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DYNAMIC PROGRAMMING AND PRICING OF CONTINGENT CLAIMS IN AN INCOMPLETE MARKET*

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Abstract. The problem of pricing contingent claims or options from the price dynamics of certain securities is well understood in the context of a complete financial market. This paper studies the same problem in an incomplete market. When the market is incomplete, prices cannot be derived from the absence of arbitrage, since it is not possible to replicate the payoff of a given contingent claim by a controlled portfolio of the basic securities. In this situation, there is a price range for the actual market price of the contingent claim. The maximum and minimum prices are studied using stochastic control methods.

The main result of this work is the determination that the maximum price is the smallest price that allows the seller to hedge completely by a controlled portfolio of the basic securities. A similar result is obtained for the minimum price (which corresponds to the purchase price).

Key words. option pricing, incomplete market, equivalent martingale measures, portfolio processes, stochastic control

AMS subject classifications. 90A09, 90C39, 93E25

Introduction. We study the problem of determining the price of a contingent claim from the price dynamics of certain securities (such as stocks and bonds). However, the price system alone cannot give a complete description of the exogenous uncertain environment; other information that is inside or outside the market is available and might influence market fluctuations. Therefore, the information structure used is as general as possible; in particular, it is not supposed to be generated by Brownian motions.

The primitive securities consist of a bond and n stocks, the latter being driven by a d-dimensional Brownian motion. Absence of arbitrage is assumed. The fluctuation in those prices is linked to the rest of the market (market fluctuations, change in rates, prices of other securities, and so on) and to other factors that are outside the market. The contingent claim is not linked only to the basic securities.

In §1, we formulate the basic problem of hedging by constructing a portfolio of the basic securities that attains (at least) the payoff of the contingent claim as its terminal wealth. Unlike in the complete market case, it is not possible to replicate the payoff of every contingent claim by a portfolio, and there are several probability measures that are equivalent to the initial probability, such that the discounted price processes are martingales. Several price systems are associated by duality to those martingale measures. Thus in this situation there is a price range for the actual market price of the contingent claim.

In §2, we study the maximum price using stochastic control methods. We show that the maximum price can be written as the difference of the (discounted) value of a portfolio and an optional increasing process that is equal to zero at time zero. We then state that the maximum price is the selling price defined as the smallest price that allows the seller to hedge completely by a controlled portfolio of the basic securities. A similar result is obtained for the minimum price (which corresponds to the purchase price). We also state that the class of contingent claims for which the supremum price is obtained for an optimal martingale measure is exactly the set of attainable claims.

In §3, we give a few methods for computing the maximum price (approximation methods, use of Bellman equations in the Markovian case). We also give two examples that illustrate the obtained results.

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1. Formulation of the problem.

1.1. The model. The basic securities consist of $n+1$ assets; they are the only assets that are available to agents for trading. One of them is a nonrisky asset (the bond), with price-per-unit $P_0(t)$ governed by the equation

$$dP_0(t) = P_0(t)r(t)dt. \tag{1}$$

The interest rate $r(t)$ is positive and bounded.

In addition to the bond, there are n risky securities (stocks). The price $P_i(t)$ for one share of the ith stock is modeled by the linear stochastic differential equation

$$dP_i(t) = P_i(t)\left[b_i(t)dt + \sum_{j=1}^{d} \sigma_{i,j}(t)dW_t^j\right]. \tag{2}$$

The information structure is modeled by a filtration $(F_t,\ 0 \le t \le T)$ that satisfies the standard hypotheses and is left quasi-continuous. The coefficients of the model r, b_i, $\sigma_{i,j}$ are taken to be predictable with respect to $\{F_t\}$. $W = (W_1, \dots, W_d)^*$ is a d-dimensional (F_t)-Brownian motion under P, with the "objective" probability taken as a primitive. We suppose that $n \le d$.

To be a reasonable model of securities markets, the prices of the basic securities should not allow one to create something out of nothing or to create free lunches. Thus, we will suppose the existence of d coefficients $\theta_1, \theta_2, \dots, \theta_d$ that are (F_t)-predictable processes such that

$$b_i(t) - r(t) = \sum_{j=1}^{d} \sigma_{ij}(t)\theta_j(t), \quad P\text{ a.s.}, \quad 1 \le i \le n.$$

$\theta_j(t)$ represents the risk premium associated with the source of uncertainty W_j; we suppose that θ_j is bounded.[1]

We adopt the following notation and make the following assumptions:

- We denote by b the column vector of stock appreciation rates $b = (b_1, \dots, b_n)^*$.
- For $1 \le i \le n$, let σ_i be the volatility row vector of the ith stock $\sigma_i = (\sigma_{i,1}, \dots, \sigma_{i,n})$
- Let $\sigma(t)$ be the volatility $(n \times d)$ matrix whose rows are $\sigma_1(t), \dots, \sigma_n(t)$. $\sigma(t)$ is supposed to be bounded.
- Let θ_t be the relative risk column vector $\theta = (\theta_1, \theta_2, \dots, \theta_d)^*$.

If we denote by **1** the vector whose every component is 1, the above equations are written as

$$b(t) - r_t\mathbf{1} = \sigma_t\theta_t, \quad P\text{ a.s.}$$

Furthermore, we will suppose that there exists a predictable vector α_t for which $\theta_t = \sigma_t^*\alpha_t$. (This hypothesis, which is not restrictive, will be explained in §1.8.)

Remark. Notice that if σ_t is taken to be injective, the above hypotheses are satisfied.

In the next section, we recall the characteristics of portfolios.

[1] For information concerning the equivalence between the assumption of no arbitrage and the existence of θ, one is referred to [Fö-Sc], [H-J-M], [An-St]. We thank the reviewers for the following references concerning this point: [Ch-Mu], [He-Pe], [Sch].

1.2. The portfolios. Let us consider an investor who can invest in the $n+1$ basic securities. At time 0, he invests the amount $x \geq 0$ in the $n+1$ securities. We shall denote by $X(t)$ the value of the amount invested at time t in the $n+1$ securities; $X(0) = x$. For $i = 1, \ldots, d$, we denote by $\pi_i(t)$ the amount that he invests in the ith stock at time t.

DEFINITION 1.2.1. *A portfolio strategy* $\pi(t) = (\pi_1(t), \ldots, \pi_n(t))$; $0 \leq t \leq T$, *is an* R^n*-valued process that is predictable with respect to* (F_t) *and satisfies*

$$\int_0^T \|\sigma_t^* \pi_t\|^2 \, dt < +\infty, \quad P \text{ a.s.}$$

The amount invested in the bond at time t is then given by

$$X(t) - \sum_{i=1}^n \pi_i(t).$$

Let $N(t)$ be the cumulative total sum of additional amounts that have been invested by the agent in the $n+1$ securities between 0 and t; N is a right-continuous left-limited (RCLL), (F_t)-optional process that satisfies $N_0 = 0$.

The value of the portfolio $X(t)$ is given by

$$dX_t = \sum_{i=1}^n \pi_i(t)[b_i(t) + \sigma_i(t) dW_t] + \left(X_t - \sum_{i=1}^n \pi_i(t) \right) r(t)dt + dN_t,$$

or equivalently,

$$dX_t = r_t X_t \, dt + \pi_t^* (b_t - r_t \mathbf{1}) dt + \pi_t^* \sigma_t \, dW_t + dN_t.$$

Remark. Loosely speaking, dN_t is the amount saved (or consumed if negative) during the time period $[t, t+dt]$.

We denote by $X^{\pi,N,x}(t)$ the value of the portfolio corresponding to the strategy (π, N) and the initial investment x.

- If $N = 0$, the portfolio is called *self-financing*, then $X^{\pi,0,x}(t)$ is the value at time t of the self-financing portfolio corresponding to the initial investment x and the portfolio π.
- If $N_t = -C_t$, where $(C_t, 0 \leq t \leq T)$ is an RCLL, (F_t)-optional increasing process that satisfies $C_0 = 0$, the portfolio is called a *portfolio strategy with consumption*; $X^{\pi,-C,x}(t)$ is the value at time t of the portfolio corresponding to the initial investment x, the portfolio π, and the process C_t that represents the cumulative amount the agent withdraws up to time t for consumption.
- If $N_t = D_t$ where $(D_t, 0 \leq t \leq T)$ is an RCLL, (F_t)-optional increasing process that satisfies $D_0 = 0$, the portfolio is called a *portfolio strategy with savings*; $X^{\pi,D,x}(t)$ is the value at time t of the portfolio corresponding to the initial investment x, the portfolio π, and the process D_t that represents the cumulative amount the agent adds in the portfolio up to time t.

1.3. Contingent claim B, selling price. Let T, a positive constant, be the terminal time for the problem.

DEFINITION 1.3.1. *A* contingent claim *B is a nonnegative*, F_T*-measurable random variable. It can be thought of as a contract or agreement that pays B at maturity T.*

The problem is to price this contingent claim. Let us consider a seller who wants to sell some contingent claim with payoff B and maturity T, between time 0 and time T. Suppose the seller chooses the price Y_t for B at any time t; more precisely, the seller must choose his price process $(Y_t,\, t \geq 0)$ (i.e., an RCLL optional nonnegative process that satisfies $Y_T = B$). Also, the seller does not want to run any risk of losing money. Therefore, he will only choose price processes that allow him to hedge completely by a controlled portfolio of the basic securities in the sense that, if at any time t, he sells the option B, and if at a later date, he buys it back, he wants to make a profit. More precisely, suppose that at time t, the seller sells the contingent claim at the price Y_t. He then invests this amount in the self-financing portfolio determined by π_t. At time $t + dt$, he buys back the contingent claim at the price Y_{t+dt} and sells the portfolio; he then makes a profit equal to

$$r_t Y_t\, dt + \pi_t^* \sigma_t (dWt + \theta_t\, dt) - dY_t.$$

Hence, we have the following definition.

DEFINITION 1.3.2. *A process Y_t is called a* price admissible for sellers *if Y_t is an* RCLL *nonnegative optional process that satisfies $Y_T = B$ and such that there exist a portfolio process π_t and an* RCLL, *(F_t)-optional increasing process $(C_t,\, 0 \leq t \leq T)$ such that $C_0 = 0$ and $dY_t = r_t Y_t\, dt + \pi_t^*\, \sigma_t(dWt + \theta_t\, dt) - dC_t$.*

Remark. It is equivalent to say that a process Y_t is a price admissible for sellers if there exists a hedging portfolio of B that is a portfolio with consumption (i.e., with withdrawals) whose value is equal to the price, that is, if there exists a portfolio process π_t and an RCLL, (F_t)-optional increasing process $(C_t,\, 0 \leq t \leq T)$ that satisfies $C_0 = 0$ and an initial investment x such that

$$X^{\pi,-C,x}(T) = B \quad \text{and} \quad Y_t = X^{\pi,-C,x}(t) \geq 0, \qquad 0 \leq t < T.$$

Note that every price admissible for sellers corresponds to one (and only one) hedging portfolio with consumption.

For the seller who sells the contingent claim B to an investor, such a strategy can be interpreted as follows:

- x is the price paid by the investor to the seller at time 0.
- π characterizes the hedging portfolio held by the seller.
- $C(t)$ represents the cumulative amount the seller withdraws from the hedging portfolio up to time t (which should be given to the investor, and since it is not, it is a profit for the seller).

Hence, it follows that x is the price paid not only for getting B at maturity T but also for getting additional amounts represented by the process $(C(t),\, 0 \leq t \leq T)$. Also, the seller's price will be the lowest price admissible for sellers. Thus, we define the selling price by Definition 1.3.3.

DEFINITION 1.3.3. *If it exists, the lowest price process admissible for sellers is called the* selling price.

Remark 1. We will see in §2 that such a process always exists, which is not obvious.

Remark 2. We could have defined the selling price as the essential infimum of the price processes admissible for sellers (but this would not define a stochastic process).

(Note that by symmetry, we can define the purchase price for B; it will be studied in §4.)

1.4. Discounting. Let β_t be the discount process given by

$$\beta_t = \exp\left\{ - \int_0^t r_s\, ds \right\}, \qquad 0 \leq t \leq T.$$

We denote by $P_i^d(t)$, $\pi^d(t)$, $C^d(t)$ the discounted price process, the discounted portfolio process associated with the portfolio $\pi(t)$, and the discounted consumption process associated with the consumption $C(t)$, respectively. For $0 \le t \le T$, we have

$$\begin{aligned} P_i^d(t) &= \beta_t P_i(t) \qquad (1 \le i \le n), \\ \pi^d(t) &= \beta_t \pi(t), \\ C_t^d &= \int_0^t \beta_s \, dC_s. \end{aligned}$$

We have the following equation for the discounted price process:

$$dP_i^d(t) = P_i^d(t)[(b_i(t) - r(t))dt + \sigma_i(t)dW_t]. \tag{3}$$

The discounted value of the portfolio with withdrawals X^d associated with the portfolio-consumption strategy (π_t, C_t) is governed by the equation

$$dX_t^d = (\pi_t^d)^*[(b(t) - r(t)\,\mathbf{1})dt + \sigma(t)dW_t] - dC_t^d,$$

that is,

$$dX_t^d = (\pi_t^d)^* \sigma_t(\theta_t \, dt + dW_t) - dC_t^d. \tag{4}$$

Also, for a contingent claim B, we denote by B^d the discounted contingent claim

$$B^d = \beta_T B.$$

We now come back to the problem of pricing the contingent claim B from the price dynamics of the $n + 1$ securities. We begin by recalling the theory of contingent claim valuation in the context of a complete market (see [Ha-Kr], [Ha-Pl], [Duf], and [Kar]). It is well known that the reference probability Q defined below has a fundamental role. Hereafter, we will often use the Girsanov theorem. (See §A.1, where we recall a general form of this theorem.)

1.5. The reference probability Q.

Notation. Let N_t be a local martingale (RCLL) under P with respect to $\{F_t\}$, such that $N_0 = 0$. We denote by $\mathcal{E}(N)_t$ the *exponential of N*, that is, the solution of the stochastic differential equation (SDE)

$$dZ_t = Z_{t^-} \, dN_t, \quad 0 \le t \le T, \quad Z_0 = 1.$$

This process is a local martingale under P.

Let $Z_0(t)$ be the exponential local martingale of

$$\left(- \int_0^t \theta_s^* \, dW_s, \, 0 \le t \le T \right)$$

that is,

$$Z_0(t) = \exp\left\{ - \int_0^t \theta_s^* \, dW_s - \frac{1}{2} \int_0^t \|\theta_s\|^2 \, ds \right\}.$$

Because θ is bounded, $Z_0(t)$, $0 \le t \le T$ is martingale under P.

We then define Q as the probability measure equivalent to P on F_T that admits the Radon–Nikodym derivative $Z_0(T)$. Let

$$\widetilde{W}_t = W_t + \int_0^t \theta_s \, ds, \qquad 0 \le t \le T.$$

By the Girsanov theorem, $(\widetilde{W}_t,\, 0 \le t \le T)$ is an (F_t)-Brownian motion under Q.

The SDE (3) relative to the discounted prices may be written as

$$dP_i^d(t) = P_i^d(t)\sigma_i(t)d\widetilde{W}_t. \tag{5}$$

Also, the discounted value X^d of the portfolio with withdrawals associated with the portfolio π_t, consumption C_t, and initial investment x is given by

$$X_t^d = x + \int_0^t (\pi_s^d)^* \sigma_s\, d\widetilde{W}_s - C_t^d, \quad Q\,\text{a.s.}, \quad 0 \le t \le T. \tag{6}$$

Notice that the prices of the basic securities are Q-martingales and the prices (of the contingent claim) admissible for sellers are Q-supermartingales.

1.6. Pricing in a complete market. We review briefly in this section important results for use later in the treatment of the incomplete market case.

DEFINITION 1.6.1. *The security market is said to be* complete *if the filtration (F_t) is that generated by the Brownian motion W_t, $n = d$, and σ has full rank. It means that all the sources of uncertainty can be explained by the price dynamics of the basic securities. If it is not complete, the market will be called* incomplete.

Recall that if the market is complete, it is possible to construct a portfolio that attains as its final wealth any contingent chain B that is integrable under Q, that is, there exist some $x \ge 0$ and some portfolio (π_t) satisfying

$$B^d = x + \int_0^T (\pi_u^d)^* \sigma_u\, d\widetilde{W}_u, \quad P\,\text{a.s.}$$

and such that the process defined by

$$x + \int_0^t (\pi_u^d)^* \sigma_u\, d\widetilde{W}_u, \qquad 0 \le t \le T$$

is a martingale under Q.

It is clear that $x = E_Q(B^d)$. This property allows us to derive the price for any contingent claim from an absence of arbitrage.

PROPOSITION 1.6.1. *In a complete market, every contingent claim (Q-integrable) is priced by arbitrage. This price is given by the expectation of the discounted contingent claim under Q, which is the unique probability measure under which the discounted prices of the basic stocks are martingales.*

Proof. Let P_B be the price for the contingent claim B. Suppose that P_B is strictly greater than $E_Q(B^d)$; then there exists some opportunity for arbitrage. For example, you can sell the contingent claim at $t = 0$ at the price P_B and invest the amount P_B in the hedging portfolio determined by π. At time T, you pay the amount B to your buyer and sell the portfolio, whose value is given by

$$\beta_T^{-1}\left(P_B + \int_0^T (\pi_u^d)^* \sigma_u\, d\widetilde{W}_u\right).$$

From an initial wealth equal to 0, at time T, you make a strictly positive profit equal to $(\beta_T)^{-1}(P_B - x)$. Also, if $P_B < E_Q(B^d)$, there exists an arbitrage opportunity. Hence, $P_B = E_Q(B^d)$.

The fact that Q is the unique probability that is equivalent to P such that the discounted price processes of the basic stocks are martingales can be easily proved using the Girsanov theorem. (The proof is similar to that of Proposition 1.8.1.) □

Remark 1. The price at time t for the contingent claim B can be determined by an absence of arbitrage and it is given by $E_Q[B^d/F_t]$.

Remark 2. It should be emphasized that the price system is associated by duality to the probability measure Q, which is equivalent to P and under which the discounted price processes are martingales.

Remark 3. In the case of a complete market, the arbitrage-free price coincides with the selling price. The hedging portfolio is a self-financing portfolio associated with the portfolio process π and the initial investment x. Clearly, if the market is complete the consumption processes are unnecessary since it is always possible to replicate the payoff of a given contingent claim by a self-financing portfolio. In the complete market contingent claim valuation, notice the fundamental roles of the construction of a hedging portfolio and the reference probability Q, which is the unique probability measure equivalent to P and under which the discounted price processes are martingales. Those remarks should be kept in mind when considering the more difficult case of an incomplete market.

We now turn to the consideration of an incomplete market. Recall that, contrary to the complete market case, the price system cannot suffice in itself to give a complete description of the environment. As a result, agents will not be able to replicate the payoff of every contingent claim by a self-financing portfolio of the basic securities and to price every contingent claim by arbitrage. Also, there exist several probability measures that are equivalent to P and under which the discounted price processes are martingales, associated by duality to different price systems.

1.7. The P-martingale measures and the attainable contingent claims.

DEFINITION 1.7.1. *Any probability measure that is equivalent to P on F_T and is such that the discounted price processes (of the basic claims) are martingales is called a P-martingale measure.*

We denote by $\mathcal{P}$ the set of all P-martingale measures. Notice that Q belongs to $\mathcal{P}$ and that, if the market is complete, $\mathcal{P} = \{Q\}$. If the market is incomplete, there are several P-martingale measures. Each martingale measure can naturally be associated by duality with a price system (for more details see [Ha-Kr]).

Assumption. Hereafter, the contingent claim B is supposed to be such that there exists a price admissible for sellers, or equivalently B is supposed to be smaller than the value of a self-financing portfolio, that is, B satisfies

$$B \le X^{H,0,y}(T), \quad P\,\text{a.s.}$$

for some portfolio strategy H and initial investment $y \ge 0$. Also, we make the following technical assumption: $(X^{H,0,y})^d$ is supposed to be a square-integrable martingale under each P-martingale measure. This assumption will allow us to work with square integrable martingales but the main results of this paper remain true under the weaker hypothesis (see §A.3).

$$\sup_{R \in \mathcal{P}} E_R(B^d) < +\infty.$$

DEFINITION 1.7.2. *Every real that can be written $E_R(B^d)$, where R is a P-martingale measure, is called a* possible price at time $t = 0$ for B.

More generally, any random variable $E_R(B^d/F_t)$, where R is a P-martingale measure, is called a possible price at time t for B. However, there exists a class of contingent claims such that the price is unique, that is, the set of possible prices contains a unique element. We will show that this class is exactly the set of contingent claims that can be synthetized by a controlled portfolio of the basic securities (called "attainable").

DEFINITION 1.7.3. *A contingent claim B is said to be* attainable *if there exist some $x \geq 0$ and some portfolio process π such that*

$$B^d = x + \int_0^T (\pi_u^d)^* \sigma_u \, d\widetilde{W}_u, \quad P\,a.s.$$

and

$$E_Q\left[\int_0^T \|\sigma_s^* \pi_s^d\|^2 \, ds\right] < +\infty.$$

As in the complete market case, an attainable contingent claim B can be priced by arbitrage; therefore, $x = E_Q(B^d)$ is the arbitrage-free price at $t = 0$. Also, we state the following property.

PROPOSITION 1.7.1. *B is attainable if and only if $E_R(B^d)$ is constant over all $R \in \mathcal{P}$, that is, there is only one possible price.*

Remark. We show only one implication. The opposite implication follows from Theorem 2.3.2.

Proof. Let $x \geq 0$ and a portfolio process π such that

$$B^d = x + \int_0^T (\pi_u^d)^* \sigma_u \, d\widetilde{W}_u, \quad P \text{ a.s.}$$

and

$$E_Q\left[\int_0^T \|\sigma_s^* \pi_s^d\|^2 \, ds\right] < +\infty.$$

Let R be a P-martingale measure. The discounted prices $P_i^d(t)$ are martingales under R. Hence, the process defined by

$$x + \int_0^t \sum_{i=1}^n \pi_i^d(u) \sigma_i(u) d\widetilde{W}_u = x + \int_0^t \sum_{i=1}^n \frac{\pi_i^d(u)}{P_i^d(u)} \, dP_i^d(u), \qquad 0 \leq t \leq T$$

is clearly a local martingale under R. It is lower than $(X^{H,0,y})^d$, the discounted value of the portfolio (associated with H and y), because it is equal to $E_Q[B^d/F_t]$. Therefore, it is a martingale under R (by Proposition 1.a in the Appendix) because $(X^{H,0,y})^d$ is a martingale under R. Hence, $E_R(B^d) = x$. □

Remark. If B is attainable, then the price for B at $t \geq 0$ can be derived by an absence of arbitrage and $E_R[B^d/F_t]$ does not depend on $R \in \mathcal{P}$, that is, the set of possible prices contains only one element given by

$$x + \int_0^t \pi_u^{d^*} \sigma_u \, d\widetilde{W}_u.$$

If B is not attainable, there are several prices for B and B cannot be priced by arbitrage. Thus, it seems interesting to determine the bounds of the set of possible prices for B. At $t = 0$, the price for B is worth not less than $\inf_{R \in \mathcal{P}} E_R(B^d)$ and not more than $\sup_{R \in \mathcal{P}} E_R(B^d)$.

Using optimal control techniques, we shall study dynamically those maximum and minimum prices. In particular, we shall show that the supremum of the possible prices is equal to the selling price.

Before proceeding with the analysis of the maximum price, let us characterize the set of P-martingale measures. Recall that Pagès [Pag] has already characterized this set in the context of a Brownian model (see also [KLSX] for utility maximization problems in that context). Also, Ansel and Stricker [An-St] have shown that the market model contains no arbitrage opportunities if and only if $\mathcal{P}$ is nonempty. Furthermore, they have characterized the set of P-martingale measures in a different context: $n = 1$ and the price process (one-dimensional) is supposed to be any continuous semimartingale (actually, the arbitrage-free hypothesis implies that it can be written $M + \int \alpha_s d\langle M\rangle_s$ for some continuous local martingale M and some predictable process α).

1.8. Characterization of the P-martingale measures.

PROPOSITION 1.8.1. *The following properties are equivalent*:

(i) *R is a P-martingale measure.*

(ii) *R is a probability equivalent to P that admits the density*

$$\left.\frac{dR}{dP}\right|_{F_T} = \varepsilon\left(-\int_0^{\cdot} \theta_s^*\, dW_s + N\right)_T$$

where N_t is a local martingale that is orthogonal (in the quadratic variation sense) to the prices of the basic securities

$$\left\langle N, \int_0 \sigma_i(s)d\widetilde{W}_s\right\rangle_T = 0 \quad \forall i \in \{1, 2, \dots, n\}, \quad Q \text{ a.s.}$$

Proof. Let us show that (i) and (ii) are equivalent. Let R be some probability measure that is equivalent to P on F_T. Put

$$L_t = \left.\frac{dR}{dP}\right|_{F_t}, \qquad 0 \le t \le T.$$

L_t is a strictly positive martingale under P. We introduce the local martingale M given by

$$M_t = \int_{0+}^t \frac{1}{L_{s-}}\, dL_s, \qquad 0 \le t \le T,$$

so that

$$L_t = \varepsilon(M)_t, \qquad 0 \le t \le T.$$

Note that, for any $i \le n$, $P_i^d(t)$ is continuous, and hence locally bounded; therefore, by Proposition 1.b in the Appendix, it follows that $\langle M, P_i^d\rangle$ exists. By the Girsanov theorem (Appendix Corollary 1.A (i)), for any $i = 1, \dots, n$, $P_i^d(t)$ is a local martingale under R if and only if

$$(b_i(t) - r_t)dt + d\left\langle M, \int_0^{\cdot} \sigma_i(s)dW_s\right\rangle_t = 0,$$

which can also be written, because $b_i - r = \sigma_i\theta$, as

$$\left\langle M + \int_0^{\cdot} \theta_s^*\, dW_s, \int_0^{\cdot} \sigma_i(s)dW_s\right\rangle_T = 0, \quad P \text{ a.s.}$$

The result therefore follows. $\square$

In particular, the reference probability Q is an equivalent martingale measure. Also, if we suppose that θ belongs to the range of σ^*, then Q is a minimal P-martingale measure, in the following sense (see [Fö-Sc] or [An-St]).

DEFINITION 1.8.1. *A P-martingale measure R will be called minimal if any local P-martingale that is orthogonal to $\int \sigma_i \, dW$, for $1 \leq i \leq n$, under P, remains a local martingale under R.*

We state the following property.

PROPOSITION 1.8.2. *The following properties are equivalent*:

(i) *Q is a minimal probability.*

(ii) *There is a predictable vector process α_t such that $\theta_t = \sigma_t^* \alpha_t$.*

Remark. Suppose that (i) or (ii) is satisfied. Let R be a P-martingale measure. By Proposition 1.8.1, there exists some local martingale N with $N_0 = 0$, orthogonal to $\int \sigma_i \, dW$, $1 \leq i \leq n$, such that

$$\left.\frac{dR}{dP}\right|_{F_T} = \varepsilon\left(-\int_0^{\cdot} \theta_s^* \, dW_s + N\right)_T .$$

Recall that

$$\left.\frac{dQ}{dP}\right|_{F_T} = \varepsilon\left(-\int_0^{\cdot} \theta_s^* \, dW_s\right)_T .$$

If N is locally square integrable, then R is minimal if and only if $N = 0$, that is, $R = Q$. Indeed, if R is minimal, then N is a local martingale under R (because it is a local P-martingale orthogonal to $\int \sigma_i \, dW$, $1 \leq i \leq n$). If N is supposed to be locally square integrable, then $\langle -\int \theta^* \, dW + N, N\rangle$ exists and we can apply the Girsanov theorem (Appendix Corollary 1.A (i))

$$\left\langle -\int \theta^* \, dW + N,\, N\right\rangle = 0,$$

that is, $\langle N, N\rangle = 0$ (because $\theta_t = \sigma_t^* \alpha_t$). Hence, $N = 0$.

Proof. The definition gives that the fact that Q is minimal is equivalent to the following property:

(**) Any local P-martingale orthogonal to $\int \sigma_i \, dW$, for $1 \leq i \leq n$, is a local martingale under Q.

By the Girsanov theorem, Corollary 1.A (i), property (**) is equivalent to the following one. Any local P-martingale orthogonal to $\int \sigma_i \, dW$, $1 \leq i \leq n$, is orthogonal to $\int \theta^* \, dW$. By some results on stable subspaces of martingales and orthogonality (see [De-Me, pp. 371, 372, VIII-46–VIII-49]), (**) is equivalent to the fact that $\int \theta^* \, dW$ belongs to the space generated by $\int \sigma_i \, dW$, $1 \leq i \leq n$, which is equivalent to the existence of n predictable processes $\alpha_i(t)$, $1 \leq i \leq n$, such that

$$\int \theta^* \, dW = \sum_{i=1}^{n} \int \alpha_i \sigma_i \, dW,$$

that is, $\theta_t = \sigma_t^* \alpha_t$ where $\alpha_t = (\alpha_1(t), \dots, \alpha_n(t))^*$. □

Hereafter, the above hypothesis ((i) or (ii)) is supposed to be satisfied. [2]

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[2] Actually, this is not restrictive because if this hypothesis is not satisfied, just replace θ_t by its orthogonal projection $\theta_t^{\mid}$ on the range of σ_t^* in all the equations (indeed, $\sigma_t \theta_t = \sigma_t \theta_t^{\mid}$).

We introduce the following notation.

Notation. We denote by D the set of local (F_t)-martingales $(N_t,\ 0 \leq t \leq T)$, with $N_0 = 0$, satisfying the following three properties.

(i) The jumps of N are strictly greater than -1 so that $\varepsilon(N)_t,\ 0 \leq t \leq T$, is a strictly positive local martingale.

(ii) $\varepsilon(N)_t,\ 0 \leq t \leq T$, is a martingale under Q.

(iii)

$$\left\langle N, \int_0^{\cdot} \sigma_i(s) d\widetilde{W}_s \right\rangle_T = 0 \quad \forall i \in \{1, 2, \ldots, n\}, \quad Q \text{ a.s.}$$

For any local (F_t)-martingale N belonging to D, define Q^N as the probability measure equivalent to Q that admits $\mathcal{E}(N)_T$ as a Radon–Nikodym derivative with respect to Q on F_T. Q^N is then a P-martingale measure.

PROPOSITION 1.8.3. *The mapping $N \longrightarrow Q^N$ is a one-to-one mapping that carries D onto $\mathcal{P}$.*

Proof. We have clearly

$$\varepsilon\left(-\int \theta_u^* \, dW_u + N\right)_T = \varepsilon\left(-\int \theta_u^* \, dW_u\right)_T \varepsilon(N)_T.$$

Indeed, $\langle N, \int \theta_u^* \, dW_u \rangle = 0$ because it is supposed that $\theta_t = \sigma_t^* \alpha_t$. The result now follows easily. □

Hereafter, to simplify notation, we will suppose that $r = 0$. Then the SDEs relative to the prices are written as

$$dP_i(t) = P_i(t)\sigma_i(t) d\widetilde{W}_t.$$

Also, the value $X^{\pi,-c,x}(t)$ of the portfolio with consumption associated with portfolio π, consumption C, and initial investment x is given by

$$X_t^{\pi,-C,x} = x + \int_0^t \pi_t^* \sigma_t \, d\widetilde{W}_t - C_t \quad \forall t \in [0, T], \quad Q \text{ a.s.}$$

This seems highly restrictive but it is not. All the results we obtain can be generalized to the case $r \neq 0$. In all the properties, just replace B by B^d and the prices, portfolio, and consumption processes by the discounted processes defined above.

We now turn to the study of the maximum price.

2. Dynamical study of the maximum price.

2.1. Predictable decomposition of the maximum price. The supremum of the possible prices for B at time 0 is given by

$$\sup_{R \in \mathcal{P}} E_R(B) = \sup_{N \in D} E_{Q^N}(B).$$

Also, the essential supremum of the possible prices for B at time t is given by

$$\operatorname{ess\,sup}_{R \in \mathcal{P}} E_R(B/F_t) = \operatorname{ess\,sup}_{N \in D} E_{Q^N}(B/F_t).$$

Using dynamical programming methods (see [ElK]), we have the following theorem. (The proof is given in the appendix.)

THEOREM 2.1.1. *There exists an* RCLL *process* $(J_t,\ 0 \le t \le T)$ *so that, for each* t

$$J_t = \operatorname{ess\ sup}_{N \in D} E_{Q^N}[B/F_t].$$

J_t *is characterized as the smallest right continuous supermartingale under* Q_N, *for every* N *belonging to* D, *which is equal to* B *at time* T. *Also,* N^* *is optimal* (i.e., $J_t = E_{Q^{N^*}}(B/F_t)$, Q *a.s.,* $0 \le t \le T$) *if and only if* J_t *is a martingale under* Q_{N^*}.

Before continuing the dynamical study of J_t, recall the hypothesis satisfied by B (introduced in §1.7):

$$B \le y + \int_0^T H_s^* \sigma_s \, d\widetilde{W}_s, \quad Q \text{ a.s.},$$

where y is a positive constant and H_t is a portfolio process that satisfies

$$E_{Q^N}\left[\int_0^T \|\sigma_s^* H_s\|^2 \, ds\right] < +\infty$$

for each $N \in D$.

This hypothesis implies the following result.

PROPOSITION 2.1.1. J_t *is of class* D *and satisfies*

$$J_t \le y + \int_0^t H_s^* \sigma_s \, d\widetilde{W}_s, \quad Q \text{ a.s.}, \quad 0 \le t \le T.$$

Proof. The process given by

$$y + \int_0^t H_s^* \sigma_s \, d\widetilde{W}_s$$

is a square integrable martingale under each P-martingale measure. Thus, for each $N \in D$,

$$E_{Q^N}[B/F_t] \le y + \int_0^t H_s^* \sigma_s \, d\widetilde{W}_s, \quad Q \text{ a.s.}, \quad 0 \le t \le T,$$

and the desired result clearly follows. □

The fact that J_t is a supermartingale under Q shows that J_t can be written under Q as the difference between a local Q-martingale and a predictable increasing process (and this decomposition is unique). The above properties relative to J_t will allow us to write the Q-martingale as the sum of a portfolio and a martingale j that may be characterized.

THEOREM 2.1.2. *There exist a portfolio process* φ_t, *a right continuous increasing predictable process* A_t *with* $A_0 = 0$, *and a purely discontinuous* Q-*martingale* j_t *such that*

$$\forall t \in [0,T],\ J_t = J_0 + \int_0^t \varphi_s^* \sigma_s \, d\widetilde{W}_s + j_t - A_t, \quad Q \text{ a.s.} \tag{7}$$

Remark. Notice that the assumption made on B is not necessary. Theorem 2.1.2 remains true under the hypothesis

$$\sup_{N \in D} E_{Q^N}(B) < \infty,$$

but in this case J_t is not generally of class D and the process j is only a local martingale. The arguments of the proof still hold, but it is a bit more complicated technically because $\langle j \rangle$ is not always defined (see §A.3).

Scheme of the proof. J_t is a Q-supermartingale; hence, it admits a unique decomposition as a local martingale M_t minus an increasing predictable process $A_t : J_t = M_t - A_t$. The martingale M_t admits the Kunita–Watanabe decomposition

$$M_t = J_0 + \int_0^t \varphi_s^* \sigma_s \, d\widetilde{W}_s + j_t \quad \forall t \in [0,T], \quad Q \text{ a.s.}$$

for some predictable process φ and some Q-local martingale j, such that

$$\left\langle j, \int_0^{\cdot} \sigma_i(s) dW_s \right\rangle_T = 0, \quad Q \text{ a.s.} \quad \forall i \in \{1, \ldots, n\}.$$

Using the fact that J_t is a supermartingale under each P-martingale measure, we show that the continuous part of j is equal to zero.

Proof. $J_t - E_Q[B/F_t]$ is a positive RCLL optional supermartingale that is equal to zero at time T and lower than the Q-martingale $X^{H,0,y}$. By the Doob–Meyer decomposition theorem (cf. [De-Me, Thm. VII.8, p. 211]), there exists a Q-integrable right continuous increasing predictable process A_t with $A_0 = 0$ such that

$$J_t - E_Q[B/F_t] = E_Q[A_T/F_t] - A_t, \qquad 0 \leq t \leq T.$$

A_T is square integrable under Q since

$$E_Q[(A_T)^2] \leq 4\, E_Q[(\sup_{0 \leq t \leq T} J_t)^2] \leq 4\, E_Q[(X_T^{H,0,y})^2]$$

(see [De-Me, inequality VII.15.1, p. 221]).

Put $M_t = E_Q[A_T + B/F_t]$, $0 \leq t \leq T$. $M_t = J_t + A_t$ is square integrablé martingale under Q; hence, $\langle M \rangle$ exists. Thus, M_t admits the Kunita–Watanabe decomposition with respect to the Q-square integrable martingales $\int \sigma_i \, dW_t$, that is, there exist a predictable process φ_t and a square integrable Q-martingale (RCLL) j_t with $j_0 = 0$ such that

$$E_Q\left[\int_0^T \|\sigma_s^* \varphi_s\|^2 \, ds\right] < +\infty,$$

$$\left\langle j, \int_0^{\cdot} \sigma_i(s) dW_s \right\rangle_T = 0, \quad Q \text{ a.s.} \quad \forall i \in \{1, \ldots, n\},$$

and

$$M_t = J_0 + \int_0^t \varphi_s^* \sigma_s \, d\widetilde{W}_s + j_t \quad \forall t \in [0,T], \quad Q \text{ a.s.}$$

(for the Kunita–Watanabe decompositions, see [De-Me, p. 374, VIII-52]).

Now, j can be written as the sum of its continuous martingale part j^c and its purely discontinuous martingale part j^d

$$j = j^c + j^d.$$

It remains to be shown that j^c is equal to zero.

This result can be obtained using the following lemma (which comes from the fact that J_t is a supermartingale under each P-martingale measure).

LEMMA 2.1.1. *For all $N \in D$, $A_t - \langle N, j\rangle_t$ is an increasing process.*

Proof. By Theorem 2.1.1, we have the following property: for each $N \in D$, J_t is a supermartingale under Q^N. By the Girsanov theorem (Corollary 1A (ii) in the Appendix), this property is equivalent to the following:

$$\forall N \in D,\ -A_t + \langle N, M\rangle_t \text{ is a decreasing process,}$$

which may be written, because $N \in D$, as

$$\forall N \in D,\ A_t - \langle N, j\rangle_t \text{ is an increasing process.} \qquad \square$$

Lemma 2.1.1, applied to some N that can be written as a stochastic integral with respect to j^c, will allow us to show that j^c is equal to zero. We take

$$N_t = \int_0^t n_s \, dj_s^c,$$

where n is a bounded predictable process. If $\mathcal{E}(N)_t$ is a martingale, then N belongs to D and Lemma 2.1.1 applied to N yields the fact that $A - \int n_s \, d\langle j^c\rangle_s$ is increasing. It now remains to decompose the measure dA_t with respect to $d\langle j^c\rangle_t$ and to choose n.

$\langle j^c\rangle$ is integrable because j is square integrable. By the Lebesgue decomposition theorem, there exist a positive predictable process h which belongs to $L^1([0,T] \times \Omega, d\langle j^c\rangle_t \, dQ)$ and an integrable predictable increasing process B such that

$$dA_t = h_t d\langle j^c\rangle_t + dB_t$$

and such that, Q almost surely, the measure dB_t is singular with respect to $d\langle j^c\rangle_t$. For each integer p, we can write the following decomposition:

$$dA_t = h_t \, \mathbf{1}_{\{h(t) \le p\}} d\langle j^c\rangle_t + dB_t^p,$$

where, Q almost surely, the measure dB^p is singular with respect to the measure $\mathbf{1}_{\{h(t) \le p\}} d\langle j^c\rangle_t$.

Let N be given by

$$N_t = \int_0^t \mathbf{1}_{\{h_s \le p\}} (1 + h_s) dj_s^c.$$

Clearly, we may choose a sequence of stopping times T_n, $n \ge 0$, such that $T_n \uparrow T$ almost surely as n tends to infinity and for each $n \in \mathbb{N}$, $\mathcal{E}(N)^{T_n}$ $(= \mathcal{E}(N^{T_n}))$ is a martingale. Thus, for each $n \in \mathbb{N}$, N^{T_n} belongs to D. Lemma 2.1.1 applied to N^{T_n} shows that for each $n \in \mathbb{N}$, $A_t - \langle N^{T_n}, j\rangle_t$ is an increasing process. Hence, for each $n \in \mathbb{N}$, $A^{T_n} - \langle N, j\rangle^{T_n}$ is an increasing process and thus $A - \langle N, j\rangle$ is an increasing process.

Because, Q almost surely, the measure dB^p is singular with respect to the measure $\mathbf{1}_{\{h(t) \le p\}} d\langle j^c\rangle_t$, it follows that the process given by

$$\int_0^t \mathbf{1}_{\{h_s \le p\}} h_s d\langle j^c\rangle_s - \int_0^t \mathbf{1}_{\{h_s \le p\}} (1 + h_s) d\langle j^c\rangle_s$$

is an increasing process. Hence, for all $p \in \mathbb{N}$, Q almost surely, $-\langle j^c\rangle_t$ is increasing on $\{t / h(t) \le p\}$ and this yields the equality

$$\langle j^c\rangle_T = 0, \quad Q \text{ a.s.}$$

Hence, $j^c = 0$. $\square$

2.2. The Brownian case. We now turn to the case when the filtration is generated by the Brownian motion. In this case, Theorem 2.1.2 takes a more simple form: j is equal to zero because every local martingale is continuous. Thus, the maximum price process can be written as the difference of a self-financing portfolio and a predictable increasing process that is equal to zero at time zero. This remarkable decomposition should be stressed; it is similar to that of the price for an American option in a complete market. (One is referred to the theory of American option pricing [Kar].)

THEOREM 2.2.1. *There exist a portfolio process φ_t and an increasing (F_t)-predictable right continuous process A_t with $A_0 = 0$ such that*

$$\forall\, t \in [0,T],\ J_t = J_0 + \int_0^t \varphi_s^* \sigma_s \, d\widetilde{W}_s - A_t, \quad Q\ q.s. \tag{8}$$

Remark. Because the process A_t is predictable, we can consider only portfolios with consumption for which the consumption process is predictable. We have the following corollaries.

COROLLARY 2.2.1. *J_t is the lowest price process admissible for sellers*; *it is the selling price for B.*

Proof. J_t is a price process admissible for sellers because
- φ_t is a portfolio process,
- A_t is an optional (because it is predictable) RCLL increasing process with $A_0 = 0$, and
- J_t is positive and $J_T = B$.

Let us show that it is the lowest, that is, that for every price process admissible for sellers X_t, we have $J_t \leq X_t$. Let X_t be a price process admissible for sellers. Suppose that X is a supermartingale under any P-martingale measure. Then, by the characterization of J_t (Theorem 2.1.1), it follows that $J_t \leq X_t$. It remains to show that X is a supermartingale under every P-martingale measure. Now, X is a positive process such that $X_T = B$ and such that there exist a portfolio process π_t and an RCLL optional increasing process C_t with $C_0 = 0$ satisfying $X_t = X^{\pi,-C,x}(t)$ (the value of the portfolio with consumption associated with (π_t, C_t)), that is,

$$X_t = x + \int_0^t \pi_u^* \sigma_u \, d\widetilde{W}_u - C_t \geq 0, \quad Q \text{ a.s.}$$

Let N be a local martingale of D. Under Q^N, $x + \int_0^t \pi_u^* \sigma_u \, d\widetilde{W}_u$ is a positive local martingale, hence, a supermartingale. Also, C_T is integrable under Q^N. Hence, X is a supermartingale under Q^N. $\square$

2.3. Optional decomposition of the maximum price. When the filtration is not that generated by the Brownian motion, j is not equal to zero, as is shown by example 2 in §3.4. As a result of the constraint $\Delta N > -1$, Lemma 2.1.1 does not imply that j is equal to zero. Thus, the predictable decomposition of J_t under Q is not the good one. The good decomposition will be the optional decomposition (Theorem 2.3.1). Using Lemma 2.1.1 (in other words, the fact that J_t is a supermartingale under each P-martingale measure), we will show that j_t is a process with negative jumps only such that the process f defined by $f_t = A_t - j_t$ is a nondecreasing process.

Let us decompose j with respect to the sign of its jumps:

$$j = j^+ + j^-,$$

where j^+ (respectively, j^-) is the compensated integral of $\mathbf{1}_{\{\Delta j(t)>0\}}$ (respectively, $\mathbf{1}_{\{\Delta j(t)<0\}}$) with respect to j, that is,

$$j^+ = \mathbf{1}_{\{\Delta j>0\}} \cdot j; \qquad j^- = \mathbf{1}_{\{\Delta j<0\}} \cdot j.$$

Notice that j^+ and j^- are square integrable martingales.

Concerning j^+, the part of j corresponding to the positive jumps of j, Lemma 2.1.1 applied to a well-chosen local martingale N will allow us to show that $j^+ = 0$. (The proof is the same as that for $j^c = 0$ in the proof of Theorem 2.1.2.)

PROPOSITION 2.3.1. *The jumps of the purely discontinuous martingale j are negative.*

Proof. We take $N_t = \int_0^t n_s \, dj_s^+$, where n is a bounded positive predictable process. Notice that N is a square integrable martingale that admits only positive jumps.

It now remains to decompose the measure dA_t with respect to $d\langle j^+\rangle_t$ and choose the process n_t. Using the same arguments as those used in the proof of Theorem 2.1.2 (replacing j^c by j^+), we obtain the desired result. □

We now come to the most important result, which will allow us to characterize the supremum of the possible prices for B.

THEOREM 2.3.1. *The process $A_t - j_t$ is an increasing process, hence, if we denote it by f_t,*

$$\forall t \in [0,T],\ J_t = J_0 + \int_0^t \varphi_s^* \sigma_s \, d\widetilde{W}_s - f_t, \quad Q\ a.s. \tag{9}$$

In particular,

$$B = J_0 + \int_0^T \varphi_s^* \sigma_s \, d\widetilde{W}_s - f_T, \quad Q\ a.s.$$

Remark. The results of Proposition 2.3.1 and Theorem 2.3.1 still hold under the weaker hypothesis

$$\sup_{N \in D} E_{Q^N}(B) < +\infty$$

(see §A.3).

Proof. Put $f_t = A_t - j_t$. By definition, f is an RCLL process. Hereafter, we will adopt the following notation: for any locally integrable finite variation RCLL adapted process V, we denote by V^P its predictable compensator.

For any $\mathcal{E} \in]0,1[$, define

$$u_t^{\mathcal{E}} = \sum_{s \le t} \Delta j_s \mathbf{1}_{\Delta j_s < -\mathcal{E}}$$

and $j^{\mathcal{E}} = u^{\mathcal{E}} - (u^{\mathcal{E}})^P$. Recall the following result (see [De-Me, p. 369, VII-44]): $j^{\mathcal{E}}$ locally converges in M^1 to j as $\mathcal{E} \to 0$, that is, there exists a sequence of stopping times T_k, $k \in \mathbb{N}$, such that $T_k \uparrow T$ almost surely as k tends to infinity and for every $k \in \mathbb{N}$,

$$E[\sup_{s \le T_k} (j_s^{\mathcal{E}})] < +\infty$$

and

$$\lim_{\mathcal{E} \to 0} E[\sup_{s \le T_k} |j_s^{\mathcal{E}} - j_s|] = 0.$$

Suppose we have shown that for all $\mathcal{E} \in]0,1[$, $A_t + (u^{\mathcal{E}})_t^P$ is an increasing process. Put $f^{\mathcal{E}} = A - j^{\mathcal{E}}$. We have $f_t^{\mathcal{E}} = (A_t + (u^{\mathcal{E}})_t^P) - u_t^{\mathcal{E}}$. The fact that the jumps of j are negative shows that $u^{\mathcal{E}}$ is a decreasing process. Hence, $f^{\mathcal{E}}$ is an increasing process.

We have $f(t) = f^{\mathcal{E}}(t) + j^{\mathcal{E}}(t) - j(t)$. Using the above property of convergence, it follows that

$$\lim_{\mathcal{E}\to 0} E[\sup_{s\le T_k} |f_s^{\mathcal{E}} - f_s|] = 0.$$

Hence, f is an increasing process as the limit of increasing processes.

It remains to be shown that for all $\mathcal{E} \in \,]0,1[$, $A_t + (u^{\mathcal{E}})_t^P$ is an increasing process.

For any $\mathcal{E} \in \,]0,1[$, define

$$N_t^{\mathcal{E}} = -\sum_{s\le t} \mathbf{1}_{\Delta j_s < -\mathcal{E}} + \left(\sum_{s\le t} \mathbf{1}_{\Delta j_s < -\mathcal{E}}\right)_t^P .$$

Because the filtration is left quasi-continuous,

$$[N^{\mathcal{E}}, j] = -u^{\mathcal{E}}, \quad \text{hence } \langle N^{\mathcal{E}}, j\rangle = -(u^{\mathcal{E}})^P.$$

Then it is clear that Lemma 2.1.1 applied to $N^{\mathcal{E}}$ shows that the process $A + (u^{\mathcal{E}})^P$ is an increasing process. However, $N^{\mathcal{E}}$ does not belong to D because $N^{\mathcal{E}}$ admits only jumps equal to -1. To solve this, put

$$N = \alpha N^{\mathcal{E}} \quad \text{for } 0 < \alpha < 1,$$

so that $\Delta N > -1$.

Now, we may choose a sequence of stopping times T_n, $n \ge 0$, such that $T_n \uparrow T$ almost surely as n tends to infinity and for each $n \in \mathbb{N}$, $\mathcal{E}(N)^{T_n}$ $(= \mathcal{E}(N^{T_n}))$ is a martingale. Thus, for each $n \in \mathbb{N}$, N^{T_n} belongs to D. Lemma 2.1.1 applied to N^{T_n} shows that for each $n \in \mathbb{N}$, $A^{T_n} - \langle N, j\rangle^{T_n}$ is an increasing process; hence, $A - \langle N, j\rangle$ is an increasing process, that is, $A + \alpha(u^{\mathcal{E}})^P$ is an increasing process for each $\alpha < 1$. Hence, $A + (u^{\mathcal{E}})^P$ is an increasing process. □

To explain the consequences of the preceding theorem, we begin with the following corollary.

COROLLARY 2.3.1. *J_t is the lowest price process admissible for sellers; it is the selling price for B.*

Proof. The proof is similar to that of Corollary 2.2.1. □

The dynamic hedging strategy adopted by the seller happens continuously in time. At $t = 0$, the seller sells the claim at J_0. He invests this amount in the hedging portfolio (determined by φ). At time $t > 0$, the value of the self-financing portfolio determined by φ and J_0 is greater than the price for B at time t; hence, between 0 and t, the seller has withdrawn from the portfolio the nonnegative amount given by

$$J_0 + \int_0^t \varphi_s^* \sigma_s \, d\widetilde{W}_s - J_t.$$

At time t, the amount invested in the portfolio is only equal to J_t. At time $s > t$, the value of the self-financing portfolio determined by φ and J_t,

$$J_t + \int_t^s \varphi_u^* \sigma_u \, d\widetilde{W}_u,$$

is greater than the price for B at time s (J_s); between t and s, the seller has withdrawn the amount

$$J_t + \int_t^s \varphi_u^* \sigma_u \, d\widetilde{W}_u - J_s$$

from his portfolio. At time s, the amount invested in the portfolio is only equal to J_s, and so on.

The readjustments have to be done continuously in time so that the withdrawals correspond to the process f_t, which represents the cumulative amount withdrawn from the portfolio between 0 and t. As time passes, the amount invested in the portfolio is too high, so the seller withdraws some money from the portfolio that should be given to the buyer, which, because it is not, is a profit for the seller.

COROLLARY 2.3.2. (1) *For each* $N \in D$, $E_{Q^N}[B] = J_0 - E_{Q^N}[f_T]$.

(2) *Let* N_n, $n \geq 0$, *be an optimizing sequence of* D, *that is, such that*

$$\lim_{n \to +\infty} E_{Q^{N_n}}[B] = J_0.$$

Then

$$\lim_{n \to \infty} E_{Q^{N_n}}(f_T) = 0.$$

Remark 1. If B is only supposed to satisfy

$$\sup_{N \in D} E_{Q^N}(B) < +\infty,$$

then the second point of this corollary still holds but the first one does not (see §A.3).

Remark 2. If Q^{N_n} converges in a certain sense to a limit probability Q^0 and if f_T is smooth enough, then we have $E_{Q^0}[f_T] = 0$. In this case, J_t is a martingale under Q^0, but this limit probability is not necessarily equivalent to Q, as is shown by example 2 in §3.4. More generally, if there exists a subsequence still denoted by N_n such that $\mathcal{E}(N_n)_T$ converges almost surely and if the limit is denoted by L^*, then $f_T = 0$ on $\{L^* > 0\}$.

Proof of (1). We denote by V_t the process

$$J_0 + \int_0^t \varphi_s^* \sigma_s \, d\widetilde{W}_s.$$

Let N be an element of D. V_t is a positive continuous Q^N-local martingale, hence, a Q^N-supermartingale. f_T is Q^N-integrable because V_T and B are Q^N-integrable. We have, by the assumption made on B,

$$\sup_{t \in [0,T]} V_t \leq \sup_t (J_t + f_t) \leq \sup_t (X^{H,0,y}(t)) + f_T.$$

Now, $X^{H,0,y}(t)$ is a square integrable Q^N-martingale; hence, $\sup_t (X^{H,0,y}(t))$ is square integrable under Q^N. Hence, V_t is a uniformly integrable local martingale under Q^N, and thus it is a Q^N-martingale. It follows that for each $N \in D$,

$$E_{Q^N}[B] = J_0 - E_{Q^N}[f_T]. \qquad \square$$

Proof of (2). Equality (1) applied to the local martingales N_n gives the equalities

$$\forall n \in N, \; E_{Q^{N_n}}(B) = J_0 - E_{Q^{N_n}}(f_T).$$

Hence, if we let n tend to $+\infty$, we obtain the desired result of

$$\lim_{n \to +\infty} E_{Q^{N_n}}[f_T] = 0. \qquad \square$$

In the following theorem, we state that the class of contingent claims for which the supremum price is obtained for an optimal N is exactly the set of attainable claims. This result generalized those of Pagès [Pag] and Karatzas et al. ([KLSX, Thm. 8.5]) in the context of a Brownian model.

THEOREM 2.3.2. *The following properties are equivalent.*

(i) $\sup_{N\in D} E_{Q^N}[B]$ *is attained by* $\hat{N} \in D$.

(ii) B *is attainable, that is, there exist a constant* x *and a portfolio* π *such that*

$$B = x + \int_0^T \pi_u^* \sigma_u \, d\widetilde{W}_u, \qquad Q \text{ a.s.}$$

and

$$E_Q\left[\int_0^T \|\sigma_s^* \pi_s\|^2 \, ds\right] < +\infty.$$

(iii) *For each local martingale* $N \in D$, $E_{Q^N}(B) = E_Q(B)$.

Remark. If B is only supposed to satisfy

$$\sup_{N\in D} E_{Q^N}(B) < +\infty$$

then only a part of this theorem still holds (see §A.3).

Proof. Let us show that (ii) $\Rightarrow$ (iii). Suppose that there exist a constant x and a portfolio π such that

$$B = x + \int_0^T \pi_u^* \sigma_u \, d\widetilde{W}_u \quad Q \text{ a.s.}$$

and

$$E_Q\left[\int_0^T \|\sigma_s^* \pi_s\|^2 \, ds\right] < +\infty.$$

Let S_t be the process

$$x + \int_0^t \pi_u^* \sigma_u \, d\widetilde{W}_u, \qquad 0 \le t \le T.$$

S_t is a square integrable Q-martingale; hence, $S_t = E_Q[B/F_t]$. Therefore, S_t is positive and lower than $X^{H,0,y}(t)$.

Let N be an element of D. Under Q^N, S_t is a local martingale of class (D), and hence, a martingale. Also, S_t is continuous; hence, the processes S and J are indistinguishable. Notice that we have also shown that each element N of D is optimal, which is equivalent to (iii).

It remains to show that (i) $\Rightarrow$ (ii). Suppose that N^0 is an optimal local martingale of D. Item (1) of Corollary 2.3.2 applied to N^0 gives

$$E_{Q^0}(B) = J_0 - E_{Q^0}(f_T),$$

that is,

$$E_{Q^0}(f_T) = 0.$$

Hence,

$$f_T = 0, \quad Q^0 \text{ a.s.},$$

that is,

$$f_T = 0 \quad Q \text{ a.s.}$$

since Q and Q^0 are equivalent. □

After the study of the maximum price, we now turn to the study of the minimum price for B.

2.4. Study of the minimum price. Recall that B is lower than M_T where $M_t = X^{H,0,y}(t)$. Let us determine the buyer's price for B. Let us consider a buyer who wants to buy some contingent claims with payoff B and maturity T, between time 0 and time T. Suppose the buyer chooses the price X_t for B at any time t; more precisely, the buyer must choose his price process $(X_t, t \geq 0)$ (that is, an RCLL optional process lower than M_t with $X_T = B$) and a portfolio process π_t. Also, the buyer does not want to run any risk of losing money. Therefore, he will only choose strategies that allow him to hedge completely by the controlled portfolio of the basic securities in the sense that, for any t, any s such that $0 \leq t \leq s \leq T$.

Suppose that, at time t, the buyer buys the claim at price X_t and sells the self-financing portfolio (determined by π) at price X_t. At time s, he sells the claim at price X_s and buys the self-financing portfolio; he then makes the profit (nonnegative) given by

$$X_s - \left(X_t + \int_t^s \pi_u^* \sigma_u \, d\widetilde{W}_u \right).$$

More precisely, the process given by

$$X_t - \left(X_0 + \int_0^t \pi_u^* \sigma_u \, d\widetilde{W}_u \right), \qquad 0 \leq t \leq T$$

is an increasing process.

DEFINITION 2.4.1. *A process X_t is called a* price process admissible for buyers *if X_t is an* RCLL *optional process lower than M_t with $X_T = B$ such that there exist a portfolio π_t and an* RCLL *optional increasing process D_t with $D_0 = 0$ satisfying $dX_t = \pi_t^* \sigma_t (dW_t + \theta_t \, dt) + dD_t$.*

Remark. It is equivalent to say that a process X_t is a price process admissible for buyers if there exists a hedging portfolio of B that is a portfolio with savings whose value is equal to the price, that is, if there exist a portfolio process π_t and an RCLL optional increasing process D_t with $D_0 = 0$ satisfying

$$X^{\pi,D,x}(T) = B \quad \text{and} \quad X_t = X^{\pi,D,x}(t), \qquad 0 \leq t \leq T.$$

The purchase price is then given by Definition 2.4.2.

DEFINITION 2.4.2. *The greatest process admissible for buyers is called the* purchase price.

We have clearly the following property.

PROPOSITION 2.4.1. (i) *A process X_t is a price process for B admissible for buyers if and only if $M_t - X_t$ is a process price for $M_T - B$ admissible for sellers.*

(ii) *The purchase price for B is equal to the difference between M_t and the selling price for $M_T - B$.*

We now turn to the study of the essential infimum of the possible prices for B. Let K_t, $0 \leq t \leq T$, be the right continuous process satisfying

$$K_t = \operatorname{ess}\inf_{N \in D} E_{Q^N}[B/F_t], \qquad 0 \leq t \leq T.$$

Notice that the minimum price for B is given by the maximum price for $M_T - B$ by the equality

$$M_t - K_t^B = J_t^{M(T)-B}, \qquad 0 \leq t \leq T, \quad Q \text{ a.s.}$$

It follows that the properties of K can be derived from those of J.

THEOREM 2.4.1. *K_t, $0 \leq t \leq T$, is characterized as the greatest right continuous submartingale under Q_N, for any $N \in D$, with $N_T = B$. Also, N is optimal if and only if K_t is a martingale under Q_N.*

THEOREM 2.4.2. *There exist a portfolio process ψ_t and a right continuous increasing optional process g_t with $g_0 = 0$ such that*

$$K_t = K_0 + \int_0^t \psi_s^* \sigma_s \, d\widetilde{W}_s + g_t, \quad Q \text{ a.s.}, \quad 0 \leq t \leq T \tag{10}$$

Also, there exist a right continuous increasing predictable process B_t with $B_0 = 0$ and a purely discontinuous martingale i_t with negative jumps such that

$$g_t = -i_t + B_t, \qquad 0 \leq t \leq T.$$

COROLLARY 2.4.1. *K_t is the greatest of the price processes admissible for buyers, that is, K_t is the purchase price for B.*

Remark 1. The properties of K_t can be clearly derived directly without using the properties of J_t (by the use of stochastic control methods).

Remark 2. The assumption made on B is not necessary to obtain the above results. One should derive the properties of K directly; the price processes should be supposed to be lower than $E_Q(B/F_t)$ (instead of M_t).

Now, let us compare the hedging strategies associated with the maximum and minimum prices to the "optimal" strategy in Föllmer and Schweizer's sense. Recall that their optimal strategy is obtained by projecting the Q-martingale $E_Q[B/F_t]$ orthogonally on the stable subspace generated by $\int \sigma_i \, dW$, $1 \leq i \leq n$, that is,

$$B = E_Q[B] + \int_0^T \pi_u^* \sigma_u \, d\widetilde{W}_u + N_T,$$

where N is a local martingale under Q orthogonal to $\int \sigma_i \, d\widetilde{W}$, $1 \leq i \leq n$ and hence a local martingale under P, because Q is minimal. $E_Q[B]$ is called the "optimal" price for B.

Let us compare the three price-portfolio strategies:

- The price-portfolio strategy associated with the "optimal" price for B, whose value of the self-financing portfolio is given by

$$E_Q(B) + \int_0^t \pi_u^* \sigma_u \, d\widetilde{W}_u, \qquad 0 \leq t \leq T$$

and whose price process is given by $E_Q(B/F_t)$, $0 \leq t \leq T$, is characterized by the fact that the difference between the value of the self-financing portfolio and the price is a local martingale under P equal to zero at time zero that is orthogonal to $\int \sigma_i \, dW$, $1 \leq i \leq n$.

• The price-portfolio strategy associated with the maximum price for B, whose value of the self-financing portfolio is given by

$$J_0 + \int_0^t \varphi_u^* \sigma_u \, d\widetilde{W}_u, \qquad 0 \leq t \leq T$$

and whose price process is given by J_t, $0 \leq t \leq T$, is characterized by the fact that the difference between the value of the self-financing portfolio and the price is an increasing process equal to zero at time zero and by the fact that the price is minimal (in the sense defined above).

• Also, the price-portfolio strategy associated with the minimum price for B is characterized by the fact that the difference between the price and the value of the self-financing portfolio is an increasing process equal to zero at time zero and by the fact that the price is maximal (in the sense defined above).

In the next section, we give a few methods for computing the maximum price and a few examples that illustrate the obtained results.

3. Methods for computing.

3.1. Determination of J_t as the limit of a sequence of processes (in the Brownian case). The selling price for B is given by J_t, the essential supremum of $E_R(B/F_t)$, for $R \in \mathcal{P}$ (the set of martingale measures). However, J_t is generally difficult to compute. We may then restrict the control set to $\mathcal{P}_n$, $n \in N$, so that the essential supremum J^n taken over all the elements of $\mathcal{P}_n$ is attained. $\mathcal{P}_n$ is chosen so that J_t is the limit of $J^n(t)$ as n tends to infinity, and we then obtain a sequence approximating J that can be calculated explicitly. We develop this method in the context of a Brownian model. In this case, every local martingale is a stochastic integral with respect to a reference martingale (the Brownian); we will see later that this property allows us to obtain some precise results.

In this section, we will suppose that $\sup_{R \in \mathcal{P}} E_R(B) < \infty$ and $E_Q(B^2) < \infty$. Let us denote by $L^2[0,T]$ the class of predictable processes ϕ satisfying

$$\int_0^T \|\Phi_t\|^2 \, dt < \infty, \quad \text{a.s.}$$

Let $K(\sigma)$ be the subset of $L^2[0,T]$ defined by

$$\nu \in K(\sigma) \Leftrightarrow \nu \in L^2[0,T]/\sigma(t)\nu(t) = 0 \quad \forall\, t \in [0,T], \quad \text{a.s.}$$

and

$$\mathcal{E}\left(\int_0^\cdot \nu_s^* \, d\widetilde{W}_s\right)_t, \qquad 0 \leq t \leq T$$

is a Q martingale.

The following result is clear.

PROPOSITION 3.1.1. *The following properties are equivalent.*

(i) *N belongs to D.*

(ii) *There exists $\nu \in K(\sigma)$ such that*

$$N_t = \int_0^t \nu_s^* \, d\widetilde{W}_s \quad \forall\, t \in [0,T], \quad \text{a.s.}$$

It follows that $J_t = \text{ess sup}_{\nu \in K(\sigma)} E_{Q^\nu}[B/F_t]$, where Q^ν denotes the probability measure that admits the following Radon–Nikodym derivative with respect to Q:

$$\exp\left\{\int_0^T \nu_s^* \, d\widetilde{W}_s - \frac{1}{2}\int_0^T \|\nu_s\|^2 \, ds\right\}.$$

In this context, ν can be interpreted as a risk premium vector associated with the risks of the market that cannot be controlled using the prices of the basic securities (loosely speaking, those risks are in an another direction). Thus, this risk premium has the role of control in the determination of the maximum price of the contingent claim.

For any $n \in N$, define $K^n(\sigma)$, a subset of $K(\sigma)$ by

$$K^n(\sigma) = \{\nu \in K(\sigma)/\|\nu(t)\| \leq n \,\forall\, t \in [0,T], \text{ a.s.}\}.$$

For any $n \in N$, let $J^n(t)$ be the right continuous process satisfying

$$J^n(t) = \text{ess} \sup_{\nu \in K^n(\sigma)} E_{Q^\nu}[B/F_t].$$

It is characterized by the following property (similar to the characterization of J).

PROPOSITION 3.1.2. *$J^n(t)$ is characterized as the smallest right continuous supermartingale under Q^ν, for every $\nu \in K^n(\sigma)$, with $J^n(T) = B$. Also, ν is optimal (i.e., $\nu \in K^n(\sigma)$ with $J^n(t) = E_{Q^\nu}[B/F_t]$, $0 \leq t \leq T$, a.s.) if and only if $J^n(t)$ is a martingale under Q^ν.*

The properties relative to J and J^n, $n \in N$, allow us to state that J is the limit of J^n as n tends to infinity.

THEOREM 3.1.1.

$$J_t = \lim_{n \to +\infty} \uparrow J^n(t) \text{ a.s.} \quad \forall\, t \in [0,T].$$

Proof. Let J° be the process defined by $J_t^\circ = \lim_{n\to+\infty} \uparrow J^n(t)$. Let us show that $J_t^\circ = J_t$. We clearly have $J_t^\circ \leq J_t$. It remains to show that $J_t^\circ \geq J_t$.

J_t° is an RCLL supermartingale under every Q^ν, $\nu \in K(\sigma)$ and bounded, because it is the increasing limit of RCLL supermartingales. Using this property, one can show quite easily, by a proof similar to that of Theorem 2.1.2, that J_t° is a price process for B admissible for sellers in the sense that there exist a portfolio process π_t° and a predictable increasing process A_t° such that

$$J_t^\circ = J_0^\circ + \int_0^t (\pi_s^\circ)^* \sigma_s \, d\widetilde{W}_s - A_t^\circ, \qquad 0 \leq t \leq T,$$

hence, $J_t \leq J_t^\circ$ (because J_t is the lowest of the admissible prices). □

Whereas J_t is a priori difficult to calculate, we have a characterization of J^n that is linked to the fact that there exists an optimal control for J^n (contrary to J).

THEOREM 3.1.2. *There is an optimal control associated with J^n.*

Proof. The proof is an application of Theorem 3.30 in [ElK]. (Indeed, the model is strongly dominated and the space to which ν_t belongs is compact.) □

$J^n(t)$ can be determined explicitly as the unique solution of a backward stochastic differential equation of the type studied by E. Pardoux and S. G. Peng (see [Pa-Pe]). Thus, we have, in the general case, a construction of the value function similar to the construction of the solution of the Hamilton–Jacobi–Bellman equation in the Markovian case. We denote by $\Pi_{\text{Ker}\sigma(s)}$ the orthogonal projection that maps $\mathbb{R}^d$ onto the kernel of σ_s.

THEOREM 3.1.3. *Let $(X^n(t),\, Y^n(t))$ be the unique solution of the backward stochastic differential equation*

$$X_t^n - \int_t^T n\,\|\pi_{\mathrm{Ker}\,\sigma_s}(Y_s^n)\|\,ds + \int_t^T Y_s^{n^*}\,d\widetilde{W}_s = B, \qquad 0 \le t \le T. \tag{11}$$

Then

(1) $X^n(t) = J^n(t)$, $0 \le t \le T$, *almost surely.*

(2) *If ν^n is an optimal control associated with $J^n(t)$, then*

$$\nu_s^n = n\frac{\Pi_{\mathrm{Ker}\,\sigma_s}(Y_s^n)}{\|\Pi_{\mathrm{Ker}\,\sigma_s}(Y_s^n)\|}\,\mathbf{1}_{\{Y_s^n \neq 0\}},\ ds\,dQ\ a.s.$$

Remark. Thus, we have $J_t = \lim_{n\to+\infty} E_{Q^{\nu^n}}(B/F_t)$ and $ds\,dQ$ almost surely. If a subsequence of $\{\|\nu^n(s)\|,\, n \in N\}$ converges, its limit is equal to 0 or $+\infty$. Loosely speaking, we see that the maximum price for the contingent claim is obtained by letting the norm of the risk premium tend to $+\infty$ or 0.

Proof. Let ν^n be an optimal control associated with J^n. Under Q, J^n is a supermartingale and admits the following decomposition

$$J_t^n = J_0^n + \int_0^t \varphi_s^{n^*}\,d\widetilde{W}_s - A_t^n, \qquad 0 \le t \le T,$$

where $\varphi^n \in L^2[0,T]$ and A^n is a predictable increasing process with $A^n(0) = 0$. Now, J^n is a martingale under Q^{ν_n}; hence, by the Girsanov theorem,

$$A_t^n = \int_0^t \varphi_s^{n^*}\,\nu_s^n\,ds, \qquad 0 \le t \le T.$$

Let $\nu \in K^n(\sigma)$. J^n is a supermartingale under Q^ν; hence, by the Girsanov theorem

$$\varphi^n(s)^*\nu^n(s) \ge \varphi^n(s)^*\nu(s), \quad ds\,dQ \text{ a.s.}$$

Because this inequality holds for each $\nu \in K^n(\sigma)$, we have

$$\varphi^n(s)^*\nu^n(s) = \operatorname*{ess\ sup}_{\nu\in K^n(\sigma)} [\varphi^n(s)^*\nu] = n\left\|\pi_{\mathrm{Ker}\,\sigma(s)}(\varphi^n(s))\right\|$$

and

$$\nu_s^n = n\frac{\Pi_{\mathrm{Ker}\sigma_s}(\varphi_s^n)}{\|\Pi_{\mathrm{Ker}\sigma_s}(\varphi_s^n)\|}\,\mathbf{1}_{\{\varphi_s^n \neq 0\}},\ ds\,dQ \text{ a.s.}$$

Hence,

$$J_t^n - \int_t^T n\,\|\pi_{\mathrm{Ker}\,\sigma_s}(\varphi_s^n)\|\,ds + \int_t^T \varphi_s^{n^*}\,d\widetilde{W}_s = B, \qquad 0 \le t \le T. \qquad \square$$

Remark. For the results of existence and uniqueness of solutions of backward equations, see [Pa-Pe].

$M^2(0,T;\mathbb{R}^d)$ will denote the normed vectorial space of $\mathbb{R}^d$-valued processes that are predictable and belong to $L^2([0,T]\times\Omega,\, dt\,dQ)$.

For any $\phi \in M^2(0,T;\mathbb{R}^d)$, define its norm by

$$\|\phi\|_{M^2}^2 = E_Q\left[\int_0^T \|\phi_s\|^2\, ds\right].$$

The backward equation (R) given by

$$X_t - \int_t^T n\, \|\pi_{\operatorname{Ker}\sigma_s}(Y_s)\|\, ds + \int_t^T Y_s^*\, d\widetilde{W}_s = B, \qquad 0 \le t \le T,$$

has Lipschitz coefficients (with respect to Y), and hence admits a unique solution $(X,Y) \in M^2(0,T;\mathbb{R}) \times M^2(0,T;\mathbb{R}^d)$. Recall that the solution can be constructed using a Picard type iteration. Y_0 is taken to be equal to 0. Let (X_p, Y_p), $p \in N^*$ be a sequence in $M^2(0,T;\mathbb{R}) \times M^2(0,T;\mathbb{R}^d)$ defined recursively by $X_p(0)$ and Y_p, which are constructed from Y_{p-1} by the representation theorem

$$E_Q\left[B - \int_0^T n\, \|\pi_{\operatorname{Ker}\sigma_s}(Y_{p-1}(s))\|\, ds/F_t\right] = X_p(0) + \int_0^t Y_p(s)^*\, d\widetilde{W}_s, \text{ a.s.}$$

$X_p(t)$ is then defined by

$$X_p(t) = \int_t^T n\, \|\pi_{\operatorname{Ker}\sigma_s}(Y_{p-1}(s))\|\, ds - \int_t^T Y_p(s)^*\, d\widetilde{W}_s + B, \qquad 0 \le t \le T.$$

Using Pardoux and Peng's result, X_p (respectively, Y_p) converges in $M^2(0,T;\mathbb{R})$ (respectively, $M^2(0,T;\mathbb{R}^d)$) to X, Y, the solution of (R) as p tends to $+\infty$.

Recall that the increasing process associated with the Q-supermartingale J is denoted by A. We denote by A^n the increasing process associated with J^n, that is,

$$A_t^n = \int_0^t n\, \|\pi_{\operatorname{Ker}\sigma_s}(\varphi_s^n)\|\, ds.$$

In general, the process A^n does not converge to A almost surely, but we have the following property (cf. [De-Me, Thm. VII.18, p. 223]). If J is of class D, then for each t, the sequence of random variables $A^n(t)$ converges to $A(t)$ weakly in L^1, that is, for each bounded F_T measurable variable U, $\lim_{n\to+\infty} E[A^n(t)U] = E[A(t)U]$.

In the next section, we study the Markovian case. We will see that, contrary to J, J_n is the solution of a usual Bellman equation (and this is linked to the existence of an optimal control associated with J_n, contrary to J).

3.2. The Bellman equation and the maximum price. In a particular case, we propose a numerical method to solve the problem. The general model is complete, that is, the filtration is that generated by the d-dimensional Brownian W. The market contains d securities whose volatility matrix has full rank, but only certain securities (the first n ones) can be traded. Therefore, the market is incomplete. We suppose that all the coefficients of the model are only functions of the time t and the price vector $P(t) = (P_1(t), \ldots, P_d(t))$, functions that are taken to be smooth enough so that the Bellman equations are satisfied. We denote by σ' (respectively, σ, δ) the volatility matrix of the d securities (respectively, the n first securities, the $(d-n)$ others). We have $\sigma' = \binom{\sigma}{\delta}$.

Under Q (the reference probability), the prices of the securities satisfy the following equations

$$\begin{cases} dP_j(t) = P_j(t)[r(t,P_t)dt + \sigma_j(t,P_t)d\widetilde{W}_t], & 1 \le j \le n, \\ dP_k(t) = P_k(t)[\mu_k(t,P_t)dt + \delta_k(t,P_t)d\widetilde{W}_t], & n+1 \le k \le d. \end{cases} \tag{12}$$

The contingent claim B is taken to be equal to $g(P_1(T), \dots, P_d(T))$ for a $\mathbb{R}^+$-valued function g on $\mathbb{R}^d$ satisfying smoothness conditions (see [Kry, p. 205]). Recall that the selling price is given by

$$J_t = \text{ess} \sup_{\nu \in K(\sigma)} E_{Q^\nu}[B/F_t],$$

where Q^ν denotes the probability measure that admits the following Radon–Nikodym derivative with respect to Q:

$$\exp\left\{ \int_0^T \nu_s^* \, d\widetilde{W}_s - \frac{1}{2} \int_0^T \|\nu_s\|^2 \, ds \right\},$$

and $K(\sigma)$ denotes the set of Ker $\sigma(t, P(t))$-valued predictable processes ν_t that belong to $L^2[0,T]$.

Note that under Q^ν, $\widetilde{W}_t - \int_0^t \nu_s \, ds$ is a Q^ν-Brownian motion.

Using some of Krylov and El Karoui's results [Kry], [ElK], we see that the maximum price J_t (which is a function J of t and $P(t)$) is the value function for a more general problem. (Ω, F_t, P, W) is a fixed probability space on which W is an F-Brownian. The controlled system is described by the following equations:

$$\text{(13)} \quad \begin{cases} dP_j(t) = P_j(t)[r(t, P_t)dt + \sigma_j(t, P_t)dW_t], & 1 \le j \le n, \\ dP_k(t) = P_k(t)[[\mu_k(t, P_t) + \delta_k(t, P_t)\nu_t]dt + \delta_k(t, P_t)dW_t], & n+1 \le k \le d, \end{cases}$$

where the control ν belongs to $K(\sigma)$.

The value function is then given by

$$J(t, x) = \sup_{\nu \in K(\sigma)} E_{\nu,t}[g(P_T)/P_t = x]$$

for $x \in (R^+)^d$ and $t \in [0, T]$.

Because the coefficient $\mu_k + \delta_k \, \nu_t$ is not bounded, $J(t, x)$ is not the solution of the classical Bellman equation. However, if we impose some smoothness conditions on the coefficients, J satisfies the following inequality in terms of generalized derivatives:

$$(\alpha) \qquad L\varphi(t, x) + G\varphi(t, x)\nu \le 0 \quad \forall \, \nu \in \operatorname{Ker} \sigma(t, x),$$

where

$$\begin{aligned} L\varphi(t, x) = {} & \frac{\partial \varphi}{\partial t}(t, x) + \frac{1}{2} \sum_{1 \le i,j \le d} x_i x_j (\sigma' \sigma'^*)_{ij} \frac{\partial^2 \varphi}{\partial x_i \partial x_j}(t, x) \\ & + \sum_{k=n+1}^{d} \frac{\partial \varphi}{\partial x_k}(t, x) \, x_k \mu_k(t, x) + \sum_{j=1}^{n} \frac{\partial \varphi}{\partial x_j}(t, x) \, x_j r(t, x). \\ G\varphi(t, x) = {} & \sum_{k=n+1}^{d} \frac{\partial \varphi}{\partial x_k}(t, x) x_k \delta_k(t, x). \end{aligned}$$

(J is characterized as the smallest solution of (α).) Also, equation (α) is clearly equivalent to the following system:

$$\text{(A)} \qquad \begin{cases} G\varphi(t, x)\nu = 0 & \forall \, \nu \in \operatorname{Ker} \sigma, \\ L\varphi(t, x) \le 0. \end{cases}$$

Thus, J is the smallest solution of system (A). It corresponds to the characterization of J as the smallest selling price (Thm. 2.2.1).

Notice that if J is $C^{1,2}([0,T] \times R^d)$, this system can be derived directly by applying Ito's formula to $J(T, P_1(T), \ldots, P_n(T))$ and using the fact that, by Theorem 2.2.1, J can be written as the difference of a portfolio (constructed from the first n securities) and an increasing process.

Notice that the above system can also be derived using some of Krylov's results on optimal control problems of Markov diffusion processes with unbounded coefficients. Using Krylov's results (cf. [Kry, p. 266]), one can show that, if the coefficients are taken to be smooth enough, $J(t,x)$ is solution of the normalized Bellman equation (in terms of generalized derivatives)

$$\sup_{\{\nu \in \operatorname{Ker} \sigma(t,x)\}} \left[\frac{1}{\lambda(\nu)} (L\varphi(t,x) + G\varphi(t,x)\nu) \right] = 0,$$

where $\lambda(\nu) = \sup(1, \|\nu\|)$. This equation is clearly equivalent to system (A).

To calculate J, we can use the following property (as in §3.1).

PROPOSITION 3.2.1.

$$J(t,x) = \lim_{n \to \infty} J_n(t,x),$$

where

$$J_n(t,x) = \sup_{\nu \in K^n(\sigma)} E_{t,\nu}[g(P_t)/P_t = x],$$

and $K^n(\sigma)$ *is the set of the processes* $\nu_t \in K(\sigma)$ *that are bounded by* n.

For each n, there exists an optimal control associated with J_n. Also, J_n is solution of the Bellman equation

$$\sup_{\{\nu \in \operatorname{Ker} \sigma(t,x), \|\nu\| \leq n\}} [L\varphi(t,x) + G\varphi(t,x)\nu] = 0.$$

Notice that

$$\sup_{\{\nu \in \operatorname{Ker} \sigma(t,x), \|\nu\| \leq 1\}} [G\varphi(t,x)\nu] = \|\pi_{t,x}(G\varphi(t,x))\|,$$

where $\Pi_{t,x}$ is the orthogonal projection from R^d onto $\operatorname{Ker} \sigma(t,x)$ ($G\varphi(t,x)$ is a row vector). Hence, we have the following property.

PROPOSITION 3.2.2. $J_n(t,x)$ *is solution of the following equation*:

$$L\varphi(t,x) + n\,\|\pi_{t,x}(G\varphi(t,x))\| = 0.$$

Remark. Notice that if

$$H(\varphi)(t,x) = \sum_{j=1}^{n} \frac{\partial \varphi}{\partial x_j}(t,x)\, x_j \sigma_j(t,x) + G(\varphi)(t,x),$$

then $\Pi_{t,x}(H\varphi(t,x)) = \Pi_{t,x}(G\varphi(t,x))$. It follows that if φ is solution of the Bellman equation with $\varphi(T,x) = g(x)$, then $(\varphi(P_t), H\varphi(t,P_t))$ is solution of the backward equation given by

$$\varphi(t,P_t) - \int_t^T n\,\left\|\pi_{\operatorname{Ker}\sigma(s,P_s)}(H\varphi(s,P_s))\right\| ds + \int_t^T H\varphi(s,P_s)\, d\widetilde{W}_s = g(P_T).$$

We recognize the backward equation obtained in §3.1.

In the next section, we give an example that shows that there exist some discontinuous solutions of the normalized Bellman equation.

3.3. Example 1. Let W' be a (unidimensional) Brownian independent of the Brownian W_t (d-dimensional). The filtration is that generated by the Brownians W and W'. The prices of the different securities and the coefficients relative to those prices (appreciation rates and volatilities) depend only on W (that is, are adapted to the filtration of W). B is taken to be a function of the terminal value of W', that is, $B = f(W'_T)$ for a positive bounded real-valued function f on $\mathbb{R}$. In this case, we show that the maximum price for B is constant on $[0, T[$, equal to the supremum of the function f and jumps at time T to reach the value $f(W'_T)$.

Let us consider a more general case. The contingent claim B is taken to be positive bounded and to depend only on W'. Let us determine its maximum price J_t. We define the model more precisely.

Let Ω_1 be the space of all $\mathbb{R}^d$-valued continuous functions on $\mathbb{R}^+$,

$$\Omega_1 = \mathcal{C}(\mathbb{R}^+, \mathbb{R}^d).$$

We denote by F^1 the σ-field generated by the coordinate process $W_t : \omega_1 \to \omega_1(t)$ for $t \geq 0$. Let $(F^1_t, t \geq 0)$ be the filtration generated by the process W_t. Let P^1 be the Wiener measure on Ω_1 constructed so that the coordinate mapping process W_t is Brownian motion.

Let Ω_2 be the space of all real-valued continuous functions on $\mathbb{R}^+$,

$$\Omega_2 = \mathcal{C}(\mathbb{R}^+, \mathbb{R}).$$

We denote by F^2 the σ-field generated by the coordinate process $W'_t : \omega_2 \to \omega_2(t)$ for $t \geq 0$. Let (F^2_t) be the filtration generated by the process (W'_t). Let P^2 be the Wiener measure on Ω_2 constructed so that the coordinate mapping process W'_t is Brownian motion.

Let (Ω, F, P) be the cross-product probability space $(\Omega_1 \times \Omega_2, F_1 \otimes F_2, P_1 \otimes P_2)$. The filtration F_t is defined by $F_t = F^1_t \otimes F^2_t$. We denote by ω the elements of Ω

$$\omega = (\omega_1, \omega_2), \quad \omega_1 \in \Omega_1, \quad \omega_2 \in \Omega_2.$$

We denote by W_t the first coordinate mapping process and by W'_t the second coordinate mapping process. W_t and W'_t are independent (F_t)-Brownian motions under P. The prices of the basic securities $P^i(t)$, the vector of stock appreciation $b(t)$, and the volatility matrix $\sigma(t)$ are taken to depend only on the first coordinate of the path (ω_1). B is taken to be F^2_T-measurable, positive bounded.

Clearly, if N is a stochastic integral with respect to W' and if the associated exponential martingale is a martingale, then N belongs to D. Hence,

$$J_t \geq \operatorname{ess}\sup_{\nu \in D'} E_{P^2}\left[B\mathcal{E}\left(\int \nu_s \, dW'_s\right)_T \Big/ F^2_t\right] \Big/ \mathcal{E}\left(\int \nu_s \, dW'_s\right)_t,$$

where D' is the set of all (F^2)-predictable processes ν defined on Ω^2 such that

$$E_{P^2}\left[\mathcal{E}\left(\int \nu_s \, dW'_s\right)_T\right] = 1.$$

Now, it follows by the representation theorem that

$$\sup_{\nu \in D'} E_{P^2}\left[B\mathcal{E}\left(\int \nu_s \, dW'_s\right)_T\right] = \sup_{\substack{X \in L^1 \\ \|X\|_1 \leq 1}} E_{P^2}[BX],$$

where $L^1 = L^1(\Omega^2, P^2, F^2_T)$ and $\|X\|_1 = E_{P^2}[|X|]$ for $X \in L^1$.

Now, because the L^∞ norm of any function on any measure space is equal to its norm as a linear functional on L^1, we have

$$\sup_{\substack{X \in L^1 \\ \|X\|_1 \le 1}} E_{P^2}[BX] = \|B\|_{L^\infty(P^2)},$$

where $\|B\|_{L^\infty(P^2)}$ denotes the essential supremum of B under P^2. Therefore,

$$\sup_{\nu \in D'} E_{P^2}\left[B\mathcal{E}\left(\int \nu_s\, dW'_s\right)_T\right] = \|B\|_{L^\infty(P^2)}.$$

Hence, $J_0 = \|B\|_{L^\infty(P^2)}$.

Also, by the same argument we have

$$\operatorname{ess\,sup}_{\nu \in D'} E_{P^2}\left[B\mathcal{E}\left(\int \nu_s\, dW_s\right)_T \Big/ F_t^2\right] \Big/ \mathcal{E}\left(\int \nu_s\, dW'_s\right)_t = \|B\|_{L^\infty(P^2/F_t^2)},$$

where $\|B\|_{L^\infty(P^2/F_t^2)}$ denotes the essential supremum of B under the conditional probability measure of P^2 given F_t^2. It follows that $J_t = \|B\|_{L^\infty(P^2/F_t^2)}$ for each $t \in [0, T]$.

Remark. This example shows that there exist some discontinuous solutions of the normalized Bellman equation.

In the next section, we give an another example but not in a Brownian model. It illustrates the fact that the purely discontinuous Q-martingale j obtained in the decomposition of J_t may not be equal to zero. Consequently, the optional decomposition of J_t is the good one.

3.4. Example 2. Let N_t be a Poisson process with intensity 1 independent of W_t. The filtration is that generated by the Brownian W (d-dimensional) and the Poisson N (one-dimensional). The prices of the different securities and the coefficients relative to those prices (appreciation rates and volatilities) depend only on W (that is, are adapted to the filtration of W). The contingent claim B is taken to be positive bounded and to depend only on N. It is a contract that pays 1 if $N_T = 0$, and 0 if $N_T \neq 0$. Note that

$$B = \mathbf{1}_{N_T = 0}.$$

The maximum price J_0 for B at time 0 is clearly equal to 1 (because it is impossible to hedge against the risk). To prove this result rigorously, choose P-martingale measure Q_α so that N_t is a Poisson process with intensity $1 + \alpha$ under $Q_\alpha (\alpha > -1)$. Then it is easy to show that

$$J_0 = \sup_{\alpha \in]-1, +\infty[} E_Q \alpha(B) = \sup_{\alpha \in]-1, +\infty[} e^{-(1+\alpha)T} = 1.$$

Also, the maximum price J_t for B at time t will be equal to 0 if $N_t \geq 1$ (because then the event $N_T = 0$ is impossible) and 1 if $N_t = 0$.

It follows that the different processes of Theorem 2.3.1 are given by

- the price for B at time 0, $J_0 = 1$,
- the portfolio process $\varphi = 0$,
- the optional increasing process $f_t = N_{t \wedge T_1} = \mathbf{1}_{N_t \geq 1}$.

We see that $f_t = A_t - j_t$, where $j_t = -N_{t\wedge T_1} + t \wedge T_1$ is a purely discontinuous martingale with negative jumps, and $A_t = t \wedge T_1$ is a predictable increasing process. By defining the model on the canonical space, it is possible to show that the sequence Q_α converges weakly in distribution, as α tends to -1, to a probability measure that is not equivalent to P and under which the Poisson process is equal to 0 almost surely. We define the model more precisely below and give explicit calculations.

Let Ω_1 be the space of all $\mathbb{R}^d$-valued continuous functions on $\mathbb{R}^+$,

$$\Omega_1 = \mathcal{C}(\mathbb{R}^+, \mathbb{R}^d).$$

We denote by F^1 the σ-field generated by the coordinate process $W_t : \omega_1 \to \omega_1(t)$ for $t \geq 0$. Let $(F^1{}_t, t \geq 0)$ be the filtration generated by the process W_t. Let P^1 be the Wiener measure on Ω_1 constructed so that the coordinate mapping process W_t is Brownian motion.

Let Ω_2 be the space of all Radon positive measures on $\mathbb{R}^+$,

$$\Omega_2 = \mathcal{M}_+(\mathbb{R}^+).$$

We denote by F^2 the σ-field generated by all the random variables $N_t : \omega_2 \to \omega_2([0,t])$ for $t \geq 0$. Let (F_t^2) be the filtration generated by the process (N_t). Let P^2 be the probability measure on Ω_2 constructed so that the coordinate mapping process N_t is Poisson process with intensity 1.

Let (Ω, F, P) be the cross-product probability space $(\Omega_1 \times \Omega_2,\ F_1 \otimes F_2,\ P_1 \otimes P_2)$. The filtration F_t is defined by $F_t = F_t^1 \otimes F_t^2$. We denote by ω the elements of Ω

$$\omega = (\omega_1, \omega_2), \quad \omega_1 \in \Omega_1, \quad \omega_2 \in \Omega_2.$$

We denote by W_t the first coordinate mapping process and by N_t the second coordinate mapping process

$$W_t(\omega) = W_t(\omega_1) = \omega_1(t); \qquad N_t(\omega) = N_t(\omega_2) = \omega_2(0,t).$$

Under P, W_t is a (F_t)-Brownian motion, and N_t is a (F_t)-Poisson process with intensity 1.

The prices of the basic securities $P^i(t)$, the vector of stock appreciation rates $b(t)$, and the volatility matrix $\sigma(t)$ are taken to depend only on the first coordinate of the path (ω_1).

Let Q^1 be the probability measure that is equivalent to P^1 on F_T^1 such that

$$\widetilde{W_t} = W_t + \int_0^t \theta_s \, ds, \qquad 0 \leq t \leq T$$

is Brownian motion under Q^1. The reference probability measure Q on F_T is equal to $Q^1 \otimes P^2$. The contingent claim is equal to

$$B = \mathbf{1}_{N_T = 0}.$$

Let us calculate

$$J_0 = \sup_{M \in D} E_{Q^M}[\mathbf{1}_{N_T=0}].$$

Let J_0' be defined by

$$J_0' = \sup_{\alpha \in]-1, +\infty[} E_{P_\alpha^2}[\mathbf{1}_{N_T=0}].$$

Let P_α^2 be the probability measure defined on F_T^2 by

$$\frac{dP_\alpha^2}{dP^2} = \mathcal{E}(\alpha \widetilde{N})_T = e^{\log(1+\alpha) N_T - \alpha T},$$

where $\widetilde{N}_t = N_t - t$, $0 \le t \le T$. Let Q_α be defined by $Q_\alpha = Q^1 \otimes P^2_\alpha$. We have

$$\frac{dQ_\alpha}{dQ} = \mathcal{E}(\alpha \widetilde{N})_T \quad \text{and} \quad \alpha \widetilde{N} \in D,$$

hence, $J'_0 \le J_0$.

Let us show that $J'_0 = 1$. From the change of measure theorem for point processes ([Br-Ja, pp. 377–379]), we have that N is a Poisson process with intensity $1 + \alpha$ under P^2_α; hence, $P^2_\alpha(N_T = 0) = e^{-(1+\alpha)T}$ and

$$J'_0 = \sup_{\alpha \in]-1,+\infty[} P^2_\alpha(N_T = 0) = 1.$$

The supremum is obtained for $\alpha = -1$; hence

$$J_0 = 1.$$

Let us calculate

$$J_t = \text{ess} \sup_{M \in D} E_{Q^M}[\mathbf{1}_{N_T=0}/F_t].$$

Let J'_t be given by

$$J'_t = \text{ess} \sup_{\alpha \in]-1,+\infty[} E_{P^2_\alpha}[\mathbf{1}_{N_T=0}/F^2_t].$$

We have $J'_t \le J_t$. Let us determine J'_t:

$$E_{P^2_\alpha}[\mathbf{1}_{N_T=0}/F^2_t] = \mathbf{1}_{N_t=0}\, E_{P^2_\alpha}[\mathbf{1}_{N_T=0}/F^2_t] = \mathbf{1}_{N_t=0}\, e^{-(1+\alpha)(T-t)},$$

hence,

$$J'_t = \mathbf{1}_{N_t=0},$$

and the supremum is obtained for $\alpha = -1$. Now,

$$J_t \le \mathbf{1}_{N_t=0};$$

hence,

$$J_t = \mathbf{1}_{N_t=0} = 1 - N_{t \wedge T_1}.$$

The seller follows the following strategy. At $t = 0$, he receives 1 from the buyer. If the Poisson process remains equal to 0 until T, the seller does not make any profit; at time T, he gives 1 to the buyer. Otherwise, at the first instant the Poisson process is different from $0(T_1)$, the seller makes profit 1.

Let us determine the limit of the probability measures Q_α as $\alpha \to -1$:

$$\forall t \in R^+ \quad \text{and} \quad \lambda \in R^+, \lim_{\alpha \downarrow -1} E_{P^2_\alpha}[e^{-\lambda N_t}] = \lim_{\alpha \downarrow -1} e^{(1+\alpha)t(e^{-\lambda}-1)} = 1.$$

Also, for all $t_1, t_2, \ldots, t_k \in (R^+)^k$, the Laplace transform of $(N_{t_1}, \ldots, N_{t_k})$ tends to 1 under P^2_α as α tends to -1. Hence, $P^2_\alpha\,(N_{t_1}, \ldots, N_{t_k})^{-1}$ converges to the Dirac measure at zero on $(R^+)^k$ as α tends to -1, and $Q_\alpha = Q^1 \otimes P^2_\alpha$ converges in a weak sense to $Q_\alpha = Q^1 \otimes \delta_0$ as

α tends to -1. Under the limit probability measure $Q^1 \otimes \delta_0$, the increasing optional process f is equal to zero almost surely, that is,

$$Q^1 \otimes \delta_0(N_{t \wedge T_1} = 0) = 1,$$

$$J_t = 1, \quad Q^1 \otimes \delta_0 \text{ a.s.}$$

Hence, J is a martingale (constant) under the limit probability measure. But this probability measure is not equivalent to P. Notice that this probability measure makes the market complete (because it annuls the Poisson process).

The essential infimum of the possible prices at $t \geq 0$, $t < T$, is given by

$$\begin{aligned} K_t &= \text{ess} \inf_{\alpha > -1} Q^\alpha(N_T = 0/F_t), \\ &= \lim_{\alpha \to +\infty} Q^\alpha(N_T = 0/F_t), \\ &= 0. \end{aligned}$$

Hence, for each $t \in [0, T]$, we have

$$K_t = \mathbf{1}_{N_T=0} \mathbf{1}_{t=T}.$$

Let us determine the limit of the probability measures Q_α as $\alpha \to +\infty$. For all $t_1, t_2, \ldots, t_k \in (R^+)^k$, the Laplace transform of $(N_{t_1}, \ldots, N_{t_k})$ tends to zero under P^2_α as α tends to $+\infty$. Hence, Q_α converges in a weak sense to the null measure as $\alpha \to +\infty$. This result shows that an optimizing sequence of P-martingale measure does not necessarily converge to a probability measure.

A. Appendix.

A.1. A few useful (well-known) properties and theorems.

PROPOSITION 1.a. *A local martingale that is of class* (D) *is a martingale* (*see* [*De-Me*, *p.* 97, VI-30]).

Recall that if M and N are local martingales, their quadratic variation process $\langle M, N\rangle$ is defined only if MN is a special semimartingale. (For the definition, see [De-Me, p. 247, VII-39].) It always exists if one of the local martingales M, N is locally bounded by the following property (see [De-Me, p. 240, VII-32]).

PROPOSITION 1.b. *If X is a special semimartingale and Y is a locally bounded semimartingale, then XY is a special semimartingale.*

Proof. Using Doob's inequality, it is easy to show that $\sup_{s \leq t} |X_s Y_s|$ is locally integrable; it follows that XY is a special semimartingale (by using Thm. 25-d, p. 234 in [De-Me]). □

Notation. Let X_t be a local martingale (RCLL) under P with respect to $\{F_t\}$, such that $X_0 = 0$. We denote by $\mathcal{E}(X)_t$ the exponential of X, that is, the solution of the SDE

$$dU_t = U_{t^-}\, dX_t \quad \text{with } U_0 = 1.$$

The process $\mathcal{E}(X)$ is a local martingale under P.

We recall a general form of the Girsanov theorem that we shall use several times. (For more details see [De-Me, p. 259, VII-49] or [Mey, p. 377].)

Let P and Q be two probabilities equivalent on F_T, such that

$$\left.\frac{dQ}{dP}\right|_{F_T} = \mathcal{E}(N)_T,$$

where N is a local martingale which satisfies $N_0 = 0$. Let Z be a special semimartingale under P. It has the unique canonical decomposition under P

$$Z_t = Z_0 + M_t + A_t, \qquad 0 \leq t \leq T,$$

where M is a local martingale that satisfies $M_0 = 0$, and A is a predictable VF (finite variation) and RCLL process that satisfies $A_0 = 0$. Note that a supermartingale is special and that in this case, A_t is a decreasing process.

THEOREM 1.A. *Suppose that $\langle M, N\rangle$ exists. Then Z is a special semimartingale under Q and its canonical decomposition under Q is given by $Z_t = Z_0 + (M_t - \langle M, N\rangle_t) + (A_t + \langle M, N\rangle_t)$. The first term between brackets is a local martingale under Q, and the second one is a predictable finite variation process.*

COROLLARY 1.A. *Suppose that $\langle M, N\rangle$ exists. Then*

(i) *Z is a local martingale under Q if and only if the process $(A_t + \langle M, N\rangle_t)$ is equal to* 0.

(ii) *Z is a supermartingale under Q if and only if $(A_t + \langle M, N\rangle_t)$ is a decreasing process.*

A.2. Characterization of the essential supremum of all the possible prices (proofs). Let J_t be the essential supremum of the possible prices for B at time t and

$$J_t = \operatorname*{ess\ sup}_{R \in \mathcal{P}} E_R[B/F_t] = \operatorname*{ess\ sup}_{N \in D} E_{Q^N}[B/F_t].$$

We use dynamic programming methods [ElK] to solve the problem. Notice that (J_t) is not defined as a process yet because for each t, it is defined Q almost surely. Now,

$$\forall N \in D \quad \text{and} \quad t \in [0,T],\ E_{Q^N}[B/F_t] = E_{Q^{\tilde N}}[B/F_t],$$

where

$$\widetilde{N}_u = N_u - N_{t\wedge u},\ 0 \leq u \leq T.$$

Hence,

$$J_t = \operatorname*{ess\ sup}_{N \in D(t)} E_{Q^N}[B/F_t],$$

where $D(t) = \{N \in D / N_u = 0\ \forall\, u \in [0,t]\}$. For any $N \in D(t)$, put

$$\Gamma(t, N) = E_{Q^N}[B/F_t] = E[\mathcal{E}(N)_T B/F_t].$$

PROPOSITION 1. *$\{\Gamma(t, N),\ N \in D(t)\}$ is stable by supremum and infimum.*

By this property, it follows that for each t, there exists a sequence $N_p \in D(t)$ so that, almost surely, $\Gamma(t, N_p)$ is an increasing sequence of random variables that converges to J_t, that is,

$$J_t = \lim_{p\to+\infty} \uparrow \Gamma(t, N_p) = \lim_{p\to+\infty} \uparrow E_{Q^{N_p}}[B/F_t].$$

This property will allow us to invert supremum and expectation (using the monotone convergence theorem).

Proof. Let $N_1,\ N_2 \in D(t)$. There exists $N \in D(t)/\Gamma(t, N) = \Gamma(t, N_1) \vee \Gamma(t, N_2)$. Indeed, put $A = \{\Gamma(t, N_2) \geq \Gamma(t, N_1)\}$. We have $A \in F_t$. Put $N = N_1 \mathbf{1}_{A^c} + N_2 \mathbf{1}_A$. We have $N \in D(t)$;

$$\begin{aligned}\Gamma(t, N) &= E[\mathcal{E}(N_1)_T B/F_t]\mathbf{1}_{A^c} + E[\mathcal{E}(N_2)_T B/F_t]\mathbf{1}_A,\\ &= \Gamma(t, N_1)\mathbf{1}_{A^c} + \Gamma(t, N_2)\mathbf{1}_A.\end{aligned}$$

Hence, $\Gamma(t,N) = \Gamma(t,N_1) \vee \Gamma(t,N_2)$. □

For each t, take a sequence $N_p \in D(t)$ such that, Q almost surely,

$$J_t = \lim_{p\to+\infty} \uparrow \Gamma(t,N_p) = \lim_{p\to+\infty} \uparrow E_{Q^{N_p}}[B/F_t].$$

Now, J_t will denote a (F_t)-adapted process that is equal to the above limit almost everywhere.

PROPOSITION 2. *For any $N \in D$, (J_t) is a supermartingale under Q^N (that is, $\mathcal{E}(N)_t\, J_t$ is a supermartingale under Q).*

Proof. Let s, t be two positive reals such that $s < t \le T$. Take a sequence $N_p \in D(t)$ such that, Q almost surely,

$$J_t = \lim_{p\to+\infty} \uparrow \Gamma(t,N_p) = \lim_{p\to+\infty} \uparrow E_{Q^{N_p}}[B/F_t].$$

Let $N \in D$. Since we can invert limit and expectation (by the monotone convergence theorem), we have

$$\begin{aligned} E\left[\frac{\mathcal{E}(N)_t}{\mathcal{E}(N)_s}\, J_t/F_s\right] &= \lim_{p\to+\infty} \uparrow E\left[\frac{\mathcal{E}(N)_t}{\mathcal{E}(N)_s}\, E[\mathcal{E}(N_p)_T B/F_t]/F_s\right], \\ &= \lim_{p\to+\infty} \uparrow E\left[\frac{\mathcal{E}(N)_t}{\mathcal{E}(N)_s}\, \mathcal{E}(N_p)_T B/F_s\right], \\ &= \lim_{p\to+\infty} \uparrow E\left[\frac{\mathcal{E}(\widetilde{N}_p)_T}{\mathcal{E}(\widetilde{N}_p)_s}\, B/F_s\right], \end{aligned}$$

where $\widetilde{N}_p(u) = N(u \wedge t) + N_p(u)$, for $u \in [0,T]$. Now, $(\widetilde{N}_p(u),\ u \in [0,T]) \in D$. Hence,

$$E\left[\frac{\mathcal{E}(N)_t}{\mathcal{E}(N)_s}\, J_t/F_s\right] \le J_s. \qquad \square$$

PROPOSITION 3. *(J_t) is the smallest supermartingale under Q^N, for any $N \in D$, which is equal to B at time T (unique up to a null set).*

Proof. Let (J'_t) be a supermartingale under Q^N, for any $N \in D$, which is equal to B at time T. Then,

$$\forall\, t \in [0,T] \quad \text{and} \quad N \in D,\ J'_t \ge E_{Q^N}[B/F_t],\ Q\text{ a.s.}$$

Hence,

$$\forall\, t \in [0,T],\ Q\text{ a.s.},\ J'_t \ge J_t. \qquad \square$$

We have also the following property.

PROPOSITION 4. *Let $\hat{N}$ be a local martingale that belongs to D. The following properties are equivalent:*

(i) *$\hat{N}$ is optimal, i.e., $\forall\, t \in [0,T]$, $J_t = E_{Q^{\hat{N}}}[B/F_t]$, Q a.s.*

(ii) *J_t is a martingale under $Q^{\hat{N}}$.*

PROPOSITION 5. *There exists an* RCLL *supermartingale still denoted by J_t so that for each $t \in [0,T]$,*

$$J_t = \operatorname*{ess\,sup}_{N \in D} E_{Q^N}[B/F_t].$$

Proof. Put $\mathbb{D} = [0,T] \cap \mathbb{Q}$. Because (J_t) is a supermartingale, we have that for P almost every ω, the mapping $t \to J_t(\omega)$ defined on $\mathbb{D}$ has at each point t of $[0,T[$ a finite right limit

$$J_{t^+}(\omega) = \lim_{s\in\mathbb{D}, s\downarrow t} J_s(\omega)$$

and at each point of $]0,T]$ a finite left limit

$$J_{t^-}(\omega) = \lim_{s\in\mathbb{D}, s\uparrow t} J_s(\omega).$$

One can show (using a well-known property) that (J_{t^+}) is an (F_{t^+})-supermartingale under Q_N for all $N \in D$. Because the filtration is right continuous, J_{t^+} is an (F_t)-supermartingale under Q_N for all $N \in D$. Hence, by Proposition 3, for all $t \in [0,T]$, Q almost surely, $J_{t^+} \geq J_t$. Also, $J_t \geq E[J_{t^+}/F_t]$. Hence, Q almost surely, $J_t = J_{t^+}$ or else,

$$\forall\, t \in [0,T],\; J_{t^+} = \operatorname{ess}\sup_{N\in D} E_{Q^N}[B/F_t].$$

The result follows by taking J_t equal to the above process J_{t^+}. □

J_t is an RCLL process that satisfies

$$J_t = \operatorname{ess}\sup_{N\in D} E_{Q^N}[B/F_t].$$

J_t is characterized as the smallest right continuous supermartingale under Q_N, for every N belonging to D, which is equal to B at time T. Also, N is optimal if and only if J_t is a martingale under Q_N.

A.3. Generalization of the results of this paper. The results of Theorem 2.1.1, Theorem 2.1.2, Theorem 2.3.1, and Corollary 2.3.1 remain under the hypothesis

$$\sup_{N\in D} E_{Q^N}(B) < \infty.$$

(In fact, we shall see below that this hypothesis is equivalent to the fact that there exists a price admissible for sellers, or equivalently that B is smaller than the value of a self-financing portfolio, that is, B satisfies

$$B \leq X^{H,0,y}(T), \quad P\,\text{a.s.}$$

for some portfolio strategy H and initial investment $y \geq 0$.)

The whole proof of Theorem 2.1.1 still holds under the above hypothesis. In this case, J_t is not generally of class D. The results of Theorem 2.1.2, Proposition 2.3.1, and Theorem 2.3.1 still hold, but j is a Q-local martingale only (but not a martingale in general). The arguments of the proof still hold, but it is a bit more complicated technically because $\langle j\rangle$ is not always defined.

Proof of Theorem 2.1.2 *under the above hypothesis.* J_t is a Q-supermartingale; hence, it admits a unique decomposition as a local martingale M_t minus an increasing predictable process $A_t : J_t = M_t - A_t$. The local martingale M_t admits the following Kunita decomposition:

$$M_t = J_0 + \int_0^t \varphi_s^* \sigma_s\, d\widetilde{W}_s + j_t \quad \forall\, t \in [0,T],\; Q\,\text{a.s.}$$

for some predictable process φ and some Q-local martingale j such that

$$\left\langle j, \int_0^{\cdot} \sigma_i(s) dW_s \right\rangle_T = 0,\ Q\text{ a.s.}, \quad \forall i \in \{1, \dots, n\}.$$

As in the proof of Theorem 2.1.2, we show that the continuous part j^c of j is equal to zero, using the following lemma (which follows from the fact that J_t is a supermartingale under each P-martingale measure).

LEMMA. *Let N be an element of D such that $\langle N, j \rangle$ exists. Then $A_t - \langle N, j \rangle_t$ is an increasing process.*

Now, $\langle j^c \rangle$ is locally integrable. By the Lebesgue Decomposition Theorem, there exist a positive predictable process h that belongs to $L^1_{\text{loc}}([0,T] \times \Omega, d\langle j^c \rangle_t \, dQ)$ and a locally integrable predictable increasing process B such that

$$dA_t = h_t d\langle j^c \rangle_t + dB_t$$

and such that, Q almost surely, the measure dB_t is singular with respect to $d\langle j^c \rangle_t$. Using the same arguments as in the proof of Theorem 2.1.2, we obtain the desired result. □

The proof of Theorem 2.3.1 under the weaker hypothesis is unchanged, and the result of Proposition 2.3.1 can be obtained by the same methods as before, but it is a bit longer because the lemma must be applied to some N such that $\langle N, j \rangle$ exists.

Remark. It follows from this, that the following properties are equivalent.

(i) $\sup_{N \in D} E_{Q^N}(B) < \infty$.

(ii) There exists a price admissible for sellers, or equivalently that B is smaller than the value of a self-financing portfolio, that is, B satisfies

$$B \leq X^{H,0,y}(T),\ P\text{ a.s.},$$

i.e.,

$$B \leq y + \int_0^T H_s^* \sigma_s \, d\tilde{w}_s,\ Q\text{ a.s.}$$

for some portfolio strategy H and initial investment $y \geq 0$.

Notice that the technical assumption on B given by

$$E_{Q^N}\left[\int_0^T \|\sigma_s^* H_s\|^2 \, ds\right] < +\infty$$

for each $N \in D$ may be interpreted as the fact that the contingent claim B is not too risky. When we do not make the technical assumption on B, Corollary 2.3.1 still holds, but Corollary 2.3.2 must be replaced by the following.

COROLLARY 2.3.2$'$. (1) *For each $N \in D$, $E_{Q^N}(B) \leq J_0 - E_{Q^N}(f_T)$.*

(2) *If N_n, $n \geq 0$, is an optimizing sequence belonging to D, that is, such that*

$$\lim_{n \to +\infty} E_{Q^{N_n}}[B] = J_0$$

then

$$\lim_{n \to \infty} E_{Q^{N_n}}(f_T) = 0.$$

Proof of (1). Let N be an element of D. The process given by

$$J_0 + \int_0^t \varphi_s^* \sigma_s \, d\widetilde{W}_s$$

is a positive continuous Q^N-local martingale, and hence a Q^N-supermartingale; this yields the inequality

$$J_0 + E_{Q^N}\left[\int_0^T \varphi_s^* \sigma_s \, d\widetilde{W}_s\right] \leq J_0.$$

Thus, f_T is Q^N-integrable and $E_{Q^N}(B) \leq J_0 - E_{Q^N}(f_T)$.

Proof of (2). Inequality (1) applied to the local martingales N_n gives the inequality

$$\forall n \in N,\ E_{Q^{N_n}}(B) \leq J_0 - E_{Q^{N_n}}(f_T).$$

Hence, if we let n tend to $+\infty$, we obtain the desired result

$$\lim_{n \to \infty} E_{Q^{N_n}}(f_T) = 0. \qquad \square$$

Also, when we do not make the technical assumption on B, Theorem 2.3.2 does not hold anymore, but we have the following result.

THEOREM 2.3.2′. *If* $\sup_{N \in D} E_{Q^N}[B]$ *is attained then* B *is attainable, that is, there exist a constant* x *and a portfolio* π *such that*

$$B = x + \int_0^T \pi_u^* \sigma_u \, d\widetilde{W}_u,\ Q \text{ a.s.}$$

Proof. Compare the proof of Theorem 2.3.2.

Thus, the contingent claims that satisfy the technical assumption (loosely speaking, those that are not too risky) are divided in two sets:

- The set of contingent claims that are attainable, which is equal to the set of contingent claims that admit a unique price.
- The set of contingent claims that are not attainable, which is equal to the set of contingent claims that admit several possible prices.

Things are not as clear for contingent claims that admit a finite selling price but do not satisfy the technical assumption. Even if they are attainable in the above sense, they may admit several possible prices. (Loosely speaking, this can be explained by the fact that, even if they are attainable, they can be too risky.) Nevertheless, we have the following properties that are equivalent.

(i) For each local martingale $N \in D$,

$$E_{Q^N}(B) = E_Q(B).$$

(ii) There exist a constant x and a portfolio π such that

$$B = x + \int_0^T \pi_u^* \sigma_u \, d\widetilde{W}_u,\ Q \text{ a.s.}$$

and such that the process given by

$$x + \int_0^{\cdot} \pi_u^* \sigma_u \, d\widetilde{W}_u$$

is a martingale under each P-martingale measure.

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