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STABILIZATION IN THE SAMPLE-AND-HOLD SENSE OF NONLINEAR RETARDED SYSTEMS*

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Abstract. In this paper, the problem of the stabilization in the sample-and-hold sense for fully nonlinear systems with an arbitrary number of arbitrary discrete as well as of distributed time delays is studied. It is shown that steepest descent feedbacks, continuous or not, induced by Lyapunov–Krasovskii functionals in a suitable (large) class, are stabilizers in the sample-and-hold sense. The fact that discontinuities are overcome by the sampling and holding process enlarges greatly the possibility of finding successful controllers for retarded nonlinear systems, by means of control Lyapunov–Krasovskii functionals.

Key words. control Lyapunov–Krasovskii functionals, retarded functional differential equations, stabilization in the sample-and-hold sense

AMS subject classifications. 34K20, 34K35, 93C10, 93C23, 93C57

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1. Introduction.

The notion of stabilization in the sample-and-hold sense has been introduced in 1997 by Clarke et al. in [9], and widely studied for systems described by ordinary differential equations. The following result holds for nonlinear systems described by ordinary differential equations (see [9], [7]): any steepest descent feedback, induced by a control Lyapunov function, is a stabilizer in the sample-and-hold sense. The main advantage of this result is the possibility of using discontinuous steepest descent feedbacks. In this paper we prove an analogous result for nonlinear retarded systems.

The stabilization problem for nonlinear systems with delays in the state and/or in the input has involved many researchers, at least in the last ten years (see, for instance, [2], [3], [4], [16], [19], [34], [38], [40], [42], [43], [44], [56], [60]). Though many methods are available in the literature, the problem of global stabilization for fully nonlinear systems with an arbitrary number of discrete delays, of any size, and of distributed delays, is far from being completely solved. This paper concerns the study of a stabilization method for retarded systems, in the path proposed by Artstein in [1] for systems described by ordinary differential equations, that is, by control Lyapunov functions (see [1], [55]). A known problem when dealing with control Lyapunov functions is that in some cases the candidate feedback control law is discontinuous (see [7]). For instance, when control Lyapunov–Razumikhin functions for retarded systems (see [22]) are used, the problem of the feedback discontinuity can arise if one makes use of universal formulas such as Sontag's one (see [55]). By suitable control Lyapunov–Krasovskii functionals, continuous feedback control laws can be provided by universal formulas (see [23]), when the small control property is satisfied (see [55] and references therein). In [26], [27], the authors prove the equivalence of the existence of a completely locally Lipschitz control Lyapunov–Krasovskii functional satisfying the small control property, and the stabilizability property by means of

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completely locally Lipschitz control laws, for fully nonlinear, retarded systems. In the paper [48], discontinuities are overcome by means of an approximation of Sontag's formula in subsets of the state space where the feedback may loose the locally Lipschitz property, and by making use of input-to-state stability redesign methods (see [54], [57], [47], [49]). It should be stressed that, in general, the aim of avoiding discontinuities may well reduce the power, and ease of use, of the control Lyapunov–Krasovskii functionals methodology, as far as the possibility of finding stabilizing feedback control laws is concerned. For instance, it may well happen that the control Lyapunov–Krasovskii functional at hand does not satisfy the small control property, thus leading often to discontinuous feedback (see [48]). Another frequent problem in practice is the actuator input saturation, which must be taken into account in the design of state feedback control laws (see, for instance, [14], [62] as far as systems with delays are concerned). As far as the sampled-data stabilization of retarded systems is concerned, many results are available in the literature (see, for instance, [13], [35] for the linear case, [29], [28], [39] for the nonlinear case). In [29], a class of uncertain nonlinear systems is considered, and it is shown for this class that sampled-data stabilization can be achieved, even in the case of measurement and input delays. In [28], the sampled-data stabilization of nonlinear systems with arbitrary delay in the input channel is obtained. The hypotheses that the system at hand is forward complete, and that either there exists a state feedback control law which globally asymptotically stabilizes the corresponding undelayed system, when it is applied on a continuous time basis, or the undelayed system at hand is stabilizable by means of a sampled-data control law, are introduced. In [39], nonlinear systems which are globally stabilizable by state feedback (applied on a continuous time basis) are considered, and robustness issues with respect to corruption of the control law, due to sampling and delay, are addressed. Here, we do not introduce the hypothesis of forward completeness, nor the one of the existence (knowledge) of any state feedback stabilizer (applied on a continuous or a discrete time basis). Instead, the existence of a control Lyapunov–Krasovskii functional and of an induced steepest descent feedback (continuous or not, see below) is assumed. On the other hand, if a stabilizer, in the continuous time basis, is available, then the theory here developed can still be used in order to guarantee stabilization in the sample-and-hold sense, when the control law is applied with a sampled-data implementation by digital devices, a typical situation in practice.

In this paper, discontinuities in the feedback are "welcome." We introduce a (large) class of control Lyapunov–Krasovskii functionals, and prove the following main result: any steepest descent feedback, regardless of whether it is continuous or not, induced by a control Lyapunov–Krasovskii functional in the above class, is a stabilizer in the sample-and-hold sense. Here, fully nonlinear systems with an arbitrary number of discrete delays of any size as well as of distributed delays in the state are considered. A discrete delay in the input channel is allowed, as long as a causal steepest descent feedback is available. State feedbacks with input dynamics (see the recent monograph [5], [2], [4], [3], [41], [25]) are not dealt with in this paper. Results about stabilizing state feedbacks which do not involve any input dynamics have been recently provided in the framework of linear time delay systems (see, for instance, [61], [58]). We recall the stabilizing effect which can be obtained by artificially introducing delays, by means of a static (memory) state feedback (see [53]). In this paper, we assume that the input belongs to a compact set, in order to adhere to the actuator saturation constraints.

The paper is organized as follows. In section 2, the retarded system plant and some preliminary results are introduced. Section 3 deals with the new notion of smoothly separable Lyapunov–Krasovskii functionals. Section 4 covers control Lyapunov–Krasovskii functionals and induced steepest descent feedbacks. In section 5, the main result of the paper is presented. In section 6, numerical examples are reported. In section 7, final conclusions are drawn and possible developments highlighted.

Notation. R denotes the set of real numbers, $R^\star$ denotes the extended real line $[-\infty, +\infty]$, R^+ denotes the set of nonnegative reals $[0, +\infty)$. The symbol $|\cdot|$ stands for the Euclidean norm of a real vector, or the induced Euclidean norm of a matrix. The essential supremum norm of an essentially bounded function is indicated with the symbol $\|\cdot\|_\infty$. For a positive integer n, for a positive real Δ (maximum involved time delay), $\mathcal{C}$ and $W^{1,\infty}$ denote the space of the continuous functions mapping $[-\Delta, 0]$ into R^n and the space of the absolutely continuous functions, with essentially bounded derivative, mapping $[-\Delta, 0]$ into R^n, respectively. For a positive real p, for $\phi \in \mathcal{C}$, $\mathcal{C}_p(\phi) = \{\psi \in \mathcal{C} : \|\psi - \phi\|_\infty \leq p\}$. The symbol $\mathcal{C}_p$ denotes $\mathcal{C}_p(0)$. For a continuous function $x : [-\Delta, c) \to R^n$, with $0 < c \leq +\infty$, for any real $t \in [0, c)$, x_t is the function in $\mathcal{C}$ defined as $x_t(\tau) = x(t + \tau)$, $\tau \in [-\Delta, 0]$. For a positive integer n, $C^1(R^n; R^+)$ ($C^1(R^n; R)$) denotes the space of the continuous functions from R^n to R^+ (to R), admitting continuous (partial) derivatives; $C^1_L(R^n; R^+)$ denotes the subset of the functions in $C^1(R^n; R^+)$ admitting locally Lipschitz (partial) derivatives; $C^1(R^+, R^+)$ denotes the space of the continuous functions from $R^+ \to R^+$, admitting continuous derivative; $C^1_L(R^+; R^+)$ denotes the subset of functions in $C^1(R^+; R^+)$ admitting locally Lipschitz derivative. For a positive integer n, a positive real q, B_q denotes the set of vectors $z \in R^n$ satisfying $|z| \leq q$. For given positive integers n, m, a map $f : \mathcal{C} \times R^m \to R^n$ is said to be Lipschitz on bounded sets if, for any positive real q, there exists a positive real L_q such that, for any $\phi_i \in \mathcal{C}_q$, $u_i \in B_q$, $i = 1, 2$, the inequality $|f(\phi_1, u_1) - f(\phi_2, u_2)| \leq L_q(\|\phi_1 - \phi_2\|_\infty + |u_1 - u_2|)$ holds. For given positive integers n, m, and a compact set $U \subset R^m$ containing the origin in the interior, a map $f : \mathcal{C} \times U \to R^n$ is said to be Lipschitz on bounded sets, uniformly in U, if, for any positive real q, there exists a positive real L_q such that, for any $\phi_i \in \mathcal{C}_q$, $i = 1, 2$, the inequality $|f(\phi_1, u) - f(\phi_2, u)| \leq L_q\|\phi_1 - \phi_2\|_\infty$ for all $u \in U$ holds. For a function $f \in C^1(R^n, R)$, the symbol $\nabla f(x)$ denotes the gradient of the function f in $x \in R^n$. Let us here recall that a continuous function $\gamma : R^+ \to R^+$ is of class $\mathcal{P}_0$ if $\gamma(0) = 0$ and $\gamma(s) \geq 0$, $s \geq 0$; of class $\mathcal{P}$ if it is of class $\mathcal{P}_0$ and $\gamma(s) > 0$, $s > 0$; of class $\mathcal{K}$ if it is of class $\mathcal{P}$ and strictly increasing; of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and unbounded. The symbols $\underline{0}$ and I_d denote the null and the identity function in R^+, respectively. The function $sgn : R \to \{-1, 0, 1\}$ is defined as $sgn(s) = \frac{s}{|s|}$, $s \neq 0$, and $sgn(0) = 0$. The function $sat : R \to [-1, 1]$ is defined as $sat(s) = sgn(s)$, $s \in (-\infty, -1) \cup (1, +\infty)$, $sat(s) = s$, $s \in [-1, 1]$. The symbols $\cup$, $\cap$, $\overline{co}$ denote union, intersection, convex hull (of sets), respectively. The symbol $\circ$ denotes composition (of functions). For a given positive integer n: for a symmetric, positive definite matrix $P \in R^{n\times n}$, $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ denote the maximum and the minimum eigenvalue of P, respectively; I_n denotes the identity matrix in $R^{n\times n}$. For a nonnegative real s, $[s]$ is the largest nonnegative integer smaller than or equal to s. The symbols H_0 and H denote Heaviside functions defined, for $s \in R$, as follows: $H_0(s) = 1$ if $s \geq 0$, $H_0(s) = 0$ if $s < 0$, $H(s) = 1$ if $s > 0$, $H(s) = 0$ if $s \leq 0$. Throughout the paper, ODE stands for ordinary differential equation, RFDE stands for retarded functional differential equation, GAS stands for globally asymptotically stable or global asymptotic stability, ISS stands for input-to-state stable or input-to-state stability, CLKF stands for control Lyapunov–Krasovskii functional.

2. Nonlinear retarded systems. Let us consider the system described by the following RFDE (see [32], [18]):

$$\begin{aligned} \dot{x}(t) &= f(x_t, u(t-\omega)), \qquad t \geq 0, \ a.e., \\ x(\tau) &= x_0(\tau), \qquad \tau \in [-\Delta, 0], \qquad x_0 \in \mathcal{C}, \end{aligned} \tag{2.1}$$

where $x(t) \in R^n$, n is a positive integer; ω is a nonnegative real (time delay in the input channel); Δ is a nonnegative integer, the maximum involved time delay; $x_t \in \mathcal{C}$; f is a map from $\mathcal{C} \times R^m$ to R^n, Lipschitz on bounded sets; m is a positive integer; $u(t) \in U$ is a measurable signal, $U \subset R^m$ is a compact set containing the origin in the interior. We assume that $f(0,0) = 0$ and $\Delta \geq \omega$. Equation (2.1) admits a locally absolutely continuous solution in a maximal time interval $[0, b)$, with $0 < b \leq +\infty$ (see [18]).

For a locally Lipschitz functional $V : \mathcal{C} \to R^+$, let, for $\phi \in \mathcal{C}$, $u \in R^m$,

$$D^+V(\phi, u) = \limsup_{h \to 0^+} \frac{V(\phi_{h,u}) - V(\phi)}{h}, \tag{2.2}$$

where, for $0 \leq h < \Delta$, $\phi_{h,u} \in \mathcal{C}$ is given by

$$\phi_{h,u}(s) = \begin{cases} \phi(s+h), & s \in [-\Delta, -h), \\ \phi(0) + (s+h) f(\phi, u), & s \in [-h, 0]. \end{cases} \tag{2.3}$$

The map $(\phi, u) \to D^+V(\phi, u)$ is the derivative in Driver's form of the functional V (see [45] and references therein). We recall the following lemma (see, for instance, [51] for the definition of the 0-GAS property).

LEMMA 2.1 (see [45], [46]). *Let $x(t)$ be the solution in a maximal time interval $[0, b)$, $0 < b \leq +\infty$, of the RFDE* (2.1). *Let $V : \mathcal{C} \to R^+$ be a locally Lipschitz functional. Then, almost everywhere in $[0, b)$, the following equality holds:*

$$\limsup_{h \to 0^+} \frac{1}{h} \left(V(x_{t+h}) - V(x_t) \right) = D^+V(x_t, u(t)). \tag{2.4}$$

Moreover, if the initial condition $x_0 \in W^{1,\infty}$, then the functional $t \to V(x_t)$ is locally absolutely continuous in $[0, b)$. Finally, the 0-GAS property of system (2.1) with $u(t) \equiv 0$, with $x_0 \in W^{1,\infty}$, is equivalent to the 0-GAS property with $x_0 \in \mathcal{C}$.

It is convenient to recall here the following direct and converse Lyapunov–Krasovskii theorem (see Lemma A.8 in [51]).

THEOREM 2.2. *The system described by (2.1) with $u(t) \equiv 0$ is 0-GAS if and only if there exist a locally Lipschitz functional $V : \mathcal{C} \to R^+$, functions α_1, α_2 of class $\mathcal{K}_\infty$, a function α_3 of class $\mathcal{K}$, such that the following conditions hold for all $\phi \in \mathcal{C}$:*

(i) $\alpha_1(\|\phi\|_\infty) \leq V(\phi) \leq \alpha_2(\|\phi\|_\infty)$;
(ii) $D^+V(\phi, 0) \leq -\alpha_3(\|\phi\|_\infty)$.

3. Smoothly-separable Lyapunov–Krasovskii functionals. The following definition denotes a very large set of functionals which will be used from here on. To my knowledge, such a set includes most of the Lyapunov–Krasovskii functionals used in the literature concerning retarded systems. For instance, standard complete quadratic Lyapunov–Krasovskii functionals (see [17], [31]), functionals defined in [23], [36] are included in the set.

DEFINITION 3.1. *A functional $V : \mathcal{C} \to R^+$ is said to be smoothly separable if there exist a function $V_1 \in C^1_L(R^n; R^+)$, a locally Lipschitz functional $V_2 : \mathcal{C} \to$*

R^+, functions β_i of class $\mathcal{K}_\infty$, $i = 1, 2$, such that, for any $\phi \in \mathcal{C}$, the following equality/inequalities hold:

$$V(\phi) = V_1(\phi(0)) + V_2(\phi), \qquad \beta_1(|\phi(0)|) \le V_1(\phi(0)) \le \beta_2(|\phi(0)|). \tag{3.1}$$

In the following, for $\phi \in \mathcal{C}$, $u \in U$, for a function $p \in C^1(R^+; R^+)$, for a function $V_1 \in C^1(R^n; R^+)$, we denote with $D^+V_1(\phi, u)$ and $D^+p \circ V_1(\phi, u)$ the following limits:

$$D^+V_1(\phi,u) = \limsup_{h\to 0^+} \frac{V_1(\phi(0)+hf(\phi,u)) - V_1(\phi(0))}{h} = \left.\frac{\partial V_1(x)}{\partial x}\right|_{x=\phi(0)} f(\phi,u),$$

$$\begin{aligned} D^+p\circ V_1(\phi,u) &= \limsup_{h\to 0^+} \frac{p\circ V_1(\phi(0)+hf(\phi,u)) - p\circ V_1(\phi(0))}{h} \\ &= \left.\frac{dp(s)}{ds}\right|_{s=V_1(\phi(0))} \left.\frac{\partial V_1(x)}{\partial x}\right|_{x=\phi(0)} f(\phi,u). \end{aligned} \tag{3.2}$$

The following lemma has been very useful in the framework of backstepping control design for retarded systems in triangular form in [26], [27]. It happens that it will be useful here for providing upper bounds in terms of the supremum norm, for the derivative of functionals suitably obtained by smoothly separable ones. Upper bounds in terms of the supremum norm will play a key role throughout the paper.

LEMMA 3.2 ([26, Lemma 5.3]; [27, Lemma 6.7, p. 333]). *Let $V_1 \in C^1(R^n; R^+)$. Let μ be a positive real. Let $V_3 : \mathcal{C} \to R^+$ be defined, for $\phi \in \mathcal{C}$, as $V_3(\phi) = \sup_{\theta\in[-\Delta,0]} e^{\mu\theta} V_1(\phi(\theta))$. Then, the following inequality holds, for any $\phi \in \mathcal{C}$, for any $u \in U$:*

$$D^+V_3(\phi,u) \le \begin{cases} -\mu V_3(\phi), & V_3(\phi) > V_1(\phi(0)), \\ \max\{-\mu V_3(\phi),\ D^+V_1(\phi,u)\}, & V_3(\phi) = V_1(\phi(0)). \end{cases} \tag{3.3}$$

By Lemma 3.2, the next theorem and corollary follow. They show how smoothly separable Lyapunov–Krasovskii functionals can play an important role in the stability theory of retarded systems (see Remark 1).

THEOREM 3.3. *Let $V : \mathcal{C} \to R^+$ be a smoothly separable functional. Let, for $\phi \in \mathcal{C}$, $V(\phi) = V_1(\phi(0)) + V_2(\phi)$, as in Definition* 3.1. *Let η, μ be positive reals, p be a function in $C^1_L(R^+; R^+)$, of class $\mathcal{K}_\infty$, $m \in \{0, 1\}$, γ_i, $i = 1, 2$, be functions of class $\mathcal{K}_\infty$, $\overline{\alpha}$ be a function of class $\mathcal{K}$ such that $I_d - \overline{\alpha}$ is of class $\mathcal{K}_\infty$, α_4 be a function of class $\mathcal{P}_0$, such that, for all $\phi \in \mathcal{C}$, $u \in U$, the following conditions hold:*

(1) $\gamma_1(|\phi(0)|) \le V(\phi) \le \gamma_2(\|\phi\|_\infty)$;

(2) $mD^+V(\phi,u) + \eta\max\{0, D^+p\circ V_1(\phi(0)) + \mu p\circ V_1(\phi(0))\} \le \alpha_4(|u|) + \overline{\alpha}\left(\eta\mu e^{-\mu\Delta} p\circ\beta_1(\|\phi\|_\infty)\right)$

with β_1 given in (3.1). *Let $V_3 : \mathcal{C} \to R^+$ be defined, for $\phi \in \mathcal{C}$, as $V_3(\phi) = \sup_{\theta\in[-\Delta,0]} e^{\mu\theta} p\circ V_1(\phi(\theta))$. Let $V_\infty : \mathcal{C} \to R^+$ be defined, for $\phi \in \mathcal{C}$, as follows:*

$$V_\infty(\phi) = mV(\phi) + \eta V_3(\phi). \tag{3.4}$$

Then, there exist functions α_i, $i = 1, 2, 3$, of class $\mathcal{K}_\infty$, such that the following conditions hold:

(i) $\alpha_1(\|\phi\|_\infty) \le V_\infty(\phi) \le \alpha_2(\|\phi\|_\infty)$ $\forall \phi \in \mathcal{C}$;

(ii) $D^+V_\infty(\phi,u) \le -\alpha_3(\|\phi\|_\infty) + \alpha_4(|u|)$ $\forall \phi \in \mathcal{C}, u \in U$.

Proof. Let β_2 be the function given in (3.1). As far as the point (i) is concerned, let us define the functions $\alpha_i, i = 1, 2$, as follows:

$$\alpha_1(s) = \eta e^{-\mu\Delta} p \circ \beta_1(s), \qquad \alpha_2(s) = m\gamma_2(s) + \eta p \circ \beta_2(s). \tag{3.5}$$

By these functions α_1, α_2 of class $\mathcal{K}_\infty$, the point (i) holds true. As far as the point (ii) is concerned, it follows from Lemma 3.2, by the inequality

$$\begin{aligned} &D^+V_3(\phi, u) \le -\mu V_3(\phi) + \max\{0, D^+p \circ V_1(\phi, u) + \mu p \circ V_1(\phi(0))\} \\ &\qquad \forall\phi \in \mathcal{C},\ \forall u \in U. \end{aligned} \tag{3.6}$$

Indeed, taking into account the inequality in point (2) and the inequality (3.6), we obtain

$$D^+V_\infty(\phi, u) \le -\eta\mu V_3(\phi) + \alpha_4(|u|) + \overline{\alpha}\left(\eta\mu e^{-\mu\Delta} p \circ \beta_1(\|\phi\|_\infty)\right) \qquad \forall\phi \in \mathcal{C}. \tag{3.7}$$

Thus, let us define the function α_3 of class $\mathcal{K}_\infty$ as follows:

$$\alpha_3(s) = (I_d - \overline{\alpha})\left(\eta\mu e^{-\mu\Delta} p \circ \beta_1(s)\right), \qquad s \ge 0. \tag{3.8}$$

The inequality in (ii) holds, for any $\phi \in \mathcal{C}$, $u \in U$, with this choice of α_3. The proof of the theorem is complete. ∎

COROLLARY 3.4. *Let $V : \mathcal{C} \to R^+$ be a smoothly separable functional. Let, for $\phi \in \mathcal{C}$, $V(\phi) = V_1(\phi(0)) + V_2(\phi)$, as in Definition* 3.1. *Let η, μ be positive reals, p be a function in $C^1_L(R^+; R^+)$, of class $\mathcal{K}_\infty$, $m \in \{0, 1\}$, γ_i, $i = 1, 2$, be functions of class $\mathcal{K}_\infty$, α_4 be a function of class $\mathcal{P}_0$, $\overline{\alpha}$ be a function of class $\mathcal{K}$ such that $I_d - \overline{\alpha}$ is of class $\mathcal{K}_\infty$, such that, for all $\phi \in \mathcal{C}$, $u \in U$, the following conditions hold:*

(1) $\gamma_1(|\phi(0)|) \le V(\phi) \le \gamma_2(\|\phi\|_\infty)$;

(2) $mD^+V(\phi, u) + \eta \max\{0, D^+p \circ V_1(\phi(0)) + \mu p \circ V_1(\phi(0))\} \le \alpha_4(|u|) + \overline{\alpha}(\eta\mu e^{-\mu\Delta} p \circ \beta_1(\|\phi\|_\infty))$ *with β_1 given in* (3.1).

Then, the system described by (2.1) *is ISS with respect to the input u.*

Proof. From Theorem 3.3, it follows that the functional $V_\infty = mV + \eta V_3$ satisfies, for suitable functions α_i, $i = 1, 2, 3$, of class $\mathcal{K}_\infty$, the inequalities in (i), (ii) in the same theorem. The result follows from Theorem 3.1 in [50]. ∎

Remark 1. Theorem 3.3 shows how to find Lyapunov–Krasovskii functionals for which the supremum norm is involved in upper and lower bounds from functionals for which the supremum norm is not involved in the upper and lower bounds. For instance, let us consider the system described by (2.1) with $u(t) \equiv 0$. Theorem 3.3 allows us to find, from a smoothly separable functional $V = V_1 + V_2$, which satisfies the inequalities (1), (2) (in the spirit of Theorem 2.1 in [18]), the functional $V_\infty = mV + \eta V_3$, which satisfies the inequalities (i), (ii) (in the spirit of Theorem 2.2). This is possible provided that the inequality (2) holds, for suitable positive reals μ, η, $m \in \{0, 1\}$ and function $p \in C^1_L(R^+; R^+)$, of class $\mathcal{K}_\infty$, in the case $u = 0$. The supremum norm for upper and lower bounds of involved functionals will be a key issue in the proof of the main result of this paper (see Theorem 5.3). As well, for ISS concerns ($u(t) \neq 0$), Theorem 3.3 allows us to satisfy, from a smoothly separable functional $V = V_1 + V_2$, which satisfies the inequalities (1), (2) (in the spirit of Theorem 2.1 in [18]), the inequalities (i), (ii) again in the spirit of Theorem 2.2. This is possible provided that the inequality in (2) holds satisfied. As reported in Corollary 3.4, the inequalities in (i), (ii) of Theorem 3.3 imply the ISS property of the system described by (2.1), with respect to the input u, according to Theorem 3.1 in [50]. The advantage of the norms used in inequalities (1), (2), with respect to the norms used in Theorem 3.1 in [50], is evident. More, the result of Theorem 3.3 can be used in the study of the stability of interconnected and networked retarded systems. Compare the results provided in [21, Theorem 8, Assumption 6], [20, Theorem 6, Assumption 1], which are in the spirit of Theorem 3.1 in [50], and the results provided in [11, Theorem 4.5, inequality (52)], which are in the spirit of Theorem 2.2. Functionals V which satisfy inequalities like, for instance, (1) in Theorem 3.3 and $D^+V(\phi,u) \le -\gamma_3(|\phi(0)|) + \gamma_4(|u|)$, for suitable functions γ_3 of class $\mathcal{K}_\infty$ and γ_4 of class $\mathcal{P}_0$, are in general inconclusive in proving stability of interconnected systems, by means of Theorem 4.5 in [11], when discrete time delays appear in interconnecting channels. Instead, functionals of the type of V_∞, obtained by V as shown in Theorem 3.3, which satisfy inequalities like (i), (ii) in the same Theorem, can well succeed in proving the above stability.

By Lemma 3.2, the following theorem is proved. It is in the spirit of Theorem 2.2 (i.e., the supremum norm of the state is involved in the upper and lower bounds of a new Lyapunov–Krasovskii functional obtained by V in Definition 3.1, as well as in the upper bound of its derivative). This theorem is fundamental to prove the main result of this paper (i.e., Theorem 5.3).

THEOREM 3.5. *Let $V : \mathcal{C} \to R^+$ be a smoothly separable functional (i.e., for $\phi \in \mathcal{C}$, $V(\phi) = V_1(\phi(0)) + V_2(\phi)$, as in Definition* 3.1*). Let the map $\phi \to D^+V_2(\phi,u)$, $\phi \in \mathcal{C}$, $u \in U$, be Lipschitz on bounded subsets of $\mathcal{C}$, uniformly in $u \in U$. Let there exist a feedback $k : \mathcal{C} \to U$ (continuous or not), positive reals η, μ, a function p in $C^1_L(R^+;R^+)$, of class $\mathcal{K}_\infty$, $m \in \{0,1\}$, functions γ_i of class $\mathcal{K}_\infty$, $i = 1,2$, a function $\overline{\alpha}$ of class $\mathcal{K}$ such that $I_d - \overline{\alpha}$ is of class $\mathcal{K}_\infty$, such that, for all $\phi \in \mathcal{C}$, the following conditions hold:*

(1) $\gamma_1(|\phi(0)|) \le V(\phi) \le \gamma_2(\|\phi\|_\infty)$;
(2) $mD^+V(\phi,k(\phi)) + \eta \max\{0, D^+p \circ V_1(\phi,k(\phi)) + \mu p \circ V_1(\phi(0))\} \le \overline{\alpha}(\eta\mu e^{-\mu\Delta} p \circ \beta_1(\|\phi\|_\infty))$

with β_1 given in (3.1)*. Let $V_3 : \mathcal{C} \to R^+$ be defined, for $\phi \in \mathcal{C}$, as $V_3(\phi) = \sup_{\theta\in[-\Delta,0]} e^{\mu\theta} p \circ V_1(\phi(\theta))$. Let $V_\infty : \mathcal{C} \to R^+$ be defined, for $\phi \in \mathcal{C}$, as*

$$V_\infty(\phi) = mV(\phi) + \eta V_3(\phi). \tag{3.9}$$

Let $\mathcal{D}_\infty : \mathcal{C} \times U \to R$ be defined, for $\phi \in \mathcal{C}$, $u \in U$, as

$$\mathcal{D}_\infty(\phi,u) = mD^+V(\phi,u) - \eta\mu V_3(\phi) + \eta \max\{0, D^+p \circ V_1(\phi,u) + \mu p \circ V_1(\phi(0))\}. \tag{3.10}$$

Then, there exist functions α_i, $i = 1,2,3$, of class $\mathcal{K}_\infty$, such that the following conditions hold:

(i) $\alpha_1(\|\phi\|_\infty) \le V_\infty(\phi) \le \alpha_2(\|\phi\|_\infty)$ $\forall \phi \in \mathcal{C}$;
(ii) *the map $\phi \to \mathcal{D}_\infty(\phi,u)$ is Lipschitz on bounded subsets of $\mathcal{C}$, uniformly in $u \in U$;*
(iii) $D^+V_\infty(\phi,u) \le \mathcal{D}_\infty(\phi,u)$ $\forall \phi \in \mathcal{C}$, $\forall u \in U$;
(iv) $\mathcal{D}_\infty(\phi,k(\phi)) \le -\alpha_3(\|\phi\|_\infty)$ $\forall \phi \in \mathcal{C}$.

Proof. Let β_2 be the function given in (3.1). As far as the point (i) is concerned, the same choice of functions α_i, $i = 1,2$, provided in (3.5) can be used here. As far as the point (ii) is concerned, the required Lipschitz property of the map $\phi \to \mathcal{D}_\infty(\phi,u)$ follows from the same property of the maps $\phi \to D^+V_1(\phi,u)$, $\phi \to D^+p \circ V_1(\phi,u)$ (recall the properties of V_1, p, and the Lipschitz property of f), $\phi \to D^+V_2(\phi,u)$, and from the Lipschitz property, on bounded subsets of $\mathcal{C}$, of the map $\phi \to V_3(\phi)$. As far as the point (iii) is concerned, it follows from Lemma 3.2, by the inequality (3.6). As far as the point (iv) is concerned, it follows from point (2) that

$$\mathcal{D}_\infty(\phi, k(\phi)) \leq -\eta\mu V_3(\phi) + \overline{\alpha}\left(\eta\mu e^{-\mu\Delta} p \circ \beta_1(\|\phi\|_\infty)\right) \qquad \forall \phi \in \mathcal{C}. \tag{3.11}$$

Thus, by the function α_3 of class $\mathcal{K}_\infty$ defined in (3.8) the inequality in (iv) holds, for any $\phi \in \mathcal{C}$. The proof of the theorem is complete. ∎

4. Control Lyapunov–Krasovskii functionals. The following definition is standard (see, for instance, [55], [23]).

DEFINITION 4.1. *A smoothly separable functional $V : \mathcal{C} \to R^+$ is said to be a CLKF if there exist functions γ_1, γ_2 of class $\mathcal{K}_\infty$ such that the following inequalities hold:*

(i) $\gamma_1(|\phi(0)|) \leq V(\phi) \leq \gamma_2(\|\phi\|_\infty)$ $\forall \phi \in \mathcal{C}$;

(ii) $\inf_{u\in U} D^+V(\phi, u) < 0$ $\forall \phi \in \mathcal{C},\ \phi(0) \neq 0$.

The following definition is an adaptation, to systems described by RFDEs, of the corresponding one given in [7], [9], for systems described by ODEs.

DEFINITION 4.2. *A map $k : \mathcal{C} \to U$ (continuous or not) is said to be a steepest descent feedback, induced by a CLKF V (i.e., V_1+V_2; see Definition 3.1, Definition 4.1), if the following conditions hold:*

(i) *there exist positive reals η and μ, $m \in \{0, 1\}$, a function $p \in C^1_L(R^+; R^+)$, of class $\mathcal{K}_\infty$, a function $\overline{\alpha}$ of class $\mathcal{K}$ such that $I_d - \overline{\alpha}$ is of class $\mathcal{K}_\infty$, such that*

$$\begin{aligned} &mD^+V(\phi, k(\phi)) + \eta \max\{0, D^+p \circ V_1(\phi, k(\phi)) + \mu p \circ V_1(\phi(0))\} \\ &\qquad \leq \overline{\alpha}\left(\eta\mu e^{-\mu\Delta} p \circ \beta_1(\|\phi\|_\infty)\right) \ \ \forall \phi \in \mathcal{C} \end{aligned} \tag{4.1}$$

with β_1 given in (3.1);

(ii) *for any $\phi_1, \phi_2 \in \mathcal{C}$, satisfying $\phi_1(\tau) = \phi_2(\tau)$, $\tau \in [-\Delta, -\omega]$, the equality holds*

$$k(\phi_1) = k(\phi_2).$$

Remark 2. In the point (i) of Definition 4.2, one may expect the standard inequality

$$D^+V(\phi, k(\phi)) \leq -\gamma_3(|\phi(0)|) \tag{4.2}$$

with γ_3 a suitable function of class $\mathcal{K}$ (see, for instance, Theorem 2.1 in [18]). It happens that the inequality (4.1) is a key issue for proving Theorem 3.5, which provides a key result for proving the main theorem of the paper (see Theorem 5.3). If the standard inequality (4.2) is satisfied, then the fact that η, μ, and p can be chosen arbitrarily is undoubtedly helpful in order to satisfy the new inequality (4.1). Recall that

$$D^+V(\phi, k(\phi)) = D^+V_1(\phi, k(\phi)) + D^+V_2(\phi, k(\phi)), \tag{4.3}$$

and thus the inequality (4.1) is implied by the inequality (4.2), when the perturbation of the term $D^+V_1(\phi, k(\phi))$, by means of adding the terms $\eta D^+p \circ V_1(\phi, k(\phi))$ and $\eta\mu p \circ V_1(\phi(0))$ (recall the inequalities in (3.1) and (4.2)), is allowed in order for the full term in the left-hand side of (4.1) to remain suitably upper bounded (nonpositive if $\overline{\alpha} = \underline{0}$ is chosen). I stress the fact that η, μ can be arbitrarily small and that the function p can be arbitrarily slowly increasing. When $V = V_1$, then the inequality (4.1) is implied by the inequality (4.2), provided that there exists a function $p \in$

$C^1_L(R^+,R^+)$, of class $\mathcal{K}_\infty$, such that $p(s)\le\gamma_3\circ\beta_2^{-1}(s)$, $s\in R^+$ (clearly γ_3 must be of class $\mathcal{K}_\infty$, and $m=\eta=\mu=1$, $\overline{\alpha}=\underline{0}$ can be chosen). Further, the inequality (4.1) is implied by the inequality (4.2) if the following hypothesis holds satisfied: there exist a function $p\in C^1_L(R^+,R^+)$ of class $\mathcal{K}_\infty$ and a positive real $\overline{p}$ such that, for all $s\in R^+$, $\frac{dp(s)}{ds}\le\overline{p}$ and $p(s)\le\gamma_3\circ\beta_2^{-1}(s)$; there exists a positive real $\overline{\eta}$ such that $(1+g)D^+V_1(\phi,k(\phi))+D^+V_2(\phi,k(\phi))\le-\gamma_3(|\phi(0)|)$ for all $\phi\in\mathcal{C}$, for all $g\in[0,\overline{\eta}]$ (a property of robustness for the inequality (4.2), which may be obtained also by suitably rearranging V_1, V_2, and γ_3). Indeed, in this case, it is sufficient to pick $m=\mu=1$, $\eta=\frac{1}{\overline{p}}\min\{\overline{\eta},\overline{p}\}$, $\overline{\alpha}=\underline{0}$ and the inequality (4.1) is satisfied. The function $p(s)=b\,log_n(1+s^j)$, $b>0$, $j\ge1$, is helpful, for instance, for polynomial systems and standard quadratic Lyapunov–Krasovskii functionals. The strange-looking inequality (4.1) is actually not that strange. For instance, if the feedback k were continuous, the control law $u(t)=k(x_t)$ were applied on a continuous time basis, and the map describing the closed-loop dynamics were Lipschitz on any bounded subset of $\mathcal{C}$, then the resulting closed-loop system would be 0-GAS. This fact will be shown in the forthcoming Corollary 4.3.

Remark 3. We do not need to know any, eventual, stabilization property, from a theoretical point of view, of the steepest descent feedback k, which may well be discontinuous. Recall that, when discontinuities appear, the Filippov or Krasovskii solutions environment should be considered (see [12], [49]) for studying the eventual stabilization property (see [9], [7], as far as systems described by ODEs are concerned, for a discussion on this argument). In other words, it is not known a priori whether the system described by (2.1), in closed loop with the steepest descent feedback k as defined in Definition (4.2), applied on a continuous time basis, is 0-GAS in the Filippov or Krasovskii sense. It is not even known a priori if at least one solution exists in the Filippov or Krasovskii sense. In the case Krasovskii solutions exist, a sufficient condition for the 0-GAS property is provided by the following inequality (V and k are defined in Definitions 4.1, 4.2):

$$\sup_{v\in F[\phi]}\limsup_{h\to0^+}\frac{V(\phi^F_{h,v})-V(\phi)}{h}\le-\gamma(|\phi(0)|)\qquad\forall\phi\in\mathcal{C},\tag{4.4}$$

where γ is a suitable function of class $\mathcal{K}$, $\phi^F_{h,v}\in\mathcal{C}$ is defined, for $h\in[0,\Delta)$, as

$$\phi^F_{h,v}(s)=\begin{cases}\phi(s+h), & s\in[-\Delta,-h),\\ \phi(0)+(s+h)v, & s\in[-h,0],\end{cases}\tag{4.5}$$

and (see [49])

$$F[\phi]=\cap_{r>0}\overline{co}\left\{f(\psi,k(\psi)),\psi\in C_r(\phi)\right\}.\tag{4.6}$$

The question arises whether or not Krasovskii or classical solutions exist for the closed-loop system and whether or not, in the positive case, the inequality (4.4) holds true, for any f as in (2.1), any V as in Definition 4.1, and any k as in Definition 4.2. The answer is provided in the literature concerning systems described by ODEs, which are a special case of the systems described by (2.1) (i.e., with $\Delta=\omega=0$), see Remarks 4 and 5.

Remark 4. When $\Delta=\omega=0$ (i.e., (2.1) is an ODE), the inequality (4.2) implies the inequality (4.1) (for $m=1$, $\overline{\alpha}=\underline{0}$, and suitable η, μ, p), provided that the following hypotheses hold: $V=V_1\in C^1_L(R^n,R^+)$; there exists a function $p_1\in$

$C_L^1(R^+;R^+)$, of class $\mathcal{K}_\infty$, such that

$$\gamma_3 \circ \gamma_2^{-1}(s) \geq p_1(s), \qquad s \geq 0. \tag{4.7}$$

Indeed, in this case, inequality (4.1) (with $m=1$, $\overline{\alpha}=\underline{0}$) returns the inequality, for $x \in R^n$ (f becomes a map from $R^n \times U$ to R^n, k becomes a map from R^n to U),

$$D^+V(x,k(x)) + \eta \max\left\{0, D^+p \circ V(x,k(x)) + \mu p \circ V(x)\right\} \leq 0. \tag{4.8}$$

Let $0<\mu<1$, $0<\eta<1$, and let $p=p_1$. From the inequality

$$D^+V(x,k(x)) \leq -\gamma_3 \circ \gamma_2^{-1} \circ V(x) \tag{4.9}$$

(i.e., the inequality (4.2), exploiting point (i) in Definition 4.1 in the case of ODEs), we obtain

$$\begin{aligned} &D^+V(x,k(x)) + \eta \max\left\{0, D^+p \circ V(x,k(x)) + \mu p \circ V(x)\right\} \\ &\quad \leq \max\left\{ -\gamma_3 \circ \gamma_2^{-1} \circ V(x), -\left(1+\eta \left.\frac{dp(s)}{ds}\right|_{s=V(x)}\right)\gamma_3 \circ \gamma_2^{-1} \circ V(x) \right. \\ &\qquad\qquad \left. + \eta\mu p \circ V(x)\right\} \leq 0. \end{aligned} \tag{4.10}$$

Thus, the inequality (4.1) holds satisfied. Now, in [33], an example of a system (described by an ODE; see [33, Example 1.3]) is provided, for which the map f, the function V, and the functions γ_2, γ_3 are as required in this remark. Therefore, also the inequality (4.1) is satisfied. For that system a stabilizing continuous feedback does not exist, because Brockett's necessary condition (see [6]) is not satisfied. For a steepest descent feedback k, induced by V, either no Filippov solutions exist for the corresponding closed-loop ODE, or the inequality (4.4) (as reformulated for ODEs) cannot hold satisfied. Indeed, by contradiction, if it were satisfied, then the feedback k would be stabilizing in the Filippov sense. But this is impossible because, in this case, Brockett's necessary condition should be satisfied as well (see [52]). Thus, it can happen that either no Filippov solutions exist for the closed loop ODE $\dot{x}(t) = f(x(t), k(x(t)))$, or the inequality (4.4) does not hold.

Remark 5. From Remark 4, since the RFDE (2.1) includes ODEs as special case, it follows that it can happen that either Krasovskii solutions do not exist for the RFDE (2.1) with $u(t)=k(x_t)$, or the inequality (4.4) does not hold. That is, in the general case, we cannot claim any stabilization property, from a theoretical point of view, for the steepest descent feedback k introduced in Definition 4.2.

Remark 6. Notice that the point (ii) in the definition of steepest descent feedback k, in Definition 4.2, is needed to assure causality of the (static) state feedback.

A consequence of Theorem 3.5 is the result provided in the following theorem, which we report here with the aim of further clarifying the meaning of the inequality (4.1).

COROLLARY 4.3. *If the steepest descent feedback k is continuous and zero at zero, the map $\phi \to f(\phi,k(\phi))$ is Lipschitz on any bounded subset of $\mathcal{C}$, and the feedback control law $u(t)=k(x_t)$ is applied to the system* (2.1), *on a continuous time basis, then the resulting closed-loop system is* 0*-GAS.*

Proof. Take the functional $V_\infty : \mathcal{C} \to R^+$ as defined in Theorem 3.5. Then, by Theorem 3.5, the following inequalities hold for any $\phi \in \mathcal{C}$,

$$\begin{gathered} \alpha_1(\|\phi\|_\infty) \leq V_\infty(\phi) \leq \alpha_2(\|\phi\|_\infty), \\ D^+V_\infty(\phi,k(\phi)) \leq -\alpha_3(\|\phi\|_\infty), \end{gathered} \tag{4.11}$$

where α_1, α_2, α_3 are functions of class $\mathcal{K}_\infty$. Then, taking into account the continuity of the feedback k and of the Lipschitz property of the map $\phi \to f(\phi, k(\phi))$, the result follows by Lemma 2.2. ∎

We introduce here the following assumption for the system described by the RFDE (2.1) which, from here on, is supposed to be satisfied.

Assumption 1. There exists a positive real q such that the initial condition $x_0 \in W^{1,\infty}$, and $ess\sup_{\theta\in[-\Delta 0]} |\frac{dx_0(\theta)}{d\theta}| \leq q$. There exist a CLKF V (i.e., $V_1 + V_2$) and an induced steepest descent feedback k (continuous or not), according to Definitions 4.1 and 4.2, respectively. The map $\phi \to D^+V_2(\phi, u)$ is Lipschitz on bounded subsets of $\mathcal{C}$, uniformly in $u \in U$.

5. Stabilization in the sample-and-hold sense. We recall here the notion of partition of $[0, +\infty)$ (see [9], [7]), and introduce a dwell time.

DEFINITION 5.1 (see [9], [7]). *A partition $\pi = \{t_i,\ i = 0, 1, \dots\}$ of $[0, +\infty)$ is a countable, strictly increasing sequence t_i, with $t_0 = 0$, such that $t_i \to +\infty$ as $i \to +\infty$. The diameter of π, denoted $diam(\pi)$, is defined as $\sup_{i\geq 0} t_{i+1} - t_i$. The dwell time of π, denoted $dwell(\pi)$, is defined as $\inf_{i\geq 0} t_{i+1} - t_i$. For any positive real $a \in (0, 1]$, $b > 0$, $\pi_{a,b}$ is any partition π with $ab \leq dwell(\pi) \leq diam(\pi) \leq b$.*

DEFINITION 5.2 (see [9], [7]). *We say that a (causal) feedback $F : \mathcal{C} \to U$ (continuous or not) stabilizes the system described by* (2.1) *in the sample-and-hold sense if, for every positive real r, R, $0 < r < R$, $a \in (0, 1]$, there exist a positive real δ depending upon r, R, q, and Δ, a positive real T, depending upon r, R, q, Δ, and a, and a positive real E, depending upon R and Δ, such that, for any partition $\pi_{a,\delta} = \{t_i,\ i = 0, 1, \dots\}$, for any initial state $x_0 \in \mathcal{C}_R$, the solution corresponding to x_0 and to the sampled-data feedback control law $u(t) = F(x_{t_k})$, $t_k - \omega \leq t < t_{(k+1)} - \omega$, $k = 0, 1, \dots,$ exists for all $t \geq 0$ and, furthermore, satisfies*

$$x_t \in \mathcal{C}_E\ \forall t \geq 0, \qquad x_t \in \mathcal{C}_r\ \forall t \geq T. \tag{5.1}$$

Remark 7. The real $a \in (0, 1]$ in Definition 5.2 is introduced in order to allow nonuniform sampling. If $a = 1$, then $dwell(\pi_{1,\delta}) = diam(\pi_{1,\delta}) = \delta$ (i.e., the sampling is uniform).

The main result of the paper is provided by the following theorem. In the following theorem no continuity hypothesis is introduced on the steepest descent feedback k, and the control law is applied by sampling and holding.

THEOREM 5.3. *The steepest descent feedback k (continuous or not, see Assumption* 1 *and Definition* 4.2*) stabilizes the system described by* (2.1) *in the sample-and-hold sense (see Definition* 5.2*).*

Proof. Thanks to Theorem 3.5, we are able to follow the main lines of the proof of the same theorem for systems described by ODEs (see [9], Theorem 20 in [7]). Let the functional $V_\infty : \mathcal{C} \to R^+$, the map $\mathcal{D}_\infty : \mathcal{C} \times U \to R$, and the functions of class $\mathcal{K}_\infty$, α_i, $i = 1, 2, 3$, be as in Theorem 3.5. Let r, R, be any positive reals, $0 < r < R$. Let $a \in (0, 1]$ be arbitrarily fixed. Let $x_0 \in \mathcal{C}_R$. Let e_2, e_1, E be positive reals satisfying

$$0 < e_2 < e_1 < r < R < E, \qquad \alpha_1(r) > \alpha_2(e_1), \qquad \alpha_1(E) > \alpha_2(R). \tag{5.2}$$

Let M, K be positive reals such that the following inequalities hold for any $\phi_i \in \mathcal{C}_E$, $i = 1, 2$, and for any $u \in U$:

$$|f(\phi_1, u)| \leq M, \qquad |\mathcal{D}_\infty(\phi_1, u) - \mathcal{D}_\infty(\phi_2, u)| \leq K\|\phi_1 - \phi_2\|_\infty. \tag{5.3}$$

Let $\overline{q} = \max\{q, M\}$ (q is the positive real in Assumption 1). Let $\beta = \alpha_3(e_2)$. Let δ be a positive real such that

$$\delta < \min(1, \Delta), \quad e_2 + \delta M < e_1, \quad R + \delta M < E, \quad \alpha_1(r) > \alpha_2(e_1) + \beta\delta,$$
$$\frac{\beta}{3} > K(2\overline{q} + M)\delta, \quad \alpha_1(E) > \alpha_2(R) + \frac{1}{3}\beta\delta. \tag{5.4}$$

Let us consider a partition $\pi_{a,\delta}$. We show first that the solution exists in $[0, t_1]$. Otherwise, by contradiction, if the solution blows up, there exists a time $\tau \in [0, t_1)$ such that $|x(t)| < E$, $t \in [0, \tau)$, and $|x(\tau)| = E$. But, by (5.4), for $t \in [0, \tau]$, the inequalities hold

$$|x(t)| \leq |x_0(0)| + \int_0^t |f(x_\theta, k(x_0))| d\theta \leq R + \delta M < E. \tag{5.5}$$

Thus, taking $t = \tau$, the absurd inequality $E < E$ arises. Therefore, the solution exists in $[0, t_1]$ and, by (5.5) for $t \in [0, t_1]$, it follows that $x_t \in \mathcal{C}_E$, $t \in [0, t_1]$. Let us now consider the following two cases: (1) $\|x_0\|_\infty \leq e_2$; (2) $\|x_0\|_\infty > e_2$. As far as case (1) is concerned, by using again the first inequality in (5.5), the following inequality holds, for any $t \in [0, t_1]$,

$$|x(t)| \leq e_2 + \delta M, \tag{5.6}$$

and thus, from (5.4), $|x(t)| < e_1$, $t \in [0, t_1]$. As far as case (2) is concerned, let $w(t) = V_\infty(x_t)$. By Assumption 1, taking into account Lemma 2.1 and Theorem 3.5, we have, for any fixed $t \in (0, t_1]$, for some $t^\star \in [0, t]$, the following equalities/inequalities:

$$\begin{aligned} w(t) - w(0) &= \int_0^t D^+V_\infty(x_\tau, k(x_0)) d\tau = t\left(\frac{1}{t}\int_0^t D^+V_\infty(x_\tau, k(x_0)) d\tau\right) \\ &\leq t\left(\frac{1}{t}\int_0^t \mathcal{D}_\infty(x_\tau, k(x_0)) d\tau\right) = \mathcal{D}_\infty(x_{t^\star}, k(x_0)) t \\ &= \mathcal{D}_\infty(x_0, k(x_0)) t + \mathcal{D}_\infty(x_{t^\star}, k(x_0)) t - \mathcal{D}_\infty(x_0, k(x_0)) t \\ &\leq -\alpha_3(\|x_0\|_\infty) t + K\|x_{t^\star} - x_0\|_\infty t, \end{aligned} \tag{5.7}$$

where α_3 is the function of class $\mathcal{K}_\infty$ in point (iv) of Theorem 3.5. Now, the following equalities/inequalities hold:

$$\begin{aligned} &\|x_{t^\star} - x_0\|_\infty \\ &\quad = \sup_{\theta \in [-\Delta, 0]} |x(t^\star + \theta) - x_0(\theta)| \\ &\quad = \max\left\{\sup_{\theta \in [-\Delta, -t^\star]} |x(t^\star + \theta) - x_0(\theta)|, \sup_{\theta \in [-t^\star, 0]} |x(t^\star + \theta) - x_0(\theta)|\right\} \\ &\quad = \max\left\{\sup_{\theta \in [-\Delta, -t^\star]} \left|x_0(-\Delta) + \int_{-\Delta}^{t^\star + \theta} \frac{dx_0(\tau)}{d\tau} d\tau - x_0(-\Delta) - \int_{-\Delta}^{\theta} \frac{dx_0(\tau)}{d\tau} d\tau\right|,\right. \\ &\qquad \left.\sup_{\theta \in [-t^\star, 0]} \left|x_0(\theta) + \int_\theta^0 \frac{dx_0(\tau)}{d\tau} d\tau + \int_0^{t^\star + \theta} f(x_\tau, k(x_0)) d\tau - x_0(\theta)\right|\right\} \\ &\quad \leq \overline{q} t^\star + \overline{q} t^\star + M t^\star \leq (2\overline{q} + M)\delta. \end{aligned} \tag{5.8}$$

From (5.7), (5.8), and taking into account that $\|x_0\|_\infty > e_2$ (and thus $\alpha_3(\|x_0\|_\infty) > \beta$), we obtain

$$w(t) - w(0) \leq -\beta t + K(2\overline{q} + M)\delta t, \tag{5.9}$$

and, from (5.4),

$$w(t) \leq w(0) - \frac{2}{3}\beta t, \qquad t \in [0, t_1]. \tag{5.10}$$

Therefore, by (5.2), we have, for $t \in [0, t_1]$,

$$|x(t)| \leq \alpha_1^{-1}\left(\alpha_2(R) - \frac{2}{3}\beta t\right) < E. \tag{5.11}$$

We will later prove the following claim.

CLAIM 1. *The solution exists in* $[0, +\infty)$ *and* $x_t \in \mathcal{C}_E$, $t \geq 0$.

Assume Claim 1 holds true. Then, in any interval $[t_k, t_{(k+1)}]$, $k = 0, 1, \ldots,$ by the same reasoning used in the interval $[0, t_1]$ (namely, by (5.7), (5.8), which hold if $x_t \in C_E$, $t \in [t_k, t_{(k+1)}]$, t_k taking the place of 0), we have

$$w(t) \leq w(t_k) - \alpha_3(\|x_{t_k}\|_\infty)(t - t_k) + \frac{1}{3}\beta(t - t_k) \tag{5.12}$$

by which, taking into account both cases $\|x_{t_k}\|_\infty \leq e_2$ and $\|x_{t_k}\|_\infty > e_2$ (see cases (1) and (2) in $[0, t_1]$), we obtain

$$w(t) \leq w(t_k) - \frac{2}{3}\beta(t - t_k)H\left(\|x_{t_k}\|_\infty - e_2\right) + \frac{1}{3}\beta(t - t_k)H_0(e_2 - \|x_{t_k}\|_\infty), \tag{5.13}$$

where H_0 and H are the Heaviside functions reported in the notation subsection. Recall that (see Theorem 3.5, inequality (5.6), and conditions (5.4))

$$\begin{aligned} &\alpha_1(\|x_t\|_\infty) \leq w(t) \leq \alpha_2(\|x_t\|_\infty), \ t \geq 0, \quad \|x_{t_k}\|_\infty \leq e_2 => |x(t)| \leq e_1, \\ &\qquad t \in [t_k, t_{(k+1)}]. \end{aligned} \tag{5.14}$$

From (5.13), (5.14) we obtain, for any integer $k \geq 0$,

$$\begin{aligned} w\left(t_{(k+1)}\right) \leq &\left(w(t_k) - \frac{2}{3}\beta\left(t_{(k+1)} - t_k\right)\right)H(\|x_{t_k}\|_\infty - e_2) \\ &+ \min\left\{\left(w(t_k) + \frac{1}{3}\beta\left(t_{(k+1)} - t_k\right)\right),\ \alpha_2(e_1)\right\}H_0(e_2 - \|x_{t_k}\|_\infty). \end{aligned} \tag{5.15}$$

Notice that, by induction reasoning with the inequality (5.15), for any $k \geq 0$,

$$w(t_k) \leq \alpha_2(R). \tag{5.16}$$

Let $W(k) = \max\{w(t_k), \alpha_2(e_1)\}$, $k = 0, 1, \ldots.$ We have, from (5.15), for $k = 0, 1, \ldots,$

$$\begin{aligned} W(k+1) \leq \max&\left\{W(k) - \frac{2}{3}\beta\left(t_{(k+1)} - t_k\right),\ \alpha_2(e_1)\right\}H(\|x_{t_k}\|_\infty - e_2) \\ &+ W(k)H_0(e_2 - \|x_{t_k}\|_\infty). \end{aligned} \tag{5.17}$$

Notice that, taking into account their opposing arguments and their definitions (see the notation subsection), either $H_0 = 1$ holds or $H = 1$ holds. We have also that, if $W(k) \leq \alpha_2(e_1) + \frac{2}{3}\beta\delta$, then $W(k+1) \leq \max\{\alpha_2(e_1), W(k)\} \leq \alpha_2(e_1) + \frac{2}{3}\beta\delta$. Therefore, if for a certain $k_1 \geq 0$, $W(k_1) \leq \alpha_2(e_1) + \frac{2}{3}\beta\delta$, then $W(k) \leq \alpha_2(e_1) + \frac{2}{3}\beta\delta$ for all $k \geq k_1$. Let k_1 be the first nonnegative integer such that $W(k) \leq \alpha_2(e_1) + \frac{2}{3}\beta\delta$. We allow $k_1 = +\infty$. Let p be the smallest positive integer such that $pa\delta \geq \Delta$. Let

$$k_0 = \left[\frac{3\alpha_2(R)}{\beta\Delta}\right] + 1. \tag{5.18}$$

Let $k_2 = pk_0$. We claim that $k_1 \leq k_2$. Indeed, by contradiction, let $k_1 > k_2$ (recall that $k_1 = +\infty$ is allowed). Then, two cases can occur:

(i) there exists a nonnegative integer $\overline{k} \in [0, k_2]$ such that $\|x_{t_{\overline{k}}}\|_\infty \leq e_2$;

(ii) an integer $\overline{k}$ such that $\|x_{t_{\overline{k}}}\|_\infty \leq e_2$ does not exist in $[0, k_2]$.

In case (ii), for any integer $k \in [0, k_2]$, we have $\|x_{t_k}\|_\infty > e_2$. Thus, by inequality (5.17) (recall that k_1 is supposed to be greater than k_2), we obtain

$$\begin{aligned} W(k_2) &\leq \alpha_2(R) - \sum_{j=1}^{k_2} \frac{2}{3}\beta a\delta = \alpha_2(R) - pk_0\frac{2}{3}\beta a\delta \\ &\leq \alpha_2(R) - \left(\left[\frac{3\alpha_2(R)}{\beta\Delta}\right] + 1\right)\frac{2}{3}\beta\Delta \leq -\alpha_2(R) < 0, \end{aligned} \tag{5.19}$$

which is absurd, since $W(k)$ is nonnegative for any $k = 0, 1, \ldots$. Therefore, the above $\overline{k}$ must exist in $[0, k_2]$ (i.e., the condition (i) must hold true). Then, we have $\|x_{t_{\overline{k}}}\|_\infty \leq e_2$, which implies $w\left(t_{\overline{k}}\right) \leq \alpha_2(e_2)$, and thus, taking into account conditions (5.2),

$$W\left(\overline{k}\right) = \alpha_2(e_1) \leq \alpha_2(e_1) + \frac{2}{3}\beta\delta. \tag{5.20}$$

But (5.20) is still absurd, because $\overline{k} \leq k_2$, and $k_1 > k_2$ was supposed to be the first nonnegative integer, satisfying $W(k_1) \leq \alpha_2(e_1) + \frac{2}{3}\beta\delta$. Therefore, it must hold $k_1 \leq k_2$. We conclude that, for $k \geq k_2$, $W(k) \leq \alpha_2(e_1) + \frac{2}{3}\beta\delta$, which implies, by (5.12), $w(t) \leq \alpha_2(e_1) + \beta\delta$, $t \geq t_{k_2}$. Let $T = \frac{2}{a}k_0\Delta$. Notice that $T \geq \frac{1}{a}k_0 pa\delta = k_0 p\delta \geq t_{k_2}$ and T does not depend on δ. We have, by (5.4), for $t \geq T$,

$$\|x_t\|_\infty \leq \alpha_1^{-1}(w(t)) \leq \alpha_1^{-1}\left(\alpha_2(e_1) + \beta\delta\right) < r. \tag{5.21}$$

Thus, for $t \geq T$, $x_t \in \mathcal{C}_r$.

The proof is over if Claim (1) holds true. By contradiction, let $\tau \geq t_1$ be the first time such that $|x(t)| = E$ (recall that $|x(t)| = E$ cannot hold for $t \in [0, t_1]$). Then, we have $w(\tau) \geq \alpha_1(E)$. Let $\overline{k}$ be a positive integer such that $t_{\overline{k}} \leq \tau < t_{(\overline{k}+1)}$. For $k = 0, 1, \ldots, \overline{k} - 1$, the inequality (5.15) holds, and thus the inequality (5.16) holds for $k = 0, 1, \ldots, \overline{k}$. In particular, the inequality holds

$$w\left(t_{\overline{k}}\right) \leq \alpha_2(R). \tag{5.22}$$

Then, for $t \in [t_{\overline{k}}, \tau]$, the inequality (5.12) holds. Therefore, taking (5.4) into account, the inequality holds, for $t \in [t_{\overline{k}}, \tau]$,

$$w(t) \leq \alpha_2(R) + \frac{1}{3}\beta\delta < \alpha_1(E), \tag{5.23}$$

which implies the absurd $E = |x(\tau)| \leq \alpha_1^{-1}(w(\tau)) < E$.

The proof of the theorem is complete. ∎

Remark 8. As far as the implementation of the sampled-data controller is concerned, further difficulties arise with respect to nonlinear systems described by ODEs due to time delays that can well be involved in the steepest descent feedback k. When the steepest descent feedback k is not memoryless, the control implementation has to take account of delayed terms of the state. This may require a faster sampling rate than the control signal updating rate, in order to make a record of the necessary past values of the state. If multiple commensurate delays are involved, then easy implementation can be achieved by choosing uniform sampling and the sampling period δ such that each involved delay is equal to such a sampling period multiplied by a suitable positive integer. If multiple noncommensurate delays are involved, then it may happen that one or more terms of the type $x(j\delta-\overline{\Delta})$, for a certain involved delay $\overline{\Delta}$, $j = 0, 1, \dots$, is not available in the buffer device. In this case, an approximation is unavoidable. If δ is suitably small, then clearly the paste state is available at a time arbitrarily near $j\delta - \overline{\Delta}$. How to overcome this approximation by suitable sampling, or by dealing with it as a disturbance with robustness issues (see [7], [8], [10]), is left for future investigations.

Let us consider now a local version of the result provided in Theorem 5.3.

DEFINITION 5.4 (see [7], [9]). *Let Q be a positive real. We say that a (causal) feedback $F : \mathcal{C}_Q \to U$ (continuous or not) stabilizes the system described by* (2.1) *in the sample-and-hold sense, in $\mathcal{C}_Q$, if, for every positive real r, R, $0 < r < R \le Q$, $a \in (0,1]$, there exist a positive real δ depending upon r, R, q, and Δ, a positive real T, depending upon r, R, q, Δ, and a, and a positive real E, depending upon R and Δ, such that, for any partition $\pi_{a,\delta} = \{t_i,\ i = 0, 1, \dots\}$, for any initial state $x_0 \in \mathcal{C}_R$, the solution corresponding to x_0 and to the sampled-data feedback control law*

$$u(t) = F(x_{t_k}), \quad t_k - \omega \le t < t_{(k+1)} - \omega, \quad k = 0, 1, \dots, \tag{5.24}$$

exists for all $t \ge 0$ and, furthermore, satisfies

$$x_t \in \mathcal{C}_E\ \forall t \ge 0, \qquad x_t \in \mathcal{C}_r\ \forall t \ge T. \tag{5.25}$$

THEOREM 5.5. *Let there exist a positive real S, a functional $V : \mathcal{C}_S \to R^+$, a map $k : \mathcal{C}_S \to U$ such that*

(i) *V is a CLKF in $\mathcal{C}_S$ (i.e., the equality/inequalities* (3.1) *in Definition* 3.1 *and the points* (i), (ii) *in Definition* 4.1 *hold for any $\phi \in \mathcal{C}_S$);*

(ii) *k is a steepest descent feedback induced by V, in $\mathcal{C}_S$ (i.e., points* (i), (ii) *in Definition* 4.2 *hold for any $\phi \in \mathcal{C}_S$, the inequality* (4.1) *holds for suitable positive reals μ, η, $m \in \{0,1\}$, function $\overline{\alpha}$ and function $p \in C^1_L([0, \beta_2(S)]; R^+)$, zero at zero, and strictly increasing, with β_2 given in* (3.1)*).*

Then, the steepest descent feedback k (continuous or not) stabilizes the system described by (2.1) *in the sample-and-hold sense, in $\mathcal{C}_Q$ (according to Definition* 5.4*), where Q is a positive real satisfying the inequality*

$$\alpha_1(S) > \alpha_2(Q) \tag{5.26}$$

with

$$\alpha_1(s) = \eta e^{-\mu\Delta} p \circ \beta_1(s), \qquad \alpha_2(s) = m\gamma_2(s) + \eta p \circ \beta_2(s), \qquad s \ge 0 \tag{5.27}$$

(see (3.5) *in the proof of Theorem* 3.3*), β_1 given in* (3.1)*, γ_2 given in Definition* 4.1*, μ, η, m, and p as given in inequality* (4.1).

Proof. The proof is equal to the one for the global case considered in Theorem 5.3. Just choose $E = S$ and take into account that: (1) conditions (5.2), (5.4) can be fulfilled because of inequality (5.26); (2) Claim 1 holds satisfied with this chosen positive real E (thus involved functionals are defined on the solution and their properties can be exploited). ∎

Remark 9. In the local case, the inequality (4.1) follows from the standard (in the sense of Theorem 2.1 in [18]) inequality (4.2). Indeed, let us assume, without loss of generality, that the functions γ_3 in (4.2) and β_2^{-1} (β_2 given in (3.1)) are Lipschitz in $[0, \max\{S, \beta_2(S)\}]$. Otherwise, just replace β_2 with the function $\hat{\beta}_2^{-1}$ of class $\mathcal{K}_\infty$, with $\hat{\beta}_2$ of class $\mathcal{K}_\infty$, globally Lipschitz, and satisfying $\hat{\beta}_2(s) \le \beta_2^{-1}(s)$, $s \in R^+$ (see [24], [27], Lemma A.3 in [51]), and $\gamma_3(s)$ with $\frac{1}{\max\{S,\beta_2(S)\}} \int_0^s \gamma_3(\tau) d\tau$, $s \in R^+$. Taking into account the differentiability property of the function V_1, the Lipschitz property of the map f, and the boundedness of the feedback k, let ρ be a positive real such that

$$\sup_{\phi \in \mathcal{C}_S} |D^+ V_1(\phi, k(\phi))| \le \rho. \tag{5.28}$$

Then, taking into account (4.2), the inequality (4.1) (with $m = 1$, $\overline{\alpha} = \underline{0}$) is satisfied, in $\mathcal{C}_S$, if the following inequality holds for suitable positive reals η, μ, and suitable, function $p \in C_L^1([0, \beta_2(S)], R^+)$, zero at zero, and strictly increasing:

$$-\gamma_3(|\phi(0)|) + \eta \left. \frac{dp(s)}{ds} \right|_{s=V_1(\phi(0))} \rho + \eta\mu p \circ V_1(\phi(0)) \le 0 \qquad \forall \phi \in \mathcal{C}_S. \tag{5.29}$$

The inequality (5.29) holds satisfied, taking into account the inequalities in (3.1) in Definition 3.1, if the following inequality holds:

$$-\gamma_3 \circ \beta_2^{-1}(s) + \eta \left. \frac{dp(\tau)}{d\tau} \right|_{\tau=s} \rho + \eta\mu p(s) \le 0 \qquad \forall s \in [0, \beta_2(S)]. \tag{5.30}$$

The inequality (5.30) is satisfied with the following choice of η, μ, and p:

$$\begin{aligned} &\eta = 1, \qquad \mu = \frac{\rho}{2\beta_2(S)}, \\ &p(s) = \frac{1}{\rho} \int_0^s e^{-\frac{1}{2\beta_2(S)}(s-\tau)} \gamma_3 \circ \beta_2^{-1}(\tau) d\tau, \qquad s \in [0, \beta_2(S)]. \end{aligned} \tag{5.31}$$

Indeed, (5.30) holds as an equality. The function p is strictly increasing in the interval of definition. To prove this, we consider the derivative of p in the open interval $(0, \beta_2(S))$ and show that this derivative is strictly positive. We have, for any $s \in (0, \beta_2(S))$, by applying the mean value theorem to the integral term,

$$\begin{aligned} \rho \frac{dp(s)}{ds} &= \gamma_3 \circ \beta_2^{-1}(s) - \frac{1}{2\beta_2(S)} \int_0^s e^{-\frac{1}{2\beta_2(S)}(s-\tau)} \gamma_3 \circ \beta_2^{-1}(\tau) d\tau \\ &\ge \gamma_3 \circ \beta_2^{-1}(s) - \frac{s}{2\beta_2(S)} \sup_{\xi \in [0,s]} \left(e^{-\frac{1}{2\beta_2(S)}(s-\xi)} \gamma_3 \circ \beta_2^{-1}(\xi) \right) \ge \frac{1}{2} \gamma_3 \circ \beta_2^{-1}(s) > 0. \end{aligned} \tag{5.32}$$

Thus, in conclusion, in the local case dealt with in Theorem 5.5, if the inequality (4.2) holds, then there exist positive reals η, μ, and a strictly increasing function $p \in C_L^1([0, \beta_2(S)]; R^+)$, zero at zero, such that the inequality (4.1), with $m = 1$, $\overline{\alpha} = \underline{0}$, holds as well.

Remark 10. From Remark 9 it follows that any causal (globally or locally) stabilizing (when applied in continuous time) feedback $k : \mathcal{C} \to R^m$ (bounded or not), Lipschitz on bounded sets, and zero at zero, for which inequality (4.2) holds with a CLKF V, as defined in Definition 4.1, stabilizes the system described by the RFDE (2.1) in the sample-and-hold sense in C_Q, for suitable positive real Q. Indeed, it is sufficient to choose a positive real S such that, for any $\phi \in \mathcal{C}_S$, $k(\phi) \in U$, thus adhering to the saturation constraints, and the positive real Q according to (5.26), with the choice of η, μ, m, p, and $\overline{\alpha}$, for instance, as provided in Remark 9. Therefore, Theorem 5.5 can be useful whenever saturation constraints appear, and digital implementation of an available control law (in continuous time) is needed, a very frequent situation in practice. The price to pay, due to the saturation constraints, is that in general only local stabilization, in the sample-and-hold sense, is achieved, by Theorem 5.5, though a globally stabilizing (in continuous time) unbounded feedback k may be available. If $k(\phi) \in U$ for all $\phi \in \mathcal{C}$ (i.e., the saturation constraints are fulfilled by the continuous time control law), and (4.1) holds (globally), then global stabilization in the sample-and-hold sense is achieved, according to Theorem 5.3. The following corollary shows how Theorem 5.5 can be applied to the particular class of nonlinear retarded systems which are input-output feedback linearizable and stabilizable by (continuous time) state feedback (see [57, Theorem 3.3], [15], [16], [44]).

COROLLARY 5.6. *Let there exist a diffeomorphism $\Psi : \Omega_x \to \Omega_z$ with $\Omega_x, \Omega_z \in R^n$ open, bounded neighborhoods of the origin, functions $\underline{\gamma}_\psi$, $\overline{\gamma}_\psi$, of class $\mathcal{K}_\infty$, a Hurwitz matrix $F \in R^{n\times n}$, a positive real S, a Lipschitz feedback $k : \mathcal{C}_S \to U$, zero at zero, such that: $B_S \subset \Omega_x$;*

$$\underline{\gamma}_\psi(|x|) \le |\Psi(x)| \le \overline{\gamma}_\psi(|x|) \qquad \forall x \in \Omega_x; \tag{5.33}$$

$$\left.\frac{\partial \Psi(x)}{\partial x}\right|_{x=\phi(0)} f(\phi, k(\phi)) = F\Psi(\phi(0)) \qquad \forall \phi \in \mathcal{C}_S. \tag{5.34}$$

Then, there exists a positive real Q such that the feedback $k : \mathcal{C}_S \to U$ stabilizes in the sample-and-hold sense, in $\mathcal{C}_Q$, the system described by (2.1).

Proof. Pick a symmetric, positive definite matrix P such that $F^TP + PF = -I_n$. Choose $V : \mathcal{C}_S \to R^+$ defined, for $\phi \in \mathcal{C}_S$, as $V(\phi) = \Psi^T(\phi(0))P\Psi(\phi(0))$. From (5.33) and from (5.34) it follows that V is a CLKF in $\mathcal{C}_S$ and the inequality (4.2) holds. Taking into account Remark (9), it follows that k is a steepest descent feedback induced by V, in $\mathcal{C}_S$. Finally, the result follows from Theorem 5.5. ∎

6. Examples.

6.1. Example 1. (See [30], [49].) We will show here an application of Theorem 5.3 to a discontinuous sliding-mode controller implemented by sampling and holding (as is frequent in practice; see [10], [7] for the delay-free case with disturbance, actuator, and observation errors). Let us consider the system described by the following RFDE (see the delay-free case presented in Example 14.1.1, pp. 552–563, in [30]; see Example 3, for a delayed case, in [49])

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= H(x_t) + G(x_t)u(t), \\ x(\tau) &= x_0(\tau), \quad \tau \in [-\Delta, 0], \end{aligned} \tag{6.1}$$

where: $x(t) = [x_1(t) \quad x_2(t)]^T \in R^2$; Δ is an arbitrary, (possibly) unknown positive real; $H : \mathcal{C} \to R$, $G : \mathcal{C} \to R^+$ are uncertain maps, Lipschitz on bounded sets; $H(0) = 0$; $x_0 \in W^{1,\infty}$, $ess\sup_{\theta\in[-\Delta,0]} |\frac{dx_0(\theta)}{d\theta}| \le q$; q is an arbitrary positive constant;

$u(t) \in U$ is the control input; $U \subset R$ is a compact set including the origin in the interior. We introduce the following assumption for the system described by the RFDE (6.1) (see [30, pp. 552–553]).

Assumption 2. For the RFDE (6.1),

(1) there exists a positive real g_0 such that, for all $\phi \in \mathcal{C}$, the inequality holds $G(\phi) \geq g_0$;

(2) there exist a positive real a_1, a function $\rho : \mathcal{C} \to R^+$ such that, for all $\phi \in \mathcal{C}$, the inequality holds $|a_1\phi_2(0) + H(\phi)| \leq \rho(\phi)G(\phi)$;

(3) there exist positive reals $\overline{\rho}$, ρ_1, $\overline{\rho} > \rho_1$, such that, for all $\phi \in \mathcal{C}$, $\rho_1 \geq \rho(\phi)$, and $[-\overline{\rho}, \overline{\rho}] \subset U$.

Let us consider the Lyapunov–Krasovskii functional $V : \mathcal{C} \to R^+$ defined, for $\phi = [\ \phi_1 \quad \phi_2\ ]^T \in \mathcal{C}$, as

$$V(\phi) = (a_1\phi_1(0) + \phi_2(0))^2 + \left\{ \begin{array}{cc} \frac{1}{2}\gamma\phi_1^2(0), & |\phi_1(0)| \leq 1, \\ \gamma\left(|\phi_1(0)| - \frac{1}{2}\right), & |\phi_1(0)| > 1, \end{array} \right\} \tag{6.2}$$

with γ a suitable positive parameter which will be chosen later. Such a functional is a CLKF as defined in Definition 4.1. Indeed, in this case, for $\phi = [\phi_1 \quad \phi_2]^T \in \mathcal{C}$, we have $V(\phi) = V_1(\phi(0))$, with $V_1 : R^2 \to R^+$ defined, for $x = [x_1 \quad x_2]^T \in R^2$, as $V_1(x) = (a_1x_1 + x_2)^2 + \left\{ \begin{smallmatrix} \frac{1}{2}\gamma x_1^2, & |x_1|\leq 1, \\ \gamma\left(|x_1| - \frac{1}{2}\right), & |x_1|>1. \end{smallmatrix} \right\}$. By Lemma 4.3, p. 145, in [30], it follows that there exist functions β_1, β_2, γ_1, γ_2 of class $\mathcal{K}_\infty$ such that the inequalities in Definition 3.1 and in point (i), Definition 4.1, hold satisfied. For any $\phi = [\phi_1 \quad \phi_2]^T \in \mathcal{C}$, $u \in U$, by the equality

$$\phi_1(0)\phi_2(0) = \phi_1(0)(a_1\phi_1(0) + \phi_2(0)) - a_1\phi_1^2(0), \tag{6.3}$$

we have

$$\begin{aligned} D^+V(\phi, u) \leq{}& 2G(\phi)\,(a_1\phi_1(0) + \phi_2(0)) \left(\frac{a_1\phi_2(0) + H(\phi)}{G(\phi)} + u \right) \\ &+ \gamma|a_1\phi_1(0) + \phi_2(0)| - \gamma a_1 \min\left\{|\phi_1(0)|, \phi_1^2(0)\right\}. \end{aligned} \tag{6.4}$$

Taking into account the possibility of choosing (a sliding control feedback; see [30, p. 553]) $u = -\overline{\rho} \cdot sgn(a_1\phi_1(0) + \phi_2(0))$, we have, for all $\phi \in \mathcal{C}$,

$$\begin{aligned} \inf_{u\in U} D^+V(\phi, u) &\leq D^+V(\phi, -\overline{\rho} \cdot sgn(a_1\phi_1(0) + \phi_2(0))) \\ &\leq -\left(2g_0\left(\overline{\rho} - \rho_1\right) - \gamma\right)|a_1\phi_1(0) + \phi_2(0)| - \gamma a_1 \min\left\{|\phi_1(0)|, \phi_1^2(0)\right\}. \end{aligned} \tag{6.5}$$

Therefore, by choosing any $\gamma \in (0, 2g_0\left(\overline{\rho} - \rho_1\right))$, it follows that $\inf_{u\in U} D^+V(\phi, u) < 0$ for all $\phi \in \mathcal{C}$, $\phi(0) \neq 0$, as required in point (ii) of Definition 4.1. Let the map $k : \mathcal{C} \to U$ be defined, for $\phi \in \mathcal{C}$, as

$$k(\phi) = -\left(\rho(\phi) + k_0\right) sgn(a_1\phi_1(0) + \phi_2(0)), \tag{6.6}$$

where $k_0 \in (0, \overline{\rho} - \rho_1]$ is a tunable control parameter. Choosing any $\gamma \in (0, 2g_0k_0)$, from (6.4), by Assumption 2, the following inequality holds:

$$D^+V(\phi, k(\phi)) \leq -(2g_0k_0 - \gamma)|a_1\phi_1(0) + \phi_2(0)| - \gamma a_1 \min\left\{|\phi_1(0)|, \phi_1^2(0)\right\}. \tag{6.7}$$

Then, according to Definition 4.2, the map k is a steepest descent feedback. Indeed, as far as inequality (4.1) is concerned, let $m = \mu = 1$, $\overline{\alpha} = \underline{0}$, $\eta = \min\left\{2g_0k_0 - \gamma,\ a_1\right\}$,

$p(s) = log_n(1+s)$, $s \geq 0$. We have, for any $\phi \in \mathcal{C}$, taking into account (6.7) and the increasing property of the function p,

$$\begin{aligned}
&D^+V(\phi, k(\phi)) + \eta \max\{0,\ D^+p \circ V_1(\phi, k(\phi)) + \mu p \circ V_1(\phi(0))\} \\
&\quad = D^+V(\phi, k(\phi)) + \eta \max\left\{0,\ \left.\frac{dp(s)}{ds}\right|_{s=V_1(\phi(0))} D^+V_1(\phi, k(\phi)) + \mu p \circ V_1(\phi(0))\right\} \\
&\quad \leq -(2g_0k_0 - \gamma)|a_1\phi_1(0) + \phi_2(0)| - \gamma a_1 \min\left\{|\phi_1(0)|,\ \phi_1^2(0)\right\} \\
&\qquad + \min\left\{2g_0k_0 - \gamma,\ a_1\right\} log_n\left(1 + V_1(\phi(0))\right) \\
&\quad \leq - \min\left\{2g_0k_0 - \gamma,\ a_1\right\}\left(|a_1\phi_1(0) + \phi_2(0)| + \gamma \min\left\{|\,\phi_1(0)|,\ \phi_1^2(0)\right\}\right) \\
&\qquad + \min\left\{2g_0k_0 - \gamma,\ a_1\right\} log_n \left(1 + \left(a_1\phi_1(0) + \phi_2(0)\right)^2 + \gamma \min\left\{|\phi_1(0)|,\ \phi_1^2(0)\right\}\right).
\end{aligned} \tag{6.8}$$

By the inequality $log_n(1 + s_1^2 + s_2) - s_1 - s_2 \leq 0$ for all $s_1, s_2 \in R^+$, it follows from (6.8) that

$$D^+V(\phi, k(\phi)) + \eta \max\{0,\ D^+p \circ V_1(\phi, k(\phi)) + \mu p \circ V_1(\phi(0))\} \leq 0 \ \forall \phi \in \mathcal{C}, \tag{6.9}$$

that is, k is a steepest descent feedback according to Definition 4.2. Assumption 1 holds satisfied. By Theorem 5.3, we conclude that the steepest descent feedback k stabilizes in the sample-and-hold sense the system described by (6.1), as described in Definition 5.2. The piecewise constant control law is defined as follows, for $t \geq 0$,

$$\begin{aligned}
&u(t) = -\left(\rho\left(x_{t_k}\right) + k_0\right) sgn(a_1x_1(t_k) + x_2(t_k)), \\
&t_k \leq t < t_{(k+1)}, \qquad k = 0, 1, \ldots, \qquad t_0 = 0.
\end{aligned} \tag{6.10}$$

Remark 11. To my best knowledge, this is the first theoretical result in the literature for nonlinear systems described by RFDEs (i.e., with state delays), concerning stabilization by means of a sliding-mode control law implemented by sampling and holding (see [59] for a recent survey on the topic). In the case that Assumption 2 holds only in $\mathcal{C}_S$, for a suitable positive real S, then, by the same kind of computations and by Theorem 5.5, it results that the feedback k in (6.6) stabilizes, in the sample-and-hold sense, the system described by (6.1), in $\mathcal{C}_Q$, for a suitable positive real Q, according to Definition 5.4.

Simulations have been performed with H, G defined, for $\phi = \left[\begin{smallmatrix}\phi_1\\ \phi_2\end{smallmatrix}\right] \in \mathcal{C}$, as $H(\phi) = b_1\phi_1(-\Delta)\phi_2(-\Delta)$, $G(\phi) = 1 + b_2 \max\{|\phi(0)|^2, |\phi(-\Delta)|^2\}$, $b_1 \in [-1, 1]$ and $b_2 \in [1, 2]$ uncertain parameters, Δ a known constant. Let $U = [-3, 3]$. We can choose, in this case, $a_1 = 1$, $\rho(\phi) = \frac{|\phi_1(-\Delta)\phi_2(-\Delta)| + |\phi_2(0)|}{1+\max\{|\phi(0)|^2, |\phi(-\Delta)|^2\}}$, $\phi \in \mathcal{C}$, $\rho_1 = 2$, $\overline{\rho} = 3$. Assumption 2 holds satisfied and Theorem 5.3 can be applied. In the performed simulations, $k_0 = 10^{-6}$, $\Delta = 1.4$, $x_0(\tau) = \left[\begin{smallmatrix}2\\ -2\end{smallmatrix}\right]$, $\tau \in -[\Delta, 0]$, $a = 1$ (uniform sampling), $b_1 = -1$, $b_2 = 1$ are chosen. Convergence of the state to a neighborhood of the origin is observed for sampling period δ equal to 0.3, 0.4, 0.6, 0.8, 0.9, and divergence is observed for δ equal to 1, 1.2, 2. In Figure 1, the behavior of the state variables of the closed-loop system, with sampling period $\delta = 0.3$, is reported. The convergence of the state variables to a neighborhood of the origin is observed. In Figure 2 the control signal, related to the same sampling period $\delta = 0.3$, is reported. The piecewise constant control signal moves towards a neighborhood of the interval $[-k_0, k_0]$ (with final suitably small oscillations), when the state moves towards a neighborhood of the origin (and thus $\rho(x_t)$ moves towards a neighborhood of zero too). If the delay

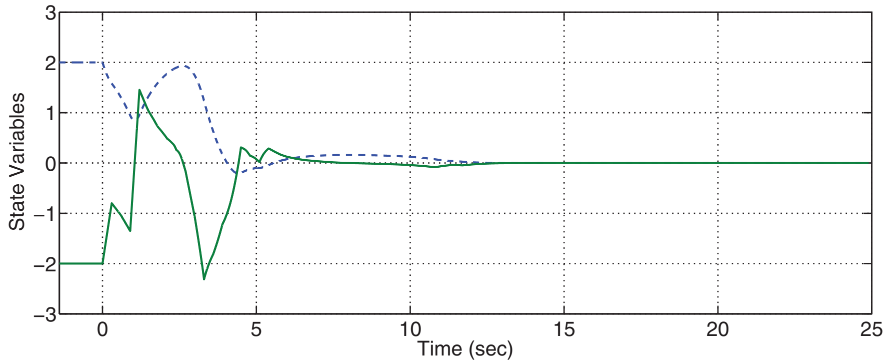

FIG. 1. *Evolution of the state variables* x_1 *and* x_2 *for the closed-loop system* (6.1), (6.10), *with sampling period* $\delta = 0.3$.

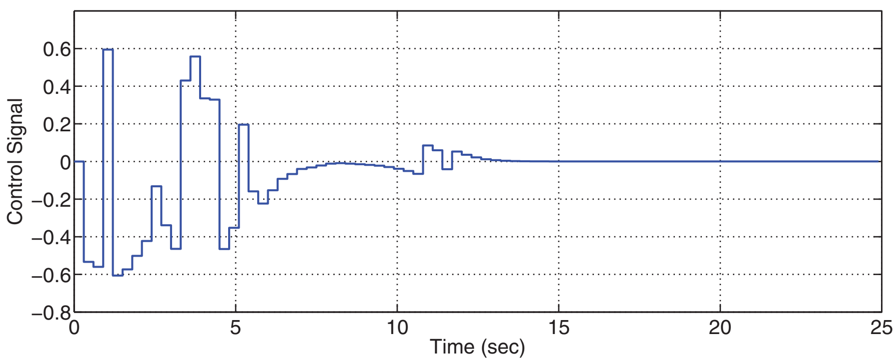

FIG. 2. *Control signal* (6.10) *with sampling period* $\delta = 0.3$.

Δ is unknown, then $\rho(\phi) = 2$ can be chosen, $\phi \in \mathcal{C}$, and in this case the feedback is memoryless. Simulations show convergence of the state to a neighborhood of the origin for sampling period δ equal to 0.01, 0.1, 0.2, and divergence for $\delta = 0.3$. In this case, the control signal cannot move towards a neighborhood of the interval $[-k_0, k_0]$, when the state moves towards a neighborhood of the origin, and, though frozen for a sampling period, continues to take the larger values $2 + k_0$, $-2 - k_0$, for all the time.

6.2. Example 2. (See [15].) We will show here an application of Theorem 5.5 to an example for which there exists a stabilizing, continuous feedback control law (in continuous time) as specified in Remark 10. Let us consider the system described by the following RFDE (introduced in [15])

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) + sech(x_1(t)) - 1, \\ \dot{x}_2(t) &= tanh(x_2(t-\Delta)) + x_1(t) + u(t), \\ x(\tau) &= x_0(\tau),\ \tau \in [-\Delta, 0], \end{aligned} \tag{6.11}$$

where $x(t) = \left[\begin{smallmatrix} x_1(t) \\ x_2(t) \end{smallmatrix}\right] \in R^2$; $\Delta > 0$ is any constant and known delay; $x_0 \in W^{1,\infty}$, and $ess\,\sup_{\theta\in[-\Delta,0]} |\frac{dx_0(\theta)}{d\theta}| \leq q$; q is an arbitrary positive constant; $u(t) \in U = [-10, 10]$. By the tools of differential geometry for retarded systems (see [16], [44]), the following state feedback control law, which, if applied in continuous time, globally stabilizes the system described by (6.11), can be found (see [15]):

$$u(t) = k(x_t), \tag{6.12}$$

where $k : \mathcal{C} \to R$ is defined, for $\phi = \left[\begin{smallmatrix}\phi_1\\ \phi_2\end{smallmatrix}\right] \in \mathcal{C}$, as

$$\begin{aligned} k(\phi) = &-\phi_1(0) + tanh(\phi_1(0))sech(\phi_1(0))\left(\phi_2(0) + sech(\phi_1(0)) - 1\right)\\ &- tanh(\phi_2(-\Delta)) - \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} \phi_1(0) \\ \phi_2(0) + sech(\phi_1(0)) - 1 \end{bmatrix}. \end{aligned} \tag{6.13}$$

Now, we would like to implement digitally the control law (6.12), and also we would like to adhere to the input saturation constraints (see Remark 10). As well, we would like to guarantee, for the system described by (6.11), the theoretical result shown in Theorem 5.5. The objective is to stabilize, in the sample-and-hold sense, the system described by (6.11) in C_Q, for suitable positive real Q (see Theorem 5.5). Let $c = sup_{s\geq 0}|\frac{d\ sech(s)}{ds}| = sup_{s\geq 0} tanh(s)sech(s) = \sqrt{0.5}sech(tanh^{-1}(\sqrt{0.5}))$. It results that $c < 0.51$. From (6.13), taking into account the constant c, it follows that, if $\phi \in C_S$, with $S = 2.30$, then $k(\phi) \in U$. Take the following functional $V : \mathcal{C} \to R^+$, defined, for $\phi \in \mathcal{C}$, as $V(\phi) = \Psi^T(\phi(0))P\Psi(\phi(0))$, where $P \in R^{2\times 2}$ is the symmetric positive definite matrix satisfying $H^TP + PH = -I_2$ with $H = \left[\begin{smallmatrix} 0 & 1\\ -1 & -2\end{smallmatrix}\right]$, and $\Psi : R^2 \to R^2$ is the function defined, for $x = \left[\begin{smallmatrix}x_1\\ x_2\end{smallmatrix}\right] \in R^2$, as $\Psi(x) = \left[\begin{smallmatrix}x_1\\ x_2+sech(x_1)-1\end{smallmatrix}\right]$. The map Ψ describes the change of coordinates by which the control law (6.12) is found and, by the control law (6.12), the following ODE $\frac{d\psi(x(t))}{dt} = H\psi(x(t))$, $t \geq 0$, holds (see [16], [44], [15]). The functional V is a CLKF, in $\mathcal{C}_S$, for the system described by (6.11), according to Definition 4.1. Indeed, we have, for $\phi \in \mathcal{C}_S$, $V_1(\phi(0)) = \Psi^T(\phi(0))P\Psi(\phi(0))$, $V_2(\phi) = 0$. We have, for $s \geq 0$, $\beta_1(s) = \gamma_1(s) = \lambda_{\min}(P)(1-c)^2s^2$, and $\beta_2(s) = \gamma_2(s) = \lambda_{\max}(P)(1+c)^2s^2$. Since, for $\phi \in \mathcal{C}_S$,

$$\inf_{u\in U} D^+V(\phi,u) \leq D^+V(\phi,k(\phi)) = -|\Psi(\phi(0))|^2 \leq -(1-c)^2|\phi(0)|^2, \tag{6.14}$$

the functional V is a $CLKF$. In order to show that k is a steepest descent feedback induced by V, in $\mathcal{C}_S$, according to Definition 4.2, we have, according to (4.1), for $p = I_d$, $\eta = 1$, $m = 0$, $\overline{\alpha} = \underline{0}$, $\mu = \frac{(1-c)^2}{\lambda_{\max}(P)(1+c)^2}$, taking into account (6.14),

$$\begin{aligned} &mD^+V(\phi,k(\phi)) + \eta\max\{0, D^+V_1(\phi,k(\phi)) + \mu V_1(\phi(0))\}\\ &\quad\leq \eta\max\{0, -(1-c)^2|\phi(0)|^2 + \mu(1+c)^2\lambda_{\max}(P)|\phi(0)|^2\} = 0. \end{aligned} \tag{6.15}$$

Therefore, k is a steepest descent feedback induced by V, in $\mathcal{C}_S$. According to (5.27), we have, in this case,

$$\alpha_1(s) = e^{-\mu\Delta}\lambda_{\min}(P)(1-c)^2s^2, \qquad \alpha_2(s) = \lambda_{\max}(P)(1+c)^2s^2, \qquad s \geq 0. \tag{6.16}$$

We obtain, from (5.26), the inequality

$$e^{-\mu\Delta}\lambda_{\min}(P)(1-c)^2S^2 > \lambda_{\max}(P)(1+c)^2Q^2, \tag{6.17}$$

which returns $Q < \frac{\sqrt{e^{-\mu\Delta}\lambda_{\min}(P)}}{\sqrt{\lambda_{\max}(P)}}\frac{1-c}{1+c}S$. With the chosen value of $S = 2.30$, and taking into account that $c < 0.51$, we obtain, for instance with $\Delta = 0.1$, that $Q = 0.30$ is

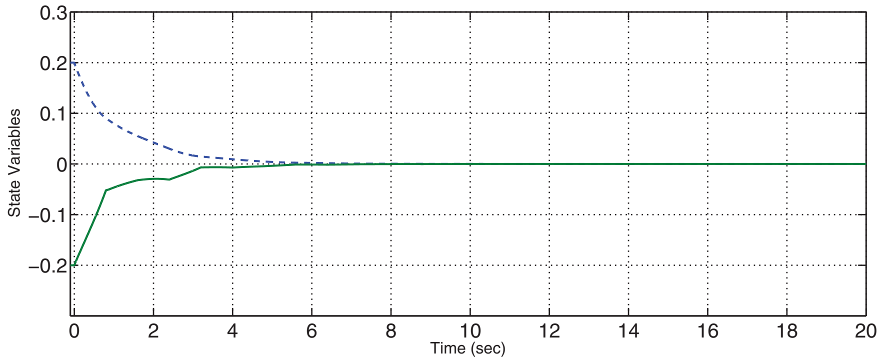

FIG. 3. *Evolution of the state variables* x_1 *and* x_2 *for the closed-loop system* (6.11), (6.18), *with sampling period* $\delta = 0.8$.

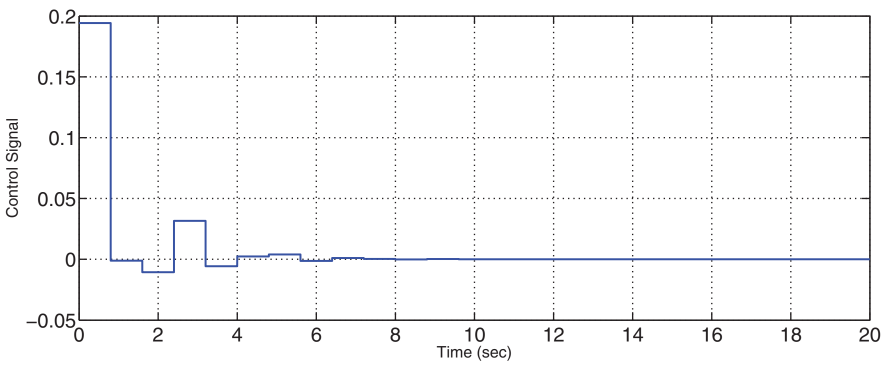

FIG. 4. *Control signal* (6.18) *with sampling period* $\delta = 0.8$.

guaranteed. The steepest descent feedback k stabilizes in the sample-and-hold sense, in $\mathcal{C}_Q$, according to Theorem 5.5, the system described by (6.11). The resulting digital control law is given by

$$\begin{aligned} u(t) = 10sat\ 0.1\Bigg(& - x_1(t_k) + tanh(x_1(t_k))sech(x_1(t_k))\,(x_2(t_k) + sech(x_1(t_k)) - 1) \\ & - tanh(x_2(t_k - \Delta)) - \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} x_1(t_k) \\ x_2(t_k) + sech(x_1(t_k)) - 1 \end{bmatrix} \Bigg), \\ & t_k \le t < t_{k+1}, \qquad t_0 = 0. \end{aligned} \tag{6.18}$$

It is required that $x_0 \in \mathcal{C}_Q$. In the performed simulations, $\Delta = 0.1$, $x_0(\tau) = \left[\begin{smallmatrix} 0.2 \\ -0.2 \end{smallmatrix}\right]$, $\tau \in -[\Delta, 0]$, $a = 1$ (uniform sampling) are chosen. Convergence of the state to a neighborhood of the origin is observed for sampling period δ equal to 0.5, 0.8, 0.9, and divergence is observed for δ equal to 1, 1.2, 1.4. In Figure 3, the behavior of the state variables of the closed-loop system, with sampling period $\delta = 0.8$, is reported. The convergence of the state variables to a neighborhood of the origin is observed. In Figure 4 the control signal, related to the sampling period $\delta = 0.8$, is reported.

7. Conclusions. In this paper we have proved, for retarded nonlinear systems, that any steepest descent feedback (continuous or not) is a stabilizer in the sample-and-hold sense. This result significantly opens the field to wide possibilities of finding stabilizers for retarded nonlinear systems by means of CLKFs, since problems of feedback discontinuities are overcome. The price to pay is that semiglobal, practical asymptotic stability is guaranteed. However, such kind of stability can be satisfactory for many practical engineering problems. In this paper, robustness issues related to actuator disturbances and/or to measurements errors (see, for instance, [37]) are not considered. The robust stabilization in the sample-and-hold sense of retarded nonlinear systems will be the topic of future investigations. In this direction, the application of the theory here developed to the (discontinuous) sliding-mode control of retarded systems in the presence of disturbances, as well as of modeling, actuator, and observation errors (see [10], [7] for the delay-free case) would be very appealing. When the sliding-mode control law is implemented digitally, in general, control chattering arises, though it may be managed if the sampling period is suitably large. The aim of avoiding control chattering due to the sampling process may be achieved by sampling the continuous time approximated sliding-mode control law (obtained, for instance, by replacing the *sgn* function with a continuous one). Such an approximation introduces a bounded disturbance which may be dealt with by mixing ISS results (in [49]) and the ones here developed, at the price of a guaranteed weaker practical stabilization than the one guaranteed in this paper. The proof of Theorem 5.3 allows us to find out, for a problem at hand, the required sampling period (see conditions (5.2), (5.4)). Problems related to a too small required sampling period, which may not be guaranteed by the sensors and actuators at hand, have not been considered in this paper, and will also be a topic of forthcoming investigations. Finally, here the delay in the input channel has been marginally addressed. Indeed, only static state feedbacks have been considered. The many, recently developed, powerful prediction-based tools (see [5] and the references therein) constitute a challenging research arena, as far as the sampled-data implementation of the controller is concerned, as well as the related theoretical stability issues (see [28]).

REFERENCES

[1] Z. Artstein, *Stabilization with relaxed controls*, Nonlinear Anal., 7 (1983), pp. 1163–1173.

[2] N. Bekiaris-Liberis, M. Jankovic, and M. Krstic, *Compensation of state-dependent state delay for nonlinear systems*, Systems Control Lett., 61 (2012), pp. 849–856.

[3] N. Bekiaris-Liberis and M. Krstic, *Compensation of state-dependent input delay for nonlinear systems*, IEEE Trans. Automat. Control, 58 (2013), pp. 275–289.

[4] N. Bekiaris-Liberis and M. Krstic, *Compensation of time-varying input and state delays for nonlinear systems*, J. Dynam. Systems Meas. Control, 134 (2012), 11009.

[5] N. Bekiaris-Liberis and M. Krstic, *Nonlinear Control Under Nonconstant Delays*, Adv. Des. Control 25, SIAM, Philadelphia, 2013.

[6] R.W. Brockett, *Asymptotic stability and feedback stabilization*, in Differential Geometric Control Theory, Progr. Math. 27, R.W. Brockett, R.S. Millman, and H.J. Sussmann, eds, Birkhäuser, Boston, 1983.

[7] F.H. Clarke, *Discontinuous feedback and nonlinear systems*, in IFAC Conference on Nonlinear Control Systems (NOLCOS), Bologna, Italy, 2010.

[8] F.H. Clarke, Yu.S. Ledyaev, L. Rifford, and R.J. Stern, *Feedback stabilization and Lyapunov functions*, SIAM J. Control Optim., 39 (2000), pp. 25–48.

[9] F.H. Clarke, Y.S. Ledyaev, E.D. Sontag, and A.I. Subbotin, *Asymptotic controllability implies feedback stabilization*, IEEE Trans. Automat. Control, 42 (1997), pp. 1394–1407.

[10] F.H. Clarke and R.B. Vinter, *Stability analysis of sliding-mode feedback control*, Control Cybernet., 38 (2009), pp. 1169–1192.

[11] S. Dashkovskiy, M. Kosmykov, A. Mironchenko, and L. Naujok, *Stability of interconnected impulsive systems with and without time delays, using Lyapunov methods*, Nonlinear Anal. Hybrid Systems, 6 (2012), pp. 899–915.

[12] A.F. Filippov, *Differential Equations with Discontinuous Righthand Sides*, Kluwer Academic, Dordrecht, 1988.
[13] E. Fridman, *A refined input delay approach to sampled-data control*, Automatica J. IFAC, 46 (2010), pp. 421–427.
[14] E. Fridman and M. Dambrine, *Control under quantization, saturation and delay: An LMI approach*, Automatica J. IFAC, 45 (2009), pp. 2258–2264.
[15] A. Germani, C. Manes, and P. Pepe, *Observer-based stabilizing control for a class of nonlinear retarded systems*, in Time Delay Systems Methods, Applications and New Trends, R. Sipahi, T. Vyhlidal, S.-I. Niculescu, and P. Pepe, eds., Lecture Notes in Control and Inform. Sci. 423, Springer, Berlin, 2012, pp. 331–342.
[16] A. Germani, C. Manes, and P. Pepe, *Input-output linearization with delay cancellation for nonlinear delay systems: The problem of the internal stability*, Internat. J. Robust Nonlinear Control, 13 (2003), pp. 909–937.
[17] K. Gu, V.L. Kharitonov, and J. Chen, *Stability of Time Delay Systems*, Birkhäuser, Boston, 2003.
[18] J.K. Hale, and S.M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Springer Verlag, New York, 1993.
[19] C. Hua, X. Guan, and P. Shi, *Robust stabilization of a class of nonlinear time-delay systems*, Appl. Math. Comput., 155 (2004), pp. 737–752.
[20] H. Ito, Z.-P. Jiang, and P. Pepe, *Construction of Lyapunov–Krasovskii functionals for networks of iISS retarded systems in small-gain formulation*, Automatica J. IFAC, 49 (2013), pp. 3246–3257.
[21] H. Ito, P. Pepe, and Z.-P. Jiang, *A small-gain condition for iISS of interconnected retarded systems based on Lyapunov–Krasovskii functionals*, Automatica J. IFAC, 46 (2010), pp. 1646–1656.
[22] M. Jankovic, *Control Lyapunov–Razumikhin functions and robust stabilization of time delay systems*, IEEE Trans. Automat. Control, 46 (2001), pp. 1048–1060.
[23] M. Jankovic, *Extension of control Lyapunov functions to time-delay systems*, in Proceedings of the 39th IEEE Conference on Decision and Control, Sydney, Australia, IEEE, Piscataway, NJ, 2000, pp. 4403–4408.
[24] I. Karafyllis, *Lyapunov theorems for systems described by retarded functional differential equations*, Nonlinear Analysis, 64 (2006), pp. 590–617.
[25] I. Karafyllis, *Stabilization by means of approximate predictors for systems with delayed input*, SIAM J. Control Optim., 49 (2011), pp. 1100–1123.
[26] I. Karafyllis and Z.-P. Jiang, *Necessary and sufficient Lyapunov-like conditions for robust nonlinear stabilization*, ESAIM Control Optim. Calc. Var., 16 (2010), pp. 887–928.
[27] I. Karafyllis and Z.P. Jiang, *Stability and Stabilization of Nonlinear Systems*, Springer, London, 2011.
[28] I. Karafyllis and M. Krstic, *Nonlinear stabilization under sampled and delayed measurements, and with inputs subject to delay and zero-order hold*, IEEE Trans. Automat. Control, 57 (2012), pp. 1141–1154.
[29] I. Karafyllis and M. Krstic, *Global stabilization of feedforward systems under perturbations in sampling schedule*, SIAM J. Control Optim., 50 (2012), pp. 1389–1412.
[30] H.K. Khalil, *Nonlinear Systems*, 3rd ed., Prentice Hall, Upper Saddle River, NJ, 2000.
[31] V.L. Kharitonov, *Time-Delay Systems, Lyapunov Functionals and Matrices*, Birkhäuser, New York, 2013.
[32] V. Kolmanovskii and A. Myshkis, *Introduction to the Theory and Applications of Functional Differential Equations*, Kluwer Academic, Dordrecht, 1999.
[33] Y.S. Ledyaev and E.D. Sontag, *A Lyapunov characterization of robust stabilization*, Nonlinear Anal. Ser. A, 37 (1999), pp. 813–840.
[34] C.-H. Lien, *Global exponential stabilization for several classes of uncertain nonlinear systems with time-varying delay*, Nonlinear Dynam. Systems Theory, 4 (2004), pp. 15–30.
[35] K. Liu and E. Fridman, *Wirtinger's inequality and Lyapunov-based sampled-data stabilization*, Automatica J. IFAC, 48 (2012), pp. 102–108.
[36] R. Mahboobi Esfanjani and S.K.Y. Nikravesh, *Stabilising predictive control of non-linear time-delay systems using control Lyapunov–Krasovskii functionals*, IET Control Theory Appl., 3 (2009), pp. 1395–1400.
[37] M. Malisoff and E.D. Sontag, *Asymptotic controllability and input-to-state stabilization: The effect of actuator errors*, in Optimal Control, Stabilization and Nonsmooth Analysis, Lecture Notes in Control and Inform. Sci. 301, Springer, Berlin, 2004, pp. 155–171.
[38] L.A. Marquez-Martinez and C.H. Moog, *Input-output feedback linearization of time-delay systems*, IEEE Trans. Automat. Control, 49 (2004), pp. 781–785.

[39] F. Mazenc, M. Malisoff, and T.H. Dinh, *Robustness of nonlinear systems with respect to delay and sampling of the controls*, Automatica J. IFAC, 49 (2013), pp. 1925–1931.

[40] F. Mazenc, S.-I. Niculescu, and M. Bekaik, *Backstepping for nonlinear systems with delay in the input revisited*, SIAM J. Control Optim., 49 (2011), pp. 2263–2278.

[41] S. Mondié and W. Michiels, *Finite spectrum assignment of unstable time-delay systems with a safe implementation*, IEEE Trans. Automat. Control, 48 (2003), pp. 2207–2212.

[42] S.-I. Niculescu, *Delay Effects on Stability, a Robust Control Approach*, Lecture Notes in Control and Inform. Sci. 269, Springer, London, 2001.

[43] T. Oguchi and J.-P. Richard, *Sliding-mode control of retarded nonlinear systems via finite spectrum assignment approach*, IEEE Trans. Automat. Control, 51 (2006), pp. 1527–1531.

[44] T. Oguchi, A. Watanabe, and T. Nakamizo, *Input-output linearization of retarded nonlinear systems by using an extension of Lie derivative*, Internat. J. Control, 75 (2002), pp. 582–590.

[45] P. Pepe, *On Liapunov-Krasovskii functionals under Carathéodory conditions*, Automatica J. IFAC, 43 (2007), pp. 701–706.

[46] P. Pepe, *The problem of the absolute continuity for Lyapunov–Krasovskii functionals*, IEEE Trans. Automat. Control, 52 (2007), pp. 953–957.

[47] P. Pepe, *Input-to-state stabilization of stabilizable, time-delay, control affine, nonlinear systems*, IEEE Trans. Automat. Control, 54 (2009), pp. 1688–1693.

[48] P. Pepe, *On Sontag's formula for the input-to-state practical stabilization of retarded control-affine systems*, Systems Control Lett., 62 (2013), pp. 1018–1025.

[49] P. Pepe and H. Ito, *On saturation, discontinuities and delays, in iISS and ISS feedback control redesign*, IEEE Trans. Automat. Control, 57 (2012), pp. 1125–1140.

[50] P. Pepe and Z.-P. Jiang, *A Lyapunov–Krasovskii methodology for ISS and iISS of time-delay systems*, Systems Control Lett., 55 (2006), pp. 1006–1014.

[51] P. Pepe and I. Karafyllis, *Converse Lyapunov–Krasovskii theorems for systems described by neutral functional differential equations in Hale's form*, Internat. J. Control, 86 (2013), pp. 232–243.

[52] E.P. Ryan, *On Brockett's condition for smooth stabilizability and its necessity in a context of nonsmooth feedback*, SIAM J. Control Optim., 32 (1994), pp. 1597–1604.

[53] R. Sipahi, S.-I. Niculescu, C.T. Abdallah, W. Michiels, and K. Gu, *Stability and stabilization of systems with time delay*, IEEE Control Systems, 31 (2011), pp. 38–65.

[54] E.D. Sontag, *Smooth stabilization implies coprime factorization*, IEEE Trans. Automat. Control, 34 (1989), pp. 435–443.

[55] E.D. Sontag, *A "universal" construction of Artstein's theorem on nonlinear stabilization*, Systems Control Lett., 13 (1989), pp. 117–123.

[56] R. Yang and Y. Wang, *Finite-time stability and stabilization of a class of nonlinear time-delay systems*, SIAM J. Control Optim., 50 (2012), pp. 3113–3131.

[57] N. Yeganefar, P. Pepe, and M. Dambrine, *Input-to-state stability of time-delay systems: A link with exponential stability*, IEEE Trans. Automat. Control, 53 (2008), pp. 1526–1531.

[58] S.Y. Yoon and Z. Lin, *Truncated predictor feedback control for exponentially unstable linear systems with time-varying input delay*, Systems Control Lett., 62 (2013), pp. 837–844.

[59] X. Yu, B. Wang, and X. Li, *Computer-controlled variable structure systems: The state-of-the-art*, IEEE Trans. Indust. Inform., 8 (2012), pp. 197–205.

[60] X. Zhang and Z. Cheng, *Global stabilization of a class of time-delay nonlinear systems*, Internat. J. Systems Sci., 36 (2005), pp. 461–468.

[61] B. Zhou, Z. Lin, and G.-R. Duan, *Truncated predictor feedback for linear systems with long time-varying input delays*, Automatica J. IFAC, 48 (2012), pp. 2387–2389.

[62] B. Zhou, Z. Lin, and G.-R. Duan, *Global and semi-global stabilization of linear systems with multiple delays and saturations in the input*, SIAM J. Control Optim., 48 (2010), pp. 5294–5332.