# THE SPACE OF EXITS OF A MARKOV PROCESS



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# THE SPACE OF EXITS OF A MARKOV PROCESS

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Martin's theory makes it possible to describe the sets of all non-negative harmonic and superharmonic functions in an arbitrary domain of euclidean space. To each Markov process there corresponds the class of so-called excessive functions, analogous in their properties to the class of non-negative superharmonic functions. The study of this class is closely connected with the study of "the space of exits of a Markov process". Corresponding results for discrete Markov chains were obtained by Doob, Hunt and Watanabe, and for certain types of processes with variable time by Kunita and Watanabe. The paper gives an account of the general theory, which includes as particular cases all the results listed.

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## Introduction

1. It is well known that any function $h$, harmonic in an open unit disc $K$ and continuous in the closure of $K$, is expressible in terms of its boundary values by Poisson's integral

$$h(x) = \int_C k_z(x)\, h(z)\, \lambda(dz), \tag{1}$$

where $C$ is the circumference of $K$, $\lambda$ is a rotation invariant measure on $C$ normalized[1] by the condition $\lambda(C) = 1$, and

$$k_z(x) = \frac{1-|x|^2}{|x-z|^2} \qquad (z \in C) \tag{2}$$

For each $z \in C$, $k_z(x)$ is itself harmonic in $K$. (It has a discontinuity at the point $z$ and cannot be represented by (1).) To each finite measure $\mu$ on $C$ there corresponds a non-negative harmonic function

$$h(x) = \int_C k_z(x)\,\mu(dx). \tag{3}$$

The converse is also true: every non-negative function $h$, harmonic in $K$, has a unique representation in the form (3). Thus, the formulae (2) and (3) give, for the case $E = K$, the complete solution of the following problem:

PROBLEM A. *To describe all non-negative harmonic functions in a domain $E$.*

2. Now let $E$ be an arbitrary simply-connected domain in the plane. If the complement of $E$ contains more than one point, then $E$ can be mapped conformally on $K$. If $E$ is bounded by a smooth contour, then the conformal mapping $\psi: E \to K$ can be extended to a continuous mapping of the closure[2] of $E$ onto $K \cup C$. It is clear that the solution of Problem A for $E$ is again given by (3), only that by $C$ we must mean the boundary of $E$, and by $k_z(x)$ the function obtained from (2) by means of $\psi$.

This method cannot be applied if the construction of the boundary is more complicated. However, the difficulty can be avoided as follows. We continue $k_z(x)$, defined by (2), inside $K$ by the formula

$$k_z(x) = -\frac{1}{\log|z|} \log\left|\frac{1-x\bar{z}}{x-z}\right|. \tag{4}$$

We obtain a function of $z$ continuous in the closed circle $K \cup C$, which is, for any $z \in K$, harmonic with respect to $x$ in $K \setminus \{z\}$. We substitute in the right-hand side of (4) for $x$ and $z$ the values $\psi(x)$ and $\psi(z)$, where $\psi$ is the function giving the conformal mapping of $E$ into $K$. We obtain for each $z \in E$, a function $k_z(x)$, harmonic in $E \setminus \{z\}$. We suppose now that the sequence of points $z_n \in E$ has the two properties: a) every compactum in $E$ contains only a finite number of points $z_n$; b) the functions $k_{z_n}(x)$ tend to some limit function. It is easy to see that the limit function is harmonic in $E$. We denote it again by $k_z(x)$, only $z$ cannot now be understood as a point of the plane: it may happen that the sequence $z_n$ has many limit points in the plane or "goes off to infinity" and has no limit point. We interpret $z$ as a point of an auxiliary topological space $\mathscr{E}$ that is a compact extension of the domain $E$. Here $z$ belongs to

[1] We denote the length of a vector $x$ by $|x|$.

[2] We are concerned with closure in the extended plane, the point $\infty$ being included.

the set $C = \mathscr{E} \setminus E$, which coincides with the boundary of $E$ in $\mathscr{E}$. This boundary is called the *Martin boundary*. For such an interpretation of $C$ and $k_z(x)$ a solution of Problem A is given once more by (3).

For multiply-connected domains we succeed also in constructing the function $k_z(x)$ and determining by means of it the Martin boundary $C$ so that all solutions of Problem A are described by (3). Generally speaking, the representation (3) will not be unique. However, we can select in $C$ a Borel subset $C_0$ such that every non-negative harmonic function $h$ is uniquely representable in the form

$$h(x) = \int_{C_0} k_z(x) \mu(dx). \tag{5}$$

3. From the foregoing it is possible to give a general presentation of the method of solution of Problem A proposed in 1941 by Martin [17]: a) *with each point of the domain $E$ a function $k_z(x)$ is associated, which is harmonic in $E \setminus \{z\}$ (the Martin function)*; b) *by means of $k_z(x)$ a compactification $\mathscr{E}$ of the set $E$ is constructed, and $k_z(x)$ is extended to values $z \in C = \mathscr{E} \setminus E$ (the Martin boundary)*; c) *a "sweeping out" of the set $C$ is effected leaving a certain subset $C_0$; formula (5) establishes a one-to-one correspondence between finite measures on $C_0$ and non-negative harmonic functions in $E$.* We analyze the constructions forming the content of steps a) and b), leaving till later a closer discussion of the methods of effecting c). The construction of b) is of a fairly general nature, but this cannot be said of a): our method of conformal mapping is entirely inapplicable if the dimension is higher than two.

Let $E$ be an arbitrary domain in $l$-dimensional euclidean space ($l \geqslant 3$), and let $z \in E$. It is known that the general form of functions bounded below and harmonic in $E \setminus \{z\}$ is given by the formula

$$a\,|x - z|^{2-l} + H(x), \tag{6}$$

where $H$ is a function bounded below and harmonic in $E$, and $a$ is a constant.[1] To construct the Martin function we have to select for each $z$ a harmonic function $H$ and a constant $a$, the choice being suitably made for different $z$. We first suppose that $E$ is bounded with a smooth boundary. Then the family (6) contains functions vanishing on the boundary.[2] All these functions differ from each other by constant factors. To remove the indeterminacy completely we may fix some value of the constant $a$, say $a = 1$. The function $g_z(x) = g(x, z)$ arising in this way is called the *Green function*. The value of (6) can be fixed at any point $x_0 \in E$. This procedure leads us to the *Martin function*[3] $k_z(x)$. The Martin function is

[1] The corresponding formulae for $l = 2$ and $l = 1$ are: $a \log |z - x|^{-1} + H(x)$ and $a\,|z - x| + H(x)$.

[2] More exactly, tending to zero as $x$ tends to any boundary point.

[3] For the Martin function defined by (4) $x_0 = 0$. Even though the Martin function depends on the choice of $x_0$, the corresponding compactifications are isomorphic.

expressed in terms of the Green function by the formula

$$k_z(x) = \frac{g(x, z)}{g(x_0, z)} \tag{7}$$

The Green function has one important advantage over the Martin function: if the domain $E$ is extended, then $g_z(x)$ increases. This enables us to construct the Green function for any domain $E$: we can always select an expanding sequence of bounded domains $E_n$ with smooth boundaries so that $E_n \uparrow E$. The corresponding Green functions increase monotonically, and their limit[1] does not depend on the choice of the approximating sequence $E_n$. It is called the Green function for the domain $E$.

For any domain the Martin function can be defined in terms of the Green function by (7).

4. For $l = 1$ the class of harmonic functions coincides with that of linear functions. An important extension of this class is the class of (upward) convex functions. For $l \geqslant 2$ the class of superharmonic functions plays a similar role.[2] A natural variant of Problem A is the following:

PROBLEM B. *To determine all non-negative superharmonic functions in a domain $E$.*

As we shall see, Problem B lends itself to generalization more readily than Problem A. In the classical case the solution of Problem B is given by the formula

$$h(x) = \int_{\mathcal{U}} k_z(x)\, \mu(dz), \tag{8}$$

which establishes a one-to-one correspondence between all finite measures on the set $\mathcal{U} = C_0 \cup E$ and all non-negative superharmonic functions. From (8) there follows a decomposition (the Riesz decomposition) of $h$ into the sum of the harmonic function

$$\int_{C_0} k_z(x)\, \mu(dz)$$

and the superharmonic function

$$\int_E k_z(x)\, \mu(dz) = \int_E g(x, z)\, \nu(dz)$$

($\nu(dz) = \mu(dz) / g(x_0, z)$), the so-called *Green potential.*

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[1] The limit mentioned is always finite for $x \neq z$ if $l \geqslant 3$. For $l < 3$ this is not the case and one must consider only domains for which the Green function is finite. (It can be proved that if $g(x, z) = \infty$ for any $x \neq z$, then $g(x, z) = \infty$ for all $x, z \in E$, and in this case no non-negative harmonic functions exist in $E$ other than constants.

[2] A function $h$ is called *superharmonic* in the domain $E$ if it is lower semi-continuous and if, for any closed sphere contained in $E$, the average value of $h$ over the surface of the sphere is less than or equal to the value at the centre of the sphere.

5. Harmonic functions may be defined as solutions of Laplace's equation. Similar properties are common to solutions of any second order elliptic differential equation with sufficiently smooth coefficients. The extension of Martin's method to such equations is in the main a matter of technique, and the qualitative picture of the solution of Problem A is unaltered. However, under weaker conditions on the coefficients of the equation of entirely new effects may appear. Although the problem in an analytical formulation requires further investigation, we can decide on the character of these effects on the basis of general probability theory, the construction of which is the subject of the present paper.

6. As was first noted by Doob [5], the natural setting for Martin's method is the theory of Markov processes.

Suppose that a point moves at random in some space $E$. We assume that after time $t$ it passes from the position $x$ into the set $\Gamma$ with probability $p(t, x, \Gamma)$, independently of how much time or in what way it has moved before. We then say that a Markov process with transition function $p(t, x, \Gamma)$ is given in $E$. We denote by $x_t$ the position of the moving point at the moment $t$. The random nature of the motion is expressed mathematically by the hypothesis that $x_t = x_t(\omega)$, where $\omega$ is an element of a certain set $\Omega$ on which the set of probability measures $\mathsf{P}_x (x \in E)$ is given. The value of $\mathsf{P}_x(A)$ is interpreted as the probability of the event $A$ under the assumption that motion begins from the point $x$. In particular, $\mathsf{P}_x\{x_t \in \Gamma\} = p(t, x, \Gamma)$.

A Markov process in $l$-dimensional euclidean space whose paths are continuous and whose transition function is invariant under all motions is called a *Wiener process* or a *Brownian motion*. For a suitable choice of the time scale its transition function is given by

$$p(t, x, \Gamma) = \int_\Gamma (2\pi t)^{-l/2} \exp\left[-\frac{1}{2t}|y-x|^2\right] m(dy),$$

where $m$ is a Lebesgue measure. Cutting off each path at the moment of its first exit from the domain $E$ we obtain a process, which is called[1] *a Brownian motion in the domain E.*

7. To the transition function $p(t, x, \Gamma)$ there corresponds the family of operators

$$P_t f(x) = \int_E p(t, x, dy) f(y),$$

acting on the set $V$ of all non-negative functions[2] in $E$. A function $f \in V$ is called *excessive* if $P_t f \leqslant f$ for any $t > 0$ and $P_t f \to f$ as $t \downarrow 0$. To each Markov process there corresponds its own class of excessive functions.

[1] Contemporary theory considers processes with random moment of cut-off $\zeta = \zeta(\omega)$. By the event $\{x_t \in \Gamma\}$ we mean the set $\{\omega: \zeta(\omega) > t, x_t(\omega) \in \Gamma\}$.

[2] Here, as almost everywhere in the introduction, we do not mention conditions regarding measurability of the functions in question. Problems of measurability are treated sufficiently fully in the main part of the paper.

For a Brownian motion in a domain $E$ this class coincides with the class of all non-negative superharmonic functions. Hence a natural generalization of Problem B is:

PROBLEM C. *To determine for a given Markov process all excessive functions.*

The class of excessive functions is described in terms of the operators $P_t$ or, what is equivalent, in terms of the transition function $p(t, x, \Gamma)$. To define the class of harmonic functions in the same simple terms is impossible. One needs certain restrictions of the process and more complicated constructions, to which we return later.

8. As in the case of Problem B, the first step towards the solution of Problem C is the construction of the Green function. The starting point is the so-called *Green kernel*, which is defined by the formula

$$g(x, \Gamma) = \int_0^\infty p(t, x, \Gamma)\, dt \tag{9}$$

and may be interpreted as the mathematical expectation of the time spent in the set $\Gamma$ of paths going out from the point $x$. It is known that for a Brownian motion in the domain $E$

$$g(x, \Gamma) = \int_\Gamma g(x, y)\, m(dy), \tag{10}$$

where $g(x, y)$ is the Green function described in 3. above and $m$ is the Lebesgue measure. It is natural also in the general case to call $g(x, y)$ the Green function for the Markov process if it satisfies (10) for some measure $m$ not depending on $x$ or $\Gamma$. Certainly the Green function is constructed less naturally than the Green kernel; it need not exist, it depends on the choice of $m$, and even for a fixed measure $m$ it can be changed arbitrarily on a set of points $y$, depending on $x$, having $m$-measure zero.

The hypothesis of the existence of the Green function is a minimal restriction on the process, necessary for the investigation of Problem C. Besides, we require a certain regularity of $g(x, y)$ as a function of $y$.

9. We first assume that:

9.A. $E$ is a separable locally compact metric space.

9.B. For each $\omega$ and for all $t \in (0, \zeta(\omega))$ there exist the limits

$$x_{t-0}(\omega) = \lim_{u \uparrow t} x_u(\omega),$$

where $P_x\{x_{t-0} \neq x_t\} = 0$ for any $t > 0$, $x \in E$.

Then the condition of regularity can be formulated as follows: for each continuous function $\varphi$ with compact support the function

$$g(y) = \int_E m(dx)\, \varphi(x)\, g(x, y) \tag{11}$$

is continuous. However, this statement obscures the essence of the matter. In fact, only the following property of $g(y)$ is important: *there exists a* *function*[1] $F(t, \omega)$, *continuous on the left with respect to* $t$ *and such that* $P_x\{F(t, \omega) \neq g[x_t(\omega)]\} = 0$ *for all* $t > 0$, $x \in E$. If this condition is satisfied, we say that $g$ is $\Lambda$-*continuous* and that $F(t, \omega)$ is *a left-continuous modification of* $g(x_t)$.

The definition of $\Lambda$-continuity does not require any topology in the space $E$. It is desirable to manage without a topology when describing the family of functions $\varphi$ for which the $\Lambda$-continuity of (11) is postulated.

We define a *base family* as any countable family of non-negative measurable functions $W$ with the following properties:

a) if $\varphi_1, \varphi_2 \in W$ and $r_1, r_2$ are non-negative rational numbers, then $r_1\varphi_1 + r_2\varphi_2 \in W$;

b) every system $\tilde{V}$ containing $W$ and closed with respect to addition, monotone (upward) passage to the limit, and subtraction (if the difference $f_2 - f_1$ is non-negative and $f_2$ is majorized by some function $\varphi \in W$) contains all non-negative measurable functions;

c) if $\mu_n$ is a sequence of measures[2] and for any $\varphi \in W$

$$\mu_n(\varphi) \to l(\varphi) < \infty,$$

then there exists a measure $\mu$ such that $\mu(\varphi) = l(\varphi)$ $(\varphi \in W)$.

We assume that in the space $E$ some base family $W$ is fixed[3] and that the function defined by (11) is $\Lambda$-continuous for all $\varphi \in W$.

**10.** To construct the Martin function we have to normalize the Green function. The normalization of 3. is not suitable in the general case, as the denominator in (7) may vanish. Also, it is essential that normalization preserves the properties of regularity we have postulated for the Green function, and for this it is necessary that the normalizing function itself is $\Lambda$-continuous.

Hunt proposed the following general normalizing condition: *the integral of a function with respect to a fixed measure* $\gamma$ is 1. Then (7) is replaced by

$$K_z(x) = \frac{g(x, z)}{q(z)}, \tag{12}$$

where

$$q(z) = \int_E \gamma(dx)\, g(x, z). \tag{13}$$

[1] Under assumptions made earlier such a function is $F(t, \omega) = g\{x_{t-0}(\omega)\}$.

[2] $\mu(\varphi)$ denotes the integral of $\varphi$ with respect to the measure $\mu$.

[3] If $E$ is a separable locally compact metric space and measurability is in the Borel sense, then a base family of functions can be constructed as follows. We consider some sequence of open sets $D_n \uparrow E$ whose closures are compact and denote by $C_0(D_n)$ the set of all non-negative continuous functions whose support lies in $D_n$. We select a countable everywhere dense subset (in the sense of uniform convergence) $\mathcal{T}_n \subset C_0(D_n)$, put $\mathcal{T} = \bigcup \mathcal{T}_n$ and denote by $W$ the set of all functions that are expressible as linear combinations with non-negative rational coefficients of elements of $\mathcal{T}$.

The definition of a base system in 5.1 is somewhat wider and is applicable with a freer interpretation of the term measurability.

If we take for $\gamma$ the unit measure concentrated at $x_0$, we return to Martin's normalization. But now we have extensive new possibilities. For example, we can put $\gamma(dx) = \varphi(x)m(dx)$, where $\varphi \in W$, and then $q(z)$ by the hypothesis already made is $\Lambda$-continuous. However, little of that. We require not only $q(x_t)$ to have a left-continuous modification, but the latter to be positive.[1] Under natural restrictions on the transition function these conditions can be satisfied by putting $\gamma(dx) = \psi(x)m(dx)$, where $\psi$ is obtained by summing the functions $\varphi \in W$ with suitable weights $d_\varphi$.

We denote by $S_\gamma$ the class of all excessive functions whose integrals with respect to the measure $\gamma$ do not exceed 1, and for each $\varphi \in W$ we put

$$c_\varphi = \sup_{h \in S_\gamma} m(h\varphi). \tag{14}$$

for a Brownian motion in $E$ and any finite non-trivial measure $\gamma$ all constants $c_\varphi$ are finite (see the proof in the footnote on p. 118). In the general case the condition that the $c_\varphi$ are finite limits the choice of $\gamma$.

Finally, we require[2] that for each $\varphi \in W$

$$\gamma(G\varphi) < \infty, \tag{15}$$

We call the measure $\gamma$ *standard* if there exists a positive left-continuous modification of $q(x_t)$, if the constants $c_\varphi$ are finite, and if (15) is satisfied. The measures $\gamma(dx) = \varphi(x)m(dx)$ constructed above are standard for a suitable choice of weights $d_\varphi$.

11. We sum up the initial assumptions. We start from a Markov process for which we fix a base system of functions, the Green function and a standard measure. We call this object an *M-process.*[3] Problem C is now stated more precisely as follows:

PROBLEM C′. *For a given M-process to describe all excessive functions that are integrable with respect to the standard measure* $\gamma$.

We have to make one important addition to the conditions given in 8.-10. In the study of excessive functions it is convenient to introduce so-called $\alpha$-excessive functions, which are defined by replacing in the definition of excessive functions the operators $P_t$ by $e^{-\alpha t}P_t$ ($\alpha$ is a non-negative number). It is convenient also to consider the kernel

$$g_\alpha(x, \Gamma) = \int_0^\infty e^{-\alpha t} p(t, x, \Gamma)\, dt,$$

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1 If conditions 9.A - 9.B are satisfied, it is sufficient to require $q$ to be continuous and positive.

2. For $l \geqslant 3$ and a Brownian motion in $l$-dimensional domain $E$, each of the functions $G\varphi(x)$ $(\varphi \in W)$ is bounded, and (15) is satisfied for any finite measure $\gamma$. In the main part of the paper, instead of (15) somewhat weaker conditions are mentioned.

3 An important class of $M$-processes consists of the so-called special $M$-processes for which 9.A - 9.B are satisfied; $W$ is defined as in the footnote on p. 95, $q$ is continuous and positive, and $p(t, x, A) \to 1$ as $t \to 0$ if $A$ is a neighbourhood of $x$.

agreeing with $g(x, \Gamma)$ for $\alpha = 0$. It is clear that $g(x, \Gamma) \geqslant g_\alpha(x, \Gamma)$ for any $\alpha \geqslant 0$. Hence, if $g(x, \Gamma)$ has a representation of the form (10), then by the Radon-Nikodym theorem $g_\alpha(x, \Gamma)$ admits a similar representation:

$$g_\alpha(x, \Gamma) = \int_\Gamma g_\alpha(x, y)\, m(dy).$$

The regularity conditions stated in 9. for $g(x, y)$ must be postulated for $g_\alpha(x, y)$ for all $\alpha \in R$, the set of all non-negative rational numbers. These properties are then satisfied automatically for the functions

$$K_z^\alpha(x) = \frac{g_\alpha(x, z)}{q(z)}, \tag{16}$$

where $q$ is defined by (13). We also have to demand that for any $\alpha \in R$ the constant $c_\varphi^\alpha$ is finite (its definition is obtained if in (14) the class $S_\gamma$ is replaced by the class of all $\alpha$-excessive functions whose integrals with respect to $\gamma$ do not exceed 1). Although the demands on a standard measure are increased, the method of constructing standard functions described in 10. still holds.

If $K_z^\alpha$ for any $\alpha \in R$ and $z \in R$ is $\alpha$-excessive, then by (14) for all $\varphi \in W$

$$\int_E K_z^\alpha(x)\, \varphi(x)\, m(dx) \leqslant c_\varphi^\alpha. \tag{17}$$

From the assumptions made it does not follow that the function $K_z^\alpha$ is $\alpha$-excessive. However, it can be proved that for $m$-almost all[1] $z$ there exists a "$\alpha$-excessive modification" of the function $K_z^\alpha$, and hence it follows that (17) is satisfied for $m$-almost all $z$. We define the *Martin function* as the function that agrees with $K_z^\alpha$ if (17) is satisfied for all $\alpha \in R$ and all $\varphi$, and is identically zero for the remaining values of $z$. The notation $K_z^\alpha$ will now be used not for the function defined by (16), but for its variant, which we call the Martin function.

12. We begin the construction of the Martin compactum as in 2. with the selection of a "convergent" sequence $z_n$ in $E$. Each sequence must contain a "convergent" subsequence. This requirement is satisfied if we call a sequence "convergent" if for any $\varphi \in W$, $\alpha \in R$ the limit

$$l^\alpha(\varphi) = \lim \int_E K_{z_n}^\alpha(x)\, \varphi(x)\, m(dx).$$

exists and is finite. By the definition of a base system $W$ the functional $l^\alpha$ has a unique representation in the form

$$l^\alpha(\varphi) = \mu^\alpha(\varphi),$$

where for all $\varphi \in W$

$$\mu^\alpha(\varphi) \leqslant c_\varphi^\alpha. \tag{18}$$

[1] That is, for all excluding a set of $m$-measure zero.

We put $\mu^\alpha(dx) = \mathcal{K}_z^\alpha(dx)$, understanding by $z$ the "limit" of the "convergent" sequence $z_n$.

These heuristic arguments are made more precise as follows: We consider the space $\mathfrak{S}$ whose points are measures $\mu^\alpha$ depending on the parameter $\alpha \in R$ and subject to the condition (18). We put $(\mu^\alpha)_n \to \mu^\alpha$ if

$$\mu_n^\alpha(\varphi) \longrightarrow \mu^\alpha(\varphi) \quad \text{for all } \alpha \in R,\ \varphi \in W.$$

In this topology $\mathfrak{S}$ is compact. We define a mapping $i$ of $E$ into $\mathfrak{S}$, by associating with a point $y$ the measure $K_y^\alpha(x)m(dx)$. The "convergence" introduced above of a sequence of points of $E$ is now convergence of their images in $\mathfrak{S}$. It is convenient to distinguish the notation for points of $\mathfrak{S}$ from that for the measures determining these points: we denote by $\mathcal{K}_z^\alpha(dx)$ measures corresponding to $z \in \mathfrak{S}$. Then for any $y \in E$

$$\mathcal{K}_{i(y)}^\alpha(dx) = K_y^\alpha(x)\, m(dx).$$

The closure[1] of $i(E)$ in $\mathfrak{S}$ is denoted by $\mathcal{E}$ and is called a *Martin compactum.*

In the classical case discussed in 2.-4. to each point of the Martin compactum there corresponds an excessive function $k_z$, and all excessive functions are obtained from these by means of integration. The position at which we have now arrived is more complicated: to the point $z \in \mathcal{E}$ there corresponds not a function but a measure, and even for $z = i(y)$ the function $K_y(x)$, which it would be natural to relate to $z$, need not at all be excessive.

However, it turns out (and this is one of the main results of the present paper) that in $\mathcal{E}$ we can distinguish a measurable subset $\mathcal{U}$ with the following properties:

a) if $z \in \mathcal{U}$ then for all $\alpha \in R$ we have $\mathcal{K}_z^\alpha(dx) = k_z^\alpha(x)m(dx)$, where $k_z^\alpha$ is a $\gamma$-integrable[2] $\alpha$-excessive function;[3]

b) every $\gamma$-integrable excessive function $h$ has a unique representation of the form

$$h(x) = \int_{\mathcal{U}} k_z(x)\,\mu(dz), \tag{19}$$

where $k_z = k_z^0$ and $\mu$ is a finite measure.

We call the set $\mathcal{U}$ *the space of exits* of the Markov process (the origin of this term will be explained later).

In the case of a Brownian motion in $E$ we regard $i$ as an embedding of $E$ in $\mathcal{U}$. In the general situation $\mathcal{U}$ need not contain $i(E)$, and distinct points of $E$ may have identical images in $\mathcal{U}$. However, it can be proved

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1 For special $M$-processes the mapping $i$ is continuous.

2 We say that a function is $\gamma$-integrable if it is integrable with respect to the measure $\gamma$.

3 It is clear that $k_{i(y)}^\alpha(x) = K_y^\alpha(x)$ for $m$-almost all $x$.

that $i(y) \in \mathcal{U}$ for $m$-almost all $y \in E$.

The result stated above gives a new solution of Problem C′.

13. The main result about excessive functions is obtained by investigating the final behaviour (the behaviour as $t \to \zeta$) of a certain random process $z_t$ in $\mathscr{E}$.

We begin with the case of a Brownian motion in $E$. Let $p(t, x, \Gamma)$ be its transition function. In 1957 Doob proved [4] that for any positive superharmonic function $h$ in $E$ a Markov process can be constructed with continuous paths and transition function

$$p^h(t, x, \Gamma) = \frac{1}{h(x)} \int_\Gamma p(t, x, dy)\, h(y).$$

This is usually called an *h-process*. (For $h = 1$ we revert to the original process.)

It may be assumed that all $h$-processes have the same paths $x_t(\omega)$ $(\omega \in \Omega,\ 0 \leqslant t < \zeta(\omega))$ and differ only in the collections of measures $\mathrm{P}_x^h$ in the space $\Omega$. Doob proved that:

a) for $\mathrm{P}_x^h$-almost all $\omega$ in the Martin topology there exists the limit

$$x_{\zeta-0}(\omega) = \lim_{t \uparrow \zeta} x_t(\omega) \in \mathscr{E};$$

b) for all Borel sets $\Gamma$ in $\mathscr{E}$

$$\mathbf{P}_x^h\{x_{\zeta-0} \in \Gamma\} = \frac{1}{h(x)} \int_\Gamma k_z(x)\, \mu(dz), \tag{20}$$

where the functions $k_z$ and the measure $\mu$ are the same as in the integral representation (8) of $h$ (in particular, $\mu(\mathscr{E} \setminus \mathcal{U}) = 0$).

(8) may be regarded as a particular case of (20) for $\Gamma = U$.

In 1959 Doob [5] extended these results to Markov processes with discrete time and a countable set of states. In 1960 Hunt [14] showed[1] that for such discrete processes (20) can be obtained before the Martin decomposition (8), so that the entire Martin theory can be deduced by means of the study of the final behaviour of $x_t$.

14. We start from an arbitrary Markov process $x_t$ with transition function $p(t, x, \Gamma)$. To speak of the final behaviour of $x_t$ has no meaning, because without changing the transition function for each $t$ the value of $x_t(\omega)$ can be changed on any set $A_t \subseteq \Omega$ such that $\mathrm{P}_x(A_t) = 0$ for all $x \in E$. However, we construct in the Martin compactum $\mathscr{E}$ a left-continuous process $z_t(\omega)$ $(\omega \in \Omega,\ 0 \leqslant t \leqslant \zeta(\omega))$ having the following properties: for any $x \in E$, $t > 0$ and any $\gamma$-integrable excessive function[2] $h$ we have $\mathrm{P}_x^h\{z_t \neq i(x_t)\} = 0$. The process $z_t$ is determined in the main uniquely, and the question of its limit behaviour as $t \to \zeta$ is completely meaningful.

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[1] A detailed account of Hunt's method is contained in [10].

[2] If $h = 0$ or $\infty$ anywhere, then certain technical complications arise. So as not to divert the reader's attention, we assume in the introduction that $h$ is everywhere positive and finite.

We prove that:

a) *the limit* $z_\zeta = \lim_{t \uparrow \zeta} z_t$ *exists in the topology of the compactum* $\mathscr{E}$;

b) *for any* $\varphi \in W$ *and any* $\alpha \in R$

$$\int_E m(dx)\,\varphi(x)\,h(x) \int_{z_\zeta \in \Gamma} e^{-\alpha\zeta}\mathbf{P}_x^h(d\omega) = \int_\Gamma \Big[\int_E \varphi(x)\,\mathscr{K}_z^\alpha(dx)\Big]\,\mu_h(dz), \tag{21}$$

*where* $\mathscr{K}_z^\alpha(dx)$ *are measures in* $E$ *corresponding to the point* $z \in \mathscr{E}$, *and* $\mu_h$ *is the so-called spectral measure defined by the formula*

$$\mu_h(A) = \int_E \gamma(dx)\,h(x)\,\mathbf{P}_x^h\{z_\zeta \in A\};$$

c) *any spectral measure* $\mu_h$ *is concentrated on a Borel subset* $\mathscr{E}_1$ *of the compactum* $\mathscr{E}$, *whose points all have the following property:* $\mathscr{K}_z^\alpha(dx) = k_z^\alpha(x)\,m(dx)$, *where* $k_z^\alpha$ *is a* $\gamma$*-integrable* $\alpha$*-excessive function*;

d) *for* $\mu_h$*-almost all points* $z \in \mathscr{E}_1$ *the spectral measure of the excessive function* $k_z = k_z^0$ *is given by*

$$\mu_{k_z}(\Gamma) = \begin{cases} 1 & \text{for } z \in \Gamma, \\ 0 & \text{for } z \notin \Gamma. \end{cases} \tag{22}$$

We denote by $\mathscr{U}$ the set of points $z \in \mathscr{E}_1$, for which (22) holds. By c) and d) it follows from (21) that

$$h(x) \int_{z_\zeta \in \Gamma} e^{-\alpha\zeta}\mathbf{P}_x^h(d\omega) = \int_{\Gamma \cap \mathscr{U}} k_z^\alpha(x)\,\mu_h(dz). \tag{23}$$

In particular, for $\Gamma = \mathscr{E}$

$$h(x)\,\mathbf{M}_x^h e^{-\alpha\zeta} = \int_{\mathscr{U}} k_z^\alpha(x)\,\mu_h(dx), \tag{24}$$

and, putting $\alpha = 0$, we have

$$h(x) = \int_{\mathscr{U}} k_z(x)\,\mu_n(dz).$$

Thus, we have arrived at the integral representation (19) of an arbitrary $\gamma$-integrable excessive function. To prove the main result stated in 12. about excessive functions completely it remains to verify that the measure $\mu$ in (19) is uniquely determined by $h$. This is done in §10.

From what has been said, the term "*space of exits*" we have introduced for the set $\mathscr{U}$ becomes meaningful: with probability 1 (with respect to any measure $P_x^h$) the path $z_t$ leaves in one of the points of $\mathscr{U}$.

15. The results presented in 12. and 14. are proved in §§7 – 10. The first six sections deal with the necessary preparation. In §1 we introduce $\alpha$-excessive functions and give the main definition not in terms of the operators $P_t$, but in terms of the Green operators

$$G_\alpha f(x)=\int_E g_\alpha(x,\ dy)\,f(y)=\int_0^\infty e^{-\alpha t}P_t f(x)\,dt.$$

The equivalence of this definition and the definition by operators $P_t$ is proved in Lemma 1.3. In §2 we give a precise definition of a Markov process, define $\Lambda$-continuity, and establish the connection between excessive functions and supermartingales. In §3 we study $L$-moments (intuitively: random moments $\tau$ that do not depend on the "past", that is, on the course of the process up to the moment $\tau$). This notion was first introduced by Nagasawa [19]. The Main Lemma 3.2 is a variant of a result of Nagasawa (a form of this lemma closer to ours is contained in [2]). In §4 we prove that if $f$ is the density of the measure[1] $\mu G$ with respect to the measure $\gamma G$ and $\tau$ is a $L$-moment, then under certain conditions $f(x_{\tau-t})$ can be regarded as a supermartingale. This is probably the main idea of the Hunt boundary theory of discrete Markov chains ([14], see also [10]). In §5 we introduce the initial notion of a $M$-process and its constituent notions of a base system, Green function, standard measure (see 8. – 11. of this Introduction). In §6 we study $h$-processes, introduce the Martin function $K_z^\alpha(x)$, prove the existence of and investigate the properties of the left-continuous modification of the function

$$\int_E K_{x_t}^\alpha(x)\,\varphi(x)\,m(dx)\qquad(\varphi\in W).$$

In §7 we construct the Martin compactum and in it the left-continuous process $z_t$ (its connection with $x_t$ is stated in 14). We prove the existence of the limit of $z_t$ for $t\to\zeta$. The main content of §§8 – 10 is described in 14. Also, in §8 the functions $g_\alpha(x,\ y)$ and $k_z^\alpha(x)$ are extended continuously to all real non-negative values of $\alpha$ so that (23) – (24) remain valid. In §10 we find the general form of minimal excessive functions (that is, functions not having non-trivial decompositions as the sum of two excessive functions). We prove that all these functions are given by the formula $ak_z$, where $z\in\mathcal{U}$ and $a$ is a non-negative constant.

In §11 the space of exits $\mathcal{U}$ is split[2] into two parts: *the set of unattainable exits* $\mathcal{U}_0$ and *the set of attainable exits* $\mathcal{U}_a$ : to within a set for which all $P_x^h$-measures are zero, $z_\zeta\in\mathcal{U}_0$ for $\zeta=\infty$ and $z_\zeta\in\mathcal{U}_a$ for $\zeta<\infty$. We prove that if $\mu_h(\mathcal{U}_a)=0$, then $P_t h=h$ for all $t$, and if

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[1] The measure $\mu G$ is defined by

$$(\mu G)(\Gamma)=\int_E \mu(dx)\,g(x,\ \Gamma).$$

[2] This decomposition plays an important role in the investigation of general boundary conditions. For processes with a countable set of states it was found by other methods in Feller's paper [11] (see also [9]).

$\mu_h(\mathcal{U}_0) = 0$, then $P_t h \to 0$ as $t \to \infty$. Next we prove that for $\alpha > 0$ the formula

$$h(x) = \int_{\mathcal{U}_a} k_z^\alpha(x)\,\mu(dx)$$

gives the general form of $\alpha$-excessive functions[1] $h$ that are integrable with respect to the measures $\gamma$ and $\gamma G$.

The concluding §12 is concerned with an analysis of the notion of a harmonic function. With each closed subset $\Gamma$ of the compactum $\mathcal{E}$ we associate an operator $P_\Gamma$ acting on excessive functions. To each system $\mathcal{A}$ of closed subsets of $\mathcal{E}$ there corresponds the class of $\mathcal{A}$-*harmonic functions*: these are excessive functions invariant with respect to all operators $P_\Gamma$ $(\Gamma \in \mathcal{A})$. We prove that if $\mathcal{A}$ is the system of all closed subsets of an open set $A$, then the class of $\mathcal{A}$-harmonic functions coincides with that of excessive functions $h$ for which the spectral measure $\mu_h$ is concentrated on the complement of $A$.

For a special $M$-process we can introduce (under certain additional restrictions) a more usual notion of a function harmonic in a domain $A$ of a space $E$; these are $\mathcal{A}$-harmonic functions, where $\mathcal{A}$ is the family of images[2] $i(\Gamma)$ of all compact sets $\Gamma \subseteq A$.

16. The transfer of the Martin theory to Markov processes with continuous time is the subject of papers of Kunita and Watanabe [15], [16]. The processes investigated by these authors are certain subclasses of special $M$-processes, and from our general theory the results of [15] and [16] follow at once.

If $i$ maps $E$ one-to-one onto $i(E)$, then $z_t$ is itself a Markov process, and can be regarded as the regularization of the initial process $x_t$. For the case when $E$ is countable, Doob [6] recently constructed a dual regularization, naturally connected with the space of entries.

We propose to devote another paper to the investigation of the space of entries and its connection with the space of exits.

17. The Appendix at the end of the paper contains the necessary information on measurability, measures and integrals, and also the main facts from the theory of supermartingales. Proofs to be found in easily available sources are omitted, but we state exactly where to find them.

For a reading of the paper a knowledge of the standard university courses on probability theory and measure theory suffices. It is also very useful as a preliminary to be acquainted with the theory of the discrete case [10]: in the general case the same guiding ideas work, but under vastly more complicated conditions.

A short account of the results of this paper was published in [12].

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1 For we have $z \in \mathcal{U}_0$, $k_z^\alpha = 0$ for all $\alpha > 0$.

2 Since $i$ is continuous, $i(\Gamma)$ is closed.

## §1. Transition Function. The Green Kernel. Excessive Functions

**1.1.** We start from a certain measurable space $(E, \mathscr{B})$. Let $V = V(E, \mathscr{B})$ be the set of all non-negative[1] $\mathscr{B}$-measurable functions from $E$, and $V^* = V^*(E, \mathscr{B})$ the set of all measures on the $\sigma$-algebra $\mathscr{B}$.

A function $u(x, \Gamma)$ $(x \in E, \Gamma \in \mathscr{B})$ is called *a kernel* if $u(x, \cdot) \in V^*$ for each $x$ and $u(\cdot, \Gamma) \in V$ for each $\Gamma$.

A one-parameter family of kernels $p(t, x, \Gamma)$ $(t \geqslant 0)$ is called *a transition function* if

**1.1.A.** $$p(t, x, E) \leqslant 1 \qquad (t \geqslant 0, x \in E).$$

**1.1.B.** $$\int_E p(s, x, dy)\, p(t, y, \Gamma) = p(s+t, x, \Gamma).$$

We also assume that the following conditions are satisfied:

**1.1.C.** For each $\Gamma \in \mathscr{B}$ $p(t, x, \Gamma)$ is $\mathscr{B}^0_\infty \times \mathscr{B}$-measurable[2] with respect to $(t, x)$.

**1.1.D.** $\lim\limits_{t \downarrow 0} p(t, x, E) = 1$ (normalization).

In the presence of **1.1.C** we can construct *the Green kernel*

$$g_\alpha(x, \Gamma) = \int_0^\infty e^{-\alpha t} p(t, x, \Gamma)\, dt \qquad (\alpha \geqslant 0). \tag{1.1}$$

Instead of $g_0(x, \Gamma)$ we write $g(x, \Gamma)$.

With each kernel $u(x, \Gamma)$ we associate the operators $f \to Uf$ in $V$ and $\mu \to \mu U$ in $V^*$. They are defined by the formulae

$$Uf(x) = \int_E u(x, dy)\, f(y), \qquad (\mu U)(\Gamma) = \int_\Gamma \mu(dx)\, u(x, \Gamma).$$

The operators corresponding to the kernels $p(t, x, \Gamma)$ and $g_\alpha(x, \Gamma)$ are denoted by $P_t$ and $G_\alpha$, respectively. We note that

$$G_\alpha = \int_0^\infty e^{-\alpha t} P_t\, dt. \tag{1.2}$$

We put $\| f \| = \sup | f(x) |$. By **1.1.A**

$$P_t f \leqslant \| f \|, \qquad \alpha G_\alpha f \leqslant \| f \|, \tag{1.3}$$

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[1] All functions considered in this paper are non-negative. Therefore we do not as a rule mention non-negativity henceforth. Apart from non-negative numerical values the function admits the value $\infty$. We take: a) $\infty + a = a + \infty = \infty$ for any $a$; b) $a \cdot \infty = \infty \cdot a = \infty$ if $a \neq 0$; c) $0 \cdot \infty = \infty \cdot 0 = 0$. We take the product of two functions to be zero wherever at least one of the factors is zero; even if the second factor is undefined, (in accordance with c)), it may be taken as $\infty$.

[2] $\mathscr{B}^0_\infty = \mathscr{B}\,[0, \infty]$ is the $\sigma$-algebra of Borel subsets on the half-line $[0, \infty]$.

so that if $f$ is bounded, then $P_t f$ and $G_\alpha f(\alpha > 0)$ are everywhere finite. By (1.1) it is evident that if $G_\alpha f(x) < \infty$ for any $\alpha$, then as $\alpha \to \infty$

$$G_\alpha f(x) \downarrow 0. \tag{1.4}$$

In operator form, 1.1.B means that $P_s P_t = P_{s+t}$. Hence we deduce the following equations for the operators $G_\alpha$:

$$G_\alpha f = G_{\lambda+\alpha} f + \lambda G_\alpha G_{\lambda+\alpha} f \qquad (\alpha,\ \lambda \geqslant 0), \tag{1.5}$$

$$G_\alpha f = G_{\lambda+\alpha} f + \lambda G_{\lambda+\alpha} G_\alpha f \qquad (\alpha,\ \lambda \geqslant 0). \tag{1.6}$$

From (1.5) and (1.4) it follows that if $f$ is bounded, then

$$\lim_{\lambda \to \infty} \lambda G_\alpha G_{\lambda+\alpha} f = G_\alpha f.$$

Consequently,[1] for any $f \in V$ and any $n$

$$G_\alpha (f \wedge n) = \lim_{\lambda \to \infty} \lambda G_\alpha G_{\lambda+\alpha} (f \wedge n) \leqslant \varliminf_{\lambda \to \infty} \lambda G_\alpha G_{\lambda+\alpha} f$$

and hence

$$G_\alpha f \leqslant \varliminf_{\lambda \to \infty} \lambda G_\alpha G_{\lambda+\alpha} f. \tag{1.7}$$

Similarly, from (1.6) we deduce that for any $f \in V$

$$G_\alpha f \leqslant \varliminf_{\lambda \to \infty} \lambda G_{\lambda+\alpha} G_\alpha f. \tag{1.8}$$

1.2. A function $f \in V$ is called *$\alpha$-excessive* if as $\lambda \to \infty$

$$\lambda G_{\lambda+\alpha} f \uparrow f. \tag{1.9}$$

0-excessive functions are briefly called *excessive*. Condition (1.9) can be broken up into two parts:

1.2.A. $\lambda G_{\lambda+\alpha} f \leqslant f$ for any $\lambda > 0$.

1.2.B. $\lambda G_{\lambda+\alpha} f \to f$ as $\lambda \to \infty$.

From (1.6) it is evident that $G_\alpha f$ satisfies 1.2.A for any $f \in V$. By (1.8) it satisfies also 1.2.B. Thus, $G_\alpha f$ is $\alpha$-excessive.

We call *null-sets* those sets $\Gamma \in \mathscr{B}$ for which $g(x, \Gamma) = 0$ for all $x \in E$. From (1.6) it follows that for such sets $g_\alpha(x, \Gamma) = 0$ for all $\alpha \geqslant 0$.

We call *$\alpha$-preexcessive* those functions that satisfy the following weaker variant of 1.2.A:

1.2.A′. For any $\lambda > 0$, $\{x\colon \lambda G_{\lambda+\alpha} f(x) > f(x)\}$ is a null-set.

LEMMA 1.1. *Let $f_1$, $f_2$ be $\alpha$-excessive functions. If $f_1 \leqslant f_2$ outside some null-set, then $f_1 \leqslant f_2$ everywhere. If $f_1 = f_2$ outside some null-set, then $f_1 = f_2$ everywhere.*

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[1] $a \wedge b$ denotes the smaller of the numbers $a$, $b$, and $a \vee b$ the larger.

For the proof it is sufficient to note that if $f_1 \leqslant f_2$ outside a certain null-set, then $\lambda G_{\lambda+\alpha} f_1 \leqslant \lambda G_{\lambda+\alpha} f_2$ everywhere for any $\lambda > 0$.

LEMMA 1.2. *If $f$ is $\alpha$-preexcessive, then $\lambda G_{\lambda+\alpha} f$ is a non-decreasing function of $\lambda$. The limit*

$$\widetilde{f} = \lim_{\lambda\to\infty} \lambda G_{\lambda+\alpha} f$$

*(which is called the regularization of the function $f$) is $\alpha$-excessive.*

PROOF. We show that for $\lambda_1 \geqslant \lambda_2 \geqslant 0$

$$\lambda_1 G_{\lambda_1+\alpha} f \geqslant \lambda_2 G_{\lambda_2+\alpha} f. \tag{1.10}$$

In (1.6), replacing $\alpha$ by $\lambda_2 + \alpha$ and $\lambda$ by $\lambda_1 - \lambda_2$, we get

$$G_{\lambda_2+\alpha} f = G_{\lambda_1+\alpha} f + (\lambda_1 - \lambda_2) G_{\lambda_1+\alpha} G_{\lambda_2+\alpha} f. \tag{1.11}$$

By 1.2.A$'$, for all $x \in E$

$$\lambda_2 G_{\lambda_1+\alpha} G_{\lambda_2+\alpha} f \leqslant G_{\lambda_1+\alpha} f. \tag{1.12}$$

(1.10) is evidently satisfied if $G_{\lambda_1+\alpha} f(x) = \infty$. But if $G_{\lambda_1+\alpha} f(x) < \infty$, then by (1.11) and (1.12)

$$\lambda_2 G_{\lambda_2+\alpha} f(x) \leqslant \lambda_2 G_{\lambda_1+\alpha} f(x) + (\lambda_1 - \lambda_2) G_{\lambda_1+\alpha} f(x) = \lambda_1 G_{\lambda_1+\alpha} f(x).$$

From (1.8)

$$G_{\lambda_2+\alpha} f \leqslant \varliminf_{\lambda_1\to\infty} (\lambda_1 - \lambda_2) G_{\lambda_2+\alpha} G_{\lambda_1+\alpha} f \leqslant \varliminf_{\lambda_1\to\infty} \lambda_1 G_{\lambda_2+\alpha} G_{\lambda_1+\alpha} f = G_{\lambda_2+\alpha} \widetilde{f}.$$

On the other hand, by (1.12) $G_{\lambda_1+\alpha} \widetilde{f} \leqslant G_{\lambda_1+\alpha} f$. Hence $G_{\lambda+\alpha} \widetilde{f} = G_{\lambda+\alpha} f$ for any $\lambda$, and as $\lambda \to \infty$

$$\lambda G_{\lambda+\alpha} \widetilde{f} = \lambda G_{\lambda+\alpha} f \uparrow \widetilde{f}.$$

COROLLARY. *For each $\alpha$-excessive function $f$ there exists a sequence of bounded $\alpha$-excessive functions $f_n$ such that $f_n \uparrow f$.*

The functions $F_n = f \wedge n$ are $\alpha$-preexcessive. Their regularizations $f_n = \widetilde{F}_n$ are bounded. From $F_n \leqslant F_{n+1} \leqslant f$ it follows that $f_n \leqslant f_{n+1} \leqslant f$. Let $f_n \uparrow F$. Evidently $F \leqslant f$. On the other hand $f_n \geqslant \lambda G_{\lambda+\alpha} F_n$ for any $\lambda > 0$, hence $F \geqslant \lambda G_{\lambda+\alpha} f$. Letting $\lambda \to \infty$ we get $F \geqslant f$. Thus, $F = f$.

1.3. The notion of an $\alpha$-excessive function can be defined not only in terms of operators $G_\alpha$, but also in terms of operators $P_t$.

LEMMA 1.3. *For $f$ to be $\alpha$-excessive it is necessary and sufficient that the following conditions are satisfied:*

1.3.A. $e^{-\alpha t} P_t f \leqslant f$ *for all* $t \geqslant 0$.

1.3.B. $e^{-\alpha t} P_t f \to f$ *for* $t \downarrow 0$.

PROOF. For brevity we put $P_t^\alpha = e^{-\alpha t}P_t$. By (1.2)

$$\lambda G_{\lambda+\alpha} f = \lambda \int_0^\infty e^{-\lambda t} P_t^\alpha f\, dt = \int_0^\infty e^{-u} P_{u/\lambda}^\alpha f\, du.$$

Hence conditions 1.2.A - 1.2.B follow from 1.3.A - 1.3.B.

We note that if $f_n \uparrow f$ and $f_n$ satisfy 1.3.A - 1.3.B, then $f$ also satisfies these conditions: it is evident that 1.3.A holds, and the validity of 1.3.B follows from

$$f_n = \lim_{t\downarrow 0} P_t^\alpha f_n \leqslant \varliminf_{t\downarrow 0} P_t^\alpha f \leqslant \varlimsup P_t^\alpha f \leqslant f.$$

We prove that if $f$ is $\alpha$-excessive, then $f$ satisfies conditions 1.3.A - 1.3.B. By the Corollary to Lemma 1.2. it is sufficient to consider the case when $f$ is bounded. We assume first that $\alpha > 0$. Then by (1.3) $G_\alpha f$ is finite. From (1.5) it follows that $f_\lambda = \lambda G_{\lambda+\alpha} f = G_\alpha \varphi_\lambda$, where $\varphi_\lambda = \lambda[f - \lambda G_{\lambda+\alpha} f] \geqslant 0$. We note that

$$P_t^\alpha f_\lambda = \int_t^\infty e^{-\alpha s} P_s \varphi_\lambda ds.$$

From this formula it is clear that $f_\lambda$ satisfies 1.3.A - 1.3.B. But $f_\lambda \uparrow f$ as $\lambda \to \infty$, and by what was proved earlier, $f$ also satisfies 1.3.A - 1.3.B.

We note that an excessive function is $\alpha$-excessive for all $\alpha > 0$. By what we have proved it satisfies 1.3.A - 1.3.B for all $\alpha > 0$. Hence it easily follows that it satisfies these conditions also when $\alpha = 0$.

COROLLARY. *If $h$ is $\alpha$-excessive, then $P_s h$ is also $\alpha$-excessive.*

This is easily seen, for if $h$ satisfies 1.3.A - 1.3.B, then $P_s h$ also satisfies these conditions.

## §2. Markov Processes

2.1. Let there be given:

a) two measurable spaces $(E, \mathscr{B})$ (phase space) and $(\Omega, \mathscr{M})$ (space of elementary events) (it is assumed that $\mathscr{B}$ contains all one-point subsets);

b) a $\mathscr{M}$-measurable function $\zeta(\omega)$ with range in $[0, \infty]$ (the moment of cut-off);

c) a function $x_t(\omega)$ defined for $\omega \in \Omega$, $t \in [0, \zeta(\omega))$ with range in $E$;

d) for each $t \geqslant 0$, a $\sigma$-algebra $\mathscr{M}_t$ in the space $\Omega_t = \{\omega: \xi(\omega) > t\}$, where $\mathscr{M}_t \subseteq \mathscr{M}$, $\mathscr{M}_t$ contains all sets of the form $\{\omega: x_t(\omega) \in \Gamma\}$ $(\Gamma \in \mathscr{B})$ and if $A \in \mathscr{M}_t$, then $A \cap \Omega_u \in \mathscr{M}_u$ for any $u \geqslant t$;

e) for each $x \in E$ a probability measure $P_x$ on the $\sigma$-algebra $\mathscr{M}$;

f) for each $t \geqslant 0$ a mapping $\omega \to \omega_t$ of $\Omega_t$ into $\Omega$ such that $\zeta(\omega_t) = \zeta(\omega) - t$ and

$$x_u(\omega_t) = x_{u+t}(\omega) \qquad (0 \leqslant u < \zeta(\omega_t)).$$

We put $\theta_t A = \{\omega: \omega_t \in A\}$. For any numerical function $\xi(\omega)$ $(\omega \in \Omega)$ we put

$$\theta_t \xi(\omega) = \xi(\omega_t) \qquad (\omega \in \Omega_t).$$

(We note that $\theta_t f(x_s) = f(x_{s+t})$ on $\Omega_{s+t}$.) We denote by $\mathcal{N}$ the $\sigma$-algebra in $\Omega$ generated by the sets $\{\omega: x_t(\omega) \in \Gamma\}$ $(t \geqslant 0,\ \Gamma \in \mathscr{B})$. The elements listed form *a Markov process* $X = (x_t, \zeta, \mathscr{M}_t, \mathbf{P}_x)$, if:

2.1.A. For any $\mathcal{N}$-measurable $\xi$ the function[1] $M_x\xi$ is $\mathscr{B}$-measurable and, whatever the $\mathscr{M}_t$-measurable $\eta$ may be,

$$\mathbf{M}_x(\eta\theta_t\xi) = \mathbf{M}_x(\eta \mathbf{M}_{x_t}\xi). \tag{2.1}$$

2.1.B.

$$\mathbf{P}_x\{x_0 \in E \setminus \{x\}\} = 0 \quad (x \in E).$$

With each finite measure $\mu$ on the $\sigma$-algebra $\mathscr{B}$ there is connected a measure $\mathbf{P}_\mu$ on the $\sigma$-algebra $\mathcal{N}$ defined by the formula

$$\mathbf{P}_\mu(A) = \int_E \mu(dx)\, \mathbf{P}_x(A). \tag{2.2}$$

The integral with respect to this measure is denoted by $\mathbf{M}_\mu$.

We denote by $\mathcal{N}_t$ the $\sigma$-algebra in $\Omega_t$ generated by the sets $\{\omega: x_s \in \Gamma\}$ $(0 \leqslant s \leqslant t,\ \Gamma \in \mathscr{B})$. We put $A \in \mathscr{F}_t$ if $A \in \mathscr{M}$ and $A \cap \Omega_t \in \mathscr{M}_t$. It is evident that $\mathscr{F}_t$ is a $\sigma$-algebra in $\Omega$, and for $\xi$ to be $\mathscr{F}_t$-measurable it is necessary and sufficient that $\xi$ is $\mathscr{M}$-measurable and its restriction to $\Omega_t$ is $\mathscr{M}_t$-measurable.

From 2.1.A and (2.2) it follows that if $\xi$ is $\mathcal{N}$-measurable and $\eta$ is $\mathcal{N}_t$-measurable, then for any measure $\mu$

$$\mathbf{M}_\mu(\eta\theta_t\xi) = \mathbf{M}_\mu(\eta \mathbf{M}_{x_t}\xi). \tag{2.3}$$

2.2. To the Markov process $X$ the corresponding transition function is

$$p(t, x, \Gamma) = \mathbf{P}_x\{x_t \in \Gamma\}. \tag{2.4}$$

*We shall discuss, without saying so each time, only processes X for which the transition function satisfies in addition to* 1.1.A – 1.1.B *also the conditions* 1.1.C – 1.1.D.

We note that for any measure $\gamma$ and any function $f \in V$

$$(\gamma P_t)(f) = \gamma(P_t f) = \mathbf{M}_\gamma f(x_t), \tag{2.5}$$

$$(\gamma G_\alpha)(f) = \gamma(G_\alpha f) = \int_0^\infty e^{-\alpha t}\mathbf{M}_\gamma f(x_t)\, dt. \tag{2.6}$$

---

[1] $M_x\xi$ denotes the integral of $\xi$ with respect to the measure $P_x$. If $\xi$ is defined not on the whole space $\Omega$ but on some set $\tilde{\Omega} \in \mathscr{M}^0$, then $M_x\xi$ means the integral of $\xi$ on its domain of definition; in other words, we extend $\xi$, putting $\xi = 0$ not on $\tilde{\Omega}$. In particular, $f(x_t)$ is taken to be zero outside $\Omega_t$ (where $x_t$ is undefined).

For the most important classes of measurable spaces $(E, \mathscr{B})$ every transition function $p(t, x, \Gamma)$ $(x \in E, \Gamma \in \mathscr{B})$ corresponds to some Markov process in $(E, \mathscr{B})$. Spaces $(E, \mathscr{B})$ having this property are called *perfect*.

The space $(E, \mathscr{B})$ is perfect, for example, in the following cases:

a) $E$ is a metrizable separable locally compact topological space, $\mathscr{B}$ is the $\sigma$-algebra of its Borel subsets or its completion with respect to the system of all finite measures $\mu$. (See [7], Theorem 4.2).

b) $E$ is generated by a countable number of sets and each $\mathscr{B}$-measurable numerical function $f$ maps $E$ onto some analytic set on the number line (the Luzin space, in the terminology of Blackwell [1]).

We note that if $(E, \mathscr{B})$ is a perfect space and if $\tilde{E} \in \mathscr{B}$, $\tilde{\mathscr{B}}$ is the family of all sets $\Gamma \in \mathscr{B}$, contained in $\tilde{E}$, then $(\tilde{E}, \tilde{\mathscr{B}})$ is also perfect.

*Henceforth we assume that the space $(E, \mathscr{B})$ is perfect.*

Among all Markov processes with a given transition function $p(t, x, \Gamma)$ there exists one *canonical* process $X = (x_t, \zeta, \mathscr{M}_t, \mathbf{P}_x)$, having the following properties:

a) the space $\Omega$ consists of all possible functions $\omega(t)$ with range in $E$, defined on all possible intervals $[0, \lambda)$ $(0 < \lambda \leqslant \infty)$;

b) if $\omega = \omega(t)$ $(0 \leqslant t < \lambda)$, then $\zeta(\omega) = \lambda$, $x_t(\omega) = \omega(t)$;

c) $\mathscr{M} = \mathscr{N}$, $\mathscr{M}_t = \mathscr{N}_t$;

d) if $\omega = \omega(t)$ $(0 \leqslant t < \lambda)$ and $s < \lambda$, then $\omega_s(t) = \omega(s + t)$ $(0 \leqslant t < \lambda - s)$.

**2.3.** Let $\mu$ be a measure in some space $\mathscr{E}$. We assume that $\mu(A) = 0$ and that all points $x \in \mathscr{E} \quad A$ have a certain property $\mathscr{P}$. Then we say that the property $\mathscr{P}$ holds *for $\mu$-almost all $x$* or *$\mu$-almost everywhere*. If the measure $\mu$ is probabilistic, we also use the term "*$\mu$-almost surely*".

If a certain condition is satisfied $P_x$-almost surely, whatever $x \in E$ may be, then we say simply that this condition is satisfied *almost surely*.

We agree to say that $f \in V$ is *$\Lambda$-continuous* if there exists a function $F(t, \omega)$ $(\omega \in \Omega, 0 < t < \zeta(\omega))$ with the following properties:

**2.3.A.** The function $F(t, \omega)$ is finite and continuous on the left with respect to $t$ on the interval $(0, \zeta(\omega))$.

**2.3.B.** For any $t > 0$, $F(t, \omega)$ is $\mathscr{N}_t$-measurable and almost surely $f[x_t(\omega)] = F(t, \omega)$ (on the set $\Omega_t$).

We call $F(t, \omega)$ a *left-continuous modification of $f(x_t)$*. Since $\mathscr{N}_t \subseteq \mathscr{N}$, from 2.3.A and[1] 0.2 we have

**2.3.C.** $F(t, \omega)$ is $\mathscr{B}^0_\infty \times \mathscr{N}$-measurable.

**2.4.** LEMMA 2.1. *Let $h$ be an $\alpha$-excessive function and $\mu$ a finite measure. Then, $(e^{-\alpha t} h(x_t), \mathscr{F}_t, \mathbf{P}_\mu)$ is a supermartingale. There exists a function $H(t, \omega)$ $(t \geqslant 0, \omega \in \Omega)$ with the following properties:*

**2.4.A.** *For all $\omega \in \Omega$, $H(t, \omega)$ is continuous on the right with respect to $t$ and is zero for $t \geqslant \zeta(\omega)$.*

**2.4.B.** *For any $t \geqslant 0$, $H(t, \omega)$ is $\mathscr{N}$-measurable. If $\mu(h) < \infty$, the set of values of $t$ for which*

$$\mathbf{P}_\mu\{H(t, \omega) \neq h(x_t)\} \neq 0$$

---

[1] Reference numbers beginning with 0 refer to the Appendix at the end of the paper.

*is at most countable.*

2.4.C. *$H(t, \omega)$ is $\mathscr{B}^0_\infty \times \mathscr{N}$-measurable.*

2.4.D. *If $\mu(h) < \infty$, then $\mathsf{P}_\mu$-almost surely*[1] *$\theta_u H(t, \omega) = H(t+u, \omega)$ for all $t \geqslant 0$.*

PROOF. It is evident that $\mathscr{F}_s \subseteq \mathscr{F}_t$ for $s \leqslant t$ and that the function $Z_t = e^{-\alpha t} h(x_t)$ is $\mathscr{F}_t$-measurable. By 1.3, $M_x Z_t = e^{-\alpha t} P_t h(x) \leqslant h(x)$. If $\xi$ is $\mathscr{F}_s$-measurable, then its restriction $\xi'$ to $\Omega_s$ is $\mathscr{M}_s$-measurable, and by (2.3) for $t \geqslant 0$

$$\mathbf{M}_x \xi Z_{s+t} = \mathbf{M}_x \xi' e^{-\alpha s} \theta_s Z_t = \mathbf{M}_x \xi' e^{-\alpha s} \mathbf{M}_{x_s} Z_t \leqslant \mathbf{M}_x \xi' e^{-\alpha s} h(x_s) = \mathbf{M}_x \xi' Z_s = \mathbf{M}_x \xi Z_s.$$

Hence $(Z_t, \mathscr{F}_t, \mathbf{P}_x)$ is a supermartingale.

We consider now the function $\hat{Z}_t$ defined in Theorem 0.2 and put $H(t, \omega) = e^{\alpha t} \hat{Z}_t$. We note that for all $t \geqslant 0$

$$\mathbf{M}_\mu Z_t = e^{-\alpha t} \mu(P_t h) \leqslant \mu(h).$$

2.4.A follows from 0.7.A. 2.4.B follows from 0.7.B and 0.7.G. 2.4.C is satisfied by 0.2.

We prove 2.4.D. If $\mu(h) < \infty$ and $\mu_u = \mu P_u$, then $\mu_u(h) = \mu(P_u h) \leqslant \mu(h) < \infty$, and by 2.4.B the set $T$ of values of $t$ for which $\mathsf{P}_{\mu_u}\{H(t, \omega) \neq h(x_t)\} \neq 0$ is at most countable. For $t \notin T$, by (2.3),

$$\mathbf{P}_\mu \{\theta_u H(t, \omega) \neq h(x_{t+u})\} = \mathbf{P}_\mu \{\theta_u [H(t, \omega) \neq h(x_t)]\} =$$
$$= \mathbf{M}_\mu \mathbf{P}_{x_u} \{H(t, \omega) \neq h(x_t)\} = \mathbf{P}_{\mu_u} \{H(t, \omega) \neq h(x_t)\} = 0.$$

On the other hand, the set of values of $t$ for which $\mathsf{P}_\mu\{H(t+u, \omega) \neq h(x_{t+u})\} \neq 0$ is also at most countable. Hence the equation $\theta_u H(t, \omega) = H(t+u, \omega)$ is satisfied $\mathsf{P}_\mu$-almost surely simultaneously for all $t$ belonging to a certain countable everywhere dense subset of the half-line $[0, \infty)$. 2.4.D now follows by 2.4.A.

## §3. $L$-moments

3.1. An $\mathscr{N}$-measurable non-negative function $\tau(\omega)$ is called a $L$-moment if:

3.1.A. $0 \leqslant \tau(\omega) \leqslant \zeta(\omega)$.

3.1.B. For any $u \geqslant 0$, almost surely[2] $\theta_u \tau = (\tau - u)^+$.

It is clear that, together with $\tau$, $(\tau - t)^+$ is a $L$-moment for any $t \geqslant 0$.

The moment of cut-off $\zeta$ can serve as an example of a $L$-moment. An important class of $L$-moments is obtained by the following construction. Let $h$ be an everywhere finite $\alpha$-excessive function and $H(t, \omega)$ the regularization of $h(x_t)$ constructed in Lemma 2.1. We put

[1] We call $H(t, \omega)$ the regularization of $h(x_t)$.
[2] We put $a^+ = a \vee 0$.

$$\left.\begin{aligned} A^t(h) &= \int_t^{\zeta} H(s, \omega)\, ds, \\ \tau_a &= \inf\{t : A^t(h) \leqslant a\} \qquad (a \geqslant 0). \end{aligned}\right\} \tag{3.1}$$

Since $A^{\zeta}(h) = 0$, $\tau_a$ satisfies 3.1.A. By 2.4.B, $\{\tau_a \leqslant t\} = \{A^t(h) \leqslant a\} \in \mathcal{N}$ for any $t \geqslant 0$, so that $\tau_a$ is $\mathcal{N}$-measurable. Next, from 2.4.D it follows that almost surely $\theta_u A^t(h) = A^{t+u}(h)$, and so, almost surely,

$$\{\theta_u \tau_a \leqslant t\} = \{\theta_u A^t(h) \leqslant a\} = \{A^{t+u}(h) \leqslant a\} = \{\tau_a \leqslant t + u\}.$$

Hence 3.1.B follows. Thus, $\tau_a$ is a $L$-moment.

LEMMA 3.1. *Let the Green kernel satisfy the following condition:* (G) *There exists a positive function* $\psi \in V$ *such that* $\gamma(G\psi) < \infty$. *Then the* 1*-excessive function* $h = G_1\psi$ *is everywhere positive and*

$$\mathbf{P}_{\gamma}\{A^0(h) = \infty\} = 0. \tag{3.2}$$

*The family of L-moments* $\tau_a$ *corresponding to this function* (see (3.1)) *satisfies the conditions*:

a) *for* $a \downarrow 0$ *almost surely* $\tau_a \uparrow \zeta$;

b) $\mathbf{P}_{\gamma}$*-almost surely,* $\tau_a < \zeta$ *for all* $a > 0$.

PROOF. Let $H(t, \omega)$ be the regularization of $h(x_t)$, constructed in Lemma 2.1. By (3.1), 2.4.C, 2.4.B, (2.6) and (1.5)

$$\mathbf{M}_{\gamma} A^0(h) = \mathbf{M}_{\gamma} \int_0^{\infty} \chi_{\zeta > t} H(t)\, dt = \int_0^{\infty} \mathbf{M}_{\gamma} H(t)\, dt =$$
$$= \int_0^{\infty} \mathbf{M}_{\gamma} h(x_t)\, dt = (\gamma G)(h) = \gamma(G G_1 \psi) \leqslant \gamma(G\psi) < \infty.$$

Hence condition (3.2) is satisfied.

If $h(x) = G_1\psi(x) = 0$, then $g_1(x, E) = 0$, since $\psi$ is positive. However, by (1.1) this contradicts 1.1.D. Thus, $h$ is everywhere positive.

We note that $\tau_a$ is a non-increasing function of $a$. Let $\tau_a \uparrow \tau$ for $a \downarrow 0$. It is clear that

$$\mathbf{P}_x\{\tau < \zeta\} \leqslant \sum_{r \in R} \mathbf{P}_x\{A^r(h) = 0,\ \zeta > r\} \leqslant$$
$$\leqslant \sum_{r \in R} \mathbf{P}_x\{H(r) = 0,\ \zeta > r\} = \sum_{r \in R} \mathbf{P}_x\{h(x_r) = 0\} = 0.$$

Thus, almost surely $\tau_a \uparrow \tau = \zeta$, and so a) is satisfied. From (3.2) it follows that $\mathbf{P}_{\gamma}$ -almost surely $A^t(h) \downarrow 0$ as $t \uparrow \zeta$. Hence condition b) is satisfied.

3.2. For each $L$-moment $\tau$ we define a $\sigma$-algebra $\mathcal{N}^{\tau}_{\infty}$ in $\Omega$ by the condition: $A \in \mathcal{N}^{\tau}_{\infty}$, if $A \in \mathcal{N}$ and for any $u \geqslant 0$ almost surely $\theta_u A \cap \{\tau > u\} = A \cap \{\tau > u\}$. It is easy to see that an $\mathcal{N}$-measurable function $\xi$ is measurable with respect to $\mathcal{N}^{\tau}_{\infty}$ if and only if for any $u \geqslant 0$ $\theta_u \xi = \xi$ almost surely on the set $\{\tau > u\}$. We note that:

3.2.A. If $\tau' \geqslant \tau$, then $\mathcal{N}^{\tau}_{\infty} \supseteq \mathcal{N}^{\tau'}_{\infty}$.

3.2.B. $x_\tau$ is measurable with respect to $\mathcal{N}^{\tau}_{\infty}$; in addition, if $\tau' \geqslant \tau$, then $x_{\tau'}$ is measurable with respect to $\mathcal{N}^{\tau}_{\infty}$.

3.3. Next we prove some Lemmas on $L$-moments.

LEMMA 3.2. *Let $\tau$ be a $L$-moment, $\xi$ be $\mathcal{N}^{\tau}_{\infty}$-measurable, $f \in V$. Then for any measure $\gamma$*

$$\int_0^\infty e^{-\alpha t}\mathbf{M}_\gamma f(x_t)\,\xi\chi_{\tau=\infty}dt = \int_E (\gamma G_\alpha)(dy)\,f(y)\,\mathbf{M}_y\xi\chi_{\tau=\infty}. \tag{3.3}$$

PROOF. For any $t \geqslant 0$, almost surely

$$\theta_t(\xi\chi_{\tau=\infty}) = \xi\chi_{\tau=\infty}$$

and from (2.3)

$$\mathbf{M}_\gamma f(x_t)\,\xi\chi_{\tau=\infty} = \mathbf{M}_\gamma f(x_t)\,\theta_t(\xi\chi_{\tau=\infty}) = \mathbf{M}_\gamma f(x_t)\,\mathbf{M}_{x_t}\xi\chi_{\tau=\infty}.$$

Using (2.6) we now obtain (3.3).

COROLLARY. *The function*

$$F(x) = \mathbf{M}_x\xi\chi_{\tau=\infty}$$

*satisfies for any $\alpha > 0$ the relation*

$$\alpha G_\alpha F(x) = F(x). \tag{3.4}$$

For a proof it is sufficient to apply[1] Lemma 3.2 to $\gamma = \delta_x$ and $f = 1$.

LEMMA 3.3. *Let $\rho$ be a $\mathcal{B}^0_\infty$-measurable function, $\gamma$ a measure on $\mathcal{B}$, $f \in V$, $\tau$ a $L$-moment, $\xi$ be $\mathcal{N}^{\tau}_{\infty}$-measurable. If $F(t) = F(t,\omega)$ is $\mathcal{B}^0_\infty \times \mathcal{M}$-measurable and for all $t > 0$*

$$\mathbf{P}_\gamma\{F(t) \neq f(x_t)\} = 0,$$

*then*[2] *for any $\alpha > 0$*

$$\int_0^\infty \rho(t)\,\mathbf{M}_\gamma e^{-\alpha(\tau-t)}F(\tau-t)\,\chi_{t<\tau<\infty}\xi\,dt = \int_E (\gamma G_\alpha)(dy)\,f(y)\,\mathbf{M}_y\rho(\tau)\,\chi_{0<\tau<\infty}\xi. \tag{3.5}$$

In particular,

$$\int_0^\infty \rho(t)\,\mathbf{M}_\gamma e^{-\alpha(\tau-t)}\chi_{t<\tau<\infty}\xi\,dt = \int_E (\gamma G_\alpha)(dy)\,\mathbf{M}_y\rho(\tau)\,\chi_{0<\tau<\infty}\xi \tag{3.6}$$

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[1] We denote by $\delta_x$ the unit measure concentrated at the point $x$, that is, the measure given by the formula $\delta_x(\Gamma) = \chi_\Gamma(x)$ ($\chi_\Gamma$ is the indicator of a set $\Gamma$, the function equal to 1 on $\Gamma$ and 0 elsewhere).

[2] By 3.1.A $\{t < \tau < \infty\} = \{0 < \tau - t < \zeta\}$. Hence, where $f(x_{\tau} - t)$ is not defined, it is multiplied by zero, and in accordance with 1.1. we take this product to be zero.

and for any $x \in E$

$$\int_0^\infty \rho(t)\, \mathbf{M}_x e^{-\alpha(\tau-t)} F(\tau-t)\, \chi_{t<\tau<\infty}\xi\, dt = \int_E g_\alpha(x, dy)\, f(y)\, \mathbf{M}_y \rho(\tau)\, \chi_{0<\tau<\infty}\xi. \quad (3.7)$$

PROOF. Interchanging (by Fubini's theorem) the operations of integration and mathematical expectation on the left-hand side of (3.5), and then making the change of variable $u = \tau - t$, we obtain

$$\mathbf{M}_\gamma \int_{-\infty}^{+\infty} \rho(t)\, e^{-\alpha(\tau-t)} F(\tau-t)\, \chi_{0<t<\tau<\infty}\xi\, dt =$$

$$= \mathbf{M}_\gamma \int_{-\infty}^{\infty} e^{-\alpha u} F(u)\, \rho(\tau-u)\, \chi_{0<u<\tau<\infty}\xi\, du = \int_0^\infty e^{-\alpha u} c(u)\, du,$$

where

$$c(u) = \mathbf{M}_\gamma F(u)\, \rho(\tau-u)\, \chi_{u<\tau<\infty}\xi = \mathbf{M}_\gamma f(x_u)\, \rho(\tau-u)\, \chi_{u<\tau<\infty}\xi.$$

But $\theta_u \{\rho(\tau)\, \chi_{0<\tau<\infty}\xi\} = \rho(\tau-u)\, \chi_{u<\tau<\infty}\xi$ almost surely. Therefore

$$c(u) = \mathbf{M}_\gamma f(x_u)\, \theta_u \{\rho(\tau)\, \chi_{0<\tau<\infty}\xi\} = \mathbf{M}_\gamma f(x_u)\, \mathbf{M}_{x_u} \rho(\tau)\, \chi_{0<\tau<\infty}\xi.$$

By (2.6), (3.5) hence follows.

To obtain (3.6) it is sufficient to apply (3.5) to $f = 1$ and $F(t, \omega) = \chi_{\zeta>t}$. Lastly, putting $\gamma = \delta_x$ in (3.5), we get (3.7).

3.4. LEMMA 3.4. *If $\xi$ is measurable with respect to $\mathcal{N}^\zeta_\infty$, then*

$$Q_\alpha(x) = \mathbf{M}_x e^{-\alpha\zeta}\xi \quad (3.8)$$

*satisfies for any $\alpha \geqslant 0$, $\lambda > 0$ the relation*

$$\lambda G_{\lambda+\alpha} Q_\alpha(x) = Q_\alpha(x) - Q_{\lambda+\alpha}(x). \quad (3.9)$$

PROOF. We suppose that $\xi$ is bounded. The passage to the general case is evident. We put

$$Q^t_\alpha(x) = \mathbf{M}_x e^{-\alpha(\zeta-t)} \chi_{t<\zeta<\infty}\xi,$$
$$a(x) = \mathbf{M}_x \rho(\zeta)\, \chi_{0<\zeta<\infty}\xi.$$

Applying (3.6) of Lemma 3.3 to $\tau = \zeta$ and the measure $\eta_{\lambda+\alpha}(dy) = g_{\lambda+\alpha}(x, dy)$, we obtain

$$\int_0^\infty \rho(t)\, G_{\lambda+\alpha} Q^t_\alpha(x)\, dt = \int_E \int_E g_{\lambda+\alpha}(x, dz)\, g_\alpha(z, dy)\, a(y) = G_{\lambda+\alpha} G_\alpha a(x).$$

By (1.8), it now follows that

$$\int_0^\infty \rho(t)\,\lambda\, G_{\lambda+\alpha} Q^t_\alpha(x)\,dt = G_\alpha a(x) - G_{\lambda+\alpha} a(x). \tag{3.10}$$

By (3.7)

$$\int_0^\infty \rho(t)\, Q^t_\alpha(x)\,dt = G_\alpha a(x). \tag{3.11}$$

From (3.10) and (3.11) $\int_0^\infty \rho(t)\,\lambda G_{\lambda+\alpha} Q^t_\alpha(x)\,dt = \int_0^\infty \rho(t)\,[Q^t_\alpha(x) - Q^t_{\lambda+\alpha}(x)]\,dt,$

and since $\rho$ is arbitrary, for almost all $t$

$$\lambda G_{\lambda+\alpha} Q^t_\alpha(x) = Q^t_\alpha(x) - Q^t_{\lambda+\alpha}(x). \tag{3.12}$$

First, let $\alpha > 0$. Then, as $t \downarrow 0$ $Q^t_\alpha \to Q_\alpha$ and $\lambda G_{\lambda+\alpha} Q^t_\alpha \to \lambda G_{\lambda+\alpha} Q_\alpha$. Taking the limit in (3.12), we get (3.9).

We now consider the case $\alpha = 0$. Let $F(x) = \mathrm{M}_x \xi \chi_{\zeta=\infty}$. We have $Q_0(x) = \mathrm{M}_x \xi = \lim_{t\downarrow 0} Q^t_0(x) + F(x)$. Hence (3.9) for $\alpha = 0$ follows from (3.12) and (3.4).

## §4. On Densities of Measures $\mu G_\alpha$

4.1. LEMMA 4.1. *Let $\mu$ and $\gamma$ be finite measures. We assume that the density $f_\alpha(x)$ of the measure $\mu G_\alpha$ with respect to the measure $\gamma G$ is $\Lambda$-continuous and bounded, and we denote by $F_\alpha(t, \omega)$ the left-continuous modification of $f_\alpha(x_t)$. Then, for any L-moment $\tau$, any $\mathscr{N}^\tau_\infty$-measurable function $\xi$ and $t > 0$*

$$\mathrm{M}_\gamma F_\alpha(\tau - t)\,\chi_{t<\tau<\infty}\xi = \mathrm{M}_\mu e^{-\alpha(\tau-t)}\xi\chi_{t<\tau<\infty}. \tag{4.1}$$

*In particular,*

$$\mathrm{M}_\gamma F_0(\tau - t)\,\chi_{t<\tau<\infty} = \mathrm{P}_\mu\{t < \tau < \infty\} \leqslant \mu(E). \tag{4.2}$$

PROOF. It is sufficient to consider the case when $\xi$ is bounded. We apply (3.6) of Lemma 3.3 to the measure $\mu$:

$$\int_0^\infty \rho(t)\,\mathrm{M}_\mu e^{-\alpha(\tau-t)}\chi_{t<\tau<\infty}\xi\,dt = \int_E (\mu G_\alpha)(dy)\,\mathrm{M}_y \rho(\tau)\,\chi_{0<\tau<\infty}\xi. \tag{4.3}$$

By hypothesis, $(\mu G_\alpha)(dy) = (\gamma G)(dy)\, f_\alpha(y)$. Hence the right-hand side of (4.3) coincides with that of (3.5) for $\alpha = 0$ (and with the change of $f$ into $f_\alpha$). Since $\rho$ is arbitrary, it follows from (4.3) and (3.5) that (4.1) is satisfied for almost all $t$. By 2.3.A and 2.3.B almost surely

$$\sup_{t>0} F_\alpha(t, \omega) = \sup_{t\in R} F_\alpha(t, \omega) = \sup_{t\in R} f_\alpha(x_t) \leqslant \|f\|.$$

Hence both sides of (4.1) are continuous on the right with respect to $t$ and this equation is satisfied for all $t > 0$.

4.2. THEOREM 4.1. *We assume that condition (G) of 3.1 is satisfied. Let $\mu$ and $\gamma$ be finite measures. We assume that for all $\alpha \in R$ the density $f_\alpha$ of the measure $\mu G_\alpha$ with respect to the measure $\gamma G$ is $\Lambda$-continuous and bounded, and we denote by $F_\alpha(t, \omega)$ the left-continuous modification of $f_\alpha(x_t)$. Then, $\mathbf{P}_\gamma$-almost surely the limits*

$$F(\zeta-0,\ \omega)=\lim_{t\uparrow\zeta} F(t,\ \omega) \tag{4.4}$$

*and*

$$F(t+0,\ \omega)=\lim_{u\downarrow t} F(u,\ \omega) \qquad (0<t<\zeta(\omega)) \tag{4.5}$$

*exist.*

PROOF. Let $\tau_a$ be an $L$-moment, as defined in Lemma 3.1. We fix some $\alpha \in R$ and $\lambda \in [0, 1]$ and we put

$$\eta_t(\omega)=F_0(t,\ \omega)-\lambda F_\alpha(t,\ \omega), \qquad Z_t(\omega)=\eta_{\tau_a-t}(\omega)\,\chi_{t<\tau_a<\infty}, \qquad \mathscr{A}_t=\mathscr{N}_\infty^{\cdot(\tau_a-t)+}$$

Applying Lemma 4.1 to the $L$-moment $(\tau_a - s)^+$ and the $\mathscr{A}_s$-measurable function $\xi$, we have

$$\mathbf{M}_\gamma\xi Z_{s+t}=\mathbf{M}_\mu\xi\,[1-\lambda e^{-\alpha(\tau_a-s-t)}]\,\chi_{s+t<\tau_a<\infty}. \tag{4.6}$$

Hence it is clear that $\mathbf{P}_\gamma$ -almost surely $Z_t \geqslant 0$ for all $t > 0$. We show that $(Z_t,\ \mathscr{A}_t,\ \mathbf{P}_\gamma)$ is a supermartingale. Since the right-hand side of (4.6) decreases for increasing $t$, 0.5.C is satisfied. From 3.2.A and 3.2.B the validity of 0.5.A and 0.5.B follows.

We fix some numbers $c < d \in R$, and denote by $\nu_a$ the number of down-crossings of $[c, d]$ for the supermartingale $Z_t$. By Theorem 0.1, taking (4.2) into account, we have

$$\mathbf{M}_\gamma\nu_a\leqslant\frac{1}{d-c}\,M_\gamma\,\sup_t Z_t\leqslant\frac{\mu(E)}{d-c}\,. \tag{4.7}$$

By Lemma 3.1. $\mathbf{P}_\gamma$ -almost surely $\tau_a < \zeta \leqslant \infty$, and hence $\nu_a$ coincides with the number of upcrossings of $[c, d]$ on the interval $(0, \tau_a)$ for the function $\eta_t$. Let $\nu$ be the number of upcrossings of $[c, d]$ for $\eta_t$ on $(0, \zeta)$. By Lemma 3.1 for $a \downarrow 0$, $\mathbf{P}_\gamma$ -almost surely $\tau_a \uparrow \zeta$ and hence $\nu_a \uparrow \nu$. Passing to the limit in (4.7), we note that $\mathbf{P}_\gamma\{\nu = \infty\} = 0$. Hence $\mathbf{P}_\gamma$ -almost surely, the function $\eta_t$ makes on the interval $(0, \zeta)$ a finite number of upcrossings of any interval with rational ends, and by Lemma 0.2, the limit $\eta_{\zeta-0}$ and the limits $\eta_{t+0}$ exist for all $t \in (0, \zeta)$. Since this has been proved for any $\lambda \in [0, 1]$, the limits (4.4) and (4.5) exist $\mathbf{P}_\gamma$-almost surely. (It must be borne in mind that the functions $F_\alpha(t, \omega)$ are almost surely bounded and thus the limits $\eta_{\zeta-0}$, $\eta_{t+0}$ almost surely are finite.)

## §5. The Green Function. $M$-processes

5.1. We say that a countable system $W$ of functions in $V$ is a *base system*, if the following conditions hold:

5.1.A. From $\varphi_1, \varphi_2 \in W$, $a_1, a_2 \in R$, it follows[1] that $a_1\varphi_1 + a_2\varphi_2 \in W$.

5.1.B. Suppose that a set of functions $\tilde{V} \supseteq W$ satisfies the conditions:

a) if $f_1, f_2 \in \tilde{V}$, then $f_1 + f_2 \in \tilde{V}$;

b) if $f_1, f_2 \in \tilde{V}$ and $f_1 \leqslant f_2 \leqslant \varphi \in W$, then $f_2 - f_1 \in \tilde{V}$;

c) if $f_n \in \tilde{V}$ and $f_n \uparrow f$, then $f \in \tilde{V}$.

Let $\mu$ be a measure on $\mathscr{B}$ such that $\mu(\varphi) < \infty$ for $\varphi \in W$. Then for any $f \in V$ there exist functions $f_1, f_2 \in \tilde{V}$, such that $f_1 \leqslant f \leqslant f_2$ and $f_1 = f_2$ $\mu$-almost everywhere.

5.1.C. If $\mu_n$ are measures on $\mathscr{B}$ and for all $\varphi \in W$

$$\mu_n(\varphi) \longrightarrow l(\varphi) < \infty,$$

then there exists a measure $\mu$ on $\mathscr{B}$ such that $l(\varphi) = \mu(\varphi)$ for all $\varphi \in W$.

We note that 5.1.B implies the following property:

5.1.B'. If $\mu_1$ and $\mu_2$ are measures on $\mathscr{B}$ and if $\mu_1(\varphi) = \mu_2(\varphi) < \infty$ for all $\varphi \in W$, then $\mu_1 = \mu_2$.

For the set $\tilde{V} = \{f: \mu_1(f) = \mu_2(f)\}$ contains $W$ and satisfies the conditions a) – c) of 5.1.B. Therefore we can construct functions $f_1, f_2 \in \tilde{V}$, such that $f_1 \leqslant f \leqslant f_2$ and $f_1 = f_2$, $(\mu_1 + \mu_2)$-a.e. It is clear $\mu_i(f_1) = \mu_i(f) = \mu_i(f_2)$ $(i = 1, 2)$, and since $\mu_1(f_1) = \mu_2(f_1)$, then $\mu_1(f) = \mu_2(f)$.

5.2. We assume that for each $\alpha \in R$ we are given a non-negative $\mathscr{B} \times \mathscr{B}$-measurable function $g_\alpha(x, y)$. We call this the Green function for the process $X$ if:

5.2.A. For any $x \in E$

$$g_\alpha(x, dy) = g_\alpha(x, y)\, m(dy),$$

where $m$ is a certain measure, subject to the condition $m(\varphi) < \infty$ $(\varphi \in W)$; $g_\alpha(x, \Gamma)$ is the Green kernel.

5.2.B. For each $\varphi \in W$ the function

$$g_\alpha(\varphi, y) = \int_E m(dx)\, \varphi(x)\, g_\alpha(x, y)$$

is $\Lambda$-continuous.

5.3. By 5.2.A all sets of $m$-measure zero are null-sets.

LEMMA 5.1. *The set*

$$D^0 = \{y: g(x, y) = 0 \text{ for } m\text{-almost all } x \in E\} \tag{5.1}$$

*belongs to $\mathscr{B}$. Without changing the values of $g_\alpha(x, y)$ the measure $m$ may be replaced by its restriction to $E \setminus D^0$, that is, we may assume that the following additional condition is satisfied:*

[1] As previously, $R$ denotes the set of all non-negative rational numbers.

5.3.A. $m(D^0) = 0$.

*When* 5.3.A *holds, the class of null-sets coincides with the class of sets of $m$-measure zero.*

PROOF. From Fubini's theorem (see 0.3.F) it follows that $D^0 \in \mathscr{B}$. For $m$-almost all $x \in E$ $g(x, D^0) = 0$. Since $g(x, D^0)$ is excessive (see §1.2), then $g(x, D^0) = 0$ for all $x \in E$ (Lemma 1.1). From (1.5) it is evident that $g_\alpha(x, D^0) = 0$ for all $\alpha \in R$, $x \in E$. If, without changing the values of $g_\alpha(x, y)$, $m$ is replaced by its restriction to $E \setminus D^0$, then, as before, conditions 5.2.A – 5.2.B are satisfied.

Let $\Gamma$ be any null-set. From 5.2.A, for any $x$, $g(x, y) = 0$ for $m$-almost all $y \in \Gamma$. By Fubini's theorem, $m$-almost all $y \in \Gamma$ belong to $D^0$, and, if 5.3.A is satisfied, then $m(\Gamma) = m(\Gamma \cap D^0) = 0$.

*Throughout the subsequent part of the paper we are concerned with a fixed measure $m$ satisfying conditions* 5.2.A – 5.2.B *and* 5.3.A. *We take the integral of $f_1 f_2$ with respect to the measure $m$ as a scalar product* $(f_1, f_2)$.

What is the degree of indeterminacy in the choice of the Green function? The answer to this question is given by the following lemma.

LEMMA 5.2. *We fix some Green function $g_\alpha(x, y)$. For a non-negative $(\mathscr{B} \times \mathscr{B})$-measurable function $g'_\alpha(x, y)$ to be a Green function it is necessary and sufficient that for any $\alpha \in R$, $x \in E$, $g'_\alpha(x, y) = g_\alpha(x, y)$ for $m$-almost all $y$, and that there exists a set $A$ such that for all $t > 0$, $x \in E$, $p(t, x, A) = 0$, and for any $\alpha \in R$, $y \in E \setminus A$, $g'_\alpha(x, y) = g_\alpha(x, y)$ for $m$-almost all $x$.*

Since this lemma is not used anywhere later, we omit the proof.

5.4. A finite measure $\gamma$ on the $\sigma$-algebra $\mathscr{B}$ is called *standard* if:

5.4.A. There exists a positive function $q(t, \omega)$ $(0 < t < \zeta(\omega))$, continuous on the left with respect to $t$ and such that for any $t > 0$, $q(t, \omega)$ is $\mathscr{N}_t$-measurable and equal to $q(x_t)$ almost surely, where[1]

$$q(y) = \int_E \gamma(dx)\, g(x, y)\ ). \tag{5.2}$$

5.4.B. There exist constants $c^\alpha_\varphi < \infty$ such that for any $\alpha$-excessive function $h$

$$(h, \varphi) \leqslant c^\alpha_\varphi \gamma(h). \tag{5.3}$$

5.4.C. There exists a positive function $\psi \in V$ such that[2] $\gamma(G\psi) < \infty$.

Let $h$ be a $\gamma$-integrable $\alpha$-excessive function. We put $\tilde{V} = \{f : f \in V,\ (f, \chi_{h=\infty}) = 0\}$. By 5.4.C, $\tilde{V} \supseteq W$ and from 5.1.B it follows that $\tilde{V} = V$. Hence

$$m\{x : h(x) = \infty\}. \tag{5.4}$$

Similarly, if the $\alpha$-excessive function is zero $\gamma$-a.e., then it is zero $m$-a.e., and from Lemma 1.1 and Lemma 5.1 it follows that it is zero everywhere.

---

[1] For $q(t, \omega)$ the value $+\infty$ is admitted.

[2] Condition 5.4.C coincides with condition $(G)$ of 3.1.

5.5. We now consider the question of the construction of standard measures $\gamma$.

LEMMA 5.3. *Suppose that the following conditions are satisfied:*

5.5.A. *For any $\varphi \in W$ the function $g(\varphi, y)$ is bounded.*

5.5.B. *For any $x \in E$, $t > 0$, $\varepsilon > 0$, there exist $\varphi \in W$ and $a > 0$ such that*

$$\mathbf{P}_x\{\zeta > t,\ g(\varphi, x_s) < a \ \text{for some}\ s \in R_t\} < \varepsilon$$

*(by $R_t$ we denote the set of all rational points of the interval $[0, t]$).*

*Then a function $\psi(y)$ can be constructed so that the measure $\gamma(dy) = \psi(y)m(dy)$ is standard. Furthermore, if $(h, \varphi) < \infty$ for all $\varphi \in W$, then $\psi$ can be chosen so that $\gamma(h) < \infty$.*

PROOF. By 5.5.A for each $\varphi \in W$ a constant $u_\varphi$ can be chosen such that $g(\varphi, y) \leqslant u_\varphi$ for all $y \in E$. We consider any convergent series[1] $\Sigma \rho_\varphi$ and put

$$d_\varphi = \{u_\varphi + (h+1,\ \varphi)\}^{-1} \rho_\varphi, \qquad \psi(y) = \sum d_\varphi \cdot \varphi(y), \qquad \gamma(dy) = \psi(y)\, m(dy).$$

Evidently $\gamma(E) < \infty$, $\gamma(h) < \infty$ and

$$q(y) = \int_E \gamma(dx)\, g(x,\ y) = \sum d_\varphi \cdot g(\varphi,\ y).$$

Let $F_\varphi(t, \omega)$ be a left-continuous modification of $g(\varphi, x_t)$. We put

$$Q(t,\ \omega) = \sum d_\varphi \cdot F_\varphi(t,\ \omega) \wedge u_\varphi.$$

The series on the right-hand side converges uniformly and defines a non-negative function, continuous on the left with respect to $t$ for any $\omega$. Clearly, for any $t > 0$, almost surely $F_\varphi(t, \omega) = g(\varphi, x_t) \leqslant u_\varphi$, and hence $Q(t, \omega) = q(x_t)$. Next, for any $\varphi \in W$, $a > 0$,

$$\begin{aligned}\mathbf{P}_x\{\zeta > t,\ \inf_{s \leqslant t} Q(s,\ \omega) = 0\} &\leqslant \mathbf{P}_x\{\zeta > t,\ \inf_{s \leqslant t} F_\varphi(s,\ \omega) = 0\} \leqslant \\ &\leqslant \mathbf{P}_x\{\zeta > t,\ F_\varphi(s,\ \omega) < a \ \text{for some}\ s \in R_t\} = \\ &= \mathbf{P}_x\{\zeta > t,\ g(\varphi,\ x_s) < a \ \text{for some}\ s \in R_t\}\end{aligned}$$

and by 5.5.B the left-hand side of this inequality is zero. Hence almost surely $Q(s, \omega) > 0$ for all $s \in (0, \zeta)$. We put $q(s, \omega) = Q(s, \omega)$ $(0 < s < \zeta)$ if $Q(s, \omega)$ is positive for all $s \in (0, \zeta)$, and $q(s, \omega) = 1$ $(0 < s < \zeta)$ in the remaining cases. It is easy to see that $q(s, \omega)$ satisfies 5.4.A.

Evidently $\gamma(h) = \sum d_\varphi\ (\varphi,\ h) \geqslant d_\varphi \cdot (\varphi,\ h)$. Hence 5.4.B is satisfied for $c_\varphi^\alpha = d_\varphi^{-1}$. Finally,

$$\gamma(G\psi) = (q,\ b) \leqslant \gamma(E) \cdot \sum \rho_\varphi < \infty.$$

Since $\psi > 0$, 5.4.C is satisfied and the lemma is proved.

---

[1] The summation here and later is over all $\varphi \in W$.

The choice of a suitable standard measure may be made easier by the following remark. For a measure $\gamma$ to satisfy 5.4.B it is sufficient that the function $\varphi/q_\lambda$ is bounded for any $\varphi \in W$, $\lambda > 0$, where

$$q_\lambda(y) = \int_E \gamma(dx)\, g_\lambda(x, y).$$

For if $h$ is $\alpha$-excessive, then[1]

$$\gamma(h) \geqslant \gamma(G_{\alpha+1}h) = (h, q_{\alpha+1}) \geqslant \left\| \frac{\varphi}{q_{\alpha+1}} \right\|^{-1} (h, \varphi).$$

5.6. LEMMA 5.4. *Let $g_y^\alpha(x) = g_\alpha(x, y)$. We put $y \in E_r$ if for all $\alpha, \lambda \in R$*

$$g_y^{\lambda+\alpha} + \lambda G_{\lambda+\alpha} g_y^\alpha = g_y^\alpha \quad \textit{for } m\textit{-almost all } x \in E. \tag{5.5}$$

*We have $m(E \setminus E_r) = 0$. For $y \in E_r$ the function $g_y^\alpha$ is $\alpha$-preexcessive. We consider its regularization*

$$\tilde{g}(x, y) = \tilde{g}_y^\alpha(x) = \lim_{\lambda \to \infty} \lambda G_{\lambda+\alpha} g_y^\alpha(x) \qquad (y \in E_r) \tag{5.6}$$

*and complete its definition by putting $\tilde{g}_\alpha(x, y) = 0$ for $y \in E \setminus E_r$. The function $\tilde{g}_\alpha(x, y)$ (which we call the regularized Green function) has the following properties:*

5.6.A. *For each $y \in E$ $\tilde{g}_\alpha(x, y)$ is $\alpha$-excessive*

5.6.B. *For any $x \in E$*

$$\tilde{g}_\alpha(x, y) = g_\alpha(x, y) \quad \textit{for } m\textit{-almost all } x \in E \tag{5.7}$$

*and hence*

$$G_\alpha f(x) = \int_E \tilde{g}_\alpha(x, y) f(y)\, m(dy). \tag{5.8}$$

PROOF. We denote by $u(x, y)$ the left-hand side of (5.5). By (1.6) for any $\varphi, \psi \in V$

$$\int_E \int_E m(dx)\, \varphi(x)\, u(x, y)\, \psi(y)\, m(dy) = \int_E m(dx)\, \varphi(x)\, [G_{\lambda+\alpha}\psi(x) + \lambda G_{\lambda+\alpha} G_\alpha \psi(x)] =$$

$$= \int_E m(dx)\, \varphi(x)\, G_\alpha \psi(x) = \int_E \int_E m(dx)\, \varphi(x)\, g_\alpha(x, y)\, \psi(y)\, m(dy).$$

Hence, for $m$-almost all $y$, $u(x, y) = g_\alpha(x, y)$ for $m$-almost all $x$ and therefore $m(E \setminus E_r) = 0$.

[1] For a Brownian motion in a domain $E$ we may choose $g_\lambda(x, y)$ so that it is positive and lower semi-continuous with respect to $y$. By Fatou's lemma $q_\lambda(y)$ is also lower semi-continuous. If $\gamma$ is a non-trivial measure, $q_\lambda(y)$ is positive and thus bounded below by a positive constant on each compactum. Hence 5.4.B is satisfied for any non-zero measure $\gamma$.

Let $y \in E_r$. Then $k_y^\alpha$ for $\lambda \in R$ satisfies 1.2.A$'$. However, as is easy to see from (1.2) $G_\lambda g_y^\alpha(x)$, and hence $\lambda G_\lambda g_y^\alpha(x)$ is continuous on the right with respect to $\lambda$. Hence 1.2.A$'$ is satisfied for all $\lambda > 0$ and $g_y^\alpha$ is $\alpha$-preexcessive. By Lemma 1.2, 5.6.A is satisfied. By §1.2, for any $f \in V$ as $\lambda \to \infty$

$$\int_E \lambda G_{\lambda+\alpha} g_y^\alpha(x) f(y)\, m(dy) = \lambda G_{\lambda+\alpha} G_\alpha f(x) \uparrow G_\alpha f(x). \tag{5.9}$$

Since $\lambda G_{\lambda+\alpha} g_y^\alpha \uparrow \widetilde{g}_y^\alpha$ for $y \in E_r$ and $m(E \setminus E_r) = 0$, (5.7) follows from (5.9). (5.8) evidently follows from (5.7).

REMARK. Let $y \in E_r$. By (5.5)

$$g_y^\alpha \leqslant g_y \quad \text{for } m\text{-almost all } x. \tag{5.10}$$

From (5.10), 5.2.A and (1.6) we conclude that for all $x \in E$

$$\lambda G_{\lambda+\alpha} g_y^\alpha \leqslant \lambda G_{\lambda+\alpha} g_y \leqslant \lambda G_\lambda g_y,$$

and hence

$$\widetilde{g}_y^\alpha \leqslant \widetilde{g}_y. \tag{5.11}$$

5.7. We denote by $E_0$ the set of all $y \in E$ such that:

5.7.A. $0 < q(y)$.

5.7.B. For all $\alpha \in R$ and all $\varphi \in W$

$$g_\alpha(\varphi, y) \leqslant c_\varphi^\alpha q(y),$$

where $c_\varphi^\alpha$ are the constants of condition 5.4.B.

We put $y \in E_1$ if $y \in E_0 \cap E_r$ for all $\alpha \in R$

$$\widetilde{g}_\alpha(x, y) = g_\alpha(x, y) \quad \text{for } (m+\gamma)\text{-almost all } x \in E \tag{5.12}$$

By Fubini's theorem it follows from (5.7) that (5.12) is satisfied for $m$-almost all $y \in E$. Next, if $y$ satisfies (5.12), then from 5.4.B and (5.11)

$$g_\alpha(\varphi, y) = \widetilde{g}_\alpha(\varphi, y) \leqslant c_\varphi^\alpha \gamma(\widetilde{g}_y^\alpha) \leqslant C_\varphi^\alpha \gamma(\widetilde{g}_y) = c_\varphi^\alpha \gamma(g_y) = c_\varphi^\alpha q(y),$$

and hence 5.7.B is satisfied. Finally, from 5.4.A it follows that for any $t > 0$ almost surely $0 < q(x_t)$. Hence the set of points $y$ for which 5.7.A is not satisfied is a null-set. So, we have proved that

$$m(E \setminus E_1) = m(E \setminus E_0) = 0. \tag{5.13}$$

5.8. The object of our investigation in what follows is the collection $\mathscr{X} = (X, W, g_\alpha(x, y), \gamma)$, where $X$ is a Markov process in a perfect measurable space $(E, \mathscr{B})$, $W$ is a base system, $g_\alpha(x, y)$ the Green function and $\gamma$ a standard measure.[1] We call this object an *M-process*.

[1] It is assumed that the transition function satisfies 1.1.C and 1.1.D and the measure $m$ condition 5.3.A, as will be mentioned at the right moment.

Let $E$ be a metrizable locally compact separable topological space. The sets belonging to the completion of the $\sigma$-algebra of Borel sets with respect to any finite measure $\mu$ are called universally measurable. Let $\mathscr{B}$ be the $\sigma$-algebra of all universally measurable sets of $E$. We denote by $C_0(E)$ the set of all non-negative continuous functions from $E$ with compact support. We consider any sequence of open sets $D_n \uparrow E$ having compact closures. In the set $C_0(D_n)$ of non-negative continuous functions whose support is contained in $D_n$ we select a countable everywhere dense (in the sense of the uniform norm $\| f \|$) subset $\mathscr{T}_n$. We put $\mathscr{T} = \bigcup \mathscr{T}_n$ and we denote by $W$ the set of functions that are representable as linear combinations with coefficients in $R$ of the elements of $\mathscr{T}$.

We show that $W$ is a base system. 5.1.A is satisfied by construction. We verify the validity of 5.1.B and 5.1.C.

Let $\tilde{V} \supseteq W$ and satisfy conditions a) – c) of 5.1.B. For any $f \in C_0(E)$ we can choose $f_n$, $\varphi \in W$ so that $\|f_n - f\| \leqslant \varphi/2^{n+2}$. We put $\tilde{f}_n = f_n + (1 - 2^{-n})\varphi$. It is evident that $\tilde{f}_n \uparrow f + \varphi$, therefore $f + \varphi \in \tilde{V}$, and hence $f \in \tilde{V}$. Thus, $\tilde{V} \supseteq C_0(E)$.

Let $\Phi \in C_0(E)$. We put $f \in \mathscr{H}$, if $f\Phi \in \tilde{V}$. It is easy to see that $\mathscr{H}$ satisfies conditions a) – d) of Lemma 0.1 and contains $C_0(E)$. It is clear that $\mathscr{H}$ contains the indicator $\chi_A$ of any open set $A$ having a compact closure.[1] By Lemma 0.1, $\mathscr{H}$ contains all non-negative Borel functions. Thus, if $f \geqslant 0$ is a Borel function and $\Phi \in C_0(E)$, then $f\Phi \in \tilde{V}$. We consider functions $\Phi_n \in C_0(E)$ such that $\Phi_n \uparrow 1$. Since $f\Phi_n \in \tilde{V}$ and $f\Phi_n \uparrow f$, we have $f \in \tilde{V}$. Thus, $\tilde{V}$ contains all non-negative Borel functions.

To prove 5.1.B it remains to note that if $\mu$ is a measure on $\mathscr{B}$ such that $\mu(\varphi) < \infty$ for all $\varphi \in W$, then for any $\mathscr{B}$-measurable function $f \geqslant 0$ there exist Borel non-negative functions $f_1$ and $f_2$ such that $f_1 \leqslant f \leqslant f_2$ and $f_1 = f_2$ $\mu$-a.e. The last statement follows at once from the definition of a universally measurable set for $f = \chi_\Gamma$, where $\Gamma$ is a set of the $\sigma$-algebra $\mathscr{B}$ with compact closure. In the general case it is sufficient to apply Lemma 0.1.

We pass now to the proof of 5.1.C. Let $\mu_n$ be measures on the $\sigma$-algebra $\mathscr{B}$ and suppose that for all $\varphi \in W$ the finite limit

$$l(\varphi) = \lim_{n\to\infty} \mu_n(\varphi)$$

exits. It is clear that this limit exists and is finite for each $\varphi \in C_0(E)$, while $l(a_1\varphi_1 + a_2\varphi_2) = a_1 l(\varphi_1) + a_2 l(\varphi_2)$, for any $a_1$, $a_2 \geqslant 0$ and any $\varphi_1$, $\varphi_2 \in C_0(E)$, and $l(\varphi) \geqslant 0$ for $\varphi \in C_0(E)$. By a well-known theorem (see, for example, [13], §56) it hence follows that $l(\varphi) = \mu(\varphi)$, where $\mu$ is the measure on $\mathscr{B}$.

Suppose that a Markov process $X = (x_t, \zeta, M_t, \mathrm{P}_x)$ in $(E, \mathscr{B})$ satisfies the conditions:

5.8.A. For any $\alpha \in R$, $x \in E$

$$g_\alpha(x, dy) = g_\alpha(x, y)\, m(dy),$$

[1] We note that if $\rho(z)$ denotes the distance from $z$ to the complement of $A$, then $f_n = (n\rho) \wedge 1 \in C_0(E)$ and $f_n \uparrow \chi_A$.

where $g_\alpha(x, y)$ is a $\mathscr{B} \times \mathscr{B}$-measurable non-negative function, and $m$ a measure that is finite on compact sets.

5.8.B. For any $\varphi \in C_0(E)$ the function

$$g_\alpha(\varphi, y) = \int_E m(dx)\, \varphi(x)\, g_\alpha(x, y)$$

is continuous and bounded.

5.8.C. For any $0 < t < \zeta(\omega)$ the left limits $x_{t-0}(\omega)$ exist, and for all $t > 0$, $x \in E$, $P_x\{x_{t-0} \neq x_t\} = 0$.

5.8.D. If $A$ is an open set containing the point $x$, then

$$\lim_{t \downarrow 0} p(t, x, A) = 1$$

Next, let $\gamma$ be a measure on $\mathscr{B}$ satisfying 5.4.B – 5.4.C and the condition:

5.8.E. The function

$$q(y) = \int_E \gamma(dx)\, g(x, y)$$

is continuous and positive.

From 5.8.A – 5.8.C conditions 5.2.A and 5.2.B evidently follow. The elements $\{X, W, g_\alpha(x, y), \gamma\}$ form an $M$-process. We call it a *special M-process.* We assume that $A$ is a non-void open set. By 5.8.D

$$\int_A g_\alpha(x, y)\, m(dy) = g_\alpha(x, A) = \int_0^\infty e^{-\alpha t} p(t, x, A)\, dt > 0.$$

Hence $m(A) > 0$. Thus, for a special $M$-process the complement of any set of $m$-measure zero is everywhere dense in $E$. This is applicable to the sets $E_r$, $E_0$, $E_1$, in particular. However, as is easy to see, $E_0$ is closed. Hence $E_0 = E$.

An important class of special $M$-processes is that of processes continuous on the right, that is, processes which satisfy the condition:

5.8.F. For almost all $\omega$, $x_{t+0} = x_t(\omega)$ for all $t \in [0, \zeta(\omega)]$. (We note[1] that 5.8.D follows from 5.8.F.)

## §6. $h$-processes

6.1. Let $\mathscr{X} = (X, W, g_\alpha(x, y), \gamma)$ be a $M$-process in a perfect phase space $(E, \mathscr{B})$. Let $h$ be a $\gamma$-integrable excessive function. We put $E^h = \{x\colon 0 \dot{<} h(x) < \infty\}$ and denote by $\mathscr{B}^h$ the family of all $\Gamma \in B$, contained in $E^h$. Then

[1] Conditions 5.8.C and 5.8.F (and hence 5.8.D) are satisfied for all so-called standard processes. (The definition of these processes and the criterion for standardness in terms of transition functions can be found, for example, in [8], §3.23.)

$$p^h(t, x, \Gamma) = \frac{1}{h(x)} \int_\Gamma p(t, x, dy)\, h(y) \qquad (x \in E^h,\ \Gamma \in \mathscr{B}^h) \tag{6.1}$$

defines in the space $(E^h, \mathscr{B}^h)$ a transition function.

Let

$$E^h_* = \{x\colon h(x) < \infty\}, \qquad E^h_0 = \{x\colon h(x) = 0\}.$$

If $x \in E^h_*$, то $0 \leqslant P_t h(x) \leqslant h(x) < \infty$. Hence

$$p(t, x, E \setminus E^h_*) = 0 \quad \text{for} \quad x \in E^h_*. \tag{6.2}$$

Similarly, if $x \in E^h_0$, then $0 \leqslant P_t h(x) \leqslant h(x) = 0$, and hence

$$p(t, x, E \setminus E^h_0) = 0 \quad \text{for} \quad x \in E^h_0. \tag{6.3}$$

(By (6.1), (6.2) and Lemma 1.3 for $x \in E^h$

$$p^h(t, x, E^h) = \frac{1}{h(x)} \int_{E^h} p(t, x, dy)\, h(y) = \frac{1}{h(x)} P_t h(x) \uparrow 1. \tag{6.4}$$

Hence, the transition function $p^h(t, x, \Gamma)$ satisfies 1.1.D.)

By §2.2 there exists a canonical Markov process $X^h = (x_t, \zeta, \mathscr{N}_t, \mathbf{P}^h_x)$ in the space $(E^h, \mathscr{B}^h)$ with transition function $p^h(t, x, \Gamma)$. We call this an *h-process*.

The Green kernel for $X^h$ is given by the formula

$$g^h_\alpha(x, dy) = \frac{1}{h(x)} g_\alpha(x, dy)\, h(y) = \frac{1}{h(x)} g_\alpha(x, y)\, h(y)\, m(dy). \tag{6.5}$$

We denote by $G^h_\alpha$ the corresponding Green operator. A function that is $\alpha$-excessive for $X^h$, in other words, satisfies the condition

$$\lambda G^h_{\lambda+\alpha} f(x) \uparrow f(x) \qquad \text{for } \lambda \to \infty \qquad (x \in E^h),$$

will be called $(h, \alpha)$-excessive.

We now prove several simple lemmas.

**6.1.A.** *For any $f \in V$*

$$P_t(fh)(x) = 0, \qquad G_\alpha(fh)(x) = 0 \qquad \text{on } E^h_0 \tag{6.6}$$

The first of these formulae follows from (6.3), the second from the first and (1.2).

**6.1.B.** *We have*

$$m(E \setminus E^h_*) = 0 \tag{6.7}$$

*and for any $f \in V$*

$$h(x)\, \mathbf{M}^h_x f(x_t) = \mathbf{M}_x(fh)(x_t), \qquad h(x)\, G^h_\alpha f(x) = G_\alpha(fh)(x) \text{ on } E^h_*. \tag{6.8}$$

*If* $f_1 = f_2$ *on* $E^h_*$, *then*

$$P_t f_1 = P_t f_2, \quad G_\alpha f_1 = G_\alpha f_2 \quad \text{on } E^h_*. \tag{6.9}$$

(6.7) follows from (5.4). By (6.2) and (1.2) the equations (6.9) hold. We show that

$$\left.\begin{aligned} h(x)\,\mathbf{M}^h_x f(x_t) &= \mathbf{M}_x (fh\chi_{E^h})(x_t), \\ h(x)\,G^h_\alpha f(x) &= G_\alpha (fh\chi_{E^h})(x) \end{aligned}\right\} \text{ on } E^h_*. \tag{6.10}$$

For by (6.1) and (6.5) these formulae are valid for $x \in E^h$, and by 6.1.A they are satisfied on $E^h_0$. Since $fh\chi_{E^h} = fh$ on $E^h_*$, (6.8) follows from (6.10) by (6.9).

6.1.C. *For any* $(h, \alpha)$*-excessive function there exists an* $\alpha$*-excessive function* $u^h$ *such that* $u^h = hu$ *on* $E^h_*$.

To prove this we consider any function $\psi \in V$ that is equal to $hu$ on $E^h_*$. By 6.1.B

$$\lambda G_{\lambda+\alpha}\psi = \lambda G_{\lambda+\alpha}(hu) = \lambda h G^h_{\lambda+\alpha} u \uparrow hu = \psi \quad \text{on } E^h_*.$$

Hence $\psi$ is $\alpha$-preexcessive and its regularization $u^h = \psi = hu$ on $E^h_*$.

6.1.D. *Let* $f$ *be* $\alpha$*-excessive,* $u$ *be* $(h, \alpha)$*-excessive and* $f = uh$ $m$*-a.e.. Then this equation is satisfied everywhere on* $E^h_*$.

The $\alpha$-excessive function $u^h$ constructed in 6.1.C is equal to $f$ m-a.e.. By Lemma 1.1 $f = u^h$ everywhere.

6.1.E. *If two* $\alpha$*-excessive functions coincide on* $E^h_*$, *they coincide everywhere.*

This follows from (6.7) and Lemma 1.1.

6.2. We denote by $\gamma^h$ the measure on $E^h$ defined by the formula

$$\gamma^h(dy) = h(y)\,\gamma(dy), \tag{6.11}$$

and put

$$\mathbf{P}^h(A) = \mathbf{P}^h_{\gamma^h}(A), \quad \mathbf{M}^h\xi = \mathbf{M}^h_{\gamma^h}\xi. \tag{6.12}$$

6.2.A. *If* $u$ *is* $h$*-excessive and is zero* $\gamma^h$-a.e. *on* $E^h$, *then* $u(x) = 0$ *for all* $x \in E^h$. *If* $\xi$ *is* $\mathscr{N}^\zeta_\infty$*-measurable and* $\mathbf{M}^h\xi = 0$, *then* $\mathbf{M}^h_x\xi = 0$ *for all* $x \in E^h$.

Let $u^h$ be the excessive function constructed in 6.1.C. By (6.7)

$$\int_E u^h(x)\,\gamma(dx) = \int_{E^h_*} u^h(x)\,\gamma(dx) = \int_{E^h} u(x)\,\gamma^h(dx) = 0.$$

Thus, $u^h = 0$ $\gamma$-a.e.. By 6.1.E $u^h = 0$ everywhere and $u = 0$ on $E^h$.

We note that $u(x) = \mathbf{M}^h_x\xi$ is $h$-excessive by Lemma 3.4 and

$$\mathbf{M}^h\xi = \int_{E^h} u(x)\,\gamma^h(dx).$$

6.2.B. *For any* $f \in V$

$$\mathbf{M}^h f(x_t) = \mathbf{M}_\gamma (fh)(x_t).$$

For by (6.8), (6.11) and (6.12)

$$\mathbf{M}^h f(x_t) = \int_{E^h} \gamma^h(dx)\, \mathbf{M}_x^h f(x_t) = \int_{E^h} \gamma(dx)\, \mathbf{M}_x (fh)(x_t).$$

Taking account of (6.6) and of the equation $\gamma\{h = \infty\} = 0$, which follows from the $\gamma$-integrability of $h$, we may replace $E^h$ by $E$ in the last integral.

6.2.C. $\gamma^h(G_\alpha^h f) = \gamma[G_\alpha(fh)]$.

By (6.11), (6.8), 6.1.A and (6.7)

$$\gamma^h(G_\alpha^h f) = \int_{E^h} \gamma^h(dx)\, G_\alpha^h f(x) = \int_{E^h} \gamma(dx)\, h(x)\, G_\alpha^h f(x) =$$

$$= \int_{E^h} \gamma(dx)\, G_\alpha(fh)(x) = \gamma[G_\alpha(fh)].$$

6.3. LEMMA 6.1. *For any* $\mathscr{N}_t$*-measurable function* $\xi$

$$h(x)\, \mathbf{M}_x^h \xi = \mathbf{M}_x \xi h(x_t) \qquad (x \in E_*^h). \tag{6.13}$$

PROOF. We first prove formula (6.13) for $\xi = f_1(x_{t_1}) \dots f_n(x_{t_n})$, where $0 \leqslant t_1 \leqslant \dots \leqslant t_n = t$ and $f_1, \dots, f_n$ are $\mathscr{B}$-measurable functions. For $n = 1$ this follows from (6.8). We use by induction of $n$. We put $t_{n-1} = u$, $s = t - u$,

$$\xi' = f_1(x_{t_1}) \dots f_{n-1}(x_{t_{n-1}}).$$

By (2.3)

$$\mathbf{M}_x \xi h(x_t) = \mathbf{M}_x \xi' (f_n h)(x_t) = \mathbf{M}_x \xi' \theta_u [(f_n h)(x_u)] = \mathbf{M}_x \xi' \mathbf{M}_{x_u} (f_n h)(x_s). \tag{6.14}$$

By (6.8) on $E_*^h$

$$\mathbf{M}_y (f_n h)(x_s) = h(y)\, \mathbf{M}_y^h f_n(x_s). \tag{6.15}$$

Noting (6.2) we have from (6.14) and (6.15)

$$\mathbf{M}_x \xi h(x_t) = \mathbf{M}_x \xi' h(x_u)\, \mathbf{M}_{x_u}^h f_n(x_s).$$

By the induction hypothesis the right-hand side is equal to

$$h(x)\, \mathbf{M}_x^h [\xi' \mathbf{M}_{x_u}^h f_n(x_s)] = h(x)\, \mathbf{M}_x^h \xi' \theta_u f_n(x_s) = h(x)\, \mathbf{M}_x^h \xi' f_n(x_t) = h(x)\, \mathbf{M}_x^h \xi.$$

We denote by $\mathscr{H}$ the system of all functions $\xi$ on $\Omega_t$ for which (6.13) is satisfied. We have proved that $\mathscr{H}$ contains the indicators of the sets

$$A = \{x_{t_1} \in \Gamma_1, \dots, x_{t_n} \in \Gamma_n\} \qquad (0 \leqslant t_1 \leqslant \dots \leqslant t_n = t,\ \Gamma_1, \dots, \Gamma_n \in \mathscr{B}).$$

The family of these sets is closed under intersections and generates a σ-algebra $\mathcal{N}_t$. From Lemma 0.1 it follows that $\mathcal{H}$ contains all $\mathcal{N}_t$-measurable functions.

6.4. We call

$$K_y^\alpha(x) = \begin{cases} \dfrac{g_\alpha(x, y)}{q(y)} & \text{for } y \in E_0, \\ 0 & \text{for } y \in E \setminus E_0 \end{cases} \tag{6.16}$$

*a Martin function.* We note that[1]

$$(K_y^\alpha, \varphi) = \begin{cases} \dfrac{g_\alpha(x, y)}{q(y)} & \text{for } y \in E_0, \\ 0 & \text{for } y \in E \setminus E_0, \end{cases} \tag{6.17}$$

and by 5.7.B

$$(K_y^\alpha, \varphi) \leqslant c_\varphi^\alpha. \tag{6.18}$$

THEOREM 6.1. *We can construct functions* $\mathrm{K}_\varphi^\alpha(t, \omega)$ $(\alpha \in R, \varphi \in W, \ 0 < t < \zeta(\omega))$, *satisfying the following conditions:*

6.4.A. *For all* $t > 0$ $\mathrm{K}_\varphi^\alpha(t, \omega)$ *is* $\mathcal{N}_t$*-measurable and for any* $\gamma$*-integrable excessive function* $h$

$$\mathbf{P}_x^h\{\mathrm{K}_\varphi^\alpha(t, \omega) \neq (K_{x_t}^\alpha, \varphi)\} = 0 \qquad (x \in E).$$

6.4.B. *For all* $\omega \in \Omega$, $\mathrm{K}_\varphi^\alpha(t, \omega)$ *is continuous on the left with respect to* $t$ *in the interval* $(0, \zeta(\omega))$.

6.4.C. *For any* $\omega \in \Omega$ *the limit*

$$\mathrm{K}_\varphi^\alpha(t+0, \omega) = \lim_{u \downarrow t} \mathrm{K}_\varphi^\alpha(u, \omega) \tag{6.19}$$

*exists for all* $t \in (0, \xi(\omega))$.

6.4.D. *For any* $\omega \in \Omega$ *the limit*

$$\mathrm{K}_\varphi^\alpha(\zeta, \omega) = \lim_{t \uparrow \zeta} \mathrm{K}_\varphi^\alpha(t, \omega). \tag{6.20}$$

*exists.*

PROOF. We consider the function $q(t, \omega)$ defined in 5.4.A. We denote by $F_\varphi^\alpha(t, \omega)$ the left-continuous modification of $g_\alpha(\varphi, x_t)$ and put

$$\mathrm{K}_\varphi^\alpha(t, \omega) = \frac{F_\varphi^\alpha(t, \omega)}{q(t, \omega)}. \tag{6.21}$$

Evidently $\mathrm{K}_\varphi^\alpha(t, \omega)$ satisfies 6.4.B. Clearly $\mathrm{K}_\varphi^\alpha(t, \omega)$ is $\mathcal{N}_t$-measurable. By (6.17), for all $t > 0$, $x \in E$

$$\mathbf{P}_x\{\mathrm{K}_\varphi^\alpha(t, \omega) \neq (K_{x_t}^\alpha, \varphi)\} = 0,$$

and 6.4.A is satisfied by Lemma 6.1.

---

[1] For $q(y) = \infty$ we take $K_y^\alpha(x) = 0$ and $(K_y^\alpha, \varphi) = 0$.

We fix $\varphi \in W$ and consider on $E^h$ the measure

$$\mu(dy) = \varphi(y) h(y) m(dy).$$

Bearing (6.7), (6.8), (6.6), (6.17) and (5.13) in mind we have for any $\Gamma \in \mathscr{B}^h$

$$(\mu G^h_\alpha)(\Gamma) = \int_{E^h} \varphi(x) h(x) m(dx) g^h_\alpha(x, \Gamma) = \int_{E^h} \varphi(x) h(x) G^h_\alpha \chi_\Gamma(x) m(dx) =$$
$$= \int_{E^h} \varphi(x) G_\alpha(h\chi_\Gamma)(x) m(dx) = \int_E \varphi(x) G_\alpha(h\chi_\Gamma)(x) m(dx) =$$
$$= \int_\Gamma g_\alpha(\varphi, y) h(y) m(dy) = \int_\Gamma (K^\alpha_y, \varphi) q(y) h(y) m(dy). \tag{6.22}$$

On the other hand, by 6.2.C for $\Gamma \in \mathscr{B}^h$

$$(\gamma^h G^h)(\Gamma) = \gamma^h(G^h \chi_\Gamma) = \gamma[G(h\chi_\Gamma)] = (\gamma G)(h\chi_\Gamma) = \int_\Gamma q(y) h(y) m(dy). \tag{6.23}$$

From (6.22) and (6.23) it is clear that $f_\alpha(y) = (K^\alpha_y, \varphi)$ is the density of the measure $\mu G^h_\alpha$ with respect to the measure $\gamma^h G^h$.

We apply Theorem 4.1 to the process $X^h$, the measures[1] $\mu$ and $\gamma^h$. Since $K^\alpha_\varphi(t, \omega)$ is a left-continuous modification of $f_\alpha(x_t)$, the limits (6.19) and (6. 20) exist $P^h$-almost surely.

We put $\omega \in \tilde{\Omega}_s$ if $\zeta(\omega) > s$ and the limit $K^\alpha_\varphi(\zeta, \omega)$ or the limit $K^\alpha_\varphi(t+0, \omega)$ does not exist for any $\alpha \in R$, $\varphi \in W$, $t > s$. Let $u(x) = P^h_x(\tilde{\Omega}_0)$. We note that by (2.3)

$$M^h_x u(x_s) = M^h_x P^h_{x_s}(\tilde{\Omega}_0) = P^h_x(\theta_s \tilde{\Omega}_0) = P^h_x(\tilde{\Omega}_s).$$

Hence it is clear that $M^h_x u(x_s) \uparrow u(x)$ for $s \downarrow 0$, and by Lemma 1.3 $u$ is $h$-excessive. It has been proved that $\gamma^h(u) = P^h(\tilde{\Omega}_0) = 0$, and by 6.2.A $0 = u(x) = P^h_x(\tilde{\Omega}_0)$ for all $x \in E^h$.

For $\omega \in \tilde{\Omega}_0$ we change the values of $K^\alpha_\varphi(t, \omega)$ putting it equal to zero on the whole interval $(0, \zeta(\omega))$. Outside $\tilde{\Omega}_0$ we keep $K^\alpha_\varphi(t, \omega)$ unaltered. It is easy to see that the modified functions $K^\alpha_\varphi$ as before, satisfy 6.4.A and 6.4.B and also 6.4.C and 6.4.D.

6.5. We put $\omega \in \Omega'$ if $K^\alpha_\varphi(t, \omega) \neq (K^\alpha_{x_t}, \varphi)$ for any $\alpha \in R$, $\varphi \in W$, $t \in R \cap (0, \zeta(\omega))$. By 6.4.A, $P^h_x(\Omega') = 0$ for all $h$ and all $x \in E^h$. Hence, we may remove a set $\Omega'$ from the space of elementary events without changing the transition functions $p^h(t, x, \Gamma)$. We make use of this fact and understand by $h$-processes $X^h$ Markov processes with restriction to the space of elementary events (which we again denote by $\Omega$). Everything stated in §6 about $h$-processes remains valid (except, of course, the definition of $\Omega$). In particular, the functions $K^\alpha_\varphi(t, \omega)$ constructed in

[1] By 5.4.C, condition $(G)$ of §3.1 is satisfied for the process $X$ and the measure $\gamma$. We put $\psi^h = \psi/h$ on $E^h$, $\psi = 0$ elsewhere. By 6.2.C, $\gamma^h[G^h\psi^h] = \gamma(G\psi) < \infty$. Since $\psi^h > 0$ on $E^h$, condition $(G)$ is satisfied for the process $X^h$ and the measure $\gamma^h$.

Theorem 6.1 satisfy 6.4.A – 6.4.D, as before. Moreover they have the following properties:

6.5.A. $K_\varphi^\alpha(t,\omega) = (K_{x_t}^\alpha, \varphi)$, if $0 < t < \zeta(\omega)$, $t \in R$.

6.5.B. For any $t > 0$

$$K_\varphi^\alpha(t,\omega) = \lim_{\substack{u \uparrow t \\ u \in R}} (K_{x_u}^\alpha, \varphi).$$

Property 6.5.A is evident, 6.5.B follows from 6.4.B and 6.5.A.

## §7. The Martin compactum

7.1. Let $\mathscr{E}$ be a compactum. Suppose that to each point $z$ on $\mathscr{E}$ and to each number $\alpha \in R$ there corresponds a measure $\mathscr{K}_z^\alpha(dx)$ on the $\sigma$-algebra $\mathscr{B}$ of the space $E$. Suppose, further, that we are given a mapping $i\colon E \to \mathscr{E}$. The collection $(\mathscr{E}, \mathscr{K}_z^\alpha, i)$ is called *a Martin compactum* for the $M$-process $\mathscr{X}$ if the following conditions are satisfied:

7.1.A. The $\mathscr{K}_z^\alpha(\varphi)$ $(\alpha \in R, \varphi \in W)$ are continuous on $\mathscr{E}$ and separate the points of $\mathscr{E}$.

7.1.B. For each $y \in E$

$$\mathscr{K}_{i(y)}^\alpha(dx) = K_y^\alpha(x)\, m(dx), \tag{7.1}$$

where $K_y^\alpha(x)$ is the Martin function for the $M$-process $\mathscr{X}$ (see (6.16)).

7.1.C. $i(E)$ is everywhere dense in $\mathscr{E}$.

We consider the $\sigma$-algebra $\mathscr{B}(\mathscr{E})$ of all Borel sets of $\mathscr{E}$ and denote by $\overline{\mathscr{B}}(\mathscr{E})$ its completion with respect to the system of all sets of finite measure. We put $V(\mathscr{E}) = V(\mathscr{E}, \mathscr{B}(\mathscr{E}))$, $\overline{V}(\mathscr{E}) = V(\mathscr{E}, \overline{\mathscr{B}}(\mathscr{E}))$ and denote by $C(\mathscr{E})$ the set of all non-negative continuous functions on $\mathscr{E}$.

We prove that 7.1.A and 5.1.B imply the following property:

7.1.D. For any $\alpha \in R$, $\varphi \in V$, $\mathscr{K}_z^\alpha(\varphi)$ is $\overline{\mathscr{B}}(\mathscr{E})$-measurable.

Let $\nu$ be a finite measure on $\mathscr{B}(\mathscr{E})$. We can define a sequence of measures $\nu_n$, each concentrated in a finite number of points, so that[1] for any $F \in C(\mathscr{E})$ $\nu_n(F) \to \nu(F)$. By 7.1.A for $\alpha \in R$, $\varphi \in W$,

$$\int_{\mathscr{E}} \mathscr{K}_z^\alpha(\varphi)\, \nu_n(dz) \to \int_{\mathscr{E}} \mathscr{K}_z^\alpha(\varphi)\, \nu(dz).$$

However, it is evident that the left-hand side can be written in the form $\mu_n^\alpha(\varphi)$, where $\mu_n^\alpha$ is a measure on $\mathscr{B}$, and by 5.1.C, there exists a measure $\mu^\alpha$ on $\mathscr{B}$ such that

$$\mu^\alpha(\varphi) = \int_{\mathscr{E}} \mathscr{K}_z^\alpha(\varphi)\, \nu(dz). \tag{7.1'}$$

---

[1] We decompose $\mathscr{E}$ into sets $\Gamma_j$ $(j = 1, 2, \ldots, k_n)$ with diameters not exceeding $1/n$ and select in each $\Gamma_j$ any point $x_j$. The measure $\nu_n$ can be defined by placing at $x_j$ a mass equal to $\nu(\Gamma_j)$.

We denote by $\tilde{V}$ the set of all functions $\varphi \in V$ such that for any $\alpha \in R$ $\mathscr{K}_z^\alpha(\varphi)$ is a Borel function and satisfies (7.1′). Evidently $\tilde{V}$ satisfies a) – c) of 5.1.B. By 5.1.B, for any $f \in V$ there exist $f_1, f_2 \in \tilde{V}$ such that $f_1 \leqslant f \leqslant f_2$ and $f_1 = f_2$ $\mu^\alpha$-a.e.. Clearly, $\mathscr{K}_z^\alpha(f_1) \leqslant \mathscr{K}_z^\alpha(f) \leqslant \mathscr{K}_z^\alpha(f_2)$. $\mathscr{K}_z^\alpha(f_1)$ and $\mathscr{K}_z^\alpha(f_2)$ are Borel functions and coincide $\nu$-a.e.. Hence $\mathscr{K}_z^\alpha(f)$ is $\overline{\mathscr{B}}(\mathscr{E})$-measurable.

THEOREM 7.1. *For each M-process $\mathscr{X}$ one and to within isomorphism only one Martin compactum can be constructed.*

*The mapping $i$ of $(E, \mathscr{B})$ into $(\mathscr{E}, \mathscr{B}_{\mathscr{E}})$, and also of $(E, \overline{\mathscr{B}})$ into $(\mathscr{E}, \overline{\mathscr{B}}(\mathscr{E}))$ is measurable.*[1]

*For a special M-process the mapping $i$ is continuous.*

PROOF. We denote by $\mathfrak{M}$ the set of measures $\mu^\alpha$, depending on a parameter $\alpha \in R$, on the $\sigma$-algebra $\mathscr{B}$. We put $\mu^\alpha \in \mathfrak{G}$, if, for any $\alpha \in R$ and $\varphi \in W$, we have $\mu^\alpha(\varphi) \leqslant c_\varphi^\alpha$ where $c_\varphi^\alpha$, are the constants of condition 5.4.B. We define a topology in $\mathfrak{M}$ by the system of neighbourhoods

$$a < \mu^\alpha(\varphi) < b \qquad (a, b, \alpha \in R, \ \varphi \in W);$$

in other words, we say that $\mu_m^\alpha \to \mu^\alpha$ if for all $\alpha \in R$, $\varphi \in W$ we have $\mu_m^\alpha(\varphi) \to \mu^\alpha(\varphi)$. We note that if $\mu_m^\alpha \in \mathfrak{G}$ and for any $\alpha \in R$, $\varphi \in W$ the numerical sequence $\mu_m^\alpha(\varphi)$ $(m = 1, 2, \ldots)$ converges, then the sequence $\mu_m^\alpha$ converges in the topology $\mathfrak{M}$ to some point of the set $\mathfrak{G}$.

We prove that $\mathfrak{G}$ is a compactum. Let $\mu_m^\alpha \in \mathfrak{G}$. Then the numbers $\mu_m^\alpha(\varphi)$ form for any $\alpha \in R$, $\varphi \in W$ a bounded sequence. By a diagonal process we can select from $\mu_m^\alpha$ a subsequence $\tilde{\mu}_m^\alpha$ such that $\tilde{\mu}_m^\alpha(\varphi)$ converges for any $\alpha \in R$, $\varphi \in W$. By what we have proved, the sequence $\tilde{\mu}_m^\alpha$ converges in the topology $\mathfrak{M}$ to the point $\mu^\alpha \in \mathfrak{G}$.

By the construction of the compactum $\mathfrak{G}$, to each point $z \in \mathfrak{G}$ there corresponds a family of measures $\mu^\alpha$. We put $\mathscr{K}_z^\alpha = \mu^\alpha$. Evidently the $\mathscr{K}_z^\alpha(\varphi)$ $(\alpha \in R, \varphi \in W)$ are continuous. If $\mathscr{K}_{z_1}^\alpha(\varphi) = \mathscr{K}_{z_2}^\alpha(\varphi)$ for all $\alpha \in R$, $\varphi \in W$, then by 5.1.B $\mathscr{K}_{z_1}^\alpha = \mathscr{K}_{z_2}^\alpha$ for all $\alpha \in R$ and hence $z_1 = z_2$. Thus, 7.1.A is satisfied.

We define the mapping $i$ by associating with $y \in E$ the function

$$\mu^\alpha(dx) = K_y^\alpha(x)\, m(dx).$$

By (6.18) and 5.7.B $\mu^\alpha(\varphi) \leqslant c_\varphi^\alpha$, so that $\mu^\alpha \in \mathfrak{G}$. The mapping $i$ so defined satisfies 7.1.B. To satisfy 7.1.C it is sufficient to select for $\mathscr{E}$ the closure of $i(E)$ in $\mathfrak{G}$.

We now suppose that $(\tilde{\mathscr{E}}, \tilde{\mathscr{K}}_z^\alpha, \tilde{i})$ is any Martin compactum. To each $z \in \tilde{\mathscr{E}}$ there corresponds the set of measures $\tilde{\mathscr{K}}_z^\alpha$, that is, a point of the topological space $\mathfrak{M}$. We denote it by $j(z)$. By 7.1.A, $j$ is a continuous one-to-one mapping of $\tilde{\mathscr{E}}$ onto $j(\tilde{\mathscr{E}})$. Since $\tilde{\mathscr{E}}$ is compact, this is a homeomorphism of $\tilde{\mathscr{E}}$ onto $j(\tilde{\mathscr{E}})$. Next, by 7.1.B, $j\tilde{i}(y) = i(y)$ and hence $j\tilde{i}(E) = i(E)$. Since $\tilde{i}(E)$ is everywhere dense in $\tilde{\mathscr{E}}$ (condition 7.1.C), we see that $i(E)$ is everywhere dense in $j(\tilde{\mathscr{E}})$, and so $j(\tilde{\mathscr{E}}) = \mathscr{E}$. Thus, $j$ is a homeomorphism of $\tilde{\mathscr{E}}$ onto $\mathscr{E}$, carrying $\tilde{i}$ into $i$. Clearly, $\mathscr{K}_{j(z)}^\alpha = \tilde{\mathscr{K}}_z^\alpha$. Therefore $j$ is an isomorphism of $(\tilde{\mathscr{E}}, \tilde{\mathscr{K}}_z^\alpha, \tilde{i})$ onto $(\mathscr{E}, \mathscr{K}_z^\alpha, i)$.

The sets $\{z\colon a < \mathscr{K}_z^\alpha(\varphi) < b\}$ $(\varphi \in W, a, \dot{\alpha}, \beta \in R)$ form a base for the topology in $\mathscr{E}$ By (7.1) and (6.17) the images of these sets under $i$

[1] $\overline{\mathscr{B}}$ is the completion of the $\sigma$-algebra $\mathscr{B}$ with respect to the system of all finite measures.

belong to the $\sigma$-algebra $\mathscr{B}$. Hence, $i$ is a measurable mapping of $(E, \mathscr{B})$ into $(\mathscr{E}, \mathscr{B}(\mathscr{E}))$. Hence it easily follows that $i^{-1}(\Gamma) \in \overline{\mathscr{B}}$ for $\Gamma \in \overline{\mathscr{B}}(\mathscr{E})$ so that $i$ is a measurable mapping of $(E, \overline{\mathscr{B}})$ into $(\mathscr{E}, \overline{\mathscr{B}}(\mathscr{E}))$.

In the case of a special $M$-process, $E_0 = E$ by 5.8. From (6.17), 5.8.B and 5.8.E it is evident that $i$ is continuous.

7.2. We say that $\tau$ is an $L_h$-moment if $\tau$ is a $L$-moment for the process $X^h$. The $\sigma$-algebra $\mathscr{N}^{\tau}_{\infty}$ constructed for such a moment $\tau$ (and with respect to the process $X^h$) is denoted by $\mathscr{N}^{\tau}_{\infty}(h)$. We note that $\zeta$ is an $L_h$-moment for any excessive function $h$.

THEOREM 7.2. *Let $(\mathscr{E}, \mathscr{K}^{\alpha}_z, i)$ be a Martin compactum for the M-process $\mathscr{X}$. We can define in $\mathscr{E}$ a random process $z_t = z_t(\omega)$ $(0 < t \leqslant \zeta(\omega))$, continuous on the left with respect to t, having a right-hand limit at each point $t \in (0, \zeta)$, and such that*

$$\mathscr{K}^{\alpha}_{z_t}(\varphi) = \mathrm{K}^{\alpha}_{\varphi}(t, \omega) \qquad (\alpha \in R, \ \varphi \in W), \tag{7.2}$$

*where $\mathrm{K}^{\alpha}_{\varphi}(t, \omega)$ are the functions defined in Theorem 6.1. Also*

$$i(x_t) = z_t \qquad \text{for} \qquad t \in R \cap (0, \zeta). \tag{7.3}$$

*If $f \in V(\mathscr{E})$, then $f(z_t(\omega))$ is $\mathscr{N}_t$-measurable with respect to $\omega$ and $\mathscr{B}^0_{\infty} \times \mathscr{N}$ -measurable with respect to the pair $t$, $\omega$.*

*For any $\gamma$-integrable excessive function h*

$$\mathrm{P}^h_x\{z_t \neq i(x_t)\} = 0 \qquad (t > 0, \ x \in E^h); \tag{7.4}$$

*for all $x \in E^h$, $u \geqslant 0$ $\mathrm{P}^h_x$-almost surely*

$$\theta_u z_t = z_{t+u} \qquad \text{for all} \qquad t \in (0, \zeta - u) \tag{7.5}$$

*and for any $f \in V(\mathscr{E})$ $f(x_{\zeta})$ is $\mathscr{N}^{\zeta}_{\infty}(h)$-measurable.*

*If $\mathscr{X}$ is a special M-process, then for $\mathrm{P}^h_x$-almost all $\omega$*

$$i(x_{t-0}) = z_t \qquad \text{for all} \qquad t \in (0, \zeta). \tag{7.6}$$

PROOF. We fix $\omega \in \Omega$ and $t \in (0, \zeta(\omega)]$. From 6.5.B, 5.1.C and (6.18) it follows that there exists a measure $\mu^{\alpha}$ such that

$$\mu^{\alpha}(\varphi) = \mathrm{K}^{\alpha}_{\varphi}(t, \omega) \qquad \text{for all} \qquad \varphi \in W. \tag{7.7}$$

The family of measures $\mu^{\alpha}$ defines a point $z \in \mathfrak{E}$. Indicating the dependence of $z$ on $t$ and $\omega$, we write $z_t(\omega)$. By definition of the measures $K^{\alpha}_z$ (7.7) is equivalent to (7.2). Now (7.3) and (7.4) follow by 7.1.B and 7.1.A, from 6.5.A and 6.4.A. From 6.4.B, 6.4.C and 6.4.D there follow the properties of the continuity of $z_t$ listed in the Theorem. Since $z_t = i(x_t) \in i(E)$ for $t \in R$, we have $z_t \in \mathscr{E}$ for any $t$.

By (2.3) and (7.4) $\mathrm{P}^h_x\{\theta_u[z_t \neq i(x_t)]\} = \mathrm{M}^h_x \mathrm{P}^h_{x_u}\{z_t \neq i(x_t)\} = 0$. Hence, $\mathrm{P}^h_x$-almost surely on $\Omega_{t+u}$

$$\theta_u z_t = \theta_u i(x_t) = i(x_{t+u}) = z_{t+u}.$$

(7.5) now follows by 6.4.B.

Let $f \in C(\mathscr{E})$. Then by (7.3)

$$f(z_t) = \lim_{u \uparrow t,\, u \in R} f[i(x_u)].$$

Hence $f(z_t)$ is $\mathscr{N}_t$-measurable. Since $\mathscr{N}_t \subseteq \mathscr{N}$ and $f(z_t)$ is continuous on the left with respect to $t$, it is $\mathscr{B}^0_\infty \times \mathscr{N}$ measurable with respect to $t$, $\omega$. Hence $f(z_\zeta)$ is $\mathscr{N}$-measurable. Next, by (7.5) $P^h_x$-almost surely on $\Omega_u$

$$\theta_u f(z_\zeta) = \theta_u \lim_{t \uparrow \zeta} f(z_t) = \lim_{t \uparrow \zeta - u} f(z_{t+u}) = f(x_\zeta). \tag{7.8}$$

Hence $f(z_\zeta)$ is $\mathscr{N}^\zeta_\infty$-measurable. Thus, the system $\mathscr{H}$ of functions for which the properties of measurability in Theorem 7.2 hold contains $C(\mathscr{E})$. By the footnote on p. 120, $\mathscr{H}$ contains the indicators of all open sets, and by Lemma 0.1 $\mathscr{H} \supseteq V(\mathscr{E})$.

For a special $M$-process by Theorem 7.1 the mapping $i$ is continuous. In this case $i(x_{t-0})$ as well as $z_t$ is continuous on the left with respect to $t$ in the interval $(0, \zeta)$. Hence (7.5) follows from (7.3), 5.8.C and Lemma 6.1.

## §8. Distribution of $\zeta$ and $z_\zeta$. Integral Representation of Excessive Functions

**8.1.** Throughout this section $h$ denotes an arbitrary $\gamma$-integrable excessive function.

LEMMA 8.1. *Let $\tau$ be an $L_h$-moment, $\xi$ be $\mathscr{N}^\tau_\infty(h)$-measurable. Then,*[1] *for any $\varphi \in V$, $\alpha \in R$, $t > 0$*

$$\int_E m(dx)\,\varphi(x)\,h(x)\,\mathbf{M}^h_x e^{-\alpha(\tau-t)}\chi_{t<\tau<\infty}\xi = \mathbf{M}^h \mathscr{K}^\alpha_{z_{\tau-t}}(\varphi)\,\chi_{t<\tau<\infty}\xi. \tag{8.1}$$

*If $\xi$ is $\mathscr{N}^\zeta_\infty(h)$-measurable, then*

$$\int_E m(dx)\,\varphi(x)\,h(x)\,\mathbf{M}^h_x e^{-\alpha\zeta}\xi = \mathbf{M}^h \mathscr{K}^\alpha_{z_\zeta}(\varphi)\,\xi. \tag{8.2}$$

PROOF. It is sufficient to consider the case when $\xi$ is bounded. (8.1) can be written in the form

$$\mu_1(\varphi) = \mu_2(\varphi), \tag{8.3}$$

where

$$\mu_1(\Gamma) = \int_\Gamma m(dx)\,h(x)\,\mathbf{M}^h_x e^{-\alpha\tau}\chi_{0<\tau<\infty}\xi \tag{8.4}$$

and

$$\mu_2(\Gamma) = \mathbf{M}^h \mathscr{K}^\alpha_{z_\tau}(\Gamma)\,\chi_{0<\tau<\infty}\xi. \tag{8.5}$$

---

[1] Although the integrands in (8.1) and (8.2) are not defined on the set $E \setminus E^h_*$, this does not matter since $m(E \setminus E^h_*) = 0$.

By 5.4.B $\mu_1(\varphi) < \infty$ for all $\varphi \in W$. By 5.1.B, to prove (8.3) for all $\varphi \in V$ it is sufficient to prove it for $\varphi \in W$.

In proving Theorem 6.1 (see 6.4) we established that $f_\alpha(y) = (K^\alpha_y, \varphi)$ is the density of the measure $\mu G^h_\alpha$ with respect to the measure $\gamma^h G^h$, where

$$\mu(dy) = \varphi(y)\, h(y)\, m(dy) \tag{8.6}$$

is a finite measure on $\mathscr{B}$. By 7.1.A, 7.1.B and (7.4), $\mathscr{K}^\alpha_{z_t}(\varphi)$ is a left-continuous modification of $f_\alpha(x_t)$. Applying Lemma 4.1 to the process $X^h$ and the measures $\mu$ and $\gamma^h$, we find

$$\mathbf{M}^h \mathscr{K}^\alpha_{z_{\tau-t}}(\varphi)\, \chi_{t<\tau<\infty}\xi = \mathbf{M}_\mu e^{-\alpha(\tau-t)}\xi\chi_{t<\tau<\infty}. \tag{8.7}$$

Evidently (8.7) is equivalent to (8.1).

By the footnote on p. 125, Lemma 3.1 can be applied to the process $X^h$ and the measure $\gamma^h$. Substituting in (8.1) the $L_h$-moment $\tau_a$ determined by this lemma and letting $\downarrow 0$, we have

$$\int_E m(dx)\, \varphi(x)\, h(x)\, \mathbf{M}^h_x(e^{-\alpha\zeta}\xi\chi_C) = \mathbf{M}^h \mathscr{K}^\alpha_{z_\zeta}(\varphi)\, \xi\chi_C, \tag{8.8}$$

where $C = \bigcap_n \{\tau_{\frac{1}{n}} < \zeta\}$. By Lemma 3.1, $\mathrm{P}^h\{\Omega \setminus C\} = 0$. It is not difficult to see that $C \in \mathscr{N}^{\cdot\zeta}_\infty(h)$ and by 6.2.A $\mathrm{P}^h_x(\Omega \setminus C) = 0$ for all $x \in E^h$. Hence (8.8) is equivalent to (8.2).

8.2. We put

$$\mu_h(\Gamma) = \mathbf{P}^h\{z_\zeta \in \Gamma\} \qquad (\Gamma \in \mathscr{B}(\mathscr{E})). \tag{8.9}$$

Let $\tilde{g}_\alpha(x, y)$ be a regularized Green function. The ratio $\tilde{K}^\alpha_y(x) = \tilde{g}_\alpha(x, y) / q(y)$ is called a regularized Martin function. (Clearly, $\tilde{K}^\alpha_y(x)$ is the regularization of the Martin function $K^\alpha_y(x)$.)

THEOREM 8.1. *In $\mathscr{E}$ a set $\mathscr{E}_1 \in \overline{\mathscr{B}}(\mathscr{E})$, can be selected so that for any $\gamma$-integrable excessive function $h$ $\mu_h(\mathscr{E} \setminus \mathscr{E}_1) = 0$ and for $z \in \mathscr{E}_1$*

$$\mathscr{K}^\alpha_z(dx) = k^\alpha_z(x)\, m(dx), \tag{8.10}$$

*where $k^\alpha_z(x)$ has the following properties:*

8.2.A. *$k^\alpha_z$ is a $\gamma$-integrable $\alpha$-excessive function and $\gamma(k^\alpha_z) \leqslant 1$.*

8.2.B. *$k^\alpha_z(x)$ is $\mathscr{B} \times \overline{\mathscr{B}}(\mathscr{E})$-measurable with respect to $x$, $z$, $(x \in E, z \in \mathscr{E}_1)$*

8.2.C. *If $y \in E_1$, then $i(y) \in \mathscr{E}_1$ u and*

$$k^\alpha_{i(y)}(x) = \tilde{K}^\alpha_y(x), \tag{8.11}$$

*where $\tilde{K}^\alpha_y(x)$ is the regularized Martin function.*

8.2.D. For $z \in \mathscr{E}_1$, and any $\alpha$, $\lambda \in R$

$$\lambda G_{\lambda+\alpha} k^\alpha_z + k^{\alpha+\lambda}_z = k^\alpha_z. \tag{8.12}$$

*On the set* $E_*^h$ *for any* $\mathcal{N}_\infty^{\cdot\zeta}$ *(h)-integrable function* $\xi$

$$h(x)\,\mathbf{M}_x^h e^{-\alpha\zeta}\xi = \mathbf{M}^h k_{z_\zeta}^\alpha(x)\,\xi \tag{8.13}$$

*and for any* $f \in \overline{V}(\mathscr{E})$

$$h(x)\,\mathbf{M}_x^h e^{-\alpha\zeta} f(z_\zeta) = \int_{\mathscr{E}_1} f(z)\,k_z^\alpha(x)\,\mu_h(dz). \tag{8.14}$$

*For all* $x \in E$

$$h(x) = \int_{\mathscr{E}_1} k_z(x)\,\mu_h(dz). \tag{8.15}$$

**PROOF.** Let $\tilde{g}_\alpha(x, y)$ be the regularized Green function and $\mu$ some measure on $\mathscr{B}$. We substitute in (8.2) the function

$$\varphi(y) = \int_E \mu(dx)\,\lambda\tilde{g}_{\lambda+\alpha}(x, y).$$

Putting

$$\left.\begin{aligned} Q_\alpha^h(x) &= \mathbf{M}_x^h e^{-\alpha\zeta}\xi, \\ r_\lambda^\alpha(x, z) &= \lambda \int_E \tilde{g}_{\lambda+\alpha}(x, y)\,\mathscr{K}_z^\alpha(dy) \end{aligned}\right\} \tag{8.16}$$

and taking account of (5.8), we have

$$\int_E \mu(dx)\,\lambda G_{\lambda+\alpha}(Q_\alpha^h h)(x) = \mathbf{M}^h\xi \int_E \mu(dx)\,r_\lambda^\alpha(x, z_\zeta). \tag{8.17}$$

By the $\mathscr{B}\times\mathscr{B}$-measurability of $\tilde{g}_{\lambda+\alpha}(x, y)$ and the $\overline{\mathscr{B}}(\mathscr{E})$-measurability of $\mathscr{K}_z^\alpha(\Gamma)$ for $\Gamma \in \mathscr{B}$ $r_\lambda^\alpha(x, z)$ is $\mathscr{B}\times\overline{\mathscr{B}}(\mathscr{E})$-measurable. By 5.6.A, $r_\lambda^\alpha(x, z)$ is $(\lambda + \alpha)$-excessive for any $z \in \mathscr{E}$.

By (6.8) and Lemma 3.4 on $E_*^h$

$$\lambda G_{\lambda+\alpha}(Q_\alpha^h h) = \lambda h G_{\lambda+\alpha}^h Q_\alpha^h = h\,(Q_\alpha^h - Q_{\alpha+\lambda}^h).$$

We substitute this expression in (8.17). Assuming that

$$\mu\{h = \infty\} = 0, \tag{8.18}$$

we obtain

$$\int_E \mu(dx)\,h(x)\,\mathbf{M}_x^h(e^{-\alpha\zeta} - e^{-(\alpha+\lambda)\zeta})\,\xi = \mathbf{M}^h\xi r_\lambda^\alpha(\mu, z_\zeta), \tag{8.19}$$

where

$$r_\lambda^\alpha(\mu, z) = \int_E \mu(dx)\, r_\lambda^\alpha(x, z).$$

Putting here $\xi = f(z_\zeta)$ and $\mu = \gamma$ we have

$$\mathbf{M}^h(e^{-\alpha\zeta} - e^{-(\alpha+\lambda)\zeta}) f(z_\zeta) = \mathbf{M}^h f(z_\zeta)\, r_\lambda^\alpha(\gamma, z_\zeta).$$

Consequently

$$\int_{\mathscr{E}} f(z)\, \mu_h(dz) = \mathbf{M}^h f(z_\zeta) \geqslant \mathbf{M}^h f(z_\zeta)\, r_\lambda^\alpha(\gamma, z_\zeta) = \int_{\mathscr{E}} f(z)\, r_\lambda^\alpha(\gamma, z)\, \mu_h(dz). \qquad (8.20)$$

Next, we put in (8.19) $\xi = f(z_\zeta)$ and $\mu(dx) = \psi(x)m(dx)$, where $\psi \in W$. Transforming the left-hand side by means of (8.2) we get

$$\mathbf{M}^h[\mathscr{K}_{z_\zeta}^\alpha(\psi) - \mathscr{K}_{z_\zeta}^{\alpha+\lambda}(\psi)] f(z_\zeta) = \mathbf{M}^h f(z_\zeta) \int_E m(dx)\, \psi(x)\, r_\lambda^\alpha(x, z_\zeta),$$

or

$$\int_{\mathscr{E}} [\mathscr{K}_z^\alpha(\psi) - K_z^{\alpha+\lambda}(\psi)] f(z)\, \mu_h(dz) = \int_{\mathscr{E}} \mu_h(dz) f(z) \int_E m(dx)\, \psi(x)\, r_\lambda^\alpha(x, z). \qquad (8.21)$$

Finally, passing to the limit in (8.2) and using Fatou's lemma we see that $P^h$-almost surely $\varliminf_{\alpha\to\infty} \mathscr{K}_{z_\zeta}^\alpha(\psi) = 0$, and hence for $\mu_h$-almost all $z$

$$\varliminf_{\alpha\to\infty} \mathscr{K}_z^\alpha(\psi) = 0 \qquad (\psi \in W). \qquad (8.22)$$

We put $z \in \mathscr{E}_1$ if:

a) for all $\psi \in W$, $\alpha, \lambda \in R$

$$\mathscr{K}_z^\alpha(\psi) = \mathscr{K}_z^{\alpha+\lambda}(\psi) + \int_E m(dx)\, \psi(x)\, r_\lambda^\alpha(x, z); \qquad (8.23)$$

b) for all $\alpha, \lambda \in R$

$$r_\lambda^\alpha(\gamma, z) \leqslant 1;$$

c) for all $\psi \in W$

$$\varliminf_{\alpha\to\infty} \mathscr{K}_z^\alpha(\psi) = 0.$$

Evidently $\mathscr{E}_1 \in \overline{\mathscr{B}}(\mathscr{E})$. By (8.20) - (8.22) $\mu_h(\mathscr{E} \setminus \mathscr{E}_1) = 0$. From a) it is clear that $\mathscr{K}_z^\alpha(\psi)$ for $z \in \mathscr{E}_1$ is a non-increasing function of $\alpha$ and hence the integral in (8.23) is a non-decreasing function of $\lambda$. Therefore, if $\lambda_1 > \lambda_2$ and $z \in \mathscr{E}_1$, then

$$r^{\alpha}_{\lambda_1}(x, z) \geqslant r^{\alpha}_{\lambda_2}(x, z)$$

for $m$-almost all $x$. Both sides of this inequality are $(\lambda_1 + \alpha)$-excessive, and by Lemma 1.1 the inequality is satisfied for all $x \in E$. We denote by $k^{\alpha}_z(x)$ the limit of $r^{\alpha}_{\lambda}(x, z)$ for $\lambda \to \infty$. Then

$$r^{\alpha}_{\lambda}(x, z) \uparrow k^{\alpha}_z(x) \qquad (x \in E,\ z \in \mathscr{E}_1,\ \alpha \in R). \tag{8.24}$$

By a) and c)

$$\mathscr{K}^{\alpha}_z(\psi) = \varliminf_{\lambda \to \infty} \int_E m(dx)\,\psi(x)\,r^{\alpha}_{\lambda}(x, z) = \int_E m(dx)\,\psi(x)\,k^{\alpha}_z(x) \qquad (z \in \mathscr{E}_1).$$

(8.10) now follows by 5.1.B. Substituting (8.10) in (8.16) and taking account of (5.8) we have

$$r^{\alpha}_{\lambda}(x, z) = \lambda G_{\lambda+\alpha} k^{\alpha}_z(x). \tag{8.25}$$

From (8.24) it is clear that $k^{\alpha}_z$ is $\alpha$-excessive, and from b) that $\gamma(k^{\alpha}_z) \leqslant 1$. From the measurability of $r^{\alpha}_{\lambda}(x, z)$ the measurability of $k^{\alpha}_z(x)$ follows, and so we have proved 8.2.A and 8.2.B.

Now we prove 8.2.C. Let $y \in E_1$ and $z = i(y)$. From (8.16), 7.1.B, (5.8) and (6.16) we have

$$r^{\alpha}_{\lambda}(x, z) = \lambda G_{\lambda+\alpha} K^{\alpha}_y(x) = \frac{\lambda C_{\lambda+\alpha} g^{\alpha}_y(x)}{q(y)}, \tag{8.26}$$

and by (5.5)

$$K^{\alpha+\lambda}_y(x) + r^{\alpha}_{\lambda}(x, z) = K^{\alpha}_y(x) \qquad \text{for } m\text{-almost all } x.$$

Hence (8.23) follows. Next, from (8.26), (5.12), 5.6.A, (5.11) and (5.2)

$$r^{\alpha}_{\lambda}(\gamma, z) = q(y)^{-1}\gamma[\lambda G_{\lambda+\alpha} g^{\alpha}_y] = q(y)^{-1}\gamma[\lambda G_{\lambda+\alpha}\tilde{g}^{\alpha}_y] \leqslant$$
$$\leqslant q(y)^{-1}\gamma(\tilde{g}^{\alpha}_y) \leqslant q(y)^{-1}\gamma(\tilde{g}_y) = q(y)^{-1}\gamma(g_y) = 1.$$

Finally, from 7.1.B, (6.16), (5.5), (5.6) and (5.12) it follows that as $\alpha \to \infty$

$$\mathscr{K}^{\alpha}_z(\psi) = q(y)^{-1}(g^{\alpha}_y, \psi) = q(y)^{-1}(g_y - \alpha G_{\alpha} g_y, \psi) \to q(y)^{-1}(g_y - \tilde{g}_y, \psi) = 0.$$

Hence $z \in \mathscr{E}_1$. Comparing (8.25) and (8.26) and letting $\lambda \to \infty$ we arrive at (8.11).

By 5.1.B it follows from (8.23), (8.10) and (8.25) that for $z \in \mathscr{E}_1$

$$k^{\alpha}_z(x)\,m(dx) = k^{\alpha+\lambda}_z(x)\,m(dx) + \lambda G_{\lambda+\alpha} k^{\alpha}_z(x)\,m(dx).$$

Since all functions occurring here are $(\lambda + \alpha)$-excessive, by Lemma 1.1 we have 8.2.D.

Let $x \in E_*^h$. Then we are justified in putting $\mu = \delta_x$ in (8.19). Taking the limit for $\lambda \to \infty$ and noting (8.24) we have (8.13).

If $f \in V(\mathscr{E})$, then by Theorem 7.2 $f(z_\zeta)$ is $\mathscr{N}^{\zeta}_{\infty}(h)$-measurable, and putting in (8.13) $\xi = f(z_\zeta)$ we obtain (8.14). However, if $f \in \overline{V}(\mathscr{E})$, then we consider $f_1, f_2 \in V(\mathscr{E})$, such that $f_1 \leqslant f \leqslant f_2$ and $f_1 = f_2$ $\mu$-almost everywhere, where

$$\mu(\Gamma) = \mathbf{P}_x^h\{z_\zeta \in \Gamma\} + \mu_h(\Gamma).$$

It is easy to see that

$$\mathbf{M}_x^h e^{-\alpha\zeta} f(z_\zeta) = \mathbf{M}_x^h e^{-\alpha\zeta} f_1(z_\zeta),$$

$$\int_{\mathscr{E}_1} f(z)\, k_z^\alpha(x)\, \mu_h(dz) = \int_{\mathscr{E}_1} f_1(z)\, k_z^\alpha(x)\, \mu_h(dz).$$

Therefore, since (8.13) holds for $f_1$, it also holds for $f$.

Equation (8.15) for $x \in E_*^h$ is obtained if we put in (8.14) $f = 1$, $\alpha = 0$. From 8.2.A and 6.1.E it follows that (8.15) is satisfied for all $x \in E$.

8.3. Formula (8.15) gives an integral representation for any $\gamma$-integrable excessive function $h$. The measure $\mu_h$ occurring there is called *the spectral measure of the excessive function h.*

Theorem 8.2 describes the decomposition of the spectral measure into two components, one corresponding to non-terminating trajectories, the other to terminating ones.

THEOREM 8.2. *We denote by $\mathscr{E}_1'$ the set of $z \in \mathscr{E}_1$, for which $k_z^\alpha = 0$ for some $\alpha \in R$, and we put $\mathscr{E}_1'' = \mathscr{E}_1 \setminus \mathscr{E}_1'$.*

*Then, on the set $E_*^h$ for any $f \in V(\mathscr{E})$*

$$h(x)\, \mathbf{M}_x^h f(z_\zeta)\, \chi_{\zeta=\infty} = \int_{\mathscr{E}_1'} f(z)\, k_z(x)\, \mu_h(dz) \tag{8.27}$$

*and for all $\alpha \in R$*

$$h(x)\, \mathbf{M}_x^h f(z_\zeta)\, e^{-\alpha\zeta} \chi_{\zeta<\infty} = \int_{\mathscr{E}_1''} f(z)\, k_z^\alpha(x)\, \mu_h(dz). \tag{8.28}$$

*If $f \in C(\mathscr{E})$ and $\tilde{f}(x) = f[i(x)]$, then*

$$\int_{\mathscr{E}_1'} f(z)\, \mu_h(dz) = \lim_{t\to\infty} \mathbf{M}_\gamma \tilde{f}(x_t)\, h(x_t) = \lim_{t\to\infty} \int_E \gamma(dx)\, P_t(\tilde{f}h)(x), \tag{8.29}$$

$$\int_{\mathscr{E}_1''} f(z)\, \mu_h(dz) = \lim_{t\downarrow 0} \int_E m(dy)\, q(y)\, \tilde{f}(y)\, \frac{h(y) - P_t h(y)}{t}. \tag{8.30}$$

PROOF. We put $\gamma(k_z^\alpha) = a_\alpha(z)$. By 5.4

$$\mathscr{E}_1' = \{z\colon a_\alpha(z) = 0 \text{ for some } \alpha > 0\}.$$

Substituting in (8.13) $\xi = \chi_{\zeta=\infty}$ and integrating with respect to $\gamma$, we find that for $\alpha > 0$, $M^h a_\alpha(z_\zeta)\chi_{\zeta=\infty} = 0$. On the other hand, if $A_\alpha = \{z\colon a_\alpha(z) = 0\}$ and $\xi_\alpha = \chi_{A_\alpha}(z_\zeta)$, then (8.14) implies that $M^h e^{-\alpha\zeta}\xi_\alpha = 0$. Hence, $P^h$-almost surely

$$a_\alpha(z_\zeta)\chi_{\zeta=\infty} = 0 \quad \text{for } \alpha > 0,$$
$$e^{-\alpha\zeta}\xi_\alpha = 0 \quad \text{for } \alpha \geqslant 0,$$

and therefore to within a set of $P^h$-measure zero

$$\{\zeta = \infty\} \subseteq \{a_\alpha(z_\zeta) = 0\} \subseteq \{z_\zeta \in \mathscr{E}_1'\} \qquad (\alpha > 0),$$
$$\{\zeta < \infty\} \subseteq \bigcap_{\alpha \in R} \{\xi_\alpha = 0\} = \bigcap_{\alpha \in R} \{a_\alpha(z_\zeta) > 0\} = \{z_\zeta \in \mathscr{E}_1''\}.$$

Hence, to within a set of $P^h$-measure zero for any $\alpha > 0$

$$\{\zeta = \infty\} = \{a_\alpha(z_\zeta) = 0\} = \{z_\zeta \in \mathscr{E}_1'\}, \tag{8.31}$$
$$\{\zeta < \infty\} = \{a_\alpha(z_\zeta) > 0 \text{ for all } \alpha \in R\} = \{z_\zeta \in \mathscr{E}_1''\}. \tag{8.32}$$

Substituting in (8.13) $\alpha = 0$, $\xi = f(z_\zeta)\chi_{\zeta=\infty}$ and noting (8.31), we have

$$h(x)\,\mathbf{M}_x^h f(z_\zeta)\chi_{\zeta=\infty} = \mathbf{M}^h k_{z_\zeta}(x) f(z_\zeta)\chi_{\zeta=\infty} = \mathbf{M}^h k_{z_\zeta}(x) f(z_\zeta)\chi_{\mathscr{E}_1'}(z_\zeta),$$

from which (8.27) follows. (8.28) follows similarly from (8.32).

We note now that for $f \in C(\mathscr{E})$, from (8.9) and (8.31),

$$\int_{\mathscr{E}_1'} f(z)\,\mu_h(dz) = \mathbf{M}^h f(z_\zeta)\chi_{\mathscr{E}_1'}(z_\zeta) = \mathbf{M}^h f(z_\zeta)\chi_{\zeta=\infty} = \lim_{t\to\infty} \mathbf{M}^h f(z_t).$$

Bearing (7.4) and 6.2.B in mind we hence have (8.29).

Finally, applying Lemma 3.3 to the process $X^h$, the functions $\rho(u) = \chi_{[0,t]}(u)$, $\xi = 1$, $\tilde{f}$ and $F(t) = f(z_t)$, the moment $\tau = \zeta$, the measure $\gamma^h$ and $\alpha = 0$:

$$\int_0^t \mathbf{M}^h f(z_{\zeta-u})\chi_{u<\zeta<\infty}\,du = \int_E m(dy)\,q(y)\,h(y)\,\tilde{f}(y)\,\mathbf{P}_y^h\{0 < \zeta \leqslant t\}. \tag{8.33}$$

We note that by (6.4)

$$\mathbf{P}_y^h\{0 < \zeta \leqslant t\} = \mathbf{P}_y^h\{\zeta > 0\} - \mathbf{P}_y^h\{\zeta > t\} = 1 - \frac{P_t h(y)}{h(y)}.$$

Substituting this expression in (8.33) we obtain

$$\int_0^t \mathbf{M}^h f(z_{\zeta-u})\,\chi_{u<\zeta<\infty}\,du = \int_E m(dy)\,q(y)\,\tilde{f}(y)\,[h(y) - P_t h(y)]. \tag{8.34}$$

Dividing by $t$ and letting $t\downarrow 0$, we have (8.30).

8.4. Theorem 8.3. *Let $z \in \mathscr{E}_1$. With any $\alpha \geqslant 0$ we can associate a function $k_z^\alpha(x)$ so that for $\alpha \in R$ it coincides with the function defined in Theorem 8.1 and that for any $\alpha \geqslant 0$ 8.2.A, 8.2.B, formulae (8.13),(8.14), (8.28) are satisfied, and also the following conditions:*

8.4.A. *For all $\lambda \geqslant 0$*

$$\lambda G_{\lambda+\alpha} k_z^\alpha + k_z^{\alpha+\lambda} = k_z^\alpha.$$

8.4.B. *For each $x \in E$, $k_z^\alpha(x)$ is a non-increasing function of $\alpha$.*

8.4.C. *For $x \in E_*^{k_z}$, $k_z^\alpha(x)$ is continuous with respect to $\alpha$ on $(0, \infty)$.*

PROOF. We consider any function $\hat{k}_z^\alpha$ that is equal to $k_z - \alpha G_\alpha k_z$ on $E_*^{k_z}$. From (6.9) and (1.6) on $E_*^{k_z}$

$$\lambda G_{\lambda+\alpha} \hat{k}_z^\alpha = \lambda G_{\lambda+\alpha} k_z - \alpha\lambda G_{\lambda+\alpha} G_\alpha k_z = (\lambda+\alpha)\, G_{\lambda+\alpha} k_z - \alpha G_\alpha k_z.$$

Clearly $\hat{k}_z^\alpha$ is $\alpha$-excessive and as $\lambda \to \infty$

$$\lambda G_{\lambda+\alpha} \hat{k}_z^\alpha \uparrow k_z - \alpha G_\alpha k_z = \hat{k}_z^\alpha \quad \text{on} \quad E_*^{k_z}. \tag{8.35}$$

We denote by $k_z^\alpha$ the regularization of $\hat{k}_z^\alpha$. From (8.35) $k_z^\alpha = \hat{k}_z^\alpha = k_z - \alpha G_\alpha k_z$ on $E_*^{k_z}$. Therefore

$$k_z = k_z^\alpha + \alpha G_\alpha k_z \tag{8.36}$$

on $E_*^{k_z}$, and hence everywhere on $E$ (Lemma 1.1). From (8.36), (8.12) and Lemma 1.1 it follows that $k_z^\alpha$ coincides for $\alpha \in R$ with the function of Theorem 8.1. Evidently 8.2.A and 8.2.B are satisfied.

From (8.36), Lemma 1.1 and (1.2) it follows that 8.4.C, 8.4.B hold for $x \in E_*^{k_z}$. By 8.2.D, 8.4.A is satisfied for $\alpha, \lambda \in R$. In view of 8.4.C it is satisfied for all $\alpha, \lambda \geqslant 0$ when $x \in E_*^{k_z}$. By Lemma 1.1 this relation is also true if $x \bar{\in} E_*^{k_z}$. From 8.4.A it follows that 8.4.B is valid for all $x \in E$. Using 8.4.B it is not difficult to extend (8.13), (8.14) and (8.28) to all $\alpha \geqslant 0$.

8.5. THEOREM 8.4. *The function $G_\alpha(x, y)$ $(\alpha \geqslant 0,\ x, y \in E)$ can be defined so that it coincides with the regularized Green function $\tilde{g}_\alpha(x, y)$ for $\alpha \in R$, $x \in E$, $y \in E_1$, and has the following properties:*

8.5.A. *For all $\alpha \geqslant 0$, $\lambda > 0$, $x, y \in E$.*

$$G_{\alpha+\lambda}(x, y) + \int_E \lambda G_{\lambda+\alpha}(x, z)\, m(dz)\, G_\alpha(z, y) = G_\alpha(x, y).$$

8.5.B. *$G_\alpha(x, y)$ for any $x, y \in E$ is a non-increasing function of $\alpha$.*

8.5.C. *If $G(x, y) < \infty$, then $G_\alpha(x, y)$ is continuous with respect to $\alpha$ on $(0, \infty)$.*

8.5.D. *For any $\alpha \geqslant 0$, $f \in V$*

$$G_\alpha f(x) = \int_E G_\alpha(x, y) f(y) m(dy). \tag{8.37}$$

PROOF. For all real $\alpha \geqslant 0$ we put

$$G_\alpha(x, y) = \begin{cases} q(y) k^\alpha_{i(y)}(x) & \text{if } x \in E,\ i(y) \in \mathscr{E}_1, \\ 0 & \text{if } x \in E,\ i(y) \overline{\in} \mathscr{E}_1. \end{cases} \tag{8.38}$$

In view of 8.2.B for $\alpha \in R$, $x \in E$, $y \in E_1$, this function coincides with $\tilde{g}_\alpha(x, y)$. 8.5.A – 8.5.C follow from 8.4.A – 8.4.C for $i(y) \in \mathscr{E}_1$ and are satisfied trivially if $i(y) \overline{\in} \mathscr{E}_1$. 8.5.D follows from (5.8) and (5.13) for $\alpha \in R$. To extend it to all real $\alpha > 0$ it is sufficient to note that the left-hand side of (8.37) is continuous with respect to $\alpha \in (0, \infty)$, provided that $f$ is bounded, and to use 8.5.B.

## §9. The Space of Exits

9.1. By Theorem 8.1 to each $z \in \mathscr{E}_1$ there corresponds the $\gamma$-integrable excessive function $k_z$. We denote its spectral measure by $\mu_z$ and we write for short $P^z$, $M^z$, $E^z, \ldots$ instead of $P^{k_z}$, $M^{k_z}$, $E^{k_z}, \ldots$ The set of points $z \in \mathscr{E}_1$, for which $\mu_z$ coincides with the unit measure $\delta_z$ concentrated at the point $z$ is called the space of exits and is denoted by $\mathscr{U}$.

THEOREM 9.1. *The space of* exits *$\mathscr{U}$ belongs to the $\sigma$-algebra $\overline{\mathscr{B}}(\mathscr{E})$ and $\mu_h(\mathscr{E} \setminus \mathscr{U}) = 0$ for all spectral measures $\mu_h$. If $z \in \mathscr{U}$, then $\gamma(k_z) = 1$.*

PROOF. From (8.9), (6.12) and 8.2.A

$$\mu_z(\mathscr{E}) = \mathbf{P}^z\{z_\zeta \in \mathscr{E}\} = \int_{E^z} \gamma^z(dx) \mathbf{P}^z_x\{z_\zeta \in \mathscr{E}\} = \int_{E^z} \gamma(dx) k_z(x) = \gamma(k_z) \leqslant 1. \tag{9.1}$$

Hence it is clear that $\gamma(k_z) = 1$ for $z \in \mathscr{U}$ and that $z \in \mathscr{E}_1$ belongs to $\mathscr{U}$ if and only if

$$\mu_z\{z\} = 1. \tag{9.2}$$

From (8.29) and (8.30) it follows that (9.2) defines a set of $\overline{\mathscr{B}}(\mathscr{E})$.

Let $\varphi$, $f \in C^+(\mathscr{E})$ and $\tilde{f}(x) = f[i(x)]$ $(x \in E)$. By (7.8) $P^h_x\{\theta_u \varphi(z_\zeta) \neq \varphi(z_\zeta),\ \zeta > u\} = 0$. By Theorem 7.2, $f(z_t)$ is $\mathscr{N}_t$-measurable, and using (2.3) we have

$$\mathbf{M}^h f(z_t) \varphi(z_\zeta) = \mathbf{M}^h f(z_t) \mathbf{M}^h_{x_t} \varphi(z_\zeta) = \mathbf{M}^h [\tilde{f}(x_t) \mathbf{M}^h_{x_t} \varphi(z_\zeta)] = \mathbf{M}^h F(x_t), \tag{9.3}$$

where $F(x) = \tilde{f}(x) M^h_x \varphi(z_\zeta)$. Therefore by 6.2.B,

$$\mathbf{M}^h f(z_t) \varphi(z_\zeta) = \mathbf{M}_\gamma (Fh)(x_t) = \mathbf{M}_\gamma [\tilde{f}(x_t) h(x_t) \mathbf{M}^h_{x_t} \varphi(z_\zeta)]. \tag{9.4}$$

From (8.14) it follows that

$$h(x)\,\mathbf{M}_x^h\varphi(z_\zeta)=\int_{\mathscr{E}_1}\varphi(z)\,k_z(x)\,\mu_h(dz).$$

Substituting this expression in (9.4), and then applying Fubini's theorem, 6.2.B and (7.4), we obtain

$$\mathbf{M}^h f(z_t)\,\varphi(z_\zeta)=\int_{\mathscr{E}_1}\mu_h(dz)\,\varphi(z)\,\mathbf{M}_\gamma k_z(x_t)\,\tilde{f}(x_t)=$$
$$=\int_{\mathscr{E}_1}\mu_h(dz)\,\varphi(z)\,\mathbf{M}^z\tilde{f}(x_t)=\int_{\mathscr{E}_1}\mu_h(dz)\,\varphi(z)\,\mathbf{M}^z f(z_t)\,\chi_{\zeta>t}. \quad (9.5)$$

As $t\to\infty$ $f(z_t)\,\chi_{\zeta>t}\to f(z_\zeta)\,\chi_{\zeta=\infty}$. Hence

$$\mathbf{M}^h f(z_\zeta)\,\varphi(z_\zeta)\,\chi_{\zeta=\infty}=\int_{\mathscr{E}_1}\mu_h(dz)\,\varphi(z)\,\mathbf{M}^z f(z_\zeta)\,\chi_{\zeta=\infty}. \quad (9.6)$$

On the other hand, applying Lemma 3.3 to $X^h$, $\rho(t)=e^{-\alpha t}$, $\tau=\zeta$, $\xi=\varphi(z_\zeta)$, the functions $\tilde{f}$ and $F(t)=f(z_t)$ and the measure $\gamma^h$, we have

$$\int_0^\infty e^{-\alpha t}\mathbf{M}^h f(z_{\zeta-t})\,\chi_{t<\zeta<\infty}\varphi(z_\zeta)\,dt=\int_E q(y)\,h(y)\,m(dy)\,\tilde{f}(y)\,\mathbf{M}_y^h e^{-\alpha\zeta}\varphi(z_\zeta). \quad (9.7)$$

From (8.14)

$$h(y)\,\mathbf{M}_y^h e^{-\alpha\zeta}\varphi(z_\zeta)=\int_{\mathscr{E}_1}\varphi(z)\,k_z^\alpha(y)\,\mu_h(dz).$$

Substituting this expression in (9.7) and changing the order of integration, we get

$$\int_0^\infty e^{-\alpha t}\mathbf{M}^h f(z_{\zeta-t})\,\chi_{t<\zeta<\infty}\varphi(z_\zeta)\,dt=\int_{\mathscr{E}_1}\mu_h(dz)\,\varphi(z)\int_E m(dy)\,q(y)\,\tilde{f}(y)\,k_z^\alpha(y). \quad (9.8)$$

By Fubini's theorem, using (6.5) we have

$$\mathbf{M}_x^h(1-e^{-\alpha\zeta})=\alpha\mathbf{M}_x^h\int_0^\infty e^{-\alpha t}\chi_{t<\zeta}\,dt=\alpha\int_0^\infty e^{-\alpha t}\mathbf{P}_x^h\{t<\zeta\}\,dt=$$
$$=\alpha G_\alpha^h 1(x)=\frac{\alpha G_\alpha h(x)}{h(x)} \qquad (x\in E^h).$$

Putting $h=k_z$ and noting (8.12) we have

$$k_z(x)\,\mathbf{M}_x^z e^{-\alpha\zeta}=k_z(x)-\alpha G_\alpha k_z(x)=k_z^\alpha(x).$$

Taking note of this equation and putting in (9.7) $h = k_z$, $\varphi = 1$, we get

$$\int_0^\infty e^{-\alpha t}\mathbf{M}^z f(z_{\zeta-t})\chi_{t<\zeta<\infty}\,dt = \int_E q(y)\,k_z(y)\,\widetilde{f}(y)\,m(dy)\,\mathbf{M}_y^z e^{-\alpha\zeta} =$$

$$= \int_E m(dy)\,q(y)\,\widetilde{f}(y)\,k_z^\alpha(y). \qquad (9.9)$$

From (9.8) and (9.9) we obtain

$$\int_0^\infty e^{-\alpha t}\mathbf{M}^h f(z_{\zeta-t})\chi_{t<\zeta<\infty}\varphi(z_\zeta)\,dt = \int_0^\infty e^{-\alpha t}\int_{\mathscr{E}_1}\mu_h(dz)\,\varphi(z)\,\mathbf{M}^z f(z_{\zeta-t})\chi_{t<\zeta<\infty}\,dt.$$

Since $\alpha$ is arbitrary,

$$\mathbf{M}^h f(z_{\zeta-t})\chi_{t<\zeta<\infty}\varphi(z_\zeta) = \int_{\mathscr{E}_1}\mu_h(dz)\,\varphi(z)\,\mathbf{M}^z f(z_{\zeta-t})\chi_{t<\zeta<\infty}$$

for almost all $t > 0$. Taking the limit as $t\downarrow 0$, we have

$$\mathbf{M}^h f(z_\zeta)\,\varphi(z_\zeta)\,\chi_{\zeta<\infty} = \int_{\mathscr{E}_1}\mu_h(dz)\,\varphi(z)\,\mathbf{M}^z f(z_\zeta)\,\chi_{\zeta<\infty}. \qquad (9.10)$$

We add (9.6) and (9.10):

$$\mathbf{M}^h f(z_\zeta)\,\varphi(z_\zeta) = \int_{\mathscr{E}_1}\mu_h(dz)\,\varphi(z)\,\mathbf{M}^z f(z_\zeta). \qquad (9.11)$$

We transform the left-hand side, using (8.9):

$$\int_{\mathscr{E}_1} f(z)\,\varphi(z)\,\mu_h(dz) = \int_{\mathscr{E}_1}\mu_h(dz)\,\varphi(z)\,\mathbf{M}^z f(z_\zeta). \qquad (9.12)$$

We select any countable everywhere dense subset $\mathfrak{N}$ in the space $C(E)$ and note that $z\in\mathscr{U}$ if and only if

$$\mathbf{M}^z f(z_\zeta) = f(z) \quad \text{for all} \quad f\in\mathfrak{N}.$$

From (9.12) it is clear that $\mu_h$-almost all $z\in\mathscr{E}_1$ have this property. Thus, $\mu_h(\mathscr{E}_1\setminus\mathscr{U}) = 0$. By Theorem 8.1, $\mu_h(\mathscr{E}\setminus\mathscr{E}_1) = 0$. Hence $\mu_h(\mathscr{E}\setminus\mathscr{U}) = 0$.

9.2. We denote by $\mathscr{B}(\mathscr{U})$ the family of Borel subsets of the compactum $\mathscr{E}$, contained in $\mathscr{U}$. Let $\mu$ be an arbitrary measure on $\mathscr{B}(\mathscr{U})$ and let

$$h(x) = \int_{\mathscr{U}} k_z(x)\,\mu(dz). \qquad (9.13)$$

By 8.2.A and 8.2.B

$$\lambda G_\lambda h(x) = \int_{\mathcal{U}} \lambda G_\lambda k_z(x)\, \mu(dz) \uparrow \int_{\mathcal{U}} k_z(x)\, \mu(dz) = h(x)$$

and consequently $h$ is excessive. By Theorem 9.1 $\gamma(k_z) = 1$ for $z \in \mathcal{U}$. Hence

$$\gamma(h) = \int_{\mathcal{U}} \gamma(k_z)\, \mu(dz) = \mu(\mathcal{U}) \tag{9.14}$$

and $h$ is $\gamma$-integrable if and only if the measure $\mu$ is finite.

From Theorems 8.2 and 9.1 it follows that *each $\gamma$-integrable excessive function can be represented in the form* (9.13) *in terms of its spectral measure* $\mu_h$. However, it still remains an open question whether some excessive function may not have two different integral representations of the form (9.13). The answer to this question is given in the next section.

## §10. Uniqueness Theorem. Minimal Excessive Functions

10.1. THEOREM 10.1. *If the $\gamma$-integrable excessive function $h$ is represented in the form*

$$h(x) = \int_{\mathcal{U}} k_z(x)\, \mu(dz) \tag{10.1}$$

($\mu$ *is a finite measure on* $\mathscr{B}(\mathcal{U})$), *then* $\mu$ *coincides with the spectral measure* $\mu_h$.

PROOF. By (8.29) and (10.1) for $f \in C(\mathscr{E})$

$$\int_{\mathscr{E}_1'} f(y)\, \mu_h(dy) = \lim_{t\to\infty} \int_{\mathcal{U}} \mu(dz)\, \mathbf{M}_\gamma \tilde{f}(x_t)\, k_z(x_t) \tag{10.2}$$

and

$$\int_{\mathscr{E}_1'} f(y)\, \mu_z(dy) = \lim_{t\to\infty} \mathbf{M}_\gamma \tilde{f}(x_t)\, k_z(x_t). \tag{10.3}$$

But when $z \in \mathcal{U}$ $\mu_z = \delta_z$, and hence from (10.3)

$$\lim_{t\to\infty} \mathbf{M}_\gamma \tilde{f}(x_t)\, k_z(x_t) = f(z)\, \chi_{\mathscr{E}_1'}(z) \qquad (z \in \mathcal{U}). \tag{10.4}$$

By 6.2.B and (9.2)

$$\mathbf{M}_\gamma \tilde{f}(x_t)\, k_z(x_t) = \mathbf{M}^z \tilde{f}(x_t) \leqslant \|\tilde{f}\|\, \mu_z(\mathscr{E}) \leqslant \|f\|.$$

Therefore we may interchange in (10.2) the limit and integration symbols. Noting (10.4) we have

$$\int_{\mathscr{E}_1'} f(y)\, \mu_h(dy) = \int_{\mathcal{U}\cap\mathscr{E}_1'} \mu(dz)\, f(z). \tag{10.5}$$

Similarly, from (8.30) we deduce

$$\int_{\mathscr{E}_1''} f(y)\,\mu_h(dy) = \int_{\mathcal{U}\cap\mathscr{E}_1''} \mu(dz)\,f(z). \tag{10.6}$$

(We have here used the estimate, which can be deduced from (8.34),

$$\int_E m(dy)\,q(y)\,\tilde{f}(y)\,[k_z(y) - P_t k_z y] \leqslant \|f\|\,\mu_z(\mathscr{E})\,t = \|f\|\,t.)$$

From (10.5) and (10.6) it follows that $\mu = \mu_h$.

10.2. A non-zero excessive function $h$ is called minimal if from the equation $h = h_1 + h_2$ where $h_1$, $h_2$ are excessive, it follows that $h_1 = a_1 h$, $h_2 = a_2 h$, where $a_1$, $a_2$ are constants.

THEOREM 10.2. *The general* $\gamma$*-integrable minimal excessive function is of the form* $ak_z$*, where* $z \in \mathcal{U}$ *and* $a$ *is a positive constant.*

PROOF. From (8.29) and (8.30) it follows that $\mu_{h_1+h_2} = \mu_{h_1} + \mu_{h_2}$. Let $z \in \mathcal{U}$ and $k_z = h_1 + h_2$. Then $\mu_{h_1} + \mu_{h_2} = \mu_z = \delta_z$. Consequently

$$0 = \delta_z(\mathscr{E} \setminus \{z\}) = \mu_{h_1}(\mathscr{E} \setminus \{z\}) + \mu_{h_2}(\mathscr{E} \setminus \{z\}).$$

Hence $\mu_{h_i}(\mathscr{E} \setminus \{z\}) = 0$, and by (8.15)

$$h_i(x) = \int_{\mathscr{E}_1} k_y(x)\,\mu_{h_i}(dy) = \mu_{h_i}(z)\,k_z(x).$$

Thus, $k_z$ is a minimal excessive function.

Now let $h$ be an arbitrary $\gamma$-integrable minimal excessive function. We have (see (9.14)) $\mu_h(\mathcal{U}) = \gamma(h)$. By 5.4, if $h$ is a non-null excessive function, then $\gamma(h) > 0$. Hence $\mu_h(\mathcal{U}) > 0$, and thus there exists a closed set $F \subset \mathcal{U}$ such that $\mu_h(F) > 0$. Since each open cover of $F$ contains a finite subcover, not every point of $F$ can have a neighbourhood with zero measure $\mu_h$. Suppose that for all neighbourhoods $A$ of $z \in \mathcal{U}$ we have $\mu_h(A) > 0$, and that $A_n$ is a sequence of neighbourhoods such that $A_n \downarrow \{z\}$. We put

$$h_n(x) = \int_{A_n} k_y(x)\,\mu_h(dy).$$

Clearly, $h_n$ and $h - h_n$ are excessive. Since $h$ is minimal, we have $h_n = a_n h$. From the relation $\gamma(h_n) = \mu_h(A_n)$ it follows that $a_n = \gamma(h_n) / \gamma(h) = \mu_h(A_n) / \gamma(h)$ and hence

$$h(x) = \frac{\gamma(A)}{\mu_h(A_n)} \int_{A_n} k_y(x)\,\mu_h(dy). \tag{10.7}$$

By 7.1.A and (8.10) $(k_y, \varphi)$ for any $\varphi \in W$ is continuous with respect to $y$. Hence, from (10.7) we have

$$(h, \varphi) = \lim_{n\to\infty} \frac{\gamma(h)}{\mu_h(A_n)} \int_{A_n} (k_y, \varphi)\,\mu_h(dy) = (k_z, \varphi)\,\gamma(h).$$

Consequently for $m$-almost all $x$ we have $h(x) = \gamma(h) k_z(x)$. This holds for all $x$, since both sides are excessive functions.

## §11. Attainable and Unattainable Exits. $P_t$-invariant and $P_t$-null Excessive Functions. Integral Representation of $\alpha$-excessive Functions

**11.1.** The space of exits $\mathcal{U}$ splits into the sum of two disjoint subsets $\mathcal{U}_0 = \mathcal{U} \cap \mathscr{E}_1'$ and $\mathcal{U}_a = \mathcal{U} \cap \mathscr{E}_1''$. We call these subsets *the space of unattainable* and *the space of attainable exits*, respectively. By Theorems 8.2 and 9.1 for any $\gamma$-integrable excessive function $h$, when $f \in V(\mathscr{E})$, $x \in E_*^h$

$$h(x)\,\mathbf{M}_x^h f(z_\zeta)\,\chi_{\zeta=\infty} = \int_{\mathcal{U}_0} f(z)\,k_z(x)\,\mu_h(dz), \tag{11.1}$$

$$h(x)\,\mathbf{M}_x^h f(z_\zeta)\,\chi_{\zeta<\infty} = \int_{\mathcal{U}_a} f(z)\,k_z(x)\,\mu_h(dz). \tag{11.2}$$

In particular

$$h(x)\,\mathbf{P}_x^h\{z_\zeta \in \Gamma,\ \zeta = \infty\} = \int_{\mathcal{U}_0 \cap \Gamma} k_z(x)\,\mu_h(dz), \tag{11.3}$$

$$h(x)\,\mathbf{P}_x^h\{z_\zeta \in \Gamma,\ \zeta < \infty\} = \int_{\mathcal{U}_a \cap \Gamma} k_z(x)\,\mu_h(dz). \tag{11.4}$$

Thus, for almost all non-terminating paths $z_\zeta \in \mathcal{U}_0$ and for almost all terminating paths $z_\zeta \in \mathcal{U}_a$.

**11.2.** LEMMA 11.1. *For all* $z \in \mathcal{U}$, $\alpha \geqslant 0$, $t \geqslant 0$,

$$G_\alpha k_z = P_t G_\alpha k_z + \int_0^t P_u k_z^\alpha\,du. \tag{11.5}$$

PROOF. By 8.4.A, on the set $E_*^z$ we have $k_z^\alpha = k_z - \alpha G_\alpha k_z$, and noting (6.9) we have on $E_*^z$

$$\int_0^t P_u k_z^\alpha\,du = \alpha \int_0^t P_u \left[\int_0^\infty e^{-\alpha s}(k_z - P_s k_z)\,ds\right] du =$$

$$= \alpha \int_0^\infty e^{-\alpha s}\left[\int_0^t (P_u k_z - P_{u+s} k_z)\,du\right] ds = \alpha \int_0^\infty e^{-\alpha s}(I_t^0 - I_{s+t}^s)\,ds, \tag{11.6}$$

where

$$I_b^a = \int_a^b P_u k_z\,du.$$

We note that

$$I_t^0 - I_{s+t}^s = I_s^0 - I_{s+t}^t = I_s^0 - P_t I_s^0 \text{ on } E_*^z \tag{11.7}$$

and

$$\alpha \int_0^\infty e^{-\alpha s} I_s^0 \, ds = \int_0^\infty du \, P_u k_z \int_u^\infty \alpha e^{-\alpha s} \, ds = G_\alpha k_z . \tag{11.8}$$

From (11.6) - (11.8) we conclude that (11.5) is satisfied on the set $E_*^z$. However, both sides of (11.5) are $\alpha$-excessive (see 1.2 and 1.3), and, by 6.1.E, (11.5) holds everywhere.

11.3. We put

$$\gamma_\alpha = \gamma + \alpha(\gamma G). \tag{11.9}$$

For $\alpha > 0$, $\gamma_\alpha$-integrability is equivalent to integrability with respect to $\gamma$ and to $\gamma G$, and hence does not depend on $\alpha$.

We say that the excessive function $h$ is $P_t$-*invariant* if $P_t h = h$ for all $t \geqslant 0$, and is $P_t$-*null* if $P_t h \downarrow 0$ for $t \to \infty$. The class of all $\gamma$-integrable $P_t$-invariant functions is denoted by $\mathscr{S}_0$, and the class of all $\gamma$-integrable $P_t$-null functions by $\mathscr{S}_a$.

THEOREM 11.1. *For $z \in \mathcal{U}_0$ the function $k_z$ is $P_t$-invariant and $k_z^\alpha = 0$ for all $\alpha > 0$.*

*For $z \in \mathcal{U}_a$ $k_z$ is a $P_t$-null function, $k_z^\alpha$ is continuous with respect to $\alpha$ for $\alpha = 0$, and satisfies the following relations*

$$Gk_z^\alpha = G_\alpha k_z, \tag{11.10}$$

$$k_z^\alpha + \alpha G k_z^\alpha = k_z, \tag{11.11}$$

$$\gamma_\alpha (k_z^\alpha) = \gamma(k_z) = 1. \tag{11.12}$$

PROOF. 1°. If $z \in \mathcal{U}_0$, then $k_z \in \mathscr{S}_0$, because by 8.3 $k_z^\alpha = 0$ for some $\alpha > 0$, and by 8.4.A $\alpha G_\alpha k_z = k_z$, it now follows from (11.5) that $k_z = P_t k_z$.

2°. If $z \in \mathcal{U}$ and $k_z \in \mathscr{S}_0$, then $k_z^\alpha = 0$ for all $\alpha > 0$ and $z \in \mathcal{U}_0$, since by (1.2) $\alpha G_\alpha k_z = k_z$ for any $\alpha > 0$ and by 8.4.A $k_z^\alpha = 0$ on $E_*^z$ and by 6.1.E $k_z^\alpha = 0$ everywhere.

3°. If $z \in \mathcal{U}_a$, then $k_z \in \mathscr{S}_a$.

Let $P_t k_z \downarrow k_z'$ for $t \to \infty$. Then $k_z' \in \mathscr{S}_0$ and $k_z' \leqslant k_z$. We put $\varphi = k_z - k_z'$ on $E_*^z$ and $\varphi = \infty$ on $E \setminus E_*^z$. Evidently $k_z = k_z' + \varphi$, and $P_t \varphi \leqslant \varphi$, $P_t k_z = k_z' + P_t \varphi$ for any $t$. Let $P_t \varphi \uparrow \varphi'$ for $t \downarrow 0$. Then $\varphi'$ is excessive and $k_z = k_z' + \varphi'$.

Since $k_z$ is minimal, $k_z' = a k_z$. We have $k_z' = P_t k_z' = a P_t k_z$, and hence $k_z' = a k_z'$. If $k_z' \neq 0$, then $a = 1$. Hence $P_t k_z \downarrow k_z$ and $k_z \in \mathscr{S}_0$. However, by 2° this is possible only if $z \in \mathcal{U}_0$. Hence $k_z' = 0$.

4°. If $z \in \mathcal{U}_a$, then $k_z^\alpha$ is continuous for $\alpha = 0$.

We put $\varphi = k_z - k_z^{+0}$ on $E_*^z$ and $\varphi = \infty$ on $E \setminus E_*^z$. Evidently $k_z = k_z^{+0} + \varphi$. By 8.4.A for $\alpha \downarrow 0$ $\alpha G_\alpha k_z \downarrow k_z - k_z^{+0} = \varphi$ on $E_*^z$. From (11.5) it follows that $\varphi = P_t \varphi$ on $E_*^z$. Evidently $P_t \varphi \leqslant \varphi$ everywhere. Just as in

$3^\circ$, we prove that $k_z = k_z^{+0} + \varphi'$, where $\varphi' = \lim_{t\downarrow 0} P_t\varphi$, is excessive. Since $k_z$ is minimal, $k_z^{+0} = ak_z$. From 8.4.A

$$\lambda G_\lambda k_z + k_z^\lambda = k_z, \qquad \lambda G_\lambda k_z^{+0} + k_z^\lambda = k_z^{+0} \qquad (\lambda > 0).$$

Therefore $ak_z^\lambda = ak_z - a\lambda G_\lambda k_z = k_z^{+0} - \lambda G_\lambda k_z^{+0} = k_z^\lambda$ on $E_*^z$. Since $k_z^\lambda \neq 0$ for $\lambda > 0$, we see that $a = 1$.

$5^\circ$. Let $z \in \mathcal{U}_a$. By $3^\circ$, $\alpha P_t G_\alpha k_z \leqslant P_t k_z \to 0$ for $t \to \infty$. Taking the limit in (11.5) we have (11.10). (11.11) follows from 8.4.A and (11.10), (11.12) follows from (11.11) and Theorem 9.1.

11.4. THEOREM 11.2. *For each $\gamma$-integrable excessive function $h$ there exists a unique decomposition into the sum of a $P_t$-invariant function $h^0$ and a $P_t$-null function $h^a$. Here*

$$h^0 = \int_{\mathcal{U}_0} k_z \mu_h(dz), \qquad h^a = \int_{\mathcal{U}_a} k_z \mu_h(dz). \tag{11.13}$$

*If $\mu$ is any finite measure on $\mathcal{U}_a$, then the function*

$$h_\alpha = \int_{\mathcal{U}_a} k_z^\alpha \mu(dz) \tag{11.14}$$

*is $\alpha$-excessive and*

$$\gamma_\alpha(h_\alpha) = \gamma(h_0) = \mu(\mathcal{U}_a). \tag{11.15}$$

*All functions of the family $h_\alpha$ satisfy the relations*

$$h_0 = h_\alpha + \alpha G_\alpha h_0, \qquad h_\alpha + \alpha G h_\alpha = h_0 \qquad (\alpha > 0). \tag{11.16}$$

*Every $\gamma_\alpha$-integrable $\alpha$-excessive function has one and only one representation in the form (11.14).*

PROOF. From Theorem 11.1 it is clear that $h^0 \in \mathcal{S}_0$ and $h^a \in \mathcal{S}_a$. We assume that $h = \tilde{h}^0 + \tilde{h}^a$, where $\tilde{h}^0 \in \mathcal{S}_0$, $\tilde{h}^a \in \mathcal{S}_a$. Clearly $\tilde{h}^0 = \lim_{t\to\infty} P_t h = h^0$ and $\tilde{h}^a = h - \tilde{h}^0 = h - h^0 = h^a$ on $E_*^h$, and hence on $E$. Since $k_z^a$ is $\alpha$-excessive, $h_\alpha$ is also excessive. By (11.12), (11.15) follows from (11.14). Formulae (11.16) follow from 8.4.A and (11.11).

Now let $\alpha > 0$ and let $f_\alpha$ be an arbitrary $\gamma_\alpha$-integrable $\alpha$-excessive function. We put $f = f_\alpha + \alpha G f_\alpha$. Evidently $\gamma(f) = \gamma_\alpha(f_\alpha) < \infty$. From (1.6) for $\lambda \geqslant \alpha$

$$\lambda G_\lambda f = \lambda G_\lambda f_\alpha + \alpha\lambda G_\lambda G f_\alpha = (\lambda - \alpha) G_\lambda f_\alpha + \alpha G f_\alpha. \tag{11.17}$$

As $\lambda \to \infty$ $\lambda G_\lambda f \uparrow f$, that is, $f$ is excessive. By 9.2

$$f = \int_{\mathcal{U}} k_z \mu_h(dz).$$

Substituting in (11.17) $\lambda = \alpha$ we have

$$\alpha G_\alpha f = \alpha G f_\alpha.$$

Thus, on $E_*^f$

$$f_\alpha = f - \alpha G f_\alpha = f - \alpha G_\alpha f = \int_{\mathcal{U}} (k_z - \alpha G_\alpha k_z)\, \mu_h(dz) = \int_{\mathcal{U}_a} k_z^\alpha \mu_h(dz).$$

In view of 6.1.E, this equation holds on all $E$.

We note, finally, that if $h_\alpha$ is represented in the form (11.14) then by (11.16) and (11.11)

$$h_0 = \int_{\mathcal{U}_a} (k_z^\alpha + \alpha G k_z^\alpha)\, \mu(dz) = \int_{\mathcal{U}_a} k_z \mu(dz),$$

and by Theorem 10.1 $\mu$ coincides with the spectral measure of the excessive function $h_0$.

## §12. Harmonic Functions

**12.1.** A non-negative function $\tau = \tau(\omega)$ is called *a moment of stopping* if $\tau \leqslant \zeta$ and for any $t \geqslant 0$ $\{\tau < t\} \in \mathcal{F}_t$ (or, what is equivalent, $\{\tau < t < \zeta\} \in \mathcal{M}_t$).

With each closed subset $\Gamma$ of the compactum $\mathscr{E}$ and each $s \geqslant 0$ we connect the moment

$$\tau_\Gamma^s = \inf\{t:\ t > s,\ z_t \bar{\in} \Gamma\}. \tag{12.1}$$

Clearly $s \leqslant \tau_\Gamma^s \leqslant \zeta$. Let $R_t^s$ be the set of all the rational numbers in the interval $(s, t)$. By Theorem 7.2 for $s < t$,

$$\{\tau_\Gamma^s < t < \zeta\} = \bigcup_{u \in R_t^s} \{z_u \bar{\in} \Gamma,\ \zeta > t\} \in \mathcal{N}_t. \tag{12.2}$$

Thus, $\tau_\Gamma^s$ is a moment of stopping. We note that

$$\tau_\Gamma^s = s + \theta_s \tau_\Gamma \tag{12.3}$$

and for $s \downarrow 0$

$$\tau_\Gamma^s \downarrow \tau_\Gamma, \tag{12.4}$$

where $\tau_\Gamma = \tau_\Gamma^0$.

**12.2.** Let $h$ be an excessive function and $H(t, \omega)$ the regularization of $h(x_t)$ constructed in Lemma 2.1. By 0.7.E and 0.7.F of Theorem 0.2, for any two moments of stopping $\sigma \leqslant \tau$,

$$\mathbf{M}_x H(\sigma) \geqslant \mathbf{M}_x H(\tau) \qquad (x \in E_*^h) \tag{12.5}$$

and

$$h(x) \geqslant \mathbf{M}_x H(\tau) \qquad (x \in E_*^h). \tag{12.6}$$

Let $\tau_n$, $\tau$ be moments of stopping and $\tau_n \downarrow \tau$. Then, by (12.5), 2.4.A and Fatou's lemma

$$\mathbf{M}_x H(\tau) \geqslant \overline{\lim}\, \mathbf{M}_x H(\tau_n) \geqslant \underline{\lim}\, \mathbf{M}_x H(\tau_n) \geqslant \mathbf{M}_x [\underline{\lim}\, H(\tau_n)] = \mathbf{M}_x H(\tau).$$

Therefore

$$\mathbf{M}_x H(\tau_n) \uparrow \mathbf{M}_x H(\tau) \qquad (x \in E_*^h). \tag{12.7}$$

12.3. LEMMA 12.1. *Let $\Gamma$ be a closed set in $\mathscr{E}$. For each $\gamma$-integrable excessive function $h$ there exists a unique excessive function $P_\Gamma h$ such that*

$$P_\Gamma h(x) = \mathbf{M}_x H(\tau_\Gamma) \qquad \textit{if} \;\; x \in E_*^h. \tag{12.8}$$

*Here*

$$P_s P_\Gamma h(x) = \mathbf{M}_x H(\tau_\Gamma^s) \qquad (x \in E_*^h). \tag{12.9}$$

*If $\tilde{\Gamma} \subseteq \Gamma$, then*

$$h \geqslant P_{\tilde{\Gamma}} h \geqslant P_\Gamma h. \tag{12.10}$$

PROOF. The uniqueness follows from 6.1.E. To prove the existence we consider any function $\tilde{h} \in V$ that is equal to $\mathbf{M}_x H(\tau_\Gamma)$ on $E_*^h$. By (6.9), (2.3), (12.3) and 2.4.D

$$P_t \tilde{h}(x) = \mathbf{M}_x \tilde{h}(x_t) = \mathbf{M}_x \mathbf{M}_{x_t} H(\tau_\Gamma) = \mathbf{M}_x \theta_t H(\tau_\Gamma) = \mathbf{M}_x H(\tau_\Gamma^t), \tag{12.11}$$

and hence

$$\lambda G_\lambda \tilde{h}(x) = \lambda \int_0^\infty e^{-\lambda t} \mathbf{M}_x H(\tau_\Gamma^t)\, dt = \int_0^\infty e^{-u} \mathbf{M}_x H(\tau_\Gamma^{u/\lambda})\, du.$$

Taking account of (12.4), (12.5) and (12.7) we conclude that as $\lambda \to \infty$

$$\lambda G_\lambda \tilde{h}(x) \uparrow \mathbf{M}_x H(\tau_\Gamma) \qquad (x \in E_*^h). \tag{12.12}$$

We denote by $P_\Gamma h$ the regularization of the preexcessive function $\tilde{h}$. By (12.12), (12.8) is satisfied, and by (12.11) and (6.9), (12.9) is valid. (12.10) is satisfied if $x \in E_*^h$ by (12.8), (12.5) and (12.6). By Lemma 1.1 it can be extended to all $x \in E$.

12.4. Let $A$ be any open set in the compactum $\mathscr{E}$ other than $\mathscr{E}$ itself. We say that the excessive function $h$ is *A-harmonic* if

$$P_\Gamma h = h \qquad \text{for all} \quad \Gamma \subseteq A \tag{12.13}$$

(we drop the indication of closure on $\Gamma$, since for other sets the operators $P_\Gamma$ are not defined).

Let $\Gamma_n$ be the set of points $z$ whose distance from $\mathscr{E} \setminus A$ is greater than or equal to $1/n$. Clearly, any closed subset $\Gamma$ of $A$ is contained in some $\Gamma_n$, and by (12.10) $h \geqslant P_\Gamma h \geqslant P_\Gamma h_n$ for any excessive function $h$.

Hence, *for an excessive function $h$ to be $A$-harmonic it is necessary and sufficient that* (12.13) *is satisfied for the sets* $\Gamma_n$.

12.5. LEMMA 12.2. *Let $\tau$ be a moment of stopping such that* $\{\tau<t<\zeta\}\in\mathcal{N}_t$ *for any* $t\geqslant 0$. *Then*

$$h(x)\,\mathbf{P}_x^h\{\tau<\zeta\}=\mathbf{M}_x H(\tau)\qquad (x\in E_*^h). \tag{12.14}$$

*If $\Gamma$ is a closed set in $\mathscr{E}$, then for all* $s>0$

$$h(x)\,\mathbf{P}_x^h\{\tau_\Gamma^s<\zeta\}=P_s P_\Gamma h(x). \tag{12.15}$$

PROOF. We fix some $x\in E_*^h$. We select a countable everywhere dense subset $T$ on the half-line $[0,\infty)$ so that for $t\in T$

$$\mathbf{P}_x\{H(t,\omega)\neq h(x_t)\}=0 \tag{12.16}$$

(this can be done by 2.4.B). We construct a finite set $T_n=\{t_1^n<t_2^n<\ldots<t_{k_n}^n\}$ so that $T_n\uparrow T$ and $\max\limits_{2\leqslant m\leqslant k_n}|t_m^n-t_{m-1}^n|\to 0$. We put

$$\tau_n=t_m^n\quad\text{for}\quad t_{m-1}^n\leqslant\tau<t_m^n.$$

Evidently $\tau_n$ is a moment of stopping and

$$A_m=\{\tau_n=t_m^n<\zeta\}\in\mathcal{N}_{t_m^n}.$$

By Lemma 6.1 and (12.16)

$$h(x)\,\mathbf{P}_x^h(A_m)=\mathbf{M}_x h(x_{t_m^n})\,\chi_{A_m}=\mathbf{M}_x H(t_m^n)\,\chi_{A_m}=\mathbf{M}_x H(\tau_n)\,\chi_{A_m}.$$

Summing these equations over $m$ we obtain

$$h(x)\,\mathbf{P}_x^h(\tau_n<\zeta)=\mathbf{M}_x H(\tau_n).$$

Noting that as $n\to\infty$ $\tau_n\downarrow\tau$ and taking (12.7) into account we now have (12.14). By (12.2) we may apply (12.14) to the moments $\tau_\Gamma^s$, and hence (12.15) follows from (12.9).

12.6. THEOREM 12.1. *For a $\gamma$-integrable excessive function $h$ to be $A$-harmonic it is necessary and sufficient that its spectral measure $\mu_h$ satisfies the condition $\mu_h(A)=0$. In particular, for each $z\in\mathcal{U}$ the general $(\mathscr{E}\setminus z)$-harmonic function is of the form $ak_z$, where $a$ is a non-negative constant.*

PROOF. By 9.2 every $\gamma$-integrable excessive function $h$ has a representation of the form

$$h(x)=\int\limits_{\mathcal{U}} k_z(x)\,\mu_h(dz). \tag{12.17}$$

We consider the sequence of sets $\Gamma_n$ constructed in 12.3. From (12.17) it is clear that if $h(x) < \infty$, then $k_z(x) < \infty$ for $\mu_h$-almost all $z$ and

$$h(x)-P_{\Gamma_n}h(x)=\int_{\mathcal{U}}[k_z(x)-P_{\Gamma_n}k_z(x)]\,\mu_h(dz) \qquad (x\in E_*^h).$$

By (12.10) the expression in square brackets is non-negative. Hence $h$ satisfies (12.12) if and only if $k_z$ satisfies this condition for $\mu_h$-almost all $z$.

To prove the Theorem it is sufficient to show that the function $k_z$ $(z\in\mathcal{U})$ is $A$-harmonic if and only if $z\notin A$.

By Lemma 12.2, the condition for $k_z$ to be $A$-harmonic can be written in the form

$$\mathbf{P}_x^z\{\tau_\Gamma<\zeta\}=1 \qquad \text{for all} \quad x\in E^z,\ \Gamma\subseteq A \tag{12.18}$$

or, what is equivalent,

$$\mathbf{P}_x^z\{\tau_\Gamma^s=\zeta\}=0 \qquad \text{for all} \quad x\in E^z,\ s\geqslant 0,\ \Gamma\subseteq A. \tag{12.19}$$

We now recall that by (8.14) and 9.1 $\mathbf{P}_x^z\{z_\zeta = z\} = 1$ for $z\in\mathcal{U}$ and hence $\mathbf{P}_x^z$-almost surely $z_t\to z$ as $t\uparrow\zeta$. Hence for $z\in\mathcal{U}\cap(\mathscr{E}\setminus A)$ (12.18) is satisfied. On the other hand, if for some $z\in\mathcal{U}$ (12.19) is satisfied, then $\mathbf{P}_x^z$-almost surely there exists a sequence $t_n\uparrow\zeta$ such that $z_{t_n}\bar{\in}\Gamma_n$ ($\Gamma_n$ are the sets in 12.3). Thus, the distance from $z_{t_n}$ to $\mathscr{E}\setminus A$ tends to zero, and $z\notin A$.

**12.7.** Let $\mathscr{A}$ be any family of closed subsets of the space $\mathscr{E}$. We say that the excessive function $h$ is *A-harmonic* if $P_\Gamma h=h$ for all $\Gamma\in\mathscr{A}$. To return to the definition of 12.4 it is sufficient to take for $\mathscr{A}$ the system of all closed subsets of the open set $A$.

Looking over the proof of Theorem 12.1 we establish without difficulty the following Theorem:

THEOREM 12.2. *Let $\mathcal{U}_{\mathscr{A}}$ denote the set of all points $z\in\mathcal{U}$, for which $k_z$ is $\mathscr{A}$-harmonic. For the $\gamma$-integrable excessive function $h$ to be $\mathscr{A}$-harmonic it is necessary and sufficient that its spectral measure $\mu_h$ is concentrated on $\mathcal{U}_{\mathscr{A}}$. For $z\in\mathcal{U}_{\mathscr{A}}$, it is sufficient that $z$ does not belong to any set of the family $\mathscr{A}$, and necessary that there exists a sequence $z_n$ converging to $z$ such that no one of the sets of the family $\mathscr{A}$ contains an infinite number of points of this sequence.*

**12.8.** We consider now a special $M$-process $\mathscr{X}$ that is continuous on the right. The moment of first exist of $\mathscr{X}$ from $\Gamma$ is defined by

$$\tau(\Gamma)=\inf\{t:\ t>0,\ x_t\bar{\in}\Gamma\}.$$

If $\Gamma$ is closed, then for any $t > 0$

$$\{\tau(\Gamma)<t<\zeta\}=\bigcup_{u\in R_t^0}\{x_u\bar{\in}\Gamma,\ \zeta>t\}\in\mathscr{N}_t, \tag{12.20}$$

so that $\tau(\Gamma)$ is a moment of stopping.

Let $D$ be an open set in $E$ and let $H(t, \omega)$ be a right-continuous modification of the $\gamma$-integrable excessive function $h$. We say that $h$ is *harmonic in* $D$ if $M_x H(\tau(\Gamma)) = h(x)$ on $E_*^h$ for all compacta

$$\Gamma \subset D. \tag{12.21}$$

THEOREM 12.3. *We assume that the special M-process* $\mathscr{X}$ *continuous on the right, satisfies the condition:*

12.8.A. *If for each* $\alpha \in R$

$$g_\alpha(x, y_1) = g_\alpha(x, y_2) \quad \textit{for m-almost all } x$$

*then*[1] $y_1 = y_2$.

*Then the class of harmonic functions in the open set D coincides with the class of* $\mathscr{A}$*-harmonic functions, where* $\mathscr{A}$ *is the family of all sets in* $\mathscr{E}$ *whose preimages are compact and are contained in D.*

*A* $\gamma$*-integrable excessive function h is harmonic in the open set D if and only if its spectral measure* $\mu_h$ *is concentrated on the set*

$$\mathcal{U}(D) = \{z\colon z \in \mathcal{U} \text{ and } k_z \text{ is harmonic in } D\}.$$

*For* $z \in \mathcal{U}$ *to belong to* $\mathcal{U}(D)$ *it is sufficient that* $z \,\overline{\in}\, i(D)$, *and necessary that there exists a sequence* $x_n \in E$ *such that* $i(x_n) \to z$ *and any compactum* $\Gamma \subset D$ *contains only a finite number of points* $x_n$.

PROOF. Let $\Gamma$ be a compact set. Since $i$ is continuous (Theorem 7.1), the set $\tilde{\Gamma} = i(\Gamma)$ is closed. From (12.2), (7.6) and 12.8.A we have

$$\{\tau_{\tilde{\Gamma}} < t < \zeta\} = \bigcup_{u \in R_t^0} \{z_u \,\overline{\in}\, \tilde{\Gamma},\ \zeta > t\} = \bigcup_{u \in R_t^0} \{x_{u-0} \,\overline{\in}\, \Gamma,\ \zeta > t\}. \tag{12.22}$$

By 5.8.C it follows from (12.20) and (12.22) that almost surely $\tau(\Gamma) = \tau_{\tilde{\Gamma}}$. Hence for any excessive function $h$ and any compact set $\Gamma$

$$P_{\tilde{\Gamma}} h(x) = M_x H(\tau(\Gamma)) \qquad (x \in E_*^h) \tag{12.23}$$

and condition (12.21) is equivalent to the condition: $P_{\tilde{\Gamma}} h = h$ for all $\Gamma \subseteq D$. The first statement of the Theorem is now proved. The remaining parts follow immediately from Theorem 12.2.

REMARK 1. We assume that for all open sets $A \subseteq E$ the moments $\tau(A)$ are moments of stopping.[2] Let $\hat{A}$ be the closure of the open set $A$ and let $\Gamma_\varepsilon = \{x\colon \rho(x, \Gamma) < \varepsilon\}$ $(\varepsilon > 0)$. Evidently $\tau(A) \leqslant \tau(\hat{A})$, $\tau(\Gamma) \leqslant \tau(\Gamma_\varepsilon)$. The set $\Gamma_\varepsilon$ is open, and if $\Gamma$ is compact, then for sufficiently small $\varepsilon$ the closure of $\Gamma_\varepsilon$ is also compact. Hence, using (12.5) and (12.6), we can state the definition for $h$ to be harmonic in $D$ in the following form: $M_x H(\tau(A)) = h(x)$ on $E_*^h$ for all open sets $A$ whose closure is compact and is contained in $D$.

---

1 Condition 12.8.A is equivalent to the condition that no two points of $E$ adhere to each other under the mapping $i$.

2 This condition is satisfied for all standard processes (see footnote on p. 121.

REMARK 2. We assume that for any excessive function $h$ almost surely $h(x_t(\omega))$ is continuous on the right with respect to $t$. Then almost surely $h(x_t(\omega)) = H(t, \omega)$ for all $t$, and in all previous definitions we may replace $M_x H(\tau)$ by $M_x h(x_\tau)$.

12.9. We consider now the class of functions that are harmonic in the whole space $E$ (we call them simply *harmonic*). By Theorem 12.3 the set $\mathcal{U}_{\text{harm}} = \mathcal{U}(E)$ contains all points of $\mathcal{U}$ that are not images of a point of $E$, and also certain images $i(x)$, mainly the limits for sequences $i(x_n)$, where $x_n$ "goes off to infinity".

We put $\mathcal{U}_{\text{pot}} = \mathcal{U} \setminus \mathcal{U}_{\text{harm}}$. An arbitrary $\gamma$-integrable excessive function $h$ has a decomposition into the sum of a harmonic function

$$h' = \int_{\mathcal{U}_{\text{harm}}} k_z \mu_h(dz)$$

and the excessive function

$$h'' = \int_{\mathcal{U}_{\text{pot}}} k_z \mu_h(dz).$$

We investigate the latter. We assume that $h'' = h_1 + h_2$, where $h_1$ is harmonic and $h_2$ is excessive. From (8.29) and (8.30) $\mu_{h''} = \mu_{h_1} + \mu_{h_2}$ and hence $\mu_{h_1}(\mathcal{U}_{\text{harm}}) = 0$. But by Theorem 12.3 $\mu_{h_1}$ is concentrated on $\mathcal{U}_{\text{harm}}$. Therefore $\mu_{h_1} = 0$ and $h_1 = 0$. Thus, it is "impossible to split off" from the function $h''$ any non-zero harmonic function. Excessive functions having this property are called *potentials*. The decomposition $h = h' + h''$ of $h$ into the harmonic function $h'$ and the potential $h''$ is called the *Riesz decomposition.*[1]

The integral representation

$$h(x) = \int_{\mathcal{U}_{\text{pot}}} k_z(x) \mu_h(dz) \tag{12.24}$$

of the potential $h$ can be transformed if we make use of $\mathcal{U}_{\text{pot}} \subseteq i(E)$. We put $E_{\text{pot}} = i^{-1}(\mathcal{U}_{\text{pot}})$. By (8.38) for $y \in E_{\text{pot}}$

$$k_{i(y)}(x) = \frac{G(x, y)}{q(y)}.$$

[1] Let $h$ be a potential. We assume that the harmonic function $h_1$ satisfies $h_1 \leqslant h$. Does there exist an excessive function $h_2$ such that $h = h_1 + h_2$? The answer to this question is in the affirmative if the process is standard (the construction of the excessive function $h_2$ is easily accomplished with the help of Theorem 12.4 of [8]). In this case there do not exist non-zero harmonic functions $h_1$ such that $h_1 \leqslant h$.

On the subsets of $E_{\text{pot}}$ that belong to $\overline{\mathscr{B}}$ we consider the measure[1]

$$\nu_h(A) = \int\limits_{i(A)} \frac{1}{q(i^{-1}(z))} \mu_h(dz).$$

We can write (12.24) in the form

$$h(x) = \int\limits_{E_{\text{pot}}} G(x, y) \nu_h(dy).$$

Thus, $h$ is a potential for the measure $\nu_h$ with kernel $G(x, y)$. From Theorem 10.1 it follows that $\nu_h$ is uniquely determined by $h$.

## Appendix I. Measurability. Measures and Integrals

**0.1.** The pair $(\Omega, \mathscr{A})$, where $\Omega$ is a certain set and $\mathscr{A}$ a $\sigma$-algebra of subsets of it, is called *a measurable space*. Let $(\Omega_1, \mathscr{A}_1)$ and $(\Omega_2, \mathscr{A}_2)$ be two measurable spaces. A mapping $f$ of $\Omega_1$ into $\Omega_2$ is called *measurable* if $f^{-1}(A_2) \in \mathscr{A}_1$ for any $A_2 \in \mathscr{A}_2$ where ($f^{-1}(A)$ denotes the complete inverse image of $A$ under $f$).

Let $f_i$ be measurable mappings of $(\Omega, \mathscr{A})$ into $(\Omega_i, \mathscr{A}_i)$, $i = 1, 2, \ldots, n$. Then the mapping $f$ of the space $(\Omega, \mathscr{A})$ into $(\Omega_1 \times \ldots \times \Omega_n, \mathscr{A}_1 \times \ldots \times \mathscr{A}_n)$, defined by the formula

$$f(\omega) = (f_1(\omega), \ldots, f_n(\omega)),$$

is measurable ([7], Lemma 1.3).

Let $f(\omega_1, \omega_2)$ be a measurable mapping of $(\Omega_1 \times \Omega_2, \mathscr{A}_1 \times \mathscr{A}_2)$ into $(\Omega, A)$. Then for any fixed $\omega_2 \in \Omega_2$, $f(\varphi_1, \omega_2)$ is a measurable mapping of $(\Omega_1, \mathscr{A}_1)$ into $(\Omega, \mathscr{A})$ ([7], Lemma 1.4).

**0.2.** We denote by $B(\Delta)$ the $\sigma$-algebra of all Borel subsets of the interval $\Delta$. A function $f$ on $\Omega$, with range in the extended number half-line $[0, \infty]$, is said to be *$\mathscr{A}$-measurable* if the mapping defined by it from $(\Omega, \mathscr{A})$ into $([0, \infty], \mathscr{B}[0, \infty])$ is measurable. The set of all such functions is denoted by $V(\Omega, \mathscr{A})$.

We assume that for each $t \in \Delta$ the function $F(t, \omega)$ is $\mathscr{A}$-measurable with respect to $\omega$ and for each $\omega \in \Omega$ it is continuous on the right with respect to $t$. Then the function $F(t, \omega)$ is $\mathscr{B}(\Delta) \times \mathscr{A}$-measurable ([7], Lemma 1.10). This statement remains valid if continuity on the right is replaced by continuity on the left.

LEMMA 0.1.[2] *Let $\mathscr{H}$ be a system of non-negative functions on the set $\Omega$ satisfying the following conditions*:

a) *if $f_1, f_2 \in \mathscr{H}$, then $a_1 f_1 + a_2 f_2 \in \mathscr{H}$ for any non-negative constants $a_1, a_2$*;

---

[1] Since the $\sigma$-algebra of Borel sets of $E$ is generated by compacta and since by 12.8.A the mapping $i$ preserves all set-theoretic operations, the images of sets belonging to the $\sigma$-algebra $\overline{\mathscr{B}}$ belong to $\overline{\mathscr{B}}(\mathscr{E})$.

[2] See [8], Lemma 0.2.

b) *if* $f_1, f_2 \in \mathscr{H}$ *and* $f_2 \leqslant f_1 < \infty$, *then* $f_1 - f_2 \in \mathscr{H}$;
c) *if* $f_n \in \mathscr{H}$ *and* $f_n \uparrow f$, *then* $f \in \mathscr{H}$;
d) $1 \in \mathscr{H}$.

*Let* $\mathscr{C}$ *be a family of subsets of* $\Omega$, *containing together with any two sets their intersection, and let* $\mathscr{A}$ *be the* $\sigma$-*algebra generated by* $\mathscr{C}$. *If* $\mathscr{H}$ *contains the indicators of all sets of* $\mathscr{C}$, *then* $\mathscr{H}$ *contains* $V(\Omega, \mathscr{A})$.

**0.3.**[1] To each measure $\mu$ on the $\sigma$-algebra $\mathscr{A}$ there corresponds the functional

$$\mu(f) = \int_\Omega f(\omega)\,\mu(d\omega)$$

in the space $V(\Omega, \mathscr{A})$, having the following properties:

**0.3.A.** $\mu(f_1 + f_2) = \mu(f_1) + \mu(f_2)$.

**0.3.B.** $\mu(cf) = c\mu(f)$, $c$ a non-negative constant.

**0.3.C.** *If* $f_n \uparrow f$, *then* $\mu(f_n) \uparrow \mu(f)$.

**0.3.D.** For any sequence $f_n \in V(\Omega, \mathscr{A})$

$$\underline{\lim}\, \mu(f_n) \geqslant \mu(\underline{\lim}\, f_n)$$

*(Fatou's lemma)*.

**0.3.E.** If $f_n \to f$ $\mu$-a.e. and there exists $\varphi \in V(\Omega, \mathscr{A})$ such that $\mu(\varphi) < \infty$ and $f_n \leqslant \varphi$ for all $n$, then $\mu(f_n) \to \mu(f)$ *(Lebesgue's Theorem)*.

**0.3.F.** Let $f \in V(\Omega_1 \times \Omega_2, \mathscr{A}_1 \times \mathscr{A}_2)$ and let $\mu_i$ be the measure on $\mathscr{A}_i$. Then the function

$$F_1(\omega_1) = \int_{\Omega_2} f(\omega_1, \omega_2)\,\mu_2(d\omega_2)$$

is $\mathscr{A}_1$-measurable, the function

$$F_2(\omega_2) = \int_{\Omega_1} f(\omega_1, \omega_2)\,\mu_1(d\omega_1)$$

is $\mathscr{A}_2$-measurable and

$$\int_{\Omega_1} F_1(\omega_1)\,\mu_1(d\omega_1) = \int_{\Omega_2} F_2(\omega_2)\,\mu_2(d\omega_2)$$

*(Fubini's Theorem)*.

From Fubini's Theorem it follows that if $A \in \mathscr{A}_1 \times \mathscr{A}_2$ and if for $\mu_1$-almost all $\omega_1$

$$(\omega_1, \omega_2) \in A \quad \text{for } \mu_2\text{-almost all } \omega_2,$$

then, for $\mu_2$-almost all $\omega_2$,

$$(\omega_1, \omega_2) \in A \quad \text{for } \mu_1\text{-almost all } \omega_1.$$

---

[1] The properties listed in 0.3 of integrals are proved, for example, in the book of Kolmogorov and Fomin, "Elements of the theory of functions and functional analysis", Fizmatgiz, Moscow 1968; English edition: Graylock Press, New York.

0.4. Let $(\Omega_1, \mathcal{A}_1)$, $(\Omega_2, \mathcal{A}_2)$, $(\Omega_3, \mathcal{A}_3)$ - be three measurable spaces and let $F(\omega_1, \omega_3)$ belong to $V\ (\Omega_1 \times \Omega_3, \mathcal{A}_1 \times \mathcal{A}_3)$. Let $\mu_{\omega_2}$ be a measure $\mathcal{A}_3$ such that for any $A \in \mathcal{A}_3$ the function $\mu_{\omega_2}(A)$ is $\mathcal{A}_2$--measurable. Then the function

$$\Phi(\omega_1, \omega_2) = \int_{\Omega_3} F(\omega_1, \omega_3)\, \mu_{\omega_2}(d\omega_3)$$

is $\mathcal{A}_1 \times \mathcal{A}_2$-measurable ([7], Lemma 1.7).

## Appendix II. Supermartingales

0.5. Let $(\Omega, \mathcal{A})$ be a measurable space, $T$ a set on the number line, P a measure on the $\sigma$-algebra $\mathcal{A}$. Suppose that to each $t \in T$ there corresponds a function $Z_t(\omega)$ on $\Omega$ and a $\sigma$-algebra $\mathcal{A}_t \subseteq \mathcal{A}$, where:

0.5.A. $\mathcal{A}_s \subseteq \mathcal{A}_t$ if $s < t$.

0.5.B. $Z_t$ is $\mathcal{A}_t$ -measurable.

0.5.C. If $\xi \geqslant 0$ is $\mathcal{A}_s$ -measurable, then for any $t \geqslant s$, $(t, s \in T)$

$$\mathrm{M}\xi Z_s \geqslant \mathrm{M}\xi Z_t.$$

($\mathrm{M}\xi$ denotes the integral of $\xi$ with respect to the measure P). Then the system $(Z_t, \mathcal{A}_t, \mathrm{P})$ is called *a supermartingale*.

0.6. Let $Z(t)$ be a non-negative function on some subset $T$ of the number line. *The number of downcrossings of the interval* $[c, d]$ for the function $Z$ is the supremum of the natural numbers $k$ having the following properties: points $t_1 < t_2 < \ldots < t_{2k-1} < t_{2k}$ can be selected in $T$ such that

$$Z(t_1) \geqslant d, \quad Z(t_2) \leqslant c, \ \ldots, Z(t_{2k-1}) \geqslant d, \quad Z(t_{2k}) \leqslant c.$$

*The number of upcrossings of* $[c, d]$ is defined similarly.

THEOREM 0.1. *Let* $(Z_t, \mathcal{A}_t, \mathrm{P})$ $(t \in T)$ *be a non-negative supermartingale, and let* $\nu$ *be the number of downcrossings of* $[c, d]$ *for the function* $Z_t$. *If* $T$ *is countable, then*

$$\mathrm{M}\nu \leqslant \frac{1}{d-c} \sup_{t \in T} \mathrm{M}Z_t. \tag{0.1}$$

*This relation remains valid when* $T$ *is an interval open on the right and* $Z_t(\omega)$ *for almost all* $\omega$ *is continuous on the right with respect to* $t$.

For the case when $T$ is finite, (0.1) is proved in [10] (Lemma 3; see also [3], Ch. 7, Theorem 3.3). Every countable set $T$ can be represented in the form of the sum of an expanding sequence of finite sets $T_n$. If $\nu_n$ is the number of downcrossings of $[c, d]$ on the set $T_n$, then $\nu_n \uparrow \nu$. Hence the countable case is obtained from the finite case by a simple limiting procedure.

Next, let $\tilde{\nu}(c, d)$ be the number of downcrossings of $[c, d]$ of the function $Z_t$ on the set of rational numbers in a right-open interval $T$. It is easy to see that $\tilde{\nu}(c, d) \leqslant \nu$ and if $Z_t$ is continuous on the right, then, $\nu \leqslant \tilde{\nu}(c', d')$ for any $c < c' < d' < d$. Hence the last part of Theorem 0.1 follows from the first.

The applications of the theorem are based on the following elementary lemma, whose proof is left to the reader.

LEMMA 0.2. *Let the function $Z(t)$, $t \in T$, make a finite number of downcrossings of any interval $[c, d]$ with rational ends. Then for any point $t$ that is a right limit for $T$ the limit of $Z(u)$ as $u \downarrow t$ on $T$ exists; for each point $t$ that is a left limit for $T$ the limit of $Z(u)$ as $u \uparrow t$ on $T$ exists.*

0.7. We now assume that $T = [0, \infty)$. Let $\mathcal{A}_t$, $t \in T$, be a system of $\sigma$-algebras satisfying 0.5.A. A function $\tau = \tau(\omega)$ with range in $T$ is called *a Markov moment with respect to* $\{\mathcal{A}_t\}$ if $\{\tau \leqslant t\} \in \mathcal{A}_t$ for all $t \in T$; it is called *a moment of stopping with respect to* $\{\mathcal{A}_t\}$ if $\{\tau < t\} \in \mathcal{A}_t$ for all $t \in T$. A Markov moment with respect to $\{\mathcal{A}_t\}$ is a moment of stopping with respect to $\{\mathcal{A}_t\}$. On the other hand, if $\tau$ is a moment of stopping with respect to $\{\mathcal{A}_i\}$, then $\tau$ is a Markov moment with respect to $\{\mathcal{A}_{t+0}\}$, where

$$\mathcal{A}_{t+0} = \bigcap_{u>t} \mathcal{A}_u$$

(evidently the $\mathcal{A}_{t+0}$ form a system of $\sigma$-algebras satisfying 0.5.A).

THEOREM 0.2. *Let $(Z_t, \mathcal{A}_t, \mathbf{P})$ $(t \geqslant 0)$ be a non-negative supermartingale and $P(\omega) < \infty$. We put $\omega \in \Omega_t$ if the limit of $Z_s$ exists, when $s$ tends to $t$ from the right on $R$, the set of non-negative rationals. Let $\omega \in \tilde{\Omega}$ if $z_t(\omega)$ makes on $R$ only a finite number of downcrossings of any interval with rational ends. We consider the function*

$$\widetilde{Z}_t(\omega) = \begin{cases} \lim\limits_{\substack{s \downarrow t \\ s \in R}} Z_s(\omega) & \text{for } \omega \in \widetilde{\Omega}_t, \\ 0 & \text{for } \omega \notin \widetilde{\Omega}_t; \end{cases}$$

$$\hat{Z}_t(\omega) = \begin{cases} \widetilde{Z}_t(\omega) & \text{for } \omega \in \widetilde{\Omega}, \\ 0 & \text{for } \omega \notin \widetilde{\Omega}. \end{cases}$$

*We have*:

0.7.A. *For each $\omega$, $\hat{Z}_t(\omega)$ is continuous on the right with respect to $t$.*

0.7.B. *For each $t \geqslant 0$, $\widetilde{Z}_t$ is $\mathcal{A}_{t+0}$-measurable and $\hat{Z}_t$ is measurable with respect to the $\sigma$-algebra $\mathcal{A} = \bigcup_{u \geqslant 0} \mathcal{A}_u$.*

*We assume that*

$$\sup_{t \geqslant 0} \mathbf{M} Z_t = Q < \infty. \tag{0.2}$$

*Then*:

0.7.C. *For almost all $\omega$*

$$\widetilde{Z}_t(\omega) = \hat{Z}_i(\omega) \quad \text{for all } t.$$

0.7.D. *$(\widetilde{Z}_t(\omega), \mathcal{A}_{t+0}, \mathbf{P})$ is a supermartingale.*

0.7.E. *If $\sigma \leqslant \tau$ are moments of stopping with respect to $\{\mathcal{A}_t\}$, then*

$$\mathrm{M}\hat{Z}_\sigma \geqslant \mathrm{M}\hat{Z}_\tau.$$

**0.7.F.** *If $\sigma$ is a moment of stopping with respect to $\{A_t\}$, then*

$$\mathrm{M}\hat{Z}_\tau \leqslant \mathrm{M}Z_0.$$

**0.7.G.** *The set of values of $t$ for which*

$$\mathbf{P}\{\hat{Z}_t \neq Z_t\} > 0,$$

*is at most countable.*

PROOF. Properties 0.7.A and 0.7.B are evident. Let $\nu$ the number of downcrossings of $[c, d]$ by the function $Z_t$ on $R$. By Theorem 0.1 and (0.2)

$$\mathrm{M}\nu \leqslant \frac{Q}{d-c}.$$

Hence it follows that

$$\mathbf{P}\{\Omega \setminus \widetilde{\Omega}\} = 0. \tag{0.3}$$

It is clear that $\hat{Z}_t(\omega) = \tilde{Z}_t(\omega)$ for $\omega \in \tilde{\Omega}$. Hence 0.7.C follows from (0.3).

Let $s < t$ and let $\xi \geqslant 0$ be $\mathscr{A}_{s+0}$-measurable. By 0.4.E, 0.5.E and (0.3) we have for any $v \in (s, t)$ and $a > 0$

$$\mathrm{M}(\widetilde{Z}_t \wedge a)\,\xi = \mathrm{M}\lim_{\substack{u\downarrow t\\ n\in R}} (Z_u \wedge a)\,\xi = \lim_{\substack{u\downarrow t\\ u\in R}} \mathrm{M}(Z_u \wedge a)\,\xi \leqslant \mathrm{M}(Z_v \wedge a)\,\xi.$$

First letting $v \downarrow s$, $v \in R$, and then $a \uparrow \infty$, we obtain

$$\mathrm{M}\widetilde{Z}_t\xi \leqslant \mathrm{M}\widetilde{Z}_s\xi.$$

Thus, the collection $(\widetilde{Z}_t, \mathscr{A}_{t+0}, \mathbf{P})$ satisfies 0.5.C. From 0.7.B, 0.5.B follows, and 0.7.D is proved.

We now use the following well-known theorem (see e.g. [18], Ch. 6, Theorem 13): if $(Z_t, \mathscr{A}_t, \mathbf{P})$ - is a non-negative supermartingale, $Z_t(\omega)$ is continuous with respect to $t$ for almost all $\omega$ and $\mathscr{A}_t = \mathscr{A}_{t+0}$ for all $t$, then for any Markov moments $\sigma \leqslant \tau$ with respect to $\{\mathscr{A}_t\}$ we have $\mathrm{M}Z_\sigma \geqslant \mathrm{M}Z_\tau$. Applying this theorem to the supermartingale $(\widetilde{Z}_t, \mathscr{A}_{t+0}, \mathbf{P})$ and taking 0.7.C into account we arrive at 0.7.E.

0.7.F follows from 0.7.E on putting $\sigma = 0$ and noting that by Fatou's lemma

$$\mathrm{M}\hat{Z}_0 = \mathrm{M}\widetilde{Z}_0 = \mathrm{M}\,\underline{\lim}\, Z_{1/n} \leqslant \underline{\lim}\, \mathrm{M}Z_{1/n} \leqslant \mathrm{M}Z_0.$$

Finally, 0.7.G is a consequence of the following theorem of Doob (see [3], Ch. 7, Theorem 11.2): for any supermartingale the set of fixed points of discontinuity is at most countable (here the point $t$ is called a fixed point of discontinuity if there exists a sequence $t_n$ such that $t_n \to t$ and $\mathbf{P}\{Z_{t_n} \not\to Z_t\} > 0$).

## References

[1] D. Blackwell, On a class of probability spaces, Proc. Third Berkeley Symp. on Math. Statistics and Probability (Berkeley and Los Angeles) 2 (1956), 1-56.

[2] P. Cartier, P.A. Meyer, et M. Weil, Le retournement du temps: compléments à l'exposé de M. Weil, Séminaire de probabilités II, Lecture notes in Math., Springer-Verlag, Berlin-Heidelberg 1968.

[3] J.L. Doob, Stochastic Processes, Wiley, New York 1953.
Translation: *Veroyatnostnye protsessy*, Inostr. Lit., Moscow 1956.

[4] J.L. Doob, Conditional Brownian motion and the boundary limits of harmonic functions, Bull. Soc. Math. France 85 (1957), 431-458.

[5] J.L. Doob, Discrete potential theory and boundaries, J. Math. Mech. 8 (1959), 433-458.

[6] J.L. Doob, Compactification of the discrete state space of a Markov process, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 10 (1968), 236-251.

[7] E.B. Dynkin, *Osnovaniya teorii markovskikh protsessov*, Izd. Fiz.-Mat., Moscow 1959.
Translation: Theory of Markov Processes, Pergamon Press, Oxford 1960.

[8] E.B. Dynkin, *Markovskie protsessy*, Izd. Fiz.-Mat., Moscow 1963.
Translation: Markov Processes, Springer-Verlag, Berlin-Heidelberg 1965.

[9] E.B. Dynkin, General boundary conditions for Markov processes with a countable set of states, Teor. Veroyatnost. i Primenen. 12 (1967), 222-257.

[10] E.B. Dynkin, Boundary theory for Markov processes (discrete case), Uspekhi Mat. Nauk 24: 2 (1969), 3-42.
= Russian Math. Surveys 24: 2 (1969), 1-42.

[11] W. Feller, On boundaries and lateral conditions for the Kolmogorov differential equations, Ann. of Math. 65 (1957), 527-570.

[12] E.B. Dynkin, Excessive functions and space of exits of a Markov process, Teor. Veroyatnost i Primenen. 14 (1969).

[13] P.R. Halmos, Measure Theory, Van Nostrand, Princeton N.J. 1950.
Translation: *Teoriya mery*, Inostr. Lit. Moscow 1953.

[14] G.A. Hunt, Markoff chains and Martin boundaries, Illinois J. Math. 4 (1960), 313-340.
= Matematika 5: 5 (1961), 121-149.

[15] H. Kunita and T. Watanabe, Markov processes and Martin boundaries, Bull. Amer. Math. Soc. 69 (1963), 386-391.

[16] H. Kunita and T. Watanabe, Markov processes and Martin boundaries, I, Illinois J. Math. 9 (1965), 485-526.

[17] R.S. Martin, Minimal positive harmonic functions, Trans. Amer. Math. Soc. 49 (1941), 137-172.

[18] P.A. Meyer, Probability and potentials, Blaisdell Publishing Co., Toronto-London 1966.

[19] M. Nagasawa, Time reversions of Markov processes, Nagoya Math. J. 24 (1964), 177-204.

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