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OPTIMAL CONTROL OF STOCHASTIC INTEGRALS AND HAMILTON–JACOBI–BELLMAN EQUATIONS. I*

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Abstract. We consider the solution of a stochastic integral control problem and we study its regularity. In particular, we characterize the optimal cost as the maximum solution of

$$\forall v \in V, \quad A(v)u \leqq f(v) \quad \text{in } \mathcal{D}'(\mathcal{O}),$$
$$u = 0 \quad \text{on } \partial\mathcal{O}, \qquad u \in W^{1,\infty}(\mathcal{O}),$$

where $A(v)$ is a uniformly elliptic second order operator and V is the set of the values of the control.

1. Introduction

1.1 General introduction. In this paper we are interested in the following problem. We consider a stochastic system governed by the stochastic differential equation

$$\begin{aligned} &dy(t) = \sigma(y(t), v(t))\, dW_t + g(y(t), v(t))\, dt, \qquad t \geqq 0, \\ &y(0) = x \in \mathbb{R}^N, \end{aligned} \tag{1.1}$$

where W_t is a Wiener process, g, σ, are given functions and $v(t)$ is a "continuous" control taking values in some set $V \subset \mathbb{R}^m$. We want to minimize the cost function.

$$J(x, v(\cdot)) = E\left\{\int_0^\tau f(y(t), v(t)) \exp\left(-\int_0^t c(y(s), v(s))\, ds\right) dt\right\} \tag{1.2}$$

over all admissible controls $v(t)$. In this formula f and c are known, given functions and τ is the exit time of the process $y(t)$ from a given domain $\bar{\mathcal{O}}$. Let us denote $u(x) = \inf_{v(\cdot)} J(x, v(\cdot))$.

At least formally, by the argument of dynamical programming, one can derive the following equation satisfied by u:

$$\begin{aligned} &\sup_{v \in V} \{A(v)u(x) - f(x, v)\} = 0 \quad \text{in } \mathcal{O}, \\ &u = 0 \quad \text{on } \partial\mathcal{O} = \Gamma, \end{aligned} \tag{1.3}$$

where $A(v) = -\frac{1}{2}\sigma\sigma^T(x, v) \cdot D^2 - g(x, v) \cdot D + c(x, v)$[1]

Thus the initial stochastic control problem is connected to some nonlinear second order elliptic problem with Dirichlet boundary conditions; problem (1.3) is called the Dirichlet problem for Hamilton–Jacobi–Bellman equations.

In the following, we are going first to build a nonlinear semigroup whose generator is essentially the nonlinear operator defined by (1.3). The optimal cost function $u(x)$ appears then to be the unique fixed point of this semigroup: this fixed-point formulation can be viewed as a weak formulation of (1.3) or as the mathematical expression of dynamical programming. These results are in the spirit of those of M. Nisio [24].

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[1] σ^T, σ is the adjoint of σ.

Next we prove under very general assumptions that u lies in $W_0^{1,\infty}(\mathcal{O})$ and that u is the maximum element of functions $w \in W_0^{1,\infty}(\mathcal{O})$ satisfying $A(v)w \leqq f(v)$ in $\mathcal{D}'(\mathcal{O})$ for all $v \in V$. Of course this is a characterization of u, and it seems very useful since in some degenerate cases it is known that (1.3) does not hold (cf. Genis and N. V. Krylov [10]).

Here in part I, after giving some general results in the construction of this nonlinear semigroup, we essentially treat the case of nondegenerate stochastic integrals ($A(v)$ is uniformly elliptic) under mild regularity assumptions. In Part II [26] (this issue, pp. 82–95) the general case is considered.

The main results of this study were announced in [21]; we also proved a result on the verification of (1.3) (including [21]) which was also proved by different methods at the same time by L. C. Evans and A. Friedman [6]. Concerning the verification of (1.3) more general results were obtained by P.-L. Lions [15], L. C. Evans and P.-L. Lions [7] (in the case of nondegenerate diffusions), P.-L. Lions [16], [17] (in the general case). Below we will recall briefly their main results. We emphasize that we give here a different characterization of the optimal cost, requiring less regularity of $\mathcal{O}$ and of the coefficients and fewer assumptions on the nondegeneracy of $\sigma(x, u)$; this must be so for an approach to be valid while the verification of (1.3) is no longer true.

Finally, we recall that this kind of problem is introduced in the book of W. H. Fleming and R. Rishel [8], and that the first general results on this problem were obtained by N. Krylov [11], [12], [14].

1.2. Summary. Our results are organized in the following way:

Section 2 Construction of a nonlinear semigroup.

Section 3. A stochastic characterization of $u(x)$.

Section 4. An analytical characterization of $u(x)$.

In § 2, following some techniques of M. Nisio [23], we build a nonlinear semigroup whose generator is related to the operator appearing in (1.3). In § 3 we give a stochastic characterization of $u(x)$, the precise way to supply dynamical programming. Finally in § 4 we prove a characterization of $u(x)$, in terms of a maximum solution of inequalities. In § 4, we shall suppose that $\sigma(x, v)$ are nondegenerate matrices. The generalization to the case of degeneracy will be developed in Part II, together with results concerning other boundary conditions, the case of optimal stopping and the case of nonhomogeneous diffusions and parabolic equations.

1.3. Assumptions and notation. We now give notation and assumptions which will remain valid in §§ 2, 3 and 4.

Let $\mathcal{O}$ be a domain in $\mathbb{R}^N$, and let V be a convex closed set in $\mathbb{R}^m$. We call an *admissible system* a set $\mathcal{A} = (\Omega, F, F_t, P, W_t, v(t), y_x(t))$, where (Ω, F, P) is a probability space, F_t is a nondecreasing right continuous family of sub σ-algebras F_t of F, W_t is a Wiener process with respect to F_t, $v(t)$ is a measurable adapted process taking values in some compact subset V_0 of V (V_0 of course may depend on $v(\cdot)$) and $y_x(t)$ is a solution of

$$(1.4)\qquad \begin{aligned} dy_x(t) &= \sigma(y_x(t), v(t))\, dW_t + g(y_x(t), v(t))\, dt, \\ y_x(0) &= x. \end{aligned}$$

We suppose that σ, g satisfy

$$(1.5)\qquad |\phi(x, v) - \phi(x', v')| \leqq C|x - x'| + \rho(|v - v'|) \quad \forall x, x' \in \mathbb{R}^N, \quad \forall v, v' \in V,$$

where $\phi = \sigma_{ij}(1 \leqq i, j \leqq n)$, $g_i(1 \leqq i \leqq n)$ and ρ is a given continuous function from $\mathbb{R}_+$ into $\mathbb{R}_+$ with $\rho(0) = 0$.

We assume also that we have

$$(1.6) \qquad |\sigma(x, v)| + |g(x, v)| \leqq C \quad \forall x \in \mathbb{R}^N, \quad \forall v \in V.$$

Now for an admissible system $\mathscr{A}$ we define a cost function

$$(1.7) \qquad \begin{aligned} J(x, \mathscr{A}, t, h) = E\bigg\{ \int_0^{t \wedge \tau_x} & f(y_x(s), v(s)) \exp\left(-\int_0^s c(y_x(\lambda), v(\lambda))\, d\lambda\right) ds \\ & + h(y_x(t \wedge \tau_x)) \cdot \exp\left(-\int_0^{t \wedge \tau_x} c(y_x(s), v(s))\, ds\right)\bigg\}, \end{aligned}$$

where h is an arbitrary measurable bounded function, τ_x is the first exit time from $\bar{\mathcal{O}}$ of $y_x(t)$, and $f(x, v)$, $c(x, v)$ are given and are assumed to satisfy (1.5) with $\phi = c$, (1.6) and

$$(1.8) \qquad |f(x, v) - f(x', v')| \leqq \rho(|x - x'| + |v - v'|) \quad \forall x, x' \in \mathbb{R}^N, \quad \forall v, v' \in V,$$

$$(1.9) \qquad c(x, v) \geqq c_0 \geqq 0 \quad \forall x \in \mathbb{R}^N, \quad \forall v \in V.$$

Finally we define for each h, an optimal cost function

$$(1.10) \qquad Q(t)h(x) = \inf J(x, \mathscr{A}, t, h) \quad \forall 0 \leqq t < +\infty.$$

Let us collect our assumptions:

(1.5) $|\phi(x, v) - \phi(x', v')| \leqq C|x - x'| + \rho(|v - v'|) \forall x, x' \in \mathbb{R}^N, \forall v, v' \in V, \forall \phi = \sigma_{ij}, g, c.$

(1.6) $|\phi(x, v)| \leqq C \ \forall \phi = \sigma_{ij}, g_i, c, f, \forall x \in \mathbb{R}^N, \forall v \in V.$

(1.8) $|f(x, v) - f(x', v')| \leqq \rho(|x - x'| + |v - v'|) \forall x, x' \in \mathbb{R}^N, \forall v, v' \in V.$

(1.9) $c(x, v) \geqq c_0 \geqq 0 \ \forall x \in \mathbb{R}^N, \forall v \in V.$

We shall denote by B_s the set of bounded functions from $\bar{\mathcal{O}}$ into $\mathbb{R}$ which are upper semicontinuous; B_s is a closed convex cone of the Banach space B of bounded measurable functions equipped with the supremum norm ($\|h\|_\infty = \sup |h(x)|$).

2. A nonlinear semigroup

2.1. The semigroup property. In this section we prove that $Q(t)$ acting on B_s is a nonlinear semigroup. This result generalizes [23] (cf. also [1]), where $\mathcal{O} = \mathbb{R}^N$. We need, in addition to (1.5-6-8-9), a technical assumption: the set of regular points is closed, i.e.,

$$(2.1) \qquad \begin{aligned} &\forall \mathscr{A}, \text{ admissible } \Gamma_0(\mathscr{A}) = \{x \in \Gamma / P(\tau_x > 0) = 0\} \text{ is closed}, \\ &\forall x \in \bar{\mathcal{O}}, \ P[y_x(\tau_x) \in \Gamma_0(\mathscr{A})] = 1. \end{aligned}$$

We shall see below that in the nondegenerate case this assumption becomes obvious, and that in many cases one can give conditions for (2.1) to be satisfied.

THEOREM 2.1. *Assume* (1.5-6-8-9) *and* (2.1). *Then* $(Q(t), t \geqq 0)$ *satisfies*:

(2.2) $Q(t): B_s \to B_s, \ Q(0) = I, \ Q(t+s) = Q(t) \circ Q(s) = Q(s) \circ Q(t),$

(2.3) $\|Q(t)h - Q(s)h\|_\infty \to 0$ *as* $t \to s$ *if* h *is uniformly continuous on* $\bar{\mathcal{O}}$,

(2.4) $\|Q(t)h_1 - Q(t)h_2\|_\infty \leqq \|h_1 - h_2\|_\infty \ \forall h_1, h_2 \in B_s, \ \forall t \geqq 0,$

(2.5) $Q(t)h_1 \leqq Q(t)h_2$ *if* $h_1 \leqq h_2$.

Remark 2.1. We shall see below that, in the case of nondegenerate σ, $Q(t)$ leaves $C_b(\bar{\mathcal{O}})$ invariant.

Remark 2.2. Let us give a heuristic justification of Theorem 2.1. By the dynamical programming argument $h(t) = Q(t)h$ is the "solution" of

$$\frac{dh}{dt}(s) + \sup_{v \in V} \{A(v)h(s, x) - f(x, v)\} = 0 \quad \forall s \in [0, t], \quad \forall x \in \mathcal{O},$$

$$h(0) = h, \qquad h(s)|_{\Gamma_0} = h|_{\Gamma_0} \quad \forall s,$$

where[2] $A(v) = -a_{ij}\, \partial^2/\partial x_i \partial x_j + b_i\, \partial/\partial x_i + c$ and $a_{ij}(x, v) = \frac{1}{2}\sigma_{ik}\sigma_{jk}(x, v)$, $b_i(x, v) = -g_i(x, v)$.

Now (2.2) appears as a classical result for some Cauchy problem, and (2.4) and (2.5) are easy consequences of the maximum principle.

The proof will be divided in several parts. First we prove some lemmas.

LEMMA 2.1. *For all* $h \in B_s$, *we have*

$$Q(t)h(x) = \inf_{\mathcal{A}_{cl}} J(x, \mathcal{A}_{cl}, t, h) \; (resp. = \inf_{\mathcal{A}_c} J(x, \mathcal{A}_c, t, h)), \tag{2.6}$$

where the infimum is taken over all admissible systems such that $v(t)$ *is right continuous with left-hand limits* (*resp. is continuous*).

Proof. Let $\mathcal{A}$ be an admissible system. We define

$$v_k(t) = \frac{1}{k}\int_{(t-k)^+}^{t} v(\lambda)\, d\lambda + \left(1 - \frac{t}{k}\right)^+ v_0 \qquad (\text{with } v_0 \in V) \tag{2.7}$$

and let $\mathcal{A}_k$ be the same system as $\mathcal{A}$ with $v(t)$ replaced by $v_k(t)$. Assuming Lemma 2.2 below for the moment,

$$J(x, \mathcal{A}_k, t, h) \to J(x, \mathcal{A}, t, h) \quad \text{as } k \to 0^+, \quad \forall h \in C_b(\bar{\mathcal{O}}).$$

Thus the equality (2.6) is proved if h is continuous. But if $h \in B_s$, there exists $h_n \in C_b(\bar{\mathcal{O}})$, $h_n(x) \downarrow h(x) \forall x \in \bar{\mathcal{O}}$. As (2.6) is true for h_n and $Q(t)\, h_n(x) \downarrow Q(t)\, h(x)$, inf $\mathcal{A}_{cl}\, J(x, \mathcal{A}_{cl}, t, h_n) \downarrow \inf_{\mathcal{A}_{cl}} J(x, \mathcal{A}_{cl}, t, h)$ and $\inf_{\mathcal{A}_c} J(x, \mathcal{A}, t, h) \downarrow \inf_{\mathcal{A}_{cl}} J(x, \mathcal{A}, t, h)$, we deduce (2.6) for h.

LEMMA 2.2. *Let* $\mathcal{A}$ *be an admissible system and let* $\mathcal{A}_k$ *be the system defined above. We have*

$$\lim_{k \to 0^+} J(x, \mathcal{A}_k, t, h) = J(x, \mathcal{A}, t, h) \quad \forall h \in C_b(\bar{\mathcal{O}}), \quad \forall x \in \bar{\mathcal{O}}, \quad \forall t \geqq 0.$$

Proof. Letting $y_k(t)$ be the solution of (1.4) corresponding to $v_k(t)$, we have

$$y_k(t) - y(t) = \int_0^t \{\sigma(y_k, v_k) - \sigma(y, v)\}\, dW_s + \int_0^t (g(y_k, v_k) - g(y, v))\, ds.$$

Thus for all $0 \leqq t \leqq T$ there exists a C_T such that

$$E\{|y_k(t) - y(t)|^2\} \leqq C_T E\left\{\int_0^t |y_k - y|^2 + \rho^2(|v_k - v|)\, ds\right\}.$$

[2] We shall always use the usual convention for sums.

By Gronwall's lemma and by a classical martingale technique, we deduce

$$E\{\sup_{0\leqq t\leqq T}|y_k(t)-y(t)|^2\}\leqq C_T^2 E\left\{\int_0^T \rho^2(|v_k-v|)\,ds\right\}. \tag{2.8}$$

But there is a $V_0\subset V$, V_0 compact, such that $v(t,\omega)\in V_0$; thus $v_k(t,\omega)\in \operatorname{conv}(V_0, v_0)$, which is also compact. Now $v_k\to v$ a.e. (t,ω), and this implies

$$E\left\{\int_0^T \rho^2(|v_k-v|)\,ds\right\}\to 0 \quad \text{as } k\to 0_+;$$

from (2.8) we have

$$\lim_{k\to 0_+} E\{\sup_{0\leqq t\leqq T}|y_k(t)-y(t)|^2\}=0. \tag{2.8$'$}$$

Finally, as in the proof of the Lemma 2.3 below, we have

$$\lim_{k\to 0_+} P\{|T\wedge\tau_k - T\wedge\tau|\geqq\varepsilon\}=0 \quad \forall\varepsilon>0, \tag{2.9}$$

where τ_k is the exit time corresponding to the process $y_k(t)$; because of (2.8′) we can extract a subsequence y_{k_m}, τ_{k_n} such that

$$y_{k_n}(t)\to y(t) \quad \text{in } C([0,T],\mathbb{R}^N) \quad \text{a.s.},$$
$$T\wedge\tau_{k_n}\to T\wedge\tau \quad \text{a.s.}$$

Thus by the Lebesgue theorem we have proved the lemma. □

LEMMA 2.3. *We have all admissible systems*

$$\lim_{\substack{x\to x_0\\ x\in\bar{\mathcal{O}}}} P\{|T\wedge\tau_x - T\wedge\tau_{x_0}|\geqq\varepsilon\}=0 \quad \forall x_0\in\bar{\mathcal{O}},\quad \forall\varepsilon>0,\quad \forall T>0.$$

Proof. We define $\tau'=\tau'_x=\inf(t\geqq 0\ y_x(t)\notin\bar{\mathcal{O}}-\Gamma_0)$ and $N_x^T=\{\omega\in\Omega/\tau_x<T, y_x(\tau_x)\notin\Gamma_0\}$. By assumption (2.1), we have

$$P(N_x^T)=0 \quad \forall x\in\bar{\mathcal{O}},\quad \forall T>0, \tag{2.10}$$

$$T\wedge\tau_x(\omega)=T\wedge\tau'_x(\omega) \quad \forall\omega\in\Omega-N_x^T. \tag{2.11}$$

The lemma is proved if we show that, for all $x_n\to x_0$ in $\bar{\mathcal{O}}$,

$$\begin{aligned} A&=\{\omega\in\Omega/\overline{\lim_n}|T\wedge\tau_{x_n}(\omega)|>0\}\\ &\subset B=\left(\bigcup_{n=1}^{\infty} N_{x_n}^T\right)\cup\{\omega\in\Omega/\overline{\lim_n}\sup_{0\leqq t\leqq T}|y_{x_n}(t,\omega)y_{x_0}(t,\omega)|>0\}, \end{aligned} \tag{2.12}$$

since from (2.10) and (2.8′) (same proof) $P(B)=0$.

In order to show (2.12), let $\omega\notin B$. First we prove $\lim_n T\wedge\tau_{x_n}(\omega)\leqq T\wedge\tau_{x_0}(\omega)$. We can suppose $\tau_{x_0}<T$: For all $\delta>0$ there is a $s_\delta<\tau_{x_0}(\omega)+\delta$ such that $y_{x_0}(s_\delta,\omega)\notin\bar{\mathcal{O}}$; hence $y_{x_n}(s_\delta,\omega)\notin\bar{\mathcal{O}}$ if n is large enough and $\tau_{x_n}(\omega)\leqq s_\delta\leqq\tau_{x_0}(\omega)+\delta$.

Next we prove $\lim_n T\wedge\tau'_{x_n}(\omega)\geqq T\wedge\tau'_{x_0}(\omega)$. We may suppose $\tau'_{x_0}(\omega)>0$, and we define, for $0<\delta<\tau'_{x_0}(\omega)$, $K_\omega=\{y_{x_0}(t,\omega)/t\in[0,\tau'_{x_0}(\omega)-\delta]\}$. K_ω is a compact set such that $K_\omega\cap\Gamma_0=\varnothing$. Now, by the choice of ω, we obtain for n large enough

$$K_\omega^n=\{y_{x_n}(t,\omega)/t\in[0,\tau'_{x_0}(\omega)-\delta]\}\cap\Gamma_0=\varnothing,$$

and this implies $\tau'_{x_n}(\omega)\geqq\tau'_{x_0}(\omega)=\delta$ for n large enough. □

Proof of Theorem 2.1. We remark first that properties (2.4), (2.5) are immediate. The steps of the proof are the following:

i) $Q(t)h \in B_s$ if $h \in B_s$.
ii) Proof of (2.3).
iii) $Q(t+s) = Q(t) \circ Q(s)$.

i) We begin by proving that if $h \in C_b(\bar{\mathcal{O}})$ then $Q(t)\, h \in B_s$. Indeed, Lemmas 2.2 and 2.3 imply that $J(x, \mathcal{A}, t, h) \in C_b(\bar{\mathcal{O}})$; thus

$$Q(t)h = \inf J(x, \mathcal{A}, t, h) \in B_s.$$

Furthermore, if $h \in B_s$, there exists $h_n \in C_b(\bar{\mathcal{O}})$, $h_n(x) \downarrow h(x)$ for all $x \in \bar{\mathcal{O}}$; therefore $Q(t)\, h_n(x) \downarrow Q(t)\, h(x)$ and $Q(t)h \in B_s$.

(ii) To prove (2.3), it is enough to prove that for all uniformly continuous

$$\sup_{\mathcal{A}} E\{|h(y_x(t \wedge \tau_x)) - h(y_x(s \wedge \tau_x))|\} \to 0 \quad (\text{as } t \to s) \text{ uniformly in } x.$$

First, remark we have $E\{|y_x(t \wedge \tau_x) - y_x(s \wedge \tau_x)|^2\} \leqq C|t-s|$ (C is independent of $\mathcal{A}$ and x); thus

$$P[|y_x(t \wedge \tau_x) - y_x(s \wedge \tau_x)| \geqq \varepsilon] \leqq C\frac{|t-s|}{\varepsilon^2} \quad \forall \varepsilon > 0.$$

Let $\mu > 0$. Then $\exists \varepsilon$, $\forall x, x' \in \bar{\mathcal{O}}$, $|x - x'| \leqq \varepsilon \Rightarrow |h(x) - h(x')| \leqq \mu$. We have

$$\sup_{\mathcal{A}} E\{|h(y_x(t \wedge \tau_x)) - h(y_x(s \wedge \tau_x))|\} \leqq \frac{C\|h\|_\infty\, |t-s|}{\varepsilon^2} + \mu,$$

and the conclusion follows easily

iii) We want to prove the semigroup property $Q(t+s) = Q(t) \circ Q(s)$. Because of Lemma 2.1, we can restrict ourselves to admissible systems with continuous $v(t)$. We can also restrict our attention to admissible systems where (Ω, F, F_t) is the canonical space $\Omega = C([0, +\infty[, \mathbb{R}^{n+m})$ (just take image measures). But at this point the proof of this property is exactly the same as the one given in [2, Thm. 5.1]. The proof depends heavily on a theorem of regular conditional probabilities proved by D. W. Stroock–S. R. S. Varadhan [25] and N. V. Krylov [11]. □

2.2. The generator of $Q(t)$. We are going to prove that the "generator" of $Q(t)$ is an extension of the operator $\phi \in C^2(\bar{\mathcal{O}}) \to \sup_{v \in V} \{A(v)\, \phi(x) - f(x, v)\}$.

THEOREM 2.2. *Under the assumptions of Theorem* 2.1, *we have for all* $h \in C_b^2(\bar{\mathcal{O}})$

$$\text{(2.13)} \qquad \frac{1}{t}\{Q(t)h(x) - h(x)\} \to -\sup_{v \in V}\{A(v)h(x) - f(x, v)\} \quad \text{as } t \to 0_+ \quad \forall x \in \mathcal{O}.$$

Moreover the convergence in (2.13) *is uniform on compact subsets of* $\mathcal{O}$.

Proof. The proof is very similar to the proof of M. Nisio [23] (see also the presentation in [2, Thm. 5.2]). We define

$$K(x, \mathcal{A}, t, h) = \int_0^{t \wedge \tau_x} f(y_x(s), v(s)) - A(v(s))\, h(y_x(s))\, ds,$$

and we prove easily (see for example [1]) that

$$\forall \varepsilon>0, \quad \exists \delta=\delta(\varepsilon, h)>0, \quad \forall t \leqq \delta, \quad \left|\frac{Q(t)\, h(x)-h(x)}{t}-\inf_{\mathscr{A}} E\left\{\frac{1}{t} K(x, \mathscr{A}, t, h)\right\}\right| \leqq \varepsilon,$$

$$\inf_{\mathscr{A}} E\left\{\frac{1}{t} K(x, \mathscr{A}, t, h)-\inf_{v \in V}[f(x, v)-A(v)\, h(x)]\right\} \geqq -C\left(1-\inf_{\mathscr{A}} E\left\{\frac{t \wedge \tau_x}{t}\right\}\right).$$

On the other hand, if $\mathscr{A}_0$ is an admissible system corresponding to $v(t)=v_0 \in V$,

$$\begin{aligned}
&\inf_{\mathscr{A}} E\left\{\frac{1}{t} K(x, \mathscr{A}, t, h)-\inf_{v \in V}[f(x, v)-A(v) h(x)]\right\} \\
&\quad \leqq E\left\{\frac{1}{t} K(x, \mathscr{A}_0, t, h)-\inf_{v \in V}[f(x, v)-A(v) h(x)]\right\} \\
&\quad \leqq C\left(1-E\left\{\frac{t \wedge \tau_x}{t}\right\}\right) \\
&\quad \leqq C\left(1-\inf_{\mathscr{A}} E\left(\frac{t \wedge \tau_x}{t}\right)\right).
\end{aligned}$$

Thus we have obtained

$$\forall t \leqq \delta, \left|\frac{Q(t)\, h(x)-h(x)}{t}-\inf_{v \in V}[f(x, v)-A(v)\, h(x)]\right| \leqq C\left\{1-\inf_{\mathscr{A}} E\left(\frac{t \wedge \tau_x}{t}\right)\right\}+\varepsilon.$$

To conclude, we just need to prove that if K is a compact subset of $\mathcal{O}$ then

$$\sup_{\mathscr{A}, x \in K} P(\tau_x<t) \underset{t \to 0}{\rightarrow} 0.$$

Letting γ be $\gamma=d(K, \Gamma)>0$, we have

$$\forall x \in \bar{\mathcal{O}}, \quad P[\tau_x<t] \leqq P\left(\sup_{0 \leqq s \leqq t}|y_x(s)-x| \geqq \gamma\right) \leqq \frac{1}{\gamma^2} E\left\{\sup_{0 \leqq s \leqq t}|y_x(s)-x|^2\right\}.$$

Since $E\{\sup_{0 \leqq s \leqq t}|y_x(s)-x|^2\} \leqq CE|y_x(t)-x|^2 \leqq C_1 t+C_2 t^2$, where C, C_1, C_2 do not depend on $\mathscr{A}$, x and t, (2.13) is easily proved. □

Remark 2.3. If we introduce

$$\Gamma_1=\left\{x \in \Gamma \ \limsup_{\varepsilon \to 0_+} \sup_{\mathscr{A}} E\left(\frac{\varepsilon \wedge \tau_x}{\varepsilon}\right)=0\right\}, \qquad \Gamma_2=\left\{x \in \Gamma \ \liminf_{\varepsilon \to 0} \inf_{\mathscr{A}} E\left(\frac{\varepsilon \wedge \tau_x}{\varepsilon}\right)=1\right\},$$

for $h \in C_b^2(\bar{\mathcal{O}})$ we have, as $t \to 0_+$,

i) $$\frac{Q(t)\, h(x) - h(x)}{t} \to 0 \quad \text{if } x \in \Gamma_1,$$

ii) $$\frac{Q(t)\, h(x) - h(x)}{t} \to -\sup \{A(v)\, h(x) - f(x, v)\} \quad \text{if } x \in \Gamma_2.$$

Remark that $\Gamma_0 \subset \Gamma_1$.

Remark 2.4. In the particular case of nondegeneracy, i.e.,

$$\exists \alpha > 0, \quad a_{ij}(x, v)\xi_i\xi_j \geqq \alpha |\xi|^2 \quad \forall \xi \in \mathbb{R}^N, \quad \forall x \in \bar{\mathcal{O}}, \quad \forall v \in V, \tag{2.14}$$

we shall see that $\Gamma_0(\mathcal{A}) = \Gamma$ for all admissible systems (if some regularity condition on Γ is assumed); hence, for all $x \in \bar{\mathcal{O}}$, as $t \to 0$

$$\frac{Q(t)\, h(x) - h(x)}{t} \to -1_{\mathcal{O}}(x) \sup_{v \in V} \{A(v)\, h(x) - f(x, v)\}.$$

Remark 2.5. We shall see below a result more precise than Theorem 2.2.

2.3. The nondegenerate case. In this section in addition to (1.5-6-8-9), we assume (2.14) and $\mathcal{O}$ has a uniform exterior sphere; i.e.,

$$\exists \rho > 0, \quad \forall x \in \Gamma, \quad \exists y \in \mathbb{R}^N - \mathcal{O}, \quad \{z / |y - z| \leqq \rho\} \cap \bar{\mathcal{O}} = \{x\}. \tag{2.15}$$

We are going to prove that under these assumptions $Q(t)$ leaves X invariant, where $X = \{h \in C_b(\bar{\mathcal{O}}),\ h \text{ is uniformly continuous on } \bar{\mathcal{O}}\}$. Before doing so or even stating the precise result, we prove a lemma which will be useful.

LEMMA 2.4. *Under assumptions* (1.5-6-8-9) *and* (2.14–15), *we have*:

(2.16) *If $\mathcal{O}$ is bounded*, $\exists \mu > 0$, $\exists C > 0$, $\forall x \in \mathcal{O}$, $\forall \mathcal{A}$ *admissible*, $E[e^{\mu \tau_x}] \leqq C$;

(2.17) $\forall \mathcal{A}$ *admissible*, $\Gamma = \Gamma_0(\mathcal{A})$.

Remark 2.6. It is clear that even if (2.14) is satisfied, $\mathcal{O}$ has to be "smooth" in order to make (2.17) true. Indeed, if $N = 1$, $V = \{v_0\}$, $y_x(t) = x + W(t)$, $\sigma(v_0) = \sqrt{2}$, $\mathcal{O} =]0, 1[\cup]1, 2[$, we have $E[\tau_1] = \frac{1}{2}$, so $1 \in \Gamma - \Gamma_0$.

Proof of Lemma 2.4. First we consider $w(x) = 1 - \exp(-k|x|^2)$ (we may always assume that $0 \in \bar{\mathcal{O}}$). We have $A(v)w(x) \geqq \{4a_{ij}(x, v)k^2 x_i x_j - 2ka_{ii}(x, v) - 2kx_i b_i(x, v)\} \exp -k|x|^2$. Thus we can choose k large enough to insure that $A(v)\, w(x) \geqq \alpha > 0$ for all $x \in \bar{\mathcal{O}}$ (because $\mathcal{O}$ is bounded), where $\tilde{A} = A - c$.

Now we take $\mu = \alpha/2$, and we have

$$\tilde{A}(v)w - \mu w \geqq \mu > 0 \quad \forall x \in \bar{\mathcal{O}}. \tag{2.18}$$

Using Ito's formula with w, it is easy to deduce (2.16) from (2.18).

Now we prove (2.17). We introduce

$$w(x, \xi) = \exp(-k\rho^2) - \exp(-k|x - \xi_1|^2), \tag{2.15'}$$

where ρ is given by (2.15), $\xi \in \Gamma$ and ξ_1 is associated to ξ by (2.15), $x \in \bar{\mathcal{O}}$ and $k > 0$. By calculation similar to the above, one shows that for k large enough

$$A(v)w(x, \xi) \geqq \alpha > 0 \quad \forall x \in \mathcal{O}. \tag{2.19}$$

Applying Ito's formula, we have

$$0 = w(\xi, \xi) = E\left\{ w(y_\xi(\tau_\xi)) + \int_0^{\tau_\xi} \alpha \exp\left(\int_0^t c(y_\xi(s), v(s))\, ds \right) dt \right\}$$

$$\geqq \alpha E\left[\int_0^{\tau_\xi} e^{-ct}\, dt \right];$$

thus $P[\tau_\xi = 0] = 1$ and $\xi \in \Gamma_0(\mathcal{A})$ for all $\xi \in \Gamma$. □

The first result concerning the regularity of $Q(t)\,h$ when h is smooth will be the following.

THEOREM 2.3. *We assume* (1.5-6-8), (2.14-15) *and*

$$|f(x, v) - f(x', v)| \leqq C|x - x'| \quad \forall x, x' \in \mathcal{O}, \tag{2.20}$$

$$c(x, v) \geqq C > [\mu_0]^+, \tag{2.21}$$

where μ_0 is given by

$$\mu_0 = \sup_{\substack{x,x' \in \mathcal{O} \\ v \in V}} \left\{ \frac{1}{2} \operatorname{Tr} \frac{(\sigma(x, v) - \sigma(x', v))(\sigma^T(x, v) - \sigma^T(x', v))}{|x - x'|^2} + \frac{(x - x') \cdot (g(x) - g(x'))}{|x - x'|^2} \right\}. \tag{2.22}$$

Then, if $h \in W^{2,\infty}(\mathcal{O})$, we have

$$|Q(t)\,h(x) - Q(t)\,h(x')| \leqq C|x - x'| \quad \forall x, x' \in \mathcal{O}, \tag{2.23}$$

where C is independent of t.

COROLLARY 2.1. *If we assume* (1.5-6-8-9) *and* (2.14-15-21) *then, for $h \in X$, $Q(t)h \in X$. Furthermore, $(Q(t)h, t \geqq 0)$ is uniformly equicontinuous.*

Proof of Corollary 2.1. By a simple approximation (uniform in v) of the function $f(v)$, one can always assume that (2.20) is satisfied and that h belongs to $W^{2,\infty}(\mathcal{O})$; then the result is obvious in view of Theorem 2.3. □

Remark 2.7. We shall see below (§ 3.1, Remark 3.5) that Corollary 2.1 is valid without assuming (2.21), and (§ 4.3) that Theorem 2.3 remains true without assuming (2.21).

Remark 2.8. If assumptions (2.14–15) are dropped, one can nevertheless prove Theorem 2.3 (and thus Corollary 2.1) with the same method if we assume

$$\begin{gathered} \exists p_0 \in W^{1,\infty}(\mathcal{O}), \quad p_0|_{\Gamma_0} = 0, \quad \forall v \in V, \quad A(v)p_0 \in L^\infty(\mathcal{O}), \\ \exists \alpha_0 > 0, \quad \forall v \in V, \quad A(v)p_0 \leqq -\alpha_0 \quad \text{in } \mathcal{O}. \end{gathered} \tag{2.24}$$

For example suppose that $g = c = 0$, $\sigma(x, v) = \sigma(v)$ and that there exists $\beta_0 > 0$ such that $\det(\sigma(v)\,\sigma^T(v)) \geqq \beta_0 > 0$. Furthermore, assume that $\mathcal{O} = \{p(x) < 0\}$ with $\partial\mathcal{O} = \{p(x) = 0\}$ and that $p \in W^{2,\infty}(\mathcal{O})$ and

$$\det\left(\frac{\partial^2 p}{\partial x_i \partial x_j}(x) \right) \geqq \alpha_0 > 0 \quad \forall x \in \bar{\mathcal{O}}.$$

Then the results above remain true. This example generalizes a result of B. Gaveau [9].

Other generalizations to the case of degenerate σ are treated in Part II.

Remark 2.9. One can generalize Corollary 2.1 to the case where $\sup_{v\in V}|f(x,v)|\in L^N(\mathcal{O})$. Indeed, this comes easily from a result of N. V. Krylov [13].

Proof of Theorem 2.3. The proof is divided into several steps:

1) Construction of a subsolution.
2) Two lemmas.
3) Conclusion.

1) We consider the function $w(x,\xi)$ defined in Lemma 2.4, and we introduce $w(x)=\inf_{\xi\in\Gamma} w(x,\xi)$. Obviously $w(x)\in W^{1,\infty}(\mathcal{O})$, $w\geqq 0$ in $\mathcal{O}$, $w=0$ on Γ. Now applying Ito's formula to $w(x,\xi)$ for fixed ξ in Γ, we have (in the proof of this theorem, we shall take $c(x,v)\equiv c_0>\mu_0$ for the sake of simplicity) that

$$w(y_x(t\wedge\tau_x),\xi)\,e^{-c_0 t\wedge\tau_x}+\alpha\int_0^{t\wedge\tau_x} e^{-c_0 s}\,ds$$

is a submartingale bounded and continuous.

Then, taking the infimum over all ξ in Γ, we have that

$$w(y_x(t\wedge\tau_x))\,e^{-c_0 t\wedge\tau_x}+\alpha\int_0^{t\wedge\tau_x} e^{-c_0 s}\,ds \tag{2.25}$$

is a submartingale bounded and continuous.

2) LEMMA 2.5. *Under the assumptions of Theorem* 2.3, *we have*

$$E[|e^{-c_0\tau_x}-e^{-c_0\tau_{x'}}|]\leqq\frac{2C_0}{\alpha}\|\nabla w\|_\infty\,|x-x'|. \tag{2.26}$$

Proof. Applying (2.25) between $\tau_x\wedge\tau_{x'}$ and τ_x, we have

$$E[w(y_x(\tau_x))\,e^{-c_0\tau_x}-w(y_x(\tau_x\wedge\tau_{x'}))\,e^{-c_0\tau_x\wedge\tau_{x'}}]\geqq-\alpha E\left[\int_{\tau_x\wedge\tau_{x'}}^{\tau_x} e^{-c_0 s}\,ds\right];$$

thus

$$\frac{\alpha}{c_0}E[e^{-c_0\tau_x\wedge\tau_{x'}}-e^{-c_0\tau_x}]\leqq\|\nabla w\|_\infty E\,\{|y_x(\tau_x\wedge\tau_{x'})-y_{x'}(\tau_x\wedge\tau_{x'})|\,e^{-c_0\tau_x\wedge\tau_{x'}}$$

and we deduce (2.26) from the following lemma. □

LEMMA 2.6. *Under the assumptions of Theorem* 2.3, *we have for all stopping times* θ

$$E\{|y_x(\theta)-y_{x'}(\theta)|^2\,e^{-2\mu_0\theta}\}\leqq|x-x'|^2. \tag{2.27}$$

Proof. We apply Ito's formula between 0 and $\theta\wedge T$ to the function $(\xi\to|\xi|^2)$ for the process $y_x(t)-y_{x'}(t)$, and obtain

$$\begin{aligned}&E\{|y_x(\theta\wedge T)-y_{x'}(\theta\wedge T)|^2\,e^{-2\mu_0\theta\wedge T}\}\\&\quad=|x-x'|^2+E\left\{\int_0^{\theta\wedge T}\mathrm{Tr}\,\{(\sigma(y_x(t))-\sigma(y_{x'}(t)))\cdot(\sigma^T(y_x(t))-\sigma^T(y_{x'}(t)))\}\,e^{-2\mu_0 t}\right.\\&\qquad\qquad+2(y_x(t)-y_{x'}(t))\cdot(g(y_x(t)-g(y_{x'}(t)))\,e^{-2\mu_0 t}\,dt\\&\qquad\qquad\left.-2\mu_0\int_0^{\theta\wedge t}|y_x(t)-y_{x'}(t)|^2\,e^{-2\mu_0 t}\,dt\right\}.\end{aligned}$$

Thus, by definition of μ_0, we have

$$E[|y_x(\theta\wedge T)-y_{x'}(\theta\wedge T)|^2\,e^{-2\mu_0\theta\wedge T}]\leqq|x-x'|^2.$$

3) *Conclusion.* Letting $x, x' \in \bar{\mathcal{O}}$, we have

$$|Q(t)\,h(x) - Q(t)\,h(x')| \leqq I + J,$$

where

$$I = \sup_{\mathcal{A}} \left| E\left[\int_0^{\tau_x \wedge t} f(y_x(s), v(s))\, e^{-c_0 s}\, ds\right] - E\left[\int_0^{\tau_{x'} \wedge t} f(y_{x'}(s), v(s))\, e^{-c_0 s}\, ds\right]\right|$$

and

$$J = \sup_{\mathcal{A}} |E[h(y_x(\tau_x \wedge t))\, e^{-c_0 \tau_x \wedge t} - h(y_{x'}(\tau_{x'} \wedge t))\, e^{-c_0 \tau_{x'} \wedge t}]|.$$

First, because of Lemma 2.5 and (2.20), we easily have $I \leqq C|x - x'|$.
Next,

$$\begin{aligned} J \leqq \sup_{\mathcal{A}} \{ &|E\{h(y_x(t \wedge \tau_x))\, e^{-c_0 t \wedge \tau_x} - h(y_x(t \wedge \tau_x \wedge \tau_{x'}))\, e^{-c_0 t \wedge \tau_x \wedge \tau_{x'}}\} \\ &+ |E\{h(y_{x'}(t \wedge \tau_{x'}))\, e^{-c_0 \tau_{x'} \wedge t} - h(y_{x'}(t \wedge \tau_x \wedge \tau_{x'}))\, e^{-c_0 t \wedge \tau_x \wedge \tau_{x'}}\}| \\ &+ |E\{h(y_x(t \wedge \tau_x \wedge \tau_{x'})) - h(y_{x'}(t \wedge \tau_x \wedge \tau_{x'})|\, e^{-c_0 t \wedge \tau_x \wedge \tau_{x'}}\}|\} \\ \leqq \sup_{v \in V} &\|A(v)\,h\|_\infty \cdot \frac{2c_0}{\alpha} \|\nabla w\|_\infty\, |x - x'| + \|\nabla h\|_\infty\, |x - x'| \end{aligned}$$

(here we have applied Ito's formula and (2.26), (2.27)). □

3. A stochastic interpretation of the minimum cost function

3.1. A stochastic control problem. We consider the optimal cost function

$$u(x) = \inf_{\mathcal{A}} E\left\{\int_0^{\tau_x} f(y_x(t), v(t)) \exp\left(-\int_0^t c(y_x(s), v(s))\, ds\right) dy\right\}. \tag{3.1}$$

We have the following;

THEOREM 3.1. *Under assumptions* (1.5-6-8), (2.1) *and*

$$c(x, v) \geqq c_0 > 0 \quad \forall x \in \bar{\mathcal{O}}, \quad \forall v \in V, \tag{3.2}$$

or under assumptions (1.5-6-8-9), (2.14–15) *if $\mathcal{O}$ is bounded* (*the nondegenerate case*), *we have*

$$u(x) = \lim_{t \to \infty} Q(t)\, h(x) \quad \textit{in } B_s \quad \forall h \in B_s,\ h|_{\cup \Gamma_0(\mathcal{A})} = 0 \tag{3.3}$$

(*in the nondegenerate case* $\forall h|_\Gamma = 0$),

$$u \in B_s, \qquad Q(t)u = u \quad \forall t \geqq 0.$$

Furthermore the equation of dynamical programming is satisfied:

$$\begin{aligned} u(x) = \inf_{\mathcal{A}} E\Big\{&\int_0^{\theta \wedge \tau_x} f(y_x(t), v(t)) \exp\left(-\int_0^t c(y_x(s), v(s))\, ds\right) dt \\ &+ u(y_x(\theta \wedge \tau_x)) \exp\left(-\int_0^{\theta \wedge \tau_x} c(y_x(t), v(t))\, dt\right)\Big\}, \end{aligned} \tag{3.5}$$

where θ is a stopping time with respect to F^t.

Finally, if $\Gamma_0(\mathcal{A})$ *is independent of* $\mathcal{A}$, $\Gamma_0(\mathcal{A})=\Gamma_0$ *for all* $\mathcal{A}$ *admissible* (*in the nondegenerate case* $\Gamma_0=\Gamma$), *then* $u(x)$ *is the unique solution of*

$$u \in B_s,\ u|_{\Gamma_0}=0,\ Q(t)\,u=u \quad \forall t \geqq 0. \tag{3.6}$$

Remark 3.1. Equality (3.5) shows that the optimal cost function $u(x)$ satisfies in some general integral sense the Bellman equation: $\sup_{v\in V}\{A(v)u-f(v)\}=0$ in $\mathcal{O}$.

Remark 3.2. i) If for all x and for all v, $f(x,v)\geqq 0$ and $\Gamma_1=\bigcup\Gamma_0(\mathcal{A})$, then it is easy to prove, by the same methods as those which follow, that $u(x)$ is the unique solution of

$$u \in B_s, \quad u|_{\Gamma_1}=0, \quad Q(t)u=u \quad \forall t \geqq 0. \tag{3.6$'$}$$

Such a case will be considered in Part II.

ii) If we assume that for each $\mathcal{A}$, $\Gamma_0(\mathcal{A})=\Gamma_0$, where Γ_0 is closed in Γ, then we can prove that $P[y_x(\tau_x)\in\Gamma_0]=1$ for all $x\in\bar{\mathcal{O}}$.

COROLLARY 3.1. *Under assumptions* (1.5–6), (2.14–15–20–21), *the optimal cost function belongs to* $W_0^{1,\infty}(\mathcal{O})$.

Proof. Since $u(x)=\lim_{t\to\infty} Q(t)0(x)$ in B_s, and by Theorem 2.3 we have $|Q(t)0(x)-Q(t)0(x')|\leqq C|x-x'|$, where C is independent of t, the result is immediate. □

Remark 3.3. If we define (cf. Dynkin [5]) the closed subset B_0 of B_s,

$$B_0=\{h\in B_s|\forall x\in\bar{\mathcal{O}},\ Q(t)\,h(x)\to Q(s)\,h(x) \text{ as } t\to s,\ h|_{\Gamma_0}=0\},$$

we can consider instead of (3.6)

$$u\in B_0, \qquad Q(t)u=u \quad \forall t\geqq 0. \tag{3.6$''$}$$

Remark 3.4. Let ϕ be given, where ϕ is the trace on Γ of some $\Phi\in B_s$; then we have $u_\phi(x)=Q(\infty)\,\Phi(x)=Q(\infty)\,h(x)$, $h\in B_s$ such that $h|_{\Gamma_0}=\phi$ (under the same hypotheses as in Theorem 3.1). Moreover, u_ϕ is the unique solution of the nonhomogeneous problem $u_\phi\in B_s$, $u_\phi|_{\Gamma_0}=\phi$, $Q(t)\,u_\phi=u_\phi$ for all $t\geqq 0$ and we also have the corresponding equation of dynamical programming.

Proof of Theorem 3.1. We prove (3.4) only for the case of nondegeneracy (hypothesis (2.14–15)) and (3.5); the other statements are obvious.

1) We know by Lemma 2.4 that there exists some $\mu>0$ such that ($\mathcal{O}$ is assumed to be bounded)

$$\exists C, \quad \forall x, \quad \forall\mathcal{A}, \quad E[e^{\mu\tau_x}]\leqq C;$$

thus

$$|Q(t)h(x)-u(x)|\leqq \sup_{\mathcal{A}} E\left[\int_{t\wedge\tau_x}^{\tau_x}\sup_{v\in V}\|f(x,v)\|_\infty\,ds\right]+\sup_{\mathcal{A}}E[\|h\|_\infty 1_{(t<\tau_x)}].$$

But $\sup_{\mathcal{A}} P[\tau_x>t]\leqq Ce^{-\mu t}$ and $\sup_{\mathcal{A}} E[\tau_x-\tau_x\wedge t]\leqq \sup_{\mathcal{A}} E[\tau_x\, 1_{(\tau_x>t)}]\leqq C'\,e^{-\mu t}$.

2) In order to prove (3.5) we need only consider admissible systems such that $v(t)$ is a continuous process (cf. Lemma 2.1). Now we define, for fixed x in $\bar{\mathcal{O}}$,

$$\xi(t)=\int_0^{t\wedge\tau_x} f(y_x(s),v(s))\exp\left(-\int_0^s c(y_x(\lambda),v(\lambda))\,d\lambda\right)dt$$
$$+u(y_x(t\wedge\tau_x))\exp\left(-\int_0^{t\wedge\tau_x} c(y_x(s),v(s))ds\right).$$

We want to prove that $\xi(t)$ is a F^t-submartingale satisfying to the property

$$\xi(\theta)\leqq E\{\xi(\theta+t)/F^\theta\},\quad \text{where } \theta \text{ is a stopping time and } t\geqq 0. \tag{3.7}$$

But the proof of that fact is exactly the same as in $\mathbb{R}^N$ (cf. [1, Thms. 5.1, 5.3]), from $u|_{\Gamma_0(\mathscr{A})}\leqq 0$ and thus $P[u(y_x(\tau_x))\leqq 0]=1$.

Therefore taking $t\to+\infty$ in (3.7) we prove that

$$E\Big[\int_0^{\theta\wedge\tau_x} f(y_x(t),v(t))\exp\Big(-\int_0^t c(y_x(s),v(s))\,ds\Big)\,dt + u(y_x(\theta\wedge\tau_x))\exp\Big(-\int_0^{\theta\wedge\tau_x} c(y_x(t),v(t))\,dt\Big)\Big]$$
$$\leqq E\int_0^{\tau_x} f(y_x(t),v(t))\exp\Big(-\int_0^t c(y_x(s),v(s))\,ds\Big)\,dt.$$

To conclude, we have to prove that

$$u(x)\leqq E\Big[\int_0^{\theta\wedge\tau_x} f(y_x(t),v(t))\exp\Big(-\int_0^t c(y_x(s),v(s))\,ds\Big)\,dt + u(y_x(\theta\wedge\tau_x))\exp\Big(-\int_0^{\theta\wedge\tau_x} c(y_x(t),v(t))\,dt\Big)\Big].$$

But $\xi(t)$ is a submartingale and this inequality is satisfied if θ is replaced by θ_k a discrete approximation of θ such that $\theta_k\to\theta$ (a.s.) as $k\to\infty$.

Since u is upper semicontinuous, the inequality remains true for θ. □

COROLLARY 3.2. *Under the assumptions of Theorem* 3.1, *we have for all* $\lambda\geqq 0$

$$u(x)=\inf_{\mathscr{A}} E\Big[\int_0^{\tau_x}\{f(y_x(t),v(t))+\lambda u(y_x(t))\}x \cdot\exp\Big(-\int_0^t (c(y_x(s),v(s))+\lambda)\,ds\Big)\,dt\Big]. \tag{3.8}$$

Proof. The proof is immediate in view of the following lemma, due to N. V. Krylov [14].

LEMMA 3.1. *Let $z(s)$, $\xi(s)$ be two bounded measurable adapted processes and assume that $z(s)+\int_0^s\xi(r)\,dr$ is a submartingale. Then for all $\lambda\geqq 0$ $z(s)\,e^{-\lambda s}+\int_0^s(\xi(r)+\lambda z(r))\,e^{-\lambda r}\,dr$ is a submartingale.*

COROLLARY 3.3. *Under assumptions* (1.5-6-8-9) *and* (2.14-15), *$u(x)$ belongs to X: $\{h\in C_b(\bar{\mathcal{O}})$, h is uniformly continuous$\}$.*

Proof. If we add the assumption (2.21), then by Corollary 3.1 $u(x)\in X$. Now let $\lambda>0$ be such that $c(x,v)+\lambda\geqq c_0>\mu_0$ is given by (2.22), and let us consider the following application T defined on B_s: if $v\in B_s$, $w=Tv$ is given by

$$w(x)=\inf_{\mathscr{A}} E\Big[\int_0^{\tau_x}\{f(y_x(t),v(t))+\lambda v(y_x(t))\} \cdot\exp\Big(-\int_0^t (c(y_x(s),v(s))+\lambda)\,ds\Big)\,dt\Big].$$

Then, by Corollary 3.2, u is a fixed point of T. To conclude, we just need to prove that T is a strict contraction on B_s. But

$$\|Tv_1 - Tv_2\|_\infty \leqq \sup_{\mathscr{A}} E[1-e^{-\lambda\tau_x}]\,\|v_1 - v_2\|_\infty$$

and by Jensen's inequality

$$\|Tv_1 - Tv_2\|_\infty \leqq (1-e^{-\lambda C})\,\|v_1 - v_2\|_\infty,$$

where $C = \sup_{\mathscr{A}} E[\tau_x] < +\infty$, by Lemma 2.4. □

Remark 3.5. With the techniques developed above, it is easy to extend Corollary 2.1 to the case where (2.21) is replaced by (1.9) (i.e., $c(x, v) \geqq 0$ instead of $c(x, v) \geqq c_0 > \mu_0$).

3.2. Application to the generator of $Q(t)$. We now prove a local version of Theorem 2.2, concerning the generator of the nonlinear semigroup $Q(t)$.

THEOREM 3.2., *Under assumptions* (1.5-6-8-9) *and* (2.1), *if* $\mathcal{O}'$ *is a bounded open set included in* $\mathcal{O}$ *and if* $h \in C^2(\mathcal{O}')$, *then*

$$\frac{Q(t)h(x) - h(x)}{t} \underset{t\to 0}{\to} \sup_{v\in V} (A(v)\,h(x) - f(x, v)) \quad \forall x \in \mathcal{O}'$$

and the convergence is uniform on compact subsets of $\mathcal{O}'$.

Proof. Let B be an open ball strictly included in $\mathcal{O}'$. We consider two open balls B_1, B_2 such that $B_2 \subset \bar{B}_2 \subset B_1 \subset \bar{B}_1 \subset B \subset \bar{B} \subset \mathcal{O}'$ and we show the convergence in B_2. We denote by τ_x^i the exit times of $\bar{B}_i$, $Q_i(t)$ the corresponding semigroups, $u_t(s, x) = Q(t-s)\,h(x)$ for $0 \leqq s \leqq t$. First, we remark that

$$\begin{aligned} u_t(s, x) = \inf_{\mathscr{A}} E\Big\{ &\int_0^{\sigma_{x,s}} f(y_x(r), v(r)) \exp\Big(-\int_0^r c(y_x(\lambda), v(\lambda))\,d\lambda\Big)\,dr \\ &+ h(y_x(\sigma_{x,s})) \exp\Big(-\int_0^{\sigma_{x,s}} c(y_x(r), v(r))\,dr\Big)\Big\}, \end{aligned} \tag{3.9}$$

where $\sigma_{x,s}$ is the exit time of the set $\mathcal{O} \times\,]0, t[$ for the $(N+1)$-dimensional process

$$z_{x,s}(r) = \begin{pmatrix} y_x(r) \\ r+s \end{pmatrix} \qquad (r \geqq 0).$$

Remark that $\Gamma_0'(\mathscr{A})$ for this process is $\Gamma_0(\mathscr{A})$ and that (2.1) is satisfied. Now by the equation of dynamical programming (3.5) we have

$$\begin{aligned} u_t(0, x) = \inf_{\mathscr{A}} E\Big\{ &\int_0^{\tau_x\wedge\theta\wedge t} f(y_x(s), v(s)) \exp\Big(-\int_0^s c(y_x(\lambda), v(\lambda))\,d\lambda\Big)\,ds \\ &+ u_t(\tau_x \wedge \theta \wedge t, y_x(\tau_x \wedge \theta \wedge t)) \exp\Big(-\int_0^{\tau_x\wedge\theta\wedge t} c(y_x(s), v(s))\,ds\Big)\Big\}. \end{aligned} \tag{3.9'}$$

Now we take $\theta = \tau_x^1$, and find

$$\begin{aligned} Q(t)\,h(x) = \inf_{\mathscr{A}} E\Big\{ &\int_0^{\tau_x^1\wedge t} f(y_x(s), v(s)) \exp\Big(-\int_0^s c(y_x(\lambda), v(\lambda))\,d\lambda\Big)\,ds \\ &+ 1_{(\tau_x^1 < t)} u_t(\tau_x^1, y_x(\tau_x^1)) \exp\Big(-\int_0^t c(y_x(s), v(s))\,ds\Big) \\ &+ 1_{(\tau_x^1 \geqq t)}\, h(y_x(t)) \exp\Big(-\int_0^t c(y_x(s), v(s))\,ds\Big)\Big\}. \end{aligned} \tag{3.9''}$$

Thus for all $x \in \bar{B}_2$, as $h \in C^2(\bar{B})$ we have (cf. proof of Theorem 2.2)

$$
\begin{aligned}
|Q(t)\,h(x)-h(x)| &\leqq |Q(t)\,h(x)-Q^1(t)\,h(x)|+|Q^1(t)\,h(x)-h(x)| \\
&\leqq \sup_{\mathscr{A}} E\{|u_t(\tau^1, y_x(\tau^1))-h(y_x(\tau_x^1))|1_{(\tau_x^1<t)}\}+C_1 t \\
&\leqq \sup_{0\leqq s\leqq t} \|Q(s)h-h\|_{\infty,B_1}\cdot \sup_{\mathscr{A}} P(\tau^1<t)+C_1 t.
\end{aligned}
$$

Now, as in the proof of Theorem 2.2, we can show that there exists $C_2>0$ such that for all $x\in\bar{B}_2$ $\sup_{\mathscr{A}} P(\tau_x^1<t)\leqq C_2\sqrt{t}$.

Thus we have finally

$$\sup_{0\leqq s\leqq t}\|Q(t)h-h\|_{\infty,\bar{B}_2}\leqq C_2\sqrt{t}\sup_{0\leqq s\leqq t}\|Q(s)h-h\|_{\infty,\bar{B}_1}+C_1 t.$$

By a similar argument we have

$$\sup_{0\leqq s\leqq t}\|Q(s)h-h\|_{\infty,\bar{B}_1}\leqq C_2\sqrt{t}\sup_{0\leqq s\leqq t}\|Q(s)h-h\|_{\infty,\bar{B}}+C_3 t;$$

hence for $t\leqq t_0$ we deduce

$$\sup_{0\leqq s\leqq t}\|Q(s)h-h\|_{\infty,\bar{B}_2}\leqq C_5 t.$$

Finally taking $\theta=\tau_x^2$ in (3.9′), we have

$$
\begin{aligned}
\forall x\in\bar{B}_2,\quad &\left|\frac{Q(t)h(x)-h(x)}{t}-\frac{Q_2(t)\,h(x)-h(x)}{t}\right| \\
&\leqq \frac{1}{t}\sup_{0\leqq s\leqq t}\|Q(s)h-h|_{\infty,\bar{B}_2}\sup_{\mathscr{A}} P(\tau_x^2<t)
\end{aligned}
$$

and we can conclude easily with the help of Theorem 2.2 and remarking that for all $x\in B_3$ a closed set $\subset B_2$, there exists C_6 such that $\sup_{\mathscr{A}} P[\tau_x^2<t]\leqq C_6\sqrt{t}$. □

4. Analytical interpretation of the optimal cost function and Hamilton–Jacobi–Bellman equations. In this section we shall always assume (1.5-6-8-9) and (2.14-15-20), i.e., the nondegenerate case, and that $\mathcal{O}$ is a regular domain. In every statement in the following, we shall call this group of hypotheses assumption A.

The main result of this section is the following. Under assumption A, $u\in W_0^{1,\infty}(\mathcal{O})$ and u is the maximum element of the set $\{\tilde{u}\in W_0^{1,\infty}(\mathcal{O}), A(v)\tilde{u}\leqq f(v) \text{ in } \mathscr{D}'(\mathcal{O}), \forall v\in V\}$.

We will also recall the main result concerning the solution of

$$\sup_{v\in V}\{A(v)u-f(v)\}=0 \quad \text{a.e. in } \mathcal{O},\quad u=0 \text{ on } \Gamma. \tag{4.1}$$

This result is obtained in L. C. Evans and P.-L. Lions [7] (see also [15]) under more smoothness assumptions on σ, b, c, f and $\mathcal{O}$ than A.

The results which we prove are organized in the following way.

§ 4.1. A first result of maximum solution.

§ 4.2. Approximation by systems of QVI.

§ 4.3. Final result for the maximum solution.

§ 4.4. Verification of H–J–B equation.

4.1. A first result of maximum solution.

THEOREM 4.1. *Under assumption* A *and if we assume in addition* (*see* (2.21))

$$c(x, v) \geqq c \geqq \mu_0, \quad \textit{where } \mu_0 \textit{ is given by } (2.22),$$

then the optimal cost function $u(x)$ *belongs to* $W_0^{1,\infty}(\mathcal{O})$ *and is the maximum element of the set s*,

$$s = \{\tilde{u} \in W_0^{1,\infty}(\mathcal{O}), \forall v \in V, A(v)\tilde{u} \leqq f(v) \text{ in } \mathcal{D}'(\mathcal{O})\}.$$

Remark 4.1. The optimal cost function $u(x)$ given by (see (3.1))

$$u(x) = \inf_{\mathcal{A}} E\left[\int_0^{\tau_x} f(y_x(t), v(t)) \exp\left(-\int_0^t c(y_x(s), v(s))\, ds\right) dt\right]$$

appears to be the solution of (3.1) in some weak sense: $u(x)$ is the upper envelope of all subsolutions of (4.1). Of course $u(x)$ itself is a subsolution.

Proof. The proof will be divided into several steps:

1) $u(x)$ belongs to s.
2) A general lemma.
3) If $\tilde{u} \in s$ then $\tilde{u}(x) \leqq u(x)$ for all $x \in \bar{\mathcal{O}}$.

1) In view of Corollary 3.1, we know that $u \in W_0^{1,\infty}(\mathcal{O})$. We have to prove that for all $v \in V$, $A(v)u \leqq f(v)$ in $\mathcal{D}'(\mathcal{O})$. To do this, we use a technique due to N. V. Krylov [11] (see a simplified version in [1]). Let $v \in V$ and let us consider an admissible system corresponding to $v(t, \omega) \equiv v$; because of Corollary 3.2 we have

$$u(x) \leqq E\left[\int_0^{\tau_x} \{f(y_x(s), v) + \lambda u(y_x(s))\} \exp\left(-\int_0^s c(y_x(t), v)\, dt - \lambda s\right) ds\right].$$

Now if we introduce u_λ, the solution of

$$A(v)u_\lambda + \lambda u_\lambda = u \quad \text{in } \mathcal{O}, \qquad u_\lambda|_\Gamma = 0,$$

we know that

$$u_\lambda(x) = E\left[\int_0^{\tau_x} u(y_x(s), v) \exp\left(-\int_0^s c(y_x(t), v)\, dt - \lambda s\right) ds\right].$$

Thus

$$A(v)u_\lambda \leqq E\left[\int_0^{\tau_x} f(y_x(s), v) \exp\left(-\int_0^s c(y_x(t), v)\, dt - \lambda s\right) ds\right] = f_\lambda(x)$$

or $A(v)(\lambda u_\lambda) \leqq \lambda f_\lambda(x)$.

To conclude, we note that λu_λ is bounded in $L^\infty(\mathcal{O})$ and that $\lambda u_\lambda - u = A(v)u_\lambda = (1/\lambda)\, A(v)\, (\lambda u_\lambda) \to 0$, as $\lambda \to +\infty$, in $\mathcal{D}'(\mathcal{O})$; $\lambda f_\lambda \to f(v)$, as $\lambda \to +\infty$, (in fact for all $x \in \bar{\mathcal{O}}$ because f is continuous) and we have in conclusion that

$$\forall v \in V \quad A(v)u \leqq f(v) \quad \text{in } \mathcal{D}'(\mathcal{O}).$$

Remark 4.2. Let us remark that even in the degenerate case (if we assume only (1.5-6-8) and (3.2)) the preceding proof remains valid, and thus we have

$$A(v)u \leqq f(v) \quad \text{in } \mathcal{D}(\mathcal{O}) \quad \forall v \in V. \tag{4.2}$$

2) Let us make precise the notation of the following lemma. Let $y(t)$ be a continuous process on the canonical Wiener space (Ω, F, F_t, P, W_t) such that

$$1_{[\theta_1(\omega),\theta_2(\omega)[}(t)\, y(t)=\left\{\int_{\theta_1}^{t} \sigma(y(t))\, dW_t+\int_{\theta_1}^{t} g(y(t))\, dt\right\} 1_{[\theta_1(\omega),\theta_2(\omega)[}(t), \tag{4.3}$$

where $\theta_1 \leqq \theta_2$ are two stopping times.

Let B be the differential operator

$$B=-\tfrac{1}{2}\, \sigma_{ik}\, \sigma_{jk} \frac{\partial^2}{\partial x_i \partial x_j}-g_i \frac{\partial}{\partial x_i}+c.$$

LEMMA 4.1. *Assume that σ, g, $c \in W^{1,\infty}(\bar{\mathcal{O}})$, that c is nonnegative and σ is uniformly nondegenerate. Let $y(t)$ be a process satisfying* (4.3), *let $f \in C(\bar{\mathcal{O}})$ and let $\tilde{u} \in W_0^{1,\infty}(\mathcal{O})$ such that*

$$B\tilde{u} \leqq f \quad \text{in } \mathcal{D}'(\mathcal{O}).$$

Then if M belongs to F_{θ_1}, and if θ is a stopping time such that $\theta_1 \leqq \theta \leqq \theta_2$, we have for all $x \in \bar{\mathcal{O}}$

$$\begin{aligned} &E\Big\{\Big(\tilde{u}(y(\theta_1 \wedge \tau)) \exp\Big(-\int_0^{\theta_1 \wedge \tau} c(y(t))\, dt\Big) \\ &\qquad -\tilde{u}(y(\theta \wedge \tau)) \exp\Big(-\int_0^{\theta \wedge \tau} c(y(t))\, dt\Big)\Big) 1_M(\omega)\Big\} \\ &\qquad \leqq E\Big\{1_M(\omega) \int_{\theta_1 \wedge \tau}^{\theta \wedge \tau} f(y(t)) \exp\Big(-\int_0^t c(y(s))\, ds\Big)\Big\}, \end{aligned} \tag{4.4}$$

where τ is the exit time from $\bar{\mathcal{O}}$ for the process $y(t)$.

Proof of Lemma 4.1. We extend $\tilde{u}$, which is zero on $\mathbb{R}^N-\mathcal{O}$; then $B\bar{u} \in W^{-1,p}(\mathbb{R}^N)$ for all $p<+\infty$. We introduce a regularizing positive convolution kernel $p_\varepsilon(\cdot) \in \mathcal{D}_+(\mathbb{R}^N)$ and we consider $\tilde{u}_\varepsilon$, a solution of

$$\begin{aligned} &Bu_\varepsilon=(p_\varepsilon * B\bar{u})|_{\mathcal{O}} \quad \text{in } \mathcal{O}, \\ &u_\varepsilon=0 \quad \text{on } \Gamma. \end{aligned}$$

Then $u_\varepsilon \in C^2(\bar{\mathcal{O}})$ and $u_\varepsilon \underset{W_0^{1,p}(\mathcal{O})}{\rightarrow} \tilde{u}$ for all $p<+\infty$; in particular, $u_\varepsilon \underset{C(\bar{\mathcal{O}})}{\rightarrow} u$.

Now if $\mathcal{O}$ is an open set such that $\mathcal{O}' \subset \bar{\mathcal{O}}' \subset \mathcal{O}$, the existence of an $\varepsilon \leqq \varepsilon_0$ implies that $Au_\varepsilon \leqq p_\varepsilon * f$ in $\mathcal{O}'$ (indeed, if $\mathcal{O}'-\operatorname{supp} p_\varepsilon \subset \mathcal{O}$, the inequality is true). Let τ' be the exit time of $\bar{\mathcal{O}}'$; then by Ito's formula we have (4.4) with $\tilde{u}$ replaced by $\tilde{u}_\varepsilon$, τ by τ' and f by $p_\varepsilon * f$. Thus when $\varepsilon \rightarrow 0$, we have (4.4) with τ replaced by τ'. But $\mathcal{O}'$ is arbitrary (with the condition $\mathcal{O}' \subset \mathcal{O}$); hence we deduce (4.4). □

3) Let $\tilde{u} \in S$. By Lemma 1.1 it is sufficient to prove that $\tilde{u}(x) \leqq J(x, \mathcal{A}, \infty, 0)$ for all admissible systems such that $v(t)$ is continuous. By taking image measure we can also assume that (Ω, F, F_t, P, W_t) is the canonical Wiener space. Let $\mathcal{A}$ be such an admissible system. We introduce

$$\begin{aligned} &\tilde{v}_n(t, \omega)=\sum_k v\Big(\frac{k}{2^n}, \omega\Big)\, 1_{[k/2^n,\,(k+1)/2^n[}(t), \\ &\exists N, \quad P(N)=0, \quad \forall \omega \notin N, \quad \forall t, \quad v_n(t, \omega) \rightarrow v(t, \omega) \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Now for k, n fixed $v((k/2^n), \omega)=$ a.s. $\lim v_j^{k,n}\, 1_{A_j}(\omega)$, where $v_j^{k,n} \in \mathbb{R}^m$, $A_j \in F_k/2^n$. Thus there exists N^1 such that $P(N^1)=0$ and

$$v(t,\omega)=\lim_n \bar{v}_n(t,\omega)\quad \forall \omega \notin N^1,\quad \forall t,$$

and

$$\bar{v}_n(t,\omega)=\sum_{j,k} v_{jk}\, 1_{A_{jk}}(\omega)\, 1_{[\theta_j,\,\theta_{j+1}[}(t),$$

where $\theta_j=j/2^n$, $\theta_{j+1}=(j+1)/2^n$, $v_{jk}\in \mathbb{R}^m$, $A_{jk}\in F_{\theta_j}$ and, for fixed j, A_{jk} are disjoint sets.

On the other hand there is a V_0 compact $\subset V_0$ such that $v(t,\omega)\in V_0$. Let W_0 be the convex envelope of V_0; W_0 is convex compact included in V. Let P_{W_0} be the Euclidean projection onto W_0, and let us finally consider

$$v^n(t,\omega)=\sum_{j,k} P_{W_0}(v_{jk})\, 1_{A_{jk}}(\omega)\, 1_{[\theta_j,\theta_{j+1}[}(t)=P_{W_0}(v_n(t,\omega)).$$

Then

$$\omega\notin N^1,\quad \forall t,\quad v^n(t,\omega)\to v(t,\omega)\quad \text{as } n\to\infty,\qquad v^n(t,\omega)\in W_0 \text{ compact of } V.$$

If we denote by $y_x^n(t)$ the process corresponding to $v^n(t)$, we have thus defined a sequence $\mathscr{A}_n$ of admissible systems on the canonical Wiener space, and by Lemma 2.2 it is sufficient to prove that

$$u(x)\leqq E\left[\int_0^{\tau_x} f(y_x^n(t),v^n(t))\exp\left(-\int_0^t c(y_x^n(s),v^n(s))\,ds\right)dt\right]$$

or

$$\forall j,\quad \forall k,\quad E\left[1_{A_{jk}}(\omega)\,\tilde{u}(y_x^n(\theta_j\wedge\tau_x))\exp\left(-\int_0^{\theta_j\wedge\tau_x} c(y_x^n(t),v^n(t))\,dt\right)\right]$$

$$\text{(4.5)}\qquad \leqq E\left[1_{A_{jk}}(\omega)\,\tilde{u}(y_x^n(\theta_{j+1}\wedge\tau_x))\exp\left(-\int_0^{\theta_{j+1}\wedge\tau_x} c(y_x^n(t),v^n(t))\,dt\right)\right.$$

$$\left.+\int_{\theta_j\wedge\tau_x}^{\theta_{j+1}\wedge\tau_x} f(y_x^n(t),v_{jk}))\exp\left(-\int_0^t c(y_x^n(s),v(s))\,ds\right)dt\right].$$

But Lemma 4.1 implies this inequality and we conclude. □

Remark 4.3. The preceding proof shows that if we do not assume (1.21), and if we know that $u(x)\in W_0^{1,\infty}(\mathcal{O})$, then u is the maximum element of S.

4.2. Approximating systems of QVI. We are going to investigate in this section the approximation of (4.1) by different systems. Following an idea of L. Tartar, introduced independently in [6], we introduce the following penalized problem P_ε: Find $u^1,\cdots,u^n$ solutions of

$$(\mathrm{P}_\varepsilon)\qquad \begin{aligned} &A_1u^1+\beta_\varepsilon(u^1-u^2)=f^1 \quad \text{in } \mathcal{O}, \qquad u^1=0 \quad \text{on } \Gamma,\\ &A_2u^2+\beta_\varepsilon(u^2-u^3)=f^2 \quad \text{in } \mathcal{O}, \qquad u^2=0 \quad \text{on } \Gamma,\\ &\cdots\\ &A_nu^n+\beta_\varepsilon(u^n-u^1)=f^n \quad \text{in } \mathcal{O}, \qquad u^n=0 \quad \text{on } \Gamma,\end{aligned}$$

where $A_i=A(v_i)$, $f_i=f(v_i)$ and $(v^1,\cdots,v^n)$ is a fixed subset of V, and $\beta_\varepsilon(t)=\beta(t/\varepsilon)$. Here β is a continuous convex nondecreasing function on R, such that $\beta(t)=0$ if $t\leqq 0$, $\beta(t)>0$ if $t>0$.

We also introduce the following system of quasivariational inequalities (in short QVI; see [2], [3], for example)

$$(\mathrm{Q}_\varepsilon)\qquad \begin{aligned} &A_1 u^1 \leqq f_1, \quad u^1 \leqq \varepsilon + u^2, \quad (A_1 u^1 - f_1)(u^1 - \varepsilon - u^2) = 0 \quad \text{in } \mathcal{O}, \\ &u^1 = 0 \quad \text{on } \Gamma, \\ &\qquad\qquad \cdots \\ &A_n u^n \leqq f_n, \quad u^n \leqq \varepsilon + u^1, \quad (A_n u^n - f_n)(u^n - \varepsilon - u^1) = 0 \quad \text{in } \mathcal{O}, \\ &u^n = 0 \quad \text{on } \Gamma. \end{aligned}$$

In this section we solve problems (P_ε), (Q_ε) (actually we shall prove just some obvious, nearly classical results which are sufficient for our goals) and we shall also give the stochastic interpretation of (Q_ε). In the next section we are going to prove that $(u^1, \cdots, u^n) \to u$, as $\varepsilon \to 0$, in $C(\mathcal{O})$ which is the optimal cost function.

THEOREM 4.2. *Under assumption* A *and if we assume in addition* (*see* (3.2))

$$\text{if } \mathcal{O} \text{ is unbounded, } c(x, v) \geqq c_0 > 0 \quad \forall x, \quad \forall v,$$

and that Γ *is regular, then there exists a unique solution* $(u^1, \cdots, u^n)$ *of* (P_ε) *in* $C^{2,\alpha}(\mathcal{O})$ $(\forall \alpha < 1)$ (resp. $C^{2,\alpha}_{\mathrm{loc}}(\mathcal{O}) \cap C_b(\bar{\mathcal{O}})$ *if* $\mathcal{O}$ *is unbounded*).

Proof. We prove just a priori estimates in the case of a bounded domain $\mathcal{O}$. First, we remark that $W^{2,p}(\mathcal{O})$ (and hence $C^{2,\alpha}$) estimates follow easily from $L^\infty(\mathcal{O})$ estimates. But $A_i u^i \leqq f^i$, for all i, and this implies that $u^i \leqq \text{const}$.

Now we consider $w(x) = w(x, \xi) = \exp(-k\rho^2) - \exp(k|x - \xi_1|^2)$, where ξ is fixed in Γ, ξ_1 is associated to ξ by (1.15) and $k > 0$. We have seen that for $k \geqq k_0 > 0$ (see (2.19))

$$A(v)\, w(x) \geqq \alpha > 0 \quad \forall x \in \mathcal{O} \quad \forall v \in V.$$

Thus, for λ large enough, we have

$$(4.6)\qquad A_i(\lambda w(x)) < f^i \quad \forall x \in \mathcal{O}, \quad \forall i, \qquad (-\lambda w)|_\Gamma \leqq 0.$$

Let x_0 be in $\mathcal{O}$, i_0 be in $\{1, \cdots, n\}$ such that

$$u_{i_0}(x_0) + \lambda w(x_0) = \min_{x,\, i} u_i(x) + \lambda w(x).$$

If $x_0 \in \Gamma$, $u_i(x) + \lambda w(x) \geqq \lambda w(x_0)$, and we conclude that $u_i(x) \geqq 0$.

If $x_0 \in \mathcal{O}$, by the maximum principle we have

$$A_{i_0}(u_{i_0}(x_0) + \lambda w(x_0)) \leqq c_{i_0}(u_{i_0}(x_0) + \lambda w(x_0));$$

since one may assume $u_{i_0}(x_0) + \lambda w(x_0) < 0$ and $A_{i_0} u_{i_0}(x_0) = f_i(x_0)$, by (4.6) we have a contradiction and this contradiction gives the L^∞ estimate. Uniqueness is proved by similar arguments. □

Remark 4.4. Actually uniqueness may be proved in the class $W^{2,n}_{\mathrm{loc}}(\mathcal{O}) \cap C_b(\bar{\mathcal{O}})$.

Remark 4.5. If β_ε is smooth then u_i are smooth.

THEOREM 4.3. *Under assumption* A *and if* $\mathcal{O}$ *is bounded, there exists a maximum weak solution of* (Q_ε) in the following sense:

$$(\mathrm{Q}_\varepsilon)\qquad \begin{aligned} &a_i(u^i, v - u^i) \geqq (f, v - u^i), \qquad v \in H^1_0(\mathcal{O}), \quad v \leqq \varepsilon + u^{i+1}, \\ &u^i \in H^1_0(\mathcal{O}), \qquad u^i \leqq \varepsilon + u^{i+1}, \end{aligned}$$

where $u^{n+1} = u^1$, *and* $a_i(u, v) = \langle A_i u, v \rangle_{H^{-1} \times H^1_0}$.

Furthermore $u^i \in C(\bar{\mathcal{O}})$ *and* $u^i = \lim_{\eta \downarrow 0} \downarrow u^i_\eta$, *where* (u^i_η) *is the solution of*

$$(\mathrm{R}_{\varepsilon,\eta}) \qquad A_i u^i_\eta + \beta_\eta(u^i_\eta - \varepsilon - u^{i+1}_\eta) = f^i \quad \text{in } \mathcal{O}, \qquad u^i_\eta = 0 \quad \text{on } \Gamma.$$

Remark 4.6. The existence of (u^i_η) is obtained in the same way as the existence of the solution (u_i) of (P_ε).

THEOREM 4.4. *Under the assumptions of Theorem* 4.3, *we have*

$$u^i(x) = \inf_\theta E\left\{ \int_0^{\tau_x} f(y_x(t), v(t)) \exp\left(-\int_0^t c(y_x(s), v(s))\, ds\right) dt \right.$$
$$\left. + \varepsilon \sum_{n \geq 1} \exp\left(-\int_0^{\theta_n} c(y_x(s), v(s))\, ds\right)\right\},$$

where $\theta = (\theta_n)_{n \in N}$ *is a sequence of stopping times such that* $\theta_0 = 0 < \theta_1 < \theta_2 < n$ $v(t, w) = v_j\, 1_{(\theta_k(\omega) \leq t < \theta_{k+1}(\omega))}$, $j \equiv i + k - 1 \pmod{\cdot\, n}$, *and* $y_x(t)$ *is the solution of*

$$dy_x(t) = \sigma(y_x(t), v(t))\, dW_t + g(y_x(t), v(t))\, dt,$$
$$y_x(0) = x$$

(*in the canonical Wiener space*).

Proofs of Theorems 4.2 *and* 4.3. As these results are just variations of results given in [2], [3], we just give hints on the proofs.

Let $u^{i,m}$ be the solution of

$$A_i u^{i,m} \leq f^i, \quad u^{i,m} \leq \varepsilon + u^{i+1,m-1}, \quad (A_i u^{i,m} - f^i)(u^{i,m} - \varepsilon - u^{i+1,m-1}) = 0 \quad \text{in } \mathcal{O},$$
$$u^{i,m}|_\Gamma = 0$$

(see [19] for the solution of this VI), and $u^{i,0}$ are given by $A_i u^{i,0} = f^i$ in $\mathcal{O}$, $u^{i,0} = 0$ on Γ.

One easily proves as in [2] that $u^{i,m} \downarrow_m$.

An argument similar to the one given in the proof of Theorem 4.2 gives

$$u^{i,m} \geq -\lambda w(x) \quad \forall i, \quad \forall m.$$

Thus $\|u^{i,m}\|_{L^\infty(\mathcal{O})} \leq$ const.

Now, since there exists λ such that $a_i(u, u) + \lambda |u|^2_{L^2(\mathcal{O})} \geq \nu \|u\|^2_{H^1_0(\mathcal{O})}$, we deduce easily from

$$a_i(u^{i,m}, -\lambda w - u^{i,m}) \geq (f^i, -\lambda w - u^{i,m})$$

that $\|u^{i,m}\|_{H^1_0(\mathcal{O})} \leq$ const.

The proof of the first part of Theorem 4.3 follows the one given in [3], for example.

Next the proof of the continuity of u^i and of Theorem 4.4 is easily obtained by methods similar to those in [3] and in [22].

Finally, by a method similar to the one given in the proof of Theorem 4.2, we prove that

$$u^i_\eta \downarrow \quad \text{when } \mu \downarrow 0, \quad \|u^i_\eta\|_{L^\infty(\mathcal{O})} \leq \text{const.}, \quad \text{and}$$
$$u^i_\eta(x) \geq -\lambda w(x) \quad \forall i, \quad \forall \mu, \quad \forall x \in \mathcal{O}.$$

Then we prove easily that $u^i_\eta \downarrow \underline{u}^i$, which is a weak solution of $(\mathrm{Q}'_\varepsilon)$, and thus $\underline{u}^i \leq u^i$. To conclude, we introduce $u^{i,m}_\eta$, the solution of

$$A_i u^{i,m}_\eta + \beta_\eta(u^{i,m}_\eta - \varepsilon - u^{i+1,m-1}_\eta) = f^i \quad \text{in } \mathcal{O}, \qquad u^{i,m}_\eta = 0 \quad \text{on } \Gamma;$$

we have

$$u_\eta^{i,m} \downarrow_{\eta\downarrow 0} u^{i,m}, \quad u_\eta^{i,m} \downarrow_{n\uparrow\infty} u_\eta^i, \quad u_\eta^i \downarrow_{\eta\downarrow 0} \underline{u}^i, \quad u^{i,m} \downarrow_{m\uparrow\infty} u^i;$$

thus $u^i = \underline{u}^i$. □

Remark 4.8. We have also that if u_ε^i is the solution of (Q_ε), u^{i,r_1} is the solution of (P_η), and $u_\varepsilon^{i,\eta}$ is the solution of $(R_{\varepsilon,\eta})$,

$$(4.7) \qquad u_\varepsilon^i \leqq u_\varepsilon^{i,\eta} \quad \forall \eta > 0, \qquad u_\varepsilon^i = \lim_{\eta\downarrow 0} \downarrow u_\varepsilon^{i,\eta},$$

$$(4.8) \qquad u^{i,\eta} \leqq u_\varepsilon^{i,\eta} \quad \forall \varepsilon > 0, \qquad u^{i,\eta} = \lim_{\varepsilon\downarrow 0} \downarrow u_\varepsilon^{i,\eta}. \qquad □$$

4.3. Final result for the maximum solution.

THEOREM 4.5. *Under assumption* A, *and if we assume* (*see* (3.2))

$$\text{if } \mathcal{O} \text{ is unbounded, } c(x, v) \geqq c > 0 \quad \forall x \in \mathcal{O}, \quad \forall v \in V,$$

then the optimal cost function $u(x)$ *belongs to* $W_0^{1,\infty}(\mathcal{O})$ *and is the maximum element of the set* S.

Proof. The proof will be divided into several parts.

1) Lipschitz estimates on $u^{i,\eta}$.

2) $u^{i,\eta} \downarrow_{\eta\downarrow 0} u_n$, $u_n \downarrow_{n\uparrow+\infty} u$ if $c(x, v) \geqq c_0 > \mu_0$.

3) Conclusion.

1) We prove that $\|u^{i,\eta}\|_{W^{1,\infty}(\mathcal{O})} \leqq$ const. (independent of i, η).

• First, we remark that, if $\mathcal{O}$ is bounded, we already know that $\|u^{i,\eta}\|_{L^\infty(\mathcal{O})} \leqq$ const. In the case of an unbounded domain, one proves by a simple limiting process ($\mathcal{O}_n \to \mathcal{O}$, $\mathcal{O}_n$ bounded) that

$$\|u^{i,\eta}\|_{L^\infty(\mathcal{O})} \leqq \sup_i \frac{\|f^i\|}{c_o}{}_{L^\infty(\mathcal{O})}.$$

• Next we prove that $|u^i, (x)| \leqq \lambda |w(x, \xi)|$ for all $\xi \in \Gamma$ and for all $x \in B(\xi, p')$, where λ, p' do not depend on i, η, ξ, and $w(x, \xi)$ is given by (2.15′). The proof is immediate if we recall that, if k is large enough,

$$A_i w(x, \xi) \geqq \alpha \exp - k|x - \xi_1|^2 \geqq \beta > 0 \quad \text{on some } B(\xi, p') = B.$$

Now on $(\partial B) \cap \mathcal{O}$ $w \geqq \gamma > 0$; thus there exists $\lambda > 0$ such that

$$A_i \lambda w(x, \xi) > \sup_i \|f_i\|_{L^\infty(\mathcal{O})} \quad \text{on } B,$$

$$\lambda w|_{(\partial B)\cap\mathcal{O}} > \max_{i,\eta} \|u^{i,\eta}\|_{L^\infty(\mathcal{O})}, \ \lambda w|_{B\cap\partial\mathcal{O}} \geqq 0.$$

From an application of the maximum principle similar to the one given in the proof of Theorem 4.2 we deduce

$$|u^{i,\eta}(x)| \leqq \lambda |w(x, \xi)| \quad \forall x \in B(\xi, p'), \quad \forall \xi \in \Gamma,$$

and this implies $|\nabla u^{i,\eta}(\xi)| \leqq$ const. for all $\xi \in \Gamma$.

• Finally we consider (as in [18]) the auxiliary function $w_i(x) = |\nabla u^{i,\eta}(x)|^2 + \lambda(C - u^{i,\eta}(x))^2$ (we shall forget about the η subscript in the following proof), where $\lambda > 0$ and $C \geqq \max_{i,\eta,x} u^{i,\eta}(x)$. We shall assume in the proof to the theorem that $\beta \in C^2(R)$; thus $u^i \in C^3(\mathcal{O})$. Differentiating (P_ε) with respect to x_j, we obtain (u_k will denote $\partial u/\partial x_k$)

$$-a^i_{k1}(x)\,u^i_{klj}(x)+b^i_k\,u^i_{kj}(x)+c^i\,u^i_j+\beta'(u^i-u^{i+1})(u^i_j-u^{i+1}_j)$$
$$=f^i_j(x)+a^i_{k1,j}(x)\,u^i_{k1}-b^i_{k,j}\,u^i_k-c^i_j\,u^i,$$

and a simple calculation shows that for all i

$$\begin{aligned}A_i\,w_i(x)+\beta'(u^i-u^{i+1})\,2(u^i_j\,u^i_j-u^{i+1}_j\,u^i_j)\\ \leqq -2\nu(u^i_{kj})^2(f^i_j+a^i_{k1,j}(x)\,u^i_{k1}-b^i_{k,j}\,u^i_k-c^i_j\,u)\,2\,u^i_j\\ +2\lambda(C-u^i)[-f^i+\beta(u^i-u^{i+1})]+C_1-2\lambda\nu(u^i_j)^2.\end{aligned}$$

Thus we have, choosing λ large enough, for all i,

$$\begin{aligned}A_i\;w_i(x)+\beta'(u^i-u^{i+1})\,2(u^i_j\,u^i_j-u^{i+1}_j\,u^i_j)\\ -\beta(u^i-u^{i+1})2\,\lambda(C-u^i)\leqq C_2-\alpha w_i(x);\end{aligned}$$

as $(C-u^i)\geqq 0$, $\beta(0)=0$ and β is convex we have

$$-\beta(u^i-u^{i+1})2\lambda(C-u^i)\geqq 2\lambda(C-u^i)\,\beta'(u^i-u^{i+1})\{(C-u^i)-(C-u^{i+1})\}.$$

Finally suppose $\mathcal{O}$ is bounded, and let i_0-x_0 be such that $w_{i_0}(x_0)=\max_{i,x} w(x)$ if x_0 belongs to Γ; we concluded that because of the above estimate if x_0 belongs to $\mathcal{O}$, at this point we have $A_{i_0}\,w_{i_0}(x_0)\geqq 0$ and

$$\beta'(u^i-u^{i+1})\,2(u^i_ju^i_j)\,2(u^i_j\,u^i_j-u^{i+1}_j\,u^i_j)-\beta(u^i-u^{i+1})\,2\lambda(C-u^i)$$
$$\geqq \beta'(u^i-u^{i+1})\,(w_i-w_{i+1})\geqq 0.$$

Hence we deduce $w_i(x)\leqq C_2/\alpha$.

The case of an unbounded domain is obtained by a limiting process, taking $\mathcal{O}_n$ a sequence of domains converging to $\mathcal{O}(\mathcal{O}_n\uparrow\mathcal{O})$.

2) Next, we suppose that $c(x,v)\geqq c_0>\mu_0$ for all $x\in\mathcal{O}$ and all $v\in V$.

We know (by the preceding estimate) that $u^{i,\eta}\downarrow u_n\in W^{1,\infty}_0(\mathcal{O})$, as $\eta\to 0$

Furthermore for all $i\leqq n$ $\quad A_i\,u_n\leqq f_i$ in $\mathcal{D}'(\mathcal{O})$. Now if we let n go to $+\infty$ such that $(v_i, i\in N)$ is dense in V, we see easily that $u^{i,\eta}\downarrow$ as $n\uparrow\infty$ we have $u_n\downarrow \underline{u}\in W^{1,\infty}(\mathcal{O})$, as $n\to\infty$ (by the preceding estimate, which is independent of n) and for all $i\in N$

$$A_i\,u\leqq f_i\quad\text{in }\mathcal{D}'(\mathcal{O}).$$

Thus

$$\forall v\in V,\quad A(v)\,u\leqq f(v)\quad\text{in }\mathcal{D}'(\mathcal{O}).$$

Now if we suppose that $c(x,v)\geqq c_0>\mu_0$ then by Theorem 4.1, $\underline{u}(x)\leqq u(x)$. But by remark 4.8 $u^i_\varepsilon\downarrow u_n$ as $\varepsilon\downarrow 0$, and from the stochastic interpretation of u^i_ε, we see that

$$\forall v\in V,\quad A(v)\,u\leqq f(v),\quad\text{in }\mathcal{D}'(\mathcal{O}).$$

Hence, if we suppose $c(x,v)\geqq c_0>\mu_0$ $u(x)=\underline{u}(x)$, and in the general case $\underline{u}(x)\in W^{1,\infty}_0(\mathcal{O})$, belongs to S and $\underline{u}(x)\geqq u(x)$ for all $x\in\bar{\mathcal{O}}$.

3) In the general case, we consider $\lambda>0$ such that $c(x,v)+\lambda\geqq c_0>\mu_0$, and we introduce a mapping T_{ε_n} defined by: if $w\in C_b(\bar{\mathcal{O}})$, $T_{\varepsilon,n}w=(T^1_{\varepsilon,n}w)_i$ is the solution of (Q_ε) where A_i is replaced by $A_i+\lambda$, f^i by $f^i+\lambda w$.

From the stochastic interpretation, we have easily

$$\|T_{\varepsilon,n} w_1 - T_{\varepsilon,n} w_2\|_{L^\infty(\mathcal{O})} \leqq \frac{\lambda}{\lambda + c_0} \|w_1 - w_2\|_{L^\infty(\mathcal{O})} \quad \text{if } \mathcal{O} \text{ is unbounded,}$$

$$\leqq \frac{1}{\lambda} \sup_{\mathcal{A},x} E[1 - e^{-\lambda\tau_x}] \quad \text{if } \mathcal{O} \text{ is bounded,}$$

$$\leqq \frac{1 - e^{-\lambda C}}{\lambda} \quad \text{where } C > 0$$

by Jensen's inequality (cf. Lemma 2.4).

Now for any $w \in C_b(\bar{\mathcal{O}})$, $T_{\varepsilon,n} w \downarrow Tw \in C_b(\bar{\mathcal{O}})$, and by step 2)

$$Tw(x) = \inf_{\mathcal{A}} E\left[\int_0^{\tau_x} \{f(y_x(t), v(t)) + \lambda w(y_x(t))\} \exp\left(-\int_0^t c(y_x(s), v(s))\, ds\right) dt\right].$$

From these two facts, we deduce that the fixed point of T_ε in $C_b(\bar{\mathcal{O}})$ converges to the fixed point of T, i.e., $u_\varepsilon^i \to u(x)$, in $C_b(\bar{\mathcal{O}})$. Thus $u \in W_0^{1,\infty}(\mathcal{O})$ and $u = \bar{u}$. To conclude, we remark that the proof of Theorem 4.1 now applies, and thus u is the maximum element of S. □

COROLLARY 4.1. *Under the assumptions of Theorem* 4.5, *we have*

$$u(x) = \inf_{\mathcal{A}_\theta} E\left[\int_0^{\tau_x} f(y_x(t), v(t)) \exp\left(-\int_0^t c(y_x(s), v(s))\, ds\right) dt\right],$$

where the infimum is taken over all admissible systems such that (Ω, F, F_t, P, W_t) *is the canonical Wiener space, and there exists* $\theta = (\theta_n)_{n \geqq 0}$, *a sequence of stopping times such that* $\theta_0 = 0 < \theta_1 < \theta_2 < \cdots < \theta_n \uparrow +\infty$ *and* $v(t, x) = v_j$ *if* $t \in [\theta_j(\omega), \theta_{j+1}(\omega)[$, *where* $(v_n)_{n \geqq 0}$ *is a sequence of elements of* V.

Proof of Corollary 4.1. Immediate in view of Theorem 4.4 and the proof of Theorem 4.5. □

4.4. Verification of H–J–B equations. We now recall a result due to L. C. Evans and P.-L. Lions [7] concerning the solution of (4.1). We will assume in this section that $\mathcal{O}$ is smooth and we have

$$\phi(\cdot, v) \in W^{2,\infty}(\mathcal{O}) \quad \text{and} \quad \sup_{v \in V} \|\phi(\cdot, v)\|_{W^{2,\infty}(\mathcal{O})} < \infty \quad \forall \phi = \sigma, b, c, f. \tag{4.9}$$

THEOREM 4.6. *Under assumptions* A *and* (4.9), *we have that* $u \in W^{2,\infty}(\mathcal{O})$ *is the unique solution in* $W^{2,\infty}(\mathcal{O})$ *of* (4.1):

$$\sup_{v \in V} \{A(v)u - f(v)\} = 0 \quad \text{a.e. } in\ \mathcal{O}, \qquad u = 0 \ on\ \Gamma.$$

Remark 4.9. This result extends previous results due to H. Brezis and L. C. Evans [4], P-L. Lions [20], L. C. Evans and A. Friedman [6], P-L. Lions and J-L. Menaldi [21], P-L. Lions [15].

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