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On the Liouville Property for Sublaplacians

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1. – Introduction

The Liouville theorem for harmonic functions states that a solution u of

$$u \geq 0, \;\; \Delta u = 0 \;\text{ in } \mathbb{R}^N$$

is a constant. This classical result has been extended to non-negative solutions of semilinear elliptic equations in $\mathbb{R}^N$ or in half-spaces by B. Gidas and J. Spruck [19]. For the case of the whole space they proved that the unique solution of

$$u \geq 0, \;\; \Delta u + Cu^\alpha = 0 \;\text{ in } \mathbb{R}^N$$

is $u \equiv 0$, provided $1 < \alpha < \frac{N+2}{N-2}$ and C is a strictly positive constant.

The Liouville property is more delicate to establish for semilinear elliptic equations or inequalities of the form

$$u \geq 0, \;\; \Delta u + h(x)u^\alpha \leq 0 \;\text{ in } \Sigma,$$

where Σ is a cone in $\mathbb{R}^N$ and $h \geq 0$ is a function which may vanish on the boundary of Σ. Liouville type theorems in this case have been established recently by H. Berestycki, L. Nirenberg and the first author. In the paper [2] they obtained, by a simpler method than in [19], a general result in this direction under some conditions relating the exponent α, the rate of growth of h at infinity, the opening of the cone Σ and the space dimension N. In the special case $\Sigma = \{x = (x_1, ..., x_N) \in \mathbb{R}^N : x_N > 0\}$ and $h(x) = x_N$, the above mentioned theorem states that the unique solution of

$$u \geq 0, \;\; \Delta u + x_N u^\alpha \leq 0 \;\text{ in } \Sigma$$

is $u \equiv 0$, provided $1 < \alpha < \frac{N+2}{N-1}$.

In [19] and [2] these non-existence results have been applied to show via a blow-up analysis the validity, under restrictions on α dictated by the Liouville theorems, of a priori estimates in the sup norm for all solutions $(u, \tau) \geq 0$ of the problem

$$\begin{aligned} & u \geq 0, \;\; \Delta u + a(x)u^\alpha + \tau = 0 \;\text{ in } \Omega \\ & u = 0 \;\text{ on } \partial\Omega, \end{aligned}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ and $\tau \in \mathbb{R}$. These estimates allow to prove, via the Leray-Schauder degree theory, the existence of non-trivial solutions of the Dirichlet problem

$$\begin{aligned} &u \geq 0, \ \Delta u + a(x)u^\alpha = 0 \ \text{ in } \Omega \\ &u = 0 \ \text{ on } \partial\Omega, \end{aligned}$$

even when the weight a may change sign in Ω (see [2] for such indefinite type problems).

The approach of [2], which works for general second order uniformly elliptic operators in non divergence form, has been adapted by I. Birindelli and the present authors to deal with the semilinear operator $\Delta_{H^n} u + a(\xi)u^\alpha$. Here, Δ_{H^n} is the second order degenerate elliptic operator

$$(1.1)\quad \Delta_{H^n} = \sum_{i=1}^{2n}\left(\frac{\partial^2}{\partial\xi_i^2} + 4\xi_i^2\frac{\partial^2}{\partial\xi_{2n+1}^2}\right) + 4\sum_{i=1}^{n}\left(\xi_{i+n}\frac{\partial^2}{\partial\xi_i\partial\xi_{2n+1}} - \xi_i\frac{\partial^2}{\partial\xi_{i+n}\partial\xi_{2n+1}}\right)$$

acting on functions $u = u(\xi)$ where $\xi = (\xi_1, \cdots, \xi_{2n}, \xi_{2n+1}) \in \mathbb{R}^{2n+1}$. In [5] and [6], the results described above for the case of the Laplace operator have been indeed extended to the operator in (1.1) under some pseudo-convexity condition on $\partial\Omega$ which allows to manage the extra difficulties posed by the presence of *characteristic points*.

The basic idea in [5] and [6] is to look at the Kohn Laplacian Δ_{H^n} as a *sublaplacian* on $\mathbb{R}^{2n+1}$ endowed with the Heisenberg group action

$$\xi \circ \eta = \left(\xi_1 + \eta_1, \ldots, \xi_{2n} + \eta_{2n}, \xi_{2n+1} + \eta_{2n+1} + 2\sum_{i=1}^{n}(\xi_{i+n}\eta_i - \xi_i\eta_{i+n})\right).$$

By this we mean that Kohn Laplacian in (1.1) can be expressed as $\Delta_{H^n} = \sum_{i=1}^{2n} X_i^2$, with

$$(1.2)\qquad X_i = \frac{\partial}{\partial\xi_i} + 2\xi_{i+n}\frac{\partial}{\partial\xi_{2n+1}}, \qquad X_{i+n} = \frac{\partial}{\partial\xi_{i+n}} - 2\xi_i\frac{\partial}{\partial\xi_{2n+1}}$$

for $i = 1, \ldots, n$. This observation allows to exploit conveniently the scaling properties of the fields X_i and of the operator Δ_{H^n} with respect to the *anistropic dilations*

$$\delta_\lambda(\xi) = (\lambda\xi_1, \ldots, \lambda\xi_{2n}, \lambda^2\xi_{2n+1}) \ (\lambda > 0\ )$$

and the action of Δ_{H^n} on functions depending only on the *homogeneous norm*

$$(1.3)\qquad \rho(\xi) = \left(\left(\sum_{i=1}^{2n}\xi_i^2\right)^2 + \xi_{2n+1}^2\right)^{\frac{1}{4}}$$

Liouville theorems, a priori estimates and the existence of non trivial solutions in Hölder-Stein spaces for the Dirichlet problem

$$\Delta_{H^n} u + a(\xi)u^\alpha = 0 \ , \qquad u = 0 \text{ on } \partial\Omega$$

are therefore obtained in the above mentioned papers under a restriction on the exponent α depending on the *homogeneous dimension* $Q = 2n + 2$ of the Heisenberg group rather than on its linear dimension $N = 2n + 1$.

The ideas and methods outlined above for the case of Δ_{H^n} can be generalized to sublaplacians L of the form $L = \sum_{i=1}^{n_1} X_i^2$ where the first order differential operators X_i in the preceding generate the whole Lie algebra of left-invariant vectorfields on a nilpotent, stratified Lie group $(G, \circ)$, see Section 2 for a quick review of the basic notions and terminology.

In Section 3 of the present paper, which originates from the graduate dissertation of the second author [10], we propose some abstract results of Liouville type for operators L as above, both in the linear and the semilinear case. The final Section 4 is devoted to the study of the semilinear Liouville property for some second order degenerate elliptic operator which do not fit in the abstract setting of Section 2, the main example being the Grushin operator which is defined on $\mathbb{R}^N = \mathbb{R}^p \times \mathbb{R}^q$ by

$$\sum_{i=1}^{p} \frac{\partial^2}{\partial {x_i}^2} + |x|^{2k} \sum_{i=1}^{q} \frac{\partial^2}{\partial y_i^2}$$

where $k \in \mathbb{N}$ and $\xi = (x_1, \dots, x_p, y_1, \dots, y_q)$ is the typical point of $\mathbb{R}^N$.

Let us mention finally that different aspects of semilinear subelliptic problems have been investigated in [17] and, more recently, in [16], [3], [8], 15], [4], [25], [30]. Liouville type theorems for linear Fuchsian or weighted elliptic operators have been established in [28], [24], [11].

2. – Sublaplacians on stratified Lie groups

In this section we recall briefly a few notions which are relevant to the analysis on Lie groups and some fundamental properties of sublaplacians on stratified, nilpotent Lie groups. For more details, see. e.g. [21], [22].

2.1. – Stratified nilpotent Lie groups

Let $\mathcal{G}$ be a real finite dimensional Lie algebra, i. e. a vector space on $\mathbb{R}$ with a *Lie bracket* $[\cdot\ ,\cdot]$, that is a bilinear map from $\mathcal{G} \times \mathcal{G}$ into $\mathcal{G}$ which is alternating

$$[X, Y] = -[Y, X] \text{ for all } X, Y \in \mathcal{G} \tag{2.1}$$

and satisfies the *Jacobi identity*

$$(2.2)\qquad [X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0 \quad \text{for all} \quad X,\ Y,\ Z\ \in \mathcal{G}.$$

$\mathcal{G}$ is called *m-nilpotent and stratified* if it can be decomposed as a direct sum of subspaces satisfying

$$(2.3)\qquad \begin{gathered} \mathcal{G}=V_1\oplus V_2\cdots\oplus V_m \text{ with } \dim V_j=n_j \\ [V_1,V_j]\subset V_{j+1} \text{ if } 1\le j<m;\ [V_1,V_m]=\{0\}. \end{gathered}$$

Therefore, V_1 generates, by means of the Lie bracket $[\cdot,\cdot]$, $\mathcal{G}$ as a Lie algebra.

Let $(G,\circ)$ be the simply connected Lie group associated to the Lie algebra $\mathcal{G}=(G,[\cdot,\cdot])$ as follows:

$$G=\mathbb{R}^N \text{ with } N=\sum_{j=1}^{m} n_j\,,$$

equipped with the group action $\circ$ defined by the Campbell-Hausdorff formula, namely

$$(2.4)\qquad \eta\circ\xi=\eta+\xi+\frac{1}{2}[\eta,\xi]+\frac{1}{12}[\eta,[\eta,\xi]]+\frac{1}{12}[\xi,[\xi,\eta]]+\dots$$

(for the other terms see e.g. [26]). Note that, in view of the nilpotency of $\mathcal{G}$, in the right hand side there is only a finite sum of terms involving commutators of ξ and η of lenght less than m.
Observe that the group law (2.4) makes $G=\mathbb{R}^N$ a Lie group whose Lie algebra of left-invariant vectorfields $Lie(G)$ coincides with $\mathcal{G}$. Recall that the Lie algebra $Lie(G)$ is the algebra of left-invariant vectorfields Y which satisfy

$$Yf(\eta\circ\xi)=(Yf)(\eta\circ\xi)\,,$$

for every smooth function f, equipped with the bracket $[[X,Y]]=XY-YX$.

Let $e_1,\dots,e_{n_1}$ be the canonical basis of the subspace $\mathbb{R}^{n_1}$ of G; then as a basis of the Lie algebra $\mathcal{G}=Lie(G)$ we can choose the vectorfields $X_1,\dots,X_{n_1}$ defined for smooth f by

$$(2.5)\qquad X_i(f)(\xi)=\lim_{t\to 0+}\frac{f(\xi\circ te_i)-f(\xi)}{t},\quad \xi\in G.$$

Since V_1 generates $\mathcal{G}$ as a Lie algebra we can define recursively, for $j=1,\dots,m$, and $i=1,\dots,n_j$, a basis $\{X_{i,j}\}$ of V_j as

$$\begin{aligned} X_{i,1}&=X_i \qquad (i=1,\dots,n_1)\\ X_\alpha&=[X_{i_1},[X_{i_2},\dots,[X_{i_{j-1}},X_{i_j}]]\dots], \end{aligned}$$

with $\alpha = (i_1, \dots, i_j)$ multi-index of length j and $X_{i_k} \in \{X_1, \dots, X_{n_1}\}$.

In terms of the decomposition $G = \mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2} \oplus \cdots \oplus \mathbb{R}^{n_m}$ one defines then a one - parameter group of *dilations* δ_λ on G by setting for

$$\xi = \xi_1 + \xi_2 + \cdots + \xi_m, \quad (\xi_j \in \mathbb{R}^{n_j}),$$

$$\delta_\lambda(\xi) = \sum_{j=1}^{m} \lambda^j \xi_j. \tag{2.6}$$

Observe that, for any $\xi \in G$, the Jacobian of the map $\xi \longrightarrow \delta_\lambda(\xi)$ equals λ^Q, where

$$Q = \sum_{j=1}^{m} j\, n_j\,. \tag{2.7}$$

The integer Q is the *homogeneous dimension* of G. Observe that the linear dimension of G is $N = \sum_{j=1}^{m} n_j$; hence $Q \geq N$ and equality holds only in the trivial case $m = 1$ and $G = \mathbb{R}^{n_1}$.

Let us recall that a *dilation - homogeneous norm* on G is, by definition, a mapping $\xi \to \rho(\xi)$ from G to $\mathbb{R}^+$ such that:

$$\begin{array}{ll} \text{i)} & \xi \to \rho(\xi) \text{ is continuous on } G \text{ and smooth on } G \setminus \{0\} \\ \text{ii)} & \rho(\xi) = 0 \text{ if and only if } \xi = 0 \\ \text{iii)} & \rho(\xi) = \rho(-\xi) \\ \text{iv)} & \rho(\delta_\lambda(\xi)) = \lambda \rho(\xi) \quad \text{for each } \lambda > 0. \end{array} \tag{2.8}$$

All homogeneous norms on G are equivalent; moreover they satisfy the triangle inequality

$$\rho(\xi \circ \eta) \leq C_0(\rho(\xi) + \rho(\eta)) \quad \text{for all } \xi, \eta \in G$$

for some constant $C_0 \geq 1$. For a given homogeneous norm and positive real R, the *Koranyi ball* centered at 0 is the set

$$B(0, R) = \{\xi \in G : \rho(\xi) < R\}.$$

These balls form, for $R > 0$, a fundamental system of neighborhoods of the origin in $(G, \circ)$. Through the group law $\circ$ one defines then the distance between $\xi, \eta \in G$ by the position

$$d(\xi, \eta) = \rho(\eta^{-1} \circ \xi)\,,$$

where η^{-1} is the inverse of η with respect to $\circ$, i.e. $\eta^{-1} = -\eta$. The Koranyi ball of radius R centered at η is defined accordingly.

It is important to point out that the Lebesgue measure is invariant for the group action and that the volumes scale as R^Q.

More precisely, if $|E|$ denotes the N - dimensional Lebesgue measure (recall that $N = \sum_{j=1}^{m} n_j$), we have

$$|B(\eta, R)| = |B(0, R)| = |B(0, 1)| R^Q.$$

as a consequence of (2.7) and (2.8).

2.2. – Sublaplacians

Let us come back now to the vectorfields X_i $(i = 1, \dots, n_1)$ defined in (2.5). The first remark is that X_i are 1 - homogeneous with respect to the dilations δ_λ, i.e.

$$X_i f(\delta_\lambda(\xi)) = \lambda (X_i f)(\delta_\lambda \xi) \tag{2.9}$$

Indeed, from the definition (2.5) of X_i we have

$$X_i f(\delta_\lambda(\xi)) = \lim_{t \to 0+} \frac{f(\delta_\lambda \xi \circ \lambda t e_i) - f(\delta_\lambda \xi)}{t}$$

Setting $\tau = t\lambda$, the righ-hand side of the preceding is

$$\lambda \lim_{\tau \to 0+} \frac{f(\delta_\lambda \xi \circ \tau e_i) - f(\delta_\lambda \xi)}{\tau} = \lambda (X_i f)(\delta_\lambda(\xi))$$

In a similar way one can check that the vectorfields of V_j are homogenous of degree j, that is

$$X_{i,j} f(\delta_\lambda(\xi)) = \lambda^j (X_{i,j} f)(\delta_\lambda \xi) \ , \ \forall i = 1, \dots, n_j \, . \tag{2.10}$$

Let us make now some simple remarks on the representation of the vectorfields X_i as first order partial differential operators. If one chooses $(e_1, \dots, e_{n_1}, \dots, e_N)$ as the canonical basis of $G = \mathbb{R}^N$, then each X_i $(i = 1, \dots, n_1)$ can be expressed in terms of the partial derivatives $\frac{\partial}{\partial x_j}$ as

$$X_i = \sum_{j=1}^{N} \sigma_{ij}(x) \frac{\partial}{\partial x_j} \tag{2.11}$$

Here, $\sigma(x) = (\sigma_{ij}(x))$ is a $n_1 \times N$ matrix of the form

$$\sigma = (I_{\mathbb{R}^{n_1}}, \ Q_2(x) \ , \ \dots, \ Q_m(x)) \tag{2.12}$$

where $I_{\mathbb{R}^{n_1}}$ denotes the identity on $\mathbb{R}^{n_1}$ and $Q_j(x)$ are $n_1 \times n_j$ matrices $(j = 2, \dots, m)$. As a consequence of (2.3) one has

$$\frac{\partial}{\partial x_j} \sigma_{ij}(x) = 0 \quad \text{for } j = 1, \dots, N. \tag{2.13}$$

The *sublaplacian* L on the group G is defined then on smooth functions u by

$$Lu = \sum_{i=1}^{n_1} X_i^2 u \tag{2.14}$$

Observe that L is 2-homogeneous with respect to the dilations δ_λ since the X_i's are 1-homogeneous; moreover, L is left-invariant with respect to the group action $\circ$, since the X_i's are such.

In view of the preceding discussion, L is a second order partial differential operator; as a consequence of (2.13) it can be expressed in divergence form as

$$Lu = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial u}{\partial x_j}) \,, \tag{2.15}$$

where $A(x) = (a_{ij}(x)) = \sigma^T(x)\sigma(x)$ is a positive semidefinite $N \times N$ matrix. When $m = 1$ the sublaplacian L coincides with the Laplace operator

$$\Delta = \sum_{i,j=1}^{N} \frac{\partial^2}{\partial {x_j}^2}$$

On the other hand, as soon as $m \geq 2$, the matrix σ has a non trivial kernel. The sublaplacian L is therefore no more uniformly elliptic but only degenerate elliptic and, more precisely, a second order operator with non-negative characteristic form according to [27]. Nevertheless, the stratification condition implies that the fields X_i $(i = 1, \dots, n_1)$ satisfy the *Hörmander condition*

$$Lie(G) = \mathcal{G} \,. \tag{2.16}$$

As a consequence of (2.16), L is *subelliptic* (see [23]). Let us just mention here that this implies the validity of Bony's Maximum Principle (see [7]).
In the sequel we will use the notation $\nabla_L u = (X_1 u, \dots, X_{n_1} u)$.

Let us conclude this section by two basic examples.

2.3. – Examples

Example 1. Take $\mathcal{G} = \mathbb{R}^N$ with the trivial Lie bracket $[X, Y] = 0$ for all X, Y and stratification $V_1 = \mathbb{R}^N$, $V_2 = \{0\}$. The dilation and the homogeneous norm in this case are, of course, isotropic. They are given, respectively, by

$$\delta_\lambda(\xi) = (\lambda\xi_1, \dots, \lambda\xi_N)\,; \qquad \rho(\xi) = \left(\sum_{i=1}^{N} \xi_i^2\right)^{\frac{1}{2}} \,.$$

The homogeneous dimension is N, the fields X_i are the partial derivatives and the sublaplacian is the standard Laplacian Δ.

Example 2. Take $\mathcal{G} = \mathbb{R}^{2n+1}$ $(n \geq 1)$ with the Lie bracket $[X, Y] = XY - YX$ and the stratification $G = \mathbb{R}^{2n} \oplus \mathbb{R}^1$. The homogeneous dimension in this case is then $Q = 2n + 2$. The dilation and the homogeneous norm on G are, respectively,

$$\delta_\lambda(\xi) = (\lambda\xi_1, \dots, \lambda\xi_{2n}, \lambda^2\xi_{2n+1}); \qquad \rho(\xi) = \left(\left(\sum_{i=1}^{2n}\xi_i^2\right)^2 + \xi_{2n+1}^2\right)^{\frac{1}{4}}.$$

It is easy to check that the group action $\circ$ defined in (2.4) is

$$\eta \circ \xi = \left(\xi_1 + \eta_1, \dots, \xi_{2n} + \eta_{2n}, \xi_{2n+1} + \eta_{2n+1} + 2\sum_{i=1}^{n}(\xi_i\eta_{i+n} - \xi_{i+n}\eta_i)\right). \tag{2.17}$$

From this it follows that the fields X_i are given in this case by (1.2) and the sublaplacian associated with the Heisenberg group $H^n = (\mathbb{R}^{2n+1}, \circ)$ is therefore given by (1.1).

3. – The Liouville property for sublaplacians on nilpotent stratified groups

3.1. – The linear case

This section is devoted to the generalization to sublaplacians L of the well-known Liouville property valid for the Laplace operator. Indeed, we prove that bounded L-harmonic functions on stratified groups G are necessarily constant.

Let $L = \sum_{i=1}^{n_1} X_i^2$ be the sublaplacian on the stratified group $(G, \circ)$. A function u is L-harmonic on G if

$$u \in \Gamma^2(G), \quad Lu = 0 \text{ in } G$$

where $\Gamma^2(G)$ is the space of functions $u : G \to \mathbb{R}$ such that

$$u, X_i u \in L^\infty(G) \cap C(G)$$

and

$$\sup_{\xi,\eta} \frac{|X_i u(\eta \circ \xi) + X_i u(\eta \circ \xi^{-1}) - 2X_i u(\eta)|}{\rho(\xi)} < \infty$$

for $i = 1, \dots, n_1$.

The basic tool in our proof of the linear Liouville theorem is the following *mean value* property for L-harmonic functions:

$$v(\xi) = \frac{C_Q}{R^Q}\int_{B_L(\xi,R)} v(\eta)|\nabla_L d_L(\xi,\eta)|^2 d\eta, \tag{3.1}$$

where $d_L(\xi, \eta) := \rho_L(\xi^{-1} \circ \eta)$, C_Q is a suitable constant and $B_L(\xi, R)$ denotes the Koranyi ball associated to an appropriate $C^\infty(G \setminus \{0\})$ homogeneous norm $\rho_L(\cdot)$. Note that

$$\rho_L^2(\xi^{-1} \circ \eta) \approx \Gamma^Q(\xi, \eta)$$

where Γ is the fundamental solution of L (see [12], [14], [18]).

THEOREM 3.1. *Let $L = \sum_{i=1}^{n_1} X_i^2$ be the sublaplacian on the nilpotent stratified group G. If u is L-harmonic on G, then u is a constant.*

PROOF. As a consequence of (2.3) the vectorfields $X_{i,m}$ commute with X_i for $i = 1, \dots, n_1$. Hence the sublaplacian L satisfies:

$$X_{i,m} L u = X_{i,m} \sum_{j=1}^{n_1} X_j^2 u = \sum_{j=1}^{n_1} X_j^2 X_{i,m} u.$$

Consequently, if u is L-harmonic the same is true for $X_{i,m} u$ $(i = 1, \dots n_m)$. Therefore, by the mean value formula (3.1) applied to $X_{i,m} u$ we get

$$X_{i,m} u = \frac{C_Q}{R^Q} \int_{B_L(\xi,R)} X_{i,m} u(\eta) |\nabla_L d_L|^2 d\eta \tag{3.2}$$

Integrating by parts the right-hand side of (3.2), we obtain:

$$\begin{aligned} X_{i,m} u = &-\frac{C_Q}{R^Q} \int_{B_L(\xi,R)} u(\eta) X_{i,m} \left(|\nabla_L d_L|^2 \right) d\eta \\ &+ \frac{C_Q}{R^Q} \int_{\partial B_L(\xi,R)} u |\nabla_L d_L|^2 \frac{X_{i,m} d_L}{|\nabla d_L|} d\Sigma. \end{aligned}$$

Here ∇ denotes the usual gradient; observe also that $\nu = \frac{\nabla d_L}{|\nabla d_L|}$ is the normal vector to ∂B_L.

Since the X_i are 1-homogeneous with respect to the intrinsic dilation, see (2.9), and are left-invariant with respect to the group action $\circ$, it follows that $X_{i,m}$ is homogeneous of degree m with respect to δ_λ and left-invariant with respect to $\circ$. The previous remark, together with the fact that d_L is homogenous of degree 1 with respect to δ_λ, provide the following estimates:

$$|X_{i,m}(d_L)| \le \frac{C}{d_L^{m-1}}, \qquad |X_{i,m} \nabla_L d_L| \le \frac{C}{d_L^m}. \tag{3.3}$$

Indeed, for the first estimate in (3.3) observe that

$$X_{i,m}(d_L(\eta)) = \frac{X_{i,m}\left(d_L \left(\frac{\eta}{\rho_L(\eta)} \right) \right)}{\rho_L^{m-1}(\eta)}$$

and that $|X_{i,m}(d_L(\frac{\eta}{\rho_L(\eta)}))|$ is bounded since d_L is C^∞ on $\partial B_L(0,1)$. The second estimate is achieved by using the same argument and the 1-homogeneity of ∇_L. Moreover, $X_{i,m}(|\nabla_L d_L|^2) = 2 X_{i,m} \nabla_L d_L \cdot \nabla_L d_L$.

Hence,

$$|X_{i,m} u(\xi)| \le \frac{C_Q}{R^{Q+m}} ||u||_{L^\infty} R^Q + \frac{C_Q}{R^{Q+m-1}} ||u||_{L^\infty} R^{Q-1} \le \frac{C_Q}{R^m} ||u||_{L^\infty}$$

for every $\xi \in G$.

Therefore, letting $R \to \infty$, one deduces that

$$X_{i,m}u \equiv 0 \quad \text{in } G \quad \text{for every } i = 1, \dots, n_m. \tag{3.4}$$

Now, from the stratification of G it follows that $X_{i,m-1}u$ is also L-harmonic. Indeed, for $k = 1, \dots n_1$,

$$[X_{i,m-1}u, X_k u] = X_\alpha u \quad \text{for some } \alpha \text{ with } |\alpha| = m\,. \tag{3.5}$$

Thus, being $X_{i,m}$ a basis of V_m, (3.4) yields $X_\alpha u \equiv 0$ in G. Repeating the same argument and using the fact that the vectorfields $X_{i,j}$ form a basis of V_j and are j-homogeneous with respect to δ_λ, see (2.10), one finally obtains that

$$X_{i,j}u \equiv 0 \quad \text{for } j = 1, \dots, m,\ i = 1, \dots, n_j.$$

Consequently, from the Hörmander condition $span\ (X_{i,j}) = Lie(X_i) = \mathcal{G}$, we deduce that $\nabla u = 0$ in G and the claim is proved. □

3.2. – The semilinear case

In this section we prove a Liouville theorem for nonnegative solutions of semilinear equations associated to sublaplacians L on stratified groups G.
The proof, which is inspired from [2], relies in particular on the behaviour of the operator L defined in (2.14) on functions which are radial with respect to the homogeneous norm $\rho_L(\cdot)$, see Section 3.1. From now we shall write, for simplicity, $\rho_L = \rho$.
One can easily check by a direct computation using (2.5) that the following holds for smooth $f : \mathbb{R} \to \mathbb{R}$ and $\rho \neq 0$

$$Lf(\rho) = f''(\rho)|\nabla_L \rho|^2 + f'(\rho)L\rho\,. \tag{3.6}$$

As recalled in the previous section, $\rho^{2-Q} \approx \Gamma$, where Γ is the fundamental solution of L. Therefore, using (3.6) with $f(\rho) = \rho^{2-Q}$, one finds that

$$0 = L\rho^{2-Q} = (2-Q)(1-Q)\rho^{-Q}|\nabla_L\rho|^2 + (2-Q)\rho^{1-Q}L\rho$$

for $\rho \neq 0$. Hence,

$$L\rho = (Q-1)\rho^{-1}|\nabla_L\rho|^2$$

yielding to the following radial expression of L:

$$Lf(\rho) = |\nabla_L\rho|^2[f''(\rho) + f'(\rho)\frac{Q-1}{\rho}]\,.$$

Let us observe that $\nabla_L\rho$ is homogeneous of degree zero and therefore is bounded in G; the same is true for $\rho L\rho$. In the sequel we will use the notation $\psi(\rho) = |\nabla_L\rho|^2$.

THEOREM 3.2. *Suppose that* $u \in \Gamma^2_{loc}(G)$ *satisfies*

$$u \geq 0, \quad Lu(\xi) + k(\xi)u^\alpha \leq 0 \quad \text{in } G \tag{3.7}$$

where k is a continuous nonnegative function such that

$$k(\xi) \geq K\rho^\gamma(\xi)$$

for sufficiently large $\rho(\xi)$ *and for some* $K > 0$, $\gamma > -2$. *Then* $u \equiv 0$, *provided* $1 < \alpha \leq \frac{Q+\gamma}{Q-2}$.

PROOF. For each $R > 0$ consider a cut-off function ϕ_R such that

$$\begin{cases} \phi_R(\rho) := \phi(\frac{\rho}{R}) & \text{with } \phi \in C^\infty[0, +\infty) \\ 0 \leq \phi \leq 1, \quad \phi \equiv 1 \quad \text{in } \left[0, \frac{1}{2}\right], \quad \phi \equiv 0 & \text{in } [1, +\infty), \\ -\frac{C}{R} \leq \frac{\partial \phi_R}{\partial \rho} \leq 0, \left|\frac{\partial^2 \phi_R}{\partial \rho^2}\right| \leq \frac{C}{R^2} & \text{for some constant } C > 0. \end{cases} \tag{3.8}$$

Set then

$$I_R := \int_G k(\xi)u^\alpha \phi_R{}^\beta d\xi = \int_{B_L(0,R)} k(\xi)u^\alpha \phi_R^\beta d\xi \tag{3.9}$$

where $\frac{1}{\beta} = 1 - \frac{1}{\alpha}$. Observe that $I_R \geq 0$ and that (3.7) implies

$$I_R \leq -\int_{B_L(0,R)} Lu\phi_R^\beta \, d\xi$$

Therefore, an integration by parts yields:

$$\begin{aligned} I_R \leq & -\int_{B_L(0,R)} uL(\phi_R^\beta)d\xi + \int_{\partial B_L(0,R)} u\nabla_L(\phi_R^\beta) \cdot \nu_L d\Sigma \\ & - \int_{\partial B_L(0,R)} \phi_R^\beta \nabla_L u \cdot \nu_L d\Sigma \leq -\int_{B_L(0,R)} uL(\phi_R^\beta)d\xi \\ & + \int_{\partial B_L(0,R)} u\beta\phi_R^{\beta-1}\phi_R' \nabla_L \rho \cdot \nu_L d\Sigma \leq -\int_{B_L(0,R)} uL(\phi_R^\beta)d\xi, \end{aligned} \tag{3.10}$$

where $\nu_L(\xi) = \sigma(\xi)\nu(\xi)$, ν being the exterior normal to ∂B_L, see (2.12), and $d\Sigma$ denotes the $(N-1)$ - dimensional Hausdorff measure.

On the other hand, (3.6) implies

$$L\phi_R^\beta = \psi \frac{\partial^2}{\partial \rho^2}\phi_R^\beta + \rho L\rho \frac{\partial}{\partial \rho}\phi_R^\beta$$

Thus, by the assumptions made on ϕ_R and taking (3.10) into account, we find, for $\Sigma_R := B_L(0,R) \setminus B_L(0,\frac{R}{2})$,

$$\begin{aligned} I_R &\leq -\int_{\Sigma_R} u[\beta\phi_R^{\beta-1}\phi_R''\psi + \beta\phi_R^{\beta-1}\phi_R' L\rho]d\xi \\ &\leq \frac{C}{R^2}\int_{\Sigma_R} u\phi_R^{\beta-1}d\xi \end{aligned}$$

since ψ and $\rho L\rho$ are bounded. Then, by Hölder inequality,

$$I_R \leq \frac{C}{R^2}\left[\int_{\Sigma_R} u^\alpha \rho^\gamma \phi_R^{(\beta-1)\alpha} d\xi\right]^{\frac{1}{\alpha}} \left[\int_{\Sigma_R} \rho^{\frac{-\gamma\beta}{\alpha}} d\xi\right]^{\frac{1}{\beta}}.$$

Choosing $R > 0$ sufficiently large so that $k \geq K\rho^\gamma$ in Σ_R, we obtain

$$I_R \leq C\left[\int_{\Sigma_R} u^\alpha k \phi_R^\beta \, d\xi\right]^{\frac{1}{\alpha}} R^{(\frac{-\gamma}{\alpha}+\frac{Q}{\beta}-2)}. \tag{3.11}$$

Therefore, for large R,

$$I_R^{1-\frac{1}{\alpha}} \leq C\, R^{(\frac{-\gamma}{\alpha}+\frac{Q}{\beta}-2)}. \tag{3.12}$$

Letting $R \to \infty$ in the above we conclude that, if $1 < \alpha < \frac{Q+\gamma}{Q-2}$, then

$$I := \lim_{R\to\infty} I_R = \int_G k u^\alpha d\xi = 0.$$

This implies $u \equiv 0$ outside a large ball since k is strictly positive there. Choose now $\overline{R} > 0$ in such a way that $k > 0$ for $\rho \geq \overline{R}$. Then, as proved above, $u \equiv 0$ on $G \setminus B_L(0, \overline{R})$.

Hence u satisfies:

$$\begin{cases} u \geq 0 & \text{in } B_L(0, \overline{R}+\delta) \\ Lu \leq 0 & \text{in } B_L(0, \overline{R}+\delta) \\ u \equiv 0 & \text{for } \overline{R} \leq \rho \leq \overline{R}+\delta \end{cases}$$

for some $\delta > 0$. Therefore, by the Bony's Maximum Principle, see [7], u has to be identically zero on G since u is not strictly positive in view of the last condition in (3.13).

Consider now the case $\alpha = \frac{Q+\gamma}{Q-2}$. In this case, from (3.12) we deduce that I_R is uniformly bounded with respect to R. Moreover, since $R \to I_R$ is increasing the integral on the right - hand side of (3.11), which coincides with $I_R - I_{\frac{R}{2}}$, goes to zero as R tends to infinity. This implies $I = 0$ and we conclude as before. □

REMARK 3.1. The claim of Theorem 3.2 holds under the less restrictive assumption that, for some $K > 0$ and $\gamma > -2$, $k(\xi) \geq K\psi\rho^{\gamma}(\xi)$ for sufficiently large $\rho(\xi)$. The proof is similar but one has to take into account that $\rho L\rho = (Q-1)\psi$ and also that ψ vanishes, by its very definition, on the characteristics points of the Koranyi's ball which are, by the way, a set of N-dimensional measure equal to zero (see [13]).

REMARK 3.2. The exponent $\frac{Q+\gamma}{Q-2}$ in Theorem 3.2 is optimal. To see this, observe first that in view of (3.6) the function $u(\rho) = (1+\rho^2)^{-\frac{p}{2}}$ satisfies

$$Lu + p(Q-p-2)\psi(1+\rho^2)^{-(\frac{p}{2}+1)} \leq 0\,.$$

Thus, were $\alpha > \frac{Q+\gamma}{Q-2}$, one could choose p such that

$$Q-2 > p\,, \qquad \alpha\frac{p}{2} - \frac{\gamma}{2} \geq \frac{p}{2} + 1\,.$$

Therefore, setting $v = Cu$ with $C = (p(Q-p-2))^{\frac{1}{\alpha-1}}$, one obtains easily that

$$-Lv \geq \psi\,(p(Q-p-2))^{\frac{\alpha}{\alpha-1}}\,(1+\rho^2)^{-\alpha\frac{p}{2}+\frac{\gamma}{2}} \geq \psi\rho^{\gamma}v^{\alpha}\,.$$

4. – Other Liouville type results

Here we prove some semilinear Liouville type results like those of the previous section for some degenerate elliptic second order operators of the form

$$L = \sum_{i=1}^{N} X_i^2 \tag{4.1}$$

which are 2-homogeneous with respect to a family of dilations but do not fit in the setting of Section 3 since they are not left-invariant with respect to any group action on $\mathbb{R}^N$.

The first example we consider is the Grushin operator L defined on $\mathbb{R}^N = \mathbb{R}^p \times \mathbb{R}^q$ by

$$L = \sum_{i=1}^{p} \frac{\partial^2}{\partial {x_i}^2} + |x|^{2k} \sum_{i=1}^{q} \frac{\partial^2}{\partial y_i^2} \tag{4.2}$$

where $k \in \mathbb{N}$ and $(x,y) = (x_1,\ldots,x_p,y_1,\ldots,y_q)$ denotes the typical point of $\mathbb{R}^N$.

This operator may be written in the form (4.1) by choosing

$$X_i = \frac{\partial}{\partial x_i} \qquad \text{for } i = 1, \dots, p$$
$$X_{i+p} = |x|^k \frac{\partial}{\partial y_i} \quad \text{for } i = 1, \dots, q.$$

It is easy to check that L satisfies the Hörmander condition (2.16) since the X_i generate $\mathbb{R}^N$ by commutators of lenght $\leq k$. It is also easy to realize that the Lie algebra generated by X_i for $k > 1$ has no constant dimension.
However, for the dilation

$$\delta_\lambda \xi = (\lambda x, \lambda^{k+1} y) \tag{4.3}$$

we have

$$X_i u(\delta_\lambda \xi) = \lambda (X_i u)(\delta_\lambda \xi) \qquad (i = 1, \dots, p\ )$$
$$X_{i+p} u(\delta_\lambda \xi) = |x|^k \lambda^{k+1} \frac{\partial u}{\partial y_i} = \lambda (X_{i+p} u)(\delta_\lambda \xi) \quad (i = 1, \dots, q)\,.$$

Hence, L is 2-homogeneous with respect to (4.3). Moreover, the norm

$$\rho(\xi) = |x| + |y|^{\frac{1}{k+1}}\,, \tag{4.4}$$

where $\xi = (x, y)$ and $|\cdot|$ denotes the euclidean norm on $\mathbb{R}^N$, is 1-homogeneous with respect to the dilation in (4.3).
It follows that N-dimensional measure of the ball

$$\Omega_R = B_p(0, R) \times B_q(0, R^{k+1})$$

associated with (4.4) (here B_p denotes the euclidean ball of $\mathbb{R}^p$) is proportional to R^Q, with

$$Q = p + (k+1)q = N + kq\,.$$

4.1. *Let u be a solution of*

$$u \geq 0\ , \quad \sum_{i=1}^{p} \frac{\partial^2 u}{\partial {x_i}^2} + |x|^{2k} \sum_{i=1}^{q} \frac{\partial^2 u}{\partial y_i^2} + u^\alpha \leq 0 \quad \textit{in } \mathbb{R}^N\,. \tag{4.5}$$

Then $u \equiv 0$, provided that $k > 1$ and $1 < \alpha \leq \frac{Q}{Q-2}$.

PROOF OF THEOREM 4.1. Set $\Omega_R := B_p(0, R) \times B_q(0, R^{k+1})$. Let φ_R and θ_R be the cut-off functions satisfying, for some constant $C > 0$,

$$(4.6)\quad \begin{cases} \varphi_R(r) := \varphi(\frac{r}{R});\ \theta_R(s) := \theta(\frac{s}{R^{k+1}}) & \text{with } \varphi, \theta \in C^\infty[0, +\infty), \\ 0 \le \varphi \le 1,\ 0 \le \theta \le 1,\ \varphi = \theta \equiv 1 & \text{in } [0, \frac{1}{2}],\ \varphi \equiv \theta \equiv 0 \text{ in } [1, +\infty), \\ -\frac{C}{R} \le \frac{\partial \varphi_R}{\partial r}' \le 0,\ -\frac{C}{R^{k+1}} \le \frac{\partial \theta_R}{\partial s} \le 0, & \left|\frac{\partial^2 \varphi_R}{\partial r^2}\right| \le \frac{C}{R^2}, \left|\frac{\partial^2 \theta_R}{\partial s^2}\right| \le \frac{C}{R^{2(k+1)}}, \end{cases}$$

where $r = |x|$ and $s = |y|$. Let us set then, for $\frac{1}{\beta} = 1 - \frac{1}{\alpha}$,

$$(4.7)\qquad I_R := \int_{\mathbb{R}^N} u^\alpha (\theta_R \varphi_R)^\beta d\xi .$$

From (4.5) we obtain

$$(4.8)\ 0 \le I_R = -\int_{\Omega_R} uL[(\theta_R\varphi_R)^\beta] dxdy + \int_{\partial\Omega_R} u\beta(\theta_R\varphi_R)^{\beta-1} \nabla_L(\theta_R\varphi_R) \cdot \nu_L d\Sigma,$$

where $\nabla_L u = (X_1 u, \dots, X_N u)$ and $\nu_L = (\nu_1, \dots, \nu_p, |x|^k \nu_{p+1}, \dots, |x|^k \nu_N)$, ν being the exterior normal to $\partial\Omega_R$.
On the other hand, simple computations show that

$$\begin{aligned} L\varphi_R &= \Delta_p \varphi_R) & L\theta_R &= |x|^{2k} \Delta_q \theta_R , \\ \nabla_L \varphi_R &= (\nabla_x \varphi_R, 0) & \nabla_L \theta_R &= (0, |x|^k \nabla_y \theta_R) \end{aligned}$$

where Δ_p, Δ_q denote the Laplacians on $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively.
The integral on the boundary in (4.8) vanishes since $\nabla_L \varphi_R \cdot \nabla_L \theta_R = \theta_R \varphi_R = 0$ on $\partial\Omega_R$ and $\beta > 1$. Therefore, by the properties of φ_R and θ_R and setting $\Sigma_R := \Omega_R \setminus (B_p(0, \frac{R}{2}) \times B_q(0, \frac{R^{k+1}}{2}))$, we obtain

$$\begin{aligned} I_R &\le -\int_{\Sigma_R} u\beta \left[\varphi_R^\beta \theta_R^{\beta-1} |x|^{2k} \left(\theta_R'' + \frac{q-1}{s}\theta_R'\right) + \varphi_R^{\beta-1}\theta_R^\beta \left(\varphi_R'' + \frac{p-1}{r}\varphi_R'\right)\right] d\xi \\ &\le \frac{C}{R^2} \int_{\Sigma_R} u\beta \varphi_R^\beta \theta_R^{\beta-1} d\xi. \end{aligned}$$

By the Hölder inequality then

$$(4.9)\qquad I_R \le \frac{C}{R^2} \left[\int_{\Sigma_R} u^\alpha (\varphi_R\theta_R)^\beta dxdy\right]^{\frac{1}{\alpha}} \left[\int_{\Sigma_R} dxdy\right]^{\frac{1}{\beta}}$$

yielding

$$I_R^{\frac{1}{\beta}} \le C R^{Q-2-\frac{Q}{\alpha}}.$$

At this point the claim follows by the same arguments as in the proof of Theorem 3.2. □

The next result concerns the k-dimensional $(1 < k < N)$ Laplace operator on $\mathbb{R}^N$, that is

$$\Delta_k = \sum_{i=1}^{k} \frac{\partial^2}{\partial x_i^2} .$$

This example shows that subellipticity is not necessary to obtain semilinear Liouville type results. The main ingredients in the proof are again the 2-homogeneity of the operator with respect to a suitable family of dilations and that the balls associated with an appropriately defined distance invade the whole space as the radius diverges.

The result is as follows:

THEOREM 4.2. *Let $u \in C^2$ be a solution of*

$$u \geq 0, \quad \Delta_k u + u^\alpha \leq 0 \quad in\, \mathbb{R}^N . \tag{4.10}$$

If $k > 2$ and $1 < \alpha < \frac{k}{k-2}$, then $u \equiv 0$. The same conclusion holds if $k = 2$ and $\alpha > 1$.

PROOF OF THEOREM 4.2. The proof is similar to that of Theorem 4.1. The first observation is that, for every $\epsilon > 0$, the operator Δ_k is 2-homogeneous with respect to the following dilations:

$$\delta_\lambda(\xi) = \delta_\lambda(x_1, \dots, x_N) = (\lambda x_1, \dots, \lambda x_k, \lambda^\epsilon x_{k+1}, \dots, \lambda^\epsilon x_N) \tag{4.11}$$

since Δ_k does not act on the variables x_j for $j = k+1, \dots, N$.
As in the proof of Theorem 4.1 one considers then the sets

$$B_k(0, R) \times B_{N-k}(0, R^\epsilon)$$

where B_j denotes the j-dimensional euclidean ball. Set $\xi = (x, y)$ with $x = (x_1, \dots, x_k)$ and $y = (x_{k+1}, \dots x_N)$ and consider the same cut-off functions φ_R, θ_R defined in (4.6) with $k+1$ replaced by ϵ.
Proceeding as in the proof of Theorem 4.1 one shows then that the integral

$$I_R := \int_{\mathbb{R}^N} u^\alpha \theta_R {\varphi_R}^\beta d\xi \,, \quad \left(\frac{1}{\alpha} + \frac{1}{\beta} = 1\right) ,$$

satisfies

$$I_R^{\frac{1}{\beta}} \leq C R^{\epsilon(N-k)+k-2-\frac{k}{\alpha}} . \tag{4.12}$$

Let $k > 2$. By assumption, $\alpha < \frac{k}{k-2}$; hence one can choose $\epsilon > 0$ so small that $\alpha < \frac{k}{k-2+\epsilon(N-k)}$. Thus (4.12) implies that I_R goes to zero as $R \to \infty$.

In the case $k = 2$, for every $\alpha > 1$ there exists $\epsilon > 0$ such that $\alpha < \frac{2}{\epsilon(N-2)}$ and we conclude again from (4.12) that $I_R \to 0$ as $R \to \infty$.
On the other hand, $B_{N-k}(0, R^\epsilon) \times B_k(0, R)$ invade $\mathbb{R}^N$ as R goes to infinity. Hence

$$I_R \to \int_{\mathbb{R}^N} u^\alpha d\xi$$

as $R \to \infty$ and this implies $u \equiv 0$. □

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