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NONLINEAR FILTERING THEORY FOR MCKEAN–VLASOV TYPE STOCHASTIC DIFFERENTIAL EQUATIONS*

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Abstract. We consider estimation theory for partially observed stochastic dynamical systems whose state equations are given by McKean–Vlasov type stochastic differential equations and hence contain a measure term corresponding to the distribution of the solution of the state process. Nonlinear filtering equations are derived in this framework based upon the classification that the measure term is either stochastic or deterministic and that either the state or the measure term is estimated. When the measure term is deterministic the standard theory holds. Further, when the measure is random, the induced functions in the dynamics of the state become random and a similar recursion for the optimal filter is obtained. The joint estimation of state and the measure term is next considered. The extended state in this setup is shown to be a Banach space valued stochastic process with random functions in its state dynamics and a nonlinear filtering equation for this setup is provided. The first step of such an analysis requires an Itô's lemma for Banach space valued stochastic processes with the given dynamics. This work is motivated by state estimation problems in mean field game theory for systems where both major (asymptotically nonnegligible in population size) agents and minor (asymptotically negligible) agents are present.

Key words. nonlinear filtering theory, Duncan–Mortensen–Zakai equation, stochastic partial differential equations, stochastic McKean–Vlasov equations

AMS subject classifications. 60G35, 93E11, 60H15, 35Q83, 35Q84

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1. Introduction.

We consider state estimation problems for stochastic dynamical systems whose state dynamics are given by McKean–Vlasov (MV) type stochastic differential equations (SDEs). This class of SDEs has the property that the drift and the diffusion coefficients of the state equation depend on the law of the state random variable. MV SDEs as well as their variants have been extensively studied and a rigorous treatment of the existence and uniqueness results for them is available in the literature; see, e.g., [34]. To the best of our knowledge, state estimation problems for MV SDEs have not been studied in the literature and consequently we first define a set of such problems in a hierarchical manner and then present some preliminary results in the form of nonlinear filtering equations.

The theory of nonlinear filtering has been developed over several decades for a large class of systems and the corresponding literature is vast; the reader is referred to [9] for a comprehensive exposition of the subject for continuous time stochastic processes as well as a broad set of applications. Two key developments in the theory of nonlinear filtering for continuous time stochastic processes are found in [12], [29], [37], [38], and [23], [24], respectively; in the first group stochastic partial differential equations (known as the Duncan–Mortensen–Zakai (DMZ) equations) are derived for the unnormalized posterior distribution and density of the Markov state

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process, while in the latter set of papers the optimal filtering equations (known as the Kushner–Stratanovich equations) are given in normalized form. Since then the theory has played an important role in many areas including stochastic control, financial modeling, speech and image processing, and Bayesian networks. Nonetheless, the theory is still being actively developed and, among many contributions, [14] discusses recent results on filter stability and [28] considers interacting particle systems, mean field theory, and its applications to nonlinear filtering.

The origin of MV type systems goes back to the analysis of interacting particles in mathemetical physics whose so-called mean field type interaction is shown to be modeled by such a nonlinear process. The theoretical analysis and application of such dynamics include the existence and uniqueness of solutions as well as connections to research areas such as mean field theory, stochastic control, and mathematical physics. MV type SDEs also appear in the recently developed theory of large population noncooperative dynamic games with mean field couplings. For such a class of game, Nash certainty equivalence theory was developed in a series of papers (see [16], [18], [19], and [17], among others) by Huang, Caines, and Malhamé, while a closely related approach for such problems was independently developed by Lasry and Lions in [25], [26], and [27], where the term mean field game (MFG) was introduced. For a summary of this work and some recent developments in MFG theory see [6]. In addition to MFG theory, MV type SDEs have some other applications; for instance in mathematical finance (see, e.g., [4]), social media models, and so on.

Recent works ([15] and [31]) consider MFGs involving a major agent and many minor agents (MM-MFG), where, by definition, a major agent has asymptotically nonvanishing influence on each minor agent as the population size goes to infinity. A fundamental feature of this setup is that, in contrast to the situation without a major agent, the mean field is stochastic due to the stochastic evolution of the state of the major agent. Therefore, the control of partially observed MM-MFG systems entails in principle the estimation of the major agent's state, the estimation of the generic minor agent's state, which has stochastic coefficient MV (SMV) dynamics in the infinite population limit, and, in some cases, the stochastic measure of the generic minor agent. However, one should notice that there is a fundamental difference between the state dynamics for the major and for the generic minor agent in the infinite population case: whereas the solution of the generic minor agent's SMV state dynamics is required to be a consistent solution in the MV sense, the major agent's state dynamics do not need to satisfy this requirement. We discuss this distinction as well as its consequences for the nonlinear filtering equations in more detail in section 2. Hence, nonlinear filtering theory for MV and SMV type systems has direct applications in the estimation problems of nonlinear MM-MFG theory and it is this which motivates one to study nonlinear filtering theory for systems with SMV dynamics (see [7], [8] for the linear MM-MFG and [11], [10] for initial investigations on the nonlinear case).

The organization of the paper is as follows. In section 2 we define MV and SMV type SDEs and state the fundamental results on the existence and uniqueness analysis of such systems. In section 3 we discuss several partially observed stochastic dynamical systems with MV dynamics and present associated nonlinear filtering equations for the conditional expectations, or conditional laws, in both normalized and unnormalized forms. These results are presented in the following hierarchy: first, we present state estimation for MV dynamics and state estimation for SMV dynamics in sections 3.1 and 3.2, respectively; then in sections 3.3 and 4.1 we discuss joint state and measure estimation for MV and SMV type systems, respectively. We conclude the paper with section 5.

Throughout the paper we use the following notation. For a matrix A, A^T, $\mathbf{tr}(A)$, and A_{ij} denote the transpose, the trace, and the corresponding entry, respectively. ∇_x and ∇^2_{xx} denote the gradient and Hessian operators with respect to the variable x, while in a one-dimensional domain ∂_x and ∂^2_{xx} will be used instead. Let $\mathbb{S}$ be a metric space. Then, $\mathcal{B}(\mathbb{S})$ denotes the Borel σ-algebra and $\mathcal{P}(\mathbb{S})$ denotes the space of probability measures on $\mathbb{S}$. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with an increasing filtration $\{\mathcal{F}_t\}$. All filtrations defined in the paper are augmented by all the P-null sets in $\mathcal{F}$. Conditional expectation with respect to a sigma algebra $\mathcal{G}$ is denoted by $\mathbb{E}\left(\cdot|\mathcal{G}\right)$. All SDEs in the paper are of Itô type. In the rest of the paper, we do not necessarily display the dependence on an underlying probability space, that is to say, we may drop the ω term in a set of variables of interest in order to simplify notation.

2. McKean–Vlasov processes. We now give a precise definition for a class of MV processes within the framework of SDEs. Let a state process $z(t)$ satisfy the following SDE:

$$dz(t,\omega) = f\left[t, z(t), \mu_t\right] dt + dw(t,\omega),\ z(0) = z_0,\ 0 \leq t \leq T, \tag{2.1}$$

where $z : [0,T]\times\Omega \to \mathbb{R}^n$, $f : [0,T]\times\mathbb{R}^n\times\mathcal{P}(\mathbb{R}^n) \to \mathbb{R}^n$, $(w(t))_{t\geq 0}$ is a standard Brownian motion in $\mathbb{R}^n$, z_0 is $\mathcal{F}^w_0$-adapted, where $\mathcal{F}^w_t = \sigma\{w(s), 0 \leq s \leq t\}$, and μ_t is a flow of measures on $\mathbb{R}^n$. Here $f\left[t, z(t,\omega), \mu_t\right] := \int_{\mathbb{R}^n} f\left(t, z(t,\omega), x\right) \mu_t(dx)$ and this notation is used in the rest of the paper. Such a system is said to have a consistent solution $(z(t), \mu_t)$ if $z(t)$ solves (2.1) with $\mu_t = \mathcal{L}\left(z(t)\right)$, where $\mathcal{L}(\cdot)$ denotes the law of a random variable. It is shown in [34] that if $f(\cdot)$ is bounded and Lipschitz a unique consistent solution exists. Since the dynamics in (2.1) involve the measure term it is not surprising that a principal proof method to establish the existence and uniqueness of a consistent solution to (2.1) depends upon a contraction argument on the space of probability measures.

The model above can be extended to the case where the measure term in (2.1) is itself random; more explicitly, the measure term is taken to be a conditional law of the state, conditioned on some other σ-algebra. Such an extension is essential in the analysis of MFG with a major agent and with nonlinear dynamics. More explicitly, let $\mu^{\omega_0}_t := \mathcal{L}\left(z(t)|\mathcal{F}^{w_0}_t\right)$, the conditional law of $z(t)$ given $\mathcal{F}^{w_0}_t = \sigma\{w_0(s), 0 \leq s \leq t\}$, where $(w_0(t))_{t\geq 0}$ is a Brownian motion independent of $(w(t))_{t\geq 0}$. Let the dynamics of the state process $z(t)$ be given by

$$dz(t,\omega) = f\left[t, z(t), \mu^{\omega_0}_t\right] dt + \sigma\left[t, z(t), \mu^{\omega_0}_t\right] dw(t) \tag{2.2}$$

with $\mathcal{F}^w_0$-adapted initial condition $z(0)$ and $\mathcal{F}^{w_0}_0$-adapted $\mu^{\omega_0}_0$. Such dynamics are referred to as SMV SDEs and the existence and uniqueness of solutions of such SDEs is proved in [31, Theorems 6.4 and 6.7] via a fixed point argument with random parameters which is satisfied under Lipschitz and boundedness assumptions on $f(\cdot)$ and $\sigma(\cdot)$ and their derivatives and restricting the class of measures to be Hölder continuous.

In the rest of the paper, we consider both standard and stochastic coefficient MV SDEs while developing nonlinear filtering theory for such systems. Notice that depending on the application one might be interested in estimating both the measure term as well as the state process itself and, consequently, we also develop the theory for such joint estimation. We also remark that in the rest of the paper, the measure induced by a random variable and its corresponding distribution function may be used interchangeably, and hence the measure estimation under consideration should be understood to be equivalent to the estimation of its distribution function and vice versa.

3. Nonlinear filtering theory for systems with McKean-Vlasov dynamics. We consider systems with controlled dynamics and hence, the state processes are defined by the following SDE:

$$dz(t,\omega) = f\left[t, z(t), \varphi(t, z(t)), \mu_t\right] dt + \sigma\left[t, z(t), \mu_t\right] dw(t,\omega), \; 0 \le t \le T, \tag{3.1}$$

with initial condition $z(0)$, where $z : [0,T] \times \Omega \to \mathbb{R}^n$, $\varphi : [0,T] \times \mathbb{R}^n \to U$, U is a compact space where the control actions take values in, $f : [0,T] \times \mathbb{R}^n \times U \times \mathcal{P}(\mathbb{R}^n) \to \mathbb{R}^n$, $\sigma : [0,T] \times \mathbb{R}^n \times \mathcal{P}(\mathbb{R}^n) \to \mathbb{R}^{n\times m}$, and $(w(t))_{t\ge 0}$ is a standard Brownian motion in $\mathbb{R}^m$. As before,

$$\begin{aligned} f\left[t, z(t), \varphi(t, z(t)), \mu_t\right] &= \int_{\mathbb{R}^n} f\left(t, z(t), \varphi(t, z(t)), x\right) \mu_t(dx) \\ \sigma\left[t, z(t), \mu_t\right] &= \int_{\mathbb{R}^n} \sigma\left(t, z(t), x\right) \mu_t(dx) \end{aligned} \tag{3.2}$$

for some probability distribution μ_t on $\mathbb{R}^n$, where here and henceforth in the paper it is assumed that all measures possess densities. We recall that given the functions $(f(\cdot), \varphi(\cdot), \sigma(\cdot))$, the pair $(\mu_t, z(t))$ is said to be a consistent pair if $z(t)$ solves (3.1) and $P\left(z(t) \le \alpha\right) = \int_{-\infty}^{\alpha} \mu_t(dx)$ for all $\alpha \in \mathbb{R}^n$ and for all $t \ge 0$. In order to extend the above MV dynamics to the situation where the measure term is also random, let $\mathcal{F}_t^{w_0} = \sigma\{w_0(s), 0 \le s \le t\}$ and let $\hat{\mu}_t^{\omega_0}$ be the conditional law of $\hat{z}(t)$ given $\mathcal{F}_t^{w_0}$. Assume now that the dynamics of the state process $\hat{z}(t)$ are given by

$$d\hat{z}(t,\omega) = \hat{f}\left[t, \hat{z}(t), \varphi\left(t, \hat{z}(t)\right), \hat{\mu}_t^{\omega_0}\right] dt + \hat{\sigma}\left[t, \hat{z}(t), \hat{\mu}_t^{\omega_0}\right] dw(t,\omega) \tag{3.3}$$

with given initial conditions $\left(\hat{z}(0), \hat{\mu}_0^{\omega_0}\right)$. For both controlled MV and controlled SMV SDEs, a unique solution exists if, in addition to $f(t,x,u,y)$ and $\sigma(t,x,y)$, $\varphi(t,x)$ is also Lipschitz continuous in x as well as $\hat{\mu}_{(\cdot)}^{\omega_0}$ being Hölder continuous [31, Theorems 6.4–6.7].

In the special case where a twice continuously space-differentiable density exists, one can show that the terms μ_t and $\hat{\mu}_t^{\omega_0}$ satisfy a special form of the Fokker–Plank–Kolmogorov (FPK) equations. More explicitly, let $\hat{p}_t^{w_0}(x)$ denote the conditional density of $\hat{z}(t)$ given $\mathcal{F}_t^{w_0}$. Then one can show that $\hat{\mu}_t^{\omega_0}$ satisfies the following stochastic coefficient FPK (SFPK) equation:

$$\begin{aligned} \frac{\partial \hat{\mu}_t^{\omega_0}(x)}{\partial t} &= \int_{-\infty}^{x} -\langle \nabla_r, \hat{f}\left[t, r, \varphi\left(t, r\right), \hat{\mu}_t^{\omega_0}\right] \hat{p}_t^{w_0}(r)\rangle + \frac{1}{2}\mathbf{tr}\langle \nabla_{rr}^2, \hat{a}(t, r, \omega_0)\hat{p}_t^{w_0}(r)\rangle dr \\ &:= \mathcal{T}_{FPK}(t, \omega_0) \end{aligned} \tag{3.4}$$

in $[0,T] \times \mathbb{R}^n$, where $\hat{a}(t,x,\omega_0) := \hat{\sigma}\left[t, x, \hat{\mu}_t^{\omega_0}\right] \hat{\sigma}^T\left[t, x, \hat{\mu}_t^{\omega_0}\right]$ and with the initial value given by $\hat{\mu}_0^{\omega_0} \in \mathcal{P}\left(\mathbb{R}^n\right)$. It should be noted that such a model represents the dynamics of a generic minor agent in the MFG setup with major and minor agents [31]. Let us now consider the dynamics given by (3.3) where the measure is deterministic. Let $a(t,x) := \sigma\left[t, x, \mu_t\right] \sigma^T\left[t, x, \mu_t\right]$ and let $p_t(x)$ denote the density of $z(t)$. Then

$$\begin{aligned} \frac{\partial \mu_t(x)}{\partial t} &= \int_{-\infty}^{x} -\langle \nabla_r, f\left[t, r, \varphi\left(t, r\right), \mu_t\right] p_t(r)\rangle + \frac{1}{2}\mathbf{tr}\langle \nabla_{rr}^2, a(t, r) p_t(r)\rangle dr \\ &:= \mathcal{T}_{FPK}(t) \end{aligned} \tag{3.5}$$

in $[0,T] \times \mathbb{R}^n$ with the initial value $\mu_0 \in \mathcal{P}\left(\mathbb{R}^n\right)$.

Before proceeding further with the filtering equations, there is one final type of dynamics that requires attention. Consider

$$dz_0(t,\omega) = f_0\left[t, z_0(t), \varphi(t, z_0(t)), \mu_t^{w_0}\right] dt + \sigma_0\left[t, z_0(t), \mu_t^{\omega_0}\right] dw_0(t,\omega) \tag{3.6}$$

with initial condition $z_0(0)$, where $z_0 : [0,T] \times \Omega \to \mathbb{R}^n$, $\varphi : [0,T] \times \mathbb{R}^n \to U_0$, U_0 is compact space where the control actions take values in, $f_0 : [0,T] \times \mathbb{R}^n \times U \times \mathcal{P}(\mathbb{R}^n) \to \mathbb{R}^n$, $\sigma_0 : [0,T] \times \mathbb{R}^n \times \mathcal{P}(\mathbb{R}^n) \to \mathbb{R}^{n\times m}$, and $(w_0(t))_{t\geq 0}$ is a standard Brownian motion in $\mathbb{R}^m$. Furthermore, let $\mathcal{F}_t^{w_0} = \sigma\{w_0(s), 0 \leq s \leq t\}$ and let $\mu_t^{\omega_0}$ be $\mathcal{F}_t^{w_0}$-adapted. As before, we set

$$f_0\left[t, z_0(t), \varphi\left(t, z_0(t)\right), \mu_t^{w_0}\right] = \int_{\mathbb{R}^n} f_0\left(t, z_0(t), \varphi\left(t, z_0(t)\right), x\right) \mu_t^{\omega_0}(dx) \tag{3.7}$$

for some random probability distribution μ_t on $\mathbb{R}^n$. Recall that SMV type SDEs typically require that the pair $(\mu_t^{\omega_0}, z_0(t))$ be consistent; i.e., $z_0(t)$ solves (3.6) and $\mu_t^{\omega_0} := \mathcal{L}\left(z_0(t)|\mathcal{F}_t^{w_0}\right)$. However, in some scenarios, while the equality (3.7) holds, one does not require the condition that the $\mu_t^{\omega_0}$ be the conditional law of the state process. In particular, the major agent's state process in the infinite population nonlinear MM-MFG theory satisfies such a dynamics [31, equation (5.6)]. We note that nonlinear filtering theory for such a setup can be developed similarly to the one described by the dynamics given in (3.3); see section 3.2.

For these SDEs, we now define several estimation problems in a hierarchical manner and obtain nonlinear filtering equations.

3.1. State estimation for MV-SDE. Assume first that the state dynamics is given by (3.1) and the observation dynamics is given as follows:

$$dy(t) = h\left(t, z(t)\right) dt + d\nu(t), \tag{3.8}$$

where $h : [0,T] \times \mathbb{R}^n \to \mathbb{R}^d$ is a bounded measurable function and $(\nu(t))_{t\geq 0}$ is a standard Brownian motion in $\mathbb{R}^d$ and it is assumed throughout the paper that it is independent of $(w(t))_{t\geq 0}$ defined in (3.1) and of $z(0)$. The nonlinear filtering problem is now defined as follows: Given the history of observations $\mathcal{F}_t^y := \sigma\{y(s) : s \leq t\}$, determine a recursive expression for $\mathbb{E}[z(t)|\mathcal{F}_t^y]$ and equivalently for $\pi(t,\cdot) := P\left(z(t) \in \cdot|\mathcal{F}_t^y\right)$. Notice that the optimality of conditional expectation for the minimum square error among all $\mathcal{F}_t^y$-measurable square-integrable random variables is well known. Observe also that $\pi(t,\cdot) : \mathcal{B}\left(\mathbb{R}^n\right) \times \Omega \to [0,1]$ is a regular conditional distribution (i.e., $\pi(t,A,\omega') = \mathbb{E}_P[1_A(z(t))|\mathcal{F}_t^y]$) and hence has the following properties: (i) for every $\omega' \in \Omega$, $\pi(t,\cdot,\omega')$ is a probability measure on $\mathbb{R}^n$; (ii) for any $A \in \mathcal{B}(\mathbb{R}^n)$, $\pi(t,A,\cdot)$ is an $\mathcal{F}_t^y$-measurable random variable; and finally (iii) for any $A \in \mathcal{B}(\mathbb{R}^n)$, $\pi(t,A,\omega') = P\left(z(t) \in A|\mathcal{F}_t^y\right)(\omega')$ a.s. The main goal in this subsection is to obtain an SDE for $\pi(t,\cdot,\omega')$. We follow the standard steps of nonlinear filtering theory that can be found in, e.g., [20], [35] and we consider the estimation of a function of $z(t)$; i.e., for ℓ in a rich enough family of test functions, $\mathbb{E}_P\left[\ell\left(z(t)\right)|\mathcal{F}_t^y\right]$.

Notice first that for fixed $(\mu_t)_{0\leq t\leq T}$, $f[t,x,u,\mu_t]$, $u \in U$, and $\sigma[t,x,\mu_t]$ become functions of (t,x,u) and (t,x), respectively, and we shall set $f^*(t,x,u) := f[t,x,u,\mu_t]$ and $\sigma^*(t,x) := \sigma[t,x,\mu_t]$. As a result, the estimation problem reduces to a standard nonlinear filtering problem. Following the approach presented in [35], we first define the exponential martingale that we will use frequently throughout the paper and invoke it for the change of measure argument. Let $\int_0^t \langle h(s,z(s)), d\nu(s)\rangle := \sum_{j=1}^d \int_0^t h_j(s,z(s))\, d\nu_j(s)$ and

$$M(t)^{-1} := \exp\left(-\int_0^t \langle h(s,z(s)), d\nu(s)\rangle - \frac{1}{2}\int_0^t |h(s,z(s))|^2 ds\right). \tag{3.9}$$

Let $\mathcal{F}_t = \sigma\{w(s), \nu(s), 0 \le s \le t\}$ and observe that $M(t)^{-1}$ is an $\mathcal{F}_t$ martingale. Define the probability measure $\hat{P}$ which is absolutely continuous with respect to P and let the Radon–Nikodym derivative on $(\Omega, \mathcal{F}_t)$ be such that $\frac{d\hat{P}}{dP}|_{\mathcal{F}_t} = M(t)^{-1}$. Hence, by Girsanov's theorem [20, Theorem 7.1.3] $y(t)$ is a $\hat{P}$-Brownian motion. We remark that since the measure P is also absolutely continuous with respect to $\hat{P}$, the measures are mutually absolutely continuous and so, the symbol a.s. following equalities between conditional expectations and conditional probabilities will be omitted in the rest of the paper. To continue, by the Kallianpur–Striebel formula we have that for all $\ell \in C_b(\mathbb{R}^n)$

$$\mathbb{E}_P\left[\ell(z(t))\,|\mathcal{F}_t^y\right] = \frac{\mathbb{E}_{\hat{P}}\left[M(t)\ell(z(t))|\mathcal{F}_t^y\right]}{\mathbb{E}_{\hat{P}}\left[M(t)|\mathcal{F}_t^y\right]}. \tag{3.10}$$

In the rest of the paper, for each problem, we first obtain a recursive expression for the term $\mathbb{E}_{\hat{P}}\left[M(t)\ell(z(t))|\mathcal{F}_t^y\right]$, which is called the DMZ equation or unnormalized filter, and we then obtain a recursion for the optimal filter (i.e., for the expectation under the original measure P).

Following the standard theory, we first note that $y(t)$ is a $\hat{P}$-Brownian motion and hence by Itô's formula it can be shown that

$$dM(t) = M(t)h^T(t, z(t))\, dy(t), \tag{3.11}$$

where

$$M(t) = \exp\left(\int_0^t h^T(s,z(s))\, dy(s) - \frac{1}{2}\int_0^t |h(s,z(s))|^2 ds\right). \tag{3.12}$$

Let $\ell \in C_b^2(\mathbb{R}^n)$, the space of all bounded twice differentiable functions with bounded derivatives up to order two on $\mathbb{R}^n$ and set $T \in \mathbb{R}^{n\times n}$ as $T := \sigma^*(t,x)\,\sigma^{*T}(t,x)$, then the process $z(t)$ possesses the following infinitesimal generator:

$$\mathcal{L}(t)\ell := \frac{1}{2}\sum_{j=1}^n \sum_{l=1}^n T_{jl}\partial_{jl}^2 \ell + \sum_{j=1}^n f_j^*(t,x,u)\,\partial_j \ell. \tag{3.13}$$

We set the following assumption for both the existence and uniqueness of solutions to the signal and observation dynamics as well as for the filtering recursions:

(A1) The functions $f(t,x,u,y), \sigma(t,x,y)$ and $h(t,x)$ are assumed to be bounded and continuous in their parameters and Lipschitz continuous in (x,u,y).

One can show that these assumptions also imply that $f^*(t,x,u)$ and $\sigma^*(t,x)$ are also bounded and Lipschitz continuous, and once we have the induced functions f^* and σ^*, the proofs of the below theorems follow from standard steps; see, e.g., [20], [35].

THEOREM 3.1. *Under assumption* (A1)*, the unnormalized filter for $z(t)$ satisfies the following stochastic integral equation: For all $\ell \in C_b^2(\mathbb{R}^n)$*

$$\begin{aligned}\mathbb{E}_{\hat{P}}\left[M(t)\ell(z(t))|\mathcal{F}_t^y\right] &= \mathbb{E}_{\hat{P}}\left[M(0)\ell(z(0))|\mathcal{F}_0^y\right] + \int_0^t \mathbb{E}_{\hat{P}}\left[M(s)\mathcal{L}(s)\ell(z(s))|\mathcal{F}_s^y\right] ds \\ &\quad + \int_0^t \mathbb{E}_{\hat{P}}\left[M(s)\ell(z(s))\,h^T(s,z(s))\,|\mathcal{F}_s^y\right] dy(s)\end{aligned} \tag{3.14}$$

with initial conditional expectation given by $\pi(0,\cdot) \in \mathcal{P}(\mathbb{R}^n)$.

With the DMZ equation in hand, we can derive the optimal filtering equation for $\pi(t)$. Let us first define the following innovations process:

$$I(t) := y(t) - \int_0^t \mathbb{E}_P\left[h(s, z(s)) | \mathcal{F}_s^y\right] ds.$$

Then $I(t)$ is an $\mathcal{F}_t^y$-Brownian motion under the original measure P.

THEOREM 3.2. *Under assumption* (A1), $\pi(t)$ *satisfies the following stochastic integral equation: For all* $\ell \in C_b^2(\mathbb{R}^n)$,

$$\begin{aligned}(3.15)\qquad \mathbb{E}_P\left[\ell\left(z(t)\right)|\mathcal{F}_t^y\right] &= \mathbb{E}_P\left[\ell\left(z(0)\right)|\mathcal{F}_0^y\right] + \int_0^t \mathbb{E}_P\left[\mathcal{L}(s)\ell\left(z(s)\right)|\mathcal{F}_s^y\right] ds \\ &+ \int_0^t \mathbb{E}_P\left[\ell\left(z(s)\right)h^T\left(s, z(s)\right)|\mathcal{F}_s^y\right] - \mathbb{E}_P\left[\ell\left(z(s)\right)|\mathcal{F}_s^y\right]\mathbb{E}_P\left[h^T\left(s, z(s)\right)|\mathcal{F}_s^y\right] dI(s)\end{aligned}$$

with initial conditional distribution $\pi(0,\cdot) \in \mathcal{P}(\mathbb{R}^n)$.

3.2. State estimation for SMV SDE. We now generalize the above result to the case where the measure term itself is random but observed. In particular, the state dynamics are now given by (3.3) and we are interested in estimating the state process $(\hat{z}(t))_{t\geq 0}$. Hence, let the observation dynamics be given as follows:

$$(3.16)\qquad d\hat{y}(t) = h\left(t, \hat{z}(t)\right) dt + d\nu(t),$$

where as before $h : [0,T] \times \mathbb{R}^n \to \mathbb{R}^d$ and $(\nu(t))_{t\geq 0}$ is a d-dimensional standard Brownian motion and it is assumed throughout that it is independent of $(w(t))_{t\geq 0}$ and $(w_0(t))_{t\geq 0}$ given in (3.3) and of the initial condition $\hat{z}(0)$. We remark that the observation process at time t, $y(t)$, has dependence on the three different Brownian sample paths and therefore, in order to further emphasize the dependence on the underlying sample paths, one can use, by abuse of notation, the following representation. Let us denote the conditional law of $z(t)$ given $\mathcal{F}_t^{w_0}$ by $\mu_t^{\omega_0}$ and denote the dynamics of the state process $z(t)$ by

$$(3.17)\qquad dz(t,\omega,\omega_0) = f\left[t, z(t), \mu_t^{\omega_0}\right] dt + \sigma\left[t, z(t,\omega), \mu_t^{\omega_0}\right] dw(t,\omega)$$

with $\mathcal{F}_0^w$-adapted initial condition $z(0,\omega)$ and $\mathcal{F}_0^{w_0}$-adapted $\mu_0^{\omega_0}$. We can then denote the observation process by

$$(3.18)\qquad d\hat{y}(t,\omega,\omega_0,\omega') = h\left(t, \hat{z}(t,\omega,\omega_0)\right) dt + d\nu(t,\omega'),$$

where $(\omega,\omega_0,\omega') \in \Omega\times\Omega\times\Omega$. However, in order to simplify the notation we usually drop the dependence on the ω terms.

To continue, consider now the dynamics of $\hat{z}(t)$. For a fixed measure flow we have

$$\begin{aligned}&\hat{f}[t, \hat{z}(t), \varphi\left(t, \hat{z}(t)\right), \hat{\mu}_t^{\omega_0}]dt + \hat{\sigma}[t, \hat{z}, \hat{\mu}_t^{\omega_0}]dw(t) \\ &\quad= \int_{\mathbb{R}^n} \hat{f}\left(t, \hat{z}, \varphi\left(t, \hat{z}(t)\right), x\right)\hat{\mu}_t^{\omega_0}(dx)dt + \int_{\mathbb{R}^n} \hat{\sigma}(t, \hat{z}, x)\hat{\mu}_t^{\omega_0}(dx)dw(t) \\ (3.19)\qquad &\quad:= \hat{f}^*\left(t, \hat{z}, \varphi(t,\hat{z}), \omega_0\right) dt + \hat{\sigma}^*(t, \hat{z}, \omega_0)dw(t),\end{aligned}$$

where $\hat{f}^* : [0,T]\times\mathbb{R}^n\times U\times\Omega \to \mathbb{R}^n$ and $\hat{\sigma}^* : [0,T]\times\mathbb{R}^n\times\Omega \to \mathbb{R}^{n\times m}$ are now random functions. Hence, the filtering problem reduces to the estimation of the state process whose dynamics are now driven by random functions. It should now be observed that the randomness of the function in the dynamics does not affect the nonlinear filtering analysis (see below for a more detailed discussion) and hence we obtain the following. Let $\mathcal{F}_t^{\hat{y}} := \sigma\{\hat{y}(s) : s \leq t\}$, $\hat{\pi}(t,\cdot) := P(\hat{z}(t) \in \cdot|\mathcal{F}_t^{\hat{y}})$ and define $M(t)^{-1}$ as in (3.9) with $\hat{z}(s)$ replacing $z(s)$ and setting $\mathcal{F}_t = \sigma\{w(s), \nu(s), w_0(s), 0 \leq s \leq t\}$. We next observe that $M(t)^{-1}$ is an $\mathcal{F}_t$ martingale and define $\hat{P}$ accordingly. Finally we let $T := \hat{\sigma}^* (t, x, \omega_0) \hat{\sigma}^{*T} (t, x, \omega_0)$ and for $\ell \in C^2 (\mathbb{R}^n)$ define the following operator on $\hat{z}(t)$:

$$\hat{\mathcal{L}}(t,\omega_0)\ell := \frac{1}{2}\sum_{j=1}^{n}\sum_{l=1}^{n} T_{jl}\partial^2_{jl}\ell + \sum_{j=1}^{n} \hat{f}^*_j (t, x, u, \omega_0)\, \partial_j \ell. \tag{3.20}$$

Assume the following standard conditions to hold:

(A2) The functions $\hat{f}(t, x, u, y), \hat{\sigma}(t, x, y)$, and $h(t, x)$ are assumed to be bounded and continuous with respect to all their parameters and Lipschitz continuous in (x, u, y).

We remark that by (A2) a consistent solution exists, and furthermore, the induced random functions $\hat{f}^*$ and $\hat{\sigma}^*$ satisfy Lipschitz continuity and boundedness a.s. when the measure is Hölder continuous [31, Proposition 6.2].

Theorem 3.3. *Under assumption* (A2)*, the unnormalized filter for $\hat{z}(t)$ satisfies the following stochastic integral equation:*

$$\begin{aligned}\mathbb{E}_{\hat{P}}\left[M(t)\ell(\hat{z}(t))|\mathcal{F}_t^{\hat{y}}\right] &= \mathbb{E}_{\hat{P}}\left[M(0)\ell(\hat{z}(0))|\mathcal{F}_0^{\hat{y}}\right] + \int_0^t \mathbb{E}_{\hat{P}}\left[M(s)\hat{\mathcal{L}}(s,\omega_0)\ell(\hat{z}(s))|\mathcal{F}_s^{\hat{y}}\right] ds\\ &\quad + \int_0^t \mathbb{E}_{\hat{P}}\left[M(s)\ell\left(\hat{z}(s)\right) h^T\left(s, \hat{z}(s)\right)|\mathcal{F}_s^{\hat{y}}\right] d\hat{y}(s)\end{aligned}$$

with initial conditional distribution $\hat{\pi}(0,\cdot) \in \mathcal{P}(\mathbb{R}^n)$.

Now, define the innovations process as $\hat{I}(t) := \hat{y}(t) - \int_0^t \mathbb{E}_P[h(s, \hat{z}(s))|\mathcal{F}_s^{\hat{y}}]ds$, which is an $\mathcal{F}_t^{\hat{y}}$-Brownian motion under the original measure P.

Theorem 3.4. *Under assumption* (A2)*, $\hat{\pi}(t)$ satisfies the following stochastic integral equation: For all $\ell \in C_b^2(\mathbb{R}^n)$,*

$$\begin{aligned}\mathbb{E}_P\left[\ell\left(\hat{z}(t)\right)|\mathcal{F}_s^{\hat{y}}\right] &= \mathbb{E}_P\left[\ell\left(\hat{z}(0)\right)|\mathcal{F}_0^{\hat{y}}\right] + \int_0^t \mathbb{E}_P\left[\hat{\mathcal{L}}(s,\omega_0)\ell\left(\hat{z}(s)\right)|\mathcal{F}_s^{\hat{y}}\right] ds\\ &+ \int_0^t \left(\mathbb{E}_P\left[\ell\left(\hat{z}(s)\right) h^T\left(s, \hat{z}(s)\right)|\mathcal{F}_s^{\hat{y}}\right] - \mathbb{E}_P\left[\ell\left(\hat{z}(s)\right)|\mathcal{F}_s^{\hat{y}}\right]\mathbb{E}_P\left[h^T\left(s, \hat{z}(s)\right)|\mathcal{F}_s^{\hat{y}}\right]\right) d\hat{I}(s)\end{aligned}$$

with initial conditional distribution $\hat{\pi}(0,\cdot) \in \mathcal{P}(\mathbb{R}^n)$.

The proofs of Theorems 3.1–3.4 are not provided as they can be obtained from the proofs of Theorems 4.2 and 4.3, which are given below. Nonetheless, we provide an argument to show why the randomness in the system dynamics does not affect the analysis. Let for all $t \in [0, T]$, $C_t([0, T]; \mathbb{R}^m) := \{x(\cdot \wedge t) : x(\cdot) \in C([0, T]; \mathbb{R}^m)\}$, where $a \wedge b := \min(a, b)$, and set

$$\mathcal{B}_{t+}\left(C\left([0, T]; \mathbb{R}^m\right)\right) := \bigcap_{s>t} \sigma\big(\mathcal{B}\left(C_s\left([0, T]; \mathbb{R}^m\right)\right)\big) \quad \forall t \in [0, T). \tag{3.21}$$

For any Polish space $\mathbb{S}$, let $\mathcal{A}_T^m(\mathbb{S})$ denote the set of all $\{\mathcal{B}_{t+}(C([0, T]; \mathbb{R}^m))\}_{t\geq 0}$-progressively measurable processes $\eta : [0, T] \times C([0, T]; \mathbb{R}^m) \to \mathbb{S}$. Consider now the random measure $\hat{\mu}^{\omega_0} : [0,T] \times \Omega \to \mathcal{P}(\mathbb{R}^n)$. By [36, Theorem 2.10], there exists $\eta \in \mathcal{A}_T^m(\mathcal{P}(\mathbb{R}^n))$ such that $\hat{\mu}_t^{\omega_0} = \eta(t, w_0(\cdot \wedge t, \omega_0))$ and hence

(3.22)
$$\begin{aligned}\hat{f}[t, \hat{z}(t), \varphi(t, \hat{z}(t)), \hat{\mu}_t^{\omega_0}] &= \hat{f}[t, \hat{z}(t), \varphi(t, \hat{z}(t)), \eta(t, w_0(\cdot \wedge t, \omega_0))],\\ \hat{\sigma}[t, \hat{z}(t), \hat{\mu}_t^{\omega_0}] &= \hat{\sigma}[t, \hat{z}(t), \eta(t, w_0(\cdot \wedge t, \omega_0))],\ P\text{-a.s. } \omega_0 \in \Omega, \forall \mathrm{t} \in [0, \mathrm{T}].\end{aligned}$$

Define now the state process $\hat{z}_0(t) = \big(\hat{z}(t), z_0(t)\big)$ by

(3.23)
$$\begin{aligned}d\hat{z}(t) &= \hat{f}[t, \hat{z}(t), \varphi(t, \hat{z}(t)), \eta(t, z_0(\cdot \wedge t, \omega_0))]dt + \hat{\sigma}[t, \hat{z}(t), \eta(t, z_0(\cdot \wedge t, \omega_0))]dw(t),\\ dz_0(t, \omega_0) &= dw_0(t, \omega_0).\end{aligned}$$

It is now clear that the dynamics in (3.23) are no more general than those considered in the literature; see section 3.1 and also [9, Part I, Chapter 3]. Notice also that one can reach the same conclusion by observing that there exist $\eta : [0,T] \times \mathbb{R}^n \times U \times C([0,T]; \mathbb{R}^m) \to \mathbb{R}^n$ and $\varsigma : [0,T] \times \mathbb{R}^n \times C([0,T]; \mathbb{R}^m) \to \mathbb{R}^{n\times m}$ such that

(3.24)
$$\begin{aligned}\hat{f}^*(t, \hat{z}(t), \varphi(t, \hat{z}(t)), \omega_0) &= \eta\big(t, \hat{z}(t), \varphi(t, \hat{z}(t)), w_0(\cdot \wedge t, \omega_0)\big)\\ \hat{\sigma}^*(t, \hat{z}(t), \omega_0) &= \varsigma\big(t, \hat{z}(t), w_0(\cdot \wedge t, \omega_0)\big),\ P\text{-a.s. } \omega_0 \in \Omega, \forall \mathrm{t} \in [0, \mathrm{T}],\end{aligned}$$

and by defining $\hat{z}_0(t) = \big(\hat{z}(t), z_0(t)\big)$ similar to (3.23) with $\hat{z}(t)$ coefficients given in (3.24).

In this regard, it is also worthwhile discussing the Markov modeling of the signal and observation process. Consider first the situation where a consistent solution to an SMV system exist. Subject to technical conditions we can then write

(3.25)
$$\begin{aligned}\frac{d}{dt}\hat{p}_t^{\omega_0}(x) =& -\left\langle \nabla_x, \int_{\mathbb{R}^n} \hat{f}[t, x, \psi(t, x), \hat{\mu}_t^{\omega_0}]\, \hat{p}_t^{\omega_0}(x) \right\rangle\\ &+ \frac{1}{2}\mathbf{tr}\langle \nabla_{xx}^2, \hat{a}(t, x, \omega_0) p_t^{\omega_0}(x)\rangle,\end{aligned}$$

where $\hat{p}_t^{\omega_0}$ is the conditional density of $\hat{z}(t)$ conditioned on $\mathcal{F}_t^{w_0}$ and for $(t,x) \in [0,T] \times \mathbb{R}^n$, $\hat{a}(t,x,\omega) := \hat{\sigma}[t, x, \mu_t^{\omega_0}]\ \hat{\sigma}^T[t, x, \mu_t^{\omega_0}]$. It should now be noted that as in section 4, such an SFPK equation can be considered as a PDE with random coefficients where the right-hand side depends only on $\hat{p}_t^{\omega_0}(x)$ at time t. Consequently, we may define the observation dynamics

$$\begin{aligned}d\hat{y}_1(t) &= h(t, \hat{z}(t))\, dt + d\nu(t),\\ d\hat{y}_2(t) &= \hat{\mu}_t^{\omega_0}(x) dt,\end{aligned} \tag{3.26}$$

and one can derive the filtering equations for $\mathbb{E}_{\hat{P}}[\ell(\hat{z}(t)) | \mathcal{F}_t^y]$, where $\mathcal{F}_t^y := \vee_{i=1,2} \mathcal{F}_t^{\hat{y}_i}$.

Consider also the situation where the density process, $\hat{p}_{(\cdot)}^{\omega_0}$ satisfies SFPK and takes values in L^2, a.s., with $\mathbb{E}\|\hat{p}_t^{\omega_0}\|_{L^2} < \infty$. This can be satisfied under additional conditions on the initial density. Assume further that the solution to the SFPK is an $\mathcal{F}_t^{w_0}$-martingale. Therefore, by the martingale representation theorem (for infinite dimensional stochastic processes, see, e.g., [21]), there exists a $\Gamma_{(\cdot)}$ such that $d\hat{p}_t^{\omega_0}(x) =$

$\Gamma_t(x)dw_0(t)$ which implies $d\hat{\mu}_t^{\omega_0}(x) = \hat{\Gamma}_t(x)dw_0(t)$ where $\hat{\Gamma}_t(x) := \int_0^x \Gamma_t(r)dr$ which holds due to the Fubini's theorem. The augmented system given by $(\hat{z}(t,\omega), \hat{\mu}_t^{\omega_0}(x))$ is Markovian on the space $(\mathbb{R}^n \times C_b(\mathbb{R}^n))$. Consequently, on this Markovian system, we can likewise define $\big(\hat{y}_1(\cdot), \hat{y}_2(\cdot)\big)$ as in (3.26) and derive the filtering equations for the conditional expectation $\mathbb{E}_{\hat{P}}\left[\ell(\hat{z}(t))|\mathcal{F}_t^y\right]$, where $\mathcal{F}_t^y := \vee_{i=1,2}\mathcal{F}_t^{\hat{y}_i}$.

We next remark that in the case the dynamics of $\hat{\mu}_t^{\omega_0} = P\left(\hat{z}(t)|\mathcal{F}_t^{w_0}\right)$ is not provided, or one does not have access to a Markov structure, the model can be refined by taking $\hat{\mu}_t^{\omega_0} = P\left(\hat{z}(t)|w_0(t)\right)$, which will also yield Markovian dynamics. Finally, in such a situation, or in the other state augmented situations, one can take the test function in the form $\ell(x, y_2) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ rather than $\ell(x)$.

3.3. State and measure estimation for MV SDEs. In the third class of problems, we consider the dynamics given in section 3.1, i.e., the measure term is taken to be deterministic, and discuss the joint estimation of the state and the measure. In this case we have the state dynamics given by (3.1) and the observation dynamics by

$$dv(t) = g\left(t, z(t), \mu_t\right)dt + d\nu(t), \tag{3.27}$$

where $g := [0,T] \times \mathbb{R}^n \times \mathcal{P}(\mathbb{R}^n) \to \mathbb{R}^d$, and we are interested in estimating $\big(z(t), \mu_t\big)$, where μ_t satisfies (3.5), and $\nu(t)$ is a standard Brownian motion in $\mathbb{R}^d$ which is independent of $w(t)$. Notice that since the measure term is deterministic, we have $P\left(z(t), \mu_t|\mathcal{F}_t^v\right) = \delta_{\mu_t}P\left(z(t)|\mathcal{F}_t^v\right)$. Hence, the filtering problem for this setup is no more general than the one defined via (3.1) and (3.8).

4. Nonlinear filtering for state and stochastic measure estimation. In this section we consider the joint estimation of the state process and a stochastic measure flow and we first present some preliminary material concerning the metric on the space of probability measures. Let $C_n := C\left([0,T];\mathbb{R}^n\right)$ be the space of continuous functions on $[0,T]$ and let $\mathcal{F}^n$ denote the σ-algebra induced by all cylindrical sets of the form $\{x(\cdot) \in C_n : x_{t_i} \in B_i,\ \ t_i \in [0,T], i = 1,\dots,l\}$, where $B_i \in \mathcal{B}(\mathbb{R}^n)$, for all i and $l \in \mathbb{N}_+$. Let $\mathcal{P}(C_n)$ denote the space of Borel probability measures μ on $(C_n, \mathcal{F}^n)$. Following the canonical representation of stochastic processes as well as using the Wasserstein metric, $D_T^\rho(\cdot,\cdot)$ on $\mathcal{P}(C_n)$, with appropriate metric ρ defined in $C\left([0,T];\mathbb{R}^n\right)$, it may be shown (see [18]) that the metric space $\mathcal{P}_\rho := \left(\mathcal{P}\left(C_n\right), D_T^\rho\right)$ is Polish. However, such a representation might not be appropriate for our problem formulation. In particular, note that the observation function $h(t,\cdot)$ is a function of $\mu_t(\omega)$ rather than the flow; $\{\mu_t(\omega), 0 \le t \le T\}$. Therefore, we shall define the $\mathbb{R}_+ \cup \{\infty\}$-valued map $\mathcal{W}(\mu_1,\mu_2) := \inf_\pi \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} d(x,y)d\pi(x,y)$, where $d = |\cdot|$ is the Euclidean norm, $\pi \in \mathcal{P}(\mathbb{R}^n \times \mathbb{R}^n)$ has marginals μ_1 and μ_2, and $\mathcal{W}(\cdot)$ is called the Wasserstein metric of order 1.

Let $\mathcal{P}_n := (\mathcal{P}(\mathbb{R}^n), \mathcal{W})$; then it is known that $\mathcal{P}_n$ is a complete, separable metric space. Notice that one could use a weaker metric, such as the Prokhorov metric on $\mathcal{P}(\mathbb{R}^n)$ induced by d, to metricize the space of probability measures and to obtain a Polish space. However, for ipotential applications to nonlinear filtering for SMV systems, such as partially observed MFG, we prefer to use $\mathcal{W}(\cdot)$. We now have two metric spaces, $(\mathbb{R}^n, d)$ and $\mathcal{P}_n$, where both of them are Polish, and we observe [32] that a countable product of Polish spaces is also Polish. Based on this, let $\mathcal{C}_\cap := \mathbb{R}^n \times \mathcal{P}_n$ and let $\left(\mathbf{z}_\mu(t,\omega)\right)_{t\ge 0}$ be a $\mathcal{C}_\cap$-valued stochastic process. Let $\xi(t,\cdot,\cdot) := P\left(\mathbf{z}_\mu(t) \in \cdot|\mathcal{F}_t^y\right)$ where $\xi(\cdot) : [0,T] \times \mathcal{B}(\mathcal{C}_\cap) \times \Omega \to [0,1]$ and as before satisfies the following: (i) for every $\bar{\omega} \in \Omega$, $\xi(t,\cdot,\bar{\omega})$ is a probability measure on $\mathcal{C}_\cap$; (ii) for any $A \in \mathcal{B}(\mathcal{C}_\cap)$, $\xi(t, A, \cdot)$

is a $\mathcal{F}_t^y$ measurable random variable; and finally (iii) for any $A \in \mathcal{B}(\mathcal{C}_\cap)$, $\xi(t, A, \bar{\omega}) = P\left(\mathbf{z}_\mu(t,\omega) \in A | \mathcal{F}_t^y\right)(\bar{\omega})$, a.s. Observe that ξ is a $\mathcal{P}(\mathcal{C}_\cap)$-valued stochastic process.

4.1. State and measure estimation for SMV SDE. We can now formulate the joint estimation of the state and the measure when the measure term is random with zero quadratic variation. Notice that such problems can be formulated as nonlinear filtering problems on the Polish space $\mathcal{C}_\cap$, as presented above, with random functionals in the dynamics. Consequently, we can use the arguments of nonlinear filtering theory for stochastic processes whose state space is Polish [20, sections 8.2, 8.3] under which the existence of regular conditional probabilities is well studied. It should also be remarked that the joint state and measure estimation scenario can be motivated by a number of situations, for instance, that where the initial unconditional distribution of the finite dimensional state is not known. In such a situation, one needs to generalize the nonlinear filtering equations so as to estimate the initial unconditional distribution.

The analysis for this case is necessarily more involved, in particular since one needs to work on infinite dimensional space valued stochastic processes, such as spaces of probability distributions and spaces of continuous functions defined on Polish spaces. This in particular involves the theory of stochastic calculus on infinite dimensional spaces, which has a vast literature and from which we cite only [21], [33], [30], [5], and [1]. Among these we note that in [1] the Itô's lemma is developed in order to obtain the Hamilton–Jacobi–Bellman equation for a partially observed stochastic control problem where the information state process is the unnormalized conditional density of nonlinear filtering and, consequently, the value function is a functional of this density process. Therefore, due to the particular form of the dynamics under consideration, the particular spaces involved in a nonlinear filtering theory for MV SDEs, as mentioned above, as well as the arbitrary differentiability of the test function on the underlying space, we first provide an appropriately generalized Itô's rule for stochastic processes taking values in infinite dimensional spaces.

In order to define the joint estimation problem precisely, consider the state dynamics given by

$$dz(t) = f\left[t, z(t), \psi\left(t, z(t)\right), \mu_t^\omega\right] dt + \sigma\left[t, z(t), \mu_t^\omega\right] dw(t,\omega) \tag{4.1}$$

with initial conditions $(z(0), \mu_0^\omega)$, where as before $z : [0,T]\times\Omega \to \mathbb{R}^n$, $\psi : [0,T]\times\mathbb{R}^n \to U$, $f : [0,T]\times\mathbb{R}^n\times U\times\mathcal{P}(\mathbb{R}^n) \to \mathbb{R}^n$, $\sigma : [0,T]\times\mathbb{R}^n\times\mathcal{P}(\mathbb{R}^n) \to \mathbb{R}^{n\times m}$, $(w(t))_{t\geq 0}$ is a standard Brownian motion in $\mathbb{R}^m$, and $\mu_t^\omega := P\left(z(t)|\mathcal{F}_t^w\right)$ with $\mathcal{F}_t^w = \sigma\{w(s), 0 \leq s \leq t\}$. Here $\psi(\cdot)$ is an $\mathcal{F}_t^w$-adapted admissible control. The observation dynamics are given by

$$d\mathbf{y}(t) = g\left(t, z(t), \mu_t^\omega\right) dt + d\nu(t), \tag{4.2}$$

where $g : [0,T]\times \to \mathbb{R}^n \times \mathcal{P}(\mathbb{R}^n) \to \mathbb{R}^d$ and $(\nu(t))_{t\geq 0}$ is a standard Brownian motion in $\mathbb{R}^d$ as before. We remark again that in order to emphasize the dependence on the underlying sample paths one shall write

$$d\mathbf{y}(t,\omega,\tilde{\omega}) = g\left(t, z(t,\omega), \mu_t^\omega\right) dt + d\nu(t,\tilde{\omega}). \tag{4.3}$$

Let $\mathbf{z}_\mu(t,\omega) := (z(t,\omega), \mu_t^\omega)$ and hence, $\mathbf{z}_\mu : [0,T]\times\Omega \to \mathcal{C}_\cap$. We also have

$$dz(t) = f^*\left(t, z(t), \psi\left(t, z(t)\right), \omega\right) dt + \sigma^*\left(t, z(t), \omega\right) dw(t). \tag{4.4}$$

We assume the following assumptions hold in the rest of this section:

(A3) The functions $f(t,x,u,y), \sigma(t,x,y)$, and $g(t,x,p)$ are assumed to be bounded and Lipschitz continuous in their parameters in the appropriate metric.

(A4) The conditional law μ_t^ω has a twice continuously space-differentiable density with respect to the Lebesgue measure and we denote it by p_t^ω.

As before, our goal is to obtain a recursive expression for $\xi(t,A) := P\left(\mathbf{z}_\mu(t) \in A | \mathcal{F}_t^{\mathbf{y}}\right)$ for $A \in \mathcal{B}(\mathcal{C}_\cap)$. Note that by assumption (A4) above, μ_t^ω satisfies the SFPK:

$$\begin{aligned}\frac{\partial \mu_t^\omega(x)}{\partial t} &= \int_{-\infty}^{x} -\langle \nabla_\alpha, f\left[t,\alpha,\psi\left(t,\alpha\right),\mu_t^\omega\right] p_t^\omega(\alpha)\rangle + \frac{1}{2}\mathbf{tr}\langle \nabla^2_{\alpha\alpha}, a(t,\alpha,\omega)p_t^\omega(\alpha)\rangle d\alpha \\ &:= \mathcal{T}^\mu_{FPK}(t,\omega) \end{aligned} \tag{4.5}$$

in $\mathbb{R}^n \times [0,T]$, where $a(t,x,\omega) := \sigma\left[t,x,\mu_t^\omega\right] \sigma^T\left[t,x,\mu_t^\omega\right]$ and the measure process μ_t has the initial value $\mu_0^\omega \in \mathcal{P}\left(\mathbb{R}^n\right)$.

Let us set $d\mathbf{w}(t) = (dw(t),0)^T$ and

$$\mathbf{F}(t,\omega) := \begin{bmatrix} f^*\left(t,z(t,\omega),\psi(t,z(t,\omega)),\omega\right) \\ \mathcal{T}^\mu_{FPK}(t,\omega) \end{bmatrix},\ \mathbf{G}(t,\omega) := \begin{bmatrix} \sigma^*\left(t,z(t,\omega),\omega\right) & \mathbf{0}^T \end{bmatrix},$$

where $\mathbf{0}$ is an n-dimensional 0-vector. Then the partially observed system (4.2) and (4.4) can be written as

$$d\mathbf{z}_\mu(t) = \mathbf{F}(t,\omega)dt + \mathbf{G}(t,\omega)d\mathbf{w}(t,\omega), \tag{4.6}$$

$$d\mathbf{y}(t) = g\left(t,\mathbf{z}_\mu(t)\right)dt + d\nu(t). \tag{4.7}$$

Therefore, the problem becomes a nonlinear filtering problem on the Polish space $\mathcal{C}_\cap$ with random functions $\mathbf{F}(t,\omega)$ and $\mathbf{G}(t,\omega)$ in the state dynamics.

In the previous sections we have shown that the standard theory holds when the state dynamics are given by random functions and hence the main technical challenge is the analysis of the infinite dimensional part; in particular the standard Itô's rule cannot be used for such processes. Consequently, the first step in the derivation of a nonlinear filtering theory for infinite dimensional processes in this paper is the development of such a result; for this purpose, recall first that $\mu : [0,T] \times \Omega \times \mathcal{B}\left(\mathbb{R}^n\right) \to [0,1]$ and since it is assumed that μ has a density it is consequently absolutely continuous with respect to the Lebesgue measure. This further implies that for $t \in [0,T]$, $\mu_t \in C_b\left(\mathbb{R}^n\right)$ a.s., where $C_b\left(\mathbb{R}^n\right)$ is the space of bounded continuous functions. For $\theta \in C_b\left(\mathbb{R}^n\right)$, set the norm $\|\theta\|_{C_b(\mathbb{R}^n)} \stackrel{\triangle}{=} \sup_{x\in\mathbb{R}^n}\theta(x)$ in $C_b\left(\mathbb{R}^n\right)$ so that it becomes a Banach space. Obviously, the product space (the space formed via direct sum) $\mathbb{R}^n \oplus C_b\left(\mathbb{R}^n\right)$ is also Banach where among other possible choices we use the max norm on this direct sum space, i.e., for $x \in \mathbb{R}^n$, $y \in C_b\left(\mathbb{R}^n\right)$, $\|(x,y)\|_\infty := \max\left(|x|, \|y\|_{C_b(\mathbb{R}^n)}\right)$, and which, by abuse of notation, we denote as $\mathcal{C}_\cap := \mathbb{R}^n \oplus C_b\left(\mathbb{R}^n\right)$. Hence, the filtering problem is now further reduced to a Banach space-valued nonlinear filtering problem. Notice also that based on this Banach space representation, the SFPK equation can now be considered as a random operator, $\mathcal{T}^\mu_{FPK} : \mathbb{R}_+ \times \Omega \to C_b(\mathbb{R}^n)$, which is $\mathcal{F}_t^w$-adapted.

We now recall the definition of Fréchet differentiation on a Banach space X (with norm $\|\cdot\|$ and dual X^*). Let $f : \mathsf{X} \to \mathbb{R}$ be a function and $\mathcal{L}\left(\mathsf{X};\mathbb{R}\right) = \mathsf{X}^*$ denote the space of linear bounded operators on the Banach space X. Then the function f is said to be Fréchet differentiable at $x \in \mathsf{X}$ if there exists an operator $\mathsf{D}f(x) \in \mathcal{L}\left(\mathsf{X};\mathbb{R}\right)$ such that

$$\lim_{0\neq\|y\|\to 0} \frac{f(x+y) - f(x) - \mathsf{D}f(x)(y)}{\|y\|} = 0. \tag{4.8}$$

Furthermore, the function is said to be continuously Fréchet differentiable if $\mathsf{D}f(\cdot) : \mathsf{X} \to \mathsf{X}^*$ is continuous under the operator norm $\|\cdot\|_{\mathsf{op}}$.

Notice that $\mathsf{D}f(x) \in \mathcal{L}(\mathsf{X};\mathbb{R})$ and hence the second order Fréchet derivative of f at $x \in \mathsf{X}$ is the operator $\mathsf{D}^2 f(x) \in \mathcal{L}(\mathsf{X};\mathcal{L}(\mathsf{X};\mathbb{R}))$ that satisfies

$$\lim_{0\neq\|y\|\to 0} \frac{\mathsf{D}f(x+y) - \mathsf{D}f(x) - \mathsf{D}^2 f(x)(y)}{\|y\|} = 0. \tag{4.9}$$

We can also define the Fréchet derivative on product Banach spaces and for $f : \mathsf{X}\times\mathsf{Y} \to \mathbb{R}$, let us denote the (total) Fréchet derivative at $(x,y) \in \mathsf{X}\times\mathsf{Y}$ by $\mathsf{D}f(x,y)$. Furthermore, for a fixed $y_0 \in \mathsf{Y}$, $f(x,y_0)$ is a function of x, and the partial Fréchet derivative of $f(x,y_0)$ with respect to x at $x_0 \in \mathsf{X}$ is denoted by $\mathsf{D}_x f(x_0,y_0)$; likewise we denote by $\mathsf{D}_y f(x_0,y_0)$ the partial Fréchet derivative of $f(x_0,y)$ with respect to y at y_0. Let us assume that $f : \mathsf{X}\times\mathsf{Y} \to \mathbb{R}$ is Fréchet differentiable at (x_0,y_0). Then, it can be shown that the partial Fréchet derivatives $\mathsf{D}_x f(x_0,y_0)$ and $\mathsf{D}_y f(x_0,y_0)$ exist and they satisfy the addition rule (see, e.g., [13, p. 447]),

$$\mathsf{D}f(x_0,y_0)\cdot(x_1,y_1) = \mathsf{D}_x f(x_0,y_0)\cdot x_1 + \mathsf{D}_y f(x_0,y_0)\cdot y_1,\ x_1 \in \mathsf{X},\ y_1 \in \mathsf{Y}, \tag{4.10}$$

where $\cdot$ notation indicates the operator nature of $\mathsf{D}f$. We can generalize the above to the actions of higher order derivatives. Namely, we can show that

$$\begin{aligned}\mathsf{D}^2 f(x_0,y_0)\cdot\big[(x_1,y_1),(x_2,y_2)\big] &= \mathsf{D}^2_x f(x_0,y_0)\cdot(x_1,x_2) + \mathsf{D}_x\mathsf{D}_y f(x_0,y_0)\cdot(x_1,y_2)\\ &+ \mathsf{D}_y\mathsf{D}_x f(x_0,y_0)\cdot(y_1,x_2) + \mathsf{D}^2_y f(x_0,y_0)\cdot(y_1,y_2),\ x_1,x_2 \in \mathsf{X},\ y_1,y_2 \in \mathsf{Y},\end{aligned} \tag{4.11}$$

where $[\cdot,\cdot]$ denotes the multilevel map for the second order Fréchet derivative.

We can now state an Itô type formula for a stochastic process valued in $\mathbb{R}^n \oplus C_b(\mathbb{R}^n)$. Notice, however, that the infinite dimensional process has zero quadratic variation.

Lemma 4.1 (Itô's lemma in a Banach space). *Let two stochastic processes $\big(z_0(t,\omega)\big)_{t\geq 0}$ and $\big(\mu_t(\omega)\big)_{t\geq 0}$ be given by*

$$z_0(t) = \int_0^t f_0\left[s, z_0(s), \psi_0\left(s, z_0(s)\right), \mu_s(\omega)\right] ds + \int_0^t \sigma_0[s, z_0(s), \mu_s(\omega)]dw(s), \tag{4.12}$$

$$\mu_t(\omega) = \int_0^t \mathcal{D}^{\mu_s}(s,\omega)ds, \tag{4.13}$$

where $f_0 : [0,T]\times\mathbb{R}^n\times U\times C_b(\mathbb{R}^n) \to \mathbb{R}^n$, $\psi_0 : [0,T]\times\mathbb{R}^n \to U$, $\sigma_0 : [0,T]\times\mathbb{R}^n\times C_b(\mathbb{R}^n) \to \mathbb{R}^{n\times m}$, $\mu : [0,T]\times\Omega \to C_b(\mathbb{R}^n)$, $(\mu_t)_{0\leq t\leq T}$ is $\mathcal{F}^w_t$-adapted, $w(t)$ is a standard Brownian motion in $\mathbb{R}^m$, and $\mathcal{D}^{\mu_t}(t,\omega)$ is a continuous, $\mathcal{F}^w_t$-adapted, and $C_b(\mathbb{R}^n)$-valued stochastic process. Assume further that the functions $f_0(\cdot)$, $\sigma_0(\cdot)$, and $\psi_0(\cdot)$ are bounded and Lipschitz continuous in their parameters. Let $\ell(z,\mu) \in C^2(\mathbb{R}^n\times C_b(\mathbb{R}^n))$, the space of twice Fréchet differentiable functions whose first and second order derivatives are continuous on bounded subsets of the Banach space $\mathbb{R}^n\times C_b(\mathbb{R}^n)$. Then, P-a.s.

$$\begin{aligned}\ell(z_0(t),\mu_t) - \ell(z_0(0),\mu_0) &= \sum_{i=1}^n \int_0^t \mathsf{D}_{z_i}\ell(z_0(s),\mu_s)\, f_{0,i}\left[s, z_0(s), \psi_0(s,z_0), \mu_s(\omega)\right] ds\\ &+ \int_0^t \mathsf{D}_\mu \ell(z_0(s),\mu_s)\cdot\mathcal{D}^{\mu_s}(s)ds + \frac{1}{2}\sum_{i,j=1}^n \int_0^t \mathsf{D}^2_{z_i z_j}\ell(z_0(s),\mu_s)\,\mathbb{G}^{\mu_s}_{i,j}(s)ds\\ &+ \sum_{i=1}^n \int_0^t \mathsf{D}_{z_i}\ell(z_0(s),\mu_s)\,\mathbf{G}^{\mu_s}(s)dw(s),\end{aligned} \tag{4.14}$$

where $\mathbb{G}(t) := \mathbf{G}^{\mu_t}\mathbf{G}^{\mu_t^T}$ *for* $\mathbf{G}^{\mu_t} = \begin{bmatrix} \sigma_0\left(t, z(t,\omega), \mu_t(\omega)\right) & \mathbf{0}^T \end{bmatrix}$, $\mathbf{0}$ *is an* n*-dimensional zero vector,* $\mathsf{D}_{z_i}\ell$ *denotes the partial Fréchet derivative of* ℓ *with respect to the* i*-th coordinate of* $\mathbb{R}^n$ *and* $\mathsf{D}_\mu\ell$ *denotes the partial Fréchet derivative with respect to* $C_b(\mathbb{R}^n)$.

Proof. For simplicity, we assume $n = 1$, and following the steps of the proof of Itô's rule in $\mathbb{R}$, we prove the lemma in three steps [22, Theorem 3.3]. Hence, we first introduce, for each $k \geq 1$, the stopping time which depends on the quadratic variation term of the Brownian motion and $\mathcal{D}^{\mu_t}(t)$ since f_0 and σ_0 are already assumed to be bounded. Therefore, let $M_t := \int_0^t \sigma_0(s, z_0(s), \mu_s(\omega))dw(s)$ and define the following stopping time:

$$\tau_k = \left\{ t \geq 0; \langle M, M \rangle_t \geq k \text{ or } \left\| \int_0^t \mathcal{D}^{\mu_s}(s,\omega)ds \right\|_{C_b(\mathbb{R})} \geq k \right\},$$

where $\langle \cdot \rangle_t$ denotes the quadratic variation of the process in $[0,t]$. Following standard reasoning, we may assume that for the $\mathbb{R}\times C_b(\mathbb{R})$-valued joint process $(z_0(t,\omega), \mu_t(\omega))$, $z_0(0,\omega)$, $\langle M, M \rangle_t$, and $\int_0^t \mathcal{D}^{\mu_s}(s,\omega)ds$ are all bounded.

As the second step, we apply Taylor's expansion in Banach spaces to the function ℓ. For $t \in [0,T]$, fix a partition $\Pi = \{t_0, \ldots, t_m\}$ with $0 = t_0 < t_1 \cdots < t_m = t$ and denote the mesh by $\|\Pi\| := \max_{1 \leq i \leq m}(t_i - t_{i-1})$. Let $\Delta_i\big(z_0(t), \mu_t\big) := \big(z_0(t_i) - z_0(t_{i-1}), \mu_{t_i} - \mu_{t_{i-1}}\big)$; then

$$\begin{aligned} \ell\left(z_0(t), \mu_t\right) - \ell\left(z_0(0), \mu_0\right) &= \sum_{i=1}^m \left(\ell\left(z_0(t_i), \mu_{t_i}\right) - \ell\left(z_0(t_{i-1}), \mu_{t_{i-1}}\right)\right) \\ &= \sum_{i=1}^m \mathsf{D}\ell\left(z_0(t_{i-1}), \mu_{t_{i-1}}\right) \cdot \left(z_0(t_i) - z_0(t_{i-1}), \mu_{t_i} - \mu_{t_{i-1}}\right) \\ &\quad + \frac{1}{2}\mathsf{D}^2\ell\left(\tilde{z}_0(i), \tilde{\mu}_i\right) \cdot \left[\Delta_i\big(z_0(t), \mu_t\big), \Delta_i\big(z_0(t), \mu_t\big)\right], \end{aligned} \tag{4.15}$$

where for some $0 \leq \varsigma_i(\omega) \leq 1$, we denote

$$(\tilde{z}_0(i), \tilde{\mu}_i) = \left(z_0(t_{i-1}), \mu_{t_{i-1}}\right) + \varsigma_i(\omega)\left(\left(z_0(t_i), \mu_{t_i}\right) - \left(z_0(t_{i-1}), \mu_{t_{i-1}}\right)\right). \tag{4.16}$$

Consider now each term in (4.15). By (4.10)

$$\begin{aligned} &\sum_{i=1}^m \mathsf{D}\ell\left(z_0(t_{i-1}), \mu_{t_{i-1}}\right) \cdot \left(z_0(t_i) - z_0(t_{i-1}), \mu_{t_i} - \mu_{t_{i-1}}\right) \\ &= \sum_{i=1}^m \mathsf{D}_z\ell\left(z_0(t_{i-1}), \mu_{t_{i-1}}\right)\left(z_0(t_i) - z_0(t_{i-1})\right) + \mathsf{D}_\mu\ell\left(z_0(t_{i-1}), \mu_{t_{i-1}}\right) \cdot \left(\mu_{t_i} - \mu_{t_{i-1}}\right). \end{aligned}$$

Similarly,

$$\begin{aligned} &\sum_{i=1}^m \mathsf{D}^2\ell\left(\tilde{z}_0(i), \tilde{\mu}_i\right) \cdot \left[\Delta_i\big(z_0(t), \mu_t\big), \Delta_i\big(z_0(t), \mu_t\big)\right] \\ &\quad = \sum_{i=1}^m \left[z_0(t_i) - z_0(t_{i-1})\right]^2 \mathsf{D}_z^2\ell\left(\tilde{z}_0(i), \tilde{\mu}_i\right) \\ &\qquad + \sum_{i=1}^m \left[z_0(t_i) - z_0(t_{i-1})\right]\mathsf{D}_z\mathsf{D}_\mu\ell\left(\tilde{z}_0(i), \tilde{\mu}_i\right) \cdot \left(\mu_{t_i} - \mu_{t_{i-1}}\right) \end{aligned}$$

$$+\sum_{i=1}^{m}[z_0(t_i)-z_0(t_{i-1})]\,\mathsf{D}_\mu\mathsf{D}_z\ell\left(\tilde z_0(i),\tilde\mu_i\right)\cdot\left(\mu_{t_i}-\mu_{t_{i-1}}\right)$$

$$+\sum_{i=1}^{m}\mathsf{D}^2_\mu\ell\left(\tilde z_0(i),\tilde\mu_i\right)\cdot\left[\mu_{t_i}-\mu_{t_{i-1}},\mu_{t_i}-\mu_{t_{i-1}}\right], \tag{4.17}$$

where the equality holds by the property of second order differentials (4.11) applied to $\mathbb{R}\times C_b(\mathbb{R})$. Let

$$J^1_1(\Pi):=\sum_{i=1}^{m}\mathsf{D}_z\ell\left(z_0(t_{i-1}),\mu_{t_{i-1}}\right)\left(z_0(t_i)-z_0(t_{i-1})\right)$$

$$J^2_1(\Pi):=\sum_{i=1}^{m}\mathsf{D}_\mu\ell\left(z_0(t_{i-1}),\mu_{t_{i-1}}\right)\cdot\left(\mu_{t_i}-\mu_{t_{i-1}}\right)$$

$$J^1_2(\Pi):=\frac{1}{2}\sum_{i=1}^{m}[z_0(t_i)-z_0(t_{i-1})]^2\,\mathsf{D}^2_z\ell\left(\tilde z_0(i),\tilde\mu_i\right)$$

$$J^2_2(\Pi):=\frac{1}{2}\sum_{i=1}^{m}[z_0(t_i)-z_0(t_{i-1})]\,\mathsf{D}_\mu\mathsf{D}_z\ell\left(\tilde z_0(i),\tilde\mu_i\right)\cdot\left(\mu_{t_i}-\mu_{t_{i-1}}\right)$$

$$J^3_2(\Pi):=\frac{1}{2}\sum_{i=1}^{m}[z_0(t_i)-z_0(t_{i-1})]\,\mathsf{D}_z\mathsf{D}_\mu\ell\left(\tilde z_0(i),\tilde\mu_i\right)\cdot\left(\mu_{t_i}-\mu_{t_{i-1}}\right)$$

$$J^4_2(\Pi):=\frac{1}{2}\sum_{i=1}^{m}\mathsf{D}^2_\mu\ell\left(\tilde z_0(i),\tilde\mu_i\right)\cdot\left[\mu_{t_i}-\mu_{t_{i-1}},\mu_{t_i}-\mu_{t_{i-1}}\right]. \tag{4.18}$$

We now analyze each term separately. It can be easily seen that

$$\lim_{\|\Pi\|\to 0}J^1_1(\Pi)=\int_0^t\mathsf{D}_z\ell\left(z_0(s),\mu_s\right)f_0\left[s,z_0,\psi_0,\mu_s\right]ds$$

$$+\int_0^t\mathsf{D}_z\ell\left[z_0(s),\mu_s\right]\sigma_0[s,z_0,\mu_s]dw(s) \tag{4.19}$$

in quadratic mean. Furthermore, since the function ℓ has continuous derivatives $J^2_1(\Pi)$ converges to the integral $\int_0^t\mathsf{D}_\mu\ell\left(z_0(s),\mu_s\right)\cdot\mathcal{D}^{\mu_s}(s)ds$ a.s. P, as $\|\Pi\|\to 0$. Consider now $J^1_2(\Pi)$. Following [22, Theorem 3.3] and using the fact that the process $z_0(t)$ is continuous, we have

$$\lim_{\|\Pi\|\to 0}J^1_2(\Pi)=\frac{1}{2}\int_0^t\mathsf{D}^2_z\ell\left(z_0(s),\mu_s\right)\sigma_0\left[s,z_0(s),\mu_s\right]\sigma_0\left[s,z_0(s),\mu_s\right]^T ds \tag{4.20}$$

in $L^1(\Omega,\mathcal{F},P)$. Consider now $J^2_2(\Pi)$ and recall that $z_0(t)$ is continuous and has finite bounded total variation; we have

$$2|J^2_2(\Pi)|=\left|\sum_{i=1}^{m}[z_0(t_i)-z_0(t_{i-1})]\,\mathsf{D}_\mu\mathsf{D}_z\ell\big(\tilde z_0(i),\tilde\mu_i\big)\cdot\left(\mu_{t_i}-\mu_{t_{i-1}}\right)\right|$$

$$=\Bigg|\sum_{i=1}^{m}\left[\int_{t_{i-1}}^{t_i}f_0\left[s,z_0,\psi_0,\mu_s\right]ds+\sigma_0\left[s,z_0,\mu_s\right]dw(s)\right]$$

$$\times\mathsf{D}_\mu\mathsf{D}_z\ell\big(\tilde z_0(i),\tilde\mu_i\big)\cdot\left(\mu_{t_i}-\mu_{t_{i-1}}\right)\Bigg|$$

$$\le Mm\max_{1\le i\le m}\left|\int_{t_{i-1}}^{t_i}f_0\left[s,z_0,\psi_0,\mu_s\right]ds+\sigma_0\left[s,z_0,\mu_s\right]dw(s)\right|\left\|\int_{t_{i-1}}^{t_i}\mathcal{D}^{\mu_s}(s)ds\right\|_{C_b(\mathbb{R})},$$

where since $\mathsf{D}_\mu \mathsf{D}_z \ell\big(\tilde{z}_0(i), \tilde{\mu}_i\big) \in \mathcal{L}\left(C_b(\mathbb{R}); \mathbb{R}\right)$ in the last step we set

$$\begin{aligned} M &:= \max_{1\leq i\leq m} \left\{\max \eta : \left|\mathsf{D}_\mu \mathsf{D}_z \ell\big(\tilde{z}_0(i), \tilde{\mu}_i\big) \cdot \big(\mu_{t_i} - \mu_{t_{i-1}}\big)\right| < \eta \|\mu_{t_i} - \mu_{t_{i-1}}\|_{C_b(\mathbb{R})}\right\} \\ &\leq \max_{1\leq i\leq m} \|\mathsf{D}_\mu \mathsf{D}_z \ell\big(\tilde{z}_0(i), \tilde{\mu}_i\big)\|_{\mathsf{op}} < \infty \end{aligned} \tag{4.21}$$

for which (4.21) holds since $\|\mathsf{D}_\mu \mathsf{D}_z \ell\big(\tilde{z}_0(i), \tilde{\mu}_i\big)\|_{\mathsf{op}} < \infty$ and $|\mathsf{D}_\mu \mathsf{D}_z \ell\big(\tilde{z}_0(i), \tilde{\mu}_i\big) \cdot x| \leq \|\mathsf{D}_\mu \mathsf{D}_z \ell\big(\tilde{z}_0(i), \tilde{\mu}_i\big)\|_{\mathsf{op}} \|x\|_{C_b(\mathbb{R})}$ for $x \in C_b(\mathbb{R})$.

Due to the continuity of f_0, σ_0, and $\mathcal{D}^{\mu_t}(t)$, it can now be shown that $J_2^2\left(\Pi\right)$ converges to zero a.s. as $\|\Pi\| \to 0$ as well as in $L^1\left(\Omega, \mathcal{F}, P\right)$ by the dominated convergence theorem.

Consider now the term $J_2^3\left(\Pi\right)$. Following similar steps one can show that

$$\lim_{\|\Pi\|\to 0} J_2^3\left(\Pi\right) = 0 \qquad \text{a.s. } P \text{ and in } \mathrm{L}^1\left(\Omega, \mathcal{F}, \mathrm{P}\right). \tag{4.22}$$

Consider the last term, $J_2^4\left(\Pi\right)$. Recall that $\mathsf{D}_\mu^2 \ell\left(\tilde{z}_0(i), \tilde{\mu}_i\right) \in \mathcal{L}\big(C_b\left(\mathbb{R}\right); \mathcal{L}\left(C_b\left(\mathbb{R}\right); \mathbb{R}\right)\big)$ and one can identify an $\alpha \in \mathcal{L}\big(C_b\left(\mathbb{R}\right); \mathcal{L}\left(C_b\left(\mathbb{R}\right); \mathbb{R}\right)\big)$ with a $\beta \in \mathcal{L}^{(2)}\big(C_b\left(\mathbb{R}\right) \times C_b\left(\mathbb{R}\right); \mathbb{R}\big)$, the Banach space of bounded bilinear maps from $C_b\left(\mathbb{R}\right)$ to $\mathbb{R}$. Therefore,

$$\begin{aligned} 2\left|J_2^4\left(\Pi\right)\right| &= \left|\sum_{i=1}^m \mathsf{D}_\mu^2 \ell\left(\tilde{z}_0(i), \tilde{\mu}_i\right) \cdot \left[\mu_{t_i} - \mu_{t_{i-1}}, \mu_{t_i} - \mu_{t_{i-1}}\right]\right| \\ &\leq \sum_{i=1}^m N \|\mu_{t_i} - \mu_{t_{i-1}}\|_{C_b(\mathbb{R})} \|\mu_{t_i} - \mu_{t_{i-1}}\|_{C_b(\mathbb{R})} \\ &= Nm \max_{1\leq i\leq m} \left\|\int_{t_{i-1}}^{t_i} \mathcal{D}^{\mu_s}(s) ds\right\|_{C_b(\mathbb{R})} \left\|\int_{t_{i-1}}^{t_i} \mathcal{D}^{\mu_s}(s) ds\right\|_{C_b(\mathbb{R})}, \end{aligned} \tag{4.23}$$

where the first inequality follows since for $\alpha \in \mathcal{L}^{(2)}\big(\mathsf{X} \times \mathsf{Y}; \mathbb{R}\big)$, $|\alpha(x, y)| \leq \upsilon \|x\|_{\mathsf{X}} \|y\|_{\mathsf{Y}}$ for some $\upsilon \geq 0$, and consequently, N is defined as

$$\begin{aligned} N := \max\big\{\upsilon : &\left|\mathsf{D}_\mu^2 \ell\left(\tilde{z}_0(i), \tilde{\mu}_i\right) \cdot \left[\mu_{t_i} - \mu_{t_{i-1}}, \mu_{t_i} - \mu_{t_{i-1}}\right]\right| \\ &< \upsilon \|\mu_{t_i} - \mu_{t_{i-1}}\|^2_{C_b(\mathbb{R})}, 1 \leq i \leq m\big\}. \end{aligned} \tag{4.24}$$

Finally, it can be shown that (4.23) goes to zero P-a.s. (as well as in $L^1\left(\Omega, \mathcal{F}, P\right)$) as $\|\Pi\| \to 0$ due to the continuous differentiability and boundedness assumptions.

To conclude the proof, we need to establish the a.s. convergence of the sum of the terms in (4.18) in $[0, t]$ for every $0 \leq t \leq T$. In that case processes on both sides of (4.14) are modifications of each other and hence indistinguishable due to the continuity of the processes. This can be shown by considering a sequence of partitions $\{\Pi^{(n)}\}_{n=1}^\infty$ of $[0, t]$ with $\|\Pi^{(n)}\| \xrightarrow{n\to\infty} 0$ and using the fact that for any convergent sequence in $L^p(\Omega, P)$, there exists a subsequence which converges P-a.s. (see [22, Theorem 3.3]). This completes the proof of the lemma. □

Remark 4.1. As in the classical Itô lemma, the result of the above lemma holds if the functions in the dynamics are themselves $\mathcal{F}_t^w$-adapted.

In the rest of this section, we first a.e. uniquely represent the measure induced by a random variable with its distribution function and use the fact that the space of such distributions is a subspace of the Banach space $\big(C_b(\mathbb{R}^n), \|\cdot\|_{C_b(\mathbb{R}^n)}\big)$. We then proceed by applying the Itô's lemma derived above in the derivation of a nonlinear filter for the joint estimation. In this regard, it should further be noticed that the measures that we consider in the paper are assumed to have a density with respect to the Lebesgue measure on $\mathbb{R}^n$, which we denote by λ, and hence, they are absolutely continuous with respect to λ. Consequently, let us consider the subspace of $C_b(\mathbb{R}^n)$ with the following norm: For $f \in C_b^1(\mathbb{R}^n)$

$$\|f\|_{C_b^1(\mathbb{R}^n)} := \sup_{x\in\mathbb{R}^n} |f(x)| + \sup_{x\in\mathbb{R}^n} |f'(x)|. \tag{4.25}$$

We further recall the total variation metric defined on $\mathcal{P}(\mathbb{R}^n)$: For $\mu, \nu \in \mathcal{P}(\mathbb{R}^n)$

$$d_{TV}(\mu,\nu) := \sup_{A\in\mathcal{B}(\mathbb{R}^n)} |\mu(A) - \nu(A)|. \tag{4.26}$$

Notice now that as a consequence of Scheffé's lemma, for a sequence of probability measures which have densities with respect to λ, if the sequence converges in $\|\cdot\|_{C_b^1(\mathbb{R}^n)}$, then it also converges in $d_{TV}(\cdot,\cdot)$. We conclude with the fact that convergence in total variation implies weak convergence and since the convergence in the Wasserstein metric is identical to the weak convergence on Polish spaces, this further implies convergence in the Wasserstein metric.

Having an Itô type result at hand, we can now continue with nonlinear filtering derivation for the joint state and measure estimation. Recall $\mathcal{C}_\cap := \mathbb{R}^n \oplus C_b(\mathbb{R}^n)$ and let $\ell \in C_b^2(\mathcal{C}_\cap)$. Following similar steps we first define an exponential martingale for the change of measure argument. Set

$$M^{-1}(t) := \exp\bigg(-\int_0^t \big\langle g\big(s,\mathbf{z}_\mu(s)\big), d\nu(s)\big\rangle - \frac{1}{2}\int_0^t |g\big(s,\mathbf{z}_\mu(s)\big)|^2 ds\bigg), \tag{4.27}$$

$\mathcal{F}_t = \sigma\{w(s),\nu(s), 0 \le s \le t\}$, and define the measure $\hat{P}$ on $(\Omega,\mathcal{F}_t)$ as $\frac{d\hat{P}}{dP}|_{\mathcal{F}_t} = M(t)^{-1}$, and therefore, $\mathbf{y}(t)$ is a $\hat{P}$-Brownian motion. Furthermore, the Kallianpur–Striebel formula is still valid and we have that

$$\mathbb{E}_P\left[\ell\big(\mathbf{z}_\mu(t)\big)|\mathcal{F}_t^{\mathbf{y}}\right] = \frac{\mathbb{E}_{\hat{P}}\left[M(t)\ell\big(\mathbf{z}_\mu(t)\big)|\mathcal{F}_t^{\mathbf{y}}\right]}{\mathbb{E}_{\hat{P}}\left[M(t)|\mathcal{F}_t^{\mathbf{y}}\right]} \tag{4.28}$$

and by the Itô formula we have

$$dM(t) = M(t)g^T\big(t,\mathbf{z}_\mu(t)\big)d\mathbf{y}(t). \tag{4.29}$$

We can now present the main result of this section on the unnormalized filtering. Let $\mathbb{G}(t,\omega) \in \mathbb{R}^{n\times n}$ be defined as $\mathbb{G}(t,\omega) := \mathbf{G}\mathbf{G}^T$ and set the following infinitesimal generator for the process $\mathbf{z}_\mu(t,\omega)$ evaluated on the test function ℓ:

$$\Lambda_{\mu_t^\omega}\ell := \frac{1}{2}\sum_{j,l=1}^n \mathbb{G}_{jl}\partial_{jl}^2\ell + \sum_{j=1}^n f_j^*\partial_j\ell + \mathsf{D}_\mu\ell\cdot\mathcal{T}_{FPK}^\mu. \tag{4.30}$$

THEOREM 4.2. *The unnormalized filter* $\mathbb{E}_{\hat{P}}\left[M(t)\ell\big(\mathbf{z}_\mu(t)\big)|\mathcal{F}_t^{\mathbf{y}}\right]$ *satisfies the following:*

$$\begin{aligned}\mathbb{E}_{\hat{P}}\left[M(t)\ell\big(\mathbf{z}_\mu(t)\big)|\mathcal{F}_t^{\mathbf{y}}\right] &= \mathbb{E}_{\hat{P}}\left[\ell\big(\mathbf{z}_\mu(0)\big)|\mathcal{F}_0^{\mathbf{y}}\right] + \int_0^t \mathbb{E}_{\hat{P}}\left[M(s)\Lambda_{\mu_s^\omega}\ell\big(\mathbf{z}_\mu(s)\big)|\mathcal{F}_s^{\mathbf{y}}\right]ds\\ &\quad + \int_0^t \mathbb{E}_{\hat{P}}\left[M(s)\ell\big(\mathbf{z}_\mu(s)\big)g^T\big(s,\mathbf{z}_\mu(s)\big)|\mathcal{F}_s^{\mathbf{y}}\right]d\mathbf{y}(s)\end{aligned} \tag{4.31}$$

with initial conditional distribution $\xi(0,\cdot) \in \mathcal{P}(\mathcal{C}_\cap)$.

Proof. The proof extends the theory of nonlinear filtering for $\mathbb{R}^p$, $1 \leq p < \infty$, valued processes to the Banach space valued signal processes whose dynamics are driven by random functions. By applying the Itô formula in Lemma 4.1 to (4.6), we obtain

$$\begin{aligned}
&\ell\big(\mathbf{z}_\mu(t)\big) - \ell\big(\mathbf{z}_\mu(0)\big) \\
&\quad = \int_0^t \mathsf{D}_z \ell\big(\mathbf{z}_\mu(s)\big) f^*\left(t, z, \varphi(s,z), \omega\right) ds + \int_0^t \mathsf{D}_\mu \ell\big(\mathbf{z}_\mu(s)\big) \cdot \mathcal{T}^\mu_{FPK}(s,\omega) ds \\
&\quad\quad + \int_0^t \mathsf{D}_z \ell\big(\mathbf{z}_\mu(s)\big) \mathbf{G}(s,\omega) d\mathbf{w}(t) + \frac{1}{2}\int_0^t \mathbf{tr}\big(\mathsf{D}^2_z \ell\big(\mathbf{z}_\mu(s)\big)\mathbb{G}(s,\omega)\big) ds
\end{aligned}$$

and hence

$$d\ell\big(\mathbf{z}_\mu(t)\big) = \Lambda_{\mu^\omega_t} \ell\big(\mathbf{z}_\mu(t)\big) dt + \nabla \ell\big(\mathbf{z}_\mu(t)\big)\mathbb{G}(t,\omega) d\mathbf{w}(t). \tag{4.32}$$

Observe now that $\ell\big(\mathbf{z}_\mu(t,\omega)\big)$ is an $\mathbb{R}$-valued stochastic process and using the Itô product rule for (4.29) and (4.32) we obtain

$$\begin{aligned}
d\big(M(t)\ell\big(\mathbf{z}_\mu(t)\big)\big) &= M(t) d\ell\big(\mathbf{z}_\mu(t)\big) + dM(t)\ell\big(\mathbf{z}_\mu(t)\big) + dM(t) d\ell\big(\mathbf{z}_\mu(t)\big) \\
&= M(t)\left(\Lambda_{\mu^\omega_t} \ell\big(\mathbf{z}_\mu(t)\big) dt + \nabla \ell\big(\mathbf{z}_\mu(t)\big)\mathbb{G}(t,\omega) d\mathbf{w}(t)\right) \\
&\quad + M(t)\ell\big(\mathbf{z}_\mu(t)\big) g^T\left(t, \mathbf{z}_\mu(t,\omega)\right) d\mathbf{y}(t),
\end{aligned} \tag{4.33}$$

where (4.33) follows due to (4.29) and by the fact that on $\mathcal{F}_t$, $\mathbf{y}(t)$ is a $\hat{P}$-Brownian motion. We can write the above equation as

$$\begin{aligned}
M(t)\ell\big(\mathbf{z}_\mu(t)\big) &= \int_0^t \Lambda_{\mu^\omega_s} \ell\big(\mathbf{z}_\mu(s)\big) ds + \int_0^t \nabla \ell\big(\mathbf{z}_\mu(s)\big)\mathbb{G}(s,\omega) d\mathbf{w}(s) \\
&\quad + \int_0^t M(s)\ell\big(\mathbf{z}_\mu(s)\big) g^T\big(s, \mathbf{z}_\mu(s)\big) d\mathbf{y}(s) + \ell\big(\mathbf{z}_\mu(0)\big).
\end{aligned} \tag{4.34}$$

Let us now take the conditional expectation of both sides conditioned on the observation σ-algebra with respect to the measure $\hat{P}$. We have

$$\begin{aligned}
\mathbb{E}_{\hat{P}}\left[M(t)\ell\big(\mathbf{z}_\mu(t)\big)|\mathcal{F}^{\mathbf{y}}_t\right] &= \mathbb{E}_{\hat{P}}\left[\ell\big(\mathbf{z}_\mu(0)\big)|\mathcal{F}^{\mathbf{y}}_0\right] + \mathbb{E}_{\hat{P}}\left[\int_0^t \Lambda_{\mu^\omega_s} \ell\big(\mathbf{z}_\mu(s)\big) ds | \mathcal{F}^{\mathbf{y}}_t\right] \\
&\quad + \mathbb{E}_{\hat{P}}\left[\int_0^t \nabla \ell\big(\mathbf{z}_\mu(s)\big)\mathbb{G}(s,\omega) d\mathbf{w}(s) | \mathcal{F}^{\mathbf{y}}_t\right] \\
&\quad + \mathbb{E}_{\hat{P}}\left[\int_0^t M(s)\ell\big(\mathbf{z}_\mu(s)\big) g^T\big(s, \mathbf{z}_\mu(s)\big) d\mathbf{y}(s) | \mathcal{F}^{\mathbf{y}}_t\right].
\end{aligned} \tag{4.35}$$

Observe that $\mathbb{E}_{\hat{P}}[\int_0^t \nabla \ell\big(\mathbf{z}_\mu(s)\big)\mathbb{G}(s,\omega) d\mathbf{w}(s)|\mathcal{F}^{\mathbf{y}}_t] = 0$ by the property of the Itô integral and the fact that under $\hat{P}$, $\mathbf{y}(t)$ is a Brownian motion independent of $w(t)$. Note also that since for all $t \in [0,T]$, $M(t)$, $g(t)$, and $\Lambda_{\mu^\omega_s}\ell$ are bounded, by [35, Lemma 5.4] we have

$$\mathbb{E}_{\hat{P}}\left[\int_0^t \Lambda_{\mu^\omega_s} \ell\big(\mathbf{z}_\mu(s)\big) ds | \mathcal{F}^{\mathbf{y}}_t\right] = \int_0^t \mathbb{E}_{\hat{P}}\left[\Lambda_{\mu^\omega_s} \ell\big(\mathbf{z}_\mu(s)\big)|\mathcal{F}^{\mathbf{y}}_s\right] ds,$$

$$\mathbb{E}_{\hat{P}}\left[\int_0^t M(s)\ell\big(\mathbf{z}_\mu(s)\big) g\left(s, \mathbf{z}_\mu\right) d\mathbf{y}(s) | \mathcal{F}^{\mathbf{y}}_t\right] = \int_0^t \mathbb{E}_{\hat{P}}\left[M(s)\ell\big(\mathbf{z}_\mu(s)\big) g\left(s, \mathbf{z}_\mu\right)|\mathcal{F}^{\mathbf{y}}_s\right] d\mathbf{y}(s),$$

which together with (4.35) completes the proof. ∎

We now provide the optimal filtering equation. Define the following innovation process: $\mathcal{I}(t) = \mathbf{y}(t) - \int_0^t \mathbb{E}_P \left[g(s, \mathbf{z}_\mu(s)) | \mathcal{F}_s^{\mathbf{y}}\right] ds$, which can be shown to be an $\mathcal{F}_t^{\mathbf{y}}$-Brownian motion under the measure P.

Theorem 4.3. *The normalized filter generating the conditional distribution $\xi(\cdot)$ satisfies the following stochastic integral equation: For all $\ell \in C_b^2(\mathcal{C}_\cap)$,*

$$\mathbb{E}_P \left[\ell\big(\mathbf{z}_\mu(t)\big) | \mathcal{F}_t^{\mathbf{y}}\right] = \mathbb{E}_P \left[\ell(\mathbf{z}_\mu(0)) | \mathcal{F}_0^{\mathbf{y}}\right] + \int_0^t \mathbb{E}_P \left[\Lambda_{\mu_s^\omega} \ell\big(\mathbf{z}_\mu(s)\big) | \mathcal{F}_s^{\mathbf{y}}\right] ds \tag{4.36}$$

$$+ \int_0^t \Big[\mathbb{E}_P \left[\ell\big(\mathbf{z}_\mu(s)\big) g^T\big(s, \mathbf{z}_\mu(s)\big) | \mathcal{F}_s^{\mathbf{y}}\right] - \mathbb{E}_P \left[\ell\big(\mathbf{z}_\mu(s)\big) | \mathcal{F}_s^{\mathbf{y}}\right] \mathbb{E}_P \left[g^T\big(s, \mathbf{z}_\mu(s)\big) | \mathcal{F}_s^{\mathbf{y}}\right]\Big] d\mathcal{I}(s)$$

with initial conditional distribution $\xi(0, \cdot) \in \mathcal{P}(\mathcal{C}_\cap)$.

Proof. Notice that by taking $\ell = 1$ in (4.31) we obtain

$$d\mathbb{E}_{\hat{P}} \left[M(t) | \mathcal{F}_t^{\mathbf{y}}\right] \stackrel{(i)}{=} \mathbb{E}_{\hat{P}} \left[M(t) g^T\big(t, \mathbf{z}_\mu(t)\big) | \mathcal{F}_t^{\mathbf{y}}\right] d\mathbf{y}(t)$$

and as a result of Itô's formula

$$\begin{aligned}\frac{1}{\mathbb{E}_{\hat{P}} \left[M(t) | \mathcal{F}_s^{\mathbf{y}}\right]} &= \int_0^t \frac{-1}{\mathbb{E}_{\hat{P}} \left[M(s) | \mathcal{F}_s^{\mathbf{y}}\right]^2} d\mathbb{E}_{\hat{P}} \left[M(s) | \mathcal{F}_s^{\mathbf{y}}\right] \\ &\quad + \int_0^t \frac{1}{\mathbb{E}_{\hat{P}} \left[M(s) | \mathcal{F}_s^{\mathbf{y}}\right]^3} d\langle \mathbb{E}_{\hat{P}} \left[M(\cdot) | \mathcal{F}_\cdot^{\mathbf{y}}\right]\rangle_s + \frac{1}{\mathbb{E}_{\hat{P}} \left[M(0) | \mathcal{F}_0^{y}\right]} \\ &= \int_0^t \frac{-1}{\mathbb{E}_{\hat{P}} \left[M(s) | \mathcal{F}_s^{\mathbf{y}}\right]^2} d\mathbb{E}_{\hat{P}} \left[M(s) | \mathcal{F}_s^{\mathbf{y}}\right] + \frac{1}{\mathbb{E}_{\hat{P}} \left[M(0) | \mathcal{F}_0^{\mathbf{y}}\right]} \\ &\quad + \int_0^t \frac{\mathbb{E}_{\hat{P}} \left[M(s) g^T\big(s, \mathbf{z}_\mu(s)\big) | \mathcal{F}_s^{\mathbf{y}}\right]}{\mathbb{E}_{\hat{P}} \left[M(s) | \mathcal{F}_s^{\mathbf{y}}\right]^3} \mathbb{E}_{\hat{P}} \left[M(s) g\big(s, \mathbf{z}_\mu(s)\big) | \mathcal{F}_s^{\mathbf{y}}\right] ds,\end{aligned} \tag{4.37}$$

where $\langle \mathbb{E}_{\hat{P}} \left[M(\cdot) | \mathcal{F}_\cdot^{\mathbf{y}}\right]\rangle_s$ denotes the quadratic variation process of $\mathbb{E}_{\hat{P}} \left[M(t) | \mathcal{F}_t^{\mathbf{y}}\right]$ and (4.37) holds due to (i) and $\mathbf{y}(t)$ being a Brownian motion. Let us now apply Itô's formula to (4.28) with $\ell \in C_b^2(\mathcal{C}_\cap)$:

$$\begin{aligned} d\mathbb{E}_P \left[\ell\big(\mathbf{z}_\mu(t)\big) | \mathcal{F}_t^{\mathbf{y}}\right] &= d\left(\frac{\mathbb{E}_{\hat{P}} \left[M(t) \ell\big(\mathbf{z}_\mu(t)\big) | \mathcal{F}_t^{\mathbf{y}}\right]}{\mathbb{E}_{\hat{P}} \left[M(t) | \mathcal{F}_t^{\mathbf{y}}\right]}\right) \\ &= \frac{d\left(\mathbb{E}_{\hat{P}} \left[M(t) \ell\big(\mathbf{z}_\mu(t)\big) | \mathcal{F}_t^{\mathbf{y}}\right]\right)}{\mathbb{E}_{\hat{P}} \left[M(t) | \mathcal{F}_t^{\mathbf{y}}\right]} + \mathbb{E}_{\hat{P}} \left[M(t) \ell\big(\mathbf{z}_\mu(t)\big) | \mathcal{F}_t^{\mathbf{y}}\right] d\left(\frac{1}{\mathbb{E}_{\hat{P}} \left[M(t) | \mathcal{F}_t^{\mathbf{y}}\right]}\right) \\ &\quad + d\left\langle \mathbb{E}_{\hat{P}} \left[M(\cdot) \ell\big(\mathbf{z}_\mu(\cdot)\big) | \mathcal{F}_\cdot^{\mathbf{y}}\right], \frac{1}{\mathbb{E}_{\hat{P}} \left[M(\cdot) | \mathcal{F}_\cdot^{\mathbf{y}}\right]}\right\rangle_t \\ &= \frac{\mathbb{E}_{\hat{P}} \left[M(t) \Lambda_{\mu_t^\omega} \ell\big(\mathbf{z}_\mu(t)\big) | \mathcal{F}_t^{\mathbf{y}}\right] dt}{\mathbb{E}_{\hat{P}} \left[M(t) | \mathcal{F}_t^{\mathbf{y}}\right]} + \frac{\mathbb{E}_{\hat{P}} \left[M(t) \ell\big(\mathbf{z}_\mu(t)\big) g^T\big(t, \mathbf{z}_\mu(t)\big) | \mathcal{F}_t^{\mathbf{y}}\right] d\mathbf{y}(t)}{\mathbb{E}_{\hat{P}} \left[M(t) | \mathcal{F}_t^{\mathbf{y}}\right]} \\ &\quad - \frac{\mathbb{E}_{\hat{P}} \left[M(t) \ell\big(\mathbf{z}_\mu(t)\big) | \mathcal{F}_t^{\mathbf{y}}\right] d\left(\mathbb{E}_{\hat{P}} \left[M(t) | \mathcal{F}_t^{\mathbf{y}}\right]\right)}{\mathbb{E}_{\hat{P}} \left[M(t) | \mathcal{F}_t^{\mathbf{y}}\right]^2} \\ &\quad + \frac{\mathbb{E}_{\hat{P}} \left[M(t) \ell\big(\mathbf{z}_\mu(t)\big) | \mathcal{F}_t^{\mathbf{y}}\right] \mathbb{E}_{\hat{P}} \left[M(t) g^T\big(t, \mathbf{z}_\mu(t)\big) | \mathcal{F}_t^{\mathbf{y}}\right]}{\mathbb{E}_{\hat{P}} \left[M(t) | \mathcal{F}_t^{\mathbf{y}}\right]^3} \mathbb{E}_{\hat{P}} \left[M(t) g\big(t, \mathbf{z}_\mu(t)\big) | \mathcal{F}_t^{\mathbf{y}}\right] dt \\ &\quad - \frac{\mathbb{E}_{\hat{P}} \left[M(t) \ell\big(\mathbf{z}_\mu(t)\big) g^T\big(t, \mathbf{z}_\mu(t)\big) | \mathcal{F}_t^{\mathbf{y}}\right]}{\mathbb{E}_{\hat{P}} \left[M(t) | \mathcal{F}_t^{\mathbf{y}}\right]^2} \mathbb{E}_{\hat{P}} \left[M(t) g\big(t, \mathbf{z}_\mu(t)\big) | \mathcal{F}_t^{\mathbf{y}}\right] dt \end{aligned} \tag{4.38}$$

$$
\begin{aligned}
&= \mathbb{E}_P\left[\Lambda_{\mu_t^\omega}\ell\big(\mathbf{z}_\mu(t)\big)|\mathcal{F}_t^{\mathbf{y}}\right] dt + \mathbb{E}_P\left[\ell\big(\mathbf{z}_\mu(t)\big)g^T\big(t,\mathbf{z}_\mu(t)\big)|\mathcal{F}_t^{\mathbf{y}}\right] d\mathbf{y}(t) \\
&\qquad - \mathbb{E}_P\left[\ell\big(\mathbf{z}_\mu(t)\big)|\mathcal{F}_t^{\mathbf{y}}\right]\mathbb{E}_P\left[g^T\big(t,\mathbf{z}_\mu(t)\big)|\mathcal{F}_t^{\mathbf{y}}\right] d\mathbf{y}(t) \\
&\qquad + \mathbb{E}_P\left[\ell\big(\mathbf{z}_\mu(t)\big)|\mathcal{F}_t^{\mathbf{y}}\right]\mathbb{E}_P\left[g^T\big(t,\mathbf{z}_\mu(t)\big)|\mathcal{F}_t^{\mathbf{y}}\right]\mathbb{E}_P\left[g\big(t,\mathbf{z}_\mu(t)\big)|\mathcal{F}_t^{\mathbf{y}}\right] dt \\
&\qquad - \mathbb{E}_P\left[\ell\big(\mathbf{z}_\mu(t)\big)g^T\big(t,\mathbf{z}_\mu(t)\big)|\mathcal{F}_t^{\mathbf{y}}\right]\mathbb{E}_P\left[g\big(t,\mathbf{z}_\mu(t)\big)|\mathcal{F}_t^{\mathbf{y}}\right] dt \qquad (4.39)\\
&= \mathbb{E}_P\left[\Lambda_{\mu_t^\omega}\ell\big(\mathbf{z}_\mu(t)\big)|\mathcal{F}_t^{\mathbf{y}}\right] dt + \mathbb{E}_P\left[\ell\big(\mathbf{z}_\mu(t)\big)g^T\big(t,\mathbf{z}_\mu(t)\big)|\mathcal{F}_t^{\mathbf{y}}\right] d\mathcal{I}(t) \\
&\qquad - \mathbb{E}_P\left[\ell\big(\mathbf{z}_\mu(t)\big)|\mathcal{F}_t^{\mathbf{y}}\right]\mathbb{E}_P\left[g^T\big(t,\mathbf{z}_\mu(t)\big)|\mathcal{F}_t^{\mathbf{y}}\right] d\mathcal{I}(t),
\end{aligned}
$$

where the first set of square brackets in (4.38) holds due to (4.31), the second one holds due to (4.37), and the last term is valid since the quadratic variation of two processes is given by

$$
\begin{aligned}
&\left\langle \mathbb{E}_{\hat{P}}\left[M(\cdot)\ell\big(\mathbf{z}_\mu(\cdot)\big)|\mathcal{F}_\cdot^{\mathbf{y}}\right], \frac{1}{\mathbb{E}_{\hat{P}}\left[M(\cdot)|\mathcal{F}_\cdot^{\mathbf{y}}\right]}\right\rangle_t \\
&\qquad = \frac{1}{4}\Bigg(\left\langle \mathbb{E}_{\hat{P}}\left[M(\cdot)\ell\big(\mathbf{z}_\mu(\cdot)\big)|\mathcal{F}_\cdot^{\mathbf{y}}\right] + \frac{1}{\mathbb{E}_{\hat{P}}\left[M(\cdot)|\mathcal{F}_\cdot^{\mathbf{y}}\right]}\right\rangle_t \\
&\qquad\qquad - \left\langle \mathbb{E}_{\hat{P}}\left[M(\cdot)\ell\big(\mathbf{z}_\mu(\cdot)\big)|\mathcal{F}_\cdot^{\mathbf{y}}\right] - \frac{1}{\mathbb{E}_{\hat{P}}\left[M(\cdot)|\mathcal{F}_\cdot^{\mathbf{y}}\right]}\right\rangle_t\Bigg), \qquad (4.40)
\end{aligned}
$$

which can be shown to be equal to the last term in (4.38). The rest of the steps can be verified via (4.28), (4.31), and (4.37), where in (4.39) we substitute the innovations process $\mathcal{I}(t) + \int_0^t \mathbb{E}_P\left[g\big(s,\mathbf{z}_\mu(s)\big)|\mathcal{F}_s^{\mathbf{y}}\right] ds$. ▯

Remark 4.2. Let $C_\infty(\mathbb{R}^n)$ be the Banach space of bounded functions such that $\theta : \mathbb{R}^n \to \mathbb{R}$ with $\lim_{\|x\|\to\infty}\theta(x) = 0$. Let $C_\infty^2(\mathbb{R}^n)$ be the Banach space of twice continuously differentiable and bounded functions $\theta : \mathbb{R}^n \to \mathbb{R}$ such that the first derivative θ' and the second derivative θ'' belong to $C_\infty(\mathbb{R}^n)$. Consider the dual space $C_\infty^{2;*}(\mathbb{R}^n)$ which has the property that $\mathcal{P}(\mathbb{R}^n)$ is a closed, bounded, and convex subset of $C_\infty^{2;*}(\mathbb{R}^n)$. We remark that instead of measure (or distribution) estimation, when the random density process lies in $C_\infty^2(\mathbb{R}^n)$ (or in another separable Banach space) one could consider the joint state and density $z_p := (z, p)$ estimation in section 4 and obtain recursion for the nonlinear filter.

5. Conclusion. In this work we have considered partially observed stochastic processes which have MV type dynamics and hence the process evolution depends on the distribution of the solution of the process. We have developed nonlinear filtering equations for such systems based on a classification where the measure term is either deterministic or random and where the state process or the joint state and measure process are being estimated. The estimation of the measure term is motivated by the recent developments in MFG theory where the state dynamics of the representative agents are of MV type SDEs. In this setting, we have characterized the nonlinear filtering equations for both normalized and unnormalized forms. A preliminary application of nonlinear filtering theory for such systems is reported in [10], where the state of the major agent is partially observed. The generalization of such a result including partial self state and the stochastic measure observation is of particular interest and is currently under development.

In certain situations, such as partially observed stochastic control problems, one needs an expression for the conditional density. For the state estimation in SMV systems such a filtering equation for the conditional density can be derived from the optimal filtering equation by extending the result in the literature to the case where the coefficients are random. Furthermore, the nonlinear filtering equations for the conditional density in the joint state and measure estimation are also of interest. In this case, however, the signal process takes values in a direct sum of Banach spaces and, consequently, one first needs to obtain the adjoint operator of the infinitesimal generator on this Banach space, which, among other technicalities, requires the development of integration by parts methods in certain infinite dimensional spaces (see [3], [2]). This extension is also under development.

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