OPTIMAL CONTROL OF STOCHASTIC INTEGRALS AND HAMILTON–JACOBI–BELLMAN EQUATIONS. II*

PIERRE-LOUIS LIONS† AND JOSÉ-LUIS MENALDI‡

Abstract. We consider the solution of a stochastic integral control problem, and we study its regularity. In particular, we characterize the optimal cost as the maximum solution of

$$\forall v \in V, \quad A(v)u \leqq f(v) \quad \text{in } \mathscr{D}'(\mathcal{O}),$$

$$u = 0 \quad \text{on } \partial\mathcal{O}, \qquad u \in W^{1,\infty}(\mathcal{O}),$$

where $A(v)$ is a uniformly elliptic second order operator and V is the set of the values of the control.

1. Introduction

1.1. General introduction. In this paper, we extend the results of part I [14] (this Journal, this issue, pp. 58–81) to the degenerate case (see also [15]).

We consider a stochastic system governed by the stochastic differential equation

$$(1.1) \qquad \begin{aligned} dy(t) &= \sigma(y(t), v(t))\, dW_t + g(y(t), v(t))\, dt, \qquad t \geqq 0, \\ y(0) &= x \in \mathbb{R}^N, \end{aligned}$$

where W_t is a Wiener process, g and σ are given functions, and $v(t)$ is a "continuous" control taking values in some set $V \subset \mathbb{R}^m$.

We want to minimize the cost function (with notational change from part I)

$$(1.2) \qquad J_x(v) = E\left\{ \int_0^{\tau_x} f(y(t), v(t)) \exp\left(-\int_0^t c(y(s), v(s))\, ds \right) dt \right\}$$

over all admissible controls $v(t)$. In this formula f and c are given functions and τ_x is the first exit time of the process $y(t)$ from a given domain $\bar{\mathcal{O}}$. Let us denote

$$(1.3) \qquad u(x) = \inf \{ J_x(v)/v = v(\cdot) \text{ admissible control} \}$$

the optimal cost function.

In part I (see also [15]), under suitable assumptions containing an assumption of nondegeneracy,

$$(1.4) \qquad \sigma\sigma^*(x, v) \geqq \alpha > 0 \quad \forall x \in \bar{\mathcal{O}}, \quad \forall v \in V, ^1$$

we proved that the function $u(x)$ is the maximum element of the set of functions $\tilde{u}$ satisfying $\tilde{u} \in W_0^{1,\infty}(\mathcal{O})$ and

$$(1.5) \qquad A(v)\tilde{u} \leqq f(v) \quad \text{in } \mathscr{D}'(\mathcal{O}) \quad \forall v \in V,$$

where $A(v) = -\frac{1}{2} \operatorname{Tr} (\sigma\sigma^*(x, v)D^2) - g(x, v)D + c(x, v).^2$

We give here results where (1.4) is relaxed and where nevertheless this approach may still be carried out.

¹ σ^* denotes the adjoint of σ. The inequality has to be understood in the sense of symmetric matrices.

² D denotes the gradient operator (we will also use the notation ∇).

In view of the principles of dynamical programming, one could expect u to solve (in a convenient sense)

$$\sup_{v \in V} \{A(v)u - f(v)\} = 0 \quad \text{a.e. in } \mathcal{O}.$$

Results in this direction (with operators $A(v)$ eventually degenerate) are given in N. V. Krylov [5], [6], [7], M. V. Safonov [18], [19], P.-L. Lions [9], [10], [11], [12], [13] (in the nondegenerate case the most general results are given in L. C. Evans and P.-L. Lions [2], P.-L. Lions [8]).

But the counterexample of I. L. Genis and N. V. Krylov [3] shows that the equation may not be satisfied (even in the weakest sense); therefore it seems useful to have a different characterization of u. Our goal here is to propose as one such characterization the superior envelope of sub-solutions. We remark that some results in this direction, for the deterministic case, are given in R. Gonzalez [4].

1.2. Summary. The results are organized in the following way:
Section 2. The degenerate case.
Section 3. The Cauchy problem.
Section 4. The obstacle problem.
In § 2, using some techniques of N. V. Krylov [7] and [8] and M. Nisio [17], as in [14], we build a nonlinear semigroup whose generator is related to the operator appearing in (1.5). Next, we give a stochastic characterization of $u(x)$, which is the precise way to apply the dynamical programming argument. Finally, we prove a characterization of $u(x)$ in terms of the maximum subsolution.

In § 3, we briefly develop the parabolic case. In § 4, we consider the obstacle problem. The case without "continuous" control was studied in J.-L. Menaldi [16].

1.3. Assumptions and notation. We now give notation and assumptions which will remain valid in §§ 2, 3 and 4. Let $\mathcal{O}$ be a domain of $\mathbb{R}^N$ and let V be a convex closed set in $\mathbb{R}^m$. We call an *admissible system* a set $\mathcal{A} = (\Omega, F, F_t, P, W_t, v(t), y_x(t))$, where (Ω, F, P) is a probability space, F_t is a nondecreasing right-continuous family of complete sub σ-algebras of F, W_t is a Wiener process with respect to F_t, $v(t)$ is a measurable adapted process taking values in some compact subset V_0 of V (V_0, of course, may depend on $v(\cdot)$) and $y_x(t)$ is a solution of Ito's equation

$$(1.6) \quad \begin{aligned} dy_x(t) &= \sigma(y_x(t), v(t)) \, dW_t + g(y_x(t), v(t)) \, dt, \qquad t \geq 0, \\ y_x(0) &= x, \end{aligned}$$

where $\sigma(x, v)$ and $g(x, v)$ are uniformly continuous and bounded functions from $\mathbb{R}^N \times V$ into $\mathbb{R}^N \otimes \mathbb{R}^N$ and $\mathbb{R}^N$ respectively which are uniformly Lipschitz continuous in x. This regularity and boundedness assumption will not be recalled in what follows (and may be relaxed in some of the results which follow).

Now, for an admissible system $\mathcal{A}$, we define a cost function

$$(1.7) \quad \begin{aligned} J(x, \mathcal{A}, t, h) = E\Big\{ &\int_0^{t \wedge \tau_x} f(y_x(s), v(s)) \exp\left(-\int_0^s c(y_x(\lambda), v(\lambda)) \, d\lambda \right) ds \\ &+ h(y_x(t \wedge \tau_x)) \exp\left(-\int_0^{t \wedge \tau_x} c(y_x(s), v(s)) \, ds \right) \Big\}, \end{aligned}$$

where h is an arbitrary measurable bounded function, τ_x is the first exit time of the process $y_x(t)$ from $\bar{\mathcal{O}}$ and $f(x, v)$, $c(x, v)$ are given uniformly continuous and bounded functions from $\mathbb{R}^N \times V$ into $\mathbb{R}$, $\mathbb{R}_+$ respectively.

Finally, we define, for each h, an optimal cost function

$$(1.8) \qquad Q(t)h(x) = \inf_{\mathscr{A}} J(x, \mathscr{A}, t, h) \quad \forall 0 \leq t < \infty.$$

Let us collect our assumptions:

$$(1.9) \qquad \begin{aligned} |\phi(x, v) - \phi(x', v')| &\leq C|x - x'| + \rho(|v - v'|) \\ &\forall x, x' \in \mathbb{R}^N, \quad \forall v, v' \in V, \quad \forall \phi = \sigma_{ij}, g_i, c, f. \end{aligned}$$

$$(1.10) \qquad \begin{aligned} |\phi(x, v)| &\leq C \quad \forall x \in \mathbb{R}^N, \quad \forall v \in V, \quad \forall \phi = \sigma_{ij}, g_i, c, f, \\ c(x, v) &\geq c_0 > 0 \quad \forall x \in \mathbb{R}^N, \quad \forall v \in V, \end{aligned}$$

where ρ is a given continuous function from $\mathbb{R}_+$ into $\mathbb{R}_+$ such that $\rho(0) = 0$.

We shall denote by B_s the set of bounded functions from $\bar{\mathcal{O}}$ into $\mathbb{R}$ which are upper semicontinuous and by B_s^+ the subset of B_s of nonnegative functions. B_s and B_s^+ are closed convex cones of the Banach space B of bounded measurable functions equipped with the supremum norm ($\|h\|_\infty = \sup\{|h(x)| : x \in \bar{\mathcal{O}}\}$).

Throughout this paper, we use an assumption which will replace the nondegeneracy assumption (1.4). We suppose that there exists a subsolution which is Lipschitz continuous, i.e.,

$$(1.11) \qquad \begin{aligned} &\text{there exists a } \bar{u} \in W_0^{1,\infty}(\mathcal{O}) \text{ such that for all } v \in V \text{ we have for} \\ &\text{some } \varepsilon > 0 \; A(v)\bar{u} \leq -1 \text{ in } \mathscr{D}'(\mathcal{O}_\varepsilon), \bar{u} \leq -\alpha < 0 \text{ in } \mathcal{O} - \mathcal{O}_\varepsilon, \end{aligned}$$

where $\mathcal{O}_\varepsilon = \{x \in \mathcal{O}, \text{dist}(x, \Gamma) > \varepsilon\}$, and the operator $A(v)$ is defined by

$$(1.12) \qquad A(v) = -a_{ij}\frac{\partial^2}{\partial x_i \, \partial x_j} + b_i\frac{\partial}{\partial x_i} + c,^3$$

with $a_{ij}(x, v) = \frac{1}{2}\sigma_{ik}\sigma_{jk}(x, v)$, $b_i(x, v) = -g_i(x, v)$.

It is easy to prove, using barrier functions as in part I or [8], that if for some $\alpha > 0$ $\Gamma = \partial^{\mathcal{O}}$ satisfies (n is the unit exterior normal to Γ)

$$\Gamma = \{x \in \Gamma / \forall v \in v, |\sigma(x, v)n(x)| \geq \alpha\}$$

$$(1.13) \qquad \bigcup \{x \in \Gamma / v \in V, 2rg(x, v)n(x) > \text{Tr}[\sigma\sigma^*(x, v)] + \alpha\},$$

r is the radius of the uniform exterior sphere associated with $\mathcal{O}$,

and

$$(1.14) \qquad \mathcal{O} \text{ is bounded, regular (i.e., the exterior uniform sphere property holds),}$$

then assumption (1.11) is satisfied for c_0 large enough.

We may also replace (1.11) by

$$(1.15) \qquad f(x, v) \geq 0 \quad \forall x \in \mathbb{R}^N, \quad \forall v \in V,$$

$$(1.16) \qquad \text{there exists } v(x) \text{ continuous on } \bar{\mathcal{O}} \text{ such that (1.13) is satisfied for } v = v(x).$$

2. Degenerate case. This section is divided into three sections. First, we study the nonlinear semigroup $Q(t)$. Next, we give a stochastic interpretation of the optimal cost. Finally, we establish an analytical interpretation.

---

[3] We will always use the usual convention for sums.

2.1. Nonlinear semigroup. In this section, we first prove that $Q(t)$ acting on B_s or B_s^+ is a nonlinear semigroup. Next we consider the generator of $Q(t)$.

THEOREM 2.1. *Assume* (1.9), (1.10) *and* (1.15). *Then* $(Q(t), t \geqq 0)$ *satisfies*

$$(2.1) \qquad Q(t) : B_s^+ \to B_s^+, \quad Q(0) = I, \quad Q(t+s) = Q(t) \circ Q(s) = Q(s) \circ Q(t),$$

$$(2.2) \qquad \|Q(t)h - Q(s)h\|_\infty \to 0 \quad \text{as } t \to s \quad \text{if } h \text{ is uniformly continuous in } \mathbb{R}^N,$$

$$(2.3) \qquad \|Q(t)h_1 - Q(s)h_2\|_\infty \leqq \|h_1 - h_2\|_\infty \quad \forall h_1, h_2 \in B_s^+, \quad \forall t \geqq 0,$$

$$(2.4) \qquad Q(t)h_1 \leqq Q(t)h_2 \quad \text{if } h_1 \leqq h_2.$$

Proof. We penalize the domain $\mathcal{O}$. Let $p(x)$ be the distance to $\mathcal{O}$, i.e.,

$$(2.5) \qquad p(x) = \inf \{|y - x| : y \in \mathcal{O}\},$$

and consider the following operator ($\varepsilon > 0$):

$$
\begin{aligned}
(2.6) \quad & Q^\varepsilon(t)h(x) \\
& = \inf E \Bigg\{ \int_0^t f(y_x(s), v(s)) \exp \left(- \int_0^s \left(c(y_x(\lambda), v(\lambda)) + \frac{1}{\varepsilon} p(y_x(\lambda)) \right) d\lambda \right) ds \\
& \qquad + h(y_x(t)) \exp \left(- \int_0^t \left(c(y_x(s), v(s)) + \frac{1}{\varepsilon} p(y_x(s)) \right) ds \right) \Bigg\}.
\end{aligned}
$$

Clearly, $Q^\varepsilon(t)$ leaves invariant the space $C_b(\mathbb{R}^N)$ of continuous and bounded functions. From Theorem 2.1 in part I, we obtain that $Q^\varepsilon(t)$ satisfies (2.1)–(2.4).

Finally, using the fact that, for all $t \geqq 0$, for $x \in \mathbb{R}^N$ and $h \in B_s^+$,

$$(2.7) \qquad Q^\varepsilon(t)h(x) \to Q(t)h(x) \quad \text{decreasing as } \varepsilon \downarrow 0,[4]$$

it is easy to conclude. $\square$

Remark 2.1. Under assumptions (1.9), (1.10) and (1.11), the semigroup $(Q(t), t \geqq 0)$ satisfies (2.1)–(2.4) with B_s instead of B_s^+. Indeed, we need to observe that, defining

$$(2.8) \qquad \Gamma_0(\mathcal{A}) = \{x \in \Gamma / P(\tau_x > 0) = 0\},$$

we deduce from (1.11) (using a lemma of [8])

$$
\begin{aligned}
(2.9) \quad & \Gamma_0(\mathcal{A}) = \Gamma \quad \forall \mathcal{A} \text{ admissible systems,} \\
& P(y_x(\tau_x) \in \Gamma_0(\mathcal{A}) \text{ if } \tau_x < \infty) = 1 \quad \forall x \in \bar{\mathcal{O}},
\end{aligned}
$$

so we can use Theorem 2.1 in part I.

We set

$$(2.10) \qquad X = \{h \in C_b(\bar{\mathcal{O}}), \ h \text{ is uniformly continuous}\}.$$

We have the following:

THEOREM 2.2. *If we assume* (1.9), (1.10) *and* (1.11), *then for each* $h \in X$ $Q(t)h \in X$. *Furthermore* $(Q(t)h, t \geqq 0)$ *is uniformly equicontinuous.*

Proof. We first consider the case where $c(x, v)$ satisfy

$$(2.11) \qquad c(x, v) \geqq c_0 > \mu_0^+ \quad \forall x \in \mathcal{O}, \quad \forall c \in V,$$

---

[4] We extend h by zero outside $\bar{\mathcal{O}}$.

where μ_0 is given by

$$\mu_0 = \sup \left\{ \frac{1}{2} \mathrm{Tr} \left[\frac{(\sigma(x, v) - \sigma(x', v))(\sigma(x, v) - \sigma(x', v))^*}{|x - x'|^2} \right] \right.$$

(2.12)

$$\left. + \frac{(x - x') \cdot (g(x, v) - g(x', v))}{|x - x'|^2} \middle/ x, x' \in \mathcal{O}, v \in V \right\}.$$

By a density argument, it is enough to prove Theorem 2.2 for smooth $f(x, v)$ and $h(x)$. By the same argument as in part I, we only have to prove that $u(t, x) = Q(t)0 \in X$ (and is uniformly equicontinuous). Let us assume that under assumption (2.11) we have proved that $|u(t, x) - u(t, x')| \leq C|x - x'|$, for all x, x' in $\bar{\mathcal{O}}$ and all $t \geq 0$. We conclude remarking that, using the dynamical programming property as in [14], we have

$$u(t, x) = \inf_{\mathcal{A}} E \left\{ \int_0^{t \wedge \tau_x} [f(y_x(s), v(s)) + k u(s, y_x(s))] \right.$$

(2.13)

$$\left. \cdot \exp \left(-\int_0^s c(y_x(\lambda), v(\lambda)) \, d\lambda - ks \right) ds \right\},$$

for all $k \geq 0$.

Thus $u(t, x)$ is a fixed point of the mapping which transforms $u(s, x)$ into the right-hand side of (2.13); but we have

$$Tu \in W^{1, \infty} \quad \text{if } u \in W^{1, \infty},$$

(2.14)

$$\|Tu - Tw\|_\infty \leq \frac{k}{c_0 + k} \|u - w\|_\infty,$$

choosing k large enough. This proves Theorem 2.2.

Now, there just remains to prove that under assumption (2.11), we have $|u(t, x) - u(t, x')| \leq C|x - x'|$. We first remark that in view of the arguments given in part I, if $c_0 > \mu_0$, then

$$E \left[\left| y_x(\theta) \exp \left(-\int_0^\theta c(y_x(s), v(s)) \, ds \right) \right. \right.$$

$$\left. \left. - y_{x'}(\theta) \exp \left(-\int_0^\theta c(y_{x'}(s), v(s)) \, ds \right) \right| \right] \leq C|x - x'|,$$

for all x, x' in $\bar{\mathcal{O}}$, and all stopping times θ.

Let us assume for the moment that u satisfies

(2.15) $|u(t, x)| \leq C|\bar{u}(x)| \quad \forall x \text{ in } \bar{\mathcal{O}}.$

Then, using the equation of dynamical programming as in the proof of Theorem 3.1 in part I, it is easy to deduce

$$|u(t, x) - u(t, x')| \leq CE \left[\left| \bar{u}(y_x(\tau_x \wedge \tau_{x'})) \exp \left(-\int_0^{\tau_x \wedge \tau_{x'}} c(y_x(s), v(s)) \, ds \right) \right. \right.$$

(2.16)

$$\left. \left. - \bar{u}(y_{x'}(\tau_x \wedge \tau_{x'})) \exp \left(-\int_0^{\tau_x \wedge \tau_{x'}} c(y_x'(s), v(s)) \, ds \right) \right| \right]$$

$$\leq C\|\nabla \bar{u}\|_{L^\infty(\mathcal{O})} |x - x'| + C|x - x'|,$$

using the inequality above (see also part I); this argument will be detailed further on.

Now to prove (2.15), we argue as follows. As in part I, we denote $u_t(s, x) = Q(t-s)0$ (where t is fixed). Using the dynamical programming property, we obtain (see also [14])

$$u(t, x) = \inf_{\mathscr{A}} E\left\{ \int_0^{\tau_x \wedge \tau_x^\varepsilon \wedge \tau} f(y_x(s), v(s)) \exp\left(-\int_0^s c(y_x(\lambda), v(\lambda))\, d\lambda\right) ds \right.$$

$$\left. + u_t(\tau_x^\varepsilon, y_x(\tau_x^\varepsilon)) \exp\left(-\int_0^{\tau_x^\varepsilon} c(y_x(s), v(s))\, ds\right) 1_{(\tau_x^\varepsilon < t)} \right\},$$

where τ_x^ε is the first time $y_x(t)$ reaches $\mathcal{O} - \mathcal{O}_\varepsilon$.

For all x in $\mathcal{O}_\varepsilon$, we deduce that

$$|u(t, x)| \leqq \sup_{\mathscr{A}} E\left\{ \int_0^{\tau_x \wedge \tau_x^\varepsilon \wedge t} |f(y_x(s), v(s))| \exp\left(-\int_0^s c(y_x(\lambda), v(\lambda))\, d\lambda\right) ds \right.$$

$$\left. + C(-\bar{u}(y_x(\tau_x^\varepsilon))) \exp\left(-\int_0^{\tau_x^\varepsilon} c(y_x(s), v(s))\, ds\right) 1_{(\tau_x^\varepsilon < t)} \right\}.$$

Now, using a method due to [13], we deduce from (1.13) that

$$(-\bar{u}(x)) \geqq \sup_{\mathscr{A}} E\left[\int_0^{\tau_x^\varepsilon \wedge t} \exp\left(-\int_0^t c(y_x(\lambda), v(\lambda))\, d\lambda\right) ds \right.$$

$$\left. + (-\bar{u}(y_x(t_x^\varepsilon))) \exp\left(-\int_0^{\tau_x^\varepsilon} c(y_x(s), v(s))\, ds\right) 1_{(\tau_x^\varepsilon < t)} \right];$$

thus, if we choose C large enough, (2.15) is proved.

For the generator of $Q(t)$ we have

THEOREM 2.3. *Under assumptions* (1.9), (1.10) *and* (1.11) *(or* (1.15)*), we have for any* $h \in C^2(\mathcal{O})$

$$(2.17) \qquad \frac{1}{t}[Q(t)h(x) - h(x)] \to -\sup_{v \in V} \{A(v)h(x) - f(v, x)\} \quad \text{as } t \to 0_+, \quad \forall x \in \mathcal{O}.$$

Moreover, the convergence in (2.17) *is uniform on compact subsets of* $\mathcal{O}$.
Proof. It is similar to that of Theorem 2.2. in part I. □

2. Stochastic interpretation. Let us consider the optimal cost function

$$(2.18) \qquad u(x) = \inf_{\mathscr{A}} E\left\{ \int_0^{\tau_x} f(y_x(t), v(t)) \exp\left(-\int_0^t c(y_x(s), v(s))\, ds\right) dt \right\}.$$

We set

$$(2.19) \qquad \Gamma_0 = \{x \in \Gamma / \exists \mathscr{A} \text{ admissible such that } P(\tau_x > 0) = 0\}.$$

Remark that if we assume (1.11), then $\Gamma_0 = \Gamma$. We have the following:

THEOREM 2.4. *Under assumptions* (1.9), (1.10) *and* (1.15) *(resp.* (1.11)*) the function* $u(x)$ *defined by* (2.18) *is the unique solution of the problem*

$$(2.20) \qquad u \in B_s^+ \text{ (resp. } B_s\text{)}, \quad u|_\Gamma = 0, \quad Q(t)u = u, \quad t \geqq 0.$$

Moreover, the equation of dynamical programming is satisfied:

$$
u(x) = \inf_{\mathscr{A}} E\left\{ \int_0^{\theta \wedge \tau_x} f(y_x(t), v(t)) \exp\left(-\int_0^t c(y_x(s), v(s))\, ds\right) dt \right.
$$

(2.21)

$$
\left. + u(y_x(\theta \wedge \tau_x)) \exp\left(-\int_0^{\theta \wedge \tau_x} c(y_x(t), v(t))\, dt\right) \right\}
$$

where θ is an arbitrary stopping time.

Furthermore, if we suppose (1.11) and (2.11), the optimal cost u defined by (2.18) belongs to $W_0^{1,\infty}(\mathcal{O})$.

Remark 2.3. Equations (2.20), (2.21) show that the optimal cost function $u(x)$ satisfies in some integral sense the Hamilton–Jacobi–Bellman equation: $\sup_{v \in V}\{A(v)u - f(v)\} = 0$ in $\mathcal{O}$.

Proof. The proof of the first part is similar to that of Theorem 3.1 in part I. We will prove that under assumptions. (1.13) and (2.11), u belongs to $W_0^{1,\infty}(\mathcal{O})$. To simplify notation we assume $c(x, v) \equiv c_0$.

We first prove that there exists some constant $C > 0$ such that

$$
|u(x)| \leqq C|-\bar{u}(x)| \quad \forall x \in \mathcal{O}.
$$

Indeed, if we choose C large enough, this inequality is obvious if $x \in \mathcal{O} - \mathcal{O}_\varepsilon$. Now if $x \in \mathcal{O}_\varepsilon$, writing (2.21) with $\theta = \tau_x^\varepsilon$ where τ_x^ε is the first time $y_x(t)$ reaches Γ_ε we deduce

$$
|u(x)| \leqq \sup_{\mathscr{A}} E\left[\int_0^{\tau_x^\varepsilon \wedge \tau_x} C e^{-c_0 t}\, dt + C(-\bar{u}(y_x(\tau_x^\varepsilon))) 1_{(\tau_x^\varepsilon < \tau_x)}\, e^{-c_0 \tau_x^\varepsilon} \right].
$$

Now, using (1.13), we have

$$
-\bar{u}(x) \geqq \sup_{\mathscr{A}} E\left[\int_0^{\tau_x^\varepsilon \wedge \tau_x} e^{-c_0 t}\, dt + (-\bar{u}(y_x(\tau_x^\varepsilon))) 1_{(\tau_x^\varepsilon < \tau_x)}\, e^{-c_0 \tau_x^\varepsilon} \right],
$$

and we conclude.

We are now able to prove that $u \in W_0^{1,\infty}(\mathcal{O})$. Let $x, x' \in \mathcal{O}$; we have, using (2.21) with $\theta = \tau_{x'} \wedge \tau_x$,

$$
|u(x) - u(x')| \leqq \sup_{\mathscr{A}} CE\left\{ \int_0^\infty |y_x(t) - y_{x'}(t)|\, e^{-c_0 t}\, dt \right\}
$$

$$
+ \sup_{\mathscr{A}} E\{|u(y_x(\tau_x \wedge \tau_{x'})) - u(y_{x'}(\tau_x \wedge \tau_{x'}))|\, e^{-c_0 \tau_x \wedge \tau_{x'}}\}.
$$

Because of (2.11) the first term is bounded by $|x - x'|$, while the second term is bounded by

$$
C \sup_{\mathscr{A}} [1_{(\tau_x \leqq \tau_{x'})}|\bar{u}(y_{x'}(\tau_x))|\, e^{-c_0 \tau_x} + 1_{(\tau_x > \tau_{x'})}|\bar{u}(y_x(\tau_{x'}))|\, e^{-c_0 \tau_{x'}}].
$$

Since $\bar{u} \in W_0^{1,\infty}(\mathcal{O})$, this quantity is less than

$$
C \sup_{\mathscr{A}} E[|y_x(\tau_x \wedge \tau_{x'}) - y_{x'}(\tau_x \wedge \tau_{x'})|\, e^{-c_0 \tau_x \wedge \tau_{x'}}] \leqq C|x - x'|,
$$

and the theorem is proved. □

We also have

THEOREM 2.5. *Assume* (1.9), (1.10), (1.15), (1.16) *and that* $\inf_{x,v} c(x, v)$ *is large enough. Then the optimal cost function u given by (2.18) belongs to* $W_0^{1,\infty}(\mathcal{O})$.

Proof. From (1.18), we may define $\bar{u}(x)$ by

$$(2.22) \qquad \bar{u}(x) = E\left\{\int_0^{\tau_x} \|f\|_\infty \, e^{-c_0 t} \, dt\right\},$$

where the admissible system considered $\mathcal{A}$ is given by the feedback $v(x)$ appearing in (1.16), which by a density argument may be assumed to be Lipschitz continuous.

Using barrier functions as in [14] or [18], it is easy to prove that $C\bar{u}(x) \leqq C \, \text{dist}\,(x, \partial \mathcal{O})$. Then, if c_0 is large enough, this implies by a proof similar to that of Theorem 2.4 that $\bar{u} \in W_0^{1,\infty}(\mathcal{O})$. We have

$$(2.23) \qquad 0 \leqq u(x) \leqq \bar{u}(x).$$

Next, using dynamical programming, we have

$$u(x) - u(x') \leqq C|x - x'| + \sup_{\mathcal{A}} E\{|u(y_x(\tau_x \wedge \tau_{x'})) - u(y_{x'}(\tau_x \wedge \tau_{x'}))| \, e^{-c_0 \tau_x \wedge \tau_{x'}}\}.$$

Hence, (2.23) gives (since $u = \bar{u}$ on $\partial \mathcal{O}$)

$$(2.24) \qquad u(x) - u(x') \leqq C|x - x'| + \sup_{\mathcal{A}} E\{|\bar{u}(y_x(\tau_{x'} \wedge \tau_x)) - \bar{u}(y_{x'}(\tau_x \wedge \tau_{x'}))| \, e^{-c_0(\tau_x \wedge \tau_{x'})}\}$$

and since $\bar{u} \in W_0^{1,\infty}(\mathcal{O})$, we deduce the result. $\quad \square$

COROLLARY 2.1. *Assume* (1.9), (1.10), (1.15) *and* (1.16). *Then the optimal cost u given by* (2.18) *is uniformly continuous in $\bar{\mathcal{O}}$. Moreover, for each $h \in X$ (given by* (2.10)) $Q(t)h \in X$, *so $Q(t)$ is a semigroup acting on X.*

Proof. It is similar to that of Theorem 2.2. $\quad \square$

Remark 2.4. Clearly, under assumptions (1.15) and (1.16), we have $\Gamma_0 = \Gamma$.

Remark 2.5. Using Theorem 2.4, we can prove a local version of Theorem 2.3 as in part I.

2.3. Analytical interpretation.

In all of what follows, u will be the optimal cost function defined by (2.18). We have already seen that, under some assumptions, u belongs to $W^{1,\infty}(\mathcal{O})$. Then, we are able to show that u is the maximum subsolution of (1.5), and that is u is the envelope (sup) of all w in $W_0^{1,\infty}(\mathcal{O})$ satisfying

$$(2.25) \qquad A(v)w \leqq f(v) \quad \text{in } \mathcal{D}'(\mathcal{O}).$$

This result may be viewed as a notion of a generalized solution of (1.5) (as is done for Monge–Ampère equations). We thus give the following result (generalizing our previous one in part I).

Throughout this section we assume

$$(2.26) \qquad \psi(\cdot, v) \in W^{2,\infty}(\mathcal{O}), \text{ and } \psi(\cdot, v) \text{ remains bounded in } W^{2,\infty}(\mathcal{O})$$
$$\text{as } v \in V \text{ for all } \psi = \sigma_{ij}, \, b_i, \, c, \, f.$$

THEOREM 2.6. *Assume* (1.9), (1.10), (2.26) *and* (1.11) (*or* (1.15)). *Then, for all w satisfying $w \in W_{\text{loc}}^{1,\infty}(\mathcal{O}) \cap C(\bar{\mathcal{O}})$, $w|_\Gamma \leqq 0$ and $A(v)w \leqq f(v)$ in $\mathcal{D}'(\mathcal{O})$, for all v in V, we have $w \leqq u$ in $\bar{\mathcal{O}}$.*

COROLLARY 2.2. *Assume* (1.9), (1.10) *and either* (1.11) *and* (2.11) *or* (1.15), (1.16) *and c_0 large enough. Then u is the maximum element of the set of functions w satisfying $w \in W_{\text{loc}}^{1,\infty} \cap C(\bar{\mathcal{O}})$, $w|_\Gamma \leqq 0$ and*

$$A(v)w \leqq f(v) \quad \text{in } \mathcal{D}'(\mathcal{O}), \quad \forall v \text{ in } V.$$

Remark 2.6. If we assume that $\mathcal{O}$ is regular and (1.13), (1.14) hold, Theorem 2.6 is still valid.

Proof of Theorem 2.6. The proof of Theorem 2.6 is very similar to the one given in part I, provided we use a lemma due to [9]. Indeed, if w satisfies the conditions listed in the above theorem, we have (using part I and [9])

$$w(x) \leq \inf_{\mathscr{A}} E\left[\int_0^{\tau_x^h} f(y_x(t), v(t)) \exp\left(-\int_0^t c(y_x(s), v(s)) \, ds\right) dt\right.$$

$$\left. + w(y_x(\tau_x^h)) \exp\left(-\int_0^{\tau_x^h} c(y_x(t), v(t)) \, dt\right)\right],$$

where τ_x^h is the first exit time of $\mathcal{O}^h = \{x \in \mathcal{O}, \text{dist}(x, \partial\mathcal{O}) \geq h\}$.

Then, if we take $h \to 0$, $\tau_x^h \uparrow \sigma_x$, where σ_x is the first exit time of $\mathcal{O}$. Thus,

$$w(x) \leq \inf_{\mathscr{A}} E\left[\int_0^{\sigma_x} f(y_x(t), v(t)) \exp\left(-\int_0^t c(y_x(s), v(s)) \, ds\right) dt\right].$$

Now, if we assume (1.11), $\sigma_x = \tau_x$ a.s. and we conclude.

On the other hand, if we assume (1.15), as $\sigma_x \leq \tau_x$ by definition, we also deduce $w \leq u$. □

Corollary 2.2 is immediately deduced from Theorem 2.6 as in part I. □

3. The Cauchy problem. We now consider the optimal control of time-dependent diffusions (or solutions of stochastic differential equations). We consider coefficients $\sigma_{ij}(x, t, v), b_i(x, t, v), c(x, t, v), f(x, t, v)$ which, for the sake of simplicity, will be assumed to belong to $W^{2,1,\infty}(\mathcal{O} \times]0, T[)$ for some $T > 0$, and for all v in V. In addition $\phi(x, t, v)$ remains bounded in $W^{2,1,\infty}(\mathcal{O} \times]0, T[)$ as $v \in V$, and $\phi(x, t, v)$ is continuous in $v \in V$ uniformly in $(x, t) \in \bar{\mathcal{O}} \times [0, T]$. These assumptions may be considerably relaxed but we will not consider such generalizations here.

We will denote $Q = \mathcal{O} \times]0, T[$. We define the optimal cost function

(3.1)
$$u(t, x) = \inf_{\mathscr{A}} E\left[\int_t^{T \wedge \tau_{x,t}} f(y_{x,t}(s), s, v(s)) \exp\left(-\int_t^s c(y_{x,y}(\lambda), \lambda, v(\lambda)) \, d\lambda\right) ds\right.$$

$$\left. + u_0(y_{x,t}(T)) \exp\left(-\int_t^T c(y_{x,t})(s), s, v(s) \, ds\right) 1_{(T < \tau_{x,t})}\right],$$

where the infimum is taken over all admissible systems $\mathscr{A}$, and where an admissible system is defined exactly as before except for $y_{x,t}$, which is the solution of:

(3.2)
$$dy_{x,t}(s) = \sigma(y_{x,t}(s), s, v(s)) \, dW_x - b(y_{x,t}(s), s, v(s)) \, ds, \qquad s \in [t, T],$$
$$y_{x,t}(t) = x.$$

Obviously $\tau_{x,t}$ denotes the exit time from $\bar{\mathcal{O}}$ of the process $y_{x,t}(s)$, and u_0 is a given function in $W^{2,\infty}(\mathcal{O})$ satisfying $u = 0$ on $\partial\mathcal{O}$.

Of course this time-dependent problem may be reduced to the general case of degenerate stochastic integrals by looking at the "space-time" diffusion $(y_{x,t}(s), s)$ starting at the point (x, t) of $\bar{Q}$; then $(\tau_{x,t} \wedge T)$ is just the first exit time from $\bar{Q}$ of this process. Instead of considering both situations (time independent and time dependent) in a same general context (and defining in particular a set Γ_0 of regular points) we prefer to give just the case of time-independent stochastic integrals and to indicate how the preceding results may be adapted to the above situation.

We will not give any proofs in this section, since they are only trivial adaptations of the methods introduced above. We only give some examples of our results.

THEOREM 3.1. *Assume either*

$$-\frac{\partial \bar{u}}{\partial t} + A(v)\bar{u} \leqq -1 \quad in \ \mathscr{D}'(\mathcal{O}_\varepsilon \times \,]0, T[) \quad \forall v \in V,$$

(3.3) $$\bar{u} \in C(\bar{Q}), \quad \bar{u}(x, T) \leqq u_0(x), \quad |\nabla_x \bar{u}(x, t)| \leqq C \quad \forall(x, t) \in Q,$$

$$u \leqq -\alpha < 0 \quad on \ (\mathcal{O}_\varepsilon) \times [0, T];$$

or

$$f(x, t, v) \geqq 0 \quad \forall(x, t, v) \in \bar{\mathcal{O}} \times [0, T] \times V,$$

$$\exists v(t, x) \ continuous \ on \ \Gamma \times [0, T] \ such \ that \ \exists \alpha > 0 \ such \ that$$

(3.4) $$\Gamma \times [0, T] = \{(x, t)/|\sigma(x, t, v)n(x)| \geqq \alpha\}$$

$$\cup \{(x, t)/-2rb(x, t, v) \cdot n(x) > \mathrm{Tr}\,[\sigma\sigma^*(x, t, v)] + \alpha\},$$

where r is the radius of the uniform exterior sphere associated to $\mathcal{O}$.

Then

  i) *we have the dynamical programming property*:

$$u(t, x) = \inf_{\mathscr{A}} E\bigg[\int_t^{\theta \wedge T \wedge \tau_{x,t}} f(y_{x,t}(s), s, v(s)) \exp\bigg(-\int_t^s c(y_{x,t}(\lambda), \lambda, v(\lambda))\, d\lambda\bigg)\, ds$$

$$+ u_0(y_{x,t}(T)) \exp\bigg(-\int_t^T c(y_{x,t}(s), s, v(s))\, ds\bigg) \cdot 1_{(T < \tau_{x,t} \wedge \theta)}$$

$$+ u(y_{x,t}(\theta)) \exp\bigg(-\int_t^\theta c(y_{x,t}(s), s, v(s))\, ds\bigg) \cdot 1_{(\theta < \tau_x)}\bigg],$$

where θ is a stopping time.

  ii)

$$u \in W^{1,\infty}(Q), \quad u = 0 \quad on \ \Gamma \times [0, T], \quad u = u_0 \ on \ \bar{\mathcal{O}} \times \{T\},$$

$$-\frac{\partial u}{\partial t} + A(v)u \leqq f(v) \quad in \ \mathscr{D}'(Q) \quad \forall v \in V.$$

  iii) *u is the maximum element of the set of functions w satisfying*

$$w \in C(\bar{Q}), \quad w \leqq 0 \ on \ \Gamma \times [0, T], \quad w \leqq u_0 \ on \ \mathcal{O} \times \{T\}, \quad \nabla_x w \in L^\infty(Q),$$

$$-\frac{\partial w}{\partial t} + A(v)w \leqq f(v) \quad in \ \mathscr{D}'(Q) \quad \forall v \in V.$$

  This result is only one example of how the results of the preceding sections adapt to this problem of control of time-dependent stochastic integrals and to this Cauchy problem for Hamilton–Jacobi–Bellman equations.

  Let us also mention that a general result concerning the verification of H–J–B equations is given in P.-L. Lions [13].

  4. The obstacle problem. This section is divided into two parts. First we give a stochastic interpretation of the optimal cost. Next, we establish an analytical interpretation.

4.1. Stochastic interpretation. Let $\Psi(x)$ be a function from $\mathbb{R}^N$ into $\mathbb{R}$ satisfying

$$|\Psi(x) - \Psi(x')| \leqq \rho(|x - x'|) \quad \forall x, x' \in \mathbb{R}^N,$$

(4.1) $\Psi(x) \geqq 0 \quad \forall x \in \Gamma_0$ (given by (2.19)),

$$|\Psi(x)| \leqq C \quad \forall x \in \mathbb{R}^N,$$

where ρ is a given continuous function from $\mathbb{R}^+$ into $\mathbb{R}^+$ such that $\rho(0) = 0$.

In some results, we will also assume

(4.2) $\Psi(x) \geqq 0 \quad \forall x \in \mathbb{R}^N.$

Let us define the cost function

$$J_x(\mathscr{A}, \theta) = E\left\{\int_0^{\theta \wedge \tau_x} f(y_x(t), v(t)) \exp\left(-\int_0^t c(y_x(s), v(s)) \, ds\right) dt\right.$$

(4.3)

$$\left. + 1_{\theta < \tau_x} \Psi(y_x(\theta \wedge \tau_x)) \exp\left(-\int_0^{\theta \wedge \tau_x} c(y_x(t), v(t)) \, dt\right)\right\},$$

where $\mathscr{A}$ is any admissible system and θ is a stopping time with respect to F_t.

The optimal cost function is given by

(4.4) $u(x) = \inf\{J_x(\mathscr{A}, \theta)/\mathscr{A}, \theta\}.$

We have the following:

THEOREM 4.1. *Under assumptions* (1.9), (1.10), (1.11) *and* (4.1) (*resp.* (1.9), (1.10), (1.15) *and* (4.2)) *the function u defined by* (4.4) *is the maximum solution of the following problem*:

$$u \in B_s \quad (\text{resp. } B_s^+), \qquad u|_{\Gamma_0} = 0,$$

(4.5) $u \leqq \Psi \quad in \ \bar{\mathcal{O}},$

$$u \leqq Q(t)u \quad \forall t \geqq 0,$$

where $Q(t)$ is the semigroup (1.8).

Proof. Let $\delta(t, \omega)$ be an adapted process such that $0 \leqq \delta(t) \leqq 1$ for all $t \geqq 0$. Let us define for $\varepsilon > 0$:

$$J_x^\varepsilon(\mathscr{A}, \delta) = E\left\{\int_0^{\tau_x} \left[f(y_x(t), v(t)) + \frac{1}{\varepsilon} \delta(t) \Psi(y_x(t))\right]\right.$$

(4.6)

$$\left. \cdot \exp\left(-\int_0^t (c(y_x(s), v(s)) + \frac{1}{\varepsilon} \delta(s)) \, ds\right) dt\right\}$$

and

(4.7) $u_\varepsilon(x) = \inf\{J_x^\varepsilon(\mathscr{A}, \delta)/\mathscr{A}, \delta\}.$

From Theorem 2.1 we have

(4.8) $u_\varepsilon \in B_s \quad (\text{resp. } B_s^+) \quad \text{and} \quad u_\varepsilon|_{\Gamma_0} = 0.$

First, we prove that

$$u_\varepsilon(x) = \inf_{\mathscr{A}} E\left\{\int_0^{\tau_x} \left[f(y_x(t), v(t)) - \frac{1}{\varepsilon}(u_\varepsilon - \Psi)^+(y_x(t))\right]\right.$$

(4.9)

$$\left. \cdot \exp\left(-\int_0^t c(y_x(s), v(s)) \, ds\right) dt\right\}.$$

Indeed, from (4.7) and the dynamical programming used for the function u_ε, we deduce that the process

$$\xi(t) = \int_0^{t\wedge\tau} \left[f(y(s), v(s)) + \frac{1}{\varepsilon}\delta(s)\Psi(s)\Psi(y(s)) \right]$$

$$\cdot \exp\left(-\int_0^s \left(c(y(\lambda), v(\lambda)) + \frac{1}{\varepsilon}\delta(\lambda) \right) d\lambda \right) ds$$

$$+ u_\varepsilon(y(t\wedge\tau)) \exp\left(-\int_0^{t\wedge\tau} \left(c(y(s), v(s)) + \frac{1}{\varepsilon}\delta(s) \right) ds \right)$$

is a submartingale for each admissible system $\mathscr{A}$. Setting

$$\xi(\infty) = \int_0^\tau \left[f(y(s), v(s)) + \frac{1}{\varepsilon}\delta(s)\Psi(y(s)) \right] \exp\left(-\int_0^s \left(c(y(\lambda), v(\lambda)) + \frac{1}{\varepsilon}\delta(\lambda) \right) d\lambda \right) ds$$

we obtain from (4.8) for $\eta(t) = E^{F_t}\xi(\infty) - \xi(t)$

$$(4.10) \qquad 0 \leqq \eta(t) \leqq C \exp\left[-\int_0^t \left(c_0 + \frac{1}{\varepsilon}\delta(s) \right) ds \right] \quad \forall t \geqq 0.$$

The process $\xi(t) + \eta(t)$ is a F_t-martingale, so the process

$$Z(t) = \eta(t) \exp\left(-\int_0^t \frac{1}{\varepsilon}\delta(s)\, ds \right) + u_\varepsilon(y(t\wedge\tau)) \exp\left(-\int_0^{t\wedge\tau} c(y(s), v(s))\, ds \right)$$

$$+ \int_0^{t\wedge\tau} \left[f(y(s), v(s)) - \frac{1}{\varepsilon}\delta(s)(u_\varepsilon - \Psi)(y(s)) \right] \exp\left(-\int_0^s c(y(\lambda), v(\lambda))\, d\lambda \right) ds$$

$$- \frac{1}{\varepsilon}\int_0^t \eta(s)\,\delta(s) \exp\left(-\int_0^s \frac{1}{\varepsilon}\delta(\lambda)\, d\lambda \right) ds$$

is a F_t-martingale too. Since $EZ(0) = EZ(t)$ and $\eta(t) \geqq 0$, choosing

$$\delta(s) = \begin{cases} 1 & \text{if } \Psi(y(s)) < u_\varepsilon(y(s)), \\ 0 & \text{if } \Psi(y(s)) \geqq u_\varepsilon(y(s)) \end{cases}$$

and taking the limit for $t \to \infty$ we deduce, using (4.10),

$$(4.11) \qquad u_\varepsilon \leqq E\left\{ \int_0^\tau \left[f(y(s), v(s)) - \frac{1}{\varepsilon}(u_\varepsilon - \Psi) + (y(s)) \right] \right.$$

$$\left. \cdot \exp\left(-\int_0^s c(y(\lambda), v(\lambda))\, d\lambda \right) ds \right\} \quad \forall\mathscr{A}.$$

Next, given a constant $k > 0$, there exists $\mathscr{A} = \mathscr{A}_{k,x}$ such that

$$E\eta(0) \leqq k.$$

Hence, using the fact that $EZ(0) = EZ(t)$, we have

$$u_\varepsilon \geqq -k\left(1 + \frac{1}{\varepsilon}t \right) - Ce^{-c_0 t} + \inf_{\mathscr{A}} E\left\{ \int_0^\tau \left[f(y(t), v(t)) - \frac{1}{\varepsilon}(u_\varepsilon - \Psi)^+(y(t)) \right] \right.$$

$$\left. \cdot \exp\left(-\int_0^t c(y(s), v(s))\, ds \right) dt \right\},$$

where C is a constant independent of $t \geqq 0$.

Thus, with $t \to \infty$ this proves (4.9).

Now, in a classical way (cf. A. Bensoussan–J.-L. Lions [1]), we deduce from (4.9) that

$$(4.12) \qquad u_\varepsilon \to u \quad \text{as } \varepsilon \to 0 \quad \text{uniformly in } \bar{\mathscr{O}}$$

and

$$(4.13) \quad u(x) \leqq u_\varepsilon(x) \leqq E\left\{ \int_0^{\theta \wedge \tau_x} f(y_x(t), v(t)) \exp\left(-\int_0^t c(y_x(s), v(s))\, ds \right) dt \right.$$
$$\left. + u_\varepsilon(y_x(\theta \wedge \tau_x)) \exp\left(-\int_0^{\theta \wedge \tau} c(y(t), v(y))\, dt \right) \right\} \quad \forall \mathscr{A}, \theta.$$

Hence, we show that $u(x)$ is a solution of problem (4.5).

Finally, the same arguments as above prove that u is the maximal solution. $\quad\square$

In order to obtain some results of regularity of the optimal cost $u(x)$ we assume

$$(4.14) \qquad |\Psi(x) - \Psi(x')| \leqq C|x - x'| \quad \forall x, x' \in \mathbb{R}^N,$$

we have

THEOREM 4.2. *Assume* (1.9), (1.11), (2.11), (4.1), (4.14) *and*

$$(4.15) \qquad C\bar{u}(x) \leqq \Psi(x) \quad \text{in } \bar{\mathscr{O}}, \quad \text{for some } C > 0,$$

or assume (1.9), (1.14), (1.15), (1.16), (4.1), (4.2), (4.14) *and c_0 large enough in* (1.10). *Then the optimal cost function u belongs to* $W_0^{1,\infty}(\mathscr{O})$.

Proof. As in [14] or [1] (and using the proof of Theorem 4.1) we have that u satisfies the dynamical programming property, i.e.,

$$(4.16) \quad u(x) = \inf_\theta E\left[\int_0^{\tau_x \wedge \theta \wedge \tau} f(y_x(t), v(t)) \exp\left(-\int_0^t c(y_x(s), v(s))\, ds \right) dt \right.$$
$$+ u(y_x(\tau)) \exp\left(-\int_0^\tau c(y_x(t), v(t))\, dt \right) 1_{(\tau < \theta \wedge \tau_x)}$$
$$\left. + \Psi(y_x(\theta)) \exp\left(+\int_0^\theta c(y_x(t), v(t))\, dt \right) 1_{(\theta < \tau \wedge \tau_x)} \right],$$

where τ is any stopping time.

Now using (4.15), we deduce as before (in similar situations)

$$|u(x)| \leqq C|\bar{u}(x)|.$$

Then the same methods as before give the Lipschitz character of u. $\quad\square$

COROLLARY 4.1. *Under assumptions* (1.9), (1.10), (1.11), (4.1) *or* (1.9), (1.10), (1.14), (1.15), (1.16) *and* (4.1), *the optimal cost function u is uniformly continuous on* $\bar{\mathscr{O}}$.

The proof of this result uses the same argument as in Theorem 2.2.

4.2. Analytical interpretation. We will in this section just state some results which are proved with the same techniques as in § 2. These are examples of how our techniques apply to the obstacle problem.

We first prove that under fairly general conditions u is a "generalized solution" of

$$(4.17) \quad \sup\left[\sup_{v \in V}\{A(v)u - f(v)\}, u - \Psi\right] = 0 \quad \text{a.e. in } \mathscr{O},$$
$$u = 0 \quad \text{in } \Gamma = \partial \mathscr{O}.$$

THEOREM 4.3. i) *Under assumptions* (1.9), (1.10), (2.26), (4.1) *and* (1.11) (*or* (1.15) *and* (4.2)), *for all w satisfying* $w \in W^{1,\infty}_{\text{loc}}(\mathcal{O}) \cap C(\bar{\mathcal{O}})$, $w|_\Gamma \leqq 0$, $A(v)w \leqq f(v)$ *in* $\mathscr{D}'(\mathcal{O})$, *for all v in V, w $\leq \Psi$ in $\mathcal{O}$, we have*

$$w \leqq u \quad in \ \bar{\mathcal{O}}.$$

ii) *Under the assumptions of Theorem* 4.2 *and if we assume in addition* (2.26), *then u is the maximum element of the set of functions w satisfying* $w \in W^{1,\infty}_{\text{loc}}(\mathcal{O}) \cap C(\bar{\mathcal{O}})$; $w|_\Gamma \leqq 0$ *and*

$$A(v)w \leqq f(v) \quad in \ \mathscr{D}'(\mathcal{O}) \quad \forall v \in V, \qquad w \leqq \Psi \quad in \ \mathcal{O}.$$

REFERENCES

[1] A. BENSOUSSAN AND J.-L. LIONS, *Applications des inéquations variationnelles en contrôle stochastique*, Dunod, Paris, 1978.

[2] L. C. EVANS AND P.-L. LIONS, *Résolutions des equations de Hamilton–Jacobi–Bellman pour des opérateurs uniformément elliptiques*, CRAS Paris, 290 (1980), pp. 1049–1052.

[3] I. L. GENIS AND N. V. KRYLOV, *An example of a one-dimensional controlled process*, Theory Prob. Appl., 21 (1976), pp. 148–152.

[4] R. GONZALEZ, *Sur l'existence d'une solution maximale de l'équation de Hamilton–Jacobi*, CRAS Paris, 282 (1976), pp. 1287–1290.

[5] N. V. KRYLOV, *Control of a solution of a stochastic integral equation*, Theory Prob. Appl., 17 (1972), pp. 114–131.

[6] ———, *On control of the solution of a stochastic integral equation with degeneration*, Izv. Akad. Nauk. USSR, 36 (1972), pp. 249–262.

[7] ———, *Controlled Diffusion Processes*, Springer-Verlag, New York, 1980.

[8] P.-L. LIONS, *Résolution analytique des problèmes de Bellman–Dirichlet*, Acta Mathematica (1981), to appear; cf. also CRAS Paris, 287 (1978), pp. 747–750.

[9] ———, *Control of diffusion processes in* $\mathbb{R}^N$, Comm. Pure Appl. Math. (1981), to appear; cf. also CRAS Paris, 288 (1979), pp. 339–342.

[10] ———, *Some problems related to the Bellman–Dirichlet equation for two elliptic operators*, Comm. PDE, to appear; cf. also thèse de 3ème cycle Paris VI, 1978.

[11] ———, *Equations de Hamilton–Jacobi–Bellman dégénérées*, CRAS Paris, 289 (1979), pp. 329–332.

[12] ———, *Equations de Hamilton–Jacobi–Bellman*, in Seminaire Goularmie-Schwartz, 1979–1980.

[13] ———, *The Dirichlet problem for Hamilton–Jacobi–Bellman equations and optimal stochastic control*, to appear.

[14] P.-L. LIONS AND J.-L. MENALDI, *Optimal control of stochastic integrals and Hamilton–Jacobi–Bellman equations*, I, this Journal, this issue, pp. 58–81.

[15] ———, *Problèmes de Bellman avec le contrôle dans les coefficients de plus haut degre*, CRAS Paris, 287 (1978), pp. 409–412.

[16] J.-L. MENALDI, *On the optimal stopping time problem for degenerate diffusions*, this Journal, 18 (1980), pp. 697–721; see also CRAS Paris, 284 (1977), pp. 1443–1446.

[17] M. NISIO, *On a nonlinear semi-group attached to stochastic optimal control*, Publ. Res. Inst. Math. Sci., 12 (1976–77), pp. 513–537.

[18] M. V. SAFONOV, *On the Dirichlet problem for Bellman's equation in a plane domain*, Math. USSR Sb., 31 (1977), pp. 231–248.

[19] ———, *On the Dirichlet problem for Bellman's equation in a plane domain*, Math. USSR Sb., 34 (1978), pp. 521–526.