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Characterization of stochastic control with optimal stopping in a Sobolev space ☆

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ABSTRACT

This work develops a new framework for a class of stochastic control problems with optimal stopping. One of our main motivations stems from dealing with the option pricing of American type. The value function is characterized as the unique solution of a partial differential equation in a Sobolev space. Together with certain regularities and estimates of the value function, the existence of the optimal strategy is established. The key ingredient is the use of the Itô formula for functions in a Sobolev space. Our approach provides a new alternative method for dealing with a class of stochastic control problems.

1. Introduction

In this paper, we develop a new framework for solving a class of stochastic control problems with optimal stopping. One of our main motivations stems from dealing with the indifference pricing of the American call options. Assuming that the American options are written on a non-traded common stock that is partially replicable by another perfectly liquid stock available in the market, we develop a new method for treating such stochastic control problems.

Using the classical verification theorem (Yong & Zhou, 1999), the value function of such a stochastic control problem may be characterized as the unique solution of an associated variational inequality (VI) under rather strong assumptions. These assumptions are: (1) the VI has the unique classical solution, i.e., unique solvability in an appropriate continuous (in fact, smooth) function space; and (2) the control problem has an underlying process and associated control process that attains the optimal value, i.e., existence of the optimal pair (π^*, τ^*). We refer to Fleming and Soner (2006) and Yong and Zhou (1999) for the verification theorem in general control theory.

In a more general setup, the aforementioned conditions for the verification theorem are difficult to verify due to the full non-linearity. As an alternative, one can use the viscosity solution approach (Crandall, Ishii, & Lions, 1992) to identify the value function. In particular, Oberman and Zariphopoulou (2003) characterizes the value function V of an investment problem in the framework of indifference pricing. A drawback of this approach is that due to the lack of information on the regularity for the viscosity solution, one cannot obtain further knowledge of optimal controls. Even the existence of the optimal control is problematic.

In this work, we develop a novel methodology, which lies between the classical approach and the viscosity solution approach. First, we show that the verification theorem holds under weaker assumptions than that of the classical verification theorem, but stronger than that of the viscosity solution methods. For the purpose of the complete characterization, we verify assumptions imposed to the verification theorem by obtaining certain regularity estimates. The benefit of this approach is that one can bypass the difficult measurability issues related to the dynamic programming principle, and obtain the existence of optimal control and the characterization of the value function via a variational inequality in a

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suitable Sobolev space. Thanks to the characterization results on control problems, one can further develop other useful results including regularity of optimal exercise boundary and monotonicity of the value function on the system parameters.

The idea of our approach can be illustrated as follows. In the standard arguments of the verification theorem, one uses the Itô formula to the classical solutions of a variational inequality. However, it is too restrictive, since the variational inequality in our case may not admit a classical solution. Therefore, we replace the assumption on the existence of a classical solution by the solution in a suitable Sobolev space. The key ingredient in the proof of the corresponding verification theorem is the utilization of the Itô formula to the functions in a Sobolev space; see Krylov (1980).

To complete the characterization of the value function, one can verify all the assumptions imposed in the verification theorem. Note that another assumption needed for the verification theorem, namely the existence of optimal pair, can be reduced to an estimation on the first derivative. To proceed, we start with a simple transformation on the backward equation (in time), which leads to an equivalent counterpart of the forward equation on standard domain $\mathcal{Q} = \mathbb{R} \times (0, T]$. Since the original domain is unbounded, we use the penalization method on the truncated functions. As a result, the truncated PDE after penalization is quasi-linear, and the Leray–Schauder fixed point theorem provides the solvability in an appropriate Sobolev space (details are to be presented in the following sections). In addition, the comparison principle is standard for the strong solution. By forcing the limit of the parameter of the penalized function $\varepsilon \to 0$, and the parameter of the truncated domain $N \to \infty$, we use local estimate to obtain the certain Hölder estimate and estimate in a certain Sobolev space on a compact domain by removing the singularity point $(\ln K, 0)$. Consequently, we obtain the existence of a solution in a certain local Sobolev space with some regularity estimates on the first order derivatives, which eventually leads to the complete characterization summarized in Theorem 11. To the best of our knowledge, the characterization of stochastic control with optimal stopping in a Sobolev function space is new and has not been done in the literature.

The rest of this paper is outlined as follows. The precise formulation of the problem is given in Section 2. Section 3 presents two weak verification theorems. Section 4 verifies regularity assumptions imposed to the verification theorem by PDE estimates for the complete characterization. Section 5 concludes the paper with some further remarks. Finally, an Appendix is placed at the end of the paper, which contains most of the long proofs and technical complements.

2. Problem formulation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a filtered probability space with a filtration $(\mathcal{F}_t)_{t\geq 0}$ that satisfies the usual conditions, and the processes B and $\widetilde{B}$ be two independent standard Brownian motions. Let $\mathcal{T}_{t,T}$ be the set of all stopping times in the interval $[t, T]$.

We consider a financial market with a risk-free interest rate, which consists of a non-traded stock issued by a firm with its price $Y_\nu^{t,y}$ at time $\nu \in [t, T]$ given by

$$dY_\nu = bY_\nu d\nu + cY_\nu(\rho dB_\nu + \sqrt{1-\rho^2}d\widetilde{B}_\nu), \qquad Y_t = y, \tag{1}$$

and one liquidly traded stock with its discounted price $S_\nu^{t,s}$ at time $\nu \in [t, T]$

$$dS_\nu = S_\nu \sigma(\lambda d\nu + dB_\nu), \qquad S_t = s. \tag{2}$$

Suppose an employee holds an American call option underlying non-traded asset $Y_\nu^{t,y}$ with strike $K > 0$ and maturity T, and the payoff $g(Y_\tau)$, where τ is the exercise time satisfying $\tau \in \mathcal{T}_{t,T}$ and g is given by the function $g(y) = (y - K)^+$.

If an employee, with initial capital w, dynamically trades in the stock S, then her wealth process $W^{t,w,\pi}$ under the self-financing rule satisfies

$$dW_\nu^\pi = \pi_\nu \sigma(\lambda d\nu + dB_\nu), \qquad W_t = w, \tag{3}$$

where π_ν represents the cash amount invested in the liquid stock $S_\nu^{t,s}$. We assume the strategy π belongs to $\mathcal{Z}_{t,T}$, which is the set of all progressively measurable processes $\pi : [t, T] \times \Omega \to \mathbb{R}$ satisfying the integrability condition

$$\mathbb{E}\left[\int_t^T \pi_\nu^2 d\nu\right] < \infty. \tag{4}$$

In what follows, we discuss the indifference price of the American option. Traditionally, a utility function is applied to the wealth at terminal time to generate an individual's price for an asset called the indifference price. One important utility function is the exponential utility function $\bar{U}(w) = -e^{-\gamma w}$, as a special case of hyperbolic absolute risk aversion (HARA). In contrast to the above static form of utility function, a class of time-dependent utility function is introduced by forward performance criteria given in Musiela and Zariphopoulou (2009, Definition 1). These time-dependent utility functions give more flexibility of the choices, as well as preserve fundamental properties of dynamic trading processes, such as the martingale property at an optimum and the supermartingale property away from it. Here and after, we adopt the following time-dependent utility function of hyperbolic absolute risk aversion (HARA) given by

$$U_t(w) = -e^{-\gamma w + \frac{1}{2}\lambda^2 t}, \tag{5}$$

which satisfies forward performance criteria by Proposition 3 of Musiela and Zariphopoulou (2009).

Under the above performance measure, we compare the following two scenarios for an employee holding the initial capital w and a unit of American call at the initial time t:

1. Find a stopping time $\tau \in \mathcal{T}_{t,T}$ to exercise the call option, while dynamically trading the capital w in liquid stock S_ν, to maximize its utility performance, and find the corresponding value
$$V(w, y, t) = \sup_{\tau\in\mathcal{T}_{t,T}, \pi\in\mathcal{Z}_{t,\tau}} \mathbb{E}\left[U_\tau(W_\tau^{t,w,\pi} + g(Y_\tau^{t,y}))|\mathcal{F}_t\right]. \tag{6}$$
2. Receive a cash payment of amount $P(w, y, t)$ by selling one unit of call at time t, the performance corresponding to the total capital $w + P(w, y, t)$ is $U_t(w + P(w, y, t))$.

Now, we are ready to define the indifference price of the American call option. The indifference price of this call is the cash payment $P(w, y, t)$, which makes the above two scenarios indifferent with respect to the given forward performance of (5). Therefore, the price $P(w, y, t)$ is the value satisfying

$$V(w, y, t) = U_t(w + P(w, y, t)). \tag{7}$$

In the rest of the paper, to determine the above indifference price of the American call $P(w, y, t) = U_t^{-1}(V(w, y, t)) - w$ implied by (7), we need to consider the stochastic optimal control in (6), i.e.,

1. characterize the value function V of (6);
2. characterize the pair of optimal strategy (τ^*, π^*) if it exists.

Moreover, we characterize directly the indifference price $P(w, y, t)$ as the unique solution of a certain PDE.

To proceed, the following assumptions are imposed throughout this paper:

(A1) Assume $\sigma, \gamma > 0$, and $\rho \in [0, 1)$.

We exclude the case $\rho = 1$, where the market is complete and the problem of indifference price can be reduced to a Black–Scholes model. For the technical convenience, we assume non-negativity of ρ, which is not really a model restriction. Otherwise if $\rho < 0$, then one can simply replace (1) by

$$dY_\nu = bY_\nu + \bar{c}Y_\nu\left(\bar{\rho}dB_\nu + \sqrt{1-\bar{\rho}^2}d\bar{B}_\nu\right),$$

where $\bar{c} = -c$, $\bar{\rho} = -\rho$, and $\bar{B}_\nu = \widetilde{B}_\nu$.

3. Two weak verification theorems

We present a version of the verification theorem for the value function V of (6) and P of (7) in this section, which generalizes the verification theorem from the classical PDE solution to the solution in the Sobolev function space with certain regularity. Recall that a Sobolev space is a function space equipped with the L_p norm of the functions together with their derivatives up to a given order. Here, we need to use derivatives $(\partial_{ww}u, \partial_{yy}u, \partial_{wy}u, \partial_t u)$ in the distribution sense, which is illustrated briefly below. Let Λ be an open set in $\mathbb{R}^l$. For a multi index $\gamma = (\gamma_1, \dots, \gamma_l)$ with $|\gamma| = \gamma_1 + \cdots + \gamma_l$ and a suitable smooth function u, denote $D^\gamma u = \frac{\partial^{|\gamma|}u}{\partial x_1^{\gamma_1}, \dots, \partial x_l^{\gamma_l}}$. If u satisfies $\int_K |u|dx < \infty$ for each compact subset $K \subset \Lambda$, then u is said to be locally integrable. The set of all such functions being in L_p is denoted by $L_{p,loc}(\Lambda)$. If for any $\varphi \in C_0^\infty(\Lambda)$ (the class of C^∞ functions with compact support), there exists a locally integrable function v such that

$$\int_\Lambda u D^\gamma \varphi dx = (-1)^{|\gamma|} \int_\Lambda \varphi v dx, \quad \varphi \in C_0^\infty(\Lambda),$$

we call v the weak γ-th partial derivative of u. The family of functions $W^k_{p,loc}(\Lambda)$ is defined to be the set of all functions $u \in L_{p,loc}(\Lambda)$ such that for every multi index γ with $|\gamma| \le k$, the weak partial derivative $D^\gamma u \in L_{p,loc}(\Lambda)$. That is,

$$W^k_{p,loc}(\Lambda) = \left\{ u \in L_{p,loc}(\Lambda) : D^\gamma u \in L_{p,loc}(\Lambda),\ \forall |\gamma| \le k \right\}.$$

For more detailed discussions and basic properties of Sobolev spaces, we refer the reader to Adams (1975), and Evans (1998) among others.

3.1. Weak verification theorem for V

In this paper, in lieu of the general multi index notation given in the above, we work with a specific form as follows. Define a three dimensional domain $\mathcal{Q}^1 = \mathbb{R} \times \mathbb{R}^+ \times [0, T)$, and denote by $W^{2,2,1}_{p,loc}(\mathcal{Q}^1)$, the collection of all functions on $\mathcal{Q}^1$ with $(w, y, t) \mapsto u(w, y, t)$ having the derivatives $(\partial_{ww}u, \partial_{yy}u, \partial_{wy}u, \partial_t u)$ in a distribution sense and $u, \partial_w u, \partial_y u, \partial_{wy}u, \partial_{ww}u, \partial_{yy}u, \partial_t u \in L_{p,loc}(\mathcal{Q}^1)$. For a scalar $\pi \in \mathbb{R}$, define a parameterized differential operator $\mathcal{L}_1^\pi : W^{2,2,1}_{p,loc} \mapsto \mathbb{R}$ as

$$\begin{aligned}\mathcal{L}_1^\pi u(w, y, t) = {} & \frac{1}{2}c^2y^2\partial_{yy}u + by\partial_y u + \frac{1}{2}\sigma^2\pi^2\partial_{ww}u \\ & + \pi\left(\rho\sigma c y \partial_{wy}u + \lambda\sigma\partial_w u\right).\end{aligned}$$

We also need to introduce the *continuation region* by

$$\mathcal{C}[v] = \{(w, y, t) \in \mathcal{Q}^1 : v(w, y, t) > U_t(w + g(y))\}. \tag{8}$$

Lemma 1 (*Verification Theorem for Function V*). *Suppose there exists $v \in W^{2,2,1}_{p,loc}(\mathcal{Q}^1)$ for some $p \ge 3$, satisfying*

$$\begin{aligned}\min\Big\{ & -\partial_t v + \inf_{\pi\in\mathbb{R}}\{-\mathcal{L}_1^\pi v(w, y, t)\}, \\ & v(w, y, t) - U_t(w + g(y))\Big\} = 0,\end{aligned} \tag{9}$$

with the terminal condition

$$v(w, y, T) = U_T(w + g(y)). \tag{10}$$

Then, $v(w, y, t) \ge V(w, y, t)$. If, in addition, there exists a pair (W^*, π^*) satisfying (3), (4), and

$$\left(\partial_t v + \mathcal{L}_1^{\pi_\nu^*} v\right)(W_\nu^*, Y_\nu, \nu) = 0, \quad \forall\, t < \nu < \tau^*, \tag{11}$$

where

$$\tau^* := \inf\left\{\nu > t : (W_\nu^*, Y_\nu, \nu) \notin \mathcal{C}[v]\right\} \wedge T. \tag{12}$$

Then the variational inequality (9)–(10) admits a unique solution in $W^{2,2,1}_{p,loc}(\mathcal{Q}^1)$, and $v(w, y, t) = V(w, y, t)$.

Proof. Fix an arbitrary stopping time $\tau \in \mathcal{T}_{t,T}$ and an admissible control $\pi \in \mathcal{Z}_{t,\tau}$, and denote $(W, Y) := (W^{t,w,\pi}, Y^{t,y})$ and $\mathbb{E}_t[\cdot] := \mathbb{E}[\cdot|\mathcal{F}_t]$ for simplicity. By virtue of the generalized Itô formula (Krylov, 1980, Theorem 2.10.1), we have

$$\begin{aligned}\mathbb{E}_t\left[v(W_\tau, Y_\tau, \tau)\right] = {} & v(w, y, t) \\ & + \mathbb{E}_t\left[\int_t^\tau (\partial_t v + \mathcal{L}_1^\pi v)(W_\nu, Y_\nu, \nu)d\nu\right].\end{aligned}$$

Applying (9), we have $v(w, y, t) \ge U_t(w + g(y))$ for all (w, y, t), hence $\mathbb{E}_t[v(W_\tau, Y_\tau, \tau)] \ge \mathbb{E}_t[U_\tau(W_\tau + g(Y_\tau))]$. Note that (9) also implies $(\partial_t v + \mathcal{L}_1^\pi v)(w, y, t) \le 0$ for all (w, y, t), which yields

$$\mathbb{E}_t\left[\int_t^\tau (\partial_t v + \mathcal{L}_1^\pi v)(W_\nu, Y_\nu, \nu)d\nu\right] \le 0.$$

Therefore, we conclude $\mathbb{E}_t[U_\tau(W_\tau + g(Y_\tau))] \le v(w, y, t)$, and arbitrariness of τ and π further implies one-sided inequality $v(w, y, t) \ge V(w, y, t)$. Furthermore, if (11) holds for some (W^*, π^*), the Itô formula together with the definition of $\mathcal{C}[v]$ gives the inequality of the opposite direction:

$$\begin{aligned}v(w, y, t) = {} & \mathbb{E}_t\left[v(W_{\tau^*}, Y_{\tau^*}, \tau^*)\right] = \mathbb{E}_t\left[U_{\tau^*}(W_{\tau^*} + g(Y_{\tau^*}))\right] \\ \le {} & V(w, y, t).\end{aligned}$$

This implies $v = V$. $\square$

3.2. Weak verification theorem for P

Next, we discuss the price P of (7). To proceed, we introduce a domain $\mathcal{Q}^2 = \mathbb{R}^+ \times [0, T)$. We denote its Sobolev space $W^{2,1}_{p,loc}(\mathcal{Q}^2)$ the collection of all functions on $\mathcal{Q}^2$ with $(y, t) \mapsto u(y, t)$ having derivatives $(\partial_{yy}u, \partial_t u)$ in a distribution sense and $u, \partial_y u, \partial_{yy}u, \partial_t u \in L_{p,loc}(\mathcal{Q}^2)$. Define also a non-linear differential operator $\mathcal{L}_2 : W^{2,1}_{p,loc}(\mathcal{Q}^2) \to \mathbb{R}$ by

$$\begin{aligned}\mathcal{L}_2 u(y, t) = {} & \frac{1}{2}c^2y^2\partial_{yy}u + (by - \rho c\lambda y)\partial_y u \\ & - \frac{1}{2}\gamma(1 - \rho^2)c^2y^2(\partial_y u)^2.\end{aligned}$$

Lemma 2 (*Verification Theorem for Function P*). *Suppose there exists a function $f \in W^{2,1}_{p,loc}(\mathcal{Q}^2)$ for some $p \ge 3$, satisfying*

$$\min\{-\partial_t f(y, t) - \mathcal{L}_2 f(y, t), f(y, t) - g(y)\} = 0, \tag{13}$$

with the terminal condition

$$f(y, T) = g(y), \quad y \in \mathbb{R}^+, \tag{14}$$

and $|\partial_y f(y, t)|$ is uniformly bounded. Then, PDE (9) together with the terminal condition (10) is uniquely solvable in $W^{2,2,1}_{p,loc}(\mathcal{Q}^1)$, and there exists a pair (W^*, π^*) satisfying (3), (4) and (11). As a result, the price function $P(w, y, t)$ of (7) is independent to the initial wealth w, and $P(w, y, t) = f(y, t)$.

Proof. Let $v(w, y, t) = U_t(w + f(y, t))$. Then, we have $v \in W^{2,2,1}_{p,loc}(\mathcal{Q}^1)$. One can directly check $\partial_{ww}v = \gamma^2 v < 0$ in $\mathcal{Q}^1$, hence the function $\pi^*[v] : \mathcal{Q}^1 \mapsto \mathbb{R}$ of the form

$$\pi^*[v](w, y, t) = -\frac{\rho c y \partial_{wy}v(w, y, t) + \lambda\partial_w v(w, y, t)}{\sigma\partial_{ww}v(w, y, t)}$$

is well defined. Note that $\pi^*[v]$ is independent to the variable w, i.e., one can rewrite

$$\pi^*[v](y,t) = -\frac{\rho c}{\sigma}\cdot y\partial_y f(y,t) + \frac{\lambda}{\sigma\gamma}.$$

Now, we can define a process π^* by, for $\nu \in (t,T)$

$$\pi^*_\nu = \pi^*[v](Y_\nu,\nu) = -\frac{\rho c}{\sigma}\cdot Y_\nu\partial_y f(Y_\nu,\nu) + \frac{\lambda}{\sigma\gamma}. \tag{15}$$

Note that Y is the unique strong solution of (1). Therefore, π^*_ν is an admissible strategy, since it satisfies the integrability (4) due to boundedness of $\partial_y f$, i.e.,

$$\mathbb{E}\left[\int_t^T(\pi^*_\nu)^2 d\nu\right] \le C + C\mathbb{E}\left[\int_t^T Y_\nu^2 d\nu\right] < \infty.$$

Since the optimal strategy π^*_ν is independent to W^*, the solution to (3) can be simply given by its integral form of (3). Calculations by change of variables leads to (9)–(11). Thus, Lemma 1 together with definition (7) implies $V(w,y,t) = U_t(w + f(y,t))$, and $f(y,t) = P(w,y,t)$. □

Remark 3 (*Boundary Condition of $f(y,t)$ at $y=0$*)**.** The verification theorem can be thought of as the probability counterpart of the uniqueness result of PDE solution (13)–(14). From the above verification theorem, the uniqueness holds without the specification of the boundary condition on $y = 0$. Similar observations are addressed in Oleĭnik and Radkevič (1973) (Fichera condition) for the linear equation, and in Bayraktar, Song, and Yang (2011) for the fully non-linear parabolic equation without obstacle.

3.3. Summary of verification theorems

To this end, we summarize the implication of Lemmas 1 and 2. The above verification theorems give conditional characterization of the control problem based on the assumptions on the solvability of the PDE. More precisely, assume the following hypotheses hold:

(H1) (13)–(14) is solvable in $W^{2,1}_{p,loc}(\mathcal{Q}^2)$;
(H2) $|\partial_y f(y,t)|$ is uniformly bounded.

Then we can conclude

1. V is the unique $W^{2,2,1}_{p,loc}$ solution of PDEs (9)–(10);
2. the pair of optimal control (π^*,τ^*) exists, and they may have representations of (12) and (15), respectively;
3. P is invariant w.r.t. w, and the unique $W^{2,1}_{p,loc}$ solution of PDEs (13)–(14) as a function of two variables of (y,t).

4. Main result: complete characterization

To obtain the complete characterization, it is crucial to study the solvability and its related estimates of the PDEs (13)–(14) to check (H1)–(H2). For the convenience in the analysis, we analyze the backward equation (in time) (13)–(14), by studying its associated forward equation of the following form.

Let $x = \ln y,\ \theta = T-t,\ u(x,\theta) = f(y,t)$ in (13)–(14), then

$$\begin{aligned}\mathcal{L}_2 f(y,t) &= \frac{1}{2}c^2y^2\partial_{yy}f + (by - \rho c\lambda y)\partial_y f \\ &\quad - \frac{1}{2}\gamma(1-\rho^2)c^2y^2(\partial_y f)^2 \\ &= \frac{1}{2}c^2(\partial_{xx}u - \partial_x u) + (b-\rho c\lambda)\partial_x u \\ &\quad - \frac{1}{2}\gamma(1-\rho^2)c^2(\partial_x u)^2 := \mathcal{L}u(x,\theta),\end{aligned}$$

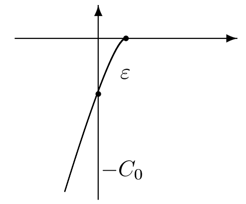

Fig. 1. $\beta_\varepsilon(t)$.

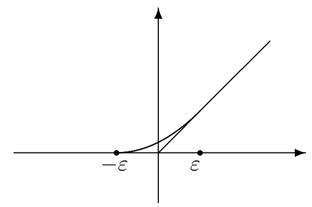

Fig. 2. $\pi_\varepsilon(t)$.

thus $u(x,\theta)$ satisfies

$$\begin{cases}\min\{(\partial_\theta u - \mathcal{L}u)(x,\theta),\ u - (e^x-K)^+\} = 0, & (x,\theta)\in\mathcal{Q},\\ u(x,0) = (e^x-K)^+, & x\in\mathbb{R},\end{cases} \tag{16}$$

where $\mathcal{Q} = \mathbb{R}\times(0,T]$. In what follows, to ensure the continuity of flow of the presentation, all longer technical proofs are postponed till the Appendix.

4.1. Solvability of problem (16)

Since $(-\infty,+\infty)\times(0,T]$ is unbounded, we first confine our attention to the truncated version of (16) in a finite domain $\mathcal{Q}_N = (-N,N)\times(0,T]$. Let $u^N(x,\theta)$ be the solution (if it exists) to the following problem

$$\begin{cases}\min\left\{\partial_\theta u^N - \mathcal{L}u^N,\ u^N - (e^x-K)^+\right\} = 0, & (x,\theta)\in\mathcal{Q}_N,\\ \partial_x u^N(-N,\theta) = 0, \quad \partial_x u^N(N,\theta) = e^N, & \theta\in(0,T],\\ u^N(x,0) = (e^x-K)^+, & x\in(-N,N).\end{cases} \tag{17}$$

In order to prove the existence of a solution to problem (17), we construct a penalty approximation of problem (17). Suppose $u^N_\varepsilon(x,\theta)$ satisfies

$$\begin{cases}\partial_\theta u^N_\varepsilon - \mathcal{L}u^N_\varepsilon + \beta_\varepsilon\left(u^N_\varepsilon - \pi_\varepsilon(e^x-K)\right) = 0, & (x,\theta)\in\mathcal{Q}_N,\\ \partial_x u^N_\varepsilon(-N,\theta) = 0, \quad \partial_x u^N_\varepsilon(N,\theta) = e^N, & \theta\in(0,T],\\ u^N_\varepsilon(x,0) = \pi_\varepsilon(e^x-K), & x\in(-N,N),\end{cases} \tag{18}$$

where $\beta_\varepsilon(t)$ (see Fig. 1) and $\pi_\varepsilon(t)$ (see Fig. 2) satisfy

$$\begin{aligned}&\beta_\varepsilon(t)\in C^2(-\infty,+\infty), \quad \beta_\varepsilon(t)\le 0,\\ &\beta'_\varepsilon(t)\ge 0, \quad \beta''_\varepsilon(t)\le 0,\\ &\beta_\varepsilon(0) = -C_0, \quad \pi_\varepsilon(t)\in C^\infty,\\ &0\le\pi'_\varepsilon(t)\le 1, \quad \pi''_\varepsilon(t)\ge 0,\\ &\lim_{\varepsilon\to0^+}\pi_\varepsilon(t) = t^+,\end{aligned}$$

$$\lim_{\varepsilon\to0}\beta_\varepsilon(t) = \begin{cases}0, & t>0,\\ -\infty, & t<0,\end{cases} \qquad \pi_\varepsilon(t) = \begin{cases}t, & t\ge\varepsilon,\\ \nearrow, & |t|\le\varepsilon,\\ 0, & t\le-\varepsilon,\end{cases}$$

where $C_0 > 0$ is to be determined.

To proceed, we state three lemmas. Their proofs are placed in the Appendix.

Lemma 4. *There exists a unique solution $u_\varepsilon^N(x,\theta) \in W_p^{2,1}(\mathcal{Q}_N)$ to problem (18) for any $p \geq 1$. Moreover, the following estimates hold,*

$$\pi_\varepsilon(e^x - K) \leq u_\varepsilon^N(x,\theta) \leq k\theta + x^2 + e^{x+(b-\rho c\lambda)^+\theta} + 1, \tag{19}$$

$$0 \leq \partial_x u_\varepsilon^N(x,\theta) \leq e^{x+(b-\rho c\lambda)^+\theta}, \tag{20}$$

$$\partial_\theta u_\varepsilon^N(x,\theta) \geq 0, \tag{21}$$

where

$$k \geq \max\left\{c^2 + \frac{(b-\rho c\lambda - \frac{1}{2}c^2)^2}{2\gamma(1-\rho^2)c^2},\right.$$
$$\left. c^2 + \frac{\left[\gamma(1-\rho^2)c^2 e^{(b-\rho c\lambda)^+T} - (b-\rho c\lambda - \frac{1}{2}c^2)\right]^2}{2\gamma(1-\rho^2)c^2}\right\}.$$

Lemma 5. *Problem (17) has a unique solution $u^N(x,\theta) \in W_{p,loc}^{2,1}(\mathcal{Q}_N) \cap C(\overline{\mathcal{Q}}_N)$ satisfying, for k of Lemma 4*

$$(e^x - K)^+ \leq u^N(x,\theta) \leq k\theta + x^2 + e^{x+(b-\rho c\lambda)^+\theta} + 1, \tag{22}$$

$$0 \leq \partial_x u^N(x,\theta) \leq e^{x+(b-\rho c\lambda)^+\theta}, \tag{23}$$

$$\partial_\theta u^N(x,\theta) \geq 0. \tag{24}$$

Lemma 6. *There exists a solution $u(x,\theta)$ uniquely in $W_{p,loc}^{2,1}(\mathcal{Q}) \cap C(\overline{\mathcal{Q}})$ to (16) satisfying, for k of Lemma 4*

$$(e^x - K)^+ \leq u(x,\theta) \leq k\theta + x^2 + e^{x+(b-\rho c\lambda)^+\theta} + 1, \tag{25}$$

$$0 \leq \partial_x u(x,\theta) \leq e^{x+(b-\rho c\lambda)^+\theta}, \tag{26}$$

$$\partial_\theta u(x,\theta) \geq 0. \tag{27}$$

4.2. The obstacle of (16)

Now we characterize the optimal exercising time of V. Problem (16) is an optimal stopping problem, which gives rise to a free boundary that can be expressed as a single-valued function of θ. For later use, we define the Stopping region ($\mathcal{S}$) and Continuation region ($\mathcal{C}$) as

$$\mathcal{S} := \{(x,\theta) \in \mathcal{Q} : u(x,\theta) = (e^x - K)^+\},$$
$$\mathcal{C} := \{(x,\theta) \in \mathcal{Q} : u(x,\theta) > (e^x - K)^+\},$$

where u is the solution of (16). Thanks to the continuity of u from Lemma 6, $\mathcal{S}$ is closed and $\mathcal{C}$ is open.

Lemma 7. *There exists $S(x) : (-\infty, +\infty) \to [0, T]$ such that*

$$\mathcal{S} = \{(x,\theta) \in \mathcal{Q} : 0 < \theta \leq S(x)\}. \tag{28}$$

Proof. If $(x_0, \theta_0) \in \mathcal{S}$, i.e., $u(x_0,\theta_0) = (e^{x_0} - K)^+$, according to (25) and (27), we have $u(x_0,\theta) = (e^{x_0} - K)^+$, $\theta \in (0, \theta_0]$. Hence $\{x_0\} \times (0, \theta_0] \subseteq \mathcal{S}$, so we can define $S(x) : (-\infty, +\infty) \to [0, T]$ as

$$S(x) = \begin{cases} 0, & \text{if } u(x,\theta) > (e^x - K)^+ \text{ for any } \theta \in (0,T], \\ \sup\{\theta \in (0,T] : u(x,\theta) = (e^x - K)^+\}. \end{cases}$$

By the definition of $S(x)$, (28) holds. □

Lemma 8. *The free boundary $S(x)$ is strictly increasing w.r.t. x on $\{x : 0 < S(x) < T\}$.*

The proof of Lemma 8 is placed in the Appendix. To proceed, we derive another lemma.

Lemma 9. *The free boundary $S(x)$ is continuous on the set $\{x : 0 < S(x) < T\}$.*

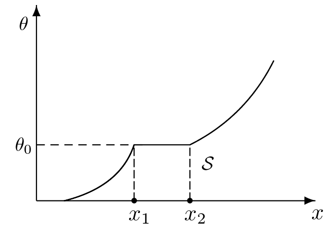

Fig. 3. Free boundary Case I (see Lemma 8).

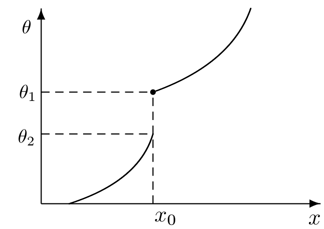

Fig. 4. Free boundary Case II (see Lemma 7).

Proof. Suppose not (see Fig. 4). There would exist an x_0 such that $S(x_0) := \theta_1 > \lim_{x\to x_0^-} S(x) := \theta_2$. Then we would have $\partial_\theta u(x_0,\theta) = \partial_{x\theta} u(x_0,\theta) = 0$ for $\theta \in (\theta_2, \theta_1)$. Since $\partial_\theta u \geq 0$ and in the domain $(x_0 - \varepsilon, x_0) \times (\theta_2, \theta_1)$

$$\partial_\theta(\partial_\theta u) - \frac{1}{2}c^2 \partial_{xx}(\partial_\theta u)$$
$$- \left[b - \rho c\lambda - \frac{1}{2}c^2 - \gamma(1-\rho^2)c^2 \partial_x u\right]\partial_x(\partial_\theta u) = 0.$$

Applying Hopf's Lemma (Friedman, 1964) we obtain $\partial_{x\theta} u(x_0,\theta) < 0$, or $\partial_\theta u \equiv 0$, $(x,\theta) \in (x_0 - \varepsilon, x_0) \times (\theta_2, \theta_1)$, which results in a contradiction. □

Since $S(x)$ is strictly increasing with respect to x on the set $\{x : 0 < S(x) < T\}$, then there exists an inverse function of $S(x)$, we denote it as $s(\theta) = S^{-1}(x)$, $0 < \theta < T$. Note that the strictly monotonicity of $S(x)$ is equivalent to the continuity of $s(\theta)$, and the continuity of $S(x)$ is equivalent to the strictly monotonicity of $s(\theta)$. Owing to Lemmas 8 and 9, we derive the following lemma.

Lemma 10. *There exists an optimal exercising boundary $s(\theta) : (0, T] \to \mathbb{R}$ such that*

$$\mathcal{S} = \{(x,\theta) \in \mathcal{Q} : x \geq s(\theta)\}. \tag{29}$$

Moreover, $s(\theta)$ is strictly increasing with respect to θ and

$$s(\theta) \geq x_0, \tag{30}$$

$$s(0) := \lim_{\theta\to 0^+} s(\theta) = x_0, \tag{31}$$

where $x_0 = \begin{cases} \ln K, & b - \rho c\lambda \leq 0, \\ \max\{\ln K,\ \ln \frac{2(b-\rho c\lambda)}{\gamma(1-\rho^2)c^2}\}, & b - \rho c\lambda > 0. \end{cases}$ *In particular, $\partial_\theta u$ is continuous across $s(\theta)$ and $s(\theta) \in C[0,T] \cap C^\infty(0,T]$.*

Proof. We can define

$$s(\theta) = \begin{cases} S^{-1}(x), & 0 < \theta < T, \\ \inf\{x : S(x) = T\}, & \theta = T. \end{cases}$$

Eq. (29) is the consequence of the definition and monotonicity of $S(x)$. According to (A.5) in the proof of Lemma 8, we know (30) is true. Lemma 8 implies $s(\theta)$ is monotonic increasing with respect to θ, then we can define $s(0) := \lim_{\theta\to 0^+} s(\theta)$, and the proof of (31) is similar to the proof of strictly monotonicity of $S(x)$ in Lemma 8.

Moreover, since $\partial_\theta u \geq 0$ and $(e^x - K)^+$ is a lower obstacle, from Friedman (1975) we know $\partial_\theta u$ is continuous across $s(\theta)$ and $s(\theta) \in C^\infty(0,T]$. □

4.3. Main result: characterization

Now, we are ready to present the complete characterization of the value function V and the indifference price P.

Theorem 11. *The following assertions hold.*

1. *The indifference price of (7) is independent of the initial capital w, and $P(w, y, t) := P(y, t)$ is the unique $W^{2,1}_{p,loc}(\mathcal{Q}^2) \cap C(\overline{\mathcal{Q}^2})$ solution of the variational inequality (13)–(14) satisfying $|\partial_y P(y, t)| < C$ for some constant C.*
2. *The value function of (6) has the form $V(w, y, t) = U_t(w + P(y, t))$, and is the unique solution of (9)–(10) in $W^{2,2,1}_{p,loc}(\mathcal{Q}^1) \cap C(\overline{\mathcal{Q}^1})$ satisfying*

$$|\partial_y V(w, y, t)| < C|V(w, y, t)|. \tag{32}$$

3. *There exists C^∞ function $y^* : [0, T) \to \mathbb{R}$, such that the strategy defined by*

$$\begin{cases} \pi^*_\nu = -\dfrac{\rho c}{\sigma} \cdot Y_\nu \partial_y P(Y_\nu, \nu) + \dfrac{\lambda}{\sigma\gamma}, \quad \nu \in (t, \tau^*), \\ \tau^* = \inf\{\nu > t : Y_\nu \geq y^*(\nu)\} \wedge T, \end{cases}$$

is optimal of the control problem (6), in the sense that

$$V(w, y, t) = \mathbb{E}[U_{\tau^*}(W^*_{\tau^*} + g(Y^{t,y}_{\tau^*}))|\mathcal{F}_t],$$

where $W^*_{t_1} = w + \int_t^{t_1} \pi^*_\nu \sigma(\lambda d\nu + dB_\nu)$.

Proof. Noting $y\partial_y f(y, t) = \partial_x u(x, \theta)$ and using estimate (26), we conclude $|\partial_y f(y, t)| \leq C$ for some constant C. Applying Lemma 6, and Lemma 2, together with Lemma 1 in order, we conclude (1)–(2) of Theorem 11. Owing to the representations of (12) and (15), the optimal control can be written as

$$\begin{cases} \pi^*_\nu = -\dfrac{\rho c}{\sigma} \cdot Y_\nu \partial_y P(Y_\nu, \nu) + \dfrac{\lambda}{\sigma\gamma}, \quad \nu \in (t, \tau^*), \\ \tau^* = \inf\{\nu > t : (W^*_\nu, Y_\nu, \nu) \notin \mathcal{C}[V]\} \wedge T. \end{cases}$$

Note that the continuation region of (8) satisfies

$$\begin{aligned} \mathcal{C}[V] &= \{(w, y, t) \in \mathcal{Q}^1 : V(w, y, t) > U_t(w + g(y))\} \\ &= \{(w, y, t) \in \mathcal{Q}^1 : U_t(w + P(y, t)) > U_t(w + g(y))\} \\ &= \{(w, y, t) \in \mathcal{Q}^1 : P(y, t) > g(y)\}. \end{aligned}$$

Therefore, the obstacles of P and V are the same, and the optimal stopping time τ^* can be written invariant to W^*:

$$\tau^* = \inf\{\nu > t : P(Y_\nu, \nu) \leq g(Y_\nu)\} \wedge T.$$

Take $y^*(t) = e^{s(T-t)}$, where $s(\cdot)$ is the function in Lemma 10. Thanks to the result of Lemma 10 together with the transformation between $P(y, t)$ and $u(x, \theta)$, we conclude the representation of τ^*. □

Next, we present some properties as a consequence of characterization Theorem 11. For instance, how does the price $P(y, t)$ change, if we scale its payoff g by n times, or if we change the risk aversion parameter γ?

Proposition 12. *$P(y, t)$ decreases with respect to γ and λ, and increases with respect to b, satisfying*

$$nP[g](y, t) \geq P[ng](y, t) \ (n \geq 1), \tag{33}$$

where $P[\varphi]$ stands for the indifference price with the payoff function φ.

The proof of Proposition 12 is given in the Appendix. In fact, it is interesting in its own right. It reveals natural economic facts. For instance, since γ can be interpreted as the absolute risk aversion coefficient, Proposition 12 claims that the employee's risk preference directly affects his exercise behavior. It implies a more prudent agent (with a larger coefficient of risk aversion γ) would be likely to exercise the option earlier to realize a cash benefit, and then invest it in other assets to earn the time value of the money obtained from the exercise of the option. So when he exercises, he gets less value than the risky agent, hence the value function decreases w.r.t. γ. In addition, (33) implies that if the agent owns n ($n > 1$) pieces of options, compared with owning one piece of the option, the agent cares less about the value of every piece of the option. So the average value of the indifference price about these pieces of options is less than the indifference price about one piece of this option.

Another consequence of (25) and (26) is given as the following estimate on $P(y, t)$ and $V(w, y, t)$.

Proposition 13. *$P(y, t)$ and $V(w, y, t)$ satisfy*

$$(y - K)^+ \leq P(y, t) \leq h(y, t), \tag{34}$$

$$0 \leq \partial_y P(y, t) \leq e^{(b-\rho c\lambda)^+(T-t)}, \tag{35}$$

$$-e^{-\gamma(w+(y-K)^+)+\frac{1}{2}\lambda^2 t} \leq V \leq -e^{-\gamma(w+h(y,t))+\frac{1}{2}\lambda^2 t}, \tag{36}$$

$$0 \leq \partial_y V(w, y, t) \leq -\gamma e^{(b-\rho c\lambda)^+(T-t)} V(w, y, t), \tag{37}$$

where k is defined in Lemma 4 and $h(y, t) = k(T - t) + (\ln y)^2 + ye^{(b-\rho c\lambda)^+(T-t)} + 1$.

5. Further remarks

This paper developed a new framework for a class of stochastic control problems with stopping. Different from the traditional approach and the viscosity solution characterization, the value function is characterized as the unique solution of a partial differential equation in a Sobolev space. Although our original motivation comes from financial engineering, the method developed will be useful to treat a larger class of stochastic control problems with stopping.

Appendix

A.1. Proofs of several results in Section 4

Proof of Lemma 4. The Leray–Schauder fixed point theorem implies the existence of the $W^{2,1}_p$ solution to problem (18), and the comparison principle holds for the strong solution. The proof of the uniqueness is standard.

First, we prove estimate (19). Note that $u_1(x, \theta) := \pi_\varepsilon(e^x - K)$ satisfies $\partial_\theta u_1 - \mathcal{L}u_1 + \beta_\varepsilon(u_1 - \pi_\varepsilon(\cdot)) \leq 0$ in $\mathcal{Q}_N$, if we choose

$$-\beta_\varepsilon(0) = C_0 = \rho c\lambda e^N + \frac{1}{2}\gamma(1 - \rho^2)c^2 e^{2N}. \tag{A.1}$$

Moreover, when N is large enough, we have $\partial_x u_1(-N, \theta) = 0$, $\partial_x u_1(N, \theta) = e^N$, $u_1(x, 0) = u^N_\varepsilon(x, 0)$. Thus $u_1(x, \theta) = \pi_\varepsilon(e^x - K)$ is a subsolution to problem (18).

On the other hand, one can check $u_2(x, \theta) := k\theta + x^2 + e^{x+(b-\rho c\lambda)^+\theta} + 1$ satisfies supersolution property $\partial_\theta u_2 - \mathcal{L}u_2 + \beta_\varepsilon(u_2 - \pi_\varepsilon(\cdot)) \geq 0$ in $\mathcal{Q}_N$. Moreover, we have

$$\partial_x u_2(-N, \theta) = -2N + e^{-N+(b-\rho c\lambda)^+\theta} \leq 0,$$
$$\partial_x u_2(N, \theta) = 2N + e^{N+(b-\rho c\lambda)^+\theta} \geq e^N,$$
$$u_2(x, 0) = x^2 + e^x + 1 \geq \pi_\varepsilon(e^x - K).$$

Applying the comparison principle, we conclude (19).

Next, we verify inequality (20). If we differentiate the equation in (18) w.r.t. x, then $v(x,\theta) = \partial_x u_\varepsilon^N(x,\theta)$ satisfies

$$\begin{cases} \partial_\theta v - \dfrac{1}{2}c^2\partial_{xx}v - \left(b - \rho c\lambda - \dfrac{1}{2}c^2\right)\partial_x v \\ + \gamma(1-\rho^2)c^2\, V\partial_x v + \beta_\varepsilon'(\cdot)(v - \pi_\varepsilon' e^x) = 0, \\ v(-N,\theta) = 0, \qquad v(N,\theta) = e^N, \quad \theta \in (0,T], \\ v(x,0) = \pi_\varepsilon'(\cdot)e^x, \quad x \in (-N,N). \end{cases} \tag{A.2}$$

Since $v_1 = 0$ and $v_2(x,\theta) = e^{x+(b-\rho c\lambda)^+\theta}$ are the subsolution and supersolution of (A.2), the estimate (20) follows by the comparison principle.

Denote $u^\delta(x,\theta) := u_\varepsilon^N(x,\theta+\delta)$ for $0 < \delta < T$, in $(-N,N) \times (0, T-\delta]$, $u^\delta(x,\theta)$ satisfies

$$\begin{cases} \partial_\theta u^\delta - \mathcal{L}u^\delta + \beta_\varepsilon(u^\delta - \pi_\varepsilon(e^x - K)) = 0, \\ \partial_x u^\delta(-N,\theta) = 0, \qquad \partial_x u^\delta(N,\theta) = e^N, \quad \theta \in (0, T-\delta], \\ u^\delta(x,0) = u_\varepsilon^N(x,\delta) \ge \pi_\varepsilon(e^x - K) = u_\varepsilon^N(x,0). \end{cases}$$

Combining with (18), applying the comparison principle, we have

$$u^\delta(x,\theta) \ge u_\varepsilon^N(x,\theta), \quad (x,\theta) \in (-N,N) \times (0, T-\delta],$$

which implies (21). □

Proof of Lemma 5. Let C_N be a generic constant independent of ε. Since $u_\varepsilon^N(x,\theta) \ge \pi_\varepsilon(e^x - K)$, then $|\beta_\varepsilon(u_\varepsilon^N - \pi_\varepsilon(e^x-K))|_{L^p(\mathcal{Q}_N)} \le C_N$. One can treat the non-linear term in the equation in (18) as a linear term with the coefficient $\frac{1}{2}\gamma(1-\rho^2)c^2\partial_x u_\varepsilon^N$. Thanks to (20), applying $C^{\alpha,\alpha/2}$ estimate (De Giorgi–Nash–Moser estimate, Lieberman, 1996) with $\alpha = 1/2$, we have $|u_\varepsilon^N|_{C^{1/2,1/4}(\overline{\mathcal{Q}}_N)} \le C_N$. Letting $\varepsilon \to 0$, we have a continuous limit (of a subsequence if necessary) $u^N(x,\theta)$ up to the boundary, that is, $u_\varepsilon^N(x,\theta) \to u^N(x,\theta)$ in $C(\overline{\mathcal{Q}}_N)$. In view of (20), applying $W_p^{2,1}$ estimate (Ladyženskaja, Solonnikov, & Ural'ceva, 1967), we have $|u_\varepsilon^N|_{W_p^{2,1}(\mathcal{Q}_N\setminus B_\delta)} \le C_N$, where B_δ is a disk with center $(\ln K, 0)$ and radius $\delta > 0$. Hence $u^N(x,\theta) \in W_{p,loc}^{2,1}(\mathcal{Q}_N)$ and $u_\varepsilon^N(x,\theta) \rightharpoonup u^N(x,\theta)$ weakly in $W_{p,loc}^{2,1}(\mathcal{Q}_N)$, which also implies $u^N(x,\theta)$ is a $W_{p,loc}^{2,1}$ solution of (17). Furthermore, (22)–(24) are consequences of (19)–(21).

In what follows, we show the uniqueness. Suppose not. We could find two $W_p^{2,1}$ solutions u_1 and u_2 satisfying (22) and (23), and $\mathcal{N} = \{(x,\theta) \in \mathcal{Q}_N : u_1 > u_2\} \ne \emptyset$. Observe that $\partial_\theta u_1 = \mathcal{L}u_1$ and $\partial_\theta u_2 \ge \mathcal{L}u_2$ in $\mathcal{N}$. Thus, in $\mathcal{N}$, $u_1 - u_2$ satisfies

$$\begin{cases} \partial_\theta(u_1-u_2) - \dfrac{1}{2}c^2\partial_{xx}(u_1-u_2) \\ -\left(b - \rho c\lambda - \dfrac{1}{2}c^2\right)\partial_x(u_1-u_2) \\ +\dfrac{1}{2}\gamma(1-\rho^2)c^2(\partial_x u_1 + \partial_x u_2)\partial_x(u_1-u_2) \le 0, \\ (u_1-u_2)(x,\theta) = 0, \quad (x,\theta) \in \partial_p\mathcal{N} \setminus \{x = \pm N\}, \\ \partial_x(u_1-u_2)(x,\theta) = 0, \quad (x,\theta) \in \partial_p\mathcal{N} \cap \{x = \pm N\}. \end{cases}$$

Owing to (23), applying maximum principle (see Tso (1985)), we have $u_1 - u_2 \le 0$ in $\mathcal{N}$, which is a contradiction. □

Proof of Lemma 6. First, we claim problem (17) is equivalent to the following problem

$$\begin{cases} \min\{\partial_\theta u^N - \mathcal{L}u^N, \quad u^N - (e^x - K)\} = 0, (x,\theta) \in \mathcal{Q}_N, \\ \partial_x u^N(-N,\theta) = 0, \qquad \partial_x u^N(N,\theta) = e^N, \quad \theta \in (0,T], \\ u^N(x,0) = (e^x-K)^+, \quad x \in (-N,N). \end{cases} \tag{A.3}$$

In fact, suppose $w(x,\theta)$ is a $W_{p,loc}^{2,1}$ solution to problem (A.3), the maximum principle (Tso, 1985) implies $w \ge 0$ combined with $w \ge e^x - K$ to get $w(x,\theta) \ge (e^x-K)^+$. Hence, w is also a solution of (17). Together with the uniqueness of (17), the equivalence follows.

Now we can rewrite problem (17) as

$$\begin{cases} \partial_\theta u^N - \mathcal{L}u^N = f(x,\theta), \quad (x,\theta) \in \mathcal{Q}_N, \\ \partial_x u^N(-N,\theta) = 0, \ \partial_x u^N(N,\theta) = e^N, \quad \theta \in (0,T], \\ u^N(x,0) = (e^x-K)^+, \quad x \in (-N,N), \end{cases} \tag{A.4}$$

where $f(x,\theta) = \chi_{\{u^N(x,\theta)=(e^x-K)^+\}}\big(-(b-\rho c\lambda)e^x + \frac{1}{2}\gamma(1-\rho^2)c^2e^{2x}\big)$. Thanks to (23), if we apply $W_p^{2,1}$ interior estimate (Ladyženskaja et al., 1967) to (A.4) for arbitrary $M < N$, it yields $|u^N|_{W_p^{2,1}(\mathcal{Q}_M\setminus B_\delta)} \le C_M$, where B_δ is a disk with center $(\ln K, 0)$ and radius δ. We emphasize C_M only depends on M, but not on N. Fix $M > 0$, and let $N \to +\infty$, u^N lead to a limit $u^{(M)}$ (possibly a subsequence) in the fixed domain $\mathcal{Q}_M$ in the sense $u^N \rightharpoonup u^{(M)}$ weakly in $W_{p,loc}^{2,1}(\mathcal{Q}_M)$ as $N \to \infty$. Moreover the Sobolev embedding theorem implies $u^N \to u^{(M)}$ in $C(\mathcal{Q}_M)$, and $\partial_x u^N \to \partial_x u^{(M)}$ in $C(\mathcal{Q}_M)$.

It is clear that $u(x,\theta) := u^{(M)}(x,\theta)$, $(x,\theta) \in \mathcal{Q}_M$ is well defined and u is the solution to problem (16). Moreover, $\partial_x u \in C(\mathcal{Q})$ and we can deduce $u \in C(\overline{\mathcal{Q}})$ from the C^α estimate. Note that (25)–(27) are consequences of (22)–(24). Lemmas 1 and 2 imply the uniqueness. □

Proof of Lemma 8. As in the proof of the equivalence between (17) and (A.3) in Lemma 6, we can show that problem (16) is equivalent to the following problem

$$\min\big\{\partial_\theta u - \mathcal{L}u, \ u - (e^x - K)\big\} = 0, \quad (x,\theta) \in \mathcal{Q},$$

with initial $u(x,0) = (e^x-K)^+$ for $x \in \mathbb{R}$. When $u(x,\theta) = (e^x-K)^+ = (e^x-K)$, we have $e^x - K \ge 0$ and $\partial_\theta u(x,\theta) - \mathcal{L}u(x,\theta) \ge 0$, which implies

$$e^x \ge \max\left\{K, \ \frac{2(b-\rho c\lambda)}{\gamma(1-\rho^2)c^2}\right\}. \tag{A.5}$$

For any $x_0 \in \{x : 0 < S(x) < T\}$, denote $S(x_0) = \theta_0 \in (0,T)$, in view of (28), we know $u(x_0,\theta) = e^{x_0} - K \ge 0$, $\theta \in (0,\theta_0]$. Denote $\mathcal{Q}_0 = (-\infty,+\infty) \times (0,\theta_0]$ and define a new function $\overline{u}(x,\theta)$ on $\mathcal{Q}_0$ by

$$\overline{u}(x,\theta) = \begin{cases} u(x,\theta), & (x,\theta) \in (-\infty, x_0] \times [0,\theta_0], \\ e^x - K, & (x,\theta) \in [x_0,+\infty) \times [0,\theta_0]. \end{cases}$$

Since $\{x_0\} \times (0,\theta_0] \subseteq \mathcal{S}$, then $\overline{u}(x,\theta)$, $\partial_x\overline{u}(x,\theta)$ are continuous in $\mathcal{Q}_0$. Now we want to prove $\overline{u}(x,\theta)$ is the solution to problem (16) in the domain $\mathcal{Q}_0$.

By the definition of $\overline{u}(x,\theta)$ we know $\overline{u}(x,0) = (e^x-K)^+$, $x \in \mathbb{R}$. According to (A.5), we can check $\overline{u}(x,\theta)$ satisfies $\min\big\{\partial_\theta\overline{u} - \mathcal{L}\overline{u}, \ \overline{u} - (e^x-K)^+\big\} = 0$, for $(x,t) \in \mathcal{Q}_0$. Hence, we know $\overline{u}(x,\theta)$ is the solution of (16) in domain $\mathcal{Q}_0$. By the uniqueness of $W_{p,loc}^{2,1}(\mathcal{Q}) \cap C(\overline{\mathcal{Q}})$ solution which satisfies (25)–(26) to problem (16) we know $u(x,\theta) = \overline{u}(x,\theta)$ for $(x,\theta) \in \mathcal{Q}_0$. In particular, $u(x,\theta) = \overline{u}(x,\theta) = (e^x-K)^+$ for $(x,\theta) \in [x_0,+\infty) \times (0,\theta_0]$. By the definition of $S(x)$ we know $S(x) \ge \theta_0 = S(x_0)$, $x \ge x_0$, therefore the monotonicity of $S(x)$ is proved.

Next we will prove the strict monotonicity of $S(x)$ on the set $\{x : 0 < S(x) < T\}$. Suppose not (see Fig. 3). There exists $x_1 < x_2$ such that $S(x) = \theta_0 \in (0,T)$, $x \in (x_1,x_2)$. Thus

$$\begin{cases} \partial_\theta u(x,\theta) - \mathcal{L}u(x,\theta) = 0, \quad (x,\theta) \in (x_1,x_2) \times (\theta_0, T], \\ u(x,\theta_0) = (e^x - K), \quad x \in (x_1,x_2). \end{cases}$$

Hence

$$\begin{aligned} \partial_\theta u|_{\theta=\theta_0} &= -\frac{1}{2}\gamma(1-\rho^2)c^2e^{2x} + (b-\rho c\lambda)e^x \\ &< \left[(b-\rho c\lambda) - \frac{1}{2}\gamma(1-\rho^2)c^2e^x\right]e^{x_1} \\ &< -\frac{1}{2}\gamma(1-\rho^2)c^2e^{2x_1} + (b-\rho c\lambda)e^{x_1} \le 0, \end{aligned} \tag{A.6}$$

which contradicts (27), the two inequalities in (A.6) are due to (A.5). □

A.2. Proof of Proposition 12

The following two lemmas imply Proposition 12.

Lemma 14. *The solution u to the problem (16) decreases with respect to γ and λ, and increases with respect to b.*

Proof. We may prove all three monotonicities in the same way by the comparison principle. Hence we only present the proof of the monotonicity of $u(x,\theta)$ with respect to γ. Suppose $\gamma_1 > \gamma_2$, and $u^N_{\varepsilon(i)}(x,\theta)(i=1,2)$ is the solution to the following problem in $\mathcal{Q}_N$,

$$\begin{cases} \partial_\theta u^N_{\varepsilon(i)} - \dfrac{1}{2}c^2\partial_{xx}u^N_{\varepsilon(i)} - \left(b-\rho c\lambda-\dfrac{1}{2}c^2\right)\partial_x u^N_{\varepsilon(i)} \\ +\dfrac{1}{2}\gamma_i(1-\rho^2)c^2(\partial_x u^N_{\varepsilon(i)})^2+\beta_\varepsilon(u^N_{\varepsilon(i)}-\pi_\varepsilon(\cdot))=0, \\ \partial_x u^N_{\varepsilon(i)}(-N,\theta)=0, \qquad \partial_x u^N_{\varepsilon(i)}(N,\theta)=e^N, \\ u^N_{\varepsilon(i)}(x,0)=\pi_\varepsilon(e^x-K). \end{cases}$$

Then $w(x,\theta) := u^N_{\varepsilon(1)}(x,\theta) - u^N_{\varepsilon(2)}(x,\theta)$ satisfies

$$\begin{aligned} &\partial_\theta w - \frac{1}{2}c^2\partial_{xx}w - \left(b-\rho c\lambda-\frac{1}{2}c^2\right)\partial_x w \\ &\quad + \frac{1}{2}\gamma_1(1-\rho^2)c^2\left(\partial_x u^N_{\varepsilon(1)}+\partial_x u^N_{\varepsilon(2)}\right)\partial_x w+\beta'_\varepsilon(\cdot)w \\ &= \frac{1}{2}(\gamma_2-\gamma_1)(1-\rho^2)c^2\left(\partial_x u^N_{\varepsilon(2)}\right)^2 \le 0. \end{aligned}$$

Combined with the initial and boundary conditions, we have $u^N_{\varepsilon(1)}(x,\theta)-u^N_{\varepsilon(2)}(x,\theta)=w(x,\theta)\le 0$. Letting $\varepsilon\to 0,\ N\to+\infty$, we know $u(x,\theta)$ is decreasing w.r.t. γ. □

Lemma 15. *The solution u to the problem (16) satisfies $nu[\widetilde{g}(x)] \ge u[n\widetilde{g}(x)]$ $(n\ge 1)$, where $\widetilde{g}(x)=(e^x-K)^+$, and $u[\widetilde{g}]$ represents the solution to problem (16) with the obstacle and initial condition $\widetilde{g}$.*

Proof. Note $\widetilde{u}(x,\theta) := nu[\widetilde{g}(x)]$ satisfies $\min\{\partial_\theta\widetilde{u}-\mathcal{L}\widetilde{u}-\frac{1}{2}\gamma n(n-1)(1-\rho^2)c^2(\partial_x u[\widetilde{g}])^2, \widetilde{u}-n\widetilde{g}\}=0$, with initial $\widetilde{u}(x,0)=n\widetilde{g}(x)$. Note that $u[n\widetilde{g}(x)]$ satisfies

$$\min\{(\partial_\theta u-\mathcal{L}u)(x,\theta),\ u-n\widetilde{g}\}=0$$

with initial $u(x,0)=n\widetilde{g}(x)$. We can confine the above problems in the bounded domain $\mathcal{Q}_N$. Suppose $\widetilde{u}^N[\widetilde{g}]$ and $u^N[n\widetilde{g}]$ are the solutions of the following problems

$$\begin{cases} \min\Big\{\partial_\theta\widetilde{u}^N[\widetilde{g}]-\mathcal{L}\widetilde{u}^N[\widetilde{g}]-\dfrac{1}{2}\gamma n(n-1)(1-\rho^2)c^2(\partial_x u^N[\widetilde{g}])^2, \\ \quad \widetilde{u}^N[\widetilde{g}]-n\widetilde{g}(x)\Big\}=0, \quad (x,\theta)\in\mathcal{Q}_N, \\ \partial_x\widetilde{u}^N[\widetilde{g}](-N,\theta)=0, \qquad \partial_x\widetilde{u}^N[\widetilde{g}](N,\theta)=ne^N, \\ \widetilde{u}^N[\widetilde{g}](x,0)=n\widetilde{g}(x). \end{cases} \tag{A.7}$$

$$\begin{cases} \min\Big\{\partial_\theta u^N[n\widetilde{g}]-\mathcal{L}u^N[n\widetilde{g}], u^N[n\widetilde{g}]-n\widetilde{g}(x)\Big\}=0, \\ \partial_x u^N[n\widetilde{g}](-N,\theta)=0, \qquad \partial_x u^N[n\widetilde{g}](N,\theta)=ne^N, \\ u^N[n\widetilde{g}](x,0)=n\widetilde{g}(x). \end{cases} \tag{A.8}$$

Comparing (A.7) with (A.8), we conclude the result. □

References

Adams, R. A. (1975). *Sobolev spaces*. New York: Academic Press.

Bayraktar, E., Song, Q., & Yang, J. (2011). On the continuity of stochastic control problems on bounded domains. *Stochastic Analysis and Applications*, *29*(1), 48–60.

Crandall, M. G., Ishii, H., & Lions, P. L. (1992). User's guide to viscosity solutions of second order partial differential equations. *Bulletin of the American Mathematical Society*, *27*(1), 1–67.

Evans, L. C. (1998). *Graduate studies in mathematics*: *Vol. 19*. *Partial differential equations*. Providence, RI: American Mathematical Society.

Fleming, W. H., & Soner, H. M. (2006). *Stochastic modelling and applied probability*: *Vol. 25*. *Controlled markov processes and viscosity solutions* (second ed.). New York: Springer.

Friedman, A. (1964). *Partial differential equations of parabolic type*. Englewood Cliffs, NJ: Prentice-Hall Inc.

Friedman, A. (1975). Parabolic variational inequalities in one space dimension and smoothness of the free boundary. *Journal of Functional Analysis*, *18*, 151–176.

Krylov, N. V. (1980). *Applications of mathematics*: *Vol. 14*. *Controlled diffusion processes*. New York: Springer-Verlag.

Ladyženskaja, O. A., Solonnikov, V. A., & Ural'ceva, N. N. (1967). *Translations of mathematical monographs*: *Vol. 23*. *Linear and quasilinear equations of parabolic type*. Providence, RI: American Mathematical Society.

Lieberman, G. M. (1996). *Second order parabolic differential equations*. River Edge, NJ: World Scientific Publishing Co. Inc.

Musiela, M., & Zariphopoulou, T. (2009). Portfolio choice under dynamic investment performance criteria. *Quantitative Finance*, *9*(2), 161–170.

Oberman, A., & Zariphopoulou, T. (2003). Pricing early exercise contracts in incomplete markets. *Computational Management Science*, *1*(1), 75–107.

Oleĭnik, O. A., & Radkevič, E. V. (1973). *Second order equations with nonnegative characteristic form*. New York: Plenum Press.

Tso, K. (1985). On an Aleksandrov-Bakel'man type maximum principle for second-order parabolic equations. *Communications in Partial Differential Equations*, *10*(5), 543–553.

Yong, J., & Zhou, X. Y. (1999). *Applications of mathematics*: *Vol. 43*. *Stochastic controls: hamiltonian systems and hjb equations*. New York: Springer-Verlag.

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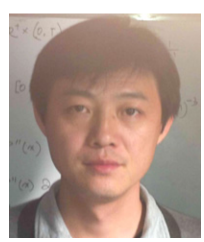

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