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ALMOST SURE EXPONENTIAL STABILITY OF DELAY EQUATIONS WITH DAMPED STOCHASTIC PERTURBATION

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ABSTRACT

In this paper, we shall first study the almost sure exponential stability for a class of stochastic differential delay equations of the form

$$\mathrm{d}\mathbf{x}(t) = f(\mathbf{x}(t), \mathbf{x}(t-\tau), t)\,\mathrm{d}t + \sigma(t)\,\mathrm{d}w(t). \tag{1}$$

This equation is regarded as a stochastically perturbed system of a nonlinear delay equation

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{x}(t-\tau), t). \tag{2}$$

Assume this delay equation is exponentially stable. The aim of this paper is to show that a damped stochastic perturbation can be tolerated by second equation without losing the property of stability, this stochastically perturbed equation (first equation) remains exponentially stable almost surely. This result is then generalized and several examples are also given for illustration.

Key Words: Ito's formula; Stability; Brownian motion; Lyapunov function.

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1. INTRODUCTION

Stochastic differential equations provide a mathematical model for sophisticated dynamical systems in physical, biological, medical, and social sciences. However in many circumstances, the future state of a system depends not only on the present state but also on its history. Stochastic differential delay or functional equations give a mathematical formulation for such systems. The stability problem for such equations has been investigated by many authors (1–5). In particular, Ladde (6) employed the Lyapunov function, together with the theory of functional differential inequalities, to study the various types of stabilities for stochastic differential functional equations. More recently, Ladde (7) used the Lyapunov function and the comparison theorem to investigate the exponential stability in the mean for linear stochastic differential functional equations.

On the other hand, little was known on the almost sure exponential stability for stochastic differential delay or functional equations until Mohammed (8) obtained the first result on the Lyapunov exponents for linear stochastic differential delay equations and Mao (9) studied the almost surely exponential stability for a class of nonlinear differential equations with stochastic delay perturbations. In (9), the equation is of the form

$$d\mathbf{x}(t) = f(\mathbf{x}(t), t)\,dt + F(\hat{\mathbf{x}}(t), t)\,dt + G(\hat{\mathbf{x}}(t), t)\,dw(t), \tag{1}$$

where $w(\cdot)$ is a multidimensional Brownian motion and $\hat{\mathbf{x}}(t) = \{\mathbf{x}(t+s): -\tau \leq s \leq 0\}$ is the history of the solution. The main idea there is as follows: Regard Equation (1) as a stochastically perturbed system of an ordinary differential equation

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), t). \tag{2}$$

Assume this equation is exponentially stable and it is showed in (9) that a certain amount of stochastic perturbation can be tolerated by this equation without losing the property of stability, that is the stochastically perturbed Equation (1) remains exponentially stable almost surely.

In this paper, we shall study the almost sure exponential stability for another class of stochastic differential delay equations of the form

$$d\mathbf{x}(t) = f(\mathbf{x}(t), \mathbf{x}(t-\tau), t)\,dt + \sigma(t)\,dw(t). \tag{3}$$

In the same point of view Equation (3) can be regarded as a stochastically perturbed system of a nonlinear delay equation

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{x}(t-\tau), t). \tag{4}$$

Assume this delay equation is exponentially stable. The aim of this paper is to show that a damped stochastic perturbation can be tolerated by Equation (4) without losing the property of stability.

This paper is organized as follows: We first study the almost sure exponential stability of Equation (3) in Section 2. We then generalize the result in Sections 3 and 4, where we shall deal with a stochastic differential delay equation

$$\mathrm{d}\mathbf{x}(t) = f(\mathbf{x}(t), \mathbf{x}(\rho(t)), t)\,\mathrm{d}t + \sigma(t)\,\mathrm{d}w(t) \tag{5}$$

and a stochastic differential functional equation

$$\mathrm{d}\mathbf{x}(t) = f(\mathbf{x}(t), \hat{\mathbf{x}}(t), t)\,\mathrm{d}t + \sigma(t)\,\mathrm{d}w(t) \tag{6}$$

respectively. For illustration, several examples shall be given in Section 5.

2. MAIN RESULTS

Throughout this paper let $(\Omega,\ F, \{F_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{F_t\}$, which is right continuous and contains all P-null sets. Denote by $|\mathbf{x}|$ the Euclidean norm of a vector $\mathbf{x} \in \mathbb{R}^d$. Denote by $\|\boldsymbol{A}\|$ the operator norm of a matrix $\boldsymbol{A}$, i.e. $\|\boldsymbol{A}\| = \sup\{|\boldsymbol{A}\mathbf{x} : \mathbf{x}| = 1\}$. Also denote by $\boldsymbol{B}^{\mathrm{T}}$ the transpose of vector of matrix $\boldsymbol{B}$. For a square matrix $\boldsymbol{A} = (a_{ij})$, trace $\boldsymbol{A} = \sum a_{ii}$. Let τ be a positive constant and $C([-\tau,\ 0]; \mathbb{R}^d)$ denote the family of all continuous $\mathbb{R}^d$-valued functions defined on $[-\tau, 0]$.

Consider a delay equation with damped stochastic perturbation of the form

$$\mathrm{d}\mathbf{x}(t) = f(\mathbf{x}(t),\ \mathbf{x}(t-\tau),\ t)\,\mathrm{d}t + \sigma(t)\,\mathrm{d}w(t) \quad \text{on } t \geq 0 \tag{7}$$

with initial data $\mathbf{x}(t) = \xi(t)$ on $-\tau \leq t \leq 0$, where $f : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^d$, $\sigma : \mathbb{R}_+ \to \mathbb{R}^{d\times m}$ and w is an m-dimensional Brownian motion, and $\xi = \{\xi(t): -\tau \leq t \leq 0\}$ is a $C([-\tau, 0]; \mathbb{R}^d)$-valued F_0-measurable random variable. Assume the equation has a unique solution that is denoted by $\mathbf{x}(t; \xi)$. For the existence and uniqueness of the solution we refer the reader to Mao (10) and Mohammed (11).

Theorem 2.1. *Let c_1–c_3 be positive constants. Assume*

(i) $2\mathbf{x}^{\mathrm{T}} f(\mathbf{x}, \mathbf{y}, t) \leq -c_1|\mathbf{x}|^2 + c_2|\mathbf{y}|^2$,
(ii) $\mathrm{trace}(\sigma(t)\sigma(t)^{\mathrm{T}}) \leq c_3\,\mathrm{e}^{-c_1 t}$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ and $t \geq 0$. If $c_1 > c_2$, then there exists a $\lambda > 0$ such that

$$\limsup_{t\to\infty} \frac{1}{t} \log|\mathbf{x}(t; \xi)| \leq -\lambda \quad \text{a.s.} \tag{8}$$

for all ξ. In other words, Equation (7) is almost surely exponentially stable.

Proof: We divide the whole proof into two steps.

Step 1. Fix ξ arbitrarily and write $\mathbf{x}(t; \xi) = \mathbf{x}(t)$ simply. Since $c_1 > c_2$ we can find a $\lambda \in (0,\ c_1)$ such that

$$\frac{c_2\,\mathrm{e}^{\lambda\tau}}{c_1 - \lambda} < 1. \tag{9}$$

We claim that for almost all $\omega \in \Omega$ there exists an integer $k_0 = k_0(\omega)$ and a positive number $C = C(\omega)$ such that for every integer $k \geq k_0$

$$\int_0^T e^{\lambda t}|\mathbf{x}(t)|^2 \, dt \leq Ck \quad \text{if } 0 \leq T \leq k. \tag{10}$$

In fact, by Itô's formula and the assumptions,

$$\begin{aligned} e^{c_1 t}|\mathbf{x}(t)|^2 &= |\mathbf{x}(0)|^2 + M(t) \\ &\quad + \int_0^t e^{c_1 s}(c_1|\mathbf{x}(s)|^2 + 2\mathbf{x}(s)^{\mathrm{T}} f(\mathbf{x}(s), \mathbf{x}(s-\tau), s) \\ &\quad + \mathrm{trace}(\sigma(s)\sigma(s)^{\mathrm{T}}) \, ds \\ &\leq |\xi(0)|^2 + M(t) + c_3 t + c_2 \int_0^t e^{c_1 s}|\mathbf{x}(s-\tau)|^2 \, ds \end{aligned} \tag{11}$$

for all $t \geq 0$, where

$$M(t) = 2\int_0^t e^{c_1 s}\mathbf{x}(s)^T \sigma(s) \, dw(s)$$

which is a continuous martingale vanishing at $t = 0$. Let $k = 1, 2, \ldots$ and, by Equation (9), one can choose $\varepsilon > 0$ so small that

$$\frac{c_2 \, e^{\lambda\tau}}{c_1 - \lambda} + \frac{2\varepsilon c_3}{c_1 - \lambda} < 1. \tag{12}$$

Applying the exponential martingale inequality (cf. Métivier (12) or Yan (13) we derive that

$$P\left(\omega : \sup_{0 \leq t \leq k}\left[M(t) - \frac{1}{2}\varepsilon\langle M(t)\rangle\right] > 2\varepsilon^{-1}\log k\right) \leq k^{-2}.$$

Hence the well-known Borel–Cantelli lemma yields that for almost all $\omega \in \Omega$ there exists an integer $k_0 = k_0(\omega)$ such that

$$\sup_{0 \leq t \leq k}\left[M(t) - \frac{1}{2}\varepsilon\langle M(t)\rangle\right] \leq 2\varepsilon^{-1}\log k \quad \text{if } k \geq k_0.$$

That is

$$M(t) \leq \frac{1}{2}\varepsilon\langle M(t)\rangle + 2\varepsilon^{-1}\log k$$

for all $0 \leq t \leq k$, $k \geq k_0$ almost surely. But

$$\langle M(t)\rangle \leq 4\int_0^t e^{2c_1 s}|\mathbf{x}(s)|^2 \, \mathrm{trace}(\sigma(s)\sigma(s)^{\mathrm{T}}) \, ds \leq 4c_3 \int_0^t e^{c_1 s}|\mathbf{x}(s)|^2 \, ds.$$

Therefore

$$M(t) \le 2\varepsilon c_3 \int_0^t e^{c_1 s} |\mathbf{x}(s)|^2 \, ds + 2\varepsilon^{-1} \log k \tag{13}$$

for all $0 \le t \le k$, $k \ge k_0$ almost surely. Substituting this into Equation (11) gives

$$\begin{aligned} e^{c_1 t} |\mathbf{x}(t)|^2 \le{} & |\xi(0)|^2 + c_3 t + 2\varepsilon^{-1} \log k \\ & + c_2 \int_0^t e^{c_1 s} |\mathbf{x}(s-\tau)|^2 \, ds + 2\varepsilon c_3 \int_0^t e^{c_1 s} |\mathbf{x}(s)|^2 \, ds \end{aligned}$$

for all $0 \le t \le k$, $k \ge k_0$ almost surely. Consequently

$$\begin{aligned} |\mathbf{x}(t)|^2 \le{} & [|\xi(0)|^2 + c_3 k + 2\varepsilon^{-1} \log k] \, e^{-c_1 t} \\ & + c_2 e^{-c_1 t} \int_0^t e^{-c_1 s} |\mathbf{x}(s-\tau)|^2 \, ds + 2\varepsilon c_3 e^{-c_1 t} \int_0^t e^{c_1 s} |\mathbf{x}(s)|^2 \, ds \end{aligned} \tag{14}$$

for all $0 \le t \le k$, $k \ge k_0$ almost surely. Therefore, for almost all $\omega \in \Omega$, if $0 \le T \le k$, $k \ge k_0$

$$\begin{aligned} \int_0^T e^{\lambda t} |\mathbf{x}(t)|^2 \, dt &\le [|\xi(0)|^2 + c_3 k + 2\varepsilon^{-1} \log k] \int_0^T e^{-(c_1-\lambda)t} \, dt + J_1 + J_2 \\ &\le \frac{1}{c_1 - \lambda} [|\xi(0)|^2 + c_3 k + 2\varepsilon^{-1} \log k] + J_1 + J_2, \end{aligned} \tag{15}$$

where

$$J_1 = c_2 \int_0^T e^{-(c_1-\lambda)t} \int_0^t e^{c_1 s} |\mathbf{x}(s-\tau)|^2 \, ds \, dt$$

and

$$J_2 = 2\varepsilon c_3 \int_0^T e^{-(c_1-\lambda)t} \int_0^t e^{c_1 s} |\mathbf{x}(s)|^2 \, ds \, dt.$$

But

$$\begin{aligned} J_1 &= c_2 \int_0^T e^{c_1 s} |\mathbf{x}(s-\tau)|^2 \int_s^T e^{-(c_1-\lambda)t} \, dt \, ds \\ &\le \frac{1}{c_1-\lambda} c_2 \int_0^T e^{\lambda s} |\mathbf{x}(s-\tau)|^2 \, ds \\ &\le \frac{1}{c_1-\lambda} c_2 \int_0^\tau e^{\lambda s} |\xi(s-\tau)|^2 \, ds + \frac{1}{c_1-\lambda} c_2 \, e^{\lambda\tau} \int_\tau^{T\vee\tau} e^{\lambda(s-\tau)} |\mathbf{x}(s-\tau)|^2 \, ds \\ &\le \frac{1}{c_1-\lambda} c_2 \int_0^\tau e^{\lambda s} |\xi(s-\tau)|^2 \, ds + \frac{1}{c_1-\lambda} c_2 \, e^{\lambda\tau} \int_0^T e^{\lambda s} |\mathbf{x}(s)|^2 \, ds \end{aligned}$$

and similarly

$$J_2 \leq \frac{1}{c_1 - \lambda} 2\varepsilon c_3 \int_0^T \mathrm{e}^{\lambda s} |\mathbf{x}(s)|^2 \, \mathrm{d}s.$$

Hence we obtain

$$\begin{aligned}
&\int_0^T \mathrm{e}^{\lambda t} |\mathbf{x}(t)|^2 \, \mathrm{d}t \\
&\quad \leq \frac{1}{c_1 - \lambda} \left[|\xi(0)|^2 + c_3 k + 2\varepsilon^{-1} \log k + c_2 \int_0^\tau \mathrm{e}^{\lambda s} |\xi(s - \tau)|^2 \mathrm{d}s \right] \\
&\qquad + \frac{1}{c_1 - \lambda} (c_2 \, \mathrm{e}^{\lambda \tau} + 2\varepsilon c_3) \int_0^T \mathrm{e}^{\lambda s} |\mathbf{x}(s)|^2 \, \mathrm{d}s.
\end{aligned} \tag{16}$$

Recalling Equation (12) we get the required Equation (10).

Step 2. By Itô's formula and the assumptions we can derive that

$$\begin{aligned}
\mathrm{e}^{\lambda t} |\mathbf{x}(t)|^2 &= |\mathbf{x}(0)|^2 + N(t) + \int_0^t \mathrm{e}^{c_1 s} (\lambda |\mathbf{x}(s)|^2 \\
&\qquad + 2\mathbf{x}(s)^{\mathrm{T}} f(\mathbf{x}(s), \mathbf{x}(s - \tau), s) + \mathrm{trace}(\sigma(s)\sigma(s)^{\mathrm{T}})) \, \mathrm{d}s \\
&\leq |\xi(0)|^2 + N(t) + \frac{c_3}{c_1 - \lambda} + c_2 \int_0^t \mathrm{e}^{\lambda s} |\mathbf{x}(s - \tau)|^2 \, \mathrm{d}s
\end{aligned} \tag{17}$$

for all $t \geq 0$, where

$$N(t) = 2 \int_0^t \mathrm{e}^{\lambda s} \mathbf{x}(s)^{\mathrm{T}} \sigma(s) \, \mathrm{d}w(s)$$

which is a continuous martingale vanishing at $t = 0$. In the same way as Step 1, we can use the exponential martingale inequality and the Borel–Cantelli lemma to show that for almost all $\omega \in \Omega$ there exists an integer $k_1 = k_1(\omega)$ such that

$$N(t) \leq \frac{1}{2} \langle N(t) \rangle + 2 \log k \leq 2 \int_0^t \mathrm{e}^{\lambda s} |\mathbf{x}(s)|^2 \, \mathrm{d}s + 2 \log k \tag{18}$$

for all $0 \leq t \leq k$. Substituting this into Equation (17) we obtain

$$\begin{aligned}
\mathrm{e}^{\lambda t} |\mathbf{x}(t)|^2 &\leq |\xi(0)|^2 + 2 \log k + \frac{c_3}{c_1 - \lambda} + c_2 \int_0^t \mathrm{e}^{\lambda s} |\mathbf{x}(s - \tau)|^2 \, \mathrm{d}s \\
&\qquad + 2 \int_0^t \mathrm{e}^{\lambda s} |\mathbf{x}(s)|^2 \, \mathrm{d}s \\
&\leq |\xi(0)|^2 + 2 \log k + \frac{c_3}{c_1 - \lambda} + c_2 \int_0^\tau \mathrm{e}^{\lambda s} |\xi(s - \tau)|^2 \, \mathrm{d}s \\
&\qquad + (2 + c_2) \int_0^t \mathrm{e}^{\lambda s} |\mathbf{x}(s)|^2 \, \mathrm{d}s
\end{aligned}$$

for all $0 \le t \le k$, $k \ge k_1$ almost surely. This, together with Equation (10) implies

$$e^{\lambda t}|\mathbf{x}(t)|^2 \le |\xi(0)|^2 + 2\log k + \frac{c_3}{c_1-\lambda} + c_2\int_0^\tau e^{\lambda s}|\xi(s-\tau)|^2\,ds + (2+c_2)\,Ck \tag{19}$$

for all $0 \le t \le k$, $k \ge k_0 \vee k_1$almost surely. In particular, for almost all $\omega \in \Omega$, if $k-1 \le t \le k$ and $k \ge k_0 \vee k_1$,

$$\frac{1}{t}\log(e^{\lambda t}|\mathbf{x}(t)|^2) \le \frac{1}{k-1}\log\left(|\xi(0)|^2 + 2\log k + \frac{c_3}{c_1-\lambda} + c_2\int_0^\tau e^{\lambda s}|\xi(s-\tau)|^2\,ds + (2+c_2)\,Ck\right)$$

and hence

$$\limsup_{t\to\infty}\frac{1}{t}\log(e^{\lambda t}|\mathbf{x}(t)|^2) \le 0 \quad \text{a.s.}$$

But

$$\limsup_{t\to\infty}\frac{1}{t}\log(e^{\lambda t}|\mathbf{x}(t)|^2) = \lambda + 2\limsup_{t\to\infty}\frac{1}{t}\log|\mathbf{x}(t)|.$$

So

$$\limsup_{t\to\infty}\frac{1}{t}\log|\mathbf{x}(t)| \le -\frac{\lambda}{2} \quad \text{a.s.}$$

as required. The proof is complete. □

In the same way we can prove the following more general result.

Theorem 2.2. *Let c_1–c_3 be three positive constant and $\boldsymbol{Q}$ be a positive definite $d \times d$ matrix. Assume for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ and $t \ge 0$:*

(i) $\mathbf{x}^T(\boldsymbol{Q}+\boldsymbol{Q}^T)f(\mathbf{x},\mathbf{y},t) \le -c_1\mathbf{x}^T\boldsymbol{Q}\mathbf{x} + c_2\mathbf{y}^T\boldsymbol{Q}\mathbf{y}$,
(ii) $\mathrm{trace}(\sigma(t)\sigma(t)^T) \le c_3\,e^{-c_1 t}$.

If $c_1 > c_2$, then the delay equation (7) *is almost surely exponentially stable.*

We shall now use this theorem to establish a useful result.

Corollary 2.3. *Let c_1–c_4 be four positive constants and let $\boldsymbol{Q}$ be a positive definite $d \times d$ matrix. Assume for all $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^d$ and $t \ge 0$:*

(i) $|\mathbf{x}|^2 \le \mathbf{x}^T\boldsymbol{Q}\mathbf{x}$,
(ii) $\mathbf{x}^T(\boldsymbol{Q}+\boldsymbol{Q}^T)f(\mathbf{x},0,t) \le -c_1\mathbf{x}^T\boldsymbol{Q}\mathbf{x}$,
(iii) $|f(\mathbf{x},0,t) - f(\mathbf{x},\mathbf{y},t)| \le c_2|\mathbf{y}|$,
(iv) $\mathrm{trace}(\sigma(t)\sigma(t)^T) \le c_3\,e^{-c_4 t}$.

If

$$(c_1/2) \wedge c_4 > c_2 \|\boldsymbol{Q}\|$$

or

$$(c_1 - c_4)\, c_4 > (c_2 \|\boldsymbol{Q}\|)^2,$$

then the delay Equation (7) *is almost surely exponentially stable*.

Proof: Let $\varepsilon \in (0, c_1)$. Then

$$\begin{aligned} \mathbf{x}^{\mathrm{T}}(\boldsymbol{Q} + \boldsymbol{Q}^{\mathrm{T}}) f(\mathbf{x}, \mathbf{y}, t) &\leq -c_1 \mathbf{x}^{\mathrm{T}} \boldsymbol{Q} \mathbf{x} + 2c_2 \|\boldsymbol{Q}\| |\mathbf{x}| |\mathbf{y}| \\ &\leq -c_1 \mathbf{x}^{\mathrm{T}} \boldsymbol{Q} \mathbf{x} + \varepsilon |\mathbf{x}|^2 + \varepsilon^{-1} (c_2 \|\boldsymbol{Q}\|)^2 \, |\mathbf{y}|^2 \\ &\leq -(c_1 - \varepsilon) \mathbf{x}^{\mathrm{T}} \boldsymbol{Q} \mathbf{x} + \varepsilon^{-1} (c_2 \|\boldsymbol{Q}\|)^2 \mathbf{y}^{\mathrm{T}} \boldsymbol{Q} \mathbf{y}. \end{aligned}$$

In the case $(c_1/2) \wedge c_4 > c_2 \|\boldsymbol{Q}\|$, we choose $\varepsilon = c_2 \|\boldsymbol{Q}\|$ and then

$$\begin{aligned} \mathbf{x}^{\mathrm{T}}(\boldsymbol{Q} + \boldsymbol{Q}^{\mathrm{T}}) f(\mathbf{x}, \mathbf{y}, t) &\leq -(c_1 - c_2 \|\boldsymbol{Q}\|)\, \mathbf{x}^{\mathrm{T}} \boldsymbol{Q} \mathbf{x} + c_2 \|\boldsymbol{Q}\| \, \mathbf{y}^{\mathrm{T}} \boldsymbol{Q} \mathbf{y} \\ &\leq -((c_1 - c_2 \|\boldsymbol{Q}\|) \wedge c_4)\, \mathbf{x}^{\mathrm{T}} \boldsymbol{Q} \mathbf{x} + c_2 \|\boldsymbol{Q}\| \mathbf{y}^{\mathrm{T}} \boldsymbol{Q} \mathbf{y}. \end{aligned}$$

But obviously

$$\operatorname{trace}(\sigma(t)\sigma(t)^{\mathrm{T}}) \leq c_3 \, \mathrm{e}^{-((c_1 - c_2 \|\boldsymbol{Q}\|) \wedge c_4) t}.$$

So by Theorem 2.2, Equation (7) is almost surely exponentially stable if

$$(c_1 - c_2 \|\boldsymbol{Q}\|) \wedge c_4 > c_2 \|\boldsymbol{Q}\|$$

which is equivalent to $(c_1/2) \wedge c_4 > c_2 \|\boldsymbol{Q}\|$ as assumed. On the other hand, in the case $(c_1 - c_4)\, c_4 > (c_2 \|\boldsymbol{Q}\|)^2$, we choose $\varepsilon = c_1 - c_4$ and then

$$\mathbf{x}^{\mathrm{T}}(\boldsymbol{Q} + \boldsymbol{Q}^{\mathrm{T}}) f(\mathbf{x}, \mathbf{y}, t) \leq -c_4 \mathbf{x}^{\mathrm{T}} \boldsymbol{Q} \mathbf{x} + \frac{(c_2 \|\boldsymbol{Q}\|)^2}{c_1 - c_4} \mathbf{y}^{\mathrm{T}} \boldsymbol{Q} \mathbf{y}.$$

Hence by Theorem 2.2 again, Equation (2.1) is almost surely exponentially stable. The proof is complete. □

3. GENERALIZATIONS

Let $\rho : \mathbb{R}_+ \to \mathbb{R}$ be a continuously differentiable function such that

$$\rho(t) \leq t \qquad \text{and} \qquad \frac{\mathrm{d}\rho(t)}{\mathrm{d}t} \geq 1 \quad \text{for all } t \geq 0. \tag{20}$$

It is easy to show that

$$t + \rho(0) \leq \rho(t) \qquad \text{and} \qquad \rho^{-1}(t) \leq t - \rho(0) \quad \text{for all } t \geq 0, \tag{21}$$

where $\rho^{-1}(\cdot)$ is the inverse function of $\rho(\cdot)$. Set $\tau = -\rho(0)$ which is assumed to be positive (otherwise $\rho(t) \equiv t$ but this is not what we are interested in here).

Consider a stochastic differential delay equation

$$\mathrm{d}\mathbf{x}(t) = f(\mathbf{x}(t), \mathbf{x}(\rho(t)), t)\,\mathrm{d}t + \sigma(t)\,\mathrm{d}w(t) \quad \text{on } t \geq 0 \tag{22}$$

with initial data $\mathbf{x}(t) = \xi(t)$ on $-\tau \leq t \leq 0$, where f, σ, w, and ξ are all the same as before. It is also assumed that the equation has a unique solution which is denoted by $\mathbf{x}(t;\xi)$ again.

Theorem 3.1. *Let Equation* (20) *hold. Then under the same conditions of Theorem* 2.1 *the delay Equation* (22) *is almost surely exponentially stable.*

Proof: We use the same notations as in the proof of Theorem 2.1. It is easy to see that Equation (11) becomes

$$\mathrm{e}^{c_1 t}|\mathbf{x}(t)|^2 \leq |\xi(0)|^2 + M(t) + c_3 t + c_2 \int_0^t \mathrm{e}^{c_1 s}|\mathbf{x}(\rho(s))|^2\,\mathrm{d}s, \tag{23}$$

where $M(t)$ is the same as before. Hence we can derive from Equations (23) and (12) that, for almost all $\omega \in \Omega$, if $0 \leq T \leq k$, $k \geq k_0$

$$\int_0^T \mathrm{e}^{\lambda t}|\mathbf{x}(t)|^2\,\mathrm{d}t \leq \frac{1}{c_1 - \lambda}[|\xi(0)|^2 + c_3 k + 2\varepsilon^{-1}\log k] + J_1 + J_2, \tag{24}$$

where J_2 is the same as before but

$$J_1 = c_2 \int_0^T \mathrm{e}^{-(c_1-\lambda)t} \int_0^t \mathrm{e}^{c_1 s}|\mathbf{x}(\rho(s))|^2\,\mathrm{d}s\,\mathrm{d}t.$$

We now compute

$$\begin{aligned} J_1 &= c_2 \int_0^T \mathrm{e}^{c_1 s}|\mathbf{x}(\rho(s))|^2 \int_s^T \mathrm{e}^{-(c_1-\lambda)t}\,\mathrm{d}t\,\mathrm{d}s. \\ &\leq \frac{1}{c_1-\lambda} c_2 \int_0^T \mathrm{e}^{\lambda s}|\mathbf{x}(\rho(s))|^2\,\mathrm{d}s. \end{aligned}$$

But by changing variable $r = \rho(s)$ and making use of Equations (20) and (21) we have

$$\begin{aligned} \int_0^T \mathrm{e}^{\lambda s}|\mathbf{x}(\rho(s))|^2\,\mathrm{d}s &\leq \int_{-\tau}^T \mathrm{e}^{\lambda(r+\tau)}|\mathbf{x}(r)|^2\,\mathrm{d}r. \\ &= \mathrm{e}^{\lambda\tau}\left(\int_{-\tau}^0 \mathrm{e}^{\lambda r}|\xi(r)|^2\,\mathrm{d}r + \int_0^T \mathrm{e}^{\lambda r}|\mathbf{x}(r)|^2\,\mathrm{d}r\right). \end{aligned} \tag{25}$$

So

$$J_1 \leq \frac{1}{c_1-\lambda} c_2\,\mathrm{e}^{\lambda\tau}\left(\int_{-\tau}^0 \mathrm{e}^{\lambda r}|\xi(r)|^2\,\mathrm{d}r + \int_0^T \mathrm{e}^{\lambda r}|\mathbf{x}(r)|^2\,\mathrm{d}r\right).$$

Hence we obtain from Equation (24) that

$$\begin{aligned}\int_0^T \mathrm{e}^{\lambda t}|\mathbf{x}(t)|^2\,\mathrm{d}t &\\ \leq \frac{1}{c_1-\lambda}&\left[|\xi(0)|^2 + c_3 k + 2\varepsilon^{-1}\log k + c_2\,\mathrm{e}^{\lambda\tau}\int_{-\tau}^0 \mathrm{e}^{\lambda s}|\xi(s)|^2\,\mathrm{d}s\right]\\ &+\frac{1}{c_1-\lambda}(c_2\,\mathrm{e}^{\lambda\tau}+2\varepsilon c_3)\int_0^T \mathrm{e}^{\lambda s}|\mathbf{x}(s)|^2\,\mathrm{d}s. \end{aligned}\tag{26}$$

Recalling Equation (12) we see Equation (10) still holds. Moreover, Equation (17) becomes

$$\mathrm{e}^{\lambda t}|\mathbf{x}(t)|^2 \leq |\xi(0)|^2 + N(t) + \frac{c_3}{c_1-\lambda} + c_2\int_0^t \mathrm{e}^{\lambda s}|\mathbf{x}(\rho(s))|^2\,\mathrm{d}s, \tag{27}$$

where $N(t)$ is the same as before. Using Equations (18) and (25) we then obtain

$$\begin{aligned}\mathrm{e}^{\lambda t}|\mathbf{x}(t)|^2 \leq{}& |\xi(0)|^2 + 2\log k + \frac{c_3}{c_1-\lambda}\\ &+ c_2\,\mathrm{e}^{\lambda\tau}\int_{-\tau}^0 \mathrm{e}^{\lambda s}|\xi(s)|^2\,\mathrm{d}s + (2+c_2\,\mathrm{e}^{\lambda\tau})\int_0^t \mathrm{e}^{\lambda s}|\mathbf{x}(s)|^2\,\mathrm{d}s\end{aligned}\tag{28}$$

for all $0 \leq t \leq k$, $k \geq k_1$ almost surely. This, together with Equation (10), implies

$$\begin{aligned}\mathrm{e}^{\lambda t}|\mathbf{x}(t)|^2 \leq{}& |\xi(0)|^2 + 2\log k + \frac{c_3}{c_1-\lambda}\\ &+ c_2\,\mathrm{e}^{\lambda\tau}\int_{-\tau}^0 \mathrm{e}^{\lambda s}|\xi(s)|^2\,\mathrm{d}s + (2+c_2\,\mathrm{e}^{\lambda\tau})\,Ck\end{aligned}\tag{29}$$

for all $0 \leq t \leq k$, $k \geq k_0 \vee k_1$ almost surely. The remainder of the proof is the same as before and the proof is complete. $\square$

Similarly we can prove the following more general result.

Theorem 3.2. *Let Equation* (20) *hold. Let c_1–c_3 be three positive constants and let $\boldsymbol{Q}$ be a positive definite $d \times d$ matrix. Assume for all* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ *and* $t \geq 0$:

(i) $\mathbf{x}^{\mathrm{T}}(\boldsymbol{Q}+\boldsymbol{Q}^{\mathrm{T}})f(\mathbf{x},\mathbf{y},t) \leq -c_1\mathbf{x}^{\mathrm{T}}\boldsymbol{Q}\mathbf{x} + c_2\mathbf{y}^{\mathrm{T}}\boldsymbol{Q}\mathbf{y}$,
(ii) $\mathrm{trace}(\sigma(t)\sigma(t)^{\mathrm{T}}) \leq c_3\,\mathrm{e}^{-c_1 t}$.

If $c_1 > c_2$, then the delay Equation (22) *is almost surely exponentially stable.*

One can also use this theorem to prove the following corollary.

Corollary 3.3. *Let Equation* (20) *hold. Let c_1–c_4 be four positive constants and let $\boldsymbol{Q}$ be a positive difinite $d \times d$ matrix. Assume for all* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ *and* $t \geq 0$:

(i) $|\mathbf{x}|^2 \leq \mathbf{x}^{\mathrm{T}}\boldsymbol{Q}\mathbf{x}$,
(ii) $\mathbf{x}^{\mathrm{T}}(\boldsymbol{Q}+\boldsymbol{Q}^{\mathrm{T}})f(\mathbf{x},0,t) \leq -c_1\mathbf{x}^{\mathrm{T}}\boldsymbol{Q}\mathbf{x}$,

(iii) $|f(\mathbf{x}, 0, t) - f(\mathbf{x}, \mathbf{y}, t)| \le c_2|\mathbf{y}|$,
(iv) $\operatorname{trace}(\sigma(t)\sigma(t)^{\mathrm{T}}) \le c_3\, \mathrm{e}^{-c_4 t}$.

If

$$(c_1/2) \wedge c_4 > c_2\|\boldsymbol{Q}\|$$

or

$$(c_1 - c_4)\, c_4 > (c_2\|\boldsymbol{Q}\|)^2,$$

then the delay Equation (22) *is almost surely exponentially stable.*

4. STOCHASTIC DIFFERENTIAL FUNCTIONAL EQUATIONS

In this section we shall study the almost sure exponential stability for a differential functional equation with damped stochastic perturbation of the form

$$\mathrm{d}\mathbf{x}(t) = f(\mathbf{x}(t), \hat{\mathbf{x}}(t), t)\, \mathrm{d}t + \sigma(t)\, \mathrm{d}w(t) \quad \text{on } t \ge 0 \tag{30}$$

with initial data $\mathbf{x}(t) = \xi(t)$ on $-\tau \le t \le 0$, where w, σ, ξ are the same as before, $\hat{\mathbf{x}}(t) = \{\mathbf{x}(t+s)\colon -\tau \le s \le 0\}$ and $f : \mathbb{R}^d \times C([-\tau, 0]; \mathbb{R}^d) \times \mathbb{R}_+ \to \mathbb{R}^d$. Assume as before that the equation has a unique solution that is denoted by $\mathbf{x}(t; \xi)$.

Theorem 4.1. *Let c_1–c_3 be positive constants and let $\beta : [-\tau, 0] \to \mathbb{R}_+$ be a Borel measurable function such that*

$$\int_{-\tau}^{0} \beta(r)\, \mathrm{d}r = 1. \tag{31}$$

Assume for all $u \in C([-\tau, 0]; \mathbb{R}^d)$ and $t \ge 0$

(i) $2u(0)^{\mathrm{T}} f(u(0), u, t) \le -c_1|u(0)|^2 + c_2 \int_{-\tau}^{0} \beta(r)|u(r)|^2\, \mathrm{d}r$,
(ii) $\operatorname{trace}(\sigma(t)\sigma(t)^{\mathrm{T}}) \le c_3\, \mathrm{e}^{-c_1 t}$.

If $c_1 > c_2$, then there exists a $\lambda > 0$ such that

$$\limsup_{t\to\infty} \frac{1}{t} \log|\mathbf{x}(t; \xi)| \le -\lambda \quad \text{a.s.} \tag{32}$$

for all ξ. In other words, Equation (30) *is almost surely exponentially stable.*

Before the proof of the theorem it should be pointed out that condition (31) is not essential and it is just a normalization. As a matter of fact, once condition

(i) is satisfied we can replace $\beta(r)$ and c_2 by $\beta(r)/\|\beta\|$ and $c_2\|\beta\|$, respectively, where

$$\|\beta\| = \int_{-\tau}^{0} \beta(r)\,\mathrm{d}r.$$

Obviously this normalization makes Equation (31) satisfied but the condition $c_1 > c_2$ should become $c_1 > c_2\|\beta\|$. Let us now begin to prove the theorem.

Proof: We use the same notations as in the proof of Theorem 2.1. It is easy to see Equation (11) becomes

$$\mathrm{e}^{c_1 t}|\mathbf{x}(t)|^2 \le |\xi(0)|^2 + M(t) + c_3 t + c_2 \int_0^t \mathrm{e}^{c_1 s} \int_{-\tau}^{0} \beta(r)|\mathbf{x}(s+r)|^2\,\mathrm{d}r\,\mathrm{d}s, \tag{33}$$

where $M(t)$ is the same as before. Hence we can derive from Equations (33) and (12) that, for almost all $\omega \in \Omega$, if $0 \le T \le k$, $k \ge k_0$

$$\int_0^T \mathrm{e}^{\lambda t}|\mathbf{x}(t)|^2\,\mathrm{d}t \le \frac{1}{c_1 - \lambda}[|\xi(0)|^2 + c_3 k + 2\varepsilon^{-1}\log k] + J_1 + J_2, \tag{34}$$

where J_2 is the same as before but

$$J_1 = c_2 \int_0^T \mathrm{e}^{-(c_1-\lambda)t} \int_0^t \mathrm{e}^{c_1 s} \int_{-\tau}^{0} \beta(r)|\mathbf{x}(s+r)|^2\,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}t.$$

We now compute

$$\begin{aligned} J_1 &= c_2 \int_0^T \mathrm{e}^{-c_1 s} \int_{-\tau}^{0} \beta(r)|\mathbf{x}(s+r)|^2\,\mathrm{d}r \int_S^T \mathrm{e}^{-(c_1-\lambda)t}\,\mathrm{d}s\,\mathrm{d}t. \\ &\le \frac{1}{c_1-\lambda} c_2 \int_0^T \mathrm{e}^{\lambda s} \int_{-\tau}^{0} \beta(r)|\mathbf{x}(s+r)|^2\,\mathrm{d}s. \end{aligned}$$

But

$$\begin{aligned} \int_0^T \mathrm{e}^{\lambda s} \int_{-\tau}^{0} \beta(r)|\mathbf{x}(s+r)|^2\,\mathrm{d}s &= \int_0^T \mathrm{e}^{\lambda s} \int_{s-\tau}^{s} \beta(r-s)|\mathbf{x}(r)|^2\,\mathrm{d}r\,\mathrm{d}s \\ &= \int_{-\tau}^{T} |\mathbf{x}(r)|^2 \int_{r\vee 0}^{(r+\tau)\wedge T} \beta(r-s)\,\mathrm{e}^{\lambda s}\,\mathrm{d}s\,\mathrm{d}r \\ &\le \int_{-\tau}^{T} |\mathbf{x}(r)|^2\,\mathrm{e}^{\lambda(r+\tau)} \int_r^{r+\tau} \beta(r-s)\,\mathrm{d}s\,\mathrm{d}r \\ &\le \int_{-\tau}^{T} |\mathbf{x}(r)|^2 \mathrm{e}^{\lambda(r+\tau)} \int_{-\tau}^{0} \beta(s)\,\mathrm{d}s\,\mathrm{d}r \le \mathrm{e}^{\lambda\tau} \int_{-\tau}^{T} \mathrm{e}^{\lambda r}|\mathbf{x}(r)|^2\,\mathrm{d}r. \end{aligned} \tag{35}$$

Hence

$$J_1 \le \frac{1}{c_1 - \lambda} c_2 e^{\lambda\tau} \left(\int_{-\tau}^{0} e^{\lambda r} |\xi(r)|^2 \, dr + \int_0^T e^{\lambda r} |\mathbf{x}(r)|^2 \, dr \right),$$

which is the same estimate as in the proof of Theorem 3.1. We therefore obtain Equation (26) and then Equation (10) follows. Moreover, Equation (17) becomes

$$e^{\lambda t} |\mathbf{x}(t)|^2 \le |\xi(0)|^2 + N(t) + \frac{c_3}{c_1 - \lambda} + c_2 \int_0^t e^{\lambda s} \int_{-\tau}^{0} \beta(r) |\mathbf{x}(s + r)|^2 \, dr \, ds, \tag{36}$$

where $N(t)$ is the same as before. Using Equations (18) and (35) we obtain Equation (28) and then Equation (29) follows because of Equation (10). The remainder of the proof is the same as before and the proof is complete. □

The following is a generalized version that can be proved in the same way.

Theorem 4.2. *Let c_1–c_2 be three positive constants and let $\boldsymbol{Q}$ be a positive definite $d \times d$ matrix. Let $\beta : [-\tau, 0] \to \mathbb{R}_+$ be a Borel measurable function such that*

$$\int_{-\tau}^{0} \beta(r) \, dr = 1.$$

Asume for all $u \in C([-\tau, 0]; \mathbb{R}^d)$ and $t \ge 0$

(i) $u(0)^T (\boldsymbol{Q} + \boldsymbol{Q}^T) f(u(0), u, t) \le -c_1 u(0)^T \boldsymbol{Q} u(0) + c_2 \int_{-\tau}^{0} \beta(r) u(r)^T \boldsymbol{Q} u(r) \, dr$,

(ii) $\mathrm{trace}(\sigma(t)\sigma(t)^T) \le c_3 \, e^{-c_1 t}$.

If $c_1 > c_2$, then Equation (30) *is almost surely exponentially stable.*

It is also not difficult to employ the above theorem to show the following corollary.

Corollary 4.3. *Let c_1–c_4 be four positive constants and let $\boldsymbol{Q}$ be a positive definite $d \times d$ matrix. Let $\beta : [-\tau, 0] \to \mathbb{R}_+$ be a Borel measurable function such that*

$$\int_{-\tau}^{0} \beta(r) \, dr = 1.$$

Assume for all $\mathbf{x} \in \mathbb{R}^d$, $u \in C([-\tau, 0]; \mathbb{R}^d)$ and $t \ge 0$:

(i) $|\mathbf{x}|^2 \le \mathbf{x}^T \boldsymbol{Q} \mathbf{x}$,

(ii) $\mathbf{x}^T (\boldsymbol{Q} + \boldsymbol{Q}^T) f(\mathbf{x}, 0, t) \le -c_1 \mathbf{x}^T \boldsymbol{Q} \mathbf{x}$,

(iii) $|f(\mathbf{x}, 0, t) - f(\mathbf{x}, u, t)| \le c_2 (\int_{-\tau}^{0} \beta(r) |u(r)|^2 \, dr)^{1/2}$,

(iv) $\mathrm{trace}(\sigma(t)\sigma(t)^T) \le c_3 \, e^{-c_4 t}$.

If

$$(c_1/2) \wedge c_4 > c_2\|\boldsymbol{Q}\|$$

or

$$(c_1 - c_4)\, c_4 > (c_2\|\boldsymbol{Q}\|)^2,$$

then Equation (30) *is almost surely exponentially stable.*

5. EXAMPLES

In this section we shall give a number of examples to illustrate our theory.

Example 5.1. Let us first consider a stochastic oscillator with delay

$$\begin{aligned}&\ddot{z}(t) + 5\dot{z}(t) + 6z(t) + \theta_1 z(t-\tau)\cos\dot{z}(t) + \theta_2\dot{z}(t-\tau)\,\mathrm{e}^{-|z(t)|}\\ &\quad = p(t)\,\mathrm{e}^{-t}\,\dot{w}(t)\end{aligned} \tag{37}$$

on $t \geq 0$ with a suitable initial data, where θ_1 and θ_2 are small parameters, $p(t)$ is a polynomial of t, and $\dot{w}(t)$ is a white noise, i.e. $w(t)$ is a scale Brownian motion. Although it will be improved later, let us first show that if

$$|\theta| := \sqrt{\theta_1^2 + \theta_2^2} < \frac{2}{225}, \tag{38}$$

then there exists a $\lambda > 0$ such that

$$\limsup_{t\to\infty} \frac{1}{t}\log(|z(t)| + |\dot{z}(t)|) \leq -\lambda \quad \text{a.s.}$$

In other words, the stochastic oscillator (Eq. (37)) is almost surely exponentially stable. In order to see this, we introduce a new variable $\mathbf{x} = (x_1, x_2)^{\mathrm{T}} = (z, \dot{z})^{\mathrm{T}}$ and write Equation (37) as an Itô equation

$$\mathrm{d}\mathbf{x}(t) = f(\mathbf{x}(t), \mathbf{x}(t-\tau))\,\mathrm{d}t + \sigma(t)\,\mathrm{d}w(t), \tag{39}$$

where

$$f(\mathbf{x}, \mathbf{y}) = \left(x_2, -6x_1 - 5x_2 - \theta_1 y_1 \cos x_2 - \theta_2 \dot{y}_2\,\mathrm{e}^{-|x_1|}\right)^{\mathrm{T}}$$

and

$$\sigma(t) = (0,\, p(t)\,\mathrm{e}^{-t})^{\mathrm{T}}.$$

Note

$$f(\mathbf{x}, 0) = \boldsymbol{A}\mathbf{x} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}\mathbf{x}.$$

It is easy to find

$$\boldsymbol{H} = \begin{bmatrix} 1 & 1 \\ -2 & -3 \end{bmatrix} \quad \text{such that } \boldsymbol{H}^{-1}\boldsymbol{A}\boldsymbol{H} = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}.$$

Set

$$\boldsymbol{Q} = 15(\boldsymbol{H}^{-1})^{\mathrm{T}}\boldsymbol{H}^{-1} = 15\begin{bmatrix} 13 & 5 \\ 5 & 2 \end{bmatrix}.$$

It is easy to check that

$$|\mathbf{x}|^2 \le \mathbf{x}^{\mathrm{T}}\boldsymbol{Q}\mathbf{x},$$
$$\mathbf{x}^{\mathrm{T}}(\boldsymbol{Q}+\boldsymbol{Q})f(\mathbf{x},0) \le -4\mathbf{x}^{\mathrm{T}}\boldsymbol{Q}\mathbf{x}.$$

Moreover,

$$|f(\mathbf{x},\mathbf{y}) - f(\mathbf{x},0)| = \left|\theta_1 y_1 \cos x_2 + \theta_2 y_2\, \mathrm{e}^{-|x_1|}\right| \le |\theta||\mathbf{y}|$$

On the other hand, for arbitrary $\varepsilon > 0$ one can find a sufficiently large C so that

$$\operatorname{trace}(\sigma(t)\sigma(t)^{\mathrm{T}}) = (p(t)\,\mathrm{e}^{-t})^2 \le C\mathrm{e}^{-(2-\varepsilon)t}.$$

Note also $\|\boldsymbol{Q}\| \le 225$. So by Corollary 2.3, Equation (39), i.e. the stochastic oscillator (Eq. (37)) is almost surely exponential stable provided $255|\theta| < 2-\varepsilon$, but this is the same as Equation (38) since ε is arbitrary.

We now start to improve the result by linear transformation. Define $\bar{x}(t) = \boldsymbol{H}^{-1}x(t)$. Then $\bar{x}(t)$ satisfies the following delay equation

$$\mathrm{d}\bar{\mathbf{x}}(t) = g(\bar{\mathbf{x}}(t), \bar{\mathbf{x}}(t-\tau))\,\mathrm{d}t + \boldsymbol{H}^{-1}\sigma(t)\,\mathrm{d}w(t), \tag{40}$$

where

$$g(\bar{\mathbf{x}},\bar{\mathbf{y}}) = \boldsymbol{H}^{-1}f(\boldsymbol{H}\bar{\mathbf{x}}, \boldsymbol{H}\bar{\mathbf{y}}).$$

Obviously the almost sure exponential stability of Equation (37) is now equivalent to that of Equation (40). We compute

$$2\bar{\mathbf{x}}^{\mathrm{T}}g(\bar{\mathbf{x}},0) = 2\bar{\mathbf{x}}^{\mathrm{T}}\boldsymbol{H}^{-1}f(\boldsymbol{H}\bar{\mathbf{x}},0) = 2\bar{\mathbf{x}}^{\mathrm{T}}\boldsymbol{H}^{-1}\boldsymbol{A}\boldsymbol{H}\bar{\mathbf{x}} \le -4|\bar{\mathbf{x}}|^2$$

and

$$\begin{aligned}|g(\bar{\mathbf{x}},\bar{\mathbf{y}}) - g(\bar{\mathbf{x}},0)| &\le \|\boldsymbol{H}^{-1}\||f(\boldsymbol{H}\bar{\mathbf{x}},\boldsymbol{H}\bar{\mathbf{y}}) - f(\boldsymbol{H}\bar{\mathbf{x}},0)| \\ &\le \|\boldsymbol{H}^{-1}\||\theta|\|\boldsymbol{H}\bar{\mathbf{y}}\| \le 15|\theta||\bar{\mathbf{y}}|.\end{aligned}$$

Hence by Corollary 2.3 with $\boldsymbol{Q}$ = identity matrix we can derive that Equation (5.4) is almost surely exponentially stable if $2 > 15|\theta|$, i.e., $|\theta| < 2/15$. So we conclude that the stochastic delay oscillator (Eq. (37)) is almost surely exponentially stable if $|\theta| \le 2/15$. Of course, $|\theta| \le 2/15$ is much improved than Equation (38). In other words, due to the special form of Equation (39), it is better to do a linear transformation and then apply our result.

Example 5.2. Let $\rho(t)$ be the same as introduced in Section 3, in particular, let Equation (20) be satisfied. If $t-\tau$ is replaced by $\rho(t)$ then Equation (37) becomes

$$\ddot{z}(t) + 5\dot{z}(t) + 6z(t) + \theta_1 z(\rho(t))\cos\dot{z}(t) + \theta_2\dot{z}(\rho(t))\,\mathrm{e}^{-|z(t)|} = p(t)\,\mathrm{e}^{-t}\,\dot{w}(t). \tag{41}$$

By Corollary 3.3 and the above arguments, we conclude that this stochastic oscillator is almost surely exponentially stable if $|\theta| < 2/15$.

Example 5.3. Let us consider a stochastic oscillator of the form

$$\begin{aligned}&\ddot{z}(t) + 5\dot{z}(t) + 6z(t) + \theta_1 \cos \dot{z}(t) \int_{-\tau}^{0} z(t+r)\,\mathrm{d}r + \theta_2 \mathrm{e}^{-|z(t)|} \int_{-\tau}^{0} \dot{z}(t+r)\,\mathrm{d}r \\ &\quad = p(t)\,\mathrm{e}^{-t}\,\dot{w}(t) \end{aligned} \tag{42}$$

on $t \geq 0$ with a suitable initial data, where $\theta_1, \theta_2,\ p(t)$ and $\dot{w}(t)$ are all the same as before. Again introduce a new variable $\mathbf{x} = (x_1, x_2)^{\mathrm{T}} = (z, \dot{z})^{\mathrm{T}}$ and write the equation as a stochastic differential functional equation

$$\mathrm{d}\bar{\mathbf{x}}(t) = f(\bar{\mathbf{x}}(t), \hat{\mathbf{x}}(t))\,\mathrm{d}t + \sigma(t)\,\mathrm{d}w(t), \tag{43}$$

where

$$f(\mathbf{x}, u) = \left(x_2, -6x_1 - 5x_2 - \theta_1 \cos x_2 \int_{-\tau}^{0} u_1(r)\,\mathrm{d}r - \theta_2\,\mathrm{e}^{-|x_1|} \int_{-\tau}^{0} u_2(r)\,\mathrm{d}r\right)^{\mathrm{T}}$$

and

$$\sigma(t) = (0, p(t)\,\mathrm{e}^{-t})^{\mathrm{T}}$$

for $\mathbf{x} \in \mathbb{R}^2$, $u \in C([-\tau, 0]; \mathbb{R}^2)$, and $t \geq 0$. In the same way as Example 5.1, define a new process $\bar{\mathbf{x}}(t) = \boldsymbol{H}^{-1}\mathbf{x}(t)$. Then $\bar{\mathbf{x}}(t)$ satisfies the following stochastic differential functional equation

$$\mathrm{d}\bar{\mathbf{x}}(t) = g(\bar{\mathbf{x}}(t), \hat{\bar{\mathbf{x}}}(t))\,\mathrm{d}t + \boldsymbol{H}^{-1}\sigma(t)\,\mathrm{d}w(t), \tag{44}$$

where $\hat{\bar{\mathbf{x}}}(t) = \{\bar{\mathbf{x}}(t+s)\colon -\tau \leq s \leq 0\}$ and

$$g(\bar{\mathbf{x}}, \bar{u}) = \boldsymbol{H}^{-1} f(\boldsymbol{H}\bar{\mathbf{x}}, \boldsymbol{H}\bar{u}) \quad \text{for } \bar{\mathbf{x}} \in \mathbb{R}^2, \quad \bar{u} \in C([-\tau, 0]; \mathbb{R}^2)$$

in which we use the notation $\boldsymbol{H}\bar{u} = \{\boldsymbol{H}\bar{u}(s)\colon -\tau \leq s \leq 0\}$. Obviously the almost sure exponential stability of Equation (42) is now equivalent to that of Equation (44). We compute

$$2\bar{\mathbf{x}}^{\mathrm{T}} g(\bar{\mathbf{x}}, 0) = 2\bar{\mathbf{x}}^{\mathrm{T}}\boldsymbol{H}^{-1} f(\boldsymbol{H}\bar{\mathbf{x}}, 0) = 2\bar{\mathbf{x}}^{\mathrm{T}}\boldsymbol{H}^{-1}\boldsymbol{A}\boldsymbol{H}\bar{\mathbf{x}} \leq -4|\bar{\mathbf{x}}|^2$$

and, setting $u = \boldsymbol{H}\bar{u}$,

$$\begin{aligned}&|\mathrm{g}(\bar{\mathbf{x}}, \bar{u}) - g(\bar{\mathbf{x}}, 0)| \\ &\quad \leq \|\boldsymbol{H}^{-1}\|\,|f(\boldsymbol{H}\bar{\mathbf{x}}, \boldsymbol{H}\bar{u}) - f(\boldsymbol{H}\bar{\mathbf{x}}, 0)| \\ &\quad \leq |\theta|\,\|\boldsymbol{H}^{-1}\| \left(\left[\int_{-\tau}^{0} u_1(r)\,\mathrm{d}r\right]^2 + \left[\int_{-\tau}^{0} u_2(r)\,\mathrm{d}r\right]^2\right)^{1/2}\end{aligned}$$

$$
\begin{aligned}
&\leq |\theta| \|\boldsymbol{H}^{-1}\| \left(\tau \int_{-\tau}^{0} |u_1(r)|^2 \,\mathrm{d}r + \tau \int_{-\tau}^{0} |u_2(r)|^2 \,\mathrm{d}r \right)^{1/2} \\
&\leq |\theta| \|\boldsymbol{H}^{-1}\| \left(\tau \int_{-\tau}^{0} |u(r)|^2 \,\mathrm{d}r \right)^{1/2} = |\theta| \|\boldsymbol{H}^{-1}\| \left(\tau \int_{-\tau}^{0} |\boldsymbol{H}\bar{u}(r)|^2 \,\mathrm{d}r \right)^{1/2} \\
&\leq |\theta| \; \|\boldsymbol{H}^{-1}\| \left(\tau \int_{-\tau}^{0} \|\boldsymbol{H}\|^2 |\bar{u}(r)|^2 \,\mathrm{d}r \right)^{1/2} \\
&\leq \tau |\theta| \; \|\boldsymbol{H}^{-1}\| \|\boldsymbol{H}\| \left(\frac{1}{\tau} \int_{-\tau}^{0} |\bar{u}(r)|^2 \,\mathrm{d}r \right)^{1/2} \\
&\leq 15\tau |\theta| \left(\frac{1}{\tau} \int_{\tau}^{0} |\bar{u}(r)|^2 \,\mathrm{d}r \right)^{1/2}.
\end{aligned}
$$

Hence, by Corollary 4.3 with $\boldsymbol{Q} =$ identity matrix and $\beta(r) = 1/\tau$, we can derive that Equation (42) is almost surely exponentially stability if $2 > 15\tau|\theta|$, i.e. $|\theta| < 15\tau/2$.

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