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STOCHASTIC PERRON'S METHOD FOR HAMILTON–JACOBI–BELLMAN EQUATIONS*

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Abstract. We show that the value function of a stochastic control problem is the unique solution of the associated Hamilton–Jacobi–Bellman equation, completely avoiding the proof of the so-called dynamic programming principle (DPP). Using the stochastic Perron's method we construct a supersolution lying below the value function and a subsolution dominating it. A comparison argument easily closes the proof. The program has the precise meaning of verification for viscosity solutions, obtaining the DPP as a conclusion. It also immediately follows that the weak and strong formulations of the stochastic control problem have the same value. Using this method we also capture the possible face-lifting phenomenon in a straightforward manner.

Key words. stochastic Perron's method, viscosity solutions, nonsmooth verification, comparison principle

AMS subject classifications. Primary, 49L20, 49L25, 60G46; Secondary, 60H30, 35Q93, 35D40

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1. Introduction.

The stochastic Perron's method was introduced in [2] for linear problems and adapted in [1] to free-boundary problems associated to Dynkin games. In the present paper, we carry out a similar line of ideas, but with significantly different technicalities, for the most important case of nonlinear problems, namely, that of Hamilton–Jacobi–Bellman equations (HJB) in stochastic control. The result presented here represents the original motivation to introduce the stochastic version of Perron's method.

The goal of the paper is

(1) to prove the general result stating that "the value function of a control problem is the unique viscosity of the associated HJB,"
(2) *without* having to first go through the proof of the dynamic programming principle (DPP) but actually obtaining it as a by-product,
(3) in as much an *elementary* manner as possible.

The motivation for such a goal is described in detail in [2] and [1]. To summarize, the program described by (1) and (2) (and, implicitly, (3)) amounts to a verification result for viscosity solutions of HJBs. Overall, we believe this to be a genuinely novel approach to stochastic control that provides a deeper understanding of the relation between controlled diffusions and (viscosity) solutions of HJBs.

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In addition to being a new method to treat a fundamental class of problems, we believe the program carried out here has two notable features, which basically amount to achieving our goals above:

1. This is a direct/verification approach to dynamic programming (similar to [19] or [20]) in that it *first* finds/constructs a solution to the HJB, *then* shows that such solution is the value function, avoiding altogether the DPP. However, this is technically very different from the verification approach in [19] or [20] and can be viewed as a probabilistic counterpart to the classical approach.
2. We believe it, indeed, to be more elementary than either going through the probabilistic proof of the DPP (which is often incomplete, as described in the recent paper [5], where some important details are fixed) and then having to prove comparison of viscosity solutions anyway or through the analytical techniques in [19] and [20]. In particular, there is no need for us to use "conditional controls" or canonical spaces, usually needed in the proof of the DPP. These arguments are still needed even in the recent proof of Bouchard and Touzi of a weaker version of the DPP [3]. While measurable selection arguments are circumvented there through the weaker formulation, the Markov arguments mentioned above are still present. Our method consists only in applying Itô's formula along the smooth test functions for viscosity solutions, plus an elementary stopping argument. In addition, arguments of the same type as we use here (maybe even more complicated) have to be used *anyway* when one uses the weak DPP to prove that the value function is a viscosity solution. Also, we avoid the technicalities related to approximation by convolution and the approximation of the state equation in [19] and [20].

We present here a fresh look at a classic problem, so some comments on the existing literature are needed. We mention briefly only those works that are most closely related, at the risk of omitting relevant but further ideas. We first start with some important work in stochastic control, which, in the same spirit as our paper, avoids the DPP.

Since our result amounts to verification without smoothness, it is conceptually closest to [19] and [20]. Using approximation by convolution of viscosity semisolutions in the deterministic case [19] and then also approximating the state equation by nondegenerate diffusions in the stochastic framework of two-player games [20], the author performs a verification argument arriving at similar conclusions (in different situations, though). The probability space needs to accommodate an additional Brownian motion in the stochastic case, and, as mentioned above, the technicalities are very different and more involved, compared to our approach. Overall, the two approaches have little, if anything, in common.

At a formal level, one of our main results, Theorem 4.1, looks very much like the main result in the seminal work of Fleming and Vermes [13] and [12] (see also Remark 5.5).[1] More precisely, while the authors in [13] and [12] show that the value function is the infimum of classical supersolutions, we show that the value function is below the infimum of stochastic supersolutions, which is a viscosity subsolution. While appearing stronger than our Theorem 4.1 (considered by itself, without the other main result, Theorem 3.1), the notable result in Fleming–Vermes has two features:

(1) It contains a sophisticated approximation/separation argument used on top of restating the optimization problem as an infinite dimensional convex program.

[1]We would like to thank Ioannis Karatzas and Mete Soner for pointing out the closely related work of Fleming and Vermes.

(2) It still uses the very definition of the value function.
(3) By itself, is not enough to show the value function is a viscosity subsolution. Even if one does not mind the complicated approximation arguments, our Theorem 4.1 is still needed on top of the very strong results in [13] and [12] to get such a conclusion. Even combining Fleming–Vermes with the Perron's method in Ishii [14] would not yield this: the infimum over viscosity supersolutions may go below the value function, unless we know a a priori that the value function is a viscosity subsolution, and we also have a comparison result (needed even for the viscosity version of Perron in [14]). A subapproximation counterpart to the work of Fleming and Vermes could close the argument, but this would still have a very different flavor than our work, since it uses, once again, the representation of the value function. Actually, the recent papers [9, 10, 11] carry along these lines for path-dependent HJBs.

If one attempts to use only the Perron's method in Ishii [14] to construct viscosity solutions, the same obvious obstacle described in relation to Fleming–Vermes arises: without additional knowledge on the properties of value function, it does not compare with the output of Perron's method.

It should also be mentioned how our result compares to other existing results about verification for viscosity solutions of HJBs, namely, [21]. The result in [21] starts from the fact that the value function is the unique viscosity solution and, using this piece of information, synthesizes the optimal control (if one exists) in terms of the generalized derivatives of the viscosity solution/value function. Our result plays a role *before* the synthesis described in [21] and proves exactly that the value function is the unique viscosity solution, *without* resorting to the use of DPP. In other words, our work addresses a different question than the one addressed in [21] (but quite similar to that of [19] and [20]).

The rest of the paper is organized as follows. In section 2, we present the basic setup of the stochastic control problem and introduce the related HJB and the terminal condition. Moreover, we state our assumptions on the Hamiltonian. In section 3, we consider the strong formulation of the stochastic control problem and introduce the class of stochastic subsolutions via which we construct a lower bound on the value function which is a viscosity supersolution. In section 4, we introduce the weak formulation of the stochastic control problem and introduce the class of stochastic supersolutions, using which we construct a viscosity subsolution to the HJB equation. Finally, in section 5, we verify that both value functions, in the weak and the strong formulation, equal the unique viscosity solution using comparison.

2. Setup. Let U be a closed subset of $\mathbb{R}^k$ (the control space) and $\mathcal{O}$ an open subset of $\mathbb{R}^d$ (the state space). Let $b : [0,T]\times\mathcal{O}\times U \to \mathbb{R}^d$ and $\sigma : [0,T]\times\mathcal{O}\times U \to \mathbb{M}_{d,d'}$ be two measurable functions. We consider the controlled diffusion

$$dX_t = b(t,X_t,u_t)dt + \sigma(t,X_t,u_t)dW_t, \quad X \in \mathcal{O}. \tag{2.1}$$

We assume that the state lives in the open domain $\mathcal{O} \subset \mathbb{R}^d$, to include the treatment of utility maximization models for positive wealth, which is popular in mathematical finance. Given a measurable function $g : \mathcal{O} \to \mathbb{R}$, our goal is to maximize the expected payoff received at a fixed time horizon $T > 0$ using predictable processes u taking values in U. Informally, we want to study the optimization problem

$$\sup_u \mathbb{E}[g(X_T^u)], \quad X_0 = x \in \mathcal{O}.$$

Remark 2.1. We choose only to maximize terminal payoffs, just to keep the notation simpler. In the literature, this is known as the Mayer formulation of stochastic control problems. The Bolza problem, which contains a running payoff as well, can be treated in an identical manner with some additional notation.

One associates the following Hamiltonian to this problem:

$$H(t,x,p,M) := \sup_{u\in U}\Big[b(x,u)\cdot p + \frac{1}{2}Tr(\sigma(x,u)\sigma(x,u)^T M)\Big],\quad 0\le t\le T,\ x\in\mathcal{O}.$$

We make the following assumption on the Hamiltonian.

Assumption 2.1. Let us denote the domain of H by

$$\mathrm{dom}(H) := \{(t,x,p,M)\in[0,T]\times\mathcal{O}\times\mathbb{R}^d\times\mathcal{S}_d : H(t,x,p,M)<\infty\}.$$

We will assume that H is continuous in the interior of $\mathrm{dom}(H)$. Moreover, we will assume that there exists a continuous function $G:[0,T]\times\mathcal{O}\times\mathbb{R}^d\times\mathcal{M}_d\to\mathbb{R}$ such that

(1) $H(t,x,p,M)<\infty \implies G(t,x,p,M)\ge 0$,

(2) $G(t,x,p,M)>0 \implies H(t,x,p,M)<\infty$.

Remark 2.2. Our assumption above on the Hamiltonian H differs from that of [16], which assumes that the domain of H is closed. This latter assumption is well suited for analyzing superreplication problems with volatility uncertainty but excludes the utility maximization problems. For example, our assumption works out well for utility maximization problems, where $\mathcal{O}=(0,\infty)$ and $G(t,x,p,M)=-M$. Of course, one may ask why not simply choose $G=e^{-H}$? This is because, in general, H is not jointly continuous everywhere as an extended-value function. For example, in the case of one-dimensional utility maximization, where $H(t,x,p,M)=-p/2M^2$ for $M<0$, one can see that the Hamiltonian is not continuous at $(p,M)=(0,0)$, even if we view it as extended-valued. If H is continuous everywhere, as an extended-valued mapping, then we can, indeed, choose $G=e^{-H}$. However, this is usually not the case.

Using the stochastic Perron's method, our goal is to show that when a comparison principle is satisfied, the value function is, immediately, the unique viscosity solution of

$$\min\{-v_t(t,x)-H(t,x,v_x(t,x),v_{xx}(t,x)),G(t,x,v_x(t,x),v_{xx}(t,x))\}=0 \tag{2.2}$$

for $(t,x)\in[0,T)\times\mathcal{O}$, with the terminal condition

$$\min[v(T,x)-g(x),G(T,x,v_x(T,x),v_{xx}(T,x))]=0\quad\text{on}\quad\mathcal{O}, \tag{2.3}$$

without having to prove the DPP.

Remark 2.3. One may question why we do not impose any kind of boundary conditions on $\partial\mathcal{O}$. This is because, as we can see from the assumptions below, we choose $\mathcal{O}$ as a natural domain, so that the controlled state process X never makes it to the boundary.

3. Stochastic subsolutions.

In this section we will consider the so-called strong formulation of the stochastic control problem.

The main goal of the paper is to outline how the stochastic Perron's method in [2] and [1] can be used for the more important problem of HJB equations. Having such a goal in mind, but wanting to keep the presentation simpler, we make *quite restrictive assumptions without losing the very interesting case when a boundary layer is present.* However, the restrictive assumptions we make are actually present in the important examples we have in mind. Our analysis can be carried through under weaker assumptions, but, as is customary in stochastic control, this would have to be done on a case-by-case basis, adapting the method to the specific optimization problem. This is particularly important as far as admissibility is concerned.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space supporting an $\mathbb{R}^{d'}$-valued Brownian motion. Given T let $\mathbb{F} := \{\mathcal{F}_t,\ 0 \le t \le T\}$ denote the completion of the natural filtration of this Brownian motion. (Note that $\mathbb{F}$ satisfies the usual conditions.)

Assumption 3.1 (state equation). For any $(t, u) \in [0, T] \times U$ and $x, y \in \mathbb{R}^d$ we have

$$\begin{aligned} |b(t,0,u)| + |\sigma(t,0,u)| &\le C(1+|u|), \\ |b(t,x,u) - b(t,y,u)| + |\sigma(t,x,u) - \sigma(t,y,u)| &\le L(|u|)|x-y| \end{aligned} \tag{3.1}$$

for some constant C and some nondecreasing function $L : [0,\infty) \to [0,\infty)$.

In what follows, we will work with controls and solutions defined on stochastic intervals. It is well known that, for deterministic intervals, one can choose integrands which are progressively measurable, optional, or predictable, as they are equal up to equivalence classes. We choose to work here with predictable controls, which are both the most general (i.e., work even for jump diffusions) and best suited to handle joint measurability in (t, ω) that is required on stochastic intervals.

Admissibility (i.e., bounds or integrability) is another very important issue, and we choose here a very small class of admissible process, namely, bounded controls, but the bound is not fixed a priori (unless the control space U is itself compact). This allows us to capture the full behavior of the value function, i.e., the face-lifting phenomenon, but the optimal control may not be admissible, if such a control exists. This choice of admissible controls is the same as in section 6 of Krylov [15] for the case of unbounded controls.

DEFINITION 3.1. *Let $0 \le \tau \le \rho \le T$ be stopping times. By $\mathcal{U}_{\tau,\rho}$ we denote the collection of predictable processes $u : (\tau, \rho] \to U$, by which we mean that the joint map*

$$(0,T] \times \Omega \ni (t,\omega) \to u_t(\omega) \times 1_{[\tau(\omega) < t \le \rho(\omega))}$$

is predictable with respect to the filtration $\mathbb{F}$ and which are uniformly bounded, i.e., there exists a positive constant $0 \le B(u) < \infty$ such that

$$\|u\| := \sup_{\tau(\omega) < t \le \rho(\omega)} |u_t(\omega)| \le B(u).$$

Our definition of admissible control is very restrictive in order to be able to deal simultaneously with a reasonably large class of problems. Of course, with this definition one does not expect an admissible optimal control to exist. However, if particular problems are considered, the definition of admissibility can be changed to a larger class that does contain the optimal control (if such exists). For example,

(1) in the case of utility maximization, controls should only be locally integrable, and admissibility is a state constraint, namely, that the wealth is nonnegative;
(2) in the case of classical quadratic-type energy minimization, controls should be square integrable.

Our proofs work verbatim in these particular cases.

Remark 3.1. Assumption 3.1 on the controlled SDE, together with Definition 3.1, ensures that there is always a unique strong (adapted to $\mathcal{F}_t$) solution to the controlled

SDE up to an explosion time. The additional Assumption 3.2 below means that there is never an explosion (for bounded controls). This is always the case if $\mathcal{O} = \mathbb{R}^d$, or in the case of utility maximization, if the control is the proportion of stocks held.

Assumption 3.2 (natural domain). For any stopping times $\tau \le \rho$ and any initial condition $\xi \in \mathcal{F}_\tau$ satisfying $\mathbb{P}(\xi \in \mathcal{O}) = 1$, if $u \in \mathcal{U}_{\tau,\rho}$, the unique strong solution $X^{u;\tau,\xi}$ of the SDE

$$\begin{cases} dX_t = b(t, X_t, u_t)dt + \sigma(t, X_t, u_t)dW_t, \ \tau \le t \le \rho, \\ X_\tau = \xi, \end{cases} \tag{3.2}$$

does not explode, i.e., $\mathbb{P}(X_t^{u;\tau,\xi} \in \mathcal{O}, \quad \tau \le t \le \rho) = 1$.

We denote $\mathcal{U}_{0,T}$ by $\mathcal{U}$. Then let us define the value function by

$$V(t,x) = \sup_{u \in \mathcal{U}_{t,T}} \mathbb{E}[g(X_T^{u;t,x})], \quad 0 \le t < T, \ x \in \mathcal{O}.$$

The goal of this section is to construct a supersolution of the HJB equation (2.2) with the terminal condition (2.3) that is smaller than the value function V. In order to do that, we need some growth property to be imposed on the payoff function g and the potential solutions of the PDE. In this direction, we make an additional assumption.

Assumption 3.3 (growth in x). There exists a continuous and strictly positive gauge function $\psi : \mathcal{O} \to (0, \infty)$ such that

(1) for any $\tau \le \rho$ and any initial condition $\xi \in \mathcal{F}_\tau$, $\mathbb{P}(\xi \in \mathcal{O}) = 1$, which satisfies $\mathbb{E}[\psi(\xi)] < \infty$, if the control u is admissible, i.e., $u \in \mathcal{U}_{\tau,\rho}$, then

$$\mathbb{E}\left[\sup_{\tau \le t \le \rho} \psi(X_t^u)\right] < \infty;$$

(2) $|g(x)| \le C\psi(x)$ for some C.

The assumption above is tailor-made to deal simultaneously with quadratic problems ($\mathcal{O} = \mathbb{R}^d, \psi(x) = |x|^2$ *or* $\psi(x) = 1 + |x|^2$) and utility maximization ($\mathcal{O} = (0,\infty)$, $\psi(x) = x^p$ or $\psi(x) = 1 + x^p$, $-\infty < p < 1$, $p \ne 0$). However, the choice of ψ does matter, especially in the comparison principle that we need for the terminal condition (see Remark 5.1).

DEFINITION 3.2. *The set of stochastic subsolutions for the parabolic PDE* (2.2)*, denoted by* $\mathcal{V}^-$*, is the set of functions* $v : [0,T] \times \mathcal{O} \to \mathbb{R}$ *which have the following properties:*

(i) *They are continuous and satisfy the terminal condition* $v(T, \cdot) \le g(\cdot)$ *together with the growth condition*

$$|v(t,x)| \le C(v)\psi(x), 0 \le t \le T, \ x \in \mathcal{O}, \quad \textit{for some} \ \ C(v) < \infty. \tag{3.3}$$

(ii) *There exists a bound* $L(v) < \infty$*, depending on* v*, such that for each stopping time* τ *and each* $\xi \in \mathcal{F}_\tau$ *such that* $\mathbb{P}(\xi \in \mathcal{O}) = 1$ *and* $\mathbb{E}[\psi(\xi)] < \infty$*, there exists a control* $u \in \mathcal{U}_{\tau,T}$ *defined on* $[\tau, T]$ *adapted to* $\mathbb{F}$*, satisfying the bound* $\|u\| \le L(v)$*, and such that for any* $\mathbb{F}$*-stopping time* $\rho \in [\tau, T]$ *we have that*

$$v(\tau, \xi) \le \mathbb{E}\big[v\big(\rho, X_\rho^{u;\tau,\xi}\big)\big|\mathcal{F}_\tau\big] \ a.s. \tag{3.4}$$

We do not expect the value function to be a stochastic subsolution, except in the situations when there exists an admissible optimal control. As mentioned, this is rarely the case, with our very restrictive definition of admissibility. However, this does not cause any problem in the course of completing the stochastic Perron's method: while the value function is not a subsolution itself, it can be approximated by subsolutions.

Remark 3.2. We ask for the submartingale property to hold only in between the fixed stopping time τ and any later $\rho \geq \tau$, which is actually less than the full submartingale property on the stochastic interval $[\tau, T]$.

Assumption 3.4. We assume that $\mathcal{V}^- \neq \emptyset$.

Remark 3.3. Assumption 3.4 is satisfied, for example, when g is bounded from below.

A crucial property of the set of stochastic subsolutions is the following stability result.

PROPOSITION 3.1. *If v^1 and v^2 are two stochastic subsolutions, then $v = v^1 \vee v^2$ is also a stochastic subsolution.*

Proof. We will only show that v satisfies item (ii) of the definition of stochastic subsolution. We can choose the uniform bound corresponding to v as

$$L(v) = L(v^1) \vee L(v^2).$$

Now, fix a stopping time τ and a random variable $\xi \in \mathcal{F}_\tau$ with $\mathbb{P}(\xi \in \mathcal{O}) = 1$ and $\mathbb{E}[\psi(\xi)] < \infty$. Then, by the definition of the stochastic subsolutions v^1 and v^2, it follows that there are two controls $\|u_1\| \leq L(v^1)$ and $\|u_2| \leq L(v^2)$ satisfying

$$v^i(\tau, \xi) \leq \mathbb{E}[v^i(\rho, X_\rho^{u_i;\tau,\xi})|\mathcal{F}_\tau], \quad i \in \{1, 2\}.$$

Now define a control u (on the stochastic interval $(\tau, T]$) by

$$u = 1_{\{v^1(\tau,\xi) \geq v^2(\tau,\xi)\}} u_1 + 1_{\{v^1(\tau,\xi) < v^2(\tau,\xi)\}} u_2. \tag{3.5}$$

Now, for each $\tau \leq \rho \leq T$, we have

1. on $\{v^1(\tau, \xi) \geq v^2(\tau, \xi)\} \in \mathcal{F}_\tau$ we have
$$v^1(\rho, X_\rho^{u_1;\tau,\xi}) = v^1(\rho, X_\rho^{u;\tau,\xi}) \leq v(\rho, X_\rho^{u;\tau,\xi});$$
2. on $\{v^1(\tau, \xi) < v^2(\tau, \xi)\} \in \mathcal{F}_\tau$ we have
$$v^2(\rho, X_\rho^{u_2;\tau,\xi}) = v^2(\rho, X_\rho^{u;\tau,\xi}) \leq v(\rho, X_\rho^{u;\tau,\xi}).$$

Applying the definition of subsolutions for v^1 and v^2 (for controls u_1 and u_2) we get

$$1_{\{v^1(\tau,\xi) \geq v^2(\tau,\xi)\}} v^1(\tau, \xi) \leq \mathbb{E}\big[1_{\{v^1(\tau,\xi) \geq v^2(\tau,\xi)\}} v^1(\rho, X_\rho^{u;\tau,\xi})\big|\mathcal{F}_\tau\big] \text{ a.s.},$$

since $\{v^1(\tau, \xi) \geq v^2(\tau, \xi)\} \in \mathcal{F}_\tau$. Therefore, according to item (1) above we have

$$1_{\{v^1(\tau,\xi) \geq v^2(\tau,\xi)\}} v^1(\tau, \xi) \leq \mathbb{E}\big[1_{\{v^1(\tau,\xi) \geq v^2(\tau,\xi)\}} v(\rho, X_\rho^{u;\tau,\xi})\big|\mathcal{F}_\tau\big] \text{ a.s.} \tag{3.6}$$

Similarly, we obtain

$$1_{\{v^1(\tau,\xi) < v^2(\tau,\xi)\}} v^2(\tau, \xi) \leq \mathbb{E}\big[1_{\{v^1(\tau,\xi) < v^2(\tau,\xi)\}} v^2(\rho, X_\rho^{u;\tau,\xi})\big|\mathcal{F}_\tau\big] \text{ a.s.},$$

and by item (2) above we have

$$1_{\{v^1(\tau,\xi) < v^2(\tau,\xi)\}} v^2(\tau, \xi) \leq \mathbb{E}\big[1_{\{v^1(\tau,\xi) < v^2(\tau,\xi)\}} v(\rho, X_\rho^{u;\tau,\xi})\big|\mathcal{F}_\tau\big] \text{ a.s.} \tag{3.7}$$

Putting (3.6) and (3.7) together we conclude. ∎

THEOREM 3.1 (the supremum of stochastic subsolutions is a viscosity supersolution). *Let Assumptions* 2.1(1), 3.1, 3.2, 3.3, *and* 3.4 *hold true. Assume also that* g *is a lower semicontinuous (LSC) function and* $V < \infty$. *Then the lower stochastic envelope*

$$v^- := \sup_{v \in \mathcal{V}^-} v \le V < \infty$$

is a viscosity supersolution of (2.2). *Moreover, if we define*

$$v^-(T-, x) := \liminf_{(t'<T, x') \to (T,x)} v^-(t', x'), \quad x \in \mathcal{O}, \tag{3.8}$$

then the function $v^-(T-, \cdot)(\ge g(\cdot))$ *is a viscosity supersolution of* (2.3).

Remark 3.4. The function v^- may not have a limit from the left at $t = T$. We therefore modify this function as described in (3.8) at $t = T$. If we consider the function v^- with the new terminal condition $v^-(T-, \cdot)$, it still is LSC, as it is used in the proof of Theorem 5.1.

Proof. Step 1. The fact that $v^- \le V$ follows directly from the definition of the class of stochastic subsolutions and by the definition of $\mathcal{U}_{t,T}$ and V.

Step 2. *The interior supersolution property.* Let $\varphi : [0, T] \times \mathcal{O} \to \mathbb{R}^d$ be a $C^{1,2}$-test function such that $v^- - \varphi$ attains a strict local minimum equal to zero at some parabolic interior point $(t_0, x_0) \in [0, T) \times \mathcal{O}$. We first prove that

$$-\varphi_t(t_0, x_0) - H(t_0, x_0, \varphi_x(t_0, x_0), \varphi_{xx}(t_0, x_0)) \ge 0 \tag{3.9}$$

by contradiction. To this end, assume that

$$(-\varphi_t - \sup_u L_t^u \varphi)(t_0, x_0) < 0.$$

But then there exists $\tilde{u} \in U$ such that

$$(-\varphi_t - L_t^{\tilde{u}} \varphi)(t_0, x_0) < 0. \tag{3.10}$$

Since the coefficients of the SDE are continuous there exists a small enough ball $B(t_0, x_0, \varepsilon)$ such that

$$-\varphi_t - L_t^{\tilde{u}} \varphi(t, x) < 0, \quad (t, x) \in B(t_0, x_0, \varepsilon),$$

and

$$\varphi(t, x) < v^-(t, x), \quad (t, x) \in B(t_0, x_0, \varepsilon) - \{(t_0, x_0)\}.$$

To be precise, throughout the paper we use the norm $\|(t, x)\| = \max\{|t|, |x|\}$, so

$$B(t_0, x_0, \varepsilon) := \{(t, x) \in [0, T) \times \mathcal{O} | \max\{|t - t_0|, |x - x_0|\} < \varepsilon\}.$$

Since $v^- - \varphi$ is LSC and $\overline{B(t_0, x_0, \varepsilon)} - B(t_0, x_0, \varepsilon/2)$ is compact, there exists a $\delta > 0$ satisfying

$$v^- - \delta \ge \varphi \quad \text{on } \overline{B(t_0, x_0, \varepsilon)} - B(t_0, x_0, \varepsilon/2).$$

Using Proposition 4.1 in [2] together with Proposition 3.1 above, we obtain a (countable) increasing sequence of stochastic subsolutions $v_n \nearrow v^-$. Now, since φ is continuous, as well as v_n's, we can use a Dini argument (identical to the one in Lemma 2.4

of [1]) to conclude that for $\delta' \in (0, \delta)$ there exists a stochastic subsolution $v = v_n$ (for some large enough n) such that

$$v - \delta' \geq \varphi \quad \text{on } \overline{B(t_0, x_0, \varepsilon)} - B(t_0, x_0, \varepsilon/2). \tag{3.11}$$

Choosing $\eta \in (0, \delta')$ small enough we have that the function

$$\varphi^\eta := \varphi + \eta$$

satisfies

$$\begin{aligned} -\varphi^\eta_t - L^{\tilde{u}}_t \varphi^\eta(t, x) < 0, \quad (t, x) \in B(t_0, x_0, \varepsilon), \\ \varphi^\eta(t, x) < v(t, x), \quad (t, x) \in \overline{B(t_0, x_0, \varepsilon)} - B(t_0, x_0, \varepsilon/2), \end{aligned}$$

and

$$\varphi^\eta(t_0, x_0) = v^-(t_0, x_0) + \eta > v^-(t_0, x_0).$$

Now we define

$$v^\eta = \begin{cases} v \vee \varphi^\eta \text{ on } \overline{B(t_0, x_0, \varepsilon)}, \\ v \text{ outside } \overline{B(t_0, x_0, \varepsilon)}. \end{cases}$$

Clearly, v^η is continuous and $v^\eta(t_0, x_0) = \varphi^\eta(t_0, x_0) > v^-(t_0, x_0)$. And since ε can be chosen so that $T > t_0 + \varepsilon$, v^η satisfies the terminal condition. In addition, the growth condition in Definition 3.2(i) holds for v^η, since such growth condition holds for the approximate supremum v (although we may not have, without additional assumptions, a similar growth condition on v^-).

We only need to show that v^η satisfies (ii) in Definition 3.2 to get a contradiction and complete the proof. Let $0 \leq \tau \leq T$ be a fixed stopping time and $\xi \in \mathcal{F}_\tau$, $\mathbb{P}(\xi \in \mathcal{O}) = 1$, such that $\mathbb{E}[\psi(\xi)] < \infty$. We need to construct a control $u \in \mathcal{U}_{\tau,T}$ that works for v^η in Definition 3.2(ii). Following the arguments in the proof of Proposition 3.1, such a control u can be constructed in a surprisingly simple way, *which represents a significant technical improvement over the previous work* [2] *or* [1].

Denote by $u_0 \in \mathcal{U}_{\tau,T}$ the control corresponding to initial time τ and initial condition ξ in Definition 3.2(ii) for the stochastic subsolution v. Denote by A the event

$$A = \{(\tau, \xi) \in B(t_0, x_0, \varepsilon/2) \text{ and } \varphi^\eta(\tau, \xi) > v(\tau, \xi)\}.$$

Recalling (3.10), we define the new admissible control $u_1 \in \mathcal{U}_{\tau,T}$ by

$$u_1 = \tilde{u} \times 1_A + u_0 \times 1_{A^c}$$

and by τ_1 the first time after τ when the diffusion started at ξ and controlled by u_1 hits the boundary of $B(t_0, x_0, \varepsilon/2)$:

$$\tau_1 = \inf\{\tau \leq t \leq T | X^{u_1;\tau,\xi}_t \in \partial B(t_0, x_0, \varepsilon/2)\}.$$

Now, denote by

$$\xi_1 = X^{u_1;\tau,\xi}_{\tau_1} \in \partial B(t_0, x_0, \varepsilon/2)$$

and by $u_2 \in \mathcal{U}_{\tau_1,T}$ the control in Definition 3.2(ii) corresponding to v for the starting time τ_1 and initial condition ξ_1. Now, we can finally define

$$u = u_1 \times 1_{\{\tau < t \leq \tau_1\}} + u_2 \times 1_{\{\tau_1 < t \leq T\}}.$$

Note that the control u is bounded by $L(v) \vee |\tilde{u}|$ and, therefore, it is admissible. Consider any stopping time ρ such that $\tau \leq \rho \leq T$. In the event A, $\varphi^\eta(\cdot, X_\cdot)$ is a submartingale up to $\rho \wedge \tau_1$ (because of Itô's formula together with the fact that φ^η is bounded in the interior ball), which reads

$$1_A \varphi^\eta(\tau, \xi) \leq \mathbb{E}[1_A \varphi^\eta(\rho \wedge \tau_1, X^{\tilde{u};\tau,\xi}_{\rho\wedge\tau_1}) | \mathcal{F}_\tau] \text{ a.s.}$$

Since

$$1_A \varphi^\eta(\rho \wedge \tau_1, X^{\tilde{u};\tau,\xi}_{\rho\wedge\tau_1}) = 1_A \varphi^\eta(\rho \wedge \tau_1, X^{u;\tau,\xi}_{\rho\wedge\tau_1}) \leq 1_A v^\eta(\rho \wedge \tau_1, X^{u;\tau;\xi}_{\rho\wedge\tau_1}),$$

we actually obtain

$$1_A v^\eta(\tau, \xi) = 1_A \varphi^\eta(\tau, \xi) \leq \mathbb{E}[1_A v^\eta(\rho \wedge \tau_1, X^{u;\tau,\xi}_{\rho\wedge\tau_1}) | \mathcal{F}_\tau] \text{ a.s.}$$

Next, we use the fact that u_1 is the "optimal" control for v, together with $v = v^\eta$ everywhere outside the open ball $B(t_0, x_0, \varepsilon/2)$, to obtain

$$1_{A^c} v^\eta(\tau, \xi) = 1_{A^c} v(\tau, \xi) \leq \mathbb{E}[1_{A^c} v(\rho \wedge \tau_1, X^{u_1;\tau.\xi}_{\rho\wedge\tau_1}) | \mathcal{F}_\tau] = \mathbb{E}[1_{A^c} v^\eta(\rho \wedge \tau_1, X^{u;\tau,\xi}_{\rho\wedge\tau_1}) | \mathcal{F}_\tau].$$

Putting the above together, we obtain

$$v^\eta(\tau, \xi) \leq \mathbb{E}\big[v^\eta\big(\rho \wedge \tau_1, X^{u;\tau,\xi}_{\rho\wedge\tau_1}\big)\big|\mathcal{F}_\tau\big] \text{ a.s.} \tag{3.12}$$

Let us introduce yet another notation: $B = \{\rho \leq \tau_1\} \in \mathcal{F}_{\tau_1}$. We know that, on the boundary $\partial B(t_0, x_0, \varepsilon/2)$, $v = v^\eta$. Applying the definition of u, together with the fact that u_2 is optimal for v starting at τ_1 with condition ξ_1, we have

$$1_{B^c} v^\eta(\tau_1, \xi_1) = 1_{B^c} v(\tau_1, \xi_1) \leq \mathbb{E}[1_{B^c} v(\rho, X^{u_2;\tau_1,\xi_1}_\rho) | \mathcal{F}_{\tau_1}] \leq \mathbb{E}[1_{B^c} v^\eta(\rho, X^{u;\tau,\xi}_\rho) | \mathcal{F}_{\tau_1}].$$

If we rewrite the right-hand side (RHS) in (3.12) as

$$\mathbb{E}\big[v^\eta\big(\rho \wedge \tau_1, X^{u;\tau,\xi}_{\rho\wedge\tau_1}\big)\big|\mathcal{F}_\tau\big] = \mathbb{E}\big[1_B v^\eta\big(\rho, X^{u;\tau,\xi}_\rho\big) + 1_{B^c} v^\eta(\tau_1, \xi_1)\big|\mathcal{F}_\tau\big]$$

and use the tower property, we get, indeed,

$$v^\eta(\tau, \xi) \leq \mathbb{E}\big[v^\eta\left(\rho, X^{u;\tau,\xi}_\rho\right)\big|\mathcal{F}_\tau\big] \text{ a.s.}$$

This completes the proof of (3.9), from which it follows that

$$H(t_0, x_0, \varphi_x(t_0, x_0), \varphi_{xx}(t_0, x_0)) < \infty.$$

Thanks to Assumption 2.1(1) we also have that

$$G(t_0, x_0, \varphi_x(t_0, x_0), \varphi_{xx}(t_0, x_0)) \geq 0, \tag{3.13}$$

finishing the proof of the interior supersolution property.

Step 3. *The terminal condition, Part* I. We will show that $v^-(T, \cdot) = g(\cdot)$. Assume that for some $x_0 \in \mathcal{O}$ we have

$$v^-(T, x_0) < g(x_0).$$

We will use this information to construct a contradiction. Since $g(\cdot)$ is LSC, there exists an $\varepsilon > 0$ such that

$$g(x) \geq v^-(T, x_0) + \varepsilon \quad \text{if} \quad |x - x_0| \leq \varepsilon.$$

Because v^- is LSC, it is bounded from below on the compact set

$$(\overline{B(T, x_0, \varepsilon)} - B(T, x_0, \varepsilon/2)) \cap ([0, T] \times \mathcal{O}).$$

For a small enough $\eta > 0$ we have that

$$v^-(T, x_0) - \frac{\varepsilon^2}{4\eta} < -\varepsilon + \inf_{(t,x) \in (\overline{B(T,x_0,\varepsilon)} - B(T,x_0,\varepsilon/2)) \cap ([0,T] \times \mathcal{O})} v^-(t, x).$$

Since the above inequality is strict, following the proof of Step 2 in Theorem 3.1, we use again Proposition 4.1 in [2] together with Proposition 3.1 above and a Dini argument to find a stochastic subsolution $v \in \mathcal{V}^-$ such that

$$v^-(T, x_0) - \frac{\varepsilon^2}{4\eta} < -\varepsilon + \inf_{(t,x) \in (\overline{B(T,x_0,\varepsilon)} - B(T,x_0,\varepsilon/2)) \cap ([0,T] \times \mathcal{O})} v(t, x). \tag{3.14}$$

For $k > 0$ define

$$\varphi^{\eta,\varepsilon,k}(t, x) = v^-(T, x_0) - \frac{|x - x_0|^2}{\eta} - k(T - t).$$

Choose k large enough, at least as large as $k \geq \varepsilon/4\eta$ but possibly much larger, such that

$$\left[-\varphi_t^{\eta,\varepsilon,k} - \sup_u L_t^u \varphi^{\eta,\varepsilon,k}\right](t, x) < 0 \quad \text{on} \quad \overline{B(T, x_0, \varepsilon)}.$$

Using (3.14) we obtain

$$\varphi^{\eta,\varepsilon,k} \leq -\varepsilon + v \quad \text{on} \quad (\overline{B(T, x_0, \varepsilon)} - B(T, x_0, \varepsilon/2)) \cap ([0, T] \times \mathcal{O}).$$

On the other hand,

$$\varphi^{\eta,\varepsilon,k}(T, x) \leq v^-(T, x_0) \leq g(x) - \varepsilon \quad \text{for} \quad |x - x_0| \leq \varepsilon.$$

Now, let $\delta < \varepsilon$ and define

$$v^{\varepsilon,\eta,k,\delta} := \begin{cases} v \vee (\varphi^{\varepsilon,\eta,k} + \delta) & \text{on} \quad \overline{B(T, x_0, \varepsilon)}, \\ v & \text{outside} \quad \overline{B(T, x_0, \varepsilon)}. \end{cases}$$

Now using the idea in Step 1 of the proof, we can show that $v^{\varepsilon,\eta,k,\delta} \in \mathcal{V}^-$ but $v^{\varepsilon,\eta,k,\delta}(T, x_0) = v^-(T, x_0) + \delta > v^-(T, x_0)$, leading to a contradiction.

The only reason we actually proved $v^-(T, \cdot) = g(\cdot)$ was to get some information about the left liminf $v^-(T-, \cdot)$. More precisely, since v^- is LSC, we know that

$$g(\cdot) = v^-(T, \cdot) \leq v^-(T-, \cdot).$$

In order to finish the proof of the theorem, we only need to show that $v^-(T-, \cdot)$ is a viscosity supersolution of (2.3), which we will do in the next step.

Step 4. *The terminal condition, Part* II. We show that the LSC function $v^-(T-,\cdot)$ is a viscosity supersolution of

$$G(T,x,v^-_x(T,x),v^-_{xx}(T,x))\geq 0,\quad x\in\mathcal{O}.$$

The arguments used below trace back to [4] and were technically refined later for more general models of superhedging in [7], [18], and others, as presented in the survey paper [17]. We basically use the notation from Lemma 4.3.2 in [16], which summarizes the existing literature.

More precisely, we rely on the fact that v^- satisfies the same equation in the interior, a fact we established in Step 2, to get information about the limit as $t\to T$. Let $y\in\mathbb{R}^d$ and $\psi(x)$ be a test function satisfying

$$0=v^-(T-,y)-\psi(y)=\min_{x\in\mathbb{R}^d}(v^-(T-,x)-\psi(x)). \tag{3.15}$$

By the very definition of $v^-(T-,\cdot)$, there exists a sequence (s_m,y_m) converging to (T,y) with $s_m<T$ such that

$$\lim_{m\to\infty}v^-(s_m,y_m)=v^-(T-,y).$$

Let us construct another test function that depends both on t and x variables,

$$\psi_m(t,x)=\psi(x)-|x-y|^4+\frac{T-t}{(T-s_m)^2},$$

and choose $(t_m,x_m)\in[s_m,T]\times\overline{B(y,\varepsilon)}$ as a minimum of $v^--\psi_m$ on $[s_m,T]\times\overline{B(y,\varepsilon)}$, where ε is chosen small enough so that $\overline{B(y,\varepsilon)}\subset\mathcal{O}$.

What we would like to do next is to show that in fact $t_m<T$ for large enough m and that $x_m\to y$. The first fact follows from the observation that

$$v^-(s_m,y_m)-\psi_m(s_m,y_m)\leq-\frac{1}{2(T-s_m)}<0$$

for large enough m and that

$$v^-(T-,x)-\psi_m(T,x)\geq v^-(T-,x)-\psi(x)\geq 0,$$

where the second inequality follows from (3.15). Let us focus on the convergence of x_m to y. The sequence (x_m) converges (up to choosing a subsequence) to some $z\in\overline{B(y,1)}$. By construction, $s_m\leq t_m$. Using this and the choice of (t_m,x_m) we obtain the following string of inequalities:

$$\begin{aligned}
0&\leq(v^-(T-,z)-\psi(z))-(v^-(T-,y)-\psi(y))\\
&\leq\liminf_{m\to\infty}\left[(v^-(t_m,x_m)-\psi(x_m))-(v^-(s_m,y_m)-\psi(y_m))\right]\\
&\leq\liminf_{m\to\infty}\Bigg[(v^-(t_m,x_m)-\psi_m(t_m,x_m))-(v^-(s_m,y_m)-\psi_m(s_m,y_m))\\
&\qquad-|x_m-y|^4+\frac{T-t_m}{(T-s_m)^2}+|y_m-y|^4-\frac{T-s_m}{(T-s_m)^2}\Bigg]\\
&\leq\liminf_{m\to\infty}\left[-|x_m-y|^4+|y_m-y|^4\right]=-|z-y|^4,
\end{aligned}$$

which proves that $z=y$.

We know that (t_m, x_m) is a minimizer of $v^- - \psi_m$ over $[s_m, T] \times \overline{B(y, \varepsilon)}$ by definition, and we also know that $s_m \leq t_m < T$ for large m. Since $x_m \to y$, we conclude that (for m large enough) we have $(v^- - \psi_m)(t_m, x_m) \leq (v^- - \psi_m)(t, x)$ for $t_m \leq t < T$, $|x - x_m| \leq \varepsilon/2$. While this does not mean that (t_m, x_m) is a local interior min for $v^- - \psi_m$ (because we may have $t_m = s_m$), it does mean that we have a local "parabolic interior minimum." It is well known that, for example, from [6], for parabolic equations a parabolic interior minimum is enough to use ψ_m as a test function at (t_m, x_m), and therefore we first conclude that

$$-D_t\psi_m(t_m, x_m) - H(t_m, x_m, D_x\psi_m(t_m, x_m), D_x^2\psi_m(t_m, x_m)) \geq 0,$$

so $H(t_m, x_m, D_x\psi_m(t_m, x_m), D_x^2\psi_m(t_m, x_m)) < \infty$ and, consequently,

$$G(t_m, x_m, D_x\psi_m(t_m, x_m), D_x^2\psi_m(t_m, x_m)) \geq 0.$$

Now the claim of this step follows from the continuity of G and the fact that $x_m \to y$, as the derivatives of ψ_m with respect to x converge to those of ψ. ∎

4. Stochastic supersolutions.

In this section we consider the weak formulation of the stochastic control problem.

Assumption 4.1. We assume that the coefficients $b : [0, T] \times \mathbb{R}^d \times U \to \mathbb{R}^d$ and $\sigma : [0, T] \times \mathbb{R}^d \times U \to \mathbb{M}_{d,d'}(\mathbb{R})$ are continuous.

DEFINITION 4.1. *For each (s, x) we denote by $\mathcal{U}_{s,x}$ the set of weak admissible controls for the* (2.1)*, by which we mean a*

$$\left(\Omega^{s,x}, \mathcal{F}^{s,x}, (\mathcal{F}_t^{s,x})_{s\leq t\leq T}, \mathbb{P}^{s,x}, (W_t^{s,x})_{s\leq t\leq T}, (X_t^{s,x})_{s\leq t\leq T}, (u_t)_{s\leq t\leq T}\right),$$

where

1. *$(\Omega^{s,x}, \mathcal{F}^{s,x}, (\mathcal{F}_t^{s,x})_{s\leq t\leq T}, \mathbb{P}^{s,x})$ is an arbitrary stochastic basis satisfying the usual conditions,*
2. *$W^{s,x}$ is a d'-dimensional Brownian motion with respect to the filtration $(\mathcal{F}_t^{s,x})_{s\leq t\leq T}$,*
3. *u is a predictable and uniformly bounded U-valued process,*
4. *$X^{s,x}$ is a continuous and adapted process satisfying* (2.1) *with initial condition $X_s = x \in \mathcal{O}$, and $\mathbb{P}^{s,x}(X_t^{s,x} \in \mathcal{O},\ s \leq t \leq T) = 1$ together with*

$$\mathbb{E}^{s,x}\left[\sup_{s\leq t\leq T} \psi(X_t^{s,x})\right] < \infty$$

for the gauge function ψ in section 3*.*

Now, for some measurable function $g : \mathcal{O} \to \mathbb{R}$, we denote by

$$\mathfrak{V}(s, x) := \sup_{\mathcal{U}^{s,x}} \mathbb{E}^{s,x}[g(X_T^{s,x})] \tag{4.1}$$

the value function of the weak control problem.

Assumption 4.2. The payoff function g is an upper semicontinuous (USC) function satisfying $|g(\cdot)| \leq C\psi(\cdot)$.

Remark 4.1.

1. Because of the growth assumption on weakly controlled solutions, $\mathbb{E}^{s,x}[g(X_T^{s,x})]$ is well defined and finite, so $\mathfrak{V} > -\infty$.
2. When both are well defined it clearly holds that $V \leq \mathfrak{V}$.

Our goal in this section is to construct an upper bound of $\mathfrak{V}$ that is a viscosity subsolution.

DEFINITION 4.2. *The set of stochastic supersolutions for the parabolic PDE* (2.2), *denoted by* $\mathcal{V}^+$, *is the set of functions* $v : [0,T] \times \mathcal{O}^d \to \mathbb{R}$ *which have the following properties:*

(1) *They are continuous and satisfy the terminal condition* $v(T,\cdot) \geq g(\cdot)$ *together with the growth condition*

$$|v(t,x)| \leq C(v)\psi(x), 0 \leq t \leq T,\ x \in \mathcal{O}.$$

(2) *For each* $(s,x) \in [0,T] \times \mathcal{O}$ *and each weak control*

$$\big(\Omega^{s,x}, \mathcal{F}^{s,x}, (\mathcal{F}^{s,x}_t)_{s\leq t\leq T}, \mathbb{P}^{s,x}, (W^{s,x}_t)_{s\leq t\leq T}, (X^{s,x}_t)_{s\leq t\leq T}, (u_t)_{s\leq t\leq T}\big),$$

the process $(u(t, X^{s,x}_t))_{s\leq t\leq T}$ *is a supermartingale on* $(\Omega^{s,x}, \mathbb{P}^{s,x})$ *with respect to the filtration* $(\mathcal{F}^{s,x}_t)_{s\leq t\leq T}$.

Assumption 4.3. $\mathcal{V}^+ \neq \emptyset$.

Remark 4.2. Assumption 4.3 is satisfied, for example, when g is bounded from above.

THEOREM 4.1 (the infimum of stochastic supersolutions is a viscosity subsolution). *Let Assumptions* 2.1(2), 4.1, 4.2, *and* 4.3 *hold true. Then* $v^+ = \inf_{v\in\mathcal{V}^+} v$ *is a viscosity subsolution of* (2.2). *Moreover, the USC function* $v^+(T,\cdot)$ *is a viscosity subsolution of* (2.3).

Proof. Step 1. The fact that $v^+ \geq \mathfrak{V}$ follows directly from the definition of the class of stochastic subsolutions and by the definition of $\mathcal{U}$.

Step 2. *The interior subsolution property.* Let $\varphi : [0,T] \times \mathcal{O} \to \mathbb{R}^d$ be a $C^{1,2}$-test function such that $v^+ - \varphi$ attains a strict local maximum equal to zero at some parabolic interior point $(t_0, x_0) \in [0,T) \times \mathbb{R}^d$, where the viscosity subsolution property fails, i.e.,

$$\min\{-\varphi_t(t_0,x_0) - H(t,x,\varphi_x(t_0,x_0), \varphi_{xx}(t_0,x_0)), G(t,x,\varphi_x(t_0,x_0),\varphi_{xx}(t_0,x_0))\} > 0.$$

Then since G is continuous and H is continuous in the interior of its domain it follows that there exists a small enough ball $B(t_0,x_0,\varepsilon)$ such that for all $(t,x) \in B(t_0,x_0,\varepsilon)$ we have

$$\min\{-\varphi_t(t,x) - H(t,x,\varphi_x(t,x),\varphi_{xx}(t,x)), G(t,x,\varphi_x(t,x),\varphi_{xx}(t,x))\} > 0.$$

Now the rest of the proof of this step is very similar to the corresponding step in the proof of Theorem 2.1 in [2] but is much simplified by following the stopping idea in the proof of Theorem 3.1 (Step 2) above. For the sake of completeness and the convenience of the reader we include the remaining part of the proof. The function $v^+ - \varphi$ is upper semicontinuous and $\overline{B(t_0,x_0,\varepsilon)} - B(t_0,x_0,\varepsilon/2)$ is compact, and there exists a $\delta > 0$ satisfying

$$v^+ + \delta \leq \varphi \quad \text{on } \overline{B(t_0,x_0,\varepsilon)} - B(t_0,x_0,\varepsilon/2).$$

Using Proposition 4.1 in [2] together with the obvious observation that the minimum of two stochastic supersolutions is also a stochastic supersolution, we obtain a (countable) decreasing sequence of stochastic supersolutions $v_n \searrow v^+$. Now, since φ is continuous, as well as v_n's, we can use once again a Dini argument (identical to the one in Lemma 2.4 of [1]) to conclude that for $\delta' \in (0, \delta)$ there exists a stochastic supersolution $v = v_n$ (for some large enough n) such that

$$v + \delta' \leq \varphi \quad \text{on } \overline{B(t_0, x_0, \varepsilon)} - B(t_0, x_0, \varepsilon/2).$$

Choosing $\eta \in (0, \delta')$ small enough we have that the function

$$\varphi^\eta := \varphi - \eta$$

satisfies

$$\begin{aligned}-\varphi_t^\eta(t,x) - H(t, x, \varphi_x^\eta(t,x), \varphi_{xx}^\eta(t,x)) > 0, (t,x) \in B(t_0, x_0, \varepsilon),\\ \varphi^\eta(t,x) > v(t,x),\ (t,x) \in \overline{B(t_0, x_0, \varepsilon)} - B(t_0, x_0, \varepsilon/2),\end{aligned}$$

and

$$\varphi^\eta(t_0, x_0) = v^+(t_0, x_0) - \eta < v^+(t_0, x_0).$$

Now we define, similarly to Step 2 above,

$$v^\eta = \begin{cases} v \wedge \varphi^\eta \text{ on } \overline{B(t_0, x_0, \varepsilon)}, \\ v \text{ outside } \overline{B(t_0, x_0, \varepsilon)}. \end{cases}$$

Clearly, v^η is continuous and $v^\eta(t_0, x_0) = \varphi^\eta(t_0, x_0) > v^-(t_0, x_0)$. And since ε can be chosen so that $T > t_0 + \varepsilon$, v^η satisfies the terminal condition. Again, the growth condition in Definition 3.2(i) holds for v^η, since such growth condition holds for the approximate infimum v. We now only need to show that v^η satisfies (ii) in Definition 4.2 to get a contradiction and complete the proof. Fix an admissible weak control

$$\left(\Omega^{s,x}, \mathcal{F}^{s,x}, (\mathcal{F}_t^{s,x})_{s \leq t \leq T}, \mathbb{P}^{s,x}, (W_t^{s,x})_{s \leq t \leq T}, (X_t^{s,x})_{s \leq t \leq T}, (u_t)_{s \leq t \leq T}\right).$$

Fix now $s \leq \tau \leq \rho \leq T$ two stopping times of the filtration $(\mathcal{F}_t^{s,x})_{s \leq t \leq T}$. Denote, similarly to Step 2, by A the event

$$A = \{(\tau, X_\tau^{s,x}) \in B(t_0, x_0, \varepsilon/2) \text{ and } \varphi^\eta(\tau, X_\tau^{s,x}) < v(\tau, X_\tau^{s,x})\}.$$

Denote by τ_1 the first time after τ when the diffusion hits the boundary of $B(t_0, x_0, \varepsilon/2)$:

$$\tau_1 = \inf\{\tau \leq t \leq T | X_t^{s,x} \in \partial B(t_0, x_0, \varepsilon/2)\}.$$

In the event A, $\varphi^\eta(\cdot, X_\cdot^{s,x})$ is a continuous supermartingale up to $\rho \wedge \tau_1$ (because of Itô's formula together with the fact that φ^η is bounded in the interior ball), which reads

$$1_A \varphi^\eta(\tau, X_\tau^{s,x}) \geq \mathbb{E}^{s,x}[1_A \varphi^\eta(\rho \wedge \tau_1, X_{\rho \wedge \tau_1}^{s,x}) | \mathcal{F}_\tau^{s,x}] \ \mathbb{P}^{s,x}\text{-a.s.}$$

Since $1_A \varphi^\eta(\rho \wedge \tau_1, X_{\rho \wedge \tau_1}^{s,x}) \geq 1_A v^\eta(\rho \wedge \tau_1, X_{\rho \wedge \tau_1}^{s,x})$, we have

$$1_A v^\eta(\tau, X_\tau^{s,x}) = 1_A \varphi^\eta(\tau, X_\tau^{s,x}) \geq \mathbb{E}^{s,x}[1_A v^\eta(\rho \wedge \tau_1, X_{\rho \wedge \tau_1}^{s,x}) | \mathcal{F}_\tau^{s,x}] \ \mathbb{P}^{s,x}\text{-a.s.}$$

Next, we use the optional sampling theorem applied to the continuous supermartingale $v(\cdot, X_\cdot^{s,x})$ in between the stopping times $\tau \leq \rho \wedge \tau_1$, together with the observation that $v = v^\eta$ everywhere outside the open ball $B(t_0, x_0, \varepsilon/2)$, to obtain

$$\begin{aligned}1_{A^c} v^\eta(\tau, X_\tau^{s,x}) = 1_{A^c} v(\tau, X_\tau^{s,x}) &\geq \mathbb{E}^{s,x}[1_{A^c} v(\rho \wedge \tau_1, X_{\rho \wedge \tau_1}^{s,x}) | \mathcal{F}_\tau^{s,x}]\\ &\geq \mathbb{E}^{s,x}[1_{A^c} v^\eta(\rho \wedge \tau_1, X_{\rho \wedge \tau_1}^{s,x}) | \mathcal{F}_\tau^{s,x}],\ \mathbb{P}^{s,x}\text{-a.s.}\end{aligned}$$

Putting the above together, we obtain

$$v^\eta(\tau, X_\tau^{s,x}) \geq \mathbb{E}^{s,x}\left[v^\eta\left(\rho \wedge \tau_1, X_{\rho\wedge\tau_1}^{s,x}\right)\big|\mathcal{F}_\tau^{s,x}\right] \ \mathbb{P}^{s,x}\text{-a.s.} \tag{4.2}$$

Let us again introduce the notation: $B = \{\rho \leq \tau_1\} \in \mathcal{F}_{\tau_1\wedge\rho}^{s,x}$. We know that on the boundary $\partial B(t_0, x_0, \varepsilon/2)$, $v = v^\eta$. Together with the optional sampling theorem applied to the continuous supermartingale $v(\cdot, X_\cdot^{s,x})$ between $\tau_1 \wedge \rho$ and ρ we have

$$\begin{aligned} 1_{B^c} v^\eta(\tau_1, X_{\tau_1}^{s,x}) = 1_{B^c} v(\tau_1, X_{\tau_1}^{s,x}) &\geq \mathbb{E}^{s,x}[1_{B^c} v(\rho, X_\rho^{s,x}) | \mathcal{F}_{\tau_1}^{s,x}] \\ &\geq \mathbb{E}[1_{B^c} v^\eta(\rho, X_\rho^{s,x}) | \mathcal{F}_{\tau_1}^{s,x}],\ \mathbb{P}^{s,x}\text{-a.s.} \end{aligned}$$

We finally rewrite the RHS in (4.2) as

$$\mathbb{E}^{s,x}\left[v^\eta\left(\rho \wedge \tau_1, X_{\rho\wedge\tau_1}^{s,x}\right)\big|\mathcal{F}_\tau^{s,x}\right] = \mathbb{E}^{s,x}\left[1_B v^\eta\left(\rho, X_\rho^{s,x}\right) + 1_{B^c} v^\eta(\tau_1, X_{\tau_1}^{s,x})\big|\mathcal{F}_\tau^{s,x}\right]$$

and use the tower property to obtain

$$v^\eta(\tau, X_\tau^{s,x}) \geq \mathbb{E}^{s,x}\left[v^\eta\left(\rho, X_\rho^{s,x}\right)\big|\mathcal{F}_\tau^{s,x}\right] \ \mathbb{P}^{s,x}\text{-a.s.}$$

Since this happens for any stopping times $s \leq \tau \leq \rho \leq T$ of the filtration $(\mathcal{F}_t^{s,x})_{s\leq t\leq T}$, we have, indeed, that v^η is a stochastic supersolution, leading to a contradiction and completing the proof.

Step 3. *The boundary condition.* Let $x_0 \in \mathcal{O}$ and ψ be a smooth function on $\mathcal{O}$ such that

$$0 = v^+(T, x_0) - \psi(x_0) = \max_{\mathcal{O}}(v^+(T, x) - \psi(x)).$$

Assume in addition, without losing generality, that the maximum is strict. Let us assume, by contradiction, that

$$G(T, x_0, \psi_x(x_0), \psi_{xx}(x_0)) > 0 \ \text{ and } v^+(T, x_0) > g(x_0). \tag{4.3}$$

Since G is continuous and, in addition, G is finite *and* continuous in the open set $G > 0$, we conclude that there exists small $\varepsilon, \delta_0 > 0$ and a finite constant C such that

$$H(t, x, \psi_x(x), \psi_{xx}(x)) < C, \quad T - t \leq \delta_0, |x - x_0| \leq \varepsilon.$$

In addition, we also have (for small enough ε)

$$\psi(x) \geq g(x) + \varepsilon, \quad |x - x_0| \leq \varepsilon.$$

Now, the whole idea is based on constructing a local supersolution

$$\psi^k(t, x) = \psi(x) + k(T - t)$$

for large k *by decoupling the bounds* δ *and* ε *in the estimate above*, then pushing it slightly down. Namely, we will make δ much smaller than ε. Fix δ_0 and ε as above. Denote by

$$h(\delta) = \sup_{T-t\leq\delta, \frac{\varepsilon}{2}\leq|x-x_0|\leq\varepsilon} \left(v^+(t, x) - \psi(x)\right),\ 0 < \delta < \delta_0.$$

Interpreting ψ as a continuous function of two variables (t, x), which actually does not depend on t and taking into account that v^+ is USC, there exist a point where the maximum above is attained, i.e.,

$$h(\delta) = v^+(t_\delta, x_\delta) - \psi(x_\delta).$$

By compactness, we can subtract a subsequence (we still denote it as $\delta \searrow 0$) such that

$$(t_\delta, x_\delta) \to (T, x^*), \quad \frac{\varepsilon}{2} \leq |x^* - x_0| \leq \varepsilon.$$

Since v^+ is USC, we conclude that

$$\begin{aligned} \limsup_{\delta \searrow 0} h(\delta) &= \limsup_{\delta \searrow 0} \big(v^+(t_\delta, x_\delta) - \psi(x_\delta)\big) \\ &\leq v^+(T, x^*) - \psi(x^*) \leq \sup_{\frac{\varepsilon}{2} \leq |x - x_0| \leq \varepsilon} \big(v^+(T, x) - \psi(x)\big) < 0, \end{aligned} \tag{4.4}$$

where the last inequality follows from the fact that we have a strict max at x_0 and the last supremum is actually attained. Therefore, we can choose $\delta < \delta_0$ small enough such that $h(\delta) < 0$. Now, for *this* fixed δ, with the notation

$$\delta' = -h(\delta) > 0$$

we have

$$v^+(t, x) \leq \psi(x) - \delta', \quad T - t \leq \delta, \frac{\varepsilon}{2} \leq |x - x_0| \leq \varepsilon. \tag{4.5}$$

Denote by D the compact "rectangular donut"

$$D = \{(t, x) | T - t \leq \delta, |x - x_0| \leq \varepsilon\} - \{(t, x) | T - t < \delta/2, |x - x_0| < \varepsilon/2\}.$$

Since, by upper semicontinuity, v^+ is bounded on $\{\delta/2 \leq T - t \leq \delta, |x - x_0| \leq \varepsilon/2\}$ we can choose k large enough such that

$$v^+ \leq \psi^k - \delta' \text{ on } \{\delta/2 \leq T - t \leq \delta, |x - x_0| \leq \varepsilon/2\}.$$

Together with (4.5), we obtain

$$v^+ \leq \psi^k - \delta' \text{ on } D.$$

In addition

$$H(t, x, \psi^k_x(t, x), \psi^k_{xx}(t, x)) = H(t, x, \psi_x(t, x), \psi_{xx}(t, x)) \leq C, \quad T - t \leq \delta, |x - x_0| \leq \varepsilon,$$

so

$$-\psi^k_t(t, x) - H(t, x, \psi^k_x(t, x), \psi^k_{xx}(t, x)) \geq k - C > 0$$

for k even larger if $T - t \leq \delta, |x - x_0| \leq \varepsilon$. Following the proof of Step 2 in Theorem 3.1, we use again Proposition 4.1 in [2] and the Dini argument to obtain a stochastic subsolution $v \in \mathcal{V}^+$ such that $v \leq \psi^k - \delta'/2$ on D.

Now let $\eta < \delta'/2 < \varepsilon$ and define

$$v^k = \begin{cases} v \wedge \big(\psi^k - \eta\big), & T - t \leq \delta, |x - x_0| \leq \varepsilon, \\ v \text{ otherwise.} \end{cases}$$

It follows, using the same stopping argument as in the proof of Theorem 3.1, that $v^k \in \mathcal{V}^+$. But we also have that $v^k(T, x_0) = v^+(T, x_0) - \eta < v^+(T, x_0)$, which contradicts the definition of the function v^+. $\square$

5. Verification by comparison. Before we go ahead, we recall that our analysis rests on the assumption of the existence of stochastic sub- and supersolutions. Such assumption may actually be nontrivial to check, especially given the choice of the gauge function ψ (see Remark 5.1 below).

Assumption 5.1. There is a comparison principle between USC subsolutions and LSC supersolutions within the class $|w| \leq C\psi$ for the PDE

$$\min[w(x) - g(x), G(T, x, w_x(x), w_{xx}(x))] = 0 \quad \text{on} \quad \mathcal{O}. \tag{5.1}$$

Remark 5.1. The choice of ψ can make a difference in whether we have a comparison result for (5.1). As mentioned, we do not have boundary conditions per se (this carries over to (5.1)), but the information on behavior of solutions near the boundary might, sometimes, be contained in the choice of ψ. Therefore, if one wants, for example, to add a constant to ψ, having an easier time checking for the existence of stochastic supersolutions or subsolutions, uniqueness may be lost in (5.1).

LEMMA 5.1. *Let us suppose that Assumption* 5.1 *and the assumptions in both Theorem* 3.1 *and Theorem* 4.1 *hold. Then*

$$v^-(T-, \cdot) = v^+(T, \cdot) = \hat{g}(\cdot), \tag{5.2}$$

where $\hat{g}$ is the unique continuous viscosity solution of (5.1). *In addition, both the strong and the weak value functions have well-defined limits at T, equal to the terminal condition $\hat{g}$:*

$$\lim_{(t<T, x') \to (T,x)} V(t, x') = \lim_{(t<T, x') \to (T,x)} \mathfrak{V}(t, x') = \hat{g}(x), \;\; x \in \mathcal{O}.$$

Proof. It follows from their definitions that $v^- \leq v^+$. Since v^+ is USC,

$$v^-(T-, x) = \liminf_{(t<T, x') \to (T,x)} v^-(t, x') \leq \limsup_{(t<T, x') \to (T,x)} v^+(t, x') \leq v^+(T, x). \tag{5.3}$$

Moreover, $v^-(T-, \cdot)$ is an LSC viscosity supersolution of (5.1) as a result of Theorem 3.1, and $v^+(T, \cdot)$ is a USC viscosity subsolution of the same PDE due to Theorem 4.1. In addition, under the assumptions that both $\mathcal{V}^-$ and $\mathcal{V}^+$ are nonempty, we have the bounds

$$|v^-|, |v^+| \leq C\psi,$$

obtaining therefore similar growth conditions for $v^+(T, \cdot)$ and $v^-(T-, \cdot)$. Thanks to the comparison assumption, it follows that $v^+(T, \cdot) = v^-(T-, \cdot)$ and the common value is the unique continuous viscosity solution of (5.1) that we denote by $\hat{g}$.

In order to prove the second statement, we only need to note that

$$v^- \leq V \leq \mathfrak{V} \leq v^+$$

and plug the equality $v^-(T-, \cdot) = v^+(T, \dot{)} = \hat{g}(\cdot)$ in (5.3). ∎

PROPOSITION 5.1 (G upper envelope of g). *Under Assumption* 5.1, *the function $\hat{g}$ is the smallest (continuous) function above g which is a viscosity supersolution of*

$$G(T, x, w_x(x), w_{xx}(x)) = 0 \quad \text{on } \mathcal{O}. \tag{5.4}$$

Proof. We know that $\hat{g} \geq g$ and that $\hat{g}$ is a viscosity supersolution of (5.4). Consider now a $w \geq g$ and w is a supersolution of (5.4). Then, w is a supersolution of (5.1). Since $\hat{g}$ is a solution of (5.1) and we have a comparison result, $\hat{g} \leq w$. ◻

Remark 5.2. When the space of controls is compact, one may take G to be equal to a positive constant. In that case $g = \hat{g}$.

DEFINITION 5.1. *We say that a comparison principle for* (2.2) *holds if whenever we have a USC viscosity subsolution v and an LSC viscosity supersolution w satisfying growth conditions $|v|, |w| \leq C\psi$ with $v(T, \cdot) \leq w(T, \cdot)$ on $\mathcal{O}$, then $v \leq w$.*

Remark 5.3. One cannot expect comparison up to time $t = 0$ for semicontinuous viscosity semisolutions, unless the viscosity property holds in the whole parabolic interior, which includes $t = 0$. This can be seen, for example, from [6] and [8]. The reader may note that we did prove the viscosity semisolution property for v^- and v^+ in the parabolic interior.

Now we are ready to state the main result of this section, which follows as a corollary of Theorems 3.1 and 4.1 and Lemma 5.1.

THEOREM 5.1. *Let us assume that a comparison principle for* (2.2) *holds. Moreover, we assume that Assumption* 5.1 *and assumptions in both Theorem* 3.1 *and Theorem* 4.1 *hold. Then, there exists a unique continuous (up to T) viscosity solution $v \in C([0, T] \times \mathcal{O})$ of the PDE* (2.2) *with terminal condition $v(T, \cdot) = \hat{g}(\cdot)$, satisfying the growth condition $|v| \leq C\psi$. Before time T we have*

$$v(t, x) = v^-(t, x) = v^+(t, x) = V(t, x) = \mathfrak{V}(t, x) \quad (t, x) \in [0, T) \times \mathcal{O}.$$

Proof. Since Assumption 5.1 holds, then $v^-(T-, \cdot) = v^+(T, \cdot) = \hat{g}(x)$. We now define the (still LSC) function

$$w(t, x) = \begin{cases} v^-(t, x), & 0 \leq t < T, x \in \mathcal{O}, \\ \hat{g}(x), & t = T, x \in \mathcal{O}. \end{cases}$$

By definition, $w \leq v^+$. At the same time, the function w is an LSC viscosity supersolution and v^+ is a USC viscosity subsolution of (2.2). Since $v^+(T \cdot) = w(T, \cdot)$ we can use comparison to conclude that $v^+ \leq w$, so

$$v^+ = w \in C([0, T] \times \mathcal{O}).$$

Denoting by $v = w = v^+$, the proof is complete. ◻

Remark 5.4. When the controls are unbounded, the value function may display a discontinuity at the terminal time T, as we expect that $v(T-, \cdot) = \hat{g}$ and $v(T, \cdot) = g$. (If $t \neq T$, it follows from the above theorem that the value function is continuous.) The discontinuity was already observed by Krylov in [15, p. 252], but the question of what the correct boundary condition should be was left open. For a particular model of superhedging, an answer was given in [4]. The technical arguments to treat such behavior close to the final time horizon were extended to more general models of superhedging in [7], [18]. A summary of such arguments can also be found in [17] or in the textbook [16]. One of our contributions is to show that this boundary condition holds without relying on the DPP. The proof of the boundary condition comes out as a simple conclusion from the stochastic Perron method.

Remark 5.5 (Fleming–Vermes). As we mentioned in the introduction, using our notation, Fleming and Vermes [13] and [12] prove that (with the notation (4.1)) we have

$$V = \mathfrak{v} = \inf\{\text{classic supersolutions}\}$$

under some technical assumptions (in particular, there is no boundary layer). The proof uses a sophisticated approximation/separation argument and the probabilistic representation of V, $\mathfrak{v}$.

The program we propose in the present paper can be summarized as

$$\boxed{\text{Theorem } 3.1 + \text{Theorem } 4.1 + \text{Comparison} \rightarrow V = \mathfrak{v} = \text{unique viscosity solution}}$$

However, in the absence of a comparison result for semicontinuous viscosity solutions, little can actually be said about the properties of the value function, following this approach.

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