Viscosity Solutions of Second Order Fully Nonlinear Elliptic Equations with State Constraints

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ABSTRACT. This work is concerned with *State Constraints* boundary value problems for second order fully nonlinear degenerate equations. Existence, uniqueness and regularity questions are addressed for viscosity solutions which are required to be supersolutions on the boundary of a domain, this last property playing the role of a boundary condition. Such problems arise when a functional is minimized over controlled, stochastic dynamics, constrained inside a domain *without* using reflection on its boundary.

Introduction. In this paper we study the class of viscosity solutions of the second order *Bellman* equation in a domain $\Omega \subset \mathbb{R}^n$ which are required to be viscosity supersolutions on $\partial\Omega$. More precisely, we seek functions $u : \bar{\Omega} \to \mathbb{R}$ such that

$$(\mathrm{SC}) \qquad \begin{cases} u + \sup\limits_{a\in U}\{-\mathrm{tr}\ A(x,a)D^2u - b(x,a)Du - h(x,a)\} = 0 & \text{in } \Omega \\ \\ u + \sup\limits_{a\in U}\{-\mathrm{tr}\ A(x,a)D^2u - b(x,a)Du - h(x,a)\} \geq 0 & \text{on } \partial\Omega \end{cases}$$

(U is a compact metric space and Du, D^2u are the first and second order differentials of u).

This last property of u on $\partial\Omega$ is not pointwise, as we explain in Section 1, and plays essentially the role of a boundary condition for the Bellman equation. Such problems have been studied by H. M. Soner, P. Loretti, I. Capuzzo-Dolcetta and P. L. Lions in the context of first order Hamilton-Jacobi equations. For more details on their work see [S], [Lo], [CdL].

Indiana University Mathematics Journal ©, Vol. 43, No. 2 (1994)

We will also consider boundary value problems of the type above, for general second order, fully nonlinear, elliptic PDE's:

$$\text{(SC)}' \qquad \begin{cases} F(x,u,Du,D^2u) = 0 \text{ in } \Omega\,, \\ F(x,u,Du,D^2u) \geq 0 \text{ on } \partial\Omega\,. \end{cases}$$

The importance of (SC) and (SC)′ lies not only in applications in control theory but it also is the major step in understanding the generalized Dirichlet problem for second order possibly degenerate, elliptic PDE's. With this problem, as well as with approximating difference schemes, we are concerned in [Ka1] using entirely analytic tools.

The motivation and applications of (SC) are mainly in stochastic optimal control problems: we want to control systems whose *state* is governed by the solution of a stochastic differential equation:

$$\text{(0.1)} \qquad \begin{cases} dX_t = b(X_t, a_t)\,dt + \sigma(X_t, a_t)\,dW_t\,, \\ X_0 = x \in \bar{\Omega}\,. \end{cases}$$

Here $W. = \{W_t \mid t \geq 0\}$ is a standard n-dimensional brownian motion, Ω is a bounded domain in $\mathbb{R}^n$, b and σ are vector and matrix valued functions respectively and $A(x,a) = \sigma(x,a)\sigma^T(x,a)$. The process $a. = \{a_t \mid t \geq 0\}$ taking values in U, is called a *control.*

Our goal is to minimize the functional

$$\text{(0.2)} \qquad J(x,a.) = E\int_0^\infty e^{-t} h(X_t, a_t)\,dt$$

over the set of *admissible* controls:

$$\text{(0.3)} \qquad \mathcal{A}_x = \{a. \mid X_t \in \bar{\Omega},\ t \geq 0,\ P\text{–a.s.}\}.$$

These are the controls keeping the state $X.$ *constrained* in $\bar{\Omega}$, hence (SC) is called a *state constraints* problem. We want to determine the optimal value of $J(x,a.)$:

$$\text{(0.4)} \qquad V(x) = \inf_{a.\in\mathcal{A}_x} J(x,a.)\,.$$

The function V is called a *value function.*

Such applications arise, for example, in financial models. An investor consumes and distributes his wealth between certain assets (stocks, bonds, etc.) continuously in time. The current wealth is given by the state $X.$ and the control is his decision to make a transaction; the objective is to maximize (or minimize) a *utility* functional. However, the wealth $X.$ of the investor has to stay nonnegative at all times and this requirement imposes a state constraint of the type described earlier.

We now return to (0.1)–(0.4). The tool for characterizing V is the *Dynamic Programming Principle* (DPP), introduced first by R. Bellman: heuristically we have that

$$\text{(DPP)} \qquad V(x) = \inf_{a.\in\mathcal{A}_x} E_x \left\{ \int_0^\tau e^{-t} h(X_t, a_t)\, dt + e^{-\tau} V(X_\tau) \right\}.$$

If V is C^2, then by Ito's formula we easily prove that it solves (SC). However, there are two fundamental difficulties. On one hand, (DPP) is very hard to prove because the set valued map $x \mapsto \mathcal{A}_x$ has a complicated structure and the stochasticity introduces measurability problems. On the other, V is not necessarily smooth; in fact, we have an example in Section 2 where the value function is discontinuous.

For this last reason, we have to work with appropriate weak solutions, namely *viscosity solutions*. They were initially introduced for first-order Hamilton-Jacobi equations by M. G. Crandall and P. L. Lions in [CL]. Later on, P. L. Lions extended the definition for stochastic control problems ([L1-3]). For a survey on viscosity solutions, we refer the reader to [CIL].

This paper is divided in four sections: In Section 1 we define the notion of a viscosity solution. We describe the stochastic control problems corresponding to (SC) and explain why degeneracies arise naturally in the second order term of the Bellman equation.

In Section 2 we state a comparison theorem for subsolutions and supersolutions of (SC)′, proved in [CIL]. We refine their theorem so that it applies to lower-semicontinuous (lsc) supersolutions and upper semicontinuous (usc) subsolutions of (SC)′. This corollary is important for the existence of continuous solutions.

In Section 3 a penalization argument is introduced in order to obtain (possibly discontinuous) viscosity solutions for (SC)′. The results in Section 2 and Section 3 are applicable to general second order fully nonlinear elliptic PDE's, including the Bellman or the Isaac equation.

Finally, in Section 4 we impose appropriate assumptions depending on the dimension n, that yield continuous (hence unique by the comparison theorem) viscosity solutions of our problem.

1. Preliminaries. We first recall the definition of a viscosity solution. Consider the second order fully nonlinear equation

$$F(x, u, Du, D^2u) = 0 \quad \text{in } E \tag{1.1}$$

where E is an arbitrary subset of $\mathbb{R}^n$ and F satisfies:

$$F(x,r,p,A), \quad \text{for } (x,r,p,A) \in E \times \mathbb{R} \times \mathbb{R}^n \times S^n$$

is uniformly continuous in all arguments (S^n is the set of $n \times n$ symmetric matrices).

$$F(x,r,p,A) \leq F(x,s,p,A), \quad \text{for } r \leq s,\ (x,p,A) \in E \times \mathbb{R}^n \times S^n, \tag{1.2}$$

$$F(x,r,p,A) \leq F(x,s,p,B), \quad \text{for } B \leq A,\ (x,r,p) \in E \times \mathbb{R} \times \mathbb{R}^n. \tag{1.3}$$

The last property is called *degenerate ellipticity.* For convenience, we denote by $LSC(\bar{E})$ (respectively $USC(\bar{E})$) the set of lower semicontinuous (resp. upper semicontinuous) real valued functions, defined on $\bar{E}$.

Definition 1.1.

(i) An upper semicontinuous (usc) function u is a viscosity subsolution of (1.1) if $\forall\ x \in E$ and $\phi \in C^2(E)$ such that x is a local maximum of $u - \phi$, we have:

$$F\big(x,\ u(x),\ D\phi(x),\ D^2\phi(x)\big) \leq 0.$$

(ii) A lower semicontinuous (lsc) function v is a viscosity supersolution of (1.1) if $\forall\ x \in E$ and $\phi \in C^2(E)$ such that x is a local minimum of $v - \phi$, we have:

$$F\big(x,\ v(x),\ D\phi(x),\ D^2\phi(x)\big) \geq 0.$$

We say that u is a viscosity solution of (1.1) if $u^*(x) = \overline{\lim\limits_{\substack{y \to x \\ y \in E}}} u(y)$ is a subsolution and $u_*(x) = \underline{\lim\limits_{\substack{y \to x \\ y \in E}}}\ u(y)$ a supersolution. □

u^* (resp. u_*) is called the upper semicontinuous (resp. lower semicontinuous) envelope of u.

A second equivalent definition of viscosity solutions for (1.1) is given using *superjets* and *subjets* (for more details see [CIL]). Given a function $u : E \to \mathbb{R}$, we define the superjet at x,

$$\begin{aligned} J^{2,+}u(x) = \big\{(p,X) \in \mathbb{R}^n \times S^n : u(y) \leq u(x) + p \cdot (x-y) \\ +\ \tfrac{1}{2}(x-y)^T X(x-y) + o(|x-y|^2) \text{ as } y \to x,\ y \in E\big\}. \end{aligned}$$

The subjet $J^{2,-}u(x)$ is defined in a symmetric way and we have the following definition.

Definition 1.2.

(i) A usc function u is a viscosity subsolution of (1.1) if

$$F(x, u(x), p, X) \leq 0 \quad \forall\ (p, X) \in J^{2,+}u(x).$$

(ii) An lsc function v is a viscosity supersolution of (1.1) if

$$F(x, v(x), p, X) \geq 0 \quad \forall\ (p, X) \in J^{2,-}v(x). \qquad \square$$

The state constraints problem for a general second order fully nonlinear equation is:

$$(\mathrm{SC})' \qquad \begin{cases} F(x, u, Du, D^2u) = 0 & \text{in } \Omega, \\ F(x, u, Du, D^2u) \geq 0 & \text{on } \partial\Omega. \end{cases}$$

In other words, we require u to be a viscosity subsolution of $F = 0$ in Ω and a viscosity supersolution on $\bar{\Omega}$.

Notice that the requirement on u, to be a viscosity supersolution on $\bar{\Omega}$, is rather strong. Assume $u \in C^2(\bar{\Omega})$ and $F(x, u(x), Du(x), D^2u(x)) \geq 0$ on $\bar{\Omega}$ in the classical sense. Then we do not necessarily have that u is a viscosity supersolution on $\partial\Omega$: for $x \in \partial\Omega$, $J^{2,-}u(x) = \big\{(Du(x) + \lambda n(x), D^2u(x) - \lambda D^2\rho + \mu n(x) \otimes n(x)) : \lambda \geq 0,\ \mu \in \mathbb{R}\big\}$, where $n(x)$ is the outer normal of $\partial\Omega$ at x and $\rho(x) = \mathrm{dist}(x, \bar{\Omega}^c)$. Therefore, u is a viscosity supersolution on $\bar{\Omega}$ if, for all $x \in \partial\Omega$,

$$(1.4) \quad F\big(x,\ u(x),\ Du(x) + \lambda n(x),\ D^2u(x) - \lambda D^2\rho + \mu n(x) \otimes n(x)\big) \geq 0$$

for all $\lambda \geq 0,\ \mu \in \mathbb{R}$.

We now go back to (SC) and describe the stochastic optimal control problem associated to the Bellman equation.

To simplify, we will assume

(1.5) $b(x, a)$, $\sigma(x, a)$, $h(x, a)$ are bounded in x and a, Lipschitz in x and continuous in a, and

(1.6) Ω is a C^3 bounded domain in $\mathbb{R}^n$.

Consider the n-dimensional diffusion process $X.$ given by the solution of the stochastic differential equation:

$$\begin{cases} dX_t = b(X_t, a_t)\,dt + \sigma(X_t, a_t)\,dW_t, \ t > 0\,, \\ X_0 = x \in \bar{\Omega}\,. \end{cases} \tag{1.7}$$

Here, $b : \mathbb{R}^n \times U \to \mathbb{R}^n$, $\sigma : \mathbb{R}^n \times U \to \mathbb{R}^{n \times n}$. Furthermore, we are given an *admissible system* ϑ, consisting of

(i) a probability space $(\Psi, \mathcal{G}, P)$ and a filtration $\mathcal{G}_t$,

(ii) a corresponding n-dimensional brownian motion $W. = \{W_t \mid t \geq 0\}$, and

(iii) a progressively measurable (with respect to $\mathcal{G}_t$) process $a. = \{a_t \mid t \geq 0\}$ called a *control*.

For each ϑ there is a unique strong solution of (1.7), $X. = \{X_t \mid t \geq 0\}$. We denote by $\mathcal{A}$ the set of all admissible systems and set

$$\mathcal{A}_x = \{\vartheta \mid X_t \in \bar{\Omega}, t \geq 0,\ P\text{–a.s.}\}. \tag{1.8}$$

These are the controls that keep the state $X.$ starting at x, *constrained* in $\bar{\Omega}$, hence "state constraints" problem.

From now on, abusing our notation, we will refer to each admissible system ϑ as the control $a.$ (given by (iii)) but we will always associate to it the corresponding probability space, filtration and brownian motion.

Our goal is to minimize the *cost* functional

$$J(x, a.) = E_x \int_0^\infty e^{-t} h(X_t, a_t)\,dt \tag{1.9}$$

(where $h : \mathbb{R}^n \times U \to \mathbb{R}$ is usually called *running cost*) over $\mathcal{A}_x$.

Finally, as in the introduction, we define the value function

$$V(x) = \inf_{a. \in \mathcal{A}_x} J(x, a.). \tag{1.10}$$

In order for $V(x)$ to be finite, we have to assume $\mathcal{A}_x \neq \emptyset\ \forall\ x \in \bar{\Omega}$; in Section 3 we impose a condition guaranteeing that. However, this requirement gives rise to an implicit degeneracy of the Bellman equation on $\partial\Omega$: if $x \in \partial\Omega$ and $\mathcal{A}_x \neq \emptyset$, there is a control $a. \in \mathcal{A}_x$. But then, for some $a \in U$, we must have $\sigma^T(x, a)n(x) = 0$, (n is the outer normal at x), otherwise the noise in the normal direction will force the state to exit from $\bar{\Omega}$. The assumption $\sigma^T(x, a)n(x) = 0$ on $\partial\Omega$ for some $a \in U$, implies that the Bellman equation is not strictly elliptic, thus it may not have classical solutions. In fact, even linear degenerate elliptic equations do not necessarily have C^2 solutions (see [OR], [Fd]).

2. Comparison Principle. The present section and the following ones will be concerned with the state constraints problem for a general second order nonlinear equation (SC)$'$. In addition to (1.2)–(1.4), we assume,

(2.1) $F(x,r,p,X) - F(x,s,p,X) \geq \gamma(r-s)$, for $r \geq s$, $x \in \bar{\Omega}$, $p \in \mathbb{R}^n$, $X \in S^n$ and γ a positive constant,

(2.2) $|F(x,r,p,X) - F(x,r,q,Y)| \leq \omega(|p-1| + |X-Y|)$,

for $x \in$ neighborhood of $\partial\Omega$, p, $q \in \mathbb{R}^n$, X, $Y \in S^n$ and ω a modulus of continuity,

(2.3) $F(y,r,p,X) - F(x,r,p,Y) \leq \omega(\alpha|x-y|^2 + |x-y|(|p|+1))$

for x, $y \in \bar{\Omega}$, $r \in \mathbb{R}$, $p \in \mathbb{R}^n$, X, $Y \in S^n$ and

$$-3\alpha \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$$

where I the $n \times n$ identity matrix and $\alpha > 0$.

One of the fundamental results for viscosity solutions of (1.1) is the comparison between subsolutions and supersolutions.

Comparison Principle. *If $u \in USC(\bar{\Omega}$ (respectively, $v \in LSC(\bar{\Omega})$) is a subsolution (resp. a supersolution) of* (1.1) *and $u \leq v$ on $\partial\Omega$, then $u \leq v$ in $\bar{\Omega}$.* □

Assumptions (1.2), (1.3) yield easily this theorem for $u, v \in C^2(\bar{\Omega})$; in its full generality it was first proved by Jensen [J] and improved by Ishii [I1]. A similar result is proved in [CIL] for (SC)$'$. In this section we will present an improved version which is crucial for the existence of continuous solutions for state constraints problems in Section 4.

Theorem 2.1. ([CIL]) *Let $u \in C(\bar{\Omega})$ (resp. $v \in C(\bar{\Omega})$) be a subsolution (resp. supersolution) of* (SC)$'$. *Then $u \leq v$ on $\bar{\Omega}$.*

Notice that we do not require additional information about u and v on $\partial\Omega$, since all we need to know is contained in the boundary condition. Furthermore, it is clear that a continuous solution of (SC)$'$ is *unique.*

We seek to refine the previous theorem for the case when u and v are *not* continuous. Let us first introduce some notation. By $K(r,\beta,e)$, we denote the cone $U_{0<t<r}B(-te,t\beta) - \{0\}$ where e is a unit vector in $\mathbb{R}^n$ and $\beta > 0$.

Theorem 2.2. *Let $u \in USC(\bar{\Omega})$ (resp. $v \in LSC(\bar{\Omega})$) be a subsolution (resp. supersolution) of* (SC)$'$. *Assume $\forall\ z \in \partial\Omega$, $\exists\ \beta > 0$, $r > 0$ such* *that* $K_z = z + K(r, \beta, n(z)) \subset \Omega$, *($n(z)$ is the outer normal of $\partial\Omega$ at z)* and $u(z) = \overline{\lim\limits_{\substack{y\to x\\ y\in K_z}}}\, u(y)$. *Then* $u \le v$ *in* $\bar\Omega$.

Remark. In this version of the comparison principle, we relax the assumption v to be continuous because it is a supersolution on $\bar\Omega$ and this gives us all the information we need. However, we require u to be upper semicontinuous on $\partial\Omega$ in a nontangential direction, contained in a cone with vertex on the boundary and cone angle ϑ, where $\tan\vartheta = \beta$. We call this property of u *nontangential upper semicontinuity*.

A similar approach to comparison theorems for first order Hamilton-Jacobi equations was taken by H. Ishii in [I2].

Proof of Theorem 2.2. We first state two lemmata, needed for the proof.

Lemma 2.1. ([CIL]) *Let* $u \in USC(\bar\Omega)$ *and* $v \in LSC(\bar\Omega)$. *If* $\Psi \in C^2(\bar\Omega\times\bar\Omega)$ *and* $u(\hat x) - v(\hat y) - \Psi(\hat x,\hat y) = \max\{u(x) - v(y) - \Psi(x,y) \mid x,\ y \in \bar\Omega\}$ *for some* $\hat x$, $\hat y \in \bar\Omega$, *then for each* $\mu > 0$ *there exist* X, $Y \in S^n$ *such that*

$$\left(-\|A\| + \frac{1}{\mu}\right) I \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le A + \mu A^2, \tag{2.4}$$

$(D_x\Psi(\hat x,\hat y), X) \in \bar J^{2,+}u(\hat x)$, $(-D_y\Psi(\hat x,\hat y), Y) \in \bar J^{2,-}v(\hat y)$, *where* $A = D^2\Psi(\hat x,\hat y)$.

Lemma 2.2. ([CIL]) *Let* (2.2) *and* (2.3) *hold. Let* $\delta \ge 0$, $r \in \mathbb{R}$, $p \in \mathbb{R}^n$, X, $Y \in S^n$ *such that*

$$-(3\alpha+\delta)\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le 3\alpha\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \delta\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{2.5}$$

Then for x, $y \in$ *neighborhood of* $\partial\Omega$,

$$F(y,r,p,Y) - F(x,r,p,X) \le \omega\left(\left(\alpha + \frac{2}{3}\delta\right)|x-y|^2 + |x-y|(|p|+1)\right) + \omega(\delta).$$

Now we can proceed with the proof of Theorem 2.2.

As it is standard by now in the theory, we will argue by contradiction. Assume that $0 < \max_{\bar\Omega}(u - v) = u(z) - v(z)$; then due to the comparison principle stated at the beginning of the section, $z \in \partial\Omega$. The nontangential upper semicontinuity of u implies that there exist $v_m \in \mathbb{R}$, $|v_m| = 1$ and $t_m \to 0$ as $m \to \infty$ such that $z - t_m v_m \in K_z \subset \Omega$ and $u(z) = \lim_{m\to 0} u(z - t_m v_m)$. Define

$$\Phi(x,y) = u(x) - v(y) - \left|\frac{t_\ell}{t_m}(x-y) + t_\ell v_m\right|^2 - t_\ell|y-z|^2. \tag{2.6}$$

Let $(\hat{v}, \hat{y}) \in \bar{\Omega} \times \bar{\Omega}$ be a maximum point of Φ. Since $z - t_m v_m \in \Omega$, we have $\Phi(\hat{x}, \hat{y}) \geq \Phi(z - t_m v_m, z)$. Thus,

$$\left| \frac{t_\ell}{t_m}(\hat{x} - \hat{y}) + t_\ell v_m \right|^2 + t_\ell |\hat{y} - z|^2 \leq u(\hat{x}) - v(\hat{y}) - u(z - t_m v_m) + v(z). \tag{2.7}$$

Keeping ℓ fixed, we let $m \to \infty$ and using the nontangential upper semicontinuity of u, we get

$$\hat{x},\ \hat{y} \to z \text{ and } \frac{t_\ell}{t_m}(\hat{x} - \hat{y}) + t_\ell v_m = o_m(1) \qquad \text{as } m \to \infty \tag{2.8}$$

for ℓ is fixed.

In fact, $\hat{x} = \hat{y} - t_m v_m + (t_m/t_\ell) o_m(1) \in \Omega$ because $\partial\Omega$ is smooth enough and $K_z \subset \Omega$. Moreover, since u is a subsolution in Ω and v a supersolution on $\bar{\Omega}$, we have

$$F(\hat{x}, u(\hat{x}), p, X) \leq 0 \text{ for } (p, X) \in \bar{J}^{2,+} u(\hat{x})$$

and

$$F(\hat{y}, v(\hat{y}), q, X) \geq 0 \text{ for } (q, X) \in \bar{J}^{2,-} v(\hat{y}).$$

Now, set $\phi(x, y) = |(t_\ell/t_m)(x - y) + t_\ell v_m|^2 + t_\ell |y - z|^2$; then

$$\begin{aligned}
D_x \phi(x, y) &= 2\frac{t_\ell}{t_m}\left(\frac{t_\ell}{t_m}(x - y) + t_\ell v_m\right), \\
-D\phi_y(x, y) &= 2\frac{t_\ell}{t_m}\left(\frac{t_\ell}{t_m}(x - y) + t_\ell v_m\right) - 2t_\ell(y - z), \\
D^2 \phi(x, y) &= 2\left(\frac{t_\ell}{t_m}\right)^2 \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + 2t_\ell \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}.
\end{aligned}$$

Using Lemmata 2.1, 2.2 and (2.8), we eventually reach a contradiction letting $m \to \infty$ (cf. Theorem 7.5 in [CIL]). □

As we said in the introduction, Ω is always assumed to have a C^3 boundary. Nevertheless, one can prove the previous theorems for domains having the uniform interior cone property (see [S], [CdL], [IL]).

Ending this section, we will give an example demonstrating that our assumptions in Theorem 2.2 cannot be relaxed, i.e., we cannot expect an "unconditional" comparison principle in (SC)′, for simply usc subsolutions and lsc supersolutions.

If this was true, then any viscosity solution of (SC)′ would be continuous. But this is not true, as shown below.

Counterexample. We consider a first order state constraints problem arising in a deterministic control theory setting.

Let $\Omega = \mathbb{R}^2 \backslash \{x = (x_1, x_2) \mid x_1 \leq 0,\ x_2 \geq 0\}$ and the control set $A = \{\alpha, \beta\}$ defining the dynamics

$$\dot{x}_t = b(x_t, i), \qquad i = \alpha, \beta$$

where

$$\begin{aligned} b(x, \alpha) &= (0, 0), \qquad & x \in \bar{\Omega}, \\ b(x, \beta) &= (0, 1), \qquad & x \in \bar{\Omega}. \end{aligned}$$

The running cost is

$$h(x) = \begin{cases} 1, & x_2 \leq 0, \\ e^{-x_2}, & x_2 > 0. \end{cases}$$

As usual, define

$$V(x) = \inf_{a.\in\mathcal{A}_x} J(x, a.), \quad \text{where } J(x, a.) = \int_0^\infty e^{-t} h(x_t)\, dt.$$

It is straightforward that V satisfies the Dynamic Programming Principle,

$$V(x) = \inf_{a.\in\mathcal{A}_x} \left\{ \int_0^\tau e^{-t} h(x_t)\, dt + e^{-\tau} V(x_\tau) \right\}, \qquad x \in \bar{\Omega}.$$

Thus, V is a viscosity solution of

$$\begin{cases} u(x) + \sup\limits_{i\in A}\{-b(x, i)Du\} = h(x), & x \in \Omega \\ u(x) + \sup\limits_{i\in A}\{-b(x, i)Du\} \geq h(x), & x \in \partial\Omega. \end{cases}$$

More precisely, V_* is a viscosity supersolution and V^* a viscosity subsolution of the boundary value problem above. If we had a comparison principle without any further assumptions, then $V^* \leq V_*$, hence $V = V^* = V_* \in C(\bar{\Omega})$. But this is not true:

$$V(x) = \begin{cases} 1, & x \in \Omega \cap \{x_1 < 0\}, \\ \dfrac{1}{2}, & x \in \Omega \cap \{x_1 \geq 0,\ x_2 = 0\}. \end{cases}$$

3. Existence. The second question we address, is the existence of (possibly discontinuous) viscosity solutions for (SC)$'$. We cannot, however, prove such a result for an arbitrary second order equation. For example, consider the linear, second order, uniformly elliptic operator L, with C^α coefficients. Then the Dirichlet problem

$$\begin{cases} Lu = f & \text{in } \Omega\,, \\ u = \phi & \text{on } \partial\Omega\,, \end{cases} \tag{3.1}$$

has a $C^{2+\alpha}(\bar{\Omega})$ solution u, for f and ϕ smooth enough.

Assume the corresponding state constraints problem

$$\begin{cases} Lu = f & \text{in } \Omega\,, \\ Lu \geq f & \text{on } \partial\Omega\,, \end{cases} \tag{3.2}$$

has a viscosity solution v.

Then by Theorem 2.2, we get that $u \leq v$ in $\bar{\Omega}$, hence $\phi \leq v$ on $\partial\Omega$ for any ϕ, which is absurd. In the same way we can prove that (SC)$'$ does not have a solution for any second order *strictly* elliptic nonlinear PDE.

The following assumption will guarantee the existence of a viscosity solution for state constraint problems:

(A1) There is a bounded lower semicontinuous supersolution v of (SC)$'$.

As a first step towards the proof, we introduce appropriate penalized approximations.

Extend F for all $x \in \mathbb{R}^n$, so that it still satisfies (1.2), (1.3), (2.1)–(2.3) and is bounded in x. Moreover, let $d \in C_b^1(\mathbb{R}^n)$ (set of bounded continuously differentiable functions on $\mathbb{R}^n$, having bounded derivatives), where $d(x) = \operatorname{dist}(x, \bar{\Omega})$ if $\operatorname{dist}(x, \bar{\Omega}) < \delta$ and $d(x) \geq \delta$ if $\operatorname{dist}(x, \bar{\Omega}) \geq \delta$ for some $\delta > 0$. We now introduce the approximating problems

$$F(x, u^\varepsilon, Du^\varepsilon, D^2u^\varepsilon) = \frac{1}{\varepsilon} d(x) \text{ in } \mathbb{R}^n. \tag{3.3}$$

Such equations have continuous viscosity solutions u^ε (see [I1], Thm. 7.2). Furthermore, by the comparison principle, we obtain that $\{u^\varepsilon\}$ is increasing in ε, as $\varepsilon \downarrow 0$ and $u^\varepsilon(x) \geq -k$ for all $x \in \mathbb{R}^n$, $\varepsilon > 0$ and a positive constant k.

Our goal is to obtain a viscosity solution of (SC)$'$ by passing to the limit in ε. Assumption (A1) gives a uniform upper bound of u^ε in $\bar{\Omega}$.

Take $C_\varepsilon = O(1/\varepsilon)$ positive and set

$$\tilde{v}_\varepsilon(x) = \begin{cases} v(x), & x \in \bar{\Omega}, \\ C_\varepsilon, & x \in \bar{\Omega}^c. \end{cases}$$

Since v is a supersolution on $\partial\Omega$, $\tilde{v}_\varepsilon$ is a lsc supersolution of the extended problem (3.3). By comparison again, $-k \le u^\varepsilon(x) \le \tilde{v}_\varepsilon(x)$ for all $x \in \mathbb{R}^n$, hence $-k \le u^\varepsilon(x) \le v(x)$ on $\bar{\Omega}$. Thus, we have the following lemma:

Lemma 3.1. *Let* (A1) *hold. Then the approximating solutions u^ε are uniformly bounded in $\bar{\Omega}$, uniformly bounded from below in $\mathbb{R}^n$ and increasing as $\varepsilon \downarrow 0$.*

Besides this a priori bound, we do not have any other information, therefore we cannot expect to get even continuous viscosity solutions for our equation. Following a by now standard argument, introduced in [BP] and [LK], we define

$$\bar{u}(x) = \overline{\lim_{\substack{\varepsilon\to 0\\ y\to x}}}\ u^\varepsilon(y), \qquad x \in \Omega,$$

$$\underline{u}(x) = \underline{\lim}_{\substack{\varepsilon\to 0\\ y\to x}}\ u^\varepsilon(y), \qquad x \in \bar{\Omega},$$

$$u(x) = \lim_{\varepsilon\to 0}\ u^\varepsilon(x), \qquad x \in \bar{\Omega}.$$

Theorem 3.1. *Given* (A1)*, u is an lsc viscosity solution and, the least lsc supersolution on $\bar{\Omega}$, of* (SC)*$'$.*

Proof. The function u is the increasing limit of continuous functions, thus it is lsc, $\underline{u} = u$ and $\bar{u} = u^*$. Using the argument described in [BP], we easily prove u is a lsc supersolution and u^* an usc subsolution in Ω. Therefore, u is a viscosity solution in Ω.

We now show that u is also a supersolution on $\partial\Omega$. Take $x_0 \in \partial\Omega$ and $\phi \in C^2(\bar{\Omega})$ where x_0 is a point of strict minimum for $u - \phi$ in $\bar{\Omega}$. Due to the uniform bound from below on u^ε, we can extend ϕ in $\mathbb{R}^n$ so that $\phi \in C^2(\mathbb{R}^n)$ and $\phi(x) - \phi(x_0) + 1 \le u^\varepsilon(x) - u^\varepsilon(x_0)$, if $\operatorname{dist}(x, \bar{\Omega}) = 1$. Let x_ε be the minimum of $u^\varepsilon - \phi$ over $\{x : \operatorname{dist}(x, \bar{\Omega}) \le 1\}$. Then it cannot occur on the boundary of this set, so it is a local minimum of $u^\varepsilon - \phi$. Hence

$$F\big(x_\varepsilon,\ u^\varepsilon(x_\varepsilon),\ D\phi(x_\varepsilon),\ D^2\phi(x_\varepsilon)\big) \ge \frac{1}{\varepsilon} d(x_\varepsilon). \tag{3.4}$$

On the other hand,

$$u^\varepsilon(x_\varepsilon) - \phi(x_\varepsilon) \le u^\varepsilon(x_0) - \phi(x_0) \tag{3.5}$$

which implies $u^\varepsilon(x_\varepsilon) \le C$ for some positive ε-independent constant, because u^ε is bounded on $\bar{\Omega}$. Passing to a subsequence, we have that $u^\varepsilon(x_\varepsilon) \to \lambda$ and $x_\varepsilon \to \bar{x}$. Going back to (3.4), we see that $\overline{\lim}_{\varepsilon\to 0}(1/\varepsilon)\, d(x_\varepsilon) < \infty$, thus $\bar{x} \in \bar{\Omega}$. But (3.5) yields $u(\bar{x}) - \phi(\bar{x}) \le u(x_0) - \phi(x_0)$. This is a contradiction unless $\bar{x} = x_0$, since

x_0 is a point of strict minimum for $u - \phi$ in $\bar{\Omega}$. Finally, $u(\bar{x}) = u(x_0) = \lambda$ and $F(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)) \geq 0$.

It remains to show that u is the least lsc supersolution on $\bar{\Omega}$. If w is a lsc supersolution of (SC)$'$ on $\bar{\Omega}$, we can extend it in $\mathbb{R}^n$, as we did earlier, so that it is a supersolution of the penalized problem. By comparison, we have $u^\varepsilon \leq w$ in $\mathbb{R}^n$, thus $u \leq w$ on $\bar{\Omega}$. □

Remark 3.1.

(a) A similar penalization argument is introduced in [CdL] to prove the existence of continuous solutions for first order problems with coercive Hamiltonians.

(b) In the context of the Bellman equation and (SC) it is easy to check that the approximation (3.3) introduces a new value function

$$V^\varepsilon(x) = \inf_{a.\in\mathcal{A}} E_x \int_0^\infty e^{-t}\left\{h(X_t, a_t) + \frac{1}{\varepsilon} d(X_t)\right\} dt$$

where $\mathcal{A}$ is defined in Section 1. By uniqueness, we have $u^\varepsilon = V^\varepsilon$ in $\mathbb{R}^n$; moreover $V^\varepsilon \leq V$ on $\bar{\Omega}$, hence $u = \lim_{\varepsilon\downarrow 0} u^\varepsilon \leq V$ on $\bar{\Omega}$.

We now return to the assumption (A1) and discuss its meaning in terms of the Bellman equation and the stochastic control applications.

As we explained in the introduction, the value function

$$V(x) = \inf_{a.\in\mathcal{A}_x} J(x, a.)$$

is expected to solve, in some weak sense, (SC). However in order for $V(x)$ to be finite, we need $\mathcal{A}_x \neq \emptyset$, $x \in \bar{\Omega}$ or equivalently $\mathcal{A}_x \neq \emptyset$ for all $x \in \partial\Omega$. For this purpose we assume the following:

(A2) There is a Lipschitz continuous function $\alpha(x)$ defined on $\partial\Omega$ such that

(i) $\operatorname{tr} A(x, \alpha(x)) D^2\rho(x) + b(x, \alpha(x)) D\rho \geq 0$,

(ii) $(D\rho(x))^T A(x, \alpha(x)) D\rho(x) = 0$,

for all $x \in \partial\Omega$, where

$$\rho(x) = \operatorname{dist}(x, \bar{\Omega}^c), \qquad \text{for } x \in \mathbb{R}^n.$$

Note that $D\rho$ is the inner normal to $\partial\Omega$.

Proposition 3.1. *Assumption* (A2) *implies that* $\mathcal{A}_x \neq \emptyset$ *for all* $x \in \bar{\Omega}$.

Proof. Extend $\alpha(x)$ on $\bar{\Omega}$ so that it is still Lipschitz. In view of (A2), the trajectories of the SDE

$$dX_t = b(X_t, \alpha(X_t))\, dt + \sigma(X_t, \alpha(X_t))\, dW_t$$

stay always in $\bar{\Omega}$ for $t \geq 0$, with probability 1. For a proof of this fact see [Fr1], Thm. 12.2.1. □

Proposition 3.2. *Assumption* (A2) *implies* (A1), *i.e., there is a bounded viscosity supersolution for the Bellman equation on* $\bar{\Omega}$.

Proof. Set $v(x) = -\tilde{\rho}(x)^{1/2} + k$, where k is a positive constant to be specified later, $\tilde{\rho}(x) \in C_b^2(\bar{\Omega})$, $\tilde{\rho}(x) = \rho(x)$ if $\rho(x) < \delta$ and $\tilde{\rho}(x) \geq \delta$ if $\rho(x) \geq \delta$. Furthermore,

$$Dv(x) = -\tfrac{1}{2}\tilde{\rho}(x)^{-1/2} D\tilde{\rho}(x), \qquad x \in \Omega,$$

$$D^2 v(x) = \tfrac{1}{2} D\tilde{\rho}(x) \otimes D\tilde{\rho}(x) - \tfrac{1}{2}\tilde{\rho}(x)^{-1/2} D^2\tilde{\rho}(x), \qquad x \in \Omega.$$

If $x \in \partial\Omega$, the set of subjets of v is empty. We will show that v is a supersolution of the Bellman equation on $\bar{\Omega}$. Clearly on $\partial\Omega$ this is true, since there is nothing to prove. If $\rho(x) \geq \delta$, all the derivatives above are bounded ($\tilde{\rho} \in C_b^2(\bar{\Omega})$). Pick k large enough so that

$$F(x, v, Dv, D^2v) = v(x) + \sup_{a \in U}\{-\text{tr } A(x,a) D^2 v - b(x,a) Dv - h(x,a)\} \geq 0.$$

From (A2) and the extension of $\alpha(x)$ introduced earlier, we get

$$\text{tr } A(x,\alpha(x)) D^2\rho(x) + b(x,\alpha(x)) D\rho(x) \geq -C\rho(x) \tag{3.6}$$

$$(D\rho(x)^T A(x,\alpha(x)) D\rho(x) = O(\rho^2(x)) \quad \text{for } \rho(x) < \delta. \tag{3.7}$$

Therefore,

$$\begin{aligned} F(x,v,Dv,D^2v) \;&\geq\; v(x) - \text{tr } A(x,\alpha(x)) D^2 v(x) \\ &\qquad - b(x,\alpha(x)) D^2 v(x) - h(x,\alpha(x)) \\ &\geq\; 0 \quad \text{when } \rho(x) < \delta. \end{aligned}$$

□

It is clear from the last two propositions that (A1) is a natural assumption imposed essentially by the problem. Furthermore, (A2)(ii) introduces a degeneracy which excludes, in general, classical solutions. This condition means that on the boundary of Ω there must be at least one control with no noise in the normal direction; otherwise, our process will always exit immediately from $\bar{\Omega}$, no matter how strong the drift b might be.

4. Continuous viscosity solutions. In this section, we are addressing the question of existence of *continuous* viscosity solutions for state constraints problems; such solutions are also unique by the theorems of Section 2.

Our basic tool is the comparison Theorem 2.2. More precisely, in the previous section we constructed an lsc solution u for (SC)$'$, i.e., u is a supersolution of $F=0$ on $\bar{\Omega}$ and u^* a subsolution of $F=0$ in Ω. If u^* satisfies the nontangential upper semicontinuity assumption of Theorem 2.2, then $u^* \leq u$. This implies $u = u^* \in C(\bar{\Omega})$, thus u is the unique continuous viscosity solution of (SC)$'$. Now the question of existence is reduced to find conditions guaranteeing that u^* has the aforementioned property. A similar idea was used by Ishii in [I2] to prove existence of continuous viscosity solutions of the Dirichlet problem, for first order Hamilton-Jacobi equations.

Here, we restrict our attention to the second order Bellman equation and (SC). We will also identify the continuous solution of (SC) with the value function V. For this reason, we will construct a new usc subsolution u^+ of the Bellman equation such that $u \leq V \leq u^+$. Then, following the strategy described above, we have to prove that u^+ is nontangentially upper semicontinuous.

Before we move on, we strengthen our assumption (A2):

(A3) $\forall\, x \in \partial\Omega\ \exists\ \alpha(x) \in W^{1,\infty}(\partial\Omega)$ and a constant c such that

- (i) tr $A(x,\alpha(x))D^2\rho(x) + b(x,\alpha(x))D\rho \geq c > 0$,
- (ii) $D\rho(x)^T A(x,\alpha(x))D\rho(x) = 0$,
- (iii) If $T(x) = \text{span}\,\{e_i(x)\}$ the tangent plane at $x \in \partial\Omega$, there are at most two i's with the property $e_i(x)^T A(x,\alpha(x))e_i(x) \neq 0$.

We use the feedback control $\alpha(x)$ given by (A3), so that the state $X.$ can escape from $\partial\Omega$ into the interior of Ω if this is optimal; thus we obtain the continuity of V on $\partial\Omega$ and eventually in $\bar{\Omega}$. In this attempt, there is a competition between the drift $b(x,\alpha(x))$ and the diffusion in the tangent directions of $\partial\Omega$. As we will see later, (A3)(iii) indicates that we cannot allow too much tangential diffusion, otherwise the control $\alpha(x)$ cannot push the state $X.$ inside Ω fast enough.

4A. One more approximation of **(SC)*.*** Define

$$v^\varepsilon(x) = \inf_{a.\in\mathcal{A}} Ex \int_0^\infty e^{-t}\{h(X_t,a_t) + \varepsilon\tilde{\rho}^{-1/2}(X_t)\}\,dt,$$

where $\tilde{\rho}$ is the distance-like function defined in Proposition 3.2.

Lemma 4.1. *If* (A3)(i)(ii) *is satisfied, v^ε is bounded in Ω and*

$$|v^\varepsilon(x)| \leq \|h\|_\infty + C\varepsilon, \qquad x \in \bar{\Omega}, \tag{4.1}$$

where C depends on $\tilde{\rho}$, $\|b\|_\infty$ and $\|\sigma\|_\infty$.

Proof. We extend the function $\alpha(\cdot)$ given by (A3), in $\bar{\Omega}$ (as in Proposition 3.1), and use it as a feedback control. Thus we obtain the SDE

$$
\left\{
\begin{array}{l}
dX_t = b(X_t, \alpha(X_t))\,dt + \sigma(X_t, \alpha(X_t))\,dW_t\,, \\
X_0 = x \in \bar{\Omega}\,.
\end{array}
\right.
\tag{4.1}
$$

By Ito's rule,

$$
\begin{aligned}
&E_x e^{-t\wedge\sigma}\tilde{\rho}^{1/2}(X_{t\wedge\sigma}) \\
&= E_x \int_0^{t\wedge\sigma} e^{-s}\Big\{-\tilde{\rho}^{1/2}(X_s) + \tfrac{1}{2}(\operatorname{tr} A(X_s, \alpha(X_s))D^2\tilde{\rho}(X_s) \\
&\qquad + b(X_s, \alpha(X_s))D\tilde{\rho}(X_s)\tilde{\rho}^{-1/2}(X_s) \\
&\qquad - \frac{1}{4}\operatorname{tr}\big(A(X_s, \alpha(X_s))D\tilde{\rho}(X_s) \otimes D\tilde{\rho}(X_s)\big)\tilde{\rho}^{-3/2}(X_s)\Big\}\,ds
\end{aligned}
$$

for $x \in \partial\Omega$ and σ the exit time from a δ-neighborhood of $\partial\Omega$. Then $\tilde{\rho}(x) = \rho(x)$ and by (A3)(i)(ii) (see (3.6), (3.7)), we have

$$
\begin{aligned}
E_x \int_0^{t\wedge\sigma} e^{-s}\rho^{-1/2}(X_s)\,ds &\le M(E_x e^{-t\wedge\sigma}\rho(X_{t\wedge\sigma})) \\
&\quad + E_x \int_0^{t\wedge\sigma} e^{-s}\rho^{1/2}(X_s)\,ds, \quad \text{for all } t \ge 0,\ x \in \partial\Omega.
\end{aligned}
\tag{4.3}
$$

M is a constant depending only on δ, $\|b\|_\infty$ and $\|\sigma\|_\infty$. On the other hand,

$$
v^\varepsilon(x) \le J(x, \alpha(X_t)) = E_x \int_0^\infty e^{-s}\big\{h(X_s, \alpha(X_s)) + \varepsilon\tilde{\rho}^{-1/2}(X_s)\big\}\,ds.
$$

Let $[\tau_i, \tau_i + \sigma_i]$, $i \in \mathbb{N}$, the time X (given by (4.2)) spends in the δ-neighborhood of $\partial\Omega$. Then,

$$
\begin{aligned}
v^\varepsilon(x) \le \sum_{i\in\mathbb{N}} E_x \int_{\tau_i}^{\tau_i+\delta_i} e^{-s}\big\{h(X_s, \alpha(X_s)) + \varepsilon\tilde{\rho}^{-1/2}(X_s)\big\}\,ds \\
+ \|h\|_\infty + \varepsilon\delta^{-1}.
\end{aligned}
$$

By (4.3) we obtain

$$
v^\varepsilon(x) \le \|h\|_\infty + \varepsilon\delta^{-1} + \varepsilon M(\delta + 1) \quad \text{for all } x \in \bar{\Omega}. \qquad \square
$$

Furthermore, a Dynamic Programming Principle holds for u^ε (for details see [Bo], Chapter III.1):

$$(\mathrm{DPP})^\varepsilon \qquad v^\varepsilon(x) = \inf_{a\in\mathcal{A}} E_x \int_0^\tau e^{-t}\{h(X_t,a_t) + \ \varepsilon\tilde{\rho}^{-1/2}(X_t)\}\,dt \\ +e^{-\tau}v^\varepsilon(X_\tau) \quad \text{for all } x \in \bar{\Omega}.$$

Proposition 4.1. *v^ε is a viscosity solution of*

$$v^\varepsilon + \sup_{a\in U}\{-tr\ A(x,a)D^2v^\varepsilon - b(x,a)Dv^\varepsilon - h(x,a)\} = \varepsilon\tilde{\rho}^{-1/2}(x) \quad in\ \Omega, \tag{4.4}$$

i.e., $(v^\varepsilon)^*$ is a u.s.c. subsolution and $(v^\varepsilon)_*$ an l.s.c. supersolution.

Proof. Immediate from $(\mathrm{DPP})^\varepsilon$. □

We notice that v^ε is a decreasing sequence. Therefore $(v^\varepsilon)^*$ is decreasing, the limit as $\varepsilon \downarrow 0$ exists, $u^+(x) = \lim_{\varepsilon\downarrow 0}(v^\varepsilon)^*(x)$, $|u^+(x)| \le \|h\|_\infty$, $x \in \bar{\Omega}$ and u^+ is a usc subsolution of

$$u + \sup_{a\in U}\{-\mathrm{tr}\ A(x,a)D^2u - b(x,a)Du - h(x,a)\} = 0 \text{ in } \Omega.$$

The last statement follows because the monotonicity and upper semicontinuity of $(v^\varepsilon)^*$ imply $u^+(x) = \overline{\lim\limits_{\substack{y\to x\\ \varepsilon\to 0}}}\ v^\varepsilon(y)$. As in Theorem 3.1, we have the following result.

Proposition 4.2. *The usc function $u^+(x) = \lim_{\varepsilon\downarrow 0}(v^\varepsilon)^*(x)$ is a subsolution of*

$$u + \sup_{a\in U}\{-tr\ A(x,a)D^2u - b(x,a)Du - h(x,a)\} = 0 \ \ in\ \Omega, \tag{4.5}$$

and

$$|u^+(x)| \le \|h\|_\infty \quad in\ \bar{\Omega}.$$ □

Remark 4.1.

(a) If $\alpha.$ is the feedback control given by (A3), then $(\mathrm{DPP})^\varepsilon$ implies:

$$v^\varepsilon(x) \le E_x \int_0^{\tau\wedge\sigma} e^{-t}\{h(X_t,\alpha(X_t) + \varepsilon\tilde{\rho}^{-1/2}(X_t)\}\,dt + e^{-\tau\wedge\sigma}v^\varepsilon(X_{\tau\wedge\sigma})$$

where σ the exist time from a δ-neighborhood of $\partial\Omega$. By the same argument as in Lemma 4.1 and by letting $\varepsilon \downarrow 0$, we obtain

$$u^+(x) \le E_x \int_0^{\tau\wedge\sigma} e^{-t}h(X_t,\alpha(X_t))\,dt + e^{\tau\wedge\sigma}u^+(X_{\tau\wedge\sigma}), \quad X \in \bar{\Omega}. \tag{4.6}$$

(b) Notice that $v^\varepsilon(x) \ge V(x)$ in $\bar{\Omega}$, by construction. Therefore $u^+(x) \ge V(x)$ for all $x \in \bar{\Omega}$.

Proposition 4.3. *If* (A3) *holds, then u^+ is nontangentially upper semicontinuous.*

Proof. By construction we have $u^+(z) = \overline{\lim\limits_{\substack{y\to z\\ y\in\bar{\Omega}}}}\ u^+(y)$ for all $z \in \bar{\Omega}$. It suffices to prove that for all $z \in \partial\Omega$, $\overline{\lim\limits_{\substack{y\to z\\ y\in K_z}}}\ u^+(y)$ where K_z is a cone with vertex at z, contained in Ω.

However, there exists a sequence $\{x_r\}_{r\ge 0}$ such that $x_r \in \bar{\Omega}$, $x_r \to z$ as $r \downarrow 0$ and $u^+(z) = \lim_{r\downarrow 0}\ u^+(x_r)$. If $\{x_r\} \subset K_z$ for some cone K_z, we have nothing to prove. Otherwise, we need to construct a sequence $\{y_r\}_{r\ge 0}$ such that $y_r \to z$ as $r \downarrow 0$, $u^+(z) \le \lim_{r\downarrow 0}\ u^+(y_r)$, and $\{y_r\} \subset K_z$ for a cone with vertex at z.

By Remark 4.1 (a), we have

$$u^+(x) \le E_x \left\{ \int_0^{\tau\wedge\sigma} e^{-t}h(X_t,\alpha(X_t))\,dt + e^{-\tau\wedge\sigma}u(X_{\tau\wedge\sigma}) \right\},$$

where $\alpha(x)$ is the feedback control defined by (A3).

We rely on this control in order to be able to hit "soon enough" a cone K_z, at times τ^y starting from any point $y \in \bar{\Omega}\backslash K_z$. Then (4.6) implies

$$u^+(\tau^{x_r}) \le E_{x_r} \left\{ \int_0^{\tau^{x_r}\wedge\sigma} e^{-t}h(X_t,\alpha(X_t))\,dt + e^{\tau^{x_r}\wedge\sigma}u^+(X_{\tau^{x_r}\wedge\sigma}) \right\} \tag{4.7}$$

If $E\tau^y \to 0$ as $y \to z$, then

$$\begin{aligned}
\lim_{r\to 0}\ u^+(x_r) = u^+(z) &\le \overline{\lim_{r\to 0}}\ Eu^+(X_{\tau^{x_r}\wedge\sigma}) \\
&\le E\ \overline{\lim_{r\to 0}} u^+(X_{\tau^{x_r}\wedge\sigma}) \\
&\le \overline{\lim_{\substack{y\to z\\ y\in K_z}}}\ u^+(y).
\end{aligned}$$
□

Therefore u^+ is nontangentially upper semicontinuous, provided the following lemma holds true:

Lemma 4.4. *Let* (A3) *hold. Then* $\forall\ z \in \partial\Omega$, *there is a cone* $K_z = z + K(r, \beta, n(z)) \subset \Omega$, *where* β *depends only on the constant* c *in* (A3)(i), *such that*

$$\lim_{\substack{y \to z \\ y \in \bar{\Omega}}} E\tau^y = 0, \tag{4.8}$$

where τ^y *the exit time from* $\bar{\Omega} \backslash K_z$ *for the process*

$$\begin{cases} dX_t = b(X_t, \alpha(X_t))\, dt + \sigma(X_t, \alpha(X_t))\, dW_t\,, \\ X_0 = y \in \Omega \backslash K_z. \end{cases} \tag{4.9}$$

□

Remark 4.2. Due to (A3)(i), (A3)(ii), $X.$ exists from $\bar{\Omega} \backslash K_z$ only through the boundary of the cone K_z and not through $\partial\Omega$. It is not hard to see that the exit time from $\bar{\Omega} \backslash K_z$ and the hitting time of K_z from $\bar{\Omega} \backslash K_z$, are equal.

We postpone the proof of the previous lemma and proceed to the existence of a continuous viscosity solution for (SC).

4B. Existence.

Theorem 4.1. *If* (A3) *holds, the value function* $V(x) = \inf_{a. \in \mathcal{A}_x} J(x, a.)$ *is the unique continuous viscosity solution of* (SC).

Proof. By Propositions 4.2 and 4.3, we have that u^+ is nontangentially upper semicontinuous, as required by Theorem 2.2. Therefore,

$$u^+(x) \leq u(x), \qquad x \in \bar{\Omega},$$

where u is the lsc solution of (SC) constructed in Section 3.

Furthermore, by Remarks 3.1(b) and 4.1(b) we have

$$V(x) \geq u(x), \qquad x \in \bar{\Omega},$$

and

$$u^+(x) \geq V(x), \qquad x \in \bar{\Omega}.$$

Hence

$$u^+(x) = V(x) = u(x), \qquad x \in \bar{\Omega},$$

and

$$V \in C(\bar{\Omega}).$$

Therefore V is the unique viscosity solution of (SC). □

In Sections 3 and 4A, we introduced two approximations of (SC):

$$u^\varepsilon + \sup_{a\in U}\left\{-\text{tr}\, A(x,a)D^2u^\varepsilon - b(x,a)Du^\varepsilon - h(x,a)\right\} = \frac{1}{\varepsilon}d(x) \tag{4.10}$$

in $\mathbb{R}^n$, where $d \in C_b^1(\mathbb{R}^n)$ and $d(x) = \text{dist}(x, \bar{\Omega})$ if $\text{dist}(x,\bar{\Omega}) < \delta$ and

$$v^\varepsilon + \sup_{\alpha\in U}\left\{-\text{tr}\, A(x,a)D^2v^\varepsilon - b(x,a)Dv^\varepsilon - h(x,a)\right\} = \varepsilon\tilde{\rho}^{-1/2}(x) \tag{4.11}$$

in Ω, where $\tilde{\rho} \in C_b^1(\bar{\Omega})$ and $\tilde{\rho}(x) = \text{dist}(x, \bar{\Omega}^c)$ if $\text{dist}(x, \bar{\Omega}^c) < \delta$ for some $\delta > 0$.
It is clear from the proof of Theorem 4.1 that the following result holds.

Theorem 4.2. *The solutions u^ε and v^ε of* (4.14) *and* (4.15)*, respectively, converge as $\varepsilon \to 0$ locally uniformly in $\bar{\Omega}$, to the value function V.*

4C. Proof of Lemma 4.2. Our arguments in this proof are local and without loss of generality we may assume $\partial\Omega$ is flat near z ($\partial\Omega$ is C^3). With an appropriate change in coordinates, $D^2\rho(x) = 0$ and $D\rho(x) = (1, 0, \dots, 0)$ for $x \in B(z,r)$, $r > 0$ and $z = (0, \dots, 0)$.

Step 1. First consider the case where the feedback control given by (A3) satisfies

$$A(x,\alpha(x)) = \begin{bmatrix} 0 & & & & \\ & \lambda_2 & & 0 & \\ & & \lambda_3 & & \\ & 0 & & \ddots & \\ & & & & \lambda_n \end{bmatrix}, x \in B(z,r),\ \lambda_i = 0 \text{ or } 1, \tag{4.12}$$

$$b(x,\alpha(x)) = (1, 0, \dots, 0),\ x \in B(z,r)$$

and $\lambda_i = 0$ except at most two i's.

Now consider the case when $n = 2$. Define a cone $K_z = \{(x_1, x_2) : x_1^2 > k^2x_2^2\}$ for some $k > 0$. If $y = (y_1, y_2) \notin K_z$, the process given by (4.9) is $X_t = (X_t^{(1)}, X_t^{(2)})$, where

$$X_t^{(1)} = y_1 + t\,,$$
$$X_t^{(2)} = y_2 + W_t\,,$$

where $W.$ is a 1-dimensional brownian motion. Assume $y_2 > 0$, hence $y \notin K_z$ implies $y_1 < ky_2$; if τ^y is the exit time from $\bar{\Omega}\backslash K_z$ starting at $y \in \bar{\Omega}\backslash K_z$, we have

$X^1_{\tau^y \wedge n} \le k X^2_{\tau^y \wedge n}$, $\forall\, n \in \mathbb{N}$. Therefore, $y_1 + E\tau^y \wedge n \le k y_2$ and $E\tau^y \le k y_2 - y_1$. If $y \to z = (0,0)$, then $E\tau^y \to 0$. The same argument applies if $y_2 < 0$.

Now consider the case when $n \ge 3$. Let the cone $K_z = \{(x_1, \dots, x_n) : x_1^2 > k(x_2^2 + x_3^2 + \cdots + x_n^2)\}$.

In the previous proof the process could not "wind" around K_z because the domain was 2-dimensional. However, in general it is higher-dimensional and the previous proof cannot be repeated.

The process $X_t = (X_t^{(1)}, \dots, X_t^{(n)})$ given by (4.6), in view of (4.12) becomes:

$$X_t^{(1)} = y_1 + t,\ X_t^{(i)} = y_i + \lambda_i W_t^{(i)}, \qquad 2 \le i \le n \tag{4.13}$$

where $y = (y_1, \dots, y_n)$ and $W_\cdot = (W_\cdot^{(2)}, \dots, W_\cdot^{(n)})$ is a $n-1$ dimensional brownian notion.

We seek to prove that $\lim_{\substack{y \to z \\ y \in \bar{\Omega}}} E\tau^y = 0$; but exit times are usc, therefore $\overline{\lim_{\substack{y \to z \\ y \in \bar{\Omega}}}}\ \tau^y \le \tau^z$, hence it suffices to show $\tau^z = 0$. For $y = z = (0, \dots, 0)$, (4.13) becomes:

$$X_t^{(1)} = t, X_t^{(i)} = \lambda_i W_t^{(i)}, \qquad 2 \le i \le n. \tag{4.14}$$

Therefore, the exit time τ^z from $\bar{\Omega} \backslash K_z$ is essentially the exit time of $kR_t^{(2)} - t$ from $[0, \infty)$, where $R_t^{(m)}$ is the m-dimensional Bessel process $[(W_t^{(1)})^2 + \cdots + (W_t^{(m)})^2]^{1/2}$.

The local behavior of a Bessel process is described by the Dvoretzky-Erdös test which is an extension of Kolmogorov's test for Bessel processes:

Dvoretzky-Erdös Test. Let $\Psi(t) \downarrow 0$ as $t \downarrow 0$. Then

$$P\big\{R_t^{(m)} < \Psi(t)\sqrt{t} \text{ i.o. as } t \downarrow 0\big\} = 1 \text{ or } 0$$

according to

$$\int_{0+} \frac{\Psi(t)^{m-2}}{t}\, dt = \infty \text{ or } < \infty \text{ if } m > 2\,,$$
$$\int_{0+} \frac{1}{t|\log \Psi(t)|}\, dt = \infty \text{ or } < \infty \text{ if } m = 2\,. \qquad \square$$

For a proof see [SW] or [ItMcK].

It is clear from the definition of τ^z that $P\big\{R_t^{(m)} < t/k \text{ i.o. as } t \downarrow 0\big\} = 1$ or 0 implies $\tau^z = 0$ or $\tau^z > 0$, P–a.s. respectively. By setting $\Psi(t) = \sqrt{t}/k$, we notice that the first integral converges for $m > 2$ and the second diverges for $m = 2$. In the latter case, $\tau^z = 0 \Rightarrow \lim_{\substack{y \to z \\ y \in \bar{\Omega}}} E\tau^y = 0$. In the former case, $\tau^z > 0$, P–a.s. and we do not necessarily have $\lim_{\substack{y \to z \\ y \in \bar{\Omega}}} E\tau^y = 0$, P–a.s. Therefore, Lemma 4.3 holds, if the diffusion matrix $A(x, \alpha(x))$ in (A3), has at most two nondegenerate tangential directions, giving rise to an at most, two dimensional Bessel process. Otherwise, the noise overpowers the drift and the process does not hit K_z "soon enough."

Step 2. We now turn to the general case in (A3). For simplicity in the presentation, we deal only with the case $n = 3$ and $X_t = (X_t^1, Y_t)$ where $Y.$ is a non-degenerate 2-dimensional process:

$$\begin{cases} X_t^1 = t\,, \\ dY_t = \displaystyle\int_0^t b_1(Y_s, \alpha(Y_s))ds + \int_0^t \sigma_1(Y_s, \alpha(Y_s))\, dW_s\,. \end{cases} \tag{4.15}$$

From Step 1 it is clear that we need an analogue of Dvoretzky-Erdös's test for general processes. In fact, we can prove the following theorem:

Theorem 4.3. *Let $X.$ be the solution of the n-dimensional, nondegenerate SDE*

$$dX_t = b(X_t)\, dt + \sigma(X_t) dW_t, \qquad X_0 = 0. \tag{4.16}$$

Then for $\Psi(t) \downarrow 0$ as $t \downarrow 0$, we have

$$P(\|X_t\| < \Psi(t)\sqrt{t} \ i.o. \ as\ t \downarrow 0) = P(R_t^{(n)} < \Psi(t)\sqrt{t} \ i.o. \ as\ t \downarrow 0).$$

Outline of Proof. We follow the proof of the Dvoretzky-Erdös Test given in [ItMcK], pp. 161-164. The Borel-Cantelli Lemma and the strong Markov property of $X.$, imply that it suffices to determine when the series $\sum_{n=1}^{\infty} a_n$ converges or diverges, where $a_n = P(X. \in A_n)$, $A_n = \{x : t_{n+1} < \|x\| < t_n\}$ and $t_n \downarrow 0$. On the other hand,

$$P(X. \in A_n) = \int_{A_n} \Gamma(y, t; 0, 0)\, dy$$

where $\Gamma(x, t\ ;\ \xi, s)$, $x,\ \xi \in \mathbb{R}^n$, $t,\ s \in \mathbb{R}^+$ is the fundamental solution of the second order differential operator $\mathrm{tr}(A(x)D^2\cdot) + b(x)D\cdot$ and $A(x) = \sigma(x)\sigma^\tau(x)$.

The fundamental solution corresponding to the n-dimensional brownian motion is

$$Z(x,t\,;\xi,s)=\frac{1}{(2\sqrt{\pi})^n}(t-s)^{-n/2}\exp\left(-\frac{\|x-\xi\|^2}{4(t-s)}\right).$$

Following [Fr2], Chapter 1, we can show, using (2.8), (4.9), (4.15) and Lemma 4.3) (in [Fr2]), that

$$\begin{aligned}(4.17)\; MZ(x,s+h\,;\xi,s) &\geq \Gamma(x,s+h\,;\xi,s)\\ &\geq \mu Z(x,s+h\,;\xi,s)-\mu h^{1/2}Z(x,s+h\,;\xi,s)\end{aligned}$$

where M, μ are constants and x, $\xi\in\mathbb{R}^n$, s, $h\in\mathbb{R}^+$. Therefore,

$$(4.18)\qquad MZ(x,s+h\,;\xi,s)\geq\Gamma(x,s+h\,;\xi,s)\geq\tilde{\mu}Z(x,s+h\,;\xi,s)$$

for h small.

Hence $\sum_{n=1}^{\infty}P(X.\in A_n)$ diverges if $\sum_{n=1}^{\infty}P(W.\in A_n)$ diverges. This concludes the proof of Theorem 4.3.

As in Step 1, it is easy to verify that $\lim\limits_{\substack{y\to z\\ y\in\bar{\Omega}}} E\tau^y=0$ for all possible cones $K_z\subset\Omega$ with vertex at z.

Here we deal with the case $Y.$ being a nondegenerate process. The stochasticity forces the process to fluctuate violently and hit all cones K_z. If $\sigma_1\sigma_1^T$ is not strictly positive (this is a "nicer" situation because there is less randomness in the tangential direction to compete with the normal drift), it is not true any more: take for example $\sigma_1=0, b_1=(1,0)$. Nevertheless, it is not hard to see that in this case there is a cone $K_z=z+K(r,\beta,n(z))$, where β depends only on c of (A3)(i), such that Lemma 4.2 holds. □

4D. Piecewise smooth domains. In the financial economics applications of state constraints problems, it is quite natural to work with a domain Ω that arises from a number of constraints. In this case Ω is only piecewise smooth as it happens even in elementary examples of mathematical programming. Our methods extend for such problems without difficulty:

Assume Ω_k $k=1,\dots,n$ is a set of smooth domains and the constraints are $X_t\in\bar{\Omega}_k,\ k=1,\dots,n,\ t\geq 0,\ P$–a.s.

We let $\Omega=\bigcap_{k=1}^n\Omega_k$ and formulate (SC) for Ω. Then Theorems 4.1 and 4.2 hold, provided the following conditions are met:

(i) Ω has the interior cone property
(ii) (A3) holds on each $\partial\Omega_k$ $k = 1, \ldots, n$.

We need (i) in order to have comparison of solutions (see the remark following the proof of Theorem 2.2). Furthermore, if (A3) holds on $\partial\Omega_k \cap$ nbhd of Ω, $k = 1, \ldots, n$, then we can enforce it on $\partial\Omega_k$ with appropriate extensions of the drift and the diffusion matrix.

5. Conclusion. The existence of a continuous viscosity solution of (SC) is guaranteed if there is a boundary control (given by (A3) for example) that can push quickly the state $X.$ on a cone strictly contained in Ω, with vertex on $\partial\Omega$. In this process, there is a competition between the normal drift and the tangential randomness of the control. The local properties of the n-dimensional brownian motion, as described by the Dvoretzky-Erdös Test, indicate how much "randomness" is allowed and this is contained in the assumption (A3)(iii).

Similar dimensional phenomena arise in the solvability of the Dirichlet Problem for Laplace's equation. If $n = 2$, the problem is solvable for any domain Ω such that $\mathbb{R}^2 \backslash \Omega$ is pathwise connected. For $n \geq 3$, this is not true any more and Ω must not have interior cusps ($\partial\Omega$ should have the exterior cone property). In the probablistic language this means that the 2-dimensional brownian motion can exit through cusps while the higher dimensional cannot.

Concluding, we present an example of a state-constrained control problem whose value function V is the unique continuous viscosity solution of (SC), if and only if the dimension is $n = 2$. Our example is not contained in (A3) but we can prove that when $n = 2$, V is continuous, while for $n \geq 3$ it is not; this phenomenon is due to the dimensional dependence of the brownian motion, as pointed out in the previous discussion. We return to the control setting described in Section 1.

Let $\Omega = \{(x_1, \ldots, x_n) : x_i \in \mathbb{R}\ i = 1, \ldots, n,\ x_1 \geq 0\}$ and the control set $A = \{\alpha, \beta\}$, defining the dynamics

$$dX_t = b(x_t, i)\, dt + \sigma(X_t, i)\, dW_t, \qquad i = \alpha, \beta,$$

where

$$\begin{aligned} b(x, \alpha) &= \Big(-\Big(\sum_{1=2}^{n} |x_i|^2 \Big)^{1/2},\ 0, \ldots, 0 \Big), \\ \sigma(x, \alpha) &= 0 \quad (\text{zero } n \times n \text{ matrix}), \end{aligned}$$

and

$$b(x, \beta) = (0, \ldots, 0),$$

$$\sigma(x,\beta) \;=\; \begin{pmatrix} 0 & & 0 \\ & 1 & \\ & & \cdots \\ 0 & & 1 \end{pmatrix}.$$

The corresponding running costs are

$$h(x,\alpha)=0,\;\; h(x,\beta)=1.$$

It is not hard to see that the optimal control is

$$\gamma_s = \begin{cases} \alpha, & 0 \le s \le \sigma^x, \\ \beta, & \sigma^x < s < \tau_{X_{\sigma^x}}, \\ \alpha, & s \ge \tau_{X_{\sigma^x}}, \end{cases}$$

where σ^x is the hitting time of $\partial\Omega$ and τ_0^x the hitting time of $(0,\ldots,0)$, starting at a point x. Notice that when the state X_t is on $\partial\Omega\backslash\{(0,\ldots,0)\}$ we cannot use the "cheap" control α because X_t will exit $\bar{\Omega}$ immediately. For $n \ge 3$ and $x \in \partial\Omega\backslash\{(0,\ldots,0)\}$, $\tau_0^x = \infty$ because the origin is unattainable for the $n-1$ dimensional brownian motion $(W^{(2)},\ldots,W^{(n)})$, given by the control β (see [KS], Proposition 3.3.22). Therefore, $V(x)=1$ if $x \in \partial\Omega,\; x \neq (0,\ldots,0)$, while $V((0,\ldots,0))=0$, hence V is discontinuous.

If $n=2$, the origin is attainable for the 1-dimensional brownian motion and $E\; e^{-\tau_0^x} = e^{-x_2}$ for $x=(0,x_2)\in\partial\Omega$, according to Remark 2.8.3 in [KS]. Since $V(x)=J(x,\gamma)$, it is immediate that V is continuous in $\bar{\Omega}$. Furthermore, it is not hard to see that for $n=2$, V is nontangentially upper semicontinuous due to the control β. However, if $n \ge 3$, this control contains "too much randomness," thus V does not have the aforementioned property.

Acknowledgments This work is part of my Ph. D. thesis under the supervision of Professor P. E. Souganidis. I would like to thank him for his guidance throughout this process and also thank Professor W. H. Fleming for many illuminating discussions.

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Received: September 17th, 1992.