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Dynamic Fund Protection

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Junichi Imai* and Phelim P. Boyle†

Abstract

Dynamic fund protection provides an investor with a floor level of protection during the investment period. This feature generalizes the concept of a put option, which provides only a floor value at a particular time. The dynamic protection feature ensures that the fund value is upgraded if it ever falls below a certain threshold level. Gerber and Pafumi (2000) have recently derived a closed-form expression for the price of this protection when the basic portfolio follows geometric Brownian motion. In this paper we examine the pricing of this feature under the constant elasticity of variance process. Two approaches are used to obtain numerical results. First, we show how to extend the basic Monte Carlo approach to handle the particular features of dynamic protection. When a discrete-time simulation approach is used to value a derivative that is subject to continuous monitoring, there is a bias. We show how to remove this bias. Second, a partial differential equation approach is used to price dynamic protection. We demonstrate that the price of the dynamic protection is sensitive to the investment assumptions. We also discuss a discrete time modification of the dynamic protection feature that is suitable for practical implementation. The paper deals just with pricing and does not consider the important question of reserving for these contracts.

1. Introduction

Both individual and institutional investors have a basic need for protection against downside risk. This need is evident in the appeal of portfolio insurance and related strategies. There are many investment protection plans designed to appeal to retail investors, ranging from basic put options to more sophisticated embedded options. These features are common in a range of equity-linked contracts in many countries. In one popular version the premiums are invested in a stock portfolio. Hardy (2001) provides a description of these types of contracts and analyzes the issues involved in pricing and reserving for them.

In the United States, equity-indexed annuities have become very popular. These contracts are deferred annuities where the benefits are based on the performance of some reference portfolio, such as the Standard and Poor's (S&P) 500 Index. Generally there is a 3% minimum guarantee provision. Tiong (2000) describes equity-indexed annuities and derives closed-form solutions for the prices of three common product designs. In Canada, insurance contracts where the benefits depend on the market value of some reference portfolio are known as segregated fund contracts because the underlying assets are kept apart from the insurance company's general account.

At first the Canadian contracts contained just an embedded put option, but the range of features was extended to include a wide variety of complicated features, such as reset (shout) options.[1] These sophisticated options often require the policyholder to make exercise decisions over the life of the contract, and it is very difficult for a policyholder to determine the optimal strategy. These features are often added as a result of marketing pressures, and even insurance companies that write them have trouble valuing them. Initial research on segregated fund contracts indicated that the premium structure was inadequate to support these complicated option features. How-

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[1] For a discussion of the complexity of these products see Boyle and Hardy (1996, 1998), Hardy (1999a,b), and Falloon (1999).

ever, the Canadian Institute of Actuaries has recently developed guidelines to put the pricing and reserving of these contracts on a scientific basis (Task Force Report 2000).

The benefits of these complex features in segregated fund contracts may be illusory for the average policyholder because they often require complicated exercise decisions. It is therefore of interest to consider simpler designs that provide floor protection for a stock portfolio and do not require sophisticated exercise strategies. A standard put option provides an investor with downside protection. The protection provided by a put option is static because it operates at a fixed time. If an investor invests in a combined package of an equity portfolio and a European put on this portfolio, there is a guaranteed minimum amount at maturity. One drawback of this strategy is that, if the portfolio has good returns followed by poor performance, so that the final value ends up below the strike price, the investor receives only the floor level at maturity and does not participate in the good returns. An advantage of the standard European put is that the investor does not need to make complicated exercise decisions.

Recently Gerber and Shiu (1998, 1999) have introduced the concept of dynamic fund protection, which extends the put option concept to provide continuous protection. The protection is incorporated automatically so there is no need for the investor to worry about the exercise decision. This is an advantage for investors, few of whom will have the expertise to make complicated exercise decisions. Dynamic fund protection guarantees that the value of the fund never falls below the floor value during the protection period. We explain the details later, but in essence the dynamic protection feature ensures that the fund value is upgraded if it ever falls below a certain threshold at any time during the life of the contract. This benefit is more valuable than a plain vanilla put, and naturally it will cost more. Gerber and Pafumi (2000) analyzed the pricing and hedging of dynamic fund protection and developed an analytical formula for the price of this protection.

This paper extends the analysis of dynamic protection in several ways. First, we use an alternative approach to the derivation of Gerber and Pafumi (2000), and we use a change of measure technique. We also explore the pricing of dynamic protection under alternative investment assumptions. Not surprisingly, the investment assumption can have a huge impact on the price of dynamic protection. We use both the Monte Carlo method and a partial differential equation (PDE) approach to develop numerical results. The numerical valuation of this feature requires some modifications of these two approaches, and these may be of independent interest. Gerber and Pafumi (2000) derive their closed-form formula under the lognormal assumption, and they directly compute the initial price of the dynamic fund protection by applying the distribution function. We also examine American-style dynamic fund protection and show that it is never optimal to exercise the dynamic protection feature early.

It is now well established in the empirical literature that equity prices do not follow a simple lognormal process. Several authors have shown that other models produce significant deviations from the Black-Scholes option prices (see Bakshi, Cao, and Chen 1997, 2000; Boyle and Tian 1999; Davydov and Linetsky 2001; and Hardy 1999a,b). For example, when the Black-Scholes model is used to price stock options, certain biases such as the well-known strike price bias (volatility smile) persist. Hence, it is of interest to investigate the price of dynamic protection under more general assumptions.

One extension of the standard assumption (geometric Brownian motion) is to assume the underlying asset follows the constant elasticity of variance (CEV) diffusion process. This process has the advantage that the volatility of the underlying asset is linked to its price level, which is consistent with the empirical observation that stock volatility tends to change as stock prices move up and down. As observed by Cox (1996), the origin of the volatility smile is the negative correlation between stock price changes and volatility changes. The CEV option-pricing model, originally developed by Cox (1975), incorporates this negative correlation. Boyle and Tian (1999) and Davydov and Linetsky (2001) demonstrate that the prices of exotic options are very different depending on whether the lognormal assumption or the CEV assumption is used.

We emphasize that the CEV model is just one of the many processes that have been proposed for stock price movements. We plan to examine the impact of other types of investment assumptions in future work.[2] We use two numerical methods to price the dynamic protection feature and illustrate them using the CEV process. First, we use Monte Carlo and indicate how to incorporate efficient variance reduction techniques. Second, we use a partial differential equation to solve for the price of the contract.

We also examine a modification of the dynamic protection feature to discrete monitoring since this seems a more practical form of contract design. To evaluate the price of the protection under discrete monitoring we use the Monte Carlo approach. In the case of the lognormal assumption, we show how to obtain a good approximation to the discrete approximation based on an approach used by Broadie, Glasserman, and Kou (1999).

This paper is organized as follows. Section 2 defines the dynamic fund protection and develops an alternative valuation approach. This section also discusses discrete monitoring. Section 3 discusses the possibility of early exercise of the dynamic fund protection. In Section 4 we use numerical approaches to explore the impact of using the CEV assumption. Both Monte Carlo simulation and a finite element PDE method are used to compute the values of the protection. And Section 5 presents our conclusions.

2. Pricing the Dynamic Fund Protection

We begin with a description of a dynamic fund protection. Then we present an alternative derivation of the Gerber-Pafumi closed-form formula for the price of dynamic fund protection. We show that there is an analogy between dynamic protection and a certain type of lookback option, and we exploit this correspondence in our derivation.

Dynamic fund protection guarantees a predetermined protection level to an investor who owns the underlying fund. We can characterize its operation as follows. Consider an investor who holds one unit of the underlying fund and overlays it with dynamic fund protection. The protection floor is similar to the strike price of a put option. The rate of return on the protected portfolio is identical to that of the underlying fund as long as the value of the fund lies above the protection floor. Once the fund value goes down below the protection level, additional money is added (or deemed to be added) instantaneously to the portfolio to bring its value up to protection level.

To define the concept more precisely we need some notation. For the most part we follow the notation of Gerber and Pafumi (2000).[3] Assume the standard Black-Scholes world. Define the probability space denoted by $(\Omega, \mathcal{F}, \mathcal{P})$, where $\{\mathcal{F}_t;\ t \in [0, T]\}$ is the filtration and T is the expiration date of the dynamic fund protection. Let $S(t)$ denote the underlying fund price. The price process is assumed to be an adapted process. Gerber and Pafumi assume that the underlying fund price follows a geometric Brownian motion:

$$dS(t) = \mu S(t)\, dt + \sigma S(t)\, dW(t), \tag{1}$$

where μ and σ are the drift and volatility coefficients, respectively; both μ and σ are constant. Let $W(t)$ be a standard Wiener process under the probability measure P. Assume that the instantaneous risk-free rate r is constant, and the money market account satisfies the following equation:

$$dB(t) = rB(t)\, dt. \tag{2}$$

Suppose an investor who holds one unit of the underlying fund protects it with the dynamic fund protection. Let K denote the protection level. Let $F(T)$ denote the payoff of this portfolio (with the protection) at time T. Then $F(T)$ is given by

$$F(T) = S(T)\ \max\left\{1, \max_{0\le s\le T} \frac{K}{S(s)}\right\}. \tag{3}$$

Note that at inception $S(0) = F(0)$. For clarity we will sometimes refer to the underlying portfolio $S(t)$ as the *naked* portfolio and the portfolio with the dynamic protection as the *protected* portfolio.

In his discussion of the Gerber-Pafumi paper, Chan (2000) notes that there is an elegant relationship between the process for the naked fund and the process for the protected fund. Let

[2] In particular the two-factor stochastic volatility model proposed recently by Chernov et al. (2001) appears to provide a good fit to the empirical data.

[3] We use $S(t)$ for the underlying fund where they use $F(t)$, and we use $F(t)$ for the protected fund where they use $\tilde{F}(t)$.

$$Z(t) = \log \frac{S(t)}{K} \quad \text{and} \quad Y(t) = \log \frac{F(t)}{K}.$$

The process $Y(t)$ can be obtained from $Z(t)$ by introducing a reflecting barrier at the origin. The processes Y and Z are connected through the local time.

2.1 The Gerber-Pafumi Approach

We now sketch the derivation used by Gerber and Pafumi. Assume that the naked fund does not pay any dividends. Let $X(T)$ denote the terminal value of the derivative that provides the dynamic fund protection. Accordingly,

$$\begin{aligned} X(T) &= F(T) - S(T) \\ &= S(T) \max\left\{1, \max_{0 \le s \le T} \frac{K}{S(s)}\right\} - S(T). \end{aligned} \tag{4}$$

Let $V(t)$ denote the value of the dynamic fund protection at time t. Then, from standard option pricing theory, we obtain

$$\begin{aligned} V(0) &= e^{-rT} \mathrm{E}^Q[X(T) | \mathcal{F}_0] \\ &= e^{-rT} \mathrm{E}^Q[F(T) | \mathcal{F}_0] - S(0), \end{aligned} \tag{5}$$

where $\mathrm{E}^Q[\cdot]$ represents the expectation under the equivalent martingale measure Q (see Harrison and Kreps 1979 and Harrison and Pliska 1981). From Girsanov's theorem, the naked fund value satisfies the following stochastic differential equation under Q:

$$dS(t) = rS(t)\, dt + \sigma S(t)\, dW^Q(t), \tag{6}$$

where $W^Q(t)$ is a standard Wiener process under the Q-measure. Gerber and Pafumi derive their solution by using the probability density function of $\log(F(T)/K)$ and compute Equation (5) directly.[4] In other words, they calculate

$$V(0) = Ke^{-rT} \int_0^\infty e^u p(u)\, du - S(0), \tag{7}$$

where $U = \log(F(T)/K)$, and $p(u)$ is the probability density function of U. The solution turns out to be not much more complicated than the standard Black-Scholes formula:

$$V(0) = Ke^{-rT}\left(1 - \frac{1}{R}\right)\Phi(h_1) + \frac{K}{R} H^R \Phi(h_2) - S(0)\Phi(h_3),$$

where

$$\begin{aligned} h_1 &= \frac{-\kappa(0) - rT + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}, \\ h_2 &= \frac{-\kappa(0) + rT + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}, \\ h_3 &= \frac{-\kappa(0) - rT - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}, \end{aligned} \tag{8}$$

and

$$R = \frac{2r}{\sigma^2}, \qquad \kappa(0) = \log \frac{S(0)}{K}, \qquad H = \frac{K}{S(0)}.$$

2.2 Another Approach to Pricing Dynamic Protection

We provide an alternative approach to evaluate the dynamic fund protection in this subsection based on the change of measure technique. This approach will enable us to derive a formula for the discretely monitored protection. It is also useful for the analysis of American-style dynamic fund protection.

The terminal payoff in Equation (4) suggests that the dynamic fund protection is closely related to a standard lookback option.[5] More precisely, it is related to a lookback call option on the minimum, in the sense that its terminal payoff is a function of the minimum value of the naked fund price up to time T. The payoff on a standard lookback call option depends on the minimum price and the terminal price of the fund, and the payoff of a lookback call option on the minimum depends on the minimum fund price and its exercise price. On the other hand, the payoff of the dynamic fund protection depends on both the

[4] The probability density function is found in Equation (2.4) in Gerber and Pafumi (2000) or in formula (91) of Section 5.7 in Cox and Miller (1965).

[5] Tiong (2000) discusses lookback options in the context of equity-indexed annuities.

terminal fund price and the exercise price in addition to the minimum fund price.

Suppose that we are now at time t. The terminal payoff of Equation (3) can be rewritten as

$$F(T) = S(T) \max\left\{M(t), \frac{K}{\min\limits_{t \le s \le T} S(s)}\right\}, \tag{9}$$

where $M(t) = \max\{1, K/\min\limits_{0 \le s \le t} S(s)\}$.

Note that $M(t)$ is a $\mathcal{F}_t$-measurable function. Because $\max\{a, b\} = a + (b - a)^+$, Equation (4) becomes

$$X(T) = S(T)\{M(t) - 1\} + S(T)\left(\frac{K}{\min\limits_{t \le s \le T} S(s)} - M(t)\right)^+, \tag{10}$$

where $(x)^+ = \max\{x, 0\}$. We now use a change of measure to evaluate the dynamic fund protection.

We select the naked fund as the new numeraire, and define a new equivalent martingale measure Q_S. From standard arguments, the value of the dynamic fund protection at time t is given by

$$V(t) = S(t)\mathrm{E}^{Q_S}\left[\frac{X(T)}{S(T)}\middle|\mathcal{F}_t\right] = S(t)\{M(t) - 1\} + S(t)\mathrm{E}^{Q_S}\left[\left(\frac{K}{\min\limits_{t \le s \le T} S(s)} - M(t)\right)^+\middle|\mathcal{F}_t\right], \tag{11}$$

where we use the $\mathcal{F}_t$-measurability of $M(t)$. From Girsanov's theorem, the price of the naked fund under Q_S follows:

$$dS(t) = (r + \sigma^2)S(t)\, dt + \sigma S(t)\, dW^{Q_S}(t). \tag{12}$$

The probability distribution function of the minimum value of the fund price is known when the fund price follows geometric Brownian motion. When we define

$$y(\tau) = \log\left(\frac{\min\limits_{t \le s \le T} S(s)}{S(t)}\right), \tag{13}$$

where $\tau = T - t$, then $y(\tau)$ represents the minimum value of the Brownian motion between time 0 and time τ. Since the distribution function of $y(\tau)$ is known, we can derive the following formula for the price of dynamic fund protection:

$$V(t) = S(t)\{M(t)\Phi(d_1) - 1\} + \frac{K}{R}\left(\frac{K'(t)}{S(t)}\right)^R \Phi(d_2) + \left(1 - \frac{1}{R}\right)Ke^{-r\tau}\Phi(d_3), \tag{14}$$

where $R = 2r/\sigma^2$, $K'(t) = K/M(t)$, and $\kappa'(t) = \log(S(t)/K'(t))$. Here d_i, $i = 1, 2, 3$ satisfy the following equations:

$$d_1 = \frac{\kappa'(t) + r\tau + \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}, \tag{15}$$

$$d_2 = \frac{-\kappa'(t) + r\tau + \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}, \tag{16}$$

$$d_3 = \frac{-\kappa'(t) - r\tau + \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}. \tag{17}$$

It is easy to check that Equation (14) is identical to the pricing formula developed by Gerber and Pafumi when we set $t = 0$.

2.3 Discretely Monitored Dynamic Fund Protection

The form of dynamic fund protection considered thus far is based on continuous monitoring. This assumption sometimes leads to elegant theoretical properties such as the closed-form expressions for the price in the case of geometric Brownian motion. From a practical perspective, it might be more useful to consider discrete monitoring, because discrete monitoring makes the contract easier to administer. We will see that discrete monitoring reduces the price.

In the case of discrete monitoring, the fund is observed at discrete time points

$$\{s_0, s_1, \ldots, s_n\}, \quad s_0 = 0,\ s_n = T.$$

If $F_n(T)$ denotes the payoff on the portfolio (under discrete monitoring) at time T, then $F_n(T)$ is given by

$$F_n(T) = S(T) \max\left\{1, \max_{0 \le j \le n} \frac{K}{S(s_j)}\right\}. \tag{18}$$

Sometimes we can get a good approximation to the discretely monitored price by adjusting the inputs to the continuously monitored formula. Of course, this works only if there is a closed-form solution for the continuously monitored contract. The idea is based on research by Broadie, Glasserman, and Kou (1999). They proposed an approximate formula for the price of a discretely monitored lookback that was based on an adjustment to the formula for a continuously monitored lookback. We are able to develop the corresponding approximation for the price of dynamic fund protection when the fund follows geometric Brownian motion.

We now develop the approximation formula to value discretely monitored dynamic fund protection. The key result required is Lemma 4 of Broadie, Glasserman, and Kou (1999).

Lemma 4

Let $B(t)$ be a Brownian motion with drift μ and volatility σ. Let Y be a maximum value of this Brownian motion up to time T; that is,

$$Y = \max_{0\le s\le T} B(s).$$

Let n denote the number of monitoring dates up to T, and $\Delta t = T/n$. The maximum value with discrete monitoring, denoted by Y_n, is $Y_n = \max_{0\le k\le n} B(k\Delta t)$. Then, for any $x > 1$,

$$\mathrm{E}[(e^{Y_n} - x)^+] = e^{-\beta_1\sigma\sqrt{\Delta t}}\mathrm{E}[(e^{Y} - e^{\beta_1\sigma\sqrt{\Delta t}}x)^+] + o(1/\sqrt{n}).$$

In Lemma 4, β_1 is a constant given by

$$\beta_1 = -\zeta(\tfrac{1}{2})/\sqrt{2\pi} \approx 0.5826,$$

where ζ is the Riemann zeta function.[6] Note that this approximation is not affected by the drift of the Brownian motion.

The value of dynamic fund protection based on n monitoring dates, $V_n(t)$, is given by

$$V_n(t) = S(t)\{M_n(t) - 1\} + K\mathrm{E}^{Q_S}[(e^{-y_n} - x)^+]. \quad (19)$$

We will need the following relation between the maximum and the minimum value of a Brownian motion:

$$-y(t) = -\min_{0\le s\le t}[B(t)] = \max_{0\le s\le t}[-B(t)] = Y(t). \quad (20)$$

In Equation (20) $y(t)$ is the minimum value of Brownian motion with drift ν and variance σ^2, while $Y(t)$ is the maximum value of Brownian motion with drift $-\nu$ and the same variance σ^2. We use y_n in the obvious notation to denote the minimum taken over discrete points. Hence, we can write the expectation in Equation (19) as

$$\begin{aligned}
&\mathrm{E}^{Q_S}[(e^{-y_n} - x)^+] \\
&\quad = \mathrm{E}^{Q_S'}[(e^{Y_n} - x)^+] \\
&\quad = e^{-\beta_1\sigma\sqrt{\Delta t}}\mathrm{E}^{Q_S'}[(e^{Y} - e^{\beta_1\sigma\sqrt{\Delta t}}x)^+] + o(1/\sqrt{n}) \\
&\quad = e^{-\beta_1\sigma\sqrt{\Delta t}}\mathrm{E}^{Q_S}[(e^{-y} - e^{\beta_1\sigma\sqrt{\Delta t}}x)^+] + o(1/\sqrt{n}).
\end{aligned}$$

Note that the underlying asset price $S(t)$ follows geometric Brownian motion with the drift $-\nu$ and volatility σ under the probability measure Q_S'. Thus, Equation (19) becomes

$$\begin{aligned}
V_n(t) &= S(t)\{M_n(t) - 1\} + Ke^{-\beta_1\sigma\sqrt{\Delta t}}\mathrm{E}^{Q_S} \\
&\quad \times [(e^{-y_n} - e^{\beta_1\sigma\sqrt{\Delta t}}x)^+] + o(1/\sqrt{n}). \quad (21)
\end{aligned}$$

From Equation (21) we can derive the approximate formula of the discretely monitored dynamic fund protection:

$$\begin{aligned}
&V_n(t) \\
&\quad = S(t)\{M(t) - 1\} + K'\left\{x'\Phi(d_1') + \frac{1}{R}x'^{-R}\Phi(d_2')\right. \\
&\qquad \left. + \left(1 - \frac{1}{R}\right)e^{-r\tau}\Phi(d_3') - x'\right\} + o(1/\sqrt{n}), \quad (22)
\end{aligned}$$

where

$$K' = Ke^{-\beta_1\sigma\sqrt{\Delta t}} \text{ and } x' = xe^{\beta_1\sigma\sqrt{\Delta t}} = \frac{S(0)}{K}e^{\beta_1\sigma\sqrt{\Delta t}}.$$

Here d_i', $i = 1, 2, 3$ satisfy the following equations:

$$d_1' = \frac{\kappa' + r\tau + \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}, \quad (23)$$

$$d_2' = \frac{-\kappa' + r\tau + \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}, \quad (24)$$

$$d_3' = \frac{-\kappa' - r\tau + \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}, \quad (25)$$

where $\kappa' = \log(x')$.

[6] The Riemann zeta function is intimately related to the distribution of prime numbers and plays a central role in the famous Riemann conjecture.

3. American-Style Dynamic Fund Protection

In this section we examine the possibility of an early exercise of an American dynamic fund protection. It is well known that it is sometimes optimal to exercise standard American puts prior to maturity if the price of the underlying asset has fallen sufficiently. However, the American dynamic fund protection option should never be exercised before maturity. This result is obvious from the nature of dynamic fund protection because no rational investor will throw away something that has a positive value. However, the proof may be of interest to some readers.

We assume that there exists an equivalent martingale measure, and we also assume that the naked fund price never becomes negative. The value of the European dynamic fund protection at any time t can be divided into two parts. One is the exercise value at time t, and the other is the value of the future benefit. The value of the future benefit is always positive.

Next we show that the value of the European dynamic fund protection is greater than (or equal to) the exercise value of the corresponding American dynamic fund protection at any time. It is clear from Equation (4) that the exercise value is given by

$$\begin{aligned} X(t) &= S(t)\ \max\left\{1, \frac{K}{\min\limits_{0\le s\le t} S(s)}\right\} - S(t) \\ &= S(t)\{M(t) - 1\}. \end{aligned} \tag{26}$$

The value of the corresponding European dynamic fund protection at time t is given by

$$\begin{aligned} V(t) &= S(t)\mathrm{E}^{Q_S}\left[\frac{X(T)}{S(T)}\middle|\mathcal{F}_t\right] = S(t)\{M(t) - 1\} \\ &\quad + S(t)\mathrm{E}^{Q_S}\left[\left(\frac{K}{\min\limits_{t\le s\le T} S(s)} - M(t)\right)^+\middle|\mathcal{F}_t\right] \\ &= X(t) + S(t)\mathrm{E}^{Q_S}\left[\left(\frac{K}{\min\limits_{t\le s\le T} S(s)} - M(t)\right)^+\middle|\mathcal{F}_t\right] \\ &\ge X(t). \end{aligned} \tag{27}$$

Inequality (27) indicates that the value of the European dynamic fund protection is greater than (or equal to) the exercise value at any time. This proof is valid for more general diffusion processes than geometric Brownian motion.

4. Numerical Evaluation of Dynamic Fund Protection

It is of interest to examine numerical methods of finding the price of dynamic fund protection. Numerical methods enable us to move beyond a standard lognormal assumption. It is most unlikely that we will have the luxury of closed-form solutions for more general stochastic processes.

We use two approaches. First, we use the Monte Carlo method. It turns out that a naive application of the standard Monte Carlo gives a very biased result. We already know from the work of Andersen and Brotherton-Ratcliffe (1996) that the standard Monte Carlo approach is biased when used to evaluate continuously monitored lookback options. The bias we encounter here comes from the same source, because we use discrete time steps to evaluate an option that is monitored continuously. We remove the bias using the Andersen and Brotherton-Ratcliffe approach. Our second approach is to solve the associated partial differential equation. This is a powerful and flexible approach when the number of dimensions is small (as is the case here). For lookback options, it turns out that one of the boundary conditions is difficult to handle. We use the modification developed by Zvan and Boyle (2000) to deal with this troublesome boundary condition.

4.1 The Monte Carlo Approach: Introduction

We now discuss the pricing of dynamic fund protection using the Monte Carlo approach. A survey of the Monte Carlo method is given by Boyle, Broadie, and Glasserman (1997). Because we already have a closed-form solution for the case when the naked fund price follows geometric Brownian motion, we will use the accurate result to benchmark the Monte Carlo estimates. In particular, we can explore how the accuracy depends on the number of simulations, the number of discretization steps, and the method used to overcome the bias. The results of this investigation will be useful when we consider other types of underlying stochastic processes for which we do not have closed-form solutions.

Table 1
Parameter Values for Base Case

Parameter	Symbol	Value
Initial fund value	$S(0)$	100
Protection level	K	80, 90, 100
Time to maturity	T	1, 3, 5 years
Riskless rate	r	4% per annum
Volatility	$\sigma = \sigma_{BS}$	20% per annum

We used different parameter values in our simulations, but to conserve space we report just the results for a few combinations. Our base case assumptions are shown in Table 1.

Table 2 displays the results of using a basic Monte Carlo simulation to estimate the current price of the dynamic fund protection. For these parameters the accurate price is 14.7931, and we can hence gauge the accuracy of the results. There are two sources of error here. First, because we discretize a continuous path-dependent derivative, there is a systematic error due to the discretization. To reduce this error, we should increase the number of time steps N. Also, because we are using the Monte Carlo method, there is the usual random error or statistical error. To reduce this error, we should increase the number of simulation trials (M). However, we also need to maintain parity between the size of these two errors. There is little point in making one of these errors very small if the other one is large. Andersen and Boyle (1999) discuss the relationship between these errors and confirm that to keep them in balance we should have M of the order of N^2.

From inspection of Table 2 we see that, if we hold the number of simulation trials constant and go down any one column of the table, the estimates increase toward the accurate value. This indicates that the more points we use to divide up the interval, the more accurate the result. The use of more points reduces the systematic error. Note that for fixed M the standard errors hardly change as we use more and more time divisions. If we look across a given row we see that the estimates do not change much as we increase the number of runs (M) but that the standard errors go down. This is the usual $1/\sqrt{M}$ convergence of standard Monte Carlo techniques. Hence, the technique's most accurate answer should be the bottom righthand corner entry. This was computed using 1,000 time steps and 1 million simulation runs.

In this case the estimate is 14.370 with a standard error of 0.015. However, the true value is 14.7931, and there is clearly a significant bias. Note that the standard error of the estimate is very low. Indeed, the estimated answer is 28 standard errors from the true value. The error arises because of the systematic bias. We have not solved the problem as stated but a different problem where monitoring takes place 1,000 times per year and not continuously. For options such as lookbacks, which depend on the extrema, the differences can be significant. The bias is especially pernicious in this case because a naive application of basic Monte Carlo techniques will give low standard errors and a false degree of confidence.

We can correct for the discretization bias using the method proposed by Andersen and Brotherton-Ratcliffe (1996). They point out that the val-

Table 2
Monte Carlo Results for Prices of Dynamic Fund Protection

	Number of Simulation Trials (M)			
Number of steps (N)	10,000 (std error)	100,000 (std error)	500,000 (std error)	1,000,000 (std error)
1	5.977	6.008	6.012	6.015
	(0.146)	(0.046)	(0.020)	(0.014)
10	11.089	11.124	11.102	11.096
	(0.146)	(0.047)	(0.021)	(0.015)
250	13.985	13.992	13.970	13.978
	(0.150)	(0.048)	(0.021)	(0.015)
1,000	14.499	14.396	14.372	14.370
	(0.152)	(0.048)	(0.022)	(0.015)

Note: Prices for dynamic fund protection using standard Monte Carlo techniques. The underlying fund is assumed to follow geometric Brownian motion. We vary both the number of time steps N and the number of simulation trials M. We also report the standard errors of the estimates. The parameters used for the dynamic fund protection are $S(0) = 100$, $K = 100$, $r = 0.04$, $\sigma = 0.2$, and $T = 1$. The accurate value from the Gerber-Pafumi closed-form solution in this case is 14.7931. Note that the bias is still significant even for large values of M and N.

Table 3
Monte Carlo Estimates with ABR Modification

	Number of Simulation Trials (M)			
Number of Steps (N)	10,000 (std error)	100,000 (std error)	500,000 (std error)	1,000,000 (std error)
1	14.767 (0.154)	14.795 (0.048)	14.793 (0.022)	14.792 (0.015)
10	14.814 (0.151)	14.830 (0.048)	14.813 (0.022)	14.811 (0.015)
250	14.809 (0.151)	14.817 (0.048)	14.796 (0.022)	14.803 (0.015)
1,000	14.918 (0.153)	14.814 (0.048)	14.789 (0.022)	14.787 (0.015)

Note: Prices for dynamic fund protection using Monte Carlo techniques and the ABR modification. The naked fund follows geometric Brownian motion. The other parameters used are $S(0) = 100$, $K = 100$, $r = 0.04$, $\sigma = 0.02$, and $T = 1$. We vary both the number of time steps N and the number of simulation trials M. We also report the standard errors of the estimates. The accurate value from the Gerber-Pafumi closed-form solution in this case is 14.7931. Note that the bias in the estimates shown in Table 2 has been removed.

ues of continuously monitored lookback options computed using a standard Monte Carlo approach are biased because this method fails to capture the true minimum asset price inside each discretization interval. We record only observations at the end points. Their modification corrects this bias by estimating the minimum (maximum) price over the entire subinterval. Because the value of the dynamic fund protection is also a function of the minimum price over the investment period, it is reasonable to expect that their modification can also reduce the bias in the case of dynamic fund protection. Their adjustment is simple to incorporate in the Monte Carlo procedure. We give the results obtained using their adjustment in Table 3, and we can see at once that the improvement over the raw Monte Carlo estimates in the previous table is dramatic: The bias is gone. With this trick the Monte Carlo method can be used to price dynamic fund protection under any type of stochastic process, and we now illustrate it for a particular assumption.

4.2 The Constant Elasticity of Variance Process

We assume that the naked fund price follows the CEV process under the Q-measure:

$$dS(t) = rS(t)\,dt + \sigma S(t)^{\alpha/2}\,dW^Q(t), \quad (28)$$

where α is a constant known as the elasticity factor. The parameter α can take any value on the real line. The behavior of the process for S_t at the boundaries (zero and infinity) depends on parameter α. Davydov and Linetsky (2001) have classified the boundary behavior of the stochastic differential equation (1) with respect to α. We briefly summarize their results.

For $1 \le \alpha < 2$ the origin is an exit boundary[7] with a positive probability that the asset price reaches zero in finite time. For $\alpha < 1$ the origin is a regular boundary,[8] and so, to ensure limited liability, we append the following condition to the stochastic differential for the naked fund price:

$$S(t) \equiv 0 \quad \text{for all } t \ge \tau, \text{ where } \tau = \inf\{t | S(t) = 0\}.$$

For $\alpha = 2$, we are back to the standard Brownian motion case. For $\alpha > 2$, the origin is a natural boundary, and the asset price does not reach 0.

The CEV process has an entrance boundary at infinity if $\alpha > 2$. As explained by Linetsky and Davydov, this means that the diffusion coefficient of the process $\sigma S^{\alpha/2}$ does not satisfy the usual growth condition. As a consequence, the expected value of the asset price at time T, given that we are now at time $t < T$, is somewhat surprisingly not equal to $S_t e^{r(T-t)}$.[9] The martingale property does not hold. To overcome this problem, the process needs to be adjusted for high asset prices as explained by Davydov and

[7] This means the fund price can reach the origin, but if it does so it remains there.

[8] This means not only that the price can reach the origin but that it also can go through the origin and become negative. This is not a desirable feature because equities have limited liability.

[9] This point was first noted by Emanuel and Macbeth in 1982.

Table 4
Monte Carlo Estimates with ABR Modification under the CEV Process

Number of Steps (N) (std error)	Number of Simulation Trials (M)			
	10,000	100,000	500,000	1,000,000
10	15.279	15.275	15.255	15.253
	(0.144)	(0.046)	(0.021)	(0.015)
250	15.363	15.356	15.334	15.341
	(0.143)	(0.045)	(0.020)	(0.014)
1,000	15.431	15.352	15.332	15.329
	(0.144)	(0.046)	(0.020)	(0.014)

Note: Prices for dynamic fund protection using Monte Carlo techniques and the ABR modification. The naked fund follows the CEV process with $\alpha = 1$. The other parameters used are $S(0) = 100$, $K = 100$, $r = 0.04$, $\sigma_{BS} = 0.2$, and $T = 1$. We vary both the number of time steps N and the number of simulation trials M. We also report the standard errors of the estimates.

Linetsky (2001) as well as by Andersen and Andreasen (1998). In our numerical work, we will consider just values of α in the range [0, 2].

To ensure that option prices based on different values of α are comparable, the value of σ in each model must be readjusted such that the initial instantaneous volatility is the same across different models. Let σ_{BS} be the instantaneous volatility for the lognormal model ($\alpha = 2$), which in this case is 0.20. Following MacBeth and Merville (1980), the value of σ to be used for models with other α values is adjusted to be

$$\sigma = (\sigma_{BS})S(0)^{1-\alpha/2}.$$

4.3 Efficient Monte Carlo Simulation of the CEV Process

In discussing the simulation of dynamic fund protection under geometric Brownian motion, we saw that the crude Monte Carlo approach gave biased estimates. We will get a similar bias under the CEV process if we use a crude Monte Carlo approach. We anticipate that the Andersen and Brotherton-Ratcliffe (ABR) method will remove the bias, and we document that this is the case. We can improve the method even more by noting that the closed-form solution for the case ($\alpha = 2$) can be used as a control variate. For a discussion of the use of control variates in Monte Carlo simulation see Boyle, Broadie, and Glasserman (1997).

Table 4 gives the price estimates for dynamic fund protection under the CEV model with $\alpha = 1$ using Monte Carlo and the ABR modification.[10] Table 5 gives the price estimates for dynamic fund protection under the CEV model with $\alpha = 1$ using Monte Carlo and the ABR modification as well as the control variate. The same base case

[10] The table of a naive application of Monte Carlo techniques is not shown because it is predictable that a naive application of Monte Carlo techniques will produce a misleading result.

Table 5
Monte Carlo Results with ABR and Control Variate under the CEV Process

Number of steps (N) (std error)	Number of Simulation Trials (M)			
	10,000	100,000	500,000	1,000,000
10	15.258	15.238	15.236	15.236
	(0.015)	(0.005)	(0.002)	(0.002)
250	15.348	15.331	15.331	15.331
	(0.016)	(0.005)	(0.002)	(0.002)
1,000	15.306	15.331	15.336	15.335
	(0.017)	(0.005)	(0.002)	(0.002)

Note: Prices for dynamic fund protection under the CEV process with $\alpha = 1$ using Monte Carlo techniques with both the ABR modification and the control variate. We used the exact results for lognormal distribution ($\alpha = 2$) as the control variate. The results indicate that the inclusion of the control variate reduces the standard errors. We vary both the number of time steps N and the number of simulation trials M. We also report the standard errors of the estimates. The parameters used for the dynamic fund protection are $S(0) = 100$, $K = 100$, $r = 0.04$, $\sigma_{BS} = 0.2$, and $T = 1$.

contract is used in both tables. For this contract the initial fund price $S(0) = 100$, the protection level $K = 100$, the volatility $\sigma = 0.2$, the risk-free rate $r = 0.04$, the time to maturity $T = 1$, and the elasticity factor $\alpha = 1$. Because we know the analytic answer when $\alpha = 2$ from the Gerber/Pafumi results, we can use this as a control variate. Comparing the standard errors in Tables 4 and 5, we see that the addition of the control variate has reduced the standard error by a factor ranging from 7 to 10. The conclusion from this analysis is that, if we combine Monte Carlo with the ABR modification and the control variate, we obtain very accurate results.

4.4 The Partial Differential Equation Approach

We also develop a numerical partial differential equation (PDE) approach to estimate the price of dynamic fund protection. This approach is of interest in its own right, and it can be used to check our Monte Carlo numbers.

First, we derive a partial differential equation of the dynamic fund protection under the CEV process. Let $S_{\min}$ denote the minimum value of the underlying fund price up to time t, namely,

$$S_{\min}(t) = \min_{0 \le u \le t} S(u).$$

Following Wilmott, Dewynne, and Howison (1993) and Forsyth, Vetzal, and Zvan (1999), the following partial differential equation is satisfied:

$$\frac{1}{2}\sigma^2 S^\alpha \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV + \frac{\partial V}{\partial t} = 0. \quad (29)$$

Equation (29) becomes identical to the standard Black-Scholes PDE when the elasticity factor $\alpha = 2$, which corresponds to the lognormal process. Note that Equation (29) does not include $S_{\min}$, which is an important feature for computation. For convenience, we convert Equation (29) into an equation that is a forward equation in time where $\tau = T - t$:

$$\frac{\partial V}{\partial \tau} = \frac{1}{2}\sigma^2 S^\alpha \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV. \quad (30)$$

The initial condition, which represents the terminal payoff of the dynamic fund protection, is given by

$$V(S, S_{\min}, \tau = 0) = S\left\{\max\left(1, \frac{K}{S_{\min}}\right) - 1\right\}. \quad (31)$$

The boundary conditions are as follows:

$$V(S, S_{\min}, \tau) \to S\left\{\max\left(1, \frac{K}{S_{\min}}\right) - 1\right\} \quad \text{as } S \to \infty, \quad (32)$$

$$\frac{\partial V}{\partial \tau} = \frac{1}{2}\sigma^2 S^\alpha \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV \quad \text{as } S_{\min} \to 0 \text{ and } S \neq S_{\min}, \quad (33)$$

and

$$\frac{\partial V}{\partial S_{\min}} = 0 \quad \text{on } S = S_{\min}. \quad (34)$$

Equation (34) means that the value of the dynamic fund protection is not sensitive to a change in $S_{\min}$ when the current fund price is close to the current minimum price. This is because we are "almost sure" that the fund price will attain the minimum, and hence the minimum will be decreased. Thus when S is close to $S_{\min}$, V does not depend on the exact value of $S_{\min}$, and Equation (34) with $S = S_{\min}$ holds. For a discussion of this condition see chapter 12 of Wilmott, Dewynne, and Howison (1993).

Because it is difficult to solve Equation (30) analytically, we solve it by a numerical approach. Although Equation (30) contains only two variables, the underlying fund price S and time τ, we must consider the minimum value of the fund price $S_{\min}$ because the minimum fund price plays a crucial role in determining the boundary conditions. In some cases we can use a change of variable technique to transform such problems into a two-dimensional problem, but it is impossible to do this under the CEV process. Thus, a three-dimensional space must be considered.

We use a finite element method (FEM) to solve the problem. The FEM approximates the partial differential equation in the weighted-integral sense, while the standard finite difference method approximates the equation in the differential sense. In the FEM, we divide the given domain into a set of subdomains, called finite elements. Next, we approximate the solution as a linear combination of nodal values and approximation functions. Then we derive the algebraic relations among the nodal values of the solution over each element and assemble the elements to obtain the approximate solution over the entire domain.

The finite element method has several advantages over the finite difference method. One of them is that the FEM is quite flexible concerning the discretization of the domain, and it can be tailored to suit the problem at hand. Because we assume that the fund price is monitored continuously here, the minimum fund price is less than or equal to the fund price at any time. Thus, the shape of the domain becomes a triangle when the time is fixed. Therefore, we use triangular meshes to discretize the domain. Figure 1 illustrates the domain of the problem and mesh points. The shape of each element is a right isosceles triangle, which is appropriate because the minimum fund price can take the same values as the fund price.

We use the Galerkin finite element approach for the space discretization: namely, we set approximation functions as weighted functions. For the time discretization, we consider the so-called θ-method where $0 \leq \theta \leq 1$; $\theta = 0$ gives the explicit scheme, $\theta = 1$ the fully implicit scheme, and $\theta = 1/2$ the Crank-Nicolson scheme.

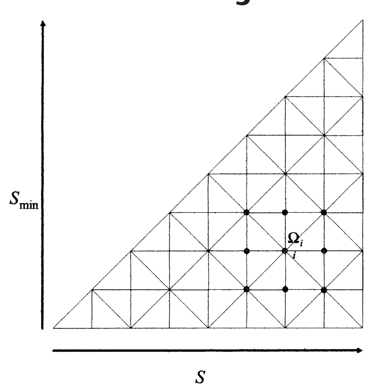
Let Ω be the domain of the problem, and N_i; $i = 1, \ldots, N_\Omega$ be C^0 Lagrange approximation functions, where N_Ω is the number of nodes in the domain. The function N_i satisfies the following properties:

$$N_i(S_j,\ (S_{\min})_j) = \delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}, \quad i, j = 1, \ldots, N_\Omega,$$

$$\sum_j^{N_\Omega} N_j(S,\ S_{\min}) = 1,$$

Figure 1

Illustration of Triangle Elements



where $i, j = 1, \ldots, N_\Omega$ represent the node number of the domain, and S_j and $(S_{\min})_j$ are the fund price and the minimum fund price on the node j. We also denote the solution of the PDE at node j at time $\tau = \tau_n$ as $U_j^n = U(S_j,\ (S_{\min})_j,\ \tau_n)$. We approximate the solution with the following linear combination of approximation functions that correspond to the values on node j:

$$U^n(S,\ S_{\min}) = \sum_j^{N_\Omega} N_j(S,\ S_{\min}) U_j^n. \tag{35}$$

From the standard finite element method, we can derive the following simultaneous equations. For any node i, we have

$$\begin{aligned}
&\int_\Omega N_i \, d\Omega \, \frac{U_i^{n+1} - U_i^n}{\Delta\tau} \\
&= \theta \Bigg[rS_i \left\{ \sum_{j \in \Omega_i} \left(\int_\Omega N_i \frac{\partial N_j}{\partial S} \, d\Omega \right) U_j^{n+1} \right\} \\
&\quad - \frac{1}{2} \sigma^2 S_i^\alpha \left\{ \sum_{j \in \Omega_i} \left(\int_\Omega \frac{\partial N_i}{\partial S} \frac{\partial N_j}{\partial S} \, d\Omega \right) U_j^{n+1} \right\} \\
&\quad - \frac{1}{2\sqrt{2}} \sigma^2 S_i^\alpha \sum_{j \in \Omega_i} \left(\int_{\partial\Omega} N_i \left(\frac{\partial N_j}{\partial S} + \frac{\partial N_j}{\partial S_{\min}} \right) d\Gamma \right) \\
&\quad \times U_j^{n+1} - rU_i^{n+1} \int_\Omega N_i \, d\Omega \Bigg] + (1 - \theta) \\
&\quad \times \Bigg[rS_i \left\{ \sum_{j \in \Omega_i} \left(\int_\Omega N_i \frac{\partial N_j}{\partial S} \, d\Omega \right) U_j^n \right\} \\
&\quad - \frac{1}{2} \sigma^2 S_i^\alpha \left\{ \sum_{j \in \Omega_i} \left(\int_\Omega \frac{\partial N_i}{\partial S} \frac{\partial N_j}{\partial S} \, d\Omega \right) U_j^n \right\} \\
&\quad - \frac{1}{2\sqrt{2}} \sigma^2 S_i^\alpha \sum_{j \in \Omega_i} \left(\int_{\partial\Omega} N_i \left(\frac{\partial N_j}{\partial S} + \frac{\partial N_j}{\partial S_{\min}} \right) d\Gamma \right) U_j^n \\
&\quad - rU_i^n \int_\Omega N_i \, d\Omega \Bigg], \quad i = 1, \ldots, N_\Omega,
\end{aligned} \tag{36}$$

where $\Delta\tau$ is a time step size, Ω_i is the set of nodes that have a connection with the node i, and $\partial\Omega$

Table 6
PDE Results for Prices of Dynamic Fund Protection

Number of Time Steps (N_t)	Number of Nodes (N_Ω)			
	8,385	33,135	131,842	525,825
10	14.750	14.719	14.511	14.430
100	14.726	14.776	14.789	14.792
500	14.726	14.776	14.789	14.792

Note: Prices for dynamic fund protection under the lognormal process using a PDE approach. We used constant time steps (N_t) and regular meshes (N_Ω) for checking its convergence. We use linear functions as N_i and an iterative method to create finer meshes, which means that the number of elements is doubled by one iteration. Thus, the numbers of elements in the table are 2^{14}, 2^{16}, 2^{18}, 2^{20}, respectively. The parameters used for the dynamic fund protection are the same as in the previous tables: $S(0) = 100$, $K = 100$, $r = 0.04$, $\sigma = 0.02$, and $T = 1$. The price boundaries are $S_0 = 1$ and $S_{N_\Omega} = 200$. We confirm that the same values are obtained when we use $S_{N_\Omega} = 400$. The theoretical value of the example is 14.793. Although this scheme might cause spurious oscillations, we confirm that they do not happen in the examples.

means the surface of the domain Ω. The derivation of these equations is described in the Appendix. Although the resulting equations may appear complicated, they become simpler when expressed in matrix form. The first part of the righthand side includes nodal values U_j^{n+1} that are unknown, while second part includes nodal values U_j^n that are known values. Therefore, let $\boldsymbol{U^{n+1}}$ denote an N_Ω-dimensional vector such that $\boldsymbol{U^{n+1}} = (\boldsymbol{U_1^{n+1}}, \ldots, U_{N_\Omega}^{n+1})'$; then Equation (36) can be written by

$$\mathbf{M}U^{n+1} = \boldsymbol{b}, \tag{37}$$

where $\mathbf{M}$ is a $N_\Omega \times N_\Omega$ matrix, and $\boldsymbol{b}$ is an N_Ω-dimensional vector. Because we know the values of the initial vector $\boldsymbol{U^o}$ from the initial conditions, we can compute the value of the vector $\boldsymbol{U^n}$ by iterations. Since the matrix $\mathbf{M}$ is a sparse matrix, we need a special treatment from a computational viewpoint. We use ILU-CGSTAB with level-one fill for the solution.[11]

Table 6 shows the accuracy of the computation. To examine the accuracy of the computation, we compare the numerical solutions obtained from the PDE approach with values under the lognormal process, because the exact price can be obtained by the closed-form formula. We used fixed time steps and regular meshes for checking its convergence. We used linear approximation functions as N_i and an iterative method to create finer meshes, which means that the number of elements is doubled by each iteration. The parameters used for the dynamic fund protection are the same as those for our earlier examples.

We used boundaries $S_0 = 1$ and $S_{N_\Omega} = 200$ and confirmed that the same results are obtained when we use $S_{N_\Omega} = 400$. We used 100 time steps and 525,825 nodes.[12]

4.5 Sensitivity of Dynamic Fund Protection to the Elasticity Parameter α

We now examine how the elasticity factor influences the value of the price of the dynamic option. In addition to estimating the values of the dynamic fund protection, we compute values of a standard put option and a standard lookback call option. These types of contracts may be more familiar and provide a comparison. Standard lookback call options are also an interesting object of comparison, because their payoffs depend on the minimum values of the underlying fund price. The payoff on a standard lookback call option is

$$S(T) - \min_{0 \le s \le T} S(s).$$

We estimate the prices of put options for different elasticity factors and different protection levels. The values are computed for α equal to {2.0,

[11] Incomplete LU preconditioner and a stabilized conjugate gradient solver.

[12] Because the finite element method is quite flexible to implement, it does not make much sense to use the computation time as a proxy of the efficiency of the method. Generally speaking, the computation time depends heavily on the size of domain, especially on the maximum value of fund price in this case, because the accuracy of the value depends on the size of the elements. It also depends on the mesh construction and on the efficiency and the accuracy of the sparse matrix solver (37).

Table 7
Sensitivity of Elasticity for European Put Option

Strike Price (K)	Method	Parameter of CEV Process (α)				
		2	1.5	1	0.5	0
100	MC	6.004	6.005	6.008	6.012	6.018
	PDE	6.005	6.006	6.008	6.012	6.018
90	MC	2.531	2.607	2.685	2.766	2.849
	PDE	2.533	2.609	2.687	2.767	2.851
80	MC	0.769	0.854	0.945	1.042	1.146
	PDE	0.770	0.855	0.946	1.043	1.147

Note: Prices for European put options using both Monte Carlo techniques and the PDE method. To obtain the Monte Carlo estimates we used 1,000 time steps and 1,000,000 simulation trials together with both the ABR modification and the control variate (except when $\alpha = 2$). The maximum standard error of these estimates is 0.0016, so this implies that the Monte Carlo estimates are very accurate to two decimal places. To obtain the numerical PDE estimates, we use a standard finite difference method. We illustrate the strong agreement between the results obtained by these two methods for different values of the CEV parameter α ranging from 0 to 2. The parameters used for the put option are $S(0) = 100$, $K = 100, 90, 80$, $r = 0.04$, $\sigma_{BS} = 0.2$, and $T = 1$.

1.5, 1.0, 0.5, 0} and protection levels of {100, 90, 80}. When the elasticity factor is 2.0, the value of the dynamic fund protection is given by Equation (14). The value of σ at Equation (28) is adjusted as before. Tables 7–9 contain prices of a standard put option, a standard lookback call option, and the dynamic fund protection, respectively. The prices of all these derivative contracts are decreasing with respect to the elasticity factor, but the impact varies across the different contracts. The elasticity factor has little effect on the price of the standard put option, especially when it is at the money. In contrast, we observe that the impact of elasticity factor α on the lookback option prices is much more significant. The value of the dynamic fund protection is also significantly affected by the change of the elasticity factor. In other words, proper model specification and accurate volatility estimates are critical for pricing the dynamic fund protection.

Table 8
Sensitivity of Elasticity for Standard Lookback Call

Method	Parameter of CEV Process (α)				
	2	1.5	1	0.5	0
MC	16.754	16.917	17.095	17.292	17.511
PDE	16.748	16.911	17.088	17.284	17.503

Note: Prices for a standard lookback call option under the CEV process with different α values. The terminal payoff of the lookback call option is $S(T) - S_{\min}(T)$, where $S_{\min}(T)$ is the minimum asset price up to time T. To obtain the Monte Carlo estimates we used 1,000 time steps and 1,000,000 simulation trials together with both the ABR modification and the control variate (except for $\alpha = 2$). When $\alpha = 2$ we used just the ABR modification. The maximum standard error of these estimates is 0.0026. To obtain the numerical PDE estimates 100 time steps and 525,825 regular nodes are used. The parameters used for the put option are $S(0) = 100$, $r = 0.04$, $\sigma_{BS} = 0.2$, and $T = 1$.

4.6 The Effect of Discrete Monitoring

We first examine the accuracy of the approximation formula that was developed in Section 2. Table 10 shows how well this approximation does in practice. We compare the prices based on daily, weekly, and monthly monitoring with those based on continuous monitoring. To obtain accurate values, we used standard Monte Carlo simulation. We used 1,092 time steps per year. To evaluate the daily monitored contract, the price is observed once per three time steps, because $364 \times 3 = 1{,}092$. Similarly, the price is observed once per 21 time steps in weekly monitoring, and once per 91 time steps in monthly monitoring. We used the same sample paths in all the examples to keep the comparisons fair. The more time steps involved, the better it does. We see that the agreement is very good except perhaps for the one-year contract with monthly monitoring.

Table 11 shows the result under the CEV process. From both Tables 10 and 11, we see that the introduction of discrete monitoring reduces the price substantially. In the case of the one-year contract with a strike price of 100, the contract price (11.375) with monthly monitoring is 70% of the corresponding continuously monitored price (14.793) in Table 10. For the five-year contract with a strike price of 80, the contract price (8.559) with monthly monitoring is 85% of the corresponding continuously monitored price (10.137).

Table 9

Sensitivity of Elasticity for Dynamic Fund Protection

Strike Price (K)	Method	Parameter of CEV Process (α)				
		2	1.5	1	0.5	0
100	MC	14.793	15.049	15.335	15.661	16.041
	PDE	14.791	15.046	15.331	15.656	16.037
90	MC	6.012	6.276	6.567	6.893	7.267
	PDE	6.012	6.275	6.564	6.889	7.264
80	MC	1.771	1.988	2.233	2.510	2.833
	PDE	1.771	1.988	2.231	2.507	2.830

Note: Prices for dynamic fund protection under the CEV process with different α as well as different strike prices. Both the Monte Carlo and PDE approaches are used. To obtain the Monte Carlo estimates we used 1,000 time steps and 1,000,000 simulation trials, and we use both the ABR modification and the control variate (except when $\alpha = 2$). When $\alpha = 2$ we used just the ABR modification. To obtain the numerical PDE estimates 100 time steps and 525,825 regular nodes are used. The parameters used for the put option are $S(0) = 100$, $K = 100$, $r = 0.04$, $\sigma = 0.2$, and $T = 1$.

5. CONCLUSIONS

Dynamic fund protection is a recent generalization of the static put option that has some appealing characteristics. This paper has explored extensions of the original concept and investigated some practical modifications. We showed how to price this discretely monitored contract and developed a simple approximation for its price. We also showed that, provided the underlying fund price is always positive, it never pays to exercise the dynamic fund protection early. We priced this protection under different investment assumptions, modified the standard Monte Carlo method to make it more efficient, and demonstrated how the protection can be valued using a PDE approach.

Armed with these methods, we examined the

Table 10

Prices for Dynamic Fund Protection with Discrete Monitoring

Strike Price (K)	Method	Frequency of Monitoring			
		Continuous	Daily	Weekly	Monthly
		$T = 1$			
100	Monte Carlo	14.793	14.119	13.053	11.375
	Approximation	—	14.098	12.977	11.096
90	Monte Carlo	6.012	5.695	5.196	4.461
	Approximation	—	5.671	5.121	4.197
80	Monte Carlo	1.771	1.666	1.494	1.254
	Approximation	—	1.648	1.451	1.119
		$T = 3$			
100	Monte Carlo	23.874	23.177	21.993	20.060
	Approximation	—	23.124	21.915	19.890
90	Monte Carlo	13.465	13.056	12.338	11.194
	Approximation	—	13.000	12.253	10.999
80	Monte Carlo	6.644	6.443	6.054	5.357
	Approximation	—	6.390	5.981	5.295
		$T = 5$			
100	Monte Carlo	29.172	28.395	27.097	25.097
	Approximation	—	28.389	27.130	25.021
90	Monte Carlo	18.026	17.517	16.709	15.395
	Approximation	—	17.511	16.682	15.294
80	Monte Carlo	10.137	9.824	9.340	8.559
	Approximation	—	9.826	9.326	8.487

Note: Prices for dynamic fund protection under the lognormal process. We assumed that the minimum price is monitored continuously, daily, weekly, and monthly, and we incorporated the control variate. We also computed the approximation value from Equation (22). The parameters used for the basic contract are $S(0) = 100$, $K = 100, 90, 80$, $r = 0.04$, $\sigma = 0.2$, and $T = 1, 3, 5$.

Table 11
Prices for Dynamic Fund Protection with Discrete Monitoring under CEV Process

Strike Price (K)	Frequency of Monitoring: Continuous (std error)	Daily (std error)	Weekly (std error)	Monthly (std error)
$\alpha = 2$				
100	14.793	14.119 (0.015)	13.053 (0.015)	11.375 (0.015)
90	6.012	5.696 (0.016)	5.197 (0.016)	4.461 (0.016)
80	1.771	1.667 (0.018)	1.495 (0.018)	1.255 (0.018)
$\alpha = 1$				
100	15.336 (0.002)	1.603 (0.014)	13.452 (0.014)	11.653 (0.014)
90	6.566 (0.002)	6.208 (0.015)	5.647 (0.015)	4.824 (0.016)
80	2.231 (0.002)	2.094 (0.017)	1.876 (0.018)	1.570 (0.018)
$\alpha = 0$				
100	16.040 (0.003)	15.232 (0.014)	13.971 (0.014)	12.014 (0.013)
90	7.264 (0.003)	6.848 (0.014)	6.203 (0.014)	5.260 (0.015)
80	2.830 (0.003)	2.647 (0.016)	2.361 (0.017)	1.963 (0.017)

Note: Prices for dynamic fund protection under the CEV process. The asset price is assumed to be monitored continuously, daily, weekly, and monthly. Monte Carlo techniques with the control variate are used in the simulation. Standard errors are also reported. We assume that default occurs when the underlying asset price goes below one. The parameter used for the dynamic fund protection are $S(0) = 100$, $K = 100, 90, 80$, $r = 0.04$, $\sigma_{BS} = 0.2$, and $T = 1$.

pricing of the contract under the CEV process. We confirmed that the CEV model gives different results from the standard lognormal model. This is not surprising given the kinship of this contract with lookback options where the same type sensitivity has already been noted. We then proposed a more practical version of the contract where the monitoring takes place at discrete time points rather than continuously.

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APPENDIX: DERIVATION OF EQUATION (36)

This appendix shows the derivation of Equation (36) with the argument of the FEM. We first describe Equation (30) with a matrix form:

$$V_\tau = \boldsymbol{G} \cdot \nabla V + (\mathbf{D}\nabla) \cdot \nabla U - rV, \qquad (38)$$

where $\boldsymbol{G}$ is a vector and $\mathbf{D}$ is a matrix such that

$$\boldsymbol{G} = \begin{pmatrix} rS \\ 0 \end{pmatrix}$$

and

$$\mathbf{D} = \begin{pmatrix} \frac{1}{2}\sigma^2 S^\alpha & 0 \\ 0 & 0 \end{pmatrix},$$

respectively. Here ∇ is the gradient operator in the two-dimensional Cartesian rectangular coordinate system, that is,

$$\nabla = \begin{pmatrix} \dfrac{\partial}{\partial S} \\ \dfrac{\partial}{\partial S_{\min}} \end{pmatrix}.$$

We multiply Equation (38) with a weight function w and integrate the resulting equation over the whole domain Ω:

$$\int_\Omega wV_\tau \, d\Omega = \int_\Omega w\boldsymbol{G} \cdot \boldsymbol{\nabla} V \, d\Omega + \int_\Omega w(\mathbf{D}\boldsymbol{\nabla}) \cdot \boldsymbol{\nabla} V \, d\Omega - \int_\Omega wrV \, d\Omega. \quad (39)$$

According to Green's formula, the second term of the equation on the righthand side is replaced by

$$\int_\Omega w(\mathbf{D}\boldsymbol{\nabla}) \cdot \boldsymbol{\nabla} V \, d\Omega = -\int_\Omega \boldsymbol{\nabla} w \cdot \mathbf{D}\boldsymbol{\nabla} V \, d\Omega + \oint_{\partial\Omega} w\mathbf{D}\boldsymbol{\nabla} V \cdot \boldsymbol{n} \, d\Gamma, \quad (40)$$

where $\partial\Omega$ is the surface of the domain Ω, and $\boldsymbol{n}$ is an outward pointing unit normal. Therefore, Equation (39) is

$$\int_\Omega wV_\tau \, d\Omega = \int_\Omega w\boldsymbol{G} \cdot \boldsymbol{\nabla} V \, d\Omega - \int_\Omega \boldsymbol{\nabla} w \cdot \mathbf{D}\boldsymbol{\nabla} V \, d\Omega + \oint_{\partial\Omega} w \, \mathbf{D}\boldsymbol{\nabla} V \cdot \boldsymbol{n} d\Gamma - \int_\Omega wrV \, d\Omega. \quad (41)$$

We apply the Galerkin FEM. In other words, we set each approximation function N_i as the weighted function w. When we denote U_j^n as an approximation value of V on node j at time τ_n, the value U can be approximated as

$$U^n(S, S_{\min}) = \sum_j N_j(S, S_{\min}) U_j^n,$$

where N_j is an approximation function. Thus, each term of Equation (42) is discretized as follows when we fix the time:

$$\int_\Omega N_i V_\tau \, d\Omega \approx \int_\Omega N_i \, d\Omega \, \frac{U_i^{n+1} - U_i^n}{\Delta\tau}, \quad (42)$$

$$\int_\Omega N_i \boldsymbol{G} \cdot \boldsymbol{\nabla} V \, d\Omega \approx rS_i \left\{ \sum_{j\in\Omega_i} \left(\int_\Omega N_i \frac{\partial N_j}{\partial S} \, d\Omega \right) U_j^n \right\}, \quad (43)$$

$$\int_\Omega \boldsymbol{\nabla} N_i \cdot \mathbf{D}\boldsymbol{\nabla} V \, d\Omega \approx \frac{1}{2} \sigma^2 S_i^\alpha \left\{ \sum_{j\in\Omega_i} \left(\int_\Omega \frac{\partial N_i}{\partial S} \frac{\partial N_j}{\partial S} \, d\Omega \right) U_j^n \right\}, \quad (44)$$

$$\int_\Omega N_i rV \, d\Omega \approx rU_i^n \int_\Omega N_i \, d\Omega, \quad (45)$$

$$\oint_{\partial\Omega} N_i \mathbf{D}\boldsymbol{\nabla} V \cdot \boldsymbol{n} \, d\Gamma \approx \frac{1}{2} \sigma^2 S_i^\alpha \oint_{\partial\Omega} N_i \frac{\partial U_i^n}{\partial S} n_S \, d\Gamma \quad (46)$$

for all $i = 1, \ldots, N_\Omega$ where n_S is an S component of vector $\boldsymbol{n}$.

Now we carefully investigate Equation (46). Note that this integral is meaningful only for the elements that connect to the boundary of the domain. Furthermore, the explicit boundary condition is available on the boundary where the fund price is equal to the maximum price of the domain. On the boundary where the minimum fund price is equal to the minimum value in the domain, n_S is equal to zero. Accordingly the integration is meaningful only on the boundary where $S = S_{\min}$. Note that the outward unit normal, $\boldsymbol{n}$, equals

$$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)',$$

on this boundary. We can incorporate the boundary condition $\partial V/\partial S_{\min} = 0$ into Equation (46). The boundary condition (34) is interpreted as

$$\frac{\partial U}{\partial S} n_S = -\boldsymbol{\nabla} U \cdot \boldsymbol{t}, \quad (47)$$

where $\boldsymbol{t}$ is a unit tangent vector such that $\boldsymbol{t} = (1/\sqrt{2}, \ 1/\sqrt{2})'$, and n_S is the S component of $\boldsymbol{n}$. Equation (47) can be confirmed by the equation

$$\left(-\frac{1}{\sqrt{2}} \right) \frac{\partial U}{\partial S} = -\left(\frac{1}{\sqrt{2}} \frac{\partial U}{\partial S} + \frac{1}{\sqrt{2}} \frac{\partial U}{\partial S_{\min}} \right)$$

and using $\partial V/\partial S_{\min} = 0$. Thus, Equation (46) can be rewritten by

$$\oint_{\partial\Omega} N_i \mathbf{D}\nabla V \cdot \boldsymbol{n}\, d\Gamma \approx -\frac{1}{2\sqrt{2}}\sigma^2 S_i^{\alpha} \sum_{j\in\Omega_i}\left(\int_{\partial\Omega} N_i\left(\frac{\partial N_j}{\partial S}+\frac{\partial N_j}{\partial S_{\min}}\right)d\Gamma\right)U_j^n. \tag{48}$$

With respect to the time discretization, we use the θ-method where $0 \le \theta \le 1$; $\theta = 0$ gives the explicit scheme, $\theta = 1$ the fully implicit scheme, and $\theta = 1/2$ the Crank-Nicolson scheme. Consequently the following equations are obtained. For any node $i = 1, \ldots, N_\Omega$,

$$\begin{aligned}\int_{\Omega} N_i\, d\Omega\, \frac{U_i^{n+1}-U_i^n}{\Delta\tau} &= \theta\Bigg[rS_i\left\{\sum_{j\in\Omega_i}\left(\int_{\Omega} N_i \frac{\partial N_j}{\partial S}\, d\Omega\right)U_j^{n+1}\right\} \\ &\quad -\frac{1}{2}\sigma^2 S_i^{\alpha}\left\{\sum_{j\in\Omega_i}\left(\int_{\Omega}\frac{\partial N_i}{\partial S}\frac{\partial N_j}{\partial S}\, d\Omega\right)U_j^{n+1}\right\} \\ &\quad -\frac{1}{2\sqrt{2}}\sigma^2 S_i^{\alpha}\sum_{j\in\Omega_i}\left(\int_{\partial\Omega} N_i\left(\frac{\partial N_j}{\partial S}+\frac{\partial N_j}{\partial S_{\min}}\right)d\Gamma\right) \\ &\quad \times U_j^{n+1} - rU_i^{n+1}\int_{\Omega} N_i\, d\Omega\Bigg] + (1-\theta) \\ &\quad \times\Bigg[rS_i\left\{\sum_{j\in\Omega_i}\left(\int_{\Omega} N_i\frac{\partial N_j}{\partial S}\, d\Omega\right)U_j^n\right\} \\ &\quad -\frac{1}{2}\sigma^2 S_i^{\alpha}\left\{\sum_{j\in\Omega_i}\left(\int_{\Omega}\frac{\partial N_i}{\partial S}\frac{\partial N_j}{\partial S}\, d\Omega\right)U_j^n\right\} \\ &\quad -\frac{1}{2\sqrt{2}}\sigma^2 S_i^{\alpha}\sum_{j\in\Omega_i}\left(\int_{\partial\Omega} N_i\left(\frac{\partial N_j}{\partial S}+\frac{\partial N_j}{\partial S_{\min}}\right)d\Gamma\right)U_j^n \\ &\quad - rU_i^n\int_{\Omega} N_i\, d\Omega\Bigg].\end{aligned} \tag{49}$$

DISCUSSION

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The authors are to be thanked for this masterful paper, generalizing results in Gerber and Pafumi (2000). A main purpose of this discussion is to suggest that, for practical implementation of the model, one might interpret the quantity

$$M(t) = \max\left(1, \max_{0\le s\le t}\frac{K}{S(s)}\right)$$

as the number of fund units in the customer's account at time t. That is, starting with one unit of the investment fund at time 0, the customer will be have $M(t)$ units at time t, $t \ge 0$. Additional units are credited to the customer's account by the company as soon as the total value of the customer's account drops below the level K. Note that the function $M(t)$ is nondecreasing. Thus, even if an early withdrawal is possible, it is optimal for the customer to leave his or her funds in the account until maturity. This illustrates the conclusion in Section 3 of the paper. Note however that there is a crucial assumption: The value of the naked fund includes the reinvestment of all dividends, so the current price for one unit of the naked fund is also the price for one unit at any future time. However, if the naked fund does pay dividends, then it might be optimal for the customer to withdraw his or her funds early and invest them without the guarantee of "dynamic fund protection."

The model can be generalized to the case where the index fund pays dividends and there is a participation rate less than 1, as follows. For $t \ge 0$, let $S(t)$ denote the value of an index (for example, S&P 500) at time t. If the index does not include the reinvestment of dividends from its underlying assets (which is the case for S&P 500), it is then assumed that the dividends are a con-

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