COMM. IN PARTIAL DIFFERENTIAL EQUATIONS, 8(10), 1101-1174 (1983)

OPTIMAL CONTROL OF DIFFUSION PROCESSES AND HAMILTON-JACOBI-BELLMAN EQUATIONS

PART 1 : THE DYNAMIC PROGRAMMING PRINCIPLE AND APPLICATIONS

P.L. Lions
Ceremade
University Paris IX-Dauphine
Place de Lattre de Tassigny
75775 Paris Cedex 16

Subject AMS classification : 93E20, 35J65, 35K60, 60J60.

Abstract : We consider the optimal control of diffusion processes. We show, using the Dynamic Programming Principle, a characterization of the optimal cost function in terms of maximum subsolution of the Hamilton-Jacobi-Bellman equations associated with the control problem. We also prove, under optimal conditions, continuity results for the optimal cost function.

Résumé : Nous considérons le problème général du contrôle optimal de processus de diffusion. Nous montrons, en utilisant le principe de la programmation dynamique, une caractérisation de la fonction coût optimum comme étant la sous-solution maximum des équations de Hamilton-Jacobi-Bellman associées au problème de contrôle. Nous démontrons également, sous des hypothèses optimales, des résultats de continuité pour la fonction coût optimum.

0360-5302/83/0810-1101$3.50/0

Key-Words : Optimal stochastic control, stochastic differential equations, diffusion processes, Hamilton-Jacobi-Bellman equations, Dynamic programming principle, subsolutions.

Mots-clés : Contrôle optimal stochastique, équations différentielles stochastiques, processus de diffusion, équations de Hamilton-Jacobi Bellman, Principe de la programmation dynamique, sous-solutions.

Introduction :

This is the first paper of a series on the subject of the optimal control of diffusion type processes and the associated Hamilton-Jacobi-Bellman equations.

We first describe briefly the type of problems we are looking at : we want to control, in an optimal way, systems which state is governed by the solution of stochastic differential equations - written in Itô's form :

$$(1) \quad dX_t = \sigma(X_t,\alpha_t)\, dB_t + b(X_t,\alpha_t)\, dt \quad , \quad X_o = x$$

where σ and b are matrix and vector-valued functions depending on $x \in R^N$, $\alpha \in A$ and A is a separable metric space (non empty !) ; where B_t is some m-dimensionnal Brownian motion ($m \geqslant 1$). In addition, α_t is a stochastic process with values in A and, loosely speaking, α_t is the control.

Next, we want to restrict the state of the system to lie inside some given open set $\mathcal{O}$ ($\subset R^N$) and thus we will stop the state X_t when it exits from $\bar{\mathcal{O}}$: let us denote by $\tau = \inf\, (t \geqslant 0\ ,\ X_t \notin \bar{\mathcal{O}})$ - of course both X_t and τ depend on x.

We now introduce the so-called cost function (or pay-off function, or reward function in the case of a maximization problem...) :

$$J(x,\alpha) = E\left[\int_0^\tau f(X_t,\alpha_t)\, e^{-\lambda t}\, dt\right] \quad , \quad \forall\, x \in \overline{\mathcal{O}}$$

(we will consider below more general types of cost functions) where $f(x,\alpha)$ is a real-valued function on $R^N \times A$, $\lambda > 0$ is the so-called <u>discount factor</u> and E denotes the expectation.

Finally we want to minimize the cost function over all possible controls α_t and therefore we introduce the optimal cost function (or minimum cost function, or value function, or criterion...) :

$$u(x) = \inf_{\alpha} J(x,\alpha) \quad , \quad \forall\, x \in \overline{\mathcal{O}} .$$

The goal of optimal stochastic control theory is two-fold : 1) determine u, 2) determine optimal (or minimum) controls α_t - possibly in some particular form. In this series of papers we will be mainly concerned with question 1), while we will come back on 2) in future publications.

The main tool for characterizations of u is the <u>dynamic programming principle</u> - a heuristic principle invented by R. Bellman [1]. This principle immediately yields - at least formally - that u should satisfy some (degenerate) nonlinear elliptic p.d.e. of second-order called the <u>Hamilton-Jacobi-Bellman equation</u> (HJB in short) :

(2) $$\sup_{\alpha \in A} [A_\alpha u(x) - f(x,\alpha)] = 0 \quad \text{in } \mathcal{O} ;$$

where A_α is the linear second-order operator given by :

(3) $$A_\alpha = - a_{ij}(x,\alpha)\, \partial_{ij} - b_i(x,\alpha)\, \partial_i + \lambda \qquad (1)\ (2)$$

and $a(x,\alpha) = \frac{1}{2}\, \sigma(x,\alpha) \cdot \sigma^T(x,\alpha)$.

Finally, let us point out that (2) "yields" a characterization of u provided some boundary condition is given : in the preceding case, we should have

$$(4) \quad u = 0 \quad \text{on} \quad \Gamma_o$$

where Γ_o is some part of $\Gamma = \partial\mathcal{O}$ ($\Gamma_o = \Gamma$ if the equation is non degenerate i.e. the operator A_α are uniformly elliptic).

At this stage let us mention that the formal derivation of (2) is, in some sense, the extension of the classical Hamilton-Jacobi theory for deterministic problems or calculus of variations problems.

The above derivation is easily justified, via Itô's formula, if we know a priori that $u \in C^2(\mathcal{O})$ - see for example W.H. Fleming and R. Rishel [7], N.V. Krylov [12]. On the other hand, if we know a solution $\tilde{u} \in C^2(\mathcal{O}) \cap C(\bar{\mathcal{O}})$ of (2) vanishing on Γ, then it is easy to check that : $\tilde{u} \equiv u$ and, in addition, it is possible to give a "rule" for building an optimal control in the so-called feedback form and which is thus Markovian. Therefore, we see that if $u \in C^2(\mathcal{O})$ or if we know a smooth solution of (2), the problem is solved. Unfortunately, in general u does not lie in $C^2(\mathcal{O})$ and thus our main concern in this work will be to solve these difficulties and therefore to give various characterizations of u, to propose several ways of checking (2) and to study uniqueness questions.

Let us point out that the main difficulties in dealing with (2) are : i) the fact that the equation is fully nonlinear i.e. the nonlinearity acts on highest order derivatives (2^{nd} order) ;

ii) the degeneracy of the problem i.e. the fact that we will try as much as possible to avoid nondegeneracy assumptions or uniform ellipticity assumptions since most applications of optimal stochastic control concern degenerate problems.

In this paper (Part 1) we will first (section I) recall the mathematical formulation of dynamic programming and we will give a first characterization of u as the maximum subsolution of (2) (-(4)). Next, we will study continuity properties of u (section II) and we will finally indicate in section III various extensions and variants of these results and methods. This paper is, in some sense, the extension of a previous work due to J.L. Menaldi and the author [29], [30].

In Part 2, we will develop the notion of viscosity solutions of (2) - extending the notion of viscosity solutions of first-order Hamilton-Jacobi equations introduced by M.G. Crandall and P.L. Lions [5]; see also M.G. Crandall, L.C. Evans and P.L. Lions [4], P.L. Lions [18], [19], [20]. In particular we will prove a very general uniqueness result for viscosity solutions of (2).

Part 3 will be devoted to regularity results for u - essentially those announced in P.L. Lions [21], [22] - which extend the previous work of N.V. Krylov [13], [14], [15]; P.L. Lions [23], [24]; L.C. Evans and A. Friedman [6]; M.V. Safonov [42], [43]. These regularity results, combined with the notion of viscosity solution, immediately imply that (2) holds in an appropriate sense.

The next parts will be devoted to stability results, and to Perron's method for HJB equations.

We want to conclude this introduction by mentionning a few problems that we do not consider here : i) the question of optimal controls - see J.P. Quadrat [41], N.V. Krylov [12], [15] for some partial results ; ii) the numerical approximation of u - see P.L. Lions and B. Mercier [31], J.P. Quadrat [41], R. Jensen and P.L. Lions [11]; iii) impulse control problems or more generally the control of jump diffusion processes - see A. Bensoussan and J.L. Lions [3], N.V. Krylov and Pragarauskas [16] for some particular results ; and B. Perthame [40] for general impulse control problems.

Let us finally point out that such optimal control problems arise frequently in many applications to management sciences, electrical engineering, finance and economy theory... We refer the reader to [7], [41] (for example).

I. Preliminary properties of the optimal cost function

I.1 Notations and assumptions :

To simplify, we will always assume (and we will not recall it) :

$$\text{(A.1)} \quad \begin{cases} \sup\limits_{\alpha \in A} \|\varphi(\cdot,\alpha)\|_{W^{2,\infty}(\mathbb{R}^N)} < \infty \ ; \ \varphi(x,\alpha) \in C(A) \ , \ \forall\, x \in \mathbb{R}^N \\ \text{for all } \varphi = \sigma_{ij} \quad (1 \leq i \leq N \ , \ 1 \leq j \leq m) \ , = b_k \\ (1 \leq k \leq N) \ , = f \ , c \end{cases}$$

$$(5) \qquad \inf_{\mathbb{R}^N \times A} c(x,\alpha) = \lambda > 0 \ .$$

Summary of Part 1

Actually this regularity is not necessary for most of the results given below but we do not want to enter into technical considerations concerning the smoothness of the coefficients.

Instead of considering controls α_t, we will work with <u>admissible systems</u> $\mathcal{A}$ given by the collection of i) a probability space (Ω,F,F^t,P) with a right-continuous increasing filtration of complete sub-σ fields, ii) a Brownian motion B_t (in R^m) F^t-adapted, iii) a progressively measurable process α_t taking its values in a compact set (depending on α_t) of A. For each admissible system $\mathcal{A}$ and for all $x \in R^N$, it is well-known that there exists a unique continuous process X_t solution of (1). For each admissible $\mathcal{A}$, we then consider the cost function :

$$(6) \qquad J(x,\mathcal{A}) = E\int_0^\tau f(X_t,\alpha_t)\exp\left\{-\int_0^t c(X_s,\alpha_s)\,ds\right\}$$

where τ is the stopping time defined by :

$$\tau = \inf\,(t \geqslant 0\ ,\ X_t \notin \bar{\mathcal{O}})$$

and $\tau = +\infty$ if $X_t \in \bar{\mathcal{O}}$, $\forall\ t \geqslant 0$.

Finally, as in the Introduction, we consider :

$$(7) \qquad u(x) = \inf_{\mathcal{A}} J(x,\mathcal{A}) \quad , \quad \forall\ x \in \bar{\mathcal{O}}\ .$$

<u>Remark I.1</u> : From time to time we will recall the dependence on x of X_t and τ by writing X_t^x (or $X_t(x)$, or $X(t,x)$), τ_x. ∎

As we explained in the Introduction, u should be related to the solution of the associated HJB equation :

$$(2) \qquad \sup_{\alpha \in A}\,[\,A_\alpha u(x) - f(x,\alpha)\,] = 0 \qquad \text{in } \mathcal{O} \quad ;$$

where A_α is now given by (3) with λ replaced by $c(x,\alpha)$.

Remark I.2 : We decided to consider only τ the exit time of X_t from $\bar{\mathcal{O}}$ and not the exit time from $\mathcal{O}$: $\tau' = \inf (t \geq 0 \, , \, X_t \notin \mathcal{O})$. Even if all the results below (in section I) hold with τ replaced by τ', we believe that our choice is more natural (from the mathematical point of view) since if we take τ' instead of τ, we have automatically : $u = 0$ on Γ and this creates, in the degenerate case, artificial discontinuities in u as a function on $\bar{\mathcal{O}}$. Of course it may happen that $\tau = \tau'$ - as it is the case in J.L. Menaldi and the author [29], [30]. ∎

Finally, in all that follows, we will assume that $\mathcal{O}$ is a bounded, smooth (say C^2) domain in R^N.

I.2 The dynamic programming principle.

The mathematical formulation of the dynamic programming principle is the following :

Theorem A : *For all* $t > 0$, *we have for all* $x \in \bar{\mathcal{O}}$:

$$(8) \qquad u(x) = \inf_{\mathcal{A}} \left\{ E \int_0^{\tau\wedge t} f(X_s,\alpha_s) \exp\left\{ - \int_0^s c(X_\lambda,\alpha_\lambda)d\lambda \right\} ds + u(X_{t\wedge\tau}) \cdot \exp\left\{ - \int_0^{\tau\wedge t} c(X_s,\alpha_s)ds \right\} \right\} .$$

This means that in order to obtain the optimal cost function, we may let evolve the system up to an arbitrary time t, pay the integral cost up to time t (actually $t\wedge\tau$), pay in addition a final cost at time t using the optimal cost function at the place where the system is at time t and finally take the infimum over all possible strategies.

Actually, a more general result is the

Theorem B : *For each admissible A, let* θ *be a stopping time* ($\theta = \theta(A)$).

i) $$M_t = u\,(X_{t\wedge\tau})\,\exp\left\{-\int_0^{t\wedge\tau} c\right\} + \int_0^{t\wedge\tau} f(X_s,\alpha_s)\,\exp\left\{-\int_0^s c\right\}ds$$

is a F_t *submartingale and we have :*

(9) $$u(x) \leqslant E\,[M_\theta] \leqslant E\,[M_\infty] = J(x,A)\ .$$

ii) *In particular we have for all* $x \in \bar{\mathcal{O}}$ *:*

(10) $$u(x) = \inf_A E\left\{\int_0^{\theta\wedge\tau} f\,\exp\left\{-\int_0^s c\right\}ds + u(X_{\theta\wedge\tau})\,\exp\left\{-\int_0^{\theta\wedge\tau} c\right\}\right\}.$$

iii) *For each admissible A and for all* $x \in \bar{\mathcal{O}}$*, we have :*

(11) $$1_{(\tau<\infty)}\,u(X_\tau) \leqslant 0 \qquad a.s.\ .$$

Of course Theorem A is an immediate consequence of Theorem B : take $\theta = t$ for all A, (10) then becomes (8).

Theorem B is due to K. Itô [10] in the general case ; in the case when $\mathcal{O} = R^N$ (and thus $\tau = +\infty$) Theorem B was first proved by N.V. Krylov [13] - see also M. Nisio [34] , [35] , A. Bensoussan and J.L. Lions [2] for simplified versions of Krylov's argument. Since we will use Theorem B in section II only with some additionnal assumption (A.2), let us mention that in this case we have a method to derive Theorem B from the case when $\mathcal{O} = R^N$: we present below briefly this method under some more restrictive assumption than A.2 - in order to simplify the presentation.

Let us first explain the main idea of the proof of (10) : let $\varepsilon > 0$, take some system realizing the infimum in (7) up to

ε ; then, considering the control $\tilde{\alpha}_t = \alpha_{\theta\wedge\tau+t}$, we deduce

$$E\left\{u(X_{\theta\wedge\tau})\ \exp\left\{-\int_0^{\theta\wedge\tau} c\right\}\right\} \leqslant E\left\{\int_{\theta\wedge\tau}^{\tau} f\ \exp\left(-\int_0^t c\right)dt\right\} \quad ;$$

and this implies :

$$u(x) \geqslant J(x,A) - \varepsilon =$$

$$= E\left\{\int_0^{\theta\wedge\tau} f\ \exp\left(-\int_0^t c\right)dt\right\} + E\left\{\int_{\theta\wedge\tau}^{\tau} f\ \exp\left(-\int_0^t c\right)dt\right\} - \varepsilon$$

$$\geqslant E\left\{\int_0^{\theta\wedge\tau} f\ \exp\left(-\int_0^t c\right)dt + u(X_{\theta\wedge\tau})\ \exp\left\{-\int_0^{\theta\wedge\tau} c\right\}\right\} - \varepsilon$$

and this proves :

$$\text{(10')} \quad u(x) \geqslant \inf_{A} E\left\{\int_0^{\theta\wedge\tau} f\ \exp\left\{-\int_0^s c\right\}ds + u(X_{\theta\wedge\tau})\ \exp\left\{-\int_0^{\theta\wedge\tau} c\right\}\right\} .$$

The reverse inequality is obtained by similar arguments involving "pasting together" various solutions of stochastic differential equations.

We now explain how under some convenient assumption (essentially (A.2) formulated below in section II) Theorem B is deduced from Krylov results in R^N. We will assume to this end :

$$\text{(12)} \qquad \exists\, \underline{u} \in C^2(\bar{\mathcal{O}})\ ,\ \underline{u} = 0 \text{ on } \Gamma\ ,\ \ A_\alpha \underline{u} \leqslant f(x,\alpha) \text{ in } \mathcal{O} \qquad \forall\, \alpha \in A.$$

We may assume that $\underline{u} \in C_b^2(R^N)$, $D^2\underline{u} \in B \cup C(R^N)$ and modifying if necessary f outside $\bar{\mathcal{O}}$ that : $A_\alpha \underline{u} \leqslant f(x,\alpha)$ in R^N.

We next introduce for all $x \in R^N$ and for all $\varepsilon > 0$:

$$u^\varepsilon(x) = \inf_A \left\{E \int_0^\infty \left[f(X_t,\alpha_t) + \frac{1}{\varepsilon}\, p(X_t)\ \underline{u}(X_t)\right]\right\} \cdot$$

$$\cdot\ \exp\left\{-\int_0^t c(X_s,\alpha_s) + \frac{1}{\varepsilon}\, p(X_s)ds\right\}\ dt \Bigg\} \ ;$$

where $p \in C_b^2(R^N)$, $D^2p \in B \cup C(R^N)$, $p \equiv 0$ in $\overline{\mathcal{O}}$, $p > 0$ in $R^N - \overline{\mathcal{O}}$. By Krylov results in R^N [12], u^ε satisfies i), ii) (conveniently adapted). On the other hand, it is easily seen – using Itô's formula and [12] – that, if $v^\varepsilon(x) = (u^\varepsilon - \underline{u})(x)$:

$$v^\varepsilon(x) = \inf_{A} E \int_0^\infty \left\{ f(X_t,\alpha_t) - A_{\alpha_t}\underline{u}(X_t) \right\} \cdot$$

$$\cdot \exp\left\{ -\int_0^t c(X_s,\alpha_s) + \frac{1}{\varepsilon} p(X_s)ds \right\} .$$

The integrand being nonnegative, this immediately implies :

$$\forall x \in \overline{\mathcal{O}} \quad , \quad v^\varepsilon(x) \underset{\varepsilon\downarrow 0}{\downarrow} v(x) =$$

$$= \inf_{A} E \int_0^\tau \left\{ f(X_t,\alpha_t) - A_{\alpha_t}\underline{u}(X_t) \right\} \exp\left\{ -\int_0^t c \right\} dt$$

and using Itô's formula, we obtain since $\underline{u} = 0$ on Γ

$$v(x) = \inf_{A} \{ J(x,A) - \underline{u}(x) \} .$$

In other words : $\quad u^\varepsilon(x) \underset{\varepsilon\downarrow 0}{\downarrow} u(x) \quad , \quad \forall x \in \overline{\mathcal{O}}$.

Since u^ε satisfies i), ii), it is now easy to pass to the limit and conclude.

<u>Remark I.3</u> : This kind of method is used in P.L. Lions [25], [26], [27] for solving various p.d.e. and in J.L. Menaldi and the author [29] in a very similar spirit.

<u>Remark I.4</u> : Actually in [10], it is also proved that u is Borel measurable on $\overline{\mathcal{O}}$.

We now give a first application of Theorems A-B :

Theorem I.1 :

i) *We have :* $u \in L^\infty(\mathcal{O})$ *and*

(13) $A_\alpha u \leqslant f(x,\alpha)$ *in* $\mathcal{D}'(\mathcal{O})$, $\forall\, \alpha \in A$.

ii) *For each bounded open set* ω *such that* $\bar{\omega} \subset \mathcal{O}$, *there exists* $C > 0$ *such that* $a^{1/2}(x,\alpha) \cdot \nabla u \in L^2(\omega)$ (4) *and*

(14) $$\sup_{\alpha \in A} \| a^{1/2}(x,\alpha) \cdot \nabla u \|_{L^2(\omega)} \leqslant C \ .$$

Remark I.5 : As we will see below, the proof of (13) is a straightforward consequence of (8) ; while (14) is a more subtle consequence of Theorem B. If we already know that u is smooth, then (14) would be an easy consequence of (13) : indeed take $\xi \in D_+(\mathcal{O})$, $\xi \equiv 1$ on $\bar{\omega}$, $0 \leqslant \xi \leqslant 1$ in $\mathcal{O}$ and let $C > \|u\|_{L^\infty(\mathcal{O})}$, multiplying (13) by $\xi^2(C+u)$, we obtain integrating by parts :

$$\int_{\mathcal{O}} \frac{\partial u}{\partial x_j} \frac{\partial}{\partial x_i} \left(a_{ij}\ \xi^2 (C+u)\right) - b_i\ \xi^2 \frac{\partial}{\partial x_i} \left(\tfrac{1}{2}\ (C+u)^2\right) dx \leqslant C$$

hence

$$\int_{\mathcal{O}} \xi^2\ a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_i}\ dx \leqslant$$

$$\leqslant C + \int_{\mathcal{O}} C\ \xi \sum_i \left| a_{ij} \frac{\partial u}{\partial x_j} \right| + \frac{1}{2}\ \xi^2 \frac{\partial a_{ij}}{\partial x_i} \frac{\partial}{\partial x_j} ((C+u)^2) dx.$$

And integrating by parts we find :

$$\int_{\mathcal{O}} \xi^2\ a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}\ dx \leqslant \int_{\mathcal{O}} \xi^2 \left\{ a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \right\}^{1/2} dx + C \ ,$$

and we conclude easily. ∎

Before going into the proof of Theorem I.1, let us indicate a few obvious consequences. First of all, in view of a general result on distributions due to L. Schwartz [44], we obtain the

Corollary I.1 : *For all* $\alpha \in A$, *the distribution* $A_\alpha u - f(\cdot,\alpha) = \mu_\alpha$ *is a measure on* $\mathcal{O}$. *Moreover* $\mu = \sup_{\alpha \in A} \mu_\alpha$ *is a nonpositive measure on* $\mathcal{O}$.

Remark I.6 : It would be of interest to decide if $\mu = 0$. This would be of course a weak form of (2). We conjecture that, at least when the coefficients are smooth, this equality holds : this conjecture is proved in a forthcoming work due to J.M. Coron and the author when $N = 1$; some results along this line will be contained in one of the following parts.

Corollary I.2 : *Let* ω *be a bounded open set such that* $\bar{\omega} \subset \mathcal{O}$. *We assume that there exist* $\nu > 0$, $p \geq 1$, $\alpha_1,\dots,\alpha_p \in A$ *such that* :

$$(15) \qquad \sum_{i=1}^{p} a(x,\alpha_i) \geq \nu I_N \quad , \quad \forall x \in \omega \quad (5)$$

Then we have : $u \in H^1(\omega)$.

Remark I.7 : Under the assumption (15), we see that

$$\tilde{A} u \in M_b(\tilde{\omega}) \quad , \quad u \in L^\infty(\tilde{\omega})$$

where $\tilde{A} = \sum_{i=1}^{p} A_{\alpha_i}$ is uniformly elliptic on $\tilde{\omega}$, open neighborhood of ω such that $\tilde{\omega} \subset \mathcal{O}$. This implies by regularity results and interpolation theory in Sobolev spaces that $u \in W^{2/p-\varepsilon,p}(\omega)$ for all $1 < p < \infty$, $\varepsilon > 0$. We see that Corollary I.2 improves a little bit the case $p = 2$.

Proof of part i) of Theorem I.1 : It is clear that u is bounded. Now in order to prove (13), we fix $\alpha \in A$ and choosing the constant control $\alpha_t = \alpha$ in equality (8), we find :

(16) $$u(x) \leqslant E\left[\int_0^{\tau\wedge t} f(X_s,\alpha_s)\,e^{-\lambda s}\,ds + u(X_{t\wedge\tau})\,e^{-\lambda\, t\wedge\tau}\right]$$

to simplify notations, we suppose $c(x,\alpha) = \lambda$ for all $(x,\alpha) \in R^N \times A$. Next, we consider $x \in \mathcal{O}$ such that : $\mathrm{dist}\,(x,\Gamma) \geqslant \delta > 0$, where $\delta > 0$ is fixed. From assumption (A.1) and martingale inequalities, we deduce immediately that we have :

$$E\,[\sup_{[0,t]} |X_s - x|^4] \leqslant C\,t^2 \qquad \text{for } t \leqslant 1$$

and C denotes here and below various constants independent of x, t, α. Therefore :

$$\begin{aligned} P\,[\tau \leqslant t] &\leqslant P\,[\sup_{[0,t]} |X_s - x| \geqslant \delta] \\ &\leqslant \frac{1}{\delta^4}\,E\,[\sup_{[0,t]} |X_s - x|^4] \leqslant C\,t^{\,2}. \end{aligned}$$

If we go back to (16), we deduce :

$$\frac{1}{t}\{u(x) - E\,[u(X_t)\,e^{-\lambda t}]\} \leqslant f(x,\alpha) + \varepsilon(t)$$

where we extend u outside $\bar{\mathcal{O}}$ by 0 and $\varepsilon(t) \to 0$ if $t \to 0_+$ uniformly in x satisfying : $\mathrm{dist}\,(x,\Gamma) \geqslant \delta > 0$.

To conclude, we just have to prove that if $u \in L^\infty(R^N)$ with compact support then :

$$\frac{1}{t}\{u(x) - E\,[u(X_t)\,e^{-\lambda t}]\} \underset{t \to 0_+}{\to} A_\alpha u \qquad \text{in } \mathcal{D}'(R^n)\,.$$

Let $\varphi \in \mathcal{D}(R^N)$, we claim that :

(17) $$\int_{R^N} E\,[u(X_t^x)\,e^{-\lambda t}]\;\varphi(x)\,dx = \int_{R^N} u(x)\;T_t\varphi(x)\,dx$$

where T_t is the semigroup corresponding to the solution of :

$$\frac{\partial\varphi}{\partial t} - \frac{\partial^2}{\partial x_i\,\partial x_j}(\varphi a_{ij}) + \frac{\partial}{\partial x_i}(b_i\varphi) + \lambda\varphi = 0 \quad \text{in } R^N \times (0,\infty)$$

- if we denote by $\tilde{X}_t$ the diffusion process with diffusion matrix a_{ij} and drift $- b_i + 2\,\dfrac{\partial a_{ij}}{\partial x_j}$ and by $c(x) = -\dfrac{\partial^2}{\partial x_i\,\partial x_j}(a_{ij}) + \dfrac{\partial b_i}{\partial x_i} + \lambda$ then we have ; $T_t\varphi(x) = E_x\left[\varphi(\tilde{X}_t)\exp\left\{-\int_0^t c(X_s)ds\right\}\right]$. To prove the above identity, it is clearly enough - by a density argument - to show it for smooth coefficients a_{ij}, b_i and for a smooth u : indeed let us observe that by results due to D.W. Stroock and S.R.S. Varadhan [47], [48] if $u = 0$ a.e. in R^N then $E[u(X_t^x)] = 0$ a.e.. Now for smooth data , the identity is almost obvious since $S_t u(x) = E[u(X_t^x)e^{-\lambda t}]$ and $T_t\varphi(x)$ are smooth functions of x and t - see O.Oleinik [39], D.W. Stroock and S.R.S. Varadhan [47] - ; and $S_t u$ is the solution $u(x,t)$ of the Cauchy problem :

$$\frac{\partial u}{\partial t} + A_\alpha u = 0 \quad \text{in } R^N \times (0,\infty)\ ,\ u(x,t)\big|_{t=0} = u(x)\ ,\ \forall\, x \in R^N.$$

Therefore we have for all $t > 0$ fixed :

$$\begin{aligned}
0 &= \int_{R^N}\int_0^t \left\{\frac{\partial u}{\partial s}(x,s) + A_\alpha u(x,s)\right\}\varphi(x,t-s)\,ds\,dx \\
&= \int_{R^N}(S_t u)\varphi\,dx - \int_{R^N} u(T_t\varphi)\,dx + \\
&\quad + \int_0^t\int_{R^N} u(s,x)\left\{\frac{\partial\varphi}{\partial s}(x,t-s) - \frac{\partial^2}{\partial x_i\,\partial x_j}(a_{ij}\varphi(x,t-s))\right. \\
&\quad \left. + \frac{\partial}{\partial x_i}(b_i\varphi(x,t-s)) + \lambda\,\varphi(x,t-s)\right\}ds\,dx \\
&= \int_{R^N}(S_t u)\varphi\,dx - \int_{R^N} u\,(T_t\varphi)\,dx
\end{aligned}$$

and the identity (17) is proved.

Next, we deduce from (17) the desired convergence since by Itô's formula :

$$T_t\varphi(x) = \varphi(x) + E\int_0^t \left\{-\frac{\partial^2}{\partial x_i\,\partial x_j}(a_{ij}\varphi) + \frac{\partial}{\partial x_i}(b_i\varphi) + \lambda\varphi\right\} \cdot \exp\left(-\int_0^s c\right) ds$$

and thus :

$$\frac{1}{t}\{u(x) - E[u(X_t)e^{-\lambda t}]\} = \int_{R^N} u(x)\,\frac{1}{t}\{T_t\varphi - \varphi\}dx \underset{t\to 0_+}{\to} \int_{R^N} u(x)\left\{-\frac{\partial^2}{\partial x_i\,\partial x_j}(a_{ij}\varphi) + \frac{\partial}{\partial x_i}(b_i\varphi) + \lambda\varphi\right\}.$$

■

Remark I.8 : Of course, when we go from (8) to (16), we lose much information that should be kept on in a tentative proof of the conjecture mentionned in Remark I.6. Let us also indicate that if we know that $u \in C^2$ then the derivation follows essentially the above scheme of proof since we write (8) in the following form :

$$\sup_A \left\{\frac{1}{t} E[u(x) - u(X_{t\wedge\tau})e^{-\lambda t\wedge\tau}] - \frac{1}{t} E\int_0^{t\wedge\tau} f(X_s,\alpha_s)e^{-\lambda s}ds\right\} = 0$$

and using Itô's formula this yields :

$$\sup_A \frac{1}{t} E\int_0^{t\wedge\tau}\left\{A_\alpha u(X_s) - f(X_s,\alpha_s)\right\} e^{-\lambda s}\,ds = 0$$

and we get easily (2), taking $t \to 0_+$ and using the above bound (uniform in A) on $P(\tau \leqslant t)$. ■

Proof of Part ii) of Theorem I.1 : We set $u(x,t) = S_t^o u(x) =$

$$= E\left[u(X_t)\,1_{(t\leqslant\tau)}e^{-\lambda t} + \int_0^{t\wedge\tau} f(X_s,\alpha)\,e^{-\lambda s}ds\right]\ ;\ \forall\, x \in \bar{\mathcal{O}}\ ,\ \forall\, t \geqslant 0.$$

where, as before, $\alpha \in A$ and X_t corresponds to the fixed control

$\alpha_t = \alpha$. In view of Theorem B, we see that : $u(x) \leqslant u(x,t)$, $\forall\, x \in \overline{\mathcal{O}}$, $\forall\, t \geqslant 0$ - use i) of Theorem B and (11).

We are first going to prove that :

$$\begin{cases} A_\alpha u(t) \leqslant f(\cdot,\alpha) & \text{in } \mathcal{D}'(\mathcal{O}) \quad , \quad \forall\, t \geqslant 0 \\ a^{1/2}\, \nabla u \in L^2(\mathcal{O} \times (0,1)) & \end{cases}$$

To prove this claim, we introduce : $u^\varepsilon(x,t) = S^\varepsilon(t)\, \tilde{u}(x) =$

$$= E\left[\tilde{u}(X_t)\exp\left(-\lambda t - \frac{1}{\varepsilon}\int_0^t p(X_s)ds\right) + \int_0^t \tilde{f}(X_s)\exp\left(-\lambda s - \int_0^s p(X_\sigma)d\sigma\right)\right]$$

where $\tilde{f} = f(\cdot,\alpha)$ in $\overline{\mathcal{O}}$ and $\tilde{f}$ has compact support, where $\tilde{u} = u$ in $\mathcal{O}$ and 0 outside, and where p is choosen as in the proof after Theorem B. As explained before $u^\varepsilon(x,t)$ is the limit of $S^\varepsilon(t)u_\delta(x)$ where $u_\delta \in \mathcal{D}(R^N)$ and $u_\delta \underset{\delta\to 0}{\to} \tilde{u}$ a.e. , $\|u_\delta\|_{L^\infty(R^N)} \leqslant C$. Denote by $v_\delta(x,t) = S^\varepsilon(t)u_\delta(x)$. It is well-known (see [46], [39]) that $v_\delta \in W^{2,1,\infty}(R^N \times [0,1])$ is the unique solution of :

$$\frac{\partial v_\delta}{\partial t} + A_\alpha v_\delta + \frac{1}{\varepsilon}\, p\, v_\delta = \tilde{f} \text{ in } R^N \times (0,\infty) \ , \ v_\delta|_{t=0} = u_\delta \text{ in } R^N .$$

Now if we multiply the equation by v_δ, we find integrating by parts as in the proof in Remark I.5 :

$$\frac{d}{dt}\,|v_\delta(t)|^2_{L^2(R^N)} + \frac{1}{2}\int_{R^N} a_{ij}(x,\alpha)\,\frac{\partial v_\delta}{\partial x_i}(x,t)\,\frac{\partial v_\delta}{\partial x_j}(x,t)dx$$
$$\leqslant C\,|v_\delta(t)|^2_{L^2} + C$$

From Gronwall Lemma, we deduce : $|v_\delta(t)|_{L^2(R^N)} \leqslant C$ if $t \leqslant 1$;

$$\int_0^1\int_{R^N} a_{ij}(x,\alpha)\,\frac{\partial v_\delta}{\partial x_i}(x,t)\,\frac{\partial v_\delta}{\partial x_j}(x,t)\,dx\,dt \leqslant C$$

where C does not depend on δ, nor on ε.

Since $u^{\varepsilon}(x,t) \to u(x,t)$ as $\varepsilon \to 0_+$ for $(x,t) \in \overline{\mathcal{O}} \times [0,\infty[$, we deduce : $\frac{\partial u}{\partial t} + A_\alpha u = f$ in $\mathcal{D}'(\mathcal{O} \times (0,\infty))$, $a^{1/2}\nabla u \in L^2(\mathcal{O} \times (0,1))$. Furthermore, we already know (proof of i)) that : $u(t) \underset{t\to 0_+}{\to} u$ in $\mathcal{D}'(\mathcal{O})$. In addition, as we observed in the beginning of the proof : $u \leqslant u(t)$ in $\overline{\mathcal{O}}$ and by the semi-group (or Markov) property of S_t^o we deduce :

$$u(x,s) \leqslant u(x,t) \qquad \forall\, x \in \overline{\mathcal{O}}\ ,\ \forall\, s \leqslant t\ .$$

This yields : $A_\alpha u(t) \leqslant f(\cdot,\alpha)$ in $\mathcal{D}'(\mathcal{O})$, $\forall\, t \geqslant 0$. Therefore, in particular, there exists a sequence $t_n \underset{n\uparrow\infty}{\downarrow} 0$ such that :

$$a^{1/2}\nabla u(t_n) \in L^2(\mathcal{O}) \quad , \quad A_\alpha u(t_n) \leqslant f(\cdot,\alpha) \quad \text{in } \mathcal{D}'(\mathcal{O}) \quad ,$$

$$u(t_n) \underset{n}{\to} u \quad \text{in } \mathcal{D}'(\mathcal{O})\ .$$

The argument given in Remark I.5 can now be easily justified for $u_n = u(t_n)$ using the results of Appendix 1 and this implies :

$$\| a^{1/2}\ \nabla u_n \|_{L^2(\omega)} \leqslant C \qquad \text{(ind. of } n)$$

We conclude sending n to $+\infty$ and observing that C does not depend on $\alpha \in A$.

■

I.3 Maximum subsolution :

In the preceding section (Theorem I.1), we have seen that u is a subsolution of the HJB equation (2) (compare (13) and (2)). The next result shows, roughly speaking, that u is the maximum subsolution of (2) having u as boundary conditions. This kind of results has been initiated by P.L. Lions and J.L. Menaldi [29], [30] - see also P.L. Lions [24], [18].

<u>Theorem I.2</u> : *Let* $v \in C(\mathcal{O})$ *satisfy* :

(13') $A_\alpha v \leqslant f(\cdot,\alpha)$ *in* $\mathcal{D}'(\mathcal{O})$, $a^{1/2}(x,\alpha)\cdot\nabla v \in L^2_{loc}(\mathcal{O})$, $\forall\, \alpha \in A$;

and for all $x \in \overline{\mathcal{O}}$:

$$\limsup_{\delta \to 0} E\left[v(X_{\tau_\delta})\exp\left(-\int_0^{\tau_\delta} c(X_t,\alpha_t)dt\right)\right] \leqslant E\left[u(X_{\tau'})\exp\left(-\int_0^{\tau'} c(X_t,\alpha_t)dt\right)\right]$$

where $\delta > 0$, τ_δ *is the first exit time of* X_t *from* $\overline{\mathcal{O}}_\delta$ *and* $\mathcal{O}_\delta = \{x \in \mathcal{O} , \operatorname{dist}(x,\Gamma) > \delta\}$. *Then we have* $v \leqslant u$ *in* $\mathcal{O}$.

<u>Remark I.9</u> : Because of the smoothness assumption on the coefficients σ, b, it can be seen (Cf. the Appendix of Part 2) that <u>the assumption</u> $a^{1/2}\nabla v \in L^2_{loc}(\mathcal{O})$ <u>may be suppressed</u>. In addition one can relax the continuity assumption on v : for example the above result still holds if v is u.s.c. on $\mathcal{O}$.

<u>Remark I.10</u> : The boundary condition on v is in particular satisfied if we have :

$$\limsup_{y \in \mathcal{O}\,,\, y \to x} v(y) \leqslant u(x) \quad , \quad \text{for all } x \in \Gamma$$

(or for all x such that for some $\mathcal{A}$, X_t "hits Γ at the point x"). In particular, if there exists Γ_0 closed subset of Γ such that for all $\mathcal{A}$ and for all $x \in \mathcal{O}$: $P[\tau' < \infty , X_{\tau'} \in \Gamma_0] = P[\tau' < \infty]$ and if $u \in C(\overline{\mathcal{O}})$ and $u = 0$ on Γ_0, then u is the maximum element of all $v \in C(\overline{\mathcal{O}})$ satisfying : $A_\alpha v \leqslant f(\cdot,\alpha)$ in $\mathcal{D}'(\mathcal{O})$, $a^{1/2}(x,\alpha)\nabla v \in L^2_{loc}(\mathcal{O})$ $\forall\, \alpha \in A$ and $v \leqslant 0$ on Γ_0. We will see in section II many instances where such assumptions are satisfied. Let us also mention that the results in [29], [30] are completely contained in this situation.

Proof of Theorem I.2 : We are first going to prove that v satisfies for all $\delta > 0$:

$$(18)\qquad v(x) \leqslant \inf_{A} E\left[\int_0^{\tau_\delta} f(X_t,\alpha_t)e^{-\lambda t}dt + v(X_{\tau_\delta})e^{-\lambda\tau_\delta}\right],\ \forall\, x \in \mathcal{O}_\delta$$

where, again to simplify notations, we assume : $c(x,\alpha) \equiv \lambda$.

We need some notations : let $\xi \in \mathcal{D}_+(\mathcal{O})$, $\xi^{1/2} \in W^{2,\infty}(\mathcal{O})$, $\xi \equiv 1$ on a neighborhood of $\overline{\mathcal{O}}_\delta$, $0 \leqslant \xi \leqslant 1$ in $\mathcal{O}$ and let $\varphi \in \mathcal{D}_+(\mathcal{O})$, $\varphi \equiv 1$ on a neighborhood of Supp ξ , $0 \leqslant \varphi \leqslant 1$ in $\mathcal{O}$.

We deduce from (13') that

$$(13'')\qquad \tilde{A}_\alpha \tilde{v} \leqslant \tilde{f}(\cdot,\alpha) \text{ in } \mathcal{D}'(R^N) \quad , \quad a^{1/2}(x,\alpha)\nabla\tilde{v} \in L^2(R^N) \ , \ \forall\, \alpha \in A$$

where $\tilde{A}_\alpha = \xi A_\alpha$; $\tilde{v} = \varphi v$ in $\mathcal{O}$, $= 0$ in $R^N - \mathcal{O}$; $\tilde{f}(\cdot,\alpha) = \xi f(\cdot,\alpha)$. Let $\varepsilon > 0$ be fixed, we set $B_\alpha = \tilde{A}_\alpha + \varepsilon$, $g_\alpha = \tilde{f}(\cdot,\alpha) + \varepsilon\tilde{v}$. We then introduce a new optimal control problem where σ is replaced by $\xi^{1/2}\sigma$, b by ξ b , c by $\xi c + \varepsilon$, f by g and $\mathcal{O}$ by R^N : we denote by $\mathcal{A}$ any such admissible system corresponding to this new problem ; we have $\mathcal{A} = (\Omega,F,F^t,P,B_t,\alpha_t)$ and the state $\tilde{X}_t$ is given by

$$d\tilde{X}_t = \xi^{1/2}(\tilde{X}_t)\sigma(\tilde{X}_t,\alpha_t)dB_t + \xi(\tilde{X}_t)b(\tilde{X}_t,\alpha_t)dt \ , \ \tilde{X}_0 = x$$

and thus

$$\tilde{X}_{t\wedge\tau_\delta} = X_{t\wedge\tau_\delta} \qquad \text{for all } t \geqslant 0 .$$

We want to prove that : $M_t = \tilde{v}(\tilde{X}_t)\exp\left(-\int_0^t (\xi c+\varepsilon)dt\right) +$

$+ \int_0^t g_{\alpha_\mu}(\tilde{X}_\mu)\exp\left[-\int_0^\mu \xi(\tilde{X}_s)\, c(\tilde{X}_s,\alpha_s)ds - \varepsilon\mu\right]d\mu$ is a continuous submartingale (with respect to F_t of course).

To this end, we fix α in A and consider the canonical diffusion process $(\tilde{\Omega},\tilde{F},\tilde{F}_t,X_t,(P_x)_{x\in R^N})$ - where $(\tilde{\Omega},\tilde{F},\tilde{F}_t,X_t)$ is the canonical space $(\tilde{\Omega} = C([0,\infty) ; R^N))$ and P_x is the solution of the martingale problem - associated with $\xi(\cdot)\, a(\cdot,\alpha)$ and $\xi(\cdot)\, b(\cdot,\alpha)$. We are now going to prove that, because of (13''), $N_t = \tilde{v}(X_t) \exp\left(-\int_0^t \tilde{c}\right) + \int_0^t g_\alpha(X_\mu) \exp\left(-\int_0^\mu \tilde{c}\right) d\mu$ is a $\tilde{F}_t$-submartingale - where $\tilde{c}(x,\alpha) = \xi(x)\, c(x,\alpha) + \varepsilon$. To prove this fact we just need to show that :

$$(19)\qquad \tilde{v}(x) \leqslant \tilde{v}(x,t) \qquad \forall\, x \in R^N ,\ \forall\, t \geqslant 0$$

where $\tilde{v}(x,t) = E_x\left\{\tilde{v}(X_t)\exp\left(-\int_0^t\tilde{c}\right) + \int_0^t g_\alpha(X_\mu) \exp\left(-\int_0^\mu\tilde{c}\right) d\mu\right\}$ is the solution (Cf. Appendix 2) of

$$\begin{cases} \dfrac{\partial\tilde{v}}{\partial t} + B_\alpha\tilde{v} = g_\alpha \quad \text{in } R^N \times (0,\infty) \ ,\ \tilde{v} \in C([0,\infty[\ ; L^2(R^N)) \\ \tilde{v} \in B \cup C(R^N \times [0,T]) \quad ,\quad a^{1/2}(\cdot,\alpha)\nabla\tilde{v} \in L^2(R^N \times (0,T)) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for all } T < \infty \ . \end{cases}$$

Indeed if such an inequality is proved, we deduce using the Markov property of X_t : if $t \geqslant s$

$$\begin{aligned} E_x[N_t \mid \tilde{F}_s] &= \int_0^s g_\alpha(X_\mu)\exp\left(-\int_0^\mu\tilde{c}\right) d\mu \ + \\ &+ \exp\left(-\int_0^s\tilde{c}\right) \cdot E_{X_s}\left\{\tilde{v}(X_{t-s})\exp\left(-\int_0^{t-s}\tilde{c}\right)+\int_0^{t-s} g_\alpha(X_\mu)\exp\left(-\int_0^\mu\tilde{c}\right)d\mu\right\} \\ &= \int_0^s g_\alpha(X_\mu)\exp\left(-\int_0^\mu\tilde{c}\right)d\mu + \tilde{v}(X_s,t-s)\exp\left(-\int_0^s\tilde{c}\right) \\ &\geqslant \int_0^s g_\alpha(X_\mu)\exp\left(-\int_0^\mu\tilde{c}\right)d\mu + \tilde{v}(X_s)\exp\left(-\int_0^s\tilde{c}\right) = N_s . \end{aligned}$$

Let us also indicate that once we know that N_t is a $\tilde{F}_t$-submartingale it is easy to deduce that M_t is a F^t-submartingale for all controls α_t taking a finite number of values (in A) and being constant between two stopping times i.e. : $\exists\ \theta_o \leq \theta_1 \leq \ldots \leq \theta_k < \infty$ such that $\alpha_t = \alpha_i$ if $\theta_i \leq t < \theta_{i+1}$, $0 \leq i \leq k$. By an easy density argument (Cf. N.V. Krylov [12]) we deduce that for any control α_t , M_t is a F^t-submartingale - and M_t is obviously continuous since $\tilde{v} \in B \cup C(R^N)$. This then immediately implies (18) since $X_{t\wedge\tau_\delta} = \tilde{X}_{t\wedge\tau_\delta}$ ($\forall\ t \geq 0$).

Therefore, in order to prove (18), there just remains to prove (19). We introduce g_α^ε , $\tilde{v}^\varepsilon$ smooth functions with compact support (uniform in ε) satisfying : $g_\alpha^\varepsilon \geq g_\alpha$, $\tilde{v}^\varepsilon \geq \tilde{v}$ in R^N and $g_\alpha^\varepsilon \underset{\varepsilon}{\to} g_\alpha$, $\tilde{v}^\varepsilon \underset{\varepsilon}{\to} \tilde{v}$ uniformly on R^N. We then denote by $\tilde{v}^\varepsilon(x,t)$ the corresponding solution of

$$\begin{cases} \dfrac{\partial \tilde{v}^\varepsilon}{\partial t} + B_\alpha \tilde{v}^\varepsilon = g_\alpha^\varepsilon \text{ in } R^N \times (0,\infty) \ , \ \tilde{v}^\varepsilon \in W^{2,1,\infty}(R^N \times (0,T)), \forall\ T < \infty \\ \tilde{v}^\varepsilon(x,0) = \tilde{v}^\varepsilon(x) \quad \text{in } R^N \quad . \end{cases}$$

The solution exists, is unique (Cf. Appendix 2) and is given by :

$$\tilde{v}^\varepsilon(x,t) = E_x\left[\tilde{v}^\varepsilon(X_t)\exp\left(-\int_o^t \tilde{c}\right) + \int_o^t g_\alpha^\varepsilon(X_\mu)\exp\left(-\int_o^\mu \tilde{c}\right) d\mu\right] .$$

In addition $\tilde{v}^\varepsilon(x,t) \underset{\varepsilon \to 0}{\to} \tilde{v}(x,t)$ uniformly on $R^N \times [\,0,T]$ ($\forall\ T < \infty$).

We finally consider : $w(x,t) = \left(\tilde{v}(x) - \tilde{v}^\varepsilon(x,t)\right)^+$. It is easily shown by usual techniques (See Appendix 1) that $a^{1/2}(\cdot,\alpha)\nabla w \in L^2(R^N)$ for $t \geq 0$ and

$$a^{1/2}(\cdot,\alpha)\nabla w = a^{1/2}(\cdot,\alpha)\ \nabla(v-v\ (t))\ 1_{(\tilde{v} \geq \tilde{v}^{\varepsilon}(t))} \quad \text{a.e.,}\quad \forall\ t \geq 0\ .$$

Next, we observe that we have :

$$\begin{cases} \dfrac{\partial \tilde{v}^{\varepsilon}}{\partial t} + B_{\alpha}\tilde{v}^{\varepsilon} = g_{\alpha}^{\varepsilon} & \text{in } R^N \times (0,\infty) \quad , \quad \tilde{v}^{\varepsilon}\big|_{t=0} \geq \tilde{v} \\ B_{\alpha}\tilde{v} \leq g_{\alpha}^{\varepsilon} & \text{in } R^N \end{cases}$$

and multiplying those equalities-inequalities by w and integrating by parts, we obtain as in the proof of Remark I.5 :

$$\frac{1}{2}\ |w(t)|^2_{L^2(R^N)} + \int_o^t \int_{R^N} a_{ij}\ \frac{\partial w(s)}{\partial x_i}\ \frac{\partial w(s)}{\partial x_j}\ ds\ dx \leq$$

$$\leq \quad C \int_o^t |w(s)|^2_{L^2(R^N)}\ ds$$

and we conclude : $w(t) = 0$, using Grönwall Lemma.

The integration by parts is allowed by a regularization argument using the results proved in Appendix 1, and totally similar to the one done in Appendix 2.

From (18), we deduce letting $\delta \to 0_+$:

$$v(x) \leq \inf_A E\left[\int_o^{\tau'} f(X_t,\alpha_t)\ e^{-\lambda t}dt + u(X_{\tau'})\ e^{-\lambda\tau'}\right]$$

and we conclude using Theorem B since we have :

$$u(x) = \inf_A E\left[\int_o^{\tau'} f(X_t,\alpha_t)\ e^{-\lambda t}dt + u(X_{\tau'})\ e^{-\lambda\tau'}\right] \quad .$$

∎

II. Continuity properties of the optimal cost function :

In this section we will prove continuity properties of u :

as it is easily seen on deterministic examples, u is not always continuous and proving continuity properties will require assumptions of the type (A.2) given below. We insist on deriving results giving the continuity of u since the notion of viscosity solution of (2) (developped in Part 2) will require the continuity of u.

II.1 Upper-semicontinuity of u.

The main difficulty in proving continuity properties of u comes from the fact that τ ($= \tau_x$) is not, in general, continuous with respect to x : nevertheless since X_t is a continuous process, $\tau = \tau(x)$ is upper semicontinuous. Therefore, for example if $f(x,\alpha) \geq 0$ then $J(x,A)$ is immediately upper semicontinuous and u being an infimum of u.s.c. functions is also u.s.c. : the result that we give now extends this simple observation. To state it, we need some assumption and notations :

$$
(A.2)\quad \left\{
\begin{array}{l}
\text{There exist } \Gamma_+ \text{ closed subset of } \Gamma,\ w \text{ borel bounded function on} \\
\bar{\mathcal{O}} \text{ such that for all admissible system } A \text{ and for all } x \in \bar{\mathcal{O}} : \\
P\,[\tau < \infty\ ,\ X_\tau \in \Gamma - \Gamma_+] = 0\ ,\ 1_{(\tau < \infty)} w(X_\tau) \leq 0 \quad \text{a.s.} \\
\forall\, x \in \Gamma_+\ ,\quad \liminf\limits_{y \in \bar{\mathcal{O}}\ ,\ y \to x} w(y) \geq 0\ ; \\
N_t = w(X_{t\wedge\tau}) \exp\left(-\int_0^{t\wedge\tau} c\right) + \int_0^{t\wedge\tau} f \exp\left(-\int_0^s c\right) ds \quad \text{is a } F_{t\wedge\tau} \\
\text{submartingale and for all bounded stopping times } \theta_1 \leq \theta_2 \\
\text{we have : } E\,[N_{\theta_2} \mid F_{\theta_1}] \geq N_{\theta_1}\ .
\end{array}
\right.
$$

In section II.2 below, we give many examples of situations when (A.2) holds : let us only mention that if $f(x,\alpha) \geq 0$ then we may take $\Gamma_+ = \Gamma$, $w \equiv 0$ and (A.2) holds.

Theorem II.1 : *Under assumption* (A.2) , *we have* :

i) *For all* A, $J(\cdot,A)$ *is u.s.c. on* $\overline{\mathcal{O}}$ *and thus* u *is u.s.c. on* $\overline{\mathcal{O}}$.

ii) $u \geq w$ *in* $\overline{\mathcal{O}}$, $u \geq 0$ *on* Γ_+ *and for all* A : $1_{(\tau > \infty)} u(X_\tau) = 0$ *a.s.*

Remark II.1 : Assumption (A.2) often implies that $P(\tau' < \infty , X_{\tau'} \notin \Gamma_+) = 0$; therefore we have in this case :

$$u(x) \geq \inf_A E \int_0^{\tau'} f(X_t,\alpha_t) \exp\left(-\int_0^t c\right) dt . \qquad \blacksquare$$

Remark II.2 : The same method than the one used in the proof below shows that in the infimum defining u(x), we may restrict our attention to dense subclasses of controls like for exemple step controls, continuous controls provided (for example) that A is a closed convex set in R^p, natural controls (in the sense of [12]).. This, in turn, shows that we may restrict the infimum over admissible systems where the probability space (Ω,F,F_t,P) and the Brownian motion B_t are fixed. ∎

Remark II.3 : It is possible to extend a little bit assumption (A.2) by replacing w, f by w_n , f_n $(n \geq 1)$ satisfying :

$$\begin{cases} 1_{(\tau < \infty)}\, w_n(X_\tau) \leq \delta_n \quad \text{a.s.} ; \\ \forall\, x \in \Gamma_+ , \quad \liminf_{y \in \overline{\mathcal{O}},\, y \to x} w_n(y) \geq -\delta_n ; \\ f_n(x,\alpha) \geq f(x,\alpha) - \delta_n \quad ; \quad f_n(x,\alpha) \underset{n\to\infty}{\to} f(x,\alpha) \end{cases}$$

where $\delta_n > 0$, $\delta_n \underset{n}{\to} 0$. ∎

Remark II.4 : We already know (Theorems A and B) that :

$M_t = u(X_{t\wedge\tau})e^{-\lambda t} + \int_0^{t\wedge\tau} f(X_s,\alpha_s)e^{-\lambda s}\,ds$ – take to simplify $c(x,\alpha) \equiv \lambda$ – is a F_t-submartingale (satisfying Doob's stopping theorems) and since f is continuous and u is u.s.c., we can deduce that M_t is a right-continuous process and this means that for all A and for all x , u is right-continuous on the trajectories of $X_{t\wedge\tau}$ a.s. . ∎

Remark II.5 : It is of course possible to take $\Gamma_+ = \emptyset$ i.e. to assume that for all A and for all $x \in \bar{\mathcal{O}}$: $X_t \in \bar{\mathcal{O}}$, $\forall\, t \geq 0$. This happens for example if on Γ : $a(x,\alpha) = 0$, $b(x,\alpha)\cdot n(x) \leq 0$ for all $\alpha \in A$; where n(x) is the unit outward normal. Remark that if we know that for all A and for all $x \in \bar{\mathcal{O}}$, X_t remains in $\bar{\mathcal{O}}$ for all $t \geq 0$ (a.s.) then (A.2) clearly holds : choose

$w = -K \leq \inf_{x,\alpha} f(x,\alpha)\, c(x,\alpha)^{-1}$. ∎

Proof of Theorem II.1 : We denote by $\tau_n = \tau(x_n)$, $X_t^n = X_t^{x_n}$ where $x_n \in \bar{\mathcal{O}}$, $x_n \xrightarrow[n]{} x$. We want to prove that : $\limsup_n J(x_n,A) \leq J(x,A)$, where A is fixed. Let us first recall that, by standard arguments on stochastic differential equations, we have for all $T < \infty$:

$$\sup_{[0,T]} |X_t^n - X_t| \xrightarrow[n]{} 0 \quad \text{a.s. and in } L^p(\Omega) \quad (\text{for all } p < \infty).$$

In addition as we observed above : $\limsup_n \tau_n \leq \tau$ and thus $\tau_n - \tau_n\wedge\tau \xrightarrow[n]{} 0$. Therefore we have ;

$$\left|J(x_n,A) - E\int_0^{\tau_n\wedge\tau} f(X_t^n,\alpha_t)\, e^{-\lambda t}\,dt\right| \xrightarrow[n]{} 0 \quad ;$$

$$|E \int_0^{\tau_n \wedge \tau} f(X_t^n, \alpha_t) e^{-\lambda t} dt - E \int_0^{\tau_n \wedge \tau} f(X_t, \alpha_t) e^{-\lambda t} dt| \leqslant$$

$$\leqslant C\, E \sup_{[0,T]} |X_t^n - X_t| + C\, e^{-\lambda T}, \text{ for all } T < \infty ;$$

since f is equi-lipschitz in x and bounded.

This shows that we just have to prove :

$$\text{(20)} \quad \varlimsup_{n \to \infty} E \int_0^{\tau_n \wedge \tau} f(X_t, \alpha_t) e^{-\lambda t} dt \leqslant E \int_0^{\tau} f(X_t, \alpha_t) e^{-\lambda t} dt .$$

Next, observe that in view of the convergence of X_t^n towards X_t we have :

$$\text{(21)} \quad 1_{(\tau_n < \tau < \infty)} \operatorname{dist}(X_{\tau_n}, \Gamma_+) \underset{n}{\to} 0 \quad \text{in proba.}.$$

Let $T < \infty$, in view of (A.2), we obtain :

$$E[N_{\tau \wedge T}] \geqslant E[N_{\tau \wedge \tau_n \wedge T}]$$

and if we let $T \to +\infty$, this yields :

$$E \int_{\tau_n \wedge \tau}^{\tau} f(X_t . \alpha_t) e^{-\lambda t} dt \geqslant E \left[w(X_{\tau_n \wedge \tau}) e^{-\lambda \tau_n} - w(X_\tau) e^{-\lambda \tau} \right]$$

$$\geqslant E \left[1_{(\tau_n < \tau)} \left\{ w(X_{\tau_n}) e^{-\lambda \tau_n} - w(X_\tau) e^{-\lambda \tau} \right\} \right]$$

and this implies (20) in view of (21) and (A.2).

Therefore for all $\mathcal{A}$, $J(\cdot, \mathcal{A})$ and thus u are u.s.c. on $\bar{\mathcal{O}}$.

Next, we observe that : $w(x) \leqslant \lim_{T \to \infty} E[N_{T \wedge \tau}] =$
$= J(x, \mathcal{A}) + E[w(X_\tau) e^{-\lambda \tau}]$ and this implies : $w \leqslant u$ in $\bar{\mathcal{O}}$; hence

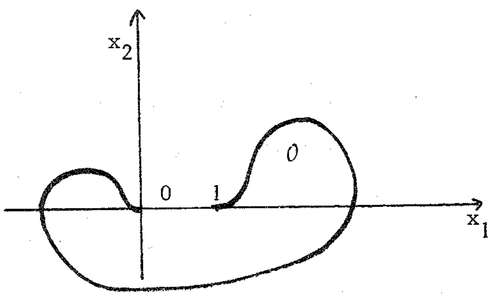
$$\forall x \in \Gamma_+ , \quad u(x) \geqslant \varlimsup_{y \in \bar{\mathcal{O}},\, y \to x} u(y)$$

$$\geqslant \limsup_{y \in \bar{\mathcal{O}},\ y \to x} w(y) \geqslant 0$$

and since $P(\tau < \infty,\ X_\tau \in \Gamma - \Gamma_+) = 0$, $1_{(\tau < \infty)} u(X_\tau) = 0$ a.s. - recall that this quantity is always nonpositive, cf. Theorem B. ■

Of course we do not claim that (A.2) is the best assumption that implies the upper semicontinuity of u : nevertheless, in view of the results of section II.2 below, it seems a general enough assumption.

Example : Let us consider a very simple case (without control) where u is not u.s.c. : take $\sigma \equiv 0$, $c \equiv 1$, $b_1 \equiv 1$, $N = 2$, $b_2 \equiv 0$, $f \equiv \lambda \in R$, then of course $X_t = x + te_1$ where $e_1 = (1,0)$, $u(x) = \lambda\ (1-e^{-\tau})$. If $\lambda \geqslant 0$, u is always u.s.c.. On the other hand, if $\lambda < 0$ and if $\mathcal{O}$ looks like the following picture :



denoting by $\Delta = \{x \in \bar{\mathcal{O}},\ x_1 \leqslant 0,\ x_2 = 0\}$, we see that u has a discontinuity on Δ and that :

$$\liminf_{\substack{x_2 > 0,\ x_2 \to 0 \\ x \to x_o,\ x \in \bar{\mathcal{O}}}} u(x) > u(x_o) \qquad \text{for } x_o \in \Delta .$$

Actually, in this case u is l.s.c. (it is possible to build more complicated examples along this line where u is neither u.s.c., nor l.s.c.). This type of examples shows that in order to have u u.s.c., one needs some kind of "subsolution".

II.2 Verification of assumption (A.2).

We want to explain in this section how it is possible to check (A.2). We will begin by choosing $\Gamma_+ = \Gamma$. Then we have the general

Proposition II.1 : *Let* $w \in W^{1,\infty}(\mathcal{O})$ *satisfy* : $w = 0$ *on* Γ *and*

$$(13) \quad A_\alpha w \leqslant f(\cdot,\alpha) \quad \text{in } \mathcal{D}'(\mathcal{O}) \quad , \quad \text{for all } \alpha \in A \ .$$

Then (A.2) *holds* (*with* $\Gamma = \Gamma_+$) *and more precisely for all* A *and for all* $x \in \overline{\mathcal{O}}$, $N_t = w(X_{t\wedge\tau})\exp\left(-\int_0^{t\wedge\tau} c\right)+\int_0^{t\wedge\tau} f(X_s,\alpha_s)\exp\left(-\int_0^s c\right)ds$ *is a continuous* F_t-*submartingale.*

Hence, a way to check (A.2) is to find w satisfying the above conditions. Before proving Proposition II.1, let us give a few examples where such a function w exists : we denote by $d(x) = \mathrm{dist}(x,\Gamma)$, let us recall that $d \in C^2(\overline{\mathcal{O}^{\delta_0}})$ for some δ_0 where $\mathcal{O}^\delta = \{x \in \mathcal{O} \ , \ d(x) < \delta\}$ and $|\nabla d| = 1$ in $\overline{\mathcal{O}^{\delta_0}}$, $\nabla d = -n$ on Γ – Cf. J. Serrin [45] or D. Gilbarg and N.S. Trudinger [8] for the proof of these assertions.

Corollary II.1 : *We assume that there exists* $\theta \geqslant 0$ *such that* :

$$(22) \quad \theta a_{ij}(x,\alpha)n_i(x)n_j(x) + b_i(x,\alpha)n_i(x) - a_{ij}(x,\alpha)\partial_{ij}d(x) \geqslant \nu > 0 \quad , \quad \forall\ (x,\alpha) \in \Gamma \times A \ .$$

Then there exists w *satisfying the conditions of Proposition* II.1 *and thus* (A.2) *holds.*

Remark II.6 : Of course (22) holds if : $a_{ij}(x,\alpha)n_i n_j \geqslant \nu > 0$, $\forall (x,\alpha) \in \Gamma \times A$ but also if : $b_i(x,\alpha)n_i(x) - a_{ij}(x,\alpha)\partial_{ij}d(x) \geqslant \nu > 0$, $\forall (x,\alpha) \in \Gamma \times A$. For example if $\mathcal{O} = \{p(x) < 0\}$ $p \in C_b^2(R^N)$, $(\partial_{ij}p) \geqslant I_N$ in R^N and if $b \equiv 0$, $\mathrm{Tr}\, a(\alpha) \geqslant \gamma > 0$; then (22) holds since, choosing $\theta = 0$, we have :

$$-a_{ij}(x,\alpha)\, \partial_{ij}d(x) \geqslant \nu\, \mathrm{Tr}(a(x,\alpha)) \geqslant \nu' > 0 .$$

Finally let us recall that $(D^2 d(x))$ is essentially given by the curvatures of Γ at x.

Remark II.7 : The conclusion of Corollary II.1 still holds if we replace (22) by the more general (and complicated) assumption :

$\exists\, \mu > 0$, $\exists\, \lambda \geqslant 0$, $\exists\, \delta_1 \in (0,\delta_o]$

$$f(x,\alpha) \geqslant -\mu c(x,\alpha)\left(\frac{1-e^{-\lambda d}}{\lambda}\right) - e^{-\lambda d}\Big\{b_i(x,\alpha)\cdot n_i(x) - a_{ij}(x,\alpha) \cdot\partial_{ij}d(x) + \lambda + \lambda a_{ij}(x,\alpha)n_i(x)n_j(x)\Big\} \text{ in } \overline{\mathcal{O}^{\delta_1}},$$

and $$-\mu\,\frac{1-e^{-\lambda\delta_1}}{\lambda} \leqslant \inf\left\{\frac{f(x,\alpha)}{c(x,\alpha)} \;/\; \alpha \in A,\ x \in \overline{\mathcal{O}},\ d(x) \geqslant \delta_1\right\}.$$

Proof of Corollary II.1 : From (22), one deduces immediately that $\tilde{w} = \frac{1}{\theta}(1 - e^{-\theta d})$ satisfies :

$$A_\alpha \tilde{w} \geqslant \nu/2 \qquad \text{in } \mathcal{O}^{\delta_1}$$

for some $\delta_1 \in (0,\delta_o]$. Let $K > 0$ be such that :

$$K(\nu/2) \geqslant \sup_{x,\alpha} |f(x,\alpha)| \quad , \quad \frac{K}{\theta}(1-e^{-\theta\delta_1}) \geqslant \sup_{x,\alpha} \frac{|f(x,\alpha)|}{c(x,\alpha)} .$$

Then if we set : $w(x) = -K\tilde{w}$ in $\overline{\mathcal{O}^{\delta_1}}$, $w(x) = -\frac{K}{\theta}(1 - e^{-\theta\delta_1})$ for $x \in \overline{\mathcal{O}}$, $d(x) \geq \delta_1$; one checks immediately :

$$A_\alpha w \leq f(\cdot,\alpha) \quad \text{in } \mathcal{O}^{\delta_1} \quad , \quad A_\alpha w \leq f(\cdot,\alpha) \quad \text{in } \mathcal{O}_{\delta_1}$$

where $\mathcal{O}_\delta = \{x \in \mathcal{O} \; , \; d(x) > \delta\}$, Finally, since $w \geq w|_{\mathcal{O}_{\delta_1}}$ in $\overline{\mathcal{O}^{\delta_1}}$, an easy computation shows that in fact we have :

$$A_\alpha w \leq f(\cdot,\alpha) \quad \text{in } \mathcal{D}'(\mathcal{O}) \quad , \quad \forall \alpha \in A . \qquad \blacksquare$$

<u>Proof of Proposition II.1</u> : We just need to show that it is possible to extend w outside $\mathcal{O}$ (in a neighborhood of $\overline{\mathcal{O}}$) in such a way that :

$$A_\alpha \tilde{w} \leq \tilde{f}(\cdot,\alpha) \quad \text{in } \mathcal{D}'(\tilde{\mathcal{O}}) \quad , \quad \forall \alpha \in A$$

where $\tilde{f}$ is u.s.c. in x, $\tilde{w} \equiv w$ in $\overline{\mathcal{O}}$ and $\tilde{\mathcal{O}}$ contains a neighborhood of $\overline{\mathcal{O}}$. If this is proved then approximating $\tilde{f}$ from above by continuous functions, we may apply the proof of Theorem I.2 and conclude easily. We first remark that by adding $c\tilde{d}$ to w (where $\tilde{d} = d$ in a neighborhood of Γ) we may assume that w is strictly decreasing along the normal variable nearby Γ :

$$\frac{\partial w}{\partial n} \leq -\nu < 0 \qquad \text{on a neighborhood of } \Gamma .$$

In addition by an easy localization argument, we may assume without loss of generality that $\mathcal{O} = \{x_N > 0\}$ and thus that :

$$\frac{\partial w}{\partial x_N} \geq \nu \geq 0 \quad , \quad \text{if} \quad 0 < x_N < h$$

for some $h > 0$. We are going to prove that we have :

$$A_\alpha \tilde{w} \leqslant \tilde{f} \quad \text{in } R^N \quad , \quad \forall\, \alpha \in A$$

where $\tilde{z} = z$ in $\{x_N > 0\}$, $= 0$ outside.

To prove this claim, let $\varphi \in \mathcal{D}_+(R^N)$ and to simplify notations assume $b = c \equiv 0$ and we denote by $a_{ij}(x) = a_{ij}(x,\alpha)$, $f(x) = f(x,\alpha)$, where α is fixed in A. Obviously, we have :

$$\langle A_\alpha \tilde{w} , \varphi\rangle = \int_{(x_N > 0)} \partial_j w \, \partial_i (a_{ij}\varphi) \, dx .$$

Next, we introduce $w_\varepsilon = w \star \rho_\varepsilon$ where $\rho_\varepsilon = \frac{1}{\varepsilon^N} \rho \left(\frac{\cdot}{\varepsilon}\right)$, $\rho \in \mathcal{D}_+(R^N)$, $\int \rho \, dx = 1$, Supp $\rho \subset B_1$: w_ε is defined on $\{x_N > \varepsilon\}$. Let $h' > 0$ be fixed : $0 < h' < h$. Then for ε small enough :

$$\partial_N w_\varepsilon(x',h') = ((\partial_N w) \star \rho_\varepsilon)(x',h') \geqslant \nu > 0 \quad , \quad \forall\, x' \in R^{N-1}$$

In addition we have :

$$\left| \int_{(x_N > 0)} \partial_j w \partial_i (a_{ij}\varphi) dx - \int_{(x_N > h')} \partial_j w \partial_i (a_{ij}\varphi) dx \right| \leqslant \delta(h')$$

$$\left| \int_{(x_N > h')} \partial_j w \partial_i (a_{ij}\varphi) dx - \int_{(x_N > h')} \partial_j w_\varepsilon \partial_i (a_{ij}\varphi) dx \right| \leqslant \delta(\varepsilon)$$

where $\varepsilon < h'$, $\delta(t) \to 0$ if $t \to 0_+$.

Next, we observe that

$$\int_{(x_N > h')} \partial_j w_\varepsilon \partial_i (a_{ij}\varphi) dx = - \int_{(x_N = h')} (\partial_j w_\varepsilon) a_{Nj} \, \varphi \, dx'$$

$$+ \int_{(x_N > h')} (A_\alpha w_\varepsilon) \, \varphi \, dx .$$

First, in view of a Lemma on the commutator of A_α and the convolution with ρ_ε in $W^{1,\infty}$ proved in P.L. Lions [24], we see that :

$$\int_{(x_N > h')} (A_\alpha w_\varepsilon)\varphi \, dx \leq \int_{(x_N > h')} f\varphi \, dx + \delta'(h',\varepsilon)$$

where $\delta'(h',t) \to 0$ if $t \to 0_+$.

In addition we have :

$$- \int_{(x_N=h')} \partial_j w_\varepsilon \, a_{Nj}\,\varphi \, dx' = -\int_{(x_N=h')} (\partial_N w_\varepsilon) a_{NN}\,\varphi \, dx' +$$

$$+ \sum_{j < N} \int_{(x_N=h')} w_\varepsilon \partial_j (a_{Nj}\varphi) dx'$$

$$\leq \gamma(h',\varepsilon) \to 0 \text{ if } \varepsilon, h' \to 0_+$$

since $w|_{x_N=0} = 0$ and $\partial_N w_\varepsilon(x',h') \geq 0$.

Therefore we proved :

$$<A_\alpha \tilde{w},\varphi> \leq \int_{(x_N > h')} f\varphi \, dx + \delta(\varepsilon) + \delta(h') + \delta'(h',\varepsilon) + \\ + \gamma(h',\varepsilon)$$

and we conclude letting $\varepsilon \to 0$, $h' \to 0$. ∎

<u>Remark II.8</u> : Of course if we know that $w \in C^2$ nearby Γ, the proof is greatly simplified since the extension of w is obvious. Remark that this is the case in Corollary II.1 and Remark II.7.

We now give a few results concerning (A.2) with $\Gamma_+ \neq \Gamma$. We will assume that $\Gamma = \Gamma_+ \cup \Gamma_-$ where Γ_+, Γ_- are closed, disjoint (Γ_+ may be empty).

<u>Proposition II.2</u> : *Let $w \in W^{1,\infty}(\mathcal{O})$ satisfy : $w = 0$ on Γ_+ and*

(13) $\quad A_\alpha w \leq f(\cdot,\alpha)$ *in* $\mathcal{D}'(\mathcal{O})$, *for all* $\alpha \in A$.

Let us assume in addition that we have :

(23) $\sigma \equiv 0$ *on* Γ_- ; *either* $b \equiv 0$ *on* Γ_- *or* $b(x,\alpha)\cdot n(x) < 0$ *on* Γ_- .

Then (A.2) *holds and for all* $x \in \bar{\mathcal{O}}$, N_t *is a continuous* F_t*-submartingale.*

<u>Remark II.9</u> : It is of course possible to give examples of situations where such a w exists : indeed results similar to those indicated in Corollary II.1, Remarks II.6.7 are easily adapted to the above situation (replace Γ by Γ_+).

<u>Proof</u> : We first show that N_t is a F_t-submartingale for each A and for each $x \in \mathcal{O} \cup \Gamma_+$. Indeed, exactly as in the proof of Proposition II.1, it is possible to extend w (and (13)) in a neighborhood of Γ_+. Then, denoting by $\mathcal{O}'_\delta = \{x \in \mathcal{O} \ , \ \mathrm{dist}(x,\Gamma_+) > \delta\}$ $(\mathcal{O}'_\delta \neq \emptyset$ for δ small enough), we apply the proof of Theorem I.2 to show that $N_{t\wedge\tau'_\delta}$ is a F_t-submartingale for $x \in \overline{\mathcal{O}'_\delta}$, where τ'_δ is the exit time of X_t from $\overline{\mathcal{O}'_\delta}$. We conclude by sending δ to 0 since in view of (23) : $\tau'_\delta \underset{\delta \to 0}{\to} \tau$ a.s..

Next, we need to show that, if $x \in \Gamma_-$, N_t is a F_t-submartingale. We will show it only in the case where $\sigma \equiv b \equiv 0$ on Γ_-. Then this claim is a consequence of the following inequality :

$$(24) \qquad w(x) \leqslant \inf_{\alpha \in A} \{f(x,\alpha)\ c(x,\alpha)^{-1}\} \quad , \quad \forall\ x \in \Gamma_- .$$

To prove (24), we fix $x_o \in \Gamma_-$ and consider : $z_n(x) = (1-n|x-x_o|)^+$. For n large enough, z_n vanishes in a neighborhood of Γ_+. Therefore multiplying (13) by z_n and integrating by parts we find :

$$\int_{\mathcal{O}} c(x,\alpha) w z_N \, dx \leqslant \int_{\mathcal{O}} f(x,\alpha) z_n \, dx + $$

$$+ C \int_{\mathcal{O}} \sup_{i,j} |a_{ij}(x,\alpha)| \, |\nabla z_n| \, dx$$

$$+ C \int_{\mathcal{O}} \sup_{i,j}\{|\partial_i a_{ij}(x,\alpha)|+|b_i(x,\alpha)|\} z_n \, dx.$$

Since $\mathcal{O}$ and thus Γ_- are smooth, we find that :

$$\left(\int_{\mathcal{O}} z_n dx\right) \left(\int_{R^N} z_n dx\right)^{-1} \underset{n}{\to} 1/2 \quad , \quad \int_{R^N} z_N dx = \frac{c_N}{n^N}$$

On the other hand, since $\sigma \equiv b \equiv 0$ on Γ_-, we deduce from the above inequality :

$$\left\{\int_{\mathcal{O}} cwz_n dx\right\} \left\{\int_{\mathcal{O}} z_n dx\right\}^{-1} \leqslant \left\{\int_{\mathcal{O}} fz_n dx\right\} \left\{\int_{\mathcal{O}} z_n dx\right\}^{-1} + \frac{C}{n}$$

If we let n go to $+\infty$, we finally obtain :

$$c(x_o,\alpha)\, w(x_o) \leqslant f(x_o,\alpha)$$

and (24) is proved. ∎

II.3 Continuity of u.

We have seen that under assumption (A.2) u is u.s.c. on $\overline{\mathcal{O}}$. We denote by $\Gamma_o = \{x \in \Gamma_+ \,/\, u(x) = 0\}$. Clearly if u is continuous on $\overline{\mathcal{O}}$, u is continuous on Γ_+ and Γ_o is closed. The next result shows that the converse is true :

Theorem II.2 : *Under assumption* (A.2), u *is continuous on* $\overline{\mathcal{O}}$ *if and only if* Γ_o *is closed.*

Corollary II.2 : *Under assumption* (A.2), *if* $u = 0$ *on* Γ_+ *then* u *is continuous on* $\overline{\mathcal{O}}$.

Indeed in this case $\Gamma_o = \Gamma_+$ and Γ_+ is closed.

Corollary II.3 : *If for all* A *and for all* $x \in \bar{\mathcal{O}}$, X_t *remains in* $\bar{\mathcal{O}}$ *for* $t \geq 0$ *then* u *is continuous on* $\bar{\mathcal{O}}$.

Indeed in this case (A.2) holds and $\Gamma_o = \Gamma_+ = \emptyset$.

Corollary II.4 : *If* (22) *holds then* $u = 0$ *on* Γ *and* $u \in C(\bar{\mathcal{O}})$.

Proof : Indeed, in the same way as in the proof of Corollary II.1, we find $w \in C^2(\overline{\mathcal{O}^{\delta_1}})$ for some $\delta_1 > 0$ satisfying :

$$A_\alpha w \geq f(x,\alpha) \quad \text{in } \mathcal{O}^{\delta_1} \quad , \quad w = 0 \text{ on } \Gamma, \ w \geq u \quad \text{if } d(x)=\delta_1 .$$

Denoting by τ' the first exit time of X_t from $\overline{\mathcal{O}^{\delta_1}}$, where X_t corresponds to the constant control $\alpha_t = \alpha_o \in A$, we deduce from Itô's formula :

$$\forall\, x \in \overline{\mathcal{O}^{\delta_1}} \ , \quad w(x) \geq E\left[\int_0^{\tau'} f(X_t,\alpha_t)e^{-\lambda t}dt + w(X_{\tau'})e^{-\lambda\tau'}\right]$$

$$\geq \inf_A E\left[\int_0^{\tau'} f(X_t,\alpha_t)e^{-\lambda t}dt + u(X_{\tau'})e^{-\lambda\tau'}\right]$$

$$= u(x)$$

and this shows : $u = 0$ on Γ. ∎

In the same way, one proves :

Corollary II.5 : *If* $\Gamma = \Gamma_+ \cup \Gamma_-$ *where* Γ_+, Γ_- *are closed, disjoint and if* (22) *holds on* Γ_+ *and* (23) *holds on* Γ_- , *then* $u \in C(\bar{\mathcal{O}})$.

More generally we have the following

Corollary II.6 : *Under assumption* (A.2) ; *if, for all* $x \in \Gamma_+$, *there exists* A *such that* $\tau = 0$ *a.s. then* $u \in C(\bar{\mathcal{O}})$.

Remark II.10 : This is the case for example if, for all $x \in \Gamma_+$, there exists $\alpha \in A$ such that

$$\theta\, a_{ij}(x,\alpha)n_i(x)n_j(x) + b_i(x,\alpha)n_i(x) - a_{ij}(x,\alpha)\partial_{ij}d(x) > 0$$

for some $\theta \geq 0$. ∎

Proof of Corollary II.6 : Indeed in this case we have for all $x \in \Gamma_+$:

$$0 \leq u(x) \leq J(x,A) = 0 \qquad \text{since } \tau = 0 \text{ a.s.}$$

Thus $u = 0$ on Γ_+ and $\Gamma_o = \Gamma_+$. ∎

Proof of Theorem II.2 : We first claim that if Γ_o is closed then for all $x \in \Gamma_o$ we have : $\lim_{y \in \bar{\mathcal{O}},\ y \to x} u(y) = 0$ uniformly in $x \in \Gamma_o$. Indeed if this were not the case, there would exist $y_n \in \bar{\mathcal{O}}$, $x_n \in \Gamma_o$ such that :

$$|y_n - x_n| \underset{n}{\to} 0 \quad , \quad |u(y_n)| \geq \delta > 0 .$$

But without loss of generality we may assume that $x_n \underset{n}{\to} x$; and since Γ_o is closed, $x \in \Gamma_o$. Then $y_n \underset{n}{\to} x$ and since u is u.s.c. we deduce : $\limsup_n u(y_n) \leq u(x) = 0$. On the other hand because of (A.2), we have :

$$0 = u(x) \geq \limsup_n u(y_n) \geq \liminf_n u(y_n) \geq \liminf_n w(y_n) \geq 0$$

and the contradiction proves our claim.

Next, let $x_n \underset{n}{\to} x$, $x_n \in \bar{\mathcal{O}}$: we denote, for each A, by $X_t^n = X_t^{x_n}$, $\tau_n = \tau(x_n)$. Let us recall that we know :

$$\sup_{[0,T]} |X_t^n - X_t| \underset{n}{\to} 0 \qquad \text{a.s. and in } L^p(\Omega) \quad (\forall\, p < \infty) .$$

Moreover :

$$\sup_{A} E\left[\sup_{[0,T]} |X_t^n - X_t|^p\right] \underset{n}{\to} 0 \qquad \forall\, T < \infty\,,\ \forall\, p < \infty\,.$$

Let $\theta = \tau_n \wedge \tau$. Applying Theorem B we find :

$$u(x_n) = \inf_{A} E\left\{\int_0^\theta f(X_t^n,\alpha_t)e^{-\lambda t}dt + u(X_\theta^n)e^{-\lambda\theta}\right\}$$

$$u(x) = \inf_{A} E\left\{\int_0^\theta f(X_t,\alpha_t)e^{-\lambda t}dt + u(X_\theta)e^{-\lambda\theta}\right\}$$

where, to simplify notations, we assume : $c \equiv \lambda$.

This yields for all $T < \infty$:

$$|u(x_n)-u(x)| \leqslant \sup_{A}\left\{E\int_0^{\theta\wedge T} C|X_t^n-X_t|e^{-\lambda t}dt + C\,e^{-\lambda T}\right\}$$
$$+ \sup_{A} |E\{(u(X_\theta^n)-u(X_\theta))\,e^{-\lambda\theta}\}| \quad .$$

To bound the last term, we argue as follows :

$$|E\{(u(X_\theta^n)-u(X_\theta))e^{-\lambda\theta}\}| \leqslant E\left\{1_{(\tau\leqslant\tau_n)}|u(X_\tau)-u(X_\tau^n)|e^{-\lambda\tau}\right\}$$
$$+ E\left\{1_{(\tau>\tau_n)}|u(X_{\tau_n})-u(X_{\tau_n}^n)|e^{-\lambda\tau_n}\right\}.$$

Next, in view of (A.2), $P(\tau<\infty\,,\ X_\tau\notin\Gamma_+) = P(\tau_n<\infty\,,\ X_{\tau_n}\notin\Gamma_+) = 0$. And by Theorem II.1, this implies : $P(\tau<\infty\,,\ X_\tau\notin\Gamma_o) = P(\tau_n<\infty\,,\ X_{\tau_n}\notin\Gamma_o) = 0$. Since X_t^n converges uniformly on $[0,T]$ to X_t a.s., we deduce from the remark made in the beginning of the proof :

$$\left|E\left\{(u(X_\theta^n)-u(X_\theta))e^{-\lambda\theta}\,1_{(\theta\leqslant T)}\right\}\right| \underset{n\to\infty}{\to} 0\,.$$

Since, on the other hand : $\left|E\left\{(u(X_\theta^n)-u(X_\theta))e^{-\lambda\theta}1_{(\theta>T)}\right\}\right| \leqslant C\,e^{-\lambda T}$;

we finally obtain :

$$\limsup_n |u(x_n) - u(x)| \leqslant C\, e^{-\lambda T} \quad , \quad \text{for all } T < \infty$$

and taking $T \to +\infty$, we conclude. ∎

Let us remark that the above proof actually shows the

Proposition II.3 : *Let ω be an open set included in $\mathcal{O}$ and let us assume that u is continuous on $\partial\omega$. Then u is continuous on $\bar{\omega}$.*

Remark II.11 : One needs only to assume that u is continuous on γ_+ closed subset of $\partial\omega$ such that : $P(\sigma < \infty\ ,\ X_\sigma \notin \gamma^+) = 0$ where σ is the exit time of X_t from $\bar{\omega}$. In particular if for all A and for all $x \in \bar{\omega}$, X_t remains in $\bar{\omega}$ for $t \geqslant 0$ then $u \in C(\bar{\omega})$ (indeed $\gamma^+ = \emptyset$).

Remark II.12 : This type of result is useful, when, for example, one knows that on a neighborhood I of $\partial\omega$ the matrices $a(x,\alpha)$ are uniformly definite positive. Then, as we will see in Part 3, this implies that $u \in C^2(I)$, therefore $u \in C(\partial\omega)$ and thus $u \in C(\bar{\omega})$.

We now conclude this section by a continuity result of a different type :

Proposition II.4 : *We assume that (A.2) holds and that there exist an open set ω included in $\mathcal{O}$, an integer $p \geqslant 1$, p elements $\alpha_1,\dots,\alpha_p$ of A such that :*

$$\sum_{i=1}^{p} a(x,\alpha_i) \geqslant \nu\, I_N \quad , \quad \forall\, x \in \omega$$

for some $\nu > 0$. Then u is continuous in measure on ω in the following sense : for every $x \in \omega$, we have :

$$\lim_{\varepsilon \to 0} \varepsilon^{-N} \operatorname{meas} \{y \in \omega \ /\ |u(y)-u(x)| > \delta\ ,\ |y-x| < \varepsilon\} = 0, \quad \forall\, \delta > 0 .$$

Remark II.13 : Let $x_o \in \omega$ and take $h_o > 0$ such that $B(x_o,h_o) \subset \omega$. We denote by $(P^i_{x_o}, X_t)$ the canonical diffusion processes associated with $a(x,\alpha_i)$, $b(x,\alpha_i)$. We already know that, if we denote by τ_o the first exit time of X_t from $B(x_o,h_o)$,

$$u(X_{t\wedge\tau_o})e^{-\lambda t\wedge\tau_o} + \int_0^{t\wedge\tau_o} f(X_s,\alpha_i)e^{-\lambda s}\,ds$$

is a $P^i_{x_o}$-submartingale. We then denote by (P_{x_o}, X_t) the canonical diffusion process associated with $\sum_{i=1}^{p} \frac{1}{p} a(x,\alpha_i)$, $\sum_{i=1}^{p} \frac{1}{p} b(x,\alpha_i)$. It is a straightforward exercize to check that, if we note $f(x) = \sum_{i=1}^{p} f(x,\alpha_i)$, we have :

$$M_t = u(X_{t\wedge\tau_o})e^{-\lambda t\wedge\tau_o} + \int_0^{t\wedge\tau_o} f(X_s)e^{-\lambda s}ds \quad \text{is a } P_{x_o}\text{-submartingale}$$

satisfying : $u(x_o) \leq E_{x_o}(M_\theta)$ for every bounded stopping time. Since u is u.s.c. on $\bar{\omega}$, we deduce easily that u is continuous at $t = 0_+$ along the trajectories of X_t. Now, since the diffusion matrix of X_t is non degenerate, the idea behind the above result is that the trajectories of X_t "fill little balls around x_o". ∎

Proof of Proposition II.4 : We keep the notations of the above remark. Let $\delta > 0$ and let $\varepsilon < h_o$. We denote by τ_ε the first exit time of X_t from $B(x_o,\varepsilon)$ and by θ the first hitting time of $\{y \in B(x_o,h_o) \,/\, |u(y) - u(x_o)| > \delta\}$. Then, we have :

$$E_{x_o} M_o = u(x_o) \leq E_{x_o}[M_{\theta\wedge\tau_\varepsilon}] =$$

$$= E_{x_o}\left[u(X_{\theta\wedge\tau_\varepsilon})e^{-\lambda\theta\wedge\tau_\varepsilon} + \int_o^{\theta\wedge\tau_\varepsilon} f(X_t)e^{-\lambda t}\,dt\right] .$$

Since u is u.s.c., we have ; $u(y) \leqslant u(x_o) + \delta(\varepsilon)$, if $|y-x_o| \leqslant \varepsilon$ where $\delta(\varepsilon) \to 0$ if $\varepsilon \to 0_+$. On the other hand, it is easy to check that $\tau_\varepsilon \downarrow 0$ if $\varepsilon \downarrow 0$. Therefore we find :

$$u(x_o) \leqslant \left(u(x_o)+\delta(\varepsilon)\right)P_{x_o}(\theta \geqslant \tau_\varepsilon) +$$

$$+ \{u(x_o)-\delta\}\, P_{x_o}[\theta < \tau_\varepsilon] + \gamma(\varepsilon)$$

where $\gamma(\varepsilon) \to 0$ if $\varepsilon \to 0_+$. This shows : $P_{x_o}[\theta < \tau_\varepsilon] \underset{\varepsilon \to 0}{\to} 0$.

We remark now that in view of fundamental results due to N.V. Krylov and M.V. Safonov [17], we have :

$$P_{x_o}(\theta < \tau_\varepsilon) \geqslant \nu\, \varepsilon^{-N} \text{meas}\,\{y \,/\, |u(y)-u(x_o)| > \delta,\ |y-x_o| < \varepsilon\};$$

and this concludes the proof. ∎

II.4 Hölder continuity of u :

Let us denote by λ_o the following quantity :

$$(24)\qquad \lambda_o = \sup_{\substack{x \neq x' \\ \alpha \in A}} \left\{\frac{1}{2}\,\mathrm{Tr}\left\{\frac{(\sigma(x,\alpha)-\sigma(x',\alpha))\cdot(\sigma^T(x,\alpha)-\sigma^T(x,\alpha'))}{|x-x'|^2}\right\} + \right.$$

$$\left. + \frac{(b(x,\alpha)-b(x',\alpha))\cdot(x-x')}{|x-x'|^2}\right\} ,$$

of course because of (A.1), $\lambda_o < \infty$.

Our main result concerning the Hölder continuity of u is the following :

Theorem II.3 : *We assume* (A.2) *and*

(25) $\exists\, C > 0\ ,\quad |u(x)| \leqslant C\ \mathrm{dist}(x,\Gamma_+) \qquad \forall\, x \in \bar{\mathcal{O}}\ ;\ u = 0$ on Γ_+.

Then $u \in C^{0,\alpha}(\bar{\mathcal{O}})$ *where* $\alpha = 1$ *if* $\lambda > \lambda_o$, α *is arbitrary in* $(0,1)$ *if* $\lambda = \lambda_o$ *and* $\alpha = \lambda/\lambda_o$ *if* $\lambda < \lambda_o$.

Remark II.14 : We could replace Γ_+ in (25) by Γ_o ; and replace $\mathrm{dist}(x,\Gamma_+)$ by $\mathrm{dist}(x,\Gamma_+)^\beta$ with $0 < \beta \leqslant 1$ then α has to be changed in $\alpha \wedge \beta$. ■

Remark II.15 : This type of result is similar to the one proved in P.L. Lions [18], [27]. Let us also mention that the exponent α is in general the best possible. Indeed take $\mathcal{O} = (-1,+1)$, $\sigma \equiv 0$, $b(x,\alpha) \equiv +x$, $f(x,\alpha) \equiv -\lambda$, $c(x,\alpha) \equiv \lambda \in (0,1)$. It is then easy to check that $u(x) = |x|^\lambda - 1$ in $[-1,+1]$. Observe that $\lambda_o = 1$ and thus $\alpha = \lambda$! ■

Remark II.16 : If $\Gamma_+ = \emptyset$, (25) is vacuous. ■

Examples : Let us mention a list of situations when the above result applies :

i) If $\Gamma_+ = \emptyset$ i.e. X_t remains in $\bar{\mathcal{O}}$ for $t \geqslant 0$ (for all A,x) then (25) is vacuous and (A.2) holds, therefore : $u \in C^{0,\alpha}(\bar{\mathcal{O}})$.

ii) If there exists $w \in W^{1,\infty}(\mathcal{O})$ satisfying (13) and $w = 0$ on Γ_+ ; and if, for example, there exists $\nu > 0$ such that for all $x \in \Gamma_+$, there exists α such that :

$$\theta\, a_{ij}(x,\alpha)n_i(x)n_j(x) + b_i(x,\alpha)n_i(x) - a_{ij}(x,\alpha)\partial_{ij}d(x) \geqslant \nu$$

for some $\theta \geqslant 0$; then (see Proposition II.1) (A.2) holds and :

$$u \geqslant w \geqslant -C\ \mathrm{dist}\,(x,\Gamma_+)$$

in addition, because of the assumption on Γ_+, it is easy to find A such that

$$u(x) \leqslant J(x,A) \leqslant C\,|x - x_o|$$

for all $x_o \in \Gamma$. This shows (25) and : $u \in C^{0,\alpha}(\bar{\mathcal{O}})$.

iii) In particular if (22) holds on Γ_+ and (23) holds on Γ_- then we may apply the above result. ∎

Remark II.17 : It is possible to extend the above result by replacing λ_o by $\lambda_o - \mu$ for all $\mu < \bar{\lambda}$, where $\bar{\lambda} = \sup\,\{\lambda \geqslant 0 \,/\, \sup\limits_{x,A} \; E(e^{\lambda\tau}) < \infty\,\}$. We will not consider here such extensions. ∎

Proof of Theorem II.3 : We first remark that in view of Itô's formula we obtain for any stopping time θ and for all A :

$$E\left[|X_\theta^x - X_\theta^{x'}|^2 \, e^{-2\lambda_o\theta}\right] \leqslant |x - x'|^2 \quad ; \quad \forall\, x,x' \in R^N .$$

Next, let $x,x' \in \bar{\mathcal{O}}$ and for each A denote by $\theta = \tau(x)\wedge\tau(x')$. We deduce from Theorem B the following identities :

$$u(x) = \inf_A E\left[\int_0^{\theta\wedge T} f(X_t^x,\alpha_t)e^{-\lambda t}dt + u(X_{\theta\wedge T}^x)e^{-\lambda\theta\wedge T}\right]$$

$$u(x') = \inf_A E\left[\int_0^{\theta\wedge T} f(X_t^{x'},\alpha_t)e^{-\lambda t}dt + u(X_{\theta\wedge T}^{x'})e^{-\lambda\theta\wedge T}\right]$$

where $T > 0$ is arbitrary and where, to simplify notations, we take $c \equiv \lambda$. This yields :

$$|u(x)-u(x')| \leqslant C\left(\int_0^T e^{(\lambda_o-\lambda)s}\,ds\right)|x-x'| + C\,e^{-\lambda T}$$

$$+ \sup_{A} E \left\{ 1_{(\theta < T)} e^{-\lambda\theta} |u(X_\theta^x) - u(X_\theta^{x'})| \right\} .$$

We first observe that either X_θ^x or $X_\theta^{x'} \in \Gamma_+$ (a.s.) and thus, using (25), we get : $|u(X_\theta^x) - u(X_\theta^{x'})| \leq C\ |X_\theta^x - X_\theta^{x'}|$ a.s.. If we go back to the above inequality, we finally deduce :

$$|u(x)-u(x')| \leq C \left(\int_0^T e^{(\lambda_o-\lambda)s} ds \right) |x-x'| + C\ e^{-\lambda T}$$
$$+ C\ E\ |X_{\theta\wedge T}^x - X_{\theta\wedge T}^{x'}|\ e^{-\lambda\theta\wedge T} .$$

Next, if $\lambda > \lambda_o$, we conclude easily sending $T \to +\infty$:

$$|u(x) - u(x')| \leq C\ |x - x'| .$$

On the other hand if $\lambda < \lambda_o$ (the case $\lambda = \lambda_o$ is treated by similar arguments) we obtain :

$$|u(x)-u(x')| \leq C\ e^{(\lambda_o-\lambda)T}\ |x-x'| + C\ e^{-\lambda T}$$
$$+ C\ E\ [\,|X_{\theta\wedge T}^x - X_{\theta\wedge T}^{x'}|\ e^{-\lambda\theta\wedge T}]$$

We observe now that we have by Itô's formula :

$$E\ [\,|X_{\theta\wedge T}^x - X_{\theta\wedge T}^{x'}|^2\ e^{-2\lambda\theta\wedge T}] \leq$$
$$\leq |x-x'|^2 + E \int_0^{\theta\wedge T} 2(\lambda_o-\lambda)\ |X_t^x - X_t^{x'}|^2\ e^{-2\lambda t}\ dt$$
$$\leq |x-x'|^2 + 2(\lambda_o-\lambda) \left(\int_0^T e^{2(\lambda_o-\lambda)t} dt \right) |x-x'|^2$$
$$\leq \{\exp\,[2(\lambda_o-\lambda)T\,]\}\ |x-x'|^2 .$$

Therefore we finally get :

$$|u(x)-u(x')| \leqslant C\, e^{(\lambda_o-\lambda)T}\, |x-x'| + C\, e^{-\lambda T}$$

where $T > 0$ is arbitrary. Taking the infimum over $T > 0$, we obtain by a straightforward computation :

$$|u(x)-u(x')| \leqslant C\, |x-x'|^{\alpha} \quad , \qquad \alpha = \lambda/\lambda_o \,. \qquad \blacksquare$$

Exactly as in the proof of Theorem II.2, we deduced the Hölder continuity of u in $\overline{\mathcal{O}}$ from that of u on Γ (or Γ_+) - assumption (25), we could give the analogue of Proposition II.3 : namely <u>if u is Hölder continuous on a neighborhood of $\partial\omega$</u>, where ω is a subdomain of $\mathcal{O}$, <u>then u is Hölder continuous on $\bar{\omega}$</u>. In particular <u>if on a neighborhood of $\partial\omega$, the matrices $a(x,\alpha)$ are uniformly definite positive</u> then u is smooth on this open set and thus $u \in C^{0,\alpha}(\bar{\omega})$. A result of the same type is given by the

<u>Proposition II.5</u> : *Let ω be an open set included in $\mathcal{O}$. We assume that on an open neighborhood I of $\partial\omega$, the following holds :*

$$(26) \quad \begin{cases} \exists \nu > 0 \,, \forall\, x \in I \,, \forall\, \xi \in R^N \,, \forall\, \alpha \in A_1 \subset A \,, \sigma \equiv 0 \,, \\ \qquad \sup_{\alpha \in A_1} (-b(x,\alpha)\cdot\xi) \geqslant \nu\, |\xi| \,. \end{cases}$$

Then $u \in W^{1,\infty}(I)$ and $u \in C^{0,\alpha}(\bar{\omega})$ where α is given as in Theorem II.3.

<u>Proof</u> : We already know that u satisfies :

$$(13) \quad A_{\alpha}u \leqslant f(\cdot,\alpha) \quad \text{in } \mathcal{D}'(\mathcal{O}) \quad , \quad \forall\, \alpha \in A \,.$$

Therefore this implies :

$$-b(x,\alpha)\cdot \nabla u \leqslant C \quad \text{in } \mathcal{D}'(I) \quad , \quad \forall\, \alpha \in A_1 \,.$$

and this implies because of (26) : $|\nabla u| \leqslant C$ in I ; that is : $u \in W^{1,\infty}(I)$. This derivation is correct if u is known to be Lipschitz ; but, in general, one regularizes u and one obtains, using a regularizing lemma in P.L. Lions [18] :

$$-b(x,\alpha) \cdot \nabla u_\varepsilon \leqslant C \quad \text{in } I_\varepsilon \quad , \quad \forall \alpha \in A_1$$

where $u_\varepsilon = u \star \rho_\varepsilon$; $\rho_\varepsilon = \frac{1}{\varepsilon^N} \rho\left(\frac{\cdot}{\varepsilon}\right)$ with $\rho \in \mathcal{D}_+(R^N)$, Supp $\rho \subset B_1$, $\int_{R^N} \rho \, dx = 1$; $I_\varepsilon = \{x \in I \text{ , dist } (x,\partial I) > \varepsilon\}$. And the above argument is justified.

Once we know that $u \in W^{1,\infty}(I)$, one obtains the Hölder continuity of u on $\bar{\omega}$ by the remark made before Proposition II.5.

∎

III. Variants and extensions.

III.1. Applications to other problems :

In this section, we want to mention briefly a few problems - related the one treated above - that can be treated by the same methods. We will not give any proof since the proofs are identical and we will only reproduce a few results (and not all) that we believe to be the most interesting ones.

First of all, let us mention that one could treat as well the case of unbounded domains $\mathcal{O}$ - and it is possible to admit unbounded coefficients satisfying for example :

$$
\text{(A.3)}\quad
\begin{cases}
\text{i) } \sigma, b \text{ have bounded first and second derivatives in } x, \\
\quad \text{uniformly in } \alpha \ ; \\
\text{ii) } \|\sigma(x,\alpha)\| + |b(x,\alpha)| \leq \varepsilon\,(1+|x|) + C_\varepsilon, \ \forall\, x \in R^N , \\
\qquad\qquad \forall\, \alpha \in A \ , \ \forall\, \varepsilon > 0 \ ; \\
\text{iii) } |D_x^\beta c| + |D_x^\beta f| \leq C(1+|x|^m) \ , \ \forall\, x \in R^N \ , \ \forall\, \alpha \in A \ , \\
\qquad\qquad \forall\, |\beta| = 1,2 \ ; \\
\text{iv) } b,\ \sigma,\ c,\ f \text{ are continuous with respect to } \alpha \in A \ .
\end{cases}
$$

(In the case of a finite horizon problem, ε can be taken arbitrary > 0).

A more interesting case is the case of coefficients bounded in x (with bounded derivatives in x) but unbounded in α : we assume for example :

$$
\text{(A.4)}\quad
\begin{cases}
\text{i) } \exists\, (A_m)_{m \geq 1} \text{ such that : } A_m \text{ is separable , } A_m \underset{m}{\uparrow} A \ , \\
\quad \text{if } K \text{ is compact } K \subset A_m \text{ for } m \text{ large} \ ; \\
\text{ii) } \sigma, b, c, f \text{ have bounded first and second derivatives in } x, \\
\quad \text{uniformly in } \alpha \ ; \\
\text{iii) } \sigma, b, c \text{ are bounded on } R^N \times A_m \ (\forall\, m \geq 1) \ ; \\
\quad f \text{ is bounded on } R^N \times A \ ; \\
\text{iv) } \sigma, b, c, f \text{ are continuous with respect to } \alpha \in A \ .
\end{cases}
$$

Then it is easy to see that most of the results above are easily adapted : indeed remark that if we denote by $u_m(x)$ the infimum over all admissible systems such that α_t takes its values in A_m, then clearly :

$$u_m(x) \underset{m \uparrow +\infty}{\downarrow} u(x) = \inf_{A} J(x,A) .$$

Let us only mention that u, formally, should satisfy instead of (2) :

$$(27) \qquad \sup_{\alpha \in A} \left\{ \frac{1}{p(x,\alpha)} (A_\alpha u(x) - f(x,\alpha)) \right\} = 0 \qquad \text{in } \mathcal{O}$$

where $p(x,\alpha) = \|a(x,\alpha)\| + |b(x,\alpha)| + c(x,\alpha)$ - see for precise results along this line N.V. Krylov [14], [12] and Part 3. We conjecture that, under (A.4), one has always : $\sup_{\alpha \in A} (p(\cdot,\alpha)^{-1} \mu_\alpha) = 0$; where $\mu_\alpha \in M(\mathcal{O})$ is given by : $\mu_\alpha = A_\alpha u - f(\cdot,\alpha)$.

Another possible extension is the case of control sets A depending on the state : assume that A depends on x - in a measurable way, and that there exists a Lipschitz selection $x \to \alpha(x) \in A(x)$ ... - . We will not develop it here.

Let us indicate that we can treat exactly in the same way the case when the cost function is given by :

$$J(x,A) = E \left\{ \int_0^\tau f(X_t,\alpha_t) \exp\left[-\int_0^t c(X_s,\alpha_s) ds \right] + \right.$$

$$\left. + \varphi(X_\tau) \exp\left[-\int_0^\tau c(X_t,\alpha_t) dt \right] \right\}$$

- the case treated above is the case $\varphi = 0$. We assume that $\varphi \in C(\Gamma)$. Then the results of section I are unchanged ; and so are those of sections II.1-2-3 provided one replaces zero boundary conditions (for example in Γ_+) by φ . Let us only mention the analogue of Theorem II.3 (recall that in (A.2) one replaces 0 by φ) :

Theorem III.1 : *Assume* (A.2) *and*

(25') $\exists C > 0\ ,\ \forall\, x \in \bar{\mathcal{O}}\ ,\ \forall\, y \in \Gamma_+\ ,\quad |u(x)-\varphi(y)| \leqslant C\ |x-y|$;

then $u \in C^{0,\alpha}(\bar{\mathcal{O}})$, *where* α *is given as in Theorem II.3.*

Remark III.1 : If $\varphi \in W^{2,\infty}(\Gamma)$, one checks (25') exactly as one checks (25).

Remark III.2 : If we assume, for example, that (22) holds on Γ_+ and (23) holds on Γ_- (Γ_+, Γ_- may be empty) then (A.2) holds for all $\varphi \in C(\Gamma)$: indeed (A.2) holds for $\varphi \in C^2(\Gamma)$ and we then argue by a density argument. And this density argument yields : $u \in C(\bar{\mathcal{O}})$. Moreover if $\varphi \in W^\beta(\Gamma)$ for $\beta \in (0,2\,]$ - where $W^\beta(\Gamma)$ means $C^{0,\beta}(\bar{\mathcal{O}})$ for $0 < \beta \leqslant 1$, $C^{1,\beta-1}(\bar{\mathcal{O}})$ for $\beta \in (1,2\,]$ - then $u \in C^{0,\gamma}(\bar{\mathcal{O}})$ with $\gamma = \alpha\beta(\beta+(2-\beta)\alpha)^{-1}$. Indeed denoting by $u = u(\varphi)$ the dependence in φ, we remark that we have by the proof of Theorem II.3 :

$$\|u(\varphi)\|_{C^{0,\alpha}(\bar{\mathcal{O}})} \leqslant C \left\{ 1 + \|\varphi\|_{W^{2,\infty}(\Gamma)} \right\}^\alpha .$$

Therefore, if $\varphi \in W^\beta(\Gamma)$, remarking that there exists, for all $\varepsilon > 0$, $\varphi_\varepsilon \in W^{2,\infty}(\Gamma)$ such that : $\|\varphi-\varphi_\varepsilon\|_{L^\infty(\Gamma)} \leqslant \varepsilon^\beta$, $\|\varphi_\varepsilon\|_{W^{2,\infty}(\Gamma)} \leqslant \frac{C}{\varepsilon^{2-\beta}} + C$; we deduce :

$$|u(x)-u(x')| \leqslant 2\|u-u_\varepsilon\|_{L^\infty(\mathcal{O})} + C|x-x'|^\alpha + \frac{C}{\varepsilon^{(2-\beta)\alpha}}\,|x-x'|^\alpha$$

- where $u=u(\varphi)$, $u_\varepsilon = u(\varphi_\varepsilon)$ -

$$\leqslant C|x-x'|^\alpha + C\left\{ \varepsilon^\beta + \frac{|x-x'|^\alpha}{\varepsilon^{(2-\beta)\alpha}} \right\} ;$$

and we conclude by minimizing over $\varepsilon > 0$.

Let us remark that if $\alpha = 1$ (i.e. $\lambda > \lambda_0$) then $\gamma = \beta/2$. ∎

We now turn to <u>the problem of continuous control with optimal stopping</u> : we keep the notations of section I and introduce a cost function depending on the initial point x, the admissible system A and a stopping time (adapted to A) θ :

$$J(x,A,\theta) = E\left\{\int_0^{\theta\wedge\tau} f(X_t,\alpha_t)\exp\left\{-\int_0^t c(X_s,\alpha_s)ds\right\} + \right.$$
$$\left. + \psi(X_\theta)\, 1_{(\theta<\tau)} \exp\left\{-\int_0^\theta c(X_t,\alpha_t)dt\right\}\right. ;$$

where $\psi \in C(\bar{\mathcal{O}})$. And we consider the optimal cost function :

$$u(x) = \inf_{A,\theta} J(x,A,\theta) .$$

In order to avoid the creation of artificial discontinuities, we will always assume :

(28) $\psi \geqslant 0$ on Γ ;

(we could assume (28) on Γ_+ only).

Then analogues of Theorems A-B, I.1 hold : in addition we have :

(29) $u \leqslant \psi$ in $\bar{\mathcal{O}}$.

We conjecture now that we have : $\sup\{\sup_\alpha \mu_\alpha , u-\psi\} = 0$ in $\mathcal{O}$. Theorem I.2 still holds : we now have to assume : $v \leqslant \psi$ in $\mathcal{O}$. In (A.2), we need to add that : $w \leqslant \psi$ in $\bar{\mathcal{O}}$; then Theorem II.1 holds. To check (A.2), the results and methods of section II.2 remain unchanged provided one assumes for example : $\psi(x) \geqslant - C\,\mathrm{dist}(x,\Gamma_+)$, $\forall x \in \bar{\mathcal{O}}$. Theorem II.2 and its applications remain unchanged : let us only remark that if (A.2) holds and $\psi = 0$ on Γ_+ , then $u = 0$ on Γ_+ and thus $u \in C(\bar{\mathcal{O}})$. Finally if $\psi \in W^{1,\infty}(\mathcal{O})$, the results of section II.4 are unchanged.

Remark III.3 : Let us remark that everything above remains true if we assume only ψ u.s.c. on $\bar{\mathcal{O}}$. In particular if (A.2) holds, ψ is u.s.c. and $\psi = 0$ on Γ_+, then $u \in C(\bar{\mathcal{O}})$. ■

We now conclude this section by considering the finite horizon problem or the control of time-dependent diffusion processes : let $T > 0$ be fixed (T is the horizon), we consider states given by :

$$dX_s = \sigma(X_s, t+s, \alpha_s)dB_s + b(X_s, t+s, \alpha_s)ds \quad , \quad X_o = x \in \bar{\mathcal{O}}$$

where $t \in [0,T]$ is fixed ; thus $X_s = X_s(x,t)$. We assume for example :

$$(\text{A.1}')\begin{cases} \sup\limits_{\alpha \in A\, ,\, t \in [0,T]} \|\varphi(\cdot,t,\alpha)\|_{W^{2,\infty}(R^N)} < \infty \; ; \\ \varphi(x,t,\alpha) \in C([0,T] \times A) \quad , \forall\, x \in R^N \\ \text{for all } \varphi = \sigma_{ij} \; (1 \leqslant i \leqslant N \; , \; 1 \leqslant j \leqslant m), = b_k \; (1 \leqslant k \leqslant N) \; , \\ \qquad = f, c \; . \end{cases}$$

An admissible system $\mathcal{A}$ is defined as before and for each $\mathcal{A}$ and for all $x,t \in \bar{\mathcal{Q}}$ - where $\mathcal{Q} = \mathcal{O} \times (0,T)$ - we introduce a cost function :

$$J(x,t,\mathcal{A}) = E\left\{\int_o^{T\wedge(T-t)} f(X_s,t+s,\alpha_s)\exp\left\{-\int_o^s c(X_\mu,t+\mu,\alpha_\mu)d\mu\right\} \right.$$
$$\left. + 1_{(T-t \leqslant \tau)} u_o(X_{T-t}) \exp\left\{-\int_o^{T-t} c(X_s,t+s,\alpha_s)ds\right\}\right\}$$

- where $\tau = \tau(x,t)$ is the first exit time of X_s from $\bar{\mathcal{O}}$ - ; and the optimal cost function is defined on $\bar{\mathcal{Q}}$ by :

$$u(x,t) = \inf_{\mathcal{A}} J(x,t,\mathcal{A}) \; .$$

To simplify we assume :

(30) $u_o \in C(\bar{\mathcal{O}})$, $u_o = 0$ on Γ .

Remark III.4 : We could consider more general cost functions, by the introduction of final costs when X_s exists from $\bar{\mathcal{O}}$ before T-t. ∎

Remark III.5 : It is possible to consider the above problem as a special case of those considered in sections I-II by the introduction of : $Y_s = t+s$. Indeed we have :

$$\begin{cases} dX_s = \sigma(X_s,Y_s,\alpha_s)dB_s + b(X_s,Y_s,\alpha_s)ds \quad , \quad X_o = x \in \bar{\mathcal{O}} \\ dY_s = 1\ ds \quad , \quad Y_o = t \in [0,T] \quad . \end{cases}$$

The state is now given by (X_s,Y_s) ; the domain is now $\mathcal{Q} = \mathcal{O} \times (0,T)$ and the first exit time of (X_s,Y_s) from $\bar{\mathcal{Q}}$ is nothing but : $\theta = \tau \wedge (T-t)$. Also remark that $P[(X_\theta,Y_\theta) \notin \partial_o\mathcal{Q}] = 0$, if $\partial_o\mathcal{Q} = (\Gamma \times [0,T]) \cup (\bar{\mathcal{O}} \times \{T\})$. ∎

Remark III.6 : It is clear that we have, if $u_o \in C^{0,\alpha}(\bar{\mathcal{O}})$ with $0 < \alpha \leq 1$:

$$|u(y,t)-u_o(x)| \leq C\,|T-t| + C\,\{|x-y|^\alpha + |T-t|^{\alpha/2}\} \; ; \quad \forall\, x,y \in \bar{\mathcal{O}} \, , \ \forall\, t \in [0,T] \; .$$

And if $u_o \in W^{2,\infty}(\mathcal{O})$, we obtain :

$$|u(y,t)-u_o(x)| \leq C\,|T-t| + C\,|x-y| \quad ; \quad \forall\, x,y \in \bar{\mathcal{O}} \, , \ \forall\, t \in [0,T] \; ;$$

and this gives an inequality of the form (25') at least for a part of $\partial_o\mathcal{Q}$ namely $\bar{\mathcal{O}} \times \{T\}$. ∎

Using Remarks III.5 and III.6, it is then very easy to adapt all the results of sections I-II to the above problem : we will not do so here. Let us only mention that when σ, b, f, c do not depend on t ; if we define for any u_o borel bounded on $\bar{\mathcal{O}}$ (for example) and for any $T > 0$:

$$\mathcal{Q}(T)u_o = u(\cdot,0)$$

then remarking that : $\mathcal{Q}(t)u_o = u(\cdot,T-t)$ for any $0 < t \leqslant T$; we deduce from the analogue of Theorem A that $\mathcal{Q}(t)$ is a semigroup. In the case when $\mathcal{O} = R^N$, this semigroup has been first considered by M. Nisio [36], [37], [38]. As a particular case of our results, let us mention that if (22) holds on Γ_+ while (23) holds on Γ_-, $\mathcal{Q}(t)$ is a strongly continuous semigroup on $X = \{u \in C(\bar{\mathcal{O}})\ ,\ u = 0 \text{ on } \Gamma_+\}$. Let us also mention that if (A.2) holds, then $\mathcal{Q}(t)$ is a semigroup on the cone $C = \{u$ upper semicontinuous on $\bar{\mathcal{O}}$; $u \geqslant w$ in $\bar{\mathcal{O}}$; $\forall A, \forall x$, $1_{(\tau < \infty)}\, u(X_\tau) = 0\}$.

<u>Remark III.7</u> : Let us finally point out that by the analogue of Theorem I.1, we know that

$$\mu_\alpha = -\frac{\partial u}{\partial t} - a_{ij}(x,t,\alpha)\partial_{ij}u - b_i(x,t,\alpha)\partial_i u + c(x,t,\alpha)u \in M(\mathcal{Q})$$

and we conjecture that : $\sup\limits_{\alpha \in A} \mu_\alpha = 0$. ∎

III.2 <u>Other boundary conditions.</u>

In the preceding sections, we have considered optimal control problems for processes which are stopped when they exit from a given domain. It is possible - and important for applications - to consider different boundary behaviors for the state processes. One case that we will not consider is the case of periodic boundary conditions, since this case is totally similar to the case when $\mathcal{O} = \mathbb{R}^N$ - at least if $\lambda > 0$, the ergodic case $\lambda = 0$ will be examined elsewhere.

We now indicate a few results concerning reflecting diffusion processes : suppose that a smooth vector-field γ (say of class C^3) is given on $\mathbb{R}^N$ such that :

(31) $\exists\, \nu > 0 \ , \ \forall\, x \in \Gamma \ , \quad \gamma(x) \cdot n(x) \geqslant \nu > 0 \ .$

We will consider state processes which, when they hit Γ, are reflected along the field γ : more precisely the state process X_t is given by the unique continuous process solution of :

$$\begin{cases} dX_t = \sigma(X_t,\alpha_t)dB_t + b(X_t,\alpha_t)dt - 1_{(X_t \in \Gamma)}\gamma(X_t)dA_t \ ; \\ X_t \in \overline{\mathcal{O}} \ , \ \forall\, t \geqslant 0 \ , \ \text{a.s.} \ ; \ X_0 = x \ ; \end{cases}$$

for some continuous nondecreasing process A_t such that $A_0 = 0$.

For the general solution of these equations, see Ikeda and Watanabe [9], D.W. Stroock and S.R.S. Varadhan [48], Tanaka [49], P.L. Lions and A.S. Sznitman [32]. We still assume (A.1) and (5).

The admissible systems are then defined as before and for each admissible system A, we define a cost function on $\overline{\mathcal{O}}$ by :

$$J(x,A) = E\int_0^\infty f(X_t,\alpha_t)\exp\left(-\int_0^t c(X_s,\alpha_s)ds\right)dt \ .$$

Finally, the optimal cost function u is given by :

$$u(x) = \inf_A J(x,A)$$

Then, formally, u should satisfy (2) and the boundary condition : $\frac{\partial u}{\partial \gamma} = 0$ on $\Gamma_o \subset \Gamma$.

Let us briefly mention two results analogous to those of section I - that we will not prove :

Theorem III.2 : *For each admissible system and for all* $x \in \bar{\mathcal{O}}$, *the process* $M_t = u(X_t) \exp\left(-\int_o^t c\right) + \int_o^t f \exp\left(-\int_o^s c\right) ds$ *is a continuous* F_t*-submartingale. In particular we have for any stopping times* $\theta = \theta(A)$ *and for any* $x \in \bar{\mathcal{O}}$

$$(10') \quad u(x) = \inf_A \left\{E \int_o^\theta f \exp\left(-\int_o^t c\right) dt + u(X_\theta) \exp\left(-\int_o^\theta c\right)\right\} .$$

Theorem III.3 : *We have :*

$$(13) \quad A_\alpha u \leq f(\cdot,\alpha) \quad \text{in } \mathcal{D}'(\mathcal{O}) \quad , \quad \forall \alpha \in A .$$

And for each bounded open set ω *such that* $\bar{\omega} \subset \mathcal{O}$, *there exists* $C > 0$ *such that* $a^{1/2}(\cdot,\alpha)\cdot\nabla u \in L^2(\omega)$ *and*

$$(14) \quad \sup_{\alpha \in A} \|a^{1/2}(\cdot,\alpha)\cdot\nabla u\|_{L^2(\omega)} \leq C .$$

Remark III.8 : It is also true that u is the maximum continuous function satisfying (13), (14) and having boundary conditions less than u. A more interesting result would be about subsolutions v satisfying boundary conditions of the type : $\frac{\partial v}{\partial \gamma} \leq 0$ on Γ_+ where $P[\tau < \infty, X_\tau \notin \Gamma_+] = 0$, unfortunately, the very writing of this boundary condition requires some smoothness which is not, in general, available. Nevertheless it is true that u is the supremum of all subsolutions satisfying (13), (14) and such that : v is smooth near Γ_+ and $\frac{\partial v}{\partial \gamma} \leq 0$ on Γ_+. ■

A more surprising result is the

Theorem III.4 : *The optimal cost function* u *belongs to* $C^{0,\alpha}(\bar{\mathcal{O}})$ *where* $0 < \alpha \leq 1$ *is given by* : $\alpha = 1$ *if* $\lambda > \bar{\lambda}_o$, α *is arbitrary in* (0,1) *if* $\lambda = \bar{\lambda}_o$ *and* $\alpha = \lambda/\bar{\lambda}_o$ *if* $\lambda < \bar{\lambda}_o$; *where* $\bar{\lambda}_o$ *depends only on* $\mathcal{O}$, γ, σ, b.

Remark III.9 : We will indicate a few precisions on $\bar{\lambda}_o$ in the proof below. ■

Remark III.10 : Similar results hold for time-dependent problems : in this case, u(x,t) is always Lipschitz in x (uniformly for $t \leq T$); and also for optimal stopping problems. In particular, in this case, this extends results obtained by J.L. Menaldi [33] where $\mathcal{O}$ is assumed to be convex and $\gamma(x) = n(x)$. ■

Remark III.11 : One can prove in addition that one may restrict the infimum to admissible systems where the probability space and the Brownian motion are fixed, and where the control process is a step-process, or a continuous process... ■

Remark III.12 : It is also possible to treat the case of boundary conditions of the form : $\frac{\partial u}{\partial \gamma} + \beta(x)u = 0$ on Γ ; but we will not do so here since the results are totally identical. ■

Proof of Theorem III.4 in the case when $\gamma \equiv n$: We first prove Theorem III.4 in the particular case when $\gamma(x) = n(x)$ on Γ (normal reflection) : this case being easier will shed some light on the general case.

The key point is the following geometric :

Lemma III.1 : *There exists* $C_o \geq 0$ *such that* :

$$\text{(32)} \quad \frac{C_o}{2} |x-y|^2 + (n(x), x-y) \geq 0 \qquad \forall\, x \in \Gamma \;,\; \forall\, y \in \bar{\mathcal{O}} \;.$$

Remark III.13 : It is possible to take $C_o = R^{-1}$ where R is the radius of the ball $B(y_o, R)$ such that $y = y_o(x)$, $\overline{B(y_o,R)} \cap \bar{\mathcal{O}} = \{x\}$ for $x \in \Gamma$. ■

Remark III.14 : If $\mathcal{O}$ is convex then clearly $C_o = 0$; and as we will see below we may take $\bar{\lambda}_o = \lambda_o$. ■

Proof of Lemma III.1 : Take C_o as in the Remark III.13. Then if $y \in \bar{\mathcal{O}}$, $|y-y_o|^2 \geq R^2$, $|x-y_o|^2 = R^2$ and $n(x) = \frac{1}{R}(y_o - x)$.
Therefore :

$$\begin{aligned}\frac{1}{2R} |y-x|^2 + (n(x), x-y) &= \frac{1}{2R} |y-y_o|^2 + \frac{1}{R}(y-y_o, y_o-x) + \\ &\quad + \frac{R}{2} + (n(x), x-y_o) + (n(x), y_o-y) \\ &\geq 0 \;. \end{aligned}$$ ■

Denoting by $A_t^x = \int_o^t 1_{(X_t^x \in \Gamma)}\, dA_t$, we now claim that we have :

$$\text{(33)} \quad E\left\{|X_t^x - X_t^{x'}|\right\} \leq e^{\lambda_o t} |x-x'| \left| E\,[e^{2C_o A_t^x}] \cdot E\,[e^{2C_o A_t^{x'}}]\right|^{1/4} .$$

Indeed from Itô's formula, we obtain :

$$E\left\{|X_t^x - X_t^{x'}|^2 \exp\{-2\lambda_o t - C_o(A_t^x + A_t^{x'})\}\right\} \leq |x-x'|^2 +$$

$$- E \int_o^t C_o \, |X_s^x - X_s^{x'}|^2 \, e^{-2\lambda_o s} \left(dA_s^x + dA_s^{x'}\right) +$$

$$- E \int_o^t 2 \left(X_s^x - X_s^{x'} , \, 1_{(X_s^x \in \Gamma)} n(X_s^x) dA_s^x - 1_{(X_s^{x'} \in \Gamma)} n(X_s^{x'}) dA_s^{x'}\right) ;$$

and using Lemma III.1, this yields :

$$E \left[|X_t^x - X_t^{x'}|^2 \exp \{- 2\lambda_o t - C_o (A_t^x + A_t^{x'})\} \right] \leq |x-x'|^2 .$$

Hence, using Cauchy-Schwarz inequality :

$$E \left[|X_t^x - X_t^{x'}| \right] \leq |x-x'| \, e^{\lambda_o t} \left\{ E \left[\exp (C_o A_t^x + C_o A_t^{x'}) \right] \right\}^{1/2}$$

and we obtain (33).

We need now to estimate $E\left[\exp\left(2C_o A_t^x\right)\right]$. To this end we introduce $\tilde{d} \in C^2(\bar{\mathcal{O}})$ such that $\tilde{d} = d$ in a neighborhood of Γ , $|\nabla \tilde{d}| \leq 1$ in $\bar{\mathcal{O}}$. Applying Itô's formula, we find :

$$\tilde{d}(X_t) = \tilde{d}(x) + \int_o^t \nabla \tilde{d}(X_s) \cdot \sigma(X_s, \alpha_s) \cdot dB_s + \int_o^t a_{ij}(X_s, \alpha_s) \partial_{ij} \tilde{d}(X_s)$$
$$+ b_i(X_s, \alpha_s) \partial_i \tilde{d}(X_s) ds - \int_o^t \nabla \tilde{d}(X_s) \cdot n(X_s) 1_{(X_s \in \Gamma)} dA_s \quad ;$$

and we deduce , since $\nabla \tilde{d} = \nabla d = -n$ on Γ :

$$A_t \leq C + C_1 t - \int_o^t \nabla \tilde{d}(X_s) \cdot \sigma(X_s, \alpha_s) dB_s$$

where $C_1 = \sup_{x,\alpha} \left[- a_{ij}(x,\alpha) \partial_{ij} \tilde{d} - b_i(x,\alpha) \partial_i \tilde{d} \right]$.

Denoting by M_t the martingale given by : $-\int_o^t \nabla \tilde{d}(X_s) \cdot \sigma(X_s, \alpha_s) dB_s$ we find : $E \left[\exp \left\{ 2C_o M_t - 2C_o^2 \int_o^t |\sigma^T(X_s, \alpha_s) \cdot \nabla \tilde{d}(X_s)|^2 ds \right\} \right] = 1$.

This yields :

$$E\,[e^{2C_oA_t}] \leqslant e^{2K_ot}$$

with $K_o = C_oC_1 + C_o^2 \sup\limits_{x,\alpha} |\sigma^T(x,\alpha)\cdot\nabla\tilde{d}(x)|^2$.

Therefore we have :

$$E\,[\,|X_t^x - X_t^{x'}|\,] \leqslant e^{\bar{\lambda}_o t}\, |x-x'|$$

where $\bar{\lambda}_o = \lambda_o + K_o$. The remainder of the proof is now identical to the proof of Theorem II.3.

<u>Proof of Theorem III.4 in the general case</u> : The main new point lies in the convenient generalization of Lemma III.1. This is done in the

<u>Lemma III.2</u> : *There exist* $\alpha_{ij} = \alpha_{ji}$ $(1 \leqslant i,j \leqslant N)$ *of class* C^2 *on* $\bar{\mathcal{O}}$ *depending only on* $\mathcal{O}$ *and* γ *such that :*

(34) $\quad (\alpha_{ij}(x)) \geqslant I_N$

(35) $\quad \frac{C_o}{2}\ (\alpha(x)\cdot(x-y),x-y) + (\gamma(x),\alpha(x)\cdot(x-y)) \geqslant 0 \qquad \forall\, x \in \Gamma\ ,\ \forall\, y \in \bar{\mathcal{O}}$.

for some $C_o \geqslant 0$ *depending only on* $\mathcal{O}$ *and* γ ,

<u>Proof</u> : The proof is long but elementary. We first observe the easy - but tedious to prove - fact that there exists $\beta_{ij} = \beta_{ji}$ of class C^2 satisfying $\beta(x) \geqslant \nu\, I_N$ in $\bar{\mathcal{O}}$ and $\beta(x)\cdot\gamma(x) = n(x)$, $\forall\, x \in \Gamma$. This is mainly due to (31). Therefore if we choose $\alpha = \lambda\,\beta^{-1}$ with λ large enough to insure (34), we find using Lemma III.1 :

$$\frac{C_o\lambda}{2}|x-y|^2 + (\gamma(x),\alpha(x)\cdot(x-y)) = \lambda\left[\frac{C_o}{2}|x-y|^2 + (n(x),x-y)\right]$$

$$\geqslant 0$$

and we conclude since we have :

$$\frac{C_o\lambda}{2}(\alpha(x)\cdot(x-y),x-y) + (\gamma(x),\alpha(x)\cdot(x-y)) \geqslant 0 ,$$

$$\forall x \in \Gamma , \forall y \in \bar{\mathcal{O}} . \blacksquare$$

We may now adapt the above proof : indeed, applying Itô's formula we find :

$$E\left\{\left[\left(\alpha(X_t^x)+\alpha(X_t^{x'})\cdot(X_t^x-X_t^{x'})\right)\right] \cdot [X_t^x-X_t^{x'}]\exp\{-2\lambda_1 t-C_1(A_t^x+A_t^{x'})\}\right\}$$

$$\leqslant \left((\alpha(x)+\alpha(x'))\cdot(x-x') , x-x'\right) ;$$

provided λ_1 is large enough depending on σ, b, α ; and C_1 is large enough depending on α, C_o.

It is now straightforward to adapt the preceding proof. $\blacksquare$

III.3 Non-positive discount factor.

In sections I-II, we always assumed (A.1) and (5). As we already said, it is possible to relax (A.1) but we will not do so here. Concerning (5), let us mention that it is possible to treat the case when $\lambda = 0$ or even $\lambda < 0$ provided the following holds :

$$(36)\quad \begin{cases} \sup\limits_{A,x\in\bar{\mathcal{O}}} E[\tau] < \infty & \text{if } \lambda = 0 ; \\ \sup\limits_{A,x\in\bar{\mathcal{O}}} E[e^{\mu\tau}] < \infty & \text{if } \lambda < 0 ; \end{cases}$$

where $\mu = -\lambda$. In what follows, $\mu \geqslant 0$ (and we take $c(x,\alpha) \equiv 0$).

A simple criterion to insure (36) is the following :

Proposition III.1 : *If there exists* $w \in C^2(\bar{\mathcal{O}})$ *satisfying :*

(37) $A_\alpha w - \mu w \geqslant \beta \geqslant 0$ *in* $\mathcal{O}$, $\forall\, \alpha \in A$; $w \geqslant \gamma \geqslant 0$ *on* Γ ;

$$w \geqslant 0 \textit{ in } \bar{\mathcal{O}} \;;$$

and if β *or* γ *is strictly positive ; then* (36) *holds.* ■

Once (36) holds, the results of sections I, II.1-2 are easily adapted.

Remark III.15 : It is possible to relax considerably the smoothness assumption on w. ■

Remark III.16 : If we have : $f(x,\alpha) \geqslant 0$ in $\bar{\mathcal{O}} \times A$, then (36) is not necessary in order to preserve the results of sections I , II.1-2 : one may assume instead of (36) :

$$(38) \quad \begin{cases} \sup\limits_{x \in \bar{\mathcal{O}}} \ \inf\limits_{A} \ E\,[\tau\,] < \infty & \text{if} \quad \lambda = 0 \quad ; \\ \sup\limits_{x \in \bar{\mathcal{O}}} \ \inf\limits_{A} \ E\,[e^{\mu\tau}\,] < \infty & \text{if} \quad \lambda < 0 \quad . \end{cases}$$

Then (38) holds as soon as, for example, there exists $w \in C^2(\bar{\mathcal{O}})$, $w \geqslant 0$ in $\bar{\mathcal{O}}$ satisfying : $\sup\limits_{\alpha \in A} [A_\alpha w - f(x,\alpha)] \geqslant \beta \geqslant 0$ in $\bar{\mathcal{O}}$, $w \geqslant \gamma \geqslant 0$ on Γ and either β or γ is strictly positive. ■

Remark III.17 : In P.L. Lions [28], the question whether (36) or (38) holds is completely solved in the case when all matrices $a(x,\alpha)$ are uniformly definite positive. ■

Proof of Proposition III.1 : Applying Itô's formula, one finds for each $\mathcal{A}$ and for each $x \in \bar{\mathcal{O}}$:

$$w(x) \geqslant E\,[w(X_{t\wedge\tau})\, e^{\mu t\wedge\tau}] + \beta\, E\left[\frac{e^{\mu t\wedge\tau}-1}{\mu}\right] .$$

If $\beta > 0$, then we deduce : $\frac{1}{\beta}\,w(x) \geqslant E\left[\frac{e^{\mu t\wedge\tau}-1}{\mu}\right]$ and sending t to $+\infty$, we conclude.

If $\gamma > 0$, then we deduce : $w(x) \geqslant E\,[w(X_{t\wedge\tau})\, e^{\mu t\wedge\tau}]$ and sending t to $+\infty$, we deduce from Fatou's Lemma :

$$\gamma\, E\,[e^{\mu\tau}] \leqslant E\,[w(X_\tau)\, e^{\mu\tau}] \leqslant w(x)$$

and we conclude. ∎

<u>Example III.1</u> : If $b \equiv 0$, and if there exists $\nu > 0$ such that $Tr(a_{ij}(x,\alpha)) \geqslant \nu > 0$ on $R^N \times A$; then

$$\sup_{x \in \bar{\mathcal{O}}} \sup_{A} E\,[e^{\mu\tau}] < \infty \quad , \quad \text{for some } \mu > 0 .$$

Indeed take $w(x) = R^2 - |x|^2$, where R is such that $\bar{\mathcal{O}} \subset B(0,R)$.
Clearly :

$$A_\alpha w = -a_{ij}(x,\alpha)\partial_{ij} w = 2\, tr(a_{ij}(x,\alpha)) \geqslant \frac{2\nu}{R^2}\, w$$

and $w > 0$ on Γ . therefore we can take $\mu = \frac{2\nu}{R^2}$.

<u>Example III.2</u> : If there exist $\nu > 0$, $k \in R$ such that :

$$\varepsilon\,[k^2\, a_{11}(x,\alpha) + k\, b_1(x,\alpha)] \geqslant \nu > 0 \ , \ \forall\, x \in \bar{\mathcal{O}} \ , \ \forall\, \alpha \in A$$

where $\varepsilon = \pm 1$, then there exists $\mu > 0$ such that

$$\sup_{x \in \bar{\mathcal{O}}} \sup_{A} E\,[e^{\mu\tau}] < \infty .$$

Indeed, if the above condition holds with $\varepsilon = 1$, $k > 0$, we consider $w(x) = 1 - e^{k(x_1 - L)}$ where L is large enough so that : $x_1 < L$ in $\overline{\mathcal{O}}$; and we obtain :

$$\begin{cases} A^{\alpha}w = \{k^2 a_{11}(x,\alpha) + k\, b_1(x,\alpha)\}\, e^{k(x_1 - L)} \geqslant \mu w \text{ for some } \mu > 0 \\ w > 0 \quad \text{on } \Gamma . \end{cases}$$

Appendix 1 : Regularization and degenerate elliptic operators.

Let $b \in W^{1,\infty}(R^N;R^N)$ and let $a = \sigma\sigma^T$ where $\sigma_{ij} \in W^{2,\infty}(R^N)$ $(1 \leqslant i \leqslant N\ ,\ 1 \leqslant j \leqslant m)$. If $u \in L^2(R^N)$, we denote by $u_\varepsilon = u \star \rho_\varepsilon$ where $\rho \in \mathcal{D}_+(R^N)$, $\int_{R^N} \rho\, dx = 1$, $\rho_\varepsilon = \frac{1}{\varepsilon^N}\, \rho\left(\frac{\cdot}{\varepsilon}\right)$.

Theorem :

i) *If* $u \in L^2(R^N)$, *then we have* :

$$(39) \qquad |b_i \partial_i u_\varepsilon - (b_i \partial_i u) \star \rho_\varepsilon|_{L^2(R^N)} \leqslant C\, |u|_{L^2(R^N)}$$

where C *depends only on* $\|b\|_{W^{1,\infty}}$.

In addition we have for all k :

$$(40) \qquad |\sigma_{ik}\partial_i u_\varepsilon - (\sigma_{ik}\partial_i u) \star \rho_\varepsilon|_{L^2(R^N)} \to 0 \quad , \quad \text{as} \quad \varepsilon \to 0_+ \ ;$$

in particular if $\sigma^T \cdot \nabla u \in L^2(R^N)$ *we obtain* :

$$(41) \qquad \sigma^T \cdot \nabla u \underset{\varepsilon}{\to} \sigma^T \cdot \nabla u \qquad \text{in} \quad L^2(R^N) \ .$$

ii) *If* $u \in L^2(R^N)$ *and* $\sigma^T \cdot \nabla u \in L^2(R^N)$ *then we have* :

$$Au_\varepsilon - (Au) \star \rho_\varepsilon = \psi_\varepsilon + \sigma_{ik}\partial_i \varphi_\varepsilon^k$$

where $\psi_\varepsilon \underset{\varepsilon}{\to} 0$ *weakly in* $L^2(R^N)$, $\varphi_\varepsilon^k \underset{\varepsilon}{\to} 0$ *in* $L^2(R^N)$ $(\forall\ k)$ *and* $A = - a_{ij}\, \partial_{ij} - b_i\, \partial_i$.

Remarks :

1) Similar results holds if $L^2(R^N)$ is replaced by $L^p(R^N)$ where $1 \leq p < \infty$.

2) Recall that $a^{1/2} \cdot \nabla u \in L^2(R^N)$ is equivalent to $\sigma^T \cdot \nabla u \in L^2(R^N)$ since we have : $\left(a_{ij}^{1/2} \xi_j\right)^2 = a_{ij} \xi_i \xi_j = \sigma_{ik} \sigma_{jk} \xi_i \xi_j = = |\sigma^T \cdot \xi|^2$.

3) Of course (40) holds if $\sigma_{ik} \in C_b^1(R^N)$ and $D\sigma_{ik} \in B \cup C(R^N)$.

4) In many cases (for example a nondegenerate), one may take $\varphi_\varepsilon^k = 0$.

Proof of i) : It is enough to prove (39) for $u \in \mathcal{D}(R^N)$. Then for all $x \in R^N$, we have :

$$(b \cdot \nabla u_\varepsilon)(x) - (b \cdot \nabla u) \star \rho_\varepsilon (x) =$$

$$= \int_{R^N} b_i(x) u(y) \partial_i \rho_\varepsilon(x-y) dy - \int_{R^N} b_i(y) \ \partial_i u(y) \rho_\varepsilon(x-y) dy$$

$$= \int_{R^N} (b_i(x) - b_i(y)) u(y) \partial_i \rho_\varepsilon(x-y) dy +$$

$$+ \int_{R^N} \partial_i b_i(y) u(y) \rho_\varepsilon(x-y) \ dy .$$

And thus we obtain :

$$|b \cdot \nabla u_\varepsilon - (b \cdot \nabla u) \star \rho_\varepsilon|_{L^2(R^N)} \leq$$

$$\leq |\{(\partial_i b_i) u\} \star \rho_\varepsilon|_{L^2(R^N)} + C \ |u \star \theta_\varepsilon|_{L^2(R^N)}$$

where $\theta_\varepsilon(x) = \frac{1}{\varepsilon^N} \theta \left(\frac{x}{\varepsilon}\right)$ and $\theta(x) = |x| \ |\nabla \rho(x)|$.

We now conclude easily in view of well-known results on convolution.

To prove (40), we remark that by the above proof we have :

$$\sigma_{ik}\partial_i u_\varepsilon - (\sigma_{ik}\partial_i u) \star \rho_\varepsilon = \{(\partial_i\sigma_{ik})u\} \star \rho_\varepsilon +$$

$$+ \int_{R^N} \left(\sigma_{ik}(x)-\sigma_{ik}(y)\right) u(y)\partial_i\rho_\varepsilon(x-y)\, dy \;.$$

And the first term converges in $L^2(R^N)$ towards $(\partial_i\sigma_{ik})(x)\, u(x)$. On the other hand since we have :

$$\sigma_{ik}(x) = \sigma_{ik}(y) + (x_\ell - y_\ell)\, \partial_\ell\, \sigma_{ik}(y) + 0(|x-y|^2)$$

we conclude that the second term, up to a function converging in L^2 to 0 , is equal to

$$\int_{R^N} (x_\ell - y_\ell)\, \partial_\ell\, \sigma_{ik}(y)\, u(y)\, \partial_i\, \rho_\varepsilon(x-y)\, dy$$

and thus converges in $L^2(R^N)$ to $\partial_\ell\, \sigma_{ik}(x)u(x)\, \varepsilon_{i\ell}$, where $\varepsilon_{i\ell}$ is given by : $\varepsilon_{i\ell} = \int_{R^N} x_\ell\, \partial_i\rho(x)\, dx = -\,\delta_{i\ell}$; and (40) is proved. Finally (41) is immediately deduced from (40).

<u>Proof of ii)</u> : We write :

$$\begin{aligned} Au_\varepsilon - (Au) \star \rho_\varepsilon &= -\,\sigma_{ik}\, \partial_i(\sigma_{jk}\partial_j u_\varepsilon) + \\ &+ \sigma_{ik}\, \partial_i\{(\sigma_{jk}\partial_j u) \star \rho_\varepsilon\} - \sigma_{ik}\, \partial_i\{(\sigma_{jk}\partial_j u) \star \rho_\varepsilon\} + \\ &+ \{\sigma_{ik}\, \partial_i(\sigma_{jk}\partial_j u)\} \star \rho_\varepsilon + \Big\{(\sigma_{ik}\, \partial_i\sigma_{jk} - b_j)\, \partial_j u_\varepsilon + \\ &- [\,(\sigma_{ik}\, \partial_i\, \sigma_{jk} - b_j)\partial_j u] \star \rho_\varepsilon\Big\} . \end{aligned}$$

In view of (39), the last difference - that we denote by φ_ε - converges weakly in $L^2(R^N)$ to 0 ; while in view of (40) the first difference may be written : $\sigma_{ik}\ \partial_i\ \varphi_\varepsilon^k$, where $\varphi_\varepsilon^k = -\sigma_{jk}\partial_j u_\varepsilon + (\sigma_{jk}\partial_j u) \star \rho_\varepsilon$ converges in $L^2(R^N)$ to 0 ($\forall$ k). Finally since $\sigma_{jk}\ \partial_j u \in L^2(R^N)$ ($\forall$ k), the second difference converges in $L^2(R^N)$ to 0 in view of (40). And we conclude. ∎

Appendix 2 : On the Cauchy problem for degenerate parabolic equations.

Let A be a second-order elliptic operator with smooth coefficients :

$$A = - a_{ij}\ \partial_{ij} - b_i\ \partial_i + c$$

where $a = \frac{1}{2}\sigma\sigma^T$, and σ_{ij}, b_k, $c \in W^{2,\infty}(R^N)$. Let $f \in B \cup C(R^N) \cap L^2(R^N)$.

If (Ω,F,F_t,P,B_t) is the canonical Wiener space, we consider $X_t = X_t^x = X(x,t)$ the continuous process unique solution of :

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt \quad , \quad X_o = x \ .$$

In the following result, essentially due to O. Oleinik [39], we consider the following Cauchy problem :

$$(C)\quad \begin{cases} \frac{\partial u}{\partial t} + Au = f & \text{in } R^N \times (0,T) \\ u|_{t=0} = u_o & \text{in } R^N \end{cases}$$

where $T > 0$ is fixed.

Theorem :

i) *If* $u_o \in L^2(R^N)$, *there exists a unique solution* u *of* (C) *in the space* $X = \{v \in C([0,T] ; L^2(R^N)) , a^{1/2} \nabla v \in L^2(R^N \times (0,T))\}$.

ii) *If* $u_o \in B \cup C(R^N)$ *then* $u \in B \cup C(R^N \times [0,T])$ *and we have :*

$$u(x,t) = E\left[\int_0^t f(X_s)\exp\left(-\int_0^s c(X_\mu)d\mu\right) + u_o(X_t)\exp\left(-\int_0^t c(X_s)ds\right)\right].$$

iii) *If* $u_o \in W^{2,\infty}(R^N)$ *then* $u \in W^{2,1,\infty}(R^N \times (0,T))$.

<u>"Proof"</u> : We will only explain why, if u,v lie in X and solve (C) for different initial conditions (u_o,v_o), we have :

$$\sup_{[0,T]} |u(s)-v(s)|_{L^2(R^N)} \leq C_T |u_o-v_o|_{L^2(R^N)} .$$

By the proof of Appendix 1, we may regularize u,v and find $u_\varepsilon,v_\varepsilon \in C^\infty(R^N \times [0,T])$ satisfying :

$$\begin{cases} D_x^\alpha D_t^\beta u_\varepsilon, D_x^\alpha D_t^\beta v_\varepsilon \in L^2(R^N \times (0,T)) \ (\forall \alpha,\beta) ; \\ \qquad u_\varepsilon, v_\varepsilon \underset{\varepsilon \to 0}{\to} u,v \ \text{in } X ; \\ \dfrac{\partial u_\varepsilon}{\partial t} + Au_\varepsilon = f + \psi_\varepsilon + \sigma_{ik}\, \partial_i \varphi_\varepsilon^k ; \\ \dfrac{\partial v_\varepsilon}{\partial t} + Av_\varepsilon = f + \tilde{\psi}_\varepsilon + \sigma_{ik}\, \partial_i \tilde{\varphi}_\varepsilon^k ; \\ \psi_\varepsilon, \tilde{\psi}_\varepsilon \underset{\varepsilon}{\to} 0 \text{ weakly in } L^2(R^N \times (0,T)) , \ \varphi_\varepsilon^k \underset{\varepsilon}{\to} 0 \text{ in } L^2(R^N \times (0,T)) . \end{cases}$$

Now, if we multiply the equations by $u_\varepsilon - v_\varepsilon$ and integrate by parts over $R^N \times (0,t)$ we find

$$\frac{1}{2}|u_\varepsilon(t)-v_\varepsilon(t)|^2_{L^2} - \frac{1}{2}|u_\varepsilon(0)-v_\varepsilon(0)|^2_{L^2} +$$

$$+ \int_0^t \int_{R^N} a_{ij}\partial_i(u_\varepsilon - v_\varepsilon)\partial_j(u_\varepsilon - v_\varepsilon)\ dx\ ds$$

$$\leqslant C \int_0^t |u_\varepsilon(s)-v_\varepsilon(s)|^2_{L^2} ds + \int_0^t \int_{R^N} \frac{1}{2}\ \partial_i a_{ij}\partial_j(u_\varepsilon - v_\varepsilon)^2\ dx\ ds\ +$$

$$+ \int_0^t \int_{R^N} (\psi_\varepsilon - \tilde{\psi}_\varepsilon)(u_\varepsilon - v_\varepsilon) ds\ dx + \int_0^t \int_{R^N} \sigma_{ik}\partial_i(\varphi^k_\varepsilon - \tilde{\varphi}^k_\varepsilon)(u_\varepsilon - v_\varepsilon) ds\ dx\ .$$

Next, remark that the last term may be written :

$$- \int_0^t \int_{R^N} (\partial_i \sigma_{ik})(\varphi^k_\varepsilon - \tilde{\varphi}^k_\varepsilon)(u_\varepsilon - v_\varepsilon) ds\ dx \quad +$$

$$- \int_0^t \int_{R^N} \sigma_{ik}\partial_i(u_\varepsilon - v_\varepsilon)(\varphi^k_\varepsilon - \tilde{\varphi}^k_\varepsilon) ds\ dx\ .$$

Therefore, if we let $\varepsilon \to 0_+$, we finally find :

$$\frac{1}{2}\ |u_\varepsilon(t)-v_\varepsilon(t)|^2_{L^2} - \frac{1}{2}\ |u_\varepsilon(0)-v_\varepsilon(0)|^2_{L^2} \quad \leqslant$$

$$\leqslant\ C \int_0^t |u_\varepsilon(s)-v_\varepsilon(s)|^2_{L^2}\ ds$$

and we conclude using Grönwall's Lemma. ∎

<u>Remark</u> : Let us mention that if $\sigma \in W^{2,\infty}(R^N)$, $b \in W^{1,\infty}(R^N)$, $c \in L^\infty(R^N)$ and $f \in L^2(R^N \times (0,T))$ (with $T > 0$ fixed), then it is possible to prove (Cf. Appendix of Part 2) that there exists a unique weak solution of (C) (for any $u_o \in L^2(R^N)$) in the following sense ; $u \in L^2(R^N \times (0,T))$ satisfies

$$\left(u\ ,\ -\frac{\partial\varphi}{\partial t} + A^\star\varphi\right)_{L^2(R^N\times(0,T))} = (f,\varphi)_{L^2(R^N\times(0,T))} + (u_o,\varphi)_{L^2(R^N)}$$

for all $\varphi \in \mathcal{D}(R^N \times [0,T])$ such that $\varphi(T) \equiv 0$; where $A^\star$ is the operator defined by : $A^\star = -a_{ij}\partial_{ij} + (b_i - 2\partial_j a_{ij})\partial_i + (c+\partial_i b_i -$
$+ (c+\partial_i b_i - \partial_{ij} a_{ij})$.

Notes

(1) Here and below, we use the convention on repeated indices.

(2) ∂_i and ∂_{ij} denote the partial derivatives with respect to x_i and x_i, x_j.

(3) We denote by $t_1 \wedge t_2 = \min(t_1, t_2)$.

(4) $a^{1/2}$ denotes the nonnegative symmetric square root of the nonnegative symmetric matrix a. In view of [46], $a^{1/2} \in W^{1,\infty}(R^N)$.

(5) in the sense of symmetric matrices.

REFERENCES :

[1] R. BELLMAN : Dynamic Programming. Princeton Univ. Press, Princeton, N.J., 1957.

[2] A. BENSOUSSAN and J.L. LIONS : Applications des inéquations variationnelles en contrôle stochastique. Dunod, Paris, 1978.

[3] A. BENSOUSSAN and J.L. LIONS : Contrôle impulsionnel et inéquations quasi-variationnelles. Dunod, Paris, 1982.

[4] M.G. CRANDALL, L.C. EVANS and P.L. LIONS : Some properties of viscosity solutions of Hamilton-Jacobi equations. To appear in Trans. Amer. Math. Soc. (1982-3).

[5] M.G. CRANDALL and P.L. LIONS : Viscosity solutions of Hamilton-Jacobi equations. Trans. Math. Soc., (1982-3).

[6] L.C. EVANS and A. FRIEDMAN : Optimal stochastic switching and the Dirichlet problem for the Bellman equation. Trans. Amer. Math. Soc., 253 (1979), p. 365-389.

[7] W.H. FLEMING and R. RISHEL : Deterministic and stochastic optimal control. Springer, Berlin, 1975.

[8] D. GILBARG and N.S. TRUDINGER : Elliptic partial differential equations of second order. Springer, Berlin, 1977.

[9] N. IKEDA and S. WATANABA : Stochastic differential equations and diffusion processes. North-Holland, Amsterdam, 1981.

[10] K. ITO : To appear.

[11] R. JENSEN and P.L. LIONS : Some asymptotic problems in fully nonlinear equations and optimal stochastic control. To appear.

[12] N.V. KRYLOV : Controlled diffusion processes. Springer, Berlin, 1980.

[13] N.V. KRYLOV : Control of a solution of a stochastic integral equation. Th. Proba. Appl., 17 (1972), p.114-131.

[14] N.V. KRYLOV : On control of the solution of a stochastic integral equation with degeneration. Math. USSR Izv., 6 (1972), p.249-262.

[15] N.V. KRYLOV : Control of the diffusion type processes. Proceedings of the International Congress of Mathematicians, Helsinki, 1978.

[16] N.V. KRYLOV and G. PRAGARAUSKAS : On a traditional derivation of the Bellman equation for general controlled stochastic processes. Liet. Mat. Rink., 21 (1981), p.101-110.

[17] N.V. KRYLOV and M.V. SAFONOV : An estimate of the probability that a diffusion process hits a set of positive measure. Soviet. Math. Dokl., 20 (1979), p.253-255.

[18] P.L. LIONS : Generalized solutions of Hamilton-Jacobi equations. Pitman, London, 1982.

[19] P.L. LIONS : Optimal stochastic control of diffusion type processes and Hamilton-Jacobi-Bellman equations. In Proceedings IFIP Conf. on Optimal Stochastic Control and Filtering in Cocoyoc, Ed. W.H. Fleming and L. Gorostiza. Springer, Berlin, 1982.

[20] P.L. LIONS : Fully nonlinear elliptic equations and applications. Proceedings "Function Spaces and Applications" Conf. in Pisek. Teubner, Berlin, 1982-3.

[21] P.L. LIONS : Equations de Hamilton-Jacobi-Bellman dégénérées. C.R. Acad. Sc. Paris, 289 (1979), p.329-332.

[22] P.L. LIONS : Optimal stochastic control and Hamilton-Jacobi-Bellman equations. In Mathematical Optimal Control Theory. Banach Center Publications, Warsaw, To appear.

[23] P.L. LIONS : Résolution analytique des problèmes de Bellman-Dirichlet. Acta Mathematica, 146 (1981), p.151-166.

[24] P.L. LIONS : Control of diffusion processes in R^N. Comm. Pure Appl. Math., 34 (1981), p.121-147.

[25] P.L. LIONS : Sur les équations de Monge-Ampère. I. To appear in Manuscripta Math..

[26] P.L. LIONS : Sur les équations de Monge-Ampère. II. To appear in Arch. Rat. Mech. Anal..

[27] P.L. LIONS : Existence results for first order Hamilton-Jacobi equations. To appear in Ricerche Mat..

[28] P.L. LIONS : Bifurcation and optimal stochastic control. To appear in Nonlinear Anal. T.M.A..

[29] P.L. LIONS and J.L. MENALDI : Optimal control of stochastic integrals and Hamilton-Jacobi-Bellman equations, I. SIAM J. Control. Optim., 20 (1982), p.58-81.

[30] P.L. LIONS and J.L. MENALDI : Optimal control of stochastic integrals and Hamilton-Jacobi-Bellman equations, II. SIAM J. Control. Optim., 20 (1982), p.82-95.

[31] P.L. LIONS and B. MERCIER : Approximation numérique des équations de Hamilton-Jacobi-Bellman. R.A.I.R.O., 14 (1980), p.369-393.

[32] P.L. LIONS and A.S. SZNITMAN : To appear.

[33] J.L. MENALDI : Sur les problèmes de temps d'arrêt, contrôle impulsionnel et continu correspondant à des opérateurs dégénérés. Thèse d'Etat, Paris, 1980.

[34] M. NISIO : Some remarks on stochastic optimal controls. Proc. Third USSR-Japan Sympos. Proba. Theory ; Lecture Notes in Math. n°550, Springer, Berlin, 1976.

[35] M. NISIO : Remarks on stochastic optimal controls. Jap. J. Math., 1 (1975), p.159-183.

[36] M. NISIO : Lectures on stochastic control theory. ISI Lecture Notes, n°9, Macmillan India Ltd, Bombay, 1981.

[37] M. NISIO : On a nonlinear semi-group attached to stochastic optimal control. Publ. R.I.M.S. Kyoto Univ., 12 (1976), p.513-537.

[38] M. NISIO : On stochastic optimal controls and envelope of Markovian semi-groups. Proc. Intern. Symp. SDE Kyoto 1976, Kinokuniya, Tokyo, 1978.

[39] O. OLEINIK : Alcuni risultati sulle equazioni lineari e quasi lineari ellitico-paraboliche a derivate parziali del second ordine. Rend. Classe Sci. Fis. Mat. Nat. Acad. Naz. Linei, 40 (1966), p.775-784.

[40] B. PERTHAME : Thèse de 3^{e} cycle, Paris VI, 1982.

[41] J.P. QUADRAT : Existence de solution et algorithme de résolution numérique, de problèmes de contrôle optimal de diffusion stochastique dégénérée ou non. SIAM J. Control Opt., 18 (1980), p.199-226.

[42] M.V. SAFONOV : On the Dirichlet problem for Bellman's equation in a plane domain. Math. USSR Sbornik, 31 (1977), p.231-248.

[43] M.V. SAFONOV : On the Dirichlet problem for Bellman's equation in a plane domain. II. Math. USSR Sbornik, 34 (1978), p.521-526.

[44] L. SCHWARTZ : Théorie des distributions. Hermann, Paris, 1950.

[45] J. SERRIN : The problem of Dirichlet for quasilinear elliptic differential equations with many independent variables. Phil. Trans. Soc. London A, 264 (1969), p.413-496.

[46] D.W. STROOCK and S.R.S. VARADHAN : Multidimensionnal diffusion processes. Springer, Berlin, 1979.

[47] D.W. STROOCK and S.R.S. VARADHAN : On degenerate elliptic-parabolic operators of second-order and their associated diffusions. Comm. Pure Appl. Math., 25 (1972), p. 651-714.

[48] D.W. STROOCK and S.R.S. VARADHAN : Diffusion processes with boundary conditions. Comm. Pure Appl. Math., 24 (1971), p.147-225.

[49] H. TANAKA : Stochastic differential equations with reflecting boundary conditions in convex regions. Hiroshima Math. J., 9 (1979), p.63-177.

Received December 1982