A Complete Representation Theorem for G-martingales

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January 13, 2012

Abstract

In this paper we establish a complete representation theorem for G-martingales. Unlike the existing results in the literature, we provide the existence and uniqueness of the second order term, which corresponds to the second order derivative in Markovian case. The main ingredient of the paper is a new norm for that second order term, which is based on an operator introduced by Song [13].

Key words: G-expectations, G-martingales, martingale representation theorem, nonlinear expectations

AMS 2000 subject classifications: 60H10, 60H30

1 Introduction

The notion of G-expectation, a type of nonlinear expectation proposed by Peng [6], [7], has received very strong attention in the literature in recent years. In Markovian case, the G-expectation and the closely related Second Order Backward SDEs introduced by Soner, Touzi, and Zhang [11], are associated with fully nonlinear PDEs, see also Peng [8]. Their typical applications include, among others, economic/financial models with volatility uncertainty and numerical methods for high dimensional fully nonlinear PDEs.

G-expectation is a typical nonlinear expectation. It can be regarded as a nonlinear generalization of Wiener probability space $(\Omega, \mathcal{F}, P)$ where $\Omega = C([0,\infty), \mathbb{R}^d)$, $\mathcal{F} = \mathcal{B}(\Omega)$

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and P is a Wiener probability measure defined on $(\Omega, \mathcal{F})$. Recall that the Wiener measure is defined such that the canonical process $B_t(\omega) := \omega_t$, $t \geq 0$ is a continuois process with stable and independent increments, namely $(B_t)_{t\geq 0}$ is a Brownian motion. G-expectation $\mathbb{E}^G$ is a sublinear expectation on the same canonical space Ω, such that the same canonical process B is a G-Brownian motion, i.e., it is a continuous process with stable and independent increaments. One important feature of this notion is its time consistency. To be precise, let ξ be a random variable and $Y_t := \mathbb{E}^G_t[\xi]$ denote the conditional G-expectation, then one has $\mathbb{E}^G_s[\xi] = \mathbb{E}^G_s[\mathbb{E}^G_t(\xi)]$ for any $s < t$. For this reason, we call the conditional G-expectation a G-martingale, or a martingale under G-expectation. It is well known that a martingale under Wiener measure can be written as a stochastic integral against the Bronwnian motion. Then a very natural and fundamental question in this nonlinear G-framework is:

$$\text{What is the structure of a } G\text{-martingale } Y? \tag{1.1}$$

Peng [6] has observed that, for $Z \in \mathcal{H}^2_G$ and $\eta \in \mathcal{M}^1_G$ (see (2.12) and (2.17) below), the following process Y is always a G-martingale:

$$dY_t = Z_t dB_t - G(\eta_t)dt + \frac{1}{2}\eta_t d\langle B\rangle_t. \tag{1.2}$$

Here G is the deterministic function Peng [6] used to define G-expectations and $\langle B\rangle$ is the quadratic variation of the G-Brownian motion B. We remark that, in a Markovian framework, we have $Y_t = u(t, B_t)$, where u is a smooth function satisfying the following fully nonlinear PDE:

$$\partial_t u + G(\partial_{xx} u) = 0. \tag{1.3}$$

Then $Z_t = \partial_x u(t, B_t)$ and $\eta_t = \partial_{xx} u(t, B_t)$. In particular, if $\xi = g(B_T)$, then by PDE arguments we see immediately that $Y_t := \mathbb{E}^G_t[\xi]$ has a representation (1.2). Peng was even able to prove this (Z, η)-representation holds if ξ is in a dense subspace $\mathcal{L}_{ip}$ of $\mathcal{L}^p_G$ (see (2.5) below). But observing that $\mathcal{L}_{ip}$ is not a complete space, a very interesting question was then raised to give a complete (Z, η)-representation theorem for $\mathbb{E}^G_t[\xi]$.

The first partial answer was provided by Xu and Zhang [14]: if Y is a symmetric G-martingale, that is, both Y and $-Y$ are G-martingales, then

$$dY_t = Z_t dB_t \quad \text{for some process } Z. \tag{1.4}$$

However, symmetric G-martingales captures only the linear part in this nonlinear framework, and it is essentially important to understand the structure of nonsymmetric G-martingales.

By introducing a new norm $\|\cdot\|_{\mathbb{L}^2_G}$ (see (2.22) below), Soner, Touzi and Zhang [10] proved a more general representation theorem: for $\xi \in \mathbb{L}^2_G$,

$$dY_t = Z_t dB_t - dK_t, \tag{1.5}$$

where K is an increasing process such that $-K$ is a G-martingale. It has been proved independently in [10] and Song [12] that $\mathbb{L}_G^p \supset \bigcap_{q>p} \mathcal{L}_G^q$, where $\|\cdot\|_{\mathcal{L}_G^q}$ is the norm introduced in [6]. In particular, [12] extended the representation (1.5) to the case $p > 1$.

Now the questions is, when does the process K in (1.5) have the structure: $dK_t = G(\eta_t)dt - \frac{1}{2}\eta_t d\langle B\rangle_t$? Several efforts have been made in this direction. Hu and Peng [4] and Pham and Zhang [9] made some progresses on the existence of η. However, there is no characterization of the process η, and in particular, they do not provide an appropriate norm for η. On the other hand, Song [13] proved the uniqueness of η in the space $\mathcal{M}_G^1$. A clever operator was introduced in this work, which successfully isolates the term $\frac{1}{2}\eta_t d\langle B\rangle_t$ from dK_t, and thus essentially captures the uncertainty of underlying distributions. This idea turns out to be the building block of the present paper.

Our main contribution of this paper is to introduce a norm for the process η, based on the work [13]. We shall prove the existence, uniqueness, and a priori norm estimates for η. In particular, given ξ_1 and ξ_2 in appropriate space, let (Y^i, Z^i, η^i), $i = 1, 2$, be the corresponding terms, we shall estimate the norms of $Z^1 - Z^2$ and $\eta^1 - \eta^2$ in terms of that of $Y^1 - Y^2$, where the latter one is more tractable due to the representation formula $Y_t = \mathbb{E}_t^G[\xi]$. Unlike [13], we prove the estimates via PDE arguments.

The rest of the paper is organized as follows. In Section 2 we introduce the G-martingales and the involved spaces. In Section 3 we propose the new norm for η and provide some estimates. Finally in Section 4 we establish the complete representation theorem for G-martingales.

2 Preliminaries

In this section we introduce G-expectations and G-martingales. We shall focus on a simple setting in which we will establish the martingale representation theorem. However, these notions can be extended to much more general framework, as in many publications in the literature.

We start with some notations in multiple dimensional setting. Fix a dimension d. Let $\mathbb{R}^d$ and $\mathbb{S}^d$ denote the sets of d-dimensional column vectors and $d \times d$-symmetric matrices, respectively. For $\sigma_1, \sigma_2 \in \mathbb{S}^d$, $\sigma_1 \le \sigma_2$ (resp. $\sigma_1 < \sigma_2$) means that $\sigma_2 - \sigma_1$ is nonnegative (resp. positive) definite, and we denote by $[\sigma_1, \sigma_2]$ the set of $\sigma \in \mathbb{S}^d$ satisfying $\sigma_1 \le \sigma \le \sigma_2$. Throughout the paper, we use $\mathbf{0}$ to denote the d-dimensional zero vector or zero matrix, and I_d the $d \times d$ identity matrix. For $x, \tilde{x} \in \mathbb{R}^d$, $\gamma, \tilde{\gamma} \in \mathbb{S}^d$, define

$$x \cdot \tilde{x} := x^T \tilde{x}, \quad |x| := \sqrt{x \cdot x}, \quad \text{and} \quad \gamma : \tilde{\gamma} := \mathrm{tr}\,(\gamma\tilde{\gamma}), \quad |\gamma| := \sqrt{\gamma : \gamma}, \tag{2.1}$$

where x^T denotes the transpose of x. One can easily check that

$$|\gamma : \tilde{\gamma}| \le |\gamma||\tilde{\gamma}|, \quad \text{and} \quad -\gamma \le \tilde{\gamma} \le \gamma \text{ implies that } |\tilde{\gamma}| \le |\gamma|. \tag{2.2}$$

2.1 Conditional G-expectations

We fix a finite time interval $[0,T]$, and two constant matrices $\mathbf{0} < \underline{\sigma} < \overline{\sigma}$ in $\mathbb{S}^d$. Define

$$G(\gamma) := \frac{1}{2}\sup_{\sigma\in[\underline{\sigma},\overline{\sigma}]}(\sigma^2 : \gamma), \quad \text{for all } \gamma \in \mathbb{S}^d. \tag{2.3}$$

Let $\Omega := \big\{\omega \in C([0,T],\mathbb{R}^d) : \omega_0 = \mathbf{0}\big\}$ be the canonical space, B the canonical process, and $\mathbb{F} := \mathbb{F}^B$ the filtration generated by B. For $\xi = \varphi(B_T)$, where $\varphi : \mathbb{R}^d \to \mathbb{R}$ is a bounded and Lipschitz continuous function, following Peng [6] we define the conditional G-expectation $\mathbb{E}^G_t[\xi] := u(t,B_t)$ where u is the (unique) classical solution of the following PDE on $[0,T]$:

$$\partial_t u + G(\partial_{xx} u) = 0, \quad u(T,x) = \varphi(x). \tag{2.4}$$

Let $\mathcal{L}_{ip}$ denote the set of random variables $\xi = \varphi(B_{t_1},\cdots,B_{t_n})$ for some $0 \le t_1 < \cdots < t_n \le T$ and some Lipschitz continuous function φ. One may define $\mathbb{E}^G_t[\xi]$ in the same spirit, by defining it backwardly over each interval $[t_i,t_{i+1}]$. In particular, when $t=0$ we define $\mathbb{E}^G[\xi] := \mathbb{E}^G_0[\xi]$.

For any $p \ge 1$, define

$$\|\xi\|^p_{\mathcal{L}^p_G} := \mathbb{E}^G[|\xi|^p], \quad \xi \in \mathcal{L}_{ip}. \tag{2.5}$$

Clearly this defines a norm in $\mathcal{L}_{ip}$. Let $\mathcal{L}^p_G$ denote the closure of $\mathcal{L}_{ip}$ under the norm $\|\cdot\|_{\mathcal{L}^p_G}$, taking the quotient as in the standard literature. One can easily extend the conditional G-expectation to all $\xi \in \mathcal{L}^1_G$.

We next provide an equivalent formulation of conditional G-expectations by using the quasi-sure stochastic analysis, initiated by Denis and Martini [2] for superhedging problem under volatility uncertainty. Let $\mathcal{A}$ denote the space of $\mathbb{F}$-progressively measurable processes taking values in $[\underline{\sigma},\overline{\sigma}]$. Denoting by $\mathbb{P}_0$ the Wiener measure, we define

$$\mathcal{P} := \Big\{\mathbb{P}^\sigma := \mathbb{P}_0 \circ (X^\sigma)^{-1} : \sigma \in \mathcal{A}\Big\} \quad \text{where} \quad X^\sigma_t := \int_0^t \sigma_s dB_s, \quad \mathbb{P}_0\text{-a.s.} \tag{2.6}$$

Then B is a $\mathbb{P}$-martingale for each $\mathbb{P} \in \mathcal{P}$. Following [2], we say

$$\text{a property holds } \mathcal{P}\text{-quasi surely, abbreviated as } \mathcal{P}\text{-q.s., if it holds } \mathbb{P}\text{-a.s. for all } \mathbb{P} \in \mathcal{P}. \tag{2.7}$$

It was proved in Denis, Hu and Peng [1] that:

$$\mathbb{E}^G[\xi] = \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}[\xi], \quad \xi \in \mathcal{L}^1_G. \tag{2.8}$$

The result was extended by Soner, Touzi and Zhang [10] to conditional G-expectations: for any $\mathbb{P} \in \mathcal{P}$ and any $t \in [0,T]$,

$$\mathbb{E}^G_t[\xi] = \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}(t,\mathbb{P})} \mathbb{E}^{\mathbb{P}'}_t[\xi], \quad \xi \in \mathcal{L}^1_G, \quad \text{where} \quad \mathcal{P}(t,\mathbb{P}) := \Big\{\mathbb{P}' \in \mathcal{P} : \mathbb{P}' = \mathbb{P} \text{ on } \mathcal{F}_t\Big\}. \tag{2.9}$$

We remark that Peng [5] had similar ideas, in the contexts of strong formulation.

We finally note that $\mathbb{E}^G_t$ obviously satisfies the following subadditivity:

$$\mathbb{E}^G_t[\xi_1+\xi_2] \le \mathbb{E}^G_t[\xi_1] + \mathbb{E}^G_t[\xi_2], \quad \text{for any} \;\; \xi_1,\xi_2 \in \mathcal{L}^1_G. \tag{2.10}$$

2.2 Stochastic integrals

First notice that, there exists a $\mathbb{S}^d$-valued process $\langle B\rangle$ such that $B_tB_t^T - \langle B\rangle_t$ is a G-martingale. In fact, under each $\mathbb{P}\in\mathcal{P}$, $\langle B\rangle$ is the same as the quadratic variation of the $\mathbb{P}$-martingale B, and consequently,

$$\underline{\sigma}^2 \le \frac{d}{dt}\langle B\rangle_t \le \overline{\sigma}^2, \quad \mathcal{P}\text{-q.s.} \tag{2.11}$$

Naturally we call $\langle B\rangle$ the quadratic variation of B. Next, we call an $\mathbb{F}$-progressively measurable process Z with appropriate dimension is an elementary process if it takes the form $Z=\sum_{i=0}^{n-1} Z_{t_i}\mathbf{1}_{[t_i,t_{i+1})}$ for some $0=t_0<\cdots<t_n\le T$ and each component of Z_{t_i} is in $\mathcal{L}_{ip}$. Let $\mathcal{H}^0_G$ denote the space of $\mathbb{R}^d$-valued elementary processes. For any $p\ge 1$, define

$$\|Z\|^p_{\mathcal{H}^p_G} \;\;:=\;\; \mathbb{E}^G\Big[\Big(\int_0^T (Z_tZ_t^T):d\langle B\rangle_t\Big)^{\frac{p}{2}}\Big], \quad Z\in\mathcal{H}^0_G; \tag{2.12}$$

and let $\mathcal{H}^p_G$ denote the closure of $\mathcal{H}^0_G$ under the norm $\|\cdot\|_{\mathcal{H}^p_G}$.

Now for each $Z\in\mathcal{H}^0_G$, we define its stochastic integral:

$$\int_0^t Z_s\cdot dB_s := \sum_{i=0}^{n-1} Z_{t_i}\cdot[B_{t_{i+1}\wedge t} - B_{t_i\wedge t}], \tag{2.13}$$

One can easily prove the Burkholder-Davis-Gundy Inequality: for any $p>0$, there exist constants $0<c_p<C_p<\infty$ such that

$$c_p\|Z\|^p_{\mathcal{H}^p_G} \le \mathbb{E}^G\Big[\sup_{0\le t\le T}|\int_0^t Z_s\cdot dB_s|^p\Big] \le C_p\|Z\|^p_{\mathcal{H}^p_G}. \tag{2.14}$$

Then one can extend the stochastic integral to all $Z\in\mathcal{H}^p_G$.

2.3 G-martingales

One important feature of conditional G-expectations is the time consistency, which can also be viewed as dynamic programming principle:

$$\mathbb{E}^G_s\Big[\mathbb{E}^G_t(\xi)\Big] = \mathbb{E}^G_s[\xi], \quad \text{for all} \quad \xi\in\mathcal{L}^1_G \;\text{ and }\; 0\le s<t\le T. \tag{2.15}$$

We recall that

$$\text{a process } Y \text{ is called a } G\text{-martingale if } \mathbb{E}^G_s[Y_t]=Y_s \text{ for all } 0\le s<t\le T. \tag{2.16}$$

Therefore, Y is a G-martingale if and only if $Y_t = \mathbb{E}^G_t[\xi]$ for $\xi = Y_T$.

It is clear that $\int_0^t Z_s dB_s$ is a G-martingale for all $Z \in \mathcal{H}^1_G$. In particular, the canonical process B is a G-martingale and is called a G-Brownian motion. However, G-martingales has a richer structure. Let $\mathcal{M}^0_G$ be the space of $\mathbb{S}^d$-valued elementary processes. Define

$$\|\eta\|^p_{\mathcal{M}^p_G} \quad := \quad \mathbb{E}^G\Big[\Big(\int_0^T |\eta_t| dt\Big)^p\Big], \quad \eta \in \mathcal{M}^0_G; \tag{2.17}$$

and let $\mathcal{M}^p_G$ denote the closure of $\mathcal{M}^0_G$ under the norm $\|\cdot\|_{\mathcal{M}^p_G}$. An interesting fact observed by Peng [6] is that the following decreasing process is also a G-martingale:

$$-K_t := \frac{1}{2}\int_0^t \eta_s : d\langle B\rangle_s - \int_0^t G(\eta_s)ds, \quad \eta \in \mathcal{M}^1_G. \tag{2.18}$$

Consequently, the following process Y is always a G-martingale:

$$Y_t = Y_0 + \int_0^t Z_s \cdot dB_s - \Big[\int_0^t G(\eta_s)ds - \frac{1}{2}\int_0^t \eta_s : d\langle B\rangle_s\Big], \quad Z \in \mathcal{H}^1_G,\ \eta \in \mathcal{M}^1_G. \tag{2.19}$$

On the other hand, for any $\xi \in \mathcal{L}_{ip}$, by Peng [7] there exist $Z \in \mathcal{H}^1_G, \eta \in \mathcal{M}^1_G$ such that $Y_t := \mathbb{E}^G_t[\xi]$ satisfies (2.19). In particular, when $\xi = \varphi(B_T)$, for the classical solution u of PDE (2.4), we have:

$$Y_t = u(t, B_t), \quad Z_t = \partial_x u(t, B_t), \quad \eta_t = \partial_{xx} u(t, B_t). \tag{2.20}$$

Our goal of this paper is to answer the following natural question proposed by Peng [7]:

$$\text{For what } \xi \text{ do there exist unique } Z \in \mathcal{H}^1_G \text{ and } \eta \in \mathcal{M}^1_G \text{ satisfying (2.19)?} \tag{2.21}$$

The problem was partially solved by Soner, Touzi and Zhang [10], which introduced the following norm:

$$\|\xi\|^p_{\mathbb{L}^p_G} := \mathbb{E}^G\Big[\sup_{0\le t\le T} \big(\mathbb{E}^G_t[|\xi|]\big)^p\Big], \quad \xi \in \mathcal{L}_{ip}. \tag{2.22}$$

Let $\mathbb{L}^p_G$ denote the closure of $\mathcal{L}_{ip}$ under the norm $\|\cdot\|_{\mathbb{L}^p_G}$. Then for any $\xi \in \mathbb{L}^2_G$, there exist unique $Z \in \mathcal{H}^2_G$ and an increasing process K with $K_0 = 0$ such that

$$Y_t := \mathbb{E}^G_t[\xi] = Y_0 + \int_0^t Z_s \cdot dB_s - K_t \quad \text{and} \quad \|Z\|_{\mathcal{H}^2_G} + \|K_T\|_{\mathcal{L}^2_G} \le C\|\xi\|_{\mathbb{L}^2_G}. \tag{2.23}$$

It was proved independently by [10] and Song [12] that $\|\xi\|_{\mathbb{L}^p_G} \le C_{p,q}\|\xi\|_{\mathcal{L}^q_G}$ for any $1 \le p < q$. Moreover, the above representation was extended by [12] to the case $p > 1$.

2.4 Summary of notations

For readers' convenience, we collect here some notations used in the paper:

- The inner product $\cdot$, the trace operator $:$, and the norms $|x|$, $|\gamma|$ are defined by (2.1).
- The function G, G^α and G_ε are defined by (2.3), (3.1), and (3.5), respectively.
- The class of probability measures $\mathcal{P}$, and the G-expectation $\mathbb{E}^G$ are defined by (2.6) and (2.8), respectively.
- The norms $\|\xi\|_{\mathcal{L}^p_G}$ and $\|\xi\|_{\mathbb{L}^p_G}$ for ξ are defined in (2.5) and (2.22), respectively.
- The norm $\|Z\|_{\mathcal{H}^p_G}$ for Z is defined in (2.12).
- The norm $\|\eta\|_{\mathcal{M}^p_G}$ for η is defined in (2.17).
- The norm $\|Y\|_{\mathbb{D}^p_G}$ for càdlàg processes Y, see also (2.22), is defined by:

$$\|Y\|^p_{\mathbb{D}^p_G} := \mathbb{E}^G\Big[\sup_{0\le t\le T}|Y_t|^p\Big]. \tag{2.24}$$

- The operator $\mathcal{E}^\alpha_{t_1,t_2}$ is defined by (3.2).
- The constants c_0, C_0 are defined by (3.4).
- The function δ_n is defined by (3.7).
- Then new norms $\|\eta\|_{\mathbb{M}_G}$ and $\|\eta\|_{\mathbb{M}^*_G}$ for η are defined by (3.11) and (3.16), respectively.
- The space $\mathcal{M}^1_{G_0}$ and class $\mathcal{P}_0$ are defined by (3.17) and (3.18), respectively.
- The new metric $d_{G,p}(\xi_1,\xi_2)$ for ξ is defined by (4.3), and $\mathbb{L}^{*p}_G$ is the corresponding closure space.
- For $0\le s\le t\le T$, the shifted canonical process B^s_t is defined by:

$$B^s_t := B_t - B_s. \tag{2.25}$$

3 A new norm for η

Our main contribution of the paper is to introduce a norm for η. For that purpose, we shall introduce two nonlinear operators, one via PDE arguments and the other via probabilistic arguments. The latter one is strongly motivated from the work Song [13], and the connection between the two operators is established in Lemma 3.4 below.

3.1 The nonlinear operator via PDE arguments

We first introduce a new nonlinear operator $\mathcal{E}^\alpha$ on Lipschitz continuous functions, with a parameter $\alpha\in\mathbb{S}^d$. Define

$$G^\alpha(\gamma) = \frac{1}{2}[G(\gamma+2\alpha)+G(\gamma-2\alpha)], \quad \gamma\in\mathbb{S}^d. \tag{3.1}$$

Given $0 \le t_1 < t_2 \le T$ and a Lipschitz continuous function φ, define $\mathcal{E}^\alpha_{t_1,t_2}(\varphi) := u^\alpha(t_1,\cdot)$, where u^α is the unique viscosity solution of PDE on $[t_1,t_2]$:

$$\partial_t u^\alpha + G^\alpha(\partial_{xx}u^\alpha) = 0, \quad u^\alpha(t_2,x) = \varphi(x). \tag{3.2}$$

Clearly G^α is strictly increasing and convex in γ. In particular, the above PDE is parabolic and is wellposed. We collect below some obvious properties of G^α and $\mathcal{E}^\alpha$, whose proofs are omitted.

Lemma 3.1 *For any $\alpha \in \mathbb{S}^d$,*

(i) $\mathcal{E}^\alpha$ satisfies the semigroup property:

$$\mathcal{E}^\alpha_{t_1,t_2}\big(\mathcal{E}^\alpha_{t_2,t_3}(\varphi)\big) = \mathcal{E}^\alpha_{t_1,t_3}(\varphi), \quad \textit{for any} \quad 0 \le t_1 < t_2 < t_3 \le T. \tag{3.3}$$

(ii) $G^{-\alpha} = G^\alpha \ge G = G^{\mathbf{0}}$.

(iii) If $\varphi = c$ is a constant, then $\mathcal{E}^\alpha_{t_1,t_2}(c) = c + G^\alpha(\mathbf{0})(t_2 - t_1)$.

The next property will be crucial for our estimates. Let

$$c_0 := \text{the smallest eigenvalue of } \frac{1}{2}[\overline{\sigma}^2 - \underline{\sigma}^2], \quad \text{and} \quad C_0 := \frac{1}{2}|\overline{\sigma}^2 - \underline{\sigma}^2|. \tag{3.4}$$

Then clearly $C_0 \ge c_0 > 0$ and $\underline{\sigma}^2 + c_0 I_d \le \overline{\sigma}^2 - c_0 I_d$. Denote, for $\varepsilon \le c_0$,

$$G_\varepsilon(\gamma) := \frac{1}{2}\sup_{\sigma \in [\underline{\sigma}_\varepsilon, \overline{\sigma}_\varepsilon]} (\sigma^2 : \gamma), \quad \text{where} \quad \underline{\sigma}^2_\varepsilon := \underline{\sigma}^2 + \varepsilon I_d, \quad \overline{\sigma}^2_\varepsilon := \overline{\sigma}^2 - \varepsilon I_d. \tag{3.5}$$

Lemma 3.2 *(i) For any $0 < \varepsilon \le c_0$ and $\alpha, \gamma \in \mathbb{S}^d$, it holds that*

$$G_\varepsilon(\gamma) + \varepsilon|\alpha| \le G^\alpha(\gamma) \le G(\gamma) + C_0|\alpha|. \tag{3.6}$$

(ii) Assume $\underline{\varphi} \le \varphi \le \overline{\varphi}$ are Lipschitz continuous functions, and $0 \le t_1 < t_2 \le T$. Then

$$\mathbb{E}^{G_\varepsilon}\Big[\underline{\varphi}(x + B^{t_1}_{t_2})\Big] + \varepsilon|\alpha|(t_2 - t_1) \le \mathcal{E}^\alpha_{t_1,t_2}(\varphi)(x) \le \mathbb{E}^G\Big[\overline{\varphi}(x + B^{t_1}_{t_2})\Big] + C_0|\alpha|(t_2 - t_1).$$

Proof. (i) We first prove the left inequality. Let $\alpha_1, \cdots, \alpha_d$ denote the eigenvalues of α, and $\hat{\alpha}$ the diagonal matrix with components $\alpha_1, \cdots, \alpha_d$. Then $|\alpha| = (\alpha_1^2 + \cdots + \alpha_d^2)^{\frac{1}{2}}$, and there exists an orthogonal matrix P such that $P^T \alpha P = \hat{\alpha}$. Let $\hat{c}_\varepsilon$ denote a diagonal matrix whose diagonal components take values ε or $-\varepsilon$. Now for any $\sigma_\varepsilon \in [\underline{\sigma}_\varepsilon, \overline{\sigma}_\varepsilon]$, by (3.5), we have

$$\sigma^2_\varepsilon + P\hat{c}_\varepsilon P^T \in [\underline{\sigma}^2, \overline{\sigma}^2] \quad \text{and} \quad \sigma^2_\varepsilon - P\hat{c}_\varepsilon P^T \in [\underline{\sigma}^2, \overline{\sigma}^2].$$

Then

$$
\begin{aligned}
2G^\alpha(\gamma) &= G(\gamma+2\alpha)+G(\gamma-2\alpha) \\
&\geq \frac{1}{2}\Big[(\sigma_\varepsilon^2+P\hat{c}_\varepsilon P^T):(\gamma+2\alpha)+(\sigma_\varepsilon^2-P\hat{c}_\varepsilon P^T):(\gamma-2\alpha)\Big] \\
&= \sigma_\varepsilon^2:\gamma+2(P\hat{c}_\varepsilon P^T):\alpha=\sigma_\varepsilon^2:\gamma+2\hat{c}_\varepsilon:(P^T\alpha P)=\sigma_\varepsilon^2:\gamma+2\hat{c}_\varepsilon:\hat{\alpha}.
\end{aligned}
$$

By the arbitrariness of σ_ε and $\hat{c}_\varepsilon$, we get

$$
G^\alpha(\gamma)\geq G_\varepsilon(\gamma)+\varepsilon\sum_{i=1}^d|\alpha_i|\geq G_\varepsilon(\gamma)+\varepsilon|\alpha|.
$$

We now prove the right inequality of (3.6). For any $\sigma_1,\sigma_2\in[\underline{\sigma},\overline{\sigma}]$, we have

$$
\sigma_1^2:(\gamma+2\alpha)+\sigma_2^2:(\gamma-2\alpha)=(\sigma_1^2+\sigma_2^2):\gamma+2(\sigma_1^2-\sigma_2^2):\alpha.
$$

Note that

$$
\underline{\sigma}^2\leq\frac{1}{2}(\sigma_1^2+\sigma_2^2)\leq\overline{\sigma}^2,\quad -[\overline{\sigma}^2-\underline{\sigma}^2]\leq\sigma_1^2-\sigma_2^2\leq\overline{\sigma}^2-\underline{\sigma}^2.
$$

Then, by (2.2),

$$
\sigma_1^2:(\gamma+2\alpha)+\sigma_2^2:(\gamma-2\alpha)\leq 4G(\gamma)+4C_0|\alpha|.
$$

Since σ_1,σ_2 are arbitrary, we prove the right inequality of (3.6), and hence (3.6).

(ii) One can easily check that

$$
\begin{aligned}
\mathbb{E}^{G_\varepsilon}\Big[\underline{\varphi}(x+B_{t_2}^{t_1})\Big]+\varepsilon|\alpha|(t_2-t_1)=\underline{v}^\alpha(t_1,x), \\
\mathbb{E}^{G}\Big[\overline{\varphi}(x+B_{t_2}^{t_1})\Big]+C_0|\alpha|(t_2-t_1)=\overline{v}^\alpha(t_1,x),
\end{aligned}
$$

where $\underline{v}^\alpha,\overline{v}^\alpha$ are the unique viscosity solution of the following PDEs on $[t_1,t_2]$:

$$
\begin{aligned}
\partial_t\underline{v}^\alpha+G_\varepsilon(\partial_{xx}\underline{v}^\alpha)+\varepsilon|\alpha|=0,\quad \underline{v}^\alpha(t_2,x)=\underline{\varphi}(x); \\
\partial_t\overline{v}^\alpha+G(\partial_{xx}\overline{v}^\alpha)+C_0|\alpha|=0,\quad \overline{v}^\alpha(t_2,x)=\overline{\varphi}(x).
\end{aligned}
$$

Then the statement follows directly from (3.6) and the comparison principle of PDEs. ■

3.2 The nonlinear operator via probabilistic arguments

For any $n\geq 1$, denote $t_i^n:=\frac{i}{n}T$, $i=0,\cdots,n$, and define

$$
\delta_n(t)=\sum_{i=0}^{n-1}(-1)^i 1_{[t_i^n,t_{i+1}^n)},\quad t\in[0,T]. \tag{3.7}
$$

This function was introduced in [13] which plays a key role for constructing a new norm. According to [13], we have

Lemma 3.3 *For any $\eta \in \mathcal{M}_G^1$, it holds that $\lim_{n\to\infty} \mathbb{E}^G\Big[\int_0^T G(\eta_t)\delta_n(t)dt\Big] = 0$.*

The next lemma establishes the connection between δ_n and G^α.

Lemma 3.4 *Let $0 \le s < t \le T$ and $\alpha \in \mathbb{S}^d$.*

(i) For any $\gamma \in \mathbb{S}^d$, we have

$$\lim_{n\to\infty} \mathbb{E}_s^G\Big[\int_s^t [\alpha\delta_n(r) + \frac{1}{2}\gamma] : d\langle B\rangle_r\Big] = G^\alpha(\gamma)(t-s). \tag{3.8}$$

(ii) For any $x \in \mathbb{R}^d$ and any Lipschitz continuous function φ, we have

$$\lim_{n\to\infty} \mathbb{E}_s^G\Big[\int_s^t \delta_n(r)\alpha : d\langle B\rangle_r + \varphi(x + B_t^s)\Big] = \mathcal{E}_{s,t}^\alpha(\varphi)(x). \tag{3.9}$$

Proof. (i) Fix n such that $\frac{2T}{n} < t - s$. Note that

$$\begin{aligned}
& \mathbb{E}^G_{t^n_{2i}}\Big[\int_{t^n_{2i}}^{t^n_{2i+2}} [\alpha\delta_n(r) + \frac{1}{2}\gamma] : d\langle B\rangle_r\Big] \\
= \ & \mathbb{E}^G_{t^n_{2i}}\Big[(\frac{1}{2}\gamma + \alpha) : [\langle B\rangle_{t^n_{2i+1}} - \langle B\rangle_{t^n_{2i}}] + (\frac{1}{2}\gamma - \alpha) : [\langle B\rangle_{t^n_{2i+2}} - \langle B\rangle_{t^n_{2i+1}}]\Big] \\
= \ & \mathbb{E}^G_{t^n_{2i}}\Big[(\frac{1}{2}\gamma + \alpha) : [\langle B\rangle_{t^n_{2i+1}} - \langle B\rangle_{t^n_{2i}}] + \mathbb{E}^G_{t^n_{2i+1}}\big[(\frac{1}{2}\gamma - \alpha) : [\langle B\rangle_{t^n_{2i+2}} - \langle B\rangle_{t^n_{2i+1}}]\big]\Big] \\
= \ & \mathbb{E}^G_{t^n_{2i}}\Big[(\frac{1}{2}\gamma + \alpha) : [\langle B\rangle_{t^n_{2i+1}} - \langle B\rangle_{t^n_{2i}}] + G(\gamma - 2\alpha)\frac{T}{n}\Big] \\
= \ & \mathbb{E}^G_{t^n_{2i}}\Big[(\frac{1}{2}\gamma + \alpha) : [\langle B\rangle_{t^n_{2i+1}} - \langle B\rangle_{t^n_{2i}}]\Big] + G(\gamma - 2\alpha)\frac{T}{n} \\
= \ & G(\gamma + 2\alpha)\frac{T}{n} + G(\gamma - 2\alpha)\frac{T}{n} = G^\alpha(\gamma)(t^n_{2i+2} - t^n_{2i}).
\end{aligned}$$

Similarly, for any $i < j$,

$$\mathbb{E}^G_{t^n_{2i}}\Big[\int_{t^n_{2i}}^{t^n_{2j}} [\alpha\delta_n(r) + \frac{1}{2}\gamma] : d\langle B\rangle_r\Big] = G^\alpha(\gamma)(t^n_{2j} - t^n_{2i}).$$

Now assume $t^n_{2i} \le s < t^n_{2i+1} \le t^n_{2j} \le t < t^n_{2j+2}$. Then

$$\begin{aligned}
& \Big|\mathbb{E}_s^G\Big[\int_s^t [\alpha\delta_n(r) + \frac{1}{2}\gamma] : d\langle B\rangle_r\Big] - G^\alpha(\gamma)(t-s)\Big| \\
\le \ & \Big|\mathbb{E}_s^G\Big[\int_s^t [\alpha\delta_n(r) + \frac{1}{2}\gamma] : d\langle B\rangle_r\Big] - \mathbb{E}_s^G\Big[\int_{t^n_{2i+2}}^{t^n_{2j}} [\alpha\delta_n(r) + \frac{1}{2}\gamma] : d\langle B\rangle_r\Big]\Big| \\
& + \Big|G^\alpha(\gamma)(t^n_{2j} - t^n_{2i+2}) - G^\alpha(\gamma)(t-s)\Big| \\
\le \ & \mathbb{E}_s^G\Big[\Big|[\int_s^{t^n_{2i+2}} + \int_{t^n_{2j}}^t][\alpha\delta_n(r) + \frac{1}{2}\gamma] : d\langle B\rangle_r\Big|\Big] + \frac{2T}{n}|G^\alpha(\gamma)| \\
\le \ & \frac{2T}{n}|\overline{\sigma}^2|[|\alpha| + \frac{1}{2}|\gamma|] + \frac{2T}{n}|G^\alpha(\gamma)| \to 0, \quad \text{as } n \to \infty,
\end{aligned}$$

where the last inequality thanks to (2.2). This proves the result.

(ii) Without loss of generality, assume $t = T$. Define

$$\begin{aligned}
\overline{u}(t,x) &:= \overline{\lim_{n\to\infty}}\,\overline{u}^n(t,x) := \overline{\lim_{n\to\infty}}\,\mathbb{E}^G_t\Big[\int_t^T \delta_n(r)\alpha : d\langle B\rangle_r + \varphi(x+B^t_T)\Big],\\
\underline{u}(t,x) &:= \overline{\lim_{n\to\infty}}\,\underline{u}^n(t,x) := \underline{\lim}_{n\to\infty}\,\mathbb{E}^G_t\Big[\int_t^T \delta_n(r)\alpha : d\langle B\rangle_r + \varphi(x+B^t_T)\Big].
\end{aligned}$$

By the structure of G-framework it is clear that $\underline{u}$ and $\overline{u}$ are deterministic functions. Obviously $\underline{u} \le \overline{u}$. We claim that $\overline{u}$ and $\underline{u}$ are viscosity subsolution and viscosity supersolution of PDE (3.2) with $t_1 = 0, t_2 = T$. Note that PDE (3.2) satisfies the comparison principle for viscosity solutions. Then $\overline{u} \le \underline{u}$ and thus $\overline{u}(t,x) = \underline{u}(t,x) = \mathcal{E}^{\alpha}_{t,T}(\varphi)(x)$. This proves the result.

We now prove that $\overline{u}$ is a viscosity subsolution, and the viscosity supersolution property of $\underline{u}$ can be proved similarly. As usual, we start from the partial dynamic programming principle: for $0 \le t < t+h \le T$,

$$\overline{u}(t,x) \le \overline{\lim_{n\to\infty}}\,\mathbb{E}^G\Big[\int_t^{t+h} \delta_n(r)\alpha : d\langle B\rangle_r + \overline{u}(t+h, x+B^t_{t+h})\Big], \tag{3.10}$$

Indeed, by the time homogeneity of the problem, we have

$$\begin{aligned}
\overline{u}^n(t,x) &= \mathbb{E}^G\Big[\int_t^{t+h} \delta_n(r)\alpha : d\langle B\rangle_r + \mathbb{E}^G_{t+h}\big[\int_{t+h}^T \delta_n(r)\alpha : d\langle B\rangle_r + \varphi(x+B^t_T)\big]\Big]\\
&= \mathbb{E}^G\Big[\int_t^{t+h} \delta_n(r)\alpha : d\langle B\rangle_r + \overline{u}^n(t+h, x+B^t_{t+h})\Big]
\end{aligned}$$

Then

$$\begin{aligned}
&\overline{u}(t,x) - \overline{\lim_{n\to\infty}}\,\mathbb{E}^G\Big[\int_t^{t+h} \delta_n(r)\alpha : d\langle B\rangle_r + \overline{u}(t+h, x+B^t_{t+h})\Big]\\
=\ & \overline{\lim_{n\to\infty}}\,\overline{u}^n(t,x) - \overline{\lim_{n\to\infty}}\,\mathbb{E}^G\Big[\int_t^{t+h} \delta_n(r)\alpha : d\langle B\rangle_r + \overline{u}(t+h, x+B^t_{t+h})\Big]\\
\le\ & \overline{\lim_{n\to\infty}}\,\mathbb{E}^G\Big[(\overline{u}^n - \overline{u})(t+h, x+B^t_{t+h})\Big].
\end{aligned}$$

Following standard arguments it is obvious that $\overline{u}$ is uniformly Lipschitz continuous in x. Moreover, $\overline{\lim}_{n\to\infty}(\overline{u}^n - \overline{u})(t+h,x) = 0$ for any $x \in \mathbb{R}$. Then (3.10) follows directly from the simple Lemma 3.5 below.

We next derive the viscosity subsolution property from (3.10). Let $(t,x) \in [0,T) \times \mathbb{R}^d$ and $\varphi \in C^{1,2}([t,T) \times \mathbb{R}^d)$ such that $0 = [\overline{u} - \varphi](t,x) = \max_{(s,y)\in[t,T]\times\mathbb{R}^d}[\overline{u} - \varphi](s,y)$. Denote

$X_s := x + B_s^t$. For any $0 < h \le T - t$, by (3.10) and then applying Itô's formula we have

$$
\begin{aligned}
&\varphi(t,x) = \overline{u}(t,x) \le \varlimsup_{n\to\infty} \mathbb{E}^G\Big[\int_t^{t+h} \delta_n(r)\alpha : d\langle B\rangle_r + \overline{u}(t+h, X_{t+h})\Big] \\
\le\ & \varlimsup_{n\to\infty} \mathbb{E}^G\Big[\int_t^{t+h} \delta_n(r)\alpha : d\langle B\rangle_r + \varphi(t+h, X_{t+h})\Big] \\
=\ & \varlimsup_{n\to\infty} \mathbb{E}^G\Big[\int_t^{t+h} \delta_n(r)\alpha : d\langle B\rangle_r + \varphi(t,x) \\
& + \int_t^{t+h} [\partial_t \varphi(r, X_r) dr + \frac{1}{2}\partial_{xx}\varphi(r, X_r) : d\langle B\rangle_r]\Big] \\
\le\ & \varlimsup_{n\to\infty} \mathbb{E}^G\Big[\int_t^{t+h} [\alpha\delta_n(r) + \frac{1}{2}\partial_{xx}\varphi(t,x)] : d\langle B\rangle_r\Big] + \varphi(t,x) + \partial_t\varphi(t,x)h \\
& + \mathbb{E}^G\Big[\int_t^{t+h} [\partial_t\varphi(r, X_r) - \partial_t\varphi(t,x)]dr + \frac{1}{2}\int_t^{t+h} [\partial_{xx}\varphi(r, X_r) - \partial_{xx}\varphi(t,x)] : d\langle B\rangle_r]\Big] \\
\le\ & G^\alpha(\partial_{xx}\varphi(t,x))h + \varphi(t,x) + \partial_t\varphi(t,x)h \\
& + \mathbb{E}^G\Big[\sup_{t\le r\le t+h} [|\partial_t\varphi(r, X_r) - \partial_t\varphi(t,x)| + \frac{|\overline{\sigma}^2|}{2}|\partial_{xx}\varphi(r, X_r) - \partial_{xx}\varphi(t,x)|\Big]h,
\end{aligned}
$$

thanks to (3.8). By standard arguments $\overline{u}$ is uniformly Lipschitz continuous in x, and note that viscosity property is a local property. Then, without loss of generality we may assume $\partial_t\varphi$ and ∂_{xx} is bounded and uniformly continuous in (t,x) with a modulus of continuity function ρ. Thus,

$$
0 \le \partial_t\varphi(t,x) + G^\alpha(\partial_{xx}\varphi(t,x)) + C\mathbb{E}^G\Big[\rho\big(C[h + \sup_{t\le r\le t+h} |B_r^t|]\big)\Big].
$$

Send $h \to 0$ we can easily get

$$
\partial_t\varphi(t,x) + G^\alpha(\partial_{xx}\varphi(t,x)) \ge 0.
$$

Clearly $\overline{u}(T,x) = \varphi$. Therefore, $\overline{u}$ is a viscosity subsolution of PDE (3.2). ■

Lemma 3.5 *Assume $\varphi_n : \mathbb{R}^d \to \mathbb{R}$ are uniformly Lipschitz continuous functions, uniformly in n, and $\varlimsup_{n\to\infty} \varphi_n(x) \le 0$ for all x. Then $\varlimsup_{n\to\infty} \mathbb{E}^G[\varphi_n(B_t)] \le 0$ for any t.*

Proof. Let L denote the uniform Lipschitz constant of φ_n. For any $\varepsilon > 0$ and $R > 0$, there exist finitely many x_i, $i = 1, \cdots, M$ and a partition $\cup_{i=1}^M O_i = O_R(\mathbf{0}) := \{x \in \mathbb{R}^d : |x| \le R\}$ such that $|x - x_i| \le \varepsilon$ for all $x \in O_i$. Denote $O_0 := \mathbb{R}^d \backslash O_R(\mathbf{0})$ and $x_0 := \mathbf{0}$. Then

$$
\begin{aligned}
\varphi_n(B_t) &= \sum_{i=0}^M \varphi_n(B_t)\mathbf{1}_{O_i}(B_t) = \sum_{i=0}^M \varphi_n(x_i)\mathbf{1}_{O_i}(B_t) + \sum_{i=0}^M [\varphi_n(B_t) - \varphi_n(x_i)]\mathbf{1}_{O_i}(B_t) \\
&\le \sum_{i=0}^M \varphi_n^+(x_i)\mathbf{1}_{O_i}(B_t) + L|B_t|\mathbf{1}_{O_0}(B_t) + L\varepsilon\sum_{i=1}^M \mathbf{1}_{O_i}(B_t) \\
&\le \sum_{i=0}^M \varphi_n^+(x_i)\mathbf{1}_{O_i}(B_t) + \frac{L}{R}|B_t|^2 + L\varepsilon.
\end{aligned}
$$

Thus, noting that our condition implies $\overline{\lim}_{n\to\infty}\varphi_n^+(x)=0$,

$$\overline{\lim_{n\to\infty}}\,\mathbb{E}^G\Big[\varphi_n(B_t)\Big] \le \overline{\lim_{n\to\infty}}\,\mathbb{E}^G\Big[\sum_{i=0}^M \varphi_n^+(x_i)\mathbf{1}_{O_i}(B_t)+\frac{L}{R}|B_t|^2+L\varepsilon\Big]$$
$$\le \sum_{i=0}^M \overline{\lim_{n\to\infty}}\,\varphi_n^+(x_i)\mathbb{E}^G[\mathbf{1}_{O_i}(B_t)]+\frac{L}{R}\mathbb{E}^G[|B_t|^2]+L\varepsilon=\frac{L}{R}\mathbb{E}^G[|B_t|^2]+L\varepsilon.$$

Send $R\to\infty$ and $\varepsilon\to 0$, we prove the result. ∎

3.3 An intermediate norm for $\eta\in\mathcal{M}_G^1$

We now use $\delta_n(t)$ to introduce the following norm for a process η.

Theorem 3.6 *For any $\eta\in\mathcal{M}_G^1$, the following limit exists:*

$$\|\eta\|_{\mathbb{M}_G} := \lim_{n\to\infty}\mathbb{E}^G\Big[\int_0^T\delta_n(t)\eta_t : d\langle B\rangle_t\Big]. \tag{3.11}$$

Proof. We first assume $\eta\in\mathcal{M}_G^0$. By otherwise considering a finer partition of $[0,T]$, without loss of generality we assume, for $0=t_0<\cdots<t_m=T$,

$$\eta=\sum_{i=0}^{m-1}\eta_{t_i}\mathbf{1}_{[t_i,t_{i+1})},\quad\text{where}\quad \eta_{t_i}=\varphi_i(B_{t_1},\cdots,B_{t_i}) \tag{3.12}$$

and φ_i is uniformly Lipschitz continuous. Denote

$$\psi_i^n(B_{t_1},\cdots,B_{t_i}) := \mathbb{E}^G_{t_i}\Big[\int_{t_i}^T\delta_n(t)\eta_t : d\langle B\rangle_t\Big].$$

We prove by backward induction that

$$\lim_n\psi_i^n=\psi_i \tag{3.13}$$

where, $\psi_m:=0$ and, for $i=m-1,\cdots,0$,

$$\psi_i(x_1,\cdots,x_i) := \mathcal{E}_{t_i,t_{i+1}}^{\varphi_i(x_1,\cdots,x_i)}(\psi_{i+1}(x_1,\cdots,x_i,\cdot))(x_i). \tag{3.14}$$

Indeed, when $i=m$, (3.13) holds obviously. Assume (3.13) holds for $i+1$. Then by (3.9) we have

$$\begin{aligned}
&\overline{\lim_{n\to\infty}}\,\psi_i^n(B_{t_1},\cdots,B_{t_i})-\psi_i(B_{t_1},\cdots,B_{t_i})\\
=\;&\overline{\lim_{n\to\infty}}\,\mathbb{E}^G_{t_i}\Big[\int_{t_i}^{t_{i+1}}\delta_n(t)\eta_{t_i}: d\langle B\rangle_t+\psi_{i+1}^n(B_{t_1},\cdots,B_{t_{i+1}})\Big]\\
&-\lim_{n\to\infty}\mathbb{E}^G_{t_i}\Big[\int_{t_i}^{t_{i+1}}\delta_n(t)\eta_{t_i}: d\langle B\rangle_t+\psi_{i+1}(B_{t_1},\cdots,B_{t_{i+1}})\Big]\Big|\\
\le\;&\overline{\lim_{n\to\infty}}\,\mathbb{E}^G_{t_i}\Big[\psi_{i+1}^n(B_{t_1},\cdots,B_{t_{i+1}})-\psi_{i+1}(B_{t_1},\cdots,B_{t_{i+1}})\Big].
\end{aligned}$$

By induction assumption, $\lim_{n\to\infty}\psi^n_{i+1}=\psi_{i+1}$. Moreover, one can easily check that ψ^n_{i+1} is uniformly continuous in x_{i+1}, uniformly in n. Then by Lemma 3.5 we obtain

$$\overline{\lim_{n\to\infty}}\,\psi^n_i(B_{t_1},\cdots,B_{t_i})-\psi_i(B_{t_1},\cdots,B_{t_i})\le 0.$$

Similarly, we can show that

$$\psi_i(B_{t_1},\cdots,B_{t_i})-\underline{\lim}_{n\to\infty}\,\psi^n_i(B_{t_1},\cdots,B_{t_i})\le 0.$$

Thus (3.13) holds for i. This completes the induction and hence proves that the limit in (3.11) for $\eta\in\mathcal{M}^0_G$.

We now consider general $\eta\in\mathcal{M}^1_G$. Let $\eta^m\in\mathcal{M}^0_G$ such that $\lim_{m\to\infty}\|\eta^m-\eta\|_{\mathcal{M}^1_G}=0$. For each m, by previous arguments we have

$$\lim_{n\to\infty}\mathbb{E}^G\Big[\int_0^T\delta_n(t)\eta^m_t:d\langle B\rangle_t\Big]\text{ exists.}$$

By (2.2), one can easily check that

$$\begin{aligned}
&\Big|\mathbb{E}^G\Big[\int_0^T\delta_n(t)\eta^m_t:d\langle B\rangle_t\Big]-\mathbb{E}^G\Big[\int_0^T\delta_n(t)\eta_t:d\langle B\rangle_t\Big]\Big|\\
\le\;&\mathbb{E}^G\Big[\Big|\int_0^T\delta_n(t)[\eta^m_t-\eta^t]:d\langle B\rangle_t\Big|\Big]\le\mathbb{E}^G\Big[\int_0^T|\eta^m_t-\eta^t||\overline{\sigma}^2|dt\Big]\\
=\;&|\overline{\sigma}^2|\|\eta^m-\eta\|_{\mathcal{M}^1_G}\to 0,\quad\text{as } m\to\infty.
\end{aligned}$$

This clearly leads to the existence of $\lim_{n\to\infty}\mathbb{E}^G\Big[\int_0^T\delta_n(t)\eta_t:d\langle B\rangle_t\Big]$. ■

We now collect some basic properties of $\|\cdot\|_{\mathbb{M}_G}$. The left inequality of (3.15) below is crucial for our purpose. We remark that, the norm $\|\cdot\|_{\mathcal{M}^1_{G_\varepsilon}}$ was introduced in Hu and Peng [4] and a similar estimate was obtained in Song [13] by using different arguments.

Theorem 3.7 $\|\cdot\|_{\mathbb{M}_G}$ *defines a norm on* $\mathcal{M}^1_G$ *(up to an equivalence class), and for any* $0<\varepsilon\le c_0$*, it holds that,*

$$\varepsilon\|\eta\|_{\mathcal{M}^1_{G_\varepsilon}}\le\|\eta\|_{\mathbb{M}_G}\le C_0\|\eta\|_{\mathcal{M}^1_G}.\tag{3.15}$$

Proof. Note that $\|\cdot\|_{\mathcal{M}^1_{G_\varepsilon}}\le\|\cdot\|_{\mathcal{M}^1_G}$. By using standard approximation arguments, it suffices to prove the statements for $\eta\in\mathcal{M}^0_G$. We now assume η takes the form (3.12) and we shall use the notations in the proof of Theorem 3.6. In particular, by (3.13) we have

$$\|\eta\|_{\mathbb{M}_G}=\psi_0.$$

We first prove (3.15). Define $\underline{\psi}^\varepsilon_i$ and $\overline{\psi}^\varepsilon_i$ by:

$$\underline{\psi}^\varepsilon_i(B_{t_1},\cdots,B_{t_i}):=\varepsilon\mathbb{E}^{G_\varepsilon}_{t_i}\Big[\int_{t_i}^T|\eta_t|dt\Big],\quad\overline{\psi}^\varepsilon_i(B_{t_1},\cdots,B_{t_i}):=C_0\mathbb{E}^G_{t_i}\Big[\int_{t_i}^T|\eta_t|dt\Big].$$

Then $\underline{\psi}^\varepsilon_m = \overline{\psi}^\varepsilon_m = 0$, and

$$\begin{aligned}
\underline{\psi}^\varepsilon_i(x_1,\cdots,x_i) &= \mathbb{E}^{G_\varepsilon}\Big[\underline{\psi}^\varepsilon_{i+1}(x_1,\cdots,x_i,x_i+B^{t_i}_{t_{i+1}})\Big] + \varepsilon|\varphi_i(x_1,\cdots,x_i)|(t_{i+1}-t_i);\\
\overline{\psi}^\varepsilon_i(x_1,\cdots,x_i) &= \mathbb{E}^{G_\varepsilon}\Big[\overline{\psi}^\varepsilon_{i+1}(x_1,\cdots,x_i,x_i+B^{t_i}_{t_{i+1}})\Big] + C_0|\varphi_i(x_1,\cdots,x_i)|(t_{i+1}-t_i).
\end{aligned}$$

Applying Lemma 3.2 (ii) and recalling (3.14), by induction one proves (3.15) immediately.

We now prove that $\|\cdot\|_{\mathbb{M}_G}$ defines a norm. First, by (3.15) we see that $\|\eta\|_{\mathbb{M}_G} \geq 0$. Next, for any $\lambda \in \mathbb{R}$, noting that $G^{-\alpha} = G^\alpha$ by Lemma 3.1 (ii), it follows from (3.14) that

$$\begin{aligned}
\|\lambda\eta\|_{\mathbb{M}_G} = \||\lambda|\eta\|_{\mathbb{M}_G} &= \lim_{n\to\infty}\mathbb{E}^G\Big[\int_0^T \delta_n(t)|\lambda|\eta_t : d\langle B\rangle_t\Big]\\
&= \lim_{n\to\infty}|\lambda|\mathbb{E}^G\Big[\int_0^T \delta_n(t)\eta_t : d\langle B\rangle_t\Big] = |\lambda|\|\eta\|_{\mathbb{M}_G}.
\end{aligned}$$

Finally, for any $\eta, \tilde\eta \in \mathcal{M}^0_G$, by the sublinearity of $\mathbb{E}^G$, we have

$$\mathbb{E}^G\Big[\int_0^T \delta_n(t)[\eta_t+\tilde\eta_t] : d\langle B\rangle_t\Big] \leq \mathbb{E}^G\Big[\int_0^T \delta_n(t)\eta_t : d\langle B\rangle_t\Big] + \mathbb{E}^G\Big[\int_0^T \delta_n(t)\tilde\eta_t : d\langle B\rangle_t\Big].$$

Send $n\to\infty$ we obtain the triangle inequality: $\|\eta+\tilde\eta\|_{\mathbb{M}_G} \leq \|\eta\|_{\mathbb{M}_G} + \|\tilde\eta\|_{\mathbb{M}_G}$. That is, up to an equivalence class, $\|\cdot\|_{\mathbb{M}_G}$ defines a norm. ■

3.4 The new norm for η

One drawback of the above norm $\|\cdot\|_{\mathbb{M}_G}$ is that we have to use different norms in the left and right sides of (3.15). Consequently, we are not able to prove the completeness of $\mathcal{M}^1_G$ under $\|\cdot\|_{\mathbb{M}_G}$. To be precise, given a Cauchy sequence $\eta^n \in \mathcal{M}^1_G$ under $\|\cdot\|_{\mathbb{M}_G}$, we are not able to prove the existence of a process η such that $\lim_{n\to\infty}\|\eta^n-\eta\|_{\mathbb{M}_G} = 0$. For this reason, we shall modify the norm $\|\cdot\|_{\mathbb{M}_G}$ slightly by using heavily the estimate (3.15). Set $\varepsilon_k := \frac{1}{1+k}c_0$, $k\geq 1$, and define

$$\|\eta\|_{\mathbb{M}^*_G} := \sum_{k=1}^\infty 2^{-k}\|\eta\|_{\mathbb{M}_{G_{\varepsilon_k}}}, \quad \eta \in \mathcal{M}^1_G. \tag{3.16}$$

Then clearly $\|\cdot\|_{\mathbb{M}^*_G}$ define a norm on $\mathcal{M}^1_G$, and we denote by $\mathbb{M}^*_G$ the closure of $\mathcal{M}^1_G$ under $\|\cdot\|_{\mathbb{M}^*_G}$. To understand the space $\mathbb{M}^*_G$, we note that $\mathcal{M}^1_{G_\varepsilon}$ is decreasing as $\varepsilon\to 0$. Set

$$\mathcal{M}^1_{G_0} := \lim_{\varepsilon\to 0}\mathcal{M}^1_{G_\varepsilon} = \bigcap_{0<\varepsilon\leq c_0}\mathcal{M}^1_{G_\varepsilon}. \tag{3.17}$$

On the other hand, recall (3.5) and let

$$\mathcal{A}_\varepsilon := \Big\{\sigma\in\mathcal{A} : \underline{\sigma}^2_\varepsilon \leq \sigma^2 \leq \overline{\sigma}^2_\varepsilon\Big\}, \quad \mathcal{P}_\varepsilon := \Big\{\mathbb{P}^\sigma : \sigma\in\mathcal{A}_\varepsilon\Big\}, \quad \mathcal{P}_0 := \lim_{\varepsilon\to 0}\mathcal{P}_\varepsilon. \tag{3.18}$$

We remark that $\mathcal{P}_0$ is a strict subset of $\{\mathbb{P}^\sigma : \sigma\in\mathcal{A}, \underline{\sigma} < \sigma < \overline{\sigma}\}$. We now have

Theorem 3.8 *Let $\eta^n \in \mathcal{M}^1_G$ be a Cauchy sequence under $\|\cdot\|_{\mathbb{M}^*_G}$. Then there exists unique (in the $\mathcal{P}_0$-q.s. sense) process $\eta \in \mathcal{M}^1_{G_0}$ such that*

$$\lim_{n\to\infty} \|\eta^n - \eta\|_{\mathbb{M}^*_G} = 0. \tag{3.19}$$

Consequently, we have

$$\mathcal{M}^1_G \subset \mathbb{M}^*_G \subset \mathcal{M}^1_{G_0}. \tag{3.20}$$

Proof. We first note that, for any $\eta \in \mathcal{M}^1_{G_0}$, by Theorem 3.7 $\|\eta\|_{\mathbb{M}_{G_\varepsilon}}$ is well defined for all $0 < \varepsilon \le c_0$, and thus $\|\eta\|_{\mathbb{M}^*_G}$ is well defined with possible value ∞.

Next, for any $0 < \varepsilon \le c_0$, there exists k large enough such that $\varepsilon_k < \varepsilon$. By the left inequality of (3.15) we see that

$$\|\eta^n - \eta^m\|_{\mathcal{M}^1_{G_\varepsilon}} \le C_{\varepsilon,\varepsilon_k}\|\eta^n - \eta^m\|_{\mathbb{M}_{G_{\varepsilon_k}}} \le 2^k C_{\varepsilon,\varepsilon_k}\|\eta^n - \eta^m\|_{\mathbb{M}^*_G} \to 0, \quad \text{as } n, m \to \infty.$$

Since $(\mathcal{M}^1_{G_\varepsilon}, \|\cdot\|_{\mathcal{M}^1_{G_\varepsilon}})$ is complete, there exists unique (in $\mathcal{P}_\varepsilon$-q.s. sense) $\eta^{(\varepsilon)} \in \mathcal{M}^1_{G_\varepsilon}$ such that $\lim_{n\to\infty}\|\eta^n - \eta^{(\varepsilon)}\|_{\mathcal{M}^1_{G_\varepsilon}} = 0$. By the uniqueness, clearly $\eta^{(\varepsilon)} = \eta^{(\tilde\varepsilon)}$, $\mathcal{P}_\varepsilon$-q.s. for any $0 < \tilde\varepsilon < \varepsilon \le c_0$. Thus there exists $\eta \in \mathcal{M}^1_{G_0}$ such that $\eta^{(\varepsilon)} = \eta$, $\mathcal{P}_\varepsilon$-q.s. for all $0 < \varepsilon \le c_0$.

We now prove (3.19). Indeed, for any $\delta > 0$, there exists N_δ such that

$$\|\eta^n - \eta^m\|_{\mathbb{M}^*_G} \le \delta, \quad \text{for all } n, m \ge N_\delta.$$

Note that, by the right inequality of (3.15),

$$\|\eta^m - \eta\|_{\mathbb{M}_{G_\varepsilon}} \le C_0\|\eta^m - \eta^{(\varepsilon)}\|_{\mathcal{M}^1_{G_\varepsilon}} \to 0, \quad \text{as } m \to \infty.$$

Then for any $n \ge N_\delta$ and $K \ge 1$,

$$\sum_{k=1}^K 2^{-k}\|\eta^n - \eta\|_{\mathbb{M}_{G_{\varepsilon_k}}} = \lim_{m\to\infty}\sum_{k=0}^K 2^{-k}\|\eta^n - \eta^m\|_{\mathbb{M}_{G_{\varepsilon_k}}} \le \varliminf_{m\to\infty}\|\eta^n - \eta^m\|_{\mathbb{M}^*_G} \le \delta.$$

Send $K \to \infty$ we obtain $\|\eta^n - \eta\|_{\mathbb{M}^*_G} \le \delta$ for all $n \ge N_\delta$. This proves (3.19).

Finally, if $\tilde\eta \in \mathcal{M}^1_{G_0}$ is another limit, then $\|\tilde\eta - \eta\|_{\mathbb{M}^*_G} = 0$. This implies that $\|\tilde\eta - \eta\|_{\mathcal{M}^1_{G_\varepsilon}} = 0$ for all ε and thus $\tilde\eta = \eta$, $\mathcal{P}_\varepsilon$-q.s. for all ε. Therefore, η is unique in $\mathcal{P}_0$-q.s. sense. ■

4 The G-martingale representation theorem

We first note that, assuming (Y^i, Z^i, η^i), $i = 1, 2$, satisfy (2.19), then

$$d(Y^1_t - Y^2_t) = (Z^1_t - Z^2_t)\cdot dB_t - [G(\eta^1_t) - G(\eta^2_t)]dt + \frac{1}{2}[\eta^1_t - \eta^2_t] : d\langle B\rangle_t. \tag{4.1}$$

By Lemma 3.3 we have

$$\|\eta^1-\eta^2\|_{\mathbb{M}_{G_\varepsilon}} = 2\lim_{n\to\infty}\mathbb{E}^{G_\varepsilon}\Big[\int_0^T \delta_n(t)d(Y_t^1-Y_t^2)\Big], \quad \text{for all } 0<\varepsilon\le c_0. \tag{4.2}$$

In light of (3.16), for any $p>1$ we define:

$$d_{G,p}(\xi_1,\xi_2) := \|Y^1-Y^2\|_{\mathbb{D}_G^p} + \sum_{k=1}^\infty 2^{-k}\lim_{n\to\infty}\mathbb{E}^{G_{\varepsilon_k}}\Big[\int_0^T \delta_n(t)d(Y_t^1-Y_t^2)\Big], \tag{4.3}$$
$$\text{where } \xi_i\in\mathcal{L}_{ip} \text{ and } Y_t^i := \mathbb{E}_t^G[\xi_i], i=1,2.$$

Then clearly $d_{G,p}$ is a metric on $\mathcal{L}_{ip}$, and we let $\mathbb{L}_G^{*p}\subset\mathcal{L}_G^p$ denote the closure of $\mathcal{L}_{ip}$ under $d_{G,p}$. We remark that

$$\|\xi_1-\xi_2\|_{\mathcal{L}_G^p} \le \|Y^1-Y^2\|_{\mathbb{D}_G^p} \le \|\xi_1-\xi_2\|_{\mathbb{L}_G^p}.$$

Remark 4.1 We remark that we allow the metric $d_{G,p}(\xi_1,\xi_2)$ to depend on Y^i, but not on Z^i or η^i explicitly. The component Y has a representation, namely as the conditional G-expectation of ξ, but in general we do not have a desirable representation for Z or η. Thus it is relatively easier to check conditions imposed on Y than those on Z or η. See also [9] for similar idea. ■

Given (Z,η) and y, let $Y^{y,Z,\eta}$ denote the G-martingale defined by (2.19) with initial value $Y_0=y$. We first have

Lemma 4.2 *For any $p>1$, $y\in\mathbb{R}$, and $(Z,\eta)\in\mathcal{H}_G^p\times\mathcal{M}_G^p$, we have $Y_T^{y,Z,\eta}\in\mathbb{L}_G^{*p}$. Moreover, for any such (y_i,Z^i,η^i), $i=1,2$, we have*

$$d_{G,p}(Y_T^{y_1,Z^1,\eta^1},Y_T^{y_2,Z^2,\eta^2}) \le C_p\Big[|y_1-y_2| + \|Z^1-Z^2\|_{\mathcal{H}_G^p} + |\eta^1-\eta^2\|_{\mathcal{M}_G^p}\Big]. \tag{4.4}$$

Proof. We first prove the a priori estimate (4.4). Denote $Y^i := Y^{y_i,Z^i,\eta^i}$, $i=1,2$. By (4.1), it is obvious that

$$\|Y^1-Y^2\|_{\mathbb{D}_G^p} \le C_p\Big[|y_1-y_2| + \|Z^1-Z^2\|_{\mathcal{H}_G^p} + \|\eta^1-\eta^2\|_{\mathcal{M}_G^p}\Big]. \tag{4.5}$$

Moreover, by (4.2) and the right inequality of (3.15), we have

$$\begin{aligned}
&\lim_{n\to\infty}\mathbb{E}^{G_{\varepsilon_k}}\Big[\int_0^T \delta_n(t)d(Y_t^1-Y_t^2)\Big] = \frac{1}{2}\|\eta^1-\eta^2\|_{\mathbb{M}_{G_{\varepsilon_k}}}\\
\le\ & C\|\eta^1-\eta^2\|_{\mathcal{M}^1_{G_{\varepsilon_k}}} \le C\|\eta^1-\eta^2\|_{\mathcal{M}_G^1} \le C\|\eta^1-\eta^2\|_{\mathcal{M}_G^p}.
\end{aligned}$$

Then,

$$\sum_{k=1}^\infty 2^{-k}\lim_{n\to\infty}\mathbb{E}^{G_{\varepsilon_k}}\Big[\int_0^T \delta_n(t)d(Y_t^1-Y_t^2)\Big] \le C\sum_{k=1}^\infty 2^{-k}\|\eta^1-\eta^2\|_{\mathcal{M}_G^p} = C\|\eta^1-\eta^2\|_{\mathcal{M}_G^p}.$$

This, together with (4.5), implies (4.4).

We now show that $Y_T := Y_T^{y,Z,\eta} \in \mathbb{L}_G^{*p}$ in two steps.

Step 1. Assume $\eta = 0$. By (4.4) and the definition of $\mathcal{H}_G^p$, we may assume without loss of generality that $Z = \sum_{i=0}^{n-1} Z_{t_i} \mathbf{1}_{[t_i, t_{i+1})} \in \mathcal{H}_G^0$. Then

$$Y_T = Y_0 + \sum_{i=0}^{n-1} Z_{t_i} B_{t_{i+1}}^{t_i} \in \mathcal{L}_{ip} \subset \mathbb{L}_G^{*p}.$$

Step 2. For the general case, by (4.4) and the definition of $\mathcal{M}_G^p$, we may assume without loss of generality that $\eta = \sum_{i=0}^{n-1} \eta_{t_i} \mathbf{1}_{[t_i, t_{i+1})} \in \mathcal{M}_G^0$. Then

$$Y_T = Y_0 + \int_0^T Z_t \cdot dB_t - \sum_{i=0}^{n-1} \Big[G(\eta_{t_i})[t_{i+1} - t_i] - \frac{1}{2} \eta_{t_i} : [\langle B \rangle_{t_{i+1}} - \langle B \rangle_{t_i}] \Big].$$

For each i, applying Itô's formula we have

$$d\Big(B_t^{t_i} (B_t^{t_i})^T \Big) = 2 B_t^{t_i} d(B_t^{t_i})^T + d\langle B^{t_i} \rangle_t = 2 B_t^{t_i} dB_t^T + d\langle B \rangle_t, \quad t \in [t_i, t_{i+1}].$$

Then

$$\eta_{t_i} : [\langle B \rangle_{t_{i+1}} - \langle B \rangle_{t_i}] \Big] = \eta_{t_i} : [B_{t_{i+1}}^{t_i} (B_{t_{i+1}}^{t_i})^T] - 2 \int_{t_i}^{t_{i+1}} (\eta_{t_i} B_t^{t_i}) \cdot dB_t.$$

Thus

$$Y_T = Y_0 + \int_0^T \tilde{Z}_t \cdot dB_t - \sum_{i=0}^{n-1} \Big[G(\eta_{t_i})[t_{i+1} - t_i] - \frac{1}{2} \eta_{t_i} : [B_{t_{i+1}}^{t_i} (B_{t_{i+1}}^{t_i})^T] \Big], \tag{4.6}$$

where

$$\tilde{Z}_t := Z_t - \sum_{i=0}^{n-1} \eta_{t_i} B_t^{t_i} \mathbf{1}_{[t_i, t_{i+1})}(t).$$

One can easily check that $\tilde{Z} \in \mathcal{H}_G^p$. Then by Step 1, $\int_0^T \tilde{Z}_t \cdot dB_t \in \mathbb{L}_G^{*p}$. Moreover, it is obvious that $\sum_{i=0}^{n-1} \Big[G(\eta_{t_i})[t_{i+1} - t_i] - \frac{1}{2} \eta_{t_i} : [B_{t_{i+1}}^{t_i} (B_{t_{i+1}}^{t_i})^T] \Big] \in \mathcal{L}_{ip}$. Then it follows from (4.6) that $Y_T \in \mathbb{L}_G^{*p}$. ■

Our main result of the paper is the following representation theorem, which is in the opposite direction of Lemma 4.2.

Theorem 4.3 *Let $p > 1$.*

(i) For any $\xi \in \mathbb{L}_G^{*p}$ and denoting $Y_t := \mathbb{E}_t^G[\xi]$, there exist unique $Z \in \mathcal{H}_G^p$ and $\eta \in \mathbb{M}_G^*$ such that (2.19) *holds $\mathcal{P}_0$-q.s. Moreover, there exists a constant $C_p > 0$ such that*

$$\|Y\|_{\mathbb{D}_G^p} + \|Z\|_{\mathcal{H}_G^p} + \|\eta\|_{\mathbb{M}_G^*} \leq C_p d_{G,p}(\xi, 0). \tag{4.7}$$

(ii) For any $\xi_1, \xi_2 \in \mathbb{L}_G^{*p}$, let (Y^i, Z^i, η^i) denote the corresponding terms. Then

$$\|Y^1 - Y^2\|_{\mathbb{D}_G^p} + \|\eta^1 - \eta^2\|_{\mathbb{M}_G^*} \le C_p d_{G,p}(\xi_1, \xi_2), \quad \|Z^1 - Z^2\|_{\mathcal{H}_G^p} \le C_p \Big(d_{G,p}(\xi_1, \xi_2)\Big)^{\frac{1}{2}}. \tag{4.8}$$

Proof. We proceed in two steps.

Step 1. We first prove a priori estimates (4.7) and (4.8) by assuming (Y, Z, η) and (Y^i, Z^i, η^i), $i = 1, 2$, are in $\mathbb{D}_G^p \times \mathcal{H}_G^p \times \mathbb{M}_G^*$ and satisfy (2.19) $\mathcal{P}_0$-q.s. Indeed, by (4.2) and (4.3) it is clear that

$$\|Y\|_{\mathbb{D}_G^p} + \|\eta\|_{\mathbb{M}_G^*} \le C_p d_{G,p}(\xi, 0), \quad \|Y^1 - Y^2\|_{\mathbb{D}_G^p} + \|\eta^1 - \eta^2\|_{\mathbb{M}_G^*} \le C d_{G,p}(\xi_1, \xi_2).$$

Moreover, combining the arguments in [10] and [3], or following the arguments in [12], one can easily prove

$$\|Z\|_{\mathcal{H}_G^p} \le C_p \|Y\|_{\mathbb{D}_G^p}, \quad \|Z^1 - Z^2\|_{\mathcal{H}_G^p} \le C_p \Big(\|Y^1 - Y^2\|_{\mathbb{D}_G^p}\Big)^{\frac{1}{2}}.$$

Then (4.7) and (4.8) hold.

Step 2. We next prove the existence of (Z, η). For any $\xi \in \mathbb{L}_G^{*p}$, by definition there exist $\xi_n \in \mathcal{L}_{ip}$ such that $\lim_{n\to\infty} \rho_G^p(\xi_n, \xi) = 0$. Let (Y^n, Z^n, η^n) be corresponding to ξ_n. As $n, m \to \infty$, by (4.8) we have

$$\|Y^n - Y^m\|_{\mathbb{D}_G^p} + \|\eta^n - \eta^m\|_{\mathbb{M}_G^*} + \|Z^n - Z^m\|_{\mathcal{H}_G^p} \le C_p \Big[d_{G,p}(\xi_n, \xi_m) + \big(d_{G,p}(\xi_n, \xi_m)\big)^{\frac{1}{2}}\Big] \to 0.$$

Then there exist $(Y, Z, \eta) \in \mathbb{D}_G^p \times \mathcal{H}_G^p \times \mathbb{M}_G^*$ such that

$$\|Y^n - Y\|_{\mathbb{D}_G^p} + \|\eta^n - \eta\|_{\mathbb{M}_G^*} + \|Z^n - Z\|_{\mathcal{H}_G^p} \to 0, \quad \text{as } n \to \infty.$$

Moreover, for any $0 < \varepsilon \le c_0$, choose k large enough so that $\varepsilon_k < \varepsilon$. Then

$$\|\eta^n - \eta\|_{\mathcal{M}^1_{G_\varepsilon}} \le C_{\varepsilon,\varepsilon_k} \|\eta^n - \eta\|_{\mathbb{M}^1_{G_{\varepsilon_k}}} \le 2^k C_{\varepsilon,\varepsilon_k} \|\eta^n - \eta\|_{\mathbb{M}_G^*} \to 0, \quad \text{as } n \to \infty.$$

Thus

$$\int_0^t G(\eta_s^n) ds \to \int_0^t G(\eta_s) ds, \quad \mathcal{P}_\varepsilon\text{-q.s.}$$

Since (Y^n, Z^n, η^n) satisfy (2.19) $\mathcal{P}_\varepsilon$-q.s., then it is clear that (Y, Z, η) also satisfy (2.19) $\mathcal{P}_\varepsilon$-q.s. By the arbitrariness of ε we see that (Y, Z, η) also satisfy (2.19) $\mathcal{P}_0$-q.s.

Finally, the uniqueness of $(Z, \eta) \in \mathcal{H}_G^p \times \mathbb{M}_G^*$ follows from (4.8). ■

We conclude this paper by providing a nontrivial example of ξ which has the representation, but is not in $\mathcal{L}_{ip}$.

Example 4.4 *Let $d = 1$ and $B_t^* := \sup_{0\le s\le t} B_s$. Then $B_T^* \in \mathbb{L}_G^{*p} \backslash \mathcal{L}_{ip}$ for any $p > 1$.*

Proof. It is clear that $B^*_T \notin \mathcal{L}_{ip}$. We prove $B^*_T \in \mathbb{L}^{*p}_G$ in several steps.

Step 1. Assume $\xi : \Omega \to \mathbb{R}$ is uniformly Lipschitz continuous and convex in ω. We show that $\mathbb{E}^G[\xi] = \mathbb{E}^{\overline{\mathbb{P}}}[\xi]$, where $\overline{\mathbb{P}} := \mathbb{P}^{\overline{\sigma}}$.

Indeed, for any n, denote $t^n_i := \frac{iT}{n}$, $i = 0, \cdots, n$, $x_0 := 0$, and define

$$\begin{aligned} g_n(x_1, \cdots, x_n) &:= \xi\Big(\sum_{i=1}^n \frac{1}{t^n_i - t^n_{i-1}}[x_{i-1}(t^n_i - t) + x_i(t - t^n_{i-1})]\mathbf{1}_{(t^n_{i-1}, t^n_i]}(t)\Big); \\ \xi_n &:= g_n(B_{t^n_1}, \cdots, B_{t^n_n}). \end{aligned}$$

Since ξ is convex, clearly g_n is convex. Then $\mathbb{E}^G[\xi_n] = \mathbb{E}^{\overline{\mathbb{P}}}[\xi_n]$. Since ξ is uniformly Lipschitz continuous, then

$$|\xi_n - \xi| \le C \max_{1\le i\le n} \sup_{t^n_{i-1}\le t\le t^n_i} |B_t - B_{t^n_i}|.$$

This implies that $\mathbb{E}^G[|\xi_n - \xi| \to 0$ and $\mathbb{E}^{\overline{\mathbb{P}}}[|\xi_n - \xi|] \to 0$ as $n \to \infty$, and therefore, $\mathbb{E}^G[\xi] = \mathbb{E}^{\overline{\mathbb{P}}}[\xi]$.

Step 2. For simplicity, we assume $\overline{\sigma} = 1$, and thus $\overline{\mathbb{P}} = \mathbb{P}_0$. Note that $\xi := B^*_T$ is uniformly Lipschitz continuous and convex in ω. Then by adapting Step 1 to conditional G-expectations we have

$$Y_t := \mathbb{E}^G_t[\xi] = \mathbb{E}^{\mathbb{P}_0}_t[B^*_T] = u(t, B_t, B^*_t),$$

where, for $x \le y$,

$$u(t, x, y) := \mathbb{E}^{\mathbb{P}_0}\Big[y \vee [x + \sup_{t\le s\le T} B^t_s]\Big] = \mathbb{E}^{\mathbb{P}_0}\Big[y \vee [x + B^*_{T-t}]\Big].$$

Note that, under $\mathbb{P}_0$, B^*_{T-t} has the same distribution as $|B_{T-t}|$. Then

$$\begin{aligned} u(t, x, y) &= \sqrt{\frac{2}{\pi}} \int_0^\infty y \vee (x + \sqrt{T-t}z)e^{-\frac{z^2}{2}} dz \\ &= \sqrt{\frac{2}{\pi}} \int_0^{\frac{y-x}{\sqrt{T-t}}} y e^{-\frac{z^2}{2}} dz + \sqrt{\frac{2}{\pi}} \int_{\frac{y-x}{\sqrt{T-t}}}^\infty (x + \sqrt{T-t}z)e^{-\frac{z^2}{2}} dz. \end{aligned}$$

For $t \in [0, T)$ and $x < y$, we have

$$\begin{aligned} &\partial_t u(t, x, y) = -\frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{(y-x)^2}{2(T-t)}}; \quad \partial_y u(t, x, y) = \sqrt{\frac{2}{\pi}} \int_0^{\frac{y-x}{\sqrt{T-t}}} e^{-\frac{z^2}{2}} dz; \\ &\partial_x u(t, x, y) = \sqrt{\frac{2}{\pi}} \int_{\frac{y-x}{\sqrt{T-t}}}^\infty e^{-\frac{z^2}{2}} dz; \quad \partial_{xx} u(t, x, y) = \sqrt{\frac{2}{\pi(T-t)}} e^{-\frac{(y-x)^2}{2(T-t)}} > 0. \end{aligned} \tag{4.9}$$

Then

$$\partial_t u + \frac{1}{2} G(\partial_{xx} u) = \partial_t u + \frac{1}{2}\partial_{xx} u = 0, \quad \text{and} \quad \partial_y u(t, y, y) = 0.$$

Note that dB^*_t has support on $\{t : B^*_t = B_t\}$. Then by Itô's formula we have

$$
\begin{aligned}
& dY_t = du(t, B_t, B^*_t) \\
= \ & \partial_t u(t, B_t, B^*_t)dt + \partial_x u(t, B_t, B^*_t)dB_t + \partial_y u(t, B_t, B^*_t)dB^*_t + \frac{1}{2}\partial_{xx}u(t, B_t, B^*_t)d\langle B\rangle_t \\
= \ & \partial_x u(t, B_t, B^*_t)dB_t - G(\partial_{xx}u(t, B_t, B^*_t))dt + \frac{1}{2}\partial_{xx}u(t, B_t, B^*_t)d\langle B\rangle_t.
\end{aligned}
$$

Thus we obtain the representation with

$$
Z_t = \partial_x u(t, B_t, B^*_t), \quad \eta_t = \partial_{xx}u(t, B_t, B^*_t). \tag{4.10}
$$

Step 3. By Lemma 4.2, it remains to show that $(Z, \eta) \in \mathcal{H}^p_G \times \mathcal{M}^p_G$. For any n, denote

$$
Z^n_t := Z_t \mathbf{1}_{[0,T-\frac{1}{n}]}, \quad \eta^n_t := \eta_t \mathbf{1}_{[0,T-\frac{1}{n}]}.
$$

Note that, in the interval $[0, T - \frac{1}{n}]$, $\partial_x u$ and $\partial_{xx}u$ are bounded and uniformly Lipschitz continuous in (t, x, y), then clearly $(Z^n, \eta^n) \in \mathcal{H}^p_G \times \mathcal{M}^p_G$. Moreover, by (4.9) we have $|\partial_x u(t, x, y)| \leq 1$ and $|\partial_{xx}u(t, x, y)| \leq \frac{C}{\sqrt{T-t}}$. Then, as $n \to \infty$,

$$
\begin{aligned}
& \mathbb{E}^G\Big[\Big(\int_0^T |Z_t - Z^n_t|^2 d\langle B\rangle_t\Big)^{\frac{p}{2}}\Big] = \mathbb{E}^G\Big[\Big(\int_{T-\frac{1}{n}}^T |Z_t|^2 d\langle B\rangle_t\Big)^{\frac{p}{2}}\Big] \\
\leq \ & \mathbb{E}^G\Big[\Big(\langle B\rangle_T - \langle B\rangle_{T-\frac{1}{n}}\Big)^{\frac{p}{2}}\Big] = \frac{C_p}{n^{\frac{p}{2}}} \to 0; \\
& \mathbb{E}^G\Big[\Big(\int_0^T |\eta_t - \eta^n_t| dt\Big)^p\Big] = \mathbb{E}^G\Big[\Big(\int_{T-\frac{1}{n}}^T |\eta_t| dt\Big)^p\Big] \\
\leq \ & C\mathbb{E}^G\Big[\Big(\int_{T-\frac{1}{n}}^T \frac{dt}{\sqrt{T-t}}\Big)^p\Big] = \frac{C_p}{n^{\frac{p}{2}}} \to 0;
\end{aligned}
$$

This proves that $(Z, \eta) \in \mathcal{H}^p_G \times \mathcal{M}^p_G$ and completes the proof. ■

References

[1] Denis, L., Hu, M. and Peng S. (2011) *Function spaces and capacity related to a sublinear expectation: application to G-Brownian motion pathes*, Potential Analysis, 34 (2), 139-161.

[2] Denis, L. and Martini, (2006) *A Theorectical Framework for the Pricing of Contingent Claims in the Presence of Model Uncertainty*, Annals of Applied Probability, 16 (2), 827-852.

[3] Hamadene, S. and Popier, A. (2008) *L^p-Solutions for Reected Backward Stochastic Differential Equations*, arXiv:0807.1846.

[4] Hu, Y. and Peng, S. (2010) *Some Estimates for Martingale Representation under G-Expectation*, arXiv:1004.1098.

[5] Peng, S. (2004) *Filtration consistent nonlinear expectations and evaluations of contingent claims*, Acta Mathematicae Applicatae Sinica, 20(2) 1.

[6] Peng, S. (2006) *G-Expectation, G-Brownian motion and related stochastic calculus of Itô's type*, Stochastic Analysis and Applications, The Abel Symposium 2005, Abel Symposia, Edit. Benth et. al., 5417, Springer-Verlag.

[7] Peng, S. (2010) *Nonlinear Expectations and Stochastic Calculus under Uncertainty*, arXiv:1002.4546.

[8] Peng, S. (2010) *Backward Stochastic Differential Equation, Nonlinear Expectation and Their Applications*, Proceedings of the International Congress of Mathematicians 2010, 393-432.

[9] Pham, T. and Zhang, J. (2011) *Some Norm Estimates for Semimartingales – Under Linear and Nonlinear Expectations*, arXiv:1107.4020.

[10] Soner, M., Touzi, N., and Zhang, J. (2011) *Martingale Representation Theorem under G-expectation*, Stochastic Processes and their Applications, 121, 265-287.

[11] Soner, M., Touzi, N., and Zhang, J. (2011) *The wellposedness of second order backward SDEs*, Probability Theory and Related Fields, to appear.

[12] Song, Y.(2011) *Some properties on G-evaluation and its applications to G-martingale decomposition.* Science China Mathematics, 54 (2), 287-300.

[13] Song, Y.(2010) *Uniqueness of the representation for G-martingales with finite variation.* arXiv:1012.1913v2.

[14] Xu, J. and Zhang B. (2009), *Martingale characterization of G-Brownian motion*, Stochastic Processes and their Applications, 119 (1), 232-248.