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BOUNDEDLY NONHOMOGENEOUS ELLIPTIC AND PARABOLIC EQUATIONS IN A DOMAIN

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ABSTRACT. In this paper the Dirichlet problem is studied for equations of the form $0 = F(u_{x^i x^j}, u_{x^i}, u, 1, x)$ and also the first boundary value problem for equations of the form $u_t = F(u_{x^i x^j} u_{x^i}, u, 1, t, x)$, where $F(u_{ij}, u_i, u, \beta, x)$ and $F(u_{ij}, u_i, u, \beta, t, x)$ are positive homogeneous functions of the first degree in (u_{ij}, u_i, u, β), convex upwards in (u_{ij}), that satisfy a uniform strict ellipticity condition. Under certain smoothness conditions on F and when the second derivatives of F with respect to (u_{ij}, u_i, u, x) are bounded above, the $C^{2+\alpha}$ solvability of these problems in smooth domains is proved. In the course of the proof, a priori estimates in $C^{2+\alpha}$ on the boundary are constructed, and convexity and restrictions on the second derivatives of F are not used in the derivation.

Bibliography: 13 titles.

This article is closely related to the author's article [1], which deals with boundedly nonhomogeneous elliptic and parabolic equations in the classes $C^{2+\alpha}$ in the whole space, along with the first boundary-value problem in a cylinder (parabolic equations) and the Dirichlet problem (elliptic equations). As in [1], we impose here the condition that the nonlinear operator be convex with respect to the highest derivatives of the unknown function. Under the condition of convexity with respect to all the derivatives, the Dirichlet problem in $C^{2+\alpha}$ was studied for nonlinear elliptic equations in [2] and [3]. The main way in which the present article differs from [1]–[3] is that here we prove boundary estimates for the solutions in $C^{2+\alpha}$, while in [1]–[3] all the constructions were based on interior estimates and did not yield solutions smooth up the boundary.

Interest in boundedly nonhomogeneous equations arose from the theory of optimal control of diffusion processes. The special case of equations convex in all the derivatives was studied for a long time with the help of probability methods (the reader can find a history of the question up to 1976 in [4]). Of recent work using probabilistic techniques we point out [5]–[8] (see also the literature cited in these articles). In [5]–[8] the derivatives of a solution of the equation are understood in various generalized senses.

The methods of the theory of differential equations developed in [1]–[3], [9], and the present article yield a solution in $C^{2+\alpha}$ when the equation is uniformly nondegenerate. In the sense of smoothness of a solution, these methods give stronger results. However, the most general results in [5], [6] and [8] have not yet been obtained by the methods of the

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theory of differential equations in the case when the equation can degenerate. It should be said that for a uniformly nondegenerate parabolic equation our Theorem 1.1 is weaker than the corresponding result from [5] in the strength of the smoothness requirements on the boundary function.

The fact that the second derivative of a solution in any direction is a subsolution of a certain equation lies at the base of the interior estimates in $C^{2+\alpha}$ in [1]–[3]. This idea has constantly been used in probability to get upper estimates of a second directional derivative. It is clear from general considerations that subsolutions are upper semicontinuous, and, since (nonlinear) elliptic operator is given by a function monotonically increasing in the second directional derivatives and yields a continuous function (equal to zero for elliptic equations) when the unknown function is substituted in it, the second directional derivatives are also continuous. The precise formulation of this idea uses results in [10] (see [1]–[3]).

The boundary estimates in $C^{2+\alpha}$ proved here are based on an idea used by the author also in the probabilistic arguments in [8]. It enables us, for example, to reduce the Dirichlet problem to a problem on a closed manifold without boundary and thereby free ourselves of the presence of a boundary. Let us clarify this idea by using the example of the equation $\Delta u - u = f$ in $E_d \cap \{x^1 > 0\}$ with the condition $u = 0$ for $x^1 = 0$.

We look for u in the form $x^1 v$. Then, since $\Delta u - u = f$, an equation of the form $Lv = f$ emerges for v. The operator L has the property that $(x^1)^{-1}$ is a supersolution for it; hence, the equation $Lv = f$ need not be supplemented by any boundary conditions for $x^1 = 0$. Consequently, the set $\{x^1 = 0\}$ loses its exclusive role. In this connection the idea arose of considering the equation $Lv = f$ as the projection of a certain equation on the manifold $\mathfrak{M} = \{(x, r): x \in E_d, r \in (-\infty, \infty), x^1 = r^2\}$.

The natural coordinates on $\mathfrak{M}$ are $(r, x^2, \dots, x^d)$. Therefore, the function $w(r, x^2, \dots, x^d) = v(r^2, x^2, \dots, x^d)$ is introduced on $\mathfrak{M}$. The equation $Lv = f$ can now be rewritten for w in the coordinates $(r, x^2, \dots, x^d)$, and we obtain the equation $\tilde{L}w = f$ acting for all real r and $x^2, \dots, x^d$. In short, we have an elliptic equation for w in the whole space. The operator $\tilde{L}$ can thus be obtained by introducing the function

$$w(r, x^2, \dots, x^d) = r^{-2} u(r^2, \dots, x^d).$$

We note that $w(0, x^2, \dots, x^d) = u_{x^1}(0, x^2, \dots, x^d)$. Therefore, the study of the second derivatives of u on $\{x^1 = 0\}$ reduces to the study of the first derivatives of w for $r = 0$. A certain unpleasant fact about the equation $\tilde{L}w = f$ is that its coefficients are locally unbounded for $r = 0$ (but we want to get estimates of the second derivatives of u on $\{x^1 = 0\}$ or estimates of the first derivatives of w for $r = 0$ by differentiating this equation). It has been noted that $\tilde{L}$ contains the expression $w_{rr} + (3/r)w_r$ (instead of $4u_{x^1x^1}$), which is the radial part of the four-dimensional Laplace operator. Therefore, it was natural to introduce four additional coordinates in place of r and assume that the r above is the length of the vector of additional coordinates. Then $w_{rr} + (3/r)w_r$ can be rewritten as the Laplace operator with respect to these additional coordinates, w now depends on $d + 3$ coordinates, and $\tilde{L}$ is transformed into some operator $\hat{L}$.

After finding the operator $\hat{L}$, we can act formally and not refer to v, L, $\tilde{L}$, nor $\mathfrak{M}$. This is done in §4, because a fairly cumbersome expression is obtained for $\hat{L}$, and the author thought it necessary to explain where it came from.

For $r = 0$ the operator $\hat{L}$ degenerates in a very special way; using this specific characteristic, we are able to estimate the norm of solutions in C^{α} and $C^{1+\alpha}$ in §§2 and 3 for operators of this type. In §4 we make an estimate on the boundary in $C^{2+\alpha}$, and in §6 it is "pasted together" with interior estimates in [1] with the help of three lemmas in §5. In §7 the basic results are proved, namely, Theorems 1.1 and 1.2, which are stated in §1.

We note especially that in estimating the second derivatives on the boundary we must differentiate an equation of the type $\hat{L}w = f$ only once, so the proof in §4 of the estimates in $C^{2+\alpha}$ on the boundary does not use any assumptions about the convexity of the nonlinear operator.

We conclude this Introduction by explaining some of our notation. Unless there is a statement to the contrary, repeated Latin indices are summed from 1 to d. A convention about Green indices is introduced and used only in the proof of Theorem 4.1. The sets $W_{T,R}$ are introduced before Theorem 2.2, and $V_{T,R}$ and $\Sigma_{T,R}$ at the beginning of §4. The other notation besides that just mentioned and that introduced in the first section is introduced and used in each section separately. Finally, $a \wedge b = \min(a, b)$, $a \vee b = \max(a, b)$, $a_+ = 0 \vee a$, $a_- = 0 \vee (-a)$ and $\|f\|_{B(\Gamma)} = \sup_\Gamma |f|$.

§1. Some notation and results

Suppose that $d \geqslant 1$ is an integer, $E_d = \{x = (x^1, \dots, x^d)\colon x^i \in (-\infty, \infty)\}$ is the Euclidean space, $T \in (0, \infty)$, $\kappa \in (0, 1)$, D is a (nonempty) open subset of E_d, $Q = (0, T) \times D$, $\bar{Q} = [0, T] \times \bar{D}$, $\partial_x Q = (0, T] \times \partial D$, $\partial_t Q = \{t = 0\} \times \bar{D}$ and $\partial' Q = \partial_t Q \cup \partial_x Q$.

Let $\rho(x) = \operatorname{dist}(x, \partial D)$, $\Delta_\rho D = \{x \in D;\ \rho(x) \leqslant \rho\}$, $D(\rho) = D \setminus \Delta_\rho D$, $Q(\rho) = (0, T) \times D(\rho)$, $Q_\varepsilon = (\varepsilon, T) \times D$ and $Q_\varepsilon(\rho) = (\varepsilon, T) \times D(\rho)$.

Let $\mathfrak{F}(\kappa, Q)$ be the collection of all real functions $F(u_{ij}, u_i, u, \beta, t, x)$ having the following properties:

1.1) F is defined for all $(t, x) \in \bar{Q}$, $\beta > 0$, and all real u_{ij} $(i, j = 1, \dots, d)$, u_i $(i = 1, \dots, d)$ and u.

1.2) In its domain, F is positive-homogeneous of first order in (u_{ij}, u_i, u, β), twice continuously (with respect to $(u_{ij}, u_i, u, \beta, t, x)$) differentiable with respect to (u_{ij}, u_i, u, x), and once continuously differentiable with respect to t.

1.3) For all $(t, x) \in \bar{Q}$ and $\beta > 0$, any symmetric matrix (u_{ij}), and any u_i and u,

$$|F_t|, |F_x| \leqslant \kappa^{-1} w, \quad \text{where } w = \left(\beta^2 + u^2 + u_i u_i + u_{ij} u_{ij}\right)^{1/2}, \tag{1.1}$$

$$\kappa|\xi|^2 \leqslant F_{u_{ij}} \xi^i \xi^j \leqslant \kappa^{-1} |\xi|^2 \quad \forall \xi \in E_d, \tag{1.2}$$

$$|F_\beta|, |F_u|, |F_{u_i}| \leqslant \kappa^{-1}, \qquad i = 1, \dots, d, \tag{1.3}$$

and the second derivative of F with respect to (u_{ij}, u_i, u, x) along any vector $(\tilde{u}_{ij}, \tilde{u}_i, \tilde{u}, \tilde{x})$ does not exceed

$$\kappa^{-1} \left[\beta^{-1} \tilde{u}_i \tilde{u}_i + \beta^{-1} \tilde{u}^2 + |\tilde{x}|^2 w + |\tilde{x}| \sum_{i,j} |\tilde{u}_{ij}| \right]. \tag{1.4}$$

Note that the last required estimate involves only an upper estimate, so that the functions F of the class $\mathfrak{F}(\kappa, Q)$ are automatically upwards convex with respect to (u_{ij}) on the set of symmetric matrices (u_{ij}). If F does not depend on t, $F \in \mathfrak{F}(\kappa, Q)$, and in addition to the inequalities (1.1)–(1.3) we have $F_u \leqslant 0$, then we write $F \in \mathfrak{F}(\kappa, D)$. Sets of the form $\mathfrak{F}(\kappa, Q_\varepsilon(\rho))$ are introduced in the obvious way.

Also, let $\bar{\mathfrak{F}}(\kappa, Q)$ be the collection of all real functions $F(u_{ij}, u_i, u, \beta, t, x)$ defined for all $(t, x) \in Q$ and $\beta > 0$, all symmetric matrices (u_{ij}), and all u_i $(i = 1, \dots, d)$ and u such that there is a sequence $F_n \in \mathfrak{F}(\kappa, Q_{1/n}(1/n))$ converging to F as $n \to \infty$ at each point where it is defined. The class $\overline{\mathfrak{F}}(\kappa, D)$ is defined similarly. The meaning of the notation $\overline{\mathfrak{F}}(\kappa, Q_\varepsilon(\rho))$ is clear.

We discuss these concepts. It follows from homogeneity arguments that condition 1.3) needs to be checked only for $\beta = 1$, and F_β exists and is continuous. The condition connected with (1.4) looks rather unusual. Therefore, we show that it holds (with some κ), for example, when F is upwards convex (or linear) with respect to the variables (u_{ij}, u_i, u), $|F_{u_k x^i}|, |F_{u_i x^i}|, |F_{ux^i}| \leqslant \kappa^{-1}$, and the matrix relation $(F_{x^i x^j}) \leqslant \kappa^{-1} w$ holds. This condition (with some κ) holds also if condition 1.5 in [1] is satisfied. Therefore, if we abstract from the continuity requirements on the derivatives of F in our condition 1.2) (which are stronger than conditions 1.2 and 1.6 in [1]), then we can say that the classes $\mathfrak{F}(\kappa, Q)$ introduced above encompass the corresponding classes in [1]. The advantages of the definition of $\mathfrak{F}(\kappa, Q)$ adopted here over the definition in [1] are very clear from the following lemma.

LEMMA 1.1. a) *Let $F \in \overline{\mathfrak{F}}(\kappa, Q)$, and fix $(t, x) \in Q$, $u_{ij} = u_{ji}$, u_i and u. Then there exist numbers $a^{ij} = a^{ji}$, b^i $(i, j = 1, \dots, d)$, c and f such that*

b) *Let $F \in \overline{\mathfrak{F}}(\kappa, Q)$, and fix $(t, x) \in Q$ and $u^k_{ij} = u^k_{ji}$, u^k_i and u^k $(k = 1, 2)$. Then there exist numbers $a^{ij} = a^{ji}$, b^i $(i, j = 1, \dots, d)$ and c satisfying* (1.5) *and such that*

$$F\left(u^1_{ij}, u^1_i, u^1, 1, t, x\right) - F\left(u^2_{ij}, u^2_i, u^2, 1, t, x\right) = a^{ij}\left(u^1_{ij} - u^2_{ij}\right) + b^i\left(u^1_i - u^2_i\right) + c\left(u^1 - u^2\right).$$

c) *The set $\overline{\mathfrak{F}}(\kappa, Q)$ of functions is uniformly bounded and equicontinuous on any set of values of the arguments of the form*

$$\{(u_{ij}, u_i, u, \beta, t, x) : u_{ij} = u_{ji}, \beta > 0, (t, x) \in Q, w \leqslant N\},$$

where $N < \infty$.

d) *If $F \in \overline{\mathfrak{F}}(\kappa, Q)$, then there is a sequence $F_n \in \mathfrak{F}(\kappa, Q_{1/n}(1/n))$ such that each F_n is infinitely differentiable on $\{(u_{ij}, u_i, u, \beta, t, x) : \beta > 0, (t, x) \in \overline{Q}_{1/n}(1/n)\}$, and $F_n \to F$ as $n \to \infty$ uniformly on each compact subset of the domain of F.*

e) *The set $\overline{\mathfrak{F}}(\kappa, Q)$ of functions is closed with respect to pointwise convergence. Moreover, $\overline{\mathfrak{F}}(\kappa, Q) = \bigcap_n \overline{\mathfrak{F}}(\kappa, Q_{1/n}(1/n))$.*

f) *If A is an index set and $F^a \in \overline{\mathfrak{F}}(\kappa, Q)$ for any $a \in A$, then $\inf\{F^a : a \in A\} \in \overline{\mathfrak{F}}(\kappa, Q)$.*

The analogous assertions hold for the functions in the class $\overline{\mathfrak{F}}(\kappa, D)$, and the inequality $c \leqslant 0$ also holds in assertions a) *and* b).

PROOF. Assertions a) and b) follow at once from the definition of $\overline{\mathfrak{F}}(\kappa, Q)$ and the fact that for $F \in \mathfrak{F}(\kappa, Q_{1/n}(1/n))$ and $u_{ij} = u_{ji}$, $(t, x) \in Q_{1/n}(1/n)$ we have

$$F = F_{u_{ij}} u_{ij} + F_{u_i} u_i + F_u u + F_\beta \beta, \qquad F_{u_{ij}} u_{ij} = \tfrac{1}{2}\left(F_{u_{ij}} + F_{u_{ji}}\right) u_{ij},$$

$$F\left(u^1_{ij}, u^1_i, u^1, \beta, t, x\right) - F\left(u^2_{ij}, u^2_i, u^2, \beta, t, x\right) = a^{ij}\left(u^1_{ij} - u^2_{ij}\right) + b^i\left(u^1_i - u^2_i\right) + c\left(u^1 - u^2\right), \tag{1.6}$$

where

$$a^{ij} = \int_0^1 F_{u_{ij}}\left(\theta u^1_{ij} + (1 - \theta) u^2_{ij}, \theta u^1_i + (1 - \theta) u^2_i, \theta u^1 + (1 - \theta) u^2, \beta, t, x\right) d\theta$$

and b^i and c are defined similarly. Assertion c) follows from a formula analogous to (1.6), in which the difference of values of F is taken at points differing also in the components (β, t, x).

To prove d) we take an $F \in \overline{\mathfrak{F}}(\kappa, Q)$ and a sequence $F_n \in \mathfrak{F}(\kappa, Q_{1/n}(1/n))$ corresponding to F. In a way similar to that for c) it can be shown that the family of F_n with $n \geqslant n_0$ is uniformly bounded and equicontinuous (in the relative topology) on sets of the form $\{u_{ij}, u_i, u, \beta, (t, x) : u_{ij} = u_{ji}, \beta \geqslant \varepsilon, (t, x) \in \overline{Q}_{1/n_0}(1/n_0), w \leqslant \varepsilon^{-1}\}$, where $\varepsilon > 0$. Therefore, $F_n \to F$ as $n \to \infty$ uniformly on such sets, and it remains for us to show that F_n can be approximated uniformly on these sets by infinitely differentiable functions. As above, the uniform convergence will follow from pointwise convergence, but since $Q_{1/n}(1/n)$ can be taken for Q, it remains for us to show that if $F \in \mathfrak{F}(\kappa, Q)$, then there is a sequence of infinitely differentiable functions $F_n \in \mathfrak{F}(\kappa, Q_{1/n}(1/n))$ such that $F_n \to F$ for $\beta > 0$ and $(t, x) \in Q$, and for any symmetric (u_{ij}) and any u_i and u.

Accordingly, we take an $F \in \mathfrak{F}(\kappa, Q)$ and any nonnegative $\zeta(t) \in C_0^\infty(-\infty, \infty)$ which equals 0 for $|t| \geqslant 1$, equals 1 near 0, and satisfies $\int \zeta \, dt = 1$, and we define $\tilde{F}_n$ by convolving $F^* = F(\frac{1}{2}(u_{ij} + u_{ji}), u_i, u, \beta, t, x)$ with $\gamma_n = \zeta_n \eta_n$ with respect to the variables u_{ij}, u_i, u, t and x, where

$$\zeta_n = \Big(\prod_{i,j} n\zeta(nu_{ij})\Big)\Big(\prod_i n\zeta(nu_i)\Big) n\zeta(nu), \qquad \eta_n = cn^{d+1}\zeta(nt)\zeta(n|x|)$$

and the constant c is chosen so that $\int \zeta(|x|)\, dx = 1$. Then we get that for $\beta > 0$

$$F_n(u_{ij}, u_i, u, \beta, t, x) = \beta \tilde{F}_n\left(\beta^{-1} u_{ij}, \beta^{-1} u_i, \beta^{-1} u, 1, t, x\right).$$

It is not hard to see that F_n is defined and infinitely differentiable with respect to all arguments when $(t, x) \in \overline{Q}_{1/n}(1/n)$. Moreover, $F_n \to F^* = F$ for $\beta > 0$, $(t, x) \in Q$ and $(u_{ij}) = (u_{ji})$. Further, for $\beta = 1$, $u_{ij} = u_{ji}$ and $(t, x) \in \overline{Q}_{1/n}(1/n)$ the function F_n satisfies condition (1.2), $|F_{nu}|, |F_{nu_i}| \leqslant \kappa^{-1}$, and

$$\begin{aligned}
|F_{nt}|, |F_{nx}| &\leqslant \kappa^{-1} \int \left[\sum_{i,j} \left(\frac{u_{ij} + u_{ji}}{2} - \frac{\tilde{u}_{ij} + \tilde{u}_{ji}}{2}\right)^2 + \sum_i (u_i - \tilde{u}_i)^2 + (u - \tilde{u})^2 + 1\right]^{1/2} \\
&\qquad \times \zeta_n(\tilde{u}_{ij}, \tilde{u}_i, \tilde{u})\, d\tilde{u} \prod_{i,j,k} d\tilde{u}_{ij} d\tilde{u}_k \\
&\leqslant \kappa^{-1}\omega + \kappa^{-1} \int \left[\tilde{u}_{ij}\tilde{u}_{ij} + \tilde{u}_i\tilde{u}_i + \tilde{u}^2\right]^{1/2} \zeta_n \, d\tilde{u} \prod_{i,j,k} d\tilde{u}_{ij} d\tilde{u}_k \\
&= \kappa^{-1}\omega + \kappa^{-1}\frac{1}{n}N \leqslant \kappa^{-1}\left(1 + \frac{1}{n}N\right) w,
\end{aligned}$$

where $N = N(d, \zeta)$. It can be verified similarly that $|F_{n\beta}| \leqslant \kappa^{-1}(1 + N/n)$ and that the second derivative of F_n with respect to (u_{ij}, u_i, u, x) along any vector $(\tilde{u}_{ij}, \tilde{u}_i, \tilde{u}, \tilde{x})$ does not exceed the expression (1.4) multiplied by $(1 + N/n)$ when $\beta > 0$, $(t, x) \in \overline{Q}_{1/n}(1/n)$ and $u_{ij} = u_{ji}$. It remains to correct F_n so as to get rid of the factor $(1 + N/n)$. This can be done, for example, by considering the functions

$$(u_{11} + \cdots + u_{dd}) N/n + (1 - N/n) F_n.$$

instead of the F_n.

Assertion e) follows immediately from c) and d). To prove f) we note first of all that, by c), the set A can be replaced by a countable subset of it without changing $\inf_A F^a$. An infimum over a countable set is a limit of infima over finite subsets. Therefore, by e), it can be assumed that A is finite; but since the infimum over a finite set can be computed successively by adjoining the points of A one after the other, it suffices to consider the case when A consists of just two points. Moreover, considering the definition of $\overline{\mathfrak{F}}(\kappa, Q)$ and e), we conclude that it suffices to prove that if $F^1, F^2 \in \mathfrak{F}(\kappa, Q)$, then $F^1 \wedge F^2 \in \overline{\mathfrak{F}}(\kappa, Q)$.

To do this, let $\Phi_n(f^1, f^2)$ be the result of convolving the functions $f^1 \wedge f^2$ and $n^2\zeta(nf^1)\zeta(nf^2)$ with respect to the variables f^1 and f^2, and let $F_n = \Phi_n(F^1, F^2)$ for $\beta = 1$. For the remaining values of $\beta > 0$ we define F_n by homogeneity. It is not hard to verify (see, for example, the proof of Theorem 5.3 in [1]) that Φ_n is upwards convex with respect to f^1 and f^2, $\Phi_{nf^1} \geqslant 0$, $\Phi_{nf^1} + \Phi_{nf^2} = 1$ and $|\Phi_n - f^1\Phi_{nf^1} - f^2\Phi_{nf^2}| \leqslant N(\zeta)/n$. This gives us at once that F_n satisfies all the conditions 1.1)–1.3) except possibly the inequality $|F_{n\beta}| \leqslant \kappa^{-1}$. Since for $\beta = 1$

$$F_{n\beta} = \Phi_n(F^1, F^2) - \Phi_{nf^1}F^1 - \Phi_{nf^2}F^2 + \Phi_{nf^1}F^1_\beta + \Phi_{nf^2}F^2_\beta,$$

it follows that $|F_{n\beta}| \leqslant \kappa^{-1} + N/n$. Correcting F_n as above and observing that $F_n \to F^1 \wedge F^2$, we get that $F^1 \wedge F^2 \in \overline{\mathfrak{F}}(\kappa, Q)$.

The assertions of the lemma can be proved similarly for the functions in the class $\overline{\mathfrak{F}}(\kappa, D)$, and the lemma is proved.

This lemma enables us to consider equations of the form

$$\inf_A F^a(u_{x^ix^j}, u_{x^i}, u, 1, t, x) = u_t,$$

which are important from the point of view of optimal control theory, as a special case of equations with functions $F \in \overline{\mathfrak{F}}(\kappa, Q)$.

To state our main results we need the spaces $C^{2+\alpha}(Q)$ introduced in [11] (or in [1]). These spaces are sometimes denoted also by $C^{1+\alpha/2,2+\alpha}(Q)$ and $H^{1+\alpha/2,2+\alpha}(Q)$ in articles by other authors. The notation $C^{2+\alpha}(D)$ has the commonly accepted meaning. In the theorems below it is assumed that there exists a function $\psi \in C^3_{\mathrm{loc}}(E_d)$ such that $D = \{x \in E_d: \psi(x) > 0\}$, $\|\psi\|_{C^3(D)} \leqslant \kappa^{-1}$, $|\psi_x| \geqslant \kappa^*$ and $|\psi_x| \geqslant \kappa$ on ∂D ($\psi_x = \operatorname{grad}\psi$).

THEOREM 1.1. *Suppose that D is a bounded domain, $F \in \overline{\mathfrak{F}}(\kappa, Q)$, $\varphi \in C(\overline{Q})$, $\varphi, \varphi_t, \varphi^{x^ix^j} \in C^2(Q)$ $(i, j = 1, \dots, d)$, and the $C^2(D)$-norms of these functions do not exceed κ^{-1}. Then the equation $u_t = F(u_{x^ix^j}, u_{x^i}, u, 1, t, x)$ in Q, with the boundary condition $u = \varphi$ on $\partial'Q$, has precisely one solution $u \in C(\overline{Q}) \cap C^2(Q)$. Moreover, the norm of u in $C^2(Q)$ does not exceed $N(\kappa, d, \|u\|_{C(Q)})$. Finally, there exists an $\alpha_1 = \alpha_1(\kappa, d) \in (0, 1)$ such that $u \in C^{2+\alpha_1}(\overline{Q}_{1/n} \cup \overline{Q}(1/n))$ for any $n \geqslant 1$, and the norm of u in $C^{2+\alpha_1}(\overline{Q}_{1/n} \cup \overline{Q}(1/n))$ does not exceed $N(\kappa, d, \|u\|_{C(Q)}, n)$.*

THEOREM 1.2. *Suppose that the domain D is bounded, $F \in \overline{\mathfrak{F}}(\kappa, D)$, $\varphi \in C(\overline{D}) \cap C^3(D)$ and $\|\varphi\|_{C^3(D)} \leqslant \kappa^{-1}$. Then the equation $F(u_{x^ix^j}, x_{x^i}, u, 1, x) = 0$ in D, with the boundary condition $u = \varphi$ on ∂D, has precisely one solution $u \in C^{2+\alpha_1}(D)$, where $\alpha_1 = \alpha_1(\kappa, d) \in (0, 1)$. Moreover, the norm of u in $C^{2+\alpha_1}(\overline{D})$ does not exceed $N(\kappa, d, \|u\|_{C(D)})$.*

These theorems are proved in §7. Concerning the nature of the dependence of α_1 on the original data, we refer the reader to Remark 6.1 for more detail. We note that the α_1 in Theorems 1.1 and 1.2 is the $\alpha_1(\kappa, d, 1/2)$ in Theorem 6.1. We note also that in the setting of Theorem 1.2 the inclusion $u \in C^3(D(1/n))$ is false in general. For example, the solution of the equation $u'' - |u'| = 1$ on $(-1, 1)$ with the boundary conditions $u(\pm 1) = 0$ has a third derivative which is discontinuous at zero.

§2. An estimate in C^α of a solution of special linear degenerate equation

In this section we prove two estimates in C^α for an auxiliary linear equation in a cylinder with axis parallel to the t-axis; one estimate is an interior estimate, and the other is valid up to lower base of the cylinder.

Suppose that d, m, n, $1 \leqslant n \leqslant m \leqslant d$ are integers, and κ, κ_1, $R \in (0, 1]$, and let $G(R) = \{(t, x): t \in [-R^4, 0], x \in E_d, |x^i| \leqslant R \text{ for } i \leqslant m, |x^i| \leqslant R^2 \text{ for } i > m\}$. Let $u \in C^2(G(2R))$, and in $G(2R)$ define an operator

$$L = a^{ij}\frac{\partial^2}{\partial x^i \partial x^j} + b^i\frac{\partial}{\partial x^i} + cg\frac{\partial}{\partial t} \tag{2.1}$$

such that for all $\lambda \in E_d$ the relations

$$\kappa^{-1}\Big(\sum_{i \leqslant m}(\lambda^i)^2 + \sum_{i \leqslant n}(x^i)^2\sum_{i > m}(\lambda^i)^2\Big) \geqslant a^{ij}\lambda^i\lambda^j$$
$$\geqslant \kappa\Big(\sum_{i \leqslant m}(\lambda^i)^2 + \sum_{i \leqslant n}(x^i)^2\sum_{i > m}(\lambda^i)^2\Big),$$
$$\kappa\sum_{i \leqslant n}(x^i)^2 \leqslant g \leqslant \kappa^{-1}\sum_{i \leqslant n}(x^i)^2, \qquad |b^i| \leqslant \kappa^{-1}, \quad i = 1, \dots, d, \qquad |c| \leqslant \kappa_1^{-1} \tag{2.2}$$

hold in $G(2R)$.

THEOREM 2.1. *There are constants $\alpha = \alpha(\kappa, d) \in (0, 1)$ and $N = N(\kappa, \kappa_1, d) < \infty$ such that for $(t_1, 0), (t_2, x) \in G(R)$*

$$|u(t_1, 0) - u(t_2, x)| \leqslant NR^{-\alpha}\big(\|u\|_{C(G(2R))} + R^2\|Lu\|_{B(G(2R))}\big)$$
$$\times\Big(\sum_{i \leqslant m}|x^i| + \sum_{i > m}|x^i|^{1/2} + |t_1 - t_2|^{1/4}\Big)^\alpha.$$

Before proving this theorem we note that the function

$$u_R(t, x) = u(R^4t, Rx^1, \dots, Rx^m, R^2x^{m+1}, \dots, R^2x^d)$$

satisfies the inequality $|L_Ru_R| \leqslant R^2\|Lu\|_{B(G(2R))}$, in $G(2R)$, where

$$L_R = \sum_{i,j \leqslant m}a^{ij}\frac{\partial^2}{\partial x^i \partial x^j} + 2\frac{1}{R}\sum_{i \leqslant m}\sum_{j > m}a^{ij}\frac{\partial^2}{\partial x^i \partial x^j} + \frac{1}{R^2}\sum_{i,j > m}a^{ij}\frac{\partial^2}{\partial x^i \partial x^j}$$
$$+ R\sum_{i \leqslant m}b^i\frac{\partial}{\partial x^i} + \sum_{i > m}b^i\frac{\partial}{\partial x^i} + R^2c - R^{-2}g\frac{\partial}{\partial t}$$

and $(R^4t, Rx^1, \dots, Rx^m, R^2x^{m+1}, \dots, R^2x^d)$ are taken as the arguments of a^{ij}, b^i, c and g. Obviously, the coefficients of L_R satisfy in $G(2)$ the conditions formulated before the theorem. Moreover, the validity of the theorem for u_R implies its validity for u. Therefore, in proving the theorem it suffices to consider the case $R = 1$.

We shall need

LEMMA 2.1. *Suppose that a parabolic operator of the form (2.1) is given in E_{d+1} with $c \equiv 0$, $|b^i| \leqslant \kappa^{-1}$ $(i \leqslant d)$, $a^{11} \geqslant \kappa$, $a^{ii} \leqslant \kappa^{-1}$ for $i \leqslant m$, $a^{ii} \leqslant \kappa^{-1}(1 + \sum_{i \leqslant n}|x^i|^2)$ for $i > m$, and $0 \leqslant g \leqslant \kappa^{-1}(1 + \sum_{i \leqslant n}|x^i|^2)$. Let*

$$G_\varepsilon(N) = \{(t, x): 0 \geqslant t \geqslant -N^4\varepsilon^4, |x^1| \leqslant \varepsilon, |x^i| \leqslant N_\varepsilon$$
$$\textit{for } i = 2, \dots, m, |x^i| \leqslant N^2\varepsilon^2 \textit{ for } i > m\}.$$

Then there exist constants $N_0 = N_0(\kappa, d) \geqslant 1$, $\varepsilon_0 = \varepsilon_0(\kappa, d) \in (0,1)$ *and* $\delta_0 = \delta_0(\kappa, d) > 0$ *such that if* $u \in C^2(G_2(N_0))$, $u \geqslant 0$, $Lu \leqslant 0$ *in* $G_2(N_0)$, *and* $u(t,x) \geqslant 1$ *for* $x^1 = 2((t,x) \in G_2(N_0))$, *then* $u \geqslant \delta_0$ *in* $G_{\varepsilon_0}(N_0)$.

PROOF. Let $\xi = \exp^{\kappa^{-2}}$, and

$$v_1(t,x) = \kappa^{-1}N^{-1/2}\int_{|x^1|}^{2}\left(\int_0^y \xi^{y-z}\,dz\right)dy, \quad v_2(t,x) = \left(\xi^2 - \xi^{x^1}\right)\left(\xi^2 - \xi^{-2}\right)^{-1},$$

$$v_3(t,x) = N^{-2}\left((x^2)^2 + \cdots + (x^m)^2\right),$$

$$v_4(t,x) = N^{-4}\sum_{i>m}(x^i)^2, \qquad v_5(t,x) = -N^{-4}t.$$

The reader can easily see that in $G_2(N)$

$$L_{v_1} \leqslant -N^{-1/2}, \quad Lv_2 \leqslant 0, \quad Lv_3 \leqslant N^{-2}2d\kappa^{-1}(2N+1),$$

$$Lv_4 \leqslant N^{-4}2d\kappa^{-1}\left(1 + 4N^2(d+1)\right), \quad Lv_5 \leqslant N^{-4}\kappa^{-1}\left(1 + 4dN^2\right).$$

Hence, $Lv < 0$ for $v = v_1 + \cdots + v_5$ in $G_2(N)$ if N is sufficiently large. From this we conclude by the maximum principle that $1 - u \leqslant v$ in $G_2(N)$. It remains to choose N, ε and δ such that $v \leqslant 1 - \delta$ in $G_\varepsilon(N)$. For this it suffices, for example, to set $\varepsilon = N^{-1}$ and take N sufficiently large. This proves the lemma.

PROOF OF THEOREM 2.1. As we saw above, we can assume that $R = 1$. Take the constants N_0, ε_0 and δ_0 from Lemma 2.1, let $\tilde{G}(\varepsilon) = G_\varepsilon(N_0)$, and consider first the special case $c = Lu = 0$ in $G(2)$.

We note that $(t,0) + \tilde{G}(\varepsilon) \subset G(2)$ for $\varepsilon \leqslant 3\varepsilon_1$ and $t \in [-1,0]$, where $\varepsilon_1 = N_0^{-1}/3$, and (as well-known standard arguments show) to prove the theorem it suffices to establish the inequality

$$\operatorname{osc}\{u, (t,0) + \tilde{G}(\varepsilon\varepsilon_0)\} \leqslant \gamma \operatorname{osc}\{u, (t,0) + \tilde{G}(3\varepsilon)\} \tag{2.3}$$

for all $t \in [-1,0]$ and $\varepsilon \leqslant \varepsilon_1$, where the constant $\gamma < 1$ depends only on κ and d. Using a dilation of the coordinates similar to what was done before Lemma 2.1, we see that it suffices to prove (2.3) for $t = 0$ and $\varepsilon = 1$ under the assumption that $u \in C^2(\tilde{G}(3))$, $Lu = 0$ in $\tilde{G}(3)$, and the coefficients of L satisfy (2.2) in $\tilde{G}(3)$. We carry out one more standard step. It is well known that (2.3) will be proved for $\varepsilon = 1$ if we prove that if $u \geqslant 0$, $u \in C^2(\tilde{G}(3))$, $Lu = 0$ in $\tilde{G}(3)$, and the Lebesgue measure $|(u \geqslant 1) \cap \tilde{G}(3)|$ of the set $(u \geqslant 1) \cap \tilde{G}(3)$ is greater than $|\tilde{G}(3)|/2$, then $u \geqslant \xi = \xi(\kappa, d) > 0$ in $\tilde{G}(\varepsilon_0)$.

Accordingly, suppose that $u \geqslant 0$, $Lu = 0$ in $\tilde{G}(3)$, and $|(u \geqslant 1) \cap \tilde{G}(3)| \geqslant |\tilde{G}(3)|/2$. Let

$$G_\pm = \left\{(t,x): t \in \left[-81N_0^4, 0\right], \pm x^1 \geqslant 0,\right.$$
$$\left.|x^1| \vee |x^2|/N_0 \vee \cdots \vee |x^m|/N_0 \in [1,3], |x^i| \leqslant 9N_0^2 \text{ for } i > m\right\},$$
$$\tilde{G}_\pm = G_\pm \cap \left\{t < -4N_0^4\right\}.$$

Obviously,

$$G_\pm \in \tilde{G}(3), \qquad |\tilde{G}(3) \setminus (G_+ \cap G_-)| = 3^{-m}|\tilde{G}(3)|,$$
$$|(u \geqslant 1) \cap (G_+ \cup G_-)| \geqslant \left(\tfrac{1}{2} - 3^{-m}\right)|\tilde{G}(3)|,$$

and one of the two inequalities

$$|(u \geqslant 1) \cap G_+| \geqslant \tfrac{1}{2}\left(\tfrac{1}{2} - 3^{-m}\right)|\tilde{G}(3)| \quad \text{or} \quad |(u \geqslant 1) \cap G_-| \geqslant \tfrac{1}{2}\left(\tfrac{1}{2} - 3^{-m}\right)|\tilde{G}(3)|$$

is valid. Both these possibilities can be handled in the same way, so we assume that the first inequality holds. We have

$$|G_+ \setminus \tilde{G}_+| = \tfrac{4}{81}|G_+| = \tfrac{4}{81}\cdot\tfrac{1}{2}\left(1 - 3^{-m}\right)|\tilde{G}(3)|,$$

$$|(u \geqslant 1) \cap \tilde{G}_+| \geqslant \left[\tfrac{1}{2}\left(\tfrac{1}{2} - 3^{-m}\right) - \tfrac{2}{81}\left(1 - 3^{-m}\right)\right]|\tilde{G}(3)| = \delta > 0. \tag{2.4}$$

Moreover, the set G_+ is a connected cylinder, and in it the operator L is uniformly parabolic. Therefore, Harnack's theorem is valid for L in G_+ (see [10]). By (2.4), for some $\delta_1 = \delta_1(N_0, d) > 0$ there is a point (t_1, x_1) lying interior to $\tilde{G}_+$ and at a distance of at least δ_1 from its boundary, at which $u \geqslant 1$. By Harnack's theorem, $u(t,x) \geqslant \varepsilon_2(\kappa, d) > 0$, if $t \in [-4N_0^4, 0]$, $x^1 = 2$, $|x^i| \leqslant 2N_0$ for $i = 2, \ldots, m$, and $|x^i| \leqslant 4N_0^2$ for $i > m$. We conclude from Lemma 2.1 that $u \geqslant \varepsilon_2\delta_0$ in $\tilde{G}(\varepsilon_0)$, and the theorem is proved in the case when $c = Lu = 0$.

The case when $c \neq 0$ is easily reduced to the case when $c = 0$, since $|Lu - cu| \leqslant |Lu| + \kappa_1^{-1}|u|$. Finally, if $c = 0$ and $Lu \neq 0$, then instead of $u(t,x)$ it is necessary to consider the function

$$v(t, x^1, \ldots, x^{d+1}) = \|Lu\|_{B(G(2))}^{-1} u(t, x^1, \ldots, x^d) + x^{d+1}$$

and observe that it satisfies the equation

$$Lv + \frac{(-Lu)}{\|Lu\|_{B(G(2))}}\frac{\partial v}{\partial x^{d+1}} + \sum_{i \leqslant n}(x^i)^2\frac{\partial^2 v}{\partial x^{d+1}\partial x^{d+1}} = 0.$$

The theorem is proved.

For convenience in using Theorem 2.1 we give it another form. Let

$$W_{T,R} = \left\{(t,x): t \in (0,T), x \in E_d, |x^i| < R \text{ for } i \leqslant d\right\}.$$

THEOREM 2.2. *Suppose that* $T > 0$, $\kappa, \kappa_1, R \in (0,1]$, *and the integers* $d, m, n \geqslant 0$ *are such that* $1 \leqslant n + 1 \leqslant m \leqslant d$. *Let L be an operator of the form* (2.1) *whose coefficients in* $W_{T,2R}$ *satisfy the conditions*

$$\kappa^{-1}\left(\sum_{i>n}(\lambda^i)^2 + \sum_{i=n+1}^{m}(x^i)^2\sum_{i\leqslant n}(\lambda^i)^2\right) \geqslant a^{ij}\lambda^i\lambda^j \geqslant \kappa\left(\sum_{i>n}(\lambda^i)^2 + \sum_{i=n+1}^{m}(x^i)^2\sum_{i\leqslant n}(\lambda^i)^2\right),$$

$$\kappa\sum_{i=n+1}^{m}(x^i)^2 \leqslant g \leqslant \kappa^{-1}\sum_{i=n+1}^{m}(x^i)^2, \quad |b^i| \leqslant \kappa^{-1}, \quad |c| \leqslant \kappa_1^{-1}.$$

Then there exist constants $\alpha = \alpha(\kappa, d) \in (0,1)$ *and* $N = N(\kappa, \kappa_1, d)$ *such that*

$$|u(t_1, x_1) - u(t_2, x_2)| \leqslant N\left(\|u\|_{C(W_{T,2R})} + \rho^2\|Lu\|_{B(W_{T,2R})}\right)$$
$$\times \rho^{-\alpha}\left(\sum_{i>n}|x_1^i - x_2^i| + \sum_{i\leqslant n}|x_1^i - x_2^i|^{1/2} + |t_1 - t_2|^{1/4}\right)^{\alpha} \tag{2.5}$$

for $u \in C^2(W_{T,2R})$, $(t_1, x_1), (t_2, x_2) \in W_{T,R}$, *and* $x_1^i = 0$ *for* $n + 1 \leqslant i \leqslant m$, *where* $\rho^4 = t_1 \wedge t_2 \wedge R^4$.

To derive (2.5) from Theorem 2.1 when

$$\frac{1}{2}\rho \geqslant \sum_{i>n}|x_1^i - x_2^i| + \sum_{i\leqslant n}|x_1^i - x_2^i|^{1/2} + |t_1 - t_2|^{1/4} \tag{2.6}$$

it suffices to suitably relabel the coordinates, observe that (after the relabelling) $(t_j, x_j) \in (t_1 \vee t_2, x_1) + G(\rho/2) \subset (t_1 \vee t_2, x_1) + G(\rho) \subset W_{T,2R}$, and transfer $(t_1 \vee t_2, x_1)$ to the origin of coordinates in E_{d+1} by a parallel translation. But if the inequality opposite to that in (2.6) holds, then (2.5) is obvious with $N = 4$.

Theorems 2.1 and 2.2 are applicable not only to parabolic operators, but also to elliptic operators. The corresponding assertions are obtained if we take a function u not depending on l. For an estimate of the Holder norm of u up to the lower base of $W_{T,R}$ we need

LEMMA 2.2. *Suppose that the assumptions of Theorem 2.2 hold, $u \in C(\overline{W}_{T,2R}) \cap C^2(W_{T,2R})$, $x_0 \in W_{0,R} = \{x: |x^i| < R,\ i = 1,\dots,d\}$, $x_0^i = 0$ for $n+1 \leqslant i \leqslant m$, and $\alpha \in (0,1)$. Let $u_0(x) = u(0,x)$. Then*

$$|u(t,x_0) - u_0(x_0)| \leqslant NR^{-\alpha\alpha/(3\alpha+4)}\Big(\|u_0\|_{C^\alpha(x_0+W_{0,R})} + \|u\|_{C((0,x_0)+W_{T,R})} + \|Lu\|_{B((0,x_0)+W_{T,R})}\Big) \tag{2.7}$$

for $t \in [0,T]$, where $N = N(\kappa, \kappa_1, d)$.

PROOF. Clearly, it suffices to prove (2.7) for $x_0 = 0$. Moreover, (2.7) is valid in an obvious way when $t > R^{3\alpha+4}$ and $N = 2$. Hence, it can be assumed that $t \leqslant R^{3\alpha+4}$. For such t the inequality (2.7) follows by the condition $R \leqslant 1$ from the inequality obtained when $R^{-\alpha}$ is omitted in (2.7). We note further that if we take $L - c$ instead of L, prove the lemma for $L - c$, and use the inequality $|Lu - cu| \leqslant |Lu| + |cu|$, then we see that there is no loss of generality in assuming that $c = 0$. If we take $u - u_0(x_0)$ in place of u with $c = 0$, then we see that there is also no loss of generality in assuming that $u_0(x_0) = 0$. Further, considering instead of u the ratio of u to the expression in parentheses in (2.7), we conclude that in proving the lemma it suffices to establish the inequality

$$|u(t,0)| \leqslant Nt^{\alpha/(3\alpha+4)} \tag{2.8}$$

under the assumption that $t \leqslant R^{3\alpha+4}$, $|u|, |Lu| \leqslant 1$ in $W_{T,R}$, and $|u_0(x)| \leqslant (|x^1| \wedge \cdots \wedge |x^d|)^\alpha$ for $x \in W_{0,R}$. Accordingly, we prove (2.8), assuming that all the additional assumptions mentioned above are satisfied.

Suppose that $\varepsilon \in (0,R]$, $\delta > 0$, and $\zeta(t) \geqslant 0$ is a smooth decreasing function such that $\zeta(t) \geqslant 1$ for $t \in [0,\delta]$, and let $\xi = \exp\kappa^{-2}$, $v_1(t,x) = \varepsilon^{-2}|x|^2$, $v_2(t,x) = \varepsilon^{-2}\delta^{-2}\kappa^{-1}t$ and

$$v_3(t,x) = \frac{1}{\varepsilon^2\kappa}\int_{|x^{n+1}|}^{\varepsilon}\int_0^y \xi^{y-z}\zeta(z)\,dz\,dy.$$

Clearly, $Lv_1 \leqslant \varepsilon^{-2}N(\kappa,d)$ in $W_{T,\varepsilon}$. Moreover, it is not hard to see that

$$Lv_2 \leqslant -\varepsilon^{-2} \quad \text{for } |x^{n+1}| \geqslant \delta, \qquad Lv_3 \leqslant -\varepsilon^{-2}\zeta \leqslant -\varepsilon^{-2} \text{ for } |x^{n+1}| \leqslant \delta.$$

This gives us the existence of a constant $N = N(\kappa,d)$ such that the function $v = v_1 + N(v_2 + v_3)$ satisfies the inequality $Lv < -1$ in $W_{T,\varepsilon}$. By the maximum principle, $|u| \leqslant \varepsilon^\alpha + v$ in $W_{T,\varepsilon}$. In particular,

$$|u(t,0)| \leqslant \varepsilon^\alpha + N\frac{1}{\varepsilon^2\kappa}\left(\frac{t}{\delta^2} + \int_0^\varepsilon\int_0^y \xi^{y-z}\zeta(z)\,dz\,dy\right).$$

In the last inequality ζ can be replaced by the indicator function of $[0,\delta]$ by passing to the limit. Then

$$|u(t,0)| \leqslant \varepsilon^\alpha + N\varepsilon^{-2}(t\delta^{-2} + \varepsilon\delta).$$

If we choose $\delta = \varepsilon^{\alpha+1}$ and $\varepsilon = t^{1/(3\alpha+4)} (\leqslant R)$, we get (2.8) here, and the lemma is proved.

Combination of Lemma 2.2 and Theorem 2.2 leads in a perfectly standard way (see, fc example, §3 in [1], or the proof of our Lemma 5.3) to the following assertion.

THEOREM 2.3. *Suppose that the assumptions of Theorem 2.2 hold, $\alpha \in (0,1)$, $u \in C(\overline{W}_{T,2R}) \cap C^2(W_{T,2R})$, $(t_j, x_j) \in \overline{W}_{T,R}$, and $x_j^i = 0$ for $n+1 \leqslant i \leqslant m$, $j = 1,2$. Then*

$$|u(t_1,x_1) - u(t_2,x_2)| \leqslant N\Big(\|u_0\|_{C^\alpha(W_{0,2R})} + \|u\|_{C(W_{T,2R})} + \|Lu\|_{B(W_{T,2R})}\Big)\Big(|t_1 - t_2|^{1/2} + |x_1 - x_2|\Big)^{\alpha_1},$$

where $\alpha_1 = \alpha_1(\kappa, d, \alpha) \in (0,1)$ and $N = N(\kappa, \kappa_1, d, R)$.

§3. An estimate in $C^{1+\alpha}$ of a solution of a special nonlinear equation

Suppose that $d, n \geqslant 1$ are integers with $n+1 \leqslant d$, $\kappa, \kappa_1 \in (0,1]$, V is a subset of E_{d+1} and the function $\Phi(u_{ij}, u_i, u, t, x)$ is defined for all $(t,x) \in E_{d+1}$ and all real u_{ij}, $j = 1,\dots,d$, u_i, $i = 1,\dots,d$, and u, and is differential with respect to u_{ij} and x. We writ $\Phi \in \Phi_n(\kappa,\kappa_1,V)$, if for any $(t,x) \in V$ and $k \leqslant n$, any symmetric matrix (u_{ij}), and any u_i and u the matrix $(\Phi_{u_{ij}})$ is nonnegative-definite and

$$|\Phi_{x^k}| \leqslant \kappa^{-1}\Big(|u| + \sum_{i\leqslant d}|u_i| + \big(\Phi_{u_{ij}}u_{ir}u_{jr}\big)^{1/2}\Big) + \kappa_1^{-1}. \tag{3.1}$$

LEMMA 3.1. *If $\Phi \in \Phi_n(\kappa,\kappa_1,V)$, then for $k \leqslant n$ and $s, r = 1,\dots,d$ there exist functior $\Phi_{sr}^k(u_{ij}, u_i, u, t, x)$ such that*

$$\big|\Phi_{ij}^k\xi^i\eta^j\big|^2 \leqslant \kappa^{-2}|\xi|^2\Phi_{u_{ij}}\eta^i\eta^j, \qquad \big|\Phi_{x^k} - \Phi_{ij}^k u_{ij}\big| \leqslant \kappa^{-1}\Big(|u| + \sum_{i\leqslant d}|u_i|\Big) + \kappa_1^{-1} \tag{3.2}$$

for any $k \leqslant n$, $(t,x) \in V$ and $\xi, \eta \in E_d$, any symmetric matrix (u_{ij}), and any u_i and u.

PROOF. We construct Φ_{sr}^k only for $(t,x) \in V$ and for symmetric (u_{ij}). For the remainin values of the arguments t, x and u_{ij} the functions Φ_{sr}^k can be defined in an arbitrary way Since the matrix $\frac{1}{2}(\Phi_{u_{ij}} + \Phi_{u_{ji}})$ is symmetric and nonnegative-definite, it has a symmetri nonnegative square root, which we denote by σ. Obviously,

$$\Phi_{u_{ij}}u_{ir}u_{jr} = \operatorname{tr}(u_{ij})\tfrac{1}{2}(\Phi_{u_{ij}} + \Phi_{u_{ji}})(u_{ij}) = \operatorname{tr}\big[(u_{ij})\sigma\big]\sigma(u_{ij})$$
$$= \big(\Phi_{u_{ij}}u_{ir}u_{jr}\big)^{1/2}\operatorname{tr}(\varepsilon_{ij})\sigma(u_{ij}),$$

where (ε_{ij}) is determined by the equality $(\varepsilon_{ij})(\Phi_{u_{st}}u_{sr}u_{tr})^{1/2} = (u_{ij})\sigma$. It is clear that ε can always be taken so that $\varepsilon_{ij}\varepsilon_{ij} = 1$.

Further, it follows from (3.1) that

$$\Phi_{x^k} = f^k + c^k u + b_i^k u_i + a^k\Phi_{ij}u_{ij},$$

where $|c^k|, |b_i^k|, |a^k| \leqslant \kappa^{-1}$, $|f^k| \leqslant \kappa_1^{-1}$ and $(\Phi_{ij}) = (\varepsilon'_{ij})\sigma$. If we now set $\Phi_{ij}^k = a^k\Phi_{ij}$, the the second inequality in (3.2) is obvious, and it remains for us to verify the first inequalit We have

$$\big|\Phi_{ij}^k\xi^i\eta^j\big|^2 \leqslant \kappa^{-2}\big((\varepsilon_{ij})^*\xi, \sigma\eta\big)^2 \leqslant \kappa^{-2}\big|(\varepsilon_{ij})^*\xi\big|^2|\sigma\eta|^2$$
$$\leqslant \kappa^{-2}|\xi|^2(\eta, \sigma^2\eta) = \kappa^{-2}|\xi|^2\Phi_{u_{ij}}\eta^i\eta^j.$$

The lemma is proved.

It is perhaps useful to have in mind that the inequalities (3.2), in turn, imply (3.1) with another κ.

In the remainder of the section we proceed from the following assumptions. Suppose that $d, n \geqslant 1$, $n+1 \leqslant d$, $\kappa, \kappa_1 \in (0,1]$, $T \in (0,\infty)$, and $\Phi(u_{ij}, u_i, u, t, x)$ is a function such that the following conditions hold:

3.1) $\Phi \in \Phi_n(\kappa, \kappa_1, W_{T,2})$.

3.2) Φ is once continuously differentiable with respect to (u_{ij}, u_i, u, x) for every $t \in (0, T]$.

3.3) For all $\lambda \in E_d$ and $(t,x) \in W_{T,2}$, all symmetric matrices (u_{ij}), and any u_i and u

$$\kappa^{-1}\Big(\sum_{i>n}(\lambda^i)^2 + \sum_{i>n}(x^i)^2\sum_{i\leqslant n}(\lambda^i)^2\Big) \geqslant \Phi_{u_{ij}}\lambda^i\lambda^j \geqslant \kappa\Big(\sum_{i>n}(\lambda^i)^2 + \sum_{i>n}(x^i)^2\sum_{i\leqslant n}(\lambda^i)^2\Big), \tag{3.3}$$

$$|\Phi_{u_i}| \leqslant \kappa^{-1}, \quad i = 1,\dots,d, \qquad |\Phi_u| \leqslant \kappa_1^{-1}.$$

Further, suppose that $g(t,x) = g(t, x^{n+1},\dots,x^d)$ is a function defined in $W_{T,2}$ such that in $W_{T,2}$

$$\kappa^{-1}\sum_{i>n}(x^i)^2 \geqslant g \geqslant \kappa\sum_{i>n}(x^i)^2.$$

Let

$$\Gamma_{T,1} = \{(t,x) \in W_{T,1}: x^i = 0 \text{ for } i > n\}.$$

THEOREM 3.1. *Suppose that* $\varepsilon \in (0,T)$, $\alpha \in (0,1)$, $v, v_{x^i} \in C(\overline{W}_{T,2}) \cap C^2(W_{T,2})$ *for* $i \leqslant n$, *and in* $W_{T,2}$

$$|v|, |v_{x^i}| \leqslant \kappa_1^{-1} \quad \textit{for } i = 1,\dots,d,$$
$$|v_{x^ix^j}| \leqslant \kappa_1^{-1} \quad \textit{for } i, j > n, \qquad gv_t = \Phi(v_{x^ix^j}, v_{x^i}, v, t, x). \tag{3.4}$$

Then for $i \leqslant n$ *and* $(t_1, x_1), (t_2, x_2) \in \Gamma_{T,1}$

$$|v_{x^i}(t_1,x_1) - v_{x^i}(t_2,x_2)| \leqslant N\big(|t_1 - t_2|^{1/2} + |x_1 - x_2|\big)^{\alpha_1}$$

in the following two cases:

a) $t_1, t_2 \geqslant \varepsilon$, $\alpha_1 = \alpha_1(\kappa, d) \in (0,1)$ *and* $N = N(\kappa, \kappa_1, d, \varepsilon)$;

b) *no restrictions on* t_1 *and* t_2, $\alpha_1 = \alpha_1(\kappa, d, \alpha) \in (0,1)$, *and* N *depends only on* $\kappa, \kappa_1, d, \alpha$, *and the norms of* $v_{x^i}(0,x)$ *in* $C^\alpha(W_{0,2})$ *for* $i \leqslant n$.

PROOF. In E_{d+n+1} we consider the function

$$u(t, x^1,\dots,x^{d+n}) = \sum_{i=1}^n x^{d+i}v_{x^i}(t, x^1,\dots,x^d). \tag{3.5}$$

Differentiating (3.4), we get that in

$$\tilde{W}_{T,2} = \{(t,x): t \in (0,T), x \in E_{d+n}, |x^i| < 2, i = 1,\dots,d+n\}$$

the function u satisfies the equation

$$Lu \equiv -gu_t + \sum_{i,j\leqslant d+n} a^{ij}u_{x^ix^j} + \sum_{i\leqslant d} b^iu_{x^i} + cu = f, \tag{3.6}$$

where $c = \Phi_u$, $b^i = \Phi_{u_i}$, $a^{ij} = \Phi_{u_{ij}}$ for $i, j = 1,\dots,d$,

$$f = \sum_{k\leqslant n} x^{d+k}\Big(\sum_{i,j\leqslant d}\Phi^k_{ij}v_{x^ix^j} - \Phi_{x^k} - \sum_{i,j=n+1}\Phi^k_{ij}v_{x^ix^j}\Big),$$
$$a^{d+i,j} = \sum_{k\leqslant n} x^{d+k}\Phi^k_{ij} \quad \text{for } i \leqslant n, \quad j = 1,\dots,d,$$
$$a^{i,d+j} = \sum_{k\leqslant n} x^{d+k}\Phi^k_{ij} \quad \text{for } i = n+1,\dots,d, \quad j \leqslant n,$$
$$a^{i,d+j} = 0 \quad \text{for } i, j \leqslant n, \qquad a^{ij} = N_0\delta^{ij} \quad \text{for } i, j = d+1,\dots,d+n,$$

N_0 is an arbitrary positive number, and the Φ^k_{ij} are the functions in Lemma 3.1. Note th

$$\sum_{i,j\leqslant d+n} a^{ij}\lambda^i\lambda^j = \sum_{i,j\leqslant d}\Phi_{u_{ij}}\lambda^i\lambda^j + N_0\sum_{i>d}(\lambda^i)^2 + \sum_{k\leqslant n}x^{d+k}\sum_{i\leqslant n, j\leqslant d}\Phi^k_{ij}\lambda^{d+i}\lambda^j$$
$$+ \sum_{k\leqslant n}x^{d+k}\sum_{i=n+1}^d\sum_{i=n+1}^d\sum_{j\leqslant n}\Phi^k_{ij}\lambda^i\lambda^{d+j},$$

and, by (3.2) and (3.3), the sum of the last two terms does not exceed

$$N\Big[\sum_{i>d}(\lambda^i)^2\sum_{i,j\leqslant d}\Phi_{u_{ij}}\lambda^i\lambda^j\Big]^{1/2} + N\Big[\sum_{j>d}(\lambda^j)^2\sum_{i=n+1}^d(\lambda^i)^2\Big]^{1/2}$$
$$\leqslant N\Big(\sum_{i>d}(\lambda^i)^2\Big)^{1/2}\Big(\sum_{i,j\leqslant d}\Phi_{u_{ij}}\lambda^i\lambda^j\Big)^{1/2}$$

in $\tilde{W}_{T,2}$.

Therefore, it is easy to choose a large $N_0 > 0$ and a small $\kappa_2 \in (0,1)$, depending only κ and d, such that in $\tilde{W}_{T,2}$ we have for all $\lambda \in E_{d+n}$

$$\kappa_2^{-1}\Big(\sum_{i,j\leqslant d}\Phi_{u_{ij}}\lambda^i\lambda^j + \sum_{i>d}(\lambda^i)^2\Big) \geqslant \sum_{i,j\leqslant d+n}a^{ij}\lambda^i\lambda^j \geqslant \kappa_2\Big(\sum_{i,j\leqslant d}\Phi_{u_{ij}}\lambda^i\lambda^j + \sum_{i>d}(\lambda^i)^2\Big).$$

Then in $\tilde{W}_{T,2}$ we have for all $\lambda \in E_{d+n}$ that for $\kappa_3 = \kappa_3(\kappa, d)$

$$\kappa_3^{-1}\Big(\sum_{i=n+1}^{d+n}(\lambda^i)^2 + \sum_{i=n+1}^d(x^i)^2\sum_{i\leqslant n}(\lambda^i)^2\Big) \geqslant \sum_{i,j\leqslant d+n}a^{ij}\lambda^i\lambda^j$$
$$\geqslant \kappa_3\Big(\sum_{i=n+1}^{d+n}(\lambda^i)^2 + \sum_{i=n+1}^d(x^i)^2\sum_{i\leqslant n}(\lambda^i)^2$$

From this it is clear that the operator L in (3.6) satisfies the conditions of Theorem if (n, m, d) is replaced by $(n, d, d+n)$ in it. Moreover, by the construction of Φ^k_{ij} and assumptions of the theorem it follows that $|f| \leqslant N(\kappa, \kappa_1, d)$ in $\tilde{W}_{T,2}$. The use of Theore 2.2 and 2.3 for the u in (3.5) now proves directly both assertions of our theorem.

Of the conditions 3.1)–3.3) only 3.2) and 3.3) are used in the next theorem, and need not require differentiability with respect to x in 3.2).

THEOREM 3.2. *Suppose that* $\varepsilon \in (0,T)$, $\alpha \in (0,1)$, $v \in C(\overline{W}_{T,2}) \cap C^2(W_{T,2})$, $|\Phi(0,0,0,t,x)| \leqslant \kappa_1^{-1}$ *in* $W_{T,2}$, *and* (3.4) *holds in* $W_{T,2}$. *Then for* $(t_1,x_1), (t_2,x_2) \in \Gamma_{T,1}$

$$|v(t_1,x_1) - v(t_2,x_2)| \leqslant N\big(|t_1-t_2|^{1/2} + |x_1-x_2|\big)^{\alpha_1}$$

in the following two cases:

a) $t_1, t_2 \geqslant \varepsilon$, $\alpha_1 = \alpha_1(\kappa, d) \in (0,1)$ *and* $N = N(\kappa, \kappa_1, d, \varepsilon)$;

b) *no restrictions on* t_1 *and* t_2, $\alpha_1 = \alpha_1(\kappa, d, \alpha) \in (0,1)$, *and* N *depends only on* $\kappa, \kappa_j, d, \alpha$ *and the norm of* $v(0,x)$ *in* $C^\alpha(W_{0,2})$.

These assertions follow easily from Theorems 2.2 and 2.3 and the fact that (3.4) can be written as a linear equation if Hadamard's formula (or Lagrange's theorem) is used to transform the first term in the sum

$$[\Phi - \Phi(0,0,0,t,x)] + \Phi(0,0,0,t,x) = \Phi.$$

§4. Estimates on the boundary in C^α for the second derivatives of a solution of a nonlinear equation

Suppose that $d \geqslant 2$, $T \in (0,\infty)$ and $\kappa, \kappa_1 \in (0,1]$. Let

$$V_{T,R} = \{(t,x): t \in (0,T), x \in E_d, 0 \leqslant x^1 < 4R^2, |x^i| < R, i \geqslant 2\},$$
$$V_{0,R} = \{x \in E_d: 0 \leqslant x^1 < 4R^2, |x^i| < R, i \geqslant 2\},$$
$$\sum_{T,R} = \{(t,x) \in V_{T,R}: x^1 = 0\}.$$

For $(t,x) \in V_{T,2}$ and for real u_{ij}, u_i $(i,j = 1,\dots,d)$ and u assume that the function $F(u_{ij}, u_i, u, t, x)$ is defined, once continuously differentiable with respect to (u_{ij}, u_i, u, x) for every $t \in (0,T)$, and such that for all $(t,x) \in V_{T,2}$, $\lambda \in E_d$, $k = 2,\dots,d$, any symmetric matrix (u_{ij}), and any u_i and u

4.1) $\kappa|\lambda|^2 \leqslant F_{u_{ij}}\lambda^i\lambda^j \leqslant \kappa^{-1}|\lambda|^2$,

4.2) $|F_{x^k}| \leqslant \kappa^{-1}(|u| + \sum_i |u_i| + \sum_{i,j}|u_{ij}|) + \kappa_1^{-1}$,

4.3) $|F_{u_i}| \leqslant \kappa^{-1}$, $i = 1,\dots,d$, $|F_u| \leqslant \kappa_1^{-1}$.

THEOREM 4.1. *Suppose that* $\varepsilon \in (0,T)$, $\alpha \in (0,1)$, u, u_x, $u_{xx} \in C(\overline{V}_{T,2}) \cap C^2(V_{T,2})$, $u_{x^i} \in C^2(\Sigma_{T,2})$ *for* $i = 2,\dots,d$, *and*

$$\begin{gathered} |u|, |u_{x^1}|, |u_{x^1x^i}| \leqslant \kappa_1^{-1} \quad \text{in } V_{T,2} \quad \text{for } i = 1,\dots,d, \\ \|u_{x^i}\|_{C^2(\Sigma_{T,2})} \leqslant \kappa_1^{-1} \quad \text{for } i = 2,\dots,d, \qquad u_t = F(u_{x^ix^j}, u_{x^i}, u, t, x) \quad \text{in } V_{T,2}. \end{gathered} \tag{4.1}$$

Then for $i = 2,\dots,d$ *and* $(t_1,x_1), (t_2,x_2) \in \Sigma_{T,1}$

$$|u_{x^1x^i}(t_1,x_1) - u_{x^1x^i}(t_2,x_2)| \leqslant N\left(|t_1 - t_2|^{1/2} + |x_1 - x_2|\right)^{\alpha_1}$$

in the following two cases:

a) $t_1, t_2 \geqslant \varepsilon$, $\alpha_1 = \alpha_1(\kappa, d) \in (0,1)$ *and* $N = N(\kappa, \kappa_1, d, \varepsilon)$;

b) *no restrictions on* t_1 *and* t_2, $\alpha_1 = \alpha_1(\kappa, d, \alpha) \in (0,1)$, *and* N *depends only on* $\kappa, \kappa_1, d, \alpha$ *and the norms of* $u_{x^1x^i}(0,x)$ *in* $C^\alpha(V_{0,2})$ *for* $i = 2,\dots,d$.

To prove this theorem we need some auxiliary construction and statements. The proof of the theorem is concluded by the proof of Lemma 4.2.

First of all, let $\varphi(t,x) = u(t,0,x^2,\dots,x^d)$ and observe that if u and F are replaced by $u - \varphi$ and $F(u_{ij} + \varphi_{x^ix^j}, u_i + \varphi_{x^i}, u + \varphi, t, x) - \varphi_t$, then it is easy to reduce the general case to the case $\varphi \equiv 0$. Therefore, it is assumed below that

$$u = 0 \quad \text{on } \Sigma_{T,2}. \tag{4.2}$$

Next, we agree to denote indices with values $d+1,\dots,d+4$ by Green letters, and repeated Greek indices are to be summed from $d+1$ to $d+4$. Recall that, unless a statement to the contrary is made, Latin indices take the values from 1 to d. We need the auxiliary space E_{d+3}, with its points written in a convenient coordinate form as $(x^2,\dots,x^{d+4})$. Let

$$W_{T,R} = \{(t,x): t \in (0,T), x \in E_{d+3}, |x^i| < R, i = 2,\dots,d+4\},$$
$$\Gamma_{T,R} = \{(t,x) \in W_{T,R}: x^\nu = 0 \text{ for } \nu > d\}.$$

For $x \in E_{d+3}$ let $[x] = (x^\nu x^\nu)^{1/2}$, and define the function v by

$$\begin{gathered} v(t,x^2,\dots,x^{d+4}) = [x]^{-2}u(t,[x]^2,x^2,\dots,x^d) \quad \text{for } [x] \neq 0, \\ v(t,x^2,\dots,x^d,0) = u_{x^1}(t,0,x^2,\dots,x^d). \end{gathered} \tag{4.3}$$

Clearly, these formulas define v in $\overline{W}_{T,2}$. Moreover, by (4.2),

$$v(t,x) = \int_0^1 u_{x^1}(t, y[x]^2, x^2,\dots,x^d)\,dy, \qquad (t,x) \in \overline{W}_{T,2}. \tag{4.4}$$

It follows from this and the assumptions of the theorem that $v, v_{x^i} \in C(\overline{W}_{T,2}) \cap C^2(W_{T,2})$ for $i = 2,\dots,d+4$, and $|v|, |v_{x^i}| \leqslant 4\kappa_1^{-1}$ for $i = 2,\dots,d+4$. Next, an uncomplicated computation with the use of (4.3) shows that for $[x] \neq 0$

$$v_{x^\nu x^\mu} = \left(\mu - [x]^2 u_{x^1}\right)\frac{2}{[x]^4}\left(\frac{4x^\nu x^\mu}{[x]^2} - \delta^{\mu\nu}\right) + 4\frac{x^\nu x^\mu}{[x]^2}u_{x^1x^1}.$$

By Taylor's formula and the assumptions of the theorem, this shows that in $W_{T,2}$ (for $[x] \neq 0$, and, by continuity, also for $[x] = 0$) we have $|v_{x^\nu x^\mu}| \leqslant 14\kappa_1^{-1}$ for $\nu, \mu > d$.

We now introduce the function Φ for $(t,x) \in W_{T,2}$ and for real u_{ij}, u_i, $i,j = 2,\dots,d+4$, and u by the formula

$$\begin{aligned} \Phi = F\Big(&\delta^{i1}u_j + \delta^{j1}u_i + [x]^2u_{ij} + \tfrac{1}{2}x^\nu\delta^{i1}u_{\nu j} + \tfrac{1}{2}x^\nu\delta^{j1}u_{i\nu} \\ &+ \tfrac{1}{4}\delta^{i1}\delta^{j1}\delta^{\mu\nu}u_{\mu\nu}, \delta^{i1}(u + \tfrac{1}{2}u_\nu x^\nu) + [x]^2u_i, [x]^2u, t, [x]^2, x^2,\dots,x^d\Big), \end{aligned}$$

where $u_1 = u_{1i} = u_{i1} = 0$ for $i = 1,\dots,d+4$ on the right-hand side.

LEMMA 4.1. a) $u_{x^1x^i}(t,0,x^2,\dots,x^d) = v_{x^i}(t,x)$ *for* $(t,x) \in \Gamma_{T,2}$, $i = 2,\dots,d$.

b) *On* $W_{T,2}$

$$[x]^2v_t = \Phi(v_{x^ix^j}, i,j = 2,\dots,d+4, v_{x^i}, i = 2,\dots,d+4, v, t, x). \tag{4.5}$$

c) *On* $W_{T,2}$

$$|v|, |v_{x^i}| \leqslant 4\kappa_1^{-1} \quad \text{for } i = 2,\dots,d+4, \qquad |v_{x^\nu x^\mu}| \leqslant 14\kappa_1^{-1} \quad \text{for } \nu, \mu > d.$$

Assertion a) follows at once from (4.4), b) can be proved by elementary computations, and c) was proved above.

By Theorem 3.1, this lemma reduces the proof of Theorem 4.1 to a check that Φ satisfies conditions 3.1)–3.3). These conditions are verified in the next lemma.

LEMMA 4.2. *Suppose that $u_{ij} = u_{ji}$ for $i, j = 2, \dots, d+4$, $(t, x) \in W_{T,2}$, $i = 2, \dots, d+4$, $k = 2, \dots, d$ and $\lambda = (\lambda^2, \dots, \lambda^{d+4}) \in E_{d+3}$. Then for $N = N(\kappa, d)$*

$$\kappa^{-1}\left([x]^2 \sum_{i=2}^{d} (\lambda^i)^2 + \frac{1}{4}[\lambda]^2\right) \geqslant \sum_{i,j=2}^{d+4} \Phi_{u_{ij}} \lambda^i \lambda^j \geqslant \kappa\left([x]^2 \sum_{i=2}^{d} (\lambda^i)^2 + \frac{1}{4}[\lambda]^2\right), \quad (4.6)$$

$$|\Phi_{x^k}| \leqslant N\left(|u| + \sum_{i=2}^{d+4} |u_i| + \left(\sum_{i,j,r=2}^{d+4} \Phi_{u_{ij}} u_{ir} u_{jr}\right)^{1/2}\right) + \kappa_1^{-1}, \quad (4.7)$$

$$|\Phi_{u_i}| \leqslant N, \qquad |\Phi_u| \leqslant \kappa^{-1} + 16\kappa_1^{-1}. \quad (4.8)$$

PROOF. The inequalities (4.6) follow from the fact that the middle member in them is equal to

$$F_{u_{ij}}\left(x^\nu \lambda^i + \tfrac{1}{2}\delta^{i1}\lambda^\nu\right)\left(x^\nu \lambda^j + \tfrac{1}{2}\delta^{j1}\lambda^\nu\right),$$

where $\lambda^1 = 0$ and $(x^\nu \lambda^i + \frac{1}{2}\delta^{i1}\lambda^\nu)(x^\nu \lambda^i + \frac{1}{2}\delta^{i1}\lambda^\nu)$ is the coefficient of κ^{-1} and κ in (4.6). To prove (4.7) note that, setting $u_{1i} = u_{i1} = 0$ $(i = 1, \dots, d+4)$ for convenience, we have

$$\begin{aligned}\sum_{i,j,n=2}^{d+4} \Phi_{u_{ij}} u_{ir} u_{jr} &= \sum_{r=2}^{d+4} F_{u_{ij}}\left(x^\nu u_{ir} + \frac{1}{2}\delta^{i1} u_{\nu r}\right)\left(x^\nu u_{jr} + \frac{1}{2}\delta^{j1} u_{\nu r}\right) \\ &\geqslant \kappa \sum_{r=2}^{d+4} \sum_{i=1}^{d} \sum_{\nu=d+1}^{d+4} \left(x^\nu u_{ir} + \frac{1}{2}\delta^{i1} u_{\nu r}\right)^2 \\ &\geqslant \kappa \sum_{i,j=1}^{d} \sum_{\nu=d+1}^{d+4} \left(x^\nu u_{ij} + \frac{1}{2}\delta^{i1} u_{\nu j}\right)^2 \\ &\quad + \kappa \sum_{i=1}^{d} \sum_{\nu=d+1}^{d+4} \left(x^\nu u_{i\nu} + \frac{1}{2}\delta^{i1} u_{\nu\nu}\right)^2.\end{aligned}$$

Moreover,

$$\begin{aligned}[x]^2 u_{ij} + \tfrac{1}{2}x^\nu \delta^{i1} v_{\nu j} + \tfrac{1}{2}x^\nu \delta^{j1} u_{i\nu} + \tfrac{1}{4}\delta^{i1}\delta^{j1} u_{\nu\nu} &= x^\nu\left(x^\nu u_{ij} + \tfrac{1}{2}\delta^{i1} u_{\nu j}\right) \\ &\quad + \tfrac{1}{2}\delta^{j1}\left(x^\nu u_{i\nu} + \frac{1}{2}\delta^{i1} u_{\nu\nu}\right).\end{aligned}$$

This and the assumption 4.2 give us (4.7). The verification of (4.8) does not present any difficulties. The lemma is thereby proved, and, as explained above, so is Theorem 4.1.

The next result is obtained from Theorem 3.2 in a similar way by passing from the function u_{x^1} to v and from equation (4.1) to (4.5).

THEOREM 4.2. *Suppose that $\varepsilon \in (0, T)$, $\alpha \in (0, 1)$, u, $u_{x^1} \in C(\overline{V}_{T,2}) \cap C^2(V_{T,2})$, $|u_{x^1}|$ $|F(0,0,0,t,x)| \leqslant \kappa_1^{-1}$ in $V_{T,2}$, and*

$$\|u\|_{C^2(\Sigma_{T,2})} \leqslant \kappa_1^{-1}.$$

Then for $(t_1, x_1), (t_2, x_2) \in \Sigma_{T,1}$

$$|u_{x^1}(t_1, x_1) - u_{x^1}(t_2, x_2)| \leqslant N\left(|t_1 - t_2|^{1/2} + |x_1 - x_2|\right)^{\alpha_1}$$

in the following two cases:

a) *$t_1, t_2 \geqslant \varepsilon$, $\alpha_1 = \alpha_1(\kappa, d) \in (0,1)$ and $N = N(\kappa, \kappa_1, d, \varepsilon)$;*

b) *no restrictions on t_1 and t_2, $\alpha_1 = \alpha_1(\kappa, d, \alpha) \in (0, 1)$, and N depends only on κ, κ_1, d, α and the norm of $u_{x^1}(0, x)$ in $C^\alpha(V_{0,2})$.*

For the validity of this theorem it is not necessary to impose condition (4.2) (cf. the remark before Theorem 3.2), and we need not require that F be differentiable with respect to x.

We shall use the following conditions several times below (this will be mentioned explicitly each time):

4.4) F is once continuously differentiable with respect to (u_{ij}, u_i, u, t, x), and for $(t, x) \in V_{T,2}$, any symmetric matrix (u_{ij}) and any u_i and u

$$|F_t| \leqslant \kappa^{-1}\left(|u| + \sum_i |u_i| + \sum_{i,j} |u_{ij}|\right) + \kappa_1^{-1}.$$

4.5) $|F_\beta| \leqslant \kappa_1^{-1}$ for all $(t, x) \in V_{T,2}$, any symmetric matrix (u_{ij}), and any u_i and u, where $F_\beta \equiv F - (F_{u_{ij}} u_{ij} + F_{u_i} u_i + F_u u)$.

THEOREM 4.3. *Suppose that in addition to the assumptions of Theorem 4.1 condition 4.4) holds, $|F(0,0,0,t,x)| \leqslant \kappa_1^{-1}$ in $V_{T,2}$, $u \in C^{2+\alpha}(\Sigma_{T,2})$, and*

$$\|u\|_{C^{2+\alpha}(\Sigma_{T,2})} \leqslant \kappa_1^{-1}. \quad (4.9)$$

Then for $i, j = 1, \dots, d$ and $(t_1, x_1), (t_2, x_2) \in \Sigma_{T,1}$

$$|u_{x^i x^j}(t_1, x_1) - u_{x^i x^j}(t_2, x_2)| \leqslant N\left(|t_1 - t_2|^{1/2} + |x_1 - x_2|\right)^{\alpha_1} \quad (4.10)$$

in the following two cases:

a) *$t_1, t_2 \geqslant \varepsilon$, $\alpha_1 = \alpha_1(\kappa, d, \alpha) \in (0,1)$ and $N = N(\kappa, \kappa_1, d, \varepsilon, \alpha)$;*

b) *no restrictions on t_1 and t_2, $\alpha_1 = \alpha_1(\kappa, d, \alpha) \in (0, 1)$, and N depends only on κ, κ_1, d, α, and the norms of $u_{x^i}(0, x)$ and $u_{x^1 x^i}(0, x)$ in $C^\alpha(V_{0,2})$ for $i = 2, \dots, d$.*

PROOF. It follows from Theorem 4.1 that only the two cases $i = j = 1$ and $i, j \geqslant 2$ need be considered. In the second case (4.10) is a consequence of (4.9). Let us analyze the first case.

By conditions 4.1)–4.4) the function F satisfies a Lipschitz condition and $F_{u_{11}} \geqslant \kappa$ on every set of the form

$$\left\{(u_{ij}, u_i, u, t, x) \colon u_{ij} = u_{ji},\ u_{ij}u_{ij} + u_i u_i + u^2 \leqslant N,\ (t, x) \in V_{T,2}\right\}.$$

Therefore, on such a set u_{11} can be expressed in terms of u_{ij} with $i \cdot j \geqslant 2$, u_i with $i \geqslant 1$, u, t, x and F by means of a function satisfying a Lipschitz condition. Hence, (4.10) with $i = j = 1$ is a consequence of (4.10) with $i \cdot j \geqslant 2$, Theorem 4.2, and (4.1). The theorem is proved.

Theorems 4.2 and 4.3 reduce the problem of estimating a Hölder constant for the functions u_x and u_{xx} on the boundary to that of estimating u_x and u_{xx} in the norm of C interior to the domain. We deduce the latter estimates from estimates of u_x and u_{xx} in the norm of C on the boundary of the domain. To estimate the maximum moduli of u_x and u_{xx} on the boundary we need

LEMMA 4.3. *Suppose that assumptions 4.1)–4.3) and 4.5) hold, $\varepsilon \in (0, T)$, and $u_x \in C^2(V_{T,2}) \cap C(\overline{V}_{T,2})$ and is a solution of (4.1).*

a) *If $u_x \in C^2(V_{T,2}) \cap C(\overline{V}_{T,2})$ and $\|u\|_{C(V_{T,2})}$, $\|u\|_{C^2(\Sigma_{T,2})} \leqslant \kappa_1^{-1}$, then*

$$\|u_x\|_{C(V_{T,1} \setminus V_{\varepsilon,1})} \leqslant N(\kappa, \kappa_1, d, \varepsilon), \qquad \|u_x\|_{C(V_{T,1})} \leqslant N, \quad (4.11)$$

where N depends only on κ, κ_1, d and the norm of u_x in $C(V_{0,2})$.

b) *If 4.4) holds along with the assumptions in* a), *and $u_t \in C(\bar V_{T,2}) \cap C^2(V_{T,2})$, then*

$$\|u_t\|_{C(V_{T,1}\setminus V_{\varepsilon,1})} \leqslant N(\kappa, \kappa_1, d, \varepsilon), \qquad \|u_t\|_{C(V_{T,1})} \leqslant N, \tag{4.12}$$

where N depends only on κ, κ_1, d and the norm of u_t in $C(V_{0,2})$.

c) *If $u_x \in C(\bar V_{T,2}) \cap C^2(V_{T,2})$, $u_{xx} \in C(\bar V_{T,2})$, and the norms of u, u_t and u_x in $C(V_{T,2})$ and of u_{x^i} in $C^2(\Sigma_{T,2})$ do not exceed κ_1^{-1} for $i = 2,\dots, d$, then for $i, j \geqslant 1$*

$$\|u_{x^ix^j}\|_{C(\Sigma_{T,1}\setminus\Sigma_{\varepsilon,1})} \leqslant N(\kappa, \kappa_1, d, \varepsilon), \qquad \|u_{x^ix^j}\|_{C(\Sigma_{T,1})} \leqslant N, \tag{4.13}$$

where N depends only on κ, κ_1, d and the norms of $u_{x^1x^i}$ in $C(V_{0,2})$ for $i \geqslant 2$.

PROOF. a) By a method of Bernstein (see, for example, the proofs of Lemma 4.1 and Theorem 4.4 in [1]) used with functions of the form $\zeta^2 u_{x^i}u_{x^i} + \alpha u^2$ it is easy to show that to prove the estimates (4.11) it suffices to establish that they are true with the indicated dependence on the constants if V is replaced by Σ in them. Since we have estimates of u_{x^i} for $i \geqslant 2$ by assumption, it remains to estimate u_{x^1}. To do this we set $\varphi(t, x) = u(t, 0, x^2,\dots, x^d)$, $v = u - \varphi$ and note that $v = 0$ on $\Sigma_{T,2}$, and

$$v_t = F_{u_{ij}} v_{x^ix^j} + F_{u_i} v_{x^i} + F_u v + \left(F_\beta + F_{u_{ij}}\varphi_{x^ix^j} + F_{u_i}\varphi_{x^i} + F_u\varphi - \varphi_t\right)$$

in $V_{T,2}$. Since the expression in parentheses is bounded in $V_{T,2}$ by assumption and $|v(0, x)| \leqslant x^1\|u_x\|_{C(V_{0,2})}$, the needed estimates of $u_{x^1} = v_{x^1}$ on Σ can be obtained with the help of elementary barriers.

b) The estimates (4.12) can also be proved by Bernstein's method with the help of a). Here a function of the form $\zeta^2 u_t + \alpha u_{x^i}u_{x^i}$ is used (see, for example, the proofs of Theorems 4.3 and 4.4 in [1]).

c) Differentiating (4.1) with respect to x^k, $k = 2,\dots, d$, we see that the function

$$v(t, x^1,\dots, x^d, x^{d+2},\dots, x^{2d}) = \sum_{k=2}^d x^{d+k}u_{x^k}(t, x^1,\dots, x^d)$$

satisfies the equation

$$v_t = F_{u_{ij}} v_{x^ix^j} + F_{u_i} v_{x^i} + F_u v + \sum_{k=2}^d x^{d+k}F_{x^k} \tag{4.14}$$

on the set $\tilde V_{T,2} = V_{T,2} \times (|x^{d+k}| \leqslant 2, k = 2,\dots, d)$. Moreover, by 4.2), the sum of the last two terms has the form

$$f_1 + \sigma_1^{ij}u_{x^ix^j} = f_1 + \sigma_1^{11}u_{x^1x^1} + \sum_{i=2,\, j=1}^d \sigma_1^{ij}v_{x^{d+i}x^j},$$

where $|f_1|, |\sigma_1^{ij}| \leqslant N(\kappa, \kappa_1, d)$ in $\tilde V_{T,2}$. From the equality $u_t = F_{u_{ij}}u_{x^ix^j} + F_{u_i}u_{x^i} + F_u u + F_\beta$ we next express $u_{x^1x^1}$ as

$$u_{x^1x^1} = f_2 + \sum_{i=2,\, j=1}^d \sigma_2^{ij}v_{x^{d+i}x^j}. \tag{4.15}$$

Then (4.14) can be given the form

$$v_t = F_{u_{ij}} v_{x^ix^j} + \sum_{i=2,\, j=1}^d \sigma^{ij}v_{x^{d+i}x^j} + N_1\sum_{k=2}^d v_{x^{d+k}x^{d+k}} + F_{u_i}v_{x^i} + f, \tag{4.16}$$

where N_1 is any constant, and $|f|, |\sigma^{ij}| \leqslant N(\kappa, d)$. We now choose N_1 large enough so that (4.16), regarded as an equation in v, is strictly parabolic in $\tilde V_{T,2}$. After this we set

$$\psi(t, x^1,\dots, x^d, x^{d+2},\dots, x^{2d}) = \sum_{k=2}^d x^{d+k}u_{x^k}(t, 0, x^2,\dots, x^d),$$

and $w = v - \psi$. Then $w = 0$ for $x^1 = 0$, and w satisfies an equation analogous to (4.16). Moreover, $|w(0, x)| \leqslant x^1N$ in $\tilde V_{0,2}$, where N depends only on d and the norms of $u_{x^1x^i}$ in $C(V_{0,2})$ for $i \geqslant 2$. The use of simple barriers shows that $|w(t, x)| \leqslant x^1N$ in $\tilde V_{T,1} \setminus \tilde V_{\varepsilon,1}$ and in $\tilde V_{T,1}$, where the constant N is different for each of these sets but depends on the initial data only as indicated in c). The inequality (4.13) for $j = 1$ and $i \geqslant 2$ follows from this. The estimates (4.13) for $i, j \geqslant 2$ are given, and the estimate of $u_{x^1x^1}$ follows from (4.15). This proves the lemma.

The next theorem is weaker than Lemma 4.3 but, as a rule, is more convenient to use. It can be proved by a simple combination of the statements in Lemma 4.3.

THEOREM 4.4. *Suppose that assumptions 4.1)–4.5) hold, $u, u_x, u_t \in C(\bar V_{T,2}) \cap C^2(V_{T,2})$, $u_{xx} \in C(\bar V_{T,2})$, (4.1) is satisfied and*

$$\|u_{x^i}\|_{C^2(\Sigma_{T,2})}, \|u\|_{C^2(\Sigma_{T,2})}, \leqslant \kappa_1^{-1} \quad \text{for } i \geqslant 2.$$

Then for any $\varepsilon \in (0, T)$ and for $i, j \geqslant 1$

$$\|u_{x^i}\|_{C(\Sigma_{T,1}\setminus\Sigma_{\varepsilon,1})}, \|u_{x^ix^j}\|_{C(\Sigma_{T,1}\setminus\Sigma_{\varepsilon,1})} \leqslant N(\kappa, \kappa_1, d, \varepsilon).$$

The same estimates hold also when $\varepsilon = 0$ if N is permitted to depend on κ, κ_1, d and the norms of $u_x(0, x)$ and $u_{xx}(0, x)$ in $C(V_{0,2})$.

§5. Three lemmas

In this section we prove some results for linear equations. We need them in §6 in order to "paste" the interior estimates of u in $C^{2+\alpha}$ in [1] together with the estimates in §4 and get estimates of u in $C^{2+\alpha}$ in the closed domain. Everywhere in this section $d \geqslant 1$, $T \in (0, \infty)$ and $\kappa, \kappa_1 \in (0, 1]$.

LEMMA 5.1. *Take h, $a > 0$ and let $M_{h,a} = (0, h) \times \{x \in E_d \colon 0 < x^1 < 2a,\ |x^i| < a,\ i \geqslant 2\}$. Suppose that $w \in C(\bar M_{h,a}) \cap C^2(M_{h,a})$ and that $w \leqslant 1$ and*

$$Lw \equiv a^{ij}w_{x^ix^j} + b^iw_{x^i} - w_t \geqslant -\kappa_1^{-1} \tag{5.1}$$

in $M_{h,a}$, where the functions a^{ij} and b^i are such that $\kappa|\lambda|^2 \leqslant a^{ij}\lambda^i\lambda^j \leqslant \kappa^{-1}|\lambda|^2$ and $|b^i| \leqslant \kappa^{-1}$ in $M_{h,a}$ for all $\lambda \in E_d$ and i. There exist constants $\delta = \delta(\kappa, d) > 0$, $a_0 = a_0(\kappa, d) \in (0, 1)$, $h_0 = h_0(\kappa, d) > 0$ and $N = N(\kappa, \kappa_1, d) > 0$ such that the following hold for $a \leqslant a_0$ and $h \leqslant h_0$:

a) *If $w \leqslant 0$ for $x^1 = 0$, then*

$$w(h, x^1, 0,\dots, 0) \leqslant I_1(h, a, x^1) \equiv N\left[1 + h^{-1}x^1(2 - x^1) - |a^{-1}x^1 + 1|^{-\delta}\right]$$

for $0 \leqslant x^1 \leqslant 2a$.

b) *If $w \leqslant 0$ for $t = 0$, then*

$$w(h, x^1, 0,\dots, 0) \leqslant I_2(h, a, x^1) \equiv a^{-2}\left[(x^1 - a)^2 + Nh\right]$$

for $0 \leqslant x^1 \leqslant 2a$.

c) *If $w \leqslant 0$ for $x^1 = 0$, and $w \leqslant 0$ for $t = 0$, then*

$$w(h, x^1, 0, \dots, 0) \leqslant I_3(h, a, x^1) \equiv N(hx^1)^{1/2} a^{-3/2}$$

for $0 \leqslant x^1 \leqslant a$.

PROOF. a) Let

$$x_0 = (-a, 0, \dots, 0), \qquad u_1(t, x) = 2\kappa^{-1} t^{-1} x^1 (2a - x^1),$$

$$u_2(t, x) = \left[a^{-\delta} - |x - x_0|^{-\delta}\right] a^{\delta} (1 - 2^{-\delta/2})^{-1}$$

and choose a_0 small enough that $-2a^{11} + 2b^1(a - x^1) \leqslant -\kappa$ in $M_{h,a}$ for $x^1 \leqslant a \leqslant a_0$. Then $Lu_1 \leqslant 0$ in $M_{h,a} \cap \{u_1 \leqslant 2\}$. Moreover, it is not hard to choose a large $\delta = \delta(\kappa, d) > 0$ such that $Lu_2 \leqslant 0$ in $M_{h,a}$ (and in $M_{h,a} \cap \{u_1 \leqslant 2\}$). Then, by the maximum principle,

$$w \leqslant u_1 + u_2 + (\kappa\kappa_1)^{-1} x^1 (2a - x^1).$$

Substitution of $(t, x) = (h, x^1, 0, \dots, 0)$ proves a).

To prove b) it is necessary to introduce the function

$$a^{-2}\left[|x + x_0|^2 + (4d\kappa^{-1} + \kappa_1^{-1})t\right]$$

and carry out analogous arguments.

To prove c) we take $b \in [0, a]$ and $\gamma > 0$, and let

$$x_b = (b, 0, \dots, 0), \qquad u_3(t, x) = a^{-2}\left[|x - x_b|^2 + tN\right]\gamma + \gamma^{-1} u_2(t, x),$$

where $N = N(\kappa, d)$ is chosen so that $Lu_3 \leqslant 0$ in $M_{h,a}$. Then, by the maximum principle,

$$w \leqslant u_3 + \kappa_1^{-1} t\gamma + \gamma^{-1}(\kappa\kappa_1)^{-1} x^1 (2a - x^1)$$

in $M_{h,a}$. Here we substitute $(t, x) = (h, x^1, 0, \dots, 0)$ and $b = x^1$. Afterwards, we take the infimum over $\gamma > 0$. It is then easy to obtain c), and the lemma is proved.

LEMMA 5.2. *Suppose that $\alpha \in (0, 1)$, $v \in C(\overline{V}_{T,2}) \cap C^2(V_{T,2} \setminus \Sigma_{T,2})$ and v satisfies (5.1) in $V_{T,2} \setminus \Sigma_{T,2}$, with a^{ij} and b^i such that the conditions of Lemma 5.1 hold in $V_{T,2}$. Then in $V_{T,1}$*

$$v(t, x) - v(t, 0, x^2, \dots, x^d) \leqslant N(x^1)^{\alpha/2}\left(\|v\|_{C^\alpha(\Sigma_{T,2})} + (x^1/t \vee 1)\nu\right), \tag{5.2}$$

$$v(t, x) - v(0, x) \leqslant Nt^{\alpha/3}\left(\|v\|_{C^\alpha(V_{0,2})} + (t^{2/3}/(x^1)^2 \vee 1)\nu\right), \tag{5.3}$$

$$v(t, x) - v(0, 0, x^2, \dots, x^d) \leqslant N\left((x^1)^{\alpha/3} + t^{\alpha/2}\right)\left(\|v\|_{C^\alpha(\Sigma_{T,2} \cup V_{0,2})} + \nu\right), \tag{5.4}$$

where $\nu = \|v\|_{C(V_{T,2})} + 1$ and $N = N(\kappa, \kappa_1, d)$.

PROOF. We first prove (5.2). Let us fix $(t_0, x_0) \in V_{T,1}$, $h \leqslant h_0 \wedge t_0$ and apply assertion a) of Lemma 5.1 to the function

$$w = \tfrac{1}{2}\nu^{-1}\left[v(t, x) - v(t_0, 0, x_0^2, \dots, x_0^d) - (a + h^{1/2})^\alpha \|v\|_{C^\alpha(\Sigma_{T,2})}\right]$$

in $M_{h,a} + (t_0 - h, 0, x_0^2, \dots, x_0^d)$. Then for $x_0^1 \leqslant 2a \leqslant 2a_0$ we get

$$v(t_0, x_0) - v(t_0, 0, x_0^2, \dots, x_0^d) \leqslant (a + h^{1/2})^\alpha \|v\|_{C^\alpha(\Sigma_{T,2})} + 2\nu I_1(h, a, x_0^1). \tag{5.5}$$

Consider the following cases: 1) $x_0^1 \geqslant a_0^2$, 2) $x_0^1 \leqslant a_0^2$, $x_0 \leqslant t_0 \wedge h_0$, 3) $x_0^1 \leqslant a_0^2$, $x_0^1 \geqslant t_0 \wedge h_0$. In the first case (5.2) is obvious. In the second case we take $a = (x_0^1)^{1/2}$, $h = x_0^1$ in (5.5), and in the third case we take $a = (x_0^1)^{1/2}$, $h = t_0 \wedge h_0$. Then we get (5.2) at the point (t_0, x_0) without difficulty. Inequalities (5.3) and (5.4) are established similarly, and the lemma is proved.

LEMMA 5.3. *Suppose that $v \in C(\overline{V}_{T,2}) \cap C^2(V_{T,2} \setminus \Sigma_{T,2})$ and $Lv + cv = f$ in $V_{T,2} \setminus \Sigma_{T,2}$, where L is the operator in (5.1), with the coefficients satisfying the conditions of Lemma 5.1 in $V_{T,2}$, and the functions c and f are such that $|c|, |f| \leqslant \kappa_1^{-1}$. Then there exists an $\alpha_0(\kappa, d) > 0$ such that of $\varepsilon > 0$ and $\alpha \in (0, \alpha_0]$ the following assertions are true:*

a) *The Hölder constant of order $\alpha/4$ for the function v in $V_{T,1} \setminus V_{\varepsilon,1}$ does not exceed*

$$N\left(\|v\|_{C^\alpha(\Sigma_{T,2})} + (\varepsilon^{-1} \vee 1)\left(\|v\|_{C(V_{T,2})} + 1\right)\right).$$

b) *The Hölder constant of order $\alpha/3$ for v in $V_{T,1} \setminus \{(t, x): x^1 \leqslant \varepsilon\}$ does not exceed*

$$N\left(\|v\|_{C^\alpha(V_{0,2})} + (\varepsilon^{-2} \vee 1)\left(\|v\|_{C(V_{T,2})} + 1\right)\right).$$

c) *The Hölder constant of order $\alpha/9$ for v in $V_{T,1}$ does not exceed*

$$N\left(\|v\|_{C^\alpha(\Sigma_{T,2} \cup V_{0,2})} + \|v\|_{C(V_{T,2})} + 1\right),$$

where N depends only on κ, κ_1 and d.

PROOF. The symbol N will denote various constants depending only on κ, κ_1 and d; let $\nu = 1 + \|v\|_{C(V_{T,2})}$. By Theorem 4.1 in [10], there exists an $\alpha_0 = \alpha_0(\kappa, d) \in (0, 1)$ such that for any $(t_1, x_1), (t_2, x_2) \in V_{T,1}$ and $\alpha \in (0, \alpha_0]$

$$|v(t_1, x_1) - v(t_2, x_2)| \leqslant \left(t_1^{1/2} \wedge t_2^{1/2} \wedge x_1^1 \wedge x_2^1\right)^{-\alpha} N\nu\left(|t_1 - t_2|^{1/2} + |x_1 - x_2|\right)^\alpha. \tag{5.6}$$

Let us show that assertions a)–c) hold with this α_0. We take $\alpha \in (0, \alpha_0]$, $0 < t_1 \leqslant t_2 \leqslant T$, $t_2 - t_1 \leqslant 1$ and $x_1, x_2 \in V_{0,2}$, $0 < x_1^1 \leqslant x_2^1$, and let

$$\sigma_t = \|v\|_{C^\alpha(\Sigma_{T,2})}, \quad \sigma_x = \|v\|_{C^\alpha(V_{0,2})}, \quad \sigma = \|v\|_{C^\alpha(\Sigma_{T,2} \cup V_{0,2})}.$$

It is not hard to see that assertions a)–c) will be proved if we prove that

$$|v(t_1, x_1) - v(t_1, x_2)| \leqslant N|x_1 - x_2|^{\alpha/4}\left(\sigma_t + (|x_1 - x_2|^{1/2}/t_1 \vee 1)\nu\right), \tag{5.7}$$

$$|v(t_1, x_1) - v(t_1, x_2)| \leqslant N|x_1 - x_2|^{\alpha/3}\left(\sigma_x + \left(|x_1 - x_2|^{2/3}/(x_1^1)^2\right| \vee 1\right)\nu\right), \tag{5.8}$$

$$|v(t_1, x_1) - v(t_2, x_1)| \leqslant N|t_2 - t_1|^{\alpha/8}\left(\sigma_t + (|t_2 - t_1|^{1/4}/t_1 \vee 1)\nu\right), \tag{5.9}$$

$$|v(t_1, x_1) - v(t_2, x_1)| \leqslant N|t_2 - t_1|^{\alpha/6}\left(\sigma_x + \left(|t_2 - t_1|^{1/3}/(x_1^1)^2 \vee 1\right)\nu\right), \tag{5.10}$$

$$|v(t_1, x_1) - v(t_2, x_1)| \leqslant N|t_2 - t_1|^{1/18}(\sigma + \nu), \tag{5.11}$$

$$|v(t_1, x_1) - v(t_1, x_2)| \leqslant N|x_1 - x_2|^{\alpha/9}(\sigma + \nu). \tag{5.12}$$

Observe first of all that (5.7) and (5.8) follow from (5.6) if

$$t_1^{1/2} \wedge x_1^1 \geqslant |x_1 - x_2|^{1/2},$$

and (5.9) and (5.10) follow from (5.6) if

$$t_1^{1/2} \wedge x_1^1 \geqslant |t_2 - t_1|^{1/4}.$$

Moreover, (5.7) and (5.9) follow from (5.6) if

$$t_1^{1/2} \wedge x_1^1 = t_1^{1/2},$$

since, for example,

$$t_1^{-\alpha/2}|x_1 - x_2|^{\alpha} = |x_1 - x_2|^{\alpha/4}\left[\left(|x_1 - x_2|^{3/2}/t_1\right)^{\alpha/2} \vee 1\right]$$
$$\leqslant N|x_1 - x_2|^{\alpha/4}\left[|x_1 - x_2|^{1/2}/t_1 \vee 1\right].$$

Similarly, (5.8) and (5.10) follow from (5.6) if

$$t_1^{1/2} \wedge x_1^1 = x_1^1.$$

Consequently, in the proof of (5.7) it remains to consider the case $x_1^1 \leqslant |x_1 - x_1|^{1/2}$. In this case

$$x_2^1 = \left(x_2^1 - x_1^1\right) + x_1^1 \leqslant 2|x_1 - x_2|^{1/2},$$

and, by (5.2), for $\bar{x}_i = (0, x_i^2, \ldots, x_i^d)$ the left-hand side of (5.7) does not exceed

$$|v(t_1, x_1) - v(t_1, \bar{x}_1)| + |v(t_1, x_2) - v(t_2, \bar{x}_2)| + |x_1 - x_2|^{\alpha}\sigma_i$$
$$\leqslant N|x_1 - x_2|^{\alpha/4}\left(\sigma_t + \left(|x_1 - x_2|^{1/2}/t_1 \vee 1\right)\nu\right) + |x_1 - x_2|^{\alpha}\sigma_t.$$

This proves (5.7). To prove (5.8) it remains to analyze the case $t_1 \leqslant |x_1 - x_2|$. In this case (5.8) follows easily from (5.3). The proofs of (5.9) and (5.10) are similar.

Inequality (5.11) follows from (5.10) if $|t_2 - t_1|^{1/3} \leqslant (x_1^1)^2$, and from (5.9) if $t_1 \geqslant |t_2 - t_1|^{1/4}$. Hence, in proving it we can assume that $(x_1^1)^2 \leqslant |t_2 - t_1|^{1/3}$ and $t_1 \leqslant |t_2 - t_1|^{1/4}$. Then, of course, $t_2 = (t_2 - t_1) + t_1 \leqslant 2|t_2 - t_1|^{1/4}$, and (5.11) is obtained in this case from (5.4). Similarly, (5.12) can be deduced from (5.4), (5.7) and (5.8). This proves the lemma.

§6. An estimate in $C^{2+\alpha}$ of a solution of a nonlinear equation in a closed domain

Suppose that $d \geqslant 1$, $T \in (0, \infty)$, $\kappa, \kappa_1 \in (0, 1]$, $\psi \in C^3_{\text{loc}}(E_d)$, $D = \{x \in E_d: \psi(x) > 0\}$, $\|\psi\|_{C^3(D)} \leqslant \kappa^{-1}$ and $|\psi_x| \geqslant \kappa$ on ∂D. Recall that $\rho(x) = \operatorname{dist}(x, \partial D)$, and for $x \in D$ let $y(x)$ denote one of the points in ∂D such that $|x - y| = \rho(x)$.

LEMMA 6.1 (see [12] or [5]). *There exist $\rho_0 = \rho_0(\kappa, d) > 0$ and $N_0 = N_0(\kappa, d) \geqslant 0$ such that $y(x)$ is uniquely determined in $D \setminus D(\rho_0)$,*

$$\|y\|_{C^2(D\setminus D(\rho_0))} \leqslant N_0 \quad \text{and} \quad \|\rho\|_{C^3(D\setminus D(\rho_0))} \leqslant N_0.$$

It is convenient for us to formulate generalizations of Lemmas 5.2 and 5.3 in terms of the function $y(x)$. These generalizations are easily obtained by means of the method of local rectification of the boundary of D, with the observation that the diameter of the part being rectified can be estimated below in terms only of an upper estimate of $|\psi_{xx}|$ and a lower estimate of $|\psi_x|$, and the observation that in some neighborhood of a point $x_0 \in \partial D$ the function $\rho(x)$ can be estimated above and below by constants multiplied by the distance from x to ∂D, measured along a straight line parallel to the normal to ∂D at x_0.

LEMMA 6.2. *Suppose that $\alpha \in (0, 1)$, $P_R = \{(x^{d+1}, \ldots, x^{2d}): |x^{d+i}| < R,\ i = 1, \ldots, d\}$, $v \in C(\bar{Q} \times \bar{P}_2) \cap C^2(Q \times P_2)$ and*

$$Lv \equiv \sum_{i,j=1}^{2d} a^{ij}v_{x^ix^j} + \sum_{i=1}^{2d} b^i v_{x^i} - v_t \geqslant -\kappa_1^{-1} \tag{6.1}$$

in $Q \times P_2$, where the functions a^{ij} and b^i are such that

$$\kappa|\lambda|^2 \leqslant \sum_{i,j\leqslant 2d} a^{ij}\lambda^i\lambda^j \leqslant \kappa^{-1}|\lambda|^2$$

and $|b^i| \leqslant \kappa^{-1}$ in $Q \times P_2$ for all $\lambda \in E_{2d}$ and i. Then for $y(x) = y(x^1, \ldots, x^d)$ and $\rho(x) = \rho(x^1, \ldots, x^d)$ we have

$$v(t, x) - v(t, y(x)) \leqslant N\rho^{\alpha/2}(x)\left(\|v\|_{C^\alpha(\partial_x Q \times P_2)} + (\rho(x)/t \vee 1)\nu\right),$$
$$v(t, x) - v(0, x) \leqslant Nt^{\alpha/3}\left(\|v\|_{C^\alpha(\partial_t Q \times P_2)} + (t^{2/3}/\rho^2(x) \vee 1)\nu\right),$$
$$v(t, x) - v(0, y(x)) \leqslant N\left(\rho^{\alpha/3}(x) + t^{\alpha/2}\right)\left(\|v\|_{C^\alpha(\partial' Q \times P_2)} + \nu\right)$$

on $Q \times P_1$, where $\nu = \|v\|_{C(Q \times P_2)} + 1$ and $N = N(\kappa, \kappa_1, d)$.

LEMMA 6.3. *Suppose that $v \in C(\bar{Q} \times \bar{P}_2) \cap C^2(Q \times P_2)$ and $Lv + cv = f$ in $Q \times P_2$, where L is the operator in the preceding lemma, and the functions c and f are such that $|c|$, $|f| \leqslant \kappa_1^{-1}$ in $Q \times P_2$. Then there exists an $\alpha_0 = \alpha_0(\kappa, d) \in (0, 1)$ such that the following hold for $\varepsilon > 0$ and $\alpha \in (0, \alpha_0]$:*

a) *The Hölder constant of order $\alpha/4$ of the function v in $Q_\varepsilon \times P_1$ does not exceed*

$$N\left(\|v\|_{C^\alpha(\partial_x Q \times P_2)} + (\varepsilon^{-1} \vee 1)\left(\|v\|_{C(Q \times P_2)} + 1\right)\right).$$

b) *The Hölder constant of order $\alpha/3$ of v in $Q(\varepsilon) \times P_1$ does not exceed*

$$N\left(\|v\|_{C^\alpha(\partial_t Q \times P_2)} + (\varepsilon^{-2} \vee 1)\left(\|v\|_{C(Q \times P_2)} + 1\right)\right).$$

c) *The Hölder constant of order $\alpha/9$ of v in $Q \times P_1$ does not exceed*

$$N\left(\|v\|_{C^\alpha(\partial' Q \times P_2)} + \|v\|_{C(Q \times P_2)} + 1\right),$$

where $N = N(\kappa, \kappa_1, d)$.

The notation

$$A_i u = u_{x^i} - \psi_{x^i}\psi_{x^j}u_{x^j}|\psi_x|^{-2}$$

is used in the next theorem.

We note that $A_i u(x)$ is the derivative of u along the projection of the ith basis vector on the plane tangent ot ∂D at a point $x \in \partial D$.

THEOREM 6.1. *Suppose that $F \in \mathfrak{F}(\kappa, Q)$, $\varepsilon \in (0, T)$, $\rho \in (0, \rho_0)$, $\alpha \in (0, 1)$, u, u_t, $u_{xx} \in C(\bar{Q}) \cap C^2(Q \cup \partial_x Q)$, $\|u\|_{C(Q)} \leqslant \kappa_1^{-1}$ and*

$$u_t = F(u_{x^ix^j}, u_{x^i}, u, 1, t, x) \tag{6.2}$$

in Q.

Then there exist $\alpha_0 = \alpha_0(\kappa, d) \in (0, 1)$ and $\alpha_1 = \alpha_1(\kappa, d, \alpha) \in (0, 1)$ such that:

a) $\|u\|_{C^{2+\alpha_0}(Q_\varepsilon(\rho))} \leqslant N(\kappa, \kappa_1, d, \rho, \varepsilon)$;

b) *if* $\|A_i u\|_{C^2(\partial_x Q)}, \|u\|_{C^{2+\alpha}(\partial_x Q)} \leqslant \kappa_1^{-1}$, $i = 1, \ldots, d$, *then*

$$\|u\|_{C^{2+\alpha_1}(Q_\varepsilon)} \leqslant N(\kappa, \kappa_1, d, \varepsilon, \alpha);$$

c) *if* $\|u_t\|_{C^\alpha(\partial_t Q)}, \|u\|_{C^{2+\alpha}(\partial_t Q)} \leqslant \kappa_1^{-1}$, *then*

$$\|u\|_{C^{2+\alpha_1}(Q(\rho))} \leqslant N(\kappa, \kappa_1, d, \rho, \alpha);$$

d) *if* b) *and* c) *hold, then*

$$\|u\|_{C^{2+\alpha_1}(Q)} \leqslant N(\kappa, \kappa_1, d, \alpha).$$

PROOF. In the proofs of a)– d) the symbols N, α_0 and α_1 denote various constants depending on the original data as indicated in a)– d).

We derive a) from the results of [1]. The definitions of the classes $\mathfrak{F}(\kappa, Q)$ here and in [1] are different. Neverthless, the proof of Theorem 4.1 in [1] for the functions in the class $\mathfrak{F}(\kappa, Q)$ introduced here goes through almost without changes (it becomes shorter, because in (4.7) of [1] some of the terms are estimated by virtue of assumption 1.3). The other results in §4 of [1] obviously remain in force. Therefore, $\|u\|_{C^2(Q_\varepsilon(\rho))} \leqslant N_1$ ($= N_1(\kappa, \kappa_1, d, \rho, \varepsilon)$). We next let $v = N_1^{-1}u$ and observe that in Q

$$v_t = F\left(v_{x^ix^j}, v_{x^i}, v, N_1^{-1}, t, x\right).$$

To this equation in $Q_\varepsilon(\rho)$ we apply the results in §2 of [1] with $M = 1$. It should be mentioned that the constant m_1 in condition 2.4 of [1] depends, of course, on N_1 (and hence on ρ, ε and κ_1). In turn, the Hölder exponent for u_t and u_{xx} (which can be obtained immediately from Theorem 2.1 in [1]), depends on the constant M_1. But here we do not want to include a dependence on ρ, ε and κ_1 in the exponent α_0. Therefore, we use the fact that the proof of Theorem 2.1 in [1] was carried out in a much more general setting than necessary for our purposes. We note that, by our assumption 1.3, we can take

$$\sigma_{ij} = \frac{1}{2}\kappa^{-1}|\xi| \operatorname{sgn}\left(\sum_k u_{x^ix^jx^k}\xi^k\right) \tag{6.3}$$

in the definition of the operator L before Lemma 2.4 in [1], and then the constant N_0 in §2 of [1] will not depend on ρ, ε nor κ_1, and the arguments in that section go through without any changes. In this way we obtain $\|v\|_{C^{2+\alpha_0}(Q_{2\varepsilon}(2\rho))} \leqslant N$, and $a)$ is proved.

b) Let $\varepsilon' = \varepsilon/2$ and $\varepsilon'' = \varepsilon/4$. First of all we show that $\|u\|_{C^2(Q_{\varepsilon'})} \leqslant N$. Repeating the arguments in §4 of [1] with a function ζ equal to 1 on $Q_{\varepsilon'}$ and to 0 when $t \leqslant \varepsilon''$, we see that to prove this estimate it suffices to show that $\|u_x, u_{xx}\|_{C(\partial x Q_{\varepsilon''})} \leqslant N$. By our assumptions about ψ, these estimates can easily be derived from Theorem 4.4 by means of local rectification of the boundary. From the estimate of u obtained in the norm of $C^2(Q_{\varepsilon'})$ we derive with the help of the same method and $a)$ that the Hölder constant of order α_1 for the functions u_{x^i} and $u_{x^ix^j}$ $(i, j \geqslant 1)$ on $\partial_x Q_\varepsilon$ does not exceed some constant N.

Differentiating (6.2) with respect to t, we then get that u_t satisfies in $Q \times P_2$ the equation

$$(u_t)_t = F_{u_{ij}}(u_t)_{x^ix^j} + \delta^{ij}(u_t)_{x^{d+i}x^{d+j}} + F(u_t)_{x^i} + F_u u_t + F_t,$$

and F_t is bounded in $Q_{\varepsilon'} \times P_2$. By Lemma 6.3$a$),

$$\|u_t\|_{C^{\alpha_1}(Q_\varepsilon)} \leqslant N.$$

The norm of u_{x^i} in $C^{\alpha_1}(Q_\varepsilon)$ can be estimated similarly.

Let us proceed to an estimate of the norm of u_{xx} in $C^{\alpha_1}(Q_\varepsilon)$. By the estimate $\|u\|_{C^2(Q_{\varepsilon'})} \leqslant N$ and by our remarks concering (6.3), Lemma 2.4 in [1] gives us the existence of an operator L of the form (6.1) whose coefficients satisfy in $Q_{\varepsilon'} \times P_2$ the conditions of Lemma 6.2 with κ replaced there by $\bar\kappa(\kappa, d) \in (0,1)$ and such that $Lv \geqslant 0$ in $Q_{\varepsilon' \times P_2}$, where

$$v(t, x^1, \dots, x^{2d}) = u_{x^ix^j}(t, x^1, \dots, x^d)x^{d+i}x^{d+j} + Nx^{d+i}x^{d+i}.$$

Since the norms of $u_{x^ix^j}$ in $C^{\alpha_1}(\partial x Q_{\varepsilon'})$ have been estimated, Lemma 6.2 tells us that

$$\left[u_{x^ix^j}(t, x) - u_{x^ix^j}(t, y(x))\right]x^{d+i}x^{d+j} \leqslant N\rho^{\alpha_1/2}(x) \tag{6.4}$$

in $Q_\varepsilon \times P_1$. From this it is proved in exactly the same way as in the proof of Theorem 3.1 of [1], with the use of (6.2) and Hölder estimates of u_t and u_{x^i} in Q, that the left-hand side of (6.4) is at most $N\rho^{\alpha_1}(x)$. After this, we have

$$\left|u_{x^ix^j}(t, x) - u_{x^ix^j}(t, y(x))\right| \leqslant N\rho^{\alpha_1}(x)$$

in Q_ε. To finish the proof of b) it remains to "paste together" this estimate in the standard way (as in Lemma 5.3) with the interior estimate in Theorem 2.1 of [1], observing once more that in our situation the Hölder exponent in Theorem 2.1 of [1] depends only on κ and d (see above), while the Hölder norms of $u_{x^ix^j}$ on $\partial_x Q_\varepsilon$ have already been estimated.

Assertions c) and d) can be proved in a completely analogous way. The theorem is proved.

REMARK 6.1. This proof shows that $\alpha_0(\kappa, d)$ and $\alpha_1(\kappa, d, \alpha)$ depend for fixed d, α and D only on estimates of the upper and lower eigenvalues of the matrix $(F_{u_{ij}} + F_{u_{ji}})$, on an estimate of $|F_{u_i}|$, and on an estimate of $|\sigma_{ij}|$ for $|\xi| = 1$, where the σ_{ij} are introduced in (6.3). Therefore, $\alpha_0(\kappa, d)$ and $\alpha_1(\kappa, d, \alpha)$ do not change if κ is replaced by κ_1 in (1.1) and $|F_\beta|, |F_u| \leqslant \kappa^{-1}$ (see (1.3)), and (1.4) is replaced by

$$\kappa_1^{-1}\left(\beta^{-1}\tilde u_i\tilde u_i + \beta^{-1}\tilde u^2 + |\tilde x|^2 + |\tilde x|^2 w\right) + \kappa^{-1}|\tilde x|\sum_{i,j}|\tilde u_{ij}|$$

in the condition connected with (1.4).

THEOREM 6.2. *Let F satisfy all the conditions* 1.1)–1.3) *except the first condition in* (1.1): $|F_t| \leqslant \kappa^{-1}w$. *Suppose that $u, u_t, u_{xx} \in C(\overline{Q}) \cap C^2(Q \cup \partial_x Q)$, with*

$$\|u\|_{C(Q)}, \|u, A_i u\|_{C^2(\partial_x Q)}, \|u_{x^i}, u_{x^ix^j}\|_{C(\partial_t Q)} \leqslant \kappa^{-1}, \qquad \|u_t\|_{C(Q)} \leqslant \kappa_1^{-1}$$

for $i, j = 1, \dots, d$, and u satisfies (6.2) *in Q. Then for i, $j = 1, \dots, d$*

$$\|u_{x^i}\|_{C(Q)} \leqslant N(\kappa, d), \qquad \|u_{x^ix^j}\|_{C(Q)} \leqslant n(\kappa, \kappa_1, d).$$

The proof of this theorem is the same as the derivation of the estimate $\|u\|_{C^2(Q_{\varepsilon'})} \leqslant N$ in b) of the preceding proof, and is easily obtained by means of the corresponding arguments in §4 of [1], Lemma 4.3a) and c), and the observation that in §4 of [1] and in Lemma 4.3a) and c) the differentiability of F with respect to t is not used in general in deriving the estimates of u_x and u_{xx}.

§7. Proofs of Theorems 1.1 and 1.2

Recall that, by the assumptions of Theorems 1.1 and 1.2, $D = \{x \in E_d: \psi(x) > 0\}$, with $\psi \in C^3_{\mathrm{loc}}(E_d)$, $\|\psi\|_{C^3(D)} \leqslant \kappa^{-1}$, and $|\psi_x| \geqslant \kappa$ on ∂D. To prove Theorem 1.1 we need two lemmas.

LEMMA 7.1. *Suppose that $\alpha \in (0,1)$, $F \in \mathfrak{F}(\kappa, Q)$, $\psi \in C^{4+\alpha}(\overline{D})$, and for $\beta = 1$ the first and second derivatives of F with respect to (u_{ij}, u_i, u, x) and the first derivative of F with respect to t are bounded along with their Hölder constants of order α with respect to (u_{ij}, u_i, u, t, x) on each set of the form*

$$\left\{(u_{ij}, u_i, u, t, x): u_{ij} = u_{ji}, \sum|u_{ij}| + \sum|u_i| + |u| \leqslant N, (t, x) \in \overline{Q}\right\};$$

where $N < \infty$. Suppose also that $F(0,0,0,1,0,x) = F_t(0,0,0,1,0,x) = 0$ in D. Then the problem

$$u_t = F(u_{x^ix^j}, u_{x^i}, u, 1, t, x) \quad \text{in } Q, \qquad u = 0 \quad \text{on } \partial' Q$$

has a unique solution u such that $u, u_t, u_{xx} \in C^{2+\alpha}(\bar{Q})$.

PROOF. The derivation of this kind of result from theorems like Theorem 6.1 d) is well known. Therefore, we dwell only on the main points of the proof.

1°. Consider a family of problems depending on a parameter $\chi \in [0,1]$:

$$u_t = (1-\chi)\Delta u + \chi F(u_{x^ix^j}, u_{x^i}, u, 1, t, x) \quad \text{in } Q, \tag{7.1}$$

$$u = 0 \quad \text{on } \partial' Q. \tag{7.2}$$

Let I be the set of all $\chi \in [0,1]$ such that problem (7.1), (7.2) has a unique solution u^χ with $u^\chi, u^\chi_t, u^\chi_{xx} \in C^{2+\alpha}(\bar{Q})$. Obviously, $0 \in I$. We prove that $I = [0,1]$ and thereby the theorem, if we prove that I is closed and open in the relative topology of $[0,1]$.

2°. To prove that I is closed it obviously suffices to establish that if u is a solution of problem (7.1), (7.2) for some fixed $\chi \in [0,1]$ and $u \in C^{2+\alpha_1}(\bar{Q})$ for some $\alpha_1 \in (0,1)$, then $u, u_t, u_{xx} \in C^{2+\alpha}(\bar{Q})$, and the norms of these functions in $C^{2+\alpha}(\bar{Q})$ can be estimated by a constant not depending on χ (nor α_1).

Let us fix a $\chi \in [0,1]$ and assume that the function u of class $C^{2+\alpha_1}(\bar{Q})$ is a solution of problem (7.1), (7.2); the symbol N denotes various constants not depending on χ, but possibly depending on α_1 and $\|u\|_{C^{2+\alpha_1}(Q)}$. We show first that for some $\alpha_2 \in (0,1)$

$$\|u_t\|_{C^{2+\alpha_2}(Q)}, \ \|u_{x^ix^j}\|_{C^{2+\alpha_2}(Q)} \leqslant N. \tag{7.3}$$

We extend u and F for $t < 0$ by setting $u(t,x) = u(0,x) = 0$ and

$$F(u_{ij}, u_i, u, \beta, t, x) = F(u_{ij}, u_i, u, \beta, 0, x).$$

Note that for $t \leqslant 0$

$$F(0,0,0,\beta,t,x) = F_t(0,0,0,\beta,t,x) = 0, \tag{7.4}$$

and consider the function

$$u^{(h)}(t,x) = (u(t,x)/h - u(t-h,x))$$

for $h > 0$ and $t \geqslant 0$. By Hadamard's formula and by (7.1) and (7.4),

$$u^{(h)}_t = a^{ij}u^{(h)}_{x^ix^j} + (1-\chi)\Delta u^{(h)} + b^i u^{(h)}_{x^i} + cu^{(h)} + f$$

in Q, where

$$\begin{aligned} a^{ij}(t,x) = \chi\int_0^1 F_{u_{ij}}\big(&\theta u_{x^ix^j}(t,x) + (1-\theta)u_{x^ix^j}(t-h,x), \theta u_{x^i}(t,x) \\ &+ (1-\theta)u_{x^i}(t-h,x), \theta u(t,x) \\ &+ (1-\theta)u(t-h,x), 1, \theta t + (1-\theta)(t-h), x\big)\, d\theta, \end{aligned}$$

and b^i, c and f are defined similarly, with $F_{u_{ij}}$ replaced by F_{u_i}, F_u and F_t, respectively. We note that $a^{ij}, b^i, c, f \in C^{\alpha_2}(\bar{Q})$ for $\alpha_2 = \alpha\alpha_1$, and the norms of these functions in $C^{\alpha_2}(Q)$ can be estimated by a constant not depending on h (the proof of this for f uses the second equation in (7.4)). Moreover, $u^{(h)} \in C^{2+\alpha_2}(\bar{Q})$. Therefore, from the theory of linear equations (see [11] and [13]) we get the first estimate in (7.3) if u_t is replaced in it by $u^{(h)}$. Passage to the limit as $h \downarrow 0$ concludes the proof of the first estimate in (7.3).

To derive the boundary estimates of u_x and u_{xx} in the norm of $C^{2+\alpha_2}$ let us consider first the case when $V_{T,2} \subset \bar{Q}$ and $\Sigma_{T,2} \subset \partial_x Q$. In this case, if we write an equation as above for the difference quotient $(u(t, x+h)/|h| - u(t,x))$, where $h^1 = 0$, and use Theorem 10.1 in Chapter IV of [13], we get $\|u_{x^i}\|_{C^{2+\alpha_2}(V_{T,1})} \leqslant N$ for $i \geqslant 2$. From this it follows that all the derivatives of u in (7.1), except perhaps $u_{x^1x^1}$, are differentiable with respect to x^1 in $V_{T,1}$, and their derivatives with respect to x^1, along with the Hölder constants of order α_2 of these derivatives, do not exceed N. By implicit function theorem, $u_{x^1x^1}$ necessarily also has this property. Consequently, $\|u_{x^i}\|_{C^{2+\alpha_2}(V_{T,1})} \leqslant N$ for all $i \geqslant 1$.

Now take some $k \geqslant 2$, write $v = u_{x^k}$, and differentiate the equation (7.1) with respect to x^k in $V_{T,1}$. Then in $V_{T,1}$

$$v_t = (1-\chi)\Delta v + \chi\big(F_{u_{ij}}v_{x^ix^j} + F_{u_i}v_{x^i} + F_u v + F_{x^k}\big), \tag{7.5}$$

where for brevity the arguments $(u_{x^ix^j}(t,x), u_{x^i}(t,x), u(t,x), 1, t, x)$ are omitted for $F_{u_{ij}}$, F_{u_i}, F_u and F_{x^k}. We observe that in $V_{T,1}$ the coefficients of (7.5) and their first derivatives with respect to x satisfy a Hölder condition with exponent α_2. Therefore, using (7.5) to write an equation for the difference quotient $(v(t,x+h) - v(t,x))/|h|$, where $h^1 = 0$ and using Theorem 10.1 in Chapter IV of [13], we get

$$\|v_{x^i}\|_{C^{2+\alpha_2}(V_{T,1/2})} \leqslant N \quad \text{for } i \geqslant 2.$$

By (7.5) itself, the same inequality also holds for $i = 1$. From this and from (7.1) we obtain

$$\|u_{x^ix^j}\|_{C^{2+\alpha_2}(V_{T,1/2})} \leqslant N \quad \text{for all } i, j.$$

Clearly, the use of this result and the method of local rectification of the boundary gives the existence of a $\rho = \rho(\kappa, d) > 0$ such that

$$\|u_{x^ix^j}\|_{C^{2+\alpha_2}(Q \setminus Q(\rho))} \leqslant N \quad \text{for all } i, j.$$

Since an interior estimate of $u_{x^ix^j}$ in $C^{2+\alpha_2}$ can be established by a similar (somewhat simpler) method, (7.3) is proved.

After this, we can apply Theorem 6.1 and take the α_1 in Theorem 6.1 in all the arguments of this step (2°). This α_1 does not depend on χ. Observe also that the norm of u in $C(\bar{Q})$ is easy to estimate with the help of the maximum principle independently of χ. Moreover, by Theorem 6.1, the norm of u in $C^{2+\alpha_1}(\bar{Q})$ can be estimated independently of χ. Hence, all the arguments carried out yield estimates independent of χ. In particular, it follows from our estimates that u, u_x and u_{xx} satisfy a Lipschitz condition with respect to (t,x), and we can write α everywhere above where we have written α_2.

3°. To prove that I is open it suffices to repeat the proof of Theorem 3.3 in [1], with S taken to be the set

$$\{v \colon \|v - u^{\chi_1}\|_{C^{2+\alpha}(Q)} \leqslant \delta, v = 0 \text{ on } \partial' Q\}$$

in the latter.

This change is due to the fact that in our situation we must require a compatibility condition in order to construct a solution of equation (3.9) in [1] with zero initial and boundary conditions. Finally, the uniqueness of the solution is easily proved with the help of the maximum principle and Hadamard's formula. The lemma is proved.

$n, m \geqslant 1$ (for example, with the help of an average operation) such that

$$\|\varphi^{nm}, \varphi_t^{n,m}, \varphi_{x^ix^j}^{n,m}\|_{C^2(Q_{1/n}(1/n))} \leqslant 2\kappa^{-1}, \qquad \|\psi^n\|_{C^3(D)} \leqslant 2\kappa^{-1},$$

$$\|\varphi^{nm} - \varphi\|_{C(Q_{1/n}(1/n))} \leqslant 1/m, \qquad \|\psi^n - \psi\|_{C^1(D)} \leqslant 1/n.$$

Then let

$$D^n = \{x \in D: \psi^n(x) > (2\kappa^{-1} + 1)/n\}, \qquad Q^n = (1/n, T) \times D^n.$$

Note that in $D \setminus D(1/n)$ we have

$$|\psi(x)| = |\psi(x) - \psi(y(x))| \leqslant (\kappa^{-1} + \kappa^{-1}/n) \leqslant 2\kappa^{-1}/n,$$

and $\psi^n(x) \leqslant (2\kappa^{-1} + 1)/n$. Therefore, $Q^n \subset Q_{1/n}(1/n)$. Moreover, since D is bounded, for sufficiently large n the boundary of D^n is sufficiently close to the boundary of D, and $|\psi_x^n| \geqslant \kappa/2$ on ∂D^n. Finally, consider the problems

$$u_t = F_n(u_{x^ix^j}, u_{x^i}, u, 1, t, x) \quad \text{in } Q^n, \qquad u = \varphi^{nm} \quad \text{on } \partial' Q^n.$$

Applying Lemma 7.2 to these problems, and then letting m and n go to ∞, we get the statement of Theorem 1.1 without difficulty.

Proof of Theorem 1.2. Note first of all that, since D is bounded, Lemma 1.1a) and b) implies the uniqueness of the solution and the a priori boundedness of its norm in $C(D)$. Therefore, arguments analogous to those in the preceding proof show that to prove Theorem 1.2 it suffices to establish the following fact.

Lemma 7.3. *Suppose that* $\alpha \in (0, 1)$, $F \in \mathfrak{F}(\kappa, D)$, D *is bounded*, $\varphi, \psi \in C^{4+\alpha}(\overline{D})$, $\|\varphi\|_{C^3(D)} \leqslant \kappa^{-1}$, *and for* $\beta = 1$ *the first and second derivatives of* f *with respect to* (u_{ij}, u_i, u, x) *along with their Hölder constants of order* α *with respect to* (u_{ij}, u_i, u, x) *are bounded on each set of the form*

$$\left\{(u_{ij}, u_i, u, x): u_{ij} = u_{ji}, \sum_{i,j} |u_{ij}| + \sum_i + |u| \leqslant N, x \in \overline{D}\right\},$$

where $N < \infty$. *Then the equation* $F(u_{x^ix^j}, u_{x^i}, u, 1, x) = 0$ *in* D *with the boundary condition* $u = \varphi$ *on* ∂D *has a unique solution* $u \in C^{4+\alpha}(\overline{D})$. *Moreover, for* $\alpha_1 = \alpha_1(\kappa, d, 1/2)$ (*where* $\alpha_1(\kappa, d, \alpha)$ *was introduced in Theorem* 5.1)

$$\|u\|_{C^{2+\alpha_1}(\overline{D})} \leqslant N(\kappa, d, \|u\|_{C(D)}).$$

The proof of the first assertion of this lemma is analogous to that of Lemma 7.1 (with considerable simplification). To prove the second assertion it suffices to apply Theorem 6.1b) for $\alpha = 1/2$, formally adding the derivative with respect to t to our elliptic equation.

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