SIAM J. CONTROL OPTIM.
Vol. 48, No. 2, pp. 551–572

A PROOF OF THE SMOOTHNESS OF THE FINITE TIME HORIZON AMERICAN PUT OPTION FOR JUMP DIFFUSIONS*

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Abstract. We give a new proof of the fact that the value function of the finite time horizon American put option for a jump diffusion, when the jumps are from a compound Poisson process, is the classical solution of a free boundary equation. We also show that the value function is C^1 across the optimal stopping boundary. Our proof, which uses only the classical theory of parabolic partial differential equations of [A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice–Hall, Englewood Cliffs, NJ, 1964] and [A. Friedman, *Stochastic Differential Equations and Applications*, Dover, Mineola, NY, 2006], is an alternative to the proof that uses the theory of viscosity solutions (see [H. Pham, *Appl. Math. Optim.*, 35 (1997), pp. 145–164]). This new proof relies on constructing a monotonous sequence of functions, each of which is a value function of an optimal stopping problem for a geometric Brownian motion, converging to the value function of the American put option for the jump diffusion uniformly and exponentially fast. This sequence is constructed by iterating a functional operator that maps a certain class of convex functions to classical solutions of corresponding free boundary equations. On the other hand, since the approximating sequence converges to the value function exponentially fast, it naturally leads to a good numerical scheme.

Key words. optimal stopping, Markov processes, jump diffusions, American options, integro-differential equations, parabolic free boundary equations

AMS subject classifications. 60G40, 62L15, 60J75

DOI. 10.1137/070686494

1. Introduction.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space hosting a Wiener process $W = \{W_t; t \geq 0\}$ and a Poisson random measure N on $\mathbb{R}_+ \times \mathbb{R}_+$, with mean measure $\lambda\nu(dx)dt$ (in which ν is a probability measure on $\mathbb{R}_+$), independent of the Wiener process. We will consider a Markov process $S = \{S_t; t \geq 0\}$ of the form

$$dS_t = \mu S_t dt + \sigma S_t dW_t + S_{t-} \int_{\mathbb{R}_+} (z-1) N(dt, dz). \tag{1.1}$$

In this model, if the stock price jumps at time t, then it moves from S_{t-} to $S_t = ZS_{t-}$, in which Z is a positive random variable whose distribution is given by ν. Note that when $Z < 1$ the stock price jumps down and when $Z > 1$ the stock price jumps up. In the Merton jump diffusion model $Z = \exp(Y)$, in which Y is a Gaussian random variable. We will take $\mu = r + \lambda - \lambda\xi$, in which $\xi = \int_{\mathbb{R}_+} xv(dx) < \infty$, so that $(e^{-rt}S_t)_{t \geq 0}$ is a martingale; i.e., $\mathbb{P}$ is a risk neutral measure. The constant $r \geq 0$ is the interest rate, and the constant $\sigma > 0$ is the volatility. We assume the risk neutral pricing measure $\mathbb{P}$, and hence the parameters of the problem, are fixed as a result of a calibration to historical data. The value function of the American put option pricing problem is

$$V(x, T) := \sup_{\tau \in \widetilde{\mathcal{S}}_{0,T}} \mathbb{E}^x\{e^{-r\tau}(K - S_\tau)^+\}, \tag{1.2}$$

*Received by the editors March 27, 2007; accepted for publication (in revised form) September 22, 2008; published electronically February 13, 2009. This research was supported in part by the National Science Foundation.
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in which $\widetilde{\mathcal{S}}_{0,T}$ is the set of stopping times (of the filtration generated by W and N) that take values in $[0,T]$, and $\mathbb{E}^x$ is the expectation under the probability measure $\mathbb{P}$, given that $S_0 = x$.

We will show that V is the classical solution of a free boundary equation and that it satisfies the *smooth fit principle*; i.e., V is continuously differentiable with respect to its first variable at the optimal stopping boundary. We argue these facts by showing that V is the fixed point of an operator, which we will denote by J, that maps a given function to the value function of an optimal stopping problem for a geometric Brownian motion. This operator acts as a regularizer: As soon as the given function f has some certain regularity properties, we show that Jf is the unique classical solution of a corresponding free boundary equation and that it satisfies the smooth fit principle. The proof of the main result concludes once we show that V has these certain regularity properties. In this last step we make use of a sequence (which is constructed by iterating J starting with the pay-off function of the put option) that converges to V uniformly and exponentially fast. Incidentally, this sequence yields a numerical procedure, whose accuracy versus speed characteristics can be controlled. Each element of this sequence is an optimal stopping problem for geometric Brownian motion and can be readily calculated using classical finite difference methods (see, e.g., [18] for the implementation of these methods). An alternative proof of the regularity of V was given in [14]. This proof used a combination of the results in [8] and the theory of viscosity solutions. In particular the proof of Proposition 3.1 in [14] is carried out (details are not provided but hinted) using arguments similar to those used in the proof of Proposition 5.3 in [15]. The latter proof uses the uniqueness results of [9] for viscosity solutions.

The infinite horizon American put options for jump diffusions were analyzed in [3] using the iterative scheme we describe here. The main technical difficulty in the current paper stems from the fact that each element in the approximating sequence solves a parabolic rather than an elliptic problem. In fact, in the infinite horizon case one can obtain a closed form representation for the value function, which is not possible in the finite horizon case. We make use of the results of [8] and Chapter 2 of [10] (also see Chapter 7 of [13]) to study the properties of the approximating sequence. For example, we show that the approximating sequence is bounded with respect to the Hölder seminorm (see page 61 in [7] for a definition), which is used to argue that the limit of the approximating sequence (which is a fixed point of J) solves a corresponding free boundary equation.

Somewhat similar approximation techniques to the one we employ were used to solve optimal stopping problems for *diffusions*; see, e.g., [2] for perpetual optimal stopping problems with nonsmooth pay-off functions and [6], [5] for finite time horizon American put option pricing problems for geometric Brownian motion. On the other hand, [1] and [11] consider the smooth fit principle for the infinite horizon American put option pricing problems for one-dimensional exponential Lévy processes using the fluctuation theory. Also see [4] for the analysis of the smooth fit principle for a multidimensional infinite horizon optimal stopping problem.

The next two sections prepare the proof of our main result, Theorem 3.1, in a sequence of lemmas and corollaries. In the next section, we introduce the functional operator J, which maps a given function to the value function of an optimal stopping problem for a geometric Brownian motion. We then analyze the properties of J. For example, J preserves convexity with respect to the first variable; the increase in the Hölder seminorm after the application of J can be controlled; J maps certain classes of functions to the classical solutions of free boundary equations. In section 3, we construct a sequence of functions that converge to the smallest fixed point of the operator J. We show that the sequence is bounded in the Hölder norm and satisfies certain regularity properties using results of section 2. We eventually arrive at the fact that the smallest fixed point of J is equal to V. As a result the regularity properties of V follow.

2. A functional operator and its properties. Let us define an operator J through its action on a test function $f : \mathbb{R}_+ \times \bar{\mathbb{R}}_+ \to \mathbb{R}_+$: The operator J takes the function f to the value function of the following optimal stopping problem:

$$Jf(x,T) = \sup_{\tau \in \mathcal{S}_{0,T}} \mathbb{E}^x \left\{ \int_0^\tau e^{-(r+\lambda)t} \lambda \cdot Pf(S^0_t, T-t)dt + e^{-(r+\lambda)\tau}(K - S^0_\tau)^+ \right\}, \tag{2.1}$$

in which

$$Pf(x, T-t) = \int_{\mathbb{R}_+} f(xz, T-t)\nu(dz), \quad x \geq 0. \tag{2.2}$$

We will extend $T \to Jf(x,T)$ onto $[0,\infty]$ by letting

$$Jf(x,\infty) = \lim_{T \to \infty} Jf(x,T). \tag{2.3}$$

Here, $S^0 = \{S^0_t; t \geq 0\}$ is the solution of

$$dS^0_t = \mu S^0_t dt + \sigma S^0_t dW_t, \quad S^0_0 = x, \tag{2.4}$$

whose infinitesimal generator is given by

$$\mathcal{A} := \frac{1}{2}\sigma^2 x^2 \frac{d^2}{dx^2} + \mu x \frac{d}{dx}. \tag{2.5}$$

In (2.1), $\mathcal{S}_{[0,T]}$ denotes the set of stopping times of S^0 which take values in $[0,T]$. Note that

$$S^0_t = xH_t, \tag{2.6}$$

where

$$H_t = \exp\left\{ \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t \right\}. \tag{2.7}$$

The next remark characterizes the optimal stopping times of (2.1) using the Snell envelope theory.

Remark 2.1. Let us denote

$$Y_t := \int_0^t e^{-(r+\lambda)s} \lambda \cdot Pf(S^0_t, T-s)ds + e^{-(r+\lambda)t}(K - S^0_t)^+. \tag{2.8}$$

Using the strong Markov property of S^0, we can determine the Snell envelope of Y as

$$\xi_t := \sup_{\tau \in \mathcal{S}_{t,T}} \mathbb{E}\{Y_\tau | \mathcal{F}_t\} \tag{2.9}$$

$$= e^{-(\lambda+r)t} Jf(S^0_t, T-t) + \int_0^t e^{-(r+\lambda)s} \lambda \, Pf(S^0_s, T-s) ds, \quad t \in [0,T].$$

Theorem D.12 in [10] implies that the stopping time

$$\tau_x := \inf\{t \in [0,T) : \xi_t = Y_t\} \wedge T = \inf\{t \in [0,T] : Jf(S^0_t, T-t) = (K - S^0_t)^+\} \tag{2.10}$$

satisfies

$$Jf(x,T) = \mathbb{E}^x \left\{ \int_0^{\tau_x} e^{-(r+\lambda)t} \lambda \cdot Pf(S^0_t, T-t) dt + e^{-(r+\lambda)\tau_x} (K - S^0_{\tau_x})^+ \right\}. \tag{2.11}$$

Moreover, the stopped process $(e^{-(r+\lambda)(t\wedge\tau_x)} Jf(S^0_{t\wedge\tau_x}, T - t\wedge\tau_x) + \int_0^{t\wedge\tau_x} e^{-(r+\lambda)s} \lambda \cdot Pf(S^0_s, T-s) ds)_{t\geq 0}$ is a martingale. The second infimum in (2.10) is less than T because $Jf(S^0_T, 0) = (K - S^0_T)^+$.

When f is bounded, it follows from the bounded convergence theorem that (using the results of [3] and arguments similar to the ones used in Corollary 7.3 in Chapter 2 of [10])

$$Jf(x,\infty) = \sup_{\tau \in \mathcal{S}_{0,\infty}} \mathbb{E}^x \left\{ \int_0^{\tau} e^{-(r+\lambda)t} \lambda \cdot Pf(S^0_t, \infty) dt + e^{-(r+\lambda)\tau} (K - S^0_\tau)^+ \right\}. \tag{2.12}$$

The next three lemmas on the properties of J immediately follow from the definition in (2.1). The first lemma states that J preserves monotonicity.

LEMMA 2.1. *Let $T \to f(x,T)$ be nondecreasing and $x \to f(x,T)$ be nonincreasing. Then $T \to Jf(x,T)$ is nondecreasing and $x \to Jf(x,T)$ is nonincreasing.*

The operator J preserves boundedness and order.

LEMMA 2.2. *Let $f : \mathbb{R}_+ \times \bar{\mathbb{R}}_+ \to \mathbb{R}_+$ be a bounded function. Then Jf is also bounded. In fact,*

$$0 \leq \|Jf\|_\infty \leq K + \frac{\lambda}{r+\lambda} \|f\|_\infty. \tag{2.13}$$

LEMMA 2.3. *For any $f_1, f_2 : \mathbb{R}_+ \times \bar{\mathbb{R}}_+ \to \mathbb{R}_+$ that satisfy $f_1(x,T) \leq f_2(x,T)$, we have that $Jf_1(x,T) \leq Jf_2(x,T)$ for all $(x,T) \in \mathbb{R}_+ \times \bar{\mathbb{R}}_+$.*

As we shall see next, the operator J preserves convexity (with respect to the first variable).

LEMMA 2.4. *If $f : \mathbb{R}_+ \times \bar{\mathbb{R}}_+ \to \mathbb{R}_+$ is a convex function in its first variable, then so is $Jf : \mathbb{R}_+ \times \bar{\mathbb{R}}_+ \to \mathbb{R}_+$.*

Proof. Note that Jf can be written as

$$Jf(x,T) = \sup_{\tau \in \mathcal{S}_{0,T}} \mathbb{E} \left\{ \int_0^{\tau} e^{-(r+\lambda)t} \lambda \cdot Pf(xH_t, T-t) dt + e^{-(r+\lambda)\tau} (K - xH_\tau)^+ \right\}. \tag{2.14}$$

Since $f(\cdot, T-t)$ is convex, so is $Pf(\cdot, T-t)$. As a result the integral with respect to time in (2.14) is also convex in x. On the other hand, note that $(K - xH_\tau)^+$ is also a convex function of x. Taking the expectation does not change the convexity with respect to x. Since the upper envelope (supremum) of convex functions is convex, the result follows. $\square$

Remark 2.2. Since $x = 0$ is an absorbing boundary for the process S^0, for any $f : \mathbb{R}_+ \times \bar{\mathbb{R}}_+ \to \mathbb{R}_+$,

$$
\begin{aligned}
Jf(0,T) &= \sup_{t \in \{0,T\}} \left\{ \int_0^t e^{-(r+\lambda)s} \lambda f(0, T-s) ds + e^{-(\lambda+r)t} K \right\} \\
&= \max \left\{ K, \int_0^T e^{-(r+\lambda)s} \lambda f(0, T-s) ds + e^{-(\lambda+r)T} K \right\}, \quad T \geq 0.
\end{aligned} \tag{2.15}
$$

If we further assume that $f \leq K$, then $Jf(0,T) = K$, $T \geq 0$.

LEMMA 2.5. *Let us assume that* $f : \mathbb{R}_+ \times \bar{\mathbb{R}}_+ \to \mathbb{R}_+$ *is convex in its first variable and* $\|f\|_\infty \leq K$. *Then* $x \to Jf(x,t)$ *satisfies*

$$|Jf(x,T) - Jf(y,T)| \leq |x-y|, \quad (x,y) \in \mathbb{R}_+ \times \bar{\mathbb{R}}_+, \tag{2.16}$$

and all $T \geq 0$.

Proof. First note that a positive convex function that is bounded from above has to be nonincreasing. Therefore f is nonincreasing. As a result of Lemma 2.1, $x \to Jf(x,t)$ is nonincreasing. This function is convex (by Lemma 2.4), and it satisfies

$$Jf(x,T) \geq (K-x)^+, \quad Jf(0,T) = K. \tag{2.17}$$

Consequently, the left and right derivatives of Jf satisfy

$$-1 \leq D_-^x Jf(x,T) \leq D_+^x Jf(x,T) \leq 0, \quad x > 0, \ T \geq 0. \tag{2.18}$$

Now, the result follows since the derivatives are bounded by 1 (also see Theorem 24.7 (on page 237) in [17]). □

Remark 2.3. Let $T_0 \in (0, \infty)$ and denote

$$F(x,T) = \sup_{\tau \in \mathcal{S}_{0,T}} \mathbb{E} \left\{ e^{-(r+\lambda)\tau} (K - xH_\tau)^+ \right\}, \quad x \in R_+, \ T \in [0, T_0]. \tag{2.19}$$

Then for $S \leq T \leq T_0$

$$F(x,T) - F(x,S) \leq C \cdot |T-S|^{1/2} \tag{2.20}$$

for all $x \in \mathbb{R}_+$ and for some C that depends only on T_0. See, e.g., equation (2.4) in [14].

The next lemma, which is very crucial for our proof of the smoothness of the American option price for jump diffusions, shows that the increase in the Hölder seminorm that the operator J causes can be controlled.

LEMMA 2.6. *Let us assume that for some* $L \in (0, \infty)$

$$|f(x,T) - f(x,S)| \leq L|T-S|^{1/2}, \quad (T,S) \in [S_0, T_0] \times [S_0, T_0], \tag{2.21}$$

for all $x \in \mathbb{R}_+$ *and for* $0 \leq S_0 < T_0 < \infty$. *Then*

$$|Jf(x,T) - Jf(x,S)| \leq (a\,L + C)\,|T-S|^{1/2}, \quad (T,S) \in [S_0, T_0] \times [S_0, T_0], \tag{2.22}$$

for some $a \in (0,1)$ *whenever*

$$|T-S| < \left(\frac{r}{r+\lambda} \frac{L}{\lambda K} \right)^2. \tag{2.23}$$

Here, $C \in (0, \infty)$ *is as in Remark* 2.3.

Proof. Without loss of generality we will assume that $T > S$. Then we can write

$$
\begin{aligned}
&Jf(x,T) - Jf(x,S) \\
&\leq \sup_{\tau \in \mathcal{S}_{0,T}} \left[\mathbb{E}\left\{ \int_0^{\tau} e^{-(r+\lambda)t} \lambda\, Pf(xH_t, T-t)dt + e^{-(r+\lambda)\tau}(K - xH_\tau)^+ \right\} \right. \\
&\left. - \mathbb{E}\left\{ \int_0^{\tau\wedge S} e^{-(r+\lambda)t} \lambda\, Pf(xH_t, S-t)dt + e^{-(r+\lambda)(\tau\wedge S)}(K - xH_{\tau\wedge S})^+ \right\} \right] \\
&= \sup_{\tau \in \mathcal{S}_{0,T}} \left[\mathbb{E}\left\{ \int_0^{\tau} e^{-(r+\lambda)t} \lambda \left(Pf(xH_t, T-t) - Pf(xH_t, S-t) \right) dt \right.\right. \\
&+ 1_{\{S<\tau\}} \left[\int_S^{\tau} e^{-(r+\lambda)t} \lambda\, Pf(xH_t, S-t)dt \right. \\
&\qquad \left.\left.\left. + \left(e^{-(r+\lambda)\tau}(K - xH_\tau)^+ - e^{-(r+\lambda)S}(K - xH_S)^+ \right) \right] \right\} \right] \\
&\leq \frac{\lambda}{r+\lambda} L\, (T-S)^{1/2} + \frac{\lambda}{r+\lambda} K \left(e^{-(r+\lambda)S} - e^{-(r+\lambda)T} \right) \\
&\qquad + \sup_{\tau \in \mathcal{S}_{S,T}} \mathbb{E}\left\{ e^{-(r+\lambda)\tau}(K - xH_\tau)^+ \right\} - \mathbb{E}\left\{ e^{-(r+\lambda)S}(K - xH_S)^+ \right\} \\
&\leq \frac{\lambda}{r+\lambda} L\, (T-S)^{1/2} + \lambda\, K\, (T-S) + e^{-(r+\lambda)S} \left(F(H_S, T-S) - F(H_S, 0) \right) \\
&\leq \left(\frac{\lambda}{r+\lambda} L + C \right) (T-S)^{1/2} + \lambda\, K\, (T-S),
\end{aligned} \tag{2.24}
$$

in which F is given by (2.19). To derive the second inequality in (2.24), we use the fact that

$$
\begin{aligned}
|Pf(xH_t, T-t) - Pf(xH_t, S-t)| &\leq \int_{\mathbb{R}_+} \nu(dz) |f(xzH_t, T-t) \\
&\qquad - f(xzH_t, S-t)| \leq L\, |T-S|^{1/2},
\end{aligned} \tag{2.25}
$$

which follows from the assumption in (2.21), and that

$$
\begin{aligned}
&\mathbb{E}\left\{ 1_{\{S<\tau\}} \int_0^{\tau\wedge S} e^{-(r+\lambda)t} \lambda\, Pf(xH_t, S-t)dt \right\} \\
&\qquad \leq \lambda K \mathbb{E}\left\{ \int_S^T e^{-(r+\lambda)t} dt \right\} \leq \frac{\lambda\, K}{\lambda + K} \left(e^{-(r+\lambda)S} - e^{-(r+\lambda)T} \right).
\end{aligned} \tag{2.26}
$$

To derive the third inequality in (2.24), we use

$$
e^{-(r+\lambda)S} - e^{-(r+\lambda)T} \leq e^{-(r+\lambda)S}(r+\lambda)(T-S) \leq (r+\lambda)(T-S). \tag{2.27}
$$

The last inequality in (2.24) follows from (2.20). Equation (2.22) follows from (2.24) whenever T and S satisfy (2.23). ∎

Let us define the continuation region and its sections by

$$\mathcal{C}^{Jf} := \{(T,x) \in (0,\infty)^2 : Jf(x,T) > (K-x)^+\}, \tag{2.28}$$

$$\mathcal{C}_T^{Jf} := \{x \in (0,\infty) : Jf(T,x) > (K-x)^+\},$$

$T > 0$, respectively.

Lemma 2.7. *Suppose that $f : \mathbb{R}_+ \times \bar{\mathbb{R}}_+ \to \bar{\mathbb{R}}_+$ is such that $x \to f(x,T)$ is a positive convex function, $T \to f(x,T)$ is nondecreasing, and $\|f\|_\infty \leq K$. Then for every $T > 0$ there exists $c^{Jf}(T) \in (0,K)$ such that $\mathcal{C}_T^{Jf} = (c^{Jf}(T),\infty)$. Moreover, $T \to c^{Jf}(T)$ is nonincreasing.*

Proof. Let us first show that if $x \geq K$, then $x \in \mathcal{C}_T^{Jf}$ for all $T \geq 0$. Let $\tau_\varepsilon := \inf\{0 \leq t \leq T : S_t^0 \leq K - \varepsilon\}$. Since $\mathbb{P}\{0 < \tau_\varepsilon < T\} > 0$ for $x \geq K$, for all $T > 0$, we have that

$$\mathbb{E}^x\left\{\int_0^{\tau_\varepsilon} e^{-(r+\lambda)t}\lambda\, Pf(S_t^0, T-t)dt + e^{-(r+\lambda)\tau_\varepsilon}(K - S_{\tau_\varepsilon}^0)^+\right\} > 0, \tag{2.29}$$

which implies that $x \in \mathcal{C}_T^{Jf}$. On the other hand, it is clear that

$$(K-x)^+ \leq Jf(x,T) \leq Jf(x,\infty), \quad (x,T) \in \mathbb{R}_+ \times \bar{\mathbb{R}}_+. \tag{2.30}$$

Thanks to Lemma 2.6 of [3], there exist $l^f \in (0,K)$ such that

$$Jf(x,\infty) = (K-x)^+,\ x \in [0,l^f]; \quad Jf(x,\infty) > (K-x)^+,\ x \in (l^f,\infty). \tag{2.31}$$

Since $x \to Jf(x,\infty)$ and $x \to Jf(x,T)$, $T \geq 0$, are convex functions (from Lemma 2.2 in [3] and Lemma 2.4, respectively), (2.29), (2.30), and (2.31) imply that there exists a point $c^{Jf}(T) \in (l^f,K)$ such that

$$Jf(x) = (K-x)^+,\ x \in [0,c^{Jf}(T)]; \quad Jf(x,T) > (K-x)^+,\ x \in (c^{Jf}(T),\infty), \tag{2.32}$$

for $T > 0$. This proves the first statement of the lemma. The fact that $T \to c(T)$ is nonincreasing follows from the fact that $T \to Jf(x,T)$ is nondecreasing. ∎

In the following lemma we will argue that if f has certain regularity properties, then Jf is the classical solution of a parabolic free boundary equation.

Lemma 2.8. *Let us assume that $f : \mathbb{R}_+ \times \bar{\mathbb{R}}_+ \to \mathbb{R}_+$ is convex in its first variable, $\|f\|_\infty \leq K$, and $T \to f(x,T)$ is nonincreasing. Moreover, we will assume that f satisfies*

$$|f(x,T) - f(x,S)| \leq A\,|T-S|^{1/2} \quad \textit{whenever} \quad |T-S| < B \tag{2.33}$$

for all $x \in \mathbb{R}_+$, where A, B are strictly positive constants that do not depend on x. Then the function $Jf : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is the unique bounded solution (in the classical sense) of

$$\mathcal{A}u(x,T) - (r+\lambda)\cdot u(x,T) + \lambda\cdot(Pf)(x,T) - \frac{\partial}{\partial T}u(x,T) = 0, \quad x > c^{Jf}(T), \tag{2.34}$$

$$u(x,T) = (K-x), \quad x \leq c^{Jf}(T), \tag{2.35}$$

in which $\mathcal{A}$ is as in (2.5) *and c^{Jf} is as in Lemma* 2.7. *Moreover,*

$$\mathcal{A}Jf(x,T)-(r+\lambda)\cdot Jf(x,T)+\lambda\cdot(Pf)(x,T)-\frac{\partial}{\partial T}Jf(x,T)\leq 0, \quad x<c^{Jf}(T). \tag{2.36}$$

Proof. The proof is motivated by Theorem 2.7.7 of [10]. Equation (2.35) is clearly satisfied by Jf. In what follows, we will first show that Jf satisfies (2.34). Let us take a point in $(t,T)\in\mathcal{C}^{Jf}$ and consider a bounded rectangle $R=(t_1,t_2)\times(x_1,x_2)$ containing this point. We will let

$$t_2-t_1<B\wedge\left(\frac{rA}{(r+\lambda)\lambda K}\right)^2. \tag{2.37}$$

Let $\partial_0 R$ be the parabolic boundary of R and consider the parabolic partial differential equation

$$\begin{aligned}&\mathcal{A}u(x,T)-(r+\lambda)\cdot u(x,T)+\lambda\cdot(Pf)(x,T)-\frac{\partial}{\partial T}u(x,T)=0 \quad \text{in } R,\\ &u(x,T)=Jf(x,T) \quad \text{on } \partial_0 R.\end{aligned} \tag{2.38}$$

As a result of Lemmas 2.5 and 2.6, Jf satisfies the uniform Lipschitz and Hölder continuity conditions, which implies that Jf is continuous. On the other hand, for any $(T,x)\in R$

$$\begin{aligned}|Pf(x,T)-Pf(y,S)|&\leq|Pf(x,T)-Pf(x,S)|+|Pf(x,S)-Pf(y,S)|\\ &\leq\int_{\mathbb{R}_+}\nu(dz)\left(|f(xz,T)-f(xz,S)|+|f(xz,S)-f(yz,S)|\right)\\ &\leq A\,|T-S|^{1/2}+\xi\,|x-y|.\end{aligned} \tag{2.39}$$

Now, Theorem 5.2 in [8] implies that (2.38) has a unique classical solution. We will show that this unique solution coincides with Jf using the optional sampling theorem. Let us introduce the stopping time

$$\tau:=\inf\{\theta\in[0,t_0-t_1):(t_0-\theta,x_0H_\theta\in\partial_0R\}\wedge(t_0-t_1), \tag{2.40}$$

which is the first time S^0 hits the parabolic boundary when S^0 starts from (x_0,t_0). Let us also define the process $N_\theta:=e^{-(r+\lambda)\theta}u(x_0H_\theta,t_0-\theta)+\int_0^\theta e^{-(r+\lambda)t}\lambda\cdot Pf(S^0_t,t_0-t)dt$, $\theta\in[0,t_0-t_1]$. From the classical Itô formula it follows that the stopped process $N_{\theta\wedge\tau}$ is a bounded martingale. As a result

$$\begin{aligned}u(x_0,t_0)=N_0&=\mathbb{E}^x\{N_\tau\}\\ &=\mathbb{E}\left\{e^{-(r+\lambda)\tau}Jf(xH_\tau,t_0-\tau))+\int_0^\tau e^{-(r+\lambda)t}\lambda\cdot Pf(S^0_t,t_0-t)dt\right\}.\end{aligned} \tag{2.41}$$

Clearly $\tau\leq\tau_x$. Since the stopped process $(e^{-(r+\lambda)(t\wedge\tau_x)}Jf(S^0_{t\wedge\tau_x},t_0-t\wedge\tau_x)+\int_0^{t\wedge\tau_x}e^{-(r+\lambda)t}\lambda\cdot Pf(S^0_s,t_0-s)ds)_{t\geq0}$ is a bounded martingale, another application of the optional sampling theorem yields

$$\mathbb{E}\left\{e^{-(r+\lambda)\tau} Jf(x_0 H_\tau, t_0-\tau) + \int_0^\tau e^{-(r+\lambda)t}\lambda \cdot Pf(S_t^0, t_0-t)dt\right\} = Jf(x_0,t_0). \tag{2.42}$$

Combining (2.41) and (2.42), we see that (2.34) is satisfied in the classical sense since the choice of $(x_0,t_0) \in \mathcal{C}^{Jf}$ is arbitrary.

We still need to show uniqueness among bounded functions. Fix $x > c^{Jf}(T)$. Let u be a bounded function satisfying (2.34) and (2.35). Let us define $M_t := e^{-(r+\lambda)t}u(xH_t, T-t) + \int_0^t e^{-(r+\lambda)t}\lambda \cdot Pf(S_s^0, T-s)ds$. Using the classical Itô formula it can be seen that $M_{t\wedge\tau_x}$ is a bounded martingale. Since τ_x is optimal (see (2.11)), by the optional sampling theorem, we have

$$\begin{aligned} u(x,T) &= M_0 = \mathbb{E}^x\{M_{\tau_x}\} \\ &= \mathbb{E}\left\{e^{-(r+\lambda)\tau}u(xH_{\tau_x}, T-\tau_x) + \int_0^{\tau_x} e^{-(r+\lambda)t}\lambda \cdot Pf(S_s^0, T-s)ds\right\} \\ &= \mathbb{E}\left\{e^{-(r+\lambda)\tau}(K-xH_{\tau_x})^+ + \int_0^{\tau_x} e^{-(r+\lambda)t}\lambda \cdot Pf(S_s^0, T-s)ds\right\} = Jf(x,T). \end{aligned} \tag{2.43}$$

Next, we will prove (2.36). To this end, let $x < c^{Jf}(t)$. Let U be a closed interval centered at x such that $U \subset (0, c^{Jf}(T))$. Let $\tau_U = \{t \geq 0 : xH_t \notin U\}$. Since $(e^{-(r+\lambda)t}Jf(S_t^0, T-t) + \int_0^t e^{-(r+\lambda)s}\lambda \cdot Pf(S_s^0, T-s)ds)_{t\geq 0}$ is a supermartingale, we can write

$$\begin{aligned} \mathbb{E}\Bigg[& e^{-(r+\lambda)(\tau_U\wedge t)}Jf(xH_{\tau_U\wedge t}, T-\tau_U\wedge t)) \\ & + \int_0^{\tau_U\wedge t} e^{-(r+\lambda)u}\lambda Pf(xH_u, T-u)du\Bigg] \leq Jf(x,T) \end{aligned} \tag{2.44}$$

for all $t \geq 0$. Since $Jf(x,t) = K - x$ when $(T,x) \in \mathbb{R}_+^2 - C^{Jf}$, we can apply Itô's formula to obtain that

$$\begin{aligned} \lim_{t\to 0}\mathbb{E}\Bigg[\frac{1}{t}\int_0^{\tau_U\wedge t} & e^{-(r+\lambda)u}\left(\left(\mathcal{A} - (r+\lambda)\cdot - \frac{\partial}{\partial T}\right)\right. \\ & \left. Jf(xH_u, t-u) + \lambda Pf(xH_u, T-u)\right)du\Bigg] \leq 0. \end{aligned} \tag{2.45}$$

Now, (2.36) follows thanks to the dominated convergence theorem, which allows us to exchange the limit and the expectation. We can apply the dominated convergence theorem thanks to the fact that U is a compact domain. ∎

LEMMA 2.9. *For a given $T > 0$, let $x \to f(x,T)$ be a convex and nonincreasing function. Then the convex function $x \to Jf(x,T)$ is of class C^1 at $x = c(T)$, i.e.,*

$$\left.\frac{\partial}{\partial x}Jf(x,T)\right|_{x=c(T)} = -1. \tag{2.46}$$

Proof. The proof is similar to the proof of Lemma 7.8 on page 74 of [10], but we will provide it here for the sake of completeness. If we let $x = c(T)$, then

$$
\begin{aligned}
&Jf(x+\varepsilon, T) \\
&= \mathbb{E}\left\{\int_0^{\tau_{x+\varepsilon}} e^{-(r+\lambda)t}\lambda \cdot Pf((x+\varepsilon)H_t, T-t)dt + e^{-(r+\lambda)\tau_{x+\varepsilon}}(K-(x+\varepsilon)H_{\tau_{x+\varepsilon}})^+\right\} \\
&= \mathbb{E}\left\{\int_0^{\tau_{x+\varepsilon}} e^{-(r+\lambda)t}\lambda \cdot Pf(xH_t, T-t)dt + e^{-(r+\lambda\tau_{x+\varepsilon})}(K-xH_{\tau_{x+\varepsilon}})^+\right\} \\
&+ \mathbb{E}\left\{\int_0^{\tau_{x+\varepsilon}} e^{-(r+\lambda)t}\lambda \cdot \left[Pf((x+\varepsilon)H_t, T-t) - Pf(xH_t, T-t)\right]dt\right\} \\
&+ \mathbb{E}\left\{e^{-(r+\lambda)\tau_{x+\varepsilon}}\left[(K-(x+\varepsilon)H_{\tau_{x+\varepsilon}})^+ - (K-xH_{\tau_{x+\varepsilon}})^+\right]\right\} \\
&\leq Jf(x,T) + \mathbb{E}\left\{1_{\{\tau_{x+\varepsilon}<T\}}e^{-(r+\lambda)\tau_{x+\varepsilon}}\left[(K-(x+\varepsilon)H_{\tau_{x+\varepsilon}}) - (K-xH_{\tau_{x+\varepsilon}})\right]\right\} \\
&+ \mathbb{E}\left\{1_{\{\tau_{x+\varepsilon}=T\}}e^{-(r+\lambda)\tau_{x+\varepsilon}}\left[(K-(x+\varepsilon)H_{\tau_{x+\varepsilon}})^+ - (K-xH_{\tau_{x+\varepsilon}})^+\right]\right\} \\
&\leq Jf(x,T) - \varepsilon\mathbb{E}^x\left\{1_{\{\tau_{x+\varepsilon}<T\}}e^{-(r+\lambda)\tau_{x+\varepsilon}}H_{\tau_{x+\varepsilon}}\right\} \\
&= Jf(x,T) - \varepsilon\mathbb{E}^x\left\{e^{-(r+\lambda)\tau_{x+\varepsilon}}H_{\tau_{x+\varepsilon}}\right\} + \varepsilon\mathbb{E}^x\left\{1_{\{\tau_{x+\varepsilon}=T\}}e^{-(r+\lambda)T}H_T\right\}.
\end{aligned}
\tag{2.47}
$$

The first inequality follows since $\tau_{x+\varepsilon}$ is not optimal when S^0 starts at x and $x \to Pf(x,T)$ is a decreasing function for any $T \geq 0$. From (2.47) it follows that

$$D_+^x Jf(x+, T) \leq -1, \tag{2.48}$$

since $e^{-(r+\lambda)t}H_t$ is a uniformly integrable martingale and $\tau_{x+\varepsilon} \downarrow 0$. Convexity of $Jf(t,x)$ (Lemma 2.4) implies that

$$-1 = D_-^x Jf(x-, t) \leq D_+^x Jf(x+, t) \leq -1, \tag{2.49}$$

which yields the desired result. ∎

3. A sequence of functions approximating *V*. Let us define a sequence of functions by the following iteration:

$$v_0(x,T) = (K-x)^+, \quad v_{n+1}(x,T) = Jv_n(x,T),\ n \geq 0, \quad \text{for all } (x,T) \in \mathbb{R}_+ \times \mathbb{R}_+. \tag{3.1}$$

We extend these functions onto $\mathbb{R}_+ \times \bar{\mathbb{R}}_+$ by letting

$$v_n(x,\infty) = \lim_{T\to\infty} v_n(x,T). \tag{3.2}$$

This sequence of functions is a bounded sequence, as the next corollary shows.

COROLLARY 3.1. *For all $n \geq 0$,*

$$(K-x)^+ \leq v_n(x,T) \leq \left(1+\frac{\lambda}{r}\right)K, \quad (x,T) \in \mathbb{R}_+ \times \bar{\mathbb{R}}_+. \tag{3.3}$$

Proof. The first inequality follows since it may not be optimal to stop immediately. Let us prove the second inequality using an induction argument: Observe that $v_0(x,T)=(K-x)^+$, $(x,T)\in\mathbb{R}_+\times\bar{\mathbb{R}}_+$, satisfies (3.3). Let us assume that (3.3) holds for n and show that it holds for $n+1$. Using (2.13), we get that

$$\|v_{n+1}\|_\infty = \|Jv_n\|_\infty \le K + \frac{\lambda}{r+\lambda}\left(1+\frac{\lambda}{r}\right)K = \left(1+\frac{\lambda}{r}\right). \qquad \square \tag{3.4}$$

As a corollary of Lemmas 2.3 and 2.4 we can state the following corollary, whose proof can be carried out by induction.

COROLLARY 3.2. *The sequence $(v_n(x,T))_{n\ge 0}$ is increasing for all $(x,T)\in\mathbb{R}_+\times\bar{\mathbb{R}}_+$. For each n, the function $x\to v_n(x,T)$, $x\ge 0$, is convex for all $T\in\bar{\mathbb{R}}_+$.*

Remark 3.1. Let us define

$$v_\infty(x,T) := \sup_{n\ge 0} v_n(x,T), \quad (x,T)\in\mathbb{R}_+\times\bar{\mathbb{R}}_+. \tag{3.5}$$

This function is well defined as a result of (3.3) and Corollary 3.2. In fact, it is convex, because it is the upper envelope of convex functions, and it is bounded by the right-hand side of (3.3).

COROLLARY 3.3. *For each $n\ge 0$ and $t\in\mathbb{R}_+$, $x\to v_n(x,T)$ is a decreasing function on $[0,\infty)$. Moreover, $T\to v_n(x,T)$ is nondecreasing. The same statements hold for $x\to v_\infty(x,T)$ and $T\to v_\infty(x,T)$, respectively.*

Proof. The behavior with respect to the first variable is a result of Corollary 3.2 and Remark 3.1 since any positive convex function that is bounded from above is decreasing. For each n, the fact that $T\to v_n(x,T)$ is nondecreasing is a corollary of Lemma 2.1. On the other hand, for any $T\ge S\ge 0$, we have that $v_\infty(x,T)=\sup_n v_n(x,T)\ge\sup_n v_n(x,S)=v_\infty(x,S)$. $\square$

Next, we will sharpen the upper bound in Corollary 3.1. This improvement has some implications for the continuity of $x\to v_n(x,T)$, $n\ge 1$, and $x\to v_\infty(x,T)$ at $x=0$.

Remark 3.2. The upper bound in (3.1) can be sharpened using Corollary 3.3 and Remark 2.2. Indeed, we have

$$\begin{aligned}(K-x)^+ &\le v_n(x,T) < K, \quad \text{for each } n, \quad \text{and}\\ (K-x)^+ &\le v_\infty(x,T) < K, \quad (x,T)\in(0,\infty)^2.\end{aligned} \tag{3.6}$$

It follows from this observation that for every $T\in\bar{\mathbb{R}}_+$, $x\to v_n(x,T)$, for every n, and $x\to v_\infty(x,T)$ are continuous at $x=0$ since $v_n(0,T)=v_\infty(0,T)=K$ and these functions are convex. (Note that convexity already guarantees continuity for $x>0$.)

LEMMA 3.1. *The function v_∞ is the smallest fixed point of the operator J.*

Proof.

$$\begin{aligned}&v_\infty(x,T-t)\\ &\quad=\sup_{n\ge 1} v_n(x,T-t)\\ &\quad=\sup_{n\ge 1}\sup_{\tau\in\mathcal{S}_{0,T}}\mathbb{E}^x\left\{\int_0^\tau e^{-(r+\lambda)t}\lambda\cdot Pv_n(S_t^0,T-t)dt+e^{-(r+\lambda)\tau}(K-S_\tau^0)^+\right\}\end{aligned} \tag{3.7}$$

$$= \sup_{\tau \in \mathcal{S}_{0,T}} \sup_{n \geq 1} \mathbb{E}^x \left\{ \int_0^\tau e^{-(r+\lambda)t} \lambda \cdot P v_n(S_t^0, T-t) dt + e^{-(r+\lambda)\tau} (K - S_\tau^0)^+ \right\}$$

$$= \sup_{\tau \in \mathcal{S}_{0,T}} \mathbb{E}^x \left\{ \int_0^\tau e^{-(r+\lambda)t} \lambda \cdot P(\sup_{n \geq 1} v_n)(S_t^0, T-t) dt + e^{-(r+\lambda)\tau} (K - S_\tau^0)^+ \right\}$$

$$= J v_\infty(x, T-t),$$

in which the fourth equality follows by applying the monotone convergence theorem three times. Let $w : \mathbb{R}_+ \times \bar{\mathbb{R}}_+ \to \mathbb{R}_+$ be another fixed point of the operator J. We will argue by induction that $w \geq v_\infty$. For $(x,t) \in \mathbb{R}_+ \times \bar{\mathbb{R}}_+$, $w(x, T-t) = Jw(x, T-t)$, which implies that $w(x, T-t) = Jw(x, T-t) \geq (K-x)^+ = v_0(\cdot)$. If we assume that $w(x, T-t) \geq v_n(x, T-t)$, then $w(x, T-t) = Jw(x, T-t) \geq J v_n(x, T-t) = v_{n+1}(x, T-t)$. Consequently $w(x, T-t) \geq v_n(x, T-t)$ for all $n \geq 0$. As a result $w(x, T-t) \geq \sup_{n \geq 0} v_n(x, T-t) = v_\infty(x, T-t)$. $\square$

LEMMA 3.2. *The sequence $\{v_n(\cdot,\cdot)\}_{n \geq 0}$ converges uniformly to v_∞. In fact, the rate of convergence is exponential:*

$$v_n(x,T) \leq v_\infty(x,T) \leq v_n(x,T) + \left(\frac{\lambda}{\lambda + r}\right)^n K, \quad (x,T) \in \mathbb{R}_+ \times \bar{\mathbb{R}}_+. \tag{3.8}$$

Proof. The first inequality follows from the definition of v_∞. The second inequality can be proved by induction. The inequality holds when we set $n = 0$ by Remark 3.2. Assume that the inequality holds for $n > 0$. Then

$$\begin{aligned} v_\infty(x,T) &= \sup_{\tau \in \mathcal{S}_{0,T}} \mathbb{E}^x \left\{ \int_0^\tau e^{-(r+\lambda)t} \lambda \cdot P v_\infty(S_t^0, T-t) dt + e^{-(r+\lambda)\tau} (K - S_\tau^0)^+ \right\} \\ &\leq \sup_{\tau \in \mathcal{S}_{0,T}} \mathbb{E}^x \left\{ \int_0^\tau e^{-(r+\lambda)t} \lambda \cdot P v_n(S_t^0, T-t) dt + e^{-(r+\lambda)\tau} (K - S_\tau^0)^+ \right\} \\ &\quad + \int_0^\infty dt\, e^{-(\lambda+r)t} \lambda \left(\frac{\lambda}{\lambda + r}\right)^n K \\ &= v_{n+1}(x,T) + \left(\frac{\lambda}{\lambda + r}\right)^{n+1} K. \qquad \square \end{aligned} \tag{3.9}$$

Remark 3.3. Note that, for a fixed $T_0 > 0$,

$$\begin{aligned} v_n(x,T) &\leq v_\infty(x,T) \\ &\leq v_n(x,T) + \left(1 - e^{-(r+\lambda)T_0}\right)^n \left(\frac{\lambda}{\lambda + r}\right)^n K, \quad x \in \mathbb{R}_+,\ T \in (0, T_0). \end{aligned} \tag{3.10}$$

This can be derived using an induction argument similar to the one used in the proof of Lemma 3.2. We simply replace (3.9) by

$$
\begin{aligned}
v_\infty(x,T) \le \sup_{\tau\in\mathcal{S}_{0,T}} \mathbb{E}^x\Big\{ \int_0^\tau e^{-(r+\lambda)t}\lambda\cdot P v_n(S_t^0, T-t)dt + e^{-(r+\lambda)\tau}(K-S_\tau^0)^+\Big\} \\
+ \int_0^{T_0} dt\, e^{-(\lambda+r)t}\left(1-e^{-(r+\lambda)T_0}\right)^n \lambda\left(\frac{\lambda}{\lambda+r}\right)^n K \\
= v_{n+1}(x,T) + K\left(1-e^{-(r+\lambda)T_0}\right)^{n+1}\left(\frac{\lambda}{\lambda+r}\right)^{n+1}.
\end{aligned}
\tag{3.11}
$$

Observe that one can replace K in (3.10) by $\|v_\infty - v_0\|_\infty$. Note that the convergence rate in (3.10) is fast. This will lead to a numerical scheme, whose error versus accuracy characteristics can be controlled, for pricing American options.

Remark 3.4. Let $T_0 \in (0,\infty)$. It can be shown using similar arguments to the ones used in the proof of Lemma 2.6 that

$$
|v_1(x,T) - v_1(x,S)| \le \lambda K\,|T-S| + C\,|T-S|^{1/2}, \quad T,S \in (0,T_0], \tag{3.12}
$$

for all $x \in \mathbb{R}_+$, in which $C \in (0,\infty)$ is as in Remark 2.3. In fact,

$$
|v_1(x,T) - v_1(x,S)| \le L\,|T-S|^{1/2} \tag{3.13}
$$

for all $x \in \mathbb{R}_+$ and for some L that depends only on T_0.

The next lemma shows that the functions v_n, $n \ge 0$, and v_∞ are locally Hölder continuous with respect to the time variable.

LEMMA 3.3. *Let $T_0 \in (0,\infty)$ and $L \in (0,\infty)$ be as in Remark* 3.4 *and $C \in (0,\infty)$ be as in Remark* 2.3. *Then, for $T,S \in (0,T_0)$, we have that*

$$
|v_n(x,T) - v_n(x,S)| \le \left(L + \frac{C}{1-a}\right)|T-S|^{1/2} \text{ whenever } |T-S| \le \left(\frac{r}{r+\lambda}\frac{L}{\lambda K}\right)^2 \tag{3.14}
$$

for all $x \in \mathbb{R}_+$ and for all $n \ge 1$. Here, $a \in (0,1)$ is as in Lemma 2.6. *Moreover,*

$$
|v_\infty(x,T) - v_\infty(x,S)| \le \left(L + \frac{C}{1-a}\right)|T-S|^{1/2} \text{ whenever } |T-S| \le \left(\frac{r}{r+\lambda}\frac{L}{\lambda K}\right)^2 \tag{3.15}
$$

for all $x \in \mathbb{R}_+$.

Proof. The proof of (3.14) will be carried out using an induction argument. Observe from Remark 3.4 that (3.14) holds for $n = 1$. Let us assume that (3.14) holds for n and show that it holds for $n+1$. Using Lemma 2.6, we have that

$$
|v_{n+1}(x,T) - v_{n+1}(x,S)| \le \left(a\left(L + \frac{C}{1-a}\right) + C\right)|T-S|^{1/2} \tag{3.16}
$$

for $|T-S| \le \left(\frac{r}{r+\lambda}\,\frac{L+C/(1-a)}{\lambda K}\right)^2$. It is clear that the right-hand side of (3.16) is less than that of (3.14), and

$$
\frac{r}{r+\lambda}\,\frac{L+C/(1-a)}{\lambda K} \ge \frac{r}{r+\lambda}\,\frac{L}{\lambda K}, \tag{3.17}
$$

from which the first statement of the lemma follows. Now, let us prove (3.15). To this end observe that

$$
\begin{aligned}
&|v_\infty(x,T)-v_\infty(x,S)| \\
&\quad\le |v_\infty(x,T)-v_n(x,T)|+|v_n(x,T)-v_n(x,S)|+|v_\infty(x,S)-v_n(x,S)| \\
&\quad\le 2\left(\frac{\lambda}{\lambda+r}\right)^n K+\left(L+\frac{C}{1-a}\right)|T-S|^{1/2}
\end{aligned}
\tag{3.18}
$$

for any $n>1$, which follows from (3.14) and Lemma 3.2. The result follows since n on the right-hand side of (3.18) is arbitrary. ∎

LEMMA 3.4. *For $n\ge 0$, $|v_n(x,T)-v_n(y,T)|\le|x-y|$, and $|v_\infty(x,T)-v_\infty(y,T)|\le|x-y|$, $(x,y)\in\mathbb{R}_+\times\bar{\mathbb{R}}_+$, for all $T\ge 0$.*

Proof. It follows from Remark 3.2 that $\|v_n\|_\infty\le K$, for all $n\ge 0$, and $\|v_\infty\|_\infty\le K$. Moreover, for each $n\ge 0$, $v_n(\cdot,T)$ is convex (for all $T\in\bar{\mathbb{R}}_+$) as a result of Corollary 3.2. On the other hand, it was pointed out in Remark 3.1 that $v_\infty(\cdot,T)$ is convex for all $T\in\mathbb{R}_+$. Since

$$
v_{n+1}(x,T)=Jv_n(x,T)\quad\text{and}\quad v_\infty(x,T)=Jv_\infty(x,T),
\tag{3.19}
$$

the statement of the lemma follows from Lemma 2.5. ∎

LEMMA 3.5. *For all $T\ge 0$ and $n\ge 0$, $\mathcal{C}_T^{v_{n+1}}=(c^{v_{n+1}}(T),\infty)$ for some $c^{v_{n+1}}(T)\in(0,K)$ and $\mathcal{C}_T^{v_\infty}=(c^{v_\infty}(T),\infty)$ for some $c^{v_\infty}\in(0,K)$. The function v_{n+1} is the unique bounded solution (in the classical sense) of*

$$
\begin{aligned}
&\mathcal{A}v_{n+1}(x,T)-(r+\lambda)\cdot v_{n+1}(x,T)+\lambda\cdot(Pv_n)(x,T) \\
&\qquad-\frac{\partial}{\partial T}v_{n+1}(x,T)=0,\quad x>c^{v_{n+1}}(T), \\
&v_{n+1}(x,T)=(K-x),\quad x\le c^{v_{n+1}}(T),
\end{aligned}
\tag{3.20}
$$

and it satisfies

$$
\left.\frac{\partial}{\partial x}v_{n+1}(x,T)\right|_{x=c^{v_{n+1}}(T)}=-1,\quad T>0.
\tag{3.21}
$$

Moreover, v_∞ is the unique bounded solution (in the classical sense) of

$$
\begin{aligned}
&\mathcal{A}v_\infty(x,T)-(r+\lambda)\cdot v_\infty(x,T)+\lambda\cdot(Pv_\infty)(x,T)-\frac{\partial}{\partial T}v_\infty(x,T)=0,\quad x>c^{v_\infty}(T), \\
&v_\infty(x,T)=(K-x),\quad x\le c^{v_\infty}(T),
\end{aligned}
\tag{3.22}
$$

and it satisfies

$$
\left.\frac{\partial}{\partial x}v_\infty(x,T)\right|_{x=c^{v_\infty}(T)}=-1,\quad T>0.
\tag{3.23}
$$

On the other hand,

$$
\mathcal{A}v_\infty(x,T)-(r+\lambda)\cdot v_\infty(x,T)+\lambda\cdot(Pv_\infty)(x,T)-\frac{\partial}{\partial T}v_\infty(x,T)\le 0,\quad x<c^{v_\infty}(T).
\tag{3.24}
$$

Proof. The fact that $\mathcal{C}^{v_{n+1}} = (c^{v_{n+1}}, \infty)$ and $C^{v_\infty} = (c^{v_\infty}, \infty)$ for some $c^{v_{n+1}} \in (0, K)$ and $c^{v_\infty} \in (0, K)$ follows from Lemma 2.7 since the assumptions in that lemma hold thanks to Corollaries 3.2 and 3.3; Remarks 3.1 and 3.2; and Lemma 3.1.

The partial differential equations (3.20), (3.22) and the inequality in (3.24) are satisfied as a corollary of Lemma 2.8; Corollaries 3.2 and 3.3, Remarks 3.1 and 3.2; and Lemmas 3.1 and 3.3.

Observe that since v_n is convex (Corollary 3.2) and nonincreasing (Corollary 3.3) with respect to its first variable, v_{n+1} $(= Jv_n)$ satisfies the smooth fit condition in (3.21) as a result of Lemma 2.9. The smooth fit condition in (3.23) holds for v_∞ as a result of Lemma 2.9 since v_∞ $(= Jv_\infty)$ (Lemma 3.1) and $x \to v_\infty(x, T)$ is nonincreasing and convex. ∎

The next lemma will be used to verify the fact that $V = v_\infty$. The classical Itô rule cannot be applied to the process $t \to v_\infty(S_t, T-t)$ since the function v_∞ may fail to be $C^{2,1}$ at $T \to c^{v_\infty}(T)$. As a result, the semimartingale decomposition of the process $t \to v_\infty(S_t, T-t)$ may contain an extra term due to the local time of the process S at the free boundary.

LEMMA 3.6. *Let $X = \{X_t; t \geq 0\}$ be a semimartingale and $b : \mathbb{R}_+ \to \mathbb{R}$ be a continuous function of bounded variation. Let $F : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ be a continuous function that is $C^{2,1}$ on $\bar{C}$ and $\bar{D}$ (it may not necessarily be $C^{1,2}$ across the boundary curve b), in which*

$$C \triangleq \{(x,t) \in \mathbb{R} \times \mathbb{R}_+ : x < b(t)\}, \quad D \triangleq \{(x,t) \in \mathbb{R} \times \mathbb{R}_+ : x > b(t)\}.$$

That is, there exist two functions $F^1, F^2 : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$, that are $C^{2,1}$ on $\mathbb{R} \times \mathbb{R}_+$, and $F(x,t) = F^1(x,t)$ when $(x,t) \in C$ and $F(x,t) = F^2(x,t)$ when $(x,t) \in D$. Moreover, $F^1(b(t),t) = F^2(b(t),t)$. Then the following generalization of Itô's formula holds:

(3.25)

$$\begin{aligned}
F(X_t,t) = F(X_0,0) &+ \int_0^t \frac{1}{2}\left[F_t(X_{s-}+,s) + F_t(X_{s-}-,s)\right] ds \\
&+ \frac{1}{2}\int_0^t \left[F_x(X_{s-}+,s) + F_x(X_{s-}-,s)\right] dX_s \\
&+ \frac{1}{2}\int_0^t 1_{\{X_{s-} \neq b(s)\}} F_{xx}(X_{s-},s) d\left\langle X, X\right\rangle_s^c \\
&+ \sum_{0<s\leq t}\left\{F(X_s,s) - F(X_{s-},s) - \frac{1}{2}\Delta X_s\left[F_x(X_{s-}-,s) + F_x(X_{s-}+,s)\right]\right\} \\
&+ \frac{1}{2}\int_0^t \left[F_x(X_{s-}+,s) - F_x(X_{s-}-,s)\right] 1_{\{X_{s-}=b(s)\}} dL_t^b,
\end{aligned}$$

where L_t^b is the local time of the semimartingale $X_t - b(t)$ at zero (see the definition on page 216 *in* [16]*).*

Lemma 3.6 was stated in Theorem 2.1 of [12] for continuous semimartingales. The generalization for the case when the underlying process is not necessarily continuous is intuitively clear and just technical, but we will prove it in the appendix for the sake of completeness. We are now ready to state the main result.

THEOREM 3.1. *The value function V is the unique bounded solution (in the classical sense) of the integro-partial differential equation in* (3.22). *Moreover, it satisfies* *the smooth fit condition at the optimal stopping boundary, i.e.,* $\frac{\partial}{\partial x}V(x,T)\big|_{x=c^{v_\infty}(T)} = -1$, $T > 0$.

Proof. The proof is a corollary of the optional sampling theorem and the generalized Itô formula given above. Let $T \in (0,\infty)$ and define

$$\widetilde{M}_t = e^{-rt}v_\infty(S_t, T-t) \quad \text{and} \quad \widetilde{\tau}_x := T \wedge \inf\{t \in [0,T] : S_t \leq c^{v_\infty}(T-t)\}. \tag{3.26}$$

It follows from (3.22) and the classical Itô lemma that $\{\widetilde{M}_{t\wedge\widetilde{\tau}_x}\}_{0\leq t\leq T}$ is a bounded $\mathbb{P}$-martingale. Using the optional sampling theorem, one obtains

$$\begin{aligned} v_\infty(x,T) = \widetilde{M}_0 = \mathbb{E}^x\left\{\widetilde{M}_{\widetilde{\tau}_x}\right\} &= \mathbb{E}^x\left\{e^{-r\widetilde{\tau}_x}v_\infty(S_{\widetilde{\tau}_x}, T-\widetilde{\tau}_x)\right\} \\ &= \mathbb{E}^x\left\{e^{-r\widetilde{\tau}_x}(K - S_{\widetilde{\tau}_x})^+\right\} \leq V(x,T). \end{aligned} \tag{3.27}$$

In the rest of the proof we will show that $v_\infty(x,T) \geq V(x,T)$. Since v_∞ satisfies the smooth fit principle across the free boundary, when we apply the generalized Itô formula to $v_\infty(S_t, T-t)$, the local time term drops. Thanks to (3.22) and (3.24), $v_\infty(S_t, T-t)$ is a positive $\mathbb{P}$-supermartingale. Again, using the optional sampling theorem, for any $\tau \in \widetilde{\mathcal{S}}_{0,T}$

$$v_\infty(x,T) = \widetilde{M}_0 \geq \mathbb{E}^x\left\{\widetilde{M}_\tau\right\} = \mathbb{E}^x\left\{e^{-r\tau}v_\infty(S_\tau, T-\tau)\right\} \geq \mathbb{E}^x\left\{e^{-r\tau}(K - S_\tau)^+\right\}. \tag{3.28}$$

As a result $v_\infty(x,T) \geq V(x,T)$. ∎

Remark 3.5. We have that

$$\mathcal{C}_T^{v_\infty} = \{x \in (0,\infty) : v_\infty > (K-x)^+\} = (c^{v_\infty}(T), \infty). \tag{3.29}$$

On the other hand, $v_\infty = K - x$ for $x \leq c^{v_\infty}$. Since $V = v_\infty$, by Theorem 3.1, it follows that

$$\mathcal{C}_T^{V} = \{x \in (0,\infty) : V > (K-x)^+\} = (c^{v_\infty}(T), \infty). \tag{3.30}$$

Appendix. Proof of Lemma 3.6. As in [12] we will define $Z_t^1 = X_t \wedge b(t)$, $Z_t^2 = X_t \vee b(t)$, and observe that

$$F(X_t, t) = F^1(Z_t^1, t) + F^2(Z_t^2, t) - F(b(t), t). \tag{A.1}$$

On the other hand, applying the Meyer–Itô formula (see Theorem 70 in [16]) to the semimartingale $X_t - b(t)$, we obtain

$$\begin{aligned} |X_t - b(t)| &= |X_0 - b(0)| + \int_0^t \operatorname{sign}(X_{s-} - b(s))d(X_s - b(s)) \\ &\quad + 2\sum_{0<s\leq t}\left[1_{\{X_{s-}>b(s)\}}(X_s - b(s))^- + 1_{\{X_{s}\leq b(s)\}}(X_s - b(s))^+\right] + L_t^b. \end{aligned} \tag{A.2}$$

Since $Z_t^1 = \frac{1}{2}\left(X_t + b(t) - |X_t - b(t)|\right)$ and $Z_t^2 = \frac{1}{2}\left(X_t + b(t) + |X_t - b(t)|\right)$, using (A.2), we get

$$\begin{aligned} dZ_t^1 &= \frac{1}{2}\left\{(1 - \operatorname{sign}(X_{t-} - b(t)))dX_t + (1 + \operatorname{sign}(X_{t-} - b(t)))db(t) - dL_t^b\right\} \\ &\quad - \left[1_{\{X_{t-}>b(t)\}}(X_t - b(t))^- + 1_{\{X_t\leq b(t)\}}(X_t - b(t))^+\right], \end{aligned} \tag{A.3}$$

$$\begin{aligned} dZ_t^2 &= \frac{1}{2}\left\{(1+\mathrm{sign}(X_{t-}-b(t)))dX_t + (1+\mathrm{sign}(X_{t-}-b(t)))db(t) - dL_t^b\right\} \\ &\quad + \left[1_{\{X_{t-}>b(t)\}}(X_t-b(t))^- + 1_{\{X_t\le b(t)\}}(X_t-b(t))^+\right]. \end{aligned} \tag{A.4}$$

It follows from the dynamics of Z^i, $i \in \{1,2\}$, that

$$d\left\langle Z^i, Z^i\right\rangle_t^c = \left(1_{\{X_{t-}<b(t)\}} + \frac{1}{4}1_{\{X_{t-}=b(t)\}}\right) d\left\langle X, X\right\rangle_t^c = 1_{\{X_{t-}<b(t)\}} d\left\langle X, X\right\rangle_t^c, \tag{A.5}$$

where the second equality follows from the occupation density formula; see, e.g., Corollary 1 on page 219 of [16]. Applying the classical Itô formula to $F^1(Z_t^1,t)$ and $F^2(Z_t^2,t)$ and using the dynamics of Z^1 and Z^2, we get

$$\begin{aligned} F^1(Z_t^1,t) &= F^1(Z_0^1,0) + \int_0^t F_t^1(Z_{s-}^1,s)ds + \int_0^t F_x^1(Z_{s-}^1,s)dZ_s^1 \\ &\quad + \frac{1}{2}\int_0^t F_{xx}^1(s,Z_{s-}^1)d\left\langle Z^1,Z^1\right\rangle_s^c \\ &\quad + \sum_{0s\le t}\left[F^1(Z_s^1,s) - F^1(Z_{s-}^1,s) - \Delta Z_s^1 F_x^1(Z_{s-}^1,s)\right] \\ &= F^1(Z_0^1,0) + \int_0^t F_t^1(Z_{s-}^1,s)ds \\ &\quad + \frac{1}{2}\int_0^t (1-\mathrm{sign}(X_{s-}-b(s)))F_x^1(Z_{s-}^1,s)dX_s \\ &\quad + \frac{1}{2}\int_0^t (1+\mathrm{sign}(X_{s-}-b(s)))F_x^1(Z_{s-}^1,s)db(s) \\ &\quad - \sum_{0<s\le t}\left[1_{\{X_{s-}>b(s)\}}(X_s-b(s))^- + 1_{\{X_s\le b(s)\}}(X_s-b(s))^+\right]F_x^1(Z_{s-}^1,s) \\ &\quad - \frac{1}{2}\int_0^t F_x^1(Z_{s-}^1,s)dL_t^b + \frac{1}{2}\int_0^t 1_{\{X_{s-}<b(s)\}}F_{xx}^1(Z_{s-}^1,s)d\left\langle X^c,X^c\right\rangle_s \\ &\quad \sum_{0<s\le t}\left[F^1(Z_s^1,s) - F(Z_{s-}^1,s) - \Delta Z_s^1 F_x^1(Z_{s-}^1,s))\right], \end{aligned} \tag{A.6}$$

(A.7)

$$
\begin{aligned}
F^2(Z_t^2,t) = F^2(Z_0^2,0) &+ \int_0^t F_t^2(Z_{s-}^2,s)ds + \int_0^t F_x^2(Z_{s-}^2,s)dZ_s^2 \\
&+ \frac{1}{2}\int_0^t F_{xx}^2(s,Z_{s-}^2)d\left\langle Z^2,Z^2\right\rangle_s^c \\
&+ \sum_{0s\le t}\left[F^2(Z_s^2,s)-F^2(Z_{s-}^2,s)-\Delta Z_s^2 F_x^1(Z_{s-}^2,s)\right] \\
= F^2(Z_0^2,0) &+ \int_0^t F_t^2(Z_{s-}^2,s)ds \\
&+ \frac{1}{2}\int_0^t (1+\mathrm{sign}(X_{s-}-b(s)))F_x^2(Z_{s-}^2,s)dX_s \\
&+ \frac{1}{2}\int_0^t (1-\mathrm{sign}(X_{s-}-b(s)))F_x^2(Z_{s-}^2,s)db(s) \\
&+ \sum_{0<s\le t}\left[1_{\{X_{s-}>b(s)\}}(X_s-b(s))^- + 1_{\{X_s\le b(s)\}}(X_s-b(s))^+\right]F_x^2(Z_{s-}^2,s) \\
&- \frac{1}{2}\int_0^t F_x^2(Z_{s-}^2,s)dL_t^b + \frac{1}{2}\int_0^t 1_{\{X_{s-}<b(s)\}}F_{xx}^2(Z_{s-}^2,s)d\left\langle X^c,X^c\right\rangle_s \\
&+ \sum_{0<s\le t}\left[F^2(Z_s^2,s)-F(Z_{s-}^2,s)-\Delta Z_s^2 F_x^2(Z_{s-}^2,s)\right].
\end{aligned}
$$

By splitting each term into its respective values on the sets $\{X_{s-}<b(s)\}$, $\{X_{s-}=b(s)\}$, and $\{X_{s-}>b(s)\}$, it can be seen that the following four equations are satisfied:

$$F^1(Z_0^1,0)+F^2(Z_0^2,0)=F(X_0,0)+F(b(0),0), \tag{A.8}$$

$$
\begin{aligned}
\int_0^t F_t^1(Z_{s-},s)ds + \int_0^t F_t^2(Z_{s-}^2,s)ds &= \frac{1}{2}\int_0^t F_t(X_{s-}+,s)+F_t(X_{s-}-,s)ds \\
+ \int_0^t \Bigg[F_t(b(s)+,s)1_{\{X_{s-}<b(s)\}} &+ \frac{1}{2}(F_t(b(s)-,s) \\
+F_t(b(s)+,s))1_{\{X_{s-}=b(s)\}} &+ F_t(b(s)-,s)1_{\{X_{s-}>b(s)\}}\Bigg]ds,
\end{aligned} \tag{A.9}
$$

$$
\begin{aligned}
&\frac{1}{2}\int_0^t (1-\mathrm{sign}(X_{s-}-b(s)))F_x^1(Z_{s-}^1,s)dX_s \\
&+\frac{1}{2}\int_0^t (1+\mathrm{sign}(X_{s-}-b(s)))F_x^2(Z_{s-}^2,s)dX_s \\
&=\frac{1}{2}\int_0^t \left[F_x(X_{s-}+,s)+F_x(X_{s-}-,s)\right]dX_s,
\end{aligned} \tag{A.10}
$$

$$
\begin{aligned}
&\frac{1}{2}\int_0^t (1+\operatorname{sign}(X_{s-}-b(s)))F_x^1(Z_{s-}^1,s)db(s)\\
&\qquad+\frac{1}{2}\int_0^t (1-\operatorname{sign}(X_{s-}-b(s)))F_x^2(Z_{s-}^2,s)db(s)\\
&=\int_0^t \Big[F_x(b(s)+,s)1_{\{X_{s-}<b(s)\}}\\
&\qquad+\frac{1}{2}\left[F_x(b(s)+,s)+F_x(b(s)-,s)\right]1_{\{X_{s-}=b(s)\}}+F_x(b(s)-,s)1_{\{X_{s-}>b(s)\}}\Big]db(s).
\end{aligned}\tag{A.11}
$$

On the other hand, equation (3.15) of [12] still holds:

$$
\begin{aligned}
F(b(t),t)=F(b(0),0)+\int_0^t &\Big[F_t(b(s)+,s)1_{\{X_{s-}<b(s)\}}\\
&+\frac{1}{2}\left[F_t(b(s)-,s)+F_t(b(s)+,s)1_{\{X_{s-}=b(s)\}}\right]\\
&+F_t(b(s)-,s)1_{\{X_{s-}>b(s)\}}\Big]ds\\
&\int_0^t \Big[F_x(b(s)+,s)1_{\{X_{s-}<b(s)\}}+\frac{1}{2}\left[F_x(b(s)-,s)+F_x(b(s)+,s)1_{\{X_{s-}=b(s)\}}\right]\\
&+F_x(b(s)-,s)1_{\{X_{s-}>b(s)\}}\Big]db(s),
\end{aligned}\tag{A.12}
$$

whose proof is carried out by using the uniqueness of finite measures on p-systems.

Let us analyze the jump terms in (A.6) and (A.7). We will denote

$$
\begin{aligned}
A:=&-\left[1_{\{X_{s-}>b(s)\}}(X_s-b(s))^-+1_{\{X_s\le b(s)\}}(X_s-b(s))^+\right]F_x^1(Z_{s-}^1,s)\\
&+\left[F^1(Z_s^1,s)-F(Z_{s-}^1,s)-\Delta Z_s^1F_x^1(Z_{s-}^1,s)\right],
\end{aligned}\tag{A.13}
$$

$$
\begin{aligned}
B:=&\left[1_{\{X_{s-}>b(s)\}}(X_s-b(s))^-+1_{\{X_s\le b(s)\}}(X_s-b(s))^+\right]F_x^2(Z_{s-}^2,s)\\
&+\sum_{0<s\le t}\left[F^2(Z_s^2,s)-F(Z_{s-}^2,s)-\Delta Z_s^2F_x^2(Z_{s-}^2,s)\right].
\end{aligned}\tag{A.14}
$$

Depending on the whereabouts of X_{s-} and X_s with respect to the boundary curve b, A and B take four different values:

1. $X_{s-}>b(s)$ and $X_t\ge b(t)$. In this case

$$A=0,\quad B=F^2(X_s,s)-F^2(X_{s-},s)-\Delta X_sF_x^2(X_{s-},s),\tag{A.15}$$

$$A+B=F(X_s,s)-F(X_{s-},s)-\Delta X_sF_x(X_{s-}+,s).\tag{A.16}$$

2. $X_{s-}>b(s)$ and $X_s<b(s)$. In this case

$$
\begin{aligned}
A&=-(b(s)-X_s)F_x^1(b(s),s)+F^1(X_s,s)\\
&\qquad-F^1(b(s),s)-(X_s-b(s))F_x^1(b(s),s)\\
&=F^1(X_s,s)-F^1(b(s),s),
\end{aligned}\tag{A.17}
$$

$$\begin{aligned} B &= (b(s) - X_s)F_x^2(b(s), s) + F^2(b(s), s) - F^2(X_{s-}, s) \\ &\quad - (b(s) - X_{s-})F_x^2(X_{s-}, s) \\ &= F^2(b(s), s) - F^2(X_{s-}, s) - \Delta X_s F_x^2(X_{s-}, s), \end{aligned} \tag{A.18}$$

$$A + B = F(X_s, s) - F(X_{s-}, s) - \Delta X_s F_x(X_s+, s). \tag{A.19}$$

3. $X_{s-} \leq b(s)$ and $X_s \geq b(s)$. We have that

$$\begin{aligned} A &= -(X_s - b(s))F_x^1(X_{s-}, s) + F^1(b(s), s) \\ &\quad - F^1(X_{s-}, s) - (b(s) - X_{s-})F_x^1(X_{s-}, s) \\ &= F^1(b(s), s) - F^1(X_{s-}, s) - \Delta X_s F^1(X_{s-}, s), \end{aligned} \tag{A.20}$$

$$\begin{aligned} B &= (X_s - b(s))F_x^2(b(s), s) + F^2(X_s, s) \\ &\quad - F^2(b(s), s) - (X_s - b(s))F_x^2(b(s), s) \\ &= F^2(X_s, s) - F^2(b(s), s). \end{aligned} \tag{A.21}$$

As a result

$$A + B = F(X_s, s) - F(X_{s-}, s) - \Delta X_s F_x(X_{s-}-, s). \tag{A.22}$$

4. $X_{s-} \leq b(s)$ and $X_s < b(s)$. Clearly,

$$A = F^1(X_s, s) - F^1(X_{s-}, s) - \Delta X_s F_x^1(X_{s-}, s) \quad \text{and} \quad B = 0. \tag{A.23}$$

As a result

$$A + B = F(X_s, s) - F(X_{s-}, s) - \Delta X_s F_x(X_{s-}, s). \tag{A.24}$$

Now, combining (A.1), (A.5), (A.6), (A.7), (A.8), (A.9), (A.10), (A.11), (A.12), (A.16), (A.19), (A.22), and (A.24), we obtain

$$\begin{aligned} F(X_t, t) &= F(X_0, 0) + \frac{1}{2}\int_0^t \left[F_t(X_{s-}+, s) + F_t(X_{s-}-, s)\right] ds \\ &\quad + \frac{1}{2}\int_0^t \left[F_x(X_{s-}+, s) + F_x(X_{s-}-, s)\right] dX_s \\ &\quad + \frac{1}{2}\int_0^t 1_{\{X_{s-} \leq b(s)\}} F_{xx}(s, X_{s-}) d\left\langle X, X\right\rangle_{s-}^c \\ &\quad + \sum_{0<s\leq t}\Big[F(X_s, s) - F(X_{s-}, s) - \Delta X_s F_x(X_{s-}-, s) 1_{\{X_{s-}\leq b(s)\}} \\ &\qquad - \Delta X_s F_x(s, X_{s-}+) 1_{\{X_{s-} > b(s)\}}\Big] \\ &\quad + \frac{1}{2}\int_0^t \left[F_x^2(Z_{s-}^2, s) - F^1(Z_{s-}^1, s)\right] dL_t^b. \end{aligned} \tag{A.25}$$

The last term on the right-hand side of (A.25) can be written as

$$\begin{aligned}\text{(A.26)}\qquad & \frac{1}{2}\int_0^t \left[F_x^2(Z_{s-}^2, s) - F^1(Z_{s-}^1, s)\right] dL_t^b \\ & \qquad = \frac{1}{2}\int_0^t \Bigg[F_x(X_{s-}+, s) - F_x(X_{s-}-, s)\Bigg] 1_{\{X_{s-}=b(s)\}} dL_t^b,\end{aligned}$$

using Theorem 69 of [16]. On the other hand, the jump term in (A.25) can be written as

(A.27)

$$\begin{aligned}&\sum_{0<s\le t}\Bigg[F(X_s, s) - F(X_{s-}, s) - \Delta X_s F_x(X_{s-}-, s) 1_{\{X_{s-}\le b(s)\}} \\ &\qquad - \Delta X_s F_x(s, X_{s-}+) 1_{\{X_{s-}>b(s)\}}\Bigg] \\ &\quad = \sum_{0<s\le t}\Bigg[F(X_s, s) - F(X_{s-}, s) - \frac{1}{2}\Delta X_s \left[F_x(X_{s-}-, s) + F_x(X_{s-}+, s)\right]\Bigg].\end{aligned}$$

This completes the proof. □

Acknowledgments. I would like to thank the two referees and the anonymous AE for their insightful comments. I also would like to express my gratitude to Farid Aitsahlia, Tom Bielecki, Masahiko Egami, Sebastian Jaimungal, Sergei Levendorskii, Mihai Sirbu, Mete Soner, Hao Xing, Virginia R. Young, Thaleia Zariphopoulou, and Gordan Zitkovic for their feedback.

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