$$\mathbf{E}[|Y^n_t - Y'_t|^2] \to 0,$$

and finally $Y'_t \geq Y_t$.

Finally, we investigate possible equality in (92). From $Y'_t \equiv Y_t$, one has

$$\mathcal{E}[X + \int_0^T f(s, Y_s) ds] = \mathcal{E}[X' + \int_0^T f(s, Y_s) ds + \int_0^T \varPhi_s ds]$$

Since $X' \geq X$ and $\int_0^T \varPhi_s ds \geq 0$, it follows from the strict monotonicity of $\mathcal{E}$ that $X' = X$ a.s., and $\int_0^T \varPhi_s ds = 0$, whence $\varPhi = 0$ $dt \times dP$ a.e. and the end of the proof. $\square$

4.4 Decomposition Theorem for $\mathcal{E}$-Supermartingales

Our next result generalizes the decomposition theorem for g-supermartingales proved in Theorem. 3.9 to continuous $\mathcal{E}$-supermartingales. The proof is very similar. It also uses mainly arguments from Theorem 3.9.

Theorem 4.3. ***(Decomposition theorem for $\mathcal{E}$-supermartingales)*** *Let $\mathcal{E}[\cdot]$ be an $\mathcal{F}$-expectation satisfying (78) and (81), and let $Y \in S^2_{\mathcal{F}}(0,T)$ be a $\mathcal{E}$-supermartingale. Then there exists an $A(\cdot) \in S^2_{\mathcal{F}}(0,T)$ with $A(0) = 0$ such that $Y + A$ is an $\mathcal{E}$-martingale.*

Proof. For $n \geq 1$, we define $y^n(\cdot)$, solution of the following BSDE:

$$y^n_t = \mathcal{E}[Y_T + \int_t^T n(Y_s - y^n_s)ds|\mathcal{F}_t]$$

We have then the following

Lemma 4.11. *We have, for each t and $n \geq 1$,*

$$Y_t \geq y^n_t, \ a.s.$$

Proof. For a $\delta > 0$ and a given integer $n > 0$, we define

$$\sigma^{n,\delta} := \inf\{t;\ y^n_t \geq Y_t + \delta\} \wedge T.$$

If $P(\sigma^{n,\delta} < T) = 0$, for all n and δ, then the proof is done. If it is not the case, then there exist $\delta > 0$ and a positive integer n such that $P(\sigma^{n,\delta} < T) > 0$. We can then define the following stopping times

$$\tau := \inf\{t \geq \sigma^{n,\delta};\ y^n_t \leq Y_t\}$$

It is clear that $\sigma^{n,\delta} \leq \tau \leq T$. Because of Theorem 4.1, $Y_t - y^n_t$ is continuous. Hence we have

$$y^n_\tau \leq Y_\tau \tag{94}$$

But since $(Y_s - y^n_s) \leq 0$ in $[\sigma^{n,\delta}, \tau]$, by monotonicity of $\mathcal{E}[\cdot]$,

$$\begin{aligned} y^n_{\sigma^{n,\delta}} &= \mathcal{E}[y^n_\tau + \int_{\sigma^{n,\delta}}^\tau n(Y_s - y^n_s)ds|\mathcal{F}_{\sigma^{n,\delta}}] \\ &\leq \mathcal{E}[y^n_\tau|\mathcal{F}_{\sigma^{n,\delta}}] \\ &\leq \mathcal{E}[Y_\tau|\mathcal{F}_{\sigma^{n,\delta}}] \end{aligned}$$

Finally, since Y is an $\mathcal{E}$-supermartingale, by (optional stopping theorem) Theorem 7.4, we have

$$Y_{\sigma^{n,\delta}} \geq y^n_{\sigma^{n,\delta}}.$$

But on the other hand, we have $P(\sigma^{n,\delta} < T) > 0$ and, by the definition of $\sigma^{n,\delta}$, $y^n_{\sigma^{n,\delta}} \geq Y_{\sigma^{n,\delta}} + \delta$ on $\left\{\sigma^{n,\delta} < T\right\}$. This induces a contradiction. The proof is complete. □

Lemma 4.11 with Theorem 4.2 above imply that $y^n(\cdot)$ monotonically converges to some $Y^0(\cdot) \leq Y(\cdot)$. Indeed, writing $\phi_t = Y_t - y_t^{(n+1)} \geq 0$ shows that $(y^n(\cdot))$ is an increasing sequence of functions.

Observe then that $y_t^n + \int_0^t n(Y_s - y_s^n)ds$ is an $\mathcal{E}$-martingale. By Lemma 4.10, there exists $(g^n, z^n) \in L^2_{\mathcal{F}}(0,T; R \times R^d)$ with

$$|g_s^n| \leq \mu |z_s^n|, \quad n = 1, 2, \cdots, \tag{95}$$

such that

$$y_t^n + \int_0^t n(Y_s - y_s^n)ds = y_T^n + \int_0^T n(Y_s - y_s^n)ds + \int_t^T g_s^n ds - \int_t^T z_s^n dB_s,$$

hence, as $y_T^n = Y_T$,

$$y_t^n = Y_T + \int_t^T [g_s^n + n(Y_s - y_s^n)]ds - \int_t^T z_s^n dB_s. \tag{96}$$

(87) also tells us that

$$|g_s^n - g_s^m| \leq \mu |z_s^n - z_s^m|, \quad n, m = 1, 2, \cdots \tag{97}$$

Let us denote, for each $n = 1, 2, \cdots$,

$$A_t^n = n \int_0^t (Y_s - y_s^n)ds$$

A^n is a continuous increasing process such that $A^n(0) = 0$.

We are now going to identify the limit of $y^n(\cdot)$. To this end, we shall use the following lemma :

Lemma 4.12. *There exists a constant C which is independent of n such that*

$$\text{(i)} \quad \mathbf{E}\int_0^T |z_s^n|^2 ds \leq C; \quad \text{(ii)} \quad \mathbf{E}[(A_T^n)^2] \leq C. \tag{98}$$

Proof. By $y_t^1 \leq y_t^n \leq y_t^{n+1} \leq Y_t$, $n = 1, 2, \cdots$ with $E[\sup_{t\in[0,T]} |Y_t|^2] < \infty$, we have $|y_t^n| \leq |y_t^1| + |Y_t|$. Thus there exists a constant C, independent of n, such that

$$\mathbf{E}[\sup_{0\leq t\leq T} |y_t^n|^2] \leq C. \tag{99}$$

We then can apply (28) and (95) to prove (98) step by step as the proof of Lemma 3.7. □

With the help of Lemma 4.12 we can now end the proof of the Decomposition Theorem.

Note first that (98)–(i) with (95) also implies

$$\mathbf{E}\int_0^T |g_s^n|^2 ds \leq \mu^2 C$$

(98)–(ii) implies that

$$y^n(\cdot) \nearrow Y(\cdot).$$

From by the monotonic limit Theorem 7.2 (in Appendix), it follows that we can write Y under the form

$$Y_t = Y_T + \int_t^T g_s ds + A_T - A_t - \int_t^T z_s dB_s \tag{100}$$

for some $(g, z) \in L^2_{\mathcal{F}}(0, T; R \times R^d)$ and an increasing process A with $A_0 = 0$ and $\mathbf{E}[A_T^2] < \infty$. Observe that $Y(\cdot)$ and then $A(\cdot)$ is continuous. It follows from Theorem 7.2 that

$$z^n(\cdot) \to z(\cdot), \quad \text{strongly in } L^2_{\mathcal{F}}(0, T; R^d).$$

It follows from (97) that

$$g^n(\cdot) \to g(\cdot), \quad \text{strongly in } L^2_{\mathcal{F}}(0, T).$$

And finally, (28) gives

$$A_t^n \longmapsto A_t, \quad \text{strongly in } L^2(\mathcal{F}_T).$$

Thanks to Lemma 4.6, we can pass to the L^2-limit in both sides of

$$y_t^n = \mathcal{E}[Y_T + A_T^n - A_t^n | \mathcal{F}_t].$$

It follows that

$$Y_t = \mathcal{E}[Y_T + A_T - A_t | \mathcal{F}_t].$$

Thus $Y_t + A_t = \mathcal{E}[Y_T + A_T | \mathcal{F}_t]$ is an $\mathcal{E}$-martingale (because of (81)). Since A is increasing, the Theorem is proved. □

4.5 Representation Theorem of an $\mathcal{F}$–Expectation by a g–Expectation

In this subsection, we will prove an important result: an $\mathcal{F}_t$–consistent nonlinear expectation can be identified as a g-expectation, provided that (78) and (81) hold.

Theorem 4.4. *We assume that an $\mathcal{F}$-expectation $\mathcal{E}[\cdot]$ satisfies (78) and (81) for some $\mu > 0$. Then there exists a function $g = g(t,z) : \Omega \times [0,T] \times R^d$ satisfying (i), (ii) and (iii") of (34) such that*

$$\mathcal{E}[X] = \mathcal{E}_g[X], \quad \forall X \in L^2(\mathcal{F}_T).$$

In particular, every $\mathcal{E}$-martingale is continuous a.s.

Moreover, we have $|g(t,z)| \le \mu|z|$ for all $t \in [0,T]$.

Proof. For each given $z \in R^d$, we consider the following forward equation

$$\begin{cases} dY_t^z = -\mu|z|dt + zdB_t, \\ Y^z(0) = 0. \end{cases}$$

We have $E[\sup_{t\in[0,T]}|Y_t^z|^2] < \infty$. It is also clear that Y^z is an $\mathcal{E}^\mu$-martingale, thus an $\mathcal{E}[\cdot]$-supermartingale. Indeed, we can write $Y_t^z = \mathcal{E}^\mu[Y_T^z|\mathcal{F}_t]$. From Theorem 4.3, there exists an increasing process $A^z(\cdot)$ with $A^z(0) = 0$ and $E[A_T^{z2}] < \infty$ such that

$$Y_t^z = \mathcal{E}[Y_T^z + A_T^z - A_t^z|\mathcal{F}_t].$$

Or

$$Y_t^z + A_t^z = \mathcal{E}[Y_T^z + A_T^z|\mathcal{F}_t], \quad t \in [0,T].$$

Then, from Lemma 4.10. there exists $(g(z,\cdot), Z^z(\cdot)) \in L^2_{\mathcal{F}}(0,T; R \times R^d)$ with $|g(z,t)| \le \mu|Z_t^z|$ such that

$$Y_t^z + A_t^z = Y_T^z + A_T^z + \int_t^T g(z,s)ds - \int_t^T Z_s^z dB_s. \tag{101}$$

We also have

$$|g(z,t) - g(z',t)| \le \mu|Z_t^z - Z_t^{z'}|. \tag{102}$$

But on the other hand, since

$$Y_t^z = Y_T^z + \int_t^T \mu|z|ds - \int_t^T zdB_s,$$

it follows that

$$\begin{aligned} A_t^z &\equiv \mu|z|t - \int_0^t g(z,s)ds, \\ Z_t^z &\equiv z. \end{aligned}$$

In particular, (102) becomes

$$|g(z,t) - g(z',t)| \le \mu|z - z'|. \tag{103}$$

Moreover,

$$Y_t^z + A_t^z = Y^z(r) + A^z(r) - \int_r^t g(z,s)ds + \int_r^t zdB_s, \quad 0 \le r \le t \le T,$$

and $Y_t^z + A_t^z$ is an $\mathcal{E}$-martingale. But with the assumption (81) one has, for each $z \in R^d$ and $r \le t$

$$\mathcal{E}[-\int_r^t g(z,s)ds + \int_r^t zdB_s|\mathcal{F}_r] = \mathcal{E}[Y_t^z + A_t^z - (Y^z(r) + A^z(r))|\mathcal{F}_r],$$

i.e.

$$\mathcal{E}[-\int_r^t g(z,s)ds + \int_r^t zdB_s|\mathcal{F}_r] = 0 \quad 0 \le r \le t \le T \tag{104}$$

Now let $\{A_i\}_{i=1}^N$ be a $\mathcal{F}_r$-measurable partition of Ω (i.e., A_i are disjoint, $\mathcal{F}_r$-measurable and $\cup A_i = \Omega$) and let $z_i \in R^d$, $i = 1, 2, \cdots, N$. From (11), it follows that

$$\begin{aligned}
&\mathcal{E}[-\int_r^t g(\sum_{i=1}^N z_i 1_{A_i}, s)ds + \int_r^t \sum_{i=1}^N z_i 1_{A_i} dB_s|\mathcal{F}_r] \\
&= \mathcal{E}[\sum_{i=1}^N 1_{A_i}\left(-\int_r^t g(z_i,s)ds + \int_r^t z_i dB_s\right)|\mathcal{F}_r] \\
&= \sum_{i=1}^N 1_{A_i}\mathcal{E}[-\int_r^t g(z_i,s)ds + \int_r^t z_i dB_s|\mathcal{F}_r] \\
&= 0
\end{aligned}$$

(because of (104)). In other words, for each simple function $\eta \in L^2(\Omega, \mathcal{F}_r, P)$,

$$\mathcal{E}[-\int_r^t g(\eta,s)ds + \int_r^t \eta dB_s|\mathcal{F}_r] = 0.$$

From this, the continuity of $\mathcal{E}[\cdot]$ in L^2 given by (80) and the fact that g is Lipschitz in z, it follows that the above equality holds for $\eta(\cdot) \in L^2_{\mathcal{F}}(0,T;R^d)$:

$$\mathcal{E}[-\int_r^t g(\eta_s,s)ds + \int_r^t \eta_s dB_s|\mathcal{F}_r] = 0. \tag{105}$$

We just have to prove now that

$$\mathcal{E}_g[X] = \mathcal{E}[X], \quad \forall X \in L^2(\mathcal{F}_T).$$

To this end we first solve the following BSDE

$$\begin{aligned}
-dy_s &= g(s,z_s)ds - z_s dB_s, \\
y_T &= X.
\end{aligned}$$

Since g is Lipschitz in z, there exists a unique solution $(y(\cdot), z(\cdot)) \in L^2_{\mathcal{F}}(0,T;R\times R^d)$. By the definition of g-expectation,

$$\mathcal{E}_g[X] = y(0).$$

On the other hand, using (105), one finds

$$\begin{aligned}\mathcal{E}[X] &= \mathcal{E}[y(0) - \int_0^T g(z_s, s)ds + \int_0^T z_s dB_s] \\ &= y(0) + \mathcal{E}[-\int_0^T g(z_s, s)ds + \int_0^T z_s dB_s] \\ &= y(0) = \mathcal{E}_g[X].\end{aligned}$$

It follows that this g-expectation $\mathcal{E}_g[\cdot]$ coincides with $\mathcal{E}[\cdot]$ and we are finished. $\square$

4.6 How to Test and Find g?

Let $g(s, z)$ be the generator of the investigated agent. An very important problem is how to find this function g. We will treat this problem for the case where g is a deterministic function: $g(t, z) : [0, \infty) \times \mathbf{R}^d \to \mathbf{R}$. We assume that

$$\begin{aligned}&|g(t, z) - g(t, z')| \le \mu|z - z'|, \ \forall t \ge 0, \ \ \forall z, z' \in \mathbf{R}^d, \\ &g(t, 0) \equiv 0, \ \ \forall t \ge 0, .\end{aligned} \tag{106}$$

In this case we can find such g by the following testing method.

Proposition 4.1. *We assume (106). Let $\bar{z} \in \mathbf{R}^d$ be given, then*

$$\int_t^T g(s, \bar{z})ds = \mathcal{E}_g[\bar{z}B_T|\mathcal{F}_t] - \bar{z}B_t \tag{107}$$

In particular

$$\int_0^T g(s, \bar{z})ds = \mathcal{E}_g[\bar{z}B_T] \tag{108}$$

Proof. We denote $Y_t := \mathcal{E}_g[\bar{z}B_T|\mathcal{F}_t]$, it is the solution of the following BSDE

$$Y_t = \bar{z}B_T + \int_t^T g(s, Z_s)ds - \int_t^T Z_s dB_s$$

Or

$$Y_t - \bar{z}B_t = \int_t^T g(s, Z_s - \bar{z} + \bar{z})ds - \int_t^T (Z_s - \bar{z})dB_s.$$

It follows that $(\bar{Y}_t, \bar{Z}_t) := (Y_t - \bar{z}B_t, Z_t - \bar{z})$ solves the BSDE

$$\bar{Y}_t = \int_t^T g(s, \bar{Z}_s + \bar{z})ds - \int_t^T \bar{Z}_s dB_s.$$

This BSDE has a unique solution $(\bar{Y}_t, \bar{Z}_t) \equiv (\int_t^T g(s, \bar{z})ds, 0)$. We thus have (107).

Remark 4.2. It is meaningful to test the generator g of an agent: at a time $t \leq T$, we let the agent evaluate $\bar{z}B_T$ and result $\mathcal{E}_g[\bar{z}B_T|\mathcal{F}_t]$. Then the deterministic data $\int_t^T g(s,\bar{z})ds$ is obtained by $\bar{Y}_t = \mathcal{E}_g[\bar{z}B_T|\mathcal{F}_t] - \bar{z}B_t$, where B_t is a known value at the time t.

Example 4.1. If g is time–invariant: $g = g(z)$, then we have

$$g(\bar{z})(T-t) = \mathcal{E}_g[\bar{z}B_T|\mathcal{F}_t] - \bar{z}B_t$$

and

$$g(\bar{z})T = \mathcal{E}_g[\bar{z}B_T], \quad \bar{z} \in \mathbf{R}^d.$$

Example 4.2. If we already know that $g = g_0(\theta, z)$, where $g_0 : [a,b]\times\mathbf{R}^d \to \mathbf{R}$ is a given function but we have to find the parameter $\theta \in [a,b]$, assume that for some $\bar{z} \in \mathbf{R}^d$, $g_0(\theta, z)$ is a strictly increasing function of θ in $[a,b]$. Then we can only test the agent once at the time, say $t = 0$. Using the formula

$$g_0(\theta, \bar{z})T = \mathcal{E}_g[\bar{z}B_T],$$

we can uniquely determine θ.

4.7 A General Situation: $\mathcal{F}_t$–Evaluation Representation Theorem

Theorem 4.4 is only valid for a part of $\mathcal{F}_t$–consistent nonlinear expectations. For a general situation we have the following result [Peng2003b]. By the limitation of the size of this lecture, we will only state the result without given the proof. We are given an $\mathcal{F}_t$–consistent nonlinear evaluation defined on $L^2(\mathcal{F}_T)$:

$$\mathcal{E}_{s,t}[\cdot] : L^2(\mathcal{F}_t) \to L^2(\mathcal{F}_s), \quad 0 \leq s \leq t \leq T.$$

It satisfies the axiomatic assumptions (A1)–(A4), with the following additional $\mathcal{E}^{g_{\mu,\mu}}$–dominated assumption ($g_{\mu,\mu}(y,z) := \mu(|y|+|z|)$), weaker than (A5):

(A5') There a sufficiently large number $\mu > 0$ such that, for each $0 \leq s \leq t \leq T$,

$$\mathcal{E}_{s,t}[X] - \mathcal{E}_{s,t}[X'] \leq \mathcal{E}_{s,t}^{g_{\mu,\mu}}[X - X'], \ \forall X, X' \in L^2(\mathcal{F}_t).$$

The g–evaluation representation theorem is as follows:

Theorem 4.5. *Let $\mathcal{E}_{s,t}[\cdot] : L^2(\mathcal{F}_t) \to L^2(\mathcal{F}_s)$, $0 \leq s \leq t \leq T$, satisfy (A1)–(A4) and (A5'). Then there exists a function $g(\omega, t, y, z)$ satisfying (34)–(i), (ii) and (iii), such that, for each $0 \leq s \leq t \leq T$,*

$$\mathcal{E}_{s,t}[X] = \mathcal{E}_{s,t}^g[X], \ \ \forall X \in L^2(\mathcal{F}_t).$$

Remark 4.3. In this result we do not need the assumption (81). Thus g may depend on (y,z).

Remark 4.4. In [Peng2003b] we also consider the situation where (A4) is weakened by (A4'): $1_A\mathcal{E}_{s,t}[X] = 1_A\mathcal{E}_{s,t}[1_AX]$, for each $A \in \mathcal{F}_s$. In this case the corresponding g satisfies only (34)–(i) and (ii) without the condition $g(s,0,0) \equiv 0$.

Remark 4.5. From the above g–evaluation reprentation theorems, we see that the dominating term, such as $\mathcal{E}^{g_{\mu,\mu}}[\cdot]$, plays an important role. A general formulation is:

$$\mathcal{E}_{s,t}[X] - \mathcal{E}_{s,t}[X'] \le \mathcal{E}^*_{s,t}[X - X'],$$

where $\mathcal{E}^*_{s,t}[X - X']$ is a given self–dominated nonlinear evaluation: i.e., it is a concrete evaluation satisfying (A1)–(A4) and

$$\mathcal{E}^*_{s,t}[X] - \mathcal{E}^*_{s,t}[X'] \le \mathcal{E}^*_{s,t}[X - X'], \ \ \forall X, X' \in L^2(\mathcal{F}_t).$$

5 Dynamic Risk Measures

Recently Rosazza Gianin [Roazza2003] considered a type of dynamic risk measures induced from g–expectations. We consider a more general situation. Let $\mathcal{E}_{s,t}[\cdot]$ be an $\mathcal{F}_t$–consistent nonlinear evaluation defined on $L^2(\mathcal{F}_T)$. It satisfies (A1)–(A4). We set, for each $0 \le s \le t \le T$, and $X \in L^2(\mathcal{F}_t)$, $\rho_{s,t}[X] := \mathcal{E}_{s,t}[-X]$. $\{\rho_{s,t}[\cdot]\}_{0\le s\le t\le T}$ is called a dynamic risk measure defined on $L^2(\mathcal{F}_T)$. We consider an $\mathcal{F}$–consistent evaluation $\{\mathcal{E}_{s,t}[\cdot]\}_{0\le s\le tT}$ satisfying some of the following axiomatic conditions: for each $0 \le s \le t \le T$ and X, $Y \in L^2(\mathcal{F}_t)$, it satisfies

(e1) subadditivity: $\mathcal{E}_{s,t}[X+Y] \le \mathcal{E}_{s,t}[X] + \mathcal{E}_{s,t}[Y]$;
(e2) positively homogeneity: $\mathcal{E}_{s,t}[\alpha X] = \alpha\mathcal{E}_{s,t}[X]$;
(**e3**) constant translability: $\mathcal{E}_{s,t}[X+\eta] = \mathcal{E}_{s,t}[X] + \eta$, $\forall \eta \in L^2(\mathcal{F}_s)$
(**e4**) convexity:

$$\mathcal{E}_{s,t}[\alpha X + (1-\alpha)Y] \le \alpha\mathcal{E}_{s,t}[X] + (1-\alpha)\mathcal{E}_{s,t}[Y], \ \forall \alpha \in [0,1].$$

Similar to [ADEH1999] and [FoSc2002] for static situations, we can define the following type of dynamic risk measures.

Definition 5.1. *A dynamic risk measure $\{\rho_{s,t}[\cdot]\}_{0\le s\le t\le T}$ is said to be coherent if the corresponding nonlinear evaluation $\{\mathcal{E}_{s,t}[\cdot]\}_{0\le s\le t\le T}$ satisfies **(e1)**–**(e3)**. It is said to be convex and constant translable if $\mathcal{E}_{s,t}[\cdot]$ satisfies (**e3**) and (**e4**).*

For the situation of $\mathcal{E}^g$–evaluation, we have the corresponding ρ^g–risk measure defined by $\rho^g_{s,t}[X] := \mathcal{E}^g_{s,t}[-X]$. A very interesting point is that the concrete function g perfectly reflexes the attitude of an investor towards risks. In fact we have the following properties:

Proposition 5.1. *We assume that g satisfies (34)–(i), (ii). Then $\mathcal{E}^g[\cdot]$ is subadditive (resp. superadditive) if g is subadditive (resp. superadditive) in $(y,z) \in R^{1+d}$. It is positively homogegeous if g is positively homogegeous in $(y,z) \in R^{1+d}$. It is convex (resp. concave) if g is convex (resp. concave) in $(y,z) \in R^{1+d}$. It has constant translability if g is independent of y. Moreover, if, for each (y,z) and P-a.s., $g(\cdot,y,z) \in D^2_{\mathcal{F}}(0,T)$, then all the above "if" can be replaced by "if and only if".*

For the proof of this proposition we refer to [EPQ1997], [BCHMP2000], [Roazza2003] and [Peng2003c]

6 Numerical Solution of BSDEs: Euler's Approximation

Let $(\epsilon_i^n)_{i=1,2,\cdots,n}$ be a Bernouil sequence, i.e., an i.i.d. sequence such that with

$$P\{\epsilon_i^n = 1\} = P\{\epsilon_i^n = -1\} = \frac{1}{2}.$$

We set

$$\begin{aligned} B_k^n &:= \sqrt{n}\sum_{i=1}^k \epsilon_i^n, \quad \mathcal{F}_k^n := \sigma\{B_k^n; 1 \le k \le n\} \\ \Delta B_{k+1}^n &:= B_{k+1}^n - B_k^n = \sqrt{n}\epsilon_k^n, \end{aligned}$$

Let ξ be $\mathcal{F}_k^n$–measurable. This implies that there exists a function: Φ : $\{1,-1\}^k \to \mathbf{R}$, such that

$$\xi^n = \Phi_n(\epsilon_1^n, \cdots, \epsilon_k^n).$$

All processes are assumed to be $\mathcal{F}_k^n$–adapted. We make the following assumption

(H1) B^n converges to B in $\mathcal{S}^2$

(H2) ξ^n converges to ξ in $L^2(P)$.

f and $f^n : [0,1] \times \Omega \times \mathbf{R} \times \mathbf{R} \longrightarrow \mathbf{R}$ such that for each $(y,z) \in \mathbf{R} \times \mathbf{R}$, $\{f^n(t,y,z)\}_{0\le t\le 1}$ (resp. $\{f(t,y,z)\}_{0\le t\le 1}$) are progressively measurable with respect to $\mathcal{F}_t^n$ (resp. to $\mathcal{F}_t$) such that

(H3)–(i):

$$\begin{aligned} |f^n(t,y,z) - f^n(t,y',z')| &\le C(|y-y'| + |z-z'|) \\ |f(t,y,z) - f(t,y',z')| &\le C(|y-y'| + |z-z'|) \end{aligned}$$

(ii) For each (y,z) paths $\{f^n(t,y,z)\}_{0\le t\le 1}$ have RCLL paths and converges to $\{f(t,y,z)\}_{0\le t\le 1}$ in $\mathcal{S}^2(R)$ with

$$|Y|_{\mathcal{S}^2} := \{E[\sup_{0\le t\le 1} |Y_t|^2]\}^{1/2}.$$

We set

$$f^n(t,y,z) \equiv g_k^n(y,z), \;\; t \in [\frac{k}{n}, \frac{k+1}{n}), \; k = 0,1,\cdots,n.$$

and

$y_n^n = \xi^n$: a given $\mathcal{F}_n^n$–measurable random variable. Then we solve backwardly

$$y_k^n = y_{k+1}^n + g_k^n(y_k^n, z_k^n)\frac{1}{n} - z_k^n \Delta B_{k+1}^n, \; k = n-1,\cdots,3,2,1.$$

Or $y_t^n \equiv y_k^n$, $z_t^n \equiv z_k^n$, $t \in [\frac{k}{n}, \frac{k+1}{n})$. We call (y^n, z^n) the solution to (g,ξ).

$$\begin{aligned} dy_t^n &= f^n(t, y_t^n, z_t^n) d\langle B^n\rangle_t - z_t^n dB_t^n, \\ y_T^n &= \xi^n. \end{aligned}$$

Theorem 6.1. ***(Existence and Uniqueness and Comparison)*** *Let*

$$g_k^n(\omega, y, z) : \Omega \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}, \; k = 1,\cdots,n-1$$

be $\mathcal{F}_k^n$–measurable and C–Lipschitz with respect to y with $n > C$. Then there exists a unique $\mathcal{F}_k^n$–adapted pair $(y_\cdot^n, z_\cdot^n)$, solution to (g,ξ). Moreover, if $(y_\cdot^{n\prime}, z_\cdot^{n\prime})$ is the solution corresponding to (g', ξ'), and if

$$g_k^{n\prime}(\omega, y, z) \geq g_k^n(\omega, y, z), \; \xi^{n\prime} \geq \xi^n,$$

then the corresponding solution $(y^{n\prime}, z^{n\prime})$ satisfies

$$y_k^{n\prime} \geq y_k^n.$$

Corollary. If $A_1(\cdot)$ and $A_2(\cdot)$ satisfies the above conditions with $A_1(y) \geq A_2(y)$, for all $y \in R$. Then $A_1^{-1}(x) \leq A_2^{-1}(x)$, for all $x \in R$.
Proof of the theorem. Assume that y_{k+1}^n are solved, we then solve (y_k^n, z_k^n).

$$y_k^n = y_{k+1}^n + g_k^n(y_k^n, z_k^n)\frac{1}{n} - z_k^n \Delta B_{k+1}^n \tag{109}$$

Since y_{k+1}^n has the form: $y_{k+1}^n = \Phi_{k+1}(\epsilon_1,\cdots,\epsilon_{k+1})$. We set

$$\begin{aligned} y_{k+1}^{(+)} &: = \Phi_{k+1}(\epsilon_1,\cdots,1), \\ y_{k+1}^{(-)} &: = \Phi_{k+1}(\epsilon_1,\cdots,-1). \end{aligned}$$

y_{k+1}^+ and y_{k+1}^- are $\mathcal{F}_k^n$–measurable. We set $\epsilon_{k+1} = \pm 1$, in (109):

$$\begin{aligned} y_k^n &= y_{k+1}^+ + g_k^n(y_k^n, z_k^n)\frac{1}{n} + -z_k^n n^{-1/2} \\ y_k^n &= y_{k+1}^- + g_k^n(y_k^n, z_k^n)\frac{1}{n} + +z_k^n n^{-1/2} \end{aligned}$$

z_k^n can be uniquely solved by $z_k^n = \frac{y_{k+1}^{(+)} - y_{k+1}^{(-)}}{2}$. The equation for y_k^n is

$$y_k^n - g_k^n(y_k^n, z_k^n)\frac{1}{n} = \frac{y_{k+1}^{(+)} + y_{k+1}^{(-)}}{2} \tag{110}$$

When $n > C$, the mapping $A(y) := y - g_k^n(y, z_k^n)\frac{1}{n}$ is strictly monotonic function of y with $A(y) \to +\infty$ (resp.$-\infty$) as $y \to +\infty$ (resp. $-\infty$). Thus the solution y_k^n of (3) exists and is unique. By the Corollary, the comparison theorem also holds. $\square$

We consider
(a) $y_t = \xi + \int_t^1 f(s, y_s, z_s)ds - \int_t^1 z_s dB_s$
$(\mathrm{b})_n$ $y_t^n = \xi^n + \int_t^1 f_n(s, y_s^n, z_s^n)d\langle B^n\rangle_t - \int_t^1 z_s^n dB_s^n$

Theorem 6.2. *(Briand, Delyon & Memin, 2001) We assume (H1), (H2) and (H3). Let (y^n, z^n) be the solution of $(b)_n$ and (y, z) be the solution of (a). Then, in $\mathcal{S}^2 \times \mathcal{S}^2$,*

$$\left(y^n, \int_0^{\cdot} z_s^n dB_s^n\right) \to \left(y, \int_0^{\cdot} z_s dB_s\right), \text{ as } n \to \infty$$

and in $\mathcal{S}^2 \times \mathcal{S}^2$

$$\left(\int_0^{\cdot} z_s^n d\langle B^n\rangle_s, \int_0^{\cdot} |z_s^n|^2 d\langle B^n\rangle_s\right) \to \left(\int_0^{\cdot} z_s^n d\langle B^n\rangle_s, \int_0^{\cdot} |z_s^n|^2 d\langle B^n\rangle_s\right) \text{ as } n \to \infty.$$

7 Appendix

7.1 Martingale Representation Theorem

The existence theorem of BSDE requires the following result: any element $\xi \in L^2(\mathcal{F}_T)$ can be represent by

$$\xi = \mathbf{E}[\xi] + \int_0^T \phi_s dB_s.$$

For notational simplification, we assume that B is 1–dimensional, i.e., $d = 1$. We need the following lemma.

Lemma 7.1. *Let $\eta \in L^2(\mathcal{F}_T)$ be given such that*

$$\mathbf{E}[\eta(1 + \int_0^T \phi_s dB_s)] = 0, \quad \forall \phi \in L^2_{\mathcal{F}}(0, T).$$

Then $\eta = 0$, a.s..

Proof. For each deterministic $\mu(\cdot) \in L^\infty(0,T;\mathbb{C})$, we denote by X^μ, the solution of the following SDE

$$dX_t^\mu = \mu(t)X_t^\mu dB_t,\ X_0^\mu = 1.$$

It suffices to prove that if, for each $\mu(\cdot) \in L^\infty(0,T;\mathbb{C})$ we have $\mathbf{E}[\eta X_T^\mu] = 0$, then $\eta = 0$, a.s.

For each $N \in \mathbb{Z}$, $x = (x_1, \cdots, x_N) \in \mathbf{R}^N$and $0 \le t_1 < \cdots < t_N \le T$, we set $\mu(t) = i\sum_{j=1}^N x_j 1_{[0,t_j]}(t)$. It is easy to check that

$$\begin{aligned} X_t^\mu &= \exp\{i\textstyle\int_0^t \mu(s)dB_s - \frac{1}{2}\int_0^t |\mu(s)|^2 ds\} \\ &= e^{i\sum_{j=1}^N x_j B_{t_j\wedge t}} \exp\{-\textstyle\frac{1}{2}\int_0^t |\mu(s)|^2 ds\} \end{aligned}$$

Thus the condition $\mathbf{E}[\eta X_T^\mu] = 0$ implies

$$\Phi_\mu(x) \ := \mathbf{E}[\eta e^{i\sum_{j=1}^N x_j B_{t_j}}] = 0.$$

Now for an arbitrary $g \in C_0^\infty(\mathbf{R}^N)$, let $\hat{g}$ be its Fourier transform. We then have

$$\begin{aligned} &\mathbf{E}[g(B_{t_1}, \cdots, B_{t_N})\eta] \\ &= \mathbf{E}[(2\pi)^{-\frac{N}{2}} \int_{\mathbf{R}^N} \hat{g}(x_1, \cdots, x_N) e^{i\sum_{j=1}^N x_j B_{t_j}} dx\eta] \\ &= (2\pi)^{-\frac{N}{2}} \int_{\mathbf{R}^N} \hat{g}(x)\ \Phi_\mu(x)dx = 0. \end{aligned}$$

Since the subset

$$\{g(B(t_1), \cdots, B(t_N)); 0 \le t_1, \cdots, t_N \le T,\ g \in C_0^\infty(\mathbf{R}^N), N \in \mathbb{Z}\}$$

is dense in $L^2(\mathcal{F}_T)$, it follows that $\eta = 0$.

We now can prove the representation theorem.

Theorem 7.1. *(Representation theorem of an element of $L^2(\mathcal{F}_T)$ by Itô's integral) For each $\xi \in L^2(\mathcal{F}_T)$ there exists a unique $z \in L^2_{\mathcal{F}}(0,T)$ such that*

$$\xi = \mathbf{E}[\xi] + \int_0^T z_s dB_s,\ a.s. \tag{111}$$

Proof. Let $\xi \in L^2(\mathcal{F}_T)$ be given. We define the following functional

$$f(\phi) := \mathbf{E}[\xi \int_0^T \phi_s dB_s],\ \phi \in L^2_{\mathcal{F}}(0,T;\mathbf{R}^d).$$

By Schwards inequality $|f(\phi)| \le \mathbf{E}[|\xi|^2]^{1/2} \cdot\ \mathbf{E}[\int_0^T |\phi_s|^2 ds]^{1/2}$. Thus f is a bounded linear functional defined on $L^2_{\mathcal{F}}(0,T)$. It follows from the well–known

Riesz representation theorem (see for example [Yosida1980] p90) that, there exists a unique process $z \in L^2_{\mathcal{F}}(0,T)$, such that

$$f(\phi) = \mathbf{E}[\int_0^T \phi_s z_s ds],\ \forall \phi \in L^2_{\mathcal{F}}(0,T),$$

or

$$\mathbf{E}[\int_0^T \phi_s dB_s(\xi - \int_0^T z_s dB_s)] = 0,\ \forall \phi \in L^2_{\mathcal{F}}(0,T).$$

Thus we have

$$\mathbf{E}[(1 + \int_0^T \phi_s dB_s)(\xi - \mathbf{E}[\xi] - \int_0^T z_s dB_s)] = 0,\ \forall \phi \in L^2_{\mathcal{F}}(0,T).$$

But by Lemma 7.1, this implies (111). □

7.2 A Monotonic Limit Theorem of Itô's Processes

We present a convergence result of a sequence of Itô processes, called "monotonic limit theorem". In this lecture we use this result to prove nonlinear supermartingale decomposition theorems. We consider the following sequence of Itô processes:

$$y^i_t = y^i_0 + \int_0^t g^i_s ds - A^i_t + \int_0^t z^i_s dB_s,\ i = 1, 2, \cdots . \tag{112}$$

for each i, the adapted process $g^i \in L^2_{\mathcal{F}}(0,T)$ are given, we also assume that, for each i,

$$A^i \in S^2_{\mathcal{F}}(0,T) \text{ is increasing with } A^i_0 = 0, \tag{113}$$

and

$$\begin{array}{l} \text{(i) } (g^i_t) \text{ and } (z^i_t) \text{ are bounded in } L^2_{\mathcal{F}}(0,T)\text{: } \mathbf{E}\int_0^T [|g^i_s|^2 + |z^i_s|^2] ds \le C; \\ \text{(ii) } (y^i_t) \text{ increasingly converges to } (y_t) \text{ with } \mathbf{E}[\sup_{0\le t\le T} |y_t|^2] < \infty, \end{array} \tag{114}$$

where the constant C is independent of i. It is clear that

$$\begin{array}{l} \text{(i) } \mathbf{E}[\sup_{0\le t\le T} |y^i_t|^2] \le C; \\ \text{(ii) } \mathbf{E}\int_0^T |y^i_t - y_t|^2 dt \to 0, \end{array} \tag{115}$$

where the constant C is independent of i.

Remark 7.1. It is not hard to check that the limit y has the following form

$$y_t = y_0 + \int_0^t g^0_s ds - A_t + \int_0^t z_s dB_s, \tag{116}$$

where g^0 and z are respectively the weak limit of $\{g^i\}_{i=1}^{\infty}$ and $\{z^i\}_{i=1}^{\infty}$ in $L^2_{\mathcal{F}}(0,T)$, $(A_t)_{t\in[0,T]}$ is an increasing process. In general, we can not prove the strong convergence of $\left\{\int_0^T z_s^i dB_s\right\}_{i=1}^{\infty}$. Our new observation is: for each $p \in [1,2)$, $\{z^i\}$ converges strongly in $L^p_{\mathcal{F}}(0,T;\mathbf{R}^d)$. This observation is crucially important, since we will treat nonlinear cases.

The limit theorem is as follows.

Theorem 7.2. *We assume (113) and (114). Then the limit y_t of $\{y^i\}_{i=1}^{\infty}$ has a form (116), where $g^0 \in L^2_{\mathcal{F}}(0,T)$ and $z \in L^2_{\mathcal{F}}(0,T;\mathbf{R}^d)$ are respectively the weak limit of $\{g^i\}_{i=1}^{\infty}$ and $\{z^i\}_{i=1}^{\infty}$ in $L^2_{\mathcal{F}}(0,T)$ and $L^2_{\mathcal{F}}(0,T;\mathbf{R}^d)$. For each $t \in [0,T]$, A_t is a weak limit of $\{A^i_t\}_{i=1}^{\infty}$ in $L^2(\mathcal{F}_T)$. $(A_t)_{t\in[0,T]}$ is an RCLL square–integrable increasing process. Furthermore, for any $p \in [0,2)$, $\{z^i\}_{i=1}^{\infty}$ strongly converges to z in $L^p_{\mathcal{F}}(0,T,\mathbf{R}^d)$, i.e.,*

$$\lim_{i\to\infty} \mathbf{E}\int_0^T |z_s^i - z_s|^p ds = 0,\ p \in [0,2). \tag{117}$$

If moreover $(y)_{t\in[0,T]}$ is continuous, then we have

$$\lim_{i\to\infty} \mathbf{E}\int_0^T |z_s^i - z_s|^2 ds = 0. \tag{118}$$

Remark 7.2. An interesting open problem is: does (118) hold without the additional continuous assumption for y?

In order to prove this theorem, we need the several Lemmas. The following lemma will be applied to prove that the limit processes y is RCLL.

Lemma 7.2. *Let $\{x^i(\cdot)\}_{i=1}^{\infty}$ be a sequence of (deterministic) RCLL processes defined on $[0,T]$ that increasingly converges to $x(\cdot)$ such that, for each $t \in [0,T]$, and $i = 1,2,\cdots$, $x^i(t) \le x^{i+1}(t)$, with $x(t) = b(t) - a(t)$, where $b(\cdot)$ is an RCLL process and $a(\cdot)$ is an increasing process with $a(0) = 0$ and $a(T) < \infty$. Then $x(\cdot)$ and $a(\cdot)$ are also RCLL processes.*

Proof. Since $b(\cdot)$, $a(\cdot)$ and thus $x(\cdot)$ have left and right limits, thus we only need to check that $x(\cdot)$ is right continuous. For each $t \in [0,T)$, since $a(t+) \ge a(t)$, thus

$$x(t+) = b(t) - a(t+) \le x(t). \tag{119}$$

On the other hand, for any $\delta > 0$, there exists a positive integer $j = j(\delta,t)$ such that $x(t) \le x^j(t) + \delta$. Since $x^j(\cdot)$ is RCLL, thus there exists a positive number $\epsilon_0 = \epsilon_0(j,t,\delta)$ such that $x^j(t) \le x^j(t+\epsilon) + \delta$, $\forall \epsilon \in (0,\epsilon_0]$. These imply that, for any $\epsilon \in (0,\epsilon_0]$,

$$x(t) \le x^j(t+\epsilon) + 2\delta \le x^{i+j}(t+\epsilon) + 2\delta \uparrow\uparrow x(t+\epsilon) + 2\delta.$$

Particularly, we have $x(t) \le x(t+) + 2\delta$ and thus $x(t) \le x(t+)$. This with (119) implies the right continuity of $x(\cdot)$. □

We need some estimates for the jumps of A. We first have

Lemma 7.3. *Let A be an increasing RCLL process defined on $[0,T]$ with $A_0 = 0$ and $\mathbf{E}(A_T)^2 < \infty$. Then, for any $\epsilon > 0$, there exists a finite number of stopping times σ_k, $k = 0,1,2,\cdots N+1$ with $\sigma_0 = 0 < \sigma_1 \le \cdots \le \sigma_N \le T = \sigma_{N+1}$ and with disjoint graphs on $(0,T)$ such that*

$$\sum_{k=0}^{N} \mathbf{E} \sum_{t\in(\sigma_k,\sigma_{k+1})} (\Delta A_t)^2 \le \epsilon. \tag{120}$$

Proof. For each $\nu > 0$, we denote

$$A_t(\nu) := A_t - \sum_{s\le t} \Delta A_s 1_{\{\Delta A_s > \nu\}}.$$

$A_\cdot(\nu)$ has jumps of $A_\cdot$ smaller than ν. Thus there is a sufficiently small $\nu > 0$ such that

$$\mathbf{E}[\sum_{s\le T} (\Delta A_s(\nu))^2] \le \frac{\epsilon}{2}.$$

Now let τ_k, $k = 1,2,\cdots$ be the successive times of jumps of A with size bigger than ν; they are stopping times, and there is N such that

$$\mathbf{E}\left(\sum_{s\in(\tau_N,T)} (\Delta A_s)^2 \right) \le \frac{\epsilon}{2}.$$

We then set $\sigma_k := \tau_k \wedge T$ for $k \le N$, and $\sigma_{N+1} = T$. It is clear that $\{\sigma_k\}_{k=0}^{N+1}$ satisfies (120). □

For applying the formula of the integral by part to the limit process y (with jumps), the above open intervals (σ_k, σ_{k+1}) is not so convenient. Thus we will cut a sufficiently small interval (σ_k, τ_k) and only work on the remaining subintervals $(\sigma_k, \tau_k]$. This is possible since our filtration is continuous. In fact we have:

Lemma 7.4. *Let $0 < \sigma \le T$ be a stopping time. Then there exists a sequence of $\mathcal{F}_t$–stopping times $\{\tau^i\}$ with $0 < \tau^i < \sigma$, a.s. for each $i = 1,2,\cdots$, such that $\tau^i \uparrow \sigma$.*

For the continuous filtration $\mathcal{F}_t$, this lemma is quite classical. The proof is omitted.

The following lemma tells that, for any given RCLL increasing process, the contribution of the jumps of A is mainly concentrated within a finite number of left–open right–closed intervals with "sufficiently small total length". Specifically, we have

Lemma 7.5. *Let A be an increasing RCLL process defined on $[0,T]$ with $A_0 = 0$ and $\mathbf{E}A_T^2 < \infty$. Then, for any $\delta, \epsilon > 0$, there exists a finite number of pairs of stopping times $\{\sigma_k, \tau_k\}$, $k = 0,1,2,\cdots N$ with $0 < \sigma_k \le \tau_k \le T$, such that*

$$\begin{array}{ll} \textit{(i)} & (\sigma_j, \tau_j] \cap (\sigma_k, \tau_k] = \emptyset \quad \textit{for each } j \neq k; \\ \textit{(ii)} & \mathbf{E} \sum_{k=0}^{N} (\tau_k - \sigma_k) \geq T - \epsilon \\ \textit{(iii)} & \sum_{k=0}^{N} \mathbf{E} \sum_{\sigma_k < t \leq \tau_k} (\Delta A_t)^2 \leq \delta \end{array}$$

Proof. We first apply Lemma 7.3 to construct a sequence of non-decreasing stopping times $\{\sigma_k\}_{k=0}^{N+1}$ with $\sigma_0 = 0$ and $\sigma_{N+1} = T$ such that, $\sigma_k < \sigma_{k+1}$ whenever $\sigma_k < T$ and that

$$\sum_{k=0}^{N} \mathbf{E} \sum_{t \in (\sigma_k, \sigma_{k+1})} (\Delta A_t)^2 \leq \delta.$$

Then for each $0 \leq k \leq N$, we apply Lemma 7.4 to construct a stopping time $0 < \tau'_k < \sigma_{k+1}$, such that

$$\mathbf{E} \sum_{k=0}^{N} (\sigma_{k+1} - \tau'_k) \leq \epsilon.$$

Finally we set

$$\tau_0 = \tau'_0, \ \tau_1 = \sigma_1 \vee \tau'_1, \quad \cdots, \ \tau_N = \sigma_N \vee \tau'_N.$$

It is clear that $\tau_k \in [\sigma_k, \sigma_{k+1}) \cap [\tau'_{k+1}, \sigma_{k+1}]$. We have also $\tau_k < \sigma_{k+1}$ whenever $\sigma_k < T$. Thus $(\sigma_k, \tau_k] \in (\sigma_k, \sigma_{k+1})$. It follows that

$$\mathbf{E} \sum_{k=0}^{N} (\sigma_{k+1} - \tau_k) \leq \epsilon,$$

or

$$\mathbf{E} \sum_{k=0}^{N} (\tau_k - \sigma_k) \geq T - \epsilon,$$

and

$$\sum_{k=0}^{N} \mathbf{E} \sum_{t \in (\sigma_k, \tau_k]} (\Delta A_t)^2 \leq \sum_{k=0}^{N} \mathbf{E} \sum_{t \in (\sigma_k, \sigma_{k+1})} (\Delta A_t)^2 \leq \delta.$$

Thus the above conditions (i)-(iii) are satisfied. □

We now give the

Proof of Theorem 7.2. Since (g^i) (resp. (z^i)) is weakly compact in $L^2_{\mathcal{F}}(0, T)$ (resp. $L^2_{\mathcal{F}}(0, T; \mathbf{R}^d)$), there is a subsequence, still denoted by (g^i) (resp. (z^i)) which converges weakly to (g^0_t) (resp. (z_t)).

Thus, for each stopping time $\tau \leq T$, the following weak convergence holds in $L^2(\mathcal{F}_\tau)$.

$$\int_0^\tau z^i_s dB_s \rightharpoonup \int_0^\tau z_s dB_s, \qquad \int_0^\tau g^i_s ds \rightharpoonup \int_0^\tau g^0_s ds.$$

Since

$$A^i_\tau = -y^i_\tau + y^i_0 + \int_0^\tau g^i_s ds + \int_0^\tau z^i_s dB_s$$

thus we also have the weak convergence

$$A^i_\tau \rightharpoonup A_\tau := -y_\tau + y_0 + \int_0^\tau g^0_s ds + \int_0^\tau z_s dB_s.$$

Obviously, $\mathbf{E}[A_T^2] < \infty$. For any two stopping times $\sigma \le \tau \le T$, we have $A_\sigma \le A_\tau$ since $A^i_\sigma \le A^i_\tau$. From this it follows that A is an increasing process. Moreover, from Lemma 7.2, both A and y are RCLL. Thus y has a form of (116). Since y is given, it is clear that z is uniquely determined. Thus not only the subsequence of $\{z^i\}_{i=1}^\infty$ but also the sequence itself converges weakly to z. Our key point is to show that $\{z^i\}_{i=1}^\infty$ converges to z in the strong sense of (117). In order to prove this we use Itô's formula applied to $(y^i_t - y_t)^2$ on a given subinterval $(\sigma, \tau]$. Here $0 \le \sigma \le \tau \le T$ are two stopping times. Observe that $\Delta y_t \equiv \Delta A_t$ and the fact that y^i and then A^i are continuous. We have

$$\begin{aligned}
&\mathbf{E}|y^i_\sigma - y_\sigma|^2 + \mathbf{E}\int_\sigma^\tau |z^i_s - z_s|^2 ds \\
&= \mathbf{E}|y^i_\tau - y_\tau|^2 - \mathbf{E}\sum_{t\in(\sigma,\tau]} (\Delta A_t)^2 - 2\mathbf{E}\int_\sigma^\tau (y^i_s - y_s)(g^i_s - g^0_s)ds \\
&\quad +2\mathbf{E}\int_{(\sigma,\tau]} (y^i_s - y_s)dA^i_s - 2\mathbf{E}\int_{(\sigma,\tau]} (y^i_s - y_{s-})dA_s \\
&= \mathbf{E}|y^i_\tau - y_\tau|^2 + \mathbf{E}\sum_{t\in(\sigma,\tau]} (\Delta A_t)^2 - 2\mathbf{E}\int_\sigma^\tau (y^i_s - y_s)(g^i_s - g^0_s)ds \\
&\quad +2\mathbf{E}\int_{(\sigma,\tau]} (y^i_s - y_s)dA^i_s - 2\mathbf{E}\int_{(\sigma,\tau]} (y^i_s - y_{s-})dA_s
\end{aligned}$$

Since $\int_{(\sigma,\tau]}(y^i_s - y_s)dA^i_s \le 0$, we then have

$$\begin{aligned}
\mathbf{E}\int_\sigma^\tau |z^i_s - z_s|^2 ds \le \mathbf{E}|y^i_\tau - y_\tau|^2 + \mathbf{E}\sum_{t\in(\sigma,\tau]} (\Delta A_t)^2 \\
+2\mathbf{E}\int_\sigma^\tau |y^i_s - y_s||g^i_s - g^0_s|ds + 2\mathbf{E}\int_{(\sigma,\tau]} |y^i_s - y_s|dA_s.
\end{aligned} \tag{121}$$

The third term on the right side tends to zero since

$$\mathbf{E}\int_0^T |y^i_s - y_s||g^i_s - g^0_s|ds \le C\left[\mathbf{E}\int_0^T |y^i_s - y_s|^2 ds\right]^{\frac{1}{2}} \to 0. \tag{122}$$

For the last term, we have, P–almost surely,

$$|y_s^1 - y_s| \geq |y_s^i - y_s| \to 0, \qquad \forall s \in [0,T].$$

Since

$$\mathbf{E}\int_0^T |y_s^1 - y_s| dA_s \leq (\mathbf{E}[\sup_s(|y_s^1 - y_s|^2])^{\frac{1}{2}}(\mathbf{E}(A_T)^2)^{\frac{1}{2}} < \infty.$$

It then follows from Lebesgue's dominated convergence theorem that

$$\mathbf{E}\int_{(0,T]} |y_s^i - y_s| dA_s \to 0. \tag{123}$$

By convergence of (122) and (123), it is clear from the estimate (121) that, once A is continuous (thus $\Delta A_t \equiv 0$) on $[0,T]$, then z^i tends to z strongly in $L^2_{\mathcal{F}}(0,T;\mathbf{R}^d)$. Thus the second assertion of the theorem, i.e., (118) follows.

But for the general case, the situation becomes complicated. Thanks to Lemma 7.5, for any positive δ and ϵ, there exist a finite number of disjoint intervals $(\sigma_k, \tau_k]$, $k = 0, 1, \cdots, N$, such that $\sigma_k \leq \tau_k \leq T$ are all stopping times satisfying

$$\begin{array}{l} \text{(i)} \quad \mathbf{E}\sum_{k=0}^N [\tau_k - \sigma_k](\omega) \geq T - \frac{\epsilon}{2}; \\ \text{(ii)} \ \sum_{k=0}^N \sum_{\sigma_k < t \leq \tau_k} \mathbf{E}(\Delta A_t)^2 \leq \frac{\delta\epsilon}{3}. \end{array} \tag{124}$$

Now, for each $\sigma = \sigma_k$ and $\tau = \tau_k$, we apply estimate (121) and then take the sum. It follows that

$$\begin{aligned} \sum_{k=0}^N \mathbf{E}\int_{\sigma_k}^{\tau_k} |z_s^i - z_s|^2 ds \leq \sum_{k=0}^N \mathbf{E}|y_{\tau_k}^i - y_{\tau_k}|^2 + \sum_{k=0}^N \mathbf{E} \sum_{t\in(\sigma_k,\tau_k]} (\Delta A_t)^2 \\ +2\mathbf{E}\int_0^T |y_s^i - y_s||g_s^i - g_s^0| ds + 2\mathbf{E}\int_{(0,T]} |y_s^i - y_s| dA_s. \end{aligned}$$

By using the convergence results (122) and (123) and taking in consideration of (124)-(ii), it follows that

$$\overline{\lim_{i\to\infty}} \sum_{k=0}^N \mathbf{E}\int_{\sigma_k}^{\tau_k} |z_s^i - z_s|^2 ds \leq \sum_{k=0}^N \mathbf{E} \sum_{t\in(\sigma_k,\tau_k]} (\Delta A_t)^2 \leq \frac{\epsilon\delta}{3}$$

Thus there exists an integer $l_{\epsilon\delta} > 0$ such that, whenever $i \geq l_{\epsilon\delta}$, we have

$$\sum_{k=0}^N \mathbf{E}\int_{\sigma_k}^{\tau_k} |z_s^i - z_s|^2 ds \leq \frac{\epsilon\delta}{2}$$

Thus, in the product space $([0,T]\times\Omega, \mathcal{B}([0,T])\times\mathcal{F}, m\times P)$ (here m stands for the Lebesgue measure on $[0,T]$), we have

$$m\times P\left\{(s,\omega)\in\bigcup_{k=0}^N(\sigma_k(\omega),\tau_k(\omega)]\times\Omega; \quad |z_s^i(\omega) - z_s(\omega)|^2 \geq \delta\right\} \leq \frac{\epsilon}{2}$$

This with (124)-(i) implies

$$m \times P\left\{(s,\omega) \in [0,T] \times \Omega; \quad |z_s^i(\omega) - z_s(\omega)|^2 \geq \delta\right\} \leq \epsilon, \qquad \forall \quad i \geq l_{\epsilon\delta}.$$

From this it follows that, for any $\delta > 0$,

$$\lim_{i\to\infty} m \times P\left\{(s,\omega) \in [0,T] \times \Omega; \quad |z_s^i(\omega) - z_s(\omega)|^2 \geq \delta\right\} = 0.$$

Thus, on $[0,T] \times \Omega$, the sequence $\{z^i\}_{i=1}^\infty$ converges in measure to z. Since $\{z^i\}_{i=1}^\infty$ is also bounded in $L^2_{\mathcal{F}}(0,T;\mathbf{R}^d)$, then for each $p \in [1,2)$, it converges strongly in $L^p_{\mathcal{F}}(0,T;\mathbf{R}^d)$. □

7.3 Optional Stopping Theorem for $\mathcal{E}^g$–Supermartingale

In this subsection the function g satisfies (i), (ii) of (34). We will discuss $\mathcal{E}^g_{\sigma,\tau}[\cdot]$ for stopping times $\sigma, \tau \in \mathcal{S}_T$. A BSDE with a given terminal condition $X \in \mathcal{F}_\tau$ at a given terminal time $\tau \in \mathcal{S}_T$ is formulated as

$$Y_s = X + \int_s^\tau g(r, Y_r, Z_r)dr - \int_s^\tau Z_r dB_r,\ s \in [0,\tau], \tag{125}$$

or equivalently, on $s \in [0,T]$,

$$Y_s = X + \int_s^T 1_{[0,\tau]}(r) g(r, Y_r, Z_r)dr - \int_s^\tau 1_{[0,\tau]}(r) Z_r dB_r. \tag{126}$$

We define

$$\mathcal{E}^g_{\sigma,\tau}[X] := Y_\sigma. \tag{127}$$

It is clear that, when $\sigma = s$ and $\tau = t$ for deterministic time parameters $s \leq t$, then $\mathcal{E}^g_{\sigma,\tau}[\cdot] = \mathcal{E}^g_{s,t}[\cdot]$. We have

Proposition 7.1. *The system of operators*

$$\mathcal{E}^g_{\sigma,\tau}[\cdot] : L^2(\mathcal{F}_\tau) \to L^2(\mathcal{F}_\sigma),\ \sigma \leq \tau,\ \sigma, \tau \in \mathcal{S}_T,$$

is an $\mathcal{F}_t$–consistent nonlinear evaluation, i.e., it satisfies (A1)–(A5) in the following sense: for each X, $X' \in L^2(\mathcal{F}_\tau)$,
(a1) *$\mathcal{E}^g_{\sigma,\tau}[X] \geq \mathcal{E}^g_{\sigma,\tau}[X']$, a.s., if $X \geq X'$, a.s.*
(a2) *$\mathcal{E}^g_{\tau,\tau}[X] = X$;*
(a3) *$\mathcal{E}^g_{\rho,\sigma}[\mathcal{E}^g_{\sigma,\tau}[X]] = \mathcal{E}^g_{\rho,\tau}[X]$, $\forall 0 \leq \rho \leq \sigma \leq \tau$;*
(a4) *$1_A \mathcal{E}^g_{\sigma,\tau}[X] = 1_A \mathcal{E}^g_{\sigma,\tau}[1_A X]$, $\forall A \in \mathcal{F}_\tau$;*
(a5) *for each $0 \leq \sigma \leq \tau \leq T$,*

$$\mathcal{E}^g_{\sigma,\tau}[X] - \mathcal{E}^g_{\sigma,\tau}[X'] \leq \mathcal{E}^{g_\mu}_{\sigma,\tau}[X - X'],\ \ \forall X, X' \in L^2(\mathcal{F}_\tau). \tag{128}$$

The proof is similar as in the case where ρ, σ and τ are deterministic. We omit it.

Another easy property is that $\mathcal{E}_{\cdot\wedge\tau,\tau}[X]$ has continuous paths:

$$(\mathcal{E}^g_{t\wedge\tau,\tau}[X])_{0\le t\le T} \in S^2_{\mathcal{F}}(0,T). \tag{129}$$

By (32) and (33) with $1_{[\sigma,\tau]}(s)g(s,y,z)$ in the place of g, we also have the following estimates

$$\mathbf{E}[|\mathcal{E}^g_{\sigma,\tau}[X]|^2] \le c\mathbf{E}[|X|^2] + c\mathbf{E}\int_\sigma^\tau |g(s,0,0)|^2)ds, \tag{130}$$

and

$$\mathbf{E}[|\mathcal{E}^g_{\sigma,\tau}[X - X']|^2] \le c\mathbf{E}[|X - X'|^2]. \tag{131}$$

where the constant c depends only on T and the Lipschitz constant C of the function g w.r.t. (y,z). As a consequence of

We also have the following estimate:

Lemma 7.6. *Let σ, $\tau \in \mathcal{S}_T$, $\sigma \le \tau$ and $X \in L^2(\mathcal{F}_\tau)$. If $X \in L^2(\mathcal{F}_\sigma)$, then we have*

$$\mathbf{E}[|\mathcal{E}^g_{\sigma,\tau}[X] - X|^2] \le c\mathbf{E}[\int_\sigma^\tau |g(s,X,0)|^2 ds].$$

where the constant c depends only on T and the Lipschitz constant C of g.

Proof. Observe that $\mathcal{E}^g_{\sigma,\tau}[X] = y_\sigma$, where $(y_t)_{t\in[0,T]}$ is the solution of the BSDE

$$y_t = X + \int_t^T 1_{[\sigma,\tau]}(s)g(s,y_s,z_s)ds - \int_t^T z_s dB_s.$$

We set $\bar{y}_t \equiv y_t - X$, $\bar{z}_t \equiv z_t$, on $[\sigma,\tau]$. This pair of adapted process is the solution of the BSDE

$$\bar{y}_t = \int_t^T 1_{[\sigma,\tau]}(s)\bar{g}(s,\bar{y}_s,\bar{z}_s)ds - \int_t^T \bar{z}_s dB_s,\ t \in [\sigma,\tau].$$

With $\bar{g}(t,y,z) := g(t,y+X,z)$, we have $\mathcal{E}^g_{\sigma,\tau}[X] - X = \mathcal{E}^{\bar{g}}_{\sigma,\tau}[0]$. From (130),

$$\begin{aligned}\mathbf{E}[|\mathcal{E}^g_{\sigma,\tau}[X] - X|^2] &= \mathbf{E}[|\mathcal{E}^{\bar{g}}_{\sigma,\tau}[0]|^2] \\ &\le c\mathbf{E}[\int_\sigma^\tau |\bar{g}(s,0,0)|^2 ds] \\ &= c\mathbf{E}[\int_\sigma^\tau |g(s,X,0)|^2 ds].\end{aligned}$$

□

We will prove the following optional stopping theorem:

Theorem 7.3. *We assume that the function g satisfies (i), (ii) of (34). Let $Y \in D^2_{\mathcal{F}}(0,T)$ be an $\mathcal{E}$–supermartingale (resp. $\mathcal{E}$–submartingale). Then for each σ, $\tau \in \mathcal{S}_T$ such that $\sigma \le \tau$, we have*

$$\mathcal{E}^g_{\sigma,\tau}[Y_\tau] \le Y_\sigma \ \textit{(resp. } \ge Y_\sigma\textit{), a.s. .} \tag{132}$$

To prove the above theorem, we need several lemmas.

Lemma 7.7. *Let $\tau \in \mathcal{S}^0_T$ be valued in $\{t_0, \cdots, t_n\}$ with $0 = t_0 \le t_1 < \cdots < t_n \le t_{n+1} = T$, and let*

$$t_i \le s < t \le t_{i+1}, \textit{ for some } i \in \{1, 2, \cdots, n\}. \tag{133}$$

Then, for each $X \in \mathcal{F}_{t\wedge\tau}$,

$$\begin{cases} \textit{(i) } \mathcal{E}^g_{t\wedge\tau,t\wedge\tau}[X] = X; \\ \textit{(ii) } \mathcal{E}^g_{s\wedge\tau,t\wedge\tau}[X] = 1_{\{t\wedge\tau\le s\}}X + 1_{\{t\wedge\tau=t\}}\mathcal{E}^g_{s,t}[X]. \end{cases} \tag{134}$$

Proof. (i) is easy. To prove (ii), we first observe that

$$\{t\wedge\tau \le s\}^C = \{t\wedge\tau = t\} \tag{135}$$

and $\{t\wedge\tau \le s\} = \{t\wedge\tau \le t_i\}$. Thus $1_{\{t\wedge\tau\le s\}}X \in \mathcal{F}_{t_i}$. We also have $1_{\{t\wedge\tau=t\}}X \in \mathcal{F}_t$. We now solve $Y_{s\wedge\tau} = \mathcal{E}^g_{s\wedge\tau,t\wedge\tau}[X]$ by, as in (126),

$$Y_{s\wedge\tau} = \ X + \int_s^T 1_{[0,t\wedge\tau]}(r)g(r,Y_r,Z_r)dr - \int_s^T 1_{[0,t\wedge\tau]}(r)Z_r dB_r. \tag{136}$$

Since $1_{[0,t\wedge\tau]} = 1_{\{t\wedge\tau\le t_i\}}1_{[0,t_i]} + 1_{\{t\wedge\tau=t\}}1_{[0,t]}$. By respectively multiplying $1_{\{t\wedge\tau\le t_i\}}$ and $1_{\{t\wedge\tau=t\}}$ on both sides of (136), we have, on $s \in [t_i, t)$,

$$Y_{s\wedge\tau}1_{\{t\wedge\tau\le t_i\}} = \ X1_{\{t\wedge\tau\le t_i\}}, \tag{137}$$

and

$$\begin{aligned} Y_{s\wedge\tau}\ 1_{\{t\wedge\tau=t\}} = \ & 1_{\{t\wedge\tau=t\}}X + \int_s^T 1_{[0,t]}(r)1_{\{t\wedge\tau=t\}}g(r,Y_r,Z_r)dr \\ & - \int_s^T 1_{[0,t]}1_{\{t\wedge\tau=t\}}(r)Z_r dB_r \\ = & 1_{\{t\wedge\tau=t\}}X + \int_s^t g(r, 1_{\{t\wedge\tau=t\}}Y_r, 1_{\{t\wedge\tau=t\}}Z_r)dr \\ & - \int_s^t 1_{[0,t_{i+1}]}Z_r dB_r. \end{aligned}$$

We observe that, the last relation implies that, on $[t_i, t]$,

$$Y_{s\wedge\tau}\ 1_{\{t\wedge\tau=t\}} = 1_{\{t\wedge\tau=t\}}\mathcal{E}^g_{s,t}[1_{\{t\wedge\tau=t\}}X] = 1_{\{t\wedge\tau=t\}}\mathcal{E}^g_{s,t}[X].$$

This with (137) and (135), we then have (ii). □

We now treat a simple situation of the above optional stopping theorem.

Lemma 7.8. *Let $Y \in D^2_{\mathcal{F}}(0,T)$ be an $\mathcal{E}^g$- martingale (respectively $\mathcal{E}^g$ - supermartingale, $\mathcal{E}^g$- submartingale). Then for each σ, $\tau \in \mathcal{S}^0_T$ such that $\sigma \leq \tau$, we have*

$$\mathcal{E}^g_{\sigma,\tau}[Y_\tau] = Y_\sigma, \text{ (resp. } \leq Y_\sigma, \ \geq Y_\sigma\text{) a.s.} \tag{138}$$

Proof. We only prove the case for $\mathcal{E}^g$–supermartingale. It is clear that, once we have

$$\mathcal{E}^g_{t\wedge\tau,\tau}[Y_\tau] \leq Y_{t\wedge\tau}, \ \ \forall t \in [0,T], \tag{139}$$

then, (138) hold for each $\sigma \in \mathcal{S}^0_T$ valued in $\{s_1, \cdots, s_m\}$ since

$$\mathcal{E}^g_{\sigma,\tau}[Y_\tau] = \sum_{i=1}^m 1_{\{\sigma=s_i\}}\mathcal{E}^g_{s_i\wedge\tau,\tau}[Y_\tau] \leq \sum_{i=1}^m 1_{\{\sigma=s_i\}}Y_{s_i\wedge\tau} = Y_\sigma.$$

We will prove (139) by deduction. Let $\tau \in \mathcal{S}^0_T$ be valued in $\{t_0, \cdots, t_n\}$ with $0 = t_0 \leq t_1 < \cdots < t_n \leq t_{n+1} = T$. Firstly, when $t \geq t_n$, (139) holds since $\mathcal{E}_{t\wedge\tau,\tau}[Y_\tau] = \mathcal{E}_{\tau,\tau}[Y_\tau] = Y_\tau$. Now suppose that for a fixed $i \in \{1, \cdots, n\}$, (139) holds for $t \geq t_i$. We shall prove that it also holds for $t \geq t_{i-1}$. We need to check the case $t \in [t_{i-1}, t_i)$.

Since $1_{\{t_i\wedge\tau=t_i\}}$ is $\mathcal{F}_t$–measurable, by (a4) we have

$$\begin{aligned}
1_{\{t_i\wedge\tau=t_i\}}\mathcal{E}^g_{t,t_i}[Y_{t_i\wedge\tau}] &= 1_{\{t_i\wedge\tau=t_i\}}\mathcal{E}^g_{t,t_i}[1_{\{t_i\wedge\tau=t_i\}}Y_{t_i}] \\
&= 1_{\{t_i\wedge\tau=t_i\}}\mathcal{E}^g_{t,t_i}[Y_{t_i}] \\
&\leq 1_{\{t_i\wedge\tau=t_i\}}Y_t.
\end{aligned}$$

It follows from (134)–(ii)

$$\begin{aligned}
\mathcal{E}^g_{t\wedge\tau,t_i\wedge\tau}[Y_{t_i\wedge\tau}] &= 1_{\{t_i\wedge\tau\leq t\}}Y_{t_i\wedge\tau} + 1_{\{t_i\wedge\tau=t_i\}}\mathcal{E}^g_{t,t_i}[Y_{t_i\wedge\tau}] \\
&\leq 1_{\{t_i\wedge\tau\leq t\}}Y_{t_i\wedge\tau} + 1_{\{t_i\wedge\tau=t_i\}}Y_t \\
&= Y_{t\wedge\tau}.
\end{aligned}$$

The last step is from $\{t_i \wedge \tau \leq t\} + \{t_i \wedge \tau = t_i\} = \Omega$ and then $t \wedge \tau = t_i \wedge \tau 1_{\{t_i\wedge\tau\leq t\}} + t1_{\{t_i\wedge\tau=t_i\}}$. From this result we derive

$$\begin{aligned}
\mathcal{E}^g_{t\wedge\tau,\tau}[Y_\tau] &= \mathcal{E}^g_{t\wedge\tau,t_i\wedge\tau}[\mathcal{E}^g_{t_i\wedge\tau,\tau}[Y_\tau]] \\
&\leq \mathcal{E}^g_{t\wedge\tau,t_i\wedge\tau}[Y_{t_i\wedge\tau}] \\
&\leq Y_{t\wedge\tau}.
\end{aligned}$$

Thus (139) holds for $t \geq t_{i-1}$. It follows by deduction that (139) holds for $t \in [0,T]$. The proof is complete. □

We now give

Proof of Theorem 7.3. We only prove the supermartingale part. For each $n = 1, 2, \cdots$, we set

$$\sigma_n := T\sum_{k=1}^{2^n-1} 2^{-n}k1_{\{2^{-n}(k-1)\le\sigma<2^{-n}k\}} + T1_{\{\sigma=T\}},$$
$$\tau_n := T\sum_{k=1}^{2^n-1} 2^{-n}k1_{\{2^{-n}(k-1)\le\tau<2^{-n}k\}} + T1_{\{\tau=T\}}.$$

It is clear that $\sigma_n \searrow \sigma$, $\tau_n \searrow \tau$ and $\sigma_n \le \tau_n$. By the above lemma, for each $m \ge n$ we have

$$\mathcal{E}^g_{\sigma_m,\tau_n}[Y_{\tau_n}] \le Y_{\sigma_m}, \text{ a.s.}$$

It follows from (129) and $Y \in D^2_{\mathcal{F}}(0,T)$ that, for each fixed n, $\mathcal{E}^g_{\sigma_m,\tau_n}[Y_{\tau_n}] \to \mathcal{E}^g_{\sigma,\tau_n}[Y_{\tau_n}]$ and $Y_{\sigma_m} \to Y_\sigma$ in $L^2(\mathcal{F}_T)$ as $m \to \infty$. We then have

$$\mathcal{E}^g_{\sigma,\tau_n}[Y_{\tau_n}] \le Y_\sigma, \text{ a.s.} \tag{140}$$

Moreover, we have

$$|\mathcal{E}^g_{\sigma,\tau_n}[Y_{\tau_n}] - \mathcal{E}^g_{\sigma,\tau}[Y_\tau]| \le |\mathcal{E}^g_{\sigma,\tau_n}[Y_{\tau_n}] - \mathcal{E}^g_{\sigma,\tau_n}[Y_\tau]| + |\mathcal{E}^g_{\sigma,\tau_n}[Y_\tau] - \mathcal{E}^g_{\sigma,\tau}[Y_\tau]|. \tag{141}$$

Since $Y_{\tau_n} \to Y_\tau$, in $L^2(\mathcal{F}_T)$, the first term on the right tends to zero in $L^2(\mathcal{F}_T)$ because of (131). For the second one,we still use (131):

$$\begin{aligned}\mathbf{E}[|\mathcal{E}^g_{\sigma,\tau_n}[Y_\tau] - \mathcal{E}^g_{\sigma,\tau}[Y_\tau]|^2] &= \mathbf{E}[|\mathcal{E}^g_{\sigma,\tau}[\mathcal{E}^g_{\tau,\tau_n}[Y_\tau]] - \mathcal{E}^g_{\sigma,\tau}[Y_\tau]|^2]\\ &\le c\mathbf{E}[|\mathcal{E}^g_{\tau,\tau_n}[Y_\tau] - Y_\tau|].\end{aligned}$$

But by Lemma 7.6 this term is bounded by $c^2\mathbf{E}[\int_\tau^{\tau_n} |g(s, Y_\tau, 0)|^2 ds]$. It follows that the term on the left side of (140) tends to $\mathcal{E}^g_{\sigma,\tau}[Y_\tau]$ in $L^2(\mathcal{F}_T)$ as $n \to \infty$. The proof is complete. □

We will also prove the following optional stopping theorem:

Theorem 7.4. *We assume that an $\mathcal{F}$-expectation $\mathcal{E}[\cdot]$ satisfies (78) and (81) for some $\mu > 0$. Let $Y \in D^2_{\mathcal{F}}(0,T)$ be an $\mathcal{E}$–supermartingale (resp. $\mathcal{E}$–submartingale). Then for each σ, $\tau \in \mathcal{S}_T$ we have*

$$\mathcal{E}[Y_\tau|\mathcal{F}_\sigma] \le Y_{\tau\wedge\sigma}, \textit{ (resp. } \ge Y_{t\wedge\tau}\textit{), a.s. .} \tag{142}$$

Proof. We only consider the supermartingale case. We first prove that

$$\mathcal{E}[Y_\tau|\mathcal{F}_t] \le Y_{t\wedge\tau} \text{ (resp. } \ge Y_{t\wedge\tau}\text{), a.s. .} \tag{143}$$

Let τ be a finite valued: $\tau = \sum_{i=1}^n 1_{\{\tau=t_i\}}t_i$, for some $0 \le t_1 \le \cdots \le t_n \le T$. If $t_n \le t$, then it is clear that

$$\mathcal{E}[Y_\tau|\mathcal{F}_t] = Y_\tau = Y_{t\wedge\tau}.$$

If $t \in [t_{n-1}, t_n]$, then both $\{\tau \le t_{n-1}\}$ and $\{\tau = t_n\}$ are $\mathcal{F}_t$–measurable. By (11) we have

$$\begin{aligned}\mathcal{E}[Y_\tau|\mathcal{F}_t] &= \mathcal{E}[Y_{t_n}1_{\{\tau=t_n\}} + Y_{\tau\wedge t_{n-1}}1_{\{\tau\le t_{n-1}\}}|\mathcal{F}_t] \\ &= 1_{\{\tau=t_n\}}\mathcal{E}[Y_{t_n}|\mathcal{F}_t] + 1_{\{\tau\le t_{n-1}\}}\mathcal{E}[Y_{\tau\wedge t_{n-1}}|\mathcal{F}_t] \\ &\le 1_{\{\tau=t_n\}}Y_t + 1_{\{\tau\le t_{n-1}\}}Y_{\tau\wedge t_{n-1}} = Y_{t\wedge\tau}.\end{aligned}$$

If $t\in[t_{n-2},t_{n-1}]$, then we have $\mathcal{E}[Y_\tau|\mathcal{F}_t] = \mathcal{E}[\mathcal{E}[Y_\tau|\mathcal{F}_{t_{n-1}}]|\mathcal{F}_t] \le \mathcal{E}[Y_{t_{n-1}\wedge\tau}|\mathcal{F}_t]$ $\le Y_{t\wedge\tau}$. We thus can prove an arbitrary case $t\in[t_i,t_{i+1}]$ by reduction. Thus (142) holds for all finite valued stopping times.

Now for $\tau\in\mathcal{S}_T$, we take τ_n as in the proof of Theorem 7.3. Since $Y\in D^2_{\mathcal{F}}(0,T)$, thus $Y_{\tau_n}\to Y_\tau$ in $L^2(\mathcal{F}_T)$. We have

$$\mathcal{E}^{-\mu}[Y_{\tau_n}-Y_\tau|\mathcal{F}_t] \le \mathcal{E}[Y_{\tau_n}|\mathcal{F}_t] - \mathcal{E}[Y_\tau|\mathcal{F}_t] \le \mathcal{E}^{\mu}[Y_{\tau_n}-Y_\tau|\mathcal{F}_t].$$

By Lemma 3.3, the right side tends to zero in $L^2(\mathcal{F}_T)$. So does the right side since

$$\mathcal{E}^{-\mu}[Y_{\tau_n}-Y_\tau|\mathcal{F}_t] = -\mathcal{E}^{\mu}[Y_\tau-Y_{\tau_n}|\mathcal{F}_t].$$

It follows that $\mathcal{E}[Y_{\tau_n}|\mathcal{F}_t]\to\mathcal{E}[Y_\tau|\mathcal{F}_t]$ in $L^2(\mathcal{F}_T)$. We then can pass two sides of the inequality

$$Y_{\tau_n\wedge t} \ge \mathcal{E}[Y_{\tau_n}|\mathcal{F}_t]$$

to the limit to get (143).
Since both $(\mathcal{E}[Y_\tau|\mathcal{F}_t])_{t\in[0,T]}\in S^2_{\mathcal{F}}(0,T)$ and $(Y_{t\wedge\tau})_{t\in[0,T]}\in D^2_{\mathcal{F}}(0,T)$ we can easily derive from (143) that for each σ, $\tau\in\mathcal{S}_T$, we have (142). □

Notes

The expectation $\mathbf{E}[\cdot]$ on the probability space $(\Omega,\mathcal{F},P)$ with $\mathcal{F}_t\subset\mathcal{F}$, $t\ge0$ is clearly $\mathcal{F}_t$–consistent. Another example of linear $\mathcal{F}_t$–consistent expectation is $\mathbf{E}_Q[\cdot]$, the expectation under Girsanov transformation dQ/dP. But it seems that the study of $\mathcal{F}_t$–consistent nonlinear expectations is still a very new subject. In 1997, [Peng1997b] (see also [Peng1997a]) introduced the notion of g–expectations which is nonlinear and $\mathcal{F}_t$–consistent. In the same year, the notion of g–evaluation was introduced in [Peng1997a] under the name "stochastic backward semigroup". See also [30]. The term "$\mathcal{F}_t$–consistent nonlinear expectation" was named in [CHMP2002].

Linear BSDE was first introduced by Bismut in [Bis1973], [Bis1978]. Bensoussan developed this approach in [Ben1981] and [Ben1982]. The existence and uniqueness theorem of a nonlinear BSDE, i.e., Theorem 3.1 was obtained in Pardoux and Peng [PP1990]. The present version of the proof was based on El Karoui, Peng and Quenez [EPQ1997]. [EPQ1997] is also a good survey of BSDE and related fields. Comparison Theorem of BSDE i.e., Theorem 3.3 was obtained in [Peng1992] for the case g is C^1 in (y,z). The present case where g is Lipschitz in (y,z) was obtained in [EPQ1997]. [EPQ1997] also observed and investigated a natural relation between BSDE theory and the problem of pricing financial derivatives. We also refer to Yong and Zhou [YZ1999] for a systematic presentation of BSDE theory. Due to the limitation of the size of this lecture, we can not present many important subjects of BSDE theory.

In 1998, Chen [Chen98] has proved the following interesting property: if $\mathcal{E}_{g^1}[X] = \mathcal{E}_{g^2}[X']$, for all $X \in L^2(\mathcal{F}_T)$, then the two generators g^1 and g^2 also coincide: $g^1(s, y, z) \equiv g^2(s, y, z)$. This result was generalized to an "inverse comparison theorem" by [BCHMP2000] and then [CHMP2001]: if $\mathcal{E}_{g^1}[X] \geq \mathcal{E}_{g^2}[X']$, for all $X \in L^2(\mathcal{F}_T)$, then $g^1 \geq g^2$.

The well - known Doob - Meyer decomposition theorem can be found in most standard text books of stochastic analysis e.g., [DM1978-1982], [HWY1992], [IW1981], [KShr1998] and [RW2000]. Decomposition theorem of g - supermartingale of Doob - Meyer's type, i.e., Theorem 3.9 was obtained by Peng [Peng1999]. A new method, i.e., penalization method, was applied to prove this nonlinear decomposition theorem. This method was firstly introduced in BSDE theory by [EKPPQ1997]. The monotonic limit theorem for Itô's processes (Theorem 7.2) as well as for BSDEs (Theorem 3.8) are also obtained in [Peng1999]. Using this penalization method, Chen and Peng [CP1998] to the L^1 case with the usual filtration, which generalizes the Meyer's result to a nonlinear situation. These penalization method and limit theorem were then applied to prove the nonlinear supermartingale decomposition theorem for an abstract $\mathcal{E}$–expectation, i.e., Theorem 4.3. Theorem 4.3 was proved in [CHMP2002]. This type of decomposition theorem for a more general situation, i.e., the case for $\mathcal{F}_t$–evaluation, was recently obtained in [Peng2003b].

The representation theorem of an $\mathcal{F}_t$–expectation by a g - expectation, i.e., Theorem 4.4 was obtained in [CHMP2002]. The more general case, i.e., Theorem 4.5 was obtained in [Peng2003b].

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