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On the Regularity Theory of Fully Nonlinear Parabolic Equations: II

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This is the second part of a series of papers to study the regularity problem of parabolic equations.

In Part I (see [15]), we proved the fundamental estimates of tangent paraboloids for solutions. From these, we obtained a Harnack inequality and $W^{2,p}$ regularity.

We now continue our investigation of interior and boundary regularity in Hölder spaces. Section 1 is devoted to $C^{2,\alpha}$ and $C^{1,\alpha}$ Schauder-type pointwise estimates under conditions in the space L^{n+1}. We use barriers to prove boundary regularities in Section 2. We establish $C^{1,\alpha}$ estimates on the lateral boundary and $C^{2,\alpha}$ estimates on the bottom and corner without any additional smoothness requirements. We develop a general machinery for global estimates in Section 3. Section 4 gives estimates for homogeneous equations. These provide a wide range of applications of both the theory developed in the previous sections and the results in Part I. We shall prove higher regularity estimates for at bottom points in Part III.

We use the notation Theorem I-2.10, for example, to indicate Theorem 2.10 in Part I.

1. Pointwise Regularity

In this section, we establish Hölder estimates.

In Section 1.1, we establish Hölder continuity estimates for the second derivatives of solutions to equation without the term Du. In Section 1.2, we prove Hölder continuity for the first-order derivatives for solutions of general parabolic equations.

From the results in Section 1.2, we can extend the results in Section 1.1 and $W^{2,p}$ estimates to general equations as well.

1.1. Schauder Estimates

First, we prove Schauder estimates, since they do not need additional machinery. In the case of the classical heat equation

$$u_t - \Delta u = g(x,t)\,,$$

our techniques can be used to prove Schauder estimates by the classical maximum principle. This is a much better way to get Schauder estimates than the standard singular integral approach.

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In this section, we consider the equation

$$u_t - F(D^2u, x, t) = g(x, t) \tag{1.1}$$

with the uniformly elliptic condition.

Recall the definition of function θ is

$$\theta(x,t) = \sup_M \frac{|F(M,x,t) - F(M,0,0)|}{|M|+1}.$$

The main result of this section is the following.

THEOREM 1.1. *Let u be a solution of (1.1) in Q_1. If any solution v of*

$$v_t - F(D^2v + C, 0, 0) = D,$$

where C and D are on the surface $F(C,0,0) = D$, has interior $C^{2,\beta}$ estimate

$$\|v\|_{C^{2,\beta}(Q_{r/2})} \leqq C\frac{\|v\|_{\infty,Q_r}}{r^{2+\beta}},$$

then u is $C^{2,\alpha}$ at $(0,0)$ for $\alpha < \beta$, provided

$$\left(⨍_{Q_r} |\theta|^{n+1}\right)^{1/(n+1)} \leqq Cr^\alpha$$

and

$$[g - g(0,0)]_{n+1,\alpha}(0,0) = \sup \frac{1}{r^\alpha}\left(⨍_{Q_r} |g - g(0,0)|^{n+1}\right)^{1/(n+1)} \leqq C.$$

Let us prove a discrete version of Theorem 1.1 first.

LEMMA 1.2. *Let u and F be as in Theorem 1.1. Then there are paraboloids*

$$P_k = A_k t + \tfrac{1}{2}x^T B_k x + C_k x + D_k,$$

such that

$$A_k = F(B_k, 0, 0)$$

$$|u - P_k|_{\infty,Q_{\lambda^k}} \leqq \lambda^{(2+\alpha)k}$$

and

$$|P_k - P_{k+1}|_{2,\lambda^k} \leqq C\lambda^{(2+\alpha)k}.$$

Moreover, A_k, B_k, C_k, and D_k are bounded. In fact,

$$|A_k| + |B_k| + |C_k| + |D_k| \leqq C(\lambda^{\alpha k} + \lambda^{\alpha(k-1)} + \cdots + 1).$$

Proof: Without loss of generality, we may suppose that $g(0,0) = 0$, $|u| \leqq 1$ in Q_1 and that $[g]$ and $[\theta]$ are small by subtracting a quadratic polynomial and by expanding the variables.

We prove it by induction. We take $A_0 = B_0 = C_0 = D_0 = 0$. Suppose now that it is true for k. Let

$$w(x,t) = \frac{(u - P_k)(\lambda^k x, \lambda^{2k} t)}{\lambda^{(2+\alpha)k}}.$$

Then $|w| \leqq 1$ and

$$w_t - F_k(D^2 w, x, t) = g_k,$$

where

$$\begin{aligned} F_k(M,x,t) &= \tfrac{1}{\lambda^{\alpha k}}\left(F\left(\lambda^{\alpha k} M + B_k, \lambda^k x, \lambda^{2k} t\right) - F\left(B_k, \lambda^k x, \lambda^{2k} t\right)\right), \\ g_k(x,t) &= \tfrac{1}{\lambda^{\alpha k}}\left(g\left(\lambda^k x, \lambda^{2k} t\right) - F\left(B_k, \lambda^k x, \lambda^{2k} t\right) + A_k\right) \end{aligned}$$

and

$$\begin{aligned} \theta_k(x,t) &= \sup\left(\left|\frac{F\left(\lambda^{\alpha k} M + B_k, \lambda^k x, \lambda^{2k} t\right) - F(B_k,0,0)}{\lambda^{\alpha k}(|M|+1)}\right.\right. \\ &\qquad \left.\left. + \frac{F\left(B_k, \lambda^k x, \lambda^{2k} t\right) - F(B_k,0,0)}{\lambda^{\alpha k}(|M|+1)}\right|\right) \\ &\leqq \sup \frac{|\lambda^{\alpha k} M + B_k| + |B_k| + 1}{\lambda^{\alpha k}(|M|+1)}\theta\left(\lambda^k x, \lambda^{2k} t\right) \\ &\leqq C\theta\left(\lambda^k x, \lambda^{2k} t\right). \end{aligned}$$

By the approximation lemma (Theorem I-5.1), there is a function v such that

$$v_t - F_k(D^2 v, 0, 0) = 0,$$

$$\|w - v\|_{\infty, Q_{1/2}} \leqq C(\varepsilon^\gamma + \|g_k\|_{n+1})$$

and

$$|w(x,t) - P(x,t)| \leqq C(\varepsilon^\gamma + \|g_k\|) + C(|x|^2 - t)^{1+\beta/2}$$
$$\leqq C(\varepsilon^\gamma + \|g_k\|) + C\lambda^{2+\beta} \quad \text{on } Q_\lambda,$$

where P is the second-order Taylor expansion of v at $(0,0)$ and C depends only on the interior $C^{2,\beta}$ estimates of v.

Now, take

$$\|g_k(x,t)\|_{n+1} = \frac{1}{\lambda^{\alpha k}} \left(⨍_{Q_{\lambda^k}} |g - F(B_k, x, t)|^{n+1} \right)^{1/(n+1)}$$

and

$$\leqq [g]_{\alpha,n+1} + C[\theta]_{\alpha,n+1} \leqq \delta,$$

$$\|\theta_k(w)\|_{n+1} = \left(⨍_{Q_{\lambda^k}} \theta_{B_k}^{n+1} \right)^{1/(n+1)}$$
$$\leqq \frac{1+|B_k|}{\lambda^{\alpha k}} \left(⨍_{Q_{\lambda^k}} |\theta|^{n+1} \right)^{1/(n+1)}$$
$$\leqq C[\theta]_{\alpha,n+1} \leqq C\delta.$$

We first choose λ and then δ so that

$$\begin{aligned} \|w - P\|_{\infty,Q_\lambda} &\leqq C(\varepsilon^\gamma + \|g_k\|) + C\lambda^{2+\beta} \\ &\leqq C(\varepsilon^\gamma + \|g_k\|) + \frac{1}{3}\lambda^{2+\alpha} \\ &\leqq \frac{2}{3}\lambda^{2+\alpha} + \frac{1}{3}\lambda^{2+\alpha} = \lambda^{2+\alpha}. \end{aligned}$$

Rescaling back, we obtain

$$\|u - P_{k+1}\| \leqq \lambda^{(k+1)(2+\alpha)},$$

where

$$P_{k+1}(x,t) = P_k(x,t) + \lambda^{k(2+\alpha)} P\left(\frac{x}{\lambda^k}, \frac{t}{\lambda^{2k}}\right).$$

Clearly P_{k+1} satisfies the required conditions.

Now, we prove Theorem 1.1.

Proof: First, we show P_k are convergent to a second-order polynomial $P_\infty(x,t)$, such that

$$[P_\infty - P_k]_{2,\lambda^k} \leqq C\lambda^{k(2+\alpha)}.$$

In fact, we have that the coefficients of P_k are convergent:

$$\begin{aligned}
&\lambda^{2k}|A_m - A_{k-1}| + \lambda^{2k}|B_m - B_{k-1}| + \lambda^k|C_m - C_{k-1}| + |D_m - D_{k-1}| \\
&\leqq C(\lambda^{2k}\left(\lambda^{\alpha k} + \lambda^{\alpha(k+1)} + \cdots + \lambda^{\alpha(m-1)}\right) \\
&\leqq C\lambda^{k(2+\alpha)}.
\end{aligned}$$

Hence, A_k, B_k, C_k, and D_k converge. Let

$$A_\infty = \lim A_k, \quad B_\infty = \lim B_k, \quad C_\infty = \lim C_k, \quad D_\infty = \lim D_k$$

and let

$$P_\infty(x,t) = A_\infty t + \tfrac{1}{2}x^T B_\infty x + C_\infty x + D_\infty.$$

Clearly, we have

$$|P_k - P_\infty| \leqq C\lambda^{(2+\alpha)k} \quad \text{on } Q_{\lambda^k}.$$

Consequently, we have

$$|u(x,t) - P_\infty(x,t)| \leqq C(|x|^2 - t)^{1+\alpha/2}.$$

This completes the proof for Theorem 1.1.

Remark. We can prove that the solution is second-order differentiable under the following Dini condition:

$$\int_{0+} \left(\fint_{Q_r} |\theta|^{n+1} \right)^{1/(n+1)} + \left(\fint_{Q_r} |g(x,t) - g(0,0)|^{n+1} \right)^{1/(n+1)} \frac{dr}{r} < \infty.$$

This is exactly the condition for the convergence of the approximation polynomials.

1.2. $C^{1,\alpha}$ Estimates

This section is the most delicate part in pointwise regularity theory. Here we do not assume that the homogeneous equations, obtained by fixing lower order variables, have second-order estimates. Consequently, we cannot apply the approximation lemma (Theorem I-5.1).

We introduce a new technique called the *compactness method* as described in Lemma 1.4, in order to obtain $C^{1,\alpha}$ estimates for general parabolic equations.

Let $F(M,p,x,t)$ be a general uniformly elliptic operator. We introduce a oscillation function for F as follows.

$$\theta_h(x,t) = \sup_{|q|\leqq h} \frac{|F(M,p+q,x,t) - F(M,p,0,0)|}{|M|+1}, \tag{1.2}$$

where the supremum is taken over all M and p.

It is worth pointing out that we always assume Du appears on the left-hand side of the equation. That is, Du appears in the expression for ellipticity constants. The case when Du appears on the right-hand side of the equation is easy to handle, as shown in Theorem 1.7, where we need a growth condition in Du for the $C^{1,\alpha}$ estimates. To the contrast, we need continuity (or small oscillation) of F in Du when Du appears in the left-hand side. See the following theorem.

THEOREM 1.3. *Let u be a bounded solution of* (1.3):

$$u_t - F(D^2u, Du, x, t) = g(x,t). \tag{1.3}$$

(a) *Suppose that the solution v of*

$$v_t - F(D^2v, P, 0, 0) = 0$$

has interior $C^{1,\beta}$ estimates, i.e.,

$$[v]_{C^{1,\beta}(Q_{r/2})} \leqq C\frac{\|v\|_{\infty,Q_r}}{r^{1+\beta}}.$$

Then there exists a $\delta_0 = \delta_0(\beta,\lambda,\Lambda)$ such that if

$$\overline{\lim_{r\to 0}}\left(\overline{\lim_{h\to 0}}\left(⨍_{Q_r} |\theta_h|^{n+1}\right)^{1/(n+1)}\right) + \|u\|_\infty \leqq \delta_0,$$

then u is $C^{1,\alpha}$ at $(0,0)$ for $\alpha < \beta$, provided

$$\left(⨍_{Q_r} |g|^{n+1}\right)^{1/(n+1)} \leqq Cr^{-1+\alpha}.$$

(b) *Suppose that the solution v of*

$$v_t - F(D^2v, Dv + P, 0, 0) = 0 \tag{1.4}$$

has interior $C^{1,\beta}$ estimates.

Then there exists a $\delta_0 = \delta_0(\beta,\lambda,\Lambda)$ such that if

$$\overline{\lim_{r\to 0}}\left(⨍_{Q_r} |\theta_\infty|^{n+1}\right)^{1/(n+1)} \leqq \delta_0,$$

then u is $C^{1,\alpha}$ at $(0,0)$ for $\alpha < \beta$, provided

$$\left(\fint_{Q_r} |g|^{n+1}\right)^{1/(n+1)} \leqq Cr^{-1+\alpha}.$$

We need the following important lemma to prove the above theorem. We prove Lemma 1.4 by the method of compactness.

LEMMA 1.4. (Approximation Lemma) *Let u be a solution of equation* (1.5)

$$u_t - F(D^2u, Du, x, t) = g(x,t) \tag{1.5}$$

in Q_1 with $|u| \leqq 1$. Then, for any $\varepsilon > 0$, there exist $\delta(\varepsilon, \lambda, \Lambda)$ and $h(\varepsilon, \lambda, \Lambda)$ such that if

$$\begin{cases} v_t - F(D^2v, 0, 0, 0) &= 0 \quad \text{on } Q_{1/2} \\ \qquad\qquad v|_{\partial_p Q_{1/2}} &= u\,, \end{cases}$$

then

$$\|u - v\|_{L^\infty(Q_{1/2})} \leqq \varepsilon\,,$$

provided that (a small oscillation condition)

$$\|\theta_h\|_{n+1} + \|g\|_{n+1} \leqq \delta(\varepsilon, \lambda, \Lambda) \tag{1.6}$$

for $h \geqq h(\varepsilon, \lambda, \Lambda)$.

Proof: We prove it by contradiction. Consider a uniformly convergent sequence u^k which are solutions of the equations

$$u_t^k - F^k(D^2u, Du^k, x, t) = g^k$$

with

$$g^k \to 0 \quad \text{and} \quad \theta^k \to 0.$$

We also assume that $F^k(\cdot, 0, 0, 0)$ is locally uniformly convergent to $F(\cdot)$. Let $u^\infty = \lim u^k$ and v^k be the corresponding solutions for the homogeneous equations with v^∞ as their limit. We want to show that both u^∞ and v^∞ are solutions of the problem

$$v_t - F(D^2v) = 0 \quad \text{in } Q_{1/2}, \quad v = u^\infty \quad \text{on } \partial_p Q_{1/2}$$

which contradict the uniqueness theorem; see [7].

Given a $W^{2,p}$ supersolution ϕ in an open set $A \subset Q_{1/2}$ of

$$\phi_t - F(D^2\phi) \geqq 0,$$

we shall show that $u^\infty - \phi$ cannot have a local maximum.

Clearly, in A, $|F^k(D^2\phi) - F(D^2\phi)| = e^k(x,t) \to 0$ in L^p. We perturb ϕ to a supersolution of $\widetilde{\phi}_t - F^k(D^2\widetilde{\phi}, \widetilde{\phi}, x, t) \geqq (g^k)^+$ by adding to ϕ a solution, ψ^k, of the extremal operators (see Lemma I-6.4)

$$\psi_t^k - \frac{1}{\varepsilon}\sum \lambda_j^+ \left(D^2\psi^k\right) + \varepsilon \sum \lambda_j^- \left(D^2\psi^k\right) = |g^k| + |\theta^k||D^2\psi^k| + e^k + C\Lambda\chi_{\{D\psi^k \geqq k\}}.$$

Here $\varepsilon = \varepsilon(n)$, λ_j^+ (or λ_k^-) are the positive (or negative) eigenvalues of $D^2\psi^k$.

We decompose the matrix $D^2\psi^k$ into positive and negative parts

$$M^k = D^2\psi^k = M^{k,+} - M^{k,-}.$$

Using the ellipticity condition, we obtain

$$\begin{aligned}
&(\psi + \phi)_t - F^k(D^2(\phi + \psi), D(\phi + \psi), x, t) \\
\geqq\ & \phi_t - F^k(D^2\phi, D(\phi + \psi), x, t) + \psi_t + \lambda\|M^{k,-}\| - \Lambda\|M^{k,+}\| \\
\geqq\ & \phi_t - F^k(D^2\phi) + \psi_t - e^k - C\theta_k^k(x,t)|D^2\psi| \\
& - \Lambda\|M^{k,+}\| + \lambda\|M^{k,-}\| - \Lambda\chi_{\{D\psi^k > k\}} \\
\geqq\ & |g^k|
\end{aligned}$$

by a proper choice of ε. Since $e^k \to 0$ and $g^k \to 0$ in L^{n+1}, we have that ψ^k goes to zero and we recover the maximum principle for u^∞. This shows that u^∞ is a solution of the homogeneous equation. So is v^∞.

Now, we prove Theorem 1.3. We only prove part (a) since part (b) can be proved similarly.

As before, we first prove a discrete version of it. Assume that solutions of the equation

(1.7) $$v_t - F(D^2v, P_0, 0, 0) = 0.$$

have $C^{1,\beta}$ estimates.

Lemma 1.5. *There exist* $0 < \varepsilon_0, \lambda_0 \leqq 1$ *such that if*

$$\theta_{r^\alpha N_0}(r) \leqq \varepsilon_0$$

and

$$\left(⨍_{Q_r} |g|^{n+1}\right)^{1/(n+1)} \leqq \varepsilon_0 r^{-1+\alpha},$$

then we can find a sequence of linear functions

$$L_k(x) = A_k x + B_k$$

such that

$$\|u - L_k(x)\|_{\infty, Q_{\lambda^k}} \leqq \lambda^{k(1+\alpha)}$$

and

$$\lambda^k |A_{k+1} - A_k| + |B_{k+1} - B_k| \leqq C\lambda^{(1+\alpha)(1+k)}.$$

Proof: We prove it by induction on k. The conclusion for $k = 0$ is satisfied by taking $L_0 = 0$. Now, suppose the lemma is true for k. Let

$$w(x,t) = \frac{(u - L_k)(\lambda^k x, \lambda^{2k} t)}{\lambda^{(1+\alpha)k}}.$$

Then, w is a solution of

$$w_t - F_k(D^2 w, Dw, x, t) = g_k(x,t),$$

where

$$\begin{aligned} F_k(M,P,x,t) &= \frac{1}{\lambda^{(-1+\alpha)k}} F\left(\lambda^{(-1+\alpha)k} M, \lambda^{\alpha k} P + DL_k, \lambda^k x, \lambda^{2k} t\right) \\ g_k(x,t) &= \frac{g(\lambda^k x, \lambda^{2k} t)}{\lambda^{(-1+\alpha)k}}. \end{aligned}$$

Let v be the solution of the following problem:

$$\begin{cases} v_t - F_k(D^2 v, DL_k, 0, 0) = 0 \\ v|_{\partial_{1/2} Q} = w. \end{cases}$$

By the approximation Lemma 1.4, we have

$$|w - v| \leqq \delta \quad \text{for } \varepsilon_0 \leqq \varepsilon(\delta)$$

(δ, ε_0 to be determined). Then, by the interior $C^{1,\beta}$ estimates, we have

$$|w - Dv(0,0)x - v(0,0)| = |w - Lx| \leqq \delta + C\lambda^{1+\beta} \quad \text{for } |(x,t)| \leqq \lambda.$$

Taking $C\lambda^{1+\beta} = \frac{1}{2}\lambda^{1+\alpha}$ and then $\delta = \frac{1}{2}\lambda^{1+\alpha}$, we obtain

$$|w - Lx| \leqq \lambda^{1+\alpha}.$$

In terms of u,

$$|u - L_{k+1}| \leqq \lambda^{(1+\alpha)(k+1)},$$

where $L_{k+1}(x) = L_k - \lambda^{(1+\alpha)k}L(x/\lambda^k)$. It is clear that L_{k+1} satisfies the required conditions.

As the proof of Theorem 1.1, we have

$$L_k(x) \to L_\infty(x).$$

Now, we scale out of N in the above lemma for flat solutions. Let $w(x,t) = Nu(x,t)$. Then w is a solution of

$$w_t - F_N(D^2w, Dw, x, t) = g_N(x,t),$$

where

$$F_N(M, P, x, t) = NF\left(\frac{M}{N}, \frac{P}{N}, x, t\right)$$

and

$$g_N(x,t) = Ng(x,t).$$

Clearly

$$\theta_{r,N}(x,t) = \theta_{r/N}(x,t),$$

$$g_N(x,t) = Ng(x,t).$$

Taking N large enough so that

$$\|\theta_{r,N}\| \leqq \varepsilon_0,$$

where ε_0 is as in Lemma 1.4, we have w is $C^{1,\alpha}$ if $|w| \leqq 1$. That is, if u is small, then u is $C^{1,\alpha}$. Theorem 1.3 follows.

Remark 1. Under similar conditions, we can prove $C^{1,\alpha}$ for flat solutions up to a linear function.

Remark 2. From Theorem I-4.11, we know that the solutions of (1.3) are flat up to a linear function almost everywhere. Hence, solutions of (1.3) are $C^{1,\alpha}$ in an open dense set.

1.3. $C^{1,\alpha}$ Estimates for Equations with Nonlinear Right-Hand Side

Let u be a solution of

$$u_t - F(D^2u, x, t) = G(Du, x, t)\,, \tag{1.8}$$

where G satisfies the condition

$$|g(P, x, t)| \leqq A|P|^2 + g(x, t) \quad \text{for some } g \in L^{n+1}\ . \tag{1.9}$$

The constant A in (1.9) is invariant by expanding the coordinates, (x, t) to (rx, r^2t) . However, we have flatter solutions in this case. In other words, we may suppose A is small in (1.9).

We want a similar regularity for the solution of (1.8) with condition (1.9). First, we prove the main step for it.

LEMMA 1.6. *Let $|u| \leqq 1$ be a solution of (1.8) as above. Then for any ε, there exist $v_i, i = 1, 2$ and a constant $\delta(\varepsilon)$ such that*

$$\begin{cases} (v_i)_t - F(D^2v_i, 0, 0) &= 0 \\ \qquad |v_i|_{\partial_p Q_{1/2}}| &\leqq 1 \end{cases}$$

and

$$-\varepsilon + v_1 \leqq u \leqq \varepsilon + v_2\,,$$

provided

$$\|g\|_{n+1} + \|\theta\|_{n+1, Q_{1/2}} \leqq \delta(\varepsilon)\,.$$

Proof: Since $u \in C^\alpha$, we can assume that u is small in L^∞ by expanding the coordinates. Equivalently, we assume that A is small.

Assume also that $u(0, 0) = 0$. Let, for some constant p,

$$w = \varphi(u) = u \exp\{pu\} \quad \text{and} \quad u_p = 1 + pu\,.$$

Then

$$w_i = u_i \exp\{pu\} u_p$$

and

$$w_{ij} = u_{ij} \exp\{pu\} u_p + u_i u_j \exp\{pu\} p(u_p + 1)\,.$$

Therefore, w is a solution of

$$w_t - \widetilde{F}(D^2w, Dw, x, t) = \widetilde{g}(Dw, x, t)\,. \tag{1.10}$$

Here

$$\widetilde{F}(M,P,x,t) =$$

$$\frac{u_p(x,t)}{\exp\{-pu(x,t)\}}\cdot F\left(\frac{\exp\{-pu(x,t)\}}{u_p(x,t)}M_{ij} - \frac{p(u_p+1)}{u_p(x,t)^2\exp\{pu(x,t)\}}P_iP_j, x, t\right)$$

and

$$\widetilde{g}(P,x,t) = \frac{u_p(x,t)}{\exp\{-pu(x,t)\}}g\left(\frac{\exp\{-pu(x,t)\}}{1+pu}P, x, t\right).$$

We have

$$|\widetilde{g}(P,x,t)| \leqq 2e|P| + e|g(x,t)|.$$

Since $\widetilde{F}(M,P,x,t)-\widetilde{F}(M,0,x,t) \approx -p|P|^2$, for small A, we can take p_0 small so that

$$w_1 = w_{p_0}$$

and

$$w_2 = w_{-p_0}$$

are supersolution and subsolution respectively of

$$v_t - \widetilde{F}(D^2v,0,x,t) = \widetilde{g}.$$

Let V_i be the solutions of

$$\begin{cases} (V_i)_t - \widetilde{F}(D^2V_i,0,0,0) &= 0 \\ V_i|_{\partial_pQ_{1/2}} &= w_i. \end{cases}$$

By the approximation lemma, we have

$$w_1 \geqq V_1 - \varepsilon \quad \text{and} \quad w_2 \leqq V_2 + \varepsilon,$$

provided

$$\|g\|_{n+1} \leqq \delta(\varepsilon) \quad \text{and} \quad \|\theta_{N\varepsilon^\alpha}\|_{n+1} \leqq \delta.$$

The lemma follows by setting

$$v_i = \varphi^{-1}(V_i).$$

By the above lemma, Theorem 1.7 follows easily. We remark that a similar theorem was proved in [13].

THEOREM 1.7. *Let u be a solution of* (1.8) *with condition* (1.9). *Then we have $C^{1,\alpha}$ estimates as in Theorem* 1.3.

Proof: This is an easy corollary of the above lemma.

2. Pointwise Boundary Regularity

In this section, we prove boundary estimates for solutions. The theory is divided into two parts. The first part is concerned with estimates without the small oscillation conditions. We have $C^{1,\alpha}$ regularity on the lateral boundary and $C^{2,\alpha}$ regularity on the bottom (including the corner), assuming only uniformly ellipticity condition. The second part is concerned with estimates with small oscillation conditions. Since the second part is parallel to the previous section and actually easier, we shall present mainly the first part.

2.1. $C^{1,\alpha}$ Regularity on the Lateral Boundary

In this section we prove $C^{1,\alpha}$ regularity on the lateral boundary. A simplified version of this is due to Krylov. We take a completely different approach, and our result is stronger and the proof is simpler than his.

Let Ω be a domain and $0 \in \partial_p\Omega$. We adopt the following notation:

$$T_r = \{x' \in \mathbb{R}^{n-1} : |x'| < r\} \tag{2.1}$$

$$S_r = T_r \times (-r^2, 0] \subset \mathbb{R}^n_x \times \mathbb{R}^1_t \tag{2.2}$$

$$Q_r^+ = T_r \times (0, r) \times (-r^2, 0]. \tag{2.3}$$

Similarly, define $T_r(x,t), S_r(x,t)$, and $Q_r^+(x,t)$ as translations of them. Let u be a continuous viscosity solution of $S(g) =: S(0,0,g)$ as in Section 2. We only need to modify the proof slightly to cover the estimates for a more general class such as $S(b,c,g)$.

Suppose $\partial_p\Omega$ is $C^{1,\alpha}$ at 0 in the following sense: there exist r_0 and C such that, for $r \leqq r_0$,

$$S_r \times \left(Cr^{1+\alpha}, Cr^{1+\alpha} + r\right) \subset \Omega \quad \text{and} \quad S_r \times \left(-Cr^{1+\alpha} - r, Cr^{1+\alpha}\right) \subset \Omega^C. \tag{2.4}$$

We need some special notation in this section:

$$\partial\Omega_r = (\partial_p\Omega) \cap S_r \times \left(-Cr^{1+\alpha} - r, Cr^{1+\alpha} + r\right) \tag{2.5}$$

$$\Omega_r = \Omega \cap S_r \times (-r, r). \tag{2.6}$$

We can define $\partial_p\Omega$ to be $C^{1,\alpha}$ up to a slope at $0 \in \partial\Omega$ in an obvious way.

In fact, we might consider the differentiability of $\partial\Omega$ at 0. We shall discuss this in the remark at the end of this section.

THEOREM 2.1. *Let u be a solution of $S(g)$ in Ω_1 and assume $\partial_p\Omega$ is $C^{1,\alpha}$ at 0 in the above sense. Let*

$$u|_{\partial\Omega_1} = \varphi .$$

Suppose φ is $C^{1,\alpha}$ at 0

$$|\varphi(x,t) - \varphi(0) - Ax'| \leqq D|x'|^{1+\alpha}$$

for some vector A denoted by $D\varphi(0)$, and

$$\left(⨍_{\Omega_r} |g|^{n+1}\, dx\right)^{1/(n+1)} \leqq Er^{\alpha-1} .$$

Then there exist B, C, and $\delta = \delta(\alpha) > 0$ such that

$$|u(x,t) - u(0) - D\varphi(0)x' - Bx_n| \leqq C(|x|^2 - t)^{\frac{1}{2}(1+\delta)}, \tag{2.7}$$

where $C = C(D,E)$ and $|B| \leqq C\|u\|_\infty(\Omega_1)$.

Before proving this theorem, let us introduce a barrier function. It is related to the construction of the function p in Section 2 of Part I.

Let $\Gamma(x,t) = (1/t)\exp\{-(|x|^2/t)\}$. First, we make some observations on the level surfaces of $\Gamma(x,t)$. Let $\eta > 0$ and

$$C_\eta = \{(x,t)|\Gamma(x,t) = \eta\} .$$

Then C_η is a smooth surface in $\mathbb{R}^{n+1} - \{0\}$. The top and the bottom of C_η are on the t-axis, C_η is inside $C_{\eta'}$ if $\eta > \eta'$ and $C_\eta \to 0$ as $\eta \to 0$. $C_\eta \cap \{t\}$ are spheres and their largest radius is $\sqrt{(n/2e)t} = 1/e\ \eta^{2/n}$; $\{(x,t)|x| = r\}$ is tangent to C_η at $t = 4r^2$; $\sup\Gamma(\frac{1}{2},t) = \Gamma(\frac{1}{2},1)$. Let $\Gamma(r,t)$ be such that $\Gamma(|x|,t) = \Gamma(x,t)$ with abuse of notation.

Let us study the parabolicity of big power Γ^N of $\Gamma(x,t)$ for large N:

$$\Gamma_r^N = -\frac{8Nr}{t}\Gamma^N. \tag{2.8}$$

Then we have

$$\Gamma_{rr}^N = \frac{(8Nr^2 - t)}{t}\frac{8N}{t}\Gamma^N \tag{2.9}$$

$$\Gamma_t^N = \frac{N}{t^2}\Gamma^N(4r^2 - t) \tag{2.10}$$

$$\Gamma_{rr}^N > 0 \quad \text{for } t < 8Nr^2. \tag{2.11}$$

Since Γ^N is radial, we have

$$\begin{aligned} e_n &= \Gamma_{rr}^N \\ e_1 &= \frac{\Gamma_r^N}{r}, \end{aligned}$$

where e_1 and e_n are the smallest and largest eigenvalues of $D^2\Gamma^N$, respectively. Taking N large enough, we have

$$\Gamma_t^N - \mathscr{M}^-\left(D^2\Gamma^N\right) = \frac{N\Gamma^N}{t}\left(-1 + \frac{4r^2}{t} - \lambda\frac{64Nr^2}{t} + 8\lambda + 8r\Lambda\right) < 0$$

for $r \geqq \frac{1}{2}, t \leqq 1$.

Let

$$P(x,t) = P(r,t) = \frac{\Gamma^N(r,t) - \Gamma^N(1,1)}{\Gamma^N(1/2,1) - \Gamma^N(1,1)}$$

and let

$$\mathbb{R} = (B_1 - B_{1/2}) \times (0,1].$$

Then

$$\begin{aligned} P_t - \mathscr{M}^-(P) &\leqq 0 \text{ in } \mathbb{R}, \\ P|_{\partial_p R - \partial B_{1/2}\times(0,1/2)} &\leqq 0 \end{aligned}$$

and

$$\begin{aligned} P(\tfrac{1}{2},1) &= 1 \\ P &\leqq 1 \\ P(1,1) &= 0 \\ P_r &\leqq 0 \\ P_{rr} &> 0 \quad \text{for } t > 0 \\ P &\leqq 0 \quad \text{for } r > 1 \\ P(r,1) &> 0 \quad \text{for } r < 1. \end{aligned}$$

In particular, we have

$$P(r,1) \geqq \frac{\partial P}{\partial r}(1,1)r > 0 \quad \text{for } r < 1. \tag{2.12}$$

We define the following:

$$P_r(x,t) = P\left(r^{-1}x, r^{-2}t + 1\right) \tag{2.13}$$

$$P_{(y,s),r}(x,t) = P_r(x-y, t-s) \tag{2.14}$$

Now we begin to prove Theorem 2.7. We can always assume that

$$\varphi(0) = |D\varphi(0)| = 0 \tag{2.15}$$

by subtracting a linear function from u.

LEMMA 2.2. *Let $u \in S$ in Ω_1 and $u|_{\partial\Omega_1} = \varphi$. Suppose that*

$$B + \beta x_n \leqq u \leqq A + \alpha x_n \quad \text{in } \Omega_2 \tag{2.16}$$

for $|A|, |B|, |\alpha - \beta| \leqq 1$ and

$$\partial\Omega \subset S_1 \times (-l, k) \quad \text{for} \quad l, k \leqq \tfrac{1}{16}.$$

Then there exist $A_1, B_1, \alpha_1, \beta_1, \varepsilon \in (0,1)$, and C such that

$$B_1 + \beta_1 x_n \leqq u \leqq A_1 + \alpha_1 x_n \quad \text{in } \Omega_{1/4}, \tag{2.17}$$

$$\begin{cases} B_1 & \leqq \quad 0 \leqq A_1 \\ \alpha_1, \beta_1 & \leqq \quad \alpha + \varepsilon|\alpha - \beta| + C|A| + C\|g\|_{n+1,\Omega_2} \\ \alpha_1, \beta_1 & \geqq \quad \beta - \varepsilon|\alpha - \beta| - C|B| - C\|g\|_{n+1,\Omega_2} \end{cases}$$

and

$$\begin{cases} |B_1 - A_1| & \leqq \quad osc_{\partial\Omega_1}\varphi + C\|g\|_{n+1,\Omega_2} + C(k + l) \\ |\alpha_1 - \beta_1| & \leqq \quad \varepsilon|\alpha - \beta| + C|B - A| + C\|g\|_{n,\Omega_2}, \end{cases}$$

where C and ε depend only on λ, Λ, and n.

Proof: Without loss of generality we may suppose that $\beta = 0$. It is obvious, from (2.16) that

$$B \leqq \inf_{\partial\Omega_2} \varphi \leqq 0 \leqq \sup_{\partial\Omega_2} \varphi \leqq A.$$

Suppose in addition that

$$(u - B)\left(0, \tfrac{1}{2}, -\tfrac{3}{4}\right) \geqq \tfrac{1}{2}(A - B) + \tfrac{1}{4}\alpha. \tag{2.18}$$

Applying Harnack's inequality to function $u - B$, we obtain

$$u - B \geqq C_1\left[\frac{1}{2}(A - B) + \frac{\alpha}{4}\right] - C_2\|g\|_{n+1,\Omega_2} \quad \text{in } T_{3/4} \times \left(\frac{3}{16}, \frac{13}{16}\right) \times \left(-\frac{1}{2}, 1\right]. \tag{2.19}$$

For any $|x_0'| \leqq \frac{1}{4}$ and $-\frac{1}{16} < t_0 \leqq 0$, let

$$\begin{aligned} v \quad = \quad & u - B - (\inf_{\partial\Omega_2}\varphi - B)\, P_{(x_0', -\frac{1}{4} - l, t_0), \frac{1}{2}} \\ & - \left\{C_1\left[\frac{1}{2}(A - B) + \frac{\alpha}{4}\right] - C_2\|g\|_{n+1,\Omega_2}\right\} P_{(x_0', \frac{1}{2} + k, t_0), \frac{1}{2}}. \end{aligned}$$

We assume

$$C_1\left[\frac{1}{2}(A-B)+\frac{\alpha}{4}\right]-C_2\|g\|_{n+1,\Omega_2}\geqq 0\,.$$

Otherwise (2.17) is obvious by choosing $A_1=A, B_1=B, \alpha_1=\alpha$, and $\varepsilon=\frac{1}{2}$.

Let

$$\begin{aligned}\widetilde{\Omega} \;=\; &\left(Q_{1/2}\left(x_0', \frac{1}{2}+k, t_0\right)\cup Q_{1/2}\left(x_0', -\frac{1}{4}-l, t_0\right)\right.\\ &\left.-\,Q_{1/4}\left(x_0', \frac{1}{2}+k, t_0\right)\right)\cap\Omega_1\,,\end{aligned}$$

we have

$$\begin{cases} v \;\geqq\; 0 & \text{on } \partial_p\widetilde{\Omega}\\ Lv \;\geqq\; g & \text{in } \widetilde{\Omega}\,.\end{cases}$$

By Theorem I-3.14 on v, we have

$$v\geqq -C_3\|g\|_{n+1}\,.$$

Using (2.12) on the line $\{x_0'\}\times(0,\frac{1}{4}]\times\{t_0\}$, we obtain:

$$\begin{aligned}u \;\geqq\; &\inf_{\partial\Omega_2}\varphi - C_3\|g\|_{n+1,\Omega_2}-C_4(\inf_{\partial\Omega_2}\varphi-B)(x_n+l)\\ &+C_5\left[\left(\frac{1}{2}(A-B)+\frac{\alpha}{4}\right)C_1-C_2\|g\|_{n+1,\Omega_2}\right](x_n-k)\,.\end{aligned}$$

Assume C_5 is a small positive number so that $0< C_5C_1/4\leqq 1/2$. Thus, since x_0' is arbitrary in $T_{1/4}$,

$$u\geqq B_1+\beta_1 x_n \quad \text{in } \Omega_{1/4}\,,$$

where

$$\begin{cases} B_1 \;=\; \inf_{\partial\Omega_2}\varphi - C_3\|g\|_{n+1,\Omega_2}-C_6(k+l)\\ \beta_1 \;=\; C_7\alpha+C_8B-C_9\|g\|_{n+1,\Omega_2}\end{cases}$$

and $0< C_7\leqq \frac{1}{2}$.

Similarly, we consider

$$w=\left(A-\sup_{\partial\Omega_2}\varphi\right)\left(1-P_{\left(x_0', -\frac{1}{4}-l, t_0\right), \frac{1}{2}}\right)+\sup_{\partial\Omega_2}\varphi+\alpha x_n-u$$

in $\widetilde{\Omega} = Q_{1/2}(x_0', -\frac{1}{4} - l, t_0) \cap \Omega_2$. Then

$$\begin{cases} Lw \geqq -g & \text{in } \widetilde{\Omega} \\ w \geqq 0 & \text{on } \partial_p\widetilde{\Omega}. \end{cases}$$

We obtain, from Theorem I-3.14,

$$w \geqq -C_{10}\|g\|_{n+1,\Omega_2}.$$

That is,

$$\begin{aligned} u &\leqq \sup\nolimits_{\partial\Omega_2} \varphi + C_{10}\|g\|_{n+1,\Omega_2} + \alpha x_n + C_{11}\left(A - \sup\nolimits_{\partial\Omega_2} \varphi\right)(x_n + l) \\ &\leqq A_1 + \alpha_1 x_n, \quad \text{in } \Omega_{1/4}. \end{aligned}$$

Taking

$$\begin{cases} A_1 = \sup_{\partial\Omega_2} \varphi + C_{10}\|g\|_{n+1,\Omega_2} + C_{11} l \\ \alpha_1 = \alpha + C_{12} A, \end{cases}$$

we obtain that

$$\begin{aligned} |\alpha_1 - \beta_1| &\leqq (1 - C_7)|\alpha| + \max(C_{12}, C_9)(A - B) + C_9\|g\|_{n+1,\Omega_2}, \\ |A_1 - B_1| &\leqq C_{13}\|g\|_{n+1,\Omega_2} + \operatorname{osc}_{\partial\Omega_1} \varphi + C_{14}(k + l). \end{aligned}$$

If (2.18) is false, substitute $\alpha x_n - u$ instead of u. Then (2.18) holds. By a symmetric argument as above, we obtain the same inequalities.

This completes the proof of the lemma.

Remark. We may expand the variables so that $|A_1|, |B_1|$, and $|\alpha_1 - \beta_1| \leqq 1$. Now let

$$osc(\partial\Omega_r) = \sup_{(x',k,t)\in\partial\Omega_r} k - \inf_{(x',k,t)\in\partial\Omega_r} k.$$

LEMMA 2.3. *Let u be as in Lemma 2.2. Then there are A_k, B_k, α_k, and β_k such that for $k = 0, 1, \ldots$*

$$\theta^k B_k + \beta_k x_n \leqq u \leqq \theta^k A_k + \alpha_k x_n \quad \textit{in } \Omega_{\theta^k},$$

$$\begin{cases} B_k \leqq 0 \leqq A_k \\ \alpha_{k+1}, \beta_{k+1} \leqq \alpha_k + \varepsilon|\alpha_k - \beta_k| + C|A_k| + C\theta^k \left(\fint_{\Omega_{\theta^k}} |g|^{n+1} \right)^{1/(n+1)} \\ \alpha_{k+1}, \beta_{k+1} \geqq \beta_k - \varepsilon|\alpha_k - \beta_k| - C|B_k| - C\theta^k \left(\fint_{\Omega_{\theta^k}} |g|^{n+1} \right)^{1/(n+1)} \end{cases}$$

and

$$|A_{k+1} - B_{k+1}| \leqq C\|g\|_{L^{n+1}(\Omega^+_{\theta^k})} + \tfrac{1}{\theta^{k+1}} \operatorname{osc}_{\partial\Omega_{\theta^k}} \varphi + C\tfrac{1}{\theta^k} \operatorname{osc}(\partial\Omega_{\theta^k})$$

$$|\alpha_{k+1} - \beta_{k+1}| \leqq \varepsilon|\alpha_k - \beta_k| + C|A_k - B_k| + C\theta^k \left(\fint_{\Omega_{\theta^k}} |g|^{n+1} \right)^{1/(n+1)} .$$

In fact, we have $\theta = \frac{1}{16}$.

Proof: We prove this inductively. For $k = 0$, take $B_0 = \inf_{\Omega_1} u, A_0 = \sup_{\Omega_1} u, \alpha_0 = \beta_0 = 0$. Then Lemma 2.2 implies the first step of the induction, i.e., the case $k = 1$. Suppose the lemma is true for k. Let $v = u(\theta^k x)/\theta^k$. Then

$$\begin{cases} \widetilde{a}_{ij} v_{ij} = \widetilde{g} \\ v|_{\partial\widetilde{\Omega}_1} = \widetilde{\varphi}, \end{cases}$$

where $\widetilde{a}_{ij}(x) = a_{ij}(\theta^k x)$, $\widetilde{g}(x) = \theta^k g(\theta^k x)$, and $\widetilde{\varphi}(x) = \varphi(\theta^k x)/\theta^k$. By Lemma 2.2, there are $B_{k+1}, A_{k+1}, \beta_{k+1}$, and α_{k+1} such that

$$\beta_{k+1} x_n + \theta B_{k+1} \leqq v \leqq \alpha_{k+1} x_n + \theta A_{k+1} \quad \text{in } \Omega_\theta ,$$

$$\begin{aligned} |B_{k+1} - A_{k+1}| &\leqq \tfrac{C}{\theta} \|\widetilde{g}\|_{n+1,\Omega_1} + \frac{\operatorname{osc}_{\partial\Omega_1} \widetilde{\varphi}}{\theta} \\ &\leqq C\theta^k \left(\fint_{\Omega_{\theta^k}} |g|^{n+1} \right)^{1/(n+1)} + \frac{\operatorname{osc}_{\partial\Omega_{\theta^k}} \widetilde{\varphi}}{\theta^{k+1}} \end{aligned}$$

and

$$\begin{aligned} |\alpha_{k+1} - \beta_{k+1}| &\leqq \varepsilon|\alpha_k - \beta_k| + C|A_k - B_k| + C\|\widetilde{g}\|_{n+1,\Omega_1} \\ &\leqq \varepsilon|\alpha_k - \beta_k| + C|A_k - B_k| + C\theta^k \left(\fint_{\Omega_{\theta^k}} |g|^{n+1} \right)^{1/(n+1)} . \end{aligned}$$

In terms of u, we have

$$|\alpha_{k+1} - \beta_{k+1}| \leqq \varepsilon|\alpha_k - \beta_k| + C|A_k - B_k| + C\theta^k \left(\fint_{\Omega_{\theta^k}} |g|^{n+1} \right)^{1/(n+1)} \quad \text{in } \Omega_{\theta^k} .$$

The other inequalities follow in the same way.

Finally, Theorem 2.1 follows from Lemma 2.3 by a standard elementary procedure.

Remark. For the differentiability of u, we do not need $u|_{\partial\Omega}$ and $\partial\Omega$ to be $C^{1,\alpha}$ as above. This involves the classical *Dini conditions.* If the Dini conditions for $g(r) = osc(\partial\Omega_r)/r$, $r^{-1/n+1}\|g\|_{n+1,\Omega_r}$ and $u|_{\partial\Omega}/r$ are satisfied, i.e.,

$$\int_{0+} \frac{g(r)}{r}\,dr < +\infty,$$

then u is differentiable at 0 and the differentiability depends only on the integrability in the Dini conditions.

We summarize the above in the following theorem.

THEOREM 2.4. *Let u be as above. Let $\varphi(x) = u|_{\partial\Omega}$ with $D\varphi(0) = 0$ and let $\varphi(r) = \sup_{\partial\Omega}\varphi$. If $osc(\partial\Omega_r), r^{-1/n+1}\|g\|_{n+1,Q_r}$ and $\varphi(r)/r$ are in the Dini class, then u is differentiable at* 0.

Proof: It is enough to prove that the approximation planes are convergent. For the convergence of A_k and B_k, we need

$$\sum_1^\infty \theta^{-k/n+1}\|g\|_{n+1,\Omega_{\theta^k}} < \infty$$

and

$$\sum_1^\infty \frac{osc\partial\Omega(\theta^k)}{\theta^k} < \infty .$$

However, for a decreasing function g, we have

$$\sum_0^\infty g(\theta^k) \simeq \int_0^\infty g(\theta^s)\,ds = \int_0^1 g(s)\frac{ds}{s}$$

by a change of variable.

2.2. C^α Estimates on the Lateral Boundary

In this section, we prove C^α regularity under the condition that the boundary data are C^α. This is an easy application of the barrier function introduced in the previous section.

THEOREM 2.5. *Let $u \in S(\lambda, \Lambda)$ in a domain Ω in $\mathbb{R}^{n+1}$. If $u|_{\partial\Omega_1}$ is C^β at $(x_0, t_0) \in \partial_x\Omega$ and $\partial_p\Omega$ is Lipschitz at (x_0, t_0), then u is C^α at (x_0, t_0), for $\alpha = \min(\beta, \alpha_0)$.*

Using the same notation as in the previous section, we prove the theorem at 0. As before, let us prove the following lemma first.
Theorem 2.5 will immediately follow.

LEMMA 2.6. *Let $u \in S$ in Ω_1 and let $u|_{\Omega_1 \cap \partial\Omega} = \varphi$ and suppose that*

$$B \leqq u \leqq A \quad \text{in } \Omega_1 \tag{2.20}$$

and

$$Q_r(0, -1/2, 0) \subset \Omega^C. \tag{2.21}$$

Then there exist A_1, B_1, and $0 < \varepsilon < 1$ such that

$$B_1 \leqq u \leqq A_1 \quad \text{in } \Omega^+_{1/4}. \tag{2.22}$$

Moreover,

$$|B_1 - A_1| \leqq osc_{T1}\varphi + C\|g\|_{n+1,\Omega_1^+} + (1-\varepsilon)|B - A|,$$

where C is a universal constant and $\varepsilon = \varepsilon(r)$.

Remark. The condition (2.21) is similar to the nondegenerate condition. If the $\partial_p\Omega$ satisfies the exterior cone condition, then (2.21) is satisfied.
We also remark that we do not need (2.21) to hold in every scale in the same direction. We need only the condition to hold in some direction in every scale.
The proof of this lemma is similar to that of Lemma 2.2.
We remark that $P_{(0,0),1}$ is subparabolic in $B_{1/2}(0) \times (-1, 0]$. Now, we define

$$P_{(0,0),1,k}(x,t) = \frac{P_{(0,0),1}(x,t) - P_{(0,0),1}(k/2, 0)}{P_{(0,0),1}(1/2, 0) - P_{(0,0),1}(k/2, 0)}. \tag{2.23}$$

Then $P_{(0,0),1,k}(x,t)$ is subparabolic and $P_{(0,0),1,k}(k/2, 0) = 0, P_{(0,0),1,k}(1/2, 0) = 1$.
Finally, we define

$$P_{(y,s),r,rk}(x,t) = P_{(0,0),1,k}\left(r^{-1}(x-y), r^{-2}(t-s)\right).$$

Now we prove the above lemma.

Proof: Consider

$$w = \left(A - \sup_{\partial\Omega_1} \varphi\right)\left(1 - P_{((0,-1/2),-\frac{1}{4}r-l,t_0),\frac{1}{2}}\right) + \sup_{\partial\Omega_1} \varphi - u$$

in $\widetilde{\Omega} = (B_1(1/2) - B_r - 1/2) \times (-r^2 + t_0, t_0) \cap \Omega_1$ for $t_0 \geqq -\frac{1}{16}$. Then

$$\begin{cases} Lw \geqq -g & \text{in } \widetilde{\Omega} \\ w \geqq 0 & \text{on } \partial_p\widetilde{\Omega}. \end{cases}$$

We obtain, from Theorem I-3.14,

$$w \geqq -C_{10}\|g\|_{n+1,\Omega_1}.$$

Noticing that P is strictly decreasing in the radial direction, we have the inequality for $t = t_0$. Since t_0 is arbitrary, the proof is completed.

Remark. As in the previous section, we can prove continuity of u under continuity of φ at 0 and the following *exterior nondegenerate condition:*

For $r > 0$, there exists a $Q_{\varphi(r)r}(x_r, 0) \subset \Omega^C$ for some x_r of $\mathrm{dis}(x_r, \partial_p\Omega) = r$ such that

$$\varphi(r) \geqq C > 0,$$

where C is universal.

2.3. Estimates for Continuous Equations

In this section, we develop $C^{1,\alpha}$ and $C^{2,\alpha}$ estimates on the boundary. Consider equations of the form

$$u_t - F(D^2u, x, t) = g(x, t). \tag{2.24}$$

The result is the following:

THEOREM 2.7. *Let u be a solution of* (2.24) *in Q_1^+. If*

$$v_t - F(D^2v + C, 0, 0) = D$$

for C, D on the surface $F(C, 0, 0) = D$ has $C^{2,\beta}$ estimates on the boundary and

$$\left(\fint_{Q_r^+} |\theta|^{n+1}\right)^{1/(n+1)} \leqq Cr^\alpha,$$

$u|_{\partial_p T_1}$ is $C^{2,\alpha}$ at P_0 and

$$[g - g(0,0)]_{n+1,\alpha}(0,0) = \sup \frac{1}{r^\alpha} \left(\fint_{Q_r^+} |g - g(0,0)|^{n+1} \right)^{1/(n+1)} \leqq C,$$

then u is $C^{2,\alpha}$ at P_0 for $\alpha < \beta$.

We also have $C^{1,\alpha}$ estimates on the boundary. This is not important for fully nonlinear equations. It is important, however, for quasi-linear equations. We state it as follows:

THEOREM 2.8. *Let u be a bounded solution of (2.25)*

$$u_t - F(D^2 u, x, t) = g(x,t). \tag{2.25}$$

Suppose that any solution v of

$$v_t - F(D^2 v, 0, 0) = 0$$

has $C^{1,\beta}$ estimates on the boundary, i.e.,

$$|(v - L)(x,t)| \leqq N d((x,t),(0,0))^{1+\beta},$$

for some constant C, where

$$N + \|L\|_{1,r} \leqq C \left(\frac{\|v\|_{\infty, Q_r^+}}{r^{1+\beta}} + \|v|_{T_r}\|_{C^{1,\beta}} \right).$$

Then there exists a constant $\delta_0 = \delta_0(\beta, \lambda, \Lambda)$ such that if

$$\varlimsup_{r \to 0} \left(\fint_{Q_r^+} |\theta|^{n+1} \right)^{1/(n+1)} + \|u\|_\infty \leqq \delta_0,$$

then u is $C^{1,\alpha}$ at P_0 for $\alpha < \beta$, provided

$$\left(\fint_{Q_r^+(P_0)} |g|^{n+1} \right)^{1/(n+1)} \leqq C r^{-1+\alpha}.$$

We use the same methods as in Section 1 to prove the above estimates.

Remark. As in Section 2.1, we can prove the above $C^{1,\alpha}$ and $C^{2,\alpha}$ for curved boundaries. It involves C^α estimates and the results for homogeneous equations on flat boundaries. We omit the proof since it is parallel to that in Section 2.1.

2.4. Estimates on the Bottom

In this section, we prove $C^{2,\alpha}$ regularity for bottom points. These results are new, even for the linear equations. A similar problem was considered by J. Kohn and L. Nirenberg in [10].

Let Ω be a domain and $0 \in \partial_b\Omega$.

DEFINITION 2.9. We say $\partial_p\Omega$ has curvature larger than l at bottom point 0 if $\partial_p\Omega$ is above $\{(x,t) : l|x|^2 = 2t\}$ for $|x|^2 + |t| \leqq C$.

Let us start out with a lemma for barrier functions.

LEMMA 2.10. *Let $p(x,t) = (|x|^2 + 2Nt)(1 - \varepsilon(1 - |x|^2))$. Then for N large enough and ε small enough, p is a supersolution on bounded domains.*

We denote $\Omega_r = \Omega \cap (B_r(0) \times (-r^2, r^2))$ in this section.

THEOREM 2.11. *Let $u \in S(g)$ in Ω, $\varphi = u|_{\partial_p\Omega}$ satisfy $\varphi(0) = d\varphi = d^2\varphi = 0$. Then u is $C^{2,\beta}$ at 0, in the sense that*

$$|u| \leqq C(|x|^2 + |t|)^{1+\beta/2}$$

for $0 < \beta < 1, \beta = \min(\beta_0, \alpha)$, provided that $\partial_p\Omega$ has curvature larger than $-1/M > -1/N$ (where N is the constant in the previous lemma) at 0 and φ is $C^{2,\alpha}$ at 0 and $g(0) = 0$. In fact,

$$[u]_{C^{2,\beta}}(0) \leqq C\left(\|u\|_{\infty,\Omega_1} + [g]_{n+1,\alpha} + [\varphi]_{C^{2,\alpha}}(0)\right).$$

LEMMA 2.12. *Let $u \in S(g)$ in Ω_1 be such that*

$$u \leqq \alpha(2Nt + |x|^2) + A$$

and

$$u = \varphi \quad \text{on } \partial_p\Omega_1, \varphi(0) = 0.$$

Assume that $\partial_p\Omega$ has curvature larger than $-1/M > -1/N$ at 0. Then there are constants α_1 and A_1 such that

$$u \leqq \alpha_1\left(2Nt + |x|^2\right) + A_1 \quad \text{in } \Omega_{1/4}$$

with

$$\alpha_1 \leqq (1-\varepsilon)\alpha + CA$$

and

$$A_1 \leqq C\|g\|_{n+1,\Omega_1} + \sup_{\partial_p\Omega_1} \varphi\,,$$

where ε *and* C *are universal constants.*

Proof: Let $p(x,t) = (\alpha + (1 - N/M)^{-1}A)(|x|^2 + 2Nt)(1 - \varepsilon(1 - |x|^2))$ as in Lemma 2.10. Then we have $p + \sup_{\partial_p\Omega_1} \varphi \geqq u$ in Ω_1. By the maximum principle,

$$u(x,t) \leqq \sup \varphi + p(x,t) + C\|g\|_{n+1,\Omega_1} \quad \text{in } \Omega_1\,.$$

The lemma follows.

The $C^{2,\alpha}$ estimates follow immediately. Let us explain it further.

Lemma 2.13. *Let* $u \in S(g)$ *where* $u = \varphi$ *on* $\partial_p\Omega_1$. *Assume* $\varphi(0) = D\varphi(0) = D^2\varphi(0) = 0$. *Then there are* $\alpha_k, \beta_k, A_k, B_k$ *such that*

$$-\beta_k \left(2Nt + |x|^2\right) - B_k\lambda^{2k} \leqq u \leqq \alpha_k \left(2Nt + |x|^2\right) + A_k\lambda^{2k} \quad \text{in } \Omega_{\lambda^k}\,,$$

with conditions

$$|\alpha_{k+1}| \leqq (1 - \varepsilon)|\alpha_k| + C|A_k|\,,$$

$$|\beta_{k+1}| \leqq (1 - \varepsilon)|\beta_k| + C|B_k|\,,$$

$$|A_{k+1}| \leqq \frac{\sup_{\partial_p\Omega_{\lambda^k}} \varphi}{\lambda^{2k}} + C\left(\fint_{\Omega_{\lambda^k}} |g|^{n+1}\right)^{1/(n+1)}$$

and

$$|B_{k+1}| \leqq \frac{-\inf_{\partial_p\Omega_{\lambda^k}} \varphi}{\lambda^{2k}} + C\left(\fint_{\Omega_{\lambda^k}} |g|^{n+1}\right)^{1/(n+1)}\,,$$

where ε *and* C *are the constants in the previous lemma and* $\lambda = 1/2$.

Proof: We proceed by induction. We take $\alpha_0,\ \beta_0 = 0,\ A_0 = \sup u,\ B_0 = -\inf u$. Suppose the lemma is true for k.

Let

$$w(x,t) = \frac{u\left(\lambda^k x, \lambda^{2k} t\right)}{\lambda^{2k}}\,.$$

Then

$$w \in S(\lambda, \Lambda, \tilde{g})\,,$$

$$-\beta_k\left(2Nt+|x|^2\right)-B_k \leqq w \leqq \alpha_k\left(2Nt+|x|^2\right)+A_k\,,$$

$$w|_{\partial_p\Omega_1}=\widetilde{\varphi}\,,$$

where

$$\widetilde{g}(x,t) = g\left(\lambda^k x, \lambda^{2k} t\right),$$

$$\widetilde{\varphi}(x,t) = \lambda^{-2k}\varphi\left(\lambda^k x, \lambda^{2k} t\right).$$

Applying the previous lemma to w, we obtain

$$-\beta_{k+1}\left(2Nt+|x|^2\right)-B_{k+1} \leqq w \leqq \alpha_{k+1}\left(2Nt+|x|^2\right)+A_{k+1} \quad \text{in } \Omega_{1/4}.$$

Clearly $\alpha_{k+1},\cdots,B_{k+1}$ satisfy the conditions. Scaling back to u, we get the $(k+1)$-th step.

We now state the pointwise version of the above lemma.

THEOREM 2.14. *Let $u \in S(g)$ in Ω_1 and let $\varphi = u|_{\partial_p\Omega}$ be $C^{2,\alpha}$ at 0. Assume $\partial_p\Omega$ has curvature larger than $-1/M > -1/N$ at 0. Then u is $C^{2,\beta}$ at 0 for $0<\beta<1, \beta=\min(\beta_0,\alpha)$, provided that φ satisfies the following consistency condition:*

There exist $\widetilde{\varphi}$ in Ω such that $\widetilde{\varphi}|_{\partial_p\Omega}-\varphi$ vanishes up to the second order at 0 and $u-\varphi \in S(\widetilde{g})$ with $\widetilde{g}(0)=0$ and

$$[\widetilde{g}]_{n+1,\alpha} \leqq C\,.$$

Actually,

$$[u]_{C^{2,\beta}}(0) \leqq C\left(\|u\|_{\infty,Q_1}+[g]+[\varphi]_{C^{2,\alpha}}(0)\right)+[\widetilde{\varphi}]_{C^{2,\alpha}}(0))\,.$$

In the case of $\Omega = Q_1$, the consistency condition only has restriction on the boundary data when $0 \in \partial_c\Omega$. For the points in $\partial_p\Omega - \partial_c\Omega$, we can take the function, $\widetilde{\varphi} = ct$ for some constant c to satisfy the consistency condition.

Clearly, $C^{2,\alpha}$ regularity is the highest regularity we could expect for functions in $S(g)$. We have various weaker (than $C^{2,\alpha}$) regularity theorems at bottom. We first establish the $C^{1,\alpha}$ estimates on the bottom points for solutions.

THEOREM 2.15. *Let $u \in S(g)$ in Q_1 and let $\varphi = u|_{\partial_b\Omega}$ be $C^{1,\alpha}$ at 0. Assume that $\partial_p\Omega$ has curvature larger than $-1/M > -1/N$ at 0. Then u is $C^{1,\alpha}$ at 0 for $0 < \alpha < 1$, provided*

$$[g]_{n+1,\alpha-1} \leqq C .$$

Moreover,

$$[u]_{C^{1,\alpha}}(0) \leqq C\left(\|u\|_{\infty,Q_1} + [g]_{n+1,\alpha-1} + [\varphi]_{C^{1,\alpha}}(0)\right) .$$

Proof: As before, we may assume that $[\varphi]_{C^{1,\alpha}}$ and $[g]_{n+1,\alpha-1}$ are small and assume $\varphi(0) = D\varphi(0) = 0$.

We prove the following by induction:

$$|u| \leqq \lambda^{k(1+\alpha)} \quad \text{in } \Omega_{\lambda^k} .$$

The case for $k = 0$ follows from a normalization. We suppose the theorem is true for k. We prove that the theorem is true for $k+1$. Let

$$w(x,t) = \frac{u\left(\lambda^k x, \lambda^{2k} t\right)}{\lambda^{k(\alpha+1)}} .$$

Then $w \in S(\widetilde{g}), |w| \leqq 1$ and $w|_{\partial_1\Omega} = \widetilde{\varphi}$, where

$$\begin{aligned} \widetilde{g}(x,t) &= \lambda^{(1-\alpha)k} g\left(\lambda^k x, \lambda^{2k} t\right) , \\ \widetilde{\varphi}(x,t) &= \lambda^{(-1-\alpha)k} \varphi\left(\lambda^k x, \lambda^{2k} t\right) . \end{aligned}$$

Applying Lemma 2.10 and taking $\varepsilon = 0$ and the maximum principle, we obtain:

$$|w| \leqq |\widetilde{\varphi}| + C\|\widetilde{g}\|_{n+1} + |x|^2 + 2Nt \leqq [\varphi](0) + C[g](0) + |x|^2 + 2Nt .$$

Now taking $|x|^2 + 2Nt \leqq \lambda^{1+\alpha}/3$, $[g](0) \leqq \lambda^{1+\alpha}/3$ and $[\varphi](0) \leqq \lambda^{1+\alpha}/3$, we have $|w| \leqq \lambda^{1+\alpha}$. Scaling back to u, we get the result for the $(k+1)$-th step.

We have C^α estimates on the bottom for any curvature condition. Let $0 \in \partial_b\Omega$. Suppose $\partial_p\Omega$ has curvature larger than $-M$ at bottom point 0. Let $\Omega_r = \Omega \cap (B_r \times (-Mr^2/2, r^2])$. We need a different barrier function now. Let p be a subsolution such that $p \leqq 0$ for $|x| = 1$ and $p > 0$ for $|x| < 1$. We remark that such a p exists, following the proof of Lemma I-3.22. For example, we can take $p = u$ where u is the function in the proof of Lemma I-3.22. Let $N = \sup\{p(x,t) : |x| \leqq 1, M(1-|x|^2) \leqq 2t \leqq 2+M\}$. Then

$$q(x,t) = \frac{N - p(x, t - M/2)}{N}$$

is a supersolution and satisfies $q \geqq 0$ and $q = 1$ for $|x| = 1$ and $q \leqq 1 - \varepsilon$ in $\Omega_{1/2}$ for some $\varepsilon > 0$.

LEMMA 2.16. *Let $u \in S(g)$ in Ω and suppose*

$$u \leqq A \text{ in } \Omega_1 .$$

Then

$$u \leqq Aq + \sup_{\partial_p \Omega_1} u + C\|f\|_{n+1} .$$

The reader can easily obtain the proof.

THEOREM 2.17. *Let u as in the above lemma. Then u is C^α at 0 if $\partial_p\Omega$ has curvature larger than $-M$ at bottom point 0 and $u|_{\partial_p\Omega}$ is C^β at 0.*

The proof follows from the above lemma immediately.

Remark 1. Actually we could also have one-sided estimates if one-sided conditions hold.

Remark 2. We can have estimates for the functions in $S(b, c, g)$. The proof is analogous to the one we gave. We need to keep track of the changes of b and c when we blow up the coordinates.

Note. We may use the solutions of Pucci's maximal operators to construct barriers on the boundary. Then the regularity of the solutions follows from the regularity of Pucci's operator and our techniques in previous sections. We hope that our methods can work for a much wider class of operators, however, including such degenerate equations as the minimal surface equations.

3. Global Regularity

In this section, we give some general machinery for obtaining global estimates from boundary estimates and interior estimates.

3.1. Machinery for Global Regularity

THEOREM 3.1. *Let S be a subset of $C(\Omega)$. Suppose the following hold:*

(1) *For any* $(x,t) \in \Omega$, *let* $s = d((x,t), \partial_p\Omega)$. *Then there is a polynomial* P_k *of degree* k *such that* $|P_k|_{k,s} \leqq C|u|_{\infty,Q_{s/2}} + Cs^{k+\alpha}$ *and*

$$|(u - P_k)(y,\tau)| \leqq \left(A\frac{|u|_\infty}{s^{k+\alpha}} + B\right) d((y,\tau),(x,t))^{k+\alpha}.$$

(2) *For any* $(x,t) \in \partial_p\Omega$, *there is a polynomial* P *of degree* k *such that*

$$|u - P| \leqq Cd((x,t),(y,\tau))^{k+\alpha}.$$

(3) *For any* $u \in S$ *and* P, *a* k*-th polynomial with* $|P|_k \leqq C$, *then, we have* $u - P \in S$.

Then $S \subset C^{k+\alpha}(\bar\Omega)$. *In fact, we have* $S \subset B_M \subset C^{k+\alpha}(\bar\Omega)$ *for* $M \leqq L(CA+B)$, *where* L *is some universal constant.*

Proof: Let $(x,t) \in \Omega$ such that $d((x,t), \partial_p\Omega) = d((x,t),(x_0,t_0)) = s$ for some $(x_0,t_0) \in \partial_p\Omega$. Now, at (x_0,t_0) we have

$$|u - P| \leqq Cd((y,\tau),(x_0,t_0))^{k+\alpha}.$$

Now applying (1), the interior regularity, on function $u - P$, we have a P_1 such that

$$\begin{aligned}
|u - P - P_1| &\leqq \left(A\frac{|u-P|}{d((x,t),\partial_p\Omega)^{k+\alpha}} + B\right) d((y,\tau),(x,t))^{k+\alpha} \\
&\leqq \left(A\frac{|u-P|}{s^{k+\alpha}} + B\right) d\left((y,\tau),(x,t)\right)^{k+\alpha} \\
&\leqq (C_1CA + B)d((y,\tau),(x,t))^{k+\alpha},
\end{aligned}$$

for $d((y,\tau),(x,t)) \leqq s/2$. Moreover, we have $|P_1|_{k,s} \leqq Cs^{k+\alpha}$. By triangle inequality and the above inequalities, we have

$$\begin{aligned}
|u - P - P_1| &\leqq |u - P| + |P_1| \\
&\leqq Cd((y,\tau),(x,t))^{k+\alpha} + C\sum s^{k+\alpha-i}d((y,\tau),(x,t))^i \\
&\leqq Cd((y,\tau),(x,t))^{k+\alpha}
\end{aligned}$$

for the points $d((y,\tau),(x,t)) \geqq s/2$.

3.2. Applications

It is easy to obtain global regularity from the above theorem. We have global $C^{2,\alpha}$ estimates in the following theorem. We omit other theorems for $C^{1,\alpha}$ and C^α norms.

THEOREM 3.2. *Let u be a solution of the following equation with F convex:*

$$\begin{cases} u_t - F(D^2u, x, t) = f(x,t) & \text{in } Q_1 \\ u = \varphi & \text{on } \partial_p Q_1 . \end{cases}$$

Suppose also that φ satisfies the consistency condition on $\partial_c Q_1$; namely,

$$\varphi_t - F(D^2\varphi, x, t) = f(x,t) \quad \text{on } \partial_c Q_1 .$$

Then $u \in C^{2,\alpha}$, provided $\theta, f \in C^\alpha$ if $\alpha < \alpha_0$, for some universal positive constant.

4. Homogeneous Equations

Heretofore, we have concentrated on estimates for equations of small perturbation from a "good" equation, namely the equation with good estimates. In this section we give estimates for good equations. Specifically, $C^{1,\alpha}$ estimates for equations with or without gradient and $C^{2,\alpha}$ and $C^{1,1}$ estimates for convex equations. This section contains the main applications of our theory. In our proofs, we use estimates for finite different quotients, which give us estimates more directly than the usual way.

4.1. Jensen's Approximate Solutions

The proof of the uniqueness theorem involves very important approximation solutions introduced by Jensen in [7]. Let us outline his construction.

Let A be any set in $\mathbb{R}^{n+2}$. We use the following notation:

$$A + B =: \{a + b : a \in A, b \in B\} \tag{4.1}$$

$$\{0\}^\varepsilon = \left\{(x,t,z) : |x|^2 - t + |z|^2 \leqq \varepsilon^2 \quad \text{for } t \leqq 0\right\} . \tag{4.2}$$

We denote $A^\varepsilon = A + \{0\}^\varepsilon$. Clearly, A^ε is the ε-neighborhood of A with respect to the parabolic metric.

LEMMA 4.1.

$$A^\varepsilon + B^\varepsilon = (A + B)^{2\varepsilon} .$$

Proof: First, notice that

$$\{0\}^\varepsilon + \{0\}^\varepsilon = \{0\}^{2\varepsilon} .$$

Then

$$\begin{aligned} A^{\varepsilon} + B^{\varepsilon} &= \{a+b+c+d : a \in A, b \in B, c \in \{0\}^{\varepsilon}, d \in \{0\}^{\varepsilon}\} \\ &= \{a+b+c : a \in A, b \in B, c \in \{0\}^{2\varepsilon}\} \\ &= (A+B)^{2\varepsilon}. \end{aligned}$$

Let u be a continuous function. Let G_u be the set of points below the graph of u, namely

$$G(u) = \{(x,t,z) : z \leqq u(x,t)\}$$

Let $G(u)^{\varepsilon}$ be the ε-neighborhood of $G(u)$. We shall refer to

$$u^{\varepsilon}(x,t) = \sup\left\{u : (x,t,z) \in G(u)^{\varepsilon}\right\}$$

as the *upper ε-envelope* of u, and

$$u_{\varepsilon}(x,t) = -(-u)^{\varepsilon}$$

as the *lower ε-envelope* of u, respectively.

Lemma 4.2. *For any point P on the graph of u^{ε}, there is a point Q on the graph of u such that the ε-neighborhood of Q is tangent with the graph of u^{ε} at P.*

Let u be a solution of

$$u_t - F(D^2u) = 0\,. \tag{4.3}$$

Lemma 4.3. *u^{ε} is a subsolution of* (4.3) *in $\Omega - \partial_p\Omega^{\varepsilon}$.*

Proof: Let $\varphi \in C^{1,1}$ such that

$$\min(\varphi - u^{\varepsilon}) = (\varphi - u)(x_0, t_0)\,.$$

By definition of u^{ε}, there is a point $(x_1, t_1, u(x_1, t_1))$ such that

$$|x_1 - x_0|^2 + |t_1 - t_0| + \left|u(x_1,t_1) - u^{\varepsilon}\right|^2 = \varepsilon^2\,.$$

Now define $\widetilde{\varphi}$ as

$$\widetilde{\varphi}(x,t) = \varphi(x - x_1 + x_0, t - t_1 + t_0) - u^{\varepsilon}(x_0,t_0) + u(x_1,t_1)\,.$$

Obviously,

$$(\widetilde{\varphi} - u)(x_1, t_1) = \min(\widetilde{\varphi} - u)\,.$$

We also have

$$\begin{cases} D\widetilde{\varphi}(x_1, t_1) &= D\varphi(x_0, t_0) \\ D^2\widetilde{\varphi}(x_1, t_1) &= D^2\varphi(x_0, t_0) \\ D_t\widetilde{\varphi}(x_1, t_1) &= D_t\varphi(x_0, t_0)\,. \end{cases}$$

Since u is a subsolution, we have

$$\varphi_t - F(D^2\varphi(x_1, t_1)) \leqq 0\,.$$

This proves the theorem.

LEMMA 4.4. *Let $\varepsilon > 0$ and $\varphi \in C^1(\Omega)$. Let u be a continuous function on Ω and u^{ε} be the upper ε-envelope of u. Assume $u^{\varepsilon} - \varphi$ has its maximum at $(x_0, t_0) \in \Omega$. Then there is a neighborhood U of (x_0, t_0) such that u^{ε} is semiconvex on U.*

The geometry of this lemma is clear. For the proof, we refer the reader to Jensen; see [7].

LEMMA 4.5. *In the above lemma, if $\varphi =$ constant and u is a solution, then $-u^{\varepsilon}_t, D^2u^{\varepsilon} < 0$ in a set of positive measure.*

Proof: Clearly $\Gamma(u^{\varepsilon})$ has a maximum. Since u^{ε} is semiconvex, $\Gamma(u^{\varepsilon})$ is $C^{1,1}$. From the maximum principle, we have

$$0 < \inf u^{\varepsilon} - \inf_{\partial_p\Omega} u^{\varepsilon} \leqq C\left(\int_{u^{\varepsilon}=\Gamma(u^{\varepsilon})} \left(-u^{\varepsilon}_t\right) \det\left(D^2u^{\varepsilon}\right)\right)^{1/(n+1)}\,.$$

Hence $\{u^{\varepsilon} = \Gamma(u^{\varepsilon})\}$ has positive measure and $-u^{\varepsilon}_t, D^2u^{\varepsilon} < 0$ in $\{u^{\varepsilon} = \Gamma(u^{\varepsilon})\}$ almost everywhere.

THEOREM 4.6. *Let u and v be two solutions of (4.3). Then $u + v$ is in $S(\lambda, \Lambda)$.*

Proof: We first prove that $u^{\varepsilon} + v^{\varepsilon}$ is in $S(\lambda, \Lambda)$. Let φ be a test function such that $\varphi - u^{\varepsilon} - v^{\varepsilon}$ has local minimum (x_0, t_0). Then φ is also a test function for $u^{\varepsilon} + v^{\varepsilon} = (u + v)^{2\varepsilon}$. Therefore, in any neighborhood of (x_0, t_0), we have that u, v are second-order differentiable and thus,

$$D^2\varphi \geqq D^2u^{\varepsilon} + D^2v^{\varepsilon}$$

in a set of positive measure. Moreover, we can arrange

$$u_t^\varepsilon - F(D^2u^\varepsilon) \leqq 0$$

$$v_t^\varepsilon - F(D^2v^\varepsilon) \leqq 0$$

pointwise in that set. On the other hand, we have

$$\left(u^\varepsilon + v^\varepsilon\right)_t - \mathscr{M}^+\left(D^2\left(u^\varepsilon + v^\varepsilon\right)\right) \leqq u_t^\varepsilon + v_t^\varepsilon - F\left(D^2u^\varepsilon\right) - F\left(D^2v^\varepsilon\right) \leqq 0\,.$$

Hence,

$$\varphi_t - \mathscr{M}\left(D^2\varphi\right) \leqq 0\,.$$

That is, $u^\varepsilon + v^\varepsilon$ is a subsolution. Since $u^\varepsilon + v^\varepsilon \to u + v$ and $\underline{S}$ is closed, we have $u + v \in \underline{S}$. By symmetry, we obtain that $u + v \in \bar{S}$.

Lemma 4.7. *The solution of the Dirichlet problem of* (4.3) *is unique.*

Proof: Let u and v be two solutions of (4.3). Then $u - v \in S$. By the maximum principle for the functions in S, we get $u = v$.

4.2. $C^{1,\alpha}$ Estimates for $u_t - F(D^2u) = 0$

Let F be any uniformly elliptic operator. Then we have the following:

Theorem 4.8. *Let u be a solution of*

$$u_t - F(D^2u) = 0 \quad \text{in } Q_1\,. \tag{4.4}$$

Then

$$\|u\|_{C^{1,\alpha}(Q_{1/2})} \leqq C(\|u\|_{\infty,Q_1} + 1)$$

for some $\alpha > 0$.

First, let us consider finite difference quotients of the solutions. Let $e \in \mathbb{R}^n$ be any unit vector and $u^h(x,t) = u(x - he, t)$ and let

$$u_h(x,t) =: \frac{u(x,t) - u^h(x,t)}{h}\,.$$

Proof: From Theorem 4.6, we know $u_h \in S(\lambda, \Lambda)$. By the interior maximum estimates Theorem I-4.16, finite difference estimates Theorem I-4.15, and C^α estimates, we have

$$\|u_h\|_{C^\alpha(Q_{1/2})} \leqq C\|u_h\|_{\infty,Q_{3/4}} \leqq C\|u\|_{\infty,Q_1}\,.$$

In particular,

$$|u_h(x+h,t) - u_h(x,t)| \leqq Ch^\alpha .$$

By Proposition 8, Chapter V, in [12], u is $C^{1,\alpha}$ in x direction. Similarly, we have estimates in t direction. Combining these, the theorem follows.

In the same way, we have $u_t \in C^\alpha$.

THEOREM 4.9. *Let u be as above. Then u_t exists in the classical sense and $u_t \in C^\alpha$ for some $\alpha > 0$.*

For the interior $C^{1,\alpha}$, Krylov (see [9]) proved the following results:

THEOREM 4.10. *Let $F(M,P)$ be uniformly elliptic and differentiable in all variables and suppose the following structure condition is satisfied:*

$$|F_{P_i}| \leqq C(|M| + 1) .$$

Then the equation

$$u_t - F(D^2u, Du) = 0$$

has interior $C^{1,\alpha}$ estimates for C^3 solutions.

Proof: We refer this to [9].

We shall present a proof of this for viscosity solutions in a forthcoming paper.

4.3. Second Derivative Estimates for Convex Equations

We follow the same lines as in the proof of $C^{1,\alpha}$ estimates to obtain $C^{2,\alpha}$ and $C^{1,1}$ estimates.

LEMMA 4.11. *Let u be a solution of*

$$u_t - F\left(D^2u\right) = 0 ,$$

for some convex uniformly elliptic F. Then for any $h \in \mathbb{R}^1, e \in S^{n-1} \subset \mathbb{R}^n$,

$$w(x,t) = u(x+he,t) + u(x-he,t) - 2u(x,t) \in \underline{S}(\lambda, \Lambda, 0) .$$

Proof: For ε small enough, we claim $w^{4\varepsilon}(x,t) = u^\varepsilon(x+he,t) + u^\varepsilon(x-he,t) - 2u_\varepsilon(x,t) \in \underline{S}(\lambda, \Lambda, 0)$.

Let φ be a test function, and let (x_0, t_0) be a minimum point of $\varphi - u$. From Lemma 4.5, $w^{4\varepsilon}$ is semi-convex. From Lemma 4.3, we have

$$u_t^{\pm h} - F\left(D^2 u^{\pm h}\right) \leqq 0$$

$$-u_{\varepsilon,t} + F\left(D^2 u_\varepsilon\right) \leqq 0\,.$$

Since F is convex, we have

$$F(M) + F(N) - 2F(Z) \geqq \mathscr{M}^-(M + N - 2Z)\,.$$

From Lemma 4.6, we have

$$w_t^\varepsilon - \mathscr{M}^-(D^2 w^\varepsilon) \leqq 0\,.$$

Hence, we have

$$\varphi_t - \mathscr{M}^-(D^2\varphi) \leqq 0\,.$$

Therefore $w^\varepsilon \in \underline{S}$. Since $\underline{S}$ is closed, $w \in \underline{S}$.

THEOREM 4.12. *Let u, F be as in the previous lemma. Then*

$$[u]_{C^{1,1}(Q_{1/2})} \leqq C(\|u\|_{\infty,Q_1} + 1)\,.$$

Proof: By Theorem I-4.15, I-4.16, and the previous lemma, we have, for any unit vector e,

$$w_h(x,t) = u(x + he, t) + u(x - he, t) - 2u(x,t) \in \underline{S}(\lambda, \Lambda, 0)\,.$$

Moreover,

$$\begin{aligned} \sup_{Q_{3/4}} \left(\frac{w_h}{h^2}\right)^+ &\leqq C\left(\left\|\frac{w_h^-}{h^2}\right\|_{\varepsilon,Q_{3/4}} + 1\right) \\ &\leqq C(\|u\|_{\infty,Q_1} + 1)\,. \end{aligned}$$

Combining this with the fact that $u_t \in C^\alpha$, we obtain:

$$C|x|^2 - Ct - u(x,t)$$

is convex in $Q_{3/4}$ for some large constant C. Now, we are in the situation as in Corollary I-3.17. Hence $C|x|^2 - Ct - u(x,t)$ is $C^{1,1}$. So is u.

Now, we begin to prove interior $C^{2,\alpha}$ estimates for convex equations.

THEOREM 4.13. *Let u be a solution of* (4.3) *with F convex. Then*

$$\|u\|_{C^{2,\alpha}(Q_{1/2})} \leqq C(\|u\|_{\infty,Q_1} + 1). \tag{4.5}$$

The following argument is due to L. Caffarelli.

Moreover we need only to prove $C^{2,\alpha}$ estimates for elliptic equations, since we know $u_t \in C^{1,\alpha}$.

LEMMA 4.14. *Let $F : M_n \to \mathbb{R}^1$ be an elliptic function and suppose*

$$F(M) = F(N) = 0$$

for two symmetric matrices M and N with $1 \leqq |M - N| \leqq 2$. Then there is a unit vector α such that

$$\alpha^T(M - N)\alpha \geqq C_0 = C(n, \lambda, \Lambda).$$

Proof: Let $m = \sup_\alpha \alpha^T(M - N)\alpha$. Then $(M - N)^+ \leqq mI$. Clearly, $|(M - N)^-| \leqq C(n)$. By the ellipticity, we have

$$\begin{aligned} 0 &= F(M) = F(N + (M - N)) \\ &\leqq F(N) + \Lambda|(M - N)^+| - \lambda|(M - N)^-| \\ &\leqq 0 + n^2 m\Lambda - \lambda C(n). \end{aligned}$$

Hence $m \geqq C(n, \lambda, \Lambda)$.

LEMMA 4.15. *Let u be as in Theorem* 4.13, $u \in C^2$, $\operatorname{diam} D^2u(B_1(0)) \geqq 1$. *Let $\{P_i\}_1^N$ be an ε-net for $D^2u(B_1)$. Then there is an ε_0 such that if $\varepsilon \leqq \varepsilon_0$, then a pure subnet of $\{P_i\}$ is an ε-net for $D^2u(B_{1/2})$.*

Proof: Clearly, we have a 2ε-net of the form $P_i = D^2u(x_i)$ for $x_i \in B_1$. Therefore, without loss of generality, we may suppose $P_i = D^2u(x_i)$.

Since we have a P_i, suppose it is $P_1 = D^2u(x_1)$, such that

$$\left|\left\{x : \left|D^2u(x) - D^2u(x_1)\right| \leqq \delta\right\}\right| \geqq C(\delta),$$

where $\delta = C_0/2$ and where the constant C_0 is as in the previous lemma.

By the lemma, there exists a P_i, say P_2, such that

$$u_{\alpha\alpha}(x_2) = \alpha^T D^2 u \alpha \geqq D_{\alpha\alpha} u(x_1) + C_0 .$$

Let $M = \sup_{x \in B_1} u_{\alpha\alpha}$. Then $M - u_{\alpha\alpha} \geqq 0$. By Theorem I-4.11, we have

$$|\{M - D_{\alpha\alpha} u \geqq C_0/2\}|^C \leqq C(M - \sup_{B_{1/2}} D_{\alpha\alpha} u) .$$

From our choice of P_1, we have

$$\left\{x : \left|D^2 u(x) - D^2 u(x_1)\right| \leqq \delta\right\} \subset \left\{M - D_{\alpha\alpha} u \geqq C_0/2\right\} .$$

Hence, we have

$$M - \sup_{B_{1/2}} D_{\alpha\alpha} \geqq C_1 .$$

If $|D_{\alpha\alpha} u(x_3) - M| \leqq \varepsilon \leqq C_1$, then

$$\{|P - P_3| \leqq \varepsilon\} \cap D^2 u(B_{1/2}) = \varnothing .$$

That completes the proof of the lemma.

LEMMA 4.16. *Let u be as in Theorem* 4.13, $u \in C^2$. *If* $\operatorname{diam} D^2 u(B_1) = 2$, *then* $\operatorname{diam} D^2 u(B_{\delta_0}) \leqq 1$ *for some universal* δ_0.

Proof: Take an ε_0-net $\{P_i\}_1^N$ for $D^2 u(B_1)$ such that $N \leqq C(n, \lambda, \Lambda)$. By the previous lemma, we have a pure subnet for $D^2 u(B_{1/2})$. This process can always proceed by change of variables $u(rx)/r^2$, unless $\operatorname{diam} D^2 B_{1/2^k} \leqq 1$. It must stop, however, at most at the N-th step. This completes the proof.

The proof of Theorem 4.13 follows immediately from this lemma.

Note. The proof of Theorem 4.13 is due to L. Caffarelli. Theorems 4.12 and 4.13 were proved in [3] for classical solutions by a different method.

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