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# Viscosity Solutions of Hamilton–Jacobi Equations in Infinite Dimensions. VII. The HJB Equation Is Not Always Satisfied

M. G. CRANDALL*

*Department of Mathematics, University of California, Santa Barbara, Santa Barbara, California 93106*

AND

P. L. LIONS

*Ceremade, Université Paris-Dauphine, Place de Lattre de Tassigny, 75775 Paris Cedex 16, France*

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Let $A = -\Delta$ with domain $H_0^1(\Omega) \cap H^2(\Omega)$ where $\Omega$ is open, smooth, and bounded. Run the state equation $dX_t/dt + AX_t = \alpha_t$ with the control $\alpha$ and the initial value $X_0 = x$ in $L^2(\Omega)$ to determine $X_t$. The values $\alpha_t$ of the control are constrained to lie in a fixed bounded subset $\mathscr{A}$ of $H^{-1}(\Omega)$. Given a running cost $f$, the value function $u(x) = \inf_\alpha \int_0^\infty e^{-t} f(X_t)\, dt$ is expected to satisfy the HJB equation of dynamic programming,

$$u + \langle Ax, \nabla u \rangle + \sup_{\gamma \in \mathscr{A}} \{ \langle -\gamma, \nabla u \rangle \} = f(x) \qquad \text{in} \quad L^2(\Omega).$$

The delicate situation in which the boundedness of $\mathscr{A}$ in $H^{-1}$ is coupled with uniform continuity of $f$ on $H^1(\Omega)$ is examined. An example is given in which $f(x) = \beta(\|x\|_{H^1(\Omega)})$ where $\beta$ is smooth and compactly supported and does not vanish identically and $\mathscr{A}$ is the unit ball, but $u \equiv 0$. On the positive side, it is proved that the value function may still be characterized as the maximal subsolution of the HJB equation. Part of the issue is to formulate the correct "viscosity" solution notions in this case. Conditions are given on $f$ for which $u$ is indeed a solution of the HJB equation, and it is shown that the value function is the uique solution when there is a solution. Various generalizations of the special cases above are presented.


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## Introduction

In this article we continue the (seeming endless) study of first-order Hamilton–Jacobi equations in infinite dimensions we began in Parts I–VI of this series ([7]). The approach used in these works involves finding a version of the notion of viscosity solutions, introduced originally in the finite dimensional setting, suitable for the prolems under discussion. See [3], [5], [6], and the recent survey papers [4], [15] for the finite dimensional origins of the theory.

Beginning with [7, Part IV], the study has focused on situations where the Hamilton–Jacobi equation is set in a real, separable, infinite dimensional Hilbert space $H$ and involves linear terms of the form $\langle Ax, \nabla u(x)\rangle$. Here, and everywhere below, $A$ is a linear and densely defined maximal monotone operator in $H$, $\langle \cdot, \cdot \rangle$ is the scalar product in $H$ (which is identified with its dual), and $\nabla u$ corresponds to the Fréchet differential of the real-valued function $u$ with respect to $x \in H$. Of course, $A$ is in general unbounded and the presence of the rather singular term $\langle Ax, \nabla u(x)\rangle$ requires some ad hoc treatment. Particular cases of some generality have been treated in [7, Parts IV and V] where, roughly speaking, solutions are requested to have some specific properties (such as weak continuity) and the rest of the equation is supposed to be nicely behaved. These behaviors and continuity properties are tuned to the type of monotone operator $A$ being dealt with.

Our tenacity in studying various cases arises from the desire to subsume as much as possible of the demanding range of situations one finds in corresponding control problems. Indeed, control problems involving the semigroup generated by $-A$ lead (see [1] and [7, Part V]) to the stationary and evolutionary Hamilton–Jacobi equations

$$u + \langle Ax, \nabla u\rangle + F(x, \nabla u) = 0 \qquad \text{in} \quad H \tag{S}$$

and

$$u_t + \langle Ax, \nabla u\rangle + F(t, x, \nabla u) = 0 \qquad \text{in} \quad (0, T] \times H, \tag{E}$$

where the solution $u(x)$ or $u(t, x)$ is a real-valued function defined on $H$ or on $[0, T] \times H$, $T > 0$ is given, and $F(t, x, p)$ and $F(x, p)$ are real-valued functions whose character depends on the details of the problem under consideration.

In order to explain the purpose of this paper and its relationship to the preceding ones of this series, we emphasize a "simple" example. Consider the evolution problem

$$\frac{dX_t}{dt} + AX_t = \alpha_t, \qquad X_0 = x, \tag{1}$$

where $X_t \in H$ represents the state of the system we are controlling at time $t$, $x \in H$ is the initial state, and $\alpha$—the control— is a bounded measurable function from $[0, \infty)$ to a given separable metric space $\mathscr{A}$ compatible with the discussion and whose structure is detailed below. We write $\mathscr{C}_{\mathscr{A}}$ for the set of control functions and $\alpha_t$ for the value at time $t$ of the control $\alpha \in \mathscr{C}_{\mathscr{A}}$. The solution of (1) is, of course, given by

$$X_t = e^{-At}x + \int_0^t e^{-A(t-s)}\alpha_s \, ds. \tag{2}$$

In addition, we have a cost functional

$$C(x, \alpha) = \int_0^\infty e^{-t} f(X_t) \, dt, \tag{3}$$

where $f$ is a given real-valued function, and a value function

$$u(x) = \inf_{\alpha \in \mathscr{C}_{\mathscr{A}}} C(x, \alpha). \tag{4}$$

The infimum is taken over all possible controls $\alpha$ that generate a solution of (1) via (2) for each initial condition $x \in H$. For example, if $f$ is bounded, then the infimum is certainly finite and it only remains to specify properties of $A$ and of $\mathscr{A}$ to make sure that the above control problem is meaningful.

If we follow the classical dynamic programming principle as in [1] or [15] or [19], we expect $u$ to solve, at least formally,

$$u + \langle Ax, \nabla u \rangle + \sup_{\gamma \in \mathscr{A}} \{ \langle -\gamma, \nabla u \rangle \} = f(x), \tag{5}$$

for some appropriate "viscosity" definition of the solution of (5).

Let us illustrate the type of situation we have in mind with a special case, which the reader may keep in mind to clarify what follows. Take $H = L^2(\Omega)$ where $\Omega$ is a smooth, bounded and open subset of $\mathbb{R}^N$ and $A = -\Delta$ on the domain $H^2(\Omega) \cap H_0^1(\Omega)$. That is, $-A$ is the $L^2(\Omega)$ realization of minus the Laplacian with Dirichlet boundary conditions. Then (1) is a controlled heat equation and (4) is the value function for the so-called infinite horizon optimal control problem with a running cost $f(x)$ and a discount factor normalized to be 1.

Included in the results of [7, Part V] is the fact that, in the above situation, if $f$ is bounded and uniformly continuous on $H$ and $\mathscr{A}$ is a bounded subset of $H$, then $u$ is the unique $B = (I + A)^{-1}$-continuous solution (see Part V for this notion) of (5) in $\mathrm{BUC}(H)$. Here and everywhere below, $\mathrm{BUC}(X)$ denotes the space of real-valued, bounded, uniformly continuous functions on a metric space $X$. (Recently, another formulation of (5) which would apply under these assumptions was proposed in Tataru [17, 18].

Moreover, this approach does not require $B$-continuity and allows $A$ to be a nonlinear maximal monotone operator. A simplified version of Tataru's approach is presented in [7, Part VI].)

However, in the case of the example above, this result is not satisfactory. Indeed, the requirement that $\mathscr{A}$ be bounded in $H$ means that only controls bounded in $L^2(\Omega)$ are allowed; in addition, the running cost is defined on all of $L^2(\Omega)$ and must be continuous on this space. In contrast, from the control point of view, one would like to allow controls which are bounded in $H^{-1}(\Omega)$ and allow $f$ to be defined only on $H_0^1(\Omega)$.

Our goal is precisely to investigate the abstract formulation of this and related situations. To begin, in Section I below we present a significant example in which $A$ is self-adjoint, $f$ is bounded, not constant, and uniformly continuous on the Hilbert space $H_1 = R((I+A)^{-1/2})$ which is equipped with the norm $\|x\|_1 = \|(I+A)^{1/2}x\|$ (where $\|x\| = \langle x, x\rangle^{1/2}$). Moreover, $\mathscr{A}$ is a bounded subset of $H_{-1} = (H_1)^*$, but our example has the striking feature that the value function $u$ is identically 0 on $H$! Of course, this $u$ enjoys all the continuity properties ever requested in [7, 12–14]. This vivid example indicates that (5) does not always hold. (However, as the rest of the investigation hints, there remains a possibility that one might yet find a subtle reinterpretation of the equation for which the value function is a solution.)

We then, in Section II, turn to the question of what inequalities of the sort that typically define viscosity solutions are in fact satisfied by the value function when $f$ in uniformly continuous on $H_1$ and $\mathscr{A}$ is bounded in $H_{-1}$. These inequalities are, as usual, implied by the "dynamic programming principle" (see [10, 15] for more information on this) and it turns out that, in particular, there are subleties indeed in the study of "supersolution inequalities."

In Section III we show that the value function may be characterized as the maximal function which satisfies the subsolution properties deduced in Section II (see [10] for the finite dimensional precursor of this result).

In Sections I–III we work in the setting of (5). In Section IV we present variants and extensions of the preceeding results to more general dynamics and running costs and to the evolution problem.

As regards generality, we note that while $A$ is always taken to be self-adjoint in this work, extensions to cases in which $A$ is not self-adjoint are possible and will be taken up in Part VIII of this series. Moreover, we observe that the situation (i.e., controls in $H_{-1}$, with running cost defined on $H_1$) considered here is extremal in comparison with the usual "bounded" situation (controls in $H = H_0$, running cost defined on $H_0$) and requires us to deal with nonlinearities in (E) or (S) which are locally unbounded "in $x$" and "in $\nabla u$"! In Part VIII, we shall close the gap between the two extreme cases by considering intermediate cases with "controls in $H_{-r}$ and running costs defined in $H_s$" with $0 \leqslant r \leqslant s \leqslant 1$. It turns out that a rather general uniqueness theory is possible if $r+s<2$. This illustrates the extremal nature of the current work again, as we deal here with the excluded case $r+s=2$.

## 1. Preliminaries and the Counterexample

In this short section, we present a few preliminaries and then the example promised in the introduction. After this, we deduce some elementary properties of more general value functions for use in later sections.

Throughout this work we assume that

$$A \text{ is a linear self-adjoint operator in } H \qquad \text{and} \qquad A \geqslant 0, \tag{A}$$

where $H$ is a real separable Hilbert space which carries the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$. We denote by $e^{-At}$ the strongly continuous contraction semigroup in $H$ generated by $-A$.

Next we define a few natural spaces and notations and collect some facts that we will use. We set

$$B=(I+A)^{-1}; \tag{6}$$

$B$ is a positive, self-adjoint, bounded operator on $H$. For $v>0$ we denote by $H_{-v}$ the Hilbert space which is the completion of $H$ in the norm

$$\|x\|_v = \langle B^v x, x \rangle^{1/2}.$$

In the natural way, $B^{v/2}$ extends to an isometry from $H_{-v}$ onto $H$ and $D(B^{-v/2})$ may be isometrically identified with the dual of $H_{-v}$ using the norm

$$\|x\|_v = \|B^{-v/2}x\|^2.$$

We denote this dual by $H_v$ and agree that $H_0 = H$. Then for any $v, \gamma \in \mathbb{R}$, $B^{\gamma/2}$ defines an isometry from $H_v$ onto $H_{v+\gamma}$. For any $v \in \mathbb{R}$, the operator $A$ extends by continuity to a bounded mapping of $H_v$ into $H_{v-2}$. In what follows, we mainly work with $H_{-1}$ and $H_1$ and we use both $\langle z, w \rangle$ and $\langle w, z \rangle$ to denote the inner product of $z, w \in H$ as well as the pairing of $z \in H_1$ and $w \in H_{-1}$. In particular, $\langle Ax, p \rangle$ is well-defined for $x \in H_1$, $p \in H_1$.

The semigroup $e^{-At}$ may be regarded as a strongly continuous semigroup on $H_v$; as such, its generator is the "part of $A$ in $H_v$." Moreover, we have $e^{-At}$: $H_v \to H_{v+\gamma}$ for all $t, \gamma > 0$ and, as a function of $t$, this mapping is continuous on $0 < t$. Finally, for each $\gamma > 0$ and $\nu \in \mathbb{R}$ there is a constant $C$ such that

$$\|e^{-At}x\|_{\nu+\gamma} \leqslant C\left(\frac{1}{t^{\gamma/2}} \vee 1\right) \|x\|_{\nu} \qquad \text{for} \quad x \in H_{\nu}. \tag{7}$$

These general considerations in hand, we introduce some notation used in the presentation of the example promised in the introduction. Let $\beta\colon \mathbb{R} \to [0, \infty)$ denote a function with the properties

$$\begin{aligned} &\beta \in C_0^{\infty}(\mathbb{R}) && \text{is even,} \\ &\beta(r) = 0 && \text{for} \quad |r| \geqslant 1, \\ &\beta > 0 && \text{on} \quad [-1, 1]. \end{aligned} \tag{8}$$

Here is the example.

PROPOSITION I.1. *Let* $\mathscr{A} = \{\gamma \in H_{-1} : \|\gamma\|_{-1} \leqslant 1\}$ *and* $f(x) = \beta(\|x\|_1)$. *If* (8) *holds and* $A$ *is unbounded, then the value function* $u$ *of* (4) *is identically* 0.

*Proof.* We begin with the relatively simple case in which $B = (I + A)^{-1}$ is compact. In view of $f \geqslant 0$, to show that $u = 0$ we need only show that $u \leqslant 0$. For this, we need to produce controls $\alpha_n$ such that $C(x, \alpha_n) \to 0$. Let $\lambda_n \to \infty$ be an increasing sequence of positive eigenvalues of $A$ and $x_n$ be corresponding unit eigenvectors (so that $Ax_n = \lambda_n x_n$, $\|x_n\| = 1$). Let $\alpha_n$ be the constant control given by

$$\alpha_n = (1 + \lambda_n)^{1/2} x_n. \tag{9}$$

Note that

$$\langle (1 + A)^{-1} \alpha_n, \alpha_n \rangle = 1$$

and

$$\left\| \int_0^t e^{-A(t-s)} \alpha_n \, ds \right\|_1^2 = \left( \frac{1 + \lambda_n}{\lambda_n} \right)^2 (1 - e^{-\lambda_n t})^2 \to 1 \qquad \text{as} \quad n \to \infty$$

if $t > 0$. In addition,

$$\int_0^t e^{-A(t-s)} \alpha_n \, ds = \frac{(1 + \lambda_n)^{1/2}}{\lambda_n} (1 - e^{-\lambda_n t}) \, x_n$$

converges weakly to zero in $H_1$ as $n \to \infty$ since $(1+\lambda_n)^{-1/2} x_n$ converges weakly to 0 in $H_1$ as $n \to \infty$. Therefore we have, for all $x \in H$, that

$$\left\| e^{-At}x + \int_0^t e^{-A(t-s)}\alpha_n \, ds \right\|_1^2 \to \|e^{-At}x\|_1^2 + 1 \qquad \text{as} \quad n \to \infty$$

for all $t > 0$. It then follows from (8) that

$$\beta\left( \left\| e^{-At}x + \int_0^t e^{-A(t-s)}\alpha_n \, ds \right\|_1 \right) \to 0$$

and, since $\beta$ is bounded,

$$\int_0^\infty \beta\left( \left\| e^{-At}x + \int_0^t e^{-A(t-s)}\alpha_n \, ds \right\|_1 \right) e^{-t} \, dt \to 0.$$

Thus $u \leqslant 0$ and the proof in this case is complete.

In the general case, we use the spectral resolution $E(\lambda)$ of $A$ to produce approximations of the above argument. Since we assumed that $A$ is unbounded, there are real numbers $\lambda_n$ and unit vectors $x_n$ such that

$$\lambda_n > 1 + \lambda_{n-1}, \qquad x_n \in R(E((\lambda_n + 1)-) - E(\lambda_n)), \ \|x_n\| = 1.$$

Let $\alpha_n$ be given by (9) as before; then

$$\|\alpha_n\|_{-1} = (1+\lambda_n)\langle (I+A)^{-1} x_n, x_n \rangle \leqslant \langle x_n, x_n \rangle = 1$$

and

$$\begin{aligned} \left\| \int_0^t e^{-A(t-s)}\alpha_n \, ds \right\|_1^2 &= (1+\lambda_n)\langle A^{-2}(I - e^{-At})^2 x_n, x_n \rangle \\ &\geqslant (1+\lambda_n)(1+\lambda_n)\frac{1}{(1+\lambda_n)^2}(1 - e^{-\lambda_n t})\langle x_n, x_n \rangle \\ &= (1 - e^{-\lambda_n t}) \end{aligned}$$

and we may conclude as above. ∎

We now establish some properties of more general value functions and various estimates of the solutions $X_t$ of (1) as given by (2). We collect some of these in the next lemma, assuming that

$$\begin{aligned} &\text{(i)} \quad \mathscr{A} \text{ is a bounded subset of } H_{-1} \text{ and carries the } H_{-1} \text{ metric} \\ &\text{(ii)} \quad \mathscr{C}_{\mathscr{A}} = \{\text{measurable maps } \alpha\colon [0, \infty) \to \mathscr{A}\}, \end{aligned} \tag{10}$$

as we do throughout this work.

PROPOSITION I.2. *Let $\alpha \in \mathscr{C}_{\mathscr{A}}$ and $x \in H$. Then (1) has a unique solution $X_t^\alpha \in L^2_{\mathrm{loc}}([0, \infty) : H_1) \cap C([0, \infty) : H)$ with $dX_t^\alpha/dt \in L^2_{\mathrm{loc}}([0, \infty) : H_{-1})$; it is given by (2). Moreover, for $\lambda > 0$ there is a constant $K$, independent of $\alpha \in \mathscr{C}_{\mathscr{A}}$, such that*

$$\int_0^\infty e^{-\lambda t} \|X_t^\alpha\|_1^2 \, dt \leqslant K < \infty. \tag{11}$$

*Next, if*

$$Y_t^\alpha = \int_0^t e^{-A(t-s)} \alpha_s \, ds, \tag{12}$$

*so that $X_t^\alpha = e^{-At}x + Y_t^\alpha$, then:*

(i) *For each $T > 0$ there is a constant $C$, independent of $\alpha \in \mathscr{C}_{\mathscr{A}}$, such that*

$$\|Y_t^\alpha\|^2 + \int_0^t \|Y_s^\alpha\|_1^2 \, ds \leqslant Ct \qquad \text{for} \quad 0 \leqslant t \leqslant T.$$

(ii) *We have*

$$\frac{1}{t} \int_0^t Y_s^\alpha \, ds \to 0 \qquad \text{weakly in} \quad H_1 \qquad \text{as} \qquad t \downarrow 0.$$

(iii) *If $\alpha_t \equiv \gamma \in H_{-1}$ is a constant, then $Y_t^\gamma \to 0$ in $H_1$ as $t \downarrow 0$.*

We precede the proof with a simple lemma, the point of which is to minimize repitition later on.

LEMMA I.3. *Let $U \in C([0, \infty) : \mathbb{R})$, $V \in L^1_{\mathrm{loc}}([0, \infty)$ be nonnegative, and $a, b, c \in [0, \infty)$. If*

$$\frac{dU}{dt} + aV \leqslant bU + c \qquad \text{on} \quad [0, \infty), \tag{13}$$

*then*

$$\begin{aligned} U(t) &\leqslant c\,\frac{e^{bt} - 1}{b} + e^{bt} U(0), \\ a \int_0^t e^{-bs} V(s) \, ds &\leqslant c\,\frac{1 - e^{-bt}}{b} + U(0), \end{aligned} \tag{14}$$

*where $(e^{bt} - 1)/b = t$ if $b = 0$, etc. In particular, for $\lambda > b > 0$,*

$$a \int_0^\infty e^{-\lambda t} (V(t) + 2U(t)) \, dt \leqslant \frac{c}{b} + U(0) + 2a\left(\frac{c}{b(\lambda - b)} + \frac{1}{\lambda - b}\, U(0)\right). \tag{15}$$

*Proof.* This is too trivial to prove, but we make some remarks anyway. Rewriting (13) as

$$\frac{d}{dt}(e^{-bt}U(t)) + e^{-bt}aV(t) \leqslant e^{-bt}c$$

and integration leads to

$$e^{-bt}U(t) + a\int_0^t e^{-bs}V(s)\,ds \leqslant U(0) + c\,\frac{e^{-bt}-1}{b}.$$

Since each term on the left is nonnegative, each is estimated by the right-hand side, leading to (14). Then (15) is proved by the use of (14) after casting out the negative terms on the right-hand side thereof and using $e^{-\lambda t} \leqslant e^{-bt}$. ∎

*Proof of Proposition* I.2. The existence and uniqueness assertions are standard. We begin with the estimate (11). Multiply the equation for $X_t^\alpha$ by $X_t^\alpha$ to find

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\|X_t^\alpha\|^2 + \langle AX_t^\alpha, X_t^\alpha\rangle &= \langle \alpha_t, X_t^\alpha\rangle \\ &\leqslant K\,\|X_t^\alpha\|_1 \\ &\leqslant \frac{K}{2\varepsilon} + \frac{K\varepsilon}{2}\,\|X_t^\alpha\|_1^2 \\ &= \frac{K}{2\varepsilon} + \frac{K\varepsilon}{2}\langle AX_t^\alpha, X_t^\alpha\rangle + \frac{K\varepsilon}{2}\,\|X_t^\alpha\|^2\end{aligned}$$

for $\varepsilon > 0$ where $K = \sup_{\gamma \in \mathscr{A}} \|\gamma\|_{-1}$. Hence

$$\frac{d}{dt}\frac{1}{2}\|X_t^\alpha\|^2 + \left(1 - \frac{K\varepsilon}{2}\right)\langle AX_t^\alpha, X_t^\alpha\rangle \leqslant \frac{K}{2\varepsilon} + \frac{K\varepsilon}{2}\,\|X_t^\alpha\|^2.$$

This is of the form of the lemma with $2U = \|X_t^\alpha\|^2$, $V = \langle AX_t^\alpha, X_t^\alpha\rangle$, $a = (1 - K\varepsilon/2)$, $b = K\varepsilon$, $c = K/2\varepsilon$. Since $\varepsilon$ may be taken arbitrarily small, (15) of Lemma I.3 provides an estimate on

$$\left(1 - \frac{K\varepsilon}{2}\right)\int_0^\infty e^{-\lambda s}(V(s) + 2U(s))\,ds = \left(1 - \frac{K\varepsilon}{2}\right)\int_0^\infty e^{-\lambda s}\,\|X_t^\alpha\|_1^2\,ds$$

when $\lambda > K\varepsilon$.

For (i), we may use the above remarks with $U(0) = 0$, noting that $(e^{bt} - 1)/b$ is bounded in the form $Ct$ when $0 \leqslant t \leqslant T$. For (ii), (i) implies that $(1/t)\int_0^t Y_s^\alpha\,ds$ is bounded in $H_1$ as $t \downarrow 0$. Since the limit exists strongly in $H$ and is 0, the result follows. For (iii), we recall the standard semigroup identity $AY_t^\alpha = (I - e^{-At})y$, which implies that $AY_t^y \to 0$ in $H_{-1}$ as $t \downarrow 0$. Since also $Y_t^\alpha \to 0$ in $H$, it follows that $(I+A)Y_t^\alpha \to 0$ in $H_{-1}$ and then $Y_t^\alpha \to 0$ in $H_1$. ∎

We examine the continuity properties of the value function $u$ of (4) under two conditions on $f$.

PROPOSITION I.4. *Let $\mathcal{A}$ be bounded in $H_{-1}$.*

(i) *If $f \in \mathrm{BUC}(H_1)$, the value function $u$ given by (4) satisfies $u \in \mathrm{BUC}(H_{-m})$ for all $m \geqslant 0$.*

(ii) *If $f \in \mathrm{UC}(H_1)$, the value function $u$ is well-defined and $u \in \mathrm{UC}(H_{-m})$ for $0 \leqslant m < 1$.*

*Remark* I.5. The value function is regarded as defined on $H$ by (4). However, when $f$ is bounded on $H_1$ the defining formulas are obviously meaningful for $x \in H_{-m}$, $m > 0$, and when $f$ satisfies $|f(x)| \leqslant C(1 + \|x\|_1)$ for some $C$ (e.g., when $f \in \mathrm{UC}(H_1)$), it is clear after what follows that the defining formulas are meaningful for $x \in H_{-m}$ for $0 \leqslant m < 1$. Here we just regard $u$ as given on $H$, and then for instance (abusing notation) "$u \in \mathrm{BUC}(H_{-m})$" means that $u$ is uniformly continuous with respect to the $H_{-m}$ norm and therefore $u$ extends by continuity to a unique function, still called $u$, on $H_{-m}$ which is uniformly continuous on this space.

*Proof of Proposition* I.4. We begin with (i). For $x, y \in H$ we have

$$|u(x) - u(y)| \leqslant \sup_{\alpha \in \mathscr{C}_{\mathcal{A}}} \int_0^\infty \left| f\left(e^{-At}x + \int_0^t e^{-A(t-s)}\alpha_s\, ds\right) - f\left(e^{-At}y + \int_0^t e^{-A(t-s)}\alpha_s\, ds\right)\right| e^{-t}\, dt$$

which implies, since $f$ is uniformly continuous on $H_1$, that

$$|u(x) - v(y)| \leqslant (2 \sup_{H_1} |f|)\, h + \int_h^\infty \omega(\|e^{-At}(x-y)\|_1)\, e^{-t}\, dt, \qquad (16)$$

where $\omega$ is the modulus of continuity of $f$ and hence is a bounded, nondecreasing, continuous function on $[0, \infty)$ such that $\omega(0) = 0$. We next observe that for $0 < h \leqslant t$ and all $z \in H$,

$$\begin{aligned}\|e^{-At}z\|_1 &= \langle (I+A)\, e^{-At}z, e^{-At}z\rangle^{1/2} \\ &\leqslant \langle (I+A)\, e^{-hA}z, e^{-hA}z\rangle^{1/2} = \|e^{-hA}z\|_1\end{aligned}$$

since $A$ is self-adjoint and nonnegative. Therefore, (16) implies that

$$|u(x)-u(y)| \leqslant (2 \sup_{H_1} |f|)h + \omega(\|e^{-hA}(x-y)\|_1). \tag{17}$$

Using (7), for $m>0$ and $h \in (0, 1]$, we have

$$\|e^{-hA}z\|_1 \leqslant Ch^{-(1+m)/2} \|z\|_{-m},$$

where $C \geqslant 0$ depends only on $m$. Hence (17) implies that

$$|u(x)-u(y)| \leqslant (2 \sup_{H_1} |f|)h + \omega(Ch^{-(1+m)/2} \|x-y\|_{-m});$$

it follows that $u$ is uniformly continuous in the $H_{-m}$ norm on $H$.

We turn to (ii). The value function is well-defined in view of the fact that $f \in \mathrm{UC}(H_1)$ implies a bound

$$|f(x)| \leqslant C(1+\|x\|_1) \qquad \text{for} \quad x \in H_1,$$

and we have (11) (which implies $\int_0^\infty e^{-\lambda t} \|X_t\|_1 \, dt$ is bounded by a constant independent of $\alpha$ since $e^{-\lambda t}\, dt$ is a finite measure). Let us record some estimates flowing from these remarks for future reference:

$$\left\{\begin{aligned}
&\text{(i)} \quad \int_0^\infty e^{-\lambda t} |f(X_t^\alpha)|^2 \, dt \\
&\qquad \leqslant C_1(\|x\|, \lambda) \qquad \text{for} \quad x \in H, \quad \alpha \in \mathscr{C}_{\mathscr{A}}, \quad \lambda > 0; \\
&\text{(ii)} \quad \int_0^\infty e^{-\lambda t} |f(X_t^\alpha)| \, dt \\
&\qquad \leqslant C_2(\|x\|, \lambda) \qquad \text{for} \quad x \in H, \quad \alpha \in \mathscr{C}_{\mathscr{A}}, \quad \lambda > 0; \\
&\text{(iii)} \quad \int_0^T |f(X_t^\alpha)|^2 \, dt \\
&\qquad \leqslant C_3(\|x\|, T)T \qquad \text{for} \quad x \in H, \quad \alpha \in \mathscr{C}_{\mathscr{A}}, \quad 0 \leqslant T < \infty; \\
&\text{(iv)} \quad \int_\Omega |f(X_t^\alpha)| \, dt \\
&\qquad \leqslant (\operatorname{meas}(\Omega))^{1/2} C_4(\|x\|)\sqrt{T} \\
&\qquad\qquad \text{for} \quad x \in H, \quad \alpha \in \mathscr{C}_{\mathscr{A}}, \quad \text{and} \quad \Omega \subset [0, T], 0 \leqslant T \leqslant 1.
\end{aligned}\right. \tag{18}$$

The character of the constants is clear.

Moreover, by the uniform continuity, for $\varepsilon > 0$ there exists $C_\varepsilon$ such that

$$|f(x) - f(y)| \leqslant \varepsilon + C_\varepsilon \|x - y\|_1. \tag{19}$$

Employing this estimate as above leads to

$$\begin{aligned}
|u(x) - u(y)| &\leqslant \varepsilon + C_\varepsilon \int_0^\infty \|e^{-At}(x-y)\|_1 e^{-t}\, dt \\
&\leqslant \varepsilon + C_\varepsilon \int_1^\infty \|e^{-At}(x-y)\|_1 e^{-t}\, dt \\
&\quad + C_\varepsilon \int_0^1 \|e^{-At}(x-y)\|_1 e^{-t}\, dt \\
&\leqslant \varepsilon + C_\varepsilon \|e^{-A}(x-y)\|_1 + C_\varepsilon \int_0^1 \|e^{-At}(x-y)\|_1 e^{-t}\, dt \\
&\leqslant \varepsilon + C_\varepsilon \|x-y\|_{-1} + C_\varepsilon \left( \int_0^1 C_m t^{1-m}\, dt \right) \|x-y\|_{-m} \\
&\leqslant \varepsilon + C_\varepsilon (1 + C_m/(1-m)) \|x-y\|_{-m}
\end{aligned}$$

for some constant $C_m > 0$ which depends only on $m$. ∎

## II. Viscosity Inequalities

Our goal in this section is to determine the "viscosity inequalities" satisfied by the value functions of the control problems under discussion. The demonstration of these inequalities, are usual, relies on a dynamic programming principle, which we formulate now.

LEMMA II.1. *Let $\mathscr{A} \subset H_{-1}$ be bounded and $f \in \mathrm{UC}(H_1)$. Then the value function $u$ of* (4) *satisfies*

$$u(x) = \inf_{\alpha \in \mathscr{C}_{\mathscr{A}}} \left\{ \int_0^h f(X_t)\, e^{-t}\, dt + u(X_h)^{-h} \right\} \tag{20}$$

*for $h > 0$ and $x \in H$ and where $X_t$ is given by* (2).

The proof of the lemma is standard (given (11)) and we skip it. Using this relation, one hopes to show (something like) that if $\varphi \in C^1(H : \mathbb{R})$ and $u - \varphi$ has a local maximum (respectively, minimum) at $x_0 \in H$, then

$$u(x_0) + \langle Ax_0, \nabla_\varphi(x_0) \rangle + \sup_{\gamma \in \mathscr{A}} \langle -\gamma, \nabla\varphi(x_0) \rangle \leqslant f(x_0) \tag{21}$$

(respectively,

$$u(x_0) + \langle Ax_0, \nabla\varphi(x_0)\rangle + \sup_{\gamma\in\mathscr{A}} \langle -\gamma, \nabla\varphi(x_0)\rangle \geqslant f(x_0)). \tag{22}$$

The various terms in these expressions are undefined in general, so restrictions must be placed on $\varphi$, one must worry about where $x_0$ lies, etc. If $x_0$, $\nabla\varphi(x_0)\in H_1$, the various terms are defined. In Theorem II.2 below, we give conditions on $\varphi$ under which it can be proved that (21) holds and this can be taken as a definition of "$u$ as a subsolution of the Hamilton–Jacobi equation"

$$u + \langle Ax, \nabla u\rangle + \sup_{\gamma\in\mathscr{A}} \langle -\gamma, \nabla u\rangle = f, \tag{23}$$

associated with the control problem under discussion. In Theorem II.4 the more delicate case of (22) is discussed and the result has a different character; $u$ is not, in general, a supersolution of (23), corresponding to the subtleties observed in the example of Proposition I.1. These results are both complex and ad hoc, but this is due to the nature of the issues at hand. In the next section, we support them by showing the extent to which our value function is determined by the results below.

We begin the discussion of "subsolution" inequalities. To facilitate writing the implications of Lemma II.1 efficiently, for each $\gamma\in\mathscr{A}$ we consider the solution $X_t^\gamma$ of

$$\frac{dX_t^\gamma}{dt} + AX_t^\gamma = \gamma \qquad \text{for} \quad t\geqslant 0, \quad X_0^\gamma = x. \tag{24}$$

We may write this solution in the form $X_t^\gamma = e^{-tA^\gamma}x$ where $A^\gamma$ is the operator in $H$ defined by

$$D(A^\gamma) = \{x\in H_1 : Ax-\gamma\in H\}, \quad A^\gamma x = Ax-\gamma \qquad \text{for} \quad x\in D(A^\gamma).$$

Here we regard $A$ as a mapping from $H_1$ into $H_{-1}$ in the natural way. It is clear that $A^\gamma$ is a densely defined maximal monotone operator in $H$. We set

$$S_t^\gamma = e^{-tA^\gamma};$$

that is, $S_t^\gamma$ is the semigroup generated by $-A^\gamma$. Estimating the right-hand side of (20) from above by choosing the constant control $\alpha_t\equiv\gamma$, we arrive at

$$\frac{1}{h}(u(x)-u(S_h^\gamma x)) + u(S_h^\gamma x)\frac{1-e^{-h}}{h}$$
$$\leqslant \frac{1}{h}\int_0^h f(S_t^\gamma x)\,dt + \frac{1}{h}\int_0^h f(S_t^\gamma x)(e^{-t}-1)\,dt; \tag{25}$$

this relation is valid for $x \in H$ and $h > 0$. By Proposition I.4 $u$ is continuous in $H$, while $S_h^\gamma x \to x$ in $H$ as $h \downarrow 0$; moreover, the last term in (25) tends to 0 as $h \downarrow 0$ for, by (18.iv),

$$\left| \int_0^h f(S_t^\gamma x)\, dt \right| \leqslant C\sqrt{h}.$$

Hence we may take limits in (25) to find

$$u(x) + \limsup_{h \downarrow 0} \frac{1}{h}(u(x) - u(S_h^\gamma x)) \leqslant \limsup_{h \downarrow 0} \frac{1}{h} \int_0^h f(S_t^\gamma x)\, dt. \tag{26}$$

Next, Proposition I.2.iii implies that

$$\|S_h^\gamma x - e^{-hA} x\|_1 \to 0 \qquad \text{as} \quad h \downarrow 0$$

Since $f \in \mathrm{UC}(H_1)$, we conclude that

$$\tilde{f}(x) \overset{\text{def}}{=} \limsup_{h \downarrow 0} \frac{1}{h} \int_0^h f(S_t^\gamma x)\, dt = \limsup_{h \downarrow 0} \frac{1}{h} \int_0^h f(e^{-At} x)\, dt \tag{27}$$

depends only on $x$. Since $e^{-At} x \to x$ in $H_1$ as $t \downarrow 0$ when $x \in H_1$, we have

$$\tilde{f}(x) = f(x) \qquad \text{for} \quad x \in H_1. \tag{28}$$

With the above notation, (26) becomes

$$u(x) + \limsup_{h \downarrow 0} \frac{1}{h}(u(x) - u(S_h^\gamma x)) \leqslant \tilde{f}(x) \qquad \text{for} \quad x \in H, \quad \gamma \in \mathscr{A}. \tag{29}$$

To obtain our first viscosity inequality, we note that if $\varphi$ is continuous on $H$ and $u - \varphi$ has a local (in $H$) maximum point $x_0$, then

$$\varphi(x_0) - \varphi(S_h^\gamma x_0) \leqslant u(x_0) - u(S_h^\gamma x_0)$$

for small $h$, and so, by (29) with $x = x_0$,

$$u(x_0) + \limsup_{h \downarrow 0} \frac{1}{h}(\varphi(x_0) - \varphi(S_h^\gamma x_0)) \leqslant \tilde{f}(x_0) \qquad \text{for} \quad \gamma \in \mathscr{A}. \tag{30}$$

Our goal now is to replace the term involving $\varphi$ in (30) by the expression

$$\langle A x_0, \nabla\varphi(x_0)\rangle + \sup_{\gamma \in \mathscr{A}} \langle -\gamma, \nabla\varphi(x_0)\rangle,$$

the "correct" Hamiltonian evaluated at $\nabla\varphi(x_0)$. To this end, we have to restrict $\varphi$. Let

$$\varphi = \varphi_1 + \varphi_2. \tag{31}$$

The required properties of the terms in this decomposition are detailed below. In what follows, $x_0$ is a fixed local maximum of $u-\varphi$. First, we assume that

$$\varphi_1 \text{ has the form } \quad \varphi_1(x) = \varphi_1(\|x\|) \tag{32}$$

and

$$\varphi_1 \in C^1([0, \infty)), \quad \varphi_1'(r) > 0 \qquad \text{for} \quad r > 0, \qquad \text{and} \qquad \varphi_1'(0) = 0; \tag{33}$$

note our use of $\varphi_1$ to mean both a function on $[0, \infty)$ and the radial function on $H$ which it defines. Correspondingly, we write $\nabla\varphi_1(x) = (\varphi_1'(\|x\|)/\|x\|)x$, understanding this to be 0 if $x = 0$. Second, we assume that

$$\begin{aligned} &\text{(i)} \quad \varphi_2 \in C^1(H), \\ &\text{(ii)} \quad \nabla\varphi_2 \in C(H, H_1). \end{aligned} \tag{34}$$

In (ii) above the meaning is that $\nabla\varphi_2$ (computed in the topology of $H$) maps $H$ into $H_1$ and it continuous from the $H$ topology into the $H_1$ topology. Hereafter, we leave it to the reader to interpret statements like this. A typical example, much used in [7, Parts IV and V], is given by

$$\varphi_2(x) = \langle Bx, x\rangle = \langle (I+A)^{-1}x, x\rangle; \qquad \nabla\varphi_2(x) = 2(I+A)^{-1}x.$$

THEOREM II.2. *Let $\mathscr{A}$ be bounded in $H_{-1}$, $f \in \mathrm{UC}(H_1)$, (31)–(34) hold, and u be the value function. If $u-\varphi$ has a local (in $H$) maximum at $x_0$, then $x_0 \in H_1$, and*

$$u(x_0) + \langle Ax_0, \nabla\varphi(x_0)\rangle + \sup_{\gamma \in \mathscr{A}} \langle -\gamma, \nabla\varphi(x_0)\rangle \leqslant f(x_0). \tag{35}$$

*Remark* II.3. Recall that we use both $\langle x, y\rangle$ and $\langle y, x\rangle$ to denote the pairing of $x \in H_m$ and $y \in H_{-m}$ as well as the inner product on $H$. Since $x_0 \in H_1$ implies

$$\nabla\varphi(x_0) = \frac{\varphi_1'(\|x_0\|)}{\|x_0\|}\,x_0 + \nabla\varphi_2(x_0) \in H_1$$

and $A\colon H_1 \to H_{-1}$, all the terms in (35) make sense.

*Proof of Theorem* II.2. We begin with the proof that $x_0 \in H_1$. Since $|f(S_h^\gamma x_0)| \leqslant C(1 + \|S_h^\gamma x_0\|_1)$ by the uniform continuity, we deduce from (25) that

$$\varphi_1(\|x_0\|) - \varphi_1(\|S_h^\gamma x_0\|) \leqslant \varphi_2(S_h^\gamma x_0) - \varphi_2(x_0) + C\left(h + \int_0^t \|S_h^\gamma x_0\|_1\, dt\right),$$

where the constant $C$ varies from occurrence to occurrence but is independent of $0 \leqslant h \leqslant 1$. Hence, using (34),

$$\begin{aligned}
&\int_0^h (\varphi_1'(\|S_t^\gamma x_0\|)/\|S_t^\gamma x_0\|)\langle S_t^\gamma x_0, AS_t^\gamma x_0 - \gamma\rangle\, dt \\
&\quad \leqslant -\int_0^h \langle \nabla\varphi_2(S_t^\gamma x_0), AS_t^\gamma x_0 - \gamma\rangle\, dt + C\left(h + \int_0^t \|S_h^\gamma x_0\|_1\, dt\right) \\
&\quad \leqslant C\left(h + \int_0^h \|S_t^\gamma x_0\|_1\, dt\right). \qquad (36)
\end{aligned}$$

We may assume that $x_0 \neq 0$ (since $0 \in H_1$) and then $\varphi_1'(\|S_t^\gamma x_0\|)/\|S_t^\gamma x_0\| \geqslant c > 0$ for small $h > 0$; moreover,

$$\begin{aligned}
&\int_0^h \langle S_t^\gamma x_0, AS_t^\gamma x_0 - \gamma\rangle\, dt \\
&\quad = \int_0^h (\langle S_t^\gamma x_0, (I + A)\, S_t^\gamma x_0\rangle - \langle S_t^\gamma x_0, \gamma\rangle - \|S_t^\gamma x_0\|^2)\, dt \\
&\quad \geqslant \int_0^h \|S_t^\gamma x_0\|_1^2\, dt - C\left(h + \int_0^h \|S_t^\gamma x_0\|_1\, dt\right).
\end{aligned}$$

Entering these considerations into (36), we deduce that

$$\int_0^h \|S_t^\gamma x_0\|_1^2\, dt \leqslant C\left(h + \int_0^h \|S_t^\gamma x_0\|_1\, dt\right)$$

which easily leads to

$$\frac{1}{h}\int_0^h \|S_t^\gamma x_0\|_1^2\, dt \leqslant C.$$

Thus $(1/h)\int_0^h S_t^\gamma x_0\, dt$ is bounded in $H_1$ and converges to $x_0$ in $H$ as $h \downarrow 0$; we conclude that $x_0 \in H_1$.

The rest of the proof is straightforward. We need only note that, since $S_t^\gamma x_0 \to x_0$ in $H_1$ by $x_0 \in H_1$ and Lemma I.2,

$$\begin{aligned}
&\lim_{h \downarrow 0} \frac{1}{h} (\varphi_1(x_0) - \varphi_1(S_h^\gamma x_0)) \\
&\quad = \lim_{h \downarrow 0} \frac{1}{h} \left( \int_0^h (\varphi_1'(\|S_t^\gamma x_0\|)/\|S_t^\gamma x_0\|) \langle S_t^\gamma x_0, AS_t^\gamma x_0 - \gamma \rangle \, dt \right) \\
&\quad = \frac{\varphi_1'(\|x_0\|)}{\|x_0\|} \langle Ax_0 - \gamma, x_0 \rangle \\
&\quad = \langle \nabla\varphi_1(x_0), Ax_0 - \gamma \rangle
\end{aligned}$$

and

$$\begin{aligned}
\lim_{h \downarrow 0} \frac{1}{h} (\varphi_2(x_0) - \varphi_2(S_h^\gamma x_0)) &= \lim_{h \downarrow 0} \frac{1}{h} \left( \int_0^h \langle \nabla\varphi_2(S_t^\gamma x_0), AS_t^\gamma x_0 - \gamma \rangle \, dt \right) \\
&= \langle \nabla\varphi_2(x_0), Ax_0 - \gamma \rangle
\end{aligned}$$

and then invoke (30). ∎

We turn to the consideration of "supersolution" inequalities. As the reader can see, a primary difference with the subsolution case is that it is neccessary to deal with general controls in this discussion; it does not suffice to take constants as above. As in the subsolution case, a new function $\check{f}$ built from the ingredients of the problem plays a role; it is given by

$$\check{f}(x) = \liminf_{h \downarrow 0} \inf_{\alpha \in \mathscr{C}_{\mathscr{A}}} \frac{1}{h} \int_0^h f\left( e^{-At}x + \int_0^t e^{-A(t-s)}\alpha_s \, ds \right) dt. \tag{37}$$

Since $\int_0^t e^{-A(t-s)}\alpha_s \, ds$ does not converge to 0 in $H_1$ as $h \downarrow 0$ in general, we may not delete this term in the definition of $\check{f}$. By uniform continuity, we do have

$$\check{f}(x) = \liminf_{h \downarrow 0} \inf_{\alpha \in \mathscr{C}_{\mathscr{A}}} \frac{1}{h} \int_0^h f\left( x + \int_0^t e^{-A(t-s)}\alpha_s \, ds \right) dt \qquad \text{for} \quad x \in H_1. \tag{38}$$

THEOREM II.4. *Let $\mathscr{A}$ be bounded in $H_{-1}$ and $f \in \mathrm{UC}(H_1)$. Let $x_0 \in H$ be a local (in $H$) minimum point of $u - \varphi$ where $\varphi = \varphi_1 + \varphi_2$, $\varphi_1, \varphi_2 \in C^1(H)$, $\varphi_1 = \varphi_1(\|x\|)$ is nonincreasing with respect to $\|x\|$, $\nabla\varphi_2 \in C(H, H_1)$. Assume that either $x_0 \in H_1$ or $\varphi_1'(\|x_0\|) < 0$. Then $x_0 \in H_1$ and we have*

$$u(x_0) + \langle Ax_0, \nabla\varphi(x_0) \rangle + \sup_{\gamma \in \mathscr{A}} (-\langle \gamma, \nabla\varphi(x_0) \rangle) \geq \check{f}(x_0). \tag{39}$$

*In particular, if*

$$\check{f}(x_0) \geqslant f(x_0), \tag{40}$$

*then*

$$u(x_0) + \langle Ax_0, \nabla\varphi(x_0)\rangle + \sup_{\gamma \in \mathscr{A}}(-\langle \gamma, \nabla\varphi(x_0)\rangle) \geqslant f(x_0). \tag{41}$$

*Finally,* (40) *holds if $f$ either is convex or satisfies the condition*

$$f(x_0) \leqslant \liminf_{n \to \infty} f(x_n) \qquad \text{whenever } x_n \to x_0 \text{ strongly in } H \text{ and weakly in } H_1. \tag{42}$$

*Remark* II.5. The terms in the above equations make sense since $x_0 \in H_1$, $\nabla\varphi(x_0) \in H_1$.

*Remark* II.6. In the example of Proposition I.1, the left-hand side of (40) is identically 0 and (40) does not hold; this "explains" the degeneracy in the example.

*Remark* II.7. The inequality (40) holds in other circumstances—e.g., if $\mathscr{A}$ is a singleton.

*Proof.* The proof that $x_0 \in H_1$ is analogous to the corresponding assertion of Theorem II.2 and we do not repeat it. With the notation of Proposition I.2, observe that $X_t^\alpha \to x_0$ as $t \downarrow 0$ uniformly in $\alpha \in \mathscr{C}_{\mathscr{A}}$ (for fixed $x_0$) by (i) of the proposition and the boundedness of $\mathscr{A}$, and thus for $h$ small enough we deduce from Lemma II.1 that

$$u(x_0) \geqslant \frac{1}{h} \inf_{\alpha \in \mathscr{C}_{\mathscr{A}}} \int_0^h f(X_t^\alpha)\,dt + \inf_{\alpha \in \mathscr{C}_{\mathscr{A}}} \frac{1}{h}(\varphi(X_h^\alpha) - \varphi(x_0)) + \varepsilon(h), \tag{43}$$

where $\varepsilon(h)$, in all that follows, denotes various constants, independent of $\alpha$, which tend to 0 with $h$.

Next we write

$$\begin{aligned} \frac{1}{h}(\varphi_1(X_h^\alpha) - \varphi_1(x_0)) &= -\frac{1}{h}\int_0^h \frac{\varphi_1'(\|X_t^\alpha\|)}{\|X_t^\alpha\|} \langle X_t^\alpha, AX_t^\alpha - \alpha_t\rangle\,dt \\ &= -\frac{\varphi_1'(\|x_0\|)}{\|x_0\|}\frac{1}{h}\int_0^h \langle X_t^\alpha, AX_t^\alpha - \alpha_t\rangle\,dt + \varepsilon(h); \end{aligned} \tag{44}$$

this is due to the fact that $X_t^\alpha = e^{-At}x_0 + Y_t^\alpha$ where $e^{-At}x_0$ converges to $x_0$ in $H_1$ while

$$\int_0^t \|Y_s^\alpha\|_1^2\,ds \leqslant Ct \tag{45}$$

for $t \leqslant 1$ by Proposition I.2.i and the boundedness of $\mathscr{A}$. Using these remarks again,

$$\begin{aligned}
&\frac{1}{h}\int_0^h \langle X_t^\alpha, AX_t^\alpha - \alpha_t\rangle\, dt \\
&\quad = \frac{1}{h}\int_0^h \langle e^{-At}x_0, AX_t^\alpha - \alpha_t\rangle + \langle Y_t^\alpha, AX_t^\alpha - \alpha_t\rangle\, dt \\
&\quad = \frac{1}{h}\int_0^h \langle x_0, Ax_0 + AY_t^\alpha - \alpha_t\rangle + \langle Y_t^\alpha, AY_t^\alpha - \alpha_t\rangle\, dt \\
&\qquad + \frac{1}{h}\int_0^h \langle Y_t^\alpha, Ax_0\rangle\, dt + \varepsilon(h).
\end{aligned} \tag{46}$$

Continuing, we note that

$$\frac{1}{h}\int_0^h Y_t^\alpha\, dt \to 0 \qquad \text{weakly in} \quad H_1 \qquad \text{as} \quad h \downarrow 0 \tag{47}$$

uniformly in $\alpha \in \mathscr{C}_{\mathscr{A}}$ by the proof of Proposition I.2.ii. In addition,

$$\frac{1}{h}\int_0^h \langle Y_t^\alpha, AY_t^\alpha - \alpha_t\rangle\, dt = \frac{1}{2h}\|Y_h^\alpha\|^2 \geqslant 0. \tag{48}$$

Therefore, we have

$$\begin{aligned}
&-\frac{\varphi_1'(\|x_0\|)}{\|x_0\|}\frac{1}{h}\int_0^h \langle X_t^\alpha, AX_t^\alpha - \alpha_t\rangle\, dt \\
&\quad \geqslant -\frac{\varphi_1'(\|x_0\|)}{\|x_0\|}\left(\langle Ax_0, x_0\rangle - \left\langle x_0, \frac{1}{h}\int_0^h \alpha_t\, dt\right\rangle\right) + \varepsilon(h).
\end{aligned} \tag{49}$$

We use this information in (44) to conclude that

$$\begin{aligned}
\frac{1}{h}(\varphi_1(X_h^\alpha) - \varphi_1(x_0)) &\geqslant -\frac{\varphi_1'(\|x_0\|)}{\|x_0\|}\left(\langle Ax_0, x_0\rangle - \left\langle x_0, \frac{1}{h}\int_0^h \alpha_t\, dt\right\rangle\right) + \varepsilon(h) \\
&= -\langle Ax_0, \nabla\varphi_1(x_0)\rangle \\
&\quad + \left\langle \nabla\varphi_1(x_0), \frac{1}{h}\int_0^h \alpha_t\, dt\right\rangle + \varepsilon(h).
\end{aligned} \tag{50}$$

Turning to the contribution from $\varphi_2$, we begin with

$$\frac{1}{h}(\varphi_2(X_h^\alpha)-\varphi_2(x_0))$$

$$=-\frac{1}{h}\int_0^h \langle \nabla\varphi_2(X_t^\alpha), AX_t^\alpha-\alpha_t\rangle\, dt$$

$$=-\frac{1}{h}\int_0^h \langle (I+A)^{1/2}\nabla\varphi_2(x_0), (I+A)^{-1/2}(AX_t^\alpha-\alpha_t)\rangle\, dt+\varepsilon(h). \tag{51}$$

This follows from the same considerations used to establish (44) and $\nabla\varphi_2\in C(H:H_1)$.

Turning to the contribution from $\varphi_2$, we begin with

$$\frac{1}{h}(\varphi_2(X_h^\alpha)-\varphi_2(x_0))=-\frac{1}{h}\int_0^h \langle \nabla\varphi_2(X_t^\alpha), AX_t^\alpha-\alpha_t\rangle\, dt$$

$$=-\frac{1}{h}\int_0^h \langle \nabla\varphi_2(x_0), (AX_t^\alpha-\alpha_t)\rangle\, dt+\varepsilon(h). \tag{52}$$

This follows from the same considerations used to establish (44) and $\nabla\varphi_2\in C(H:H_1)$

Next consider

$$\frac{1}{h}\int_0^h \langle \nabla\varphi_2(X_0), (AX_t^\alpha-\alpha_t)\rangle\, dt$$

$$=\left\langle \nabla\varphi_2(X_0), \frac{1}{h}\int_0^h (Ae^{-At}x_0+AY_t^\alpha-\alpha_t)\, dt\right\rangle.$$

Since $x_0\in H_1$, $Ae^{-At}x_0\to Ax_0$ in $H_{-1}$ as $t\downarrow 0$. Also, it follows from (47) that

$$\frac{1}{h}\int_0^h AY_t^\alpha\, dt\to 0 \qquad \text{weakly in} \quad H_{-1} \qquad \text{as} \quad h\downarrow 0.$$

Thus

$$\frac{1}{h}\int_0^h \langle \nabla\varphi_2(X_0), (AX_t^\alpha-\alpha_t)\rangle\, dt=\left\langle \nabla\varphi_2(X_0), \left(Ax_0-\frac{1}{h}\int_0^h \alpha_t\, dt\right)\right\rangle+\varepsilon(h),$$

and using this in (52) we end up with

$$\frac{1}{h}(\varphi_2(X_h^\alpha)-\varphi_2(x_0))=-\left\langle \nabla\varphi_2(x_0), \left(Ax_0-\frac{1}{h}\int_0^h \alpha_t\, dt\right)\right\rangle+\varepsilon(h).$$

Returning to (43), using $\varphi = \varphi_1 + \varphi_2$, the relation just above, and (50), we finally obtain

$$u(x_0) \geqslant \frac{1}{h} \inf_{\alpha \in \mathscr{C}_{\mathscr{A}}} \left\{ \int_0^h f(X_t^\alpha)\, dt \right\} - \langle A x_0, \nabla\varphi(x_0) \rangle$$

$$+ \inf_{\alpha \in \mathscr{C}_{\mathscr{A}}} \left\{ - \left\langle \nabla\varphi(x_0), \frac{1}{h} \int_0^h \alpha_t\, dt \right\rangle \right\} + \varepsilon(h). \tag{53}$$

Next we observe that

$$\inf_{\alpha \in \mathscr{C}_{\mathscr{A}}} \left\{ - \left\langle p, \frac{1}{h} \int_0^h \alpha_t\, dt \right\rangle \right\} = \inf_{\gamma \in \mathscr{A}} \{ -\langle p, \gamma \rangle \}$$

holds for $p \in H_1$ (and, in particular, for $p = \nabla\varphi(x_0)$), while

$$\begin{cases} f(X_t^\alpha) = f(e^{-At}x_0 + Y_t^\alpha) - f(x_0 + Y_t^\alpha) + f(x_0 + Y_t^\alpha) = f(x_0 + Y_t^\alpha) + \varepsilon(h) \\ \text{uniformly in } \alpha \in \mathscr{C}_{\mathscr{A}},\ 0 \leqslant t \leqslant h, \end{cases}$$

since $f \in \mathrm{UC}(H_1)$ and $e^{-At}x_0 \to x_0$ in $H_1$. Thus (53) implies (39).

We conclude by establishing (40) under the stated conditions. While convexity of $f$ implies (42), we show directly that convexity implies (40). Convexity (and continuity) imply that there is a $p \in H_{-1}$ such that

$$f(x) \geqslant f(x_0) + \langle x - x_0, p \rangle;$$

thus

$$\frac{1}{h} \int_0^h f\left( x_0 + \int_0^t e^{-A(t-s)} \alpha_s\, ds \right) dt$$

$$= \frac{1}{h} \int_0^h f(x_0 + Y_t^\alpha)\, dt \geqslant f(x_0) + \left\langle \frac{1}{h} \int_0^h Y_t^\alpha\, dt, p \right\rangle$$

and we have (47).

The proof that (42) implies (40) is less immediate; in particular, $Y_t^\alpha$ itself is not necessarily even bounded in $H_1$ unformly in $\alpha$. What we must show is that (40) implies that if $h_n \downarrow 0$, $\alpha^n \in \mathscr{C}_{\mathscr{A}}$, and

$$Y_t^{\alpha_n} = \int_0^t e^{-A(t-s)} \alpha_s^n\, ds,$$

then

$$\liminf_{n \to \infty} \frac{1}{h_n} \int_0^{h_n} f(x_0 + Y_t^{\alpha_n})\, dt \geqslant f(x_0).$$

The weak topology on bounded subsets of $H_1$ is metrizable, and we let $d_R$ be a metric for this topology on the $H_1$ ball $B^1_R$ of radius $R$ about $x_0$. The assumption (42) implies that

$$f(x_0+z) \geqslant f(x_0) - \omega_r(d_R(z+x_0, x_0) + \|z\| \qquad \text{for} \quad \|z\|_1 \leqslant R$$

for some real-valued function $\omega_R(r)$ which satisfies $\omega_R(0+)=0$.

The estimate of Proposition I.2.i yields

$$\|Y^{\alpha_n}_t\|^2 + \int_0^t \|Y^{\alpha_n}_s\|_1^2 \, ds \leqslant Ct$$

and therefore $A^n_R = \{t \in (0, h_n) : \|Y^{\alpha_n}_t\|_1 > R\}$ satisfies

$$\text{meas}(A^n_R) \leqslant \frac{Ch_n}{R^2}. \tag{54}$$

We claim that

$$\sup_{\substack{t \notin A^n_R \\ 0 \leqslant t \leqslant h_n}} d_R(x_0, x_0 + Y^{\alpha_n}_t) = \varepsilon^n_R \to 0 \qquad \text{as} \quad n \to \infty;$$

if not, there exists $t_n \downarrow 0$ such that

$$\|x_0 + Y^{\alpha_n}_{t_n}\|_1 \leqslant R \qquad \text{but} \quad Y^{\alpha_n}_{t_n} \nrightarrow 0 \qquad \text{weakly in } H_1,$$

which is impossible since $Y^{\alpha_n}_{t_n} \to 0$ in $H$. We therefore have

$$f(x_0 + Y^{\alpha_n}_t) \geqslant f(x_0) - \omega_R(\varepsilon^n_R + C\sqrt{h_n}) \qquad \text{for} \quad t \in A^n_R$$

which, together with (18.iii), implies

$$\begin{aligned}
\frac{1}{h_n}\int_0^{h_n} f(x_0 + Y^{\alpha_n}_t)\, dt &= \int_{[0,h_n)\setminus A^n_R} f(x_0 + Y^{\alpha_n}_t)\, dt + \frac{1}{h_n}\left(\int_{A^n_R} f(x_0 + Y^{\alpha_n}_t)\, dt\right) \\
&\geqslant \left(1 - \frac{\text{meas}(A^n_R)}{h_n}\right)(f(x_0) - \omega_R(\varepsilon^n_R + C\sqrt{h_n})) \\
&\quad + \frac{1}{h_n}\left(\int_{A^n_R} f(x_0 + Y^{\alpha_n}_t)\, dt\right) \\
&\geqslant \left(1 - \frac{\text{meas}(A^n_R)}{h_n}\right)(f(x_0) - \omega_R(\varepsilon^n_R + C\sqrt{h_n})) - \frac{C}{R^2}.
\end{aligned}$$

Now, using (54), we may let $n \to \infty$ and then $R \to \infty$ to complete the proof. ∎

## III. Maximal Subsolutions and Comparison

We continue our analysis of the model problem formulated in the Introduction. In the preceeding section we showed that certain subsolution and supersolution inequalities were satisfied by the value function. Dynamic programming yields the correct subsolution inequalities (at least in $H_1$), while the subsolution inequalities involve a function $\check{f}$ which "should" coincide with $f$ on $H_1$ but does no do so in general.

In this section we first show that the value function is the maximal subsolution of the subsolution inequalities. Some of the arguments used to establish this also can be used to prove that the value function is the *unique* viscosity solution whenever it is a viscosity solution, in particular, if $\check{f} = f$ on $H_1$ (see Theorem III.4 below).

We begin by recalling the notion of a $B$-continuous function on $H$ as introduced in [7, Part V]: we say that $u$ is $B$-continuous if $u(x_n) \to u(x)$ whenever $x_n \to x$ weakly in $H$ and $Bx_n \to Bx$ strongly in $H$ (this is convergence in the intersection of the weak topology on $H$ and the strong topology of $H_{-2}$). We assume here, as usual, that

$$\mathscr{A} \text{ is bounded in } H_{-1} \qquad \text{and} \qquad f \in \mathrm{UC}(H_1). \tag{55}$$

Definition III.1. *If $v \in \mathrm{UC}(H)$ is B-continuous, then it is a viscosity subsolution of*

$$v + \langle Ax, \nabla v \rangle + \sup_{\gamma \in \mathscr{A}} \langle -\gamma, \nabla v \rangle = f(x) \qquad \text{in} \quad H \tag{HJE}$$

*if, whenever $\varphi \in C^1(H)$ is weakly sequentially lower semicontinuous, $\nabla\varphi \in C(H, H_1)$, $g = g(\|x\|)$ is radial, $C^1$ is on $H$ with $g'(r) > 0$ for $r > 0$, and $x_0 \in H$ is a local maximum point of $v - (\varphi + g)$ in $H$, we have $x_0 \in H_1$ and*

$$v(x_0) + \langle Ax_0, \nabla(\varphi + g)(x_0) \rangle + \sup_{\gamma \in \mathscr{A}} \langle -\gamma, \nabla(\varphi + g)(x_0) \rangle \leqslant f(x_0). \tag{56}$$

Observe that all terms above are sensible, for $\nabla\varphi(x_0) \in H_1$, $\nabla g(x_0) = (g'(x_0)/\|x_0\|) x_0 \in H_1$ (understood as 0 if $x_0 = 0$), and $Ax_0 \in H_{-1}$.

We have seen, in the preceeding section, that the value function is indeed a viscosity subsolution in the sense of (56). According to the next result, it is maximal subsolution, a well-known fact in control theory in finite dimensions—see, for instance, [10].

Theorem III.2. *Let* (55) *hold and $v \in \mathrm{UC}(H)$ be a B-continuous viscosity subsolution of* (HJE). *Then $v \leqslant u$, where $u$ is the value function.*

*Proof.* We begin by explaining the strategy of the proof; it is an adaptation of a method introduced for stochastic control problems in Lions [11]. For $\gamma \in \mathscr{A}$, let $S_t^\gamma = e^{-tA^\gamma}$ be the semigroup which solves (24) and define the operator $T_t^\gamma$ for $t \geqslant 0$ on functions $\varphi$ by

$$(T_t^\gamma \varphi)(x) = \int_0^t f(S_s^\gamma x) e^{-s}\, ds + \varphi(S_t^\gamma x) e^{-t}; \tag{57}$$

using the fact that $S_t^\gamma$ is a semigroup, a simple calculation shows that $T_t^\gamma$ also has the semigroup property and we interpret it as a semigroup on $\mathrm{UC}(H)$ (its generator is formally $\varphi \to \varphi + \langle Ax - \gamma, \nabla\varphi \rangle - f$); however, $T_t^\gamma$ is not a strongly continuous semigroup on $\mathrm{UC}(H)$. We show that under the conditions of Theorem III.2, we have

$$v \leqslant T_t^\gamma v \qquad \text{for all} \quad \gamma \in \mathscr{A}, \quad x \in H, \quad t \geqslant 0. \tag{58}$$

Indeed, once this fact is establish, the rest of Theorem III.2 follows easily: iterating (58) with various choices of $\gamma$ and using that $T_t^\gamma$ is order preserving, we obtain

$$v(x) \leqslant \inf \left\{ \int_0^T f(X_t^\alpha) e^{-t}\, dt + v(X_T^\alpha) e^{-T} \colon \alpha \in \mathscr{C}_{\mathscr{A}} \right.$$
$$\left. \text{is piecewise constant on } [0, T] \right\},$$

where $\alpha$ is piecewise constant on $[0, T]$ if there exists a partition $0 = t_0 < t_1 < \cdots < t_m = T$ and $\gamma_1, ..., \gamma_m \in \mathscr{A}$ such that $\alpha_t = \gamma_j$ on $(t_{j-1}, t_j)$. Now by a simple density argument (recalling that $\mathscr{A}$ is separable and bounded in $H_{-1}$), we deduce from (59) that

$$v(x) \leqslant \inf_{\alpha \in \mathscr{C}_{\mathscr{A}}} \left\{ \int_0^T f(X_t^\alpha) e^{-t}\, dt + v(X_T^\alpha) e^{-T} \right\}.$$

The conclusion is reached then by sending $T \to \infty$. This is the "control" part of the proof.

The heart of the matter is thus the *proof of* (58). The intuitive reason why this inequality holds is that $w(t, x) = T_t^\gamma v(x)$ is formally the solution of the Cauchy problem

$$\frac{\partial w}{\partial t} + w + \langle Ax, \nabla w \rangle - \langle \gamma, \nabla w \rangle = f, \quad w|_{t=0} = v \qquad \text{on } H. \tag{60}$$

Now $v$, being a subsolution of the stationary problem, is also a subsolution of the Cauchy problem and thus we expect the comparison $v \leqslant w$.

Let $\lambda > 0$ and set

$$(J_\lambda^\gamma \varphi)(x) = \int_0^\infty (\lambda\varphi(S_s^\gamma x) + f(S_s^\gamma x)) e^{-(\lambda+1)s}\, ds;$$

$J_\lambda^\gamma$ is formally the resolvent operator of the operator $\psi \to \psi + \langle Ax - \gamma, \nabla\psi \rangle - f$. We first establish the "exponential formula"

$$(J_{n/t}^\gamma)^n \varphi(x) \to T_t^\gamma \varphi(x) \qquad \text{for} \quad x \in H. \tag{61}$$

We compute

$$\begin{aligned}
&(J_\lambda^\gamma)^2 \varphi(x) \\
&= \int_0^\infty \left[ f(S_s^\gamma x) + \lambda \left( \int_0^\infty (f(S_{s+\sigma}^\gamma x) + \lambda\varphi(S_{s+\sigma}^\gamma x)) e^{-(\lambda+1)\sigma}\, d\sigma \right) \right] e^{-(\lambda+1)s}\, ds \\
&= \int_0^\infty \left[ f(S_s^\gamma x)\, e^{-(\lambda+1)s} + \lambda \left( \int_s^\infty (f(S_\sigma^\gamma x) + \lambda\varphi(S_\sigma^\gamma x)) e^{-(\lambda+1)\sigma}\, d\sigma \right) \right] ds \\
&= \int_0^\infty [f(S_s^\gamma x)\, e^{-(\lambda+1)s} + \lambda s f(S_s^\gamma x) + \lambda^2 s \varphi(S_s^\gamma x))]\, ds,
\end{aligned}$$

and then, by induction,

$$\begin{aligned}
(J_\lambda^\gamma)^n \varphi(x) &= \int_0^\infty f(S_s^\gamma x) \left( 1 + \lambda s + \cdots + \frac{\lambda^{n-1} s^{n-1}}{(n-1)!} \right) e^{-(\lambda+1)s}\, ds \\
&\quad + \int_0^\infty \frac{\lambda^n s^{n-1}}{(n-1)!} e^{-(\lambda+1)s} \varphi(S_s^\gamma x)\, ds.
\end{aligned} \tag{62}$$

Now we set $\lambda = n/t$ and observe that $(1 + \lambda s + \cdots + \lambda^{n-1} s^{n-1}/(n-1)!)\, e^{-\lambda s}$ is nonnegative and bounded by 1 for $s \geqslant 0$ while $(\lambda^n s^{n-1}/(n-1)!) e^{-\lambda s}$ is nonnegative and integrates to 1 over $0 \leqslant s$. It is then a "simple" and interesting) classical exercise in measure theory to check that $(1 + \lambda s + \cdots + \lambda^{n-1} s^{n-1}/(n-1)!) e^{-\lambda s}$ converges weakly in $L^\infty([0, \infty)$ to $1_{[0,t]}(s)$ while $(\lambda^n s^{n-1}/(n-1)!) e^{-\lambda s}$ converges weakly in the sense of measures to $\delta_t(s)$. Then (62) leads easily to (61).

Therefore, in order to conclude, we only need to show that for all $\lambda > 0$

$$v(x) \leqslant (J_\lambda^\gamma v)(x) \qquad \text{for} \quad x \in H. \tag{63}$$

Indeed, the inequality $v \leqslant J_\lambda^\gamma v$ may be interated to obtain $v \leqslant (J_\lambda^\gamma)^n v$ and then we use (61) to find $v \leqslant T_t^\gamma v$, completing the proof. To this end, we first observe that $v$, which is assumed to be $B$-continuous and to lie in $\mathrm{UC}(H)$, is a viscosity subsolution of

$$(\lambda + 1)v + \langle Ax, \nabla v \rangle - \langle \gamma, \nabla v \rangle \leqslant f + \lambda v, \tag{64}$$

where the definition of a subsolution is as in Definition III.1 choosing $\mathscr{A} = \{\gamma\}$. Moreover, $J_\lambda^\gamma v \in \mathrm{UC}(H)$ is $B$-continuous and is a viscosity solution of (64) by Theorems II.2 and II.4 and Remark II.7. Here "viscosity supersolution" is defined as in Definition III.1 with maximum replaced by minimum and $g$ decreasing instead of increasing and $\leqslant$ replaced by $\geqslant$ in (56), and a viscosity solution is a function which is both a subsolution and a supersolution. Thus (64), and therefore Theorem III.2, follows at once from Theorem III.3 below. ∎

THEOREM III.3. *Let* $\mu > 0$, $\gamma \in H_{-1}$, $f \in \mathrm{UC}(H_1)$, *and* $v, w \in \mathrm{UC}(H)$ *be B-continuous. Let v be a viscosity subsolution and w be a viscosity supersolution of*

$$\mu u + \langle Ax, \nabla u\rangle - \langle \gamma, \nabla u\rangle = f. \tag{65}$$

*Then* $v \leqslant w$ *on H.*

*Proof.* The proof is a nontrivial modification of the method introduced in [7, Part V]. We consider first the expression.

$$\Phi(x, y) = v(x) - w(y) - \frac{1}{2\varepsilon}\|x-y\|^2 - \frac{\delta}{2}(\|x\|^2 + \|y\|^2), \tag{66}$$

where $\varepsilon, \delta > 0$. It follows from the general perturbation results of Ekeland [8], Bourgain [2], Stegall [16], ..., that for all $\nu > 0$ there exists $p$, $q$, $\hat{x}$, $\hat{y} \in H$ such that $\|p\|, \|q\| \leqslant 1$ and $(\hat{x}, \hat{y})$ is a global maximum point of $\Phi(x, y) + \nu(\langle Bq, x\rangle + \langle Bq, y\rangle)$. To see this, one observes that $\Phi$ is uppersemicontinuous on sets of the form $K_R = \{(x, y) \in H \times H : \|x\|^2 + \|y\|^2 \leqslant R^2\}$ in the $H_{-2} \times H_{-2}$ topology (because of the $B$-continuity of $v$, $w$) and thus can be perturbed by a continuous linear functional on $H_{-2} \times H_{-2}$ of arbitrarily small norm to a function which attains its maximum on $K_R$ (because $K_R$ is also bounded and closed in $H_{-2}$). In view of the growth of $\Phi$, this maximum on $K_R$ also is a global maximum on $H \times H$ if $R$ is large. (These arguments are from [7, Part V].)

Now it becomes more complicated—with $(\hat{x}, \hat{y})$ as above, we define

$$\Psi_\lambda(x, y) = v(x) - w(y) - \frac{1}{2\varepsilon}\langle (I + \lambda A)^{-1}(x-y), (x-y)\rangle - \frac{\delta}{2}(\|x\|^2 + \|y\|^2)$$

$$+ \nu(\langle Bq, x\rangle + \langle Bq, y\rangle) - \frac{1}{2}(\|x-\hat{x}\|_{-1}^2 + \|y-\hat{y}\|_{-1}^2).$$

Then, by the arguments above, for all $\kappa > 0$ there exists $p'$, $q'$, $\hat{x}_\lambda$, $\hat{y}_\lambda \in H$ such that $\|p'\|$, $\|q'\| \leqslant 1$ and $(\hat{x}_\lambda, \hat{y}_\lambda)$ is a global maximum of

$\Psi_\lambda(x, y)+\kappa(\langle Bp', x\rangle+\langle Bq', y\rangle)$ over $H\times H$. Now, by the assumptions, $(\hat{x}_\lambda, \hat{y}_\lambda)\in H_1\times H_1$ and then, using the definitions of subsolution and supersolution,

$$\mu v(\hat{x}_\lambda)+\Big\langle A\hat{x}_\lambda-\gamma, (I+\lambda A)^{-1}\frac{(\hat{x}_\lambda-\hat{y}_\lambda)}{\varepsilon}$$

$$+\delta\hat{x}_\lambda+\nu Bp+\kappa Bp'+B(\hat{x}_\lambda-\hat{x})\Big\rangle\leqslant f(\hat{x}_\lambda) \tag{67}$$

$$\mu w(\hat{y}_\lambda)+\Big\langle A\hat{y}_\lambda-\gamma, (I+\lambda A)^{-1}\frac{(\hat{x}_\lambda-\hat{y}_\lambda)}{\varepsilon}$$

$$-\delta\hat{y}_\lambda-\nu Bq-\kappa Bq'-B(\hat{y}_\lambda-\hat{y})\Big\rangle\geqslant f(\hat{y}_\lambda). \tag{68}$$

Next, since $v, w\in \mathrm{UC}(H)$, the extremal property of $(\hat{x}_\lambda, \hat{y}_\lambda)$ implies that $\hat{x}_\lambda, \hat{y}_\lambda$ are bounded in $H$ uniformly in $0<\lambda$, $0<\nu\leqslant 1$, $0<\kappa\leqslant 1$ for $\varepsilon, \delta>0$ and fixed.

Combining (67) and (68) and putting $A_\lambda=A(I+\lambda A)^{-1}$, we have

$$\begin{aligned}\mu(v(\hat{x}_\lambda)-w(\hat{y}_\lambda))&+\frac{1}{\varepsilon}\langle A_\lambda(\hat{x}_\lambda-\hat{y}_\lambda), (\hat{x}_\lambda-\hat{y}_\lambda)\rangle+\delta\langle A\hat{x}_\lambda, \hat{x}_\lambda\rangle+\delta\langle A\hat{y}_\lambda, \hat{y}_\lambda\rangle\\ &\leqslant f(\hat{x}_\lambda)-f(\hat{y}_\lambda)+\delta(\langle\gamma, \hat{x}_\lambda\rangle+\langle\gamma, \hat{y}_\lambda\rangle)\\ &\quad-\langle B(A\hat{x}_\lambda-\gamma), \nu p+\kappa p'+(\hat{x}_\lambda-\hat{x})\rangle\\ &\quad-\langle B(A\hat{y}_\lambda-\gamma), \nu q+\kappa q'+(y_\lambda-\hat{y})\rangle.\end{aligned}$$

From this we deduce that

$$\delta(\|\hat{x}_\lambda\|_1^2+\|\hat{y}_\lambda\|_1^2)\leqslant 2\delta\,\|\gamma\|_{-1}\,(\|\hat{x}_\lambda\|_1+\|\hat{y}_\lambda\|_1)+C$$

for some $C>0$ depending only on $\delta, \varepsilon>0$. Reaching this conclusion involves using the bound on $(\hat{x}_\lambda, \hat{y}_\lambda)$ in $H\times H$ to estimate the effect of $u(\hat{x}_\lambda)$, $v(\hat{y}_\lambda)$, and the linear growth of $f$; linear terms in $\|\hat{x}_\lambda\|_1$, $\|\hat{y}_\lambda\|_1$ can be absorbed by quadratics in the usual way. Therefore $\hat{x}_\lambda, \hat{y}_\lambda$ are bounded in $H_1$ uniformly in $\lambda, \nu, \kappa$.

Next we show that $(\hat{x}_\lambda, \hat{y}_\lambda)$ converges to $(\hat{x}, \hat{y})$ in $H$ as $\lambda, \kappa\downarrow 0$. To this end, we first observe that the $H_1$ bound on $\hat{x}_\lambda, \hat{y}_\lambda$ implies

$$|\langle(I+\lambda A)^{-1}(\hat{x}_\lambda-\hat{y}_\lambda), (\hat{x}_\lambda-\hat{y}_\lambda)\rangle-\langle\hat{x}_\lambda-\hat{y}_\lambda, \hat{x}_\lambda-\hat{y}_\lambda\rangle|\leqslant C\sqrt{\lambda}$$

and then

$$\begin{aligned}\Psi_\lambda(\hat{x}_\lambda, \hat{y}_\lambda)\leqslant \Phi(\hat{x}_\lambda, \hat{y}_\lambda)+C(\sqrt{\lambda}+\kappa)-\|\hat{x}_\lambda-\hat{x}\|_{-1}^2&\\ -\|\hat{y}_\lambda-\hat{y}\|_{-1}^2+\nu(\langle Bp, \hat{x}_\lambda\rangle+\langle Bq, \hat{y}_\lambda\rangle).&\end{aligned} \tag{69}$$

The properties of $(\hat{x}_\lambda, \hat{y}_\lambda)$ and $(\hat{x}, \hat{y})$ imply

$$\Psi_\lambda(\hat{x}, \hat{y}) - C_\kappa \leqslant \Psi_\lambda(\hat{x}_\lambda, \hat{y}_\lambda) \tag{70}$$

and

$$\Phi(\hat{x}_\lambda, \hat{y}_\lambda) + v(\langle Bp, \hat{x}_\lambda \rangle + \langle Bq, \hat{y}_\lambda \rangle) \leqslant \Phi(\hat{x}, \hat{y}) + v(\langle Bp, \hat{x} \rangle + \langle Bq, \hat{y} \rangle). \tag{71}$$

In these inequalities and those to follow, $C$ denotes various constants independent of $\lambda$ and $\kappa$. Finally, we remark that

$$\Phi(\hat{x}, \hat{y}) + v(\langle Bp, \hat{x} \rangle + \langle Bq, \hat{y} \rangle) - C_\kappa - \beta(\lambda) \leqslant \Psi_\lambda(\hat{x}, \hat{y}) \tag{72}$$

where

$$\beta(\lambda) \to 0 \qquad \text{as} \quad \lambda \downarrow 0.$$

Combining the inequalities (69)–(72), we deduce that

$$\langle B(\hat{x}_\lambda - \hat{x}), (\hat{x}_\lambda - \hat{x}) \rangle + \langle B(\hat{y}_\lambda - \hat{y}), (\hat{y}_\lambda - \hat{y}) \rangle \leqslant (C_\kappa + \sqrt{\lambda}) + k(\lambda) \tag{73}$$

where $k(\lambda) \to 0$ as $\lambda \downarrow 0$.

Therefore, we see that $(\hat{x}_\lambda, \hat{y}_\lambda) \to (\hat{x}, \hat{y})$ in $H_{-2}$ as we let $\lambda, \kappa \downarrow 0$. Now, since $\hat{x}_\lambda, \hat{y}_\lambda$ are bounded in $H_1$, this convergence also holds weakly in the $H_1$ topology and strongly in the $H$ topology. In particular, $\hat{x}, \hat{y} \in H_1$.

We have produced the following information:

$$\begin{gathered} v - \varphi - g \text{ has a maximum at } \hat{x} \in H_1 \text{ where} \\ \varphi(x) = \frac{1}{\varepsilon} \langle x, \hat{y} \rangle - v \langle Bp, x \rangle \text{ and } g(x) = \left( \frac{1}{2\varepsilon} + \frac{\delta}{2} \right) \|x\|^2 \end{gathered} \tag{74}$$

and

$$\begin{gathered} w - \varphi - g \text{ has a maximum at } \hat{x} \in H_1 \text{ where} \\ \varphi(x) = \frac{1}{\varepsilon} \langle \hat{x}, y \rangle - v \langle Bq, x \rangle \text{ and } g(y) = \left( \frac{1}{2\varepsilon} + \frac{\delta}{2} \right) \|y\|^2. \end{gathered} \tag{75}$$

By the assumption that $v$, $w$ are sub- and supersolutions, (74), (75) imply

$$\mu v(\hat{x}) + \left\langle A\hat{x} - \gamma, \frac{\hat{x} - \hat{y}}{\varepsilon} + \delta\hat{x} + vBp \right\rangle \leqslant f(\hat{x})$$

and

$$\mu u(\hat{y}) + \left\langle A\hat{y} - \gamma, \frac{\hat{x} - \hat{y}}{\varepsilon} - \delta\hat{y} - vBq \right\rangle \geqslant f(\hat{y}).$$

Substracting these relations and using the definition of $\Phi$, we obtain

$$\mu\Phi(\hat{x}, \hat{y}) + \frac{\mu}{2\varepsilon}\|\hat{x}-\hat{y}\|^2 + \frac{1}{\varepsilon}\langle A(\hat{x}-\hat{y}), \hat{x}-\hat{y}\rangle + \mu\frac{\delta}{2}(\|x\|^2 + \|y\|^2)$$
$$+ \delta(\langle A\hat{x}-\gamma, \hat{x}\rangle + \langle A\hat{y}-\gamma, \hat{y}\rangle)$$
$$\leqslant f(\hat{x}) - f(\hat{y}) + Cv, \tag{76}$$

where $C$ now denotes constants depending only on $\varepsilon$ and $\delta$. Now

$$\frac{\mu_0}{\varepsilon}\|\hat{x}-\hat{y}\|_1^2 \leqslant \frac{\mu}{2\varepsilon}\|\hat{x}-\hat{y}\|^2 + \frac{1}{\varepsilon}\langle A(\hat{x}-\hat{y}), \hat{x}-\hat{y}\rangle \tag{77}$$

where $\mu_0 = \min(\mu/2, 1)$ and, since $f \in \mathrm{UC}(H_1)$,

$$f(\hat{x}) - f(\hat{y}) \leqslant \frac{\mu_0}{\varepsilon}\|\hat{x}-\hat{y}\|_1^2 + m(\varepsilon) \tag{78}$$

where $m(\varepsilon)$ is independent of $\delta$, $v$, and $m(0+) = 0$.

We also have, clearly,

$$-\frac{\|\gamma\|_{-1}^2\,\delta}{2\mu_0} \leqslant \mu_0\delta(\|\hat{x}\|_1^2 + \|\hat{y}\|_1^2) - \delta\,\|\gamma\|_{-1}(\|\hat{x}\|_1 + \|\hat{y}\|_1)$$
$$\leqslant \mu\frac{\delta}{2}(\|\hat{x}\|^2 + \|\hat{y}\|^2) + \delta(\langle A\hat{x}-\gamma, \hat{x}\rangle + \langle A\hat{y}-\gamma, \hat{y}\rangle).$$

This and (76) imply

$$\Phi(\hat{x}, \hat{y}) \leqslant m(\varepsilon) + Cv + \frac{\delta\,\|\gamma\|_{-1}^2}{2\mu_0}.$$

Recalling the definition of $(\hat{x}, \hat{y})$, we first let $\kappa, v \downarrow 0$ to obtain

$$\Phi(x, y) \leqslant m(\varepsilon) + \frac{\delta\,\|\gamma\|_{-1}^2}{2\mu_0}.$$

Putting $x = y$ this yields

$$v(x) - w(y) \leqslant m(\varepsilon) + \frac{\delta\,\|\gamma\|_{-1}^2}{2\mu_0}.$$

We conclude upon letting $\delta, \varepsilon \downarrow 0$. ∎

The next result concerning the uniqueness of solutions of the stationary Hamilton–Jacobi equation is proved via simple modifications of the arguments which were used to established Theorem III.3. Note that there is a solution of the equation *only* if the value function is a supersolution;

this follows from Theorem III.2, which implies that $u$ lies above any solution, and the theorem below, which implies that $u$ lies below any solution and hence is the unique solution if a solution exists.

THEOREM III.4. *Let $\mu > 0$, $\mathscr{A}$ be a bounded subset of $H_{-1}$, and $f \in \mathrm{UC}(H_1)$ and $v, w \in \mathrm{UC}(H)$ be B-continuous. Let $v$ be a viscosity subsolution and $w$ be a viscosity supersolution of*

$$\mu u + \langle Ax, \nabla u \rangle + \sup_{\gamma \in \mathscr{A}} \{ -\langle \gamma, \nabla u \rangle \} = f.$$

*Then $v \leqslant w$ on $H$.*

## IV. GENERALIZATIONS

We proceed to consider more complex control problems. The dynamic now have the general form

$$\frac{dX_t}{dt} + AX_t = b(X_t, \alpha_t) \qquad \text{for} \quad t \geqslant 0, \quad X_0 = x \in H. \tag{79}$$

Here $\mathscr{A}$ is an arbitrary separable metric space, $b: H \times \mathscr{A} \to H_{-1}$, and $\mathscr{C}_{\mathscr{A}}$ is the set of measurable mappings from $[0, \infty)$ into $\mathscr{A}$. The value function is given by

$$u(x) = \inf_{\alpha \in \mathscr{C}_{\mathscr{A}}} \left\{ \int_0^\infty f(X_t^\alpha, \alpha_t)\, e^{-\lambda t}\, dt \right\}, \tag{80}$$

where $f: H_1 \times \mathscr{A} \to \mathbb{R}$ is the running cost, $\lambda > 0$ is chosen according to the problem, and $X_t^\alpha$ denotes the solution (when it exists) of (79) when we display $\alpha \in \mathscr{C}_{\mathscr{A}}$.

PROPOSITION IV.1. *Let $b(x, \gamma) - b(y, \gamma) \in H$ for $x, y \in H$, $\gamma \in \mathscr{A}$, and let there be constants $C_0$, $C_1$ such that*

$$\begin{aligned} &\text{(i)} \quad \|b(x,\gamma) - b(y,\gamma)\| \leqslant C_0 \|x - y\| \qquad \textit{for} \quad x, y \in H, \quad \gamma \in \mathscr{A}, \\ &\text{(ii)} \quad \|b(0,\gamma)\|_{-1} \leqslant C_1 \qquad \textit{for} \quad \gamma \in \mathscr{A}. \end{aligned} \tag{81}$$

*Let $f$ satisfy*

$$\begin{aligned} &f: H_1 \times \mathscr{A} \to \mathbb{R} \textit{ and} \\ &f(x, \gamma) \textit{ is uniformly continuous in } x \in H_1 \textit{ uniformly for } \gamma \in \mathscr{A} \end{aligned} \tag{82}$$

*and let $C$ be a constant such that*

$$|f(0, \gamma)| \leqslant C \qquad \text{for} \quad \gamma \in \mathscr{A}. \tag{83}$$

*We have*

(i) *for $\alpha \in \mathscr{C}_{\mathscr{A}}$ and $x \in H$, (79) has a unique solution $X_t^\alpha$ with*

$$X_t^\alpha \in C([0, \infty, H) \cap L^2_{\text{loc}}(0, \infty : H_1) \qquad \text{and} \qquad \frac{dX_t^\alpha}{dt} \in L^2_{\text{loc}}(0, \infty : H_{-1}).$$

(ii) *If $\lambda > C_0$, then the integrals appearing in the definition (80) of the value function $u$ are convergent and $u \in \text{UC}(H)$. Moreover, for all $\varepsilon > 0$ there exists $C_\varepsilon$ such that*

$$|u(x) - u(y)| \leqslant \varepsilon(\|x - y\| + 1) + C_\varepsilon \|x - y\|_{-1} \qquad \text{for} \quad x, y \in H. \tag{84}$$

(iii) *If (83) is replaced by*

$$|f(x, \gamma)| \leqslant C \qquad \text{for} \quad x \in H_1, \quad \gamma \in \mathscr{A}, \tag{83'}$$

*then for all $\lambda > 0$ we have $u \in \text{UC}(H_{-1})$.*

*Proof.* The existence and uniqueness is routine and we do not discuss it here (key estimates occur below in any case). We begin the proof by showing that the value function is well-defined in the case of (ii). This requires various estimates on the solution $X_t$ of (79). Writing $b(X_t, \alpha_t) = b(0, \alpha_t) + b(X_t, \alpha_t) - b(0, \alpha_t)$ in (79), multiplication by $X_t$ and use of (81) yields

$$\frac{d}{dt}\frac{1}{2}\|X_t\|^2 + \langle AX_t, X_t\rangle \leqslant C_1\|X_t\|_1 + C_0\|X_t\|^2$$

and if $\varepsilon > 0$ this implies

$$\begin{aligned} \frac{d}{dt}\frac{1}{2}\|X_t\|^2 + \langle AX_t, X_t\rangle &\leqslant \varepsilon\|X_t\|_1^2 + \frac{C_1^2}{4\varepsilon} + C_0\|X_t\|^2 \\ &= \varepsilon\langle AX_t, X_t\rangle + \frac{C_1^2}{4\varepsilon} + (C_0 + \varepsilon)\|X_t\|^2. \end{aligned} \tag{85}$$

By (15) of Lemma I.3 (with $V = \langle AX_t, X_t\rangle$, $a = (1 - \varepsilon)$, etc.) and with $\lambda = 2C_0 + 3\varepsilon$,

$$\int_0^\infty (\langle AX_t, X_t\rangle + \|X_t\|^2)\, e^{-(2C_0 + 3\varepsilon)t}\, dt \leqslant C_\varepsilon(\|x\|^2 + 1), \tag{86}$$

where $C_\varepsilon$ depends only on $\varepsilon$ (regarding $C_1$, $C_0$ as fixed). Thus if $\lambda > \kappa > C_0$,

$$\int_0^\infty \|X_t\|_1 e^{-\lambda t}\,dt = \int_0^\infty \|X_t\|_1 e^{-\kappa t} e^{-(\lambda-\kappa)t}\,dt$$
$$\leqslant \left(\int_0^\infty \|X_t\|_1^2 e^{-2\kappa t}\,dt\right)^{1/2} \left(\int_0^\infty e^{-2(\lambda-\kappa)t}\,dt\right)^{1/2}$$

and we conclude that

$$\int_0^\infty \|X_t\|_1 e^{-\lambda t}\,dt \leqslant C_\lambda(\|x\|+1) \qquad \text{for} \quad \lambda > C_0. \tag{87}$$

The properties (82), (83) of $f$ guarantee that

$$|f(X_t, \alpha_t)| \leqslant C + |f(X_t, \alpha_t) - f(X_t, 0)| \leqslant C + K(1 + \|X_t\|_1)$$

for some constant $K$ independent of $\alpha$ which, together with (87), shows the convergence of $\int_0^\infty f(X_t, \alpha_t) e^{-\lambda t}\,dt$ for $\lambda > C_0$ and that the value function $u$ is well-defined by (80).

To study the continuity of $u$ we fix $x, y \in H$ and let $X_t$, $Y_t$ be, respectively, the solutions of (79) with these initial values. Setting $Z_t = X_t - Y_t$, forming the difference of the equations, multiplying by $Z_t$, and using (81.i) yields

$$\frac{d}{dt}\frac{1}{2}\|Z_t\|^2 + \langle AZ_t, Z_t\rangle \leqslant C_0 \|Z_t\|^2. \tag{88}$$

The same process, but multiplying by $(I+A)^{-1} Z_t$ instead of $Z_t$, yields

$$\frac{d}{dt}\|Z_t\|_{-1}^2 + 2\langle AZ_t, (I+A)^{-1} Z_t\rangle \leqslant 2C_0 \|Z_t\|\,\|Z_t\|_{-1}.$$

Since $A(I+A)^{-1} = I - (I+A)^{-1}$, this implies

$$\frac{d}{dt}\|Z_t\|_{-1}^2 + 2\|Z_t\|^2 \leqslant 2(C_0+1)\|Z_t\|\,\|Z_t\|_{-1}$$
$$\leqslant (C_0+1)(\|Z_t\|^2 + \|Z_t\|_{-1}^2). \tag{89}$$

Now we let $\lambda > 0$ be arbitrary and establish that $u \in \mathrm{UC}(H_{-1})$ when (83′) holds. Because of (83′), for $\varepsilon > 0$ we have

$$|u(x) - u(y)| \leqslant C\varepsilon + \sup_{\alpha \in \mathscr{C}_{\mathscr{A}}} \int_\varepsilon^T |f(X_t, \alpha_t) - f(Y_t, \alpha_t)|\, e^{-\lambda t}\,dt + Ce^{-\lambda T}. \tag{90}$$

However, (82) implies that for $\delta > 0$ there is a constant $C_\delta$ such that

$$|f(X_t, \alpha_t) - f(Y_t, \alpha_t)| \leqslant \delta + C_\delta \, \|Z_t\|_1$$

and then

$$\int_\varepsilon^T e^{-\lambda t} \, |f(X_t, \alpha_t) - f(Y_t, \alpha_t)| \, dt \leqslant \frac{\delta}{\lambda} + C_\delta \int_\varepsilon^T \|Z_t\|_1 \, e^{-\lambda t} \, dt.$$

Continuing, (88) and (15) (with $c = 0$ and $\varepsilon$ in place of 0) imply that

$$\int_\varepsilon^T \|Z_t\|_1 \, dt \leqslant \left( \int_\varepsilon^T \|Z_t\|_1^2 \, dt \right)^{1/2} T^{1/2} \leqslant C(T, \varepsilon) \, \|Z_\varepsilon\|. \tag{91}$$

We denote by $C$("list") various "constants," changing from line to line, which remain bounded as the entries in the list stay bounded or away from 0 as appropriate. Combining (90) and (91) we have

$$|u(x) - u(y)| \leqslant \frac{\delta}{\lambda} + C\varepsilon + C(\delta, T, \varepsilon) \, \|Z_\varepsilon\| + Ce^{-\lambda T}. \tag{92}$$

We integrate this relation over $0 < \eta \leqslant \varepsilon \leqslant 2\eta$ to find

$$\begin{aligned}
|u(x) - u(y)| &\leqslant \frac{\delta}{\lambda} + \frac{3}{2} C_\eta + C(\delta, T, \eta) \int_\eta^{2\eta} \|Z_\varepsilon\| \, d\varepsilon + Ce^{-\lambda T} \\
&\leqslant \frac{\delta}{\lambda} + \frac{3}{2} C\eta + C(\delta, T, \eta) \int_0^{2n} \|Z_\varepsilon\| \, d\varepsilon + Ce^{-\lambda T} \\
&\leqslant \frac{\delta}{\lambda} + \frac{3}{2} C_\eta + C(\delta, T, \eta) \left( \int_0^{2\eta} \|Z_\varepsilon\|^2 \, d\varepsilon \right)^{1/2} (2\eta)^{1/2} + Ce^{-\lambda T}.
\end{aligned}$$

Finally, we use (89) to estimate

$$\int_0^{2\eta} \|Z_\varepsilon\|^2 \, d\varepsilon \leqslant \|x - y\|_{-1}^2 \, e^{2\eta C_2}$$

and combine this with the above to end up with

$$|u(x) - u(y)| \leqslant \frac{\delta}{\lambda} + \frac{3}{2} C\eta + C(\delta, T, \eta) \, \|x - y\|_{-1} \, e^{\eta C_2} (2\eta)^{1/2} + Ce^{-\lambda T}.$$

Since the right-hand side may be made as desired by first taking $\eta$ sufficiently small and $T$ sufficiently large and then $\|x - y\|_{-1}$ sufficiently small, we have $u \in \mathrm{UC}(H_{-1})$.

The outline above needs only to be supplemented with new estimates on

$$\int_0^\varepsilon \|Z_t\|_1 \, dt \qquad \text{and} \qquad \int_T^\infty \|Z_t\|_1 \, e^{-\lambda t} \, dt$$

to prove (84). For $\varepsilon \in (0, 1]$ we have

$$\int_0^\varepsilon \|Z_t\|_1 \, dt \leqslant \varepsilon^{1/2} \left( \int_0^\varepsilon \|Z_t\|_1^2 \, dt \right)^{1/2} \leqslant C_3 \varepsilon^{1/2} \, \|x - y\|$$

by virtue of (88). The second integral is bounded by choosing $\lambda > \kappa > C_0$ and writing

$$\begin{aligned} \int_T^\infty \|Z_t\|_1 \, e^{-\lambda t} \, dt &= \int_T^\infty \|Z_t\|_1 \, e^{-\kappa t} e^{-(\lambda - \kappa) t} \, dt \\ &\leqslant \left( \int_T^\infty \|Z_t\|_1^2 \, e^{-2\kappa t} \, dt \right)^{1/2} \left( \int_T^\infty e^{-2(\lambda - \kappa) t} \, dt \right)^{1/2} < \infty \\ &\leqslant \left( \int_T^\infty \|Z_t\|_1^2 \, e^{-2\kappa t} \, dt \right)^{1/2} \frac{e^{-(\lambda - k) T}}{(2(\lambda - \kappa))^{1/2}}. \end{aligned}$$

Again, Lemma I.3 with $c = 0$ implies

$$\int_0^\infty \|Z_t\|_1^2 \, e^{-2\kappa t} \, dt = \int_0^\infty (\|Z_t\|^2 + \langle AZ_t, Z_t \rangle) \, e^{-2\kappa t} \, dt \leqslant C_4 \, \|x - y\|^2.$$

Altogether then,

$$\int_T^\infty \|Z_t\|_1 \, e^{-\lambda t} \, dt \leqslant C_5 e^{-(\lambda - \kappa) T} \, \|x - y\|$$

and

$$|u(x) - v(y)| \leqslant \frac{\delta}{\lambda} + C_6(\varepsilon^{1/2} + e^{-(\lambda - \kappa) T}) \, \|x - y\| + C(\delta, \varepsilon, T) \, \|x - y\|_{-1}. \quad \blacksquare$$

We next formulate variants of the results above in the evolution context of a "finite horizon" problem. The ingredients are now a state equation

$$\frac{d}{dt} X_t + AX_t = b(t, X_t, \alpha_t) \qquad \text{for} \quad s \leqslant t \leqslant T, \quad X_s = x \in H, \tag{93}$$

where $0 \leqslant s \leqslant T$, $T$ is the fixed "horizon" (final time), and $b$: $[0, T] \times H \times \mathscr{A} \to H_{-1}$ satisfies that

$$\left\{\begin{array}{ll}\text{(i)} & b(t,x,\gamma)-b(t,0,\gamma) \text{ is continuous from } [0,T]\times H\times\mathscr{A} \\ & \text{into } H \text{ and the continuity in } t \text{ is uniform in } \gamma \text{ for fixed} \\ & x\in H \\ \text{(ii)} & b(t,0,\gamma) \text{ is continuous from } [0,T]\times\mathscr{A} \text{ into } H_{-1} \text{ and} \\ & \text{the continuity in } t \text{ is uniform in } \gamma \\ \text{(iii)} & \|b(t,x,\gamma)-b(t,y,\gamma)\| \leqslant C_0\,\|x-y\| \quad \text{for} \quad t\in[0,T], \\ & x,y\in H,\ \gamma\in\mathscr{A}, \\ \text{(iv)} & \|b(t,0,\gamma)\|_{-1}\leqslant C_1 \text{ for } t\in[0,T],\ \gamma\in\mathscr{A}.\end{array}\right. \tag{94}$$

The function $\alpha$ in (93) is taken from $\mathscr{C}_{\mathscr{A}}(s,T)$ where

$$\mathscr{C}_{\mathscr{A}}(s,T)=\{\text{measurable } \alpha\colon [s,T]\to\mathscr{A}\} \tag{95}$$

and $\mathscr{A}$ is a separable metric space as before.

For the running cost, we take a function $f\colon [0,T]\times H_1\times\mathscr{A}\to\mathbb{R}$ which satisfies that

$$\begin{gathered}f \text{ is continuous, and } f(t,x,\gamma) \text{ is continuous in } t \text{ uniformly in } \gamma \\ \text{for fixed } x\in H_1 \text{ and is uniformly continuous in } x\in H_1 \text{ uniformly in } (t,\gamma),\end{gathered} \tag{96}$$

and we have a terminal cost

$$g\in \mathrm{UC}(H). \tag{97}$$

The cost function is

$$J(s,x,\alpha)=\int_s^T f(t,X_t,\alpha_t)\,dt+g(X_T) \tag{98}$$

and the value function is

$$u(s,x)=\inf_{\alpha\in\mathscr{C}_{\mathscr{A}}(s,T)} J(s,x,\alpha). \tag{99}$$

First of all, we explain the continuity properties of the value function.

THEOREM IV.2. *Let* (94), (96), *and* (97) *hold. Then the value function u of* (99) *is uniformly continuous in* $x\in H$ *uniformly in* $s\in[0,T]$ *and is uniformly continuous in* $s\in[0,T]$ *uniformly for x in any bounded subset of H. In addition,* $u(T,x)=g(x)$ *for* $x\in H$ *and u satisfies that for* $\varepsilon>0$ *there exists* $C_\varepsilon$ *such that*

$$\begin{aligned}|u(s,x)-u(s,y)| &\leqslant \varepsilon(1+\|x-y\|) \\ &\quad + C_\varepsilon\,\|x-y\|_{-1} \qquad \textit{for} \quad x,y\in H,\quad s\in[0,T-\varepsilon].\end{aligned} \tag{100}$$

The proof of this result may be given by modifying the arguments already given. As with the results below, we therefore do not present these proofs, regarding them as straightforward even though writing them out would take quite a while.

Next we explain what the partial differential equation of dynamic programming is in this situation and the associated "viscosity" notions of subsolutions and supersolutions. These notions are adaptations of ones from [7, Part V]. The equation is

$$-\frac{\partial u}{\partial s} + \langle Ax, \nabla u\rangle + \sup_{\gamma\in\mathscr{A}} (-\langle b(s, x, \gamma), \nabla u\rangle - f(s, x, \gamma)) = 0 \quad \text{on} \quad [0, T)\times H. \tag{101}$$

DEFINITION IV.3. We say $\varphi = \psi + g$ is a subset function (for (101)) if

$$\begin{aligned} &\psi\in C^1([0, T]\times H) \text{ and is weakly sequentially lower semicontinuous,}\\ &\nabla\psi \in C([0, T]\times H : H_1),\\ &g\in C^1(H) \text{ is a function of } \|x\| \text{ alone and } g'(r)>0 \text{ for } t>0. \end{aligned}\tag{102}$$

DEFINITION IV.4. We say $v\in C([0, T)\times H)$ is a viscosity subsolution if for every subtest function $\varphi = \psi + g$ and local maximum $(\hat{s}, \hat{x})\in [0, T)\times H$ of $v-\varphi$ we have $\hat{x}\in H_1$ and

$$-\frac{\partial\varphi}{\partial s}(\hat{s}, \hat{x}) + \langle A\hat{x}, \nabla\varphi(\hat{s}, \hat{x})\rangle + \sup_{\gamma\in\mathscr{A}} (-\langle b(\hat{s}, \hat{x}, \gamma), \nabla\varphi(\hat{s}, \hat{x})\rangle - f(\hat{s}, \hat{x}, \gamma)) \leqslant 0. \tag{103}$$

We denote by $\mathrm{UC}_s([0, T]\times H)$ the space of continuous functions $u(t, x)$ on $[0, T]\times H$ which are continuous in $x$ uniformly in $t\in[0, T]$. Moreover, $u$ is $B$-continuous on $[0, T)\times H$ if $u(t_n, x_n)\to u(t, x)$ whenever $[0, T)\ni t_n\to t\in[0, T)$, $x_n\to x$ weakly in $H$, and $Bx_n\to Bx$ strongly in $H$.

THEOREM IV.5. *Let $v\in \mathrm{UC}_s([0, T]\times H)$ be $B$-continuous on $[0, T)\times H$ and a subsolution of* (101), *and*

$$(v(t_n, x) - g(x))^+ \to 0 \qquad \textit{uniformly on bounded subsets of } H \textit{ as } \quad t_n\uparrow T.$$

*Then $v(t, x)\leqslant u(t, x)$ on $[0, T]\times H$ where value function $u$ is given by* (99).

Viscosity supersolutions $w\in C([0, T)\times H)$ of (101) are defined by replacing $v$ with $w$, maximum with minimum, and $\leqslant$ with $\geqslant$, and by requiring $-\varphi$ to be a subtest function in Definition IV.4.

THEOREM IV.6. *Let $v, w \in UC_s([0, T] \times H)$ be B-continuous on $[0, T) \times H$ and, respectively, a subsolution and a supersolution of* (101). *Assume, moreover, that*

$$(v(t_n, x) - w(t_n, x))^+ \to 0 \qquad \text{uniformly on bounded subsets of } H \text{ as} \quad t_n \uparrow T. \tag{104}$$

*Then $v \leqslant w$ on $[0, T] \times H$.*

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