Exponential Stability of Stochastic Delay Interval Systems With Markovian Switching

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***Abstract*—In the past few years, a lot of research has been dedicated to the stability of interval systems as well as the stability of systems with Markovian switching. However, little research is on the stability of interval systems with Markovian switching, which is the topic of this paper. The system discussed in this paper is the stochastic delay interval system with Markovian switching. It is a very advanced system and takes all the features of interval systems, Itô equations, and Markovian switching, as well as time lag, into account. The theory developed here is applicable in many different and complicated situations so the importance of this paper is clear.**

***Index Terms*—Brownian motion, generalized Itô's formula, interval matrix, linear matrix inequality, Markov chain.**

I. Introduction

STOCHASTIC modeling has come to play an important role in many branches of science and industry. An area of particular interest has been the automatic control of stochastic systems, with consequent emphasis being placed on the analysis of stability in stochastic models, and here we mention Arnold [1], Hale and Lunel [6], Has'minskii [8], Kolmanovskii and Myshkis [10], Ladde and Lakshmikantham [11], Mao [13]–[15], Mohammed [19] among others.

Recently, the hybrid systems driven by continuous-time Markov chains have been used to model many practical systems where they may experience abrupt changes in their structure and parameters. The hybrid systems combine a part of the state that takes values continuously and another part of the state that takes discrete values. Such hybrid systems have been considered for the modeling of electric power systems by Willsky and Levy [27] as well as for the control of a solar thermal central receiver by Sworder and Rogers [24]. Athans [2] suggested that the hybrid systems would become a basic framework in posing and solving control-related issues in battle management command, control, and communications (BM/C^3) systems. An important class of hybrid systems is the jump linear systems,

$$\dot{x}(t) = A(r(t))x(t) \tag{1.1}$$

where a part of the state $x(t)$ takes values in R^n while another part of the state $r(t)$ is a Markov chain taking values in $S = \{1, 2, \ldots, N\}$. One of the important issues in the study of hybrid systems is the automatic control, with consequent emphasis being placed on the analysis of stability. Ji and Chizeck [9] and Mariton [18] studied the stability of such jump linear systems. Basak *et al.* [3] discussed the stability of a semi-linear stochastic differential equation with Markovian switching while Mao [16] investigated the stability of a nonlinear stochastic differential equation with Markovian switching. Shaikhet [20] took the time delay into account and considered the stability of a semi-linear stochastic differential delay equation with Markovian switching, while Mao *et al.* [17] investigated the stability of a nonlinear stochastic differential delay equation with Markovian switching.

The abrupt changes of structure and parameters in the hybrid systems are usually caused by phenomena such as component failures or repairs, changing subsystem interconnections, and abrupt environmental disturbances. When we model such systems it is therefore necessary to take parameter uncertainty and environmental noise as well as time delay into account. Recalling the linear jump system (1.1), if we consider the effect of time delay, the system should be described by

$$\dot{x}(t) = A(r(t))x(t) + B(r(t))x(t-\delta). \tag{1.2}$$

If we also take the environmental noise into account, the system becomes a stochastic differential delay equation with Markovian switching

$$\begin{aligned} dx(t) = {} & [A(r(t))x(t) + B(r(t))x(t-\delta)]\,dt \\ & + [C(r(t))x(t) + D(r(t))x(t-\delta)]\,dw(t) \end{aligned} \tag{1.3}$$

where $w(t)$ is a Brownian motion. However, when we estimate systems parameter matrices $A(\cdot)$, $B(\cdot)$, etc. there are always some uncertainty and error. If we further take these uncertainty and error into account we arrive at the stochastic delay interval system with Markovian switching (SDISwMS)

$$\begin{aligned} dx(t) = {} & [(A(r(t)) + \Delta A(r(t)))x(t) \\ & + (B(r(t)) + \Delta B(r(t)))x(t-\delta)]\,dt \\ & + [(C(r(t)) + \Delta C(r(t)))x(t) \\ & + (D(r(t)) + \Delta D(r(t)))x(t-\delta)]\,dw(t). \end{aligned} \tag{1.4}$$

The main aim of this paper is to discuss the exponential stability of this SDISwMS. Such systems take all the features of interval systems, Itô equations, hybrid systems as well as time-lag into account so they are very advanced. The theory developed here is applicable in many different and complicated situations and, hence, the importance of this paper is clear.

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It should also be pointed out that, in the past few years, a lot of research has been dedicated to the robustness of stable deterministic interval systems

$$\dot{x}(t) = (A + \Delta A)x(t).$$

In fact, this type of systems has been examined in several papers, for instance, [7], [25], and [26]. Similar systems which incorporate time delays have also been studied, for example, in [23]. The stability of stochastic interval systems was discussed by Šiljak [21]. More recently, Liao and Mao [12] investigated the stability of stochastic interval systems with time delays using the Razumikhin technique. However, so far there are few papers that not only deal with delay interval systems but also take the Markovian switching into account. The aim of this paper is to close this gap.

II. Notations and SDISwMS

Throughout this paper, unless otherwise specified, we use the following notations. Let $|\cdot|$ be the Euclidean norm in R^n. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\text{trace}(A^T A)}$ while its operator norm is denoted by $\|A\| = \sup\{|Ax|: |x| = 1\}$. If A is a symmetric matrix, denote by $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ its largest and smallest eigenvalue, respectively. If A and B are symmetric matrices, by $A > B$ and $A \geq B$ we means that $A - B$ is positive and nonnegative definite, respectively.

Let $R_+ = [0, \infty)$ and $\tau > 0$. Let $C([-\tau, 0]; R^n)$ denote the family of continuous functions φ from $[-\tau, 0]$ to R^n with the norm $\|\varphi\| = \sup_{-\tau \leq \theta \leq 0} |\varphi(\theta)|$ (without any confusion with the matrix operator norm $\|A\|$). Let δ: $R_+ \to [0, \tau]$ be a continuous function which will stand for the time lag of the systems discussed in this paper. As a standing hypothesis, we shall always assume that δ is differentiable and its derivative is bounded by a constant less than one, namely

$$\dot{\delta}(t) \leq \delta_0 < 1 \qquad \forall\, t \geq 0.$$

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all P-null sets). Denote by $C^b_{\mathcal{F}_0}([-\tau, 0]; R^n)$ the family of all bounded, $\mathcal{F}_0$-measurable, $C([-\tau, 0]; R^n)$-valued random variables. If $x(t)$ is a continuous R^n-valued stochastic process on $t \in [-\tau, \infty)$, we let $x_t = \{x(t+\theta): -\tau \leq \theta \leq 0\}$ for $t \geq 0$ which is regarded as a $C([-\tau, 0]; R^n)$-valued stochastic process. Let $w(t)$, $t \geq 0$, be a 1-dimensional Brownian motion defined on the probability space. Let $r(t)$, $t \geq 0$, be a right-continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, \ldots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N \times N}$ given by

$$P\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta), & \text{if } i \neq j \\ 1 + \gamma_{ii}\Delta + o(\Delta), & \text{if } i = j \end{cases}$$

where $\Delta > 0$. Here, $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while

$$\gamma_{ii} = -\sum_{j \neq i} \gamma_{ij}.$$

We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $w(\cdot)$. It is well known that almost every sample path of $r(t)$ is a right-continuous step function with a finite number of simple jumps in any finite subinterval of R_+.

For $A^m = [a^m_{ij}]_{n\times n}$ and $A^M = [a^M_{ij}]_{n \times n}$ satisfying $a^m_{ij} \leq a^M_{ij}\ \forall\, 1 \leq i, j \leq n$, the interval matrix $[A^m, A^M]$ is defined by $[A^m, A^M] = \{A = [a_{ij}]: a^m_{ij} \leq a_{ij} \leq a^M_{ij}, 1 \leq i, j \leq n\}$. For $A, \overline{A} \in R^{n\times n}$, where $\overline{A}$ is a nonnegative matrix, we use the notation $[A \pm \overline{A}]$ to denote the interval matrix $[A - \overline{A}, A + \overline{A}]$. In fact, any interval matrix $[A^m, A^M]$ has a unique representation of the form $[A \pm \overline{A}]$, where $A = (1/2)(A^m + A^M)$ and $\overline{A} = (1/2)(A^M - A^m)$.

Given, for each $i \in S$, the interval matrices

$$[A_i \pm \overline{A}_i] \quad [B_i \pm \overline{B}_i] \quad [C_i \pm \overline{C}_i] \quad [D_i \pm \overline{D}_i]$$

let us consider the n-dimensional stochastic delay interval system with Markovian switching (SDISwMS)

$$\begin{aligned} dx(t) &= [(A_r + \Delta A_r)x(t) + (B_r + \Delta B_r)x(t-\delta)]\,dt \\ &\quad + [(C_r + \Delta C_r)x(t) + (D_r + \Delta D_r)x(t-\delta)]\,dw(t). \end{aligned} \tag{2.1}$$

Here

$$\begin{aligned} \Delta A_i &\in [-\overline{A}_i, \overline{A}_i] \quad \Delta B_i \in [-\overline{B}_i, \overline{B}_i] \\ \Delta C_i &\in [-\overline{C}_i, \overline{C}_i] \quad \Delta D_i \in [-\overline{D}_i, \overline{D}_i] \end{aligned}$$

and throughout this paper, unless otherwise necessarily, we drop t from $r(t)$ and $\delta(t)$ for the sake of simplicity. It is not difficult to show (see [17]) that given any initial data $x_0 = \xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; R^n)$, (2.1) has a unique continuous solution, denoted here by $x(t; \xi)$, on $t \geq -\tau$. Moreover, the solution has the property that

$$E\left[\sup_{-\tau \leq s \leq t} |x(s;\xi)|^2\right] < \infty, \qquad \text{on } t \geq 0. \tag{2.2}$$

The main aim of this paper is to discuss the exponential stability of the SDISwMS (2.1). It should be pointed out that the theory developed in this paper can be generalized to the more general SDISwMS driven by multidimensional Brownian motion

$$\begin{aligned} dx(t) = {}& [(A_r + \Delta A_r)x(t) + (B_r + \Delta B_r)x(t-\delta)]\,dt \\ & + \sum_{k=1}^{m} [(C_{r,k} + \Delta C_{r,k})x(t) \\ & \qquad + (D_{r,k} + \Delta D_{r,k})x(t-\delta)]\,dw_k(t). \end{aligned} \tag{2.3}$$

The reason why we concentrate on (2.1) rather than (2.3) is to avoid the notations becoming too complicated. Once understanding the theory developed in this paper, the reader should be able to cope with (2.3) without any difficulty.

III. Simplified System

However, in order to make the theory more understandable, let us first consider a simplified system

$$dx(t) = [A_r x(t) + B_r x(t-\delta)]\,dt + [C_r x(t) + D_r x(t-\delta)]\,dw(t) \quad (3.1)$$

on $t \geq 0$ with initial data $x_0 = \xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; R^n)$. System (2.1) reduces to this simplified one if we let $\Delta A_i = 0$ and so on. In this equation, the system parameters are given by matrices rather than interval matrices so it is in fact a linear stochastic differential delay equations with Markovian switching (SDDEwMS). The stability of SDDEwMS has its own right. Although Mao *et al.* [17] discussed the stability of nonlinear SDDEwMS, the time delay there is a constant while it is here a time-varying function. Moreover, the linear structure of the system enables us to establish more effective criteria on stability.

We still denote by $x(t;\xi)$ the solution of (3.1) but we will, most of time, write $x(t;\xi) = x(t)$ unless otherwise we need to emphasize the role of initial data ξ. It is known (see [22]) that $\{x_t, r(t)\}_{t\geq 0}$ is a $C([-\tau, 0]; R^n) \times S$-valued Markov process. Its infinitesimal operator L, acting on functional V: $C([-\tau, 0]; R^n) \times S \times R_+ \to R$, is defined by

$$\begin{aligned} LV(x_t, i, t) &= \lim_{\Delta\to 0+} \frac{1}{\Delta}[E(V(x_{t+\Delta}, r(t+\Delta), t+\Delta)\,|\,x_t, r(t)=i) \\ &\quad - V(x_t, i, t)]. \end{aligned} \quad (3.2)$$

For example, if

$$\begin{aligned} &V(\varphi, i, t) = \varphi^T(0)Q_i\varphi(0) \\ &(\varphi, i, t) \in C([-\tau, 0]; R^n) \times S \times R_+ \end{aligned}$$

where Q_is are symmetric matrices, then (see [17])

$$\begin{aligned} LV(x_t, i, t) &= 2x^T(t)Q_i[A_i x(t) + B_i x(t-\delta)] \\ &\quad + [C_i x(t) + D_i x(t-\delta)]^T Q_i [C_i x(t) + D_i x(t-\delta)] \\ &\quad + \sum_{j=1}^N \gamma_{ij} x^T(t)Q_j x(t). \end{aligned} \quad (3.3)$$

In the study of stability that follows, we will define the Lyapunov functional involving the form

$$\begin{aligned} &V(\varphi, i, t) = \int_{-\delta(t)}^0 \varphi^T(\theta)H_i\varphi(\theta)\,d\theta \\ &(\varphi, i, t) \in C([-\tau, 0]; R^n) \times S \times R_+ \end{aligned} \quad (3.4)$$

where H_is are symmetric matrices. To our best knowledge there is no reference on the explicit formula for the infinitesimal operator acting on this functional so we present a lemma here.

Lemma 3.1: The infinitesimal operator acting on the functional defined by (3.4) has the form

$$\begin{aligned} LV(x_t, i, t) &= x^T(t)H_i x(t) - \left(1-\dot{\delta}\right)x^T(t-\delta)H_i x(t-\delta) \\ &\quad + \sum_{j=1}^N \gamma_{ij}\int_{-\delta}^0 x^T(t+\theta)H_j x(t+\theta)\,d\theta. \end{aligned} \quad (3.5)$$

Proof: Compute, for sufficiently small $\Delta > 0$

$$\begin{aligned} &E(V(x_{t+\Delta}, r(t+\Delta), t+\Delta)\,|\,x_t, r(t)=i) \\ &= E\left(\int_{-\delta(t+\Delta)}^0 x^T(t+\Delta+\theta) \times H_{r(t+\Delta)}x(t+\Delta+\theta)\,d\theta \,\middle|\, x_t, r(t)=i\right) \\ &= E\left(\int_{-\delta(t)}^0 x^T(t+\theta)H_{r(t+\Delta)}x(t+\theta)\,d\theta \,\middle|\, x_t, r(t)=i\right) \\ &\quad + E\left(\int_{-\delta(t+\Delta)}^0 x^T(t+\Delta+\theta)H_{r(t+\Delta)}x(t+\Delta+\theta)\,d\theta \right. \\ &\quad \left. - \int_{-\delta(t)}^0 x^T(t+\theta)H_{r(t+\Delta)}x(t+\theta)\,d\theta \,\middle|\, x_t, r(t)=i\right) \\ &= \sum_{j=1}^N P(r(t+\Delta)=j|r(t)=i) \\ &\quad \times \int_{-\delta(t)}^0 x^T(t+\theta)H_j x(t+\theta)\,d\theta \\ &\quad + E\left(\int_{\Delta-\delta(t+\Delta)}^{\Delta} x^T(t+\theta)H_{r(t+\Delta)}x(t+\theta)\,d\theta \right. \\ &\quad \left. - \int_{-\delta(t)}^0 x^T(t+\theta)H_{r(t+\Delta)}x(t+\theta)\,d\theta \,\middle|\, x_t, r(t)=i\right) \\ &= \sum_{j=1}^N (\gamma_{ij}\Delta + o(\Delta))\int_{-\delta(t)}^0 x^T(t+\theta)H_j x(t+\theta)\,d\theta \\ &\quad + \int_{-\delta(t)}^0 x^T(t+\theta)H_i x(t+\theta)\,d\theta \\ &\quad + E\left(\int_0^{\Delta} x^T(t+\theta)H_{r(t+\Delta)}x(t+\theta)\,d\theta \right. \\ &\quad - \int_{-\delta(t)}^{\Delta-[\delta(t)+\dot{\delta}\Delta+o(\Delta)]} x^T(t+\theta) \\ &\quad \left. \times H_{r(t+\Delta)}x(t+\theta)\,d\theta \,\middle|\, x_t, r(t)=i\right) \\ &= \sum_{j=1}^N \gamma_{ij}\Delta\int_{-\delta(t)}^0 x^T(t+\theta)H_j x(t+\theta)\,d\theta + V(x_t, i, t) \\ &\quad + \Delta x^T(t)H_i x(t) - \Delta(1-\dot{\delta})x^T(t-\delta(t)) \\ &\quad \times H_i x(t-\delta(t)) + o(\Delta). \end{aligned}$$

Substituting this into (3.2) yields the required formula (3.5). $\square$

We can now state our first result on the exponential stability in mean square.

Theorem 3.1: Assume that there are two constants λ_1 and λ_2 such that

$$\lambda_1 > \tau\lambda_2.$$

Assume also that there are symmetric matrices $Q_i > 0$, $H_i \geq 0$ and constants $\varepsilon_i > 0$ ($1 \leq i \leq N$) such that

$$H_i \geq \frac{1}{1-\delta_0}\left(\varepsilon_i^{-1}B_i^T Q_i B_i + 2D_i^T Q_i D_i\right)$$

$$\lambda_{\max}\left(Q_iA_i + A_i^TQ_i + 2C_i^TQ_iC_i + \varepsilon_iQ_i + \sum_{j=1}^N \gamma_{ij}Q_j + H_i\right) \leq -\lambda_1$$

$$\lambda_{\max}\left(\sum_{j=1}^N \gamma_{ij}H_j\right) \leq \lambda_2$$

for all $i \in S$. Then, for any initial data $\xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; R^n)$, the solution of (3.1) has the property that

$$\limsup_{t\to\infty} \frac{1}{t}\log\left(E|x(t;\xi)|^2\right) \leq -\lambda < 0. \tag{3.6}$$

In other words, (3.1) is exponentially stable in mean square. Moreover, the positive number λ is the unique root to

$$\alpha\lambda + (\lambda_2 + \alpha_1\lambda)\tau e^{\lambda\tau} = \lambda_1 \tag{3.7}$$

where

$$\alpha = \max_{1\leq i\leq N}\lambda_{\max}(Q_i) \quad \alpha_1 = \max_{1\leq i\leq N}\lambda_{\max}(H_i).$$

Proof: Let us first show that $\lambda_2 \geq 0$. Choose i for $\lambda_{\min}(H_i)$ to be the smallest of $\lambda_{\min}(H_j)$ ($1 \leq j \leq N$), i.e.,

$$\lambda_{\min}(H_i) = \min_{1\leq j\leq N}\lambda_{\min}(H_j)$$

and let $v \neq 0$ be the corresponding eigenvector of H_i, i.e., $H_iv = \lambda_{\min}(H_i)v$. Then

$$v^TH_iv = \lambda_{\min}(H_i)|v|^2.$$

Moreover

$$\begin{aligned} v^T\left(\sum_{j=1}^N \gamma_{ij}H_j\right)v &= \sum_{j\neq i}^N \gamma_{ij}v^TH_jv + \gamma_{ii}v^TH_iv \\ &\geq \sum_{j\neq i}^N \gamma_{ij}\lambda_{\min}(H_j)|v|^2 + \gamma_{ii}\lambda_{\min}(H_i)|v|^2 \\ &\geq \lambda_{\min}(H_i)|v|^2\sum_{j=i}^N \gamma_{ij} = 0. \end{aligned}$$

Thus

$$\lambda_{\max}\left(\sum_{j=1}^N \gamma_{ij}H_j\right)|v|^2 \geq v^T\left(\sum_{j=1}^N \gamma_{ij}H_j\right)v \geq 0.$$

Since $|v| > 0$, we obtain

$$\lambda_{\max}\left(\sum_{j=1}^N \gamma_{ij}H_j\right) \geq 0$$

so we must have $\lambda_2 \geq 0$. Since $\lambda_1 > \tau\lambda_2$, we see that $\lambda_1 > 0$ and that (3.7) does have a unique root $\lambda > 0$.

Now, fix any initial data ξ and write $x(t;\xi) = x(t)$. Let us define the Lyapunov functional V_1: $C([-\tau, 0]; R^n) \times S \times R_+ \to R$ by

$$V_1(\varphi, i, t) = e^{\lambda t}V(\varphi, i, t)$$

with

$$V(\varphi, i, t) = \varphi^T(0)Q_i\varphi(0) + \int_{-\delta(t)}^0 \varphi^T(\theta)H_i\varphi(\theta)\,d\theta.$$

By the generalized Itô formula (see [22]), we have

$$\begin{aligned} &EV_1(x_t, r(t), t) \\ &\quad= EV_1(\xi, r(0), 0) + E\int_0^t LV_1(x_s, r(s), s)\,ds. \end{aligned} \tag{3.8}$$

It is straightforward to see that

$$LV_1(x_t, i, t) = e^{\lambda t}[\lambda V(x_t, i, t) + LV(x_t, i, t)]$$

while by (3.3) and Lemma 3.1

$$\begin{aligned} &LV(x_t, i, t) \\ &\quad= 2x^T(t)Q_i[A_ix(t) + B_ix(t-\delta)] \\ &\qquad+ [C_ix(t) + D_ix(t-\delta)]^TQ_i[C_ix(t) + D_ix(t-\delta)] \\ &\qquad+ \sum_{j=1}^N \gamma_{ij}x^T(t)Q_jx(t) \\ &\qquad+ x^T(t)H_ix(t) - \left(1-\dot{\delta}\right)x^T(t-\delta)H_ix(t-\delta) \\ &\qquad+ \sum_{j=1}^N \gamma_{ij}\int_{-\delta}^0 x^T(t+\theta)H_jx(t+\theta)\,d\theta. \end{aligned} \tag{3.9}$$

Using the elementary inequality

$$2x^TQ_iB_iy \leq \varepsilon_ix^TQ_ix + \varepsilon_i^{-1}y^TB_i^TQ_iB_iy$$

as well as the assumptions, we compute

$$\begin{aligned} &LV(x_t, i, t) \\ &\quad\leq x^T(t)\left(Q_iA_i + A_i^TQ_i + 2C_i^TQ_iC_i + \varepsilon_iQ_i + \sum_{j=1}^N \gamma_{ij}Q_j + H_i\right)x(t) + x^T(t-\delta) \\ &\qquad\times\left(\varepsilon_i^{-1}B_i^TQ_iB_i + 2D_i^TQ_iD_i - (1-\delta_0)H_i\right)x(t-\delta) \\ &\qquad+ \int_{-\delta}^0 x^T(t+\theta)\left(\sum_{j=1}^N \gamma_{ij}H_j\right)x(t+\theta)\,d\theta \\ &\quad\leq -\lambda_1|x(t)|^2 + \lambda_2\int_{-\tau}^0 |x(t+\theta)|^2\,d\theta. \end{aligned}$$

Note also that

$$V(x_t, i, t) \le \alpha |x(t)|^2 + \alpha_1 \int_{-\tau}^{0} |x(t+\theta)|^2 \, d\theta.$$

Substituting these into (3.8) we obtain that

$$\begin{aligned} &EV_1(x_t, r(t), t) \\ &\quad \le EV_1(\xi, r(0), 0) - (\lambda_1 - \alpha\lambda) E \int_0^t e^{\lambda s} |x(s)|^2 \, ds \\ &\qquad + (\lambda_2 + \alpha_1 \lambda) \int_0^t e^{\lambda s} \left(\int_{-\tau}^{0} |x(s+\theta)|^2 \, d\theta \right) ds. \end{aligned} \tag{3.10}$$

Compute

$$\begin{aligned} \int_0^t e^{\lambda s} \left(\int_{-\tau}^{0} |x(s+\theta)|^2 \, d\theta \right) ds &= \int_0^t e^{\lambda s} \left(\int_{s-\tau}^{s} |x(u)|^2 \, du \right) ds \\ &\le \int_{-\tau}^{t} \left(\int_u^{u+\tau} e^{\lambda s} \, ds \right) |x(u)|^2 \, du \\ &\le \tau e^{\lambda \tau} \int_{-\tau}^{t} e^{\lambda u} |x(u)|^2 \, du. \end{aligned}$$

Substituting this into (3.10), and recalling (3.7), we get

$$\begin{aligned} &EV_1(x_t, r(t), t) \\ &\quad \le EV_1(\xi, r(0), 0) + (\lambda_2 + \alpha_1 \lambda) \tau e^{\lambda \tau} E \int_{-\tau}^{0} |\xi(\theta)|^2 \, d\theta. \end{aligned}$$

On the other hand, we note that

$$\begin{aligned} EV_1(x_t, r(t), t) &\ge e^{\lambda t} E \left(x^T(t) Q_{r(t)} x(t) \right) \\ &\ge e^{\lambda t} E \left(\min_{1 \le i \le N} x^T(t) Q_i x(t) \right) \\ &\ge e^{\lambda t} \min_{1 \le i \le N} \lambda_{\min}(Q_i) E |x(t)|^2 \end{aligned}$$

and, hence, the assertion (3.6) follows. □

If we let $\varepsilon_i = 1$ for all i in Theorem 3.1, we obtain the following useful result.

Corollary 3.1: Assume that there are two constants λ_1 and λ_2 such that

$$\lambda_1 > \tau \lambda_2.$$

Assume also that there are symmetric matrices $Q_i > 0$ and $H_i \ge 0$ ($1 \le i \le N$) such that

$$H_i \ge \frac{1}{1-\delta_0} \left(B_i^T Q_i B_i + 2 D_i^T Q_i D_i \right)$$

$$\lambda_{\max} \left(Q_i A_i + A_i^T Q_i + 2 C_i^T Q_i C_i + Q_i + \sum_{j=1}^{N} \gamma_{ij} Q_j + H_i \right) \le -\lambda_1$$

$$\lambda_{\max} \left(\sum_{j=1}^{N} \gamma_{ij} H_j \right) \le \lambda_2$$

for all $i \in S$. Then, (3.1) is exponentially stable in mean square.

It is useful to point out that Corollary 3.1 is stated without ε_i so it looks neat, but Theorem 3.1 is more general since it allows to choose different ε_i for different situations in practice, for example, in the proof of Theorem 4.1 we will choose $\varepsilon_i = \|B_i\| + \|\overline{B}_i\|$.

IV. Stability of SDISwMS

After presenting Theorem 3.1, we can now return to the stability study for the SDISwMS (2.1). In what follows, we denote by I the $n \times n$ identity matrix. We shall set $\|A\|^{-1} A^T Q A = 0$ when $A = 0$ as usual.

Theorem 4.1: Assume that there are symmetric matrices $Q_i > 0$ and $H_i \ge 0$ such that

$$\begin{aligned} H_i \ge \frac{1}{1-\delta_0} \Big(&\|B_i\|^{-1} B_i^T Q_i B_i + 2\big(1 + \|\overline{D}_i\| / \|D_i\|\big) D_i^T Q_i D_i \\ &+ \|Q_i\| \left[\|\overline{B}_i\| + 2 \|\overline{D}_i\| \big(\|D_i\| + \|\overline{D}_i\| \big) \right] I \Big) \end{aligned}$$

for $i \in S$. Suppose we can verify

$$\lambda_1 > \tau \lambda_2$$

where

$$\begin{aligned} \lambda_1 = -\max_{1 \le i \le N} \Bigg\{ &\lambda_{\max} \Bigg(Q_i A_i + A_i^T Q_i + 2\big(1 + \|\overline{C}_i\| / \|C_i\|\big) \\ &\times C_i^T Q_i C_i + \big(\|B_i\| + \|\overline{B}_i\| \big) Q_i + \sum_{j=1}^{N} \gamma_{ij} Q_j + H_i \Bigg) \\ &+ 2 \|Q_i\| \left[\|\overline{A}_i\| + \|\overline{C}_i\| \big(\|C_i\| + \|\overline{C}_i\| \big) \right] \Bigg\} \end{aligned}$$

and

$$\lambda_2 = \max_{1 \le i \le N} \lambda_{\max} \left(\sum_{j=1}^{N} \gamma_{ij} H_j \right).$$

Then, for any initial data $\xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; R^n)$, the solution of (2.1) has the property

$$\limsup_{t \to \infty} \frac{1}{t} \log \left(E |x(t; \xi)|^2 \right) \le -\lambda < 0. \tag{4.1}$$

Moreover, the positive number λ can be determined in the same way as stated in Theorem 3.1.

To prove this theorem, we present a useful lemma.

Lemma 4.1: Let $[A \pm \overline{A}]$ be an interval matrix and $\Delta A \in [-\overline{A}, \overline{A}]$. Let Q be a nonnegative–definite symmetric matrix. Then $\|\Delta A\| \le \|\overline{A}\|$ and

$$\begin{aligned} (A + \Delta A)^T Q (A + \Delta A) \le &\left(1 + \|\overline{A}\| / \|A\|\right) A^T Q A \\ &+ \|Q\| \|\overline{A}\| \big(\|A\| + \|\overline{A}\| \big) I. \end{aligned} \tag{4.2}$$

Proof: The result $\|\Delta A\| \le \|\overline{A}\|$ is well known, so we only need to show (4.2). Note that (4.2) holds if either $A = 0$ or $\overline{A} = 0$. Otherwise

$$\begin{aligned} &(A + \Delta A)^T Q (A + \Delta A) \\ &\quad = A^T Q A + \Delta A^T Q A + A^T Q \Delta A + \Delta A^T Q \Delta A \\ &\quad \le A^T Q A + \Delta A^T Q A + A^T Q \Delta A + \|Q\| \, \|\overline{A}\|^2 I \end{aligned}$$

while

$$\begin{aligned}\Delta A^T QA + A^T Q\Delta A &\le \varepsilon A^T QA + \frac{1}{\varepsilon}\,\Delta A^T Q\Delta A \\ &\le \varepsilon A^T QA + \frac{1}{\varepsilon}\,\|Q\|\,\|\overline{A}\|^2 I\end{aligned}$$

for any $\varepsilon > 0$. Choosing $\varepsilon = \|\overline{A}\|/\|A\|$, we have

$$\Delta A^T QA + A^T Q\Delta A \le \left(\|\overline{A}\|/\|A\|\right) A^T QA + \|Q\|\,\|A\|\,\|\overline{A}\| I$$

and then (4.2) follows. □

We can now prove Theorem 4.1 easily.

Proof: We first prove the theorem in the case when

$$\|B_i\| + \|\overline{B}_i\| > 0 \qquad \forall i \in S.$$

In this case, to apply Theorem 3.1, we let $\varepsilon_i = \|B_i\| + \|\overline{B}_i\|$. By Lemma 4.1 and the assumptions, we see easily that for every $i \in S$,

$$\begin{aligned}H_i \ge \frac{1}{1-\delta_0}\big[&\varepsilon_i^{-1}(B_i+\Delta B_i)^T Q_i (B_i+\Delta B_i) \\ &+ 2(D_i+\Delta D_i)^T Q_i (D_i+\Delta D_i)\big].\end{aligned}$$

Moreover, compute

$$\begin{aligned}&\lambda_{\max}\Bigg(Q_i(A_i+\Delta A_i) + (A_i+\Delta A_i)^T Q_i + 2(C_i+\Delta C_i)^T \\ &\qquad \times Q_i(C_i+\Delta C_i) + \varepsilon_i Q_i + \sum_{j=1}^N \gamma_{ij} Q_j + H_i\Bigg) \\ &\le \lambda_{\max}\Bigg(Q_iA_i + A_i^T Q_i + 2\|Q_i\|\,\|\overline{A}_i\| I \\ &\qquad + 2\big(1+\|\overline{C}_i\|/\|C_i\|\big)\, C_i^T Q C_i + 2\|Q_i\|\,\|\overline{C}_i\| \\ &\qquad \times \big(\|C_i\| + \|\overline{C}_i\|\big) I + \varepsilon_i Q_i + \sum_{j=1}^N \gamma_{ij} Q_j + H_i\Bigg) \\ &\le \lambda_{\max}\Bigg(Q_iA_i + A_i^T Q_i + 2\big(1+\|\overline{C}_i\|/\|C_i\|\big) C_i^T Q_i C_i \\ &\qquad + \varepsilon_i Q_i + \sum_{j=1}^N \gamma_{ij} Q_j + H_i\Bigg) \\ &\quad + 2\|Q_i\|\,\big[\|\overline{A}_i\| + \|\overline{C}_i\|\,\big(\|C_i\| + \|\overline{C}_i\|\big)\big] \le -\lambda_1.\end{aligned}$$

Therefore, the results follow from Theorem 3.1. In the case when some $\|B_i\| + \|\overline{B}_i\| = 0$ we can choose the corresponding ε_i sufficiently small and then apply Theorem 3.1 to show the exponential stability and to get (4.1) by letting $\varepsilon_i \to 0$. □

Let us now employ Theorem 4.1 to establish a couple of useful criteria. Theorem 4.1 depends on the choices of matrices Q_i and H_i while the corollary below depends only on the choices of Q_is. The advantage of Theorem 4.1 is that it gives more flexibility in applications, but the drawback is that one needs to construct more matrices required while the corollary below is in the opposite way.

Corollary 4.1: Assume that there are positive–definite symmetric matrices Q_i ($1 \le i \le N$) such that

$$\begin{aligned}&\lambda_{\max}\Bigg(Q_iA_i + A_i^T Q_i + 2\big(1+\|\overline{C}_i\|/\|C_i\|\big) C_i^T Q_i C_i \\ &\qquad + \big(\|B_i\| + \|\overline{B}_i\|\big)\, Q_i + \sum_{j=1}^N \gamma_{ij} Q_j\Bigg) \\ &\quad + 2\|Q_i\|\,\big[\|\overline{A}_i\| + \|\overline{C}_i\|\,\big(\|C_i\| + \|\overline{C}_i\|\big)\big] < -\beta \\ &\quad \forall i = 1, \ldots, N\end{aligned}$$

where

$$\begin{aligned}\beta = \frac{1}{1-\delta_0} \max_{1\le i\le N} \big\{&\lambda_{\max}\big(\|B_i\|^{-1} B_i^T Q_i B_i \\ &+ 2\left(1+\|\overline{D}_i\|/\|D_i\|\right) D_i^T Q_i D_i\big) \\ &+ \|Q_i\|\,\big[\|\overline{B}_i\| + 2\|\overline{D}_i\|\,\big(\|D_i\| + \|\overline{D}_i\|\big)\big]\big\}.\end{aligned}$$

Then the conclusion (4.1) of Theorem 4.1 still holds but the $\lambda > 0$ is now the unique root to

$$\lambda\left(\alpha + \beta\tau e^{\lambda\tau}\right) = \lambda_1$$

where α is the same as defined in Theorem 3.1 but

$$\begin{aligned}\lambda_1 = -\max_{1\le i\le N}\Bigg\{&\lambda_{\max}\Bigg(Q_iA_i + A_i^T Q_i + 2\big(1+\|\overline{C}_i\|/\|C_i\|\big) \\ &\times C_i^T Q_i C_i + \big(\|B_i\| + \|\overline{B}_i\|\big)\, Q_i + \sum_{j=1}^N \gamma_{ij} Q_j\Bigg) \\ &+ 2\|Q_i\|\,\big[\|\overline{A}_i\| + \|\overline{C}_i\|\,\big(\|C_i\| + \|\overline{C}_i\|\big)\big]\Bigg\} - \beta.\end{aligned}$$

Proof: To apply Theorem 4.1, we let $H_i = \beta I$ for all $i \in S$. Then

$$\sum_{j=1}^N \gamma_{ij} H_j = 0 \qquad \forall i \in S$$

so $\lambda_2 = 0$, while, by the assumption

$$\begin{aligned}\lambda_1 &= -\max_{1\le i\le N}\Bigg\{\lambda_{\max}\Bigg(Q_iA_i + A_i^T Q_i + 2\big(1+\|\overline{C}_i\|/\|C_i\|\big) \\ &\qquad \times C_i^T Q_i C_i + \big(\|B_i\| + \|\overline{B}_i\|\big)\, Q_i + \sum_{j=1}^N \gamma_{ij} Q_j + H_i\Bigg) \\ &\qquad + 2\|Q_i\|\,\big[\|\overline{A}_i\| + \|\overline{C}_i\|\,\big(\|C_i\| + \|\overline{C}_i\|\big)\big]\Bigg\} \\ &\ge -\max_{1\le i\le N}\Bigg\{\lambda_{\max}\Bigg(Q_iA_i + A_i^T Q_i + 2\big(1+\|\overline{C}_i\|/\|C_i\|\big) \\ &\qquad \times C_i^T Q_i C_i + \big(\|B_i\| + \|\overline{B}_i\|\big)\, Q_i + \sum_{j=1}^N \gamma_{ij} Q_j\Bigg) \\ &\qquad + 2\|Q_i\|\,\big[\|\overline{A}_i\| + \|\overline{C}_i\|\,\big(\|C_i\| + \|\overline{C}_i\|\big)\big]\Bigg\} - \beta > 0.\end{aligned}$$

Therefore, the result follows from Theorem 3.1. □

The following corollary will take a further step. It will not rely on the choices of N matrices Q_i, but only need to verify a specified matrix to be an M-matrix. For the convenience of the reader, let us cite some useful results on M-matrices. For more detailed information, please see [4]. We will need a few more notations. If B is a vector or matrix, by $B \gg 0$ we mean all elements of B are positive. If B_1 and B_2 are vectors or matrices with same dimensions, we write $B_1 \gg B_2$ if and only if $B_1 - B_2 \gg 0$. Moreover, we also adopt here the traditional notation by letting

$$Z^{N\times N} = \{A = [a_{ij}]_{N\times N}: a_{ij} \le 0,\ i \ne j\}.$$

Definition 4.1: A square matrix $A = [a_{ij}]_{N\times N}$ is called a nonsingular M-matrix if A can be expressed in the form $A = sI - B$ with $s > \rho(B)$ while all the elements of B are nonnegative, where I is the identity matrix and $\rho(B)$ the spectral radius of B.

It is easy to see that a nonsingular M-matrix A has nonpositive off-diagonal and positive-diagonal entries, that is

$$a_{ii} > 0, \quad \text{while } a_{ij} \le 0,\ i \ne j.$$

In particular, $A \in Z^{N\times N}$. There are many conditions which are equivalent to the statement that A is a nonsingular M-matrix, and we now cite some of them for the use of this paper.

Lemma 4.2: If $A \in Z^{N\times N}$, then the following statements are equivalent.

1) A is a nonsingular M-matrix.
2) A is semipositive; that is, there exists $x \gg 0$ in R^N such that $Ax \gg 0$.
3) A^{-1} exists and its elements are all nonnegative.
4) All the leading principal minors of A are positive; that is

$$\begin{vmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kk} \end{vmatrix} > 0, \qquad \text{for every } k = 1, 2, \ldots, N.$$

Let us now establish a new criterion, in terms of an M-matrix, on the exponential stability for the SDISwMS (2.1).

Corollary 4.2: Define the matrix

$$\begin{aligned} K = \operatorname{diag}\Big(&-\lambda_{\max}(A_1 + A_1^T) - 2\|\overline{A}_1\| \\ &- 2\big(\|C_1\| + \|\overline{C}_1\|\big)^2 - \|B_1\| - \|\overline{B}_1\|, \ldots \\ &-\lambda_{\max}(A_N + A_N^T) - 2\|\overline{A}_N\| \\ &- 2\big(\|C_N\| + \|\overline{C}_N\|\big)^2 - \|B_N\| - \|\overline{B}_N\|\Big) \end{aligned}$$

and the vector

$$\kappa = (1-\delta_0)\begin{bmatrix} \Big[\|B_1\| + \|\overline{B}_1\| + 2\big(\|D_1\| + \|\overline{D}_1\|\big)^2\Big]^{-1} \\ \vdots \\ \Big[\|B_N\| + \|\overline{B}_N\| + 2\big(\|D_N\| + \|\overline{D}_N\|\big)^2\Big]^{-1} \end{bmatrix}.$$

(Set $a^{-1} = \infty$ when $a = 0$ as usual.) If $K - \Gamma$ is a nonsingular M-matrix and

$$\kappa \gg (K - \Gamma)^{-1}\vec{1} \tag{4.3}$$

where $\vec{1} = (1, \ldots, 1)^T$, then the SDISwMS (2.1) is exponentially stable in mean square.

Proof: Set

$$\vec{q} = (q_1, \ldots, q_N)^T = (K - \Gamma)^{-1}\vec{1}. \tag{4.4}$$

By Lemma 4.2, we observe that all the elements of $(K - \Gamma)^{-1}$ are nonnegative. Since $(K - \Gamma)^{-1}$ is invertible, its each row must have at least one nonzero, and, hence, positive element. Therefore, we must have

$$\vec{q} \gg 0.$$

By (4.3)

$$\frac{q_i}{1-\delta_0}\Big[\|B_i\| + \|\overline{B}_i\| + 2\big(\|D_i\| + \|\overline{D}_i\|\big)^2\Big] < 1 \qquad \forall i \in S.$$

To apply Corollary 4.1 we let $Q_i = q_i I$ for $i \in S$. Then

$$\begin{aligned} &\lambda_{\max}\big(\|B_i\|^{-1}B_i^T Q_i B_i + 2\big(1 + \|\overline{D}_i\|/\|D_i\|\big)\,D_i^T Q_i D_i\big) \\ &\quad + \|Q_i\|\big[\|\overline{B}_i\| + 2\|\overline{D}_i\|\big(\|D_i\| + \|\overline{D}_i\|\big)\big] \\ &\le q_i\big[\|B_i\| + 2\big(1 + \|\overline{D}_i\|/\|D_i\|\big)\,\|D_i\|^2 \\ &\qquad + \|\overline{B}_i\| + 2\|\overline{D}_i\|\big(\|D_i\| + \|\overline{D}_i\|\big)\big] \\ &= q_i\Big[\|B_i\| + \|\overline{B}_i\| + 2\big(\|D_i\| + \|\overline{D}_i\|\big)^2\Big] < 1 - \delta_0. \end{aligned}$$

So the β defined in Corollary 4.1 is less than 1. On the other hand

$$\begin{aligned} &\lambda_{\max}\Bigg(Q_iA_i + A_i^TQ_i + 2\big(1 + \|\overline{C}_i\|/\|C_i\|\big)\,C_i^TQ_iC_i \\ &\qquad + \big(\|B_i\| + \|\overline{B}_i\|\big)\,Q_i + \sum_{j=1}^N \gamma_{ij}Q_j\Bigg) \\ &\quad + 2\|Q_i\|\,\big[\|\overline{A}_i\| + \|\overline{C}_i\|\big(\|C_i\| + \|\overline{C}_i\|\big)\big] \\ &\le q_i\Big[\lambda_{\max}(A_i + A_i^T) + 2\|\overline{A}_i\| + 2\big(\|C_i\| + \|\overline{C}_i\|\big)^2 \\ &\qquad + \|B_i\| + \|\overline{B}_i\|\Big] + \sum_{j=1}^N \gamma_{ij}q_j \\ &= -[(K-\Gamma)\vec{q}]_i \end{aligned}$$

where $[(K-\Gamma)\vec{q}]_i$ stands for the ith element of the vector $(K - \Gamma)\vec{q}$ and it is 1 by (4.4). Therefore

$$\begin{aligned} &\lambda_{\max}\Bigg(Q_iA_i + A_i^TQ_i + 2\big(1 + \|\overline{C}_i\|/\|C_i\|\big)\,C_i^TQ_iC_i \\ &\qquad + \big(\|B_i\| + \|\overline{B}_i\|\big)\,Q_i + \sum_{j=1}^N \gamma_{ij}Q_j\Bigg) \\ &\quad + 2\|Q_i\|\,\big[\|\overline{A}_i\| + \|\overline{C}_i\|\big(\|C_i\| + \|\overline{C}_i\|\big)\big] \le -1 < -\beta \end{aligned}$$

for all $i \in S$. The result now follows from Corollary 4.1. □

V. Examples

Let us now discuss some examples to illustrate our theory. The examples demonstrate that our theory can be used in two ways: 1) when the system parameter matrices and the bounds for the uncertainties are all known our theory can be used to verify whether the underlying system is stable and 2) when some bounds for the uncertainties are unknown our theory can be used to estimate these bounds so that the underlying system will remain stable should we be able to control the uncertainties within the bounds.

Example 5.1: An important class of hybrid systems is the linear jump systems, where continuous-time Markov chains are used to model the abrupt changes in system structure and parameters. The mathematical model for the linear jump systems is described by the linear differential equation with Markovian switching

$$\dot{x}(t) = A_r x(t) \tag{5.1}$$

where $r = r(t)$ is a Markov chain. Such hybrid systems have been intensively discussed in control and systems community and we mention [2], [9], [18], [24], and [27], among others. To illustrate our theory, let a part of the state $x(t)$ take values in R^3 while let another part of the state $r(t)$ be a right-continuous Markov chain taking values in $S = \{1, 2, 3\}$ with generator

$$\Gamma = \begin{bmatrix} -2 & 1 & 1 \\ 3 & -4 & 1 \\ 1 & 1 & -2 \end{bmatrix}.$$

Moreover, we specify the system matrices as follows:

$$A_1 = \begin{bmatrix} -2 & 1 & -2 \\ 2 & -2 & 1 \\ 1 & -2 & -3 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0.5 & 1 & 0.5 \\ -0.8 & 0.5 & 1 \\ -0.7 & -0.9 & 0.2 \end{bmatrix}$$
$$A_3 = \begin{bmatrix} -0.5 & -0.9 & -1 \\ 1 & -0.6 & -0.7 \\ 0.8 & 1 & -1 \end{bmatrix}.$$

Given these parameters, by [16, Corollary 5.1] we observe that (5.1) is exponentially stable in mean square if

$$\tilde{\mathcal{A}} := \mathrm{diag}(-\lambda_1, -\lambda_2, -\lambda_3) - \Gamma$$

is a nonsingular M-matrix, where

$$\lambda_i = \lambda_{\max}\left(A_i + A_i^T\right), \qquad i = 1, 2, 3.$$

To verify this is the case, we compute

$$\lambda_1 = -2.4385 \quad \lambda_2 = 1.207\,18 \quad \lambda_3 = -0.950\,67.$$

Therefore, the matrix $\tilde{\mathcal{A}}$ becomes

$$\tilde{\mathcal{A}} = \begin{bmatrix} 4.4385 & -1 & -1 \\ -3 & 2.792\,82 & -1 \\ -1 & -1 & 2.950\,67 \end{bmatrix}.$$

Using Maple, we can easily compute the inverse matrix of $\tilde{\mathcal{A}}$

$$\tilde{\mathcal{A}}^{-1} = \begin{bmatrix} 0.439\,02 & 0.239\,54 & 0.229\,97 \\ 0.597\,35 & 0.733\,44 & 0.451\,01 \\ 0.351\,23 & 0.329\,75 & 0.569\,69 \end{bmatrix}.$$

This implies, by Lemma 4.2, that $\tilde{\mathcal{A}}$ is a nonsingular M-matrix. We can therefore conclude that (5.1) with the parameters previously specified is indeed exponentially stable in mean square.

However, as explained in Section I, the abrupt changes of structure and parameters in the hybrid systems are usually caused by phenomena such as component failures or repairs, changing subsystem interconnections, abrupt environmental disturbances and the effects of time delay. The question is: *If we take parameter uncertainty and environmental noise as well as time delay into account, how large the uncertainty and environmental noise can the system tolerate so that it remains stable?*

To explain how we can apply the theory developed in this paper to answer the question, assume the system under the environmental noise, etc., is described by a three-dimensional SDISwMS

$$\begin{aligned} dx(t) = {} & [A_r x(t) + \Delta B_r x(t-\delta)]\,dt \\ & + [\Delta C_r x(t) + \Delta D_r x(t-\delta)]\,dw(t) \end{aligned} \tag{5.2}$$

on $t \geq 0$, where A_rs are the same as previously shown, and

$$\Delta B_i \in [-\overline{B}_i, \overline{B}_i] \quad \Delta C_i \in [-\overline{C}_i, \overline{C}_i] \quad \Delta D_i \in [-\overline{D}_i, \overline{D}_i].$$

For illustration, we let $\delta = \delta(t) = 0.1\sin^2 t$ and assume that the bounds for the uncertainties are known as follows:

$$\begin{aligned} \|\overline{B}_1\| &= \|\overline{B}_2\| = \|\overline{B}_3\| = 0.1 \\ \|\overline{C}_1\| &= \|\overline{C}_2\| = \|\overline{C}_3\| = 0.2 \\ \|\overline{D}_1\| &= \|\overline{D}_2\| = \|\overline{D}_3\| = 0.3. \end{aligned} \tag{5.3}$$

So the matrix K defined in Corollary 4.2 becomes

$$K = \mathrm{diag}(2.2585, -1.387\,18, 0.770\,67)$$

giving

$$K - \Gamma = \begin{bmatrix} 4.4585 & -1 & -1 \\ -3 & 2.612\,82 & -1 \\ -1 & -1 & 2.770\,67 \end{bmatrix}.$$

Compute

$$(K-\Gamma)^{-1} = \begin{bmatrix} 0.483\,93 & 0.292\,46 & 0.280\,22 \\ 0.722\,26 & 0.880\,56 & 0.578\,50 \\ 0.435\,34 & 0.423\,37 & 0.670\,85 \end{bmatrix}$$

which implies, by Lemma 4.2, that $K - \Gamma$ is a nonsingular M-matrix. Also, compute

$$(K-\Gamma)^{-1}\vec{1} = (1.056\,61, 2.181\,32, 1.529\,57)^T.$$

On the other hand, noting $\dot{\delta}(t) = 0.2\sin t\cos t \leq 0.1 = \delta_0$, the vector κ defined in Corollary 4.2 is now

$$\kappa = (3.214\,29, 3.214\,29, 3.214\,29)^T.$$

Clearly, $\kappa \gg (K-\Gamma)^{-1}\vec{1}$. Therefore, by Corollary 4.2, we can conclude that the SDISwMS (5.2) is exponentially stable in mean square under the bounds for uncertainties specified by (5.3).

Example 5.2: However, we may not know all the bounds for the uncertainties. For example, we may only know the following bounds:

$$\begin{aligned} \|\overline{B}_1\| &= \|\overline{B}_2\| = \|\overline{B}_3\| = 0.05 \\ \|\overline{C}_1\| &= \|\overline{C}_2\| = \|\overline{C}_3\| = 0.1 \end{aligned} \tag{5.4}$$

but the bounds for ΔD_is are unknown. In this case, if we can estimate the bounds for ΔD_is so that the SDISwMS (5.2) will still be exponentially stable in mean square, we then know that the system will be stable as long as we can control the uncertainties ΔD_is within the bounds. To explain how our theory can be applied to solve this problem, we note that the matrix K defined in Corollary 4.2 now becomes

$$K = \mathrm{diag}(2.3685, -1.277\,18, 0.880\,67)$$

and the vector κ defined in Corollary 4.2 is

$$\kappa = 0.9\Big(\left(0.05 + 2\|\overline{D}_1\|^2\right)^{-1}, \left(0.05 + 2\|\overline{D}_2\|^2\right)^{-1}, \left(0.05 + 2\|\overline{D}_3\|^2\right)^{-1}\Big)^T.$$

Compute

$$K-\Gamma=\begin{bmatrix} 4.3685 & -1 & -1 \\ -3 & 2.72282 & -1 \\ -1 & -1 & 2.88067 \end{bmatrix}$$

which has its inverse matrix

$$(K-\Gamma)^{-1}=\begin{bmatrix} 0.47095 & 0.26706 & 0.25619 \\ 0.66354 & 0.79720 & 0.50708 \\ 0.39383 & 0.36945 & 0.61211 \end{bmatrix}.$$

By Lemma 4.2, $K-\Gamma$ is a nonsingular M-matrix. Also compute

$$(K-\Gamma)^{-1}\vec{1}=(0.9942,\, 1.9678,\, 1.3754)^T.$$

For (5.2) to be exponentially stable, we require

$$\kappa \gg (K-\Gamma)^{-1}\vec{1}$$

that is

$$0.9\left(0.05+2\|\overline{D}_1\|^2\right)^{-1}>0.9942$$
$$0.9\left(0.05+2\|\overline{D}_2\|^2\right)^{-1}>1.9678$$
$$0.9\left(0.05+2\|\overline{D}_3\|^2\right)^{-1}>1.3754.$$

Solving these inequalities gives

$$\|\overline{D}_1\|<0.6539 \quad \|\overline{D}_2\|<0.4513 \quad \|\overline{D}_3\|<0.5497.$$

By Corollary 4.2, we can therefore conclude that the SDISwMS (5.2) is exponentially stable in mean square as long as (5.4) and (5.5) are satisfied.

VI. SUMMARY

Our aim in this paper was to investigate the exponential stability for a class of stochastic delay interval systems with Markovian switching. Given that the abrupt changes of structure and parameters in the hybrid systems are usually caused by phenomena such as component failures or repairs, changing subsystem interconnections, and abrupt environmental disturbances, we should take parameter uncertainty and environmental noise as well as time delay into account when we model such systems. By doing so, we arrive at the stochastic delay interval systems with Markovian switching. To understand how such systems can tolerate the effects of uncertainty, noise, and time delay, we establish a number of sufficient criteria on exponential stability in mean square. To cope with the difficulties which arise from the variable time delay, some new techniques have been developed which make the paper particularly appealing to both the control and systems audience, as well as mathematicians. We also discuss two examples to demonstrate how our theory can be used in two ways, depending on whether or not the system parameter matrices and the bounds for the uncertainties are known.

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