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Ergodicity and strong limit results for two-time-scale functional stochastic differential equations

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ABSTRACT

This article focuses on a class of two-time-scale functional stochastic differential equations, where the phase space of the segment processes is infinite-dimensional. The systems under consideration have a fast-varying component and a slowly varying one. First, the ergodicity of the fast-varying component is obtained. Then inspired by the Khasminskii's approach, an averaging principle, in the sense of convergence in the pth moment uniformly in time within a finite time interval, is developed.

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1. Introduction

Having a wide range of applications in science and engineering [32], manufacturing systems and production planning [31], control and optimization [5], as well as in finance and economics [11, 23], singularly perturbed systems have received continuing and growing attention. Such systems have been investigated extensively. We refer to [13, 35], and references therein for some recent advances. Singularly perturbed systems usually exhibit multi-scale behavior owing to inherent rates of changes of the systems or different rates of interactions of subsystems and components. To reflect the slow and fast motions in the underlying systems, one introduces a time-scale separation parameter $\varepsilon \in (0, 1)$. Due to the multi-scale property, the systems are highly complex, thus difficult to analyze. As a result, it is foremost important to reduce their complexity. The averaging principle pioneered by Khasminskii [20] for a class of diffusions provides an effective way to reduce the complexity of the systems involved. For systems in which both fast and slow components co-exist, the idea of the averaging approach reveals that there is a limit dynamical system given by the average of the slow component w.r.t. the invariant probability measure of the fast component that is an ergodic process. The averaging equation approximates the slow component in a suitable sense whenever $\varepsilon \downarrow 0$ leading to a substantial reduction of computational complexity. The work [20] due to Khasminskii inspired much of the subsequent development. To date, there has been vast literature on the study for multi-scale stochastic dynamical systems; see the monograph [18]. For weak and strong convergence and the associated averaging principle, we refer to [15, 21, 22, 27, 35] for

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stochastic differential equations (SDEs), and [4, 6, 7, 14, 24] for stochastic partial differential equations (SPDEs). For numerical methods, we refer to [12, 16]. As for related control and filtering problems, see [25, 26]. Concerning large deviations, see [25, 33].

The aforementioned references are all concerned with systems without "memory." Nevertheless, more often than not, dynamical systems with delays are unavoidable in a wide variety of applications in science and engineering, where the dynamics are subject to propagation of memory. In response to the great needs, there is also extensive literature on functional SDEs; see for example, [28, chapter 5] and [29]. Nevertheless, except some developments such as [34], multi-scale structures and singularly perturbed systems have not been considered extensively. Moreover, the work on two-time-scale functional SDEs is even more scarce to the best of our knowledge.

In contrast to the rapid progress in two-time-scale systems and differential delay equations, the study on limit theorems for averaging principles for functional SDEs is still in its infancy. Compared with the existing literature, for such systems, one of the outstanding issues is the phase space of the segment processes is infinite-dimensional, which makes the goal of obtaining a strong limit theorem for the averaging principle a very difficult task. This work aims to take the challenges and to establish a strong limit theorem for the averaging principles for a class of two-time-scale functional SDEs.

The rest of the article is organized as follows. Section 2 presents the setup of the problem we wish to study. The ergodicity of the frozen equation with memory is obtained in Section 3. Section 4 constructs some auxiliary two-time-scale stochastic systems with memory and provides a number of preliminary lemmas. Section 5 derives a strong limit theorem for the averaging principle in the spirit of Khasminskii's approach for the slow component.

Before proceeding further, a word of notation is in order. Throughout the article, generic constants will be denoted by c; we use the shorthand notation $a \lesssim b$ to mean $a \le cb$, and we use $a \lesssim_T b$ to emphasize that the constant c depends on T for a constant $T > 0$.

2. Formulation

For integers $n, m \ge 1$, let $\mathbb{R}^n$ be an n-dimensional Euclidean space with norm $|\cdot|$ and inner product $\langle \cdot, \cdot \rangle$, and $\mathbb{R}^n \otimes \mathbb{R}^m$ denote the collection of all $n \times m$ matrices with real entries. For an $A \in \mathbb{R}^n \otimes \mathbb{R}^m$, $\|A\|$ stands for its Frobenius matrix norm. For an interval $I \subset (-\infty, \infty)$, $C(I; \mathbb{R}^n)$ means the family of all continuous functions $I \to \mathbb{R}^n$. For a fixed $\tau > 0$, let $\mathscr{C} = C([-\tau, 0]; \mathbb{R}^n)$, endowed with the uniform norm $\|\cdot\|_\infty$. For $h(\cdot) \in C([-\tau, \infty); \mathbb{R}^n)$ and $t \ge 0$, define the segment $h_t \in \mathscr{C}$ by $h_t(\theta) = h(t + \theta), \theta \in [-\tau, 0]$.

Introducing a time-scale separation parameter $\varepsilon \in (0, 1)$, we consider two-time-scale systems of functional SDEs of the following form.

$$\mathrm{d}X^\varepsilon(t) = b_1\big(X_t^\varepsilon, Y_t^\varepsilon\big)\mathrm{d}t + \sigma_1(X_t^\varepsilon)\mathrm{d}W_1(t), \quad t > 0, \quad X_0^\varepsilon = \xi \in \mathscr{C}, \tag{2.1}$$

and

$$\mathrm{d}Y^\varepsilon(t) = \frac{1}{\varepsilon} b_2\big(X_t^\varepsilon, Y^\varepsilon(t), Y^\varepsilon(t-\tau)\big)\mathrm{d}t + \frac{1}{\sqrt{\varepsilon}}\sigma_2\big(X_t^\varepsilon, Y^\varepsilon(t), Y^\varepsilon(t-\tau)\big)\mathrm{d}W_2(t), \ t > 0 \tag{2.2}$$

with the initial value $Y_0^\varepsilon = \eta \in \mathscr{C}$, where $b_1 : \mathscr{C} \times \mathscr{C} \to \mathbb{R}^n$, $b_2 : \mathscr{C} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma_1 : \mathscr{C} \to \mathbb{R}^n \otimes \mathbb{R}^m$, $\sigma_2 : \mathscr{C} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \otimes \mathbb{R}^m$ are Gâteaux differentiable, $(W_1(t))_{t\ge 0}$ and $(W_2(t))_{t \ge 0}$ are two mutually independent m-dimensional Brownian motions defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, equipped with $(\mathscr{F}_t)_{t\ge 0}$, a family of filtration satisfying the usual conditions (i.e., for each $t \ge 0, \mathscr{F}_t = \mathscr{F}_{t+} := \bigcap_{s>t} \mathscr{F}_s$, and $\mathscr{F}_0$ contains all $\mathbb{P}$-null sets). As usual, for two-time-scale systems (2.1) and (2.2), $X^\varepsilon(t)$ is called the slow component, while $Y^\varepsilon(t)$ is called the fast component.

We denote by $\nabla^{(i)}$ the gradient operators for the i-th component. Throughout the article, for any $\chi,\phi\in\mathscr{C}$ and $x,x',y,y'\in\mathbb{R}^n$, we assume that,

(A1) $\nabla b_1=(\nabla^{(1)}b_1,\nabla^{(2)}b_1)$ is bounded, and there exists an $L>0$ such that,

$$|b_1(\chi,\phi)|\le L(1+\|\chi\|_\infty)\quad\text{and}\quad \|\sigma_1(\phi)-\sigma_1(\chi)\|\le L\|\phi-\chi\|_\infty.$$

(A2) $\nabla b_2=(\nabla^{(1)}b_2,\nabla^{(2)}b_2,\nabla^{(3)}b_2)$ and $\nabla\sigma_2=(\nabla^{(1)}\sigma_2,\nabla^{(2)}\sigma_2,\nabla^{(3)}\sigma_2)$ are bounded.

(A3) There exist $\lambda_1>\lambda_2>0$, independent of χ, such that,

$$\begin{aligned}&2\langle x-x',b_2(\chi,x,y)-b_2(\chi,x',y')\rangle+\|\sigma_2(\chi,x,y)-\sigma_2(\chi,x',y')\|^2\\&\quad\le-\lambda_1|x-x'|^2+\lambda_2|y-y'|^2.\end{aligned}$$

(A4) For the initial value $X_0^\varepsilon=\xi\in\mathscr{C}$ of (2.1), there exists a $\lambda_3>0$ such that,

$$|\xi(t)-\xi(s)|\le\lambda_3|t-s|,\quad s,t\in[-\tau,0].$$

Let us comment the assumptions (**A1**)-(**A4**) above. From (**A1**) and (**A2**), the gradient operators ∇b_1, ∇b_2, and $\nabla\sigma_2$ are bounded, respectively, so that b_1, b_2, and σ_2 are Lipschitz. Then, both (2.1) and (2.2) are well posed (see, e.g., [28, Theorem 2.2, P.150]). While, (**A3**) is imposed to analyze the ergodic property of the frozen equation (see Theorem 3.1 below), guarantee the Lipschitz property of $\overline{b}_1$ (see Corollary 3.3 below), defined in (3.3), and provide a uniform bound of the segment process $(Y_t^\varepsilon)_{t\in[0,T]}$ (see Lemma 4.3 below). Next, (**A4**) ensures that the displacement of the segment process $(X_t^\varepsilon)_{t\in[0,T]}$ is continuous in the L^p sense; see Lemma 4.1 in what follows.

3. Ergodicity of the Frozen Equation with Memory

Consider an SDE with memory associated with the fast motion and the frozen slow component in the following form.

$$\mathrm{d}Y(t)=b_2(\zeta,Y(t),Y(t-\tau))\mathrm{d}t+\sigma_2(\zeta,Y(t),Y(t-\tau))\mathrm{d}W_2(t),\quad t>0,\quad Y_0=\eta\in\mathscr{C}.\tag{3.1}$$

Under (**A2**), (3.1) has a unique strong solution $(Y(t))_{t\ge-\tau}$ (see, e.g., [28, Theorem 2.2, P.150]). To highlight the initial value $\eta\in\mathscr{C}$ and the frozen segment $\zeta\in\mathscr{C}$, we write the corresponding solution process $(Y^\zeta(t,\eta))_{t\ge-\tau}$ and the segment process $(Y_t^\zeta(\eta))_{t\ge0}$ instead of $(Y(t))_{t\ge-\tau}$ and $(Y_t)_{t\ge0}$, respectively.

Our main result in this section is stated as below. It is concerned with the ergodicity of the frozen SDE with memory.

Theorem 3.1. *Under (**A2**) and (**A3**), $Y_t^\zeta(\eta)$ has a unique invariant probability measure μ^ζ, and there exists $\lambda>0$ such that,*

$$|\mathbb{E}b_1(\zeta,Y_t^\zeta(\eta))-\overline{b}_1(\zeta)|\lesssim \mathrm{e}^{-\lambda t}(1+\|\eta\|_\infty+\|\zeta\|_\infty),\quad t\ge0,\ \eta\in\mathscr{C},\tag{3.2}$$

where

$$\overline{b}_1(\zeta):=\int_{\mathscr{C}}b_1(\zeta,\varphi)\mu^\zeta(\mathrm{d}\varphi),\quad \zeta\in\mathscr{C}.\tag{3.3}$$

Proof. The main idea of the proof concerning existence of an invariant probability measure goes back to [1, Lemma 2.4], which, nevertheless, involves functional SDEs with additive noises.

Let $\mathscr{P}(\mathscr{C})$ be the set of all probability measures on $\mathscr{C}$. W_2 denotes the L^2-Wasserstein distance on $\mathscr{P}(\mathscr{C})$ induced by the bounded distance $\rho(\xi,\eta) := 1 \wedge \|\xi-\eta\|_\infty$, i.e.,

$$W_2(\mu_1,\mu_2) = \inf_{\pi\in\mathscr{C}(\mu_1,\mu_2)} (\pi(\rho^2))^{\frac{1}{2}}, \quad \mu_1,\mu_2 \in \mathscr{P}(\mathscr{C}),$$

where $\mathscr{C}(\mu_1,\mu_2)$ is the set of all coupling probability measures with marginals μ_1 and μ_2. It is well known that $\mathscr{P}(\mathscr{C})$ is a complete metric space w.r.t. the distance W_2 [8, Lemma 5.3 & Theorem 5.4], and the convergence in W_2 is equivalent to the weak convergence [8, Theorem 5.6, P.179]. Let $P_t^{\zeta,\eta}$ be the law of the segment process $Y_t^\zeta(\eta)$. According to the Krylov-Bogoliubov existence theorem ([9, Theorem 3.1.1, P.21]), if $P_t^{\zeta,\eta}$ converges weakly to a probability measure μ_η^ζ, then μ_η^ζ is an invariant probability measure. So, in the light of the previous discussion, it suffices to prove the assertions below.

(i) $\{P_t^{\zeta,\eta}\}_{t\geq 0}$ is a Cauchy sequence w.r.t. the distance W_2. If so, by the completeness of $\mathscr{P}(\mathscr{C})$ w.r.t. the distance W_2, there is a $\mu_\eta^\zeta \in \mathscr{P}(\mathscr{C})$ such that $\lim_{t\to\infty} W_2(P_t^{\zeta,\eta}, \mu_\eta^\zeta) = 0$.

(ii) $W_2(\mu_\eta^\zeta, \mu_{\eta'}^\zeta) = 0$ for any $\eta,\eta' \in \mathscr{C}$ and frozen $\zeta \in \mathscr{C}$, that is, μ_η^ζ is independent of η.

In the sequel, we shall claim that (i) and (ii) hold, one-by-one. For any $t_2 > t_1 > \tau$ and the frozen segment $\zeta \in \mathscr{C}$, consider the following SDE with memory.

$$\mathrm{d}\overline{Y}(t) = b_2(\zeta, \overline{Y}(t), \overline{Y}(t-\tau))\mathrm{d}t + \sigma_2(\zeta, \overline{Y}(t), \overline{Y}(t-\tau))\mathrm{d}W_2(t), \ t \in [t_2-t_1, t_2] \tag{3.4}$$

with the initial value $\overline{Y}_{t_2-t_1} = \eta$. The solution process and the segment process associated with (3.4) are denoted by $(\overline{Y}^\zeta(t,\eta))$ and $(\overline{Y}_t^\zeta(\eta))$, respectively. Observe that the laws of $Y_{t_2}^\zeta(\eta)$ and $\overline{Y}_{t_2}^\zeta(\eta)$ are $P_{t_2}^{\zeta,\eta}$ and $P_{t_1}^{\zeta,\eta}$, respectively. By (**A2**), there exists an $\alpha > 0$ such that,

$$\|\sigma_2(\chi,x,y) - \sigma_2(\chi,x',y')\| \leq \alpha\,(|x-x'| + |y-y'|), \tag{3.5}$$

and

$$|b_2(\chi,0,0)| + \|\sigma_2(\chi,0,0)\| \leq \alpha\,(1+\|\chi\|_\infty) \tag{3.6}$$

for any $\chi \in \mathscr{C}$ and $x, x', y, y' \in \mathbb{R}^n$. Accordingly, (3.5) and (3.6), together with (**A3**), yield that there exist $\lambda_1' > \lambda_2' > 0$, independent of χ, such that,

$$2\langle x, b_2(\chi,x,y)\rangle + \|\sigma_2(\chi,x,y)\|^2 \leq -\lambda_1'|x|^2 + \lambda_2'|y|^2 + c\big(1+\|\chi\|_\infty^2\big) \tag{3.7}$$

for any $\chi \in \mathscr{C}$ and $x, y \in \mathbb{R}^n$. For a sufficiently small $\lambda' > 0$ obeying $\lambda_1' - \lambda' - \lambda_2' e^{\lambda'\tau} = 0$ due to $\lambda_1' > \lambda_2' > 0$, applying Itô's formula, we infer from (3.7) that,

$$\begin{aligned} e^{\lambda' t}\mathbb{E}|Y^\zeta(t,\eta)|^2 &\leq |\eta(0)|^2 + \int_0^t e^{\lambda' s}\mathbb{E}\left\{c\left(1+\|\zeta\|_\infty^2\right) + \lambda'|Y^\zeta(s,\eta)|^2\right. \\ &\qquad \left. -\ \lambda_1'|Y^\zeta(s,\eta)|^2 + \lambda_2'|Y^\zeta(s-\tau,\eta)|^2\right\} \mathrm{d}s \\ &\lesssim \|\eta\|_\infty^2 + e^{\lambda' t}\left(1+\|\zeta\|_\infty^2\right), \quad t > 0. \end{aligned}$$

Consequently, we arrive at,

$$\mathbb{E}|Y^\zeta(t,\eta)|^2 \lesssim e^{-\lambda' t}\|\eta\|_\infty^2 + 1 + \|\zeta\|_\infty^2, \quad t > 0. \tag{3.8}$$

Also, by the Itô formula, in addition to the Burkhold-Davis-Gundy (B-D-G for abbreviation) inequality, we derive from (3.5) and (3.8) that, for any $t \geq \tau$,

$$\mathbb{E}\left\|Y_t^\zeta(\eta)\right\|_\infty^2$$

$$
\begin{aligned}
&\lesssim 1+\|\zeta\|_\infty^2+\mathbb{E}|Y^\zeta(t-\tau,\eta)|^2+\int_{t-2\tau}^t \mathbb{E}|Y^\zeta(s,\eta)|^2\mathrm{d}s\\
&\quad+2\,\mathbb{E}\left(\sup_{t-\tau\le s\le t}\left|\int_{t-\tau}^s \langle Y^\zeta(s,\eta),\sigma_2(\zeta,Y^\zeta(s,\eta),Y^\zeta(s-\tau,\eta))\mathrm{d}W_2(s)\rangle\right|\right)\\
&\le \frac{1}{2}\,\mathbb{E}\|Y_t^\zeta(\eta)\|_\infty^2+c\left\{1+\|\zeta\|_\infty^2+\mathbb{E}|Y^\zeta(t-\tau,\eta)|^2+\int_{t-2\tau}^t \mathbb{E}|Y^\zeta(s,\eta)|^2\mathrm{d}s\right\}. \qquad (3.9)
\end{aligned}
$$

On the other hand, following the argument leading to (3.9), one has, for any $t\in[0,\tau]$,

$$
\mathbb{E}\|Y_t^\zeta(\eta)\|_\infty^2\le \frac{1}{2}\,\mathbb{E}\|Y_t^\zeta(\eta)\|_\infty^2+c\left\{1+\|\zeta\|_\infty^2+\|\eta\|_\infty^2+\int_0^t \mathbb{E}|Y^\zeta(s,\eta)|^2\mathrm{d}s\right\}. \qquad (3.10)
$$

Thus, combining (3.8) with (3.9) and (3.10) leads to,

$$
\mathbb{E}\|Y_t^\zeta(\eta)\|_\infty^2\le c\,(\mathrm{e}^{-\lambda' t}\|\eta\|_\infty^2+1+\|\zeta\|_\infty^2). \qquad (3.11)
$$

In what follows, we assume $t\in[t_2-t_1,t_2]$, and set $\Gamma^\zeta(t,\eta):=Y^\zeta(t,\eta)-\overline{Y}^\zeta(t,\eta)$ for the sake of notational simplicity. Again, for a sufficiently small $\lambda>0$ such that $\lambda_1-\lambda-\lambda_2\mathrm{e}^{\lambda\tau}=0$ owing to $\lambda_1>\lambda_2$, by the Itô formula, it follows from (**A3**) that,

$$
\begin{aligned}
\mathrm{e}^{\lambda t}\mathbb{E}|\Gamma^\zeta(t,\eta)|^2 &\le \mathrm{e}^{\lambda(t_2-t_1)}\mathbb{E}|\Gamma^\zeta(t_2-t_1,\eta)|^2\\
&\quad+\int_{t_2-t_1}^t \mathrm{e}^{\lambda s}\mathbb{E}\{(\lambda-\lambda_1)|\Gamma^\zeta(s,\eta)|^2+\lambda_2|\Gamma^\zeta(s-\tau,\eta)|^2\}\mathrm{d}s\\
&\le \mathrm{e}^{\lambda(t_2-t_1)}\mathbb{E}|\Gamma^\zeta(t_2-t_1,\eta)|^2+\mathrm{e}^{\lambda\tau}\int_{t_2-t_1-\tau}^{t_2-t_1}\mathrm{e}^{\lambda s}\mathbb{E}|\Gamma^\zeta(s,\eta)|^2\mathrm{d}s\\
&\lesssim \mathrm{e}^{\lambda(t_2-t_1)}\left(\|\eta\|_\infty^2+\mathbb{E}\left\|Y_{t_2-t_1}^\zeta(\eta)\right\|_\infty^2\right).
\end{aligned}
$$

This, together with (3.11), yields that,

$$
\mathbb{E}|\Gamma^\zeta(t,\eta)|^2\lesssim \mathrm{e}^{-\lambda(t+t_1-t_2)}\left(1+\|\eta\|_\infty^2+\|\zeta\|_\infty^2\right). \qquad (3.12)
$$

Imitating a similar procedure to derive (3.9), in particular, we obtain from (**A3**), (3.5), and (3.12) that,

$$
\mathbb{E}\left\|\Gamma_{t_2}^\zeta(\eta)\right\|_\infty^2\lesssim \mathrm{e}^{-\lambda t_1}\left(1+\|\eta\|_\infty^2+\|\zeta\|_\infty^2\right). \qquad (3.13)
$$

This further implies that,

$$
W_2\big(P_{t_1}^{\zeta,\eta},P_{t_2}^{\zeta,\eta}\big)\le \mathbb{E}\{1\wedge\|\Gamma_{t_2}^\zeta(\eta)\|_\infty^2\}\lesssim \mathrm{e}^{-\lambda t_1}\big(1+\|\eta\|_\infty^2+\|\zeta\|_\infty^2\big),
$$

which goes to zero as t_1 (hence t_2) tends to ∞. Thus, claim (i) holds. By carrying out a similar argument to obtain (3.13), one finds that,

$$
\mathbb{E}\left\|Y_t^\zeta(\eta)-Y_t^\zeta(\eta')\right\|_\infty^2\lesssim \mathrm{e}^{-\lambda t}\|\eta-\eta'\|_\infty^2. \qquad (3.14)
$$

For fixed $\zeta\in\mathscr{C}$ and arbitrary $\eta,\eta\in\mathscr{C}$, observe that,

$$
W_2\big(\mu_\eta^\zeta,\mu_{\eta'}^\zeta\big)\le W_2\big(P_t^{\zeta,\eta},\mu_\eta^\zeta\big)+W_2\big(P_t^{\zeta,\eta'},\mu_{\eta'}^\zeta\big)+W_2\big(P_t^{\zeta,\eta},P_t^{\zeta,\eta'}\big). \qquad (3.15)
$$

Consequently, claim (ii) follows by taking (3.14) and (3.15) into consideration.

Let K be a compact subset of $\mathscr{C}$. By virtue of (3.11) and the invariance of μ^ζ, it then follows that,

$$
\int_K \|\psi\|_\infty^2\pi^\zeta(\mathrm{d}\psi)\le c\left\{1+\|\zeta\|_\infty^2+\mathrm{e}^{-\lambda t}\int_K \|\psi\|_\infty^2\pi^\zeta(\mathrm{d}\psi)\right\}.
$$

Thus, choosing $t > 0$ sufficiently large such that $\delta := c\,\mathrm{e}^{-\lambda t} < 1$, one finds that,

$$\int_K \|\psi\|_\infty^2 \pi^\zeta(\mathrm{d}\psi) \lesssim 1 + \|\zeta\|_\infty^2.$$

This, by letting K increasing to $\mathscr{C}$ and applying Fatou's lemma, implies that,

$$\int_{\mathscr{C}} \|\psi\|_\infty^2 \pi^\zeta(\mathrm{d}\psi) \lesssim 1 + \|\zeta\|_\infty^2. \tag{3.16}$$

Next, with the aid of the invariance of π^ζ, (3.14), and (3.16), we deduce from (**A1**) that,

$$\begin{aligned}|\mathbb{E}b_1(\zeta, Y_t^\zeta(\eta)) - \overline{b}_1(\zeta)| &\lesssim \int_{\mathscr{C}} \mathbb{E}\|Y_t^\zeta(\eta) - Y_t^\zeta(\psi)\|_\infty \pi^\zeta(\mathrm{d}\psi) \lesssim \mathrm{e}^{-\frac{\lambda t}{2}} \int_{\mathscr{C}} \|\eta - \psi\|_\infty \pi^\zeta(\mathrm{d}\psi)\\ &\lesssim \mathrm{e}^{-\frac{\lambda t}{2}}(1 + \|\eta\|_\infty + \|\zeta\|_\infty).\end{aligned}$$

As a result, (3.2) follows. □

Remark 3.2. It should be noted that there are other alternative approaches to obtain existence and uniqueness of invariant probability measures for functional SDEs. Regarding to the existence of invariant probability measures, see [10,19] by Arzelà–Ascoli's tightness characterization, [2] using a remote start method, [3] adopting Kurtz's tightness criterion, and [30] by considering the semi-martingale characteristics. As for uniqueness of invariant probability measures, we refer to [17,19] by utilizing an asymptotic coupling method.

The next corollary, which plays a crucial role in discussing strong limit theorem for the averaging principle, states that $\overline{b}_1$, defined by (3.3), enjoys a Lipschitz property.

Corollary 3.3. *Under (**A1**)-(**A3**), $\overline{b}_1 : \mathscr{C} \to \mathbb{R}^n$, defined in (3.3), is Lipschitz.*

Proof. For arbitrary $\phi, \zeta \in \mathscr{C}$, let,

$$\nabla_\phi \overline{b}_1(\zeta) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\overline{b}_1(\zeta + \varepsilon\phi)\Big|_{\varepsilon=0}$$

be the direction derivative of $\overline{b}_1$ at ζ along the direction ϕ. By Theorem 3.1, we have,

$$\begin{aligned}\nabla_\phi \overline{b}_1(\zeta) &= \lim_{t\to\infty} \mathbb{E}\nabla_\phi b_1\left(\zeta, Y_t^\zeta(\eta)\right)\\ &= \lim_{t\to\infty} \mathbb{E}\left\{\left(\nabla_\phi^{(1)} b_1\right)(\zeta, Y_t^\zeta(\eta)) + \left(\nabla^{(2)}_{\nabla_\phi Y_t^\zeta(\eta)} b_1\right)(\zeta, Y_t^\zeta(\eta))\right\}, \quad \phi, \zeta, \eta \in \mathscr{C}.\end{aligned}$$

According to (**A1**), to verify that $\overline{b}_1 : \mathscr{C} \to \mathbb{R}^n$ is Lipschitz, it remains to verify,

$$\sup_{t\geq 0} \mathbb{E}\|\nabla_\phi Y_t^\zeta(\eta)\|_\infty^2 < \infty. \tag{3.17}$$

Observe that $\nabla_\phi Y^\zeta(t, \eta)$ satisfies the following linear SDE with memory.

$$\begin{aligned}\mathrm{d}\left(\nabla_\phi Y^\zeta(t,\eta)\right) = &\left\{\left(\nabla_\phi^{(1)} b_2\right)(\zeta, Y^\zeta(t,\eta), Y^\zeta(t-\tau,\eta))\right.\\ &+ \left(\nabla^{(2)}_{\nabla_\phi Y^\zeta(t,\eta)} b_2\right)(\zeta, Y^\zeta(t,\eta), Y^\zeta(t-\tau,\eta))\\ &\left.+ \left(\nabla^{(3)}_{\nabla_\phi Y^\zeta(t-\tau,\eta)} b_2\right)(\zeta, Y^\zeta(t,\eta), Y^\zeta(t-\tau,\eta))\right\}\mathrm{d}t\\ &+ \left\{(\nabla_\phi^{(1)}\sigma_2)(\zeta, Y^\zeta(t,\eta), Y^\zeta(t-\tau,\eta))\right.\end{aligned}$$

$$+\left(\nabla^{(2)}_{\nabla_\phi Y^\zeta(t,\eta)}\sigma_2\right)(\zeta, Y^\zeta(t,\eta), Y^\zeta(t-\tau,\eta))$$
$$+\left(\nabla^{(3)}_{\nabla_\phi Y^\zeta(t-\tau,\eta)}\sigma_2\right)(\zeta, Y^\zeta(t,\eta), Y^\zeta(t-\tau,\eta))\Big\}\,\mathrm{d}W_2(t), \quad t>0$$

with the initial data $\nabla_\phi Y_0^\zeta(\eta)=0$. In the sequel, let $\chi\in\mathscr{C}$ and $x,x',y,y'\in\mathbb{R}^n$. For any $\varepsilon>0$, it is trivial to see from (**A3**) that,

$$2\varepsilon\langle x, b_2(\chi, x'+\varepsilon x, y'+\varepsilon y)-b_2(\chi,x',y')\rangle+\|\sigma_2(\chi,x'+\varepsilon x,y'+\varepsilon y)-\sigma_2(\chi,x',y')\|^2$$
$$\le -\lambda_1\varepsilon^2|x|^2+\lambda_2\varepsilon^2|y|^2.$$

Multiplying ε^{-2} on both sides, followed by sending $\varepsilon\downarrow 0$, gives that,

$$\begin{aligned}&2\big\langle x, \big(\nabla_x^{(2)}b_2\big)(\chi,x',y')+\big(\nabla_y^{(3)}b_2\big)(\chi,x',y')\big\rangle\\&+\left\|\big(\nabla_x^{(2)}\sigma_2\big)(\chi,x',y')+\big(\nabla_y^{(3)}\sigma_2\big)(\chi,x',y')\right\|^2\\&\le -\lambda_1|x|^2+\lambda_2|y|^2.\end{aligned}\tag{3.18}$$

On the other hand, by virtue of (3.5), for any $\varepsilon>0$, one has,

$$\|\sigma_2(\chi,x'+\varepsilon x,y'+\varepsilon y)-\sigma_2(\chi,x',y')\|^2\le\alpha\varepsilon^2(|x|^2+|y|^2),$$

which further yields that,

$$\|(\nabla_x^{(2)}\sigma_2)(\chi,x',y')+(\nabla_y^{(3)}\sigma_2)(\chi,x',y')\|^2\le\alpha(|x|^2+|y|^2).\tag{3.19}$$

Thus, with (3.18) and (3.19) in hand, (3.17) holds by repeating the argument which (3.11) is obtained. □

4. Preliminary Results

In this article, we study the strong deviation between the slow component $X^\varepsilon(t)$ and the averaged component $\overline{X}(t)$, which satisfies the following functional SDE.

$$\mathrm{d}\overline{X}(t)=\overline{b}_1(\overline{X}_t)\mathrm{d}t+\sigma_1(\overline{X}_t)\mathrm{d}W_1(t),\quad \overline{X}_0=\xi\in\mathscr{C},\tag{4.1}$$

where $\overline{b}_1:\mathscr{C}\to\mathbb{R}^n$ is defined in (3.3). To achieve this goal, we need to construct some auxiliary two-time-scale stochastic systems with memory and provide a number of preliminary lemmas.

Throughout this article, we fix $T>0$ and set $\delta:=\frac{\tau}{N}\in(0,1)$ for a positive integer N sufficiently large. For any $t\in[0,T]$, consider the following auxiliary two-time-scale systems of functional SDEs.

$$\mathrm{d}\mathfrak{x}^\varepsilon(t)=b_1\big(X^\varepsilon_{t_\delta},\mathfrak{y}^\varepsilon_t\big)\mathrm{d}t+\sigma_1\big(X^\varepsilon_{t_\delta}\big)\mathrm{d}W_1(t),\ \ \mathfrak{x}^\varepsilon_0=X^\varepsilon_0=\xi\in\mathscr{C},\tag{4.2}$$

and

$$\begin{cases}\mathrm{d}\mathfrak{y}^\varepsilon(t)=\dfrac{1}{\varepsilon}b_2\big(X^\varepsilon_{t_\delta},\mathfrak{y}^\varepsilon(t),\mathfrak{y}^\varepsilon(t-\tau)\big)\mathrm{d}t+\dfrac{1}{\sqrt{\varepsilon}}\sigma_2\big(X^\varepsilon_{t_\delta},\mathfrak{y}^\varepsilon(t),\mathfrak{y}^\varepsilon(t-\tau)\big)\mathrm{d}W_2(t),\\ \mathfrak{y}^\varepsilon(t_\delta)=Y^\varepsilon(t_\delta)\end{cases}\tag{4.3}$$

with the initial value $\mathfrak{y}^\varepsilon_0=Y^\varepsilon_0=\eta\in\mathscr{C}$, where $t_\delta:=\lfloor t/\delta\rfloor\delta$, the nearest breakpoint preceding t, with $\lfloor t/\delta\rfloor$ being the integer part of t/δ.

To proceed, we present several preliminary lemmas. The first lemma concerns the continuity in the L^p sense for the displacement of the segment process $(X_t^\varepsilon)_{t\in[0,T]}$.

Lemma 4.1. *Under (**A1**) and (**A4**),*

$$\sup_{t\in[0,T]} \mathbb{E}\left\|X_t^\varepsilon - X_{t_\delta}^\varepsilon\right\|_\infty^p \lesssim_T \delta^{\frac{p-2}{2}}, \quad p > 2.$$

Proof. From (**A1**), it follows that,

$$|b_1(\chi,\phi)| \le L(1+\|\chi\|_\infty) \ \text{ and } \ \|\sigma_1(\phi)\| \le (L\vee\|\sigma_1(0)\|)(1+\|\phi\|_\infty).$$

Thus, in accordance with [28, Theorem 4.1, P.160], we have,

$$\mathbb{E}\Big(\sup_{0\le t\le T}\|X_t^\varepsilon\|_\infty^p\Big) \lesssim_T 1+\|\xi\|_\infty^p. \tag{4.4}$$

Observe that,

$$\begin{aligned}
\mathbb{E}\|X_t^\varepsilon - X_{t_\delta}^\varepsilon\|_\infty^p &\le \mathbb{E}\left(\max_{m=0,\dots,N-1}\ \sup_{-(m+1)\delta\le\theta\le-m\delta}|X^\varepsilon(t+\theta)-X^\varepsilon(t_\delta+\theta)|^p\right)\\
&\le N\max_{m=0,\dots,N-1}\mathbb{E}\left(\sup_{-(m+1)\delta\le\theta\le-m\delta}|X^\varepsilon(t+\theta)-X^\varepsilon(t_\delta+\theta)|^p\right)\\
&:= N\max_{m=0,\dots,N-1}J_p(t,m,\delta),
\end{aligned}$$

where $N=\tau/\delta$ by the definition of δ. To complete the proof of Lemma 4.1, it is sufficient to show,

$$J_p(t,m,\delta) \lesssim_T \delta^{\frac{p}{2}}. \tag{4.5}$$

For any $t\in[0,T]$ and any $\theta\in[-\tau,0]$, there exist $k,m\ge 0$ such that $t\in[k\delta,(k+1)\delta)$ and $\theta\in[-(m+1)\delta,-m\delta]$. Thus, one has,

$$t+\theta\in[(k-m-1)\delta,(k+1-m)\delta] \ \text{ and } \ t_\delta+\theta\in[(k-m-1)\delta,(k-m)\delta].$$

In what follows, we consider three cases.

Case 1. $m\le k-1$. Invoking Hölder's inequality and B-D-G's inequality, we obtain from (**A1**) and (4.4) that,

$$\begin{aligned}
&J_p(t,m,\delta)\\
&\lesssim \delta^{p-1}\int_{(k-m-1)\delta}^{t-m\delta}\mathbb{E}|b_1(X_s^\varepsilon,Y_s^\varepsilon)|^p\mathrm{d}s + \mathbb{E}\Big(\sup_{-(m+1)\delta\le\theta\le-m\delta}\Big|\int_{k\delta+\theta}^{t+\theta}\sigma_1(X_s^\varepsilon)\mathrm{d}W_1(s)\Big|^p\Big)\\
&\lesssim \delta^{p-1}\int_{(k-m-1)\delta}^{t-m\delta}\mathbb{E}|b_1(X_s^\varepsilon,Y_s^\varepsilon)|^p\mathrm{d}s + \mathbb{E}\Big(\Big|\int_{(k-m-1)\delta}^{t-(m+1)\delta}\sigma_1(X_s^\varepsilon)\mathrm{d}W_1(s)\Big|^p\Big)\\
&\quad+\mathbb{E}\left(\sup_{-(m+1)\delta\le\theta\le-m\delta}\left|\int_{t-(m+1)\delta}^{t+\theta}\sigma_1(X_s^\varepsilon)\mathrm{d}W_1(s)\right|^p\right)\\
&\quad+\mathbb{E}\left(\sup_{-(m+1)\delta\le\theta\le-m\delta}\left|\int_{(k-m-1)\delta}^{k\delta+\theta}\sigma_1(X_s^\varepsilon)\mathrm{d}W_1(s)\right|^p\right)\\
&\lesssim \delta^{p-1}\int_{(k-m-1)\delta}^{t-m\delta}\mathbb{E}|b_1(X_s^\varepsilon,Y_s^\varepsilon)|^p\mathrm{d}s + \delta^{\frac{p-2}{2}}\mathbb{E}\left(\int_{(k-m-1)\delta}^{t-(m+1)\delta}\|\sigma_1(X_s^\varepsilon)\|^p\mathrm{d}s\right)\\
&\quad+\mathbb{E}\left(\int_{t-(m+1)\delta}^{t-m\delta}\|\sigma_1(X_s^\varepsilon)\|^2\mathrm{d}s\right)^{p/2}+\mathbb{E}\left(\int_{(k-m-1)\delta}^{(k-m)\delta}\|\sigma_1(X_s^\varepsilon)\|^2\mathrm{d}s\right)^{p/2}\\
&\lesssim_T \delta^{\frac{p}{2}}.
\end{aligned}\tag{4.6}$$

Case 2. $m \geq k+1$. In view of (**A5**), it follows that,

$$|X^\varepsilon(t+\theta) - X^\varepsilon(t_\delta+\theta)|^p = |\xi(t+\theta) - \xi(t_\delta+\theta)|^p \lesssim \delta^p.$$

Case 3. $m = k$. Also, by Hölder's inequality and B-D-G's inequality, we deduce from (**A1**) and (4.4) that,

$$\begin{aligned}
J_p(t,m,\delta) &= \mathbb{E}\left(\sup_{-(k+1)\delta\leq\theta\leq-k\delta} |X^\varepsilon(t+\theta) - X^\varepsilon(k\delta+\theta)|^p\right)\\
&\lesssim \delta^p + \mathbb{E}\left(\sup_{-(k+1)\delta\leq\theta\leq-k\delta} (|X^\varepsilon(t+\theta) - X^\varepsilon(0)|^p \mathbf{1}_{\{t+\theta>0\}})\right)\\
&\lesssim \delta^p + \mathbb{E}\left(\sup_{-t\leq\theta\leq-k\delta}\left|\int_0^{t+\theta} b_1(X_s^\varepsilon, Y_s^\varepsilon)\mathrm{d}s\right|^p\right)\\
&\quad + \mathbb{E}\left(\sup_{-t\leq\theta\leq-k\delta}\left|\int_0^{t+\theta} \sigma_1(X_s^\varepsilon)\mathrm{d}W_1(s)\right|^p\right)\\
&\lesssim \delta^p + \delta^{p-1}\int_0^{t-k\delta} \mathbb{E}\big|b_1\big(X_s^\varepsilon, Y_s^\varepsilon\big)\big|^p \mathrm{d}s + \delta^{\frac{p-2}{2}}\int_0^{t-k\delta}\mathbb{E}\big\|\sigma_1\big(X_s^\varepsilon\big)\big\|^p\mathrm{d}s\\
&\lesssim {}_T\delta^{\frac{p}{2}}.
\end{aligned} \tag{4.7}$$

Consequently, the desired assertion (4.5) is finished by taking the discussions above into account. □

The lemma below provides an error bound of the difference in the strong sense between the slow component $(X^\varepsilon(t))$ and its approximation $(\mathfrak{x}^\varepsilon(t))$.

Lemma 4.2. *Assume that (**A1**) and (**A2**) hold and suppose further $\varepsilon/\delta \in (0,1)$. Then, there exists $\beta > 0$ such that,*

$$\mathbb{E}\left(\sup_{0\leq s\leq T} |X^\varepsilon(t) - \mathfrak{x}^\varepsilon(t)|^p\right) \lesssim_T \delta^{\frac{p-2}{2}}\left(1+\varepsilon^{-1}\mathrm{e}^{\frac{\beta\delta}{\varepsilon}}\right), \quad p>2.$$

Proof. In view of Hölder's inequality and B-D-G's inequality, it follows from (**A1**) and Lemma 4.1 that,

$$\begin{aligned}
\mathbb{E}\left(\sup_{0\leq s\leq t}|X^\varepsilon(s) - \mathfrak{x}^\varepsilon(s)|^p\right) &\lesssim {}_T\int_0^t \mathbb{E}\{\|X_s^\varepsilon - X_{s_\delta}^\varepsilon\|_\infty^p + \|Y_s^\varepsilon - \mathfrak{y}_s^\varepsilon\|_\infty^p\}\mathrm{d}s\\
&\lesssim {}_T\delta^{\frac{p-2}{2}} + \int_0^t \mathbb{E}\|Y_s^\varepsilon - \mathfrak{y}_s^\varepsilon\|_\infty^p \mathrm{d}s, \quad t\in(0,T].
\end{aligned}$$

Therefore, to finish the argument of Lemma 4.2, it suffices to show that there exists $\beta > 0$ such that,

$$\sup_{t\in[0,T]} \mathbb{E}\|Y_t^\varepsilon - \mathfrak{y}_t^\varepsilon\|_\infty^p \lesssim_T \varepsilon^{-1}\delta^{\frac{p-2}{2}}\mathrm{e}^{\frac{\beta\delta}{\varepsilon}}. \tag{4.8}$$

In what follows, we verify (4.8) by an induction argument. For any $t\in[0,\tau)$, due to $Y_0^\varepsilon = \mathfrak{y}_0^\varepsilon = \eta$, it is readily to check that,

$$\mathbb{E}\|Y_t^\varepsilon - \mathfrak{y}_t^\varepsilon\|_\infty^p \leq \sum_{j=0}^{\lfloor t/\delta\rfloor}\mathbb{E}\left(\sup_{j\delta\leq s\leq((j+1)\delta)\wedge t}|Y^\varepsilon(s) - \mathfrak{y}^\varepsilon(s)|^p\right) := I(t,\delta).$$

By means of Itô's formula and B-D-G's inequality, together with $\mathfrak{y}^\varepsilon(t_\delta) = Y^\varepsilon(t_\delta)$, we obtain from (**A2**) that,

$$\begin{aligned}
&\mathbb{E}\left(\sup_{j\delta\le s\le((j+1)\delta)\wedge t}|Y^\varepsilon(s)-\mathfrak{y}^\varepsilon(s)|^p\right)\\
&\quad\le\frac{c}{\varepsilon}\int_{j\delta}^{((j+1)\delta)\wedge t}\left\{\mathbb{E}\|X_s^\varepsilon-X_{s_\delta}^\varepsilon\|_\infty^p+\mathbb{E}|Y^\varepsilon(s)-\mathfrak{y}^\varepsilon(s)|^p\right\}\mathrm{d}s\\
&\quad\quad+\frac{1}{2}\mathbb{E}\left(\sup_{j\delta\le s\le((j+1)\delta)\wedge t}|Y^\varepsilon(s)-\mathfrak{y}^\varepsilon(s)|^p\right),\quad t\in[0,\tau].
\end{aligned}$$

Consequently, we conclude that,

$$\begin{aligned}
I(t,\delta)&\lesssim\frac{1}{\varepsilon}\int_0^t\mathbb{E}\|X_s^\varepsilon-X_{s_\delta}^\varepsilon\|_\infty^2\mathrm{d}s+\frac{1}{\varepsilon}\sum_{j=0}^{\lfloor t/\delta\rfloor}\int_{j\delta}^{((j+1)\delta)\wedge t}\mathbb{E}|Y^\varepsilon(s)-\mathfrak{y}^\varepsilon(s)|^p\mathrm{d}s\\
&\le\frac{1}{\varepsilon}\int_0^t\mathbb{E}\|X_s^\varepsilon-X_{s_\delta}^\varepsilon\|_\infty^2\mathrm{d}s+\frac{1}{\varepsilon}\int_0^\delta\sum_{j=0}^{\lfloor t/\delta\rfloor}\mathbb{E}\left(\sup_{j\delta\le r\le((j\delta+s))\wedge t}|Y^\varepsilon(r)-\mathfrak{y}^\varepsilon(r)|^p\right)\mathrm{d}s\\
&\lesssim\frac{1}{\varepsilon}\int_0^t\mathbb{E}\|X_s^\varepsilon-X_{s_\delta}^\varepsilon\|_\infty^2\mathrm{d}s+\frac{1}{\varepsilon}\int_0^\delta I(t,s)\mathrm{d}s.
\end{aligned}\tag{4.9}$$

This, combined with Lemma 4.1 and Gronwall's inequality, gives that,

$$\mathbb{E}\|Y_t^\varepsilon-\mathfrak{y}_t^\varepsilon\|_\infty^p\lesssim\varepsilon^{-1}\delta^{\frac{p-2}{2}}\mathrm{e}^{\frac{c\delta}{\varepsilon}},\qquad t\in[0,\tau)\tag{4.10}$$

for some $c>0$. Next, for any $t\in[\tau,2\tau)$, thanks to (4.10), it is immediate to note that,

$$\begin{aligned}
\mathbb{E}\|Y_t^\varepsilon-\mathfrak{y}_t^\varepsilon\|_\infty^p&\le\mathbb{E}\left(\|Y_\tau^\varepsilon-\mathfrak{y}_\tau^\varepsilon\|_\infty^p\right)+\mathbb{E}\left(\sup_{\tau\le s\le t}|Y^\varepsilon(s)-\mathfrak{y}^\varepsilon(s)|^p\right)\\
&\le c\left\{\varepsilon^{-1}\delta^{\frac{p-2}{2}}\mathrm{e}^{\frac{c\delta}{\varepsilon}}+\sum_{j=0}^{\lfloor t-\tau\rfloor}\mathbb{E}\left(\sup_{(N+j)\delta\le s\le((N+j+1)\delta)\wedge t}|Y^\varepsilon(s)-\mathfrak{y}^\varepsilon(s)|^p\right)\right\}\\
&:=c\left\{\varepsilon^{-1}\delta^{\frac{p-2}{2}}\mathrm{e}^{\frac{c\delta}{\varepsilon}}+M(t,\tau,\delta)\right\}.
\end{aligned}$$

Carrying out a similar argument to derive (4.9), we deduce from (4.10) that,

$$\begin{aligned}
M(t,\tau,\delta)&\lesssim\frac{1}{\varepsilon}\int_\tau^t\mathbb{E}\|X_s^\varepsilon-X_{s_\delta}^\varepsilon\|_\infty^2\mathrm{d}s\\
&\quad+\frac{1}{\varepsilon}\int_0^\delta\sum_{j=0}^{\lfloor t-\tau\rfloor}\mathbb{E}\left(\sup_{(N+j)\delta\le r\le((N+j)\delta+s)\wedge t}|Y^\varepsilon(r)-\mathfrak{y}^\varepsilon(r)|^p\right)\mathrm{d}s\\
&\quad+\frac{1}{\varepsilon}\int_0^\delta\sum_{j=0}^{\lfloor t-\tau\rfloor}\mathbb{E}\left(\sup_{j\delta\le s\le((j+1)\delta)\wedge(t-\tau)}|Y^\varepsilon(s)-\mathfrak{y}^\varepsilon(s)|^p\right)\mathrm{d}s\\
&\lesssim\frac{\delta^{\frac{p-2}{2}}}{\varepsilon}+\frac{\delta}{\varepsilon}\cdot\frac{\delta^{\frac{p-2}{2}}}{\varepsilon}\mathrm{e}^{\frac{c\delta}{\varepsilon}}+\frac{1}{\varepsilon}\int_0^\delta M(t,\tau,s)\mathrm{d}s.
\end{aligned}$$

Thus, the Gronwall inequality reads,

$$M(t,\tau,\delta) \lesssim \left\{ \frac{\delta^{\frac{p-2}{2}}}{\varepsilon} + \frac{\delta}{\varepsilon} \cdot \frac{\delta^{\frac{p-2}{2}}}{\varepsilon} \mathrm{e}^{\frac{c\delta}{\varepsilon}} \right\} \mathrm{e}^{\frac{c\delta}{\varepsilon}} \lesssim \frac{\delta}{\varepsilon} \cdot \frac{\delta^{\frac{p-2}{2}}}{\varepsilon} \mathrm{e}^{\frac{c\delta}{\varepsilon}} \lesssim \frac{\delta^{\frac{p-2}{2}}}{\varepsilon} \mathrm{e}^{\frac{c\delta}{\varepsilon}},$$

where we have used $\varepsilon/\delta \in (0,1)$ in the second step. Finally, (4.8) follows by repeating the previous procedure. □

The following consequence explores a uniform estimate w.r.t. the parameter ε for the segment process associated with the auxiliary fast motion.

Lemma 4.3. *Assume that (**A1**) and (**A3**) hold. Then, there exists a $C_T > 0$, independent of ε such that,*

$$\sup_{t\in[0,T]} \mathbb{E}\|\mathfrak{y}_t^\varepsilon\|_\infty^2 \le C_T. \tag{4.11}$$

Proof. From (2.2), it follows that,

$$\begin{aligned} Y^\varepsilon(t) &= \eta(0) + \int_0^{t/\varepsilon} b_2\big(X_{\varepsilon s}^\varepsilon, Y^\varepsilon(\varepsilon s), Y^\varepsilon(\varepsilon s - \tau)\big)\mathrm{d}t \\ &\quad + \int_0^{t/\varepsilon} \sigma_2\big(X_{\varepsilon s}^\varepsilon, Y^\varepsilon(\varepsilon s), Y^\varepsilon(\varepsilon s - \tau)\big)\mathrm{d}\overline{W}_2(s), \quad t > 0, \end{aligned} \tag{4.12}$$

where we used the fact that $\overline{W}(t) := \frac{1}{\sqrt{\varepsilon}} W_2(\varepsilon t)$ is a Brownian motion. For fixed $\varepsilon > 0$ and $t \ge 0$, let $\overline{Y}^\varepsilon(t+\theta) = Y^\varepsilon(\varepsilon t + \theta)$, $\theta \in [-\tau, 0]$. So, one has $\overline{Y}_t^\varepsilon = Y_{\varepsilon t}^\varepsilon$. Observe that (4.12) can be rewritten as follows.

$$\begin{aligned} \overline{Y}^\varepsilon(t/\varepsilon) &= \eta(0) + \int_0^{t/\varepsilon} b_2\left(X_{\varepsilon s}^\varepsilon, \overline{Y}^\varepsilon(s), \overline{Y}^\varepsilon(s-\tau)\right) \mathrm{d}s \\ &\quad + \int_0^{t/\varepsilon} \sigma_2\left(X_{\varepsilon s}^\varepsilon, \overline{Y}^\varepsilon(s), \overline{Y}^\varepsilon(s-\tau)\right) \mathrm{d}\overline{W}_2(s). \end{aligned}$$

Then, following the argument to obtain (3.11), for any $s > 0$ we can deduce that,

$$\mathbb{E}\|\overline{Y}_s^\varepsilon\|_\infty^2 \lesssim 1 + \|\eta\|_\infty^2 \mathrm{e}^{-\lambda s} + \mathbb{E}\left(\sup_{0\le r\le \varepsilon s} \|X_r^\varepsilon\|_\infty^2\right).$$

This, together with $\overline{Y}_t^\varepsilon = Y_{\varepsilon t}^\varepsilon$, gives that,

$$\mathbb{E}\|Y^{\varepsilon}_{\varepsilon s}\|^2_\infty \lesssim 1 + \|\eta\|^2_\infty e^{-\lambda s} + \mathbb{E}\left(\sup_{0\le r\le \varepsilon s} \|X^{\varepsilon}_r\|^2_\infty\right).$$

In particular, taking $s = t/\varepsilon$ we arrive at,

$$\mathbb{E}\|Y^{\varepsilon}_{t}\|^2_\infty \lesssim 1 + \|\eta\|^2_\infty + \mathbb{E}\left(\sup_{0\le r\le t} \|X^{\varepsilon}_r\|^2_\infty\right).$$

This, together with (4.4), yields that,

$$\sup_{t\in[0,T]} \mathbb{E}\|Y^{\varepsilon}_{t}\|^2_\infty \le C_T$$

for some $C_T > 0$. Observe from (4.8) and Hölder's inequality that,

$$\begin{aligned}\mathbb{E}\|\mathfrak{y}^{\varepsilon}_{t}\|^2_\infty &\le 2\mathbb{E}\|Y^{\varepsilon}_{t} - \mathfrak{y}^{\varepsilon}_{t}\|^2_\infty + 2\mathbb{E}\|Y^{\varepsilon}_{t}\|^2_\infty \\ &\lesssim_T 1 + \left(\varepsilon^{-1}\delta^{\frac{p-2}{2}} e^{\frac{\beta\delta}{\varepsilon}}\right)^{2/p}, \quad p > 4.\end{aligned}$$

Next, taking $\delta = \varepsilon(-\ln\varepsilon)^{\frac{1}{2}}$ in the estimate above and letting $y = (-\ln\varepsilon)^{\frac{1}{2}}$, we have,

$$\mathbb{E}\|\mathfrak{y}^{\varepsilon}_{t}\|^2_\infty \lesssim_T 1 + \left(e^{y^2}(e^{-y^2}y)^{\frac{p-2}{2}} e^{\beta y}\right)^{2/p}, \quad p > 4.$$

Then, the desired assertion follows since the leading term $e^{y^2}(e^{-y^2}y)^{\frac{p-2}{2}}e^{\beta y} \to 0$ as $y \uparrow \infty$ whenever $p > 4$. □

5. A Strong Limit Theorem for the Slow Component

With several preliminary lemmas at our hands, we are in position to present our main result.

Theorem 5.1. *Under (**A1**)-(**A4**), one has,*

$$\lim_{\varepsilon\to 0} \mathbb{E}\Big(\sup_{0\le t\le T} |X^{\varepsilon}(t) - \overline{X}(t)|^p\Big) = 0, \quad p > 0.$$

Proof. For any $t \in [0, T]$ and $p > 0$, set,

$$\Lambda(t) := \mathbb{E}\left(\sup_{0\le s\le t} |X^{\varepsilon}(s) - \overline{X}(s)|^p\right) \quad \text{and} \quad \Gamma(t) := \mathbb{E}\left(\sup_{0\le s\le t} |\mathfrak{x}^{\varepsilon}(s) - \overline{X}(s)|^p\right).$$

By Hölder's inequality, it is sufficient to verify that,

$$\lim_{\varepsilon\to 0} \Lambda(T) = 0, \quad p > 4. \tag{5.1}$$

In what follows, let $t \in [0, T]$ be arbitrary and assume $p > 4$. For any $t \in [0, T]$, it follows from Lemma 4.2 that,

$$\Lambda(t) \lesssim \mathbb{E}\left(\sup_{0\le s\le t} |X^{\varepsilon}(s) - \mathfrak{x}^{\varepsilon}(s)|^p\right) + \Gamma(t) \lesssim \delta^{\frac{p-2}{2}}\left(1 + \frac{1}{\varepsilon}e^{\frac{\beta\delta}{\varepsilon}}\right) + \Gamma(t). \tag{5.2}$$

Next, if we can show that,

$$\Gamma(t) \lesssim \delta^{\frac{p-2}{2}}\left(1 + \frac{1}{\varepsilon}e^{\frac{\beta\delta}{\varepsilon}}\right) + \varepsilon^{\frac{1}{2}}\delta^{p-\frac{3}{2}} + \int_0^t \Lambda(s)\mathrm{d}s, \tag{5.3}$$

by inserting (5.3) back into (5.2) and utilizing Gronwall's inequality, we deduce that

$$\Lambda(t) \lesssim \delta^{\frac{p-2}{2}}\left(1 + \frac{1}{\varepsilon}e^{\frac{\beta\delta}{\varepsilon}}\right) + \varepsilon^{\frac{1}{2}}\delta^{p-\frac{3}{2}}.$$

Thus, the desired assertion (5.1) follows by choosing $\delta = \varepsilon(-\ln\varepsilon)^{\frac{1}{2}}$. Indeed, it is easy to see that $\varepsilon/\delta \in (0,1)$, which is prerequisite in Lemma 4.2, for $\varepsilon \in (0,1)$ small enough, and that $\delta \to 0$ as $\varepsilon \downarrow 0$. Furthermore, let $y = (-\ln\varepsilon)^{\frac{1}{2}}$ (hence $\varepsilon = \mathrm{e}^{-y^2}$), which goes into infinity as ε tends to zero. Then, we have,

$$\Lambda(t) \lesssim (\mathrm{e}^{-y^2}y)^{\frac{p-2}{2}}\left(1+\mathrm{e}^{y^2+\beta y}\right) + \mathrm{e}^{-(p-1)y^2}y^{p-\frac{3}{2}},$$

which goes to zero by taking $p > 4$ and letting $y \uparrow \infty$.

Next, we aim to claim (5.3). Set,

$$\Gamma_p(t,\delta,\varepsilon) := \mathbb{E}\left(\sup_{0\le s\le t}\left|\int_0^s \{b_1(X^\varepsilon_{r_\delta},\mathfrak{y}^\varepsilon_r) - \bar{b}_1(X^\varepsilon_{r_\delta})\}\mathrm{d}r\right|^p\right),\quad t\in[0,T].$$

Applying Hölder's inequality, B-D-G's inequality, Lipschitz property of $\bar{b}_1$ due to Corollary 3.3, as well as Lemmas 4.1 and 4.2, we derive that,

$$\begin{aligned}
\Gamma(t) &\lesssim \mathbb{E}\left(\sup_{0\le s\le t}\left|\int_0^s \left\{b_1\left(X^\varepsilon_{r_\delta},\mathfrak{y}^\varepsilon_r\right) - \bar{b}_1(\bar{X}_r)\right\}\mathrm{d}r\right|^p\right) + \int_0^t \mathbb{E}\left\|\sigma_1\left(X^\varepsilon_{s_\delta}\right) - \sigma_1\left(\bar{X}_s\right)\right\|^p\mathrm{d}s \\
&\lesssim \Gamma_p(t,\delta,\varepsilon) + \int_0^t \mathbb{E}\left|\bar{b}_1(X^\varepsilon_{s_\delta}) - \bar{b}_1\left(X^\varepsilon_s\right)\right|^p\mathrm{d}s + \int_0^t \mathbb{E}\left|\bar{b}_1\left(X^\varepsilon_s\right) - \bar{b}_1\left(\mathfrak{x}^\varepsilon_s\right)\right|^p\mathrm{d}s \\
&\quad + \int_0^t \mathbb{E}\left|\bar{b}_1\left(\mathfrak{x}^\varepsilon_s\right) - \bar{b}_1\left(\bar{X}_s\right)\right|^p\mathrm{d}s + \int_0^t \mathbb{E}\left\|\sigma_1\left(X^\varepsilon_{s_\delta}\right) - \sigma_1\left(\bar{X}_s\right)\right\|^p\mathrm{d}s \\
&\lesssim \Gamma_p(t,\delta,\varepsilon) + \int_0^t \mathbb{E}\left\|X^\varepsilon_s - \mathfrak{x}^\varepsilon_s\right\|_\infty\mathrm{d}s + \int_0^t \mathbb{E}\left\|X^\varepsilon_{s_\delta} - X^\varepsilon_s\right\|^p_\infty\mathrm{d}s + \int_0^t \Gamma(s)\mathrm{d}s + \int_0^t \Lambda(s)\mathrm{d}s \\
&\lesssim \delta^{\frac{p-2}{2}} + \frac{1}{\varepsilon}\delta^{\frac{p-2}{2}}\mathrm{e}^{\frac{c\delta}{\varepsilon}} + \Gamma_p(t,\delta,\varepsilon) + \int_0^t \Gamma(s)\mathrm{d}s + \int_0^t \Lambda(s)\mathrm{d}s,
\end{aligned}$$

which, together with Gronwall's inequality, leads to,

$$\Gamma(t) \lesssim \delta^{\frac{p-2}{2}}\left(1+\frac{1}{\varepsilon}\mathrm{e}^{\frac{\beta\delta}{\varepsilon}}\right) + \Gamma_p(t,\delta,\varepsilon) + \int_0^t \Lambda(s)\mathrm{d}s, \tag{5.4}$$

where we have utilized the fact that $\Gamma_p(t,\delta,\varepsilon)$ is nondecreasing with respect to t. By comparing (5.3) with (5.4), we only need to prove,

$$\Gamma_p(t,\delta,\varepsilon) \lesssim \varepsilon^{\frac{1}{2}}\delta^{p-\frac{3}{2}}. \tag{5.5}$$

For any $p > 0$, let,

$$\Upsilon_p(k,\delta,\varepsilon) = \mathbb{E}\left(\left|\int_{k\delta}^{((k+1)\delta)\wedge t} \{b_1(X^\varepsilon_{k\delta},\mathfrak{y}^\varepsilon_s) - \bar{b}_1(X^\varepsilon_{k\delta})\}\mathrm{d}s\right|^p\right).$$

In the sequel, we show that (5.5) holds. By Hölder's inequality, we obtain that,

$$\begin{aligned}
\Gamma_p(t,\delta,\varepsilon) &= \mathbb{E}\left(\sup_{0\le s\le t}\left|\sum_{k=0}^{\lfloor s/\delta\rfloor}\int_{k\delta}^{((k+1)\delta)\wedge t}\left\{b_1\left(X^\varepsilon_{k\delta},\mathfrak{y}^\varepsilon_r\right) - \bar{b}_1\left(X^\varepsilon_{k\delta}\right)\right\}\mathrm{d}r\right|^p\right) \\
&\le \mathbb{E}\left(\sup_{0\le s\le t}\left((\lfloor s/\delta\rfloor+1)^{p-1}\sum_{k=0}^{\lfloor s/\delta\rfloor}\Upsilon_p(k,\delta,\varepsilon)\right)\right)
\end{aligned}$$

$$\begin{aligned}&\leq (\lfloor t/\delta\rfloor+1)^{p-1}\sum_{k=0}^{\lfloor t/\delta\rfloor}\Upsilon_p(k,\delta,\varepsilon)\\&\leq (\lfloor t/\delta\rfloor+1)^{p}\max_{0\leq k\leq\lfloor t/\delta\rfloor}\Upsilon_p(k,\delta,\varepsilon).\end{aligned}\tag{5.6}$$

For any $p'\in(1,2)$, by Hölder's inequality, (**A1**), and (4.4), observe that,

$$\begin{aligned}\Upsilon_p(k,\delta,\varepsilon)&\leq\Upsilon_2(k,\delta,\varepsilon)^{\frac{p'}{2}}\left(\mathbb{E}\left(\left|\int_{k\delta}^{((k+1)\delta)\wedge t}\{b_1(X^\varepsilon_{k\delta},\mathfrak{y}^\varepsilon_s)-\bar{b}_1(X^\varepsilon_{k\delta})\}\mathrm{d}s\right|^{\frac{2(p-p')}{2-p'}}\right)\right)^{\frac{2-p'}{2}}\\&\leq\Upsilon_2(k,\delta,\varepsilon)^{\frac{p'}{2}}\left(\delta^{\frac{2(p-p')}{2-p'}-1}\mathbb{E}\left(\left|\int_{k\delta}^{((k+1)\delta)\wedge t}\left|b_1(X^\varepsilon_{k\delta},\mathfrak{y}^\varepsilon_s)-\bar{b}_1(X^\varepsilon_{k\delta})\right|^{\frac{2(p-p')}{2-p'}}\mathrm{d}s\right|\right)\right)^{\frac{2-p'}{2}}\\&\lesssim\Upsilon_2(k,\delta,\varepsilon)^{\frac{p'}{2}}\delta^{\frac{2(p-p')}{2-p'}\times\frac{2-p'}{2}}\\&\lesssim\Upsilon_2(k,\delta,\varepsilon)^{\frac{p'}{2}}\delta^{p-p'},\quad p>4.\end{aligned}$$

Substituting this into (5.6), we arrive at,

$$\Gamma_p(t,\delta,\varepsilon)\lesssim\Upsilon_2(k,\delta,\varepsilon)^{\frac{p'}{2}}\delta^{-p'}.\tag{5.7}$$

Thus, to complete the argument by taking $p'\downarrow 1$ in (5.7), it remains to show that,

$$\Upsilon_2(k,\delta,\varepsilon)\lesssim\frac{\varepsilon}{\delta}.$$

For any $r\in[k\delta,(k+1)\delta)$, by the definition of $\mathfrak{y}^\varepsilon$, defined as in (4.3), it follows that,

$$\begin{aligned}\mathfrak{y}^\varepsilon(r)&=\mathfrak{y}^\varepsilon(k\delta)+\frac{1}{\varepsilon}\int_{k\delta}^{r}b_2(X^\varepsilon_{k\delta},\mathfrak{y}^\varepsilon(u),\mathfrak{y}^\varepsilon(u-\tau))\mathrm{d}u\\&\quad+\frac{1}{\sqrt{\varepsilon}}\int_{k\delta}^{r}\sigma_2(X^\varepsilon_{k\delta},\mathfrak{y}^\varepsilon(u),\mathfrak{y}^\varepsilon(u-\tau))\mathrm{d}W_2(u)\\&=\mathfrak{y}^\varepsilon(k\delta)+\int_0^{\frac{r-k\delta}{\varepsilon}}b_2(X^\varepsilon_{k\delta},\mathfrak{y}^\varepsilon(k\delta+\varepsilon u),\mathfrak{y}^\varepsilon(k\delta+\varepsilon u-\tau))\mathrm{d}u\\&\quad+\int_0^{\frac{r-k\delta}{\varepsilon}}\sigma_2(X^\varepsilon_{k\delta},\mathfrak{y}^\varepsilon(k\delta+\varepsilon u),\mathfrak{y}^\varepsilon(k\delta+\varepsilon u-\tau))\mathrm{d}\mathfrak{W}_2(u),\end{aligned}\tag{5.8}$$

where the shift $\mathfrak{W}_2(u):=(W_2(\varepsilon u+k\delta)-W_2(k\delta))/\sqrt{\varepsilon}$ is also a Wiener process. For fixed $\varepsilon>0$ and $u\geq 0$, let,

$$\bar{Y}^{X^\varepsilon_{k\delta}}(u+\theta)=\mathfrak{y}^\varepsilon(k\delta+\varepsilon u+\theta),\quad\theta\in[-\tau,0].$$

Then (5.8) can be rewritten as,

$$\begin{aligned}\bar{Y}^{X^\varepsilon_{k\delta}}\left(\frac{r-k\delta}{\varepsilon}\right)&=\mathfrak{y}^\varepsilon(k\delta)+\int_0^{\frac{r-k\delta}{\varepsilon}}b_2\left(X^\varepsilon_{k\delta},\bar{Y}^{X^\varepsilon_{k\delta}}(u),\bar{Y}^{X^\varepsilon_{k\delta}}(u-\tau)\right)\mathrm{d}u\\&\quad+\int_0^{\frac{r-k\delta}{\varepsilon}}\sigma_2\left(X^\varepsilon_{k\delta},\bar{Y}^{X^\varepsilon_{k\delta}}(u),\bar{Y}^{X^\varepsilon_{k\delta}}(u-\tau)\right)\mathrm{d}\mathfrak{W}_2(u).\end{aligned}$$

Consequently, by the weak uniqueness of solution, we arrive at,

$$\mathscr{L}(\mathfrak{y}^\varepsilon_r)=\mathscr{L}\left(\bar{Y}^{X^\varepsilon_{k\delta}}_{(r-k\delta)/\varepsilon}(\mathfrak{y}^\varepsilon_{k\delta})\right),\quad r\in[k\delta,(k+1)\delta),\tag{5.9}$$

where $\mathscr{L}(\zeta)$ denotes the law of random variable ζ. It can be seen that,

$$\Upsilon_2(k,\delta,\varepsilon)=\mathbb{E}\left(\left|\int_0^{\frac{((k+1)\delta)\wedge t-k\delta}{\varepsilon}}\{b_1(X_{k\delta}^\varepsilon,\mathfrak{y}_{\varepsilon s+k\delta}^\varepsilon)-\overline{b}_1(X_{k\delta}^\varepsilon)\}\mathrm{d}s\right|^2\right)$$
$$=2\int_0^{\frac{((k+1)\delta)\wedge t-k\delta}{\varepsilon}}\int_s^{\frac{((k+1)\delta)\wedge t-k\delta}{\varepsilon}}\mathbb{E}\big\langle\Lambda_0\big(\mathfrak{y}_{\varepsilon s+k\delta}^\varepsilon\big),\Lambda_0\big(\mathfrak{y}_{\varepsilon r+k\delta}^\varepsilon\big)\big\rangle\mathrm{d}r\mathrm{d}s,\qquad(5.10)$$

where, for $u\in[k\delta,(k+1)\delta]$,

$$\Lambda_0(\mathfrak{y}_u^\varepsilon):=b_1(X_{k\delta}^\varepsilon,\mathfrak{y}_u^\varepsilon)-\overline{b}_1(X_{k\delta}^\varepsilon).$$

Then, taking (5.9) and (5.10) into account and applying the property of conditional expectation yields that,

$$\begin{aligned}\Upsilon_2&(k,\delta,\varepsilon)\\&=2\int_0^{\frac{((k+1)\delta)\wedge t-k\delta}{\varepsilon}}\int_s^{\frac{((k+1)\delta)\wedge t-k\delta}{\varepsilon}}\mathbb{E}\big\langle\Lambda\big(\overline{Y}_s^{X_{k\delta}^\varepsilon}(\mathfrak{y}_{k\delta}^\varepsilon)\big),\Lambda\big(\overline{Y}_r^{X_{k\delta}^\varepsilon}(\mathfrak{y}_{k\delta}^\varepsilon)\big)\big\rangle\mathrm{d}r\mathrm{d}s\\&=2\int_0^{\frac{((k+1)\delta)\wedge t-k\delta}{\varepsilon}}\int_s^{\frac{((k+1)\delta)\wedge t-k\delta}{\varepsilon}}\mathbb{E}\Big(\mathbb{E}\big(\big\langle\Lambda\big(\overline{Y}_s^{X_{k\delta}^\varepsilon}(\mathfrak{y}_{k\delta}^\varepsilon)\big),\Lambda\big(\overline{Y}_r^{X_{k\delta}^\varepsilon}(\mathfrak{y}_{k\delta}^\varepsilon)\big)\big\rangle|\mathscr{F}_s\big)\Big)\mathrm{d}r\mathrm{d}s\\&=2\int_0^{\frac{((k+1)\delta)\wedge t-k\delta}{\varepsilon}}\int_s^{\frac{((k+1)\delta)\wedge t-k\delta}{\varepsilon}}\mathbb{E}\Big(\big\langle\Lambda\big(\overline{Y}_s^{X_{k\delta}^\varepsilon}(\mathfrak{y}_{k\delta}^\varepsilon)\big),\mathbb{E}\big(\Lambda\big(\overline{Y}_r^{X_{k\delta}^\varepsilon}(\mathfrak{y}_{k\delta}^\varepsilon)\big)|\mathscr{F}_s\big)\big\rangle\Big)\mathrm{d}r\mathrm{d}s\end{aligned}\qquad(5.11)$$

where, for $u\in(0,\frac{((k+1)\delta)\wedge t-k\delta}{\varepsilon})$,

$$\Lambda\big(\overline{Y}_u^{X_{k\delta}^\varepsilon}\big):=b_1\big(X_{k\delta}^\varepsilon,\overline{Y}_u^{X_{k\delta}^\varepsilon}(\mathfrak{y}_{k\delta}^\varepsilon)\big)-\overline{b}_1\big(X_{k\delta}^\varepsilon\big),$$

and $\mathscr{F}_s$ is the σ-algebra generated by $\overline{Y}_s^{X_{k\delta}^\varepsilon}(\mathfrak{y}_{k\delta}^\varepsilon)$. By the Markov property of $\overline{Y}_t^{X_{k\delta}^\varepsilon}(\mathfrak{y}_{k\delta}^\varepsilon)$, for any $r,s\in(0,\frac{((k+1)\delta)\wedge t-k\delta}{\varepsilon})$, we deduce from (5.11) that,

$$\begin{aligned}&\mathbb{E}\Big(\big\langle\Lambda\big(\overline{Y}_s^{X_{k\delta}^\varepsilon}(\mathfrak{y}_{k\delta}^\varepsilon)\big),\mathbb{E}\big(\Lambda\big(\overline{Y}_r^{X_{k\delta}^\varepsilon}(\mathfrak{y}_{k\delta}^\varepsilon)\big)|\mathscr{F}_s\big)\big\rangle\Big)\\&\quad=\mathbb{E}\Big(\big\langle\Lambda\big(\overline{Y}_s^{X_{k\delta}^\varepsilon}(\mathfrak{y}_{k\delta}^\varepsilon)\big),\mathbb{E}^{\overline{Y}_s^{X_{k\delta}^\varepsilon}(\mathfrak{y}_{k\delta}^\varepsilon)}\big(\Lambda\big(\overline{Y}_{r-s}^{X_{k\delta}^\varepsilon}(\mathfrak{y}_{k\delta}^\varepsilon)\big)\big)\big\rangle\Big)\end{aligned}\qquad(5.12)$$

Finally, we obtain from (3.2), (5.12), and Lemma 4.3 that,

$$\begin{aligned}\Upsilon_2(k,\delta,\varepsilon)&\lesssim\big(1+\mathbb{E}\|X_{k\delta}^\varepsilon\|_\infty^2+\mathbb{E}\|\mathfrak{y}_{k\delta}^\varepsilon\|_\infty^2\big)\int_0^{\frac{((k+1)\delta)\wedge t-k\delta}{\varepsilon}}\int_s^{\frac{((k+1)\delta)\wedge t-k\delta}{\varepsilon}}\mathrm{e}^{-\lambda(r-s)}\mathrm{d}r\mathrm{d}s\\&\lesssim\frac{\varepsilon}{\delta}.\end{aligned}$$

The proof is therefore complete. □

Remark 5.2. In this article, we only focus on the case that the diffusion coefficient of the slow component is independent of the fast motion. For the case that the slow component fully depends on the fast one, there is an illustrative counterexample [27, P. 1011] in which the weak convergence holds, but there is no strong convergence.

Remark 5.3. In the present article, we demonstrate a strong limit theorem for the averaging principle for a class of two-time-scale SDEs with memory under certain dissipative conditions. Nevertheless, our main result can be generalized to some cases, where the fast motion does not satisfy a dissipative condition. Indeed, by a close inspection of the argument of Theorem 5.1, to cope with the non-dissipative case, one of the crucial procedures is to discuss the ergodic property of the frozen equation without dissipativity. However, for some special cases, this problem has been addressed in [3].

Remark 5.4. Because time delay is ubiquitous, the results obtained here will be applicable to a wide range of applications in the areas as alluded to in the introduction section. They will encourage further investigation and study of two-time-scale delay systems.

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