Stochastic Processes and their Applications 99 (2002) 31–51

stochastic processes and their applications

www.elsevier.com/locate/spa

A leavable bounded-velocity stochastic control problem☆

Ioannis Karatzas[a], Daniel Ocone[b,*]

[a] *Departments of Mathematics and Statistics, 619 Mathematics Building, Columbia University, New York, NY 10027, USA*

[b] *Department of Mathematics, Rutgers University, Hill Center, Busch Campus, Piscataway, NJ 08854-8019, USA*

Received 11 October 2000; received in revised form 20 November 2001; accepted 10 December 2001

Abstract

This paper studies bounded-velocity control of a Brownian motion when discretionary stopping, or 'leaving', is allowed. The goal is to choose a control law and a stopping time in order to minimize the expected sum of a running and a termination cost, when both costs increase as a function of distance from the origin. There are two versions of this problem: the *fully observed case*, in which the control multiplies a known gain, and the *partially observed case*, in which the gain is random and unknown. Without the extra feature of stopping, the fully observed problem originates with Beneš (Stochastic Process. Appl. 2 (1974) 127–140), who showed that the optimal control takes the 'bang–bang' form of pushing with maximum velocity toward the origin. We show here that this same control is optimal in the case of discretionary stopping; in the case of power-law costs, we solve the variational equation for the value function and explicitly determine the optimal stopping policy.

We also discuss qualitative features of the solution for more general cost structures. When no discretionary stopping is allowed, the partially observed case has been solved by Beneš et al. (Stochastics Monographs, Vol. 5, Gordon & Breach, New York and London, pp. 121–156) and Karatzas and Ocone (Stochastic Anal. Appl. 11 (1993) 569–605). When stopping is allowed, we obtain lower bounds on the optimal stopping region using stopping regions of related, fully observed problems.

MSC: primary 93E20; 60G40; secondary 60J65

Keywords: Stochastic control; Optimal stopping; Partial observations; Control with discretionary stopping

☆ Research supported in part by the National Science Foundation under Grant NSF-DMS-00-99690.

* Corresponding author.

E-mail addresses: ik@math.columbia.edu (I. Karatzas), ocone@math.rutgers.edu (D. Ocone).

0304-4149/02/$ - see front matter
PII: S0304-4149(01)00157-0

1. Introduction and summary

In the standard model of stochastic control in continuous time, the controller may influence the state dynamics, but must operate over a prescribed time-horizon. If the controller is also allowed to choose a quitting time adaptively, at the expense of incurring a termination cost, one has a problem of *control with discretionary stopping* (or 'leavable control' problem, in the terminology of Dubins and Savage (1976)). Such problems arise, for example, in target-tracking models in which one must decide when to stop and engage the target.

Theoretical results on control with discretionary stopping are set forth in Krylov (1980) and Bensoussan and Lions (1982), Maitra and Sudderth (1996), Morimoto (2000). Recently, a literature has developed on explicit solutions to particular models. In the area of mathematical finance, Karatzas and Wang (2001) treat utility maximization with discretionary stopping, while Karatzas and Kou (1998) and Karatzas and Wang (2000) study hedging of American contingent claims under constraints. Work on target-tracking models begins with Beneš (1992), who provides explicit solutions to linear–quadratic–gaussian (LQG) problems when stopping is allowed. Davis and Zervos (1994) and Karatzas et al. (2000) solve, respectively, infinite- and finite-fuel versions of the singular control problem of Beneš et al. (1980), with the extra feature of discretionary stopping.

The explicit solutions obtained in these papers share an interesting feature. The qualitative nature of the optimal policy changes significantly as the parameters weighing the relative importance of continuation cost, stopping cost, and discount rate pass through certain, precisely identified, critical values.

This paper presents the solution to a problem of target-tracking type that involves both *control* and discretionary *stopping*. Without the extra feature of stopping, the problem was originally formulated and solved by Beneš (1974), and involves the control of Brownian motion through drift constrained to lie in a bounded set. The state-process is

$$X_y^u(t) = y + \int_0^t \theta u(s)\,\mathrm{d}s + B(t), \quad 0 \leqslant t < \infty, \tag{1.1}$$

where θ is a fixed constant, $B(\cdot)$ is scalar Brownian motion, and the control process $u(\cdot)$ is appropriately adapted and satisfies

$$-1 \leqslant u(t) \leqslant 1 \quad \text{for all } t \geqslant 0. \tag{1.2}$$

The goal of control is to track the origin, and then to stop when 'sufficiently' close, in such a way as to balance running and stopping costs. We model this by the problem of minimizing the expected discounted cost

$$J(y;u,\tau) := E\left[\int_0^\tau \mathrm{e}^{-\alpha t} k_1(|X_y^u(t)|)\,\mathrm{d}t + \mathrm{e}^{-\alpha\tau} k_2(|X_y^u(\tau)|) 1_{\{\tau<\infty\}}\right] \tag{1.3}$$

over control processes $u(\cdot)$ satisfying (1.2) and over stopping times τ. Here and in the sequel, $k(\cdot)$ (with or without a lower-case index) denotes a function which is

$$\text{non-negative, non-decreasing and continuous on } [0,\infty), \quad \text{with } k(0)=0. \tag{1.4}$$

For convenience of notation, we shall always extend such a function $k(\cdot)$ by even symmetry $k(-y)=k(y)$, $y>0$, to the entire real line, and write $k(y)$ instead of $k(|y|)$.

For the dynamics of (1.1), *Assumption* (1.4) *determines the form of the optimal control*. Indeed, Beneš (1975) shows that the intuitively natural, feedback control law

$$\bar{u}(t) = -\mathrm{sgn}(\theta X_y^{\bar{u}}(t)), \quad 0 \leqslant t < \infty, \tag{1.5}$$

where $\mathrm{sgn}(x) := \mathbf{1}_{(0,\infty)}(x) - \mathbf{1}_{(-\infty,0]}(x)$, minimizes any cost functional of the form $E[k(X_y^u(T))]$, when $T>0$ is fixed and no discretionary stopping is allowed. Ikeda and Watanabe (1977) develop a comparison argument that reproves Beneš's result and shows that $\bar{u}(\cdot)$ of (1.5) is in fact optimal for minimizing any cost functional of the form $E[\int_0^T k(t,X_y^u(t))\,\mathrm{d}t]$, where $0<T\leqslant\infty$ and $k(t,\cdot)$ is of the form (1.4) for every $t\in[0,T]$. In Section 2, it is shown that *the feedback control law* $\bar{u}(\cdot)$ *of* (1.5) *is still optimal, even when discretionary stopping is allowed.*

Suppose now that the drift parameter θ in (1.1) is replaced by a random variable Θ independent of $B(\cdot)$, with known distribution μ, and observable by the controller only indirectly (through observation of the state-process $X(\cdot)$). Then we obtain a *partially observed* problem of adaptive control that combines features of *filtering*, *control*, as well as *stopping*. Without the extra feature of stopping, this problem was posed by Beneš and Rishel, and was solved in progressively greater generality by Beneš et al. (1991) and Karatzas and Ocone (1992,1993). Roughly speaking, these results establish a *certainty-equivalence principle*. For cost-functions $k(\cdot)$ of the form (1.4), and for a fairly large class of random variables Θ, the optimal control becomes

$$u^*(t) = -\mathrm{sgn}(\hat{\Theta}(t) X_y^{u^*}(t)), \tag{1.6}$$

where $\hat{\Theta}(t)$ is the mean-square optimal estimate of Θ given observations of the state up to time t. We conjecture that the control law $u^*(\cdot)$ of (1.6) will still be optimal when stopping is allowed. This conjecture, and the problem of finding the optimal policy in the partially observed problem, are open in general. However, it is possible to obtain bounds on the optimal stopping region for the partially observed problem in terms of optimal stopping regions of the fully observed case. These results appear in Section 3. With the exception of Chapter 4, Section 6 in Bensoussan and Lions (1982), and to the best of our knowledge, these results represent one of the first attempts in the stochastic optimization literature to study models that combine all three features of filtering, control and stopping.

2. The fully observed problem

This section develops the results on the fully observed problem (1.1)–(1.3). To be precise, the expected cost J of (1.3) is to be minimized over the class of *admissible policies*, which we define rigorously now. An admissible policy $((\Omega,\mathscr{F},\mathbb{P}),\mathbb{F},B,u,\tau)$

consists of:

(i) a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$ equipped with a filtration $\mathbb{F} = \{\mathscr{F}_t\}_{0 \leqslant t < \infty}$ that satisfies the 'usual conditions' and with an $\mathbb{F}$-adapted, scalar Brownian motion $B(\cdot)$ such that $\{B(t+s) - B(t);\ s \geqslant 0\}$ is independent of $\mathscr{F}_t$, for every $t \geqslant 0$;

(ii) an $\mathbb{F}$-progressively measurable process $u(\cdot)$ with values in $[-1, 1]$; and,

(iii) an $\mathbb{F}$-stopping time τ.

We often abbreviate the notation for an admissible policy to (u, τ). A system $((\Omega, \mathscr{F}, \mathbb{P}), \mathbb{F}, B)$ satisfying condition (i) is said to be *policy-supporting*. We shall assume in the rest of the paper that $\theta > 0$, where θ is the drift parameter of the state equation (1.1). Since the set $[-1, 1]$ of control actions is symmetric about the origin, this entails no loss of generality.

The value function for the control problem is denoted

$$V(y) := \inf\{J(y; u, \tau)/(u, \tau) \text{ is admissible}\}, \tag{2.1}$$

where $J(y; u, \tau)$ is defined in (1.3). A policy $(\tilde{u}, \tilde{\tau})$ is said to be *optimal*, if $J(y; \tilde{u}, \tilde{\tau}) = V(y)$.

2.1. Reduction to an optimal stopping problem

It turns out that optimal policies have a common, specific form. Consider a policy-supporting system $((\Omega, \mathscr{F}, \mathbb{P}), \mathbb{F}, B)$, and let S be a closed subset of $\mathbb{R}$ *containing the origin*. Then, for any $y \in \mathbb{R}$, there is an admissible policy (u_S, τ_S) on $((\Omega, \mathscr{F}, \mathbb{P}), \mathbb{F}, B)$ satisfying

$$\tau_S = \inf\{t \geqslant 0 / X_y^{u_S}(t) \in S\}, \quad u_S(t) = -\operatorname{sgn}(X_y^{u_S}(t)) \ \text{ for } t \leqslant \tau_S. \tag{2.2}$$

(Properly speaking, we should also index $u_S(\cdot)$ and τ_S by the initial condition $y \in \mathbb{R}$; we omit this dependence in the interest of simplicity, especially since S will be the same for all $y \in \mathbb{R}$ in the optimal policies.)

It is easy to show that such a policy (u_S, τ_S) *as in* (2.2) *exists*. If $y = 0$, set $\tau_S = 0$ and let u_S be an arbitrary admissible control. If $y \neq 0$, let

$$\bar{X}_y(t) := y - \theta t \operatorname{sgn}(y) + B(t), \quad 0 \leqslant t < \infty \tag{2.3}$$

be Brownian motion with drift $-\theta \operatorname{sgn}(y)$ started at y, and set $\tau_S := \inf\{t \geqslant 0 / \bar{X}_y(t) \in S\}$. Let $u_S(\cdot)$ be any $\mathbb{F}$-progressively measurable process satisfying $u_S(t) = -\operatorname{sgn}(y)$ if $t \leqslant \tau_S$. There are many such processes $u_S(\cdot)$; for example, simply set $u_S(t) = 1$ for times $t \geqslant \tau_S$. Because S contains the origin, $\bar{X}_y(\cdot)$ hits the set S before hitting the origin, and $u_S(t) = -\operatorname{sgn}(\bar{X}_y(t))$ if $t \leqslant \tau_S$. Hence $X_y^{u_S}(t) = \bar{X}_y(t)$ on $\{t \leqslant \tau_s\}$. If a control of the form (u_S, τ_S) is optimal, then S is called an *optimal stopping region*.

When the optimal stopping time τ^* for the problem of (2.1), (1.3) is positive with positive probability, the optimal control is of the feedback-form (1.5), at least until time τ^*. This is the same control shown by Beneš (1975) and Ikeda and Watanabe (1989) to be optimal when no discretionary stopping is allowed. The fact that it remains optimal in the presence of discretionary stopping is intuitively reasonable and, indeed, a simple argument shows that it is true for cost-functions of form (1.4). This result is stated next, and is the starting point for our analysis; it allows us effectively to reduce the mixed control/stopping problem of (2.1), (1.3) to one of pure optimal stopping.

For the statement of this result, it is convenient to say that a stopping time τ is *admissible*, if it comes from an admissible policy $((\Omega,\mathscr{F},\mathbb{P}),\mathbb{F},B,u,\tau)$. Given such a policy, recall the definition of the process $\bar{X}_y(\cdot)$ in (2.3), and define

$$\sigma_0 := \inf\{t \geqslant 0 / \bar{X}_y(t) = 0\}, \quad \bar{u}_y(t) := -\operatorname{sgn}(y) \text{ for } 0 \leqslant t \leqslant \sigma_0. \tag{2.4}$$

Proposition 1. *In the notation of* (2.4), (2.1), (1.3), *we have*

$$V(y) = \inf\{J(y;\bar{u},\tau)/\tau \textit{ is admissible}\}.$$

Proof. Let (u,τ) be any admissible policy. Without loss of generality, assume $y \geqslant 0$. If $y = 0$, it is clearly optimal to stop immediately (i.e., $\tau \equiv 0$). If $y > 0$, we have $\bar{X}_y(t) \leqslant X_y^u(t)$ for all $0 \leqslant t \leqslant \sigma_0$ almost surely, because constraint (1.2) implies

$$\bar{X}_y(t) - X_y^u(t) = -\int_0^t \theta(u(s)+1)\,\mathrm{d}s \leqslant 0 \quad \text{for all } 0 \leqslant t \leqslant \sigma_0.$$

Hence, $0 \leqslant \bar{X}_y(t) \leqslant X_y^u(t)$ for all $0 \leqslant t \leqslant \sigma_0$, almost surely. Therefore, using $k_2(0)=0$, we obtain $\mathrm{e}^{-\alpha(\tau\wedge\sigma_0)}k_2(\bar{X}_y(\tau\wedge\sigma_0)) \leqslant \mathrm{e}^{-\alpha\tau}k_2(X_y^u(\tau))$, almost surely on $\{\tau < \infty\}$. Because $k_1(\cdot)$ is increasing and non-negative on $[0,\infty)$, it also follows that

$$\int_0^{\tau\wedge\sigma_0} k_1(\bar{X}_y(t))\,\mathrm{d}t \leqslant \int_0^{\tau} k_1(X_y^u(t))\,\mathrm{d}t, \quad \text{a.s.}$$

Thus $J(y;\bar{u},\tau\wedge\sigma_0) \leqslant J(y;u,\tau)$, and taking infima completes the proof. □

In conjunction with the symmetry about the origin inherent in the problem of (2.1) and (1.3), Proposition 1 has the following immediate implication.

Corollary 1 (Reduction to optimal stopping). *For the problem of* (2.1), (1.3), *the value-function* $V(\cdot)$ *coincides with the value of the pure optimal stopping problem*

$$V(y) = \inf E\left[\int_0^{\tau} \mathrm{e}^{-\alpha t}k_1(\bar{X}_y(t))\,\mathrm{d}t + \mathrm{e}^{-\alpha\tau}k_2(\bar{X}_y(\tau))\cdot 1_{\{\tau<\infty\}}\right], \quad 0 \leqslant y < \infty, \tag{2.5}$$

where the infimum is taken over admissible stopping times. For $y < 0$, *we have* $V(y) = V(-y)$.

2.2. Analytical preliminaries

By Corollary 1, it suffices to study the optimal stopping problem (2.5) for $y \geqslant 0$. The formal variational equation for the function $V(\cdot)$ of (2.5) is thus

$$\min\{(1/2)V''(y) - \theta V'(y) - \alpha V(y) + k_1(y),\ k_2(y) - V(y)\} = 0 \quad \text{for } y > 0, \tag{2.6}$$

$$V(0) = 0. \tag{2.7}$$

Established optimal stopping theory specifies in what sense $V(\cdot)$ solves this equation; see, for instance, Krylov (1980), Bensoussan and Lions (1982), or Salminen (1985), Alvarez (1995a,b). In particular, value functions for optimal stopping of a diffusion are typically of class C^1 (continuous and continuously differentiable), subject to mild regularity conditions, and the requirement of C^1-smoothness imposes a smooth-fit constraint on the function $V(\cdot)$, between

(i) the stopping region, where $V(y) = k_2(y)$ holds, and
(ii) the continuation region, where the following equation prevails:

$$(1/2)V''(y) - \theta V'(y) - \alpha V(y) + k_1(y) = 0. \tag{2.8}$$

Rather than invoking theory, *we shall impose the C^1-smoothness condition as an Ansatz*, solve (2.6) directly, and then apply the Verification Lemma 1 below, in order to identify our solution as the value function. Solutions to similar stopping problems are developed in Taylor (1968). For notational convenience, we shall introduce the infinitesimal generator

$$L := \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \theta\frac{\mathrm{d}}{\mathrm{d}x} \tag{2.9}$$

of the process $\bar{X}(\cdot)$ in (2.3). A function $\varphi(\cdot)$ on $[0,\infty)$ is said to be *piecewise-C^2*, if it is continuous and continuously differentiable, as well as twice-continuously differentiable at all but a finite number of points; at such points, the right- and left-limits of $\varphi''(\cdot)$ exist and are finite.

Lemma 1 (Verification result). *Let $W(\cdot)$ be a non-negative, piecewise-C^2, solution to the equation*

$$\min\{[L-\alpha]W(y) + k_1(y),\ k_2(y) - W(y)\} = 0 \quad \textit{for a.e. } y \geqslant 0. \tag{2.10}$$

Then $W(\cdot) = V(\cdot)$, the closed set $S = \{y \in \mathbb{R}/W(|y|) = k_2(y)\}$ is the optimal stopping region, and hence (u_S, τ_S) is the optimal policy for the discretionary stopping problem.

Note that the condition $W(0)=0$, found in (2.6)–(2.7) but not explicitly in (2.10), is in fact imposed by the twin demands that $W(\cdot)$ be non-negative and that $W(\cdot) \leqslant k_2(\cdot)$.

Proof of Lemma 1. This is entirely standard; an application of Itô's rule shows that $W(\cdot) \leqslant V(\cdot)$. The fact that $W(\cdot)$ is of class C^1, but only piecewise-C^2, is no impediment, because Itô's rule extends to this case; see Karatzas and Shreve (1991, p. 219). Applying Itô's rule again, together with the observation that $0 \in S$, one finds that $W(y) = J(y; \bar{u}, \tau_S)$. Hence $W(\cdot) = V(\cdot)$. The details are omitted. □

2.3. Qualitative features of the solution

In the interest of concreteness, we shall assume henceforth that

$$k_1(\cdot) = k(\cdot), \quad k_2(\cdot) = \delta k(\cdot) \text{ for some } \delta \in (0, \infty) \tag{2.11}$$

and some function $k : \mathbb{R} \to [0, \infty)$ of class C^1, which is evenly symmetric on $\mathbb{R}$ and satisfies condition (1.4). In this context, the following notation will be used:

$$\beta_1 := \theta - \sqrt{\theta^2 + 2\alpha}, \quad \beta_2 := \theta + \sqrt{\theta^2 + 2\alpha}, \tag{2.12}$$

$$P(y) := \frac{2}{\beta_2 - \beta_1}\left[\int_y^\infty e^{\beta_2(y-\xi)} k(\xi)\,d\xi + \int_{-\infty}^y e^{\beta_1(y-\xi)} k(\xi)\,d\xi\right], \tag{2.13}$$

$$Q_1(c) := \frac{\beta_2 e^{-\beta_1 c}}{\alpha(\beta_2 - \beta_1)}\left[\delta\left(\alpha k(c) + \frac{\beta_1}{2} k'(c)\right) - \left(\alpha P(c) + \frac{\beta_1}{2} P'(c)\right)\right], \tag{2.14}$$

$$Q_2(c) := \frac{-\beta_1 e^{-\beta_2 c}}{\alpha(\beta_2 - \beta_1)}\left[\delta\left(\alpha k(c) + \frac{\beta_2}{2} k'(c)\right) - \left(\alpha P(c) + \frac{\beta_2}{2} P'(c)\right)\right] \tag{2.15}$$

as well as

$$r(y) := [L - \alpha](\delta k(y)) + k(y) = (\delta/2)k''(y) - \delta\theta k'(y) + (1 - \alpha\delta)k(y), \tag{2.16}$$

whenever $k(\cdot)$ is of class C^2. Observe that $\beta_1 < 0$, $\beta_2 > 0$ of (2.12) are the roots of the quadratic equation $\beta^2 - 2\theta\beta - 2\alpha = 0$.

Subject to suitable regularity conditions on the function $k(\cdot)$ (see (2.18) in Proposition 2(a) below), the function $P(\cdot)$ is real-valued and is a particular solution of Eq. (2.8), whose general solution can then be written in the form

$$W(y) = P(y) + A_1 e^{\beta_1 y} + A_2 e^{\beta_2 y} \tag{2.17}$$

for real constants A_1 and A_2. For instance, when solving Eq. (2.8) on an interval (a, c) subject to the boundary conditions $W(c) = \delta k(c)$ and $W'(c) = \delta k'(c)$, these constants are given by $A_1 = Q_1(c)$ and $A_2 = Q_2(c)$ as in (2.14) and (2.15).

In the remainder of this subsection, we study some qualitative features of the solution to the optimal stopping problem, under suitable conditions on the cost-function $k(\cdot)$.

Proposition 2. *Consider the problem of* (2.1), (1.3) *under the assumptions of this subsection.*

(a) *Suppose that for every* $\eta > 0$, *there exists* $M_\eta > 0$ *such that*

$$k(x + \xi) \leqslant k(x)e^{\eta\xi}, \quad \forall \xi \geqslant 0, \ x \geqslant M_\eta. \tag{2.18}$$

If $\alpha\delta < 1$ *and* S *is an optimal stopping region for* (2.1), *then* S *cannot be bounded. On the other hand, if* $\alpha\delta > 1$, *then* S *must be bounded.*

(b) *Suppose that* $k(\cdot)$ *is of class* C^2, *and satisfies*

$$[L-\alpha]k(x)+mk(x)>0, \quad \forall x>0 \tag{2.19}$$

for some $m>0$. *Then there is a number* $\delta^*>0$ *such that*, *for* $0\leqslant\delta<\delta^*$, *an optimal policy for problem* (2.1) *is to stop immediately at any* $y\in\mathbb{R}$.

(c) *Suppose that* $k(\cdot)$ *is of class* C^2, *and satisfies*

$$r(x)=[L-\alpha](\delta k(x))+k(x)<0, \quad \forall x>M \tag{2.20}$$

for some $M>0$; *in particular*, *this is the case if* $\alpha\delta=1$ *and*

$$2\theta k'(x)>k''(x), \quad \forall x>M \tag{2.21}$$

holds for some $M>0$. *Then* S *must be bounded.*

The 'sub-exponential growth condition' (2.18) guarantees that the function $P(\cdot)$ of (2.13) is well-defined, of class C^2, solves Eq. (2.8), namely

$$[L-\alpha]P(y)+k(y)=0 \tag{2.22}$$

and satisfies

$$P(y)\sim\frac{k(y)}{\alpha} \quad \text{as } y\to\infty. \tag{2.23}$$

For a function $k(\cdot)$ of class C^1 satisfying (1.4), condition (2.18) holds if $\lim_{x\to\infty}(k'(x)/k(x))=0$; if, in addition, $k(\cdot)$ is of class C^2 and satisfies $\lim_{x\to\infty}(k''(x)/k'(x))=0$, then (2.21) also holds.

Proof of Proposition 2. Let $y>0$. We know by Proposition 1 that the optimal control process is $\bar{u}_y(\cdot)\equiv-\mathrm{sgn}(y)$ and that it is optimal to stop at or before the first time the process $\bar{X}_y(\cdot)$ of (2.3) hits the origin.

Assume that $k(\cdot)$ satisfies condition (2.18). Fix any $c>0$, and let τ_c be the first time that the process $\bar{X}_y(\cdot)$ hits c. If $y>c$, the cost of using the policy of stopping at time τ_c is

$$U_c(y):=E\left[\int_0^{\tau_c}\mathrm{e}^{-\alpha t}k(\bar{X}_y(t))\,\mathrm{d}t+\delta\mathrm{e}^{-\alpha\tau_c}k(\bar{X}_y(\tau_c))\right].$$

Using the sub-exponential growth of $k(\cdot)$ and the standard argument for the proof of the Feynman–Kac formula, it can be shown that $U_c(y)=A\mathrm{e}^{\beta_1 y}+P(y)$ for $y\geqslant c$, where A is chosen so that $U_c(c)=\delta k(c)$. It follows from (2.23) and from $\beta_1<0$ that

$$U_c(y)\sim\frac{k(y)}{\alpha} \quad \text{as } y\to\infty. \tag{2.24}$$

Now the proof of Part (a) follows easily. If $\alpha\delta>1$, then (2.24) implies that for all large enough y, we have $V(y)\leqslant U_c(y)<\delta k(y)$ and hence y is not in the optimal stopping region. Conversely, if S is bounded with $c^*=\sup S$, then $V(y)=U_{c^*}(y)$ for $y\geqslant c^*$, and it follows from (2.24) that $\alpha\delta\geqslant 1$.

To prove Part (b), simply note that, given (2.19), there exists $\delta^* > 0$ such that we have $[L-\alpha](\delta k(x)) + k(x) \geqslant 0$ for all $x \geqslant 0$ and all $\delta < \delta^*$. Using a verification argument similar to Lemma 1, it follows that $\delta k(\cdot)$ is the value function for the problem of (2.1), (1.3), and that the optimal policy is to stop immediately.

For Part (c), apply Itô's rule to the process $e^{-\alpha t}\delta k(\bar{X}_y(t))$, $0 \leqslant t \leqslant \sigma$ where σ is the first time the process $\bar{X}_y(\cdot)$ of (2.3) exits an interval (x,z) with $M < x < y < z < \infty$. From (1.3), (2.3)–(2.5), (2.20) we obtain then

$$V(y) \leqslant J(y;\bar{u},\sigma) = \delta k(y) + E\int_0^{\sigma} e^{-\alpha t} r(\bar{X}_y(t))\,dt < \delta k(y),$$

showing that $y \notin S$. In other words, $S \subset [-M, M]$. $\square$

The next result will be useful in Section 3.

Proposition 3. *Consider the problem of* (2.1), (1.3) *in the setting* (2.11) *of this section, and assume that an optimal stopping region $S(\alpha,\delta,\theta)$ exists for every $\alpha>0$, $\delta>0$ and $\theta>0$. Then $S(\alpha,\delta,\theta)$ decreases as θ increases, as δ increases, and as α increases.*

Proof. By Proposition 1, all optimal stopping regions include the origin. With δ and θ held fixed, let $V_\alpha(\cdot)$ denote the value function corresponding to parameter values (α,δ,θ). It is clear that $V_\alpha(\cdot)$ decreases as α increases. Thus if $\alpha_1 > \alpha_0$ and $y \in S(\alpha_1,\delta,\theta)$, one finds that $\delta k(y) = V_{\alpha_1}(y) \leqslant V_{\alpha_0}(y)$. But since $V_{\alpha_0}(y) \leqslant \delta k(y)$ is always true, it follows that $V_{\alpha_0}(y) = \delta k(y)$ and, hence, that $y \in S(\alpha_0,\delta,\theta)$. In other words, $S(\alpha_1,\delta,\theta) \subseteq S(\alpha_0,\delta,\theta)$.

Next, fix α and θ and consider the value functions $V_{\delta_1}(\cdot)$ and $V_{\delta_0}(\cdot)$ for $\delta_1 > \delta_0 > 0$. Let $y > 0$ and assume $y \notin S_0 := S(\alpha,\delta_0,\theta)$. Then, from Proposition 1,

$$\delta_0 k(y) > V_{\delta_0}(y) = E\left[\int_0^{\tau_{S_0}} e^{-\alpha t} k(\bar{X}_y(t))\,dt + \delta_0 e^{-\alpha\tau_{S_0}} k(\bar{X}_y(\tau_{S_0}))\right].$$

(Recall that the process $\bar{X}_y(\cdot)$ of (2.3) is Brownian motion with negative drift, so that $\tau_{S_0} := \inf\{t \geqslant 0 / X_y(t) \in S_0\}$ is a.s. finite.) In particular, $E[e^{-\alpha\tau_{S_0}} \cdot k(\bar{X}_y(\tau_{S_0}))] < k(y)$, and thus

$$\begin{aligned} V_{\delta_1}(y) &\leqslant E\left[\int_0^{\tau_{S_0}} e^{-\alpha t} k(\bar{X}_y(t))\,dt + \delta_1 e^{-\alpha\tau_{S_0}} k(\bar{X}_y(\tau_{S_0}))\right] \\ &< \delta_0 k(y) + (\delta_1 - \delta_0)\cdot E[e^{-\alpha\tau_{S_0}} k(\bar{X}_y(\tau_{S_0}))] < \delta_1 k(y). \end{aligned}$$

Therefore, $y \notin S(\alpha,\delta_1,\theta)$ also.

Finally, fix (α,δ) and consider the value functions $V_{\theta_0}(\cdot)$ and $V_{\theta_1}(\cdot)$ corresponding to $0 < \theta_0 < \theta_1$. Let S_0 and S_1 denote the corresponding optimal stopping regions, and let $\bar{X}_{y,0}(t) = y - \theta_0 t + B(t)$ and $\bar{X}_{y,1}(t) = y - \theta_1 t + B(t)$. It is clear that $\bar{X}_{y,1}(t) < \bar{X}_{y,0}(t)$ for $t \leqslant \tau_1 := \tau_{S_0} \wedge \sigma$ where σ is the first time $\bar{X}_{y,1}(\cdot)$ hits the origin. From this and the assumption that $k(0)=0$, we deduce the a.s. comparison

$$\int_0^{\tau_{S_0}} e^{-\alpha t}k(\bar{X}_{y,0}(t))\,dt + \delta e^{-\alpha\tau_{S_0}}k(\bar{X}_{y,0}(\tau_{S_0}))$$
$$\geqslant \int_0^{\tau_1} e^{-\alpha t}k(\bar{X}_{y,1}(t))\,dt + \delta e^{-\alpha\tau_1}k(\bar{X}_{y,1}(\tau_1)).$$

Since $\bar{X}_{y,0}(\cdot)$ is the optimal state-process and τ_{S_0} is the optimal stopping time for parameter values (α,δ,θ_0), it follows that $V_{\theta_1}(\cdot)\leqslant V_{\theta_0}(\cdot)$, and hence, repeating the argument in the case of α, that $S_1\subseteq S_0$. □

2.4. Concrete results

If one imposes sufficiently strong conditions on the cost-function $k(\cdot)$ of (2.11), then a particularly "crisp" picture emerges, as shown by the results of Theorems 1 and 2 below. Their proofs are collected in Section 4.

Theorem 1. *Assume that the function $k(\cdot)$ of (2.11) is also of class C^2 and*

$$k(0)=k'(0)=0<k''(0+),\quad k(\infty)=k'(\infty)=\infty,\quad k''(\cdot)>0, \tag{2.25}$$

$$\lim_{x\to\infty}\frac{k'(x)}{k(x)}=0,\quad \lim_{x\to\infty}\frac{k''(x)}{k'(x)}=0 \tag{2.26}$$

are satisfied. Under these conditions, we have

$$\delta_* := \sup\{\delta>0/r(y;\delta)\geqslant 0,\ \forall y\geqslant 0\}\in\left(0,\frac{1}{\alpha}\right), \tag{2.27}$$

where we denote by $r(\cdot\,;\delta)$ the function of (2.16) in order to highlight its dependence on the parameter $\delta>0$.

(a) *If $0<\delta\leqslant\delta_*$, then $V(y)=\delta k(y)$ and $\tau\equiv 0$ is optimal for all $y\in\mathbb{R}$. Thus, $S=\mathbb{R}$ is the optimal stopping region.*

(b) *If $\delta\geqslant(1/\alpha)$, assume that the function*

$$r(\cdot\,;\delta) \text{ of (2.16) has a unique root } y_* \text{ in } (0,\infty). \tag{2.28}$$

Then the function $Q_2(\cdot)$ of (2.15) also has a unique positive root c^*; in terms of it, the value-function $V(\cdot)$ of (2.1) is given by

$$V(y)=\begin{cases}\delta k(y) & \text{if } |y|\leqslant c^*,\\ P(|y|)+(\delta k(c^*)-P(c^*))e^{\beta_1(|y|-c^*)} & \text{if } |y|>c^*,\end{cases} \tag{2.29}$$

and the optimal policy is (u_S,τ_S) in the notation of (2.2), where $S=[-c^*,c^*]$.

(c) *If $\delta_*<\delta<(1/\alpha)$, assume that*

$$r(\cdot\,;\delta) \text{ of (2.16) has exactly two roots } \underline{c}<\bar{c} \text{ in } (0,\infty). \tag{2.30}$$

Then there is a unique pair of numbers $0<\ell_*<\ell^*<\infty$ such that $(Q_1(\ell_*), Q_2(\ell_*))=(Q_1(\ell^*),Q_2(\ell^*))$; these satisfy $\ell_*<\underline{c}<\bar{c}<\ell^*$, the value-function $V(\cdot)$ is *given by*

$$V(y) = \begin{cases} \delta k(y) & \text{if } y \in S, \\ P(|y|) + Q_1(\ell_*)e^{\beta_1 |y|} + Q_2(\ell_*)e^{\beta_2 |y|} & \text{if } y \notin S, \end{cases} \tag{2.31}$$

and the optimal policy is (u_S, τ_S) *in the notation of* (2.2) *with* $S = (-\infty, -\ell^*] \cup [-\ell_*, \ell_*] \cup [\ell^*, \infty)$.

It is straightforward to check, that all conditions (2.25)–(2.26) and assumptions (2.28), (2.30) are satisfied by functions of the form $k(x) = x^p$, for $p > 1$. In this case, the critical constants of (2.27) and (2.30) are given by

$$\delta_* = \left(\alpha + \frac{p\theta^2}{2(p-1)} \right)^{-1} \quad \text{and}$$

$$\bar{c}, \underline{c} = \frac{p\delta\theta}{2(1-\alpha\delta)} \left[1 \pm \sqrt{1 - \frac{2(1-\alpha\delta)}{\theta^2} \frac{(p-1)}{\delta p}} \right]. \tag{2.32}$$

These computations are especially clean in the important *quadratic case* $k(x) = x^2$. Note that in this case the constant δ_* and the functions of (2.13)–(2.15) become, respectively,

$$\delta_* = (\theta^2 + \alpha)^{-1} \quad \text{and} \quad P(y) = \frac{y^2}{\alpha} - \frac{2\theta}{\alpha^2} y + \frac{2\theta^2 + \alpha}{\alpha^3},$$

$$Q_1(c) = \frac{\beta_2}{\alpha(\beta_2 - \beta_1)} \left[(\alpha\delta - 1)c^2 + \left(\frac{(1-\alpha\delta)\beta_2}{\alpha} + 2\theta\delta \right) c - \frac{1}{2} \left(\frac{\beta_2}{\alpha} \right)^2 \right] e^{-\beta_1 c},$$

$$Q_2(c) = \frac{-\beta_1}{\alpha(\beta_2 - \beta_1)} \left[(\alpha\delta - 1)c^2 + \left(\frac{(1-\alpha\delta)\beta_1}{\alpha} + 2\theta\delta \right) c - \frac{1}{2} \left(\frac{\beta_1}{\alpha} \right)^2 \right] e^{-\beta_2 c}.$$

We summarize the results for this case in a separate theorem.

Theorem 2. *Consider the problem of* (2.1), (1.3), (2.11) *with quadratic* $k(x) = x^2$.

(a) *If* $0 < \delta \leqslant (\alpha + \theta^2)^{-1}$, *then* $V(y) = \delta y^2$ *and* $\tau \equiv 0$ *is optimal for all* $y \in \mathbb{R}$. *Thus* $S = \mathbb{R}$ *is the optimal stopping region.*

(b) *If* $\alpha\delta \geqslant 1$, *then* $V(\cdot)$ *is given as in* (2.29), *where* c^* *is the unique positive root of* $Q_2(c) = 0$, *or equivalently, of*

$$(\alpha\delta - 1)c^2 + \left(\frac{(1-\alpha\delta)\beta_1}{\alpha} + 2\theta\delta \right) c - \frac{1}{2} \left(\frac{\beta_1}{\alpha} \right)^2 = 0. \tag{2.33}$$

The optimal policy is (u_S, τ_S) *in the notation of* (2.2), *where* $S = [-c^*, c^*]$.

(c) *If* $(\theta^2 + \alpha)^{-1} < \delta < \alpha^{-1}$, *there is a unique pair of values* $0 < \ell_* < \ell^* < \infty$ *such that* $(Q_1(\ell_*), Q_2(\ell_*)) = (Q_1(\ell^*), Q_2(\ell^*))$. *Let* $S = (-\infty, -\ell^*] \cup [-\ell_*, \ell_*] \cup [\ell^*, \infty)$. *Then* $V(\cdot)$ *is given as in* (2.31), *and the optimal policy is* (u_S, τ_S) *in the notation of* (2.2).

3. Partial observations

This section formulates carefully the problem of control with discretionary stopping in the case of partial observations, and presents bounds for the optimal stopping region. In particular, a sufficient condition is given for the optimality of $\tau \equiv 0$ from all starting points $y \in \mathbb{R}$.

In the work that follows, it is useful to define a finite-horizon version of the problem (2.1)–(1.3) in form (2.11). We shall assume from now on, that, in addition to (1.4), the cost-function $k(\cdot)$ in (2.11) satisfies a polynomial-growth condition of the type

$$k(y) \leqslant C(1+|y|^p), \quad \forall y > 0 \text{ for some real constants } C > 0, p > 1.$$

Fix a time-horizon $T > 0$ and set

$$\begin{aligned} V_T(y) &:= \inf E\left[\int_0^{\tau\wedge T} \mathrm{e}^{-\alpha t} k(X_y^u(t))\,\mathrm{d}t + \delta \mathrm{e}^{-\alpha\tau\wedge T} k(X_y^u(\tau\wedge T))\right] \\ &= \inf J(y; u, \tau\wedge T), \end{aligned} \tag{3.1}$$

in the notation of Section 2, where the infimum is taken over admissible policies (u,τ). The problem of finding the value-function $V_T(\cdot)$ and the associated optimal policy, is called the *finite-horizon problem*. The problem solved in Section 2 is, by way of contrast, called the *infinite-horizon problem*.

Since $\inf J(y;u,\tau\wedge T) \geqslant \inf J(y;u,\tau)$, we see that $V_T(\cdot) \geqslant V(\cdot)$. Combining this with the fact that $\delta k(\cdot)$ is always an upper bound on $V_T(\cdot)$, one has the following simple result.

Lemma 2. *If $V(y) = \delta k(y)$, then $V_T(y) = \delta k(y)$ for all $T \in (0,\infty)$. Thus, if it is optimal to stop at y in the infinite-horizon problem, it is optimal to do so for any finite-horizon problem as well.*

In this section the parameters α and δ are fixed, but it will be necessary to vary the drift parameter θ. Let J_z, V_z and $V_{T,z}$, respectively, denote the cost, the infinite-horizon value function, and the finite-horizon value function, when θ takes on the value z.

We turn now to the partially observed problem. Let μ be a probability measure on $\mathbb{R}$ with bounded support, and set

$$\theta := \inf\{|x| / \mu([-x,x]) = 1\}. \tag{3.2}$$

In treating the partially observed case, we shall change notation slightly and write the state-equation (1.1) as

$$X_y(t) = y + \int_0^t \Theta u(s)\,\mathrm{d}s + B(t), \quad 0 \leqslant t < \infty, \tag{3.3}$$

where Θ is to be a random variable with distribution μ, independent of the Brownian motion $B(\cdot)$. Thus, θ is now an upper bound on the possible values of the drift. As usual in partially observed control, solutions to the state-equation are constructed by Girsanov transformation on the process $X(\cdot)$. This necessitates reformulating the notion of admissibility. An *admissible policy* for the partially observed control problem consists of

(i) a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$ equipped with a filtration $\mathbb{F}$ that satisfies the usual conditions, a random variable Θ that is independent of $\mathbb{F}$ and has distribution μ, and an $\mathbb{F}$-adapted, scalar Brownian motion $X_y(\cdot)$ starting at y, such that $\{X_y(t+s) - X_y(t);\ s \geqslant 0\}$ is independent of $\mathscr{F}_t$, for all $t \geqslant 0$.
(ii) an $\mathbb{F}$-progressively measurable process $u(\cdot)$ with values in $[-1, 1]$, as well as
(iii) an $\mathbb{F}$-stopping time τ.

For such an admissible system and for any $T > 0$, we construct on $\mathscr{F}_T$ the probability measure $\mathbb{P}^u_T$ by the Girsanov formula

$$\frac{\mathrm{d}\mathbb{P}^u_T}{\mathrm{d}\mathbb{P}} = \exp\left\{ -\int_0^T \Theta u(s)\,\mathrm{d}X_y(s) - \frac{1}{2}\int_0^T \Theta^2 u^2(s)\,\mathrm{d}s \right\}. \tag{3.4}$$

If $B^u(\cdot)$ is defined via

$$B^u(t) := X_y(t) - y - \int_0^t \Theta u(s)\,\mathrm{d}s, \quad 0 \leqslant t < \infty \tag{3.5}$$

then $B^u(t)$, $0 \leqslant t \leqslant T$ is Brownian motion with respect to $\mathbb{F}$ under $\mathbb{P}^u_T$, and is independent of Θ. At least up to time T, $((\Omega, \mathscr{F}, \mathbb{P}^u_T), \mathbb{F}, B^u, u, \tau)$ is an admissible policy in the sense of Section 1. Let $E^u_T[\,\cdot\,]$ denote expectation with respect to $\mathbb{P}^u_T$.

In the sequel, the following additional definitions related to the Girsanov measure change are also useful. For $z \in \mathbb{R}$, introduce the probability measure $\mathbb{P}^u_{T,z}$ on $\mathscr{F}_T$ via

$$\frac{\mathrm{d}\mathbb{P}^u_{T,z}}{\mathrm{d}\mathbb{P}} = \Lambda^u_T(z) := \exp\left\{ -\int_0^T z u(s)\,\mathrm{d}X_y(s) - \frac{1}{2}\int_0^T z^2 u^2(s)\,\mathrm{d}s \right\}. \tag{3.6}$$

Use $E^u_{T,z}[\,\cdot\,]$ to denote expectation with respect to the probability measure $\mathbb{P}^u_{T,z}$ of (3.6). Also, define

$$B^u(z, t) := X_y(t) - y - \int_0^t z u(s)\,\mathrm{d}s, \quad 0 \leqslant t < \infty. \tag{3.7}$$

Then for each z, the process $B^u(z, t)$, $0 \leqslant t \leqslant T$ is Brownian motion under $\mathbb{P}^u_{T,z}$. Moreover, $\mathrm{d}\mathbb{P}^u_T/\mathrm{d}\mathbb{P} = \Lambda^u_T(\Theta)$. For any $t < T$, the independence of Θ and $\mathbb{F}$ under $\mathbb{P}$ implies

$$N(t) := E\left[\left. \frac{\mathrm{d}\mathbb{P}^u_T}{\mathrm{d}\mathbb{P}} \right| \mathscr{F}_t \right] = \int \Lambda^u_t(z)\mu(\mathrm{d}z). \tag{3.8}$$

This expression does not depend on T. Clearly, $N(\cdot)$ is a positive $\mathbb{F}$-martingale under $\mathbb{P}$.

Given that $B^u(\cdot)$ is an $\mathbb{F}$-Brownian motion up to time T and is independent of Θ under $\mathbb{P}^u_T$, it makes sense to define the finite- and infinite-horizon costs for the partially observed problem, as follows:

$$\mathscr{J}_T(y; u, \tau) := E^u_T\left[\int_0^{\tau\wedge T} \mathrm{e}^{-\alpha t} k(X_y(t))\,\mathrm{d}t + \delta \mathrm{e}^{-\alpha(\tau\wedge T)} k(X_y(\tau \wedge T)) \right], \tag{3.9}$$

and

$$\mathscr{J}(y;u,\tau) := \lim_{T\to\infty} \mathscr{J}_T(y;u,\tau).$$

It can be shown that $\mathscr{J}$ is well-defined and

$$\mathscr{J}(y;u,\tau) = E\left[\int_0^\tau \mathrm{e}^{-\alpha t} k(X_y(t))N(t)\,\mathrm{d}t + \delta \mathrm{e}^{-\alpha\tau} k(X_y(\tau))N(\tau)\right]. \tag{3.10}$$

This formula is derived by using (3.14) and (3.15) below and taking limits. In the formula, $\tau = \infty$ is allowed if one interprets $\mathrm{e}^{-\alpha\tau}k(X_y(\tau))N(\tau) = 0$, on $\{\tau = \infty\}$. This makes sense, because $N(\cdot)$ is a positive $\mathbb{P}$-martingale, which implies that $\lim_{t\to\infty} N(t)$ exists in $[0,\infty)$, $\mathbb{P}$-a.s., and because $X_y(\cdot)$ is Brownian motion under $\mathbb{P}$, which implies that $\lim_{t\to\infty} \mathrm{e}^{-\alpha t}k(X_y(t))=0$, $\mathbb{P}$-a.s. thanks to the polynomial growth condition on $k(\cdot)$.

Let $\mathscr{V}_T(y)$ and $\mathscr{V}(y)$ denote, respectively, the infima of $\mathscr{J}_T(y;u,\tau)$ and $\mathscr{J}(y;u,\tau)$ over the class of admissible policies for the partially observed problem.

Finally, let $S_T(\alpha,\delta,z)$ and $S(\alpha,\delta,z)$ denote the optimal stopping regions in the finite- and infinite-horizon, fully observed problems when the drift parameter is z. Of course $S(\alpha,\delta,z) = \{y \in \mathbb{R}/V_z(y) = \delta k(y)\}$, and similarly for $S_T(\alpha,\delta,z)$. Also, let

$$\Sigma(\alpha,\delta,\theta) := \{y \in \mathbb{R}/\mathscr{V}(y) = \delta k(y)\} \tag{3.11}$$

be the optimal stopping region for the partially observed problem, when the quantity θ of (3.2) is finite. Define $\Sigma_T(\alpha,\delta,\theta)$ similarly.

Theorem 3. *Assume that the quantity θ of* (3.2) *is finite. Then*

$$S_T(\alpha,\delta,\theta) \subseteq \Sigma_T(\alpha,\delta,\theta) \quad \text{and} \quad S(\alpha,\delta,\theta) \subseteq \Sigma(\alpha,\delta,\theta). \tag{3.12}$$

Thus, if it is optimal to stop at y in the fully observed problem with drift θ, then it is also optimal to stop at y in the partially observed problem with $\mathbb{P}(|\Theta| \leqslant \theta) = 1$.

In particular, if $k(y) = y^p$ for $p > 1$, and if $0 < \delta \leqslant \delta_*$ as in (2.33), *then the optimal policy in both finite and infinite-horizon partially observed problems is to stop immediately, irrespective of the initial condition.*

The heart of the proof is the following lemma.

Lemma 3. *For any $T > 0$, we have*

$$\mathscr{V}_T(y) \geqslant \int_{\mathbb{R}} V_{T,z}(y)\mu(\mathrm{d}z). \tag{3.13}$$

Proof. By conditioning on $\mathscr{F}_t$ inside the expectation, one obtains

$$E_T^u[\mathbf{1}_{\{t\leqslant\tau\wedge T\}} k(X_y(t))] = E[\mathbf{1}_{\{t\leqslant\tau\wedge T\}} k(X_y(t))N(t)] \tag{3.14}$$

and

$$E_T^u[\mathrm{e}^{-\alpha(\tau\wedge T)} k(X_y(\tau\wedge T))] = E[\mathrm{e}^{-\alpha(\tau\wedge T)} k(X_y(\tau\wedge T))N(\tau\wedge T)] \tag{3.15}$$

for $t \leqslant T$. Using the formula for $N(t)$ in (3.8) and Fubini, we get

$$E_T^u[\mathbf{1}_{\{t\leqslant\tau\wedge T\}}k(X_y(t))] = \int_{\mathbb{R}} E[\mathbf{1}_{\{t\leqslant\tau\wedge T\}}k(X_y(t))\Lambda_t^u(z)]\mu(\mathrm{d}z)$$

$$= \int_{\mathbb{R}} E_{T,z}^u[\mathbf{1}_{\{t\leqslant\tau\wedge T\}}k(X_y(t))]\mu(\mathrm{d}z).$$

Similarly,

$$E_T^u[\mathrm{e}^{-\alpha(\tau\wedge T)}k(X_y(\tau\wedge T))] = \int_{\mathbb{R}} E_{T,z}^u[\mathrm{e}^{-\alpha(\tau\wedge T)}k(X_y(\tau\wedge T))]\mu(\mathrm{d}z).$$

Using these identities, we find that

$$\mathscr{J}_T(y;u,\tau) = \int_{\mathbb{R}} E_{T,z}^u\left[\int_0^{\tau\wedge T} \mathrm{e}^{-\alpha t}k(X_y(t))\,\mathrm{d}t + \delta\mathrm{e}^{-\alpha(\tau\wedge T)}k(X_y(\tau\wedge T))\right]\mu(\mathrm{d}z). \tag{3.16}$$

Now, for each fixed $z\in\mathbb{R}$, the system $((\Omega,\mathscr{F},\mathbb{P}_{T,z}^u),\mathbb{F},B^u(z,\cdot),(u,\tau))$ constitutes an admissible policy for the fully observed problem. Thus, the integrand of the last expression dominates $V_{T,z}(y)$, for each $z\in\mathbb{R}$. As a result,

$$\mathscr{J}_T(y;u,\tau) \geqslant \int_{\mathbb{R}} V_{T,z}(y)\mu(\mathrm{d}z). \tag{3.17}$$

(Note: The measurability of $z \mapsto V_{T,z}(y)$ follows, because $V_{T,z}(y)$ decreases as $|z|$ increases; see the proof of Proposition 3 for the infinite-horizon case, and note that this proof works for $V_T(\cdot)$ as well.) Taking an infimum over admissible policies (u,τ) for the partially observed control problem, leads to (3.13).

Proof of Theorem 3. Let us first note that $S_T(\alpha,\delta,z)$ is decreasing as $|z|$ increases. This is proved in Proposition 3 for the infinite-horizon problem. That proof uses an almost-sure comparison argument, and hence it applies as well to the finite-horizon case.

Assume that $y\in S_T(\alpha,\delta,\theta)$. It then follows that $y\in S_T(\alpha,\delta,z)$, for all z such that $|z|\leqslant\theta$. As a consequence, we have $V_{T,z}(y)=\delta k(y)$ for all z with $|z|\leqslant\theta$, and so by Lemma 3,

$$\mathscr{V}_T(y) \geqslant \int_{-\theta}^{\theta} V_{T,z}(y)\mu(\mathrm{d}z) = \delta k(y).$$

But $\mathscr{V}_T(y)\leqslant\delta k(y)$, since stopping immediately has cost $\delta k(y)$. Thus $\mathscr{V}_T(y)=\delta k(y)$, which implies $y\in\Sigma(\alpha,\delta,\theta)$. This argument shows $S_T(\alpha,\delta,\theta)\subseteq\Sigma_T(\alpha,\delta,\theta)$.

Now let $y\in S(\alpha,\delta,\theta)$. Then, by Proposition 3, $V_z(y)=\delta k(y)$ for all z with $|z|\leqslant\theta$. By Lemma 2, we have $V_{T,z}(y)=\delta k(y)$ for all $T>0$ and $|z|\leqslant\theta$. Thus, using the inequality (3.17),

$$\mathscr{J}_T(y;u,\tau)\geqslant\delta k(y) \quad \text{for any } T>0 \text{ and admissible } (u,\tau).$$

By letting $T \to \infty$, we obtain $\mathscr{J}(y;u,\tau) \geqslant \delta k(y)$ for any admissible (u,τ), and thus $\mathscr{V}(y) \geqslant \delta k(y)$. Again, it follows that $y \in \Sigma(\alpha,\delta,\theta)$. □

4. Proofs of Theorems 1 and 2

We shall provide here in careful detail the proof for Theorem 2, the more concrete of our two main results, as it contains all of the main ideas. We shall then sketch the proof of Theorem 1, which mirrors it closely.

The C^1-smooth-fit Ansatz of Section 2.2 has the following consequence. Suppose that $W(\cdot)$ solves the equation $[L-\alpha]W(y)+k(y)=0$ on the interval (a,c), and satisfies $W(y)=\delta k(y)$ on the interval (c,b), where $a<c<b$. If $W'(\cdot)$ is continuous across $y=c$, it is straightforward to derive that $W(\cdot)$ must be of the form

$$W(y)=P(y)+Q_1(c)\mathrm{e}^{\beta_1 y}+Q_2(c)\mathrm{e}^{\beta_2 y} \quad \text{for } a<y<c. \tag{4.1}$$

In other words, $A_1=Q_1(c)$ and $A_2=Q_2(c)$, where A_1 and A_2 are the constants in Formula (2.17) and $Q_1(\cdot)$, $Q_2(\cdot)$ are the functions of (2.14), (2.15). The exact same expression as in (4.1) prevails, if instead $W(\cdot)$ satisfies $[L-\alpha]W(y)+k(y)=0$ on (c,b) and $W(y)=\delta k(y)$ on (a,c), and if $W'(\cdot)$ is continuous across $y=c$.

Proof of Theorem 2(a). *Small termination cost*. In this case $(\alpha+\theta^2)\delta \leqslant 1$, and a fortiori $\alpha\delta<1$. As a consequence, the function $r(\cdot)$ of (2.16) takes the form

$$\begin{aligned} r(y) &= [L-\alpha](\delta y^2)+y^2 \\ &= (1-\alpha\delta)\left(y-\frac{\delta\theta}{1-\alpha\delta}\right)^2+\frac{\delta}{1-\alpha\delta}[1-(\alpha+\theta^2)\delta] \geqslant 0 \end{aligned}$$

for all $y \geqslant 0$. Thus $W(y)=\delta y^2$ is a solution of Eq. (2.10), and satisfies all the conditions of the verification Lemma 1. This proves that $S=\mathbb{R}$ is the optimal stopping region, and that δy^2 is the value function for the stopping problem of (2.5), and so completes the proof of case (a).

Proof of Theorem 2(b). *Large termination cost*. Here $\alpha\delta \geqslant 1$. It follows easily that there is a unique positive root c^* to Eq. (2.33), or, equivalently, $Q_2(c^*)=0$. Set

$$W(y):=\begin{cases} \delta y^2 & \text{if } 0 \leqslant y \leqslant c^*, \\ P(y)+Q_1(c^*)\mathrm{e}^{\beta_1 y} & \text{if } y>c^*. \end{cases}$$

As a consequence of (4.1), this function $W(\cdot)$ is of class C^1. Continuity of $W(\cdot)$ at the point $y=c^*$ is easily shown to imply

$$Q_1(c^*)\mathrm{e}^{\beta_1 y}=(\delta(c^*)^2-P(c^*))\mathrm{e}^{\beta_1(y-c^*)},$$

and hence $W(\cdot)$ coincides on $[0,\infty)$ with the function on the right-hand side of Eq. (2.29), part (b) of Theorem 1. Thus, in order to complete the proof, it suffices to show that $W(\cdot)$ satisfies the hypotheses of the verification Lemma 1. To do this, it is necessary to prove that $W(\cdot)$ is non-negative and that

$$[L-\alpha](\delta y^2)+y^2 \geqslant 0 \quad \text{if } 0 \leqslant y \leqslant c^*, \tag{4.2}$$

and

$$\delta y^2 \geqslant W(y) \quad \text{if } y > c^*. \tag{4.3}$$

To prove $W(\cdot) \geqslant 0$, one can use the minimum principle. Since $W(c^*) > 0$, and since $W(y) > 0$ for all sufficiently large y by virtue of $\beta_1 < 0$, it follows that if $W(\cdot)$ is ever negative, it must achieve a negative absolute minimum at some point $x \in (c^*, \infty)$. But this is impossible, because $W(\cdot)$ solves $[L-\alpha]W(y) = -y^2$ on (c^*, ∞).

For (4.2), note that the function $r(\cdot)$ of (2.16) is strictly decreasing on $[0, \infty)$, since $\alpha\delta \geqslant 1$ and $\theta\delta > 0$; direct calculation and the definition of c^* show that for $0 \leqslant y \leqslant c^*$,

$$\begin{aligned} r(y) = (1-\alpha\delta)y^2 - 2\theta\delta y + \delta &\geqslant (1-\alpha\delta)(c^*)^2 - 2\theta\delta c^* + \delta \\ &= (\beta_1/\alpha)(1-\alpha\delta)c^* + [\delta - (\beta_1^2/2\alpha^2)]. \end{aligned}$$

This last expression is positive. Indeed, since $\beta_1 < 0$ and $\alpha\delta \geqslant 1$, its first term is non-negative; by using the identity $\beta_1^2 = 2\theta\beta_1 + 2\alpha$, we see that the second term is

$$\delta - (\beta_1^2/2\alpha^2) = \frac{\alpha\delta - 1}{\alpha} - \frac{\theta\beta_1}{\alpha^2} > 0.$$

To check (4.3), observe that for $y > c^*$ we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}y}[\delta y^2 - W(y)] &= \frac{2}{\alpha}(\alpha\delta - 1)y + \frac{2\theta}{\alpha^2} - \mathrm{e}^{\beta_1(y-c^*)}\left[\frac{2}{\alpha}(\alpha\delta-1)c^* + \frac{2\theta}{\alpha^2}\right] \\ &> \frac{2}{\alpha}(\alpha\delta-1)(y-c^*) \geqslant 0. \end{aligned}$$

Since $W(c^*) = \delta(c^*)^2$, it follows that $\delta y^2 > W(y)$ for $y > c^*$, as we wished to show. (The factor in front of $\mathrm{e}^{\beta_1(y-c^*)}$ in the computation of the derivative comes from the fact that W is C^1 at c^*.)

Proof of Theorem 2(c). *Moderate termination cost*. In this case we have $\alpha\delta < 1$ but $(\theta^2 + \alpha)\delta > 1$. The first step establishes the existence of ℓ_* and ℓ^*, together with bounds on their values.

Lemma 4. *If* $(\theta^2+\alpha)\delta > 1$ *and* $\alpha\delta < 1$, *then there is a unique pair of values* $0 < \ell_* < \ell^* < \infty$ *such that* $(Q_1(\ell_*), Q_2(\ell_*)) = (Q_1(\ell^*),\ Q_2(\ell^*))$. *Moreover*,

$$0 < \ell_* < \underline{c} < \bar{c} < \ell^*, \tag{4.4}$$

where

$$\underline{c} := \frac{\theta\delta - \sqrt{\delta((\theta^2+\alpha)\delta - 1)}}{1-\alpha\delta} \quad \textit{and} \quad \bar{c} := \frac{\theta\delta + \sqrt{\delta((\theta^2+\alpha)\delta - 1)}}{1-\alpha\delta}$$

are the roots of the equation $r(c) := (1-\alpha\delta)c^2 - 2\theta\delta c + \delta = 0$.

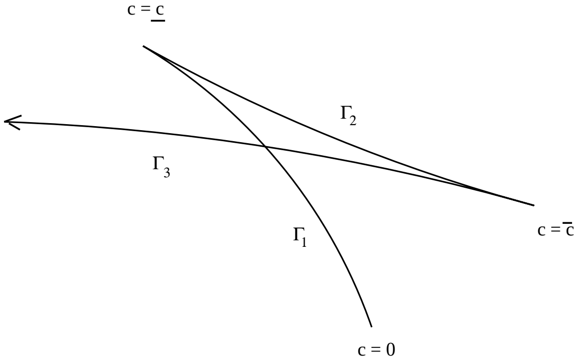

Fig. 1.

Proof. Let us decompose the planar curve $\Gamma := \{(Q_1(c),\ Q_2(c))/0 \leqslant c < \infty\}$ into three pieces Γ_1, Γ_2, and Γ_3, corresponding, respectively, to the parameter intervals $0 \leqslant c < \underline{c}$, $\underline{c} \leqslant c \leqslant \bar{c}$, and $\bar{c} < c$. The lemma states that Γ has a unique point of self-intersection, which is an intersection point of Γ_1 and Γ_3. To prove this, we study the qualitative shape of the curve Γ. The first observation is that

$$Q_2(c) > Q_2(0) \quad \text{for all } c > 0. \tag{4.5}$$

Indeed, introduce $K(c) := -\beta_1 \mathrm{e}^{-\beta_2 c}/(\alpha(\beta_2 - \beta_1)) > 0$; using the formula for $Q_2(\cdot)$, the identity $\beta_1/\alpha = -2/\beta_2$, and the inequality $(2/\beta_2^2)[\mathrm{e}^{\beta_2 c} - 1] > (2c/\beta_2) + c^2$, one finds that

$$\begin{aligned} Q_2(c) - Q_2(0) &= K(c)\left[(\alpha\delta - 1)c^2 + 2c\left(\theta\delta - \frac{1-\alpha\delta}{\beta_2}\right) + \frac{2}{\beta_2^2}(\mathrm{e}^{\beta_2 c} - 1)\right] \\ &\geqslant K(c)\left[\alpha\delta c^2 + 2\delta c\left(\theta + \frac{\alpha}{\beta_2}\right)\right] > 0. \end{aligned}$$

Secondly, direct computation shows that

$$Q_1'(c) = \frac{\beta_1\beta_2}{\alpha(\beta_2 - \beta_1)}\, r(c)\mathrm{e}^{-\beta_1 c}, \tag{4.6}$$

$$\frac{Q_2'(c)}{Q_1'(c)} = -\mathrm{e}^{(\beta_1 - \beta_2)c} = -\mathrm{e}^{-2c\sqrt{\theta^2 + 2\alpha}}, \quad c \neq \underline{c},\ \ c \neq \bar{c}. \tag{4.7}$$

When $c \neq \underline{c}$ and $c \neq \bar{c}$, the fraction $Q_2'(c)/Q_1'(c)$ represents the slope of the curve Γ at the point c, and (4.7) shows that this slope takes values in $(-1, 0)$ and is strictly increasing in c. From (4.6), $Q_1(\cdot)$ decreases and $Q_2(\cdot)$ increases in c on $\Gamma_1 \cup \Gamma_3$, while $Q_1(\cdot)$ increases and $Q_2(\cdot)$ decreases on Γ_2. Because the slope is decreasing in c, it follows that Γ_2 lies above and to the right of Γ_1, while Γ_3 lies below and to the left of Γ_2. Since $Q_2(\bar{c}) > Q_2(0)$, Γ_1 and Γ_3 must thus intersect; and since the slope of the curve is everywhere greater along Γ_3 than along Γ_1, they can intersect at most once. Fig. 1 illustrates the situation. □

Let $W(\cdot)$ denote the function on the right-hand side of (2.32), Theorem 1(c). By the smooth-fit condition (4.1), $W(\cdot)$ is automatically of class C^1. We now show that $W(\cdot)$ verifies the remaining hypotheses of Lemma 1.

First, $W(\cdot)$ must again be non-negative by the same argument used in the proof of part (b). Secondly, we claim that

$$[L-\alpha]W(y)+y^2 \geqslant 0 \quad \text{if } 0 \leqslant y \leqslant \ell_* \text{ or } y \geqslant \ell^*.$$

Clearly, $[L-\alpha]W(y)+y^2=0$ on (ℓ_*,ℓ^*) by construction. But on $[0,\ell_*]\cup[\ell^*,\infty)$, we have $W(y)=\delta y^2$ and thus $[L-\alpha]W(y)+y^2=r(y)=(1-\alpha\delta)(y-\underline{c})(y-\bar{c})>0$, because $\ell_*<\underline{c}$ and $\ell^*>\bar{c}$ as shown in Lemma 4.

Finally, we must show that $W(y)<\delta y^2$ on (ℓ_*,ℓ^*). Set $g(y):=\delta y^2-W(y)$ and observe that on $[\ell_*,\ell^*]$ we have

$$[L-\alpha]g(y)=r(y), \quad g(\ell_*)=g'(\ell_*)=g(\ell^*)=g'(\ell^*)=0.$$

The variation-of-parameters formula gives the representations:

$$g(y)=\int_{\ell_*}^{y}\frac{e^{\beta_2(y-\xi)}-e^{\beta_1(y-\xi)}}{\beta_2-\beta_1}\,r(\xi)\,d\xi=-\int_{y}^{\ell^*}\frac{e^{\beta_2(y-\xi)}-e^{\beta_1(y-\xi)}}{\beta_2-\beta_1}\,r(\xi)\,d\xi.$$

Since $r(\cdot)>0$ on $(\ell_*,\underline{c})$ and on $(\bar{c},\ell^*)$, one easily sees from these representations that $g(\cdot)$ is also positive on these intervals. Moreover, $g(\cdot)$ cannot achieve a non-positive minimum on $(\underline{c},\bar{c})$ by the maximum principle applied to $[L-\alpha]g(y)=r(y)$, because $r(y)<0$ for $y\in(\underline{c},\bar{c})$. Thus $g(\cdot)$ remains positive on $[\ell_*,\ell^*]$.

This completes the proof of Theorem 2. □

Sketch of Proof for Theorem 1. The conditions of (2.26) imply that both $k(\cdot)$ and $k'(\cdot)$ satisfy the 'sub-exponential growth condition' of (2.18); in particular, we have then the equation (2.22) for the function $P(\cdot)$ of (2.13), as well as

$$\alpha P(y)+\frac{\beta_2}{2}P'(y)=\beta_2\int_y^{\infty}e^{\beta_2(y-\xi)}k(\xi)\,d\xi.$$

Coupled with the conditions $k(0)=k'(0)=0$, this gives $Q_2(0)<0$ for the function of (2.15), as well as

$$\begin{aligned}&Q_2(c)-Q_2(0)\\&\quad=\frac{-\beta_1 e^{-\beta_2 c}}{\alpha(\beta_2-\beta_1)}\left[\delta\left(\alpha k(c)+\frac{\beta_2}{2}k'(c)\right)+\beta_2\int_0^{\infty}e^{\beta_2(c-\xi)}k(\xi)\,d\xi\right]>0\end{aligned}$$

for all $c>0$, just as in the proof of Lemma 4. It is also straightforward to check, that *Computations* (4.6), (4.7) *hold in this more general context as well.*

With these observations in place, the argument follows exactly the same lines as in the proof of Theorem 2. For $0<\delta\leqslant\delta_*$, we have $[L-\alpha](\delta k(y))+k(y)\geqslant 0$ for all $y\geqslant 0$, so $S=\mathbb{R}$ and $V(\cdot)=\delta k(\cdot)$.

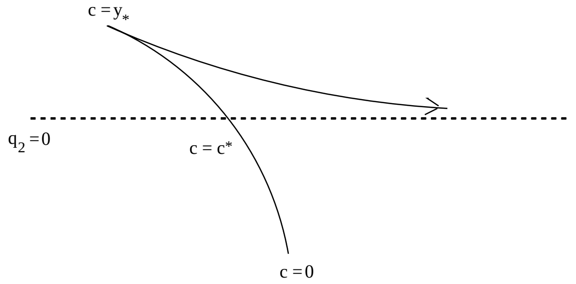

Fig. 2.

For $\delta_* < \delta < (1/\alpha)$, we have $r(\underline{c}) = r(\bar{c}) = 0$ and $r(\cdot) < 0$ on $(\underline{c}, \bar{c})$, $r(\cdot) > 0$ on $[0,\infty) \setminus (\underline{c}, \bar{c})$; thus, the plane curve $\Gamma = \{(Q_1(c),\ Q_2(c))/0 \leqslant c < \infty\}$ has exactly the same shape as in Fig. 1; in particular, there are only two values $\ell_* < \ell^*$ of c with $(Q_1(\ell_*), Q_2(\ell_*)) = (Q_1(\ell^*), Q_2(\ell^*))$, corresponding to the unique point of self-intersection for the curve Γ.

Finally, for $\alpha\delta \geqslant 1$, the planar curve Γ decomposes into two pieces, Γ_1 (corresponding to $0 \leqslant c < y_*$) and Γ_2 (corresponding to $c \geqslant y_*$). On Γ_1, the function $Q_1(\cdot)$ decreases, while $Q_2(\cdot)$ increases and eventually becomes positive; on Γ_2, the function $Q_1(\cdot)$ increases, and $Q_2(\cdot)$ decreases asymptotically to zero but stays always positive. Thus, there is exactly one point at which the curve Γ crosses the horizontal axis (equivalently, exactly one value $c^* > 0$ with $Q_2(c^*) = 0$); see Fig. 2. The details of these derivations are left to the diligent reader. □

Remark. It is now clear, from the proofs of Theorems 1 and 2, that the curve Γ contains all the essential information for constructing the optimal stopping region S in the case of quite arbitrary $k(\cdot)$. We suspect that there should be a theorem, deriving S from the location and sequence of self-intersection points for the curve Γ and from the roots of the equation $Q_2(c) = 0$. It seems rather messy to attempt a general statement, but Theorems 1 and 2 illustrate an instance of the principle.

References

Alvarez, L.R.H., 1995a. Optimal stopping of one-dimensional diffusions. Preprint, University of Turku, Finland, November.

Alvarez, L.R.H., 1995b. Reward functionals and the optimal stopping of linear diffusions. Preprint, University of Turku, Finland, November.

Beneš, V.E., 1974. Girsanov functionals and optimal bang–bang laws for final-value stochastic control. Stochastic Process. Appl. 2, 127–140.

Beneš, V.E., 1975. Composition and invariance methods for solving some stochastic control problems. Adv. in Appl. Probab. 7, 299–329.

Beneš, V.E., 1992. Some combined control and stopping problems. Paper Presented at the CRM Workshop on Stochastic Systems, Montréal, November.

Beneš, V.E., Karatzas, I., Rishel, R.W., 1991. The separation principle for a Bayesian adaptive control problem with no strict-sense optimal law. In: Davis, M.H.A., Elliott, R.J. (Eds.), Applied Stochastic Analysis, Stochastics Monographs, Vol. 5. Gordon & Breach, New York, London, pp. 121–156.

Beneš, V.E., Shepp, L.A., Witsenhausen, H.S., 1980. Some solvable stochastic control problems. Stochastics 4, 38–83.

Bensoussan, A., Lions, J.L., 1982. Applications of Variational Inequalities in Stochastic Control. North-Holland, Amsterdam, New York.

Davis, M.H.A., Zervos, M., 1994. A problem of singular stochastic control with discretionary stopping. Ann. Appl. Probab. 4, 226–240.

Dubins, L.E., Savage, L.J., 1976. Inequalities for Stochastic Processes: How to gamble if you must, 2nd Edition. Dover, New York.

Ikeda, N., Watanabe, S., 1977. A comparison theorem for solutions of stochastic differential equations, and its applications. Osaka J. Math. 14, 619–633.

Ikeda, N., Watanabe, S., 1989. Stochastic Differential Equations and Diffusion Processes, 2nd Edition. North-Holland, Amsterdam, New York.

Karatzas, I., Kou, S.G., 1998. Hedging American contingent claims with constrained portfolios. Finance Stochastics 3, 215–258.

Karatzas, I., Ocone, D.L., 1992. The resolvent of a degenerate diffusion on the plane, with application to partially observed stochastic control. Ann. Appl. Probab. 2, 629–668.

Karatzas, I., Ocone, D.L., 1993. The finite-horizon version for a partially observed stochastic control problem of Beneš and Rishel. Stochastic Anal. Appl. 11, 569–605.

Karatzas, I., Shreve, S.E., 1991. Brownian Motion and Stochastic Calculus, 2nd Edition. Springer, New York.

Karatzas, I., Wang, H., 2000. A barrier option of American type. Appl. Math. Optim. 42, 259–279.

Karatzas, I., Wang, H., 2001. Utility maximization with discretionary stopping. SIAM J. Control Optim. 39, 306–329.

Karatzas, I., Ocone, D.L., Wang, H., Zervos, M., 2000. Finite-fuel singular control with discretionary stopping. Stochastics 71, 1–50.

Krylov, N.V., 1980. Controlled Diffusion Processes. Springer, New York.

Maitra, A., Sudderth, W.D., 1996. Discrete Gambling and Stochastic Games. Springer, New York.

Morimoto, H., 2000. Variational inequalities for combined control and stopping. Preprint, Ehime University, Japan, February.

Salminen, P., 1985. Optimal stopping of one-dimensional diffusions. Math. Nachr. 124, 85–101.

Taylor, H.M., 1968. Optimal stopping in a Markov process. Ann. Math. Statist. 39, 1333–1344.