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NORMAL AND INTEGRAL CURRENTS[1]

BY HERBERT FEDERER AND WENDELL H. FLEMING

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1. Introduction

Long has been the search for a satisfactory analytic and topological formulation of the concept "k dimensional domain of integration in euclidean n-space." Such a notion must partake of the smoothness of differentiable manifolds and of the combinatorial structure of polyhedral chains with integer coefficients. In order to be useful for the calculus of variations, the class of all domains must have certain compactness properties. All these requirements are met by the *integral currents* studied in this paper.

The use of de Rham's (odd) currents, Schwartz distributions in case $k = 0$, allows a flexible theory in which suitable norms emerge naturally with individual problems. All currents occurring here have compact support, and most of them have finite mass; the mass $\mathbf{M}(T)$ of a k-dimensional current T is its norm when considered as a linear function of continuous k forms, which may be thought of as k-covector valued functions normed by maximum comass. A current T for which

$$\mathbf{N}(T) = \mathbf{M}(T) + \mathbf{M}(\partial T) < \infty$$

is called *normal*; here $\partial T = T \circ d$ is the boundary of T. Each locally Lipschitzian map f induces a linear map $f_\#$ of normal currents, with $\partial \circ f_\# = f_\# \circ \partial$. Another important norm, in addition to $\mathbf{M}$ and $\mathbf{N}$, is the norm $\mathbf{F}$ termed "flat" by Whitney.

The Lipschitzian images of finite polyhedral chains with integer coefficients are called *integral Lipschitz chains*; the $\mathbf{M}$ limits of integral Lipschitz chains are called *rectifiable currents*. If both T and ∂T are rectifiable currents, then T is called an *integral current*. Clearly every integral current is normal.

Integral currents are actually much smoother than one might expect from the preceding definition. This is shown by the following strong approximation theorem (8.22): *Every integral current is the* $\mathbf{N}$ *limit of a sequence of currents* $f_{i\#}(P_i)$ *where* P_i *are polyhedral chains with integer coefficients and* f_i *are diffeomorphisms of class* 1 *converging to the identity*

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map. On the other hand the class of all integral currents has very strong closure properties exemplified by the following two propositions (8.13, 8.14): *Every weak limit of an* **N**-*bounded sequence of integral currents is an integral current. Every normal rectifiable current is an integral current.* Another result of the theory is the general isoperimetric inequality (6.1): *Every k-dimensional integral cycle X bounds a $(k + 1)$-dimensional integral current Y such that*

$$\mathbf{M}(Y)^{k/(k+1)} \leqq c\mathbf{M}(X) ,$$

where c is a positive number depending only on the dimensions k and n. The discussion of problems involving convergence is greatly simplified by the connectivity theorem (7.1): *If T is the weak limit of an* **N**-*bounded sequence of normal currents T_i, then there exist normal currents G_i and K_i such that*

$$T_i - T = G_i + \partial K_i \quad \textit{for } i = 1, 2, 3, \cdots ,$$
$$\mathbf{M}(G_i) + \mathbf{M}(K_i) \longrightarrow 0 \quad \textit{as } i \to \infty ;$$

in case all T_i are integral currents, so are all G_i and K_i; in case all T_i are integral cycles, then $G_i = 0$ for large i. These facts are related with homological properties of any compact Lipschitz neighborhood retract A in n-space (5.11, 9.6): *The singular homology groups of A with integer coefficients are isomorphic with the homology groups of the complex of all integral currents with support in A; in each integral homology class there is a cycle of least mass; for each $\zeta > 0$ only finitely many homology classes contain cycles of mass less than ζ.*

The application of the theory of integral currents to the calculus of variations, in particular to the problem of Plateau, has been carried through only in small part. Powerful existence and continuity theorems (9.6-9.12) have been obtained, but the results (9.13-9.16) concerning the local structure of minimal currents represent merely an encouraging start toward the solution of a very complex problem.

In work to be published separately[2], the theory of integral currents has been applied to the study of continuous maps of finite area from a compact k-manifold into n-space, and has enriched and simplified the theory of Lebesgue area.

The basic techniques of this paper are measure theoretic, extended from ordinary functions to differential forms of arbitrary degree. The elegant linear function approach to measure theory, which almost ignores that

[2] Abstracts were presented to the American Mathematical Society on June 20, 1959 (558-33, 558-34) and October 31, 1959 (560-45).

functions are defined on sets and have values at points--and that spheres are round, is followed throughout the first seven sections. However in § 8 it appears necessary to use more refined local techniques from the theory of relative differentiation of measures, a covering theorem of Vitali type, and the fundamental structure theorem from the theory of Hausdorff measure. Since the latter techniques are less widely known, the relevant facts have been summarized (8.2, 8.3, 8.7, 8.9) for the reader's convenience.

Of previous related work, responsible for the origins of the present concepts, special mention is due to the theory of generalized surfaces ([Y 1], [FY 1, 2]), the general theory of Hausdorff measure and its application to the Gauss-Green theorem ([FE 2, 3, 4]), and the theory of distributions whose partial derivatives are measures ([G 1, 2], [FL 1, 2], [K], [P], [FE 5]). However, knowledge of this earlier work is not prerequisite for understanding the present paper.

2. Preliminaries

This section serves to recall some familiar facts, mostly in order to explain terminology and notation, sometimes to point out trivial but useful extensions of classical results. For more details the reader may consult standard books such as [W 2], [BO 1] concerning Grassmann algebra; [SA], [BO 2] concerning measure theory; [SC], [R], [W 2] concerning distributions, currents and differential forms; [ES] concerning algebraic topology.

2.1 *Euclidean spaces.* R is the real number field and R^n is the vector space consisting of all n-tuples $x = (x_1, \cdots, x_n)$ of real numbers. To each $a \in R^n$ corresponds the translation

$$\tau_a : R^n \longrightarrow R^n\ , \quad \tau_a(x) = a + x \ \text{ for } x \in R^n\ ,$$

and to $\varepsilon > 0$ corresponds the homothetic transformation

$$\mu_\varepsilon : R^n \longrightarrow R^n\ , \quad \mu_\varepsilon(x) = \varepsilon x \ \text{ for } x \in R^n\ .$$

The space R^n is equipped with the inner product

$$x \cdot y = \sum_{i=1}^n x_i y_i$$

and the norm $|x| = (x \cdot x)^{1/2}$.

L_n is the Lebesgue measure over R^n.

$$\alpha(n) = L_n(R^n \cap \{x : |x| < 1\})\ .$$

If f maps an open subset of R^n into another normed vector space and x is a point where f is differentiable (in the sense of Fréchet), then $Df(x)$ is the differential of f at x. Furthermore $D_1 f, \cdots, D_n f$ are the first order partial derivatives of f.

If γ is a measure and g is a function, then

$$\gamma(g) = \int g\,d\gamma = \int g(x)d\gamma x \ .$$

2.2 *Grassman algebra.* With each finite dimensional real vector space W one associates the dual spaces

$$\bigwedge_k(W) \text{ and } \bigwedge^k(W)$$

of k-vectors and k-covectors; here $\bigwedge^0(W) = R = \bigwedge_0(W)$, and $\bigwedge_k(W) = \{0\} = \bigwedge^k(W)$ in case $k > \dim W$ or $k < 0$. The direct sums

$$\bigwedge_*(W) = \bigoplus_k \bigwedge_k(W) \text{ and } \bigwedge^*(W) = \bigoplus_k \bigwedge^k(W)$$

are the contravariant and covariant Grassman algebras of W, with the exterior multiplication $\wedge$.

An inner product $\cdot$ of W, with the corresponding norm $|\ \ |$, induces inner products and norms on $\bigwedge_*(W)$ and $\bigwedge^*(W)$, which will also be denoted $\cdot$ and $|\ \ |$; dual orthogonal bases for $\bigwedge_*(W)$ and $\bigwedge^*(W)$ are obtained by exterior multiplication from dual orthogonal bases for W and $\bigwedge^1(W)$.

The vector spaces $\bigwedge_k(W)$ and $\bigwedge^k(W)$ are dually paired not only with respect to the norms $|\ \ |$, but also relative to the *mass* and *comass* $\|\ \ \|$ defined as follows:

If $\xi \in \bigwedge_k(W)$, then

$$\|\xi\| = \inf \{\textstyle\sum_{\beta\in B} |\beta| : B \text{ is a finite set of simple } k\text{-vectors and } \xi = \textstyle\sum_{\beta\in B}\beta\} \ .$$

If $\omega \in \bigwedge^k(W)$, then

$$\|\omega\| = \sup \{\omega(\gamma) : \gamma \text{ is a simple } k\text{-vector and } |\gamma| \leqq 1\} \ .$$

Since the set

$$S = \{\gamma : \gamma \text{ is a simple } k\text{-vector and } |\gamma| \leqq 1\}$$

is compact, the supremum in the definition of $\|\omega\|$ is attained for some γ. Furthermore[3] the infimum in the definition of $\|\xi\|$ is attained for some set B with at most $\binom{\dim W}{k}$ elements. In fact S is connected and $\xi/\|\xi\|$ belongs to the convex hull of S, because if B is a finite set of simple k-vectors with

$$\xi = \textstyle\sum_{\beta\in B}\beta \ , \qquad \lambda = \textstyle\sum_{\beta\in B}|\beta| \ ,$$

then

[3] This answers a question raised in [**W** 2, I, 13], but will not be needed in this paper.

$$\xi/\lambda = \sum_{\beta\in B}(|\beta|/\lambda)(\beta/|\beta|)\,, \quad 1 = \sum_{\beta\in B}|\beta|/\lambda\,,$$

and because the convex hull of S is closed.

2.3 *Forms and currents.* Assuming that U is an open subset of R^n, let $\mathbf{E}^k(U)$ be the real vector space of differential k-forms of class ∞ on U; these forms may be regarded as infinitely differentiable maps of U into $\bigwedge^k(R^n)$. Convergence in $\mathbf{E}^k(U)$ means uniform convergence, on each compact subset of U, of each partial derivative of any order. The direct sum

$$\mathbf{E}^*(U) = \bigoplus_k \mathbf{E}^k(U)$$

is a graded differential algebra with the exterior multiplication $\wedge$ and the exterior differentiation d. One writes

$$\operatorname{spt}\varphi = U \cap \operatorname{Clos}\{x : \varphi(x) \neq 0\} \quad \text{for } \varphi \in \mathbf{E}^k(U)\,.$$

Let $\mathbf{E}_k(U)$ be the space of real-valued continuous linear functions on $\mathbf{E}^k(U)$, with the weak topology. For each $T \in \mathbf{E}_k(U)$, $\operatorname{spt} T$ is the smallest set C such that C is closed relative to U, and $T(\varphi) = 0$ whenever $\varphi \in \mathbf{E}^k(U)$ with $\operatorname{spt}\varphi \subset U - C$; $\operatorname{spt} T$ is compact and one calls T a *current with compact support.* If $T_i \in \mathbf{E}_k(U)$ for $i = 1, 2, 3, \cdots$ and $T(\varphi) = \lim_{i\to\infty} T_i(\varphi)$ exists whenever $\varphi \in \mathbf{E}^k(U)$, then $T \in \mathbf{E}_k(U)$ and U has a compact subset containing the supports of all the currents T_i. The direct sum

$$\mathbf{E}_*(U) = \bigoplus_k \mathbf{E}_k(U)$$

is a chain complex with the boundary operator ∂ defined by the formula $\partial T = T \circ d$. Clearly d and ∂ are continuous.

If $T \in \mathbf{E}_k(U)$ and $\omega \in \mathbf{E}^m(U)$ with $k \geqq m$, then

$$T \wedge \omega \in \mathbf{E}_{k-m}(U)\,,$$
$$(T\wedge\omega)(\varphi) = T(\omega\wedge\varphi) \quad \text{for } \varphi \in \mathbf{E}^{k-m}(U)\,,$$
$$\partial(T\wedge\omega) = (-1)^m(\partial T)\wedge\omega + (-1)^{m-1}T\wedge(d\omega)\,.$$

If $\psi: R^n \to \bigwedge^m(R^n)$ is an L_n summable function with compact support, one associates with ψ the current

$$\mathbf{C}\psi \in E_{n-m}(R^n)\,,$$

$$(\mathbf{C}\psi)(\varphi) = \int_{R^n} \psi\wedge\varphi \quad \text{for } \varphi \in \mathbf{E}^{n-m}(R^n)\,,$$

where R^n has its usual orientation. In case ψ is the characteristic function of a bounded open set A, then $\mathbf{C}\psi$ will be identified with A; thus

$$A(\varphi) = \int_A \varphi \quad \text{for } \varphi \in \mathbf{E}^n(R^n)\,.$$

Similarly an oriented k-dimensional convex cell A in R^n is identified with the operation of integrating over A; hence $A \in \mathbf{E}_k(R^n)$ and

$$A(\varphi) = \int_A \varphi \quad \text{for } \varphi \in \mathbf{E}^k(R^n) .$$

Finite linear combinations of convex cells are called *polyhedral chains*; they will occur both with real and with integer coefficients.

According to the preceding conventions the interval

$$I = \{t : 0 \leqq t \leqq 1\}$$

is identified with the current $I \in \mathbf{E}_1(R)$ such that

$$I(\varphi) = \int_0^1 \varphi \quad \text{for } \varphi \in \mathbf{E}^1(R) .$$

If $T \in \mathbf{E}_k(U)$, then

$$D_j T = - T \circ D_j \in \mathbf{E}_k(U) \quad \text{for } j = 1, \cdots, n .$$

In case $k > 0$ one finds that

$$\partial T = - \textstyle\sum_{j=1}^n D_j T \wedge dX_j ,$$

where $X_1, \cdots, X_n$ are the usual coordinate functions.

If $\psi \in \mathbf{E}^m(R^n)$ and spt ψ is compact, then

$$D_j \mathbf{C}\psi = \mathbf{C} D_j \psi \quad \text{for } j = 1, \cdots, n .$$

One defines an isomorphism of $\mathbf{E}_n(R^n)$ onto $\mathbf{E}_0(R^n)$, the space of Schwartz distributions with compact support, by mapping T onto $T \wedge dX_1 \wedge \cdots \wedge dX_n$. Inasmuch as

$$\begin{aligned} D_j(T \wedge dX_1 \wedge \cdots \wedge dX_n) &= D_j T \wedge dX_1 \wedge \cdots \wedge dX_n \\ &= (-1)^j \partial T \wedge dX_1 \wedge \cdots \wedge dX_{j-1} \wedge dX_{j+1} \wedge \cdots \wedge dX_n \end{aligned}$$

for $j = 1, \cdots, n$, the theorem characterizing distributions whose partial derivatives vanish in a region may be restated as follows:

If $T \in \mathbf{E}_n(R^n)$, V is a subregion of R^n and spt $\partial T \subset R^n - V$, then there exists a real number r such that $T(\varphi) = r \int_V \varphi$ whenever $\varphi \in \mathbf{E}^n(R^n)$ with spt $\varphi \subset V$.

2.4 *Currents of finite mass.* For $\varphi \in \mathbf{E}^k(U)$ let

$$\mathbf{M}(\varphi) = \sup \{\| \varphi(x) \| : x \in U\} .$$

For $T \in \mathbf{E}_k(U)$ let

$$\mathbf{M}(T) = \sup \{T(\varphi) : \varphi \in \mathbf{E}^k(U) \text{ and } \mathbf{M}(\varphi) \leqq 1\} .$$

With $\varphi \in \mathbf{E}^k(U)$ one associates the continuous function $\|\varphi\|$ such that

$$\|\varphi\|(x) = \|\varphi(x)\| \quad \text{for } x \in U .$$

With any $T \in \mathbf{E}_k(U)$ for which $\mathbf{M}(T) < \infty$ one associates the *variation measure* $\|T\|$ such that

$$\|T\|(f) = \sup \{T(\varphi) : \varphi \in \mathbf{E}^k(U) \text{ and } \|\varphi\| \leqq f\}$$

whenever f is a nonnegative real-valued continuous function on U.

If $T \in \mathbf{E}_k(U)$, $\mathbf{M}(T) < \infty$, $\omega \in \mathbf{E}^m(U)$, $m \leqq k$, then

$$\mathbf{M}(T \wedge \omega) \leqq \binom{k}{m} \|T\|(\|\omega\|) ,$$

because

$$\|\omega \wedge \psi\| \leqq \binom{k}{m} \|\omega\| \cdot \|\psi\| \quad \text{for } \psi \in \mathbf{E}^{k-m}(U) ;$$

in case $\omega(x)$ is simple whenever $x \in$ spt T, the factor $\binom{k}{m}$ may be omitted.

A current $T \in \mathbf{E}_k(U)$ for which $\mathbf{M}(T) < \infty$ has a unique extension (also denoted by T) to the class of all bounded Borel measurable k-forms (k-covector-valued Baire functions) on U such that Lebesgue's theorem on bounded convergence holds. This extension may also be interpreted as integration with respect to a k-vector-valued measure. One defines $T \wedge \omega$ by the usual formula in case ω is a bounded Baire form. Bounded convergence of a sequence of Baire forms ω_i implies convergence of the corresponding sequence of currents $T \wedge \omega_i$. If A is a Borel subset of U, one writes

$$T \cap A$$

for $T \wedge \omega$, where ω is the characteristc function of A; clearly

$$\mathbf{M}(T \cap A) = \|T\|(A) .$$

If $\lim_{i\to\infty} T_i = T \in \mathbf{E}_k(U)$ and $\liminf_{i\to\infty} \mathbf{M}(T_i) < \infty$, then $\mathbf{M}(T) < \infty$ and

$$\lim_{i\to\infty} T_i(\varphi) = T(\varphi)$$

for every continuous k-form φ on U,

$$\liminf_{i\to\infty} \|T_i\|(A) \geqq \|T\|(A)$$

for every open set $A \subset U$.

If A is compact subset of R^n and $0 \leqq c < \infty$, then $\mathbf{E}_k(R^n) \cap \{T : \text{spt } T \subset A \text{ and } \mathbf{M}(T) \leqq c\}$ is compact.

If ψ is a real valued L_n summable function with compact support, then

$$\mathbf{M}(\mathbf{C}\psi) = \int |\psi| \, dL_n .$$

If A is an oriented k-dimensional convex cell, then $\mathbf{M}(A)$ equals the k-dimensional measure of A.

2.5 *Induced maps*. Suppose U and V are open subset of R^m and R^n. An infinitely differentiable map

$$f : U \longrightarrow V$$

induces continuous linear transformations

$$f^\# : \mathbf{E}^*(V) \longrightarrow \mathbf{E}^*(U) \text{ and } f_\# : \mathbf{E}_*(U) \longrightarrow \mathbf{E}_*(V)$$

such that $d \circ f^\# = f^\# \circ d$, $\partial \circ f_\# = f_\# \circ \partial$ and $f^\#$ preserves $\wedge$.

If $\varphi \in \mathbf{E}^k(V)$ and $x \in U$, then $f^\#(\varphi)(x) \in \bigwedge^k(R^m)$ is the image of $\varphi[f(x)] \in \bigwedge^k(R^n)$ under the map induced by $Df(x)$.

If $T \in \mathbf{E}_k(U)$, then $f_\#(T) = T \circ f^\# \in \mathbf{E}_k(V)$. Clearly

$$\operatorname{spt} f^\#(\varphi) \subset f^{-1}(\operatorname{spt} \varphi)\ , \quad \operatorname{spt} f_\#(T) \subset f(\operatorname{spt} T)\ ,$$

$$f_\#(T) \wedge \omega = f_\#[T \wedge f^\#(\omega)] \quad \text{for } \omega \in \mathbf{E}^m(V),\ m \leqq k.$$

If $\mathbf{M}(T) < \infty$, then

$$\mathbf{M}[f_\#(T)] \leqq ||\, T \,||\, (|\, Df \,|^k) < \infty\ ,$$

because $||\, f^\#(\varphi) \,|| \leqq |\, Df \,|^k \cdot ||\, \varphi \circ f \,||$ whenever $\varphi \in \mathbf{E}^k(V)$; the definition of $f^\#(\omega)$ and the formula for $f_\#(T) \wedge \omega$ extend readily to the case where ω is a bounded Baire form on V; in particular $f_\#(T) \cap B = f_\#[T \cap f^{-1}(B)]$ for every Borel set $B \subset V$.

If $g : V \to W$ is another infinitely differentiable map, then $(g \circ f)^\# = f^\# \circ g^\#$ and $(g \circ f)_\# = g_\# \circ f_\#$.

If $U \subset V$ and f is the inclusion map, then $f_\#$ is a one-to-one map of $\mathbf{E}_k(U)$ onto

$$\mathbf{E}_k(V) \cap \{T : \operatorname{spt} T \subset U\}\ ,$$

with $\mathbf{M}[f_\#(T)] = \mathbf{M}(T)$ for $T \in \mathbf{E}_k(U)$. While $f_\#^{-1}$ is usually not continuous, convergence of a sequence in $\mathbf{E}_k(U)$ is equivalent to convergence of its $f_\#$ image sequence provided the supports of all currents in the sequence are contained in a fixed compact subset of U.

2.6 *Cartesian products*. Suppose V and W are open subsets of R^m and R^n, and

$$p : V \times W \longrightarrow V\ , \quad q : V \times W \longrightarrow W$$

are the usual projections. Then

$$\bigoplus_{k=0}^{l} \{p^\#(\varphi) \wedge q^\#(\psi) : \varphi \in \mathbf{E}^k(V),\ \psi \in \mathbf{E}^{l-k}(W)\}$$

is dense in $\mathbf{E}^l(V \times W)$. One defines

$$S \times T \in \mathbf{E}_{i+j}(V \times W) \quad \text{for } S \in \mathbf{E}_i(V) \text{ and } T \in \mathbf{E}_j(W)$$

in such a way that, for $\varphi \in \mathbf{E}^k(V)$ and $\psi \in \mathbf{E}^{i+j-k}(W)$,

$$(S \times T)[p^\#(\varphi) \wedge q^\#(\psi)] = \begin{cases} S(\varphi)T(\psi) & \text{in case } k = i \\ 0 & \text{in case } k \neq i \, . \end{cases}$$

Then

$$\partial(S \times T) = (\partial S) \times T + (-1)^i S \times (\partial T) \, ,$$
$$\mathbf{M}(S \times T) \leqq \binom{m}{i} \mathbf{M}(S)\mathbf{M}(T) \, .$$

2.7 *Homotopies.* Suppose U is an open subset of R^m,

$$f : U \longrightarrow R^n \text{ and } g : U \longrightarrow R^n \, .$$

If $h : R \times U \rightarrow R^n$ is an infinitely differentiable map such that $h(0, x) = f(x)$ and $h(1, x) = g(x)$ whenever $x \in U$, then, for $T \in \mathbf{E}_k(U)$,

$$\partial h_\#(I \times T) + h_\#(I \times \partial T) = g_\#(T) - f_\#(T) \, .$$

Particularly useful for our purpose is the linear homotopy from f to g defined by the formula

$$h(t, x) = (1 - t)f(x) + tg(x) \quad \text{for } t \in R, \, x \in U \, .$$

If f and g are infinitely differentiable, so is h, and

$$\mathbf{M}[h_\#(I \times T)] \leqq \| T \| \left(| g - f | \cdot \sup \{ | Df |^k, | Dg |^k \} \right)$$

whenever $T \in \mathbf{E}_k(U)$ and $\mathbf{M}(T) < \infty$. This inequality follows easily from the statement that

$$\mathbf{M}[h_\#(I \times T)] \leqq \varepsilon \lambda^k \mathbf{M}(T)$$

provided $| g(x) - f(x) | \leqq \varepsilon$, $| Df(x) | \leqq \lambda$, $| Dg(x) | \leqq \lambda$ for $x \in \operatorname{spt} T$. To prove the statement consider the maps

$$\xi : R \times U \longrightarrow R \times U \, , \quad \eta : R \times U \longrightarrow R^n \, ,$$
$$\xi(t, x) = (t\varepsilon/\lambda, x) \, , \qquad \eta(t, x) = h(t\lambda/\varepsilon, x)$$

for $(t, x) \in R \times U$, and observe that

$$h = \eta \circ \xi \, , \quad \xi_\#(I \times T) = \{ t : 0 \leqq t \leqq \varepsilon/\lambda \} \times T \, ,$$
$$| D\eta(t, x) | \leqq \lambda \quad \text{for } (t, x) \in R \times \operatorname{spt} T \, ,$$

hence

$$\mathbf{M}[h_\#(I \times T)] \leqq \lambda^{k+1} \mathbf{M}[\xi_\#(I \times T)] \leqq \lambda^{k+1}(\varepsilon/\lambda)\mathbf{M}(T) \, .$$

3. Basic properties of normal and integral currents

Here various special types of currents are defined, and the chain maps and homotopy operators induced by locally Lipschitzian maps are constructed. It is shown in 3.9 that slicing a normal current by level surfaces of a Lipschitzian function usually yields normal currents; the inequality obtained there was suggested by a result of Eilenberg [E] concerning Hausdorff measures, though the proofs are entirely different. Furthermore normal currents which, together with their boundaries, have supports in the appropriate skeletons of a rectilinear cell complex are proved to be elementary chains of that complex.

3.1 DEFINTION. Abbreviating

$$\mathbf{N}(T) = \mathbf{M}(T) + \mathbf{M}(\partial T) ,$$

we associate with $U \subset R^n$ the linear spaces of *normal currents*

$$\mathbf{N}_k(U) = \mathbf{E}_k(R^n) \cap \{T : \mathbf{N}(T) < \infty \text{ and spt } T \subset U\}$$

which form the chain complex

$$\mathbf{N}_*(U) = \textstyle\bigoplus_k \mathbf{N}_k(U) .$$

In case U is open one may identify $\mathbf{N}_k(R^n)$ with

$$\mathbf{E}_k(U) \cap \{T : \mathbf{N}(T) < \infty\}$$

according to 2.5.

3.2 DEFINITION. For $\varphi \in \mathbf{E}^k(R^n)$ let

$$\mathbf{F}(\varphi) = \sup \{\mathbf{M}(\varphi), \mathbf{M}(d\varphi)\} .$$

Dually, for $T \in \mathbf{E}_k(R^n)$ let

$$\mathbf{F}(T) = \sup \{T(\varphi) : \varphi \in \mathbf{E}_k(R^n) \text{ and } \mathbf{F}(\varphi) \leqq 1\} .$$

3.3 REMARK. Some significant properties of the *flat norm* $\mathbf{F}$ will be discussed in § 7. Here we use only the obvious fact that every $\mathbf{F}$-Cauchy sequence of currents in $\mathbf{E}_k(R^n)$, with supports contained in a fixed compact set is convergent in $\mathbf{E}_k(R^n)$.

3.4 LEMMA. *Suppose U is an open subset of R^m, $T \in \mathbf{N}_k(U)$, $\lambda \geqq 0$ and $\varepsilon \geqq 0$. If f and g are infinitely differentiable maps of U into R^n such that*

$$|Df(x)| \leqq \lambda, \ |Dg(x)| \leqq \lambda, \ |f(x) - g(x)| \leqq \varepsilon \quad \textit{for } x \in \text{spt } T ,$$

then

$$\mathbf{F}[g_\#(T) - f_\#(T)] \leqq \varepsilon \sup \{\lambda^k, \lambda^{k-1}\} \mathbf{N}(T) .$$

PROOF. For each $\varphi \in \mathbf{E}^k(R^n)$ with $\mathbf{F}(\varphi) \leqq 1$ one sees from 2.7 that

$$\begin{aligned}[g_\#(T) - f_\#(T)](\varphi) &= h_\#(I \times T)(d\varphi) + h_\#(I \times \partial T)(\varphi) \\ &\leqq \mathbf{M}[h_\#(I \times T)] + \mathbf{M}[h_\#(I \times \partial T)] \leqq \varepsilon\lambda^k \mathbf{M}(T) + \varepsilon\lambda^{k-1}\mathbf{M}(\partial T) .\end{aligned}$$

3.5 DEFINITION. Suppose U and V are open subsets of R^m and R^n. Each locally Lipschitzian map

$$f : U \longrightarrow V$$

induces a chain map

$$f_\sharp : \mathbf{N}_*(U) \longrightarrow \mathbf{N}_*(V)$$

defined for $T \in \mathbf{N}_k(U)$ by the formula

$$f_\sharp(T) = \lim_{i\to\infty} f_{i\sharp}(T)$$

where $f_1, f_2, f_3, \cdots$ are infinitely differentiable maps of U into V, which converge to f uniformly on spt T, and whose differentials are bounded uniformly on spt T.

The limit exists by 3.4 and 3.3; linearity of $f_\sharp$ is obvious; $\partial \circ f_\sharp = f_\sharp \circ \partial$ because ∂ is continuous. Clearly

$$\operatorname{spt} f_\sharp(T) \subset f(\operatorname{spt} T) \quad \text{for } T \in \mathbf{N}_k(U) .$$

If $g : V \to W$ is another locally Lipschitzian map, then

$$(g \circ f)_\sharp = g_\sharp \circ f_\sharp .$$

3.6 REMARK. If f and g are locally Lipschitzian maps of U into R^n, h is the linear homotopy from f to g, $T \in \mathbf{N}_k(U)$ and λ is a continuous real-valued function on U such that

$$| Df(x) | \leqq \lambda(x) \text{ and } | Dg(x) | \leqq \lambda(x)$$

for L_m almost all x in some neighborhood of spt T, then

$$\begin{aligned} \partial h_\sharp(I \times T) + h_\sharp(I \times \partial T) &= g_\sharp(T) - f_\sharp(T) , \\ \mathbf{M}[f_\sharp(T)] &\leqq \| T \| (\lambda^k) , \\ \mathbf{M}[h_\sharp(I \times T)] &\leqq \| T \| \left(| g - f | \cdot \lambda^k\right) . \end{aligned}$$

This follows readily from 2.5 and 2.7 through regularization of f and g. Observe also that

$$\mathbf{M}[g_\sharp(T) - f_\sharp(T)] \leqq 2 \int_{\{x : f(x) \neq g(x)\}} \lambda^k d \, \| T \| ,$$

because f and g can be approximated by infinitely differentiable maps which agree in a neighborhood of $\{x : f(x) = g(x)\}$.

3.7 DEFINITION. A *Lipschitz chain* in R^n is a current of the type $f_\sharp(P)$ where P is a polyhedral chain with real coefficients in some R^m and $f : R^m \to R^n$ is Lipschitzian; in case the coefficients of P are integers, $f_\sharp(P)$ is an *integral Lipschitz chain*.

Similarly *chains of class* r and *integral chains of class* r are defined by requiring f to be of class r.

A current $T \in \mathbf{E}_k(R^n)$ is termed *rectifiable* if and only if for every $\varepsilon > 0$ there exists an integral Lipschitz chain Q of R^n such that $\mathbf{M}(T - Q) < \varepsilon$; clearly one may also require that $\operatorname{spt} Q$ be contained in any assigned neighborhood of $\operatorname{spt} T$.

An *integral current* is a current T such that both T and ∂T are rectifiable. For $U \subset R^n$ let

$$\mathbf{I}_k(U) = \mathbf{N}_k(U) \cap \{T : T \text{ is an integral current}\},$$
$$\mathbf{I}_*(U) = \bigoplus_k \mathbf{I}_k(U) .$$

Note that if T is an integral current and $\varepsilon > 0$, then there are integral Lipschitz chains Q_1 and Q_2 such that $\mathbf{M}(T - Q_1) < \varepsilon$ and $\mathbf{M}(\partial T - Q_2) < \varepsilon$; the fact that one may actually choose $Q_2 = \partial Q_1$ will not be proved until 5.8.

3.8 REMARK. Several alternate geometric characterizations of rectifiable and integral currents may be found in §8. Until then we need only the following elementary properties:

(1) *$\mathbf{I}_k(U)$ is a subgroup of the vector space $\mathbf{N}_k(U)$; $\mathbf{I}_*(U)$ is a subcomplex of $\mathbf{N}_*(U)$.*

(2) *If $f: U \to V$ is locally Lipschitzian, $T \in \mathbf{N}_k(U)$ and T is rectifiable, then $f_\#(T)$ is rectifiable; hence $f_\#[\mathbf{I}_k(U)] \subset \mathbf{I}_k(V)$.*

(3) *If T is rectifiable and A is open, then $T \cap A$ is rectifiable.*

(4) *If S and T are rectifiable, then $S \times T$ is rectifiable.*

(5) *If T is a k-dimensional rectifiable current in R^k, then $T = \mathbf{C}\psi$ for some integer-valued L_k summable function ψ.*

(6) *Every 0-dimensional rectifiable current is polyhedral.*

Regarding (3) observe that for P and f as in 3.7 there exist polyhedral chains P_i with integer coefficients in R^m such that

$$f[\operatorname{spt}(P - P_i)] \subset \{y : \text{distance}\,(y, R^n - A) < i^{-1}\} ,$$
$$\lim_{i\to\infty} \mathbf{M}[P_i - P \cap f^{-1}(A)] = 0 .$$

Consequently

$$Q = \lim_{i\to\infty} f_\#(P_i)$$

is a rectifiable current such that $f_\#(P)(\varphi) = Q(\varphi)$ whenever $\varphi \in \mathbf{E}^k(R^n)$, $\operatorname{spt} \varphi \subset A$; hence $f_\#(P) \cap A = Q$.

To verify (5), let C be the class of all currents $\mathbf{C}\psi$ corresponding to integer-valued L_k summable functions ψ with compact support, and observe:

(a) All polyhedral k-chains with integer coefficients in R^k belong to C.

(b) If $T_1, T_2, T_3, \cdots$ belong to C and

$$T = \lim_{i\to\infty} T_i \,, \quad \lim_{i\to\infty} \mathbf{M}(T_i - T) = 0 \,,$$

then T belongs to C.

(c) If P and f are as in 3.7, then f may be approximated uniformly by simplicial maps f_i with a common Lipschitz constant λ. Each $f_{i\#}(P)$ belongs to C by (a). Furthermore 3.6, applied with T and g replaced by P and f_i, hence $h_\#(I \times P) \in \mathbf{E}_{k+1}(R^k) = \{0\}$, yields

$$\mathbf{M}[f_{i\#}(P) - f_\#(P)] \leqq \|\partial P\| \left(|f_i - f| \cdot \lambda^{k-1}\right) \to 0 \quad \text{as } i \to \infty \,.$$

3.9 THEOREM. *Suppose $T \in \mathbf{N}_k(R^n)$, u is a real-valued function on R^n with Lipschitz constant ξ, and*

$$A_r = \{x : u(x) > r\} \quad \textit{for } r \in R \,.$$

If $s \in R$ and

$$\liminf_{h\to 0+} h^{-1}[\|T\|(A_s) - \|T\|(A_{s+h})] < \infty \,,$$

then $T \cap A_s \in \mathbf{N}_k(R^n)$ and

$$\|\partial(T \cap A_s) - (\partial T) \cap A_s\|(W)$$
$$\leqq \xi \liminf_{h\to 0+} h^{-1}[\|T\|(W \cap A_s) - \|T\|(W \cap A_{s+h})]$$

for every open subset W of R^n, hence

$$\operatorname{spt}[\partial(T \cap A_s) - (\partial T) \cap A_s] \subset (\operatorname{spt} T) \cap \{x : u(x) = s\} \,.$$

PROOF. Suppose $k \geqq 1$ and $\eta > \xi$. For each $h > 0$ the distance between A_{s+h} and $R^n - A_s$ is at least h/ξ, and consequently one may choose $f_h \in \mathbf{E}^0(R^n)$ with Lipschitz constant η/h so that $f_h(x) = 1$ for $x \in A_{s+h}$, $f_h(x) = 0$ for $x \in R^n - A_s$, $0 \leqq f_h(x) \leqq 1$ for $x \in R^n$, $\operatorname{spt}(df_h) \subset A_s - A_{s+h}$. Using Lebesgue's theorem on bounded convergence one obtains

$$\lim_{h\to 0+} \mathbf{M}(T \wedge f_h - T \cap A_s) = 0 \,,$$
$$\lim_{h\to 0+} \mathbf{M}[(\partial T) \wedge f_h - (\partial T) \cap A_s] = 0 \,,$$
$$T \cap A_s = \lim_{h\to 0+} T \wedge f_h, \qquad \mathbf{M}(T \cap A_s) < \infty \,,$$
$$\partial(T \cap A_s) = \lim_{h\to 0+} \partial(T \wedge f_h) = \lim_{h\to 0+} [(\partial T) \wedge f_h - T \wedge df_h] \,,$$
$$(\partial T) \cap A_s = \lim_{h\to 0+} (\partial T) \wedge f_h, \qquad \mathbf{M}[(\partial T) \cap A_s] < \infty \,,$$
$$\partial(T \cap A_s) - (\partial T) \cap A_s = -\lim_{h\to 0+} T \wedge df_h \,.$$

Since

$$\|T \wedge df_h\|(W) \leqq \eta h^{-1} \|T\|[W \cap (A_s - A_{s+h})]$$

whenever $h > 0$ and W is an open subset of R^n, it follows that

$$\mathbf{M}[\partial(T\cap A_s)-(\partial T)\cap A_s]<\infty\ ,\quad \mathbf{M}[\partial(T\cap A_s)]<\infty\ ,$$

and the conclusion of the theorem holds with η replacing ξ.

3.10 COROLLARY. *$T\cap A_s\in \mathbf{N}_k(R^n)$ for L_1 almost all s. If $-\infty\leqq a<b\leqq\infty$, then*

$$\int_a^b \mathbf{M}[\partial(T\cap A_s)-(\partial T)\cap A_s]ds\leqq \xi\,\|T\|\left(\{x:a<u(x)\leqq b\}\right)\ .$$

PROOF. The non-increasing function γ, defined by

$$\gamma(s)=\|T\|(A_s)\quad \text{for } s\in R\ ,$$

is differentiable L_1 almost everywhere, with

$$\mathbf{M}[\partial(T\cap A_s)-(\partial T)\cap A_s]\leqq -\xi\gamma'(s)\ .$$

3.11 THEOREM. *Suppose*

$$e:R^m\longrightarrow R^n \text{ and } p:R^n\longrightarrow R^m$$

are an isometric embedding and an orthogonal projection such that $p\circ e$ is the identity map of R^m, and $T\in\mathbf{N}_k(R^n)$.

(1) *If* spt $T\subset e(R^m)$, *then $T=e_\sharp[p_\sharp(T)]$.*

(2) *If U is an open subset of R^n and $U\cap$ spt $T\subset e(R^m)$,*

then

$$T\cap U=e_\sharp[p_\sharp(T\cap U)]\ ,$$
$$\operatorname{spt}[\partial p_\sharp(T\cap U)-p_\sharp((\partial T)\cap U)]\subset p[e(R^m)-U]\ .$$

PROOF OF (1). Applying 3.4 with f and g replaced by $e\circ p$ and the identity map of R^n, $\lambda=1$, $\varepsilon=0$, one finds that

$$\mathbf{F}[T-(e\circ p)_\sharp(T)]=0\ .$$

PROOF OF (2). Letting

$$A_s=\{x:\text{distance }(x,R^n-U)>s\}\quad \text{for } s>0\ ,$$

one sees that

$$T\cap U=\lim_{s\to 0+}T\cap A_s\ ,\quad (\partial T)\cap U=\lim_{s\to 0+}(\partial T)\cap A_s\ .$$

Moreover 3.9, 3.10 and (1) imply that, for L_1 almost all $s>0$,

$$T\cap A_s\in\mathbf{N}_k(R^n)\ ,\ T\cap A_s=(e\circ p)_\sharp(T\cap A_s)\ ,$$
$$\operatorname{spt}[\partial(T\cap A_s)-(\partial T)\cap A_s]\subset e(R^m)-A_s\ .$$

Using continuity properties of $p_\sharp$, $e_\sharp$, ∂, spt one concludes that

$$T\cap U=\lim_{s\to 0+}(e\circ p)_\sharp(T\cap A_s)=(e\circ p)_\sharp(T\cap U)\ ,$$

$\operatorname{spt} p_\sharp[\partial(T\cap U)-(\partial T)\cap U]\subset p(\bigcap_{s>0}[e(R^m)-A_s])=p[e(R^m)-U].$

3.12 COROLLARY. *If $T\in \mathbf{N}_k(R^n)$ and* spt T *is contained in the union of finitely many $(k-1)$-dimensional planes, then $T=0$.*

PROOF. It is sufficient to show that if spt $T\subset C\cup Y$, where C is closed set and Y is a $(k-1)$-dimensional plane, then spt $T\subset C$. For this purpose choose e and p as in 3.11 with $m=k-1$ and $e(R^{k-1})=Y$, and let $U=R^n-C$; it follows that

$$p_\sharp(T\cap U)\in E_k(R^{k-1})=\{0\}\ ,\quad T\cap U=e_\sharp(0)=0\ .$$

The preceding argument fails when $T\in\mathbf{N}_1(R^n)$ and spt T is a finite set, but it is very easy to treat this case directly.

3.13 COROLLARY. *If $T\in\mathbf{N}_k(R^n)$, U is a bounded open subset of R^n, Y is an oriented k-dimensional plane in R^n, $Y\cap U$ is connected and*

$$U\cap\operatorname{spt} T\subset Y\ ,\quad (\partial T)\cap U=0\ ,$$

then there exists a real number r such that

$$T\cap U=r\cdot(Y\cap U)\ ;$$

in case T is rectifiable, r is an integer.

PROOF. Choose e and p as in 3.11 with $m=k$ and $e(R^k)=Y$. Since $p(Y\cap U)=e^{-1}(U)$ is a bounded subregion of R^k and $p_\sharp(T\cap U)\in\mathbf{E}_k(R^k)$, spt $\partial p_\sharp(T\cap U)\subset p(Y-U)=R^k-e^{-1}(U)$, there exists a real number r such that

$$p_\sharp(T\cap U)(\psi)=r\int_{e^{-1}(U)}\psi$$

whenever $\psi\in\mathbf{E}^k(R^k)$ and spt $\psi\subset e^{-1}(U)$. In case T is rectifiable, so are $T\cap U$ and $p_\sharp(T\cap U)$ by 3.8, hence r is an integer.

If $\varphi\in\mathbf{E}^k(R^n)$ and spt $\varphi\subset U$, then spt $e^\sharp(\varphi)\subset e^{-1}(U)$ and

$$\begin{aligned}(T\cap U)(\varphi)&=p_\sharp(T\cap U)[e^\sharp(\varphi)]=r\int_{e^{-1}(U)}e^\sharp(\varphi)\\&=r\int_{Y\cap U}\varphi=r\cdot(Y\cap U)(\varphi)\ .\end{aligned}$$

In order to conclude that $T\cap U=r\cdot(Y\cap U)$ it is therefore sufficient to observe that

$$\mathbf{M}(T\cap U)<\infty\ ,\quad \|T\cap U\|(R^n-U)=0\ ,$$
$$\mathbf{M}(Y\cap U)<\infty\ ,\quad \|Y\cap U\|(R^n-U)=0\ .$$

3.14 COROLLARY. *If C is a rectilinear finite cell complex in R^n, with* *the i-skeletons C_i, and $T \in \mathbf{N}_k(R^n)$,* spt $T \subset C_k$, spt $\partial T \in C_{k-1}$, *then T is a polyhedral chain of C with real coefficients; in case T is rectifiable, the coefficients are integers.*

PROOF. Let $B_1, \cdots, B_q$ be the k-cells of C, each open relative to C_k and with a definite orientation. If Y_j is the k-plane containing B_j, then

$$U_j = R^n - [(C_k \cup Y_j) - B_j]$$

is an open subset of R^n for which

$$U_j \cap C_k = B_j = U_j \cap Y_j ,$$

and 3.13 yields a real number r_j such that

$$T \cap U_j = r_j B_j ;$$

in case T is rectifiable, r_j is an integer. Accordingly

$$S = T - \textstyle\sum_{j=1}^{q} r_j B_j \in \mathbf{N}_k(R^n) ,$$

$$S \cap U_l = 0 \quad \text{for } l = 1, \cdots, q, \text{ spt } S \subset C_{k-1} ,$$

and 3.12 implies that $S = 0$.

4. Maps with singularities

For later use in 5.5, induced chain maps and homotopies are shown to exist even in certain cases where the inducing locally Lipschitzian map has essential singularities on the support of the currents whose images are being sought.

4.1 DEFINITION. Suppose u is a Lipschitzian non-negative real-valued function on R^m and

$$X = \{x : u(x) > 0\} .$$

By a *u-admissible map* we mean a locally Lipschitzian map f of an open subset of R^m containing X into some R^n, carrying bounded sets into bounded sets, with $| Df(x) | \leqq u(x)^{-1}$ for L_n almost all x in X.

We call T a *u-admissible current* if and only if

$$T \in \mathbf{N}_k(R^n) , \quad || T || (R^n - X) = 0 , \quad || \partial T || (R^n - X) = 0 ,$$
$$|| T || (u^{-k}) < \infty , \quad || \partial T || (u^{-k+1}) < \infty .$$

4.2 THEOREM. *Suppose u is a Lipschitzian non-negative real-valued function on R^m and*

$$A_s = \{x : u(x) > s\} \quad \textit{for } s \in R .$$

To each u-admissible map f into R^n corresponds a chain map

$$f_{\sharp u}$$

of the complex of u-admissible currents into $\mathbf{N}_*(R^n)$, *and to any two u-admissible maps f and g into* R^n *corresponds a chain homotopy*

$$H_u(f, g)$$

such that

$$\partial \circ H_u(f, g) + H_u(f, g) \circ \partial = g_{\sharp u} - f_{\sharp u} .$$

If T is a k-dimensional u-admissible current and

$$G = \{s : s > 0,\ T \cap A_s \in \mathbf{N}_k(R^n),\ (\partial T) \cap A_s \in \mathbf{N}_{k-1}(R^n)\} ,$$

then L_1 *almost all positive numbers belong to G, and*

(1) $f_{\sharp u}(T) = \lim_{G \ni s \to 0} f_\sharp(T \cap A_s)$;

(2) $\mathbf{M}[f_{\sharp u}(T)] \leqq \| T \| (u^{-k})$;

(3) $H_u(f, g)(T) = \lim_{G \ni s \to 0} h_\sharp[I \times (T \cap A_s)]$ *where h is the linear homotopy from f to g*;

(4) $\mathbf{M}[H_u(f, g)(T)] \leqq \| T \| (| g - f | \cdot u^{-k})$;

(5) *in case T is rectifiable,* $f_{\sharp u}(T)$ *and* $H_u(f, g)(T)$ *are rectifiable.*

PROOF. Recall 3.10 and observe that if $s \in G$, $t \in G$, $s < t$, then

$$\begin{aligned} \mathbf{M}[f_\sharp(T \cap A_s) - f_\sharp(T \cap A_t)] &= \mathbf{M}[f_\sharp(T \cap (A_s - A_t))] \\ &\leqq \| T \cap (A_s - A_t) \| (u^{-k}) \leqq \int_{R^m - A_t} u^{-k} d \| T \| \end{aligned}$$

and

$$\begin{aligned} &\mathbf{M}(h_\sharp[I \times (T \cap A_s)] - h_\sharp[I \times (T \cap A_t)]) \\ &\quad = \mathbf{M}(h_\sharp[I \times (T \cap (A_s - A_t))]) \leqq \| T \cap (A_s - A_t) \| (| g - f | \cdot u^{-k}) \\ &\quad \leqq \int_{R^m - A_t} | g - f | u^{-k} d \| T \| \end{aligned}$$

according to 3.6. Since the integrals on the right approach 0 with t, and the supports of $f_\sharp(T \cap A_s)$ and $h_\sharp[I \times (T \cap A_s)]$ are contained in the closure of the convex hull of the bounded set

$$f(\operatorname{spt} T) \cup g(\operatorname{spt} T) ,$$

there exist currents $f_{\sharp u}(T)$ and $H_u(f, g)(T)$ satisfying (1), (3) and (5). Moreover (2) and (4) follow from the fact that

$$\begin{aligned} \mathbf{M}[f_\sharp(T \cap A_s)] &\leqq \| T \cap A_s \| (u^{-k}) , \\ \mathbf{M}(h_\sharp[I \times (T \cap A_s)]) &\leqq \| T \cap A_s \| (| g - f | \cdot u^{-k}) \end{aligned}$$

whenever $s \in G$.

To complete the proof one must show that $f_{\sharp u}$ commutes with ∂, and verify the homotopy formula for $g_{\#u} - f_{\#u}$.

Applying (1) to T and ∂T, and using the continuity of ∂, one obtains

$$\begin{aligned}\partial f_{\#u}(T) - f_{\#u}(\partial T) &= \lim_{G \ni s \to 0} [\partial f_{\#}(T \cap A_s) - f_{\#}((\partial T) \cap A_s)] \\ &= \lim_{G \ni s \to 0} f_{\#}(Q_s) ,\end{aligned}$$

where

$$Q_s = \partial(T \cap A_s) - (\partial T) \cap A_s \quad \text{for } s > 0 .$$

Similarly (1), (3) and 3.6 yield

$$\begin{aligned}g_{\#u}(T) - f_{\#u}(T) &= \lim_{G \ni s \to 0} [g_{\#}(T \cap A_s) - f_{\#}(T \cap A_s)] \\ &= \lim_{G \ni s \to 0} \left(\partial h_{\#} I[\ \times (T \cap A_s)] + h_{\#}[I \times \partial(T \cap A_s)]\right)\end{aligned}$$

and

$$\begin{aligned}&\partial[H_u(f, g)(T)] + H_u(f, g)(\partial T) \\ &\qquad = \lim_{G \ni s \to 0} \left(\partial h_{\#}[I \times (T \cap A_s)] + h_{\#}[I \times ((\partial T) \cap A_s)]\right) ,\end{aligned}$$

hence

$$[g_{\#u} - f_{\#u} - \partial \circ H_u(f, g) - H_u(f, g) \circ \partial](T) = \lim_{G \ni s \to 0} h_{\#}(I \times Q_s) .$$

Letting ξ be a Lipschitz constant for u and defining γ as in 3.10, one sees from 3.9 that, for L_1 almost all $s > 0$,

$$\begin{gathered}\mathbf{M}(Q_s) \leqq -\xi\gamma'(s) , \\ u(x) = s \quad \text{for } x \in \operatorname{spt}(Q_s) ,\end{gathered}$$

and hence from 3.6 that

$$\begin{gathered}\mathbf{M}[f_{\#}(Q_s)] \leqq s^{-k+1}\mathbf{M}(Q_s) \leqq -\xi s^{-k+1}\gamma'(s) , \\ \mathbf{M}[h_{\#}(I \times Q_s)] \leqq \varepsilon s^{-k+1}\mathbf{M}(Q_s) \leqq -\varepsilon\xi s^{-k+1}\gamma'(s) ,\end{gathered}$$

where $\varepsilon = \sup \{|g(x) - f(x)| : x \in \operatorname{spt} T\} < \infty$.

Furthermore, if $\delta > 0$, then

$$\begin{aligned}-\delta^{-1}\int_0^\delta s^{-k+1}\gamma'(s)\, ds &\leqq \int_0^\delta s^{-k} \cdot -\gamma'(s)\, ds \\ &\leqq \int_0^\delta s^{-k} d_s[\mathbf{M}(T) - \gamma(s)] = \int_{\{x : u(x) \leqq \delta\}} u^{-k} d\, \|T\|\end{aligned}$$

because $\mathbf{M}(T) - \gamma(s) = \|T\|(\{x : u(x) \leqq s\})$ for $s \in R$; the next to the last integral is a Riemann-Stieltjes integral, improper at 0. Since the last integral approaches 0 with δ, one finds that $-s^{-k+1}\gamma'(s)$ has the approximate limit 0, as $s \to 0+$; the same then follows for $\mathbf{M}[f_{\#}(Q_s)]$ and $\mathbf{M}[h_{\#}(I \times Q_s)]$.

4.3 THEOREM. *Suppose u and v are Lipschitzian non-negative real-valued functions on R^m. If the maps f and g into R^n and the k-dimensional current T are both u- and v-admissible, then*

$$f_{\#u}(T) = f_{\#v}(T) \text{ and } H_u(f, g)(T) = H_v(f,g)(T) .$$

PROOF. Writing $A_s = \{x : u(x) > s\}$ and $B_t = \{x : v(x) > t\}$ one finds that for L_2 almost all (s, t) in R^2 the currents $T \cap A_s$, $T \cap B_t$ and $T \cap A_s \cap B_t$ are normal, with

$$\begin{aligned}\mathbf{M}[f_\#(T \cap A_s) - f_\#(T \cap B_t)] \\ &\leqq \mathbf{M}(f_\#[T \cap (A_s - A_s \cap B_t)]) + \mathbf{M}(f_\#[T \cap (B_t - A_s \cap B_t)]) \\ &\leqq \int_{R^m - B_t} v^{-k} d \, \| T \| + \int_{R^m - A_s} u^{-k} d \, \| T \|\end{aligned}$$

and similarly

$$\begin{aligned}\mathbf{M}(h_\#[I \times (T \cap A_s)] - h_\#[I \times (T \cap B_t)]) \\ &\leqq \int_{R^m - B_t} | g - f | \, v^{-k} d \, \| T \| + \int_{R^m - A_s} | g - f | \, u^{-k} d \, \| T \| .\end{aligned}$$

5. The deformation theorem

Here it is shown how one can deform normal and integral currents into like currents with supports in the appropriate skeletons of a standard cubical cell complex with prescribed mesh, while keeping bounds on the masses of the image and deformation currents. This process is extremely useful throughout the following sections.

5.1 *Construction*. For some fixed positive integer n, let Z be the set of all n-termed sequences of integers; thus

$$Z = \bigcup_{j=0}^n Z_j$$

where Z_j consists of the sequences with j even and $n - j$ odd coordinates. To $\zeta \in Z_j$ correspond the j-dimensional cube $\zeta' = R^n \cap \{x : | x_i - \zeta_i | < 1$ whenever ζ_i is even and $x_i = \zeta_i$ whenever ζ_i is odd$\}$ and the $(n-j)$-dimensional cube $\zeta'' = R^n \cap \{x : | x_i - \zeta_i | < 1$ whenever ζ_i is odd and $x_i = \zeta_i$ whenever ζ_i is even$\}$. Then

$$C' = \{\zeta' : \zeta \in Z\} \text{ and } C'' = \{\zeta'' : \zeta \in Z\}$$

are the familiar dual cubical complexes subdividing R^n, with the k-skeletons

$$C'_k = \bigcup_{j=0}^k \bigcup_{\zeta \in Z_j} \zeta' \text{ and } C''_k = \bigcup_{j=n-k}^n \bigcup_{\zeta \in Z_j} \zeta'' .$$

Assuming that $k = 0, 1, \cdots, n - 1$ let

$$\theta_k(x) = \inf_{\lambda \in \Lambda} \sup_{i=1,\cdots,k+1} | x_{\lambda(i)} | \quad \text{for } x \in R^n ,$$

where Λ is the set of all permutations of $\{1, \cdots, n\}$. Furthermore let $u_k: R^n \to R$ and $\sigma_k: R^n - C''_{n-k-1} \to C'_k$ be defined so as to satisfy the following condition:

If $x \in R^n$, $\zeta \in Z_n$ and $| x_i - \zeta_i | \leqq 1$ for $i = 1, \cdots, n$, then

$$u_k(x) = \theta_k(x - \zeta) \leqq 1 ;$$

in case $x \notin C''_{n-k-1}$, then x has at most k even coordinates, hence $u_k(x) > 0$, and $\sigma_k(x)$ is the point y of R^n such that

$$y_i = \zeta_i + (x_i - \zeta_i)/u_k(x) \quad \text{whenever } | x_i - \zeta_i | < u_k(x) ,$$
$$y_i = \zeta_i + \operatorname{sign}(x_i - \zeta_i) \quad \text{whenever } | x_i - \zeta_i | \geqq u_k(x) ;$$

the second alternative occurs for at least $n - k$ values of i, hence $y \in C'_k$.

This rule for computing $u_k(x)$ and $\sigma_k(x)$ is unambiguous, because if also $\eta \in Z_n$ and $| x_i - \eta_i | \leqq 1$ for $i = 1, \cdots, n$, then for each i either $\eta_i = \zeta_i$ or $| x_i - \eta_i | = 1 = | x_i - \zeta_i |$.

One readily verifies that u_k is Lipschitzian and σ_k is locally Lipschitzian, with

$$| \sigma_k(x) - x | < n \quad \text{for } x \in R^n - C''_{n-k-1} ,$$
$$| D\sigma_k(x) | \leqq n/u_k(x) \quad \text{for } L_n \text{ almost all } x .$$

Regarding the last statement observe that if

$$B = R^n \cap \{x : 0 < x_1 < x_2 < \cdots < x_n < 1\} ,$$

then $u_k(x) = x_{k+1}$ and

$$\sigma_k(x) = (x_1/x_{k+1}, \cdots, x_k/x_{k+1}, 1, \cdots, 1)$$

whenever $x \in B$; accordingly, for $x \in B$ and $h \in R^n$, the i^{th} coordinate of $D\sigma_k(x)(h)$ equals $[h_i - (x_i/x_{k+1})h_{k+1}]/x_{k+1}$ in case $i \leqq k$, and equals 0 in case $i > k$; therefore

$$| D\sigma_k(x)(h) | \leqq (| h | + k | h_{k+1} |)/x_{k+1} \leqq n | h | /u_k(x) .$$

By suitable isometries the result may then be extended from B to L_n almost all of R^n.

Accordingly σ_k is (u_k/n)-admissible, in the sense of 4.1.

If $A = R^n \cap \{x : | x_i | < 1 \text{ for } i = 1, \cdots, n\}$, then

$$\int_A (u_k)^{-k}\, dL_n = 2^n n! \int_B (x_{k+1})^{-k}\, dL_n x$$
$$= 2^n n! \int_0^1 t^{-k} L_{n-1}(B_t)\, dt ,$$

where $B_t = R^{n-1} \cap \{w : 0 < w_1 < \cdots < w_k < t < w_{k+1} < \cdots < w_{n-1} < 1\}$,

$$L_{n-1}(B_t) = t^k (k!)^{-1} (1-t)^{n-1-k} [(n-1-k)!]^{-1}$$

for $0 < t < 1$; consequently

$$\int_A (u_k)^{-k} \, dL_n = 2^n \binom{n}{k} .$$

Inasmuch as $u_k \circ \tau_\zeta = u_k$ for $\zeta \in Z_n$ and A is a fundamental region modulo Z_n, one concludes that

$$\int_A [u_k(x+y)]^{-k} \, dL_n x = 2^n \binom{n}{k} \quad \text{whenever } y \in R^n .$$

This construction is consistently extended to $k = n$ by setting

$$u_n(x) = 1 \text{ and } \sigma_n(x) = x \quad \text{for } x \in R^n .$$

The functions u_k and σ_k have a very simple geometric meaning. When R^n is remetrized by the norm θ_{n-1}, then $u_k(x)$ becomes the distance from x to C''_{n-k-1}. The points $\sigma_{n-1}(x), \sigma_{n-2}(x), \cdots, \sigma_0(x)$ are obtained from x by successive central projections of the cubes of C' onto their boundaries.

5.2 LEMMA. *If γ is a Radon measure over R^n and*

$$A = R^n \cap \{a : |a_i| < 1 \text{ for } i = 1, \cdots, n\} ,$$

then $\gamma(\{x : (u_k \circ \tau_a)(x) = 0\}) = 0$ for L_n almost all a in A, and

$$\int_A \gamma[(u_k \circ \tau_a)^{-k}] \, dL_n a = \binom{n}{k} \gamma(R^n) L_n(A) .$$

PROOF. Assuming γ non-negative and setting $u_k(x)^{-k} = \infty$ whenever $u_k(x) = 0$, one finds with the help of Fubini's theorem that

$$\begin{aligned} L_n(A)\binom{n}{k}\gamma(R^n) &= \int_{R^n} 2^n \binom{n}{k} \, d\gamma y \\ &= \int_{R^n}\int_A [u_k(a+y)]^{-k} \; dL_n a \; d\gamma y \\ &= \int_A \int_{R^n} [(u_k \circ \tau_a)(y)]^{-k} \; d\gamma y \; dL_n a . \end{aligned}$$

5.3 REMARK. If $f_j \geqq 0$ and $\int_A f_j \, dL_n = c_j L_n(A)$ for $j = 1, \cdots, q$, then

$$L_n(A \cap \{a : f_j(a) \leqq q c_j \text{ for } j = 1, \cdots, q\}) > 0 .$$

In fact otherwise $L_n(A) = \int_A \sum_{j=1}^q (f_j / q c_j) \, dL_n > \int_A 1 \, dL_n$.

5.4 REMARK. If $f : R^l \to R^n$ and $g : R^m \to R^n$ are Lipschitzian maps and

$l + m < n$, then $f(R^l) \cap (\tau_a \circ g)(R^m)$ is empty for L_n almost all a. In fact the set of all points a in R^n which lack this property is the image of the Lipschitzian map

$$\psi : R^l \times R^m \to R^n \;, \quad \psi(x, y) = f(x) - g(y) \quad \text{for } x \in R^l,\, y \in R^m \;.$$

5.5 THEOREM. *If $T \in \mathbf{N}_k(R^n)$ and $\varepsilon > 0$, then there exist*

$$P \in \mathbf{N}_k(R^n) \;, \quad Q \in \mathbf{N}_k(R^n) \;, \quad S \in \mathbf{N}_{k+1}(R^n)$$

with the following properties:

(1) $T = P + Q + \partial S$.

(2) $$\frac{\mathbf{M}(P)}{\varepsilon^k} \leqq 2n^k\left[\binom{n}{k}\frac{\mathbf{M}(T)}{\varepsilon^k} + \binom{n}{k-1}\frac{\mathbf{M}(\partial T)}{\varepsilon^{k-1}}\right],$$

$$\frac{\mathbf{M}(\partial P)}{\varepsilon^{k-1}} \leqq 2n^{k-1}\binom{n}{k-1}\frac{\mathbf{M}(\partial T)}{\varepsilon^{k-1}},$$

$$\frac{\mathbf{M}(Q)}{\varepsilon^k} \leqq 6n^k\binom{n}{k-1}\frac{\mathbf{M}(\partial T)}{\varepsilon^{k-1}},$$

$$\frac{\mathbf{M}(S)}{\varepsilon^{k+1}} \leqq 4n^{k+1}\binom{n}{k}\frac{\mathbf{M}(T)}{\varepsilon^k}.$$

(3) $\operatorname{spt}(P) \cup \operatorname{spt}(S) \subset \{x : \text{distance}\,(x, \operatorname{spt} T) \leqq 2n\varepsilon\}$,
$\operatorname{spt}(\partial P) \cup \operatorname{spt}(Q) \subset \{x : \text{distance}\,(x, \operatorname{spt} \partial T) \leqq 2n\varepsilon\}$.

(4) *In case T is an integral current, so are P, Q, S.*

(5) *P is a polyhedreal chain of $\mu_\varepsilon(C')$ with real coefficients; in case T is an integral current, the coefficients of P are integers.*

(6) *In case T is a Lipschitz chain [an integral Lipschitz chain], so are P, Q, S.*

(7) *In case ∂T is a Lipschitz chain [an integral Lipschitz chain], so is Q.*

PROOF. Since the general theorem is easily reduced to the special case $\varepsilon = 1$ by means of the homothetic transformation μ_ε, we assume that $\varepsilon = 1$.

According to 5.2 and 5.3 we may choose $a \in A$ so that

$$\|T\|\big(\{x : (u_k \circ \tau_a)(x) = 0\}\big) = 0 \;,$$
$$\|\partial T\|\big(\{x : (u_{k-1} \circ \tau_a)(x) = 0\}\big) = 0 \;,$$
$$\|T\|\,[(u_k \circ \tau_a)^{-k}] \leqq 2\binom{n}{k}\mathbf{M}(T) \;,$$
$$\|\partial T\|\,[(u_{k-1} \circ \tau_a)^{-k+1}] \leqq 2\binom{n}{k-1}\mathbf{M}(\partial T) \;.$$

Abbreviating

$$u = (u_k \circ \tau_a)/n \;, \quad v = (u_{k-1} \circ \tau_a)/n$$

and noting that $u \geqq v$, we see that T, ∂T, $\sigma_k \circ \tau_a$ and σ_n are u-admissible, ∂T, $\sigma_{k-1} \circ \tau_a$ and $\sigma_k \circ \tau_a$ are v-admissible, and use 4.2 to define

$$\begin{aligned} P_1 &= (\sigma_k \circ \tau_a)_{\sharp u}(T) , \\ Q_1 &= H_u(\sigma_k \circ \tau_a, \sigma_n)(\partial T) , \\ S &= H_u(\sigma_k \circ \tau_a, \sigma_n)(T) , \\ P_2 &= - Q_2 = H_v(\sigma_k \circ \tau_a, \sigma_{k-1} \circ \tau_a)(\partial T) , \\ P &= P_1 + P_2, \qquad Q = Q_1 + Q_2 . \end{aligned}$$

We immediately obtain (1) from the equation

$$\partial S + Q_1 = T - P_1 ,$$

and also note that (4) holds. Moreover

$$\begin{aligned} \partial P_1 &= (\sigma_k \circ \tau_a)_{\sharp u}(\partial T) = (\sigma_k \circ \tau_a)_{\sharp v}(\partial T) , \\ \partial P_2 &= (\sigma_{k-1} \circ \tau_a)_{\sharp v}(\partial T) - (\sigma_k \circ \tau_a)_{\sharp v}(\partial T) , \\ \partial P &= (\sigma_{k-1} \circ \tau_a)_{\sharp v}(\partial T) . \end{aligned}$$

Since the images of σ_k and σ_{k-1} are C'_k and C'_{k-1}, and since for $x \in R^n - C''_{n-k}$ the line segment from $\sigma_k(x)$ to $\sigma_{k-1}(x)$ is contained in C'_k, we find that

$$\operatorname{spt} P \subset C'_k, \ \operatorname{spt} \partial P \subset C'_{k-1} ;$$

hence (5) follows from 3.14. Using the fact that

$$| (\sigma_k \circ \tau_a)(x) - x | < 2n \quad \text{for } x \in R^n - \tau_a^{-1}(C''_{n-k-1}) ,$$

$$| (\sigma_k \circ \tau_a)(x) - (\sigma_{k-1} \circ \tau_a)(x) | < n \quad \text{for } x \in R^n - \tau_a^{-1}(C''_{n-k}) ,$$

we obtain (3) and the estimates

$$\mathbf{M}(P_1) \leqq || T || (u^{-k}) \leqq 2n^k \binom{n}{k} \mathbf{M}(T) ,$$

$$\mathbf{M}(P_2) \leqq || \partial T || (nv^{-k+1}) \leqq 2n^k \binom{n}{k-1} \mathbf{M}(\partial T) ,$$

$$\mathbf{M}(Q_1) \leqq || \partial T || (2nu^{-k+1}) \leqq 4n^k \binom{n}{k-1} \mathbf{M}(\partial T) ,$$

$$\mathbf{M}(S) \leqq || T || (2nu^{-k}) \leqq 4n^{k+1} \binom{n}{k} \mathbf{M}(T) ,$$

$$\mathbf{M}(\partial P) \leqq || \partial T || (v^{-k+1}) \leqq 2n^{k-1} \binom{n}{k-1} \mathbf{M}(\partial T) ,$$

which imply (2).

In case T is a Lipschitz chain, one may by 5.4 choose a so that

$$\tau_a(\operatorname{spt} T) \subset R^n - C''_{n-k-1} .$$

Similarly, if ∂T is a Lipschitz chain, one may require that

$$\tau_a(\operatorname{spt}\partial T)\subset R^n - C''_{n-k} .$$

Then (6) and (7) are evident.

5.6 REMARK. The numerical coefficients in 5.5 (2) are far from the best possible. However their precise values are not important for applications in this paper. Only the exponents of ε really matter.

By a slight modification one can make sure that if ∂T is a polyhedral chain [with integer coefficients], so is Q. In fact one may assume that $\tau_a(\operatorname{spt}\partial T)$ does not meet C''_{n-k}, and replace σ_k and σ_{k-1} by suitable approximations which are simplicial on a neighborhood of $\tau_a(\operatorname{spt}\partial T)$.

5.7 LEMMA. *If $X \in \mathbf{I}_k(R^n)$, ∂X is an integral Lipschitz chain and $\eta > 0$, then there exists $S \in \mathbf{I}_{k+1}(R^n)$ such that $X - \partial S$ is an integral Lipschitz chain, $\mathbf{N}(S) \leqq \eta$, $\operatorname{spt} S \subset \{x : \text{distance}(x, \operatorname{spt} X) \leqq \eta\}$.*

PROOF. Let Y be an integral Lipschitz chain for which

$$\left[1 + 2n^k\binom{n}{k}\right]\mathbf{M}(X - Y) \leqq \eta/2 ,$$

$$\operatorname{spt} Y \subset \{x : \text{distance}(x, \operatorname{spt} X) \leqq \eta/2\} ,$$

choose ε so that $0 < 2n\varepsilon \leqq \eta/2$ and

$$\varepsilon\left[4n^{k+1}\binom{n}{k}\mathbf{M}(X - Y) + 8n^k\binom{n}{k-1}\mathbf{M}(\partial X - \partial Y)\right] \leqq \eta/2 ,$$

and apply 5.5 with $T = X - Y$. Since $\partial T = \partial X - \partial Y$ is an integral Lipschitz chain, so are Q and

$$X - \partial S = Y + P + Q .$$

Furthermore

$$\mathbf{N}(S) \leqq \mathbf{M}(S) + \mathbf{M}(X - Y) + \mathbf{M}(P) + \mathbf{M}(Q) \leqq \eta .$$

5.8 THEOREM. *For each integral current T there exists a sequence of integral Lipschitz chains T_i such that*

$$\lim_{i\to\infty} T_i = T \text{ and } \lim_{i\to\infty} \mathbf{N}(T_i - T) = 0 .$$

PROOF. Suppose i is a positive integer. First apply 5.7 with $X = \partial T$ to obtain an integral current S such that $\partial T - \partial S$ is an integral Lipschitz chain, $\mathbf{N}(S) \leqq i^{-1}$, $\operatorname{spt} S \subset \{x : \text{distance}(x, \operatorname{spt}\partial T) \leqq i^{-1}\}$. Then apply 5.7 with $X = T - S$ to obtain an integral current S' such that $T - S - \partial S'$ is an integral Lipschitz chain, $\mathbf{N}(S') \leqq i^{-1}$, $\operatorname{spt} S' \subset \{x : \text{distance}(x, \operatorname{spt} T - S) \leqq i^{-1}\}$. Letting $T_i = T - S - \partial S'$ one finds that $\mathbf{N}(T_i - T) \leqq \mathbf{M}(S) + \mathbf{M}(\partial S') + \mathbf{M}(\partial S) \leqq 2i^{-1}$, $\operatorname{spt}(T_i) \cup \{x : \text{distance}(x, \operatorname{spt} T) \leqq 2i^{-1}\}$.

5.9 DEFINITION. A is a *local Lipschitz neighborhood retract* in R^n if and only if there exist a neighborhood U of A in R^n and a locally Lipschitzian map $f : U \to A$ such that $f(x) = x$ for $x \in A$.

5.10 REMARK. A subset of R^n is a local Lipschitz neighborhood retract if and only if it is locally compact and locally contractible by Lipschitzian deformations.

5.11 THEOREM. *If A and B are local Lipschitz neighborhood retracts in R^n, with $A \supset B$, then the homology groups of the chain complex*

$$\mathbf{I}_*(A)/\mathbf{I}_*(B)$$

are isomorphic with the singular homology groups of (A, B) with integer coefficients. This isomorphism is induced by the inclusion map of the complex of integral Lipschitz chains into the complex of integral currents.

Similarly the homology groups of the chain complex

$$\mathbf{N}_*(A)/\mathbf{N}_*(B)$$

are isomorphic with the singular homology groups of (A, B) with real coefficients.

PROOF. By virtue of the five-lemma one need only consider the case when B is the empty set.

Suppose $T \in \mathbf{I}_k(A)$, choose U and f according to 5.9, and use 5.5 with

$$2n\varepsilon < \text{distance}\,(\operatorname{spt} T,\, R^n - U)\,.$$

Then $f_\#(P) \in \mathbf{I}_k(A)$, $f_\#(Q) \in \mathbf{I}_k(A)$, $f_\#(S) \in \mathbf{I}_{k+1}(A)$ and $f_\#(P)$ is an integral Lipschitz chain. Furthermore 3.6 implies that $f_\#(T) = T$.

If $\partial T = 0$, then $Q = 0$ and $T = f_\#(P) + \partial f_\#(S)$.

If ∂T is an integral Lipschitz chain, so is $f_\#(P+Q)$, and $\partial T = \partial f_\#(P+Q)$.

6. Isoperimetric inequalities

A general relative isoperimetric inequality is established for pairs of local Lipschitz neighborhood retracts. There follows a treatment of normal n-dimensional currents in n-space which includes and extends the known basic theory of distributions whose partial derivatives are measures.

6.1 THEOREM. *Suppose $A \supset B$, U and V are neighborhoods of A and B in R^n, $f : U \to A$ and $g : V \to B$ are retractions, Γ is a compact subset of A, $a > 0$, $b > 0$,*

f has the Lipschitz constant ξ on $\{x : \text{distance}\,(x, \Gamma) \leqq a\}$,

g has the Lipschitz constant η on $\{x : \text{distance}\,(x, B \cap \Gamma) \leqq b\}$.

If $X \in \mathbf{I}_k(\Gamma)$, $\operatorname{spt} \partial X \subset B$ *and*

$$c_1 \, \| X \| (A - B) \leqq [\inf \{b, (\eta + 2)^{-1}a\}]^k ,$$

then there exists $Y \in \mathbf{I}_{k+1}(A)$ *such that*

$$\operatorname{spt}(X - \partial Y) \subset B ,$$
$$\mathbf{M}(Y)^{k/(k+1)} \leqq c_2 \xi^k (\eta + 2)^k \, \| X \| (A - B) ,$$
$$\mathbf{M}(\partial Y) \leqq c_3 (\eta + 2)^k \, \| X \| (A - B) .$$

Here

$$c_1 = 2^{-k} 3^{k+1} n^{2k} \binom{n+1}{k}, \; c_3 = 3 + 6n^2 \binom{n}{k-1} ,$$
$$c_2 = \left(\left[4n^{k+1} \binom{n}{k} + 3n \right]^k 3n^k \binom{n+1}{k} 2^{-k} \right)^{1/(k+1)} .$$

PROOF. Assume $\| X \| (A - B) > 0$, let ε be the positive number such that

$$3n^k \binom{n+1}{k} \| X \| (A - B) = 2^k \varepsilon^k ,$$

and let

$$X_s = X \cap \{x : \text{distance}(x, B \cap \Gamma) > s\} \quad \text{for } s \in R .$$

Since $\operatorname{spt} \partial X \subset B \cap \Gamma$ one sees from 3.10 that

$$\int_0^\varepsilon \mathbf{M}(\partial X_s) \, ds \leqq \| X \| (A - B) ,$$

and one may therefore choose s so that $0 < s < \varepsilon$, $\varepsilon \mathbf{M}(\partial X_s) \leqq \| X \| (A - B)$. Clearly $\mathbf{M}(X_s) \leqq \| X \| (A - B)$ and

$$\operatorname{spt} \partial X_s \subset \{x : \text{distance}(x, B \cap \Gamma) = s\} .$$

Applying 5.5 with $T = X_s$ one obtains

$$\mathbf{M}(P) \leqq 2n^k \left[\binom{n}{k} + \binom{n}{k-1} \right] \| X \| (A - B) < 2^k \varepsilon^k .$$

But P is an integral linear combination of disjoint k-dimensional cubes with side length 2ε, hence $P = 0$. Accordingly

$$X = X - X_s + Q + \partial S ,$$
$$\operatorname{spt}(X - X_s + Q) \subset \{x : \text{distance}(x, B \cap \Gamma) \leqq 3n\varepsilon\} ,$$
$$\operatorname{spt}(S) \subset \{x : \text{distance}(x, \Gamma) \leqq 2n\varepsilon\} .$$

Next observe that $3n\varepsilon \leqq b$, and that

$$| g(x) - x | \leqq (\eta + 1) 3n\varepsilon \quad \text{whenever distance}(x, B \cap \Gamma) \leqq 3n\varepsilon ,$$

because $y \in B \cap \Gamma$ implies $| g(x) - x | \leqq | g(x) - g(y) | + | y - x |$. Hence one may construct a map $G : R^n \to R^n$ with Lipschitz constant $\eta + 2$ such that

$$G(x) = g(x) \text{ whenever distance } (x, B \cap \Gamma) \leqq 3n\varepsilon ,$$
$$| G(x) - x | \leqq (\eta + 1)3n\varepsilon \quad \text{whenever } x \in R^n .$$

For this purpose one may first procure a map $\gamma: R^n \to R^n$ with Lipschitz constant η such that

$$\gamma(x) = g(x) \quad \text{whenever distance } (x, B \cap \Gamma) \leqq 3n\varepsilon ,$$

let

$$\beta : R^n \to R^n \cap \{y : | y | \leqq (\eta + 1)3n\varepsilon\}$$

be the radial retraction, and define

$$G(x) = x + \beta[\gamma(x) - x] \quad \text{for } x \in R^n .$$

Letting h be the linear homotopy from G to the identity map of R^n, one sees from 3.6 that

$$X - G_\#(X) = \partial h_\#(I \times X)$$

because $G(x) = x$ for $x \in \operatorname{spt} \partial X$. Consequently

$$X = G_\#(X - X_s + Q) + \partial[G_\#(S) + h_\#(I \times X)] ,$$
$$\operatorname{spt} G_\#(X - X_s + Q) \subset B \cap \{x : \text{distance } (x, \Gamma) \leqq (\eta + 2)3n\varepsilon\} ,$$
$$\operatorname{spt} [G_\#(S) + h_\#(I \times X)] \subset \{x : \text{distance } (x, \Gamma) \leqq (\eta + 2)3n\varepsilon\} ,$$

with $(\eta + 2)3n\varepsilon \leqq a$, and one may define

$$Y = f_\#[G_\#(S) + h_\#(I \times X)]$$

to obtain

$$\partial Y = f_\#[X - G_\#(X - X_s + Q)] = X - G_\#(X - X_s + Q)$$

because $f(x) = x$ for $x \in A$. Furthermore

$$\begin{aligned} \mathbf{M}[G_\#(S)] &\leqq (\eta + 2)^{k+1}\mathbf{M}(S) \\ &\leqq (\eta + 2)^{k+1} 4n^{k+1}\binom{n}{k}\varepsilon \, || X || \, (A - B) , \end{aligned}$$
$$\mathbf{M}[h_\#(I \times X)] \leqq (\eta + 1)3n\varepsilon(\eta + 2)^k \, || X || \, (A - B) ,$$

because $G(x) = x$ for $x \in B \cap \operatorname{spt} X$. Inasmuch as

$$[\varepsilon \, || X || \, (A - B)]^{k/(k+1)} = \left[3n^k\binom{n+1}{k}2^{-k}\right]^{1/(k+1)} || X || \, (A - B)$$

one finds that

$$\mathbf{M}[G_\#(S) + h_\#(I \times X)]^{k/(k+1)} \leqq c_2(\eta + 2)^k \,\|X\|\,(A - B)\ ,$$
$$\mathbf{M}(Y)^{k/(k+1)} \leqq \xi^k c_2(\eta + 2)^k \,\|X\|\,(A - B)\ .$$

Finally

$$\partial Y = X_s - G_\#(Q) + [X - X_s - G_\#(X - X_s)]\ ,$$
$$\mathbf{M}[G_\#(Q)] \leqq (\eta + 2)^k 6n^2\binom{n}{k-1}\varepsilon\mathbf{M}(\partial X_s)\ ,$$
$$\mathbf{M}[X - X_s - G_\#(X - X_s)] \leqq 2(\eta + 2)^k \,\|X\|\,(\{x : x \neq G(x)\})\ ,$$
$$\mathbf{M}(\partial Y) \leqq (\eta + 2)^k c_3 \,\|X\|\,(A - B)\ .$$

6.2 REMARK. In case B is empty, so are V and g; one may then take b arbitrarily large and $\eta = 0$, to arrive at the following proposition:

If $X \in \mathbf{I}_k(\Gamma)$ *with*

$$\partial X = 0 \text{ and } c_1\mathbf{M}(X) \leqq a^k\ ,$$

then there exists $Y \in \mathbf{I}_{k+1}(A)$ *such that*

$$\partial Y = X \text{ and } \mathbf{M}(Y)^{k/(k+1)} \leqq 2^k c_2 \xi^k \mathbf{M}(X)\ .$$

In case A is convex, one can take a arbitrarily large and $\xi = 1$.

Of course the constants c_1, c_2, c_3 are far from the best possible.

6.3 COROLLARY. *If* $T \in \mathbf{I}_n(R^n)$, *then*

$$\mathbf{M}(T)^{(n-1)/n} \leqq c_4\mathbf{M}(\partial T)\ .$$

Here $c_4 = 2^{n-1}c_2$ *and* c_2 *is as in* 6.1 *with* $k = n - 1$.

PROOF. Let $X = \partial T$, $A = R^n$ to obtain $Y \in \mathbf{I}_n(R^n)$ with

$$\partial Y = \partial T \text{ and } \mathbf{M}(Y)^{(n-1)/n} \leqq c_4\mathbf{M}(\partial T)\ .$$

Furthermore $Y - T \in \mathbf{I}_n(R^n)$ and $\partial(Y - T) = 0$, hence $Y - T = 0$.

6.4 THEOREM. *If* $T \in \mathbf{N}_n(R^n)$ *with* $n > 1$, *then there exists an* L_n *summable function* ψ *with compact support such that*

$$T = \mathbf{C}\psi \text{ and } \left(\int |\psi|^{n/(n-1)}\, dL_n\right)^{(n-1)/n} \leqq c_5\mathbf{M}(\partial T)\ .$$

Here $c_5 = 2n^n c_4$ *and* c_4 *is as in* 6.3.

PROOF. Let $s = n/(n-1)$.

First consider, for some $\varepsilon > 0$, an n-chain P of the complex $\mu_\varepsilon(C')$, say

$$P = \textstyle\sum_{i=1}^q r_i A_i$$

where A_i are distinct n-cubes of $\mu_\varepsilon(C')$ oriented like R^n and r_i are real coefficients. Letting φ_i be the characteristic function of A_i, one sees that

$$P = \mathbf{C}\psi \text{ with } \psi = \sum_{i=1}^{q} r_i \varphi_i \ .$$

By induction with respect to the number of distinct coefficients of P it will be shown that

$$\left(\int |\psi|^s \, dL_n\right)^{1/s} \leqq c_4 \mathbf{M}(\partial P) \ .$$

For this purpose one may assume that the coefficients of P are positive, because obviously

$$\mathbf{M}(\partial \textstyle\sum_{i=1}^{q} |r_i| A_i) \leqq \mathbf{M}(\partial \textstyle\sum_{i=1}^{q} r_i A_i) \ .$$

In case all r_i are equal, then

$$P = r_1 X \text{ with } X = \sum_{k=1}^{q} A_i \in \mathbf{I}_n(R^n) \ ,$$

and 6.3 implies that

$$\begin{aligned} \left(\int |\psi|^s \, dL_n\right)^{1/s} &= |r_1| \cdot \left(\int \textstyle\sum_{i=1}^{q} \varphi_i \, dL_n\right)^{1/s} \\ &= |r_1| \cdot \mathbf{M}(X)^{1/s} \leqq |r_1| \cdot c_4 \mathbf{M}(\partial X) = c_4 \mathbf{M}(\partial P) \ . \end{aligned}$$

In case $r_i = r_1$ for $i < m$, $r_i \leqq r_m < r_1$ for $i \geqq m$, then one may write

$$\begin{aligned} P &= P' + P'' \ , & \psi &= \psi' + \psi'' \ , \\ P' &= \mathbf{C}\psi' \ , & \psi' &= \textstyle\sum_{i=1}^{m-1} (r_1 - r_m)\varphi_i \ , \\ P'' &= \mathbf{C}\psi'' \ , & \psi'' &= \textstyle\sum_{i=1}^{m} r_m \varphi_i + \sum_{i=m+1}^{m} r_i \varphi_i \ , \end{aligned}$$

where P' and P'' have fewer distinct coefficients than P, hence by induction

$$\begin{aligned} \left(\int |\psi|^s \, dL_n\right)^{1/s} &\leqq \left(\int |\psi'|^s \, dL_n\right)^{1/s} + \left(\int |\psi''|^s \, dL_n\right)^{1/s} \\ &\leqq c_4 \mathbf{M}(\partial P') + c_4 \mathbf{M}(\partial P'') \ . \end{aligned}$$

Furthermore $\mathbf{M}(\partial P') + \mathbf{M}(\partial P'') = \mathbf{M}(\partial P)$. In fact suppose B is an $(n-1)$-dimensional common face of two n-cubes U and V of $\mu_{\varepsilon}(C')$, with incidence numbers 1 and -1. Assume $U = A_i$ for some $i < m$. Then the coefficients of B in $\partial P'$ and $\partial P''$ are 0 and 0 if $V = A_j$ for some $j < m$; $r_1 - r_m$ and $r_m - r_j$ if $V = A_j$ for some $j \geqq m$; $r_1 - r_m$ and r_m if $V \neq A_j$ for $j = 1, 2, \cdots, q$. All these coefficients are non-negative.

Now consider any $T \in \mathbf{N}_n(R^n)$. For each $\varepsilon > 0$, Theorem 5.5 supplies a chain P_ε of $\mu_\varepsilon(C')$ and a normal current Q_ε such that

$$\begin{gathered} T = P_\varepsilon + Q_\varepsilon, \text{ spt } P_\varepsilon \subset \{x : \text{distance } (x, \text{spt } T) \leqq 2n\varepsilon\} \ , \\ \mathbf{M}(\partial P_\varepsilon) \leqq 2n^n \mathbf{M}(\partial T), \quad \mathbf{M}(Q_\varepsilon) \leqq 6n^{n+1} \varepsilon \mathbf{M}(\partial T) \ . \end{gathered}$$

Then one uses the result of the preceding discussion to obtain a step function ψ_ε for which

$$P_\varepsilon = \mathbf{C}\psi_\varepsilon \text{ and } \left(\int |\psi_\varepsilon|^s \, dL_n\right)^{1/s} \leqq c_4 \mathbf{M}(\partial P_\varepsilon) .$$

Observing that, for $\varepsilon > 0$ and $\varepsilon' > 0$,

$$\begin{aligned} \int |\psi_\varepsilon - \psi_{\varepsilon'}| \, dL_n = \mathbf{M}(P_\varepsilon - P_{\varepsilon'}) &= \mathbf{M}(Q_{\varepsilon'} - Q_\varepsilon) \\ &\leqq 6n^{n+1}(\varepsilon' + \varepsilon)\mathbf{M}(\partial T) , \end{aligned}$$

one infers the existence of L_n summable function ψ such that

$$\int |\psi_\varepsilon - \psi| \, dL_n \longrightarrow 0 \quad \text{as } \varepsilon \to 0+ .$$

It follows that $T = \lim_{\varepsilon\to 0+} P_\varepsilon = \mathbf{C}\psi$ and

$$\left(\int |\psi|^s \, dL_n\right)^{1/s} \leqq \liminf_{\varepsilon\to 0+} \left(\int |\psi_\varepsilon|^s \, dL_n\right)^{1/s} \leqq c_4 2n^n \mathbf{M}(\partial T) .$$

6.5 COROLLARY. *If $T \in \mathbf{N}_n(R^n)$, $n > 1$, and T is rectifiable, then*

$$\mathbf{M}(T)^{(n-1)/n} \leqq c_5 \mathbf{M}(\partial T) .$$

PROOF. According to 3.8 (5) one may take ψ integer valued, hence $|\psi| \leqq |\psi|^{n/(n-1)}$.

6.6 REMARK. The above proof of 6.4 has been selected because it exhibits the theorem as a natural consequence of the general theory developed here. Of course the constant c_5 is much larger than necessary, but this is irrelevent for the applications in the present paper.

The best constant, $n^{-1}\alpha(n)^{-1/n}$, may be obtained by the following argument, through which the theorem was actually first discovered:

By regularization the theorem is reduced to the case where $T = \mathbf{C}f$ with $f \in \mathbf{E}^0(R^n)$. For $t \geqq 0$ let

$$A_t = \{x : |f(x)| > t\} , \quad B_t = \{x : |f(x)| = t\} ,$$

let a_t be the characteristic function of A_t, let f_t be the function obtained from f by truncation at heights t and $-t$, and let

$$u(t) = \left(\int |f_t|^{n/(n-1)} \, dL_n\right)^{(n-1)/n} .$$

If $h > 0$, then

$$\begin{aligned} |f_{t+h}| &\leqq |f_t| + h a_t , \\ u(t+h) &\leqq u(t) + h L_n(A_t)^{(n-1)/n} . \end{aligned}$$

One knows from A.P. Morse's theorem [M 1] that, for L_1 almost all $t>0$, B_t is an $(n-1)$-dimensional regular submanifold of class ∞ of R^n, hence the classical isoperimetric inequality (obtained from the Brunn-Minkowski inequality in [DS]) implies that

$$L_n(A_t)^{(n-1)/n} \leqq n^{-1}\alpha(n)^{-1/n}\mathbf{H}^{n-1}(B_t) ,$$

where $\mathbf{H}^{n-1}$ is $(n-1)$-dimensional Hausdorff measure. Therefore

$$\begin{aligned}\left(\int |f|^{n/(n-1)}\, dL_n\right)^{(n-1)/n} &= u(\infty) - u(0) = \int_0^\infty u'(t)\, dt \\ &\leqq n^{-1}\alpha(n)^{-1/n}\int_0^\infty \mathbf{H}^{n-1}(B_t)\, dt ,\end{aligned}$$

and using the coarea formula [FE 6, 3.1] one finds that the last integral equals

$$\int |\operatorname{grad} f|\, dL_n = \mathbf{M}(\partial T) .$$

Note that Theorem 6.4 answers the question raised in [SC, vol. 2, page 41, fourth paragraph]. The results of Kryloff, concerning distributions with compact support whose partial derivatives are representable by p^{th} power summable functions, can also be obtained by the present method. Replacing f by f^v, where $v = p(n-1)/(n-p)$, one finds that

$$\left(\int |f|^{np/(n-p)}\, dL_n\right)^{(n-1)/n} \leqq n^{-1}\alpha(n)^{-1/n}v\int |f|^{v-1}\,|\operatorname{grad} f|\, dL_n$$

and Hölder's inequality with exponents $p/(p-1)$ and p implies that the right integral does not exceed

$$\left(\int |f|^{np/(n-p)}\, dL_n\right)^{(p-1)/p} \cdot \left(\int |\operatorname{grad} f|^p\, dL_n\right)^{1/p} ;$$

hence division yields the inequality

$$\left(\int |f|^{np/(n-p)}\, dL_n\right)^{(n-p)/np} \leqq c\left(\int |\operatorname{grad} f|^p\, dL_n\right)^{1/p}$$

with $c = \alpha(n)^{-1/n}p(n-1)/n(n-p)$.

6.7 REMARK. If $T \in \mathbf{N}_1(R)$, then there exists a real valued function ψ on R such that $\mathbf{C}\psi = T$ and the total variation of ψ equals $\mathbf{M}(\partial T)$; obviously

$$2\,|\psi(x)| \leqq \mathbf{M}(\partial T) \quad \text{for } x \in R .$$

In case T is rectifiable and $T \neq 0$, then ψ is integer valued, hence $2 \leqq \mathbf{M}(\partial T)$.

7. Weak and flat convergence

In this section the deformation theorem (5.5) is shown to imply a connectivity theorem (7.1), which presently explains the relation between normal currents and the flat chains of [W 2], and which will later play an important role in the theory of integral currents.

7.1 THEOREM. *Suppose A and B are local Lipschitz neighborhood retracts in R^n, with $A \supset B$, and Ω is a compact subset of A. If*

$$T_i \in \mathbf{N}_k(\Omega), \quad \operatorname{spt} \partial T_i \subset B \quad \textit{for } i = 1, 2, 3, \cdots,$$
$$\sup \{\mathbf{N}(T_i) : i = 1, 2, 3, \cdots\} < \infty ,$$
$$\lim_{i \to \infty} T_i = T \in \mathbf{N}_k(\Omega) ,$$

then there exist

$$G_i \in \mathbf{N}_k(A) \textit{ and } K_i \in \mathbf{N}_{k+1}(A) \quad \textit{for } i = 1, 2, 3, \cdots$$

with the following properties:

(1) $T_i - T = G_i + \partial K_i$ *for* $i = 1, 2, 3, \cdots$.

(2) $\mathbf{M}(G_i) + \mathbf{M}(K_i) \to 0$ *as* $i \to \infty$.

(3) *In case all* $T_i - T$ *are integral currents, so are all* G_i *and* K_i; *moreover* $\operatorname{spt} G_i \subset B$ *for large* i.

PROOF. One may assume that $T = 0$. Furthermore it is obviously sufficient to show that the conclusions hold for a subsequence of the given sequence. Transitions to subsequences, which occur often in what follows, will be indicated by words but not notationally.

Let λ be the supremum of the set of numbers

$$4n^{k+1}\binom{n}{k}\mathbf{M}(T_i) \text{ and } 6n^k\binom{n}{k-1}\mathbf{M}(\partial T_i)$$

corresponding to $i = 1, 2, 3, \cdots$. Suppose U and V are neighborhoods of A and B in R^n, and

$$f : U \longrightarrow A , \quad g : V \longrightarrow B$$

are locally Lipschitzian retractions. Furthermore suppose

$$0 < 2n\rho < \text{distance}(\Omega, R^n - U)$$

and note that f maps $\{x : \text{distance}(x, \Omega) \leqq 2n\rho\}$ in Lipschitzian fashion onto a compact subset Γ of A.

For each positive number $\varepsilon < \rho$ one may use 5.5 to obtain sequences of normal currents P_i, Q_i, S_i whose supports are contained in

$$\{x : \text{distance}(x, \Omega) \leqq 2n\varepsilon\}$$

and which satisfy the conditions

$$T_i = P_i + Q_i + \partial S_i\ , \quad P_i \text{ is a chain of } \mu_\varepsilon(C')\ ,$$
$$\mathbf{M}(P_i) \leqq \lambda(1+\varepsilon)\ , \quad \mathbf{M}(Q_i) \leqq \lambda\varepsilon\ , \quad \mathbf{M}(S_i) \leqq \lambda\varepsilon\ .$$

The subcomplex of $\mu_\varepsilon(C')$ consisting of all cells within $2n\varepsilon$ of Ω is finite, hence its k-dimensional chains with real coefficients form a finite dimensional real vector space normed by $\mathbf{M}$. Since the chains P_i belong to a bounded subset of this vector space, one may replace the given sequences by a subsequence such that, for all positive integers i and j,

$$\mathbf{M}(P_i - P_j) \leqq \lambda\varepsilon$$

hence

$$T_i - T_j = (P_i - P_j + Q_i - Q_j) + \partial(S_i - S_j)\ ,$$
$$\mathbf{M}(P_i - P_j + Q_i - Q_j) \leqq 3\lambda\varepsilon\ , \quad \mathbf{M}(S_i - S_j) \leqq 2\lambda\varepsilon\ .$$

In case all T_i are integral currents, so are all P_i, Q_i, S_i.

This construction may be applied successively with $\varepsilon = \rho/2, \rho/4, \rho/8, \cdots$, each time yielding subsequences of the preceding. Then Cantor's diagonal process supplies sequences whose i^{th} terms belong to the subsequences corresponding to $\varepsilon = \rho/2, \cdots, \rho/2^i$. Accordingly the Cantor sequence has the following property:

There exist normal currents F_i and H_i whose supports are contained in

$$\{x : \text{distance}\,(x, \Omega) \leqq n\rho 2^{1-i}\}$$

and for which

$$T_i - T_{i+1} = F_i + \partial H_i\ ,$$
$$\mathbf{M}(F_i) \leqq 3\lambda\rho 2^{-i}\ , \quad \mathbf{M}(H_i) \leqq \lambda\rho 2^{1-i}\ .$$

In case all T_i are integral currents, so are all F_i and H_i. It follows that

$$\Phi_j = \textstyle\sum_{i=j}^\infty F_i \in \mathbf{E}_k(U)\ , \qquad \mathbf{M}(\Phi_j) \leqq 3\lambda\rho 2^{1-j}\ ,$$

$$\Psi_j = \textstyle\sum_{i=j}^\infty H_i \in \mathbf{E}_{k+1}(U)\ , \quad \mathbf{M}(\Psi_j) \leqq \lambda\rho 2^{2-j}\ ,$$

$$T_j = \lim_{q\to\infty}(T_j - T_{q+1}) = \lim_{q\to\infty} \textstyle\sum_{i=j}^q (F_i + \partial H_i) = \Phi_j + \partial\Psi_j\ ,$$

and hence also that $\partial\Psi_j = \Phi_j - T_j$ and $\partial\Phi_j = \partial T_j$ have finite mass; thus Φ_j and Ψ_j are normal currents. In case all T_i are integral currents, the Φ_j, Ψ_j, $\partial\Psi_j$, $\partial\Phi_j$ are rectifiable, hence Φ_j and Ψ_j are integral currents.

Inasmuch as

$$T_j = f_\#(T_j) = f_\#(\Phi_j) + \partial f_\#(\Psi_j) \quad \text{for } j = 1, 2, 3, \cdots ,$$

$$\mathbf{M}[f_\sharp(\Phi_j)] + \mathbf{M}[f_\sharp(\Psi_j)] \longrightarrow 0 \quad \text{as } j \to \infty ,$$

the conclusions (1), (2) and the first half of (3) hold with $G_j = f_\sharp(\Phi_j)$ and $K_j = f_\sharp(\Psi_j)$. Moreover, in case all T_i are integral currents one can improve the choice of G_j and K_j for large j; one applies 6.1 with $X_j = f_\sharp(\Phi_j)$ to obtain

$$Y_j \in \mathbf{I}_{k+1}(A) \quad \text{with spt}\,(X_j - \partial Y_j) \subset B ,$$
$$\mathbf{M}(Y_j) + \mathbf{M}(\partial Y_j) \longrightarrow 0 \quad \text{as } j \to \infty ,$$

and one takes $G_j = X_j - \partial Y_j$, $K_j = Y_j + f_\sharp(\Psi_j)$.

7.2 THEOREM. *Suppose* $T \in \mathbf{E}_k(R^n)$ *and* U *is a convex neighborhood of* spt T *in* R^n. *Then*

$$\mathbf{F}(T) \leqq \mathbf{M}(T - \partial K) + \mathbf{M}(K) \textit{ whenever } K \in \mathbf{E}_{k+1}(R^n) ,$$

and equality holds for some K *with* spt $K \subset$ Closure U; *in case* $\mathbf{M}(T) < \infty$, K *is normal.*

PROOF. If $\varphi \in \mathbf{E}^k(R^n)$, $\mathbf{F}(\varphi) \leqq 1$ and $K \in \mathbf{E}_{k+1}(R^n)$, then

$$T(\varphi) = (T - \partial K)(\varphi) + K(d\varphi) \leqq \mathbf{M}(T - \partial K) + \mathbf{M}(K) .$$

In case $\mathbf{F}(T) = \infty$, equality holds for $K = 0$.

Assuming that $\mathbf{F}(T) < \infty$ and U is bounded, consider the vector spaces

$$V^m = \mathbf{E}^m(U) \cap \{\varphi \mid U : \varphi \in \mathbf{E}^m(R^n)\}$$

on which $\mathbf{M}$ is a finite norm, and let W_m be the space of all $\mathbf{M}$-bounded real-valued linear functions on V^m, with the conjugate norm

$$\mathbf{M}'(a) = \sup\,\{a(\psi) : \psi \in V^m,\ \mathbf{M}(\psi) \leqq 1\} \quad \text{for } a \in W_m .$$

Then $W_k \times W_{k+1}$ acts as conjugate space of $V^k \times V^{k+1}$, with norms and pairing defined by

$$\nu(\psi, \chi) = \sup\,\{\mathbf{M}(\psi), \mathbf{M}(\chi)\} ,$$
$$\nu'(a, b) = \mathbf{M}'(a) + \mathbf{M}'(b) ,$$
$$(a, b)\cdot(\psi, \chi) = a(\psi) + b(\chi)$$

for $\psi \in V^k$, $\chi \in V^{k+1}$, $a \in W_k$, $b \in W_{k+1}$. Let

$$e : \mathbf{E}^k(R^n) \longrightarrow V^k \times V^{k+1} ,$$
$$e(\varphi) = (\varphi \mid U, (d\varphi) \mid U) \quad \text{for } \varphi \in \mathbf{E}^k(R^n) ,$$

and note that

$$T(\varphi) \leqq \mathbf{F}(T)\cdot\nu[e(\varphi)] \quad \text{for } \varphi \in \mathbf{E}^k(R^n) .$$

It follows from the Hahn-Banach theorem that $T = t \circ e$ for some linear functional t on $V^k \times V^{k+1}$ with norm $\leqq \mathbf{F}(T)$, and hence there exist $a \in W_k$ and $b \in W_{k+1}$ such that

$$T(\varphi) = a(\varphi \mid U) + b[(d\varphi) \mid U)] \quad \text{for } \varphi \in \mathbf{E}^k(R^n) ,$$
$$\mathbf{M}'(a) + \mathbf{M}'(b) \leqq \mathbf{F}(T) .$$

Defining

$$G \in \mathbf{E}_k(R^n) , \qquad G(\varphi) = a(\varphi \mid U) \quad \text{for } \varphi \in \mathbf{E}^k(R^n) ,$$
$$K \in \mathbf{E}_{k+1}(R^n) , \qquad K(\omega) = b(\omega \mid U) \quad \text{for } \omega \in \mathbf{E}^{k+1}(R^n) ,$$

one finds that

$$T = G + \partial K , \quad \mathbf{M}(G) \leqq \mathbf{M}'(a) , \quad \mathbf{M}(K) \leqq \mathbf{M}'(b) .$$

7.3 COROLLARY. *Suppose A is a compact subset of R^n and*

$$T_i \in \mathbf{N}_k(A) \quad \text{for } i = 1, 2, 3, \cdots ,$$
$$\sup \{\mathbf{N}(T_i) : i = 1, 2, 3, \cdots\} < \infty .$$

Under these conditions $\lim_{i\to\infty} T_i = T \in \mathbf{E}_k(R^n)$ *if and only if*

$$\lim_{i\to\infty} \mathbf{F}(T_i - T) = 0 .$$

7.4 REMARK. Let $\mathbf{F}_k(R^n)$ be the real vector space consisting of those currents $T \in \mathbf{E}_k(R^n)$ for which there exists a sequence of polyhedral chains P_i with real coefficients such that

$$\lim_{i\to\infty} P_i = T \text{ and } \lim_{i\to\infty} \mathbf{F}(P_i - T) = 0 .$$

Note that $\mathbf{F}_k(R^n)$ is a proper subset of

$$\mathbf{E}_k(R^n) \cap \{T : \mathbf{F}(T) < \infty\} ;$$

for example the classical doublet $T \in \mathbf{E}_0(R)$, defined by the formula $T(\varphi) = \varphi'(0)$ for $\varphi \in \mathbf{E}^0(R)$, does not belong to $\mathbf{F}_0(R)$ even though $\mathbf{F}(T) = 1$.

From 5.5 and 7.2 one sees that $\mathbf{N}_k(R^n) \subset \mathbf{F}_k(R^n)$.

If P is a polyhedral chain, then $\mathbf{F}(P)$ equals Whitney's flat norm of P; this fact may be verified with the help of Wolfe's theorem [W2, IX 7C] and regularization. It follows that $\mathbf{F}_k(R^n)$ may be regarded as the space of Whitney's flat chains with compact support. From [W2, VII 5A] one therefore obtains the following approximation theorem:

For every normal current T there exists a sequence of polyhedral chains P_i such that

$$\lim_{i\to\infty} P_i = T \text{ and } \lim_{i\to\infty} \mathbf{N}(P_i) = \mathbf{N}(T) .$$

The analogous proposition concerning integral currents will be proved by an entirely different method in 8.23.

8. Structure and closure theorems for integral currents

In addition to the principal theorems on integral currents (8.12, 8.13, 8.14, 8.22, 8.23) this section contains basic facts (8.8) on the differentiation of currents of finite mass, the result (8.5) that every k-dimensional normal current is absolutely continuous with respect to the k-dimensional Hausdorff and integral geometric measures, and a theorem (8.16) describing the tangential and density properties of rectifiable currents.

8.1 *Projections.* An orthogonal projection of R^n onto R^k is a linear transformation which maps the orthogonal complement of its kernel isometrically onto R^k. If p is such a projection, so is $p \circ g$ for every orthogonal transformation g of R^n; in this way the orthogonal group of R^n operates transitively on the space of all orthogonal projections of R^n onto R^k. The theory of Haar measure therefore gives meaning to the phrase "almost all projections".

Let $\Lambda(k, n)$ be the set of all increasing functions

$$\lambda : \{1, \cdots, k\} \longrightarrow \{1, \cdots n\} \ ;$$

to each such λ corresponds an orthogonal projection

$$p^\lambda \colon R^n \longrightarrow R^k \ , \quad p^\lambda(x) = (x_{\lambda(1)}, \cdots, x_{\lambda(k)}) \quad \text{for } x \in R^n \ .$$

Defining $\omega_k = dX_1 \wedge \cdots \wedge dX_k \in \mathbf{E}^k(R^k)$, where $X_1, \cdots, X_k$ are the usual coordinate functions on R^k, one may write each $\varphi \in \mathbf{E}^k(R^n)$ with coefficients $\varphi_\lambda \in \mathbf{E}^0(R^n)$ in the form

$$\varphi = \textstyle\sum_{\lambda \in \Lambda(k,n)} \varphi_\lambda \wedge p^{\lambda\#}(\omega_k) \ .$$

Accordingly, if $T \in \mathbf{E}_k(R^n)$, then

$$T(\varphi) = \textstyle\sum_{\lambda \in \Lambda(k,n)} [T \wedge p^{\lambda\#}(\omega_k)](\varphi_\lambda) \ .$$

A subset A of R^n is called a *k-dimensional nonparametric Lipschitz manifold* if and only if there exists an orthogonal projection $p : R^n \to R^k$ such that p maps A homeomorphically onto R^k and $(p \mid A)^{-1}$ is Lipschitzian.

8.2 *Hausdorff measure. If $A \subset R^n$*, then

$$\mathbf{H}^k(A)$$

equals the limit, as $r \to 0+$, of the infimum of the sums

$$\textstyle\sum_{B \in F} 2^{-k}\alpha(k) \operatorname{diameter}(S)^k$$

corresponding to all countable coverings F of A such that $\operatorname{diameter}(S) < r$ for $S \in F$.

8.3 *Densities*. If γ is a Carathéodory measure over R^n, $A \subset R^n$ and $z \in R^n$, then

$$\Theta^k(\gamma, A, z) = \lim_{r \to 0+} \alpha(k)^{-1} r^{-k} \gamma(A \cap \{x : |x - z| < r\})$$

is the *k-dimensional* γ *density of* A *at* z; similarly the *upper* and *lower densities*

$$\Theta^{*k}(\gamma, A, z) \text{ and } \Theta^k_*(\gamma, A, z)$$

are defined as the corresponding lim sup and lim inf.

One abbreviates $\Theta^k(\gamma, R^n, z) = \Theta^k(\gamma, z), \cdots$.

The following statements are well known [FE 3, Section 3]:

(1) *If* $A \subset R^n$, $B \subset R^n$, $t > 0$ *and* $\Theta^{*k}(\gamma, A, z) > t$ *for* $z \in B$, *then* $\gamma(A) \geqq t\mathbf{H}^k(B)$.

(2) *If* $A \subset R^n$, $t > 0$ *and* $\Theta^{*k}(\gamma, A, z) < t$ *for* $z \in A$, *then* $\gamma(A) \leqq 2^k t \mathbf{H}^k(A)$.

(3) *If* B *is a Borel subset of* R^n *and* $\gamma(B) < \infty$, *then* $\Theta^k(\gamma, B, z) = 0$ *for* $\mathbf{H}^k$ *almost all* z *in* $R^n - B$.

(4) *If* $A \subset R^n$ *and* $\mathbf{H}^k(A) < \infty$, *then* $2^{-k} \leqq \Theta^{*k}(\mathbf{H}^k, A, z) \leqq 1$ *for* $\mathbf{H}^k$ *almost all* z *in* A.

In case A *is a k-dimensional proper regular submanifold of class* 1 *of* R^n, *then* $\Theta^k(\mathbf{H}^k, A, z) = 1$ *whenever* $z \in A$.

8.4 LEMMA. *If* B *is a Borel subset of* R^n, $p : R^n \to R^k$ *is an orthogonal projection and* $L_k[p(B)] = 0$, *then*

$$(T \cap B) \wedge p^\sharp(\omega_k) = 0 \quad \textit{whenever} \quad T \in \mathbf{N}_k(R^n) .$$

PROOF. Suppose $T \in \mathbf{N}_k(R^n)$ and $A = p(B)$.

If $f \in \mathbf{E}^0(R^n)$, then $p_\sharp(T \wedge f) \in \mathbf{N}_k(R^k)$ and since $L_k(A) = 0$ one sees from 6.4 that

$$0 = p_\sharp(T \wedge f) \cap A = p_\sharp[T \wedge f \cap p^{-1}(A)] ,$$
$$0 = [T \wedge f \cap p^{-1}(A)][p^\sharp(\omega_k)] = [T \wedge p^\sharp(\omega_k) \cap p^{-1}(A)](f) .$$

Accordingly

$$[T \wedge p^\sharp(\omega_k)] \cap p^{-1}(A) = 0 ,$$

and the lemma follows from the fact that $B \subset f^{-1}(A)$.

8.5 THEOREM. *If* B *is a Borel subset of* R^n *such that*

$$L_k[p(B)] = 0$$

for almost all orthogonal projections p *of* R^n *onto* R^k, *then*

$$\| T \| (B) = 0 \quad \textit{whenever} \quad T \in \mathbf{N}_k(R^n) .$$

PROOF. Choose an orthogonal transformation g of R^n such that

$$L_k[(p^\lambda \circ g)(B)] = 0 \quad \text{for all } \lambda \in \Lambda(k, n) .$$

From 8.4 it follows that, if $T \in \mathbf{N}_k(R^n)$, then

$$[T \cap g(B)] \wedge p^{\lambda\#}(\omega_k) = 0 \quad \text{for all } \lambda \in \Lambda(k, n) ,$$

hence $T \cap g(B) = 0$. Replacing T by $g_\#(T)$ one obtains

$$T \cap B = g_\#^{-1}[g_\#(T) \cap g(B)] = 0 .$$

8.6 COROLLARY. *If $T \in \mathbf{N}_k(R^n)$, then*

$$\Theta^{*k}(\| T \|, z) < \infty \quad \textit{for } \| T \| \textit{ almost all } z \textit{ in } R^n .$$

PROOF. Letting

$$B = R^n \cap \{z : \Theta^{*k}(\| T \|, z) = \infty\}$$

one sees from 8.3 (1), with $A = R^n$, that $\mathbf{H}^k(B) = 0$. Consequently

$$L_k[p(B)] = \mathbf{H}^k[p(B)] = 0$$

for every orthogonal projection p of R^n onto R^k, and 8.5 implies that $\| T \|(B) = 0$.

8.7 *Covering and differentiation.* From [BE] and [M 3] one knows the following propositions:

(1) *If $A \subset R^n$, F is a family of closed spherical balls in R^n such that each point of A is the center of arbitrarily small members of F, and γ is a nonnegative Radon measure over R^n, then F has a disjointed subfamily covering γ almost all of A.*

(2) *If β and γ are Radon measures over R^n, with γ nonnegative, then the symmetrical derivate*

$$\frac{d\beta}{d\gamma}(z) = \lim_{r \to 0+} \frac{\beta(\{x : | x - z | < r\})}{\gamma(\{x : | x - z | < r\})}$$

exists and is finite for γ almost all x in R^n. In case β is absolutely continuous with respect to γ, then

$$\beta(A) = \int_A \frac{d\beta}{d\gamma}(z)\, d\gamma z \quad \textit{for every Borel set } A \subset R^n .$$

8.8 THEOREM. *Suppose $T \in \mathbf{E}_k(R^n)$ and $\mathbf{M}(T) < \infty$.*

(1) *For $\| T \|$ almost every z in R^n there exists a k-vector*

$$\vec{T}(z) \in \bigwedge\nolimits_k(R^n)$$

characterized by the property that, for all $\varphi \in \mathbf{E}^k(R^n)$,

$$\varphi(z)[\vec{T}(z)] = \frac{d(T \wedge \varphi)}{d\| T \|}(z) .$$

(2) $\|\vec{T}(z)\| \leqq 1$ *whenever* $\vec{T}(z)$ *exists, and equality holds for* $\|T\|$ *almost all* z *in* R^n.

(3) *For* $\|T\|$ *almost every* z *in* R^n *there exists an orthogonal projection* p *of* R^n *onto* R^k *such that*

$$\left|\frac{d[T\wedge p^\sharp(\omega_k)]}{d\,\|T\|}(z)\right| \geqq \binom{n}{k}^{-1}.$$

PROOF. The uniqueness of $T(z)$ satisfying (1) is evident, and its existence may be shown by prescribing its coordinates to be

$$\frac{d[T\wedge p^{\lambda\sharp}(\omega_k)]}{d\,\|T\|}(z)$$

wherever these derivates exist for all $\lambda \in \Lambda(k, n)$.

The first half of (2) is obvious from (1). To prove the second half observe that if $\varphi \in \mathbf{E}^k(R^n)$ with $\mathbf{M}(\varphi) \leqq 1$, then

$$T(\varphi) = \int \varphi(z)[\vec{T}(z)]\, d\,\|T\|\, z \leqq \int \|\vec{T}(z)\|\, d\,\|T\|\, z\,,$$

hence the right integral equals $\mathbf{M}(T)$.

Finally (3) follows from (2) with $p = p^\lambda$ for some $\lambda \in \Lambda(k, n)$.

8.9 (γ, k)*-rectifiable sets*. Suppose γ is a Carathéodory measure over R^n, and $A \subset R^n$. One calls A a (γ, k)*-rectifiable set* if and only if for each $\varepsilon > 0$ there exists a Lipschitzian map $f: R^k \to R^n$ such that $\gamma[A - f(R^k)] < \varepsilon$.

The following facts were proved in [FE 3; 5.8, 4.6, 4.3, 8.7, 4.7]:

(1) *If* A *is* (γ, k)*-rectifiable,* $\gamma(A) < \infty$ *and*

$$\Theta^{*k}(\gamma, A, a) < \infty \quad \textit{for } \gamma \textit{ almost all } a \textit{ in } A\,,$$

then there exists a countable family of k*-dimensional nonparametric Lipschitz manifolds whose union contains* γ *almost all of* A.

(2) *If* A *is a Borel set,* $\gamma(A) < \infty$,

$$0 < \Theta^{*k}(\gamma, A, x) < \infty \quad \textit{for } \gamma \textit{ almost all } x \textit{ in } A\,,$$

and A *has no Borel subset* B *such that* $\gamma(B) > 0$ *and*

$$L_k[p(B)] = 0$$

for almost all orthogonal projections p *of* R^n *onto* R^k, *then* A *is* (γ, k)*-rectifiable.*

8.10 DEFINITION. A current $T \in \mathbf{N}_k(R^n)$ has the *rectifiable projection property* provided the following condition holds:

If $p: R^n \to R^k$ is an orthogonal projection and $z \in R^n$, then $p_\sharp(T \cap \{x : |x - z| < s\})$ is rectifiable for L_1 almost all positive numbers s.

In case $k = 0$ this condition should be interpreted through the convention $R^0 = \{0\}$; the condition is then clearly equivalent to the statement: $T(\{x : |x - z| < s\})$ is an integer for L_1 almost all s.

8.11 THEOREM. *If $T \in \mathbf{N}_k(R^n)$, $\partial T \in \mathbf{I}_{k-1}(R^n)$ and T has the rectifiable projection property, then $T \in \mathbf{I}_k(R^n)$.*

PROOF.[4] In case $k = 0$, the theorem is trivial. Assume $k \geqq 1$. Since $\partial\partial T = 0$, there exists an integral current S such that $\partial S = \partial T$, for instance by 6.2. Then $T - S$ has the rectifiable projection property with $\partial(T-S) = 0$, and it is sufficient to prove that $T - S$ is an integral current. Replacing T by $T - S$, one may therefore assume that $\partial T = 0$.

First suppose z and p are as in 8.8 (3). For $r > 0$ let

$$A_r = \{x : |z - x| < r\} , \quad m(r) = \| T \| (A_r) ,$$

and note that

$$p_\sharp(T \cap A_r)(\omega_k) = (T \cap A_r)[p^\sharp(\omega_k)] = [T \wedge p^\sharp(\omega_k)](A_r) ,$$

with $\mathbf{M}(\omega_k) = 1$, hence

$$\liminf_{r \to 0+} \frac{\mathbf{M}[p_\sharp(T \cap A_r)]}{m(r)} \geqq \binom{n}{k}^{-1} .$$

Using 3.9, the rectifiable projection property of T, 6.5 and 6.7 one finds, for L_1 almost all $r > 0$, that:

$$m'(r) \geqq \mathbf{M}[\partial(T \cap A_r)] \geqq \mathbf{M}[\partial p_\sharp(T \cap A_r)] ,$$

$$p_\sharp(T \cap A_r) \text{ is a normal rectifiable current} ,$$

$$\mathbf{M}[\partial p_\sharp(T \cap A_r)] \geqq c\mathbf{M}[p_\sharp(T \cap A_r)]^{(k-1)/k} ,$$

where c is a positive constant; in case $k = 1$, r must be taken so small that $p_\sharp(T \cap A_r) \neq 0$. The resulting inequality

$$m(r)^{1/k-1} m'(r) \geqq c\left(\frac{\mathbf{M}[p_\sharp(T \cap A_r)]}{m(r)}\right)^{(k-1)/k}$$

may be integrated, because $m^{1/k}$ is nondecreasing, to give

$$\liminf_{r \to 0+} r^{-1} m(r)^{1/k} \geqq (c/k)\binom{n}{k}^{(1-k)/k} ,$$

$$\Theta^k_*(\| T \|, z) \geqq (c/k)^k \binom{n}{k}^{1-k} \alpha(k)^{-1} > 0 .$$

It now follows from 8.6, 8.5 and 8.9 (2) that R^n is $\| T \|$ rectifiable.

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[4] Regarding the argument showing that $\Theta^k_*(\| T \|, z) > 0$, acknowledgment is made of the previous use of an isoperimetric inequality for a similar purpose in [**FY 1**, page 132].

According to 8.9 (1) there exist k-dimensional nonparametric Lipschitz manifolds B_j such that

$$\| T \| (R^n - \textstyle\bigcup_{j=1}^{\infty} B_j) = 0 .$$

Choose orthogonal projections p_j: $R^n \to R^k$ such that p_j maps B_j homeomorphically onto R^k and $(p_j \mid B_j)^{-1}$ has a finite Lipschitz constant η_j.

Suppose $\varepsilon > 0$. Abbreviating

$$K(z, r) = \{x : | z - x | < r\} ,$$

let G_j be the set of all pairs (z, r) for which $z \in B_j$, $r > 0$,

$$\| T \| [K(z, r) - B_j] \leqq \varepsilon\eta_j^{-k} \| T \| [K(z, r)] ,$$
$$\| T \| \big(\{x : | z - x | = r\}\big) = 0 ,$$

$p_{j\#}[T \cap K(z, r)]$ is a rectifiable current. Since 8.7 (2) and the rectifiable projection property of T imply that

$$\inf \{r : (z, r) \in G_j\} = 0$$

for $\| T \|$ almost all z in B_j, one infers from 8.7 (1) the existence of a countable set

$$H \subset \textstyle\bigcup_{j=1}^{\infty} G_j$$

such that the family

$$\{K(z, r) : (z, r) \in H\}$$

is disjointed and covers $\| T \|$ almost all of R^n.

For each $(z, r) \in H \cap G_j$ the current

$$S(z, r) = [(p_j \mid B_j)^{-1} \circ p_j]_\#[T \cap K(z, r)]$$

is rectifiable; inasmuch as

$$[(p_j \mid B_j)^{-1} \circ p_j](x) = x \quad \text{for } x \in B_j ,$$

one sees from 3.6 that

$$\begin{aligned} \mathbf{M}[S(z, r) - T \cap K(z, r)] &\leqq 2\eta_j^k \| T \| [K(z, r) - B_j] \\ &\leqq 2\varepsilon \| T \| [K(z, r)] . \end{aligned}$$

Therefore the current

$$S = \textstyle\sum_{(z,r) \in H} S(z, r)$$

is rectifiable and

$$\mathbf{M}(S - T) \leqq \textstyle\sum_{(z,r) \in H} 2\varepsilon \| T \| K(z, r) = 2\varepsilon \mathbf{M}(T) .$$

8.12 THEOREM. *If*

$$T_i \in \mathbf{I}_k(R^n) \quad for\ i = 1, 2, 3, \cdots ,$$
$$\lim_{i\to\infty} T_i = T \in \mathbf{N}_k(R^n) ,$$
$$\lim_{i\to\infty} \mathbf{F}(T_i - T) = 0 ,$$

then $T \in \mathbf{I}_k(R^n)$.

PROOF. Since $\mathbf{F}(\partial T_i - \partial T) \leqq \mathbf{F}(T_i - T) \to 0$ as $i \to \infty$, one may assume, by induction with respect to k, that ∂T is an integral current. In view of 8.11 it is therefore sufficient to verify that T has the rectifiable projection property.

Suppose $p : R^n \to R^k$ is an orthogonal projection, $z \in R^n$, and let

$$A_s = \{x : |x - z| < s\} \quad \text{for } s > 0 .$$

Use 7.2 to choose normal currents G_i and K_i such that

$$T_i - T = G_i + \partial K_i \quad \text{for } i = 1, 2, 3, \cdots ,$$
$$\mathbf{M}(G_i) + \mathbf{M}(K_i) \longrightarrow 0 \quad \text{as } i \to \infty .$$

Note that, for each i and L_1 almost all $s > 0$,

$T_i \cap A_s - T \cap A_s = G_i \cap A_s + \partial(K_i \cap A_s) - [\partial(K_i \cap A_s) - (\partial K_i) \cap A_s]$,

$p_\#(T_i \cap A_s)$ is rectifiable,

$p_\#(K_i \cap A_s) \in \mathbf{N}_{k+1}(R^k) = \{0\}$.

For each $b > 0$ it follows from 3.10 that

$$\int_0^b \mathbf{M}[p_\#(T_i \cap A_s) - p_\#(T \cap A_s)]\, ds$$
$$\leqq \int_0^b (\mathbf{M}[G_i \cap A_s] + \mathbf{M}[\partial(K_i \cap A_s) - (\partial K_i) \cap A_s)]\, ds$$
$$\leqq b\mathbf{M}(G_i) + \mathbf{M}(K_i)$$

for each i, hence Fatou's lemma yields

$$\liminf_{i\to\infty} \mathbf{M}[p_\#(T_i \cap A_s) - p_\#(T \cap A_s)] = 0$$

for L_1 almost all s between 0 and b.

8.13 COROLLARY. *If A is a compact subset of R^n and c is a positive number, then $\mathbf{I}_k(A) \cap \{T : \mathbf{N}(T) \leqq c\}$ is compact.*

PROOF. By 7.3 and 8.12 the above set is closed in

$$\mathbf{N}_k(A) \cap \{T : \mathbf{N}(T) \leqq c\} ,$$

whose compactness is evident.

8.14 COROLLARY. *Every normal rectifiable current is an integral current.*

PROOF. Apply 8.12 to the case when the T_i are integral Lipschitz chains and

$$\lim_{i\to\infty} T_i = T \in \mathbf{N}_k(R^n) \;, \quad \lim_{i\to\infty} \mathbf{M}(T_i - T) = 0 \;.$$

8.15 REMARK. Suppose γ is a nonnegative Radon measure over R^n with bounded support,

$$v : R^n \longrightarrow \bigwedge_k(R^n)$$

is a Baire function such that

$$\int \| v(x) \| \, d\gamma x < \infty \;,$$

and let

$$T(\varphi) = \int \varphi(x)[v(x)] \, d\gamma x \quad \text{for } \varphi \in \mathbf{E}^k(R^n) \;.$$

Then $T \in \mathbf{E}_k(R^n)$, $\mathbf{M}(T) < \infty$,

$$\| T \| (B) = \int_B \| v(x) \| \, d\gamma x \quad \text{for every Borel set } B \subset R^n \;,$$

$$\Theta^k(\| T \|, z) = \| v(z) \| \; \Theta^k(\gamma, z) \quad \text{for } \gamma \text{ almost all } z \;,$$

$$\vec{T}(z) = \frac{v(z)}{\| v(z) \|} \qquad \text{for } \| T \| \text{ almost all } z \;.$$

8.16 THEOREM. *If T is a k-dimensional rectifiable current in R^n, then:*
(1) *For $\mathbf{H}^k$ almost all z in R^n, $\Theta^k(\| T \|, z)$ is an integer.*
(2) *For every Borel subset B of R^n,*

$$\| T \| (B) = \int_B \Theta^k(\| T \|, z) \, d\mathbf{H}^k z \;.$$

(3) *There exists a countable family F of k-dimensional proper regular submanifolds of class 1 of R^n such that, for $\| T \|$ almost every z in R^n, $\vec{T}(z)$ is a simple k-vector whose k-space is the tangent space at z of some member of F.*

PROOF. Let C be the class of all $T \in \mathbf{E}_k(R^n)$ which satisfy the conditions (1), (2), (3). The theorem will be proved through the verification of the following six statements; in (i), (iv), (v) it is assumed that U is a k-dimensional open convex cell in R^k and f maps R^k into R^n.

(i) If f is continuously differentiable, f is univalent on the closure of U, and $Df(u)$ is univalent for $u \in U$, then $f_\#(U) \in C$.

(ii) If $T_i \in C$ for $i = 1, 2, 3, \cdots$ and

$$\textstyle\sum_{i=1}^{\infty} T_i = T \in \mathbf{E}_k(R^n) \;, \quad \sum_{i=1}^{\infty} \mathbf{M}(T_i) < \infty \;,$$

then $T \in C$.

(iii) If $T_i \in C$ for $i = 1, 2, 3, \cdots$ and

$$\lim_{i\to\infty} T_i = T \in \mathbf{E}_k(R^n) , \qquad \lim_{i\to\infty} \mathbf{M}(T_i - T) = 0 ,$$

then $T \in C$.

(iv) If f is continuously differentiable, then $f_\sharp(U) \in C$.

(v) If f is Lipschitzian, then $f_\sharp(U) \in C$.

(vi) Every k-dimensional rectifiable current in R^n belongs to C.

To verify (i) suppose $\xi \in \bigwedge_k(R^k)$, with $|\xi| = 1$, orients R^k. For $u \in R^k$ let

$$j(u) \in \textstyle\bigwedge_k(R^n)$$

be the image of ξ under the map induced by $Df(u)$, and define

$$v : R^n \to \textstyle\bigwedge_k(R^n) , \quad v(x) = 0 \quad \text{whenever } x \in R^n - f(U) ,$$
$$v(x) = j(u)/|\,j(u)\,| \quad \text{whenever } x = f(u),\ u \in U .$$

If $\varphi \in \mathbf{E}^k(R^n)$, then

$$f_\sharp(U)(\varphi) = \int_U f^\sharp(\varphi) = \int_U \varphi[f(u)][j(u)]\, dL_k u$$
$$= \int_U \varphi[f(u)](v[f(u)])\cdot|\,j(u)\,|\, dL_k u \ = \int_{f(U)} \varphi(x)[v(x)]\, d\mathbf{H}^k x$$

by [FE 3, 5.9], [FE 2, 2.1]. Letting $F = \{f(U)\}$ one therefore obtains (i) from 8.3 (4) and 8.15 with $\gamma = \mathbf{H}^k \cap f(U)$.

To verify (ii) let

$S_i = \{x : \Theta^k(\|\, T_i \,\|, x) = 0\}$,

$A_i = \{x : \Theta^k(\|\, T_i \,\|, x)$ is a positive integer, $\|\, \vec{T}_i(x) \,\| = 1\}$,

associate F_i with T_i according to (3), and for $M \in F_i$ let

$P_i(M) = A_i \cap M \cap \{x : \vec{T}_i(x)$ is a simple k-vector tangent to M at $x\}$.

Also let

$$A = \textstyle\bigcup_{i=1}^\infty A_i , \quad F = \bigcup_{i=1}^\infty F_i ,$$

$$Q_i(M) = P_i(M) \cap \{x : \Theta^k[\mathbf{H}^k, P_i(M), x] = 1,\ \Theta^k(\mathbf{H}^k, A, x) = 1\} .$$

Since $\mathbf{H}^k(B) \leqq \|\, T_i \,\|\,(B)$ for every Borel set $B \subset A_i$, one sees that

$$\mathbf{H}^k(A) \leqq \textstyle\sum_{i=1}^\infty \mathbf{M}(T_i) < \infty ,$$

$$\mathbf{H}^k(R^n - [S_i \cup \textstyle\bigcup_{M\in F_i} P_i(M)]) = 0 \quad \text{for each } i ,$$

and from 8.3 (3), (4) that

$$\mathbf{H}^k[P_i(M) - Q_i(M)] = 0 \ \text{ whenever } M \in F_i .$$

If $\varphi \in \mathbf{E}^k(R^n)$, then

$$T(\varphi) = \sum_{i=1}^{\infty} \int \varphi(x)[\vec{T}_i(x)]\Theta^k(\| T_i \|, x)\, d\mathbf{H}^k x = \int \varphi(x)[v(x)]\, d\gamma x$$

where $\gamma = \mathbf{H}^k \cap A$ and

$$v(x) = \sum_{i=1}^{\infty} \Theta^k(\| T_i \|, x)\vec{T}_i(x) \ .$$

For $\mathbf{H}^k$ almost all x in R^n,

$$x \in \bigcap_{i=1}^{\infty} [S_i \cup \bigcup_{M \in F_i} Q_i(M)]$$

and in fact $x \in S_i$ for all but finitely many i; if

$$M \in F_i, \quad N \in F_j, \quad x \in Q_i(M) \cap Q_j(N) \ ,$$

then

$$\Theta^{*k}[\mathbf{H}^k, P_i(M) \cup P_j(N), x] \leqq \Theta^k(\mathbf{H}^k, A, x) = 1 \ ,$$
$$\Theta^k_*(\mathbf{H}^k, M \cap N, x) \geqq \Theta^k[\mathbf{H}^k, P_i(M) \cap P_j(N), x] = 1 \ ,$$

hence M and N have the same tangent space at x and

$$\vec{T}_i(x) = \pm\ \vec{T}_j(x) \ ;$$

consequently $v(x)$ is a simple k-vector and $\| v(x) \|$ is an integer, with $\Theta^k(\gamma, x) = 1$. Now (ii) follows from 8.15.

Obviously (ii) implies (iii).

To verify (iv) let S_i be the i^{th} barycentric subdivision of U and let P_i be the sum of all those k-dimensional simplices V of S_i for which $V \subset \{u : Df(u)$ is univalent$\}$, $f \mid \text{Closure } V$ is univalent. Then (i) and (ii) imply that $f_\sharp(P_i) \in C$. Defining $j(u)$ as in the proof of (i) one also obtains

$$\mathbf{M}[f_\sharp(U) - f_\sharp(P_i)] \leqq \int_{U - \operatorname{spt} P_i} | j(u) |\, dL_k u \ .$$

Since this integral approaches 0 as $i \to \infty$, (iv) now follows from (iii).

To verify (v) apply the method of [FE 1, 4.3] and [W 1] to construct continuously differentiable maps f_i of R^k into R^n, with a common Lipschitz constant, such that

$$L_k(U \cap \{u : f_i(u) \neq f(u)\}) \longrightarrow 0 \quad \text{as } i \to \infty \ .$$

Then 3.6 implies that

$$\mathbf{M}[f_{i\sharp}(U) - f_\sharp(U)] \longrightarrow 0 \quad \text{as } i \to \infty \ ,$$

and consequently (v) follows from (iv).

Obviously (v), (iii), (iv) imply (vi).

8.17 REMARK. The converse of the preceding theorem is true. This may be verified by means of the construction occurring in the second half of the proof of 8.11.

8.18 LEMMA. *For any two positive integers $k \leqq n$ there exist positive numbers ζ and ν with the following property:*

If P is an oriented k-dimensional plane in R^n, $z \in P$, $r > 0$ and

$$X \in \mathbf{I}_k(\{x : |x - z| \leqq r\}) ,$$
$$\operatorname{spt} \partial X \subset \{x : |x - z| = r\} ,$$
$$r^{-k} \| X \| (R^n - P) \leqq \zeta ,$$

then there exists an integer m such that

$$\mathbf{M}[X - m\cdot(P \cap \{x : |x - z| \leqq r\})] \leqq \nu \| X \| (R^n - P) .$$

PROOF. In view of the obvious isometric and homothetic transformations one need only consider the special case where $z = 0$, $r = 1$ and

$$P = R^n \cap \{x : x_i = 0 \text{ for } i = k+1, \cdots, n\} .$$

One may apply 6.1 with

$$A = \{x : |x| \leqq 1\} , \quad B = \{x : |x| = 1\} \cup (P \cap A) ,$$
$$f(x) = x/\sup\{1, |x|\} \quad \text{for } x \in R^n ,$$
$$U = R^n , \quad \xi = 1 , \quad a = \infty , \quad \Gamma = A$$

and suitable g, V, η, b whose existence follows from [ES, II 9.9]. Letting

$$\zeta = b^k/c_1 , \quad \nu = 1 + c_3(\eta + 2)^k$$

one finds that for every X satisfying the above conditions there exists $Y \in \mathbf{I}_{k+1}(A)$ such that

$$\operatorname{spt}(X - \partial Y) \subset B, \ \mathbf{M}(\partial Y) \leqq c_3(\eta + 2)^k \| X \| (A - B) ;$$

moreover, if $W = \{x : |x| < 1\}$, then

$$W \cap \operatorname{spt}(X - \partial Y) \subset P, \ [\partial(X - \partial Y)] \cap W = 0 ,$$

and 3.3 implies the existence of an integer m for which

$$(X - \partial Y) \cap W = m\cdot(P \cap W) = m\cdot(P \cap A) ;$$

it follows that

$$X - m\cdot(P \cap A) = X \cap (A - W) + (\partial Y) \cap W ,$$
$$\mathbf{M}[X - m\cdot(P \cap A)] \leqq \nu \| X \| (A - P) ,$$

because $\mathbf{H}^k[(A - W) \cap P] = 0$, hence $\| X \| [(A - W) \cap P] = 0$ by 8.5.

8.19 LEMMA. *Suppose M is a k-dimensional proper regular submanifold of class* 1 *of R^n, $z \in M$, P is the k-dimensional plane tangent to M at z, and $0 < t < 1$. For each sufficiently small positive number r there exists a diffeomorphism f of class* 1, *mapping R^n onto R^n, such that*
f and f^{-1} have the Lipschitz constant t^{-1},
$f(x) = x$ whenever $|x - z| \geqq r$,
$f(M \cap \{x : |x - z| \leqq tr\}) \subset P$.

PROOF. Assume $k < n$, $z = 0$ and

$$P = R^n \cap \{x : x_i = 0 \text{ for } i = k+1, \cdots, n\} .$$

Choose $\rho > 0$ and continuously differentiable real-valued functions $g_{k+1}, \cdots, g_n$ on R^k such that

$$M \cap \{x : |x| \leqq \rho\} \subset \{x : x_i = g_i(x_1, \cdots, x_k) \text{ for } i = k+1, \cdots, n\} .$$

Note that the functions g_i and their differentials vanish at the origin of R^k, let

$$\varepsilon = (1-t)^2(3-t)^{-1}(n-k)^{-1} ,$$

and suppose $0 < r < \rho$ with

$$|Dg_i(y)| < \varepsilon \quad \text{whenever} \quad y \in R^k,\ |y| \leqq r,\ i = k+1, \cdots, n .$$

Letting u be a continuously differentiable real-valued function on R^n such that

$$\begin{aligned} u(x) &= 0 \quad \text{whenever } |x| \geqq r , \\ u(x) &= 1 \quad \text{whenever } |x| \leqq tr , \\ |Du(x)| &< 2(r - tr)^{-1} \quad \text{whenever } x \in R^n , \end{aligned}$$

define the functions f_i and f on R^n by the formulae

$$\begin{aligned} f_i(x) &= x_i - u(x) g_i(x_1, \cdots, x_k) \qquad \text{for } i = k+1, \cdots, n , \\ f(x) &= (x_1, \cdots, x_k, f_{k+1}(x), \cdots, f_n(x)) \end{aligned}$$

whenever $x \in R^n$. Since

$$|g_i(y)| \leqq r\varepsilon \quad \text{whenever} \quad y \in R^k,\ |y| \leqq r,\ i = k+1, \cdots, n ,$$

one finds that, if $x \in R^n$ and $h \in R^n$, then

$$\begin{aligned} |Df(x)(h) - h| &\leqq (n-k)[2(r-tr)^{-1} r\varepsilon + \varepsilon] \cdot |h| \\ &= (1-t)|h| \leqq (t^{-1} - 1)|h| . \end{aligned}$$

Hence for $a \in R^n$ and $h \in R^n$ the equation

$$f(a+h)-f(a)-h=\int_0^1 [Df(a+th)(h)-h]\,dt$$

yields the inequalities

$$|f(a+h)-f(a)| \geqq |h|-(1-t)|h| = t|h| ,$$
$$|f(a+h)-f(a)| \leqq |h|+(t^{-1}-1)|h| = t^{-1}|h| .$$

8.20 LEMMA. *If $Y \in \mathbf{I}_k(R^n)$ and $\eta > 0$, then $\|Y\|$ almost every point z in $R^n - \operatorname{spt}\partial Y$ is the center of arbitrarily small closed spherical balls B for which*

$$\|Y\|(\operatorname{Bdry} B)=0 , \qquad Y \cap B \in \mathbf{I}_k(R^n)$$

and there exist a polyhedral chain K with integer coefficients, and a diffeomorphism f of class 1 mapping R^n onto R^n, such that
f and f^{-1} have the Lipschitz constant $1+\eta$,
$f(x)=x$ whenever $x \in R^n - B$,
$\operatorname{spt} K \subset B$,
$\mathbf{M}[f_\sharp(Y\cap B)-K] < \eta\|Y\|(B)$.

PROOF. It follows from 8.16 and 8.3 (3) that, for $\|Y\|$ almost all z in R^n,

$$0 < \Theta^k(\|Y\|, z) < \infty$$

and there exists a k-dimensional proper regular submanifold M of class 1 of R^n such that $z \in M$ and

$$\Theta^k(\|Y\|, R^n - M, z) = 0 .$$

Thus fix z and M, let ζ and ν be in 8.18, and choose a number t satisfying the conditions

$$1 < t^{-1} < 1+\eta , \quad t^{-k}-1 < \nu^{-1}\eta ,$$
$$(t^{-k}-1)\Theta^k(\|Y\|, z)\alpha(k) < \zeta .$$

For $r > 0$ let

$$G_r = \{x : tr < |x-z| \leqq r\} , \quad H_r = \{x : |x-z| \leqq r\} - M$$

and note that

$$\lim_{r\to 0+} \alpha(k)^{-1}r^{-k}\|Y\|(G_r) = (1-t^k)\Theta^k(\|Y\|, z) ,$$
$$\lim_{r\to 0+} \alpha(k)^{-1}r^{-k}\|Y\|(H_r) = 0 ,$$

hence

$$t^{-k}\lim_{r\to 0+} \alpha(k)^{-1}r^{-k}\|Y\|(G_r\cup H_r) = (t^{-k}-1)\Theta^k(\|Y\|, z) .$$

Assuming $z \notin \operatorname{spt}\partial Y$ one sees with the help of 3.9, 3.10, 3.8 (3), 8.14 that,

for L_1 almost all sufficiently small positive numbers r,
distance $(z, \operatorname{spt} \partial Y) > r$,
$Y \cap \{x : |x - z| < r\} \in \mathbf{I}_k(R^n)$,
$\operatorname{spt} \partial\big(Y \cap \{x : |x - z| < r\}\big) \subset \{x : |x - z| = r\}$,
$\|Y\|\big(x\{ : |x - z| = r\}\big) = 0$,
$t^{-k}\,\|Y\|\,(G_r \cup H_r) < \nu^{-1}\eta\,\|Y\|\,\big(\{x : |x - z| \leqq r\}\big)$,
$t^{-k}r^{-k}\,\|Y\|\,(G_r \cup H_r) < \zeta$.

Fix such an r, let P and f be as in 8.19, and let

$$B = \{x : |x - z| \leqq r\}\ , \quad X = f_\#(Y \cap B)\ .$$

Then $X \in \mathbf{I}_k(B)$, $\operatorname{spt} \partial X \subset \operatorname{Bdry} B$ with

$$f^{-1}(B - P) \subset G_r \cup H_r\ ,$$
$$r^{-k}\,\|X\|\,(B - P) \leqq r^{-k}t^{-k}\,\|Y\|\,(G_r \cup H_r) < \zeta\ ,$$

and 8.18 yields an integer m for which

$$\begin{aligned}\mathbf{M}[X - m\cdot(P \cap B)] &< \nu\,\|X\|\,(B - P)\\ &\leqq \nu t^{-k}\,\|Y\|\,(G_r \cup H_r) < \eta\,\|Y\|\,(B)\ .\end{aligned}$$

Finally one chooses a polyhedral chain K with integer coefficients such that

$$\mathbf{M}[m\cdot(P \cap B) - K] < \eta\,\|Y\|\,(B) - \mathbf{M}[X - m\cdot(P \cap B)]\ .$$

8.21 LEMMA. *If $Y \in \mathbf{I}_k(R^n)$, $\delta > 0$ and ∂Y is a polyhedral chain with integer coefficients; then there exist $S \in \mathbf{I}_{k+1}(R^n)$ and a diffeomorphism f of class 1 mapping R^n onto R^n such that*
f and f^{-1} have the Lipschitz constant $1 + \delta$,
$|f(x) - x| \leqq \delta$ for $x \in R^n$,
$f(x) = x$ whenever $x \in \operatorname{spt} \partial Y$ or distance $(x, \operatorname{spt} Y) \geqq \delta$,
$\operatorname{spt} S \subset \{x : \text{distance}\,(x, \operatorname{spt} Y) \leqq \delta\}$,
$\mathbf{N}(S) < \delta$,
$f_\#(Y) - \partial S$ is a polyhedral chain with integer coefficients.

PROOF. Choose η so that $0 < \eta \leqq \delta$ and $4n^k\binom{n}{k}\mathbf{M}(Y)\eta < \delta/3$. Using 8.20 and 8.7 (1) one constructs a sequence of disjoint closed spherical balls B_i contained in $R^n - \operatorname{spt} \partial Y$, with associated diffeomorphism f_i and polyhedral chains K_i, such that

$$\text{diameter}\,(B_i) \leqq \delta/2 \quad \text{for } i = 1, 2, 3, \cdots\ ,$$
$$\|Y\|\,\big[(R^n - \operatorname{spt} \partial Y) - \textstyle\bigcup_{i=1}^{\infty} B_i\big] = 0\ .$$

Assured by 8.5 that

$$\| Y \| (\operatorname{spt} \partial Y) = 0 ,$$

choose a positive integer m for which

$$\| Y \| \big(R^n - \textstyle\bigcup_{i=1}^m B_i\big) \leqq (1 + \eta)^{-k} \eta \mathbf{M}(Y) ,$$

and let

$$f = f_1 \circ f_2 \circ \cdots \circ f_m ,$$

$$T = f_\#(Y) - \textstyle\sum_{i=1}^m K_i .$$

Then $\operatorname{spt} T \subset \{x : \text{distance}(x, \operatorname{spt} Y) \leqq \delta/2\}$,

$$\mathbf{M}(T) \leqq \mathbf{M}[f_\#(Y - \textstyle\sum_{i=1}^m Y \cap B_i)] + \textstyle\sum_{i=1}^m \mathbf{M}[f_\#(Y \cap B_i) - K_i]$$

$$\leqq (1 + \eta)^k \| Y \| \big(R^n - \textstyle\bigcup_{i=1}^m B_i\big) + \textstyle\sum_{i=1}^m \eta \| Y \| (B_i)$$

hence $\mathbf{M}(T) \leqq 2\eta \mathbf{M}(Y)$ and

$$2n^k \binom{n}{k} \mathbf{M}(T) < \delta/3 .$$

Moreover $f_\#(\partial Y) = \partial Y$; hence ∂T is a polyhedral chain with integer coefficients. Applying 5.5 and 5.6 with a number ε such that

$$0 < 2n\varepsilon < \delta/2 , \quad \varepsilon 4 n^{k+1} \binom{n}{k} \mathbf{M}(T) < \delta/3 ,$$

$$2n^k \binom{n}{k} \mathbf{M}(T) + \varepsilon 8 n^k \binom{n}{k-1} \mathbf{M}(\partial T) < \delta/3 ,$$

one obtains polyhedral chains P and Q with integer coefficients, and an integral current S, such that

$$f_\#(Y) - \partial S = P + Q + \textstyle\sum_{i=1}^m K_i ,$$

$$\mathbf{N}(S) \leqq \mathbf{M}(S) + \mathbf{M}(T) + \mathbf{M}(P) + \mathbf{M}(Q) < \delta .$$

8.22 Theorem. *If* $T \in \mathbf{I}_k(R^n)$ *and* $\varepsilon > 0$, *then there exists a polyhedral chain* P *with integer coefficients in* R^n, *and a diffeomorphism* f *of class* 1 *mapping* R^n *onto* R^n, *such that*

f *and* f^{-1} *have the Lipschitz constant* $1 + \varepsilon$,

$|f(x) - x| \leqq \varepsilon$ *for* $x \in R^n$,

$f(x) = x$ *whenever* distance $(x, \operatorname{spt} T) \geqq \varepsilon$,

$\operatorname{spt} P \subset \{x : \text{distance}(x, \operatorname{spt} T) \leqq \varepsilon\}$,

$\mathbf{N}[P - f_\#(T)] < \varepsilon$.

Proof. Choose δ so that

$$0 < \delta \leqq \varepsilon/2 , \quad (1 + \delta)^2 \leqq 1 + \varepsilon , \quad (1 + \delta)^k \delta + \delta \leqq \varepsilon .$$

First apply 8.21 with $Y_1 = \partial T$ to obtain $S_1 \in \mathbf{I}_k(R^n)$ and f_1 such that $\mathbf{N}(S_1) < \delta$ and

$$\partial[f_{1\#}(T) - S_1] = f_{1\#}(\partial T) - \partial S_1$$

is a polyhedral chain with integer coefficients. Then apply 8.21 with $Y_2 = f_{1\#}(T) - S_1$ to obtain $S_2 \in \mathbf{I}_{k+1}(R^n)$ and f_2 such that $\mathbf{N}(S_2) < \delta$ and

$$P = f_{2\#}[f_{1\#}(T) - S_1] - \partial S_2$$

is a polyhedral chain with integer coefficients. Letting

$$f = f_2 \circ f_1$$

one finds that

$$\mathbf{N}[P - f_\#(T)] \leqq \mathbf{N}[f_{2\#}(S_1)] + \mathbf{N}(\partial S_2) < (1 + \delta)^k \delta + \delta \leqq \varepsilon .$$

8.23 COROLLARY. *For each integral current T there exists a sequence of polyhedral chains P_i with integer coefficients such that*

$$\lim_{i\to\infty} P_i = T \text{ and } \lim_{i\to\infty} \mathbf{N}(P_i) = \mathbf{N}(T) .$$

PROOF. Clearly

$$\mathbf{N}(P) < \mathbf{N}[f_\#(T)] + \varepsilon \leqq (1 + \varepsilon)^k \mathbf{N}(T) + \varepsilon .$$

Furthermore, if h is the linear homotopy from f to the identity map of R^n, then

$$\begin{aligned}\mathbf{F}[T - f_\#(T)] &\leqq \mathbf{M}[h_\#(I \times T)] + \mathbf{M}[h_\#(I \times \partial T)] \\ &\leqq \varepsilon(1 + \varepsilon)^k \mathbf{M}(T) + \varepsilon(1 + \varepsilon)^{k-1}\mathbf{M}(\partial T) ,\end{aligned}$$

hence

$$\mathbf{F}(P - T) \leqq \mathbf{M}[P - f_\#(T)] + \mathbf{F}[f_\#(T) - T] < \varepsilon[1 + (1 + \varepsilon)^k \mathbf{N}(T)] .$$

9. Minimal currents

Here existence and continuity theorem concerning integral currents minimizing mass, subject to certain boundary conditions, are derived from the general theory developed in earlier sections; the only serious new problem, caused by the potentially bad behavior of free boundaries, is overcome in the proof of 9.8. This section also contains some partial results—and some conjectures—about the local structure of minimal currents.

9.1 DEFINITION. Suppose $B \subset A \subset R^n$ and $T \in \mathbf{I}_k(A)$ with $k > 0$. One calls T a *current of least mass relative to* (A, B) if and only if

$$\mathbf{M}(T) \leqq \mathbf{M}(T + S)$$

whenever $S \in \mathbf{I}_k(A)$, $\operatorname{spt} \partial S \subset B$.

One calls T a *minimal current relative to* (A, B) if and only if there exists an open covering F of $\operatorname{spt} T$ such that

$$\mathbf{M}(T) \leqq \mathbf{M}(T + S)$$

whenever $W \in F$, $S \in \mathbf{I}_k(A \cap W)$, $\operatorname{spt} \partial S \subset B$.

In particular, in case B is empty, one thus defines currents of least mass and minimal currents relative to A. If also $A = R^n$, then reference to A is omitted.

This terminology is a natural extension of language used in classical differential geometry and calculus of variations. In fact 5.7 implies that an oriented surface of least area, with prescribed Lipschitzian boundary and minimizing area in competition with arbitrary topological types, is a current of least mass. Moreover it follows from the theory of geodesic fields [V] that an oriented minimal surface, with sufficiently smooth boundary, is a minimal current.

9.2 *Cones*. The *cone* $zT \in \mathbf{E}_{k+1}(R^n)$ *with vertex* $z \in R^n$ *and base* $T \in \mathbf{E}_k(R^n)$ is defined by the formula

$$zT = h_\sharp(I \times T)$$

where $h : R \times R^n \to R^n$, $h(t, x) = (1 - t)z + tx$ is the linear homotopy from the constant map $R^n \to \{z\}$ to the identity map of R^n. Clearly

$$\partial zT + z\partial T = T \quad \text{if } k \geqq 1 \,,$$
$$\partial zT + T(1)z = T \quad \text{if } k = 0 \,.$$

Moreover, if $r = \sup\{| x - z | : x \in \operatorname{spt} T\}$, then

$$\mathbf{M}(zT) \leqq r\,\mathbf{M}(T)/(k + 1) \,,$$

because $0 = t_0 < t_1 < \cdots < t_q = 1$ implies

$$zT = \textstyle\sum_{i=1}^{q} h_\sharp\big(\{t : t_{i-1} \leqq t \leqq t_i\} \times T\big) \,,$$

$$\mathbf{M}(zT) \leqq \textstyle\sum_{i=1}^{q} r(t_i - t_{i-1})t_i^k \mathbf{M}(T) \,,$$

whence $\mathbf{M}(zT) \leqq r \int_0^1 t^k \, dt \, \mathbf{M}(T)$.

9.3 *Assumptions*. It is assumed in what follows that A and B are compact Lipschitz neighborhood retracts in R^n, with $A \supset B$; hence there exist $a > 0$, $b > 0$ and Lipschitzian retractions

$$f : R^n \cap \{x : \text{distance}\,(x, A) < a\} \to A \,,$$
$$g : R^n \cap \{x : \text{distance}\,(x, B) < b\} \to B \,,$$

with Lipschitz constants ξ and η. In accordance with 5.11 the k-dimensional integral homology classes of (A, B) will be regarded as cosets belonging to the factorgroup

$$[\mathbf{I}_k(A) \cap \partial^{-1}\mathbf{I}_{k-1}(B)]/[\partial\mathbf{I}_{k+1}(A) + \mathbf{I}_k(B)] .$$

9.4 THEOREM. *If* $X \in \mathbf{I}_k(A)$, Γ *is a k-dimensional integral homology class of* (A, B), $Y \in X + \Gamma$ *and*

$$\mathbf{M}(Y) = \inf \{\mathbf{M}(T) : T \in X + \Gamma\} ,$$

then Y *is a minimal current relative to* (A, B).

PROOF. Let $F = \{V\} \cup \{W_z : z \in A - B\}$, where

$$V = \{x : \text{distance}\,(x, B) < \inf \{b, a/(\eta + 1)\}\} ,$$
$$W_z = \{x : |x - z| < \inf \{a, \text{distance}\,(z, B)\}\} .$$

If $S \in \mathbf{I}_k(A \cap V)$, $\operatorname{spt} \partial S \subset B$, and h is the linear homotopy from g to the identity map, then $h_\#(I \times \partial S) = 0$,

$$S = g_\#(S) + \partial(f \circ h)_\#(I \times S) \in \mathbf{I}_k(B) + \partial\mathbf{I}_{k+1}(A) ,$$

hence $Y + S \in X + \Gamma$, $\mathbf{M}(Y + S) \geqq \mathbf{M}(Y)$. On the other hand if $z \in A - B$, $S \in \mathbf{I}_k(A \cap W_z)$, $\operatorname{spt} \partial S \subset B$, then

$$\partial S = 0 , \quad S = \partial(zS) , \quad S = f_\#(S) = \partial f_\#(zS) \in \partial\mathbf{I}_{k+1}(A) ,$$

hence $Y + S \in X + \Gamma$, $\mathbf{M}(Y + S) \geqq \mathbf{M}(Y)$.

9.5 THEOREM. If $X \in \mathbf{I}_k(A)$, *c is a positive number, and*

$$\begin{aligned} \Xi &= \mathbf{I}_k(A) \cap \{T : \operatorname{spt} \partial(T - X) \subset B,\ \mathbf{N}(T) \leqq c\} \\ &= [X + \mathbf{I}_k(A) \cap \partial^{-1}\mathbf{I}_{k-1}(B)] \cap \{T : \mathbf{N}(T) \leqq c\} , \end{aligned}$$

then:

(1) Ξ *is compact.*

(2) *For each k-dimensional integral homology class* Γ *of* (A, B) *the set*

$$\mathbf{I}_k(A) \cap \{T : T - X \in \Gamma,\ \mathbf{N}(T) \leqq c\} = (X + \Gamma) \cap \Xi$$

is open and closed in Ξ, *hence compact.*

(3) *There are only finitely many k-dimensional integral homology classes* Γ *of* (A, B) *such that* $(X + \Gamma) \cap \Xi$ *is nonempty.*

PROOF. Clearly (1) follows from 8.13.

To verify (2) suppose $T_i \in \Xi$, $\lim_{i\to\infty} T_i = T \in \Xi$. Applying 7.1 with T_i replaced by $T_i - X$ one finds that, for large i,

$$(T_i - X) - (T - X) \in \mathbf{I}_k(B) + \partial\mathbf{I}_{k+1}(A) .$$

If $T - X \in \Gamma$, then $T_i - X \in \Gamma$ for large i; hence $(X + \Gamma) \cap \Xi$ is open in Ξ. If $T_i - X \in \Gamma$ for all i, then $T - X \in \Gamma$; hence $(X + \Gamma) \cap \Xi$ is closed in Ξ.

Finally (3) follows from (1) and (2), because Ξ is the union of the disjoint sets $(X + \Gamma) \cap \Xi$.

9.6 COROLLARY. *If $X \in \mathbf{I}_k(A)$, then:*

(1) *There exists a $Y \in \mathbf{I}_k(A)$ such that $\partial Y = \partial X$ and Y is a current of least mass relative to A.*

(2) *For each k-dimensional integral homology class of A there exists a $Y \in X + \Gamma$ such that*

$$\mathbf{M}(Y) = \inf \{\mathbf{M}(T) : T \in X + \Gamma\} \ .$$

(3) *For each positive number ζ there are only finitely many k-dimensional integral homology classes Γ of A such that*

$$\inf \{\mathbf{M}(T) : T \in X + \Gamma\} \leqq \zeta \ .$$

PROOF. Assume B is the empty set. Since

$$\mathbf{N}(T) = \mathbf{M}(T) + \mathbf{M}(\partial X)$$

whenever $T \in \mathbf{I}_k(A)$ and $\partial(T - X) = 0$, the problems of minimizing $\mathbf{M}(T)$ and $\mathbf{N}(T)$ are equivalent for any class of such currents T. Therefore 9.5 is applicable, with $c = \zeta + \mathbf{M}(\partial X)$.

9.7 LEMMA. *If $m_1, m_2, m_3, \cdots$ are real-valued nondecreasing functions on $\{s : s > 0\}$ such that*

$$\lim_{i\to\infty} m_i(s) = m(s)$$

exists whenever $s > 0$, then

$$\liminf_{i\to\infty} m_i'(s) \leqq m'(s) \quad \textit{for } L_1 \textit{ almost all } s > 0 \ .$$

PROOF. For $0 < t < t + h$ Fatou's lemma implies that

$$\begin{aligned}\int_t^{t+h} \liminf_{i\to\infty} m_i'(s)\,ds &\leqq \liminf_{i\to\infty} \int_t^{t+h} m_i'(s)\,ds \\ &\leqq \liminf_{i\to\infty} [m_i(t+h) - m_i(t)] = m(t+h) - m(t) \, .\end{aligned}$$

9.8 THEOREM. *Suppose $X \in \mathbf{I}_k(A)$, p and q are positive numbers, and*

(i) $|f(x) - f(y)| \leqq (1 + ps^q)\,|x - y|$ *whenever $s < a$*, distance$(x, A) \leqq s$, distance$(y, A) \leqq s$;

(ii) $|g(x) - x| \leqq s(1 + ps^q)$,

$|g(x) - g(y)| \leqq (1 + ps^q)\,|x - y|$

whenever $s < b$, distance$(x, B) \leqq s$, distance$(y, B) \leqq s$;

(iii) $\|\partial X\| \big(\{x : 0 <$ distance$(x, B) < s\}\big) \leqq ps^q$ *for* $s > 0$.

If Γ *is a k-dimensional integral homology class of* (A, B), *then there exists a* $Y \in X + \Gamma$ *such that* $\| Y \| (B) = 0$,

$$\mathbf{M}(Y) = \inf \{\mathbf{M}(T) : T \in X + \Gamma\} ,$$
$$\mathbf{M}(\partial Y) \leqq (1 + 2^k)[\mathbf{M}(\partial X) + \delta e] ,$$

where $\delta \leqq \sup \{2^{2k/q} r^{-1}\mathbf{M}(Y),\ 2^{2k(q+1)/q}\}$,

$$r = \inf \{a,\ b,\ p^{-1/q}\} .$$

PROOF. Let $\zeta = \inf \{\| T \| (A - B) : T \in X + \Gamma\}$ and choose a sequence of currents $T_i \in X + \Gamma$ such that

$$\| T_i \| (A - B) \longrightarrow \zeta \quad \text{as } i \to \infty .$$

Defining $u(x) = \text{distance}(x, B)$ for $x \in R^n$,

$$m_i(s) = \| T_i \| \big(\{x : 0 < u(x) < s\}\big) \quad \text{for } s > 0,\ i = 1, 2, 3, \cdots ,$$

one may assume after passage to a subsequence by Helly's theorem that $m(s) = \lim_{i \to \infty} m_i(s)$ exists whenever $s > 0$. Also let h be the linear homotopy from g to the identity map.

If $x \in A$ and $u(x) \leqq s < r$, then $| g(x) - x | \leqq 2s$ because $ps^q \leqq 1$, hence

$$\text{distance}\,[h(x, t), A] \leqq s \quad \text{for } 0 \leqq t \leqq 1 .$$

Therefore, in case $0 < s < r$ and $m_i'(s) < \infty$, the currents[5]

$$T_{i,s} = T_i \cap \{x : u(x) < s\} ,$$
$$Q_{i,s} = T_i - T_{i,s} + (f \circ h)_\sharp(I \times \partial T_{i,s})$$

belong to $\mathbf{I}_k(A)$, according to 3.9 and 8.14, with

$$\begin{aligned} \| \partial T_{i,s} \| (R^n - B) &\leqq \| \partial T_i \cap \{x : u(x) < s\} \| (R^n - B) + m_i'(s) \\ &= \| \partial X \| \big(\{x : 0 < u(x) < s\}\big) + m_i'(s) \leqq ps^q + m_i'(s) , \end{aligned}$$

and 3.6 implies that

$$\begin{aligned} \mathbf{M}(Q_{i,s}) &\leqq \| T_i \| \big(\{x : u(x) \geqq s\}\big) + (1 + ps^q)^k \mathbf{M}[h_\sharp(I \times \partial T_{i,s})] \\ &\leqq \| T_i \| (A - B) - m_i(s) + (1 + ps^q)^{2k} s \| \partial T_{i,s} \| (R^n - B) \\ &\leqq \| T_i \| (A - B) - m_i(s) + (1 + \beta s^q) s [ps^q + m_i'(s)] , \end{aligned}$$

where $\beta = (2^{2k} - 1)p$. Furthermore

$$\begin{aligned} T_i - T_{i,s} &= T_i \cap \{x : u(x) > s\} , \\ \mathbf{M}[\partial(T_i - T_{i,s})] &\leqq \| \partial T_i \| \big(\{x : u(x) > s\}\big) + m_i'(s) \\ &\leqq \mathbf{M}(\partial X) + m_i'(s) , \end{aligned}$$

[5] It has come to our attention that a similar construction occurs in [**L**, Section 4].

$$\partial(f\circ h)_\sharp(I \times \partial T_{i,s}) = \partial T_{i,s} - g_\sharp(\partial T_{i,s}) ,$$
$$\mathbf{M}[\partial(f\circ h)_\sharp(I \times \partial T_{i,s})] \leqq 2(1 + ps^q)^{k-1} \| \partial T_{i,s} \| (R^n - B) ,$$
$$\mathbf{M}(\partial Q_{i,s}) \leqq (1 + 2^k)[\mathbf{M}(\partial X) + m_i'(s)] .$$

Inasmuch as

$$\begin{aligned} Q_{i,s} - T_i &= (f\circ h)_\sharp(I \times \partial T_{i,s}) - T_{i,s} \\ &= - \partial(f\circ h)_\sharp(I \times T_{i,s}) - g_\sharp(T_{i,s}) \in \partial\mathbf{I}_{k+1}(A) + \mathbf{I}_k(B) , \end{aligned}$$

one finds that $Q_{i,s} \in X + \Gamma$, hence

$$\zeta \leqq \| Q_{i,s} \| (A - B) \leqq \mathbf{M}(Q_{i,s}) .$$

Combining this inequality with the previous estimate of $\mathbf{M}(Q_{i,s})$, letting $i \to \infty$, and applying 9.7 one obtains

$$\zeta \leqq \zeta - m(s) + (1 + \beta s^q)s[ps^q + m'(s)] ,$$
$$\frac{1}{s} - \frac{\beta s^{q-1}}{1 + \beta s^q} - \frac{ps^q}{m(s)} \leqq \frac{m'(s)}{m(s)}$$

for L_1 almost all s between 0 and r. Since $\log\circ m$ is a nondecreasing function, integration gives

$$-\int_v^w ps^q m(s)^{-1}\, ds \leqq \log[\varphi(w)] - \log[\varphi(v)]$$

for $0 < v < w \leqq r$, where

$$\varphi(s) = (1 + \beta s^q)^{1/q} s^{-1} m(s) \quad \text{whenever } s > 0 .$$

Of course this reasoning becomes invalid if $m(s) = 0$ for some $s > 0$, but that case can be treated very easily in what follows.

It will be shown that

$$\liminf_{v\to 0+} \varphi(v) \leqq \delta e ,$$

where $\delta = \sup\{\varphi(r), 2^{2k(q+1)/q}\}$. Otherwise

$$w = \inf\{s : 0 < s \leqq r, \varphi(s) \leqq \delta\} > 0$$

with $\varphi(w) \leqq \delta$, and $0 < v < w$ implies

$$\begin{aligned} \log[\varphi(v)/\delta] &\leqq \log[\varphi(v)/\varphi(w)] \\ &\leqq \int_v^w ps^{q-1}(1 + \beta s^q)^{1/q}\varphi(s)^{-1}\, ds \\ &\leqq \delta^{-1}\int_0^r ps^{q-1}(1 + \beta s^q)^{1/q} ds \\ &= \delta^{-1}p\beta^{-1}(q + 1)^{-1}[(1 + \beta r^q)^{(q+1)/q} - 1] \\ &\leqq \delta^{-1}2^{2k(q+1)/q} \leqq 1 , \end{aligned}$$

hence $\varphi(v)/\delta \leqq e$.

Defining

$$S = \{s : 0 < s < r,\ m'(s) < \delta e\}$$

one then uses the inequality

$$v^{-1}\int_0^v m'(s)\,ds \leqq (1 + \beta v^q)^{-1/q}\varphi(v)$$

to infer that

$$L_1\big(S \cap \{s : 0 < s < \varepsilon\}\big) > 0 \quad \text{whenever } \varepsilon > 0 .$$

Now let

$$\Psi = (X + \Gamma) \cap \{T : \mathbf{M}(T) \leqq 2\zeta ,\ \mathbf{M}(\partial T) \leqq (1 + 2^k)[\mathbf{M}(\partial X) + \delta e]\}$$

and observe that

$$\inf \{\mathbf{M}(T) : T \in \Psi\} = \zeta .$$

In fact for L_1 almost all s in S there exist, by 9.7, arbitrarily large integers i such that

$$m_i'(s) < \delta e ;$$

then $Q_{i,s} \in \Psi$ and $\mathbf{M}(Q_{i,s})$ is close to ζ in case i is large and s is small.

Since Ψ is compact, by 9.5 (2), there exists a Y for which $\mathbf{M}(Y) = \zeta$; moreover

$$\varphi(r) \leqq 2^{2k/q} r^{-1} m(r) , \quad m(r) \leqq \zeta .$$

9.9 COROLLARY. *If the conditions of* 9.8 *hold, then*:

(1) *There exists a* $Y \in \mathbf{I}_k(A)$ *such that*

$$\operatorname{spt}(\partial Y - \partial X) \subset B$$

and Y *is a current of least mass relative to* (A, B).

(2) *For each positive number* ζ *there are only finitely many k-dimensional integral homology classes* Γ *of* (A, B) *such that*

$$\inf \{\mathbf{M}(T) : T \in X + \Gamma\} \leqq \zeta .$$

PROOF. To verify (2) let

$$r = \inf \{a, b, p^{-1/q}\} , \quad t = \sup \{2^{2k/q} r^{-1}\zeta,\ 2^{2k(q+1)/q}\} ,$$
$$c = \zeta + (1 + 2^k)[\mathbf{M}(\partial X) + te] ,$$

use 9.8 to infer that $(X + \Gamma) \cap \{T : \mathbf{N}(T) \leqq c\}$ is nonempty whenever

$$\inf \{\mathbf{M}(T) : T \in X + \Gamma\} \leqq \zeta ,$$

and apply 9.5 (3).

Clearly (1) follows from (2) and 9.8.

9.10 REMARK. The conditions (i), (ii) of 9.8 are satisfied in case A, B are sets with positive reach and f, g are the unique nearest point retractions [FE 6, 4.8 (8)]; in particular these conditions hold whenever A, B are compact regular submanifolds of class 2 of R^n.

9.11 THEOREM. *For* $Q \in [\partial \mathbf{I}_k(A) + \mathbf{I}_{k-1}(B)]$ *let*

$$\Omega(Q) = \inf \{\mathbf{M}(T) : T \in \mathbf{I}_k(A),\ \operatorname{spt}(\partial T - Q) \subset B\} .$$

If c is any positive number, then the function Ω is continuous on the set

$$[\partial \mathbf{I}_k(A) + \mathbf{I}_{k-1}(B)] \cap \{Q : \mathbf{N}(Q) \leqq c\} .$$

PROOF. If Q_1 and Q_2 belong to this set, and if there exist $G \in \mathbf{I}_{k-1}(B)$ and $K \in \mathbf{I}_k(A)$ such that

$$Q_1 - Q_2 = G + \partial K ,$$

then

$$| \Omega(Q_1) - \Omega(Q_2) | \leqq \mathbf{M}(K) .$$

In fact, for any $T \in \mathbf{I}_k(A)$ with $\operatorname{spt}(\partial T - Q_1) \subset B$,

$$\partial(T - K) - Q_2 = \partial T - Q_1 + G , \quad \Omega(Q_2) \leqq \mathbf{M}(T) + \mathbf{M}(K) .$$

Accordingly the theorem follows from 7.1.

9.12 COROLLARY. *Suppose* $T_i \in \mathbf{I}_k(A)$ *for* $i = 1, 2, 3, \cdots$ *and*

$$\lim_{i \to \infty} T_i = T \in \mathbf{E}_k(R^n) , \quad \limsup_{i \to \infty} \mathbf{M}(\partial T_i) < \infty .$$

If each T_i is a current of least mass relative to (A, B), *then*

$$\lim_{i \to \infty} \mathbf{M}(T_i) = \mathbf{M}(T)$$

and T is a current of least mass relative to (A, B).

PROOF. $\mathbf{M}(T) \geqq \Omega(\partial T) = \lim_{i \to \infty} \Omega(\partial T_i) = \lim_{i \to \infty} \mathbf{M}(T_i) \geqq \mathbf{M}(T)$.

9.13 THEOREM. *If T is a k-dimensional current of least mass relative to A, then*

$$r^{-k} \| T \| (\{x : | x - z | < r\}) \geqq \xi^{k(1-k)} c$$

whenever $z \in \operatorname{spt} T$ *and* $0 < r < \inf \{a, \text{distance}\,(z, \operatorname{spt} \partial T)\}$.

Here $c = (2^{k-1} c_2 k)^{-k}$, c_2 *is as in* 6.1 *with k replaced by* $k - 1$.

PROOF. For $0 < s \leqq r$ let

$$T_s = T \cap \{x : | x - z | < s\} , \quad m(s) = \mathbf{M}(T_s) ,$$

and use 3.9, 8.14 to infer that, for L_1 almost all such s,

$$\mathbf{M}(\partial T_s) \leqq m'(s) < \infty , \quad T_s \in \mathbf{I}_k(R^n) ;$$

applying 6.2 with X and A replaced by ∂T_s and $\{x : | x - z | \leqq s\}$ one obtains $Y_s \in \mathbf{I}_k(\{x : |x - z| \leqq r\})$ such that

$$\partial Y_s = \partial T_s \,, \quad \mathbf{M}(Y_s)^{(k-1)/k} \leqq 2^{k-1} c_2 \mathbf{M}(\partial T_s) \,,$$

whence it follows that

$$\partial f_\sharp(Y_s) = \partial T_s \,, \quad m(s) \leqq \mathbf{M}[f_\sharp(Y_s)] \leqq \xi^k \mathbf{M}(Y_s) \,,$$
$$m(s)^{(k-1)/k} \leqq \xi^{k-1} 2^{k-1} c_2 m'(s) \,,$$
$$(\xi^{k-1} 2^{k-1} c_2 k)^{-1} \leqq [m^{1/k}]'(s) \,.$$

Then integration gives $(\xi^{k-1} 2^{k-1} c_2 k)^{-1} r \leqq [m(r)]^{1/k}$.

9.14 COROLLARY.[6] *If T is a k-dimensional current of least mass relative to A, then $\|T\|(X) \geqq \mathbf{H}^k(X)$ for every Borel set $X \subset$ spt $T -$ spt ∂T, and for $\mathbf{H}^k$ almost every z in* spt $T -$ spt ∂T *the tangent cone (contingent) of* spt T *at z is contained in the k-space of the simple k-vector $\vec{T}(z)$.*

PROOF. Regarding the first statement observe that, by 8.16 (1) and 9.13,

$$\Theta^k(\|T\|, z) \geqq 1 \quad \text{for } \mathbf{H}^k \text{ almost all } z \text{ in } X \,,$$

and apply 8.16 (2).

To verify the second statement recall 8.16 (3) and 8.3 (3), and consider a $z \in$ spt $T -$ spt ∂T for which there exists a k-dimensional proper regular submanifold M of class 1 of R^n such that $z \in M$, the tangent space Y of M at z is the k-space of the simple k-vector $\vec{T}(z)$, and

$$\Theta^k(\|T\|, R^n - M, z) = 0 \,.$$

Assume $z = 0$ and let $u(x) =$ distance (x, Y) whenever $y \in R^n$. For each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\{x : |x| < 2\delta\} \subset R^n - \operatorname{spt} \partial T \,,$$
$$M \cap \{x : |x| < 2\delta\} \subset \{x : u(x) \leqq \varepsilon |x|\} \,,$$
$$s^{-k} \|T\| \big(\{x : |x| < s\} - M\big) < \varepsilon^k (1 + 2\varepsilon)^{-k} \xi^{k(1-k)} c$$

whenever $0 < s < 2\delta$. It will be shown that

$$(\operatorname{spt} T) \cap \{x : |x| < \delta\} \subset \{x : u(x) \leqq 2\varepsilon |x|\} \,.$$

In fact suppose $w \in$ spt T, $|w| < \delta$, $u(w) > 2\varepsilon |w|$, let

$$r = |w| \varepsilon (1 + \varepsilon)^{-1} \,, \quad s = |w| + r = |w| (1 + 2\varepsilon)(1 + \varepsilon)^{-1} \,,$$

and note that if $|x - w| < r$, then

[6] The second half of 9.14, and also parts of 9.16, were suggested by similar results in [Y 2]. Unfortunately L. C. Young found an error in the proof of his inequality (2.1), leaving his main theorem conjectural, but this error does not affect the results we refer to.

$$u(x) > u(w) - r > 2\varepsilon\,|\,w\,| - r = \varepsilon(|\,w\,| + r) > \varepsilon\,|\,x\,|\,;$$

accordingly 9.13, applied with z replaced by w, gives

$$\begin{aligned} r^k\xi^{k(1-k)}c &\leqq \|\,T\,\|\big(\{x : |\,x - w\,| < r\}\big) \\ &\leqq \|\,T\,\|\big(\{x : |\,x\,| < s,\ u(x) > \varepsilon\,|\,x\,|\}\big) \\ &\leqq \|\,T\,\|\big(\{x : |\,x\,| < s\} - M\big) < s^k\varepsilon^k(1 + 2\varepsilon)^{-k}\xi^{k(1-k)}c\,, \end{aligned}$$

which is false because $r = s\varepsilon(1 + 2\varepsilon)^{-1}$.

9.15 REMARK. Suppose $R^n = P \oplus Q$, where P and Q are mutually orthogonal subspaces of R^n, and

$$p : R^n \longrightarrow P\,, \quad q : R^n \longrightarrow Q$$

are the corresponding projections.

If v is a simple k-vector of R^n, then

$$|\,v\,|^{2/k} \geqq |\,p(v)\,|^{2/k} + |\,q(v)\,|^{2/k}\,.$$

Since this inequality is positively homogeneous of degree $2/k$, it suffices to verify it in case $v = w_1 \wedge w_2 \wedge \cdots \wedge w_k$ where $w_1, w_2, \cdots, w_k$ are orthonormal; then

$$\begin{aligned} |\,v\,|^{2/k} = 1 &= \textstyle\sum_{i=1}^k k^{-1}\,|\,p(w_i)\,|^2 + \sum_{i=1}^k k^{-1}\,|\,q(w_i)\,|^2 \\ &\geqq \textstyle\prod_{i=1}^k |\,p(w_i)\,|^{2/k} + \prod_{i=1}^k |\,q(w_i)\,|^{2/k} \\ &\geqq |\,\textstyle\bigwedge_{i=1}^k p(w_i)\,|^{2/k} + |\,\bigwedge_{i=1}^k q(w_i)\,|^{2/k} \end{aligned}$$

because the geometric mean never exceeds the algebraic mean. Equality holds if and only if the restrictions of p and q to the k-space of v are conformal.

In case $k \geqq 2$, then $|\,v\,| \geqq |\,p(v)\,| + |\,q(v)\,|$. Equality holds if and only if the k-space of v is contained in P or Q, or $k = 2$ and the restrictions of p and q to the k-space of v are conformal.

Now suppose $\dim P = \dim Q = k \geqq 2$ and let

$$X = P \cap \{z : |\,z\,| < 1\}\,, \quad Y = Q \cap \{z : |\,z\,| < 1\}\,.$$

Then $X + Y$ is a current of least mass, because if $S \in \mathbf{I}_k(R^n)$ with $\partial S = \partial(X + Y)$, then

$$\partial p_\#(S) = \partial X\,, \quad p_\#(S) = X\,, \quad \partial q_\#(S) = \partial Y\,, \quad q_\#(S) = Y\,,$$

$$\begin{aligned} \mathbf{M}(S) &= \int |\,\vec{S}(z)\,|\,d\,\|\,S\,\|\,z \\ &\geqq \int |\,p[\vec{S}(z)]\,|\,d\,\|\,S\,\|\,z + \int |\,q[\vec{S}(z)]\,|\,d\,\|\,S\,\|\,z \\ &\geqq \mathbf{M}[p_\#(S)] + \mathbf{M}[q_\#(S)] = \mathbf{M}(X + Y)\,. \end{aligned}$$

Note that $\operatorname{spt}(X + Y) - \operatorname{spt} \partial(X + Y)$ is not a manifold, because the tangent cone of this set at 0 is $P \cup Q$.

A much deeper reason for the complicated nature of the supports of minimal currents appears through confrontation of 9.6 (2) with Thom's theorem [T, III. 9]: There exists a 14-dimensional compact real analytic manifold A with a 7-dimensional homology class Γ such that the support of no integral cycle in Γ is a manifold; yet Γ contains a minimal current.

9.16 REMARK.[7] Suppose $T \in \mathbf{I}_k(A)$. It is not difficult to show that, for $\|T\|$ almost all z in R^n,

$$\lim_{s \to 0+} (\mu_{1/s} \circ \tau_{-z})_\#\big(T \cap \{x : |x - z| < s\}\big) = \Theta^k(\|T\|, z) \cdot \big(Y_z \cap \{y : |y| < 1\}\big)$$

where Y_z is the k-space of the simple k-vector $\vec{T}(z)$.

Now assume T is a current of least mass relative to A, and the retraction f satisfies the condition 9.8 (i). Letting

$$z \in \operatorname{spt} T - \operatorname{spt} \partial T ,$$

$$0 < r \leqq \inf\{a,\ p^{-1/q},\ \operatorname{distance}(z, \operatorname{spt} \partial T)\} ,$$

$$T_s = T \cap \{x : |x - z| < s\} , \quad m(s) = \mathbf{M}(T_s)$$

whenever $s > 0$, one sees by means of the cone construction that, for L_1 almost all s between 0 and r,

$$\begin{aligned} m(s) \leqq \mathbf{M}[f_\#(z \partial T_s)] &\leqq (1 + ps^q)^k \mathbf{M}(z \partial T_s) \\ &\leqq (1 + \beta s^q) s \mathbf{M}(\partial T_s)/k \leqq (1 + \beta s^q) s m'(s)/k \end{aligned}$$

where $\beta = (2^k - 1)p$, hence

$$\frac{k}{s} - \frac{k\beta s^{q-1}}{1 + \beta s^q} \leqq \frac{m'(s)}{m(s)} .$$

Integration of this inequality shows that

$$\varphi(s) = (1 + \beta s^q)^{k/q} s^{-k} m(s)$$

is nondecreasing on $\{s : 0 < s < r\}$, which implies existence of the limit

$$L = \lim_{s \to 0+} \varphi(s) = \lim_{s \to 0+} s^{-k} m(s) = \lim_{s \to 0+} \mathbf{M}(\mu_{1/s} \circ \tau_{-z})_\#(T_s) .$$

Inasmuch as

$$\begin{aligned} s^{1-k}\mathbf{M}(\partial T_s) &\geqq k s^{-k} m(s)(1 + \beta s^q)^{-1} \\ &= k\varphi(s)(1 + \beta s^q)^{-1-k/q} \geqq kL(1 + \beta r^q)^{-1-k/q} \end{aligned}$$

for L_1 almost all s between 0 and r, and also

[7] See footnote 6

$$r^{-k}\int_0^r s^{1-k}\mathbf{M}(\partial T_s)ks^{k-1}ds \leqq kr^{-k}\int_0^r m'(s)ds \leqq kr^{-k}m(r) \; ,$$

one finds, by letting $r \to 0+$, that kL is the approximate limit, as $s \to 0+$, of

$$s^{1-k}\mathbf{M}(\partial T_s) = \mathbf{M}[\partial(\mu_{1/s}\circ\tau_{-z})_\#(T_s)] \; .$$

These facts are compatible with the conjecture that the currents $(\mu_{1/s}\circ\tau_{-z})_\#(T_s)$ converge, as $s \to 0+$, to a current P; if so, P should be a current of least mass relative to the tangent cone C of A at z, and $P=0\partial P$; moreover it seems plausible that ∂P would be a minimal current relative to $C \cap \{y : |y| = 1\}$.

9.17 REMARK. With each continuous map

$$F : R^n \times \textstyle\bigwedge_k(R^n) \longrightarrow R$$

and each $T \in \mathbf{E}_k(R^n)$ such that $\mathbf{M}(T) < \infty$ one associates the integral

$$\int F[x, \vec{T}(x)\, d]\, \| T \| \, x \; .$$

In case $F(x, \cdot)$ is a norm on $\bigwedge_k(R^n)$ for each $x \in R^n$, classical arguments show that the integral is lower semicontinuous with respect to T, and that the ratio of the integral to $\mathbf{M}(T)$ has positive upper and lower bounds when spt T is restricted within a compact subset of R^n. Hence one easily obtains several results (analogous to 9.4, 9.6, 9.11, 9.12, 9.13, 9.14) concerning integral currents minimizing the above integral.

BROWN UNIVERSITY

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