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MINIMIZING EXPECTED LOSS OF HEDGING IN INCOMPLETE AND CONSTRAINED MARKETS*

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Abstract. We study the problem of minimizing the expected discounted loss

$$E\left[e^{-\int_0^T r(u)du}(C - X^{x,\pi}(T))^+\right]$$

when hedging a liability C at time $t = T$, using an admissible portfolio strategy $\pi(\cdot)$ and starting with initial wealth x. The existence of an optimal solution is established in the context of continuous-time Ito process *incomplete* market models, by studying an appropriate dual problem. It is shown that the optimal strategy is of the form of a knock-out option with payoff C, where the "domain of the knock-out" depends on the value of the optimal dual variable. We also discuss a dynamic measure for the risk associated with the liability C, defined as the supremum over different scenarios of the minimal expected loss of hedging C.

Key words. expected loss, hedging, incomplete markets, portfolio constraints, dynamic measures of risk

AMS subject classifications. Primary, 90A09, 90A46; Secondary, 93E20, 60H30

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1. Introduction.

In a *complete* financial market which is free of arbitrage opportunities, any sufficiently integrable random payoff (*contingent claim*) C, whose value has to be delivered and is known at time $t = T$, can be hedged perfectly: starting with a large enough initial capital x, an agent can find a trading strategy π that will allow his wealth $X^{x,\pi}(\cdot)$ to hedge the liability C *without risk* at time $t = T$, that is,

$$X^{x,\pi}(T) \geq C \quad \text{almost surely (a.s.)} \quad \text{for some portfolio } \pi(\cdot), \tag{1.1}$$

while maintaining "solvency" throughout $[0, T]$. (For an overview of standard results in complete and some incomplete markets in continuous-time Ito process models, see, for example, Cvitanić (1997), or a recent book by Karatzas and Shreve (1998).) This is either no longer possible or too expensive to accomplish in a market which is incomplete due to various "market frictions," such as insufficient number of assets available for investment, transaction costs, portfolio constraints, problems with liquidity, presence of a "large investor," and so on. In this paper we concentrate on the case in which incompleteness arises due to some assets not being available for investment and the more general case of portfolio constraints. Popular approaches to the problem of hedging a claim C in such contexts have been to either maximize the expected utility of the difference $-D := X^{x,\pi}(T) - C$ or minimize the risk of D. In particular, one of the most studied approaches is to minimize $E[D^2]$, so-called quadratic hedging of Föllmer–Schweizer–Sondermann (for recent results and references see Pham,

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Rheinländer, and Schweizer (1998), for example). An obvious disadvantage of this approach is that one is penalized for high profits and not just high losses. On the other hand, Artzner, Delbaen, Eber, and Heath (1999) have shown in a static hedging setting that the only measure of risk that satisfies certain natural "coherence" properties is of the type $E[\bar{D}^+]$ (or a supremum of these over a set of probability measures), where $\bar{D}^+$ is the discounted value of the positive part of D. Motivated by this work, Cvitanić and Karatzas (1999) solve the problem of minimizing $E[\bar{D}^+]$ in a context of a *complete* continuous-time Ito process model for the financial market. We solve in this paper the same problem in a more difficult context of incomplete or constrained markets. Recently, Pham (1998) has solved the problem of minimizing $E[(D^+)^p]$ for $p > 1$ in discrete-time models and under cone constraints. Moreover, independently from Pham and the present paper, Föllmer and Leukert (1999b) analyze the problem of minimizing $E[l(D^+)]$ for a general loss function l and in general incomplete semimartingale models, emphasizing the Neyman–Pearson lemma approach, as opposed to the duality approach. The former approach was used by the same authors in Föllmer and Leukert (1999a) to solve the problem of maximizing the probability of perfect hedge $P[D \leq 0]$. Some early work on problems like these is presented in Dembo (1997), in a one-period setting. A very general study of the the duality approach and its use in the utility maximization context can be found in Kramkov and Schachermayer (1999).

As mentioned above, another approach would be to try to hedge away all the risk of the agent by superreplicating the claim he has to deliver at time T, namely to have $X^{x,\pi}(T) \geq C$, a.s. This has been done in the framework of constraints by Cvitanić and Karatzas (1993), Broadie, Cvitanić, and Soner (1998), and Cvitanić, Pham, and Touzi (1999). However, the cost $x = x_C$ of the least expensive strategy accomplishing the superreplication is typically very high, and hence the strategy is appropriate neither for pricing nor for hedging purposes. For example, if the agent sells a call option C on one share of stock S, and he cannot borrow money, then his cost of superreplicating the option is equal to the price of one share of S. Nobody would pay this much for the option if they can buy the stock itself. More interesting examples include newly deregulated energy markets, reinsurance markets, and emerging markets. The cost of superreplication in these markets is usually too high (even infinite), and one is forced to introduce preferences, typically in terms of a loss or a utility function. This is the approach taken also in this paper, with a linear loss function.

Suppose now that, in addition to the genuine risk that the liability C represents, the agent also faces some uncertainty regarding the model for the financial market itself. Following Cvitanić and Karatzas (1999), we capture such uncertainty by allowing a family $\mathcal{P}$ of possible "real-world probability measures," instead of just one measure. Thus, the "max-min" quantity

$$\underline{V}(x) := \sup_{P\in\mathcal{P}} \inf_{\pi} E^P[\bar{D}^+] \tag{1.2}$$

represents the maximal risk that the agent can encounter when faced with the "worst possible scenario" $P \in \mathcal{P}$. In the special case of incomplete markets and under the condition that all equivalent martingale measures are included in the set of possible real-world measures $\mathcal{P}$, we show that

$$\underline{V}(x) = \overline{V}(x) := \inf_{\pi} \sup_{P\in\mathcal{P}} E^P[\bar{D}^+]. \tag{1.3}$$

In other words, the corresponding fictitious "stochastic game" between the market and the agent has a value. The trading strategy attaining this value is shown to be the one that corresponds to borrowing just enough money from the bank at time $t = 0$ as to be able to have at least the amount C at time $t = T$.

We describe the market model in section 2 and introduce the optimization problem in section 3. As is by now standard in financial mathematics, we define a dual problem, whose optimal solution determines the optimal terminal wealth $X^{x,\hat{\pi}}(T)$. It turns out that this terminal wealth is of the "knock-out" option type—namely, it is either equal to C or to 0 or to a certain (random) value $0 \leq B \leq C$, depending on whether the optimal dual variable is less than, larger than, or equal to one, respectively. What makes the dual problem more difficult than in the usual utility optimization problems (as in Cvitanić and Karatzas (1992)) is that the objective function fails to be everywhere differentiable, and the optimal dual variable (related to the Radon–Nikodym derivative of an "optimal change of measure") can be zero with positive probability. Nevertheless, we are able to solve the problem using nonsmooth optimization techniques for infinite dimensional problems, which can be found in Aubin and Ekeland (1984). We discuss in section 4 the stochastic game associated with (1.2) and (1.3).

2. The market model. We recall here the standard Ito process model for a financial market $\mathcal{M}$. It consists of one *bank account* and d *stocks*. Price processes $S_0(\cdot)$ and $S_1(\cdot), \ldots, S_d(\cdot)$ of these instruments are modeled by the equations

$$dS_0(t) = S_0(t)r(t)dt\ , \quad S_0(0) = 1,$$

$$dS_i(t) = S_i(t)\left[b_i(t)dt + \sum_{j=1}^{d}\sigma_{ij}(t)dW^j(t)\right], \quad S_i(0) = s_i > 0\ ;\ i = 1, \ldots, d. \tag{2.1}$$

Here $W(\cdot) = (W^1(\cdot), \ldots, W^d(\cdot))'$ is a standard d-dimensional Brownian motion on a complete probability space $(\Omega, \mathcal{F}, P)$, endowed with a filtration $\mathbf{F} = \{\mathcal{F}(t)\}_{0\leq t\leq T}$, the P-augmentation of $\mathcal{F}^W(t) := \sigma(W(s);\ 0 \leq s \leq t)\ ,\ \ 0 \leq t \leq T$, the filtration generated by the Brownian motion $W(\cdot)$. The *coefficients* $r(\cdot)$ (interest rate), $b(\cdot) = (b_1(\cdot), \ldots, b_d(\cdot))'$ (vector of stock return rates) and $\sigma(\cdot) = \{\sigma_{ij}(\cdot)\}_{1\leq i,j\leq d}$ (matrix of stock-volatilities) of the model $\mathcal{M}$ are all assumed to be progressively measurable with respect to $\mathbf{F}$. Furthermore, the matrix $\sigma(\cdot)$ is assumed to be invertible, and all processes $r(\cdot)$, $b(\cdot)$, $\sigma(\cdot)$, $\sigma^{-1}(\cdot)$ are assumed to be bounded uniformly in $(t, \omega) \in [0, T] \times \Omega$.

The "risk premium" process

$$\theta_0(t) := \sigma^{-1}(t)[b(t) - r(t)\tilde{\mathbf{1}}]\ ,\quad 0 \leq t \leq T, \tag{2.2}$$

where $\tilde{\mathbf{1}} = (1, \ldots, 1)' \in \mathbb{R}^d$, is then bounded and $\mathbf{F}$-progressively measurable. Therefore, the process

$$Z_0(t) := \exp\left[-\int_0^t \theta_0'(s)dW_0(s) - \frac{1}{2}\int_0^t ||\theta_0(s)||^2 ds\right],\ 0 \leq t \leq T, \tag{2.3}$$

is a P-martingale, and

$$P_0(\Lambda) := E[Z_0(T)1_\Lambda],\quad \Lambda \in \mathcal{F}(T) \tag{2.4}$$

is a probability measure equivalent to P on $\mathcal{F}(T)$. Under this *risk-neutral equivalent martingale measure* P_0, the discounted stock prices $\frac{S_1(\cdot)}{S_0(\cdot)}, \ldots, \frac{S_d(\cdot)}{S_0(\cdot)}$ become martingales, and the process

$$W_0(t)\ :=\ W(t) + \int_0^t \theta_0(s)ds\ ,\quad 0 \leq t \leq T, \tag{2.5}$$

becomes Brownian motion, by the Girsanov theorem.

Consider now an agent who starts out with initial capital x and can decide, at each time $t \in [0,T]$, what proportion $\pi_i(t)$ of his (nonnegative) wealth to invest in each of the stocks $i = 1, \dots, d$. However, the portfolio process $(\pi_1(\cdot), \dots, \pi_d(\cdot))'$ has to take values in a given closed convex set $K \subset \mathbb{R}^d$ of constraints, for almost everywhere (a.e.) $t \in [0,T]$, a.s. We will also assume that K contains the origin. For example, if the agent can hold neither short nor long positions in the last $d-m$ stocks $S_{m+1}(\cdot), \dots, S_d(\cdot)$, we get a typical example of an incomplete market, in the sense that not all square-integrable payoffs can be exactly replicated. (One of the best known examples of incomplete markets, the case of stochastic volatility, is included in this framework.) Another typical example is the case of an agent who has limits on how much he can borrow from the bank, or how much he can go short or long in a particular stock.

With $\pi(t) = (\pi_1(t), \dots, \pi_d(t))' \in K$ chosen, the agent invests the amount $X(t)(1 - \sum_{i=1}^d \pi_i(t))$ in the bank account, at time t, where we have denoted $X(\cdot) \equiv X^{x,\pi,\kappa}(\cdot)$ his wealth process. Moreover, for reasons of mathematical convenience, we allow the agent to spend money outside of the market, and $\kappa(\cdot) \geq 0$ denotes the corresponding *cumulative consumption process*. The resulting wealth process satisfies the equation

$$\begin{aligned} dX(t) &= -d\kappa(t) + \left[X(t)(1 - \sum_{i=1}^d \pi_i(t)) \right] r(t)dt \\ &\quad + \sum_{i=1}^d \pi_i(t)X(t) \left[b_i(t)dt + \sum_{j=1}^d \sigma_{ij}(t)dW^j(t) \right] \\ &= -d\kappa(t) + r(t)X(t)dt + \pi'(t)\sigma(t)X(t)dW_0(t); \quad X(0) = x. \end{aligned}$$

Denoting

$$\bar{X}(t) = e^{-\int_0^t r(u)du} X(t), \tag{2.6}$$

the discounted version of a process $X(\cdot)$, we get the equivalent equation

$$d\bar{X}(t) = -e^{-\int_0^t r(u)du} d\kappa(t) + \pi'(t)\sigma(t)\bar{X}(t)dW_0(t) \; ; \quad X(0) = x. \tag{2.7}$$

It follows that $\bar{X}(\cdot)$ is a nonnegative local P_0-supermartingale, hence also a P_0-supermartingale, by Fatou's lemma. Therefore, if τ_0 is defined to be the first time it hits zero, we have $X(t) = 0$ for $t \geq \tau_0$, so that the portfolio values $\pi(t)$ are irrelevant after that happens. Accordingly, we can and do set $\pi(t) = 0$ for $t \geq \tau_0$.

More formally, we have the following definition.

DEFINITION 2.1. (i) *A portfolio process* $\pi : [0,T] \times \Omega \to \mathbb{R}^d$ *is* $\mathbf{F}$*-progressively measurable and satisfies* $\int_0^T ||\pi(t)||^2 dt < \infty$*, a.s., as well as*

$$\pi(t) \in K \quad \textit{for a.e. } t \in [0,T] \tag{2.8}$$

a.s. A consumption process $\kappa(\cdot)$ *is a nonnegative, nondecreasing, progressively measurable process with right-continuous with left limits (RCLL) paths, with* $\kappa(0) = 0$ *and* $\kappa(T) < \infty$*.*

(ii) *For a given portfolio and consumption processes* $\pi(\cdot)$*,* $\kappa(\cdot)$*, the process* $X(\cdot) \equiv X^{x,\pi,\kappa}(\cdot)$ *defined by* (2.7) *is called the wealth process corresponding to strategy* (π, κ) *and initial capital* x*.*

(iii) *A portfolio-consumption process pair* $(\pi(\cdot), \kappa(\cdot))$ *is called admissible for the initial capital* x*, and we write* $(\pi, \kappa) \in \mathcal{A}(x)$*, if*

$$X^{x,\pi,\kappa}(t) \geq 0, \qquad 0 \leq t \leq T, \tag{2.9}$$

holds a.s.

We refer to the lower bound of (2.9) as a *margin requirement*. The no-arbitrage price of a contingent claim C in a complete market is unique and is obtained by multiplying ("discounting") the claim by $H_0(T) := Z_0(T)/S_0(T)$ and taking expectation. Since the market here is incomplete, there are more relevant *stochastic discount factors* other than $H_0(T)$. We identify them along the lines of Cvitanić and Karatzas (1993), hereafter [CK93], and Karatzas and Kou (1996), hereafter [KK96], as follows: Introduce the *support function*

$$\delta(\nu) := \sup_{\pi \in K} \{-\pi' \nu\} \tag{2.10}$$

of the set $-K$, as well as its *barrier cone*

$$\tilde{K} := \{\nu \in \mathbb{R}^d \ / \ \delta(\nu) < \infty\}. \tag{2.11}$$

For the rest of the paper we assume the following mild conditions.

Assumption 2.2. The closed convex set $K \subset \mathbb{R}^d$ contains the origin; in other words, the agent is allowed not to invest in stocks at all. In particular, $\delta(\cdot) \geq 0$ on $\tilde{K}$. Moreover, the set K is such that $\delta(\cdot)$ is continuous on the barrier cone $\tilde{K}$ of (2.11).

Denote by $\mathcal{D}$ the set of all *bounded* progressively measurable process $\nu(\cdot)$ taking values in $\tilde{K}$ a.e. on $\Omega \times [0, T]$. In analogy with (2.2)–(2.5), introduce

$$\theta_\nu(t) := \sigma^{-1}(t)[\nu(t) + b(t) - r(t)\tilde{\mathbf{1}}] \ , \quad 0 \leq t \leq T, \tag{2.12}$$

$$Z_\nu(t) := \exp\left[-\int_0^t \theta'_\nu(s) dW(s) - \frac{1}{2}\int_0^t ||\theta_\nu(s)||^2 ds\right], \ 0 \leq t \leq T, \tag{2.13}$$

$$P_\nu(\Lambda) := E[Z_\nu(T) 1_\Lambda], \quad \Lambda \in \mathcal{F}(T), \tag{2.14}$$

$$W_\nu(t) \ := \ W(t) + \int_0^t \theta_\nu(s) ds \ , \quad 0 \leq t \leq T, \tag{2.15}$$

a P_ν-Brownian motion. Also denote

$$H_\nu(t) := e^{-\int_0^t \delta(\nu(u)) du} Z_\nu(t). \tag{2.16}$$

Note that

$$dZ_\nu(t) = -Z_\nu(t)\theta'_\nu(t) dW(t). \tag{2.17}$$

From this and (2.7) we get, by Ito's rule,

$$\begin{aligned} d(H_\nu(t)\bar{X}(t)) = -e^{-\int_0^t r(u)du} H_\nu(t) d\kappa(t) - [\delta(\nu(t)) + \pi'(t)\nu(t)] H_\nu(t) \bar{X}(t) dt \\ + [\pi'(t)\sigma(t) - \theta_\nu(t)] H_\nu(t)\bar{X}(t) dW(t) \ ; \quad X(0) = x \end{aligned} \tag{2.18}$$

for all $\nu \in \mathcal{D}$. Therefore, $H_\nu(\cdot)\bar{X}(\cdot)$ is a P-local supermartingale (note that $\delta(\nu) + \pi'\nu \geq 0$ for $\pi \in K$ and $\nu \in \tilde{K}$), and from (2.9) thus also a P-supermartingale, by Fatou's lemma. Consequently,

$$E[H_\nu(T)\bar{X}^{x,\pi,\kappa}(T)] \leq x \quad \forall \ (\pi, \kappa, \nu) \ \in \ \mathcal{A}(x) \times \mathcal{D} \ . \tag{2.19}$$

3. The minimization problem and its dual. Suppose now that, at time $t = T$, the agent has to deliver a payoff given by a contingent claim C, a random variable in $\mathbf{L}^2(\Omega, \mathcal{F}(T), P)$, with

$$P[C \geq 0] = 1 \quad \text{and} \quad P[C > 0] > 0. \tag{3.1}$$

Introduce a (possibly infinite) process

$$\bar{C}(t) := ess \sup_{\nu \in \mathcal{D}} E\left[H_\nu(T)\bar{C} \,\middle|\, \mathcal{F}(t) \right], \qquad 0 \leq t \leq T, \tag{3.2}$$

a.s., the discounted version of the process

$$C(\cdot) := S_0(\cdot)\bar{C}(\cdot). \tag{3.3}$$

We have denoted

$$\bar{C} := \frac{C}{S_0(T)} \tag{3.4}$$

the discounted value of the $\mathcal{F}(T)$-measurable random variable C. We impose the following assumption throughout the rest of the paper (see Remark 3.14 for a discussion on the relevance of this assumption).

Assumption 3.1. We assume

$$C(0) = \sup_{\nu \in \mathcal{D}} E[H_\nu(T)\bar{C}] < \infty. \tag{3.5}$$

The following theorem is taken from the literature on constrained financial markets (see, for example, [CK93], [KK96], or Cvitanić (1997)).

THEOREM 3.2. (Cvitanić and Karatzas (1993)). *Let $C \geq 0$ be a given contingent claim. Under Assumption* 3.1*, the process $C(\cdot)$ of* (3.3) *is finite, and it is equal to the minimal admissible wealth process hedging the claim C. More precisely, there exists a pair $(\pi_C, \kappa_C) \in \mathcal{A}(C(0))$ such that*

$$C(\cdot) \equiv X^{C(0), \pi_C, \kappa_C}(\cdot), \tag{3.6}$$

and, if for some $x \geq 0$ and some pair $(\pi, \kappa) \in \mathcal{A}(x)$ we have

$$X^{x,\pi,\kappa}(T) \geq C, \;\; P - a.s., \tag{3.7}$$

then

$$X^{x,\pi,\kappa}(t) \geq C(t), \quad 0 \leq t \leq T, \;\; P - a.s.$$

Consequently, if $x \geq C(0)$, there exists then an admissible pair $(\pi, \kappa) \in \mathcal{A}(x)$ such that $X^{x,\pi,\kappa}(T) \geq C$. Achieving a "hedge without risk" is not possible for $x < C(0)$. Motivated by results of Artzner et al. (1999) (and similarly as in a complete market setting of Cvitanić and Karatzas (1999)) we choose the following risk function to be minimized:

$$V(x) \equiv V(x; C) := \inf_{(\pi,\kappa) \in \mathcal{A}(x)} E\left[\bar{C} - \bar{X}^{x,\pi,\kappa}(T)\right]^+. \tag{3.8}$$

In other words, we are minimizing the expected discounted net loss, over all admissible trading strategies.

If $x \geq C(0)$, we have $V(x) = 0$, because, as mentioned above, we can find a wealth process that hedges C. Moreover, the margin requirement (2.9) implies that $x \geq 0$, so we assume from now on that

$$0 < x < C(0). \tag{3.9}$$

Note that we can (and do) assume $X^{x,\pi,\kappa}(T) \leq C$, $P-$a.s., in our optimization problem (3.8), since the agent can always consume down to the value of C, in case he has more than C at time T. In particular, if $C(\omega) = 0$, we can (and do) assume $X^{x,\pi,\kappa}(T,\omega) = 0$, too. This means that the set $\{C = 0\} \in \mathcal{F}(T)$ is not relevant for the problem (3.8), which motivates us to define a new probability measure

$$P^C(\Lambda) = \frac{1}{E[\bar{C}]} E[\bar{C}\mathbf{1}_\Lambda], \quad \Lambda \in \mathcal{F}(T) \tag{3.10}$$

(see also Remark 3.14 (ii)). Denote by E^C the associated expectation operator.

The problem (3.8) has then an equivalent formulation

$$V(x) = E[\bar{C}] \inf_{(\pi,\kappa)\in\mathcal{A}(x)} E^C \left[1 - \frac{X^{x,\pi,\kappa}(T)}{C}\right]^+ . \tag{3.11}$$

We approach the problem (3.11) by recalling familiar tools of convex duality: starting with the convex loss function $R(y) = (1-y)^+$, consider its Legendre–Fenchel transform

$$\tilde{R}(z) := \min_{0\leq y\leq 1} [R(y) + yz] = z \wedge 1, \quad z \geq 0 \tag{3.12}$$

(where $z \wedge 1 = \min\{z, 1\}$). The minimum in (3.12) is attained by any number $I(z; b)$ of the form

$$I(z;b) := \left\{ \begin{array}{ccc} 0 & ; & z > 1 \\ 1 & ; & 0 \leq z < 1 \\ b & ; & z = 1 \end{array} \right\}, \tag{3.13}$$

where $0 \leq b \leq 1$.

Consequently, denoting

$$Y^{x,\pi,\kappa} := \frac{X^{x,\pi,\kappa}(T)}{C}, \; P^C - a.s., \tag{3.14}$$

we conclude from (3.12) that for any initial capital $x \in (0, C(0))$ and any $(\pi,\kappa) \in \mathcal{A}(x)$, $\nu \in \mathcal{D}$, $z \geq 0$ we have

$$(1 - Y^{x,\pi,\kappa})^+ \geq \tilde{R}(zH_\nu(T)) - zH_\nu(T)Y^{x,\pi,\kappa}, \; P^C - a.s. \tag{3.15}$$

Thus, multiplying by $E[\bar{C}]$, taking expectations, and in conjunction with (2.19), we obtain

$$\begin{aligned} E[\bar{C}]E^C \left[1 - Y^{x,\pi,\kappa}\right]^+ &\geq E[\bar{C}]E^C \left[\tilde{R}(zH_\nu(T))\right] - zE[\bar{C}]E^C \left[H_\nu(T)Y^{x,\pi,\kappa}\right] \\ &\geq E[\bar{C}]E^C \left[\tilde{R}(zH_\nu(T))\right] - xz. \end{aligned} \tag{3.16}$$

This is a type of a duality relationship that has proved to be very useful in constrained portfolio optimization studied in Cvitanić and Karatzas (1992). The difference here is that we have to extend it to the random variables in the set

$$\begin{aligned} \mathcal{H} := \{ H \in \mathbf{L}^1(\Omega, \mathcal{F}(T), P^C) \ / \ H \geq 0 \ P^C - a.s., \\ E[\bar{C}]E^C[HY^{x,\pi,\kappa}] \leq x, \ \forall\, (\pi,\kappa) \in \mathcal{A}(x) \}. \end{aligned} \tag{3.17}$$

Remark 3.3. As the referee points out, it should be noted that the set $\mathcal{H}$ does not actually depend on x. This is because $X^{x,\pi,\kappa}(\cdot) = xX^{1,\pi,\kappa/x}(\cdot)$, so that the inequality in the definition (3.17) of $\mathcal{H}$ is equivalent to

$$E[\bar{C}]E^C[HY^{1,\pi,\kappa}] \leq 1 \quad \forall\, (\pi,\kappa) \in \mathcal{A}(1). \tag{3.18}$$

It is clear that $\mathcal{H}$ is a convex set. It is also closed in $\mathbf{L}^1(\Omega, \mathcal{F}(T), P^C)$. Indeed, if $H_n \to H$ in $\mathbf{L}^1(\Omega, \mathcal{F}(T), P^C)$, then there exists a (relabeled) subsequence $\{H_n\}_{n\in\mathbb{N}}$ converging to H, P^C-a.s.; therefore $H \geq 0$, $P^C - a.s.$, and, by Fatou's lemma, $E[\bar{C}]E^C[HY^{x,\pi,\kappa}] \leq x \quad \forall (\pi,\kappa) \in \mathcal{A}(x)$.

By Theorem 3.2 we have $C(\cdot) = X^{C(0),\pi_C,\kappa_C}(\cdot)$ for some $(\pi_C, \kappa_C) \in \mathcal{A}(x)$. Consequently, we have $Y^{C(0),\pi_C,\kappa_C} = 1$, $P^C - a.s.$, and therefore

$$E[\bar{C}]E^C[H] = E[\bar{C}H] \leq C(0) \quad \forall\, H \in \mathcal{H}, \tag{3.19}$$

where we extend a random variable H to the probability space $(\Omega, \mathcal{F}(T), P)$ by setting $H = 0$ on $\{C = 0\}$. Similarly, since $0 \in K$, taking $\bar{X}^{x,0,0}(T) = x$ in the definition (3.17) of $\mathcal{H}$, we see that

$$E[\bar{C}]E^C[H/C] = E[H] \leq 1 \quad \forall\, H \in \mathcal{H}. \tag{3.20}$$

Moreover, since $E[\bar{C}]E^C[H_\nu(T)] \leq C(0) < \infty \quad \forall \nu \in \mathcal{D}$, and by (2.19), we get

$$\mathcal{H}_{\mathcal{D}} := \{H_\nu(T) \ / \ \nu \in \mathcal{D}\} \ \subset \ \mathcal{H}. \tag{3.21}$$

Remark 3.4. The idea of introducing the set $\mathcal{H}$ is similar to and inspired by the approach of Kramkov and Schachermayer (1999), who work with the set of all nonnegative processes $G(\cdot)$ such that $G(\cdot)\bar{X}(\cdot)$ is a P-supermartingale for all admissible wealth processes $X(\cdot)$.

Next, arguing as above (when deducing (3.16)), we obtain

$$E[\bar{C}]E^C\left[1 - Y^{x,\pi,\kappa}\right]^+ \geq E[\bar{C}]E^C\left[\tilde{R}(zH)\right] - xz =: \tilde{J}(H;z) - xz \quad \forall\, H \in \mathcal{H},\ z \geq 0, \tag{3.22}$$

where we have denoted

$$\tilde{J}(H;z) := E[\bar{C}]E^C[(zH) \wedge 1]. \tag{3.23}$$

It is easily seen that $-\tilde{J}(\cdot;z) : \mathbf{L}^1(\Omega, \mathcal{F}(T), P^C) \to \mathbb{R}$ is a convex, lower-semicontinuous and proper functional, in the terminology of convex analysis; see, for example, Aubin and Ekeland (1984), henceforth [AE84].

Remark 3.5. It is straightforward to see that the inequality of (3.22) holds as equality for some $(\hat{\pi}, \hat{\kappa}) \in \mathcal{A}(x)$ and $\hat{z} \geq 0$, $\hat{H} \in \mathcal{H}$, *if and only if* we have

$$E[\bar{C}]E^C\left[\hat{H}Y^{x,\hat{\pi},\hat{\kappa}}\right] = x \tag{3.24}$$

and

$$Y^{x,\hat{\pi},\hat{\kappa}} = I(\hat{z}\hat{H};\hat{B}) = 1_{\{\hat{z}\hat{H}<1\}} + \hat{B}1_{\{\hat{z}\hat{H}=1\}}, \; P^C - a.s. \tag{3.25}$$

for some $\mathcal{F}(T)$-measurable random variable $\hat{B}$ that satisfies $0 \leq \hat{B} \leq 1$, P^C- a.s. We also set

$$Y^{x,\hat{\pi},\hat{\kappa}} = 0 \;\text{ on }\; \{C = 0\}.$$

If (3.24) and (3.25) are satisfied, then $(\hat{\pi},\hat{\kappa})$ is optimal for the problem (3.11), under the "change of variables" (3.14), since the lower bound of (3.22) is attained. Moreover, $\hat{H} \in \mathcal{H}$ is optimal for the auxiliary dual problem

$$\tilde{V}(z) = \sup_{H\in\mathcal{H}} \tilde{J}(H;z) \tag{3.26}$$

with $z = \hat{z}$. If we let

$$X^{x,\hat{\pi},\hat{\kappa}}(T) = CY^{x,\hat{\pi},\hat{\kappa}}, \quad P - a.s., \tag{3.27}$$

the conditions (3.24) and (3.25) become

$$E\left[\hat{H}\bar{X}^{x,\hat{\pi},\hat{\kappa}}(T)\right] = x \tag{3.28}$$

and

$$X^{x,\hat{\pi},\hat{\kappa}}(T) = C\left(1_{\{\hat{z}\hat{H}<1\}} + \hat{B}1_{\{\hat{z}\hat{H}=1\}}\right), \; P - a.s. \tag{3.29}$$

for some $\mathcal{F}(T)$-measurable random variable $\hat{B}$ that satisfies $0 \leq \hat{B} \leq 1$, a.s., and $X^{x,\hat{\pi},\hat{\kappa}}(T)$ is the terminal wealth of the strategy $(\hat{\pi},\hat{\kappa})$ which is optimal for the problem (3.8). □

In light of the preceding remark, our approach will be the following: we will try to find a number $\hat{z} > 0$, a solution $\hat{H}$ to the auxiliary dual problem (3.26) with $z = \hat{z}$, a number $\hat{z} > 0$, a random variable $\hat{B}$ as above, and a pair $(\hat{\pi},\hat{\kappa}) \in \mathcal{A}(x)$ such that (3.24) and (3.25) (or, equivalently, (3.28) and (3.29)) are satisfied.

THEOREM 3.6. *For any given $z > 0$, there exists an optimal solution $\hat{H} = \hat{H}_z \in \mathcal{H}$ for the auxiliary dual problem* (3.26)*.*

Proof. Let $H_n \in \mathcal{H}$ be a sequence that attains the supremum in (3.26), so that

$$\lim_n \tilde{J}(H_n;z) = \tilde{V}(z).$$

Note that, by (3.19), $\mathcal{H}$ is a bounded set in $\mathbf{L}^1(\Omega,\mathcal{F}(T),P^C)$, so that by Komlós theorem (see Schwartz (1986), for example) there exists a random variable $\hat{H} \in \mathbf{L}^1(\Omega,\mathcal{F}(T),P^C)$ and a (relabeled) subsequence $\{H_i\}_{i\in\mathbb{N}}$ such that

$$G_n := \frac{1}{n}\sum_{i=1}^n H_i \to \hat{H}, \quad P^C - a.s.$$

Fatou's lemma then implies $\hat{H} \in \mathcal{H}$. Since $0 \leq (zG_n) \wedge 1 \leq 1$, by the dominated convergence theorem and concavity of $\tilde{J}(\cdot;z)$ we get

$$\tilde{J}(\hat{H};z) = \lim_n \tilde{J}\left(\frac{1}{n}\sum_{i=1}^n H_i;z\right) \geq \lim_n \left[\frac{1}{n}\sum_{i=1}^n \tilde{J}(H_i;z)\right] = \tilde{V}(z).$$

Thus, $\hat{H} \in \mathcal{H}$ is optimal. ◻

LEMMA 3.7. *The function $\tilde{V}(z)$ is continuous on* $[0, \infty)$.

Proof. Let $H \in \mathcal{H}$ and assume first $z_1, z_2 > 0$. We have

$$\begin{aligned}
\tilde{J}(H; z_1) &= E[\bar{C}]E^C[(z_1 H) \wedge 1] = E[\bar{C}]E^C\big[(z_2 H) \wedge 1 + (z_1 H) \wedge 1 - (z_2 H) \wedge 1\big] \\
&= \tilde{J}(H; z_2) + E[\bar{C}]E^C\big[H(z_1 - z_2)\mathbf{1}_{\{z_1 H < 1, z_2 H < 1\}} + (z_1 H - 1)\mathbf{1}_{\{z_1 H < 1, z_2 H \geq 1\}} \\
&\qquad + (1 - z_2 H)\mathbf{1}_{\{z_1 H \geq 1, z_2 H < 1\}}\big] \\
&\leq \tilde{V}(z_2) + 2E[\bar{C}](1 - z_2/z_1)^+.
\end{aligned}$$

Taking the supremum over $H \in \mathcal{H}$ we get $\tilde{V}(z_1) - \tilde{V}(z_2) \leq 2E[\bar{C}](1 - z_2/z_1)^+$. Since we can do the same while interchanging the roles of z_1 and z_2, we have shown continuity on $(0, \infty)$. To prove continuity at $z_2 = 0$, note that, by duality and (3.20), we have

$$\tilde{J}(H; z_1) = E[\bar{C}]E^C[(z_1 H) \wedge 1] \leq E[(\bar{C} - y)^+] + y z_1 E[H] \leq E[(\bar{C} - y)^+] + y z_1$$

for all $z_1 > 0, y > 0$. Choosing first y large enough and then z_1 small enough, we can make the two terms on the right-hand side arbitrarily close to zero, uniformly in $H \in \mathcal{H}$. ◻

PROPOSITION 3.8. *For every $0 < x < C(0)$ there exists $\hat{z} = \hat{z}_x \in (0, \infty)$ that attains the supremum* $\sup_{z \geq 0}[\tilde{V}(z) - xz]$.

Proof. Denote

$$\alpha(z) := \tilde{V}(z) - xz.$$

Note that $\alpha(0) = 0$. It is clear that

$$\limsup_{z \to \infty} \alpha(z) < 0, \tag{3.30}$$

so that the supremum of $\alpha(z)$ over $[0, \infty)$ cannot be attained at $z = \infty$. Consequently, being continuous by Lemma 3.7, function $\alpha(z)$ either attains its supremum at some $\hat{z} > 0$, or else $\alpha(z) \leq \alpha(0) = 0$ for all $z > 0$. Suppose that the latter is true. We have then

$$x \geq \frac{\tilde{V}(z)}{z} \geq E[\bar{C}]E^C\left[H \wedge \frac{1}{z}\right] \tag{3.31}$$

for all $z > 0$ and $H \in \mathcal{H}$. In particular, we can use the dominated convergence theorem while letting $z \to 0$ to get

$$x \geq E[H\bar{C}]$$

for all $H \in \mathcal{H}_{\mathcal{D}}$. Taking the supremum over $H \in \mathcal{H}_{\mathcal{D}}$ we obtain $x \geq C(0)$, a contradiction. ◻

Denote $\hat{H} = \hat{H}_{\hat{z}}$ the optimal dual variable for problem (3.26), corresponding to $z = \hat{z}$ of Proposition 3.8. We want to show that there exists an $\mathcal{F}(T)$-measurable random variable $0 \leq \hat{B} \leq 1$ such that the optimal wealth for the primal problem is given by $CI(\hat{z}\hat{H}, \hat{B})$, where $I(z; b)$ is given in (3.13). In order to do that, we recall some notions and results from convex analysis, as presented, for example, in [AE84].

First, introduce the space

$$\mathbf{L} := \mathbf{L}^1(\Omega, \mathcal{F}(T), P^C) \times \mathbb{R} \tag{3.32}$$

with the norm

$$\|(Z,z)\| := E[\bar{C}]E^C|Z| + |z|$$

and its subset

$$\mathcal{G} := \{(zH, z) \in \mathbf{L} \ / \ z \geq 0, H \in \mathcal{H}\}. \tag{3.33}$$

It is easily seen that $\mathcal{G}$ is convex, by the convexity of $\mathcal{H}$. It is also closed in $\mathbf{L}$. Indeed, if we are given subsequences $z_n \geq 0$ and $H_n \in \mathcal{H}$ such that $(z_n H_n, z_n) \to (Z, z)$ in $\mathbf{L}$, then $z_n \to z$; we also have, from (3.19),

$$E^C|z_n H_n - zH_n| \leq |z_n - z| E^C[H_n] \leq \frac{C(0)}{E[\bar{C}]}|z_n - z|,$$

so that $zH_n \to Z$ in $\mathbf{L}^1(\Omega, \mathcal{F}(T), P^C)$. If $z = 0$ we get $Z = 0$, and we are done. If $z > 0$, we get $H_n \to Z/z$ in $\mathbf{L}^1(\Omega, \mathcal{F}(T), P^C)$, therefore $Z/z \in \mathcal{H}$ because $\mathcal{H}$ is closed in $\mathbf{L}^1(\Omega, \mathcal{F}(T), P^C)$, and we are done again. The closedness of $\mathcal{G}$ has been confirmed.

We now define a functional $\tilde{U} : \mathbf{L} \to \mathbb{R}$ by

$$\tilde{U}(Z, z) := -E[\bar{C}]E^C[Z \wedge 1] + xz = -\tilde{J}(Z; 1) + xz. \tag{3.34}$$

It is easy to check that $\tilde{U}$ is convex, lower-semicontinuous, and proper on $\mathbf{L}$. Moreover, since we have

$$\begin{aligned} \tilde{J}(\hat{H}, \hat{z}) - x\hat{z} &= \tilde{V}(\hat{z}) - x\hat{z} \geq \tilde{V}(z) - xz \\ &\geq \tilde{J}(H, z) - xz \quad \forall\ (H, z) \in \mathcal{H} \times [0, \infty) \end{aligned}$$

from Proposition 3.8 and in the notation of Theorem 3.6, it follows that the pair $\hat{G} := (\hat{z}\hat{H}, \hat{z}) \in \mathcal{G}$ is optimal for the *dual problem*

$$\inf_{(Z,z) \in \mathcal{G}} \tilde{U}(Z, z). \tag{3.35}$$

Let $\mathbf{L}^* := \mathbf{L}^\infty(\Omega, \mathcal{F}(T), P^C) \times \mathbb{R}$ be the dual space to $\mathbf{L}$ and let $N(\hat{z}\hat{H}, \hat{z})$ be the *normal cone* to the set $\mathcal{G}$ at the point $(\hat{z}\hat{H}, \hat{z})$, given by

$$N(\hat{z}\hat{H}, \hat{z}) = \{(Y, y) \in L^* \ / \ E[\bar{C}]E^C[\hat{z}\hat{H}Y] + \hat{z}y = \max_{(zH,z) \in \mathcal{G}} (E[\bar{C}]E^C[zHY] + zy)\} \tag{3.36}$$

by Proposition 4.1.4 in [AE84]. Let $\partial\tilde{U}(\hat{z}\hat{H}, \hat{z})$ denote the subdifferential of $\tilde{U}$ at $(\hat{z}\hat{H}, \hat{z})$, which, by Proposition 4.3.3 in [AE84], is given by

$$\begin{aligned} \partial\tilde{U}(\hat{z}\hat{H}, \hat{z}) = \{&(Y, y) \in \mathbf{L}^* \ / \ \tilde{U}(\hat{z}\hat{H}, \hat{z}) - \tilde{U}(Z, z) \\ &\leq E[\bar{C}]E^C[Y(\hat{z}\hat{H} - Z)] + y(\hat{z} - z) \quad \forall\ (Z, z) \in \mathbf{L}\}. \end{aligned} \tag{3.37}$$

Then, by Corollary 4.6.3 in [AE84], since $(\hat{z}\hat{H}, \hat{z})$ is optimal for the problem (3.35), we obtain the following proposition.

Proposition 3.9. *The pair $(\hat{z}\hat{H}, \hat{z}) \in \mathcal{G}$ is a solution to*

$$0 \in \partial\tilde{U}(\hat{z}\hat{H}, \hat{z}) + N(\hat{z}\hat{H}, \hat{z}). \tag{3.38}$$

In other words, there exists a pair $(\hat{Y}, \hat{y}) \in \mathbf{L}^*$ *which belongs to the normal cone* $N(\hat{z}\hat{H}, \hat{z})$ *and such that* $-(\hat{Y}, \hat{y})$ *belongs to the subdifferential* $\partial\tilde{U}(\hat{z}\hat{H}, \hat{z})$*.*

From (3.36) and (3.37), this is equivalent to

$$E[\bar{C}]E^C[\hat{z}\hat{H}\hat{Y}] + \hat{z}\hat{y} \geq E[\bar{C}]E^C[zH\hat{Y}] + z\hat{y} \quad \forall z \geq 0, H \in \mathcal{H}, \tag{3.39}$$

and

$$\begin{aligned} &E[\bar{C}]E^C[\hat{Y}(\hat{z}\hat{H} - Z)] + \hat{y}(\hat{z} - z) \\ &\quad \leq E[\bar{C}]E^C[(\hat{z}\hat{H}) \wedge 1] - E[\bar{C}]E^C[Z \wedge 1] + x(z - \hat{z}) \quad \forall\ (Z, z) \in \mathbf{L}. \end{aligned} \tag{3.40}$$

It is clear from (3.40) (by letting $z \to \pm\infty$ while keeping Z fixed) that, necessarily,

$$\hat{y} = -x.$$

On the other hand, if we let $\hat{z} = z$ in (3.39), we get

$$E^C[\hat{Y}\hat{H}] \geq E^C[\hat{Y}H] \quad \forall H \in \mathcal{H}. \tag{3.41}$$

Moreover, letting $z = (\hat{z} + \varepsilon)$ for some $\varepsilon > 0$, $H = \hat{H}$ in (3.39), and recalling $\hat{y} = -x$, we obtain

$$x \geq E[\bar{C}]E^C[\hat{Y}\hat{H}].$$

Similarly, we get the reverse inequality by letting $\hat{z} = z - \varepsilon$ and $H = \hat{H}$ in (3.39) (recall that $\hat{z} > 0$ by Proposition 3.8), to obtain, finally,

$$E[\bar{C}]E^C[\hat{Y}\hat{H}] = x. \tag{3.42}$$

This last equality will correspond to (3.24) with $\hat{Y} = Y^{x,\hat{\pi},\hat{\kappa}}$, if we can show the following result and recall (3.14).

PROPOSITION 3.10. *There exists an admissible pair* $(\hat{\pi}, \hat{\kappa}) \in \mathcal{A}(x)$ *such that*

$$X^{x,\hat{\pi},\hat{\kappa}}(T) = C\hat{Y},\ P - a.s.$$

and such that (3.28) *is satisfied.*

(Here we set $\hat{Y} = 0$ on $\{C = 0\}$.)

Proof. This follows immediately from (3.41) and (3.42), which can be written as

$$x = E[\bar{C}\hat{Y}\hat{H}] = \sup_{H \in \mathcal{H}} E[\bar{C}\hat{Y}H]$$

(with $H = 0$ on $\{C = 0\}$). Indeed, Theorem 3.2 tells us that the right-hand side is no smaller than the minimal amount of initial capital needed to hedge $C\hat{Y}$; thus, there exists a pair $(\hat{\pi}, \hat{\kappa}) \in \mathcal{A}(x)$ that does the hedge. □

In order to "close the loop," it only remains to show (3.25).

PROPOSITION 3.11. *Let* $-(Y, y) \in \partial\tilde{U}(\hat{z}\hat{H}, \hat{z})$*. Then* $y = -x$ *and* Y *is of the form*

$$Y = 1_{\{\hat{z}\hat{H} < 1\}} + B1_{\{\hat{z}\hat{H} = 1\}}\ ,\ P^C - a.s. \tag{3.43}$$

for some $\mathcal{F}(T)$*-measurable random variable* B *that satisfies* $0 \leq B \leq 1$*,* P^C *a.s.*

Proof. We have already seen that $y = -x$. Define a random variable A by

$$Y = \mathbf{1}_{\{\hat{z}\hat{H}<1\}} + A. \tag{3.44}$$

From (3.40) with $\hat{y} = -x$ and $\hat{Y} = Y$, we get

$$\begin{aligned} &E^C[A(\hat{z}\hat{H} - Z)] - E^C[Z\mathbf{1}_{\{\hat{z}\hat{H}<1\}}] \\ &\quad \leq E^C[\mathbf{1}_{\{\hat{z}\hat{H}\geq 1\}}] - E^C[Z \wedge 1] \quad \forall\, Z \in \mathbf{L}^1(\Omega, \mathcal{F}(T), P^C). \end{aligned} \tag{3.45}$$

Let $Z \in \mathbf{L}^1(\Omega, \mathcal{F}(T), P^C)$ be such that

$$\{\hat{z}\hat{H} < 1\} = \{Z < 1\}.$$

Then,

$$E^C[A(\hat{z}\hat{H} - Z)] = E^C[A(\hat{z}\hat{H} - Z)\mathbf{1}_{\{\hat{z}\hat{H}<1\}}] + E^C[A(\hat{z}\hat{H} - Z)\mathbf{1}_{\{\hat{z}\hat{H}\geq 1\}}] \leq 0, \tag{3.46}$$

by (3.45). This implies

$$A \leq 0 \;\text{ on }\; \{\hat{z}\hat{H} < 1\}, \quad A \geq 0 \;\text{ on }\; \{\hat{z}\hat{H} \geq 1\}, \quad P^C - a.s., \tag{3.47}$$

for otherwise we could make Z arbitrarily small (respectively, large) on $\{\hat{z}\hat{H} < 1\} \cap \{A > 0\}$ (respectively, on $\{\hat{z}\hat{H} \geq 1\} \cap \{A < 0\}$) to get a contradiction in (3.46).

Suppose now that $P^C[A < 0, \hat{z}\hat{H} < 1] > 0$. There exists then $\delta > 0$ such that $E^C[A(\hat{z}\hat{H} - 1)\mathbf{1}_{\{\hat{z}\hat{H}<1\}}] > \delta$, because of (3.47). For a given $\varepsilon > 0$, let $Z = 1 - \varepsilon$ on $\{\hat{z}\hat{H} < 1\}$ and $Z = 1$ on $\{\hat{z}\hat{H} \geq 1\}$, in (3.46). This gives

$$E^C[A(\hat{z}\hat{H} - 1 + \varepsilon)\mathbf{1}_{\{\hat{z}\hat{H}<1\}}] + E^C[A(\hat{z}\hat{H} - 1)\mathbf{1}_{\{\hat{z}\hat{H}\geq 1\}}] \leq 0. \tag{3.48}$$

The left-hand side is greater than $\delta + \varepsilon E^C[A\mathbf{1}_{\{\hat{z}\hat{H}<1\}}]$ for all $\varepsilon > 0$ (recall (3.47) again), a contradiction to (3.48). Thus, we have shown

$$A = 0 \;\text{ on }\; \{\hat{z}\hat{H} < 1\}, \quad P^C - a.s. \tag{3.49}$$

Going back to (3.46), this implies

$$E^C[A(\hat{z}\hat{H} - Z)\mathbf{1}_{\{\hat{z}\hat{H}\geq 1\}}] \leq 0 \tag{3.50}$$

for all $Z \in \mathbf{L}^1(\Omega, \mathcal{F}(T), P^C)$ such that $\{Z < 1\} = \{\hat{z}\hat{H} < 1\}$. If we set now $Z = 1$ on $\{\hat{z}\hat{H} \geq 1\}$, we get from (3.50) and (3.47)

$$A = 0 \;\text{ on }\; \{\hat{z}\hat{H} > 1\}, \quad P^C - a.s. \tag{3.51}$$

Using (3.49) and (3.51) in (3.45), we obtain

$$\begin{aligned} &E^C[A(1 - Z)\mathbf{1}_{\{\hat{z}\hat{H}=1\}}] - E^C[Z(\mathbf{1}_{\{\hat{z}\hat{H}<1\}} - \mathbf{1}_{\{Z<1\}})] \\ &\quad \leq E^C[\mathbf{1}_{\{\hat{z}\hat{H}\geq 1\}} - \mathbf{1}_{\{Z\geq 1\}}] \quad \forall\, Z \in \mathbf{L}^1(\Omega, \mathcal{F}(T), P^C). \end{aligned} \tag{3.52}$$

Suppose now that $P^C[A > 1, \hat{z}\hat{H} = 1] > 0$. There exists then $\delta > 0$ such that $E^C[A\mathbf{1}_{\{\hat{z}\hat{H}=1, A>1\}}] > \delta + P^C[\hat{z}\hat{H} = 1, A > 1]$. Setting $Z = 0$ on $\{\hat{z}\hat{H} = 1, A > 1\}$, $Z = 1$ on $\{\hat{z}\hat{H} = 1, A \leq 1\}$, and $Z = 1 - \varepsilon$ otherwise (for a given $\varepsilon > 0$), (3.52) implies

$$\begin{aligned} &E^C[A\mathbf{1}_{\{\hat{z}\hat{H}=1, A>1\}}] - (1 - \varepsilon)E^C[\mathbf{1}_{\{\hat{z}\hat{H}<1\}} - \mathbf{1}_{\{\hat{z}\hat{H}\neq 1\}}] \\ &\quad \leq P^C[\hat{z}\hat{H} \geq 1] - P^C[\hat{z}\hat{H} = 1, A \leq 1]. \end{aligned} \tag{3.53}$$

The left-hand side is greater than $\delta + P^C[\hat{z}\hat{H} = 1, A > 1] + (1-\varepsilon)(P^C[\hat{z}\hat{H} \neq 1] - P^C[\hat{z}\hat{H} < 1])$, so that from (3.53) we conclude $\delta - \varepsilon(P^C[\hat{z}\hat{H} \neq 1] - P^C[\hat{z}\hat{H} < 1]) \leq 0$ for all $\varepsilon > 0$, a contradiction. Therefore,

$$A \leq 1 \ , \quad \text{on} \quad \{\hat{z}\hat{H} = 1\}, \quad P^C - a.s. \tag{3.54}$$

Together with (3.44), (3.47), (3.49), and (3.51), this completes the proof. ∎

We now state the main result of the paper.

THEOREM 3.12. *For any initial wealth x with $0 < x < C(0) < \infty$, there exists an optimal pair $(\hat{\pi}, \hat{\kappa}) \in \mathcal{A}(x)$ for the problem* (3.8) *of minimizing the expected loss of hedging the claim C. It can be taken as that strategy for which the terminal wealth $X^{x,\hat{\pi},\hat{\kappa}}(T)$ is given by* (3.29), *i.e.,*

$$X^{x,\hat{\pi},\hat{\kappa}}(T) = C\left(1_{\{\hat{z}\hat{H}<1\}} + \hat{B}1_{\{\hat{z}\hat{H}=1\}}\right) \ , \ P - a.s. \tag{3.55}$$

Here $(\hat{z}, \hat{H})$ is an optimal solution for the dual problem (3.35), *and $\hat{B}$ can be taken as the random variable B in Proposition* 3.11, *with (Y, y) replaced by some $(\hat{Y}, \hat{y}) \in \{-\partial\tilde{U}(\hat{z}\hat{H}, \hat{z}) \cap N(\hat{z}\hat{H}, \hat{z})\}$, which exists by Proposition* 3.9.

Proof. It follows from Remark 3.5. Indeed, it was observed in that remark that a pair $(\hat{\pi}, \hat{\kappa}) \in \mathcal{A}(x)$ is optimal for the problem (3.8) if it satisfies (3.28) and (3.55) for some $\mathcal{F}(T)$-measurable random variable $\hat{B}$, $0 \leq \hat{B} \leq 1$, and some $\hat{z} \geq 0$, $\hat{H} \in \mathcal{H}$. The existence of such a pair $(\hat{\pi}, \hat{\kappa}) \in \mathcal{A}(x)$ was established in Proposition 3.10 in conjunction with Proposition 3.11, with $\hat{B}$, $\hat{z}$, and $\hat{H}$ as in the statement of the theorem. ∎

The following simple example is mathematically interesting from several points of view. It shows that the optimal dual variable $\hat{H}$ can be equal to zero with positive probability, unlike the case of classical utility maximization under constraints (as in Cvitanić and Karatzas (1992)). Moreover, $\hat{z}\hat{H}$ can be equal to one with positive probability, so that the use of nonsmooth optimization techniques and subdifferentials for the dual problem is really necessary. It also shows why it can be mathematically convenient to allow nonzero consumption. Finally, it confirms that condition (3.5) is not always necessary for the dual approach to work.

Example 3.13. Suppose $r(\cdot) \equiv 0$ for simplicity, and let $C \geq 0$ be any contingent claim such that $P[C \geq x] > 0$. We consider the trivial primal problem for which $K = \{0\}$, so that there is only one possible admissible portfolio strategy $\hat{\pi}(\cdot) \equiv 0$ (in other words, the agent can invest only in the riskless asset). We do not assume condition (3.5), which, for these constraints, is equivalent to C being bounded. It is clear that the value $V(x)$ of the primal problem is $E[C-x]^+$, and duality implies

$$E[C-x]^+ \geq E[C((zH) \wedge 1)] - xz \tag{3.56}$$

for all $z \geq 0$, $H \in \mathcal{H}$ (see (3.22)). Here we can take $\mathcal{H}$ to be the set of all nonnegative random variables such that $E[H] \leq 1$. Let $\hat{z} := P[C \geq x] > 0$ and $\hat{z}\hat{H} := \mathbf{1}_{\{C \geq x\}}$. It is then easily checked that $\hat{H} \in \mathcal{H}$ and that the pair $(\hat{H}, \hat{z})$ attains equality in (3.56), so that the pair $(\hat{z}\hat{H}, \hat{z}) \in \mathcal{G}$ is optimal for the dual problem (3.35). One possible choice for the optimal terminal wealth is

$$X^{x,\hat{\pi},\hat{\kappa}}(T) = x\mathbf{1}_{\{C \geq x\}} + C\mathbf{1}_{\{C < x\}}.$$

According to (3.55), this corresponds to $\hat{B} = x/C$ on $\{C \geq x\}$, and $\hat{\kappa}(t) = 0$ for $t < T$, while $\hat{\kappa}(T) = (x - C)\mathbf{1}_{\{C < x\}}$.

Remark 3.14. (i) Assumption 3.1 is satisfied, for example, if C is bounded. We need it in order to get existence for the dual problem (3.35), due to our use of the Komlós theorem. Example 3.13 shows that this assumption is not always necessary: in this example the dual problem has a solution and there is no gap between the primal and the dual problem, even when (3.5) is not satisfied.

(ii) If we, in fact, assumed that C is bounded, the switch to the equivalent formulation (3.11) from (3.8) would not be necessary. (The reason for this is that the dual spaces of $\mathbf{L}^1(\Omega, \mathcal{F}(T), P)$ and $\mathbf{L}^1(\Omega, \mathcal{F}(T), P^C)$ are then the same, up to the equivalence class determined by the set $\{C = 0\}$.)

Remark 3.15. Since there are almost no examples with explicit solutions to this problem, it is of interest to study possible numerical algorithms. In Markovian continuous-time models this would involve solving Hamilton–Jacobi–Bellman PDEs, while in discrete models one could apply standard linear or convex programming techniques; see Blumenstein (1999) for some of these issues.

4. Dynamic measures of risk. Suppose now that we are not quite sure whether our subjective probability measure P is equal to the real-world measure. We would like to measure the risk of hedging the claim C under constraints given by set K, and under uncertainty about the real-world measure. According to Artzner et al. (1999), and Cvitanić and Karatzas (1999), it makes sense to consider the following quantities as the lower and upper bounds for the measure of such a risk, where we denote by $\mathcal{P}$ a set of possible real-world measures:

$$\underline{V}(x) := \sup_{Q \in \mathcal{P}} \inf_{(\pi,\kappa) \in \mathcal{A}(x)} E^Q \left[\bar{C} - \bar{X}^{x,\pi,\kappa}(T) \right]^+, \tag{4.1}$$

the *maximal risk that can be incurred, over all possible real-world measures*, dominated by its "min-max" counterpart

$$\overline{V}(x) := \inf_{(\pi,\kappa) \in \mathcal{A}(x)} \sup_{Q \in \mathcal{P}} E^Q \left[\bar{C} - \bar{X}^{x,\pi,\kappa}(T) \right]^+, \tag{4.2}$$

the upper-value of a fictitious *stochastic game* between an agent (who tries to choose $(\pi, \kappa) \in \mathcal{A}(x)$ so as to minimize his risk) and "the market" (whose "goal" is to choose the real-world measure that is least favorable for the agent). Here, E^Q is expectation under measure Q. A question is whether the "upper-value" (4.2) and the "lower-value" (4.1) of this game coincide and, if they do, to compute this common value. We shall answer this question only in a very specific setting as follows. Let P be the "reference" probability measure, as in the previous sections. We first change the margin requirement (2.9) to a more flexible requirement

$$\bar{X}^{x,\pi,\kappa}(t) \geq -k, \qquad 0 \leq t \leq T, \quad P - a.s., \tag{4.3}$$

where k is a constant such that $\infty > k \geq C(0) - x > 0$. We still assume $0 < x < C(0)$, and we look at the special case of the constraints given by

$$K = \{\pi \in \mathbb{R}^d \ / \ \pi_{m+1} = \cdots = \pi_d = 0\} \tag{4.4}$$

for some $m < d$. In other words, we only consider the case of a market which is incomplete due to the insufficient number of assets available for investment. In this case,

$$\tilde{K} = \{\nu \in \mathbb{R}^d \ / \ \nu_1 = \cdots = \nu_m = 0\}$$

and

$$\mathcal{D} = \{\text{bounded progress. meas. processes } \nu(\cdot) \ / \ \nu_1(\cdot) \equiv \cdots \equiv \nu_m(\cdot) \equiv 0\}.$$

We define the set $\mathcal{P}$ of possible real-world probability measures as follows. Let $\mathcal{E}$ be a set of progressively measurable and bounded processes $\nu(\cdot)$ and such that

$$\mathcal{D} \subset \mathcal{E}. \tag{4.5}$$

We set

$$\mathcal{P} := \{P_\nu \ / \ \nu \in \mathcal{E}\}, \tag{4.6}$$

in the notation of (2.14) (note that the reference measure P is not necessarily in $\mathcal{P}$). In other words, our set of all possible real-world probability measures includes all the "equivalent martingale measures" for our market, corresponding to bounded "kernels" $\nu(\cdot)$. This way, under a possible real-world probability measure $P_\nu \in \mathcal{P}$, the model $\mathcal{M}$ of (2.1) becomes

$$\begin{aligned} dS_0(t) &= S_0(t)r(t)dt, \quad S_0(0) = 1, \\ dS_i(t) &= S_i(t)\left[(r(t) - \nu_i(t))dt + \sum_{j=1}^{d} \sigma_{ij}(t)dW_\nu^j(t)\right], \\ S_i(0) &= s_i \in (0, \infty) \ ; \quad i = 1, \dots, d, \end{aligned} \tag{4.7}$$

in the notation of (2.15). The resulting modified model $\mathcal{M}_\nu$ is similar to that of (2.1); now $W_\nu(\cdot)$ plays the role of the driving Brownian motion (under P_ν), but the stock return rates are different for different "model measures" P_ν.

The following theorem shows that, if the uncertainty about the real-world probability measure is large enough (in the sense that all equivalent martingale measures corresponding to bounded kernels are possible candidates for the real-world measure), then the optimal thing to do in order to minimize the expected risk of hedging a claim C in the market is the following: *borrow exactly as much money from the bank as is needed to hedge C.*

THEOREM 4.1. *Under the above assumptions we have*

$$\overline{V}(x) = \underline{V}(x) = C(0) - x. \tag{4.8}$$

In other words, the stochastic game defined by (4.1) *and* (4.2) *has a value that is equal to the expected loss of the strategy which borrows $C(0) - x$ from the bank and then invests according to the least expensive strategy for hedging the claim C.*

Proof. Let (π^*, κ^*) be the strategy from the statement of the theorem, namely the one for which we have

$$\bar{X}^*(t) := \bar{X}^{x,\pi^*,\kappa^*}(t) = \bar{C}(t) - (C(0) - x), \ P - a.s.,$$

in the notation of (3.2). Such a strategy exists by Theorem 3.2. It is clear that (4.3) is then satisfied, so that $(\pi^*, \kappa^*) \in \mathcal{A}(x)$. Since for this strategy $E^Q[\bar{C} - \bar{X}^{x,\pi^*,\kappa^*}(T)]^+ = C(0) - x$ for all $Q \in \mathcal{P}$, it also follows that

$$\overline{V}(x) \leq C(0) - x. \tag{4.9}$$

On the other hand, we have here $\delta(\nu) = 0$ for $\nu \in \tilde{K}$, so that $H_\nu(\cdot) = Z_\nu(\cdot)$ for $\nu \in \mathcal{D}$, and Ito's rule gives, in analogy to (2.7) and in the notation of (2.15),

$$(4.10)\quad d\bar{X}^*(t) = -e^{-\int_0^t r(u)du} d\kappa^*(t) + (\pi^*(t))'\sigma(t)\bar{X}^*(t)dW_\nu(t)\ ;\quad X^*(0) = x$$

for all $\nu \in \mathcal{D}$, since $\nu'(\cdot)\pi^*(\cdot) \equiv 0$. Therefore, $\bar{X}^*(\cdot)$ is a P_ν-local supermartingale bounded from below, thus also a P_ν-supermartingale, by Fatou's lemma. Consequently,

$$(4.11)\qquad E_\nu[\bar{X}^*(T)] \le x \quad \forall\ \nu\ \in\ \mathcal{D},$$

where E_ν is the expectation under P_ν measure. Since $P_\nu \in \mathcal{P}$ for all $\nu \in \mathcal{D}$, (4.11) and Jensen's inequality imply

$$(4.12)\qquad \underline{V}(x) \ge \sup_{\nu\in\mathcal{D}} \inf_{(\pi,\kappa)\in\mathcal{A}(x)} (E_\nu[\bar{C}] - x)^+ = \sup_{\nu\in\mathcal{D}} (E_\nu[\bar{C}] - x)^+ = C(0) - x.$$

Since $\underline{V}(x) \le \overline{V}(x)$, (4.8) is a consequence of (4.9) and (4.12). □

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