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Cubature on Wiener space

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It is well known that there is a mathematical equivalence between 'solving' parabolic partial differential equations (PDEs) and 'the integration' of certain functionals on Wiener space. Monte Carlo simulation of stochastic differential equations (SDEs) is a naive approach based on this underlying principle. In finite dimensions, it is well known that cubature can be a very effective approach to integration. We discuss the appropriate extension of this idea to Wiener space. In the process we develop high-order numerical schemes valid for high-dimensional SDEs and semi-elliptic PDEs.

Keywords: cubature formulae; stochastic analysis; Chen series

1. Introduction

Let $C_b^\infty(\mathbb{R}^N, \mathbb{R}^N)$ be the space of $\mathbb{R}^N$-valued smooth functions defined in $\mathbb{R}^N$ whose derivatives of any order are bounded. We regard elements of $C_b^\infty(\mathbb{R}^N, \mathbb{R}^N)$ as vector fields on $\mathbb{R}^N$. Let $V_0, \dots, V_d$ be such vector fields, and define the differential operator $L = V_0 + \frac{1}{2}(V_1^2 + \cdots + V_d^2)$. Consider the parabolic partial differential equation (PDE)

$$\left.\begin{aligned} \frac{\partial u}{\partial t}(t, x) &= -Lu(t, x), \\ u(T, x) &= f(x), \end{aligned}\right\} \tag{1.1}$$

for a given Lipschitz function f. This paper proposes a new method to approximate $u(0, x)$ for a given x.

The classical theory of parabolic operators tends to address those equations which describe irreducible diffusive systems where 'heat' will be transported to all open sets. Solutions then tend to have regularity. The usual condition on L is the Hörmander condition. However, there are many situations where the Hörmander condition must fail as the diffusion respects certain invariants, and thus remains something similar to a leaf in a foliation. The existence of such diffusions is clear from probabilistic arguments, but in general the solution will not be regular along the leaf. This creates substantial difficulties for conventional numerical methods, which tend to 'bleed' off the leaf. The method we discuss here, almost by definition, avoids this problem and is well adapted to cases where the Hörmander condition fails. Many problems in physics and mathematical finance require such numerical techniques.

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Let us make more precise the probabilistic representation of parabolic PDEs. Consider the probability space $(C_0^0([0,T],\mathbb{R}^d),\mathcal{F},\mathbb{P})$, where $C_0^0([0,T],\mathbb{R}^d)$ is the space of $\mathbb{R}^d$-valued continuous functions defined in $[0,T]$ and which starts at zero (i.e. the Wiener space), $\mathcal{F}$ is its Borel σ-field and $\mathbb{P}$ is the Wiener measure. By convention, for a path $\omega\in C_0^0([0,T],\mathbb{R}^d)$, we will set $\omega^0(t)=t$.

Now define the coordinate mapping process $B_t^i(\omega)=\omega^i(t)$ for $t\in[0,T]$, $\omega\in\Omega$. Then, under the Wiener measure, $B=(B_t^1,\dots,B_t^d)_{t\in[0,T]}$ is a d-dimensional Brownian motion starting at zero. Moreover, $B^0(t)=t$.

Let $\xi_{t,x}$, $t\in[0,T]$, $x\in\mathbb{R}^N$, be the solution of the Stratonovich stochastic differential equation (SDE)

$$\mathrm{d}\xi_{t,x}=\sum_{i=0}^{d}V_i(\xi_{t,x})\circ\mathrm{d}B_t^i,\quad \xi_{0,x=x}. \tag{1.2}$$

We take a version of $\xi_{t,x}$ which coincides on continuous, bounded variation paths with the pathwise solution. The function $u:(t,x)\to\mathbb{E}(f(\xi_{T-t,x}))$ is solution of (1.1) (Ikeda & Watanabe 1981). Hence, to approximate the solution of a parabolic PDE, we can approximate weakly the solution of a SDE. Many approximations of the solution of parabolic PDEs exploit this fact (Kloeden & Platen 1992; Kusuoka 1998).

Let us write $\Phi_{T,x}$ for the almost surely defined mapping

$$C_0^0([0,T],\mathbb{R}^d)\to\mathbb{R}^N,\qquad \omega\mapsto\xi_{T,x}(\omega).$$

Then

$$u(0,x)=\int_\Omega f(\Phi_{T,x}(\omega))\mathbb{P}(\mathrm{d}\omega).$$

Hence, to approximate the solution of a parabolic PDE, we actually have to approximate an integral over the Wiener space. As we have to approximate an integral over an infinite-dimensional space, it is probably wise to come back for a moment to techniques which approximate an integral over a finite-dimensional space. We will focus on one of these techniques, called cubature (Stroud 1971). We will denote by $\mathbb{R}_m[X_1,\dots,X_d]$ the space of polynomials in d variables and of degree less than m.

Definition 1.1. Let μ be a positive measure on $\mathbb{R}^d$, and m be a natural number. We will say that the points $x_1,\dots,x_n$ in the support of μ, and the positive weights $\lambda_1,\dots,\lambda_n$ define a cubature formula of degree m with respect to μ if and only if, for all polynomials $P\in\mathbb{R}_m[X_1,\dots,X_d]$,

$$\int_{\mathbb{R}^d}P(x)\mu(\mathrm{d}x)=\sum_{i=1}^{n}\lambda_iP(x_i). \tag{1.3}$$

When $d=1$, one talks about quadrature formulae rather than cubature formulae. Tchakaloff proved that such a formula exists (Putinar 1997; Stroud 1971). Moreover, one should be able to find such a formula with its number of points being less than or equal to the dimension of the space of polynomials of degree less than or equal to m.

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Theorem 1.2. *Let m be positive integers and let μ be a positive measure on $\mathbb{R}^d$ with the property that $\int |P(x)|\mu(\mathrm{d}x) < \infty$ for all $P \in \mathbb{R}_m[X_1, \dots, X_d]$. One can then find n points $x_1, \dots, x_n \in \operatorname{supp}(\mu)$ and n positive real numbers $\lambda_1, \dots, \lambda_n$, with*

$$n \leqslant \dim \mathbb{R}_m[X_1, \dots, X_d],$$

such that the cubature relation (1.3) holds for all $P \in \mathbb{R}_m[X_1, \dots, X_d]$.

One would then use the expression

$$\sum_{i=1}^{n} \lambda_i f(x_i)$$

to approximate the integral

$$\int_{\mathbb{R}^d} f(x)\mu(\mathrm{d}x).$$

In other words, to approximate the integral of a function f with respect to μ, one integrates f with respect to the positive measure with finite support $\mathbb{Q} = \sum_{i=1}^{n} \lambda_i \delta_{x_i}$. The accuracy of the estimate depends on whether or not f can be approximated well by polynomials of degree m. That shows, using Taylor's formula, that cubature is an efficient approach to numerical integration of smooth functions.

In § 2, we will recall the stochastic Taylor formula (Kloeden & Platen 1992), which generalizes Taylor's formula. The polynomials are in this context replaced by the Stratonovich-iterated integrals. We will say that the paths $\omega_1, \dots, \omega_n$ and the weights $\lambda_1, \dots, \lambda_n$ define a cubature formula on Wiener space of degree m if the expectations of the Stratonovich-iterated integrals of degree less than m are, under the Wiener measure and under the probability measure

$$\mathbb{Q} = \sum_{i=1}^{n} \lambda_i \delta_{\omega_i},$$

the same. In § 3, we will construct from this probability measure, by concatenation of paths, a new probability measure with finite support on the Wiener space $\tilde{\mathbb{Q}}$, such that the expectations of $f(\xi_{T-t,x})$ under $\tilde{\mathbb{Q}}$ and under the Wiener measure are close. That means that, to approximate the solution of the PDE (1.1), we solve many ordinary differential equations (ODEs), and take a weighted average of these solutions. Section 4 will be devoted to the algebra which will allow us to understand the algebraic relations between iterated integrals. We will then be able to give concrete examples of cubature formulae on Wiener space of degree 3 and degree 5.

2. Stochastic Taylor formula and definition of cubature on the Wiener space

(*a*) *Stochastic Taylor expansion*

We recall that the stochastic process ξ is defined by its integral expression: for all smooth functions f,

$$f(\xi_{t,x}) = f(x) + \int_0^t \sum_{i=0}^{d} V_i f(\xi_{s,x}) \circ \mathrm{d}B^i(s).$$

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Let $\mathcal{A}_m = \{(i_1, \dots, i_k) \in \{0, \dots, d\}^k, k + \mathrm{card}\{j, i_j = 0\} \leqslant m\}$. The definition of $\mathcal{A}_m$ comes from the fact that

$$\int_{0<t_1<\cdots<t_k<t} \circ \, \mathrm{d}B_{t_1}^{i_1} \cdots \circ \, \mathrm{d}B_{t_k}^{i_k}$$

is equal, in law, to

$$\sqrt{t}^{k+\mathrm{card}\{j,i_j=0\}} \int_{0<t_1<\cdots<t_k<1} \circ \, \mathrm{d}B_{t_1}^{i_1} \cdots \circ \, \mathrm{d}B_{t_k}^{i_k}. \tag{2.1}$$

The following is a weak form of the stochastic Taylor formula (see Kloeden & Platen (1992) for a more detailed proof).

Proposition 2.1. *Let f be a bounded smooth function, and let m be a natural number. Then*

$$f(\xi_{t,x}) = \sum_{(i_1,\dots,i_k)\in\mathcal{A}_m} V_{i_1} \cdots V_{i_k} f(x) \int_{0<t_1<\cdots<t_k<t} \circ \, \mathrm{d}B_{t_1}^{i_1} \cdots \circ \, \mathrm{d}B_{t_k}^{i_k} + R_m(t, x, f), \tag{2.2}$$

where the remainder process R_m satisfies, for some constant C depending only on d and m,

$$\sup_{x\in\mathbb{R}^N} \sqrt{E(R_m(t,x,f)^2)} \leqslant C t^{(m+1)/2} \sup_{(i_1,\dots,i_k)\in\mathcal{A}_{m+2}\setminus\mathcal{A}_m} \|V_{i_1} \cdots V_{i_k} f\|_\infty.$$

Proof. By induction, one can show that

$$R_m(t,x,f) = \sum_{\substack{(i_1,\dots,i_k)\in\mathcal{A}_m \\ (i_0,i_1,\dots,i_k)\notin\mathcal{A}_m}} \int_{0<t_0<\cdots<t_k<t} V_{i_0} \cdots V_{i_k} f(\xi_{t_0,x}) \circ \mathrm{d}B_{t_0}^{i_0} \cdots \circ \mathrm{d}B_{t_k}^{i_k}.$$

We obtain the predicted upper bound once we notice that, by Itô's formula, if $i_0 \neq 0$,

$$\begin{aligned}\int_0^t V_{i_0} \cdots V_{i_k} f(\xi_{t_0,x}) \circ \mathrm{d}B_{t_0}^{i_0} \\ = \int_0^t V_{i_0} \cdots V_{i_k} f(\xi_{t_0,x}) \, \mathrm{d}B_{t_0}^{i_0} + \frac{1}{2} \int_0^t V_{i_0}^2 \cdots V_{i_k} f(\xi_{t_0,x}) \, \mathrm{d}t_0.\end{aligned}$$

∎

So we see from the stochastic Taylor formula that $f(\xi_{t,x})$ is approximated by the sum of Stratonovich-iterated integrals

$$\sum_{(i_1,\dots,i_k)\in\mathcal{A}_m} V_{i_1} \cdots V_{i_k} f(x) \int_{0<t_1<\cdots<t_k<t} \circ \, \mathrm{d}B_{t_1}^{i_1} \cdots \circ \, \mathrm{d}B_{t_k}^{i_k}$$

in the same way that a smooth function g is approximated at the point x by

$$\sum_{k\leqslant m} g^{(m)}(x_0) \frac{(x-x_0)^m}{m!}.$$

It now seems natural to give a definition of a cubature formula on Wiener space, replacing the polynomials by the iterated integrals, and fixing the positive measure to the Wiener measure.

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(*b*) *Cubature formula on Wiener space*

We will denote by $C^0_{0,\mathrm{bv}}([0,T],\mathbb{R}^d)$ the subset of $C^0_0([0,T],\mathbb{R}^d)$ made of bounded variation paths.

Definition 2.2. Let m be a natural number. We will say that the paths

$$\omega_1,\dots,\omega_n \in C^0_{0,\mathrm{bv}}([0,T],\mathbb{R}^d)$$

and the positive weights $\lambda_1,\dots,\lambda_n$ define a cubature formula on Wiener space of degree m at time T, if and only if, for all $(i_1,\dots,i_k)\in\mathcal{A}_m$,

$$E\biggl(\int_{0<t_1<\cdots<t_k<T}\circ\,\mathrm{d}B^{i_1}_{t_1}\cdots\circ\,\mathrm{d}B^{i_k}_{t_k}\biggr)=\sum_{j=1}^{n}\lambda_j\int_{0<t_1<\cdots<t_k<T}\mathrm{d}\omega^{i_1}_j(t_1)\cdots\mathrm{d}\omega^{i_k}_j(t_k).$$

Remark 2.3. That also means that the expectations under the Wiener measure and under the probability measure $\mathbb{Q}=\sum_{j=1}^n\lambda_j\delta_{\omega_j}$ of the Stratonovich-iterated integrals of degree less than m are the same.

We now extend Tchakaloff's theorem to our case. We will prove it in the appendix.

Theorem 2.4. *Let m be a natural number. Then one can find n paths $\omega_1,\dots,\omega_n\in C^0_{0,\mathrm{bv}}([0,T],\mathbb{R}^d)$ and n positive weights $\lambda_1,\dots,\lambda_n$, with $n\leqslant\operatorname{card}\mathcal{A}_m$, such that these paths and weights define a cubature formula on Wiener space of degree m at time T.*

Using the scaling properties of the Brownian motion, we obtain the following simple proposition.

Proposition 2.5. *Assume the paths $\omega_1,\dots,\omega_n\in C^0_{0,\mathrm{bv}}([0,1],\mathbb{R}^d)$ and the weights $\lambda_1,\dots,\lambda_n\in\mathbb{R}_+$ define a cubature formula on Wiener space of degree m at time 1. Define, for $i=1,\dots,n$ the paths $\omega_{T,i}\in C^0_{0,\mathrm{bv}}([0,1],\mathbb{R}^d)$ by $\omega^j_{T,i}=\sqrt{T}\omega^j_i(t/T)$ for $j=1,\dots,d$. The paths $\omega_{T,i}$ and the weights λ_i, $i=1,\dots,n$, then define a cubature formula on Wiener space of degree m at time T.*

In view of this proposition, it is enough to construct cubature formulae on Wiener space at time 1. From now on, unless otherwise stated, all cubature formulae on Wiener space will be at time 1.

3. The algorithm

Assume that the paths $\omega_1,\dots,\omega_n$ in $C^0_{0,\mathrm{bv}}([0,T],\mathbb{R}^d)$ and the weights $\lambda_1,\dots,\lambda_n$ define a cubature formula on Wiener space of degree m. Define, for $i=1,\dots,n$, the paths $\omega_{T,i}:[0,T]\to\mathbb{R}^{d+1}$ by $\omega^0_{T,i}(t)=t$ and $\omega^j_{T,i}=\sqrt{T}\omega^j_i(t/T)$ for $j=1,\dots,d$. We then have a new probability measure with finite support on the Wiener space Ω,

$$\mathbb{Q}_T=\sum_{i=1}^{n}\lambda_i\delta_{\omega_{T,i}}.$$

By construction of the ω_i and by the scaling property, for all $(i_1,\dots,i_k)\in\mathcal{A}_m$,

$$E\biggl(\int_{0<t_1<\cdots<t_k<T}\circ\,\mathrm{d}B^{i_1}_{t_1}\cdots\circ\,\mathrm{d}B^{i_k}_{t_k}\biggr)=E_{\mathbb{Q}_T}\biggl(\int_{0<t_1<\cdots<t_k<T}\circ\,\mathrm{d}B^{i_1}_{t_1}\cdots\circ\,\mathrm{d}B^{i_k}_{t_k}\biggr).$$

Let us first adapt the stochastic Taylor formula to the probability measure $\mathbb{Q}_T$.

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Lemma 3.1. *$R_m(T,x,f)$, the process defined in (2.2), satisfies*

$$\sup_x \mathbb{E}_{\mathbb{Q}_T}[|R_m(T,x,f)|] \leqslant C_{d,m,\mathbb{Q}_1} T^{(m+1)/2} \sup_{(i_1,\ldots,i_k)\in\mathcal{A}_{m+2}\setminus\mathcal{A}_m} \|V_{i_1}\cdots V_{i_k} f\|_\infty. \quad (3.1)$$

Proof. We saw that

$$R_m(T,x,f) = \sum_{\substack{(i_1,\ldots,i_k)\in\mathcal{A}_m \\ (i_0,i_1,\ldots,i_k)\notin\mathcal{A}_m}} \int_{0<t_0<\cdots<t_k<T} V_{i_0}\cdots V_{i_k} f(\xi_{t_0,x}) \circ \mathrm{d}B_{t_0}^{i_0}\cdots\circ \mathrm{d}B_{t_k}^{i_k}.$$

Hence, $E_{\mathbb{Q}_T}(|R_m(T,x,f)|)$ is bounded by

$$\sum_{j=1}^n \lambda_j \sum_{\substack{(i_1,\ldots,i_k)\in\mathcal{A}_m \\ (i_0,i_1,\ldots,i_k)\notin\mathcal{A}_m}} \left|\int V_{i_0}\cdots V_{i_k} f(\xi_{t_0,x}(\omega_{T,i}))\,\mathrm{d}\omega_{T,i_0}(t_0)\cdots \mathrm{d}\omega_{T,i_k}(t_k)\right|.$$

Changing variables in the integrals to get an expression with the original paths $\omega_1,\ldots,\omega_n$, we obtain the predicted upper bound. ■

(a) *Approximation of $\mathbb{E}(f(\xi_{T,x}))$ when f is smooth*

Recall that $\Phi_{T,x}(\omega)$ is, for $\omega\in C^0_{0,\mathrm{bv}}([0,T],\mathbb{R}^d)$, the solution at time T of the ODE

$$\mathrm{d}y_{t,x} = \sum_{i=0}^d V_i(y_{t,x})\,\mathrm{d}\omega^i(t), \quad y_{0,x}=x.$$

Proposition 3.2.

$$\sup_{x\in\mathbb{R}^N}\left|\mathbb{E}\left(f(\xi_{T,x}) - \sum_{i=1}^n \lambda_i f(\Phi_{T,x}(\omega_{T,i}))\right)\right| \leqslant C\sqrt{T}^{m+1} \sup_{(i_1,\ldots,i_k)\in\mathcal{A}_{m+2}\setminus\mathcal{A}_m} \|V_{i_1}\cdots V_{i_k} f\|_\infty, \quad (3.2)$$

where C is a constant independent of T.

Proof.

$$\sum_{i=1}^n \lambda_i f(\Phi_{T,x}(\omega_{T,i})) = \mathbb{E}_{\mathbb{Q}_T}(f(\xi_{T,x}))$$

and

$$\begin{aligned} |(\mathbb{E}-\mathbb{E}_{\mathbb{Q}_T})(f(\xi_{T,x}))| &\leqslant \mathbb{E}(|R_m(t,x,f)|) + \mathbb{E}_{\mathbb{Q}_T}(|R_m(t,x,f)|) \\ &\quad + \left|(\mathbb{E}-\mathbb{E}_{\mathbb{Q}_T})\left(\sum_{(i_1,\ldots,i_k)\in\mathcal{A}_m} V_{i_1}\cdots V_{i_k} f(x)\int_{0<t_1<\cdots<t_k<t}\circ\,\mathrm{d}B_{t_1}^{i_1}\cdots\circ\mathrm{d}B_{t_k}^{i_k}\right)\right|. \end{aligned}$$

The last term of the right-hand side is zero (by definition of $\mathbb{Q}_T$). The proof is then finished using the equations (2.2) and (3.1). ■

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The upper bound in proposition (3.2) is usually not small, unless T or the noise (or volatility in financial terms) is small. To obtain an algorithm with a small error, we divide the interval $[0,T]$ into smaller subintervals: consider some reals

$$0 = t_0 < t_1 < t_2 < \cdots < t_k = T,$$

and let $s_l = t_l - t_{l-1}$. Define the Markov random variable $(Y_i)_{0\leqslant i\leqslant k}$ by

$$\mathbb{P}(Y_{l+1} = \Phi_{s_l,x}(\omega_{s_l,i}) \mid Y_l = x) = \lambda_i.$$

Theorem 3.3.

$$\sup_{x\in\mathbb{R}^N} |\mathbb{E}(f(Y_k)/Y_0 = x) - \mathbb{E}(f(\xi_{T,x}))| \leqslant C\sum_{j=1}^{k} s_j^{(m+1)/2} \sup_{(i_1,\dots,i_k)\in\mathcal{A}_{m+2}\backslash\mathcal{A}_m} \|V_{i_1}\cdots V_{i_k}P_{T-t_j}f\|_\infty, \qquad (3.3)$$

where $P_t f(x) = \mathbb{E}(f(\xi_{t,x}))$.

Proof. First of all, let us remark that

$$P_T f(x) - \mathbb{E}(f(Y_k)/Y_0 = x) = \sum_{j=0}^{k-1} \mathbb{E}\biggl(P_{T-t_j}f(Y_j) - P_{T-t_{j+1}}\frac{f(Y_{j+1})}{Y_0} = x\biggr).$$

Then, by (3.2),

$$\begin{aligned}
\sup_x |\mathbb{E}(P_{T-t_j}f(Y_j) &- P_{T-t_{j+1}}f(Y_{j+1})/Y_0 = x)| \\
&\leqslant \sup_x |\mathbb{E}(P_{s_{j+1}}P_{T-t_{j+1}}f(Y_j) - P_{T-t_{j+1}}f(Y_{j+1})/Y_0 = x)| \\
&\leqslant \sup_x \mathbb{E}(|\mathbb{E}(P_{s_{j+1}}P_{T-t_{j+1}}f(Y_j) - P_{T-t_{j+1}}f(Y_{j+1})/Y_j)|/Y_0 = x) \\
&\leqslant \sup_x |\mathbb{E}(P_{s_{j+1}}P_{T-t_{j+1}}f(Y_j) - P_{T-t_{j+1}}f(Y_{j+1})/Y_j = x)| \\
&\leqslant Cs_{j+1}^{(m+1)/2} \sup_{(i_1,\dots,i_k)\in\mathcal{A}_{m+2}\backslash\mathcal{A}_m} \|V_{i_1}\cdots V_{i_k}P_{T-t_{j+1}}f\|_\infty.
\end{aligned}$$

■

We could have stated the previous proposition slightly differently: let

$$\mathbb{Q}_T^k = \sum_{i_1=1}^{n}\cdots\sum_{i_k=1}^{n} \lambda_{i_1}\cdots\lambda_{i_k}\delta_{\omega_{s_1,i_1}\otimes\cdots\otimes\omega_{s_k,i_k}}. \qquad (3.4)$$

Then

$$\begin{aligned}
\mathbb{E}(f(Y_k)/Y_0 = x) &= \sum_{i_1=1}^{n}\cdots\sum_{i_k=1}^{n} \lambda_{i_1}\cdots\lambda_{i_k} f(\Phi_{T,x}(\omega_{s_1,i_1}\otimes\cdots\otimes\omega_{s_k,i_k})) \\
&= \mathbb{E}_{\mathbb{Q}_T^k}(f(\xi_{T,x})).
\end{aligned}$$

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So to approximate the expectation of the solution of a SDE numerically, we solve many carefully chosen ODEs and take a weighted average of these solutions. One can see the algorithm as a n-ary tree. At each node, we save a weight and an element of $\mathbb{R}^N$. $(1, x)$ is saved in the root node. (υ, y), saved in a node of level $j \in \{0, \ldots, k-1\}$, leads to n new nodes $(\upsilon\lambda_i, \Phi_{s_{j+1},y}(\omega_{s_{j+1},i}))$. In practice, one can compute $\Phi_{s_{j+1},y}(\omega_{s_{j+1},i})$ with an adaptive step-size Runge–Kutta method for example (Press *et al.* 1992). Once we have filled the tree, the sum of $\upsilon f(y)$ for (υ, y) spanning over all nodes of level k is our approximation of the solution of the PDE (1.1). When $(n^{k+1}-1)/(n-1)$ (the number of ODEs to solve) is too big, one is forced to do some Monte-Carlo simulation on the probability measure $\mathbb{Q}_T^k$ to simulate $\mathbb{E}_{\mathbb{Q}_T^k}(f(\xi_{T,x}))$.

Example 3.4. Assume that $n = 2$ (cubature formula with two paths) and $k = 2$ ($[0, T] = [0, s_1] \cup [s_1, s_1 + s_2]$). Let

$$y_{i,j} = \Phi_{s_2,\Phi_{s_1,x}(\omega_{s_1,i})}(\omega_{s_2,j}).$$

Then

$$\mathbb{E}_{\mathbb{Q}_T^2}(f(\xi_{T,x})) = \lambda_1\lambda_1 y_{1,1} + \lambda_1\lambda_2 y_{1,2} + \lambda_2\lambda_1 y_{2,1} + \lambda_2\lambda_2 y_{2,2},$$

$$\begin{tikzcd}[column sep=tiny]
 & & & (1,x) \arrow[dll, dash] \arrow[drr, dash] & & & \\
 & (\lambda_1, \Phi_{s_1,x}(\omega_{s_1,1})) \arrow[dl, dash] \arrow[dr, dash] & & & & (\lambda_2, \Phi_{s_1,x}(\omega_{s_1,2})) \arrow[dl, dash] \arrow[dr, dash] & \\
(\lambda_1\lambda_1, y_{1,1}) & & (\lambda_1\lambda_2, y_{1,2}) & & (\lambda_1\lambda_2, y_{2,1}) & & (\lambda_2\lambda_2, y_{2,2})
\end{tikzcd}$$

(b) *Approximation of* $\mathbb{E}(f(\xi_{T,x}))$ *when* f *is Lipschitz*

In the upper bound (3.3), since $V_{i_1} \cdots V_{i_k}$ is a differential operator of order k, we need $P_{T-t_j} f$ to be smooth. Using the Malliavin calculus in Kusuoka & Stroock (1987), we understand, loosely speaking, that even if f is not smooth, $P_t f$ is.

Condition 3.5 (UFG). The $C_b^\infty(\mathbb{R}^N, \mathbb{R})$-module

$$\sum_{(i_1,\ldots,i_k) \in \mathcal{A}_m \setminus \{\emptyset,(0)\}} C_b^\infty(\mathbb{R}^N, \mathbb{R})[V_{i_1}, [V_{i_2}[\cdots, V_{i_k}]\cdots]]$$

is finitely generated as a $C_b^\infty(\mathbb{R}^N, \mathbb{R})$-module.

Under this assumption, which is weaker than the Hörmander condition, following Kusuoka (1998), we obtain the following estimate, which shows that the algorithm will converge even when f is just Lipschitz continuous and, as long as (UFG) is satisfied, when $L = V_0 + \frac{1}{2}(V_1^2 + \cdots + V_d^2)$ is not elliptic.

Proposition 3.6.

$$\sup_{x \in \mathbb{R}^N} |\mathbb{E}(f(Y_k)/Y_0 = x) - \mathbb{E}(f(\xi_{T,x}))| \leqslant K \|\nabla f\|_\infty \left(s_k^{1/2} + \sum_{i=1}^{k-1} \frac{s_i^{(m+1)/2}}{(T - t_i)^{m/2}} \right).$$

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Proof. We have already seen that the error of the algorithm is bounded by

$$C\sum_{j=1}^{k-1} s_j^{(m+1)/2} \sup_{(i_1,\dots,i_k)\in\mathcal{A}_{m+2}\setminus\mathcal{A}_m} \|V_{i_1}\cdots V_{i_k}P_{T-t_j}f\|_\infty + \left\|\mathbb{E}\left(\frac{f(Y_k)}{Y_{k-1}}=x\right)-\mathbb{E}(f(\xi_{s_k,x}))\right\|_\infty.$$

The inequality

$$\begin{aligned}\|\mathbb{E}(f(Y_k)/Y_{k-1}=x)-\mathbb{E}(f(\xi_{s_k,x}))\|_\infty &\leqslant \left\|\mathbb{E}\left(\frac{f(Y_k)}{Y_{k-1}}=x\right)-f(x)\right\|_\infty+\left\|\mathbb{E}\left(\frac{f(Y_k)}{Y_{k-1}}=x\right)-f(x)\right\|_\infty \\ &\leqslant K\sqrt{s_k}\|\nabla f\|_\infty\end{aligned}$$

explains the term $s_k^{1/2}$ in the upper bound of the error of the algorithm. As, under the Hörmander condition (Kusuoka & Stroock 1987; Malliavin 1997),

$$\|V_{i_1}\cdots V_{i_k}P_s f\|_\infty \leqslant \frac{Ks^{1/2}}{s^{(k+\mathrm{card}\{j,i_j=0\})/2}}\|\nabla f\|_\infty,$$

we have

$$\sup_{(i_1,\dots,i_k)\in\mathcal{A}_{m+2}\setminus\mathcal{A}_m}\|V_{i_1}\cdots V_{i_k}P_{T-t_j}f\|_\infty \leqslant \frac{K}{s^{m/2}}\|\nabla f\|_\infty.$$

Hence, under the Hörmander condition,

$$\sup_{x\in\mathbb{R}^N}|\mathbb{E}(f(Y_k)/Y_0=x)-\mathbb{E}(f(\xi_{T,x}))| \leqslant K\|\nabla f\|_\infty\left(s_k^{1/2}+\sum_{i=1}^{k-1}\frac{s_i^{(m+1)/2}}{(T-t_i)^{m/2}}\right).$$

The result can be extended to the not as restrictive condition (UFG) using similar arguments to those in Kusuoka (1998). ■

Example 3.7. We define the reals $0=t_0<t_1<t_2<\cdots<t_k=T$ by

$$t_j = T\left(1-\left(1-\frac{j}{k}\right)^\gamma\right),$$

where γ is a given parameter. Then for $0<\gamma<m-1$,

$$\sup_{x\in\mathbb{R}^N}|\mathbb{E}(f(Y_k)/Y_0=x)-\mathbb{E}(f(\xi_{T,x}))| \leqslant Kn^{-\gamma/2}\|\nabla f\|_\infty.$$

For $\gamma=m-1$,

$$\sup_{x\in\mathbb{R}^N}|\mathbb{E}(f(Y_k)/Y_0=x)-\mathbb{E}(f(\xi_{T,x}))| \leqslant Kn^{-(m-1)/2}\log(n)\|\nabla f\|_\infty.$$

For $\gamma>m-1$,

$$\sup_{x\in\mathbb{R}^N}|\mathbb{E}(f(Y_k)/Y_0=x)-\mathbb{E}(f(\xi_{T,x}))| \leqslant Kn^{-(m-1)/2}\|\nabla f\|_\infty.$$

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4. Free Lie algebras and iterated integrals

To construct cubature formulae on Wiener space, we will need to understand the algebraic relation between the iterated integrals. This was first understood by Chen (1957) and is at the base of rough-paths theory (Lyons 1998).

(a) Tensor algebra

If W is a vector space, then one can define a tensor algebra over W:

$$T(W) = \bigoplus_{k=0}^{\infty} W^{\otimes k}.$$

As we will only be interested in the projection to $\bigoplus_{k=0}^{m} W^{\otimes k}$ of elements of $T(W)$, we will not make the distinction between the tensor algebra of polynomials and the tensor algebra of series (Reutenauer 1993). $T(W)$ is an associative algebra with a unit. This space happens to be extremely useful in the studies of the iterated integrals of a smooth path with values in W, as its kth iterated integral is an element of $W^{\otimes k}$ (see Lyons 1998). Our situation is a bit more complex because the zeroth coordinate of our path (which is just $t \to t$) plays a different role from the other coordinates, as we saw in § 3.

Recall that, because of the convention $\omega^0(t) = t$, a path $\omega \in C_0^0([0,T], \mathbb{R}^d)$ is $\mathbb{R} \oplus \mathbb{R}^d$-valued.

For some given vector spaces $W_1, \dots, W_k$, we define the symmetrized product

$$(W_1, \dots, W_k) = \bigoplus_{\sigma \in S_k} W_{\sigma(1)} \otimes \cdots \otimes W_{\sigma(k)},$$

where S_k denotes the group of permutation of order k. Then we define

$$(W_1, W_2)^{p,q} = (W_1, \dots, W_1, W_2, \dots, W_2),$$

where one has p times the term W_1 and q times the term W_2 in the bracketed term on the right-hand side.

Now define

$$U_k(\mathbb{R}, \mathbb{R}^d) = \bigoplus_{\substack{(i,j)\in\mathbb{N} \\ 2i+j=k}} (\mathbb{R}, \mathbb{R}^d)^{i,j}.$$

For example, $U_0(\mathbb{R}, \mathbb{R}^d) = \mathbb{R}$, $U_1(\mathbb{R}, \mathbb{R}^d) = \mathbb{R}^d$, $U_2(\mathbb{R}, \mathbb{R}^d) = \mathbb{R} \oplus (\mathbb{R}^d \otimes \mathbb{R}^d)$. Finally, let

$$T(\mathbb{R}, \mathbb{R}^d) = \bigoplus_{k=0}^{\infty} U_k(\mathbb{R}, \mathbb{R}^d).$$

Let $\lambda \in \mathbb{R}$, and $a = (a_0, a_1, a_2, \dots)$ and $b = (b_0, b_1, b_2, \dots)$ be some elements of $T(\mathbb{R}, \mathbb{R}^d)$, where a_i, b_i are the components of a, b in $U_i(\mathbb{R}, \mathbb{R}^d)$. We then define the sum, tensor product and the action of scalars in the following way:

$$a + b = (a_0 + b_0, a_1 + b_1, a_2 + b_2, \dots),$$

$$(a \otimes b)_i = \sum_{j=0}^{i} a_j \otimes b_{i-j},$$

$$\lambda a = (\lambda a_0, \lambda a_1, \lambda a_2, \dots).$$

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$(T(\mathbb{R},\mathbb{R}^d),\otimes,\oplus,\cdot)$ is an associative algebra. Note that, by construction, if $a\in U_k(\mathbb{R},\mathbb{R}^d)$ and $b\in U_l(\mathbb{R},\mathbb{R}^d)$, then $a\otimes b\in U_{k+l}(\mathbb{R},\mathbb{R}^d)$. For $i=0,\dots,d$, by $\varepsilon_i=(0,\dots,0,1,0,\dots,0)$, where the 1 is in the ith position. Then $\varepsilon_0,\varepsilon_1,\dots,\varepsilon_d$ is a basis of $\mathbb{R}\oplus\mathbb{R}^d$. That implies that

$$\{\varepsilon_{i_1}\otimes\cdots\otimes\varepsilon_{i_k},(i_1,\dots,i_k)\in\mathcal{A}_n\setminus\mathcal{A}_{n-1}\} \tag{4.1}$$

is a basis of $U_n(\mathbb{R},\mathbb{R}^d)$.

One can define the exponential on $T(\mathbb{R},\mathbb{R}^d)$ using its power series

$$\exp(a)=\sum_{k\geqslant 0}\frac{a^{\otimes k}}{k!}.$$

Also, if $a_0\neq 0$, there exists $c=(0,c_1,\dots)\in T(\mathbb{R},\mathbb{R}^d)$ such that $a=a_0(1+c)$. We then define the inverse and the logarithm as

$$a^{-1}=\frac{\sum_{k\geqslant 0}(-1)^k c^{\otimes k}}{a_0},$$
$$\log(a)=\log(a_0)+\sum_{k\geqslant 1}(-1)^{k-1}\frac{c^{\otimes k}}{k}.$$

Quotienting $T(\mathbb{R},\mathbb{R}^d)$ by the ideal

$$\bigoplus_{k=n+1}^{\infty}U_k(\mathbb{R},\mathbb{R}^d),$$

we obtain the truncated tensor algebra of degree n, which we will denote by $T^{(n)}(\mathbb{R},\mathbb{R}^d)$. We identify $T^{(n)}(\mathbb{R},\mathbb{R}^d)$ with

$$\bigoplus_{k=0}^{n}U_k(\mathbb{R},\mathbb{R}^d).$$

We will denote by π_n the natural projection of $T(\mathbb{R},\mathbb{R}^d)$ onto $T^{(n)}(\mathbb{R},\mathbb{R}^d)$. It is an algebra homomorphism which commutes with the inverse, the logarithm and the exponential.

$[a,b]=a\otimes b-b\otimes a$ defines a Lie bracket on $T(\mathbb{R},\mathbb{R}^d)$ and $T^{(n)}(\mathbb{R},\mathbb{R}^d)$. We will denote by $\mathcal{U}$ the space of linear combinations of finite sequences of Lie brackets of elements of $W=\mathbb{R}\oplus\mathbb{R}^d$,

$$W\oplus[W,W]\oplus[W,[W,W]]\oplus\cdots.$$

$\mathcal{U}$ is the free Lie algebra generated by $\mathbb{R}\oplus\mathbb{R}^d$ (Reutenauer 1993). We will denote $\pi_n(\mathcal{U})$ by $\mathcal{U}^{(n)}$. An element of $\mathcal{U}^{(n)}$ will be called a Lie polynomial of degree n, while an infinite sequence of Lie brackets will be called a Lie series. Finally, we introduce the free nilpotent group of degree n, $(G_n(\mathbb{R},\mathbb{R}^d)=\exp\mathcal{U}^{(n)}\subset T^{(n)}(\mathbb{R},\mathbb{R}^d),\otimes)$.

(*b*) *Lie basis and the Poincaré–Birkhoff–Witt theorem*

We will denote by $\mathcal{B}_{\mathcal{U}}$ a basis of the Lie algebra $\mathcal{U}$, i.e. a set of Lie polynomials such that $\mathcal{L}\in\mathcal{U}$ if and only if there exists a unique family of reals $(c_\ell)_{\ell\in\mathcal{B}_{\mathcal{U}}}$, with the

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c_ℓ being zero all but finitely many ℓ, such that

$$\mathcal{L} = \sum_{\ell \in \mathcal{B}_\mathcal{U}} c_\ell \ell.$$

We will also assume that our basis is provided with a total order $\preccurlyeq$, and that, for all $\ell \in \mathcal{B}_\mathcal{U}$, there exists a natural number k such that $\ell \in U_k(\mathbb{R}, \mathbb{R}^d)$. Such a basis can be described, for example, by the Lyndon words basis (Reutenauer 1993).

Theorem 4.1 (Poincaré–Birkhoff–Witt). *The set*

$$\bigcup_{n \geqslant 0} \{\ell_1 \otimes \cdots \otimes \ell_n, \ell_1, \ldots, \ell_n \in \mathcal{B}_\mathcal{U}, \ell_1 \preccurlyeq \cdots \preccurlyeq \ell_n\} \tag{4.2}$$

forms a basis of $T(\mathbb{R}, \mathbb{R}^d)$.

If $P_1, \ldots, P_n$ are Lie polynomials, we define their symmetrized product by

$$(P_1, \ldots, P_n) = \frac{1}{n!} \sum_{\sigma \in S_n} P_{\sigma(1)} \otimes \cdots \otimes P_{\sigma(n)}.$$

From the previous theorem, we immediately obtain the following corollary.

Corollary 4.2. *The set*

$$\bigcup_{n \geqslant 0} \{(\ell_1, \ldots, \ell_n), \ell_1, \ldots, \ell_n \in \mathcal{B}_\mathcal{U}, \ell_1 \preccurlyeq \cdots \preccurlyeq \ell_n\} \tag{4.3}$$

forms a basis of $T(\mathbb{R}, \mathbb{R}^d)$.

This symmetrized product is practical for developing an exponential.

Proposition 4.3.

$$\exp\left(\sum_{i=1}^n \beta_i \ell_i\right) = \sum_{k=0}^\infty \sum_{i_1 + \cdots + i_n = k} \frac{\beta_1^{i_1} \cdots \beta_n^{i_n}}{i_1! \cdots i_n!} (\ell_1, \ldots, \ell_1, \ldots, \ell_n, \ldots, \ell_n), \tag{4.4}$$

where the term ℓ_j in the symmetrized product appears i_j times.

(*c*) *Chen's theorem*

Let us consider a path $\omega \in C^0_{0,\mathrm{bv}}([0, T], \mathbb{R}^d)$, and define its series of iterated integrals (an element of $T(\mathbb{R}, \mathbb{R}^d)$) by

$$\boldsymbol{X}_{s,t}(\omega) = \sum_{k=0}^\infty \int_{s<t_1<\cdots<t_k<t} \mathrm{d}\omega(t_1) \otimes \cdots \otimes \mathrm{d}\omega(t_k).$$

$\boldsymbol{X}_{s,t}(\omega)$ will be referred to as the Chen series of the path ω. Its truncated series of degree n of iterated integrals $\pi_n(\boldsymbol{X}_{s,t}(\omega))$ will be denoted by $\boldsymbol{X}^{(n)}_{s,t}(\omega)$. Using the basis $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_d$ introduced in § 4 *b*, we see that

$$\boldsymbol{X}_{s,t}(\omega) = \sum_{k=0}^\infty \sum_{(i_1,\ldots,i_k) \in \mathcal{A}_k \setminus \mathcal{A}_{k-1}} \int_{s<t_1<\cdots<t_k<t} \mathrm{d}\omega^{i_1}(t_1) \cdots \mathrm{d}\omega^{i_k}(t_k) \varepsilon_{i_1} \otimes \cdots \otimes \varepsilon_{i_k}. \tag{4.5}$$

We can now state Chen's theorem. We will sketch its proof, but we refer the reader to Chen (1957), Lyons (1998) or Reutenauer (1993) for a more detailed one.

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Theorem 4.4. *The process $\boldsymbol{X}_{s,t}(\omega)$ is multiplicative, i.e.*

$$\boldsymbol{X}_{r,s}(\omega) \otimes \boldsymbol{X}_{s,t}(\omega) = \boldsymbol{X}_{r,t}(\omega).$$

Moreover, $\log(X_{s,t}(\omega))$ *is a Lie series.*

Reciprocally, let $\mathcal{L} \in \mathcal{U}^{(n)}$ *be a Lie polynomial. There then exists a continuous path of bounded variation* ω *such that*

$$\pi_n(\log(\boldsymbol{X}_{s,t}(\omega))) = \mathcal{L}.$$

Proof. Let $x^k_{s,t}(\omega)$ be the component of $\boldsymbol{X}_{s,t}(\omega)$ in $U_k(\mathbb{R}, \mathbb{R}^d)$. The multiplicativity of the Chen series is equivalent to

$$x^n_{r,t}(\omega) = \sum_{k=0}^{n} x^k_{r,s}(\omega) \otimes x^{n-k}_{s,t}(\omega),$$

which can be shown easily by induction from the equality

$$x^{n+1}_{s,t}(\omega) = \int_s^t x^n_{s,u}(\omega)\, \mathrm{d}\omega(u).$$

Let $\boldsymbol{X}^{(n)}_{s,t}(\omega) = \pi_n(\boldsymbol{X}_{s,t}(\omega))$. $\boldsymbol{X}^{(n)}_{0,t}(\omega)$ satisfies the differential equation

$$\mathrm{d}\boldsymbol{X}^{(n)}_{0,t}(\omega) = \boldsymbol{X}^{(n)}_{0,t}(\omega) \otimes \mathrm{d}\omega(t), \quad \boldsymbol{X}^{(n)}_{0,0}(\omega) = 1.$$

Any differential equation of the form $\mathrm{d}g_t = g_t \otimes \mathrm{d}\omega(t)$ which starts in $G_n(\mathbb{R}, \mathbb{R}^d)$ remains in $G_n(\mathbb{R}, \mathbb{R}^d)$ (Lyons 1998). Therefore, for all n,

$$\boldsymbol{X}^{(n)}_{s,t}(\omega) = (\boldsymbol{X}^{(n)}_{0,s}(\omega))^{-1} \otimes \boldsymbol{X}^{(n)}_{0,t}(\omega)$$

belongs to $G_n(\mathbb{R}, \mathbb{R}^d)$, i.e. $\pi_n(\log(X_{s,t}(\omega)))$ belongs to $\mathcal{U}^{(n)}$ for all n, i.e. $\log(X_{s,t}(\omega))$ is a Lie series.

The converse statement can easily be seen to be equivalent to a problem of sub-Riemannian geometry, and is actually Chow–Rashevesky's theorem (Gromov 1996). It can also be proved using the Baker–Campbell–Hausdorff formula. The latter is constructive, and we will use this method to construct explicit cubature formulae at the end of this section. ∎

Example 4.5.

$$\pi_2(\log(\boldsymbol{X}_{0,1}(\omega))) = \varepsilon_0 + \sum_{i=1}^{d} \omega^i(1)\varepsilon_i + \frac{1}{2} \sum_{1 \leqslant i < j \leqslant d} \int_0^1 (\omega^i(t)\, \mathrm{d}\omega^j(t) - \omega^j(t)\, \mathrm{d}\omega^i(t))[\varepsilon_i, \varepsilon_j]. \tag{4.6}$$

The term

$$\frac{1}{2} \int_0^1 (\omega^i(t)\, \mathrm{d}\omega^j(t) - \omega^j(t)\, \mathrm{d}\omega^i(t))$$

is the Lévy area.

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Remark 4.6. One can do the same thing replacing the path of bounded variation ω by the Brownian motion B, using Stratonovich's integration. We will denote by $\boldsymbol{X}_{s,t}(\circ B)$ the series of Stratonovich-iterated integrals. It is not too difficult to adapt the proof of Chen's theorem to see that $\log(\boldsymbol{X}_{s,t}(\circ B))$ is a (random) Lie series. That leads to the small time asymptotic of the solution of Stratonovich SDE (Ben Arous 1989; Kunita 1980; Yamato 1979).

Remark 4.7. For $\lambda \in \mathbb{R}$, $a = (a_0, a_1, a_2, \dots)$ an element of the tensor algebra $T(\mathbb{R}, \mathbb{R}^d)$, we define the following product

$$\langle \lambda, a \rangle = (a_0, \lambda a_1, \lambda^2 a_2, \dots).$$

Equation (2.1) can then be rewritten in the following simple way:

$$\langle \sqrt{t-s}, \boldsymbol{X}_{0,1}(\circ B) \rangle \overset{\mathcal{L}}{=} \boldsymbol{X}_{s,t}(\circ B).$$

Remark 4.8. If Γ denotes the algebra homomorphism which sends ε_i to V_i, then it is straightforward to check that, for a smooth function f,

$$f(\xi_{t,x}) = \Gamma(\boldsymbol{X}_{0,t}^{(m)}(\circ B)) f(x) + R_m(t, x), \tag{4.7}$$

where $R_m(t, x)$ is the remainder in the stochastic Taylor formula (2.2).

(*d*) *Back to cubature on Wiener space*

Using the notations introduced above, one sees that the paths $\omega_1, \dots, \omega_n$ and the positive weights $\lambda_1, \dots, \lambda_n$ define a cubature formula on Wiener space of degree m, if and only if

$$E(\boldsymbol{X}_{0,1}^{(m)}(\circ B)) = \sum_{j=1}^{n} \lambda_j \boldsymbol{X}_{0,1}^{(m)}(\omega_j).$$

We now give a new definition of a cubature formula on Wiener space, no longer in terms of path but in terms of Lie polynomials.

Definition 4.9. Let m be a natural number. We will say that the Lie polynomials $\mathcal{L}_1, \dots, \mathcal{L}_n \in \mathcal{U}^{(m)}$ and the positive weights $\lambda_1, \dots, \lambda_n$ define a cubature formula on Wiener space of degree m, if and only if

$$E(\boldsymbol{X}_{0,1}^{(m)}(\circ B)) = \sum_{j=1}^{n} \lambda_j \pi_m(\exp(\mathcal{L}_j)). \tag{4.8}$$

Using Chen's theorem, one can go from the definition of cubature on Wiener space using paths to the one using Lie polynomials. Indeed, suppose that the Lie polynomials $\mathcal{L}_1, \dots, \mathcal{L}_n \in \mathcal{U}^{(m)}$ and the positive weights $\lambda_1, \dots, \lambda_n$ define a cubature formula on Wiener space of degree m. Define some paths $\omega_1, \dots, \omega_n \in C^0_{0,\mathrm{bv}}([0,1], \mathbb{R}^d)$ such that $\pi_m(\log(\boldsymbol{X}_{0,1}(\omega_i))) = \mathcal{L}_i$ for $i = 1, \dots, n$. Then the paths $\omega_1, \dots, \omega_n$ and the weights $\lambda_1, \dots, \lambda_n$ define a cubature formula on Wiener space of degree m.

Reciprocally, if the paths $\omega_1, \dots, \omega_n \in C^0_{0,\mathrm{bv}}([0,1], \mathbb{R}^d)$ and the weights $\lambda_1, \dots, \lambda_n$ define a cubature formula on Wiener space of degree m, then the Lie polynomials

$$\pi_m(\log(\boldsymbol{X}_1(\omega_1))), \dots, \pi_m(\log(\boldsymbol{X}_{0,1}(\omega_n)))$$

and the weights $\lambda_1, \dots, \lambda_n$ define a cubature formula on Wiener space of degree m.

To find these Lie polynomials, one obviously needs to know the expectation under the Wiener measure of the series of Stratonovich-iterated integrals.

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(*e*) *Expectation of the Stratonovich-iterated integrals*

First note that B_t can be expressed in the form

$$t\varepsilon_0 + \sum_{i=1}^{d} B_t^i \varepsilon_i.$$

The following identity was first understood by Fawcett (2003). We provide here an analytical proof.

Proposition 4.10.

$$E(\boldsymbol{X}_{0,1}(\circ B)) = \exp\biggl(\varepsilon_0 + \frac{1}{2}\sum_{i=1}^{d}\varepsilon_i\otimes\varepsilon_i\biggr). \tag{4.9}$$

Proof. There are a few possible approaches. One can calculate explicitly

$$E\biggl(\int_{0<t_1<\dots<t_k<t} \circ\, \mathrm{d}B_{t_1}^{i_1}\cdots\circ\,\mathrm{d}B_{t_k}^{i_k}\biggr)$$

for all $i_1,\dots,i_k$, and use these values together with (4.5) to find the value of $E(\boldsymbol{X}_{0,1}(\circ B))$. A prettier approach uses the relation between PDEs and expectation of SDEs. Using (4.7), we get

$$\mathbb{E}(f(\xi_{t,x})) = \Gamma(\mathbb{E}(\boldsymbol{X}_{0,t}^{(2m)}(\circ B)))f(x) + o(t^m),$$

and using the scaling property of the Brownian motion,

$$\mathbb{E}(f(\xi_{t,x})) = \Gamma\langle\sqrt{t}, \mathbb{E}(\boldsymbol{X}_{0,1}^{(2m)}(\circ B))\rangle f(x) + o(t^m).$$

But we also know that $(t,x)\to\mathbb{E}(f(\xi_{T-t,x}))$ is the solution of the equation (1.1). Hence, we can use the classical notation,

$$\mathbb{E}(f(\xi_{t,x})) = \exp(tL)f(x),$$

which means that, for all m,

$$\mathbb{E}(f(\xi_{t,x})) = \sum_{k=0}^{m}\frac{t^kL^k}{k!}f(x) + o(t^m).$$

Note that

$$L = \Gamma\biggl(\varepsilon_0 + \frac{1}{2}\sum_{i=1}^{d}\varepsilon_i\otimes\varepsilon_i\biggr).$$

Hence, we have the following polynomial (in t) equality,

$$\begin{aligned}
\mathbb{E}(f(\xi_{t,x})) &= \sum_{k=0}^{m}\biggl\{t^k\Gamma\biggl(\varepsilon_0 + \frac{1}{2}\sum_{i=1}^{d}\varepsilon_i\otimes\varepsilon_i\biggr)^k(k!)^{-1}\biggr\}f(x) + o(t^m)\\
&= \Gamma\biggl(\pi_{2m}\biggl(\exp\biggl(t\varepsilon_0 + \tfrac{1}{2}t\sum_{i=1}^{d}\varepsilon_i\otimes\varepsilon_i\biggr)\biggr)\biggr)f(x) + o(t^m)\\
&= \Gamma\biggl(\pi_{2m}\biggl\langle t, \exp\biggl(\varepsilon_0 + \frac{1}{2}\sum_{i=1}^{d}\varepsilon_i\otimes\varepsilon_i\biggr)\biggr\rangle\biggr)f(x) + o(t^m).
\end{aligned}$$

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This proves that, for all m,

$$\langle\sqrt{t},\mathbb{E}(\boldsymbol{X}_{0,1}^{(2m)}(\circ B))\rangle=\pi_{2m}\left\langle t,\exp\left(\varepsilon_0+\frac{1}{2}\sum_{i=1}^{d}\varepsilon_i\otimes\varepsilon_i\right)\right\rangle.$$

Since we are now working in $T(\mathbb{R},\mathbb{R}^d)$, we can now take $t=1$ and let m tend to infinity to obtain our result. ■

This proposition tells us that the expectation $\pi_m(\boldsymbol{X}_{0,1}(\circ B))$ does not belong to $G_m(\mathbb{R},\mathbb{R}^d)$. Nonetheless, $\pi_m(E(\boldsymbol{X}_{0,1}(\circ B)))$ can be expressed as a positive linear combination of elements of $G_m(\mathbb{R},\mathbb{R}^d)$. This is what Tchakaloff's theorem, adapted to the Wiener space, tells us.

5. Explicit constructions of cubature formulae on Wiener space

(a) *Equations to get a cubature formula*

We have understood that, to find a cubature formula on Wiener space of a given degree m, one needs to find some Lie polynomials $\mathcal{L}_1,\dots,\mathcal{L}_n\in\mathcal{U}^{(m)}$ and some positive weights $\lambda_1,\dots,\lambda_n$ such that

$$\pi_m\left(\exp\left(\varepsilon_0+\frac{1}{2}\sum_{i=1}^{d}\varepsilon_i\otimes\varepsilon_i\right)\right)=\sum_{j=1}^{n}\lambda_j\pi_m(\exp(\mathcal{L}_j)).$$

Denote by $\ell_1,\dots,\ell_{k_m}$ the elements of $\mathcal{B}_{\mathcal{U}}\cap T^{(m)}(\mathbb{R},\mathbb{R}^d)$; moreover we take them such that $\ell_i=\varepsilon_i$ for $i=1,\dots,d$. So we need to find some reals $(\beta_{i,j})_{i=1,\dots,k_m,\,j=1,\dots,n}$ such that

$$\pi_m\left(\exp\left(\varepsilon_0+\frac{1}{2}\sum_{i=1}^{d}\varepsilon_i\otimes\varepsilon_i\right)\right)=\sum_{j=1}^{n}\lambda_j\pi_m\left(\exp\left(\sum_{i=1}^{k_m}\beta_{i,j}\ell_i\right)\right).\tag{5.1}$$

We understand (equation (4.4)) how to develop the right-hand side of the previous expression. One now has to develop the left-hand side in the basis described in the corollary of the Poincaré–Birkhoff–Witt theorem. But, even before doing so, we can describe the $\beta_{i,j}$ for $i=1,\dots,d$.

Proposition 5.1. *Assume that the points and weights $(\beta_{i,j})_{i=1,\dots,k_m,\,j=1,\dots,n},\lambda_j$ satisfy (5.1). Then $(\beta_{1,j},\dots,\beta_{d,j})_{j=1,\dots,n}\in\mathbb{R}^d$ and $\lambda_1,\dots,\lambda_n$ define a cubature formula of degree m with respect to the d-dimensional Gaussian measure, i.e. for all polynomials P in $\mathbb{R}_m[X_1,\dots,X_d]$,*

$$\frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d}P(x)\exp(-\tfrac{1}{2}\|x\|^2)\,\mathrm{d}x=\sum_{j=1}^{n}\lambda_jP(\beta_{1,j},\dots,\beta_{d,j}).\tag{5.2}$$

Proof. Let us consider the commutative algebra $TC(\mathbb{R}^d)$ with a basis

$$(\varrho_{i_1}\varrho_{i_2}\cdots\varrho_{i_k})_{k\geqslant 0,\,1\leqslant i_1\leqslant\cdots\leqslant i_k\leqslant d},$$

and the algebra homomorphism $\varDelta$

$$\begin{aligned}T(\mathbb{R},\mathbb{R}^d)&\to TC(\mathbb{R}^d),\\ \ell\to 0,&\quad\text{if }\ell\in\mathcal{B}_{\mathcal{U}}\setminus\{\varepsilon_1,\dots,\varepsilon_d\},\\ \varepsilon_i\to\varrho_i,&\quad\text{if }i\neq 0.\end{aligned}$$

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Equation (5.1) implies

$$\Delta\biggl(\sum_{j=1}^{n}\lambda_j\pi_m\biggl(\exp\biggl(\sum_{i=1}^{k_m}\beta_{i,j}\ell_i\biggr)\biggr)\biggr)=\Delta\biggl(\pi_m\biggl(\exp\biggl(\frac{1}{2}\sum_{i=1}^{d}\varepsilon_i\otimes\varepsilon_i\biggr)\biggr)\biggr),$$

i.e.

$$\sum_{j=1}^{n}\lambda_j\pi_m\biggl(\exp\biggl(\sum_{i=1}^{d}\beta_{i,j}\varrho_i\biggr)\biggr)=\pi_m\biggl(E\biggl(\exp\biggl(\sum_{i=1}^{d}N_i\varrho_i\biggr)\biggr)\biggr),$$

where $(N_1,\dots,N_d)$ is a d-dimensional normal random variable. It is then straightforward that (5.2) is satisfied by all monomials of degree less than or equal to m. ■

(*b*) *Degree 3*

First of all, note that

$$\pi_3\biggl(\exp\biggl(\varepsilon_0+\frac{1}{2}\sum_{i=1}^{d}\varepsilon_i\otimes\varepsilon_i\biggr)\biggr)=\varepsilon_0+\frac{1}{2}\sum_{i=1}^{d}\varepsilon_i\otimes\varepsilon_i.$$

In this subsection, $z_1,\dots,z_n\in\mathbb{R}^d$ and $\lambda_1,\dots,\lambda_n\in\mathbb{R}_+$ will be some points and weights defining a cubature formula of degree 3 with respect to the d-dimensional Gaussian measure. The most classical formula is characterized by $n=2^d$, $z_1,\dots,z_n=\{-1,+1\}^d$ and $\lambda_i=2^{-d}$. But by Tchakaloff's theorem, we know that we should be able to find a cubature formula of degree 3 with no more than $\frac{1}{6}d(d+1)(d+2)$ points. See Stroud (1971) and Victoir (2004) for some constructions of cubature formulae of degree 5 with respect to the Gaussian measure, with $2d$ points.

Proposition 5.2. *Define $\mathcal{L}_1,\dots,\mathcal{L}_n$ to be the Lie polynomials $\mathcal{L}_i=\varepsilon_0+z_i^1\varepsilon_1+\dots+z_i^d\varepsilon_d$. Then $\mathcal{L}_1,\dots,\mathcal{L}_n$ and $\lambda_1,\dots,\lambda_n$ define a cubature formula on Wiener space of degree* 3.

Proof.

$$\begin{aligned}\pi_3(\exp(\mathcal{L}_i))=1+\varepsilon_0+\tfrac{1}{2}\varepsilon_0\otimes\varepsilon_0+\sum_{j=1}^{d}(z_i^j\varepsilon_j+\tfrac{1}{2}(z_i^j)^2\varepsilon_j\otimes\varepsilon_j)\\ +\frac{1}{2}\biggl(\varepsilon_0\otimes\sum_{j=1}^{d}z_i^j\varepsilon_j+\sum_{j=1}^{d}z_i^j\varepsilon_j\otimes\varepsilon_0\biggr)\\ +\frac{1}{2}\sum_{j<k=1}^{d}z_i^jz_i^k(\varepsilon_j\otimes\varepsilon_j+\varepsilon_j\otimes\varepsilon_j)\\ +\frac{1}{6}\biggl(\sum_{j=1}^{d}z_i^j\varepsilon_j\biggr)^3.\end{aligned}$$

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By definition of the weights λ_i and the points $(z_i^1, \dots, z_i^d)$,

$$\begin{aligned}\sum_{i=1}^{n}\lambda_i\biggl(\sum_{j=1}^{d} z_i^j\varepsilon_j\biggr)^3 &= \sum_{i=1}^{n}\lambda_i\biggl(\sum_{j=1}^{d} z_i^j\varepsilon_j\biggr)\\ &= \sum_{i=1}^{n}\lambda_i\sum_{j<k=1}^{d} z_i^j z_i^k(\varepsilon_j\otimes\varepsilon_j+\varepsilon_j\otimes\varepsilon_j)\\ &= 0,\end{aligned}$$

and

$$\sum_{i=1}^{n}\lambda_i\sum_{j=1}^{d}\tfrac{1}{2}(z_i^j)^2\varepsilon_j\otimes\varepsilon_j = \frac{1}{2}\sum_{j=1}^{d}\varepsilon_j\otimes\varepsilon_j.$$

Hence,

$$\begin{aligned}\sum_{i=1}^{n}\lambda_i\pi_3(\exp(\mathcal{L}_i)) &= 1+\varepsilon_0+\tfrac{1}{2}\varepsilon_0\otimes\varepsilon_0+\frac{1}{2}\sum_{j=1}^{d}\varepsilon_j\otimes\varepsilon_j\\ &= \pi_3\biggl(\exp\biggl(\frac{1}{2}\sum_{i=1}^{d}\varepsilon_i\otimes\varepsilon_i\biggr)\biggr).\end{aligned}$$

■

Proposition 5.3. *Define $\omega_1, \dots, \omega_n$ to be the paths $\omega_i : t \mapsto t(1, z_i^1, \dots, z_i^d)$. Then $\omega_1, \dots, \omega_n$ and $\lambda_1, \dots, \lambda_n$ define a cubature formula on Wiener space of degree 3.*

Proof. The log of the Chen series between time 0 and 1 of the path ω_i is equal to $\varepsilon_0 + z_i^1\varepsilon_1 + \cdots + z_i^d\varepsilon_d$. ■

(c) Degree 5

Things start to get complicated. First of all, we need to develop the expression

$$\pi_5\biggl(\exp\biggl(\varepsilon_0+\frac{1}{2}\sum_{i=1}^{d}\varepsilon_i\otimes\varepsilon_i\biggr)\biggr)$$

in the basis (4.3).

Proposition 5.4. $\pi_5(\exp(\varepsilon_0 + \frac{1}{2}(\varepsilon_1^2 + \cdots + \varepsilon_d^2)))$ *is equal to*

$$\begin{aligned}&1+\varepsilon_0+\tfrac{1}{2}(\varepsilon_0,\varepsilon_0)+\sum_{i=1}^{d}\tfrac{1}{2}(\varepsilon_i,\varepsilon_i)+\sum_{i=1}^{d}\tfrac{1}{8}(\varepsilon_i,\varepsilon_i,\varepsilon_i,\varepsilon_i)\\ &\quad+\sum_{i=1}^{d}\sum_{j=i+1}^{d}\tfrac{1}{4}(\varepsilon_i,\varepsilon_i,\varepsilon_j,\varepsilon_j)+\sum_{i=1}^{d}\tfrac{1}{4}(2(\varepsilon_0,\varepsilon_i,\varepsilon_i)+\tfrac{1}{3}[[\varepsilon_0,\varepsilon_i],\varepsilon_i])\\ &\qquad+\sum_{i=1}^{d}\sum_{j=i+1}^{d}\tfrac{1}{8}(([\varepsilon_i,\varepsilon_j],[\varepsilon_i,\varepsilon_j])+\tfrac{2}{3}([[\varepsilon_i,\varepsilon_j],\varepsilon_j],\varepsilon_i)+\tfrac{2}{3}([[\varepsilon_j,\varepsilon_i],\varepsilon_i],\varepsilon_j)).\end{aligned} \tag{5.3}$$

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This will follow from the following lemmas.

Lemma 5.5. *Let a and b be two non-commutative variables. Then*

$$ab^2 + b^2a = 2(a, b, b) + \tfrac{1}{3}[[a, b], b].$$

Proof. It is immediate from the following two equalities:

$$2(a, b, b) = \tfrac{2}{3}(abb + bab + bba), \qquad \tfrac{1}{3}[[a, b], b] = \tfrac{1}{3}abb + \tfrac{1}{3}bba - \tfrac{2}{3}bab.$$

■

Lemma 5.6. *Let a and b be two non-commutative variables. Then*

$$a^2b^2 + b^2a^2 = [a, b]^2 + \tfrac{2}{3}([[a, b], b], a) + \tfrac{2}{3}([[b, a], a], b) + 2(a, a, b, b).$$

Proof. It is also immediate from the following equalities:

$$\begin{aligned}
[a, b]^2 &= abab - baab - abba + baba;\\
2(a, a, b, b) &= \tfrac{1}{3}(aabb + bbaa + baba + abab + abba + baab);\\
-\tfrac{2}{3}([[b, a], a], b) &= \tfrac{2}{3}((-baa - aab + 2aba), b)\\
&= \tfrac{2}{3}\tfrac{1}{2}(-bbaa - baab + 2baba - baab - aabb + 2abab)\\
&= -\tfrac{1}{3}(bbaa + aabb - 2baba - 2abab + 2baab);\\
\tfrac{2}{3}([[a, b], b], a) &= \tfrac{2}{3}((abb + bba - 2bab), a)\\
&= \tfrac{1}{3}(aabb + bbaa + 2abba - 2abab - 2baba);
\end{aligned}$$

so that

$$\tfrac{2}{3}([[a, b], b], a) + \tfrac{2}{3}([[b, a], a], b) = \tfrac{2}{3}(a^2b^2 + b^2a^2 - 2baba - 2abab + ab^2a + ba^2b).$$

■

In this subsection, $z_1, \ldots, z_n \in \mathbb{R}^d$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}_+$ will now be some points and weights defining a cubature formula of degree 5 with respect to the d-dimensional Gaussian weight. For $d = 1$, we can take $n = 3$ with $(z_1, z_2, z_3) = (-\sqrt{3}, 0, \sqrt{3})$, and $(\lambda_1, \lambda_2, \lambda_3) = (\frac{1}{6}, \frac{2}{3}, \frac{1}{6})$. That leads to the most classical cubature formula of degree 5 with respect to the Gaussian measure, where the z_i are the points in $\{-\sqrt{3}, 0, \sqrt{3}\}^d$. But this formula involves 3^d points, far more than the Tchakaloff upper bound of $\binom{d+5}{5}$. See Victoir (2004), for some constructions of cubature formulae of degree 5 with respect to the Gaussian measure, with $O(d^3)$ points.

Proposition 5.7. *For $k = 1, \ldots, n$ and $\eta = \pm 1$, let $\mathcal{L}_{k,\eta}$ to be the Lie polynomials*

$$\begin{aligned}
\mathcal{L}_{k,1} = \varepsilon_0 &+ \sum_{i=1}^{d} \tfrac{1}{12}(z_k^i)^2[[\varepsilon_0, \varepsilon_i], \varepsilon_i] + \sum_{i=1}^{d} z_k^i \varepsilon_i + \sum_{1 \leqslant i < j \leqslant d} \tfrac{1}{2} z_k^i z_k^j [\varepsilon_i, \varepsilon_j]\\
&+ \sum_{1 \leqslant i < j \leqslant d} (\tfrac{1}{6} z_k^i (z_k^j)^2 x[[\varepsilon_i, \varepsilon_j], \varepsilon_j] + \tfrac{1}{6} z_k^j (z_k^i)^2 (1 - x)[[\varepsilon_j, \varepsilon_i], \varepsilon_i]),
\end{aligned}$$

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$$\mathcal{L}_{k,-1} = \varepsilon_0 + \sum_{i=1}^{d} \tfrac{1}{12}(z_k^i)^2[[\varepsilon_0, \varepsilon_i], \varepsilon_i] + \sum_{i=1}^{d} z_k^i \varepsilon_i - \sum_{1 \leqslant i < j \leqslant d} \tfrac{1}{2} z_k^i z_k^j [\varepsilon_i, \varepsilon_j]$$
$$+ \sum_{1 \leqslant i < j \leqslant d} (\tfrac{1}{6} z_k^j (z_k^i)^2 x[[\varepsilon_j, \varepsilon_i], \varepsilon_i] + \tfrac{1}{6} z_k^i (z_k^j)^2 (1-x)[[\varepsilon_i, \varepsilon_j], \varepsilon_j]),$$

where x is a constant. Define also $\lambda_{k,\eta} = \frac{1}{2}\lambda_k$. The $\mathcal{L}_{k,\eta}$ and $\lambda_{k,\eta}$ then define a cubature formula on Wiener space of degree 5.

Proof. Define the random variables $Z_1, \ldots, Z_d$ such that $(Z_1, \ldots, Z_d)$ is equal to $(z_k^1, \ldots, z_k^d)$ with probability λ_k, and Λ independent of the Z_i, such that $\Lambda = \pm 1$ with probability $\frac{1}{2}$.

We have to check that

$$\sum_{k=1}^{n} \tfrac{1}{2} \lambda_k \pi_5(\exp(\mathcal{L}_{k,1}) + \exp(\mathcal{L}_{k,-1}))$$

is equal to the expression (5.3). Since

$$\sum_{k=1}^{n} \lambda_k (z_k^i)^p = 0 \quad \text{for } p = 1, 3, 5,$$

we only have to check that the term in front of the basis elements of $U_k(\mathbb{R}, \mathbb{R}^d)$, $k = 0, 2, 4$, of these two expressions coincide. Using (5.2), we only have to check that the terms in front of the basis (4.3) elements involve some Lie brackets. (4.4) quickly gives us the results in table 1.

Using the basic properties of the random variables Λ, Z_i, we obtain that the expressions in the second and third rows of the table are equal.

The coefficient in front of the basis element that we have not yet checked is clearly zero for both

$$\sum_{k=1}^{n} \tfrac{1}{2} \lambda_k \pi_5(\exp(\mathcal{L}_{k,1}) + \exp(\mathcal{L}_{k,-1}))$$

and (5.3), which completes the proof. ■

Remark 5.8. We approximate weakly the Brownian motion and its Lévy area

$$\left((B_1^i)_{1 \leqslant i \leqslant d}, \left(\int_0^1 B_s^i \, \mathrm{d}B_s^j - B_s^j \, \mathrm{d}B_s^i \right)_{1 \leqslant i < j \leqslant d} \right), \tag{5.4}$$

by

$$Z = ((Z_i)_{1 \leqslant i \leqslant d}, (\Lambda Z_i Z_j)_{1 \leqslant i < j \leqslant d}).$$

(and the higher-order iterated integrals as a function of this random variables). Z takes only $O(d^3)$ different values.

That can be compared with the weak simplified approximation schemes of order 5 described in Kloeden & Platen (1992). Let $X_1, \ldots, X_d$ be i.i.d. random variables with law

$$P(X_i = -\sqrt{3}) = P(X_i = \sqrt{3}) = \tfrac{1}{4} P(X_i = 0) = \tfrac{1}{6}$$

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Table 1. *Results for proof of proposition 5.7*

basis element	(5.3)	$\sum_{k=1}^{n} \frac{1}{2}\lambda_k \pi_5(\exp(\mathcal{L}_{k,1}) + \exp(\mathcal{L}_{k,-1}))$
$[\varepsilon_i, \varepsilon_j]$, $i < j$	0	$E(\Lambda Z_i Z_j)$
$[[\varepsilon_0, \varepsilon_i], \varepsilon_i]$	$\frac{1}{12}$	$\frac{1}{12}E(Z_i^2)$
$([\varepsilon_i, \varepsilon_j], [\varepsilon_i, \varepsilon_j])$, $i < j$	$\frac{1}{8}$	$\frac{1}{8}E(\Lambda^2 Z_i^2 Z_j^2)$
$([[\varepsilon_i, \varepsilon_j], \varepsilon_j], \varepsilon_i)$, $i \neq j$	$\frac{1}{12}$	$E(Z_i^2 Z_j^2)\left(\dfrac{x}{12} + \dfrac{1-x}{12}\right)$

and let $(Y_{i,j})_{1\leqslant i<j\leqslant d}$ also be i.i.d. random variables, independent of $X_i, \dots, X_i$, and with law $P(Y_{i,j} = -1) = P(Y_{i,j} = 1)$. They approximate (5.4) by

$$((X_i)_{1\leqslant i\leqslant d}, (Y_{i,j})_{1\leqslant i<j\leqslant d}),$$

which takes $2^{(d(d-1))/2}3^d$ different values.

We now want to obtain, from the cubature formula on Wiener space of degree 5 involving Lie elements, a formula involving paths.

Let $\mathcal{U}_1$ denote the subspace of $\mathcal{U}$ spanned by ε_0, $[\varepsilon_0, \varepsilon_i]$, $[\varepsilon_i, [\varepsilon_i, \varepsilon_0]]$, ε_i, $[\varepsilon_i, \varepsilon_j]$ and $[\varepsilon_i, [\varepsilon_i, \varepsilon_j]] + [\varepsilon_j, [\varepsilon_j, \varepsilon_i]]$, for $i < j = 1, \dots, d$. Denote by $\pi_{\mathcal{U}_1}$ the natural projection from $\mathcal{U}$ onto $\mathcal{U}_1$.

Theorem 5.9. *Let* $\omega \in C^0_{0,\mathrm{bv}}([0,1], \mathbb{R}^d)$ *be such that*

$$\pi_{\mathcal{U}_1}(\log(\boldsymbol{X}_{0,1}(\omega))) = \varepsilon_0 + \tfrac{1}{12}\sum_{i=1}^{d}[[\varepsilon_0, \varepsilon_i], \varepsilon_i] + \sum_{i=1}^{d}\varepsilon_i + \sum_{1\leqslant i<j\leqslant d}\tfrac{1}{2}[\varepsilon_i, \varepsilon_j] + \sum_{1\leqslant i<j\leqslant d}\tfrac{1}{12}([[\varepsilon_i, \varepsilon_j], \varepsilon_j] + [[\varepsilon_i, \varepsilon_j], \varepsilon_j]). \quad (5.5)$$

For $i = 1, \dots, n$, *we then define the weights* $\tilde{\lambda}_i = \tilde{\lambda}_{i+n} = \frac{1}{2}\lambda_i$ *and the paths*

$$\omega_i(t) = (t, z_i^1\omega^1(t), \dots, z_i^d\omega^d(t)),$$
$$\omega_{n+i}(t) = (t, z_i^1\omega^d(t), \dots, z_i^d\omega^1(t)).$$

The paths $\omega_1, \dots, \omega_{2n}$ *and the weights* $\tilde{\lambda}_1, \dots, \tilde{\lambda}_{2n}$ *define a cubature formula on Wiener space of degree* 5.

Proof. $\pi_{\mathcal{U}_1}(\log(\boldsymbol{X}_{0,1}(\omega_k))) = \mathcal{L}_{k,1}$ and $\pi_{\mathcal{U}_1}(\log(\boldsymbol{X}_{0,1}(\omega_{k+n}))) = \mathcal{L}_{k,-1}$. We do not find the terms $\pi_5(\log(\boldsymbol{X}_{0,1}(\omega_k))) - \mathcal{L}_{k,1}$ and $\pi_5(\log(\boldsymbol{X}_{0,1}(\omega_{k+n}))) - \mathcal{L}_{k,-1}$ in

$$\sum_{k=1}^{n}\tfrac{1}{2}\lambda_k\pi_5(\boldsymbol{X}_{0,1}(\omega_k) + \boldsymbol{X}_{0,1}(\omega_{k+n})),$$

thanks to the q and p such that $E(\Lambda^q Z^p) = 0$. ∎

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So, we now need to construct a path ω satisfying (5.5). We see with a few integrations by parts that (5.5) is equivalent to the following conditions:

$$\left.\begin{aligned}\int_0^1 \omega^i(t)\,\mathrm{d}t = \int_0^1 \omega^i(t)^2\,\mathrm{d}t = \tfrac{1}{2}\\ \omega^i(0) = 0\\ \omega^i(1) = 1\end{aligned}\right\}\quad \forall i \in \{1,\dots,d\},$$

$$\left.\begin{aligned}\int_0^1 \omega^i(t)\,\mathrm{d}\omega^j(t) - \int_0^1 \omega^j(t)\,\mathrm{d}\omega^i(t) = 1\\ \int_0^1 \omega^i(t)^2\,\mathrm{d}\omega^j(t) + \int_0^1 \omega^j(t)^2\,\mathrm{d}\omega^i(t) = 1\end{aligned}\right\}\quad \text{for } 1 \leqslant i < j \leqslant d.$$

The path ω satisfying the above relation and minimizing

$$\int_0^1 \sqrt{\sum_{i=1}^d \left(\frac{\mathrm{d}\omega^i}{\mathrm{d}t}(t)\right)^2}$$

(path of shortest length) can be found theoretically, using some classical calculus of variation theory (Gelfand & Fomin 1963). Unfortunately, the number of parameters is too high and the equations are not simple enough to construct such an ω explicitly. We will provide piecewise linear solutions. The idea is to find an integer L and some coefficients $\theta_{i,j} \in \mathbb{R}$ for $i = 1,\dots,d$ and $j = 1,\dots,L$ such that

$$\pi_{\mathcal{U}_1}\left(\log\left(\prod_{j=1}^L \exp\left(\frac{\varepsilon_0}{L} + \sum_{i=1}^d \theta_{i,j}\varepsilon_i\right)\right)\right)$$

is equal to the expression (5.5). This obviously involves extensive use of the Baker–Campbell–Hausdorff formula.

Theorem 5.10 (Baker–Campbell–Hausdorff formula).

$$\begin{aligned}&\pi_{\mathcal{U}_1}(\log(\mathrm{e}^{\varepsilon_1}\cdots\mathrm{e}^{\varepsilon_d}))\\ &\qquad = \sum_{i=1}^d \varepsilon_i + \sum_{1\leqslant i<j\leqslant d} \tfrac{1}{2}[\varepsilon_i,\varepsilon_j] + \sum_{1\leqslant i<j\leqslant d} \tfrac{1}{12}([[\varepsilon_i,\varepsilon_j],\varepsilon_j] + [[\varepsilon_j,\varepsilon_i],\varepsilon_i]).\end{aligned}$$

Once we have the coefficients $\theta_{i,j}$, we define the piecewise linear path ω with derivatives equal to $(1, L\theta_{1,j},\dots,L\theta_{d,j})$ in the interval

$$\left[\frac{j-1}{L},\frac{j}{L}\right].$$

By the multiplicativity of the Chen series, the series of iterated integrals of ω is equal to

$$\prod_{j=1}^L \exp\left(\frac{\varepsilon_0}{L} + \sum_{i=1}^d \theta_{i,j}\right).$$

Hence, ω will be a solution to equation (5.5).

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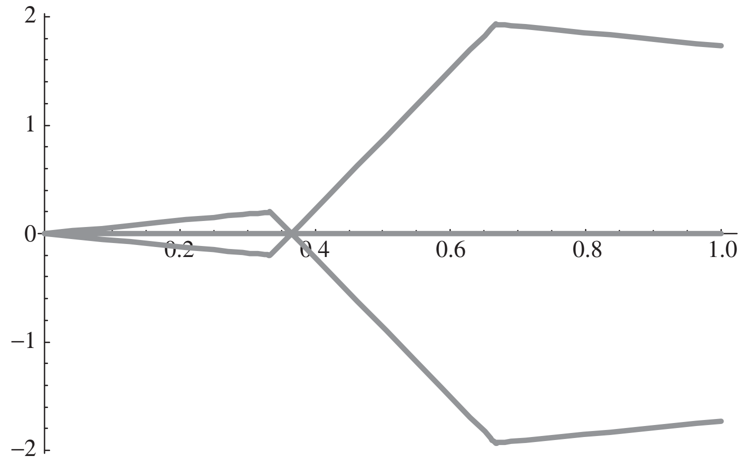

Figure 1. Quadrature formula of degree 5 with one Brownian motion.

(i) *Construction of the path for $d = 1$*

We programmed the Baker–Campbell–Hausdorff formula in MATHEMATICA and set it to find the coefficients $\theta_{i,j}$, with $L = 3$ (which gives three unknowns for three equations). It gave us

$$\theta_{1,1} = \tfrac{1}{6}(4 - \sqrt{22}), \qquad \theta_{1,2} = \tfrac{1}{6}(-2 + 2\sqrt{22}), \qquad \theta_{1,2} = \tfrac{1}{6}(4 - \sqrt{22}).$$

In other words, if we define

$$\omega_1^{[1]} : [0,1] \to \mathbb{R},$$
$$t \mapsto \begin{cases} \tfrac{1}{2}(4 - \sqrt{22})t, & 0 \leqslant t \leqslant \tfrac{1}{3}, \\ \tfrac{1}{6}(4 - \sqrt{22}) + (-1 + \sqrt{22})(t - \tfrac{1}{3}), & \tfrac{1}{3} \leqslant t \leqslant \tfrac{2}{3}, \\ \tfrac{1}{6}(2 + \sqrt{22}) + \tfrac{1}{2}(4 - \sqrt{22})(t - \tfrac{2}{3}), & \tfrac{2}{3} \leqslant t \leqslant 1. \end{cases}$$

Then the path $\omega^{[1]}(t) = (t, \omega_1^{[1]}(t))$ satisfies (5.5).

Hence, we obtain the results that the paths $(t, -\sqrt{3}\omega_1^{[1]}(t)), (t, 0), (t, +\sqrt{3}\omega_1^{[1]}(t))$ with the weights $\frac{1}{6}$, $\frac{2}{3}$, $\frac{1}{6}$ define a cubature formula on Wiener space of degree 5 (see figure 1).

(ii) *Construction of the path for $d = 2$*

This time, we take $L = 4$ to have the number of unknowns equal to the number of equations (eight). MATHEMATICA gives us an expression, which starts to be quite complicated, so we only give the numerical values in table 2.

We define $\omega_i^{[2]}$, $i = 1, 2$, as the piecewise linear continuous path from $[0, 1]$ to $\mathbb{R}$, starting at 0 and such that

$$\frac{\mathrm{d}\omega_i}{\mathrm{d}t} = 4\theta_{i,j} \quad \text{in } [\tfrac{1}{4}(j-1), \tfrac{1}{4}j].$$

The path $t \mapsto (t, \omega_1^{[2]}(t), \omega_2^{[2]}(t))$ (see figure 2) then satisfies (5.5).

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Table 2. *Numerical values for $d = 2$, $L = 4$*

θ_{11}	0.048 077 867 969 717
θ_{12}	0.089 335 677 121 310
θ_{13}	1.177 095 041 847 874
θ_{14}	−0.314 508 586 938 901
θ_{21}	1.012 368 507 495 347
θ_{22}	−1.619 604 763 795 00
θ_{23}	1.702 104 005 103 968
θ_{24}	−0.094 867 748 804 311

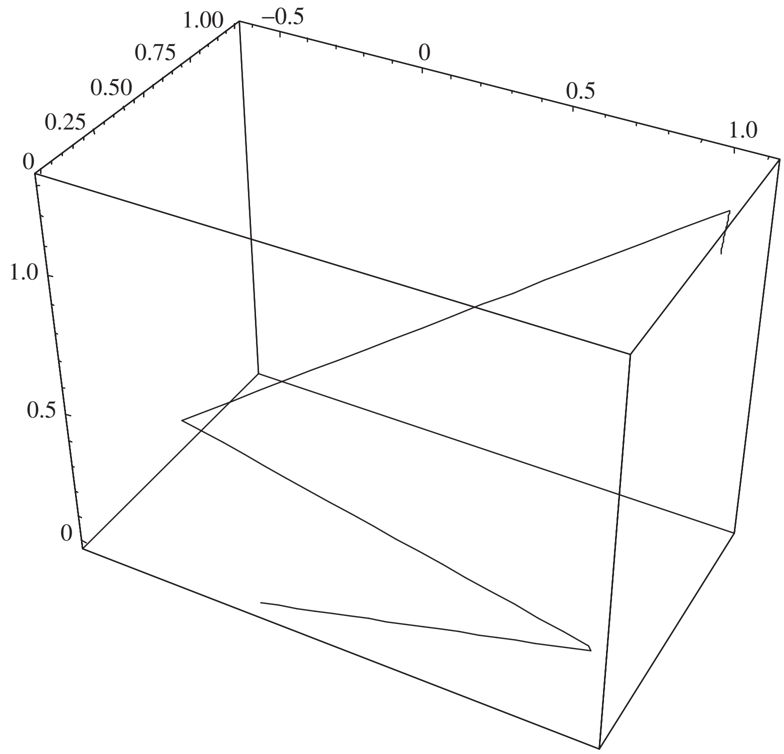

Figure 2. $t \mapsto (t, \omega_1^{[2]}(t), \omega_2^{[2]}(t))$.

The points

$$(0,0),\ (2,0),\ (-2,0),\ (1,\sqrt{3}),\ (-1,\sqrt{3}),\ (-1,-\sqrt{3}),\ (-1,-\sqrt{3})$$

and the weights

$$\tfrac{1}{2},\ \tfrac{1}{12},\ \tfrac{1}{12},\ \tfrac{1}{12},\ \tfrac{1}{12},\ \tfrac{1}{12},\ \tfrac{1}{12}$$

define a cubature formula of degree 5 with respect to the Gaussian weight. From the above path and these points and weights, we get, using theorem (5.9), a cubature formula on Wiener space of degree 5, with 13 paths (see table 3).

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Table 3. *Paths and weights for $d = 2$, degree 5*

path	weight
$(t, 0, 0)$	$\frac{1}{2}$
$(t, \omega_1^{[2]}(t), \sqrt{3}\omega_2^{[2]}(t))$	$\frac{1}{24}$
$(t, -\omega_1^{[2]}(t), \sqrt{3}\omega_2^{[2]}(t))$	$\frac{1}{24}$
$(t, \omega_1^{[2]}(t), -\sqrt{3}\omega_2^{[2]}(t))$	$\frac{1}{24}$
$(t, -\omega_1^{[2]}(t), -\sqrt{3}\omega_2^{[2]}(t))$	$\frac{1}{24}$
$(t, \omega_2^{[2]}(t), \sqrt{3}\omega_1^{[2]}(t))$	$\frac{1}{24}$
$(t, -\omega_2^{[2]}(t), \sqrt{3}\omega_1^{[2]}(t))$	$\frac{1}{24}$
$(t, \omega_2^{[2]}(t), -\sqrt{3}\omega_1^{[2]}(t))$	$\frac{1}{24}$
$(t, -\omega_2^{[2]}(t), -\sqrt{3}\omega_1^{[2]}(t))$	$\frac{1}{24}$
$(t, 2\omega_1^{[1]}(t), 0)$	$\frac{1}{24}$
$(t, -2\omega_1^{[1]}(t), 0)$	$\frac{1}{24}$
$(t, 0, 2\omega_1^{[1]}(t))$	$\frac{1}{24}$
$(t, 0, -2\omega_1^{[1]}(t))$	$\frac{1}{24}$

(iii) *Construction of the path for $d \geqslant 3$*

It is difficult to extend previous computational techniques to higher d. Nevertheless, we still propose a piecewise linear solution. We will find, for an integer L, some coefficients $\theta_{i,j}$ such that

$$\pi_{\mathcal{U}_1}\left(\log\left(\prod_{j=1}^{L}\exp\left(\sum_{i=1}^{d}\theta_{i,j}\varepsilon_i\right)\right)\right)$$
$$= \sum_{i=1}^{d}\varepsilon_i + \sum_{1\leqslant i<j\leqslant d}\tfrac{1}{2}[\varepsilon_i,\varepsilon_j] + \sum_{1\leqslant i<j\leqslant d}\tfrac{1}{12}([[\varepsilon_i,\varepsilon_j],\varepsilon_j] + [[\varepsilon_j,\varepsilon_i],\varepsilon_i]),$$

and such that, if we define the piecewise linear path ω with derivatives equal to $(L\theta_{1,j},\dots,L\theta_{d,j})$ in the interval

$$\left[\frac{j-1}{L},\frac{j}{L}\right],$$

we have

$$\forall i\in\{1,\dots,d\},\quad \int_0^1 \omega^i(t)\,\mathrm{d}t = \int_0^1 \omega^i(t)^2\,\mathrm{d}t = \tfrac{1}{2}.$$

Let

$$\Theta(\varepsilon_1,\varepsilon_2,\dots,\varepsilon_d) = \mathrm{e}^{\varepsilon_1}\mathrm{e}^{\varepsilon_2}\cdots\mathrm{e}^{\varepsilon_d}\mathrm{e}^{-\varepsilon_1}\mathrm{e}^{-\varepsilon_2}\cdots\mathrm{e}^{-\varepsilon_d}.$$

Then, using the Baker–Campbell–Hausdorff formula, we make the following propositions.

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Proposition 5.11. *Let*

$$\Phi(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_d) = \Theta(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_d)\Theta(-\varepsilon_1, -\varepsilon_2, \dots, -\varepsilon_d).$$

Then

$$\log(\Phi(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_d)) = 2 \sum_{1 \leqslant i < j \leqslant d} [\varepsilon_i, \varepsilon_j].$$

Let

$$f_\Phi : [0,1] \to \mathbb{R}^d, \qquad t \to (f_\Phi^1(t), \dots, f_\Phi^d(t)),$$

be a bounded variation path defined by

$$f_\Phi^i(t) = \begin{cases} 0, & 0 \leqslant t \leqslant \dfrac{i-1}{4d}, \\ 4d\left(t - \dfrac{i-1}{4d}\right), & \dfrac{i-1}{4d} \leqslant t \leqslant \dfrac{i}{4d}, \\ 1, & \dfrac{i}{4d} \leqslant t \leqslant \dfrac{d+i-1}{4d}, \\ 1 - 4d\left(t - \dfrac{d+i-1}{4d}\right), & \dfrac{d+i-1}{4d} \leqslant t \leqslant \dfrac{d+i}{4d}, \\ 0, & \dfrac{d+i}{4d} \leqslant t \leqslant \dfrac{1}{2}, \\ -f_\Phi^i(t - \tfrac{1}{2}), & \tfrac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

Then the Chen series of f_Φ is equal to $\Phi(a_1, a_2, \dots, a_d)$. Moreover, for all $i = 1, \dots, d$,

$$\int_0^1 f_\Phi^i(t)\,\mathrm{d}t = 0 \quad \text{and} \quad \int_0^1 (f_\Phi^i(t))^2\,\mathrm{d}t = \frac{1}{2} - \frac{1}{6d}.$$

Proposition 5.12. *Let*

$$\begin{aligned}\Psi(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_d) = {} & \Theta(-\varepsilon_1, \dots, -\varepsilon_d)\Theta(2\varepsilon_d, \dots, 2\varepsilon_1)\Theta(-\varepsilon_1, \dots, -\varepsilon_d) \\ & \times \Theta(-\varepsilon_d, \dots, -\varepsilon_1)\Theta(2\varepsilon_1, \dots, 2\varepsilon_d)\Theta(-\varepsilon_d, \dots, -\varepsilon_1).\end{aligned}$$

Then

$$\pi_{\mathcal{U}_1}(\log(\Psi(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_d))) = \sum_{1 \leqslant i < j \leqslant d} ([\varepsilon_i, [\varepsilon_i, \varepsilon_j]] + [\varepsilon_j, [\varepsilon_j, \varepsilon_i]]).$$

Let

$$f_\Psi : [0,1] \to \mathbb{R}^d, \qquad t \to (f_\Psi^1(t), \dots, f_\Psi^d(t)),$$

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be a bounded variation path defined by

$$f^i_\Psi(t) = \begin{cases} 0, & 0 \leqslant t \leqslant \dfrac{i-1}{8d}, \\ -8d\Big(t - \dfrac{i-1}{8d}\Big), & \dfrac{i-1}{8d} \leqslant t \leqslant \dfrac{i}{8d}, \\ -1, & \dfrac{i}{8d} \leqslant t \leqslant \dfrac{d+2(i-1)}{8d}, \\ -1 + 8d\Big(t - \dfrac{d+2(i-1)}{8d}\Big), & \dfrac{d+2(i-1)}{8d} \leqslant t \leqslant \dfrac{d+2i}{8d}, \\ 1, & \dfrac{d+2i}{8d} \leqslant t \leqslant \dfrac{3d+i-1}{8d}, \\ 1 - 8d\Big(t - \dfrac{3d+i-1}{8d}\Big), & \dfrac{3d+i-1}{8d} \leqslant t \leqslant \dfrac{3d+i}{8d}, \\ 0, & \dfrac{3d+i}{8d} \leqslant t \leqslant \tfrac{1}{2}, \\ f^{d+1-i}_\Psi(t-\tfrac{1}{2}), & \tfrac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

Then the Chen series of f_Ψ is equal to $\Psi(a_1, a_2, \dots, a_d)$. Moreover, for all $i = 1, \dots, d$,

$$\int_0^1 f^i_\Psi(t)\,\mathrm{d}t = 0 \quad \textit{and} \quad \int_0^1 (f^i_\Psi(t))^2\,\mathrm{d}t = \frac{3}{4} - \frac{5}{12d}.$$

A final use of the Baker–Campbell–Hausdorff formula gives us the following lemma.

Lemma 5.13. *For all ν,*

$$\begin{aligned} \pi_{\mathcal{U}_1}\Bigg(\log\Bigg(\exp\Bigg(-\nu\sum_{i=1}^d \varepsilon_i\Bigg)\Phi(\tfrac{1}{2}\varepsilon_1, \dots, \tfrac{1}{2}\varepsilon_d)\exp\Bigg((1+\nu)\sum_{i=1}^d \varepsilon_i\Bigg) \\ \times \Psi\Bigg(\frac{\varepsilon_1}{12^{1/3}}, \dots, \frac{\varepsilon_d}{12^{1/3}}\Bigg)\exp\Bigg(-\nu\sum_{i=1}^d \varepsilon_i\Bigg)\Bigg)\Bigg) \\ = \sum_{i=1}^d \varepsilon_i + \sum_{1\leqslant i<j\leqslant d} \tfrac{1}{2}[\varepsilon_i, \varepsilon_j] + \sum_{1\leqslant i<j\leqslant d} \tfrac{1}{12}([[\varepsilon_i, \varepsilon_j], \varepsilon_j] + [[\varepsilon_j, \varepsilon_i], \varepsilon_i]). \end{aligned} \tag{5.6}$$

Let

$$\left.\begin{aligned} f_\nu : [0,1] \to \mathbb{R}, \\ t \mapsto \begin{cases} -4\nu t, & 0 \leqslant t \leqslant \tfrac{1}{4}, \\ -\nu + (4\nu+2)(t - \tfrac{1}{4}), & \tfrac{1}{4} \leqslant t \leqslant \tfrac{3}{4}, \\ 1 + \nu - 4\nu(t - \tfrac{3}{4}), & \tfrac{3}{4} \leqslant t \leqslant 1, \end{cases} \end{aligned}\right\} \tag{5.7}$$

where ν satisfies the quadratic equation (with positive discriminant)

$$\nu^2(16 + 48d) + \nu(48d + 20) + d(18\kappa^2 + 3) - \frac{10}{12^{2/3}} - 5 = 0.$$

Then, for all $i = 1, \dots, d$,

$$\int_0^1 (\omega^i_{x,y,\nu}(t))^2\,\mathrm{d}t = \tfrac{1}{2}.$$

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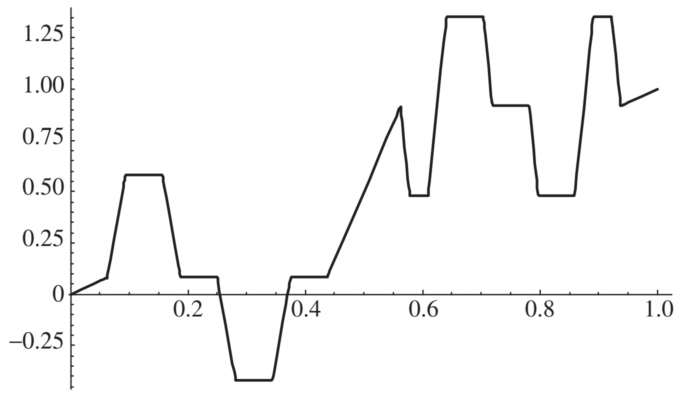

Figure 3. The graph of $\omega^{[3],1}_{x,y,\nu}$.

Proof. The value of the Chen series of $(\omega^1_{x,y,\nu},\dots,\omega^d_{x,y,\nu})$ is a consequence of the multiplicativity of the Chen series:

$$\int_0^1 \omega^i_{x,y,\nu}(t)\,\mathrm{d}t = 4x\int_0^1 f_\nu - \nu y + (1+\nu)y = y + 2x = \tfrac{1}{2}.$$

Basic calculus can be used to show that $\int_0^1(\omega^i_{x,y,\nu}(t))^2\,\mathrm{d}t$ is equal to

$$4x\left(\frac{4\nu^2+5\nu+5}{12}\right) + y\left(\frac{3}{4.12^{2/3}} - \frac{5}{12^{5/3}d} + \frac{1}{8} - \frac{1}{24d} + (1+\nu)^2 + \nu^2\right).$$

Using the values of x, y, we obtain that $\int_0^1(\omega^i_{x,y,\nu}(t))^2\,\mathrm{d}t$ is equal to

$$\frac{1}{1+d}\left(\frac{4\nu^2+5\nu+5}{12} + d\left(\frac{3}{8.12^{2/3}} + \frac{9}{16} + \nu + \nu^2\right) - \frac{1}{48} - \frac{5}{2.12^{5/3}}\right).$$

Hence,

$$\int_0^1 (\omega^i_{x,y,\nu}(t))^2\,\mathrm{d}t = \tfrac{1}{2}$$

if and only if

$$\frac{1}{1+d}\left(\frac{4\nu^2+5\nu+5}{12} + d\left(\frac{3}{8.12^{2/3}} + \frac{9}{16} + \nu + \nu^2\right) - \frac{1}{48} - \frac{5}{2.12^{5/3}} - \frac{d+1}{2}\right) = 0.$$

Multiplying by $48(1+d)$ and rearranging, we get

$$\nu^2(16+48d) + \nu(48d+20) + d(18\kappa^2+3) - \frac{10}{12^{2/3}} - 5 = 0.$$

■

Define $\omega^{[d]}$ as the path $t \mapsto (t, \omega^{[d],1}_{x,y,\nu}(t),\dots,\omega^{[d],d}_{x,y,\nu}(t))$ with the above values of x, y, ν (figure 3). We immediately obtain the following corollary.

Corollary 5.14. *The natural projection on $\mathcal{U}_1$ of the log of the Chen series of the path $\omega^{[d]}$ is equal to (5.5).*

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Appendix A. Proof of Tchakaloff's theorem for the Wiener space

We give here a proof of theorem 2.4. We understand by Chen's theorem that it is equivalent to show the following.

Theorem A 1. *Let m be a natural number. One can then find n Lie polynomials $\mathcal{L}_1, \dots, \mathcal{L}_n \in \mathcal{U}^{(m)}$ and n positive weights $\lambda_1, \dots, \lambda_n$, with $n \leqslant \operatorname{card} \mathcal{A}_m$, such that these Lie polynomials and weights define a cubature formula on Wiener space of degree m, i.e. such that*

$$E(\boldsymbol{X}_{0,1}^{(m)}(\circ B)) = \sum_{j=1}^{n} \lambda_j \pi_m(\exp(\mathcal{L}_j)).$$

Proof. By remark 4.6, $\boldsymbol{X}_{0,1}^{(m)}(\circ B) = \pi_m(\exp(\mathcal{L}))$, where $\mathcal{L}$ is obviously random. We define the real-valued random variables X_ℓ by

$$\pi_m(\mathcal{L}) = \sum_{\ell \in \mathcal{B}_{\mathcal{U}} \cap \mathcal{U}^{(m)}} X_\ell \ell$$

and let

$$X = (X_\ell)_{\ell \in \mathcal{B}_{\mathcal{U}} \cap \mathcal{U}^{(m)}}.$$

Note that $X \in \mathbb{R}^{\operatorname{card} \mathcal{B}_{\mathcal{U}} \cap \mathcal{U}^{(m)}}$. By Tchakaloff's theorem, one can find n elements

$$(\tilde{X}_1 = (\tilde{X}_{1,\ell})_{\ell \in \mathcal{B}_{\mathcal{U}} \cap \mathcal{U}^{(m)}}, \dots, \tilde{X}_n = (\tilde{X}_{n,\ell})_{\ell \in \mathcal{B}_{\mathcal{U}} \cap \mathcal{U}^{(m)}})$$

of $\mathbb{R}^{\operatorname{card} \mathcal{B}_{\mathcal{U}} \cap \mathcal{U}^{(m)}}$ and n positive weights $\lambda_1, \dots, \lambda_n$ such that for all

$$(p_\ell)_{\ell \in \mathcal{B}_{\mathcal{U}} \cap \mathcal{U}^{(m)}} \in \mathbb{N}^{\mathcal{B}_{\mathcal{U}} \cap \mathcal{U}^{(m)}}$$

satisfying $\sum p_\ell \leqslant m$,

$$E\biggl(\prod_\ell X_\ell^{p_\ell}\biggr) = \sum_{j=1}^{n} \lambda_j \prod_\ell \tilde{X}_{j,\ell}^{p_\ell}.$$

We then define $\mathcal{L}_j = \sum_{\ell \in \mathcal{B}_{\mathcal{U}} \cap \mathcal{U}^{(m)}} \tilde{X}_{j,\ell} \ell$ for $j = 1, \dots, n$. It is straightforward to check that

$$E(\pi_m(\exp(\mathcal{L}))) = \sum_{j=1}^{n} \lambda_j \pi_m(\exp(\mathcal{L}_j)),$$

i.e. the n Lie polynomials $\mathcal{L}_1, \dots, \mathcal{L}_n \in \mathcal{U}^{(m)}$ and the n positive weights $\lambda_1, \dots, \lambda_n$ define a cubature formula on Wiener space of degree m.

We are now going to show that, if n is greater than $\dim U_m(\mathbb{R}, \mathbb{R}^d) = \operatorname{card} \mathcal{A}_m$, then we can find $n' < n$ Lie polynomials in $\{\mathcal{L}_1, \dots, \mathcal{L}_n\}$ and n' new weights that defines a cubature formula on Wiener space of degree m. Reiterating the argument will show that there exists a cubature formula of degree m with no more than $\operatorname{card} \mathcal{A}_m$ points. So assume that $n > \operatorname{card} \mathcal{A}_m$. The n vectors $\pi_m(\exp(\mathcal{L}_1)), \dots, \pi_m(\exp(\mathcal{L}_n))$ are in the vector space $U_m(\mathbb{R}, \mathbb{R}^d)$. As n is greater than the dimension of the vector space that they lie in, they are not independent, i.e. there exists $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ such that

$$\sum_{i=1}^{n} \alpha_i \exp(\mathcal{L}_i) = 0.$$

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With no loss of generality, we assume that $|\lambda_i/\alpha_i|$ attains its minimum for $i = n$. We can also assume that α_n is positive (otherwise we replace all the α_i by $-\alpha_i$). Then

$$E(\pi_m(\exp(\mathcal{L}))) = \pi_m\left(\sum_{i=1}^{n} \lambda_i \exp(\mathcal{L}_i) - \frac{\lambda_n}{\alpha_n} \sum_{i=1}^{n} \alpha_i \exp(\mathcal{L}_i)\right)$$
$$= \pi_m\left(\sum_{i=1}^{n-1} \left(\lambda_i - \frac{\lambda_n}{\alpha_n}\alpha_i\right) \exp(\mathcal{L}_i)\right),$$

i.e. the Lie polynomials $\mathcal{L}_1, \ldots, \mathcal{L}_{n-1} \in \mathcal{U}^{(m)}$ and the $n-1$ non-negative weights

$$\lambda_1 - \frac{\lambda_n}{\alpha_n}\alpha_1, \ldots, \lambda_{n-1} - \frac{\lambda_n}{\alpha_n}\alpha_{n-1}$$

define a cubature formula on Wiener space of degree m. ∎

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