A PROBABILISTIC APPROACH TO CLASSICAL SOLUTIONS OF THE MASTER EQUATION FOR LARGE POPULATION EQUILIBRIA

JEAN-FRANÇOIS CHASSAGNEUX*, DAN CRISAN† AND FRANÇOIS DELARUE‡

Abstract. We analyze a class of nonlinear partial differential equations (PDEs) defined on $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$, where $\mathcal{P}_2(\mathbb{R}^d)$ is the Wasserstein space of probability measures on $\mathbb{R}^d$ with a finite second-order moment. We show that such equations admit a classical solutions for sufficiently small time intervals. Under additional constraints, we prove that their solution can be extended to arbitrary large intervals. These nonlinear PDEs arise in the recent developments in the theory of large population stochastic control. More precisely they are the so-called *master equations* corresponding to asymptotic equilibria for a large population of controlled players with mean-field interaction and subject to minimization constraints. The results in the paper are deduced by exploiting this connection. In particular, we study the differentiability with respect to the initial condition of the flow generated by a forward-backward stochastic system of McKean-Vlasov type. As a byproduct, we prove that the decoupling field generated by the forward-backward system is a classical solution of the corresponding master equation. Finally, we give several applications to mean-field games and to the control of McKean-Vlasov diffusion processes.

Keywords: Master equation; McKean-Vlasov SDEs; forward-backward systems; decoupling field; Wasserstein space; master equation.

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1. Introduction

The theory of large population stochastic control describes asymptotic equilibria among a large population of controlled players with mean field interaction and subject to minimization constraints. It has received a lot of interest since the earlier works on mean-field games of Lasry and Lions [23, 24, 25] and of Huang, Caines and Malhamé [20]. Mean-field game theory is the branch of large population stochastic control theory that corresponds to the case when equilibria inside the population are understood in the sense of Nash and thus describe consensus between the players that make the best decision they can, taking into account the current states of the others in the game. We cover this class of control problems in Section 5.2. There are other types of large population equilibria in the literature yielding different types of asymptotic control problems. As an example, the case when players obey a common policy controlled by a single center of decision is investigated in [5, 9]. We cover this distinct control problem in Section 5.3.

Lasry and Lions described equilibria by means of a fully-coupled forward-backward system consisting of two partial differential equations: a (forward) Fokker-Planck equation describing the dynamics of the population and a (backward) Hamilton-Jacobi-Bellman equation describing the optimization constraints. In his seminal lectures at

*Department of Mathematics, Imperial College London. j.chassagneux@imperial.ac.uk
†Department of Mathematics, Imperial College London. d.crisan@imperial.ac.uk
‡Laboratoire Jean-Alexandre Dieudonné, Université de Nice Sophia-Antipolis. delarue@unice.fr

the *Collège de France*, Lions noticed that the flow of measures solving the Fokker-Planck equation (that is the forward part of the system) can be interpreted as the characteristic trajectories of a nonlinear PDE. The equilibrium of a large population of players with mean field interaction is characterized through a nonlinear partial differential equation set on an enlarged state space that contains both the private position of a typical player and the distribution of the population. The solution of the PDE contains all the necessary information to entirely describe the equilibria of the game and, on the model of the Chapman-Kolmogorov equation for the evolution of a Markov semi-group, it is called the *master equation* of the game. This equation has the form [1]

$$\partial_t u(t,x,\mu) = Au(t,x,\mu) + f\big(x, u(t,x,\mu), Bu(t,x,\mu), \nu\big) + \int_{\mathbb{R}^d} \big[Cu(t,x,\mu)\big](\cdot) \mathrm{d}\mu(\cdot), \quad (1.1)$$

for $t > 0$ and $(x,\mu) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$, where $\mathcal{P}_2(\mathbb{R}^d)$ is the Wasserstein space of probability measures on $\mathbb{R}^d$ with a finite second-order moment. In (1.1), ν is the image of μ by the mapping $\mathbb{R}^d \ni x \mapsto (x, u(t,x,\mu))$; moreover, A and B are differential operators that differentiate in the x variable, respectively at the second and first order, whilst C is a nonlocal operator that involves differentiation in the μ variable. The notion of differentiation in the measure variable follows Lions' definition (see [4]).

Since its introduction in Lions' lectures, there have been only a few papers on the master equation. In the notes he wrote following Lions' lectures (see [4]), Cardaliaguet discusses the particular case when players have deterministic trajectories, and where the solutions to the master equation is understood in the viscosity sense. In this framework, the existence of classical solutions has just been investigated for short time horizons by Gangbo and Swiech in the preprint [16]. Recently, in the independent works [1, 8, 17, 18] and with different approaches, several authors revisited, mostly heuristically, the master equation in the case when the dynamics of the players are stochastic. A few months ago, in a lecture at the *Collège de France* [27], Lions gave an outline of a proof, based on PDE arguments, for investigating the master equation rigorously in the latter case. In [1, 8], the notion of master equation is extended to other types of stochastic control problem with players that obey a common policy controlled by a single center of decision.

The goal of this paper is to develop a probabilistic analysis of the class of equations (1.1). We seek *classical* solutions for a class of PDEs that incorporates the master equations for both types of policies (individual or collective) and for players with dynamics that can be either deterministic or stochastic. Beyond their purely theoretical interest, classical solutions (as opposed to viscosity solutions) are expected to be of use when handling approximated equilibria in a variety of situations: For instance, they help in proving the convergence of the equilibria, when computed over finite systems of players, toward the equilibria of the asymptotic game. This is indeed a challenging question that remains partially open.[2] Similarly, the analysis of numerical schemes for computing the equilibria certainly benefits from robust regularity estimates for the solution of the master equation.

One of the reason for using a probabilistic approach is that there has been an expanding literature in probability theory on forward-backward systems, which have been widely used in stochastic control. Although mostly limited to the finite dimension, the existing theory gives a helpful insight into the general mechanism for deriving the master equation. One of the most noticeable results is that a forward-backward system may be

[1]The master equation is introduced here in its forward form. However in its application to mean field games it is used in its backward form, see equation (2.12).

[2]See however the recent advances in [13, 22].

decoupled by means of a *decoupling field* provided the system is uniquely solvable, see e.g. [29, 30]. More precisely, the decoupling field allows one to express the backward component of the solution as a function of the forward one. When the coefficients of the forward-backward system are deterministic, the decoupling field satisfies (in a suitable sense) a quasilinear PDE. In the case of mean-field games, the forward-backward system consists of two coupled PDEs, one of Fokker-Planck type and another one of Hamilton-Jacobi-Bellman type, and the corresponding quasilinear PDE is nothing but the master equation.

Another reason for analysing the master equation by means of probabilistic arguments is that equilibria in large population stochastic control problems driven by either individual or collective policies may be characterized as solutions of finite-dimensional forward-backward systems of the McKean-Vlasov type, see [5, 7]. The reformulation is based either on the connection between Hamilton-Jacobi-Bellman equations and backward SDEs or on the stochastic Pontryagin principle, see [14, 35] for the basic mechanisms in the non McKean-Vlasov framework. This reformulation has a crucial role as it allows one to reduce the infinite-dimensional system made of the Fokker-Planck equation and of the Hamilton-Jacobi-Bellman equation to a finite dimensional system. The price to pay is that the coefficients of the finite dimensional system may depend upon the law of the solution, in the spirit of McKean's theory of nonlinear SDEs. Inspired by Pardoux and Peng's work [31] on the connection between backward SDEs and classical solutions to semilinear PDEs, we develop a systematic approach for analyzing the smoothness of the solution of the master equation by investigating the smoothness of the flow generated by the solution of the McKean-Vlasov forward-backward system with respect to the initial input. However, because of the McKean-Vlasov nonlinearity, the analysis is far from a straightforward adaptation of the classical result of Pardoux and Peng [31]. The main issue is that the independent variable includes a probability measure, which requires a non-trivial extension of the notion of differentiability with respect to a probability measure.

Several notions of derivatives with respect to a probability measure have been introduced in the literature. For example, the notion of Wasserstein derivative has been discussed within the context of optimal transport, see the monograph by Villani [34]. An alternative, though connected, approach was suggested by Lions, see [4]. Generally speaking, Lions' approach consists in lifting (in a canonical manner) functions defined on the Wasserstein space $\mathcal{P}_2(\mathbb{R}^d)$ (the space of probability measures on $\mathbb{R}^d$, with finite second-order moments endowed with the Wasserstein metric) into functions defined on $L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^d)$, the space of square integrable d-dimensional random variables defined on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$. In this way, the operation of differentiation with respect to a probability measure is defined as the Fréchet differentiation in $L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^d)$. This approach is especially suited to the mean field games framework. Indeed, the probabilistic representation we use yields a canonical lifted representation of the equilibria on $L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^d)$ that carries the underlying noise. The McKean-Vlasov forward-backward system that models the equilibria consists of a forward component describing the dynamics of the population and a backward one describing the dynamics of the solution of the master equation along the state of the population. Any perturbation in $L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^d)$ of the initial condition of the forward component thus generates a perturbation in $L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^d)$ of the solution of the master equation. Using this strategy, the smoothness of the solution of the master equation is deduced by investigating the smoothness of the flow generated by the McKean-Vlasov forward-backward system with respect to an initial condition in $L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^d)$.

In the sequel, we apply this strategy to general forward-backward systems of equations of McKean-Vlasov type. Under suitable assumption, we prove that existence and uniqueness of solutions holds for the system and that the corresponding *decoupling field* is the unique classical solution of the time reversed version of the PDE (1.1). To do this we prove first the smoothness of the decoupling field by using the notion of differentiation described above. Next, we apply a tailor-made chain rule on the Wasserstein space to identify the structure of the PDE from the coefficients of the forward-backward system. In general, the result holds for sufficiently small time intervals, as it is usually the case with forward-backward processes.

Inspired by [11], we then show that, provided we have an *a priori* estimate for the gradient of the solution of the master equation, existence and uniqueness of a classical solution may be extended, via an inductive argument, to arbitrary large time intervals. This requires the identification of a suitable space of solutions that is left invariant along the induction, which is one of the most technical issues of the paper. In the framework of large population stochastic control, we identify three classes of examples under which the *a priori* bound for the gradient is shown to hold. The first two belong to the framework of mean-field games. To bound the gradient in each of them, we combine either convexity (in the first example) or ellipticity (in the second example) with the so-called Lasry-Lions condition, used for guaranteeing uniqueness of the equilibria, see [4]. To the best of our knowledge, except the aforementioned video by Lions [27], the solvability of the master equation in the classical sense is, in both cases, a new result[3]. The third example concerns the situation when players obey a common center of decision, in which case the stochastic control problem may be reformulated as an optimization problem over controlled McKean-Vlasov diffusion processes. In this last example, the proof mainly relies on convexity.

In a parallel work to ours made available recently, Buckdahn *et al.* [3] adopted a similar approach to study forward flows, proving that the semigroup of a standard McKean-Vlasov stochastic differential equation with smooth coefficients is the classical solution of a linear PDE defined on $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$. The results in [3] do not cover nonlinear PDEs of the type (1.1) that include master equations for large population equilibria. It must be also noticed that a crucial assumption is made therein on the smoothness of the coefficients, which restrict rather drastically the scope of applications. We avoid this, however, we do pay a heavy price for working under more tractable assumptions, see Remark 2.5 below.

We treat here systems of players driven by idiosyncratic (or independent) noises. Motivated by practical applications, see [8, 19], in subsequent work, the players will be driven by an additional common source of noise, in which case the McKean-Vlasov interaction in the forward-backward equations under consideration becomes random itself, as it then stands for the conditional distribution of the population given the common source of randomness.

The paper is organized as follows. The general set-up together with the main results are described in Section 2. The chain rule on the Wasserstein space is discussed in Section 3. The smoothness of the flow of a McKean-Vlasov forward-backward system is investigated in small time in Section 4. In Section 5, we provide some applications

[3]As far as we understand the sketch of the proof in [27], the underlying arguments are reminiscent of the way in which we use convexity in the first class of examples.

to large population stochastic control. The proofs of some technical results are given in Appendix.

2. General step-up and overview of the results

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space supporting a d-dimensional Brownian motion $(W_t)_{t\geqslant 0}$ and a square integrable random variable ξ, independent of $(W_t)_{t\geqslant 0}$. We denote by $(\mathcal{F}_t^{\xi,W})_{t\geqslant 0}$ the augmented filtration generated by ξ and $(W_t)_{t\geqslant 0}$. For a given terminal time $T > 0$, we consider the following system of equations:

$$\left\{\begin{array}{lcl} X_s & = & \xi + \int_0^s b(X_r, Y_r, Z_r, \mathbb{P}_{(X_r,Y_r)})\mathrm{d}r + \int_0^s \sigma(X_r, Y_r, \mathbb{P}_{(X_r,Y_r)})\mathrm{d}W_r, \\ Y_s & = & g(X_T, \mathbb{P}_{X_T}) + \int_s^T f(X_r, Y_r, Z_r, \mathbb{P}_{(X_r,Y_r)})\mathrm{d}r - \int_s^T Z_r \mathrm{d}W_r, \end{array}\right. \quad s \in [t,T] \tag{2.1}$$

The processes X, Y and Z are d, m and $m \times d$ dimensional, respectively. The coefficients $b : \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^m) \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \times \mathbb{R}^m \times \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^m) \to \mathbb{R}^{d\times d}$, $f : \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^m) \to \mathbb{R}^m$ and $g : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^m$ are measurable functions that satisfy conditions that will be imposed below. $\mathbb{P}_{(X_r,Y_r)}$ denotes the law of (X_r, Y_r). The system (2.1) is called a forward-backward system of McKean-Vlasov type. Notice that, for simplicity, the coefficients b, σ and f are time homogeneous and X has same dimension as the noise W. These constraints can however be lifted and a similar analysis will apply.

In the following, we will show that, under convenient assumptions, there exists a unique solution of the forward-backward system (2.1) together with a *decoupling field* $U : [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^m$ to (2.1). Namely, U is a function such that

$$Y_s = U(s, X_s, \mathbb{P}_{X_s})\,,\ 0 \leqslant s \leqslant T\,. \tag{2.2}$$

Finally, we will show

$$(t, x, \mu) \in [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to U(T-t, x, \mu)$$

is a classical solution of the equation (1.1).

2.1. Definition of U.

The construction of the decoupling field U is typically discussed under the assumption that the existence and uniqueness of the solution of the system (2.1) is holds. See, e.g. [5, 6, 7], for conditions under which this holds for an arbitrary time horizon T. We adopt here a different approach: We first focus on the case where T is sufficiently small so that the existence and uniqueness of the solution of the system (2.1) hold. This helps us construct the decoupling field U for the same time horizon and, therefore deduce the existence of a unique local solution of PDE (1.1). Secondly we use results from [5, 7] to pass from a small time to an arbitrary time horizon and there justify the existence of a unique global solution to (1.1).

A common strategy to introduce the decoupling field consists in letting the initial time in (2.1) vary. Without any loss of generality, we can assume that $(\Omega, \mathcal{A}, \mathbb{P})$ is equipped with a filtration $(\mathcal{F}_t)_{t\geqslant 0}$ (satisfying the usual condition) such that $(\Omega, \mathcal{F}_0, \mathbb{P})$ is rich enough to carry $\mathbb{R}^d$-valued random variables with any arbitrary distribution in $\mathcal{P}_2(\mathbb{R}^d)$ and $(W_t)_{t\geqslant 0}$ is an $(\mathcal{F}_t)_{t\geqslant 0}$-Brownian motion. In particular, ξ in (2.1) may be taken as an $\mathcal{F}_0$-measurable square-integrable random variable.

In the sequel, we often use the symbol μ to denote the law of ξ. We will use the notation $[\Theta] := \mathbb{P}_\Theta$ to denote the law of the random variable Θ (so then $\mu = [\xi]$). Within this set-up, we consider the following version of (2.1) with the forward component starting at time t from $\xi \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)$:

$$\begin{cases} X_s^{t,\xi} &= \xi + \int_t^s b\big(\theta_r^{t,\xi}, [\theta_r^{t,\xi,(0)}]\big)\mathrm{d}r + \int_t^s \sigma\big(\theta_r^{t,\xi,(0)}, [\theta_r^{t,\xi,(0)}]\big)\mathrm{d}W_r, \\ Y_s^{t,\xi} &= g\big(X_T^{t,\xi}, [X_T^{t,\xi}]\big) + \int_s^t f\big(\theta_r^{t,\xi}, [\theta_r^{t,\xi,(0)}]\big)\mathrm{d}r - \int_t^s Z_r^{t,\xi}\mathrm{d}W_r, \end{cases} \quad s \in [t,T] \tag{2.3}$$

with $\theta^{t,\xi} = (X^{t,\xi}, Y^{t,\xi}, Z^{t,\xi})$ and $\theta^{t,\xi,(0)} = (X^{t,\xi}, Y^{t,\xi})$.

A crucial remark for the subsequent analysis is to notice that the Yamada–Watanabe theorem extends to equations of the same type as (2.3). More precisely, one can prove that, whenever pathwise uniqueness holds, solutions are also unique in law [21, Example 2.14]. As a consequence, it follows that the law of $(X^{t,\xi}, Y^{t,\xi})$ only depends upon the law of ξ. In other words, $[(X_r^{t,\xi}, Y_r^{t,\xi})]$ is a function of $[\xi] = \mu$. Given $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, it thus makes sense to consider $([(X_r^{t,\xi}, Y_r^{t,\xi})])_{t \leqslant r \leqslant T}$ without specifying the choice of the lifted random variable ξ that has μ as distribution. We then introduce, for any $x \in \mathbb{R}^d$, a stochastic flow associated to the system (2.3), defined as

$$\begin{cases} X_s^{t,x,\mu} &= x + \int_t^s b\big(\theta_r^{t,x,\mu}, [\theta_r^{t,\xi,(0)}]\big)\mathrm{d}r + \int_t^s \sigma\big(\theta_r^{t,x,\mu,(0)}, [\theta_r^{t,\xi,(0)}]\big)\mathrm{d}W_r, \\ Y_s^{t,x,\mu} &= g\big(X_T^{t,x,\mu}, [X_T^{t,\xi}]\big) + \int_s^t f\big(\theta_r^{t,x,\mu}, [\theta_r^{t,\xi,(0)}]\big)\mathrm{d}r - \int_t^s Z_r^{t,x,\mu}\mathrm{d}W_r, \end{cases} \tag{2.4}$$

with $\theta^{t,x,\mu} = (X^{t,x,\mu}, Y^{t,x,\mu}, Z^{t,x,\mu})$ and $\theta^{t,x,\mu,(0)} = (X^{t,x,\mu}, Y^{t,x,\mu})$.

We now have all the ingredients to give the definition of a decoupling field to (2.3) on $[0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$. For the following definition, assume for the moment that, for any $(t,x,\mu) \in [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$ and any random variable $\xi \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)$ with distribution μ, (2.3) has a unique (progressively-measurable) solution $(X_s^{t,\xi}, Y_s^{t,\xi}, Z_s^{t,\xi})_{s \in [t,T]}$ such that

$$\sup_{s \in [t,T]} |X_s^{t,\xi}|^2, \quad \sup_{s \in [t,T]} |Y_s^{t,\xi}|^2, \quad \text{and} \quad \int_t^T |Z_s^{t,\xi}|^2 \mathrm{d}s,$$

are integrable. Assume also that (2.4) has a unique (progressively-measurable) solution $(X_s^{t,x,\xi}, Y_s^{t,x,\xi}, Z_s^{t,x,\xi})_{s \in [t,T]}$ such that

$$\sup_{s \in [t,T]} |X_s^{t,x,\xi}|^2, \quad \sup_{s \in [t,T]} |Y_s^{t,x,\xi}|^2, \quad \text{and} \quad \int_t^T |Z_s^{t,x,\xi}|^2 \mathrm{d}s,$$

are integrable. Then, we may let:

Definition 2.1 (The decoupling field U). *The function $U : [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \mapsto \mathbb{R}^m$ defined as*

$$U(t,x,\mu) = Y_t^{t,x,\mu}, \quad (t,x,\mu) \in [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \tag{2.5}$$

is called the decoupling field of the forward-backward system (2.3) *(or, equivalently, of the corresponding stochastic flow* (2.4)*).*

The decoupling property (2.2) of U is proved in Proposition 2.2 below, under assumptions that guarantee existence and uniqueness to (2.3) and (2.4).

Recall now that the 2-Wasserstein distance W_2, defined on $\mathcal{P}_2(\mathbb{R}^k)$, $k \geqslant 1$ is given by

$$W_2(\mu,\nu) = \inf_\gamma \left[\int_{(\mathbb{R}^k)^2} |u - v|^2 \gamma(\mathrm{d}u, \mathrm{d}v); \ \gamma(\cdot \times \mathbb{R}^k) = \mu, \ \gamma(\mathbb{R}^k \times \cdot) = \nu \right]^{1/2}.$$

As already mentioned, a very convenient way to prove strong existence and uniqueness to (2.3) and (2.4) consists in working first with small time horizons. For T sufficiently small, there exists a unique solution to the systems (2.3) and (2.4) under the following assumption:

Assumption ((***H*0**)(i))**.** *There exists a constant $L > 0$ such that the mappings b, σ, f and g are L-Lipschitz continuous in all the variables, the distance on $\mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^m)$, respectively $\mathcal{P}_2(\mathbb{R}^d)$ being the 2-Wasserstein distance.*

The existence of a local solution to the systems (2.3) and (2.4) under assumption ($\boldsymbol{H0}$)(i) is not new (see for instance [6, Proof of Lemma 2]). The proof consists of a straightforward adaption of the results in [11] for classical forward backward stochastic differential equations (FBSDEs). To be precise, one shows that the systems (2.3) and (2.4) are uniquely solvable under assumption ($\boldsymbol{H0}$)(i) provided $T \leqslant c$ for a constant $c := c(L) > 0$. Examples where the result can be extended to long time horizons will be discussed in Section 5.

It is quite illuminating to observe that the system (2.4) can be rewritten as a classical coupled FBSDE with time dependent coefficients, as follows

$$\left\{ \begin{array}{lcl} X_s^{t,x,\mu} & = & x + \int_t^s \hat{b}_{t,\mu}(r, \theta_r^{t,x,\mu}) \mathrm{d}r + \int_t^s \hat{\sigma}_{t,\mu}(r, \theta_r^{t,x,\mu,(0)}) \mathrm{d}W_r, \\ Y_s^{t,x,\mu} & = & \hat{g}_{t,\mu}(X_T^{t,x,\mu}) + \int_s^t \hat{f}_{t,\mu}(r, \theta_r^{t,x,\mu}) \mathrm{d}r - \int_t^s Z_r^{t,x,\mu} \mathrm{d}W_r, \end{array} \right. \tag{2.6}$$

with $(\hat{b}_{t,\mu}, \hat{f}_{t,\mu}, \hat{\sigma}_{t,\mu}, \hat{g}_{t,\mu})(r,x,y,z) := (b,f,\sigma,g)(x,y,z,[\theta_r^{t,\xi,(0)}])$. Basically, for this new set of coefficients, the dependence upon the measure is frozen since μ and $[\theta^{t,\xi,(0)}]$ are fixed and do not depend on x. In particular, when replacing x by ξ in (2.4) and (2.6), for some random variable ξ with μ as distribution, uniqueness of solutions to the classical (time-inhomogeous) FBSDE (2.6) implies that $(X^{t,\xi,\mu}, Y^{t,\xi,\mu}, Z^{t,\xi,\mu}) = \theta^{t,\xi}$. Then, the representation (2.6) allows us to characterize the decoupling field of the system (2.3) as follows:

Under ($\boldsymbol{H0}$)(i), we know from the classical theory of coupled FBSDEs [11] that, for T sufficiently small, for any $t \in [0,T]$, there exists a continuous decoupling field $\hat{U}_{t,\mu} : [t,T] \times \mathbb{R}^d \ni (s,x) \mapsto \hat{U}_{t,\mu}(s,x)$ to (2.6) such that $Y_s^{t,x,\mu} = \hat{U}_{t,\mu}(s, X_s^{t,x,\mu})$ for $s \in [t,T]$, the representation remaining true when x is replaced by an $\mathcal{F}_t$-measurable square-integrable random variable (see [11, Corollary 1.5]). In particular, choosing $s = t$, we get $U(t,x,\mu) = \hat{U}_{t,\mu}(t,x)$. We deduce that

Proposition 2.2. *Given $(t,x,\mu) \in [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$ and $\xi \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)$, with $[\xi] = \mu$, we have, for T small enough, for all $s \in [t,T]$,*

$$\hat{U}_{t,\mu}(s,x) = U(s,x,[X_s^{t,\xi}])\,,\ Y_s^{t,x,\mu} = U(s, X_s^{t,x,\mu}, [X_s^{t,\xi}]) \ \text{ and } Y_s^{t,\xi} = U(s, X_s^{t,\xi}, [X_s^{t,\xi}]).$$

Proof. By uniqueness of the solution to (2.3), the processes

$$(X_u^{s,X_s^{t,\xi}}, Y_u^{s,X_s^{t,\xi}})_{u \in [s,T]} \quad \text{and} \quad (X_u^{t,\xi}, Y_u^{t,\xi})_{u \in [s,T]}$$

coincide, so that $[(X_u^{s,X_s^{t,\xi}}, Y_u^{s,X_s^{t,\xi}})] = [(X_u^{t,\xi}, Y_u^{t,\xi})]$ for $u \in [s,T]$. We deduce that

$$\hat{U}_{t,\mu}(s,\cdot) = \hat{U}_{s,[X_s^{t,\xi}]}(s,\cdot),$$

which is the first equality. Now, the second equality follows from the fact that

$$Y_s^{t,x,\mu} = \hat{U}_{t,\mu}(s, X_s^{t,x,\mu}) = \hat{U}_{s,[X_s^{t,\xi}]}(s, X_s^{t,x,\mu}).$$

The last inequality is obtained by inserting $X_s^{t,\xi}$ instead of x in the first equality and observing $\hat{U}_{t,\mu}(s, X_s^{t,\xi}) = \hat{U}_{t,\mu}(s, X_s^{t,\xi,\mu}) = Y_s^{t,\xi,\mu} = Y_s^{t,\xi}$. □

2.2. **Smoothness of U.** Introduce now the additional assumption:

Assumption((**$H0$**)(ii))**.** *The functions b, σ, f and g are twice differentiable in the variables x, y and z, the derivatives of order 1 and 2 being uniformly bounded and uniformly Lipschitz-continuous in the variables x, y and z, uniformly in the parameter μ.*

Under (**$H0$**)(i–ii), we know from the classical theory of coupled FBSDEs [11] that the decoupling field $\hat{U}_{t,\mu}$ is once continuously differentiable in time and twice continuously differentiable in the x variable on $[t,T]\times\mathbb{R}^d$. It also satisfies $Z_s^{t,x,\mu} = \hat{V}_{t,\mu}(s, X_s^{t,x,\mu})$ with

$$\hat{V}_{t,\mu}(s,x) := \partial_x \hat{U}_{t,\mu}(s,x)\hat{\sigma}_{t,\mu}\big(s,x,\hat{U}_{t,\mu}(s,x)\big)\ , \quad s \in [t,T],\ x \in \mathbb{R}^d. \tag{2.7}$$

Notice that, throughout the paper, gradients of real-valued functions are expressed as row vectors. In particular, the term $\partial_x \hat{U}_{t,\mu}(s,x)$ is thus an $m\times d$ matrix as $\hat{U}_{t,\mu}$ takes values in $\mathbb{R}^m$.

Moreover, the decoupling field is a classical solution of the following quasi-linear PDE (or system of quasi-linear PDEs since m may be larger than 1)[4]:

$$\begin{aligned}
&\partial_s \hat{U}_{t,\mu}(s,x) + \partial_x \hat{U}_{t,\mu}(s,x)\hat{b}_{t,\mu}\big(s,x,\hat{U}_{t,\mu}(s,x),\hat{v}_{t,\mu}(s,x)\big) \\
&\quad + \frac{1}{2}\mathrm{Tr}\big[\partial^2_{xx}\hat{U}_{t,\mu}(s,x)\big(\hat{\sigma}_{t,\mu}\hat{\sigma}^\dagger_{t,\mu}\big)\big(s,x,\hat{U}_{t,\mu}(s,x)\big)\big] + \hat{f}_{t,\mu}\big(s,x,\hat{U}_{t,\mu}(s,x),\hat{v}_{t,\mu}(s,x)\big) = 0,
\end{aligned} \tag{2.8}$$

on $[t,T]\times\mathbb{R}^d$ with the final boundary condition $\hat{U}_{t,\mu}(T,x) = \hat{g}_{t,\mu}(x)$, $x\in\mathbb{R}^d$. In (2.8), the trace reads as the m dimensional vector $(\mathrm{Tr}[\partial^2_{xx}\hat{U}^i_{t,\mu}(s,x)(\hat{\sigma}_{t,\mu}\hat{\sigma}^\dagger_{t,\mu})(s,x,\hat{U}_{t,\mu}(s,x))])_{1\leqslant i\leqslant m}$. Recalling the link between $\hat{U}_{t,\mu}(t,\cdot)$ and $U(t,\cdot,\mu)$, it is then clear that the function $U(t,\cdot,\mu)$ is twice continuously differentiable in the x variable.

A more challenging question is the smoothness in the direction of the measure. Generally speaking, we will show that $U(t,x,\cdot)$ is twice differentiable in the measure direction, in a suitable sense, provided that the coefficients of the system (2.1) are also regular in the measure direction. For the reader's convenience, we provide next a brief introduction to the notion of regularity with respect to the measure argument, further details being given in Section 3.

Given a function $V : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$, we call the *lift of* V on the probability space $(\Omega,\mathcal{A},\mathbb{P})$[5] the mapping $\mathcal{V} : L^2(\Omega,\mathcal{A},\mathbb{P}) \to \mathbb{R}$ defined by

$$\mathcal{V}(X) = V\big([X]\big), \quad X \in L^2(\Omega,\mathcal{A},\mathbb{P}).$$

Following Lions (see [4]), the mapping V is then said to be differentiable (resp. $\mathcal{C}^1$) on the Wasserstein space $\mathcal{P}_2(\mathbb{R}^d)$ if the lift $\mathcal{V}$ is Fréchet differentiable (resp. Fréchet differentiable with continuous derivatives) on $L^2(\Omega,\mathcal{A},\mathbb{P})$. The main feature of this approach is that the Fréchet derivative $D\mathcal{V}(X)$, when seen as an element of $L^2(\Omega,\mathcal{A},\mathbb{P})$ via Riesz' theorem, can be represented as

$$D\mathcal{V}(X) = \partial_\mu V\big([X]\big)(X),$$

where $\partial_\mu V([X]) : \mathbb{R}^d \ni v \mapsto \partial_\mu V([X])(v) \in \mathbb{R}^d$ is in $L^2(\mathbb{R}^d,[X];\mathbb{R}^d)$. In this way the tangent space to $\mathcal{P}_2(\mathbb{R}^d)$ at a probability measure μ is identified with a subspace of $L^2(\mathbb{R}^d,\mu;\mathbb{R}^d)$.

Note that the map $\partial_\mu V(\mu) : \mathbb{R}^d \ni v \mapsto \partial_\mu V(\mu)(v) \in \mathbb{R}^d$ is uniquely defined up to a μ-negligible Borel subset. Choosing a version for each μ might be a problem for handling

[4]For any matrix a we will denote its transpose by $a^\dagger$ and its trace by $\mathrm{Tr}(a)$.

[5]For notational convenience, the lifting procedure is done onto the same probability space that carries the driving Brownian motion W. Alternatively, one can use an arbitrary rich enough atomless probability space, see [4] and Section 3 for details.

it as a function of the joint variable (v, μ). In the next section, we will present conditions under which a continuous version of $\partial_\mu U(\mu)(\cdot)$ can be identified, such a version being uniquely defined on the support of μ. The next step is then to discuss the smoothness of the map $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \ni (v, \mu) \mapsto \partial_\mu V(\mu)(v)$. We say that V is *partially* $\mathcal{C}^2$ if the mapping $\mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \ni (\mu, v) \mapsto \partial_\mu V(\mu)(v)$ is continuous at any point (μ, v) such that $v \in \mathrm{Supp}(\mu)$ and if, for any $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, the mapping $\mathbb{R}^d \ni v \mapsto \partial_\mu V(\mu)(v)$ is differentiable, its derivative being jointly continuous with respect to μ and v at any point (μ, v) such that $v \in \mathrm{Supp}(\mu)$. The gradient is then denoted by $\partial_v[\partial_\mu V(\mu)](v) \in \mathbb{R}^{d\times d}$.

Note that, $\partial_\mu V(\mu)(v)$ is a d-dimensional row vector and $\partial_v[\partial_\mu V(\mu)](v)$ is a $d \times d$ matrix.

2.3. **Solution of a Master PDE.** In Section 3, we prove a chain rule for functions defined on the space $\mathcal{P}_2(\mathbb{R}^d)$ which are *partially* $\mathcal{C}^2$ in the above sense. Applying the chain rule to $U(t, x, \cdot)$, we get:

$$\begin{aligned} &U\big(t,x,[X_s^{t,\xi}]\big) - U\big(t,x,[\xi]\big) \\ &= \int_t^s \widehat{\mathbb{E}}\Big[\partial_\mu U\big(t,x,[X_r^{t,\xi}]\big)\big(\langle X_r^{t,\xi}\rangle\big) b\big(\langle \theta_r^{t,\xi}\rangle, [\theta_r^{t,\xi,(0)}]\big)\Big]\,\mathrm{d}r \\ &\quad + \frac{1}{2}\int_t^s \widehat{\mathbb{E}}\Big[\mathrm{Tr}\big[\partial_v\big[\partial_\mu U\big(t,x,[X_r^{t,\xi}]\big)\big]\big(\langle X_r^{t,\xi}\rangle\big)\big(\sigma\sigma^\dagger\big)\big(\langle\theta_r^{t,\xi,(0)}\rangle, [\theta_r^{t,\xi,(0)}]\big)\big]\Big]\,\mathrm{d}r. \end{aligned} \tag{2.9}$$

The above identity relies on new notations. Indeed, in order to distinguish the original randomness in the dynamics of (2.3), which has a physical meaning, from the randomness used to represent the derivatives on the Wasserstein space, we will represent the derivatives on the Wasserstein space on another probability space, denoted by $(\hat{\Omega}, \hat{\mathcal{A}}, \hat{\mathbb{P}})$. $(\hat{\Omega}, \hat{\mathcal{A}}, \hat{\mathbb{P}})$ is a copy of the original space $(\Omega, \mathcal{A}, \mathbb{P})$. In particular, for a random variable ξ defined on $(\Omega, \mathcal{A}, \mathbb{P})$, we denote by $\langle \xi \rangle$ its copy on $\hat{\Omega}$. All the expectations in the above expression may be translated into expectations under $\mathbb{E}$. Nevertheless, we will refrain from doing this to avoid ambiguities between "lifts" and random variables constructed on the original space $(\Omega, \mathcal{A}, \mathbb{P})$. We will state conditions under which the expectations in (2.9) are indeed well-defined.

Notice that, in (2.9), we used the same convention as in (2.8) for denoting gradients. The term $\partial_\mu U(t, x, [X_r^{t,\xi}])(\langle X_r^{t,\xi}\rangle)$ is thus seen as an $m \times d$ matrix and the trace term $\mathrm{Tr}[\partial_v[\partial_\mu U](t, x, [X_r^{t,\xi}])(\langle X_r^{t,\xi}\rangle)(\sigma\sigma^\dagger)(\langle\theta_r^{t,\xi,(0)}\rangle, [\theta_r^{t,\xi,(0)}])]$ as a vector of dimension m.

Combined with the analysis of the smoothness of U, we will then show that the function $[0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \ni (t, x, \mu) \mapsto U(t, x, \mu)$ solves, up to a time reversal, a PDE of the form (1.1). For the time being, we present a formal calculation to deduce this claim, the complete argument being given in Section 4. The basic observation is that, in the framework of Proposition 2.2, the time-increments of U may be expanded as

$$\begin{aligned} U(s+h, x, [X_s^{t,\xi}]) - U(s, x, [X_s^{t,\xi}]) &= U(s+h, x, [X_s^{t,\xi}]) - U(s+h, x, [X_{s+h}^{t,\xi}]) \\ &\quad + \hat{U}_{t,\mu}(s+h, x) - \hat{U}_{t,\mu}(s, x), \end{aligned} \tag{2.10}$$

for $t \leqslant s \leqslant s+h \leqslant T$. Applying the chain rule to the difference term $U(s+h, x, [X_s^{t,\xi}]) - U(s+h, x, [X_{s+h}^{t,\xi}])$ on the right hand side of the previous equality and assuming that the derivatives in the chain rule are continuous in time so that we can let h tend to 0, we obtain

$$\Big[\frac{\mathrm{d}}{\mathrm{d}h}\Big]_{|h=0} U(s+h,x,[X_s^{t,\xi}]) = -\widehat{\mathbb{E}}\Big[b\big(\langle\theta_s^{t,\xi}\rangle,[\theta_s^{t,\xi,(0)}]\big)\partial_\mu U(s,x,[X_s^{t,\xi}])\big(\langle X_s^{t,\xi}\rangle\big)\Big] \tag{2.11}$$
$$-\frac{1}{2}\widehat{\mathbb{E}}\Big[\mathrm{Tr}\big[\partial_v[\partial_\mu U(s,x,[X_s^{t,\xi}])]\big(\langle X_s^{t,\xi}\rangle\big)\big(\sigma\sigma^\dagger\big)\big(\langle\theta_s^{t,\xi,(0)}\rangle,[\theta_s^{t,\xi,(0)}]\big)\big]\Big] + \partial_s\hat{U}_{t,\mu}(s,x).$$

Choosing $s=t$, we deduce that U is right differentiable in time (it is then differentiable in time provided that the right-hand side is continuous in time). Recalling (2.8) together with the notation $\mu=[\xi]$ and using the transfer theorem to express the expectations that appear in the chain rule as integrals over $\mathbb{R}^d$, we then get that U is a solution to the equation

$$\begin{aligned}
&\partial_t U(t,x,\mu) + \partial_x U(t,x,\mu) b\big(x,U(t,x,\mu),\partial_x^\sigma U(t,x,\mu),\nu\big)\\
&\quad + \frac{1}{2}\mathrm{Tr}\big[\partial^2_{xx}U(t,x,\mu)\big(\sigma\sigma^\dagger\big)\big] + f\big(x,U(t,x,\mu),\partial_x^\sigma U(t,x,\mu),\nu\big)\\
&\quad + \int_{\mathbb{R}^d}\partial_\mu U(t,x,\mu)(v) b\big(v,U(t,v,\mu),\partial_x^\sigma U(t,v,\mu),\nu\big)\mathrm{d}\mu(v)\\
&\quad + \frac{1}{2}\int_{\mathbb{R}^d}\mathrm{Tr}\big[\partial_v\big[\partial_\mu U(t,x,\mu)\big](v)\big(\sigma\sigma^\dagger\big)\big(v,U(t,v,\mu),\nu\big)\big]\mathrm{d}\mu(v) = 0,
\end{aligned} \tag{2.12}$$

for $(t,x,\mu)\in[0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)$, with the terminal condition $U(T,x,\mu)=g(x,\mu)$, where ν is the law of $(\xi,U(t,\xi,\mu))$ when $[\xi]=\mu$ and

$$\partial_x^\sigma U(t,x,\mu) = \partial_x U(t,x,\mu)\sigma(x,U(t,x,\mu),\nu).$$

In particular, $u(t,\cdot,\cdot)=U(T-t,\cdot,\cdot)$ satisfies the equation (1.1), the operators A, B and C therein being defined as follows:

$$\begin{aligned}
Au(t,x,\mu) &= \partial_x u(t,x,\mu) b\big(x,u(t,x,\mu),\partial_x^\sigma u(t,x,\mu),\nu\big)\\
&\quad + \frac{1}{2}\mathrm{Tr}\big[\partial^2_{xx}u(t,x,\mu)\big(\sigma\sigma^\dagger\big)\big(x,u(t,x,\mu),\nu\big)\big]\\
Bu(t,x,\mu) &= \partial_x u(t,x,\mu)\sigma(x,u(t,x,\mu),\nu)\\
Cu(t,x,\mu)(v) &= \partial_\mu u(t,x,\mu)(v) b\big(v,u(t,v,\mu),\partial_x^\sigma u(t,v,\mu),\nu\big)\\
&\quad + \frac{1}{2}\mathrm{Tr}\big[\partial_v\big[\partial_\mu u(t,x,\mu)\big](v)\big(\sigma\sigma^\dagger\big)\big(v,u(t,v,\mu),\nu\big)\big],\ v\in\mathbb{R}^d,
\end{aligned}$$

with the initial condition $u(0,x,\mu)=g(x,\mu)$, and with the same convention as above for the meaning of ν and of $\partial_x^\sigma u$.

Our first main result is that, for small time horizons, all the partial derivatives that appear above make sense as continuous functions whenever the coefficients are sufficiently smooth. In this sense, U is a "classical" solution of (2.12), see Theorem 2.7 right below. We can actually prove that it is the unique one to satisfy suitable growth conditions, see Theorem 2.8. Our second main result is the extension to arbitrarily large time horizons for three classes of population equilibria. We refer the reader to Subsection 2.5 for a short account of the second result and to Section 5 for complete statements.

2.4. **Assumptions.** For an L^2 space, we use the notation $\|\cdot\|_2$ as a generic notation for the corresponding L^2-norm. For a linear mapping Υ on an L^2 space, we let $|||\Upsilon||| := \sup_{\|v\|_2=1}\|\Upsilon(v)\|_2$, and for a bilinear form on an L^2 space, we let in the same way $|||\Upsilon||| := \sup_{\|v_1\|_2=1,\|v_2\|_2=1}\|\Upsilon(v_1,v_2)\|_2$.

For a function h from a product space of the form $\mathbb{R}^k\times\mathcal{P}_2(\mathbb{R}^l)$ into $\mathbb{R}$, where $k,l\geqslant 1$, we denote by $\partial_w h(w,\mu)$ the derivative (if it exists) of h with respect to the Euclidean variable

w and by $D\mathcal{H}(w,\chi)$ the Fréchet derivative of the lifted mapping $\mathcal{H} : L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^l) \ni \chi \mapsto h(w,[\chi])$. The Fréchet derivative is seen as a linear form on L^2.

Concerning the first order differentiability of the coefficients, we shall assume:

Assumption. ($\boldsymbol{H1}$) *In addition to* ($\boldsymbol{H0}$)(i)*, the mappings b, f, σ, g are differentiable in $(w=(x,y,z),\mu)$*[6] *with jointly continuous derivatives in (w,μ)*[7] *in the following sense: There exist a constant $\tilde{L}$ (in addition to the constant L defined in* ($\boldsymbol{H0}$)(i)*), a constant $\alpha \geqslant 0$ and a functional $\Phi_\alpha : [L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^l)]^2 \ni (\chi,\chi') \mapsto \Phi_\alpha(\chi,\chi') \in \mathbb{R}_+$, continuous at any point (χ,χ) of the diagonal, such that, for all $\chi,\chi' \in L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^l)$,*

$$\Phi_\alpha(\chi,\chi') \leqslant \mathbb{E}\Big\{\big(1+|\chi|^{2\alpha}+|\chi'|^{2\alpha}+\|\chi\|_2^{2\alpha}\big)|\chi-\chi'|^2\Big\}^{1/2} \quad \textit{when } \chi \sim \chi', \tag{2.13}$$

and, for h matching any of the coordinates[8] *of b, f, σ or g, for all $w,w' \in \mathbb{R}^k$ and $\chi,\chi' \in L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^l)$,*

$$|||D\mathcal{H}(w,\chi)||| \leqslant L\ , \quad |||D\mathcal{H}(w,\chi)-D\mathcal{H}(w',\chi')||| \leqslant \tilde{L}\big(|w-w'|+\Phi_\alpha(\chi,\chi')\big),$$

and

$$|\partial_w h(w,[\chi])| \leqslant L\ , \quad |\partial_w h(w,[\chi])-\partial_w h(w',[\chi'])| \leqslant \tilde{L}\big(|w-w'|+\Phi_\alpha(\chi,\chi')\big).$$

Moreover, for any $\chi \in L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^l)$, the family $(D\mathcal{H}(w,\chi))_{w\in\mathbb{R}^k}$, identified by Riesz' theorem with a collection of elements in $L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^l)$, is uniformly square integrable.

Remark 2.3. *(i) In particular we note that $|D\mathcal{H}(w,\chi)\cdot\chi'| \leqslant L\|\chi'\|_2$ (where '$\cdot$' denotes the action of the duality).*

(ii) Notice that the right-hand side in (2.13) *might not be finite. Actually, we shall make use of* (2.13) *when χ and χ' coincide outside a bounded subset $\mathbb{R}^l$, namely $\chi(\omega)=\chi'(\omega)$ whenever $|\chi(\omega)|$ and $|\chi'(\omega)|$ are larger than some prescribed $R \geqslant 0$, in which cases the right-hand side in* (2.13) *is finite. For instance, choosing $\chi=\chi'$, we get from* (2.13) *that Φ_α is zero on the diagonal. Notice also that, when $\alpha=0$, we can directly choose $\Phi_\alpha(\chi,\chi')=\mathbb{E}[|\chi-\chi'|^2]^{1/2}$.*

(iii) Proposition 3.8 below shows that, under ($\boldsymbol{H1}$)*, the function $\mathbb{R}^l \ni v \mapsto \partial_\mu h(w,\mu)(v)$ admits, for any $w \in \mathbb{R}^k$ and $\mu \in \mathcal{P}_2(\mathbb{R}^l)$, a continuous version. It allows to represent $D\mathcal{H}(w,\chi)$, when identified with an element of $L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^l)$ by Riesz' theorem, in the form $\partial_\mu h(w,[\chi])(\chi)$. We stress that such a continuous version of $\mathbb{R}^l \ni v \mapsto \partial_\mu h(w,\mu)(v)$ is uniquely defined on the support of μ. Reexpressing the bounds in* ($\boldsymbol{H1}$)*, it satisfies*

$$\begin{aligned} &\mathbb{E}\big[\big|\partial_\mu h(w,[\chi])(\chi)\big|^2\big]^{1/2} \leqslant L, \\ &\mathbb{E}\big[\big|\partial_\mu h(w,[\chi])(\chi)-\partial_\mu h(w',[\chi'])(\chi')\big|^2\big]^{1/2} \leqslant \tilde{L}\big\{|w-w'|+\Phi_\alpha(\chi,\chi')\big\}, \end{aligned} \tag{2.14}$$

Moreover, the uniform square integrability property is equivalent to say that the family $(\partial_\mu h(w,[\chi])(\chi))_{w\in\mathbb{R}^k}$ is uniformly square integrable for any $\chi \in L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d)$.

(iv) The uniform integrability assumption plays a major role in our analysis. Taking into account the fact that all the $(D\mathcal{H}(w,\chi))_{w\in\mathbb{R}^k}$ have a norm less than L, this amounts

[6]Here μ stands for the generic symbol to denote the measure argument.

[7]Under the standing assumptions on the joint continuity of the derivatives, it is easily checked that the joint differentiability is equivalent to partial differentiability in each of the two directions w and μ.

[8]For the presentation of the assumption, it is here easier to take h as a real-valued function, which explains why we identify h with a coordinate of b, f, σ or g; however, we will sometimes say –rather abusively– that h matches b, f, σ or g.

to require that

$$\lim_{\mathbb{P}(A)\to 0, A\in\mathcal{A}} \sup_{w\in\mathbb{R}^k} \sup_{\Lambda\in L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^l):\|\Lambda\|_2\leqslant L} \big|D\mathcal{H}(w,\chi)\cdot\big(\Lambda\mathbf{1}_A\big)\big| = 0.$$

We stress the fact that it is automatically satisfied when $\alpha = 0$ in (2.13). Indeed, we shall prove in (4.14) below that, whenever $\alpha = 0$, there exists a constant $C \geqslant 0$ such that, for all $w \in \mathbb{R}^k$, $|D\mathcal{H}(w,\chi)|$ (identified with a random variable) is less than $C(1+|\chi|+\|\chi\|_2)$.

Concerning the second order differentiation of the coefficients, we shall assume:

Assumption (($\boldsymbol{H2}$))**.** *In addition to ($\boldsymbol{H1}$), all the mappings $(x,y,z)\mapsto b(x,y,z,\mu)$, $(x,y,z)\mapsto f(x,y,z,\mu)$, $(x,y)\mapsto\sigma(x,y,\mu)$ and $x\mapsto g(x,\mu)$ are twice differentiable for any $\mu\in\mathcal{P}_2(\mathbb{R}^l)$ the second-order derivatives being jointly continuous in (x,y,z) and μ. Moreover, for h equal to any of the coordinates of b, f, σ or g, for any $w\in\mathbb{R}^k$ and $\mu\in\mathcal{P}_2(\mathbb{R}^l)$, with the appropriate dimensions k and l, there exists a continuously differentiable version of the mapping $\mathbb{R}^l\ni v\mapsto\partial_\mu h(w,\mu)(v)$ such that the mapping $\mathbb{R}^k\times\mathbb{R}^l\ni(w,v)\mapsto\partial_\mu h(w,\mu)(v)$ is differentiable (in both variables) at any point (w,v) such that $v\in\mathrm{Supp}(\mu)$, the partial derivative $\mathbb{R}^k\times\mathbb{R}^l\ni(w,v)\mapsto\partial_v[\partial_\mu h(w,\mu)](v)$ being continuous at any (w,v) such that $v\in\mathrm{Supp}(\mu)$ and the partial derivative $\mathbb{R}^k\times\mathrm{Supp}(\mu)\ni(w,v)\mapsto\partial_w[\partial_\mu h(w,\mu)](v)$ being continuous in (w,v). With the same constants $\tilde{L}$ and α as in ($\boldsymbol{H1}$), for $w\in\mathbb{R}^k$ and $\chi\in L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^l)$,*

$$\big|\partial^2_{ww}h(w,[\chi])\big| + \mathbb{E}\big[\big|\partial_w\big[\partial_\mu h(w,[\chi])\big](\chi)\big|^2\big]^{1/2} + \mathbb{E}\big[\big|\partial_v\big[\partial_\mu h(w,[\chi])\big](\chi)\big|^2\big]^{1/2} \leqslant \tilde{L},$$

and, for $w,w'\in\mathbb{R}^k$ and $\chi,\chi'\in L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^l)$,

$$\begin{aligned}
&\big|\partial^2_{ww}h(w,[\chi]) - \partial^2_{ww}h(w',[\chi'])(\chi')\big| \\
&\quad + \mathbb{E}\big[\big|\partial_w\big[\partial_\mu h(w,[\chi])\big](\chi) - \partial_w\big[\partial_\mu h(w',[\chi'])\big](\chi')\big|^2\big]^{1/2} \\
&\quad + \mathbb{E}\big[\big|\partial_v\big[\partial_\mu h(w,[\chi])\big](\chi) - \partial_v\big[\partial_\mu h(w',[\chi'])\big](\chi')\big|^2\big]^{1/2} \\
&\leqslant \tilde{L}\big\{|w-w'| + \Phi_\alpha(\chi,\chi')\big\},
\end{aligned}$$

In ($\boldsymbol{H2}$), we include the assumption:

Assumption (($\boldsymbol{H\sigma}$))**.** *The function σ is bounded by $\tilde{L}$.*

Note that ($\boldsymbol{H2}$) contains ($\boldsymbol{H0}$)(ii) (and obviously ($\boldsymbol{H0}$)(i) and ($\boldsymbol{H1}$)).

Remark 2.4. *The specific form of ($\boldsymbol{H1}$) and ($\boldsymbol{H2}$) is dictated by our desire to establish results for arbitrary large horizons. Generally speaking, such results are established by means of a recursive argument, which consists in using the current value $U(t,\cdot,\cdot)$ of the decoupling field at time t as a new boundary condition, or put it differently in letting $U(t,\cdot,\cdot)$ play at time t the role of g at time T when the FBSDEs (2.3) and (2.4) are considered on $[0,t]$ instead of $[0,T]$. A delicate point in this construction is to choose a space of boundary conditions which is stable, namely in which $U(t,\cdot,\cdot)$ remains along the recursion. We remark that we cannot prove that boundary conditions with globally Lipschitz derivatives in the measure argument are stable, even in small time. One of the contribution of the paper is thus to identify a space of terminal conditions which are indeed stable and which permits to apply the recursion method.*

Remark 2.5. *The reader may compare ($\boldsymbol{H0}$), ($\boldsymbol{H1}$) and ($\boldsymbol{H2}$) with the assumptions in [3]. We first point out that, in [3], the first L^2 bound in (2.14) is assumed to hold in L^∞. The example $h([\xi]) = \|\xi\|_2$, for which the derivative has the form $\partial_\mu h([\xi])(v) = v/\|\xi\|_2$,* *shows that asking $\partial_\mu h$ to be in L^∞ is rather restrictive. We also observe that, differently from* [3]*, we do not require the coefficients to admit second-order derivatives of the type $\partial^2_{\mu\mu}$. The reason is that we here establish the chain rule for functions from $\mathcal{P}_2(\mathbb{R}^d)$ to $\mathbb{R}$ that may not have second-order derivatives of the type $\partial^2_{\mu\mu}$, see Theorem 3.5.*

2.5. **Main results: from short to long time horizons and application to control.** Inspired by Assumptions ($\boldsymbol{H0}$)(i), ($\boldsymbol{H1}$) and ($\boldsymbol{H2}$), we let:

Definition 2.6. *Given non-negative real numbers β, a, b, with $a < b$, we denote by $\mathcal{D}_\beta([a,b])$ the space of functions $V : [a,b] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \ni (t,x,\mu) \mapsto V(t,x,\mu) \in \mathbb{R}^m$ for which we can find a constant $C \geqslant 0$ such that*

(i) For any $t \in [a,b]$, the function $V(t,\cdot,\cdot) : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \ni (x,\mu) \mapsto V(t,x,\mu)$ satisfies the same assumption as g in ($\boldsymbol{H0}$)(i), ($\boldsymbol{H1}$) *and* ($\boldsymbol{H2}$)*, but with α replaced by β and with L and $\tilde{L}$ replaced by C (and thus with $w = x \in \mathbb{R}^d$, $v \in \mathbb{R}^d$ and $\chi \in L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^d)$ in the various inequalities where these letters appear);*

(ii) For any $x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, the function $[a,b] \ni t \mapsto V(t,x,\mu)$ is differentiable, the derivative being continuous with respect to (t,x,μ) on the set $[a,b] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$. Moreover, the functions

$$
\begin{aligned}
[a,b] \times \mathbb{R}^d \times L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^d) \ni (t,x,\xi) &\mapsto \partial_x V(t,x,[\xi]) \in \mathbb{R}^d, \\
[a,b] \times \mathbb{R}^d \times L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^d) \ni (t,x,\xi) &\mapsto \partial_\mu V(t,x,[\xi])(\xi) \in L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^d), \\
[a,b] \times \mathbb{R}^d \times L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^d) \ni (t,x,\xi) &\mapsto \partial^2_x V(t,x,[\xi]) \in \mathbb{R}^d, \\
[a,b] \times \mathbb{R}^d \times L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^d) \ni (t,x,\xi) &\mapsto \partial_x[\partial_\mu V(t,x,[\xi])](\xi) \in L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^d) \\
[a,b] \times \mathbb{R}^d \times L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^d) \ni (t,x,\xi) &\mapsto \partial_v[\partial_\mu V(t,x,[\xi])](\xi) \in L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^d)
\end{aligned}
$$

are continuous.

For the reader's convenience, when $[a,b] = [0,T]$, we will simply use the notation $\mathcal{D}_\beta$ for $\mathcal{D}_\beta([0,T])$.

The set $\bigcup_{\beta \geqslant 0} \mathcal{D}_\beta$ is the space we use below for investigating existence and uniqueness of a "classical" solution to (2.12). For short time horizons, our main result takes the following form (see Theorem 4.33):

Theorem 2.7. *Under Assumption* ($\boldsymbol{H2}$)*, there exists a constant $c = c(L)$ (c not depending upon $\tilde{L}$ nor α) such that, for $T \leqslant c$, the function U defined in* (2.5) *is in $\mathcal{D}_{\alpha+1}$ and satisfies the PDE* (2.12)*.*

Uniqueness holds in the class $\bigcup_{\beta \geqslant 0} \mathcal{D}_\beta$:

Theorem 2.8. *Under* ($\boldsymbol{H0}$)(i) *and* ($\boldsymbol{H\sigma}$)*, there exists at most one solution to the PDE* (2.12) *in the class $\bigcup_{\beta \geqslant 0} \mathcal{D}_\beta$ (regardless of how large the time horizon T is).*

The extension to arbitrarily large time horizons will be discussed in Section 5. The principle for extending the result from small to long horizons has been already covered in several papers, including [11, 29]. Basically, the principle is to prove that, following the recursive step, the decoupling field remains in a space of admissible boundary conditions for which the length of the interval of solvability can be bounded from below. Generally speaking, this requires, first, to identify a class of functions on $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$ left invariant by the recursive step and, second, to control the Lipschitz constant of the decoupling field, uniformly along the recursion. In the current framework, the Lipschitz constant means the Lipschitz constant in both the space variable and the measure argument.

As suggested by Theorem 2.7, we are not able to prove that the space $\mathcal{D}_\beta$, for a fixed $\beta \geqslant 0$ is left invariant by the recursion step. In particular, for the case $\beta = 0$, this means that, even in short time, we cannot prove that the decoupling field has Lipschitz derivatives in the direction of the measure when the terminal condition g has Lipschitz derivatives. This difficulty motivates the specific form of the local Lipschitz assumption in $(\boldsymbol{H1})$ and $(\boldsymbol{H2})$. Indeed, Theorem 2.7 shows that the set $\bigcup_{\beta\geqslant 0}\mathcal{D}_\beta$ is preserved by the dynamics of the master PDE under $(\boldsymbol{H2})$ although none of the sets $\mathcal{D}_\beta$ has been shown to be stable. More precisely, we allow the exponent α in $(\boldsymbol{H1})$ and $(\boldsymbol{H2})$ to increase by 1 at each step of the recursion, Theorem 2.7 guaranteeing that the set $\bigcup_{\beta\geqslant 0}\mathcal{D}_\beta$ is indeed stable along the induction.

In Section 5, we give three examples when the Lipschitz constant of the decoupling field can be indeed controlled. First, we consider the forward-backward system deriving from the tailor-made version of the stochastic Pontryagin principle for mean-field games. Then, we establish a Lipschitz estimate of U, in the case when the extended Hamiltonian of the control problem is convex in both the state and control variables and when the Lasry-Lions monotonicity condition that guarantees uniqueness of the equilibrium is satisfied (see [4]). We then interpret U as the gradient in space of the solution of the master equation that arises in the theory of mean-field games and, as a byproduct, we get that, in this framework, the master equation for mean-field games is solvable. Second, we propose another approach to handle the master equation for mean-field games when the extended Hamiltonian is not convex in x. We directly express the solution of the master equation as the decoupling field of a forward-backward system of the McKean-Vlasov type. We then prove the required Lipschitz estimate of U when the cost functionals are bounded in x and are linear-quadratic in α, the volatility is non-degenerate and the Lasry-Lions condition is in force. Third, we consider the forward-backward system deriving from the stochastic Pontryagin principle, when applied to the control of McKean-Vlasov diffusion processes. Then, we establish a similar estimate for the Lipschitz control of U, but under a stronger convexity assumption of the extended Hamiltonian –namely, convexity must hold in the state and control variables and also in the direction of the measure– (in which case there is no need of the Lasry-Lions condition). Again, this permits us to deduce that the master equation associated to the control problem has a global classical solution.

We may summarize with the following statement (again, we refer to Section 5 for a complete account):

Theorem 2.9. *We can find general examples taken from large population stochastic control such that, for a given $T > 0$,* (2.3) *and* (2.4) *have a unique solution and the decoupling field U belongs to $\bigcup_{\beta\geqslant 0}\mathcal{D}_\beta$ and satisfies the PDE* (2.12). *In particular, the corresponding forward equation* (1.1) *has a unique classical solution on* $[0,\infty)$.

2.6. **Frequently used notations.** For two random variables X and X', the relationship $X \sim X'$ means that X and X' have the same distribution. The conditional expectation given $\mathcal{F}_t$ is denoted by $\mathbb{E}_t$. Let $t \in [0,T)$. For a progressively-measurable process $(X_s)_{s\in[t,T]}$ with values in $\mathbb{R}^l$, for some integer $l \leqslant 1$, we let

$$
\begin{aligned}
&\|X\|_{\mathcal{H}^p,t} := \mathbb{E}_t\Big[\Big(\int_t^T |X_s|^2 \mathrm{d}s\Big)^{p/2}\Big]^{1/p}, \quad \|X\|_{\mathcal{S}^p,t} := \mathbb{E}_t\Big[\sup_{s\in[t,T]}|X_s|^p\Big]^{1/p}, \\
&\|X\|_{\mathcal{H}^p} := \mathbb{E}\Big[\Big(\int_t^T |X_s|^2 \mathrm{d}s\Big)^{p/2}\Big]^{1/p}, \quad \|X\|_{\mathcal{S}^p} := \mathbb{E}\Big[\sup_{s\in[t,T]}|X_s|^p\Big]^{1/p}.
\end{aligned}
\tag{2.15}
$$

In particular, we denote by $\mathcal{S}^p([t,T];\mathbb{R}^l)$ the space of continuous and adapted random processes from $[t,T]$ to $\mathbb{R}^l$ with a finite norm $\|\cdot\|_{\mathcal{S}^p}$ and by $\mathcal{H}^p([t,T];\mathbb{R}^l)$ the space of progressively-measurable processes from $[t,T]$ to $\mathbb{R}^l$ with a finite norm $\|\cdot\|_{\mathcal{H}^p}$.

In the sequel, the generic letter C is used for denoting constants the value of which may often vary from line to line. Constants whose precise values have a fundamental role in the analysis will be denoted by letters distinct from C.

3. Chain rule – application to the proof of Theorem 2.8

In this section, we discuss the chain rule used in (2.9) and apply it to prove Theorem 2.8. Namely, we provide a chain rule for $(U(\mu_t))_{t\geqslant 0}$ where U is an $\mathbb{R}$-valued smooth functional defined on the space $\mathcal{P}_2(\mathbb{R}^d)$ and $(\mu_t)_{t\geqslant 0}$ is the flow of marginal measures of an $\mathbb{R}^d$-valued Itô process $(X_t)_{t\geqslant 0}$.

There are two strategies to expand $(U(\mu_t))_{t\geqslant 0}$. The first one consists, for a given $t>0$, in dividing the interval $[0,t]$ into sub-intervals of length $h=t/N$, for some integer $N\geqslant 1$, and then in splitting the difference $U(\mu_t)-U(\mu_0)$ accordingly:

$$U(\mu_t)-U(\mu_0)=\sum_{i=0}^{N-1}\big[U(\mu_{ih})-U(\mu_{(i-1)h})\big].$$

The differences $U(\mu_{ih})-U(\mu_{(i-1)h})$ are expanded by applying Taylor's formula at order 2. Since the order of the remaining terms in the Taylor expansion are expected to be smaller than the step size h, we can derive the chain rule by letting h tend to 0. This strategy fits the original proof of Itô's differential calculus and is presented in details in [3, Section 6] and in [8, Section 6].

An alternative strategy consists in approximating the dynamics differently. Instead of discretizing in time as in the previous strategy, it is conceivable to reduce the space dimension by approximating the flow $(\mu_t)_{t\geqslant 0}$ with the flow of empirical measures

$$\Big(\bar{\mu}_t^N=\frac{1}{N}\sum_{\ell=1}^N\delta_{X_t^\ell}\Big)_{t\geqslant 0},\quad N\geqslant 1,$$

where $(X_t^1)_{t\geqslant 0},\dots,(X_t^N)_{t\geqslant 0}$ stand for N independent copies of $(X_t)_{t\geqslant 0}$. Letting

$$\forall x^1,\dots,x^N\in\mathbb{R}^d,\quad u^N(x^1,\dots,x^N)=U\Big(\frac{1}{N}\sum_{\ell=1}^N\delta_{x^\ell}\Big),\tag{3.1}$$

we expand $(u^N(X_t^1,\dots,X_t^N))_{t\geqslant 0}$ by standard Itô's formula. Letting N tend to the infinity, we then expect to recover the same chain rule as the one obtained by the first method. Here u^N is interpreted as a finite dimensional projection of U.

The first strategy mimics the proof of the standard chain rule. The second one gives an insight into the significance of the differential calculus on the space of probability measures introduced by Lions in [4]. Both strategies require some smoothness conditions on U: Clearly, U must be twice differentiable in some suitable sense. From this viewpoint, the strategy by particle approximation is advantageous: Taking benefit of the finite dimensional framework, by using a standard mollification argument it works under weaker smoothness conditions required on the coefficients. In particular, differently from [3, 8], we do not require the existence of $\partial^2_{\mu\mu}U$ to prove the chain rule, see Theorem 3.5.

3.1. **Full $\mathcal{C}^2$ regularity.** We first remind the reader of the notion of lifted version of U. On $L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^d)$ (the σ-field $\mathcal{A}$ being prescribed), we let

$$\mathcal{U}(X) = U\big([X]\big), \quad X \in L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^d).$$

Instead of $(\Omega, \mathcal{A}, \mathbb{P})$, we could use $(\hat{\Omega}, \hat{\mathcal{A}}, \hat{\mathbb{P}})$, but since no confusion is possible with the "physical" random variables that appear in (2.3) and (2.4), we continue to work on $(\Omega, \mathcal{A}, \mathbb{P})$.

Following Lions' approach (see [4, Section 6]), the mapping U is said to be differentiable on the Wasserstein space if the lift $\mathcal{U}$ is differentiable in the sense of Fréchet on $L^2(\Omega, \mathcal{A}, \mathbb{P})$. By Riesz' theorem, the Fréchet derivative $D\mathcal{U}(X)$, seen as an element of $L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^d)$, can be represented as

$$D\mathcal{U}(X) = \partial_\mu U\big([X]\big)(X),$$

where $\partial_\mu U([X]) : \mathbb{R}^d \ni v \mapsto \partial_\mu U([X])(v) \in \mathbb{R}^d$ is in $L^2(\mathbb{R}^d, \mu; \mathbb{R}^d)$, see [4, Section 6]. Recall that, as a gradient, $\partial_\mu U([X])(v)$ will be seen as a row vector.

A natural question to investigate is the joint regularity of the function $\partial_\mu U$ with respect to the variables μ and v. This requires a preliminary analysis for choosing a 'canonical version' of the mapping $\partial_\mu U(\mu) : \mathbb{R}^d \ni v \mapsto \partial_\mu U(\mu)(v) \in \mathbb{R}^d$, which is *a priori* defined just as an element of $L^2(\mathbb{R}^d, \mu; \mathbb{R}^d)$. In this perspective, a reasonable strategy consists in choosing a continuous version of the derivative if such a version exists. For instance, whenever $D\mathcal{U}$ is Lipschitz continuous, we know from [5] that, for any $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, there exists a Lipschitz continuous version of the function $\mathbb{R}^d \ni v \mapsto \partial_\mu U(\mu)(v)$, with a Lipschitz constant independent of μ. This result is made precise in Proposition 3.8 below.

The choice of a continuous version is especially meaningful when the support of μ is the entire $\mathbb{R}^d$, in which case the continuous version is uniquely defined. Whenever the support of μ is strictly included in $\mathbb{R}^d$, some precaution is however needed, as the continuous version may be arbitrarily defined outside the support of μ. To circumvent this difficulty, one might be tempted to look for a version of $\partial_\mu U(\mu) : \mathbb{R}^d \ni v \mapsto \partial_\mu U(\mu)(v) \in \mathbb{R}^d$, for each $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, such that the global mapping

$$\mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \ni (\mu, v) \mapsto \partial_\mu U(\mu)(v) \in \mathbb{R}^d \tag{3.2}$$

is continuous. Noticing, by means of a convolution argument, that the set $\{\mu' \in \mathcal{P}_2(\mathbb{R}^d) : \mathrm{supp}(\mu') = \mathbb{R}^d\}$ is dense in $\mathcal{P}_2(\mathbb{R}^d)$, this would indeed permit to uniquely determine the value of $\partial_\mu U(\mu)(v)$ for v outside the support of μ (when it is strictly included in $\mathbb{R}^d$).

Unfortunately, in the practical cases handled below, the best we can do is to find a version of $\partial_\mu U(\mu) : \mathbb{R}^d \ni v \mapsto \partial_\mu U(\mu)(v) \in \mathbb{R}^d$, for each $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, such that the global mapping (3.2) is continuous at the points (μ, v) such that $v \in \mathrm{supp}(\mu)$.

The fact that $\partial_\mu U(\mu) : \mathbb{R}^d \ni v \mapsto \partial_\mu U(\mu)(v) \in \mathbb{R}^d$ is not uniquely determined outside the support of μ is not a problem for investigating the differentiability of $\partial_\mu U(\mu)$ in v. It is an issue only for investigating the differentiability in μ. We thus say that the chosen version of $\mathbb{R}^d \ni v \mapsto \partial_\mu U(\mu)(v)$ is differentiable in v, for a given $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, if the mapping $\mathbb{R}^d \ni v \mapsto \partial_\mu U(\mu)(v)$ is differentiable in the standard sense, the derivative being denoted by $\mathbb{R}^d \ni v \mapsto \partial_v[\partial_\mu U(\mu)](v)$ (which belongs to $\mathbb{R}^{d\times d}$). Note that *there is no Schwarz' theorem for exchanging the derivatives as* U does not depend on v.

Now, if we can find a jointly continuous version of the global mapping (3.2), $\partial_\mu U$ is said to be differentiable in μ, at $v \in \mathbb{R}^d$, if the lifted mapping $L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^d) \ni X \mapsto \partial_\mu U([X])(v) \in \mathbb{R}^d$ is differentiable in the Fréchet sense. Then, according to the previous discussion, the derivative can be interpreted as a mapping $\mathbb{R}^d \ni v' \mapsto \partial_\mu[\partial_\mu U([X])(v)](v') \in \mathbb{R}^{d\times d}$ in $L^2(\mathbb{R}^d,\mu;\mathbb{R}^{d\times d})$, which we will denote by $\mathbb{R}^d \ni v' \mapsto \partial^2_\mu U([X])(v,v')$. In a first step, we will prove Itô's formula when this additional assumption on the smoothness of $\partial_\mu U$ in μ is in force. More precisely, we will say that U is *fully* $\mathcal{C}^2$ if the global mapping $\partial_\mu U$ in (3.2) is continuous and the mappings

$$\begin{aligned}
&\mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \ni (\mu,v) \mapsto \partial_\mu U(\mu)(v),\\
&\mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \ni (\mu,v) \mapsto \partial_v[\partial_\mu U(\mu)](v),\\
&\mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \times \mathbb{R}^d \ni (\mu,v,v') \mapsto \partial^2_\mu U(\mu)(v,v'),
\end{aligned}$$

are continuous for the product topologies, the space $\mathcal{P}_2(\mathbb{R}^d)$ being endowed with the 2-Wasserstein distance.

Under suitable assumptions, it can be checked that *full* $\mathcal{C}^2$ regularity implies twice Fréchet differentiability of the lifting $\mathcal{U}$. As we won't make use of such a result, we refrain from providing its proof in the paper. We will be much more interested in a possible converse: Can we expect to recover that U is $\mathcal{C}^2$ regular (with respect to v and μ), given the fact that $\mathcal{U}$ has some Fréchet or Gâteaux differentiability properties at the second-order? We answer this (more challenging) question in Subsection 3.4 below.

To clarify the significance of the notion of *full* $\mathcal{C}^2$ regularity, we now make the connection between the derivatives of u^N and those of U:

Proposition 3.1. *Assume that U is $\mathcal{C}^1$. Then, for any $N \geqslant 1$, the function u^N is differentiable on $\mathbb{R}^N$ and, for all $x^1,\dots,x^N \in \mathbb{R}^d$, the mapping*

$$\mathbb{R}^d \ni x^i \quad\mapsto\quad \partial_{x^i} u^N(x^1,\dots,x^N) \in \mathbb{R}^d$$

reads

$$\partial_{x^i} u^N(x^1,\dots,x^N) = \frac{1}{N}\partial_\mu U\biggl(\frac{1}{N}\sum_{\ell=1}^N \delta_{x^\ell}\biggr)(x^i).$$

If, moreover, U is fully $\mathcal{C}^2$, then, for any $N \geqslant 1$, the function u^N is $\mathcal{C}^2$ on $\mathbb{R}^N$ and, for all $x^1,\dots,x^N \in \mathbb{R}^d$, the mapping

$$\mathbb{R}^d \times \mathbb{R}^d \ni (x^i,x^j) \quad\mapsto\quad \partial^2_{x^i x^j} u^N(x^1,\dots,x^N) \in \mathbb{R}^{d\times d}$$

satisfies

$$\partial^2_{x^i x^j} u^N(x^1,\dots,x^N) = \frac{1}{N}\partial_v\biggl[\partial_\mu U\biggl(\frac{1}{N}\sum_{\ell=1}^N \delta_{x^\ell}\biggr)\biggr](x^i)\delta_{i,j} + \frac{1}{N^2}\partial^2_\mu U\biggl(\frac{1}{N}\sum_{\ell=1}^N \delta_{x^\ell}\biggr)(x^i,x^j).$$

Proof. The formula for the first order derivative has been already proved in [5]. It remains to deduce the formula for the second order derivative. When $i \neq j$, it is a direct consequence of the first order formula. When $i = j$, the computations require some precaution as differentiability is simultaneously investigated in the directions of μ and v in $\partial_\mu U(\mu)(v)$, but, by the joint continuity of the second-order derivatives $\partial^2_\mu U(\mu)(v)$ and $\partial_v\partial_\mu U(\mu)(v,v')$, they are easily handled. □

Remark 3.2. *Assume that U is fully $\mathcal{C}^2$. Then, for any $X \in L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d)$ and $Y,Z \in L^\infty(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d)$, the mapping*

$$\vartheta : \mathbb{R}^2 \ni (h,k) \mapsto U\bigl(\bigl[X + hY + kZ\bigr]\bigr) \in \mathbb{R}$$

is of class $\mathcal{C}^2$ on $\mathbb{R}^2$, with

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}h}\Big[\frac{\mathrm{d}\vartheta}{\mathrm{d}k}\Big](h,k) &= \frac{\mathrm{d}}{\mathrm{d}h}\mathbb{E}\big[\partial_\mu U\big([X+hY+kZ]\big)(X+hY+kZ)Z\big] \\ &= \mathbb{E}\big[\mathrm{Tr}\big(\partial_v\partial_\mu U\big([X+hY+kZ]\big)(X+hY+kZ)Z\otimes Y\big)\big] \\ &\quad + \mathbb{E}\hat{\mathbb{E}}\big[\mathrm{Tr}\big(\partial^2_\mu U\big([X+hY+kZ]\big)(X+hY+kZ,\hat{X}+h\hat{Y}+k\hat{Z})Z\otimes\hat{Y}\big)\big],\end{aligned}$$

the triplet $(\hat{X},\hat{Y},\hat{Z})$ denoting a copy of (X,Y,Z) on $(\hat{\Omega},\hat{\mathcal{A}},\hat{\mathbb{P}})$ and the tensorial product operating on $\mathbb{R}^d$.

By Schwarz' Theorem, the roles of Z and Y can be exchanged, which means (choosing $h=k=0$) that

$$\begin{aligned}&\mathbb{E}\big[\mathrm{Tr}\big(\partial_v\partial_\mu U\big([X]\big)(X)Z\otimes Y\big)\big] + \mathbb{E}\hat{\mathbb{E}}\big[\mathrm{Tr}\big(\partial^2_\mu U\big([X]\big)(X,\hat{X})Z\otimes\hat{Y}\big)\big] \\ &\quad = \mathbb{E}\big[\mathrm{Tr}\big(\partial_v\partial_\mu U\big([X]\big)(X)Y\otimes Z\big)\big] + \mathbb{E}\hat{\mathbb{E}}\big[\mathrm{Tr}\big(\partial^2_\mu U\big([X]\big)(X,\hat{X})Y\otimes\hat{Z}\big)\big].\end{aligned} \tag{3.3}$$

Choosing Y of the form $\varepsilon\varphi(X)$ and Z of the form $\varepsilon\psi(X)$, with $\mathbb{P}(\varepsilon=1)=\mathbb{P}(\varepsilon=-1)=1/2$ and ε independent of X, and considering two bounded Borel measurable functions φ and $\psi:\mathbb{R}^d\to\mathbb{R}^d$, we deduce that

$$\mathbb{E}\big[\mathrm{Tr}\big(\partial_v\partial_\mu U\big([X]\big)(X)\varphi(X)\otimes\psi(X)\big)\big] = \mathbb{E}\big[\mathrm{Tr}\big(\partial_v\partial_\mu U\big([X]\big)(X)\psi(X)\otimes\varphi(X)\big)\big], \tag{3.4}$$

from which we deduce that $\partial_v\partial_\mu U([X])(X)$ takes values in the set of symmetric matrices of size d. By continuity, it means that $\partial_v\partial_\mu U(\mu)(v)$ is a symmetric matrix for any $v\in\mathbb{R}^d$ when μ has the entire $\mathbb{R}^d$ as support. By continuity in μ, we deduce that $\partial_v\partial_\mu U(\mu)(v)$ is a symmetric matrix for any $v\in\mathbb{R}^d$ and any $\mu\in\mathcal{P}_2(\mathbb{R}^d)$.

Now, choosing Y and Z of the form $\varphi(X)$ and $\psi(X)$ respectively and plugging (3.4) into (3.3), we deduce that

$$\begin{aligned}\mathbb{E}\hat{\mathbb{E}}\big[\mathrm{Tr}\big(\partial^2_\mu U\big([X]\big)(X,\hat{X})\varphi(X)\otimes\psi(\hat{X})\big)\big] &= \mathbb{E}\hat{\mathbb{E}}\big[\mathrm{Tr}\big(\partial^2_\mu U\big([X]\big)(X,\hat{X})\psi(X)\otimes\varphi(\hat{X})\big)\big] \\ &= \mathbb{E}\hat{\mathbb{E}}\big[\mathrm{Tr}\big(\partial^2_\mu U\big([X]\big)(\hat{X},X)\psi(\hat{X})\otimes\varphi(X)\big)\big] \\ &= \mathbb{E}\hat{\mathbb{E}}\big[\mathrm{Tr}\big(\big[\partial^2_\mu U\big([X]\big)(\hat{X},X)\big]^\dagger\varphi(X)\otimes\psi(\hat{X})\big)\big],\end{aligned}$$

from which we deduce that $\partial^2_\mu U([X])(X,\hat{X}) = (\partial^2_\mu U([X])(\hat{X},X))^\dagger$. By the same argument as above, we finally deduce that $\partial^2_\mu U(\mu)(v,v') = (\partial^2_\mu U(\mu)(v',v))^\dagger$, for any $v,v'\in\mathbb{R}^d$ and any $\mu\in\mathcal{P}_2(\mathbb{R}^d)$.

3.2. **The chain rule for U fully $\mathcal{C}^2$.** We consider an $\mathbb{R}^d$-valued Itô process

$$\mathrm{d}X_t = b_t\mathrm{d}t + \sigma_t\mathrm{d}W_t, \quad X_0\in L^2(\Omega,\mathcal{A},\mathbb{P}),$$

where $(b_t)_{t\geqslant 0}$ and $(\sigma_t)_{t\geqslant 0}$ are progressively-measurable processes with values in $\mathbb{R}^d$ and, respectively, $\mathbb{R}^{d\times d}$ respectively with respect to the (augmented) filtration generated by W, such that

$$\forall T>0, \quad \mathbb{E}\bigg[\int_0^T\big(|b_t|^2+|\sigma_t|^4\big)\mathrm{d}t\bigg] < +\infty. \tag{3.5}$$

The following is the main result of this section

Theorem 3.3. *Assume that U is fully $\mathcal{C}^2$ and that, for any compact subset $\mathcal{K}\subset\mathcal{P}_2(\mathbb{R}^d)$,*

$$\sup_{\mu\in\mathcal{K}}\bigg[\int_{\mathbb{R}^d}\big|\partial_\mu U(\mu)(v)\big|^2\mathrm{d}\mu(v) + \int_{\mathbb{R}^d}\Big|\partial_v\big[\partial_\mu U(\mu)\big](v)\Big|^2\mathrm{d}\mu(v)\bigg] < +\infty, \tag{3.6}$$

Then, letting $\mu_t := [X_t]$ and $a_t := \sigma_t(\sigma_t)^\dagger$, for any $t \geqslant 0$,

$$U(\mu_t) = U(\mu_0) + \int_0^t \mathbb{E}\big[\partial_\mu U(\mu_s)(X_s) b_s\big] \mathrm{d}s + \frac{1}{2}\int_0^t \mathbb{E}\Big[\mathrm{Tr}\Big(\partial_v\big(\partial_\mu U(\mu_s)\big)(X_s) a_s\Big)\Big]\mathrm{d}s. \quad (3.7)$$

The proof relies on a mollification argument captured in the proof of the following result:

Proposition 3.4. *Assume that the chain rule* (3.7) *holds for any function U that is fully $\mathcal{C}^2$ with first and second order derivatives that are bounded and uniformly continuous (with respect to both the space and measure variables). Then Theorem 3.3 holds, in other words, the chain rule* (3.7) *is valid for any fully $\mathcal{C}^2$ function U satisfying* (3.6).

Proof. [Proof of Proposition 3.4.] Let U be a *fully* $\mathcal{C}^2$ function that satisfies (3.6). We 'mollify' U in such a way that its mollification is bounded with bounded first and second order derivatives. Let $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ be a smooth function with compact support and, for arbitrary $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ define

$$\forall \mu \in \mathcal{P}_2(\mathbb{R}^d), \quad \big(U \star \varphi\big)(\mu) := U\big(\varphi \sharp \mu\big),$$

where $\varphi\sharp\mu$ denotes the image of μ by φ. The lifted version of $U \star \varphi$ is nothing but $\mathcal{U} \circ \varphi$, where (with an abuse of notation) φ is canonically lifted as $\varphi : L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^d) \ni X \mapsto \varphi(X)$. It is then quite standard to check that:

$$\begin{aligned}
&\partial_\mu\big[U \star \varphi\big](\mu)(v) = \Big(\sum_{k=1}^d \Big[\partial_\mu U\big(\varphi\sharp\mu\big)\big(\varphi(v)\big)\Big]_k \frac{\partial \varphi_k}{\partial x_i}(v)\Big)_{i=1,\dots,d},\\
&\partial_\mu^2\big[U \star \varphi\big](\mu)(v,v') = \Big(\sum_{k,\ell=1}^d \Big[\partial_\mu^2 U\big(\varphi\sharp\mu\big)\big(\varphi(v),\varphi(v')\big)\Big]_{k,\ell} \frac{\partial \varphi_k}{\partial x_i}(v)\frac{\partial \varphi_\ell}{\partial x_j}(v')\Big)_{i,j=1,\dots,d},\\
&\partial_v\Big[\partial_\mu\big[U \star \varphi\big](\mu)(v)\Big] = \Big(\sum_{k=1}^d \Big[\partial_\mu U\big(\varphi\sharp\mu\big)\big(\varphi(v)\big)\Big]_k \frac{\partial^2 \varphi_k}{\partial x_i \partial x_j}(v)\\
&\qquad\qquad + \sum_{k,\ell=1}^d \Big[\partial_v\big[\partial_\mu U\big(\varphi\sharp\mu\big)\big]\big(\varphi(v)\big)\Big]_{k,\ell} \frac{\partial \varphi_k}{\partial x_i}(v)\frac{\partial \varphi_\ell}{\partial x_j}(v)\Big)_{i,j=1,\dots,d}.
\end{aligned} \quad (3.8)$$

Recall from Remark 3.2 that the second-order derivatives that appear in (3.8) have some symmetric structure. Now, since φ is compactly supported, the mapping $\mathcal{P}_2(\mathbb{R}^d) \ni \mu \mapsto \varphi\sharp\mu$ has a relatively compact range[9] (in $\mathcal{P}_2(\mathbb{R}^d)$). By the continuity of U and its derivatives, we deduce that $U \star \varphi$ and its first and second order derivatives are bounded and uniformly continuous on the whole space.

Assume now that the chain rule has been proved for any bounded and uniformly continuous U with bounded and uniformly continuous derivatives of order 1 and 2. Then, for some U just satisfying the assumption of Theorem 3.3, we can apply the chain rule to $U \star \varphi$, for any φ as above. In particular, we can apply the chain rule to $U \star \varphi_n$ for any $n \geqslant 1$, where $(\varphi_n)_{n \geqslant 1}$ is a sequence of compactly supported smooth functions such that $(\varphi_n, \partial_x \varphi_n, \partial_{xx}^2[\varphi_n]_1, \dots, \partial_{xx}^2[\varphi_n]_d)(v) \to (v, I_d, 0, \dots, 0)$ uniformly on compact sets as $n \to \infty$, I_d denoting the identity matrix of size d. In order to pass to the limit in the chain rule (3.7), the only thing is to verify some almost sure (or pointwise) convergence in the underlying expectations and to check the corresponding uniform integrability argument.

[9] Tightness is obvious. By boundedness of φ, any subsequence converging in the weak sense is also convergent with respect to W_2.

Without any loss of generality, we can assume that there exists a constant C such that

$$|\varphi_n(v)| \leqslant C|v|, \quad |\partial_x \varphi_n(v)| \leqslant C \quad \text{and} \quad |\partial^2_{xx}[\varphi_n(v)]_k| \leqslant C\ ,\ 1 \leqslant k \leqslant d\,, \tag{3.9}$$

for any $n \geqslant 1$ and $v \in \mathbb{R}^d$ and that $\varphi_n(v) = v$ for any $n \geqslant 1$ and any v with $|v| \leqslant n$. Then, for any $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ and any random variable X with μ as distribution, it holds

$$W_2^2\big(\varphi_n \sharp \mu, \mu\big) \leqslant \mathbb{E}\big[|\varphi_n(X) - X|^2 \mathbf{1}_{\{|X| \geqslant n\}}\big] \leqslant C\mathbb{E}\big[|X|^2 \mathbf{1}_{\{|X| \geqslant n\}}\big],$$

which tends to 0 as $n \to \infty$. By continuity of U and its partial derivatives and by (3.8), it is easy to deduce that, a.s.,

$$\begin{aligned} &U \star \varphi_n(\mu) \to U(\mu), \quad \partial_\mu\big[U \star \varphi_n\big](\mu)(X) \to \partial_\mu U(\mu)(X), \\ &\partial_v\big[\partial_\mu\big(U \star \varphi_n\big)\big](\mu)(X) \to \partial_v\big[\partial_\mu U(\mu)\big](X). \end{aligned} \tag{3.10}$$

Moreover, we notice that

$$\sup_{n \geqslant 1} \mathbb{E}\Big[\big|\partial_\mu\big[U \star \varphi_n\big](\mu)(X)\big|^2 + \big|\partial_v\big[\partial_\mu\big(U \star \varphi_n\big)(\mu)\big](X)\big|^2\Big] < \infty. \tag{3.11}$$

Indeed, by (3.8) and (3.9), it is enough to check that

$$\sup_{n \geqslant 1}\bigg[\int_{\mathbb{R}^d}\Big|\partial_\mu U\big(\varphi_n \sharp \mu\big)(v)\Big|^2 \mathrm{d}\big(\varphi_n \sharp \mu\big)(v) + \int_{\mathbb{R}^d}\Big|\partial_v\big[\partial_\mu U\big(\varphi_n \sharp \mu\big)\big](v)\Big|^2 \mathrm{d}\big(\varphi_n \sharp \mu\big)(v)\bigg] < \infty,$$

which follows directly from (3.6), noticing that the sequence $(\varphi_n \sharp \mu)_{n \geqslant 1}$ lives in a compact subset of $\mathcal{P}_2(\mathbb{R}^d)$ as it is convergent.

By (3.10) and (3.11) and by a standard uniform integrability argument, we deduce that, for any $t \geqslant 0$ and any $s \in [0,t]$ such that $\mathbb{E}[|b_s|^2 + |\sigma_s|^4] < \infty$,

$$\begin{aligned} &\lim_{n \to +\infty} \mathbb{E}\big[\partial_\mu(U \star \varphi_n)([X])(X) b_s\big] = \mathbb{E}\big[\partial_\mu U([X])(X) b_s\big], \\ &\lim_{n \to +\infty} \mathbb{E}\big\{\mathrm{Tr}\big[\partial_v\big(\partial_\mu(U \star \varphi_n)([X])\big)(X) a_s\big]\big\} = \mathbb{E}\big\{\mathrm{Tr}\big[\partial_v\big(\partial_\mu U([X])\big)(X) a_s\big]\big\}. \end{aligned}$$

Recall that the above is true for any $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ and any $X \in L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^d)$ with μ as distribution. In particular, we can choose $\mu = \mu_s$ and $X = X_s$ in the above limits. As the bound $\mathbb{E}[|b_s|^2 + |\sigma_s|^4] < \infty$ is satisfied for almost every $s \in [0,t]$, this permits to pass to the limit inside the integrals appearing in the chain rule applied to each of the $(U \star \varphi_n)_{n \geqslant 1}$. In order to pass to the limit in the chain rule itself, we must exchange the pathwise limit that holds for almost every $s \in [0,t]$ and the integral with respect to the time variable s. The argument is the same as in (3.11). Indeed, since the flow of measures $([X_s])_{0 \leqslant s \leqslant t}$ is continuous for the 2-Wasserstein distance, the family of measures $(([\varphi_n(X_s)])_{0 \leqslant s \leqslant t})_{n \geqslant 1}$ is relatively compact and thus

$$\sup_{n \geqslant 1} \sup_{s \in [0,t]} \mathbb{E}\Big[\big|\partial_\mu U\big([\varphi_n(X_s)]\big)\big(\varphi_n(X_s)\big)\big|^2 + \big|\partial_v\big[\partial_\mu U\big([\varphi_n(X_s)]\big)\big]\big(\varphi_n(X_s)\big)\big|^2\Big] < \infty,$$

which is enough to prove that the functions

$$\Big([0,t] \ni s \mapsto \mathbb{E}\big[\partial_\mu(U \star \varphi_n)([X_s])(X_s) b_s\big] + \mathbb{E}\big\{\mathrm{Tr}\big[\partial_v\big(\partial_\mu(U \star \varphi_n)([X_s])\big)(X_s) a_s\big]\big\}\Big)_{n \geqslant 1}$$

are uniformly integrable on $[0,t]$. □

We now turn to the proof of Theorem 3.3. We give just a sketch of the proof, as a refined version of Theorem 3.3 is given later, see Theorem 3.5 in the next subsection.

Proof. [Proof of Theorem 3.3.] By Proposition 3.4, we can replace U by $U \star \varphi$, for some compactly supported smooth function φ. Equivalently, we can replace $(X_t)_{t \geqslant 0}$

by $(\varphi(X_t))_{t\geqslant 0}$. In other words, we can assume that U and its first and second order derivatives are bounded and uniformly continuous and that $(X_t)_{t\geqslant 0}$ is a bounded Itô process.

Finally by the same argument as in the proof of Proposition 3.4, we can also assume that $(b_t)_{t\geqslant 0}$ and $(\sigma_t)_{t\geqslant 0}$ are bounded. Indeed, it suffices to prove the chain rule when $(X_t)_{t\geqslant 0}$ is driven by truncated processes and then to pass to the limit along a sequence of truncations that converges to $(X_t)_{t\geqslant 0}$.

Let $((X_t^\ell)_{t\geqslant 0})_{\ell\geqslant 1}$ a sequence of i.i.d. copies of $(X_t)_{t\geqslant 0}$. That is, for any $\ell \geqslant 1$,

$$\mathrm{d}X_t^\ell = b_t^\ell \mathrm{d}t + \sigma_t^\ell \mathrm{d}W_t^\ell, \quad t \geqslant 0,$$

where $((b_t^\ell, \sigma_t^\ell, W_t^\ell)_{t\geqslant 0}, X_0^\ell)_{\ell\geqslant 1}$ are i.i.d copies of $((b_t, \sigma_t, W_t)_{t\geqslant 0}, X_0)$.

Recalling the definition of the flow of marginal empirical measures:

$$\bar{\mu}_t^N = \frac{1}{N}\sum_{\ell=1}^N \delta_{X_t^\ell},$$

the standard Itô's formula yields together with Proposition 3.1, $\mathbb{P}$-a.s., for any $t \geqslant 0$

$$\begin{aligned}
&u^N\big(X_t^1, \ldots, X_t^N\big) = u^N\big(X_0^1, \ldots, X_0^N\big) \\
&+ \frac{1}{N}\sum_{\ell=1}^N \int_0^t \partial_\mu U\big(\bar{\mu}_s^N\big)(X_s^\ell) b_s^\ell \mathrm{d}s + \frac{1}{N}\sum_{\ell=1}^N \int_0^t \partial_\mu U\big(\bar{\mu}_s^N\big)(X_s^\ell)\sigma_s^\ell \mathrm{d}W_s^\ell \qquad (3.12)\\
&+ \frac{1}{2N}\sum_{\ell=1}^N \int_0^t \mathrm{Tr}\big\{\partial_v\big[\partial_\mu U\big(\bar{\mu}_s^N\big)\big](X_s^\ell) a_s^\ell\big\}\mathrm{d}s + \frac{1}{2N^2}\sum_{\ell=1}^N \int_0^t \mathrm{Tr}\big\{\partial_\mu^2 U\big(\bar{\mu}_s^N\big)(X_s^\ell, X_s^\ell) a_s^\ell\big\}\mathrm{d}s,
\end{aligned}$$

with $a_s^\ell := \sigma_s^\ell(\sigma_s^\ell)^\dagger$.

We take expectation on both sides of the previous equality and obtain (the stochastic integral has zero expectation due to the boundedness of the coefficients), recalling (3.1),

$$\begin{aligned}
\mathbb{E}\big[U\big(\bar{\mu}_t^N\big)\big] &= \mathbb{E}\big[U\big(\bar{\mu}_0^N\big)\big] + \frac{1}{N}\sum_{\ell=1}^N \mathbb{E}\bigg[\int_0^t \partial_\mu U\big(\bar{\mu}_s^N\big)(X_s^\ell) b_s^\ell \mathrm{d}s\bigg] \\
&+ \frac{1}{2N}\sum_{\ell=1}^N \mathbb{E}\bigg[\int_0^t \mathrm{Tr}\big\{\partial_v\big[\partial_\mu U\big(\bar{\mu}_s^N\big)\big](X_s^\ell) a_s^\ell\big\}\mathrm{d}s\bigg] \\
&+ \frac{1}{2N^2}\sum_{\ell=1}^N \mathbb{E}\bigg[\int_0^t \mathrm{Tr}\big\{\partial_\mu^2 U\big(\bar{\mu}_s^N\big)(X_s^\ell, X_s^\ell) a_s^\ell\big\}\mathrm{d}s\bigg].
\end{aligned}$$

All the above expectations are finite, due to the boundedness of the coefficients. Using the fact that the processes $((a_s^\ell, b_s^\ell, X_s^\ell)_{s\in[0,t]})_{\ell\in\{1,\ldots,N\}}$ are i.i.d., we deduce that

$$\begin{aligned}
\mathbb{E}\big[U\big(\bar{\mu}_t^N\big)\big] &= \mathbb{E}\big[U\big(\bar{\mu}_0^N\big)\big] + \mathbb{E}\bigg[\int_0^t \partial_\mu U\big(\bar{\mu}_s^N\big)(X_s^1) b_s^1 \mathrm{d}s\bigg] && (3.13)\\
&+ \frac{1}{2}\mathbb{E}\bigg[\int_0^t \mathrm{Tr}\big\{\partial_v\big[\partial_\mu U\big(\bar{\mu}_s^N\big)\big](X_s^1) a_s^1\big\}\mathrm{d}s\bigg] && (3.14)\\
&+ \frac{1}{2N}\mathbb{E}\bigg[\int_0^t \mathrm{Tr}\big\{\partial_\mu^2 U\big(\bar{\mu}_s^N\big)(X_s^1, X_s^1) a_s^1\big\}\mathrm{d}s\bigg], && (3.15)
\end{aligned}$$

In particular, because of the additional $1/N$, the term in (3.15) converges to 0. Moreover, the coefficients $(a_s)_{s\in[0,t]}$ and $(b_s)_{s\in[0,t]}$ being bounded, we know from [32, Theorem

10.2.7]:

$$\lim_{N\to+\infty}\mathbb{E}\big[\sup_{0\leqslant s\leqslant t}W_2^2(\bar{\mu}_s^N,\mu_s)\big]=0. \tag{3.16}$$

This implies together with the uniform continuity of U with respect to the distance W_2, that $\mathbb{E}\big[U\big(\bar{\mu}_t^N\big)\big]$ (resp. $\mathbb{E}\big[U\big(\bar{\mu}_0^N\big)\big]$) converges to $U(\mu_t)$ (resp. $U(\mu_0)$). Combining the uniform continuity of $\partial_\mu U$ on $\mathcal{P}_2(\mathbb{R}^d)\times\mathbb{R}^d$ with (3.16), the second term in the right-hand side of (3.13) converges. Similar arguments lead to the convergence of the term in (3.14). $\square$

The notion of differentiation as defined by Lions plays an essential role in the chain rule formula. It is the right differentiation procedure to give the natural extension from the chain rule for empirical distribution processes to the chain rule for measure valued processes.

3.3. **The chain rule for U partially $\mathcal{C}^2$.** We observe that, in the formula for chain rule (3.7), the second order derivative $\partial_\mu^2 U$ does not appear. It is thus a quite natural question to study its validity when $\partial_\mu^2 U$ does not exist. This is what we refer to as 'partial $\mathcal{C}^2$ regularity'. More precisely, we will say that U is partially $\mathcal{C}^2$ (in v) if the lift $\mathcal{U}$ is Fréchet differentiable and, for any $\mu\in\mathcal{P}_2(\mathbb{R}^d)$, we can find a continuous version of the mapping $\mathbb{R}^d\ni v\mapsto\partial_\mu U(\mu)(v)$ such that:

• the mapping $\mathcal{P}_2(\mathbb{R}^d)\times\mathbb{R}^d\ni(\mu,v)\mapsto\partial_\mu U(\mu)(v)$ is jointly continuous at any (μ,v) such that $v\in\mathrm{Supp}(\mu)$,

• for any $\mu\in\mathcal{P}_2(\mathbb{R}^d)$, the mapping $\mathbb{R}^d\ni v\mapsto\partial_\mu U(\mu)(v)\in\mathbb{R}^d$ is continuously differentiable and its derivative is jointly continuous with respect to μ and v at any point (μ,v) such that $v\in\mathrm{Supp}(\mu)$, the derivative being denoted by $\mathbb{R}^d\ni v\mapsto\partial_v[\partial_\mu U(\mu)](v)\in\mathbb{R}^{d\times d}$.

Recall from the discussion in Subsection 3.1 that, for each $\mu\in\mathbb{R}^d$, the mapping $\partial_\mu U(\mu):v\mapsto\partial_\mu U(\mu)(v)$ is uniquely defined on the support of μ.

The following is the chain rule for is partially $\mathcal{C}^2$:

Theorem 3.5. *Assume that U is partially $\mathcal{C}^2$ and that, for any compact subset $\mathcal{K}\subset\mathcal{P}_2(\mathbb{R}^d)$,* (3.6) *holds true. Then, the chain rule holds for an Itô process satisfying* (3.5).

Notice that, in the chain rule, the mapping $\partial_\mu U:\mathcal{P}_2(\mathbb{R}^d)\times\mathbb{R}^d\ni(\mu,v)\mapsto\partial_\mu U(\mu)(v)$ is always evaluated at points (μ,v) such that v belongs to the support of μ and thus for which $\partial_\mu U(\mu)(v)$ is uniquely defined.

Proof. *First step.* We start with the same mollification procedure as in the proof of Theorem 3.3, see (3.8).

Repeating the computations, $U\star\varphi$ and its first and partial second order derivatives are bounded. Nevertheless, contrary to the argument in the proof of Theorem 3.3, we cannot claim here that $\partial_\mu(U\star\varphi)$ and $\partial_v[\partial_\mu(U\star\varphi)]$ are continuous on the whole space since $\partial_\mu U$ and $\partial_v[\partial_\mu U]$ are only continuous at points (μ,v) such that v is in the support of μ. In order to circumvent this difficulty, we first notice, from (3.8), that $\partial_\mu(U\star\varphi)$ and $\partial_v[\partial_\mu(U\star\varphi)]$ are also continuous at points (μ,v) such that v is in the support of μ, the reason being that $v\in\mathrm{Supp}(\mu)$ implies $\varphi(v)\in\mathrm{Supp}(\varphi\sharp\mu)$. We then change $\mathcal{P}_2(\mathbb{R}^d)\ni\mu\mapsto(U\star\varphi)(\mu)$ into $\mathcal{P}_2(\mathbb{R}^d)\ni\mu\mapsto(U\star\varphi)(\mu\star\rho)$ where ρ is a smooth convolution kernel, with the entire $\mathbb{R}^d$ as support and with exponential decay at infinity, and $\mu\star\rho$ stands for the probability measure with density given by

$$\mathbb{R}^d\ni x\mapsto\int_{\mathbb{R}^d}\rho(x-y)\mathrm{d}\mu(y).$$

We then observe that

$$\partial_\mu\big[(U\star\varphi)(\mu\star\rho)\big](v)=\int_{\mathbb{R}^d}\partial_\mu(U\star\varphi)(\mu\star\rho)(v-v')\rho(v')\mathrm{d}v',$$
$$\partial_v\big[\partial_\mu\big[(U\star\varphi)(\mu\star\rho)\big]\big](v)=\int_{\mathbb{R}^d}\partial_v\big[\partial_\mu(U\star\varphi)(\mu\star\rho)\big](v-v')\rho(v')\mathrm{d}v'.$$

Since the support of ρ is the whole $\mathbb{R}^d$, the measure $\mu\star\rho$ also has $\mathbb{R}^d$ as support, so that, for any $v\in\mathbb{R}^d$, $(\mu\star\rho,v)$ is a continuity point of both $\partial_\mu(U\star\varphi)$ and $\partial_v[\partial_\mu(U\star\varphi)]$. Since $\partial_\mu(U\star\varphi)$ and $\partial_v[\partial_\mu(U\star\varphi)]$ are bounded, we deduce from Lebesgue's theorem that the maps $(\mu,v)\mapsto\partial_\mu(U\star\varphi)(\mu\star\rho)(v)$ and $(\mu,v)\mapsto\partial_v[\partial_\mu(U\star\varphi)(\mu\star\rho)](v)$ are continuous on the whole $\mathcal{P}_2(\mathbb{R}^d)\times\mathbb{R}^d$.

Moreover, whenever ρ is chosen along a sequence that converges to the Dirac mass at 0 (for the W_2 distance), it is also easy to check that, for any $\mu\in\mathcal{P}_2(\mathbb{R}^d)$ and any $v\in\mathrm{Supp}(\mu)$, $\partial_\mu(U\star\varphi)(\mu\star\rho)(v)$ and $\partial_v[\partial_\mu(U\star\varphi)](\mu\star\rho)(v)$ converge to $\partial_\mu(U\star\varphi)(\mu)(v)$ and $\partial_v[\partial_\mu(U\star\varphi)](\mu)(v)$. In particular, if Itô's formula holds true for functionals of the type $\mathcal{P}_2(\mathbb{R}^d)\ni\mu\mapsto(U\star\varphi)(\mu\star\rho)$, it also holds true for functionals of the type $\mathcal{P}_2(\mathbb{R}^d)\ni\mu\mapsto(U\star\varphi)(\mu)$ and then for functionals of the type $\mathcal{P}_2(\mathbb{R}^d)\ni\mu\mapsto U(\mu)$ by the same approximation argument as in the proof of Theorem 3.3.

Therefore, without any loss of generality, we can assume that U and its first and partial second order derivatives are bounded and uniformly continuous on the whole space. As in the proof of Theorem 3.3, we can also assume that $(X_t)_{t\geqslant 0}$ is a bounded Itô process.

Second step. The proof requires another mollification argument. Taking now ρ as a smooth compactly supported density on $\mathbb{R}^d$ and using the same notations as above, we define the convolution u_n^N of u^N:

$$\begin{aligned}u_n^N(x^1,\dots,x^N)&=n^{Nd}\int_{(\mathbb{R}^d)^N}u^N(x^1-y^1,\dots,x^N-y^N)\prod_{\ell=1}^N\rho\big(ny^\ell\big)\prod_{\ell=1}^N\mathrm{d}y^\ell\\&=\mathbb{E}\Big[U\Big(\frac{1}{N}\sum_{i=1}^N\delta_{x^i-Y^i/n}\Big)\Big],\end{aligned}\tag{3.17}$$

where $Y^1,\dots,Y^N$ are N i.i.d. random variables with density ρ. Recalling that

$$W_2^2\Big(\frac{1}{N}\sum_{i=1}^N\delta_{x^i-Y^i/n},\frac{1}{N}\sum_{i=1}^N\delta_{x^i}\Big)\leqslant\frac{1}{N}\sum_{i=1}^N\big(\frac{Y^i}{n}\big)^2,$$

we notice that

$$\mathbb{E}\Big[W_2^2\Big(\frac{1}{N}\sum_{i=1}^N\delta_{x^i-Y^i/n},\frac{1}{N}\sum_{i=1}^N\delta_{x^i}\Big)\Big]\leqslant\frac{C}{n^2},\tag{3.18}$$

as ρ has compact support. Above and in the rest of the proof, the constant C is a general constant that is allowed to increase from line to line. Importantly, it does not depend on n nor N.

Observe now that, for two random variables $X,X'\in L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d)$, we can find $t\in[0,1]$ such that

$$\begin{aligned}|U([X])-U([X'])|&=\big|\mathbb{E}\big[\partial_\mu U\big(\big[tX+(1-t)X'\big]\big)\big(tX+(1-t)X'\big)(X-X')\big]\big|\\&\leqslant\big\|\partial_\mu U\big(\big[tX+(1-t)X'\big]\big)\big(tX+(1-t)X'\big)\big\|_2\|X-X'\|_2\\&\leqslant C\|X-X'\|_2,\end{aligned}$$

the last line following from the fact that the function $\mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \ni (\mu, v) \mapsto \partial_\mu U(\mu)(v)$ is bounded. Therefore, we deduce from (3.17) and (3.18) that

$$\begin{aligned} \left| u_n^N(x^1,\dots,x^N) - u^N(x^1,\dots,x^N) \right| &= \left| \mathbb{E}\left[U\left(\frac{1}{N} \sum_{i=1}^N \delta_{x^i - Y^i/n} \right) - U\left(\frac{1}{N} \sum_{i=1}^N \delta_{x^i} \right) \right] \right| \\ &\leqslant C n^{-1}. \end{aligned} \tag{3.19}$$

Given a bounded random variable X with law μ, we know from [32, Theorem 10.2.1] that the quantity $\mathbb{E}[W_2^2(\mu, \bar{\mu}^N)]$ tends to 0 as N tends to the infinity, $\bar{\mu}^N$ denoting the empirical measure of a sample of size N of the same law as X. Moreover, the rate of convergence of $(\mathbb{E}[W_2^2(\mu, \bar{\mu}^N)])_{N \geqslant 1}$ towards 0 only depends upon the bounds for the moments of X. Together with (3.19), this says that we can find a sequence $(\varepsilon_\ell)_{\ell \geqslant 1}$ converging to 0 as ℓ tends to ∞ such that, for any $n, N \geqslant 1$ and for any $t \geqslant 0$,

$$\begin{aligned} &\mathbb{E}\big[\big| u_n^N(X_t^1,\dots,X_t^N) - U\big(\mu_t\big) \big| \big] \\ &\leqslant \mathbb{E}\big[\big| u_n^N(X_t^1,\dots,X_t^N) - u^N(X_t^1,\dots,X_t^N) \big| \big] + \mathbb{E}\big[\big| U\big(\bar{\mu}_t^N\big) - U\big(\mu_t\big) \big| \big] \\ &\leqslant \varepsilon_n + \varepsilon_N. \end{aligned} \tag{3.20}$$

(It is worth mentioning that the sequence $(\varepsilon_\ell)_{\ell \geqslant 1}$ may be assumed to be independent of t.) By boundedness of U, we deduce that, for any $p \geqslant 1$ and any $t \geqslant 0$,

$$\mathbb{E}\big[\big| u_n^N(X_t^1,\dots,X_t^N) - U\big(\mu_t\big) \big|^p \big]^{1/p} \leqslant \varepsilon_n^{(p)} + \varepsilon_N^{(p)}, \tag{3.21}$$

for a sequence $(\varepsilon_\ell^{(p)})_{\ell \geqslant 1}$ that tends to 0 as ℓ tends to ∞ (and the terms of which are allowed to increase from line to line).

Now, by the first part in Proposition 3.1, we compute

$$\begin{aligned} \partial_{x_i} u_n^N(x^1,\dots,x^N) &= n^{Nd} \int_{(\mathbb{R}^d)^N} \partial_{x_i} u^N(x^1 - y^1, \dots, x^N - y^N) \prod_{\ell=1}^N \rho(n y^\ell) \prod_{\ell=1}^N \mathrm{d}y^\ell \\ &= \frac{n^{Nd}}{N} \int_{(\mathbb{R}^d)^N} \partial_\mu U\left(\frac{1}{N} \sum_{\ell=1}^N \delta_{x^\ell - y^\ell} \right)(x^i - y^i) \prod_{\ell=1}^N \rho(n y^\ell) \prod_{\ell=1}^N \mathrm{d}y^\ell \\ &= \frac{1}{N} \mathbb{E}\left[\partial_\mu U\left(\frac{1}{N} \sum_{\ell=1}^N \delta_{x^\ell - Y^\ell/n} \right)(x^i - Y^i/n) \right]. \end{aligned}$$

Using the uniform continuity of $\partial_\mu U$ on the whole space and following the proof of (3.20), we deduce that, for any $t \geqslant 0$,

$$\mathbb{E}\big[\big| N \partial_{x_i} u_n^N(X_t^1,\dots,X_t^N) - \partial_\mu U(\mu_t)(X_t^i) \big| \big] \leqslant \varepsilon_n + \varepsilon_N. \tag{3.22}$$

Again, by boundedness of $\partial_\mu U$, we deduce that, for any $p \geqslant 1$ and any $t \geqslant 0$,

$$\mathbb{E}\big[\big| N \partial_{x_i} u_n^N(X_t^1,\dots,X_t^N) - \partial_\mu U(\mu_t)(X_t^i) \big|^p \big]^{1/p} \leqslant \varepsilon_n^{(p)} + \varepsilon_N^{(p)}. \tag{3.23}$$

Now, we differentiate once more in x_i:

$$\begin{aligned} &\partial_{x_i x_i}^2 u_n^N(x^1,\dots,x^N) \\ &= \frac{n^{Nd+1}}{N} \int_{(\mathbb{R}^d)^N} \left\{ \partial_\mu U\left(\frac{1}{N} \sum_{\ell=1}^N \delta_{x^\ell - y^\ell} \right)(x^i - y^i) \right\} \otimes \nabla \rho(n y^i) \prod_{\ell \neq i} \rho(n y^\ell) \prod_{\ell=1}^N \mathrm{d}y^\ell, \end{aligned}$$

the tensorial product operating on elements of $\mathbb{R}^d$. We then split the derivative into two pieces:

$$N\partial^2_{x_i x_i} u_n^N(x^1,\dots,x^N) = T^{1,N}_{n,i}(x^1,\dots,x^N) + T^{2,N}_{n,i}(x^1,\dots,x^N),$$

with

$$\begin{aligned}
&T^{1,N}_{n,i}(x^1,\dots,x^N)\\
&\quad = n^{Nd+1}\int_{(\mathbb{R}^d)^N}\biggl\{\partial_\mu U\biggl(\frac{1}{N}\sum_{\ell\neq i}\delta_{x^\ell-y^\ell}+\frac{1}{N}\delta_{x^i}\biggr)(x^i-y^i)\biggr\}\otimes\nabla\rho(ny^i)\prod_{\ell\neq i}\rho(ny^\ell)\prod_{\ell=1}^N \mathrm{d}y^\ell\\
&T^{2,N}_{n,i}(x^1,\dots,x^N)\\
&\quad = n^{Nd+1}\int_{(\mathbb{R}^d)^N}\biggl\{\biggl[\biggl(\partial_\mu U\biggl(\frac{1}{N}\sum_{\ell=1}^N\delta_{x^\ell-y^\ell}\biggr)\\
&\qquad\qquad -\partial_\mu U\biggl(\frac{1}{N}\sum_{\ell\neq i}\delta_{x^\ell-y^\ell}+\frac{1}{N}\delta_{x^i}\biggr)\biggr]\bigl(x^i-y^i\bigr)\biggr\}\otimes\nabla\rho(ny^i)\prod_{\ell\neq i}\rho(ny^\ell)\prod_{\ell=1}^N \mathrm{d}y^\ell.
\end{aligned}$$

By integration by parts (recall that $\mathbb{R}^d \ni v \mapsto \partial_\mu U(\mu)(v)$ is differentiable), we can split $T^{1,N}_{n,i}$ into

$$T^{1,N}_{n,i}(x^1,\dots,x^N) = T^{11,N}_{n,i}(x^1,\dots,x^N) + T^{12,N}_{n,i}(x^1,\dots,x^N),$$

with

$$\begin{aligned}
T^{11,N}_{n,i}(x^1,\dots,x^N) &= n^{Nd}\int_{(\mathbb{R}^d)^N}\biggl\{\partial_v\biggl[\partial_\mu U\biggl(\frac{1}{N}\sum_{\ell=1}^N\delta_{x^\ell-y^\ell}\biggr)\biggr](x^i-y^i)\biggr\}\prod_{\ell=1}^N\rho(ny^\ell)\prod_{\ell=1}^N\mathrm{d}y^\ell\\
T^{12,N}_{n,i}(x^1,\dots,x^N) &= n^{Nd}\int_{(\mathbb{R}^d)^N}\biggl\{\partial_v\biggl[\partial_\mu U\biggl(\frac{1}{N}\sum_{\ell\neq i}\delta_{x^\ell-y^\ell}+\frac{1}{N}\delta_{x^i}\biggr)\\
&\qquad -\partial_\mu U\biggl(\frac{1}{N}\sum_{\ell=1}^N\delta_{x^\ell-y^\ell}\biggr)\biggr](x^i-y^i)\biggr\}\prod_{\ell=1}^N\rho(ny^\ell)\prod_{\ell=1}^N\mathrm{d}y^\ell.
\end{aligned}$$

The first term is treated as per (3.20) and (3.22). Namely, we have, for any $t \geqslant 0$,

$$\mathbb{E}\bigl[\bigl|T^{11,N}_{n,i}(X^1_t,\dots,X^N_t) - \partial_v\bigl[\partial_\mu U(\mu_t)\bigr](X^i_t)\bigr|\bigr] \leqslant \varepsilon_n + \varepsilon_N. \tag{3.24}$$

Then, by boundedness of $\partial_v[\partial_\mu U]$ for any $p \geqslant 1$ and any $t \geqslant 0$,

$$\mathbb{E}\bigl[\bigl|T^{11,N}_{n,i}(X^1_t,\dots,X^N_t) - \partial_v\bigl[\partial_\mu U(\mu_t)\bigr](X^i_t)\bigr|^p\bigr]^{1/p} \leqslant \varepsilon^{(p)}_n + \varepsilon^{(p)}_N. \tag{3.25}$$

To handle the second term, we use uniform continuity of $\partial_v[\partial_\mu U]$. Indeed, we have $|T^{12,N}_{n,i}(x^1,\dots,x^N)| \leqslant \varepsilon_N$ as

$$W_2^2\biggl(\frac{1}{N}\sum_{\ell\neq i}\delta_{x^\ell-y^\ell}+\frac{1}{N}\delta_{x^i},\frac{1}{N}\sum_{\ell=1}^N\delta_{x^\ell-y^\ell}\biggr) \leqslant \frac{1}{N}|y^i|^2 \leqslant \frac{C}{N},$$

since, in $T^{12,N}_{n,i}(x^1,\dots,x^N)$, ny^i belongs to the (compact) support of ρ. This says that, for any $t \geqslant 0$,

$$\mathbb{E}\bigl[\bigl|T^{12,N}_{n,i}(X^1_t,\dots,X^N_t)\bigr|\bigr] \leqslant \varepsilon_N. \tag{3.26}$$

And, then, for any $p \geqslant 1$ and any $t \geqslant 0$,

$$\mathbb{E}\bigl[\bigl|T^{12,N}_{n,i}(X^1_t,\dots,X^N_t)\bigr|^p\bigr]^{1/p} \leqslant \varepsilon^{(p)}_N. \tag{3.27}$$

We finally handle $T_{n,i}^{2,N}$. Following the proof of (3.27), we have, for any $p \geqslant 1$ and any $t \geqslant 0$,

$$\mathbb{E}\big[\big|T_{n,i}^{2,N}(X_t^1,\dots,X_t^N)\big|^p\big]^{1/p} \leqslant n\varepsilon_N^{(p)}, \tag{3.28}$$

the additional n coming from the differentiation of the regularization kernel.

Third step. In order to complete the proof, we apply Itô's formula to $(u_n^N(X_t^1,\dots,X_t^N))_{t\geqslant 0}$ for given values of n and N. We obtain

$$\begin{aligned}0 &= u_n^N\big(X_t^1,\dots,X_t^N\big) - u_n^N\big(X_0^1,\dots,X_0^N\big) - \sum_{\ell=1}^N \int_0^t \partial_{x^\ell} u_n^N\big(X_s^1,\dots,X_s^N\big) b_s^\ell \mathrm{d}s \\ &\quad - \sum_{\ell=1}^N \int_0^t \partial_{x^\ell} u_n^N\big(X_s^1,\dots,X_s^N\big)\sigma_s^\ell \mathrm{d}W_s^\ell - \frac{1}{2}\sum_{\ell=1}^N \int_0^t \mathrm{Tr}\big\{\partial_{x_\ell}^2 u_n^N\big(X_s^1,\dots,X_s^N\big) a_s^\ell\big\}\mathrm{d}s,\end{aligned}$$

with $a_s^\ell := \sigma_s^\ell(\sigma_s^\ell)^\dagger$. To compare with the expected result, we take the difference with

$$\begin{aligned}\Delta_t^N &= U(\mu_t) - U(\mu_0) - \frac{1}{N}\sum_{\ell=1}^N \int_0^t \partial_\mu U(\mu_s)(X_s^\ell) b_s^\ell \mathrm{d}s \\ &\quad - \frac{1}{N}\sum_{\ell=1}^N \int_0^t \partial_\mu U(\mu_s)(X_s^\ell)\sigma_s^\ell \mathrm{d}W_s^\ell - \frac{1}{2N}\sum_{\ell=1}^N \int_0^t \mathrm{Tr}\big\{\partial_v\big[\partial_\mu U(\mu_s)\big](X_s^\ell) a_s^\ell\big\}\mathrm{d}s.\end{aligned} \tag{3.29}$$

From (3.21), (3.23), (3.25), (3.27) and (3.28), we obtain, for any $T > 0$,

$$\sup_{0\leqslant t\leqslant T} \mathbb{E}\big[|\Delta_t^N|\big] \leqslant \varepsilon_n + (1+n)\varepsilon_N,$$

the sequence $(\varepsilon_\ell)_{\ell\geqslant 1}$ now depending on T. Letting N tend to ∞, we deduce from Fatou's lemma and the law of large numbers that

$$\sup_{0\leqslant t\leqslant T} |\Delta_t| \leqslant \varepsilon_n, \tag{3.30}$$

where

$$\Delta_t = U(\mu_t) - U(\mu_0) - \int_0^t \mathbb{E}\big[\partial_\mu U(\mu_s)(X_s) b_s\big]\mathrm{d}s - \frac{1}{2}\int_0^t \mathbb{E}\Big[\mathrm{Tr}\big\{\partial_v\big[\partial_\mu U(\mu_s)\big](X_s) a_s\big\}\Big]\mathrm{d}s.$$

Letting n tend ∞ in (3.30), we deduce that $\Delta \equiv 0$, which completes the proof.

3.4. **A sufficient condition for partial $\mathcal{C}^2$ regularity.** The following is a sufficient criterion for partial $\mathcal{C}^2$ regularity used in the next section:

Theorem 3.6. *Let $U : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ be a function such that its lifted version $\mathcal{U} : L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d) \ni \xi \mapsto U([\xi]) \in \mathbb{R}$ is once continuously Fréchet differentiable. Assume also that for any continuously differentiable map $\mathbb{R} \ni \lambda \mapsto X^\lambda \in L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d)$, with the property that all the $(X^\lambda)_{\lambda\in\mathbb{R}}$ have the same distribution and that $|[\mathrm{d}/\mathrm{d}\lambda]X^\lambda| \leqslant 1$ (in L^∞), the mapping*

$$\mathbb{R} \ni \lambda \mapsto D\mathcal{U}(X^\lambda)\cdot\chi = \mathbb{E}\big[\partial_\mu U([X^\lambda])(X^\lambda)\chi\big] \in \mathbb{R} \tag{3.31}$$

is continuously differentiable for any $\chi \in L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d)$. Moreover assume that the derivative of the mapping $\mathbb{R} \ni \lambda \mapsto D\mathcal{U}(X^\lambda)\cdot\chi$ at $\lambda = 0$ depends on the family $(X^\lambda)_{\lambda\in\mathbb{R}}$ only through the value of X^0 and of $[\mathrm{d}/\mathrm{d}\lambda]_{|\lambda=0}X^\lambda$ (see footnote[10] below for more details),

[10] This means that for two families $(X^\lambda)_{\lambda\in\mathbb{R}}$ and $(X^{\lambda,\prime})_{\lambda\in\mathbb{R}}$ with $X^0 = X^{0,\prime}$ and $[\mathrm{d}/\mathrm{d}\lambda]_{|\lambda=0}X^\lambda = [\mathrm{d}/\mathrm{d}\lambda]_{|\lambda=0}X^{\lambda,\prime}$, the derivatives $[\mathrm{d}/\mathrm{d}\lambda]_{|\lambda=0}[D\mathcal{U}(X^\lambda)\cdot\chi]$ and $[\mathrm{d}/\mathrm{d}\lambda]_{|\lambda=0}[D\mathcal{U}(X^{\lambda,\prime})\cdot\chi]$ are the same (the variable χ being given).

so that we can denote

$$\partial^2_{\zeta,\chi}\mathcal{U}(X) := \frac{\mathrm{d}}{\mathrm{d}\lambda_{|\lambda=0}}\big[D\mathcal{U}(X^\lambda)\cdot\chi\big],$$

whenever $X := X^0$ *and* $\zeta := [\mathrm{d}/\mathrm{d}\lambda]_{|\lambda=0}X^\lambda$. *Finally, assume that there exist a constant* C *and an exponent* $\alpha \geqslant 0$ *such that, for any* X, χ *and* ζ *in* $L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d)$, *with* $|\zeta| \leqslant 1$ *(in* L^∞*), it holds (with* Φ_α *as in* (***H*1**) *and in particular satisfying* (2.13)*):*

(*i*) $|D\mathcal{U}(X)\cdot\chi| + |\partial^2_{\zeta,\chi}U(X)| \leqslant C\|\chi\|_2$,

(*ii*) $|D\mathcal{U}(X)\cdot\chi - D\mathcal{U}(X')\cdot\chi| + |\partial^2_{\zeta,\chi}\mathcal{U}(X) - \partial^2_{\zeta,\chi}\mathcal{U}(X')| \leqslant C\Phi_\alpha(X,X')\|\chi\|_2$.

Then U *is partially* $\mathcal{C}^2$ *and satisfies for any compact subset* $\mathcal{K} \subset \mathcal{P}_2(\mathbb{R}^d)$*:*

$$\sup_{\mu\in\mathcal{K}}\bigg[\int_{\mathbb{R}^d}\big|\partial_\mu U(\mu)(v)\big|^2\mathrm{d}\mu(v) + \int_{\mathbb{R}^d}\big|\partial_v\big[\partial_\mu U(\mu)\big](v)\big|^2\mathrm{d}\mu(v)\bigg] < \infty,$$

so that the chain rule applies to an Itô process satisfying (3.5)*.*

Remark 3.7. *The thrust of Theorem 3.6 is to study the smoothness of the mapping* $v \mapsto \partial_\mu U(\mu)(v)$ *independently of the smoothness in the direction* μ *by restricting the 'test' random variables* $(X^\lambda)_{\lambda\in\mathbb{R}}$ *to an identically distributed family. One of the issue in the proof is precisely to construct such a family of test random variables.*

Proof. In the proof, we use quite often the following result, which is a refinement of [5, Lemma 3.3] (see the adaptation of the proof in Subsection 6.1 in Appendix):

Proposition 3.8. *Consider a collection* $(V(\mu) : \mathbb{R}^d \ni v \mapsto V(\mu)(v))_\mu$ *of Borel functions from* $\mathbb{R}^d$ *into* $\mathbb{R}^d$ *indexed by elements* $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ *such that, for any* $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, *the mapping* $\mathbb{R}^d \ni v \mapsto V(\mu)(v) \in \mathbb{R}^d$ *belongs to* $L^2(\mu,\mathbb{R}^d;\mathbb{R}^d)$. *Assume also that there exist a constant* C *and an exponent* α *such that, for any* $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ *and any* $\xi,\xi' \in L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d)$, *such that* ξ *and* ξ' *have distribution* μ, *and*

$$\mathbb{E}\big[|V(\mu)(\xi) - V(\mu)(\xi')|^2\big]^{1/2} \leqslant C\mathbb{E}\big[\big(1+|\xi|^{2\alpha}+|\xi'|^{2\alpha}+\|\xi\|_2^{2\alpha}\big)|\xi-\xi'|^2\big]^{1/2}. \tag{3.32}$$

Then, for any $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, *the mapping* $v \mapsto V(\mu)(v)$ *admits a locally Lipschitz continuous version, that satisfies*

$$|V(\mu)(v) - V(\mu)(v')| \leqslant C\bigg[1 + 2\max\big(|v|^{2\alpha},|v'|^{2\alpha}\big) + \bigg(\int_{\mathbb{R}^d}|x|^2\mathrm{d}\mu(x)\bigg)^\alpha\bigg]^{1/2}|v-v'|.$$

As a warm-up, we discuss what Proposition 3.8 says in the framework of Theorem 3.6. Representing $D\mathcal{U}(X)\cdot\chi$ as $\mathbb{E}[\partial_\mu U([X])(X)\chi]$, we can write (choosing $X = \xi$ and $X' = \xi'$, with $[\xi] = [\xi'] = \mu$, in part *(ii)* of the statement of Theorem 3.6)

$$\begin{aligned}&\big|\mathbb{E}\big[\big(\partial_\mu U(\mu)(\xi') - \partial_\mu U(\mu)(\xi)\big)\chi\big]\big|\\ &\quad\leqslant C\mathbb{E}\big[\big(1+|\xi|^{2\alpha}+|\xi'|^{2\alpha}+\|\xi\|_2^{\alpha}\big)|\xi-\xi'|^2\big]^{1/2}\mathbb{E}\big[|\chi|^2\big]^{1/2}.\end{aligned}$$

This says that, for any $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, we can find a locally Lipschitz continuous version of the mapping $\mathbb{R}^d \ni v \mapsto \partial_\mu U(\mu)(v)$, the local Lipschitz constant being at most of α-polynomial growth, uniformly with respect to μ in W_2-balls. In addition, Proposition 3.8 gives us a bit more. Consider a sequence $(\mu_n)_{n\geqslant 0}$ with values in $\mathcal{P}_2(\mathbb{R}^d)$ such that $\mu_n \to \mu$ in the 2-Wasserstein distance. Then, the functions $(\mathbb{R}^d \ni v \mapsto \partial_\mu U(\mu_n)(v))_{n\geqslant 0}$

are uniformly continuous on compact sets. Moreover, we notice, by Markov inequality that $\mathbb{P}(|\xi_n| \geqslant 2\|\xi_n\|_2) \leqslant 1/4$, so that

$$\begin{aligned}\frac{3}{4}\inf_{|v|\leqslant 2\|\xi_n\|_2}|\partial_\mu U(\mu_n)(v)| &\leqslant \mathbb{E}\big[\mathbf{1}_{\{|\xi_n|\leqslant 2\|\xi_n\|_2\}}|\partial_\mu U(\mu_n)(\xi_n)|^2\big]^{1/2} \\ &\leqslant \mathbb{E}\big[|\partial_\mu U(\mu_n)(\xi_n)|^2\big]^{1/2} \leqslant C,\end{aligned} \tag{3.33}$$

where ξ_n has distribution μ_n, the last inequality following from *(i)* in the statement of Theorem 3.6. This says that the family $(\inf_{|v|\leqslant 2\|\xi_n\|_2}|\partial_\mu U(\mu_n)(v)|)_{n\geqslant 0}$ is bounded. As the sequence $(\|\xi_n\|_2)_{n\geqslant 0}$ is bounded and the mappings $(\mathbb{R}^d \ni v \mapsto \partial_\mu U(\mu_n)(v))_{n\geqslant 0}$ are uniformly locally Lipschitz continuous, the sequence $(|\partial_\mu U(\mu_n)(0)|)_{n\geqslant 0}$ is also bounded. Therefore, the family $(\mathbb{R}^d \ni v \mapsto \partial_\mu U(\mu_n)(v))_{n\geqslant 0}$ is relatively compact for the topology of uniform convergence on compact subsets. Passing to the limit [11] (up to a subsequence) into the relationship

$$D\mathcal{U}(\xi_n)\cdot\chi = \mathbb{E}\big[\partial_\mu U(\mu_n)(\xi_n)\chi\big],$$

we deduce, by identification, that the limit of $\partial_\mu U(\mu_n)$ must coincide with $\partial_\mu U(\mu)$ on the support of μ. This says that the function $\mathcal{P}_2(\mathbb{R}^d)\times\mathbb{R}^d \ni (\mu,v)\mapsto \partial_\mu U(\mu)(v)$ is (jointly) continuous at any point (μ,v) such that $v\in \mathrm{Supp}(\mu)$. Moreover, by point *(i)* in the statement of Theorem 3.6, we have $\int_{\mathbb{R}^d}|\partial_\mu U(\mu)(v)|^2 d\mu(v)\leqslant C$, for a constant C independent of μ, which is the first part in the condition (3.6) for applying the chain rule to partially $\mathcal{C}^2$ functions.

To complete the proof we have two main steps. The first one uses a new mollification argument. The second consists in a coupling lemma, which permits to choose relevant versions of the random variables along which the differentiation is performed.

First step. Given a distribution μ and a random variable ξ with distribution μ, we introduce the convoluted version μ^n of μ:

$$\mu^n = \mu \star \mathcal{N}_d(0,\tfrac{1}{n}I_d),$$

n denoting an integer larger than 1 and $\mathcal{N}_d(0,(1/n)I_d)$ denoting the d-dimensional Gaussian distribution with covariance matrix $(1/n)I_d$, where I_d is the identity matrix of dimension d. Then, we can define the mapping

$$\mathcal{V}^n(\mu,v) = \int_{\mathbb{R}^d}\partial_\mu U\big(\mu^n\big)(v-u)n^{d/2}\rho\big(n^{1/2}u\big)\mathrm{d}u, \tag{3.34}$$

where ρ stands for the standard d-dimensional Gaussian kernel. The mapping $\mathcal{V}^n$ is the convolution of $\partial_\mu U(\mu^n)(\cdot)$ with the measure $\mathcal{N}_d(0,(1/n)I_d)$. By the warm-up, the sequence $(\partial_\mu U(\mu^n)(0))_{n\geqslant 1}$ is bounded and the functions $(\partial_\mu U(\mu^n) : \mathbb{R}^d \ni v \mapsto \partial_\mu U(\mu^n)(v)\in\mathbb{R}^d)_{n\geqslant 1}$ are locally Lipschitz, the Lipschitz constant being at most of α-polynomial growth, uniformly in $n\geqslant 1$. In particular, the sequence of functions $(\mathcal{V}^n(\mu,\cdot))_{n\geqslant 0}$ is relatively compact for the topology of uniform convergence on compact subsets. Any limit must coincide with $\partial_\mu U(\mu)$ at points v in the support of μ or, put it differently, any limit provides a version of $\partial_\mu U(\mu)$ which is locally Lipschitz continuous, the Lipschitz constant being at most of α-polynomial growth, uniformly in μ in bounded subsets of $\mathcal{P}_2(\mathbb{R}^d)$. When μ has full support, the sequence $(\mathcal{V}^n(\mu,\cdot))_{n\geqslant 0}$ converges to the unique continuous version of $\partial_\mu U(\mu)$, the convergence being uniform on compact subsets.

[11] From [34, Theorem 6.9], μ_n converges weakly to μ. Using the Skorokhod representation theorem, we can find a sequence (ξ_n) converging almost surely to ξ. The convergence holds also in L^2 since this sequence is uniformly square integrable, recall [34, Definition 6.8(iii)].

Letting $\xi^n = \xi + n^{-1/2}G$, where G is an $\mathcal{N}_d(0, I_d)$ Gaussian variable independent of ξ, we then observe that, for any $\mathbb{R}^d$-valued square integrable random variable χ such that the pair (ξ, χ) is independent of G,

$$\begin{aligned} D\mathcal{U}(\xi^n) \cdot \chi = \mathbb{E}\big[\partial_\mu U\big(\mu^n\big)\big(\xi^n\big)\chi\big] &= \mathbb{E}\bigg[\bigg(\int_{\mathbb{R}^d} \partial_\mu U\big(\mu^n\big)(\xi - u) n^{d/2} \rho(n^{1/2}u)\mathrm{d}u\bigg)\chi\bigg] \\ &= \mathbb{E}\big[\mathcal{V}^n(\mu, \xi)\chi\big], \end{aligned} \tag{3.35}$$

where $\mathcal{V}^n(\mu, \xi)$ is viewed as a row vector. We note that the mapping $\mathbb{R}^d \ni v \mapsto \mathcal{V}^n(\mu, v)$ is differentiable with respect to v (this was not the case for the original mapping $\mathbb{R}^d \ni v \mapsto \partial_\mu U(\mu)(v)$ at this stage of the proof).

Second step. We construct now, independently of the measure μ considered above, a family $(Y^\lambda)_{\lambda \in \mathbb{R}}$ that is differentiable with respect to λ in $L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R})$ but which is, at the same time, invariant in law, all the Y^λ, for $\lambda \in \mathbb{R}$, being uniformly distributed on $[-\pi/2, \pi/2]$. The strategy consists in starting with the uniform distribution:

Given two independent $\mathcal{N}(0,1)$ random variables Z and Z', we let, for any $\lambda \in \mathbb{R}$,

$$Z^\lambda = \cos(\lambda) Z + \sin(\lambda) Z', \quad Z'^{,\lambda} = -\sin(\lambda) Z + \cos(\lambda) Z',$$

so that $(Z^\lambda, Z'^{,\lambda})$ has the same law as (Z, Z') (because of the invariance of the Gaussian distribution by rotation). We then let

$$Y^\lambda = \arcsin\Big(\frac{Z^\lambda}{\sqrt{(Z^\lambda)^2 + (Z'^{,\lambda})^2}}\Big) = \arcsin\Big(\frac{Z^\lambda}{\sqrt{Z^2 + (Z')^2}}\Big).$$

It is easy to check that Y^λ has a uniform distribution on $[-\pi/2, \pi/2]$ for any $\lambda \in \mathbb{R}$. Pathwise, the mapping $\mathbb{R} \ni \lambda \mapsto Y^\lambda$ is differentiable at any λ such that $Z'^{,\lambda} \neq 0$. Noticing that $[\mathrm{d}/\mathrm{d}\lambda] Z^\lambda = Z'^{,\lambda}$ (pathwise), we get:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} Y^\lambda = \frac{Z'^{,\lambda}}{\sqrt{Z^2 + (Z')^2}} \Big(1 - \frac{(Z^\lambda)^2}{(Z^\lambda)^2 + (Z'^{,\lambda})^2}\Big)^{-1/2} = \mathrm{sign}\big(Z'^{,\lambda}\big).$$

On the event $\{Z'^{,0} \neq 0\} = \{Z' \neq 0\}$, which is of probability 1, the set of λ's such that $Z'^{,\lambda} = 0$ is locally finite. The above derivative being bounded by 1, this says that, pathwise, the mapping $\mathbb{R} \ni \lambda \mapsto Y^\lambda$ is 1-Lipschitz continuous. Therefore, the random variables $(Y^\lambda - Y^0)/\lambda$, $\lambda \neq 0$, are bounded by 1. Moreover, still on the event $\{Z' \neq 0\}$, the above computation shows that

$$\lim_{\lambda \to 0} \frac{Y^\lambda - Y^0}{\lambda} = \mathrm{sign}\big(Z'\big). \tag{3.36}$$

Therefore, by Lebesgue's dominated convergence theorem, the mapping $\mathbb{R} \ni \lambda \mapsto Y^\lambda \in L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R})$ is differentiable at $\lambda = 0$ with $\mathrm{sign}(Z')$ as its derivative. In the sequel, we will denote Y^0 by Y.

Actually, by a rotation argument, differentiability holds at any $\lambda \in \mathbb{R}$, with $[\mathrm{d}/\mathrm{d}\lambda] Y^\lambda = \mathrm{sign}(Z'^{,\lambda})$. It is then clear that $\mathbb{R} \ni \lambda \mapsto \mathrm{sign}(Z'^{,\lambda}) \in L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^d)$ is continuous. Indeed, the path $\mathbb{R} \ni \lambda \mapsto Z'^{,\lambda}$ is continuous. Composition by the function *sign* preserves continuity since, for any $\lambda \in \mathbb{R}$, the set of zero points of $Z'^{,\lambda}$ is of zero probability.

Third step. Assume now that μ denotes a given distribution as in the first step. We then choose a random variable ξ with μ as distribution, ξ being independent of the pair (Z, Z'). Given the same $(Y^\lambda)_{\lambda \in \mathbb{R}}$ as above and some parameter $\delta > 0$, we let

$$\forall \lambda \in \mathbb{R}, \quad \xi^\lambda = (\delta \times Y^\lambda) e + \xi,$$

where e is an arbitrary deterministic unitary vector in $\mathbb{R}^d$. (We omit the dependence upon δ in the notation ξ^λ.) Then, we know that the mapping $\mathbb{R} \ni \lambda \mapsto \xi^\lambda$ is continuously differentiable in $L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^d)$, with

$$\frac{\mathrm{d}}{\mathrm{d}\lambda_{|\lambda=0}} \xi^\lambda = (\delta \times \mathrm{sign}(Z'))e.$$

Going back to (3.35), we get for another random variable $\chi \in L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^d)$, with (ξ, χ, Z, Z') independent of G,

$$D\mathcal{U}\big(\xi^\lambda + \tfrac{1}{\sqrt{n}} G\big) \cdot \chi = \mathbb{E}\big[\mathcal{V}^n\big([\xi^\lambda], \xi^\lambda\big)\chi\big],$$

where $\mathcal{V}^n(\mu, v)$ is seen as a row vector. As the mapping $\mathbb{R} \ni \lambda \mapsto \xi^\lambda$ is continuously differentiable in $L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^d)$ and since all the random variables $(\xi^\lambda)_{\lambda \in \mathbb{R}}$ have the same distribution, we deduce that (for (ξ, χ, Z, Z') independent of G)

$$\begin{aligned}
\partial^2_{\mathrm{sign}(Z')e,\chi} \mathcal{U}\big(\xi + \delta Y e + \tfrac{1}{\sqrt{n}} G\big) &= \frac{\mathrm{d}}{\mathrm{d}\lambda_{|\lambda=0}} \big[D\mathcal{U}\big(\xi^{\lambda/\delta} + \tfrac{1}{\sqrt{n}} G\big) \cdot \chi\big] \\
&= \frac{1}{\delta} \frac{\mathrm{d}}{\mathrm{d}\lambda_{|\lambda=0}} \big[D\mathcal{U}\big(\xi^{\lambda} + \tfrac{1}{\sqrt{n}} G\big) \cdot \chi\big] \\
&= \mathbb{E}\big[\mathrm{Tr}\big\{\partial_v \mathcal{V}^n\big([\xi + \delta Y e], \xi + \delta Y e\big)\big((\mathrm{sign}(Z')\chi) \otimes e\big)\big\}\big].
\end{aligned}$$

Noticing that the random variable $|\mathrm{sign}(Z')|$ is equal to 1 almost surely, we can replace χ by $\mathrm{sign}(Z')\chi$ (recall that $|\chi|$ must be less than 1) with (ξ, χ) independent of (Z, Z'), so that

$$\partial^2_{\mathrm{sign}(Z')e,\mathrm{sign}(Z')\chi} \mathcal{U}\big(\xi + \delta Y e + \tfrac{1}{\sqrt{n}} G\big) = \mathbb{E}\big[\mathrm{Tr}\big\{\partial_v \mathcal{V}^n\big([\xi + \delta Y e], \xi + \delta Y e\big)\big(\chi \otimes e\big)\big\}\big].$$

Finally, we let

$$\mathcal{W}^{n,\delta}(\mu, v) = \int_{\mathbb{R}} \partial_v \mathcal{V}^n\big(\mu \star p^\delta, v + \delta u e\big) p(u) \mathrm{d}u, \tag{3.37}$$

where p is the uniform density on $[-\pi/2, \pi/2]$ and $p^\delta(\cdot) = p(\cdot/\delta)/\delta$ is the uniform density on $[-\delta\pi/2, \delta\pi/2]$. Moreover $\mu \star p^\delta$ is an abbreviated notation for denoting the convolution of μ with the uniform distribution on the segment $[-(\delta\pi/2)e, (\delta\pi/2)e]$. Since the pair (ξ, χ) is independent of (Z, Z'), we end up with the duality formula:

$$\partial^2_{\mathrm{sign}(Z')e,\mathrm{sign}(Z')\chi} \mathcal{U}\big(\xi + \delta Y e + \tfrac{1}{\sqrt{n}} G\big) = \mathbb{E}\big[\mathrm{Tr}\big\{\mathcal{W}^{n,\delta}(\mu, \xi)\big(\chi \otimes e\big)\big\}\big]. \tag{3.38}$$

By the smoothness assumption on $\partial^2_{\zeta,\chi}\mathcal{U}$ (see *(ii)* in the statement of Theorem 3.6), we deduce that, for another ξ', with distribution μ as well, such that the triple (ξ, ξ', χ) is independent of (Z, Z') and the 5-tuple (ξ, ξ', χ, Z, Z') is independent of G,

$$\begin{aligned}
&\big|\mathbb{E}\big[\mathrm{Tr}\big\{\big(\mathcal{W}^{n,\delta}(\mu, \xi) - \mathcal{W}^{n,\delta}(\mu, \xi')\big)\big(\chi \otimes e\big)\big\}\big]\big| \\
&\leqslant C\mathbb{E}\Big[\big(1 + |\xi|^{2\alpha} + |\xi'|^{2\alpha} + |\delta Y|^{2\alpha} + |\tfrac{1}{\sqrt{n}} G|^{2\alpha} + \|\xi\|_2^{2\alpha}\big)|\xi - \xi'|^2\Big]^{1/2} \mathbb{E}\big[|\chi|^2\big]^{1/2} \\
&\leqslant C\mathbb{E}\big[\big(1 + |\xi|^{2\alpha} + |\xi'|^{2\alpha} + \|\xi\|_2^{2\alpha}\big)|\xi - \xi'|^2\big]^{1/2} \mathbb{E}\big[|\chi|^2\big]^{1/2},
\end{aligned} \tag{3.39}$$

where we used the independence of (ξ, ξ') and (Z, Z') to pass from the second to the third line, the value of C varying from the second to the third line (but remaining independent of δ and n, when δ is taken in a bounded set). The above is true for any $\sigma(\xi, \xi')$-measurable $\chi \in L^2(\Omega, \mathcal{A}, \mathbb{P})$. We deduce that, for any other $e' \in \mathbb{R}^d$ with $|e'| = 1$,

$$\begin{aligned}
&\mathbb{E}\big[\big|\mathrm{Tr}\big\{\big(\mathcal{W}^{n,\delta}(\mu, \xi) - \mathcal{W}^{n,\delta}(\mu, \xi')\big)\big(e' \otimes e\big)\big\}\big|^2\big] \\
&\qquad \leqslant C\mathbb{E}\big[\big(1 + |\xi|^{2\alpha} + |\xi'|^{2\alpha} + \|\xi\|_2^{2\alpha}\big)|\xi - \xi'|^2\big]^{1/2}.
\end{aligned}$$

By Proposition 3.8, this says that $\mathbb{R}^d \ni v \mapsto \mathrm{Tr}\{(\mathcal{W}^{n,\delta}(\mu,v))(e' \otimes e)\}$ has a locally Lipschitz version, the local Lipschitz constant on a ball of center 0 and radius γ is less than $C(1+\gamma^\alpha)$, the constant C being uniform with respect to ξ in L^2 balls.

Fourth step. From (3.34) and (3.37), we know that

$$\begin{aligned}\mathcal{W}^{n,\delta}(\mu,v) &= \int_{\mathbb{R}} \partial_v \mathcal{V}^n\big(\mu \star p^\delta, v + \delta u e\big) p(u) \mathrm{d}u \\ &= n^{(d+1)/2} \int_{\mathbb{R}\times\mathbb{R}^d} \partial_\mu U\big(\mu \star p^\delta \star \mathcal{N}_d(0, \tfrac{1}{n} I_d), w + \delta u e\big) p(u) \rho'(n^{1/2}(v-w)) \mathrm{d}u \mathrm{d}w.\end{aligned}$$

Since $\mu \star \mathcal{N}_d(0,(1/n)I_d$ has full support, we know from the warm-up that $\partial_\mu U(\mu \star p^\delta \star \mathcal{N}_d(0,(1/n)I_d), \cdot)$ converges towards $\partial_\mu U(\mu \star \mathcal{N}_d(0,(1/n)I_d), \cdot)$ as δ tends to 0, uniformly on compact subsets. We deduce that, as δ tends to 0, $\mathcal{W}^{n,\delta}(\mu,v)$ converges to

$$\begin{aligned}\mathcal{W}^n(\mu,v) &= n^{(d+1)/2} \int_{\mathbb{R}^d} \partial_\mu U\big(\mu \star \mathcal{N}(0, \tfrac{1}{n} I_d), w\big) \rho'\big(n^{1/2}(v-w)\big) \mathrm{d}w \\ &= \partial_v \left(n^{d/2} \int_{\mathbb{R}} \partial_\mu U\big(\mu \star \mathcal{N}(0, \tfrac{1}{n} I_d), w\big) \rho(n^{1/2}(v-w)) \mathrm{d}w \right) = \partial_v \mathcal{V}^n(\mu,v).\end{aligned}$$

Therefore, we deduce that the mappings $(\mathbb{R}^d \ni v \mapsto \mathrm{Tr}\{(\partial_v \mathcal{V}^n(\mu,v))(e' \otimes e)\})_{n\geqslant 1}$ are locally Lipschitz continuous, uniformly in μ (the local Lipschitz constant being at most of α-polynomial growth). Since $\partial_v \mathcal{V}^n(\mu,v)$ is independent of e, this says that the mappings $(\mathbb{R}^d \ni v \mapsto \partial_v \mathcal{V}^n(\mu,v))_{n\geqslant 1}$ are locally Lipschitz continuous, uniformly with respect to μ in sets of probability measures with uniformly bounded second-order moments.

By (3.38) and *(i)* in the statement of Theorem 3.6,

$$\sup_{n\geqslant 1, \delta\in[0,1]} \mathbb{E}\big[\big|\mathrm{Tr}\big\{\big(\mathcal{W}^{n,\delta}(\mu,\xi)\big)(e' \otimes e)\big\}\big|^2\big] \leqslant C, \tag{3.40}$$

for a possibly new value of C. Letting δ tend to 0, we deduce from Fatou's lemma that $\sup_{n\geqslant 1} \mathbb{E}[|\mathrm{Tr}\{(\partial_v \mathcal{V}^n(\mu,\xi))(e' \otimes e)\}|^2] \leqslant C$ and thus that $\sup_{n\geqslant 1} \mathbb{E}[|\partial_v \mathcal{V}^n(\mu,\xi)|^2] \leqslant C$, which implies that, by local Lipschitz property of $\partial_v \mathcal{V}^n(\mu,\cdot)$ (the local Lipschitz constant being at most of α-polynomial growth),

$$\forall n \geqslant 1, \qquad \inf_{|v|\leqslant 2|\xi|_2} |\partial_v \mathcal{V}^n(\mu,v)| \leqslant C, \tag{3.41}$$

where we used the same argument as in (3.33). This says that the sequence of mappings $(\mathbb{R}^d \ni v \mapsto \partial_v \mathcal{V}^n(\mu,v))_{n\geqslant 1}$ is relatively compact for the topology of uniform convergence. By the warm-up, the sequence of functions $(\mathbb{R}^d \ni v \mapsto (\mathcal{V}^n(\mu,v), \partial_v \mathcal{V}^n(\mu,v)))_{n\geqslant 1}$ is relatively compact for the topology of uniform convergence. As any limit of the sequence $(\mathbb{R}^d \ni v \mapsto \mathcal{V}^n(\mu,v))_{n\geqslant 1}$ provides a version of $\partial_\mu U(\mu)$, we deduce that there exists a version of $\partial_\mu U(\mu) : \mathbb{R}^d \ni v \mapsto \partial_\mu U(\mu)(v)$ which is continuously differentiable with respect to v. By (3.40), we deduce that, for any $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ and any $\xi \in L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^d)$ with μ as distribution, $\mathbb{E}[|\partial_v[\partial_\mu U(\mu)](\xi)|^2] \leqslant C$, for a constant C independent of μ. Moreover, passing to the limit in (3.38) (first on δ and then on n), we get

$$\partial^2_{\mathrm{sign}(Z')e, \mathrm{sign}(Z')\chi} \mathcal{U}(\xi) = \mathbb{E}\big[\mathrm{Tr}\big\{\big(\partial_v[\partial_\mu U(\mu)](\xi)\big)\big(\chi \otimes e\big)\big\}\big]. \tag{3.42}$$

Combining the above identity and point *(i)* in the statement of Theorem 3.6, we recover the fact that $\int_{\mathbb{R}^d} |\partial_v[\partial_\mu U(\mu)](v)|^2 d\mu(v) \leqslant C$, for a constant C independent of μ, which is a required condition for applying the chain rule.

Last step. We have just found, for any $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, a version of the mapping $\partial_\mu U(\mu)$ that is differentiable in the variable v, with $\partial_v[\partial_\mu U(\mu)]$ denoting its derivative. In order to complete the proof, it remains to prove that the resulting mapping $\mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \ni (\mu, v) \mapsto \partial_v[\partial_\mu U(\mu)](v)$ is continuous in the joint variable (μ, v) at any point $v \in \mathrm{Supp}(\mu)$. We already know that it is locally Lipschitz continuous with respect to v, the local Lipschitz constant being at most of α-polynomial growth, uniformly in μ in sets of probability measures with uniformly bounded second-order moments. For a sequence $(\mu^n)_{n\geqslant 1}$ in $\mathcal{P}_2(\mathbb{R}^d)$ converging for the 2-Wasserstein distance to some μ, we deduce from the local Lipschitz property and by the same argument as in (3.41) that the sequence of functions $(\mathbb{R} \ni v \mapsto \partial_v[\partial_\mu U(\mu^n)](v))_{n\geqslant 1}$ is relatively compact for the topology of uniform convergence on compact subsets. By means of the bound $\sup_{n\geqslant 1} \mathbb{E}[|\partial_v[\partial_\mu U(\mu^n)](\xi^n)|^2] \leqslant C$, with $\xi^n \sim \mu^n$, it is quite easy to pass to the limit in the right-hand side of (3.42). By *(ii)* in the statement of the theorem, we can also pass to the limit in the left-hand side. Equation (3.42) then permits to identify any limit with $\partial_v[\partial_\mu U(\mu)]$ on the support of μ. Since the mappings $(\partial_v[\partial_\mu U(\mu^n)])_{n\geqslant 1}$ are uniformly continuous on compact subsets, we deduce that, for an additional sequence $(v^n)_{n\geqslant 1}$, with values in $\mathbb{R}^d$, that converges to some $v \in \mathrm{Supp}(\mu)$, the sequence $(\partial_v[\partial_\mu U(\mu^n)](v^n))_{n\geqslant 1}$ converges, up to a subsequence, to $\partial_v[\partial_\mu U(\mu)](v)$. Now, by relative compactness of the sequence $(\mathbb{R} \ni v \mapsto \partial_v[\partial_\mu U(\mu^n)](v))_{n\geqslant 1}$, the sequence $(\partial_v[\partial_\mu U(\mu^n)](v^n))_{n\geqslant 1}$ is bounded. By a standard compactness argument, the sequence $(\partial_v[\partial_\mu U(\mu^n)](v^n))_{n\geqslant 1}$ must be convergent with $\partial_v[\partial_\mu U(\mu)](v)$ as limit. □

3.5. **Proof of Theorem 2.8.** In order to prove Theorem 2.8, we first need an extension of the chain rule to functions that depend on time, space and measure:

Proposition 3.9. *Consider an Itô process $(X_t)_{t\in[0,T]}$ driven by $(b_t, \sigma_t)_{t\in[0,T]}$ satisfying* (3.5) *and a function $V : [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^m$ belonging to $\bigcup_{\beta\geqslant 0} \mathcal{D}_\beta$, see Definition 2.6. Then, $\mathbb{P}$ almost surely, for any $t \in [0,T]$,*

$$
\begin{aligned}
&V\big(t, X_t, [X_t]\big) - V\big(0, X_0, [X_0]\big) \\
&= \int_0^t \Big(\partial_t V(r, X_r, [X_r]) + \partial_x V(r, X_r, [X_r]) b_r + \widehat{\mathbb{E}}\big[\partial_\mu V\big(r, X_r, [X_r]\big)\big(\langle X_r \rangle\big)\langle b_r \rangle\big]\Big) \mathrm{d}r \\
&\quad + \frac{1}{2} \int_0^t \Big(\mathrm{Tr}\big[\partial_{xx}^2 V\big(r, X_r, [X_r]\big)\big(\sigma_r \sigma_r^\dagger\big)\big] \\
&\qquad\qquad + \widehat{\mathbb{E}}\big[\mathrm{Tr}\big[\partial_v\big[\partial_\mu V\big]\big(r, X_r, [X_r]\big)\big(\langle X_r \rangle\big)\langle \sigma_r (\sigma_r)^\dagger \rangle\big)\big]\big]\Big) \mathrm{d}r \\
&\quad + \int_0^t \partial_x V\big(r, X_r, [X_r]\big) \sigma_r \mathrm{d}W_r.
\end{aligned}
$$

Remark 3.10. *In comparison with Theorem 3.3, the formula is stated here in terms of the expectation $\hat{\mathbb{E}}$ on the auxiliary probability space $(\hat{\Omega}, \hat{\mathcal{A}}, \hat{\mathbb{P}})$. The goal is to distinguish the random variables X_r, b_r and $\sigma_r \sigma_r^\dagger$, observed on the "physical space" $(\Omega, \mathcal{A}, \mathbb{P})$, from the random variables $\langle X_r \rangle$, $\langle b_r \rangle$ and $\langle \sigma_r \sigma_r^\dagger \rangle$ that are used to express the derivatives in the direction μ.*

Proof. The proof is similar to that outlined in Subsection 2.3. As in the proof of Theorem 3.3, we can assume that the processes $(b_t)_{t\in[0,T]}$ and $(\sigma_t)_{t\in[0,T]}$ are bounded.

Given $s \in [0,T]$ and $h > 0$ such that $s+h \in [0,T]$, we then expand

$$
\begin{aligned}
&V\big(s+h, X_{s+h}, [X_{s+h}]\big) - V\big(s, X_s, [X_s]\big) \\
&\quad = V\big(s+h, X_{s+h}, [X_{s+h}]\big) - V\big(s+h, X_{s+h}, [X_s]\big) \\
&\qquad + V\big(s+h, X_{s+h}, [X_s]\big) - V\big(s, X_s, [X_s]\big).
\end{aligned}
\tag{3.43}
$$

Thanks to the regularity assumptions in ($\boldsymbol{H1}$) and ($\boldsymbol{H2}$), we notice that, almost surely, the map $\mathcal{P}_2(\mathbb{R}^d) \ni \mu \mapsto V(s+h, X_{s+h}, \mu)$ satisfies the assumption of Theorem 3.6. Therefore, we can write

$$
\begin{aligned}
&V\big(s+h, X_{s+h}, [X_{s+h}]\big) - V\big(s+h, X_{s+h}, [X_s]\big) \\
&= \int_s^{s+h} \hat{\mathbb{E}}\big[\partial_\mu V(s+h, X_{s+h}, [X_r])(\langle X_r\rangle)\langle b_r\rangle\big]\mathrm{d}r \\
&\quad + \frac{1}{2}\int_s^{s+h} \hat{\mathbb{E}}\big[\mathrm{Tr}\big(\partial_v\big[\partial_\mu V(s+h, X_{s+h}, [X_r])\big](\langle X_r\rangle)\langle \sigma_r\sigma_r^\dagger\rangle\big)\big]\mathrm{d}r.
\end{aligned}
$$

Recall that any versions of $\mathbb{R}^d \ni v \mapsto \partial_\mu V(s+h, X_{s+h}, \mu)(v)$ and $\mathbb{R}^d \ni v \mapsto \partial_v[\partial_\mu V(s+h, X_{s+h}, \mu)](v)$ may be used in the writing of the above formula. In particular, we can choose the versions of $\partial_\mu V$ and $\partial_v[\partial_\mu V]$ that satisfy ($\boldsymbol{H1}$) and ($\boldsymbol{H2}$). By the assumption we have on the regularity of $\partial_\mu V$ and $\partial_v[\partial_\mu V]$ in the variable x, see ($\boldsymbol{H1}$) and ($\boldsymbol{H2}$), and in the variable t, see *(ii)* in Definition (2.6), we deduce that there exists a sequence of non-negative random variables $(\varepsilon_h)_{h>0}$ that tends to 0 in probability with h, such that

$$
\begin{aligned}
&\bigg| V\big(s+h, X_{s+h}, [X_{s+h}]\big) - V\big(s+h, X_{s+h}, [X_s]\big) \\
&- \int_s^{s+h} \hat{\mathbb{E}}\big[\partial_\mu V(r, X_r, [X_r])(\langle X_r\rangle)\langle b_r\rangle\big]\mathrm{d}r \\
&- \frac{1}{2}\int_s^{s+h} \hat{\mathbb{E}}\big[\mathrm{Tr}\big(\partial_v\big[\partial_\mu V(r, X_r, [X_r])\big](\langle X_r\rangle)\langle \sigma_r\sigma_r^\dagger\rangle\big)\big]\mathrm{d}r \bigg| \leqslant h\varepsilon_h.
\end{aligned}
$$

It must be noticed that the family $(\varepsilon_h)_{h>0}$ may be chosen independently of $s \in [0,T]$. The reason is that, thanks to ($\boldsymbol{H1}$) and ($\boldsymbol{H2}$), for any continuous $\mathbb{R}^d$-valued path $(x_t)_{t\in[0,T]}$,

$$
\lim_{\delta \to 0} \sup_{r,s\in[0,T]:|r-s|\leqslant\delta} \hat{\mathbb{E}}\Big[\big|\partial_\mu V\big(s, x_s, [X_r]\big)\big(\langle X_r\rangle\big) - \partial_\mu V\big(r, x_r, [X_r]\big)\big(\langle X_r\rangle\big)\big|\Big] = 0,
$$

with a similar result when $\partial_\mu V$ is replaced by $\partial_v[\partial_\mu V]$. By means of the standard Itô formula, the second difference on the right hand side of (3.43) can be handled in a similar way, yielding a similar bound (for the relevant expansion) on an interval of length h. We then easily complete the proof by dividing any interval $[0,t] \subset [0,T]$ into pieces of length less than h, applying the above bound on each piece of the subdivision and then by letting h tend to 0. $\square$

We now turn to

Proof. [Proof of Theorem 2.8.] The proof is a variant of the four-step-scheme used in [28]. We divide it into two steps.

First step. Given a solution U to (2.12) in the class $\bigcup_{\beta\geqslant 0}\mathcal{D}_\beta$ and given $t \in [0,T]$ and $\xi \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)$, we build a solution to (2.3).

Letting

$$
\partial_x^\sigma U(t,x,\mu) = \partial_x U(t,x,\mu)\sigma\big(x, U(t,x,\mu), [(\xi, U(t,\xi,\mu))]\big),
$$

with $\xi \sim \mu$, we indeed claim that the McKean-Vlasov SDE

$$\begin{aligned} dX_s &= b\big(X_s, U(s, X_s, [X_s]), \partial_x^\sigma U(s, X_s, [X_s]), \big[X_s, U(s, X_s, [X_s])\big]\big)\mathrm{d}s \\ &\quad + \sigma\big(X_s, U(s, X_s, [X_s]), \big[X_s, U(s, X_s, [X_s])\big]\big)\mathrm{d}W_s, \quad X_t = \xi, \end{aligned} \tag{3.44}$$

has a solution (the idea that we shall exploit in the proof being that the triplet process $(X_s, U(s, X_s, [X_s]), \partial_x^\sigma U(s, X_s, [X_s]))_{s\in[t,T]}$ solves the system (2.3)). The proof is not completely straightforward as $\partial_x^\sigma U$ is not Lipschitz continuous in the direction of the measure (see $(\boldsymbol{H1})$). In particular, we cannot apply Sznitman's result in [33], which relies on a contraction argument. Instead, we make use of Schauder's theorem, applying the same strategy as in [6].

The argument is as follows. Let $\mathcal{C}([t,T], \mathcal{P}_2(\mathbb{R}^d))$ be the family of marginal measures $(\mu_r)_{r\in[t,T]}$ with finite second-order moments such that the mapping $[t,T] \ni r \mapsto \mu_r \in \mathcal{P}_2(\mathbb{R}^d)$ is continuous. For $(\mu_r)_{r\in[t,T]} \in \mathcal{C}([t,T], \mathcal{P}_2(\mathbb{R}^d))$, we may solve

$$\begin{aligned} dX_s &= b\big(X_s, U(s, X_s, \mu_s), \partial_x^\sigma U(s, X_s, \mu_s), \big[X_s, U(s, X_s, \mu_s)\big]\big)\mathrm{d}s \\ &\quad + \sigma\big(X_s, U(s, X_s, \mu_s), \big[X_s, U(s, X_s, \mu_s)\big]\big)\mathrm{d}W_s, \quad X_t = \xi. \end{aligned}$$

By Sznitman's result, the above equation admits a unique solution, which we will denote by $(X_s^{(\mu_r)_{r\in[t,T]}})_{s\in[t,T]}$. We then consider the mapping

$$\Phi : \big(\mu_r\big)_{r\in[t,T]} \mapsto \big(\big[X_s^{(\mu_r)_{r\in[t,T]}}\big]\big)_{s\in[t,T]},$$

which maps $\mathcal{C}([t,T], \mathcal{P}_2(\mathbb{R}^d))$ into itself. By standard stability arguments, it is quite clear that the mapping Φ is continuous, $\mathcal{C}([t,T], \mathcal{P}_2(\mathbb{R}^d))$ being endowed with the supremum distance $d((\mu_r)_{r\in[t,T]}, (\tilde{\mu}_r)_{r\in[t,T]}) := \sup_{r\in[t,T]} W_2(\mu_r, \tilde{\mu}_r)$. Moreover, by boundedness of $\partial_x U$ and σ and by the Lipschitz property of U, we can find a constant C (independent of the input $(\mu_r)_{r\in[t,T]}$) such that, for any $S \in [t,T]$

$$\mathbb{E}_t\big[\sup_{s\in[t,S]} |X_s^{(\mu_r)_{r\in[t,T]}}|^4\big]^{1/2} \leqslant C\bigg(1 + |\xi|^2 + \int_t^S \int_{\mathbb{R}^d} |x|^2 d\mu_s(x)\mathrm{d}s\bigg).$$

This proves that, when

$$\forall s \in [t,T], \quad \int_{\mathbb{R}^d} |x|^2 d\mu_s(x) \leqslant C(1 + \|\xi\|_2^2)\exp\big(C(s-t)\big), \tag{3.45}$$

the same holds for $\mathbb{E}[|X_s^{(\mu_r)_{r\in[t,T]}}|^2]$ for all $s \in [t,T]$. In such a case, we also have

$$\mathbb{E}_t\big[\sup_{s\in[t,S]} |X_s^{(\mu_r)_{r\in[t,T]}}|^4\big]^{1/2} \leqslant C\big(1 + |\xi|^2\big) + C\big(1 + \|\xi\|_2^2\big)\exp(CT).$$

so that, for any event $A \in \mathcal{A}$ and any real $R > 0$, Cauchy-Schwarz inequality yields

$$\begin{aligned} \mathbb{E}\big[\mathbf{1}_A \sup_{s\in[t,S]} |X_s^{(\mu_r)_{r\in[t,T]}}|^2\big] &\leqslant C\mathbb{E}\Big[\mathbb{E}_t[\mathbf{1}_A]^{1/2}\Big(\big(1 + |\xi|^2\big) + \big(1 + \|\xi\|_2^2\big)\exp(CT)\Big)\Big], \\ &\leqslant C\Big(\big(1 + R^2\big) + \big(1 + \|\xi\|_2^2\big)\exp(CT)\Big)\mathbb{P}(A)^{1/2} \\ &\qquad + C\mathbb{E}\Big[\mathbf{1}_{\{|\xi|\geqslant R\}}\Big(\big(1 + |\xi|^2\big) + \big(1 + \|\xi\|_2^2\big)\exp(CT)\Big)\Big]. \end{aligned}$$

In particular, choosing $A = \{|X_s^{(\mu_r)_{r\in[t,T]}}| > R^4\}$ for some $s \in [t,T]$, applying Markov inequality and using the fact that (3.45) is also satisfied by $\mathbb{E}[|X_s^{(\mu_r)_{r\in[t,T]}}|^2]$, we get that

$$\begin{aligned}
&\sup_{s\in[t,S]} \mathbb{E}\Big[\mathbf{1}_{\{|X_s^{(\mu_r)_{r\in[t,T]}}|>R^4\}}|X_s^{(\mu_r)_{r\in[t,T]}}|^2\Big] \\
&\leqslant C^{3/2}R^{-4}\Big(\big(1+R^2\big)+\big(1+\|\xi\|_2^2\big)\exp(CT)\Big)\Big(\big(1+\|\xi\|_2^2\big)\exp(CT)\Big)^{1/2} \\
&\quad + C\mathbb{E}\Big[\mathbf{1}_{\{|\xi|\geqslant R\}}\Big(\big(1+|\xi|^2\big)+\big(1+\|\xi\|_2^2\big)\exp(CT)\Big)\Big].
\end{aligned} \tag{3.46}$$

For a given $s \in [t,T]$, we now denote by $\mathcal{K}_s$ the subset of $\mathcal{P}_2(\mathbb{R}^d)$ made of probability measures such that $\int_{\mathbb{R}^d}|x|^2\mathrm{d}\mu(x)$ is less than the right-hand side in (3.45) and $\int_{\{|x|>R^4\}}|x|^2\mathrm{d}\mu(x)$ is less than the right-hand side in (3.46) for any $R>0$. It is easy to checked that $\mathcal{K}_s$ is a compact subset of $\mathcal{P}_2(\mathbb{R}^d)$. Indeed, any sequence in $\mathcal{K}_s$ is tight and admits a subsequence that converges in the weak sense. Using (3.46), the subsequence is square-uniformly integrable. Using Skorohod's representation theorem, we deduce that the sequence converges in the W_2-Wasserstein sense. By Fatou's lemma, $\mathcal{K}_s$ is closed. Below, we let $\mathcal{K} = \{(\mu_r)_{r\in[t,T]} \in \mathcal{C}([t,T],\mathcal{P}_2(\mathbb{R}^d)) : \forall r \in [t,T],\ \mu_r \in \mathcal{K}_r\}$.

Notice now that, under (3.45), we have, for all $s,s' \in [t,T]$,

$$\mathbb{E}\big[|X_{s'}^{(\mu_r)_{r\in[t,T]}} - X_s^{(\mu_r)_{r\in[t,T]}}|^2\big] \leqslant C'|s'-s|,$$

for a constant C' depending upon C, $\|\xi\|_2$ and T. This says that the map is $[t,T] \ni s \mapsto X_s^{(\mu_r)_{r\in[t,T]}} \in L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d)$, is continuous, uniformly in $(\mu_r)_{r\in[t,T]} \in \mathcal{K}$. Using the Arzelà-Ascoli theorem, we deduce that the restriction of Φ to $\mathcal{K}$ has a relatively compact range. Since $\mathcal{K}$ is closed and convex, Schauder's theorem applies and (3.44) has a solution.

Second step. We consider another solution U' to (2.12) in the class $\bigcup_{\beta\geqslant 0}\mathcal{D}_\beta$. With X a solution of (3.44), we can apply the chain rule to both $(Y_s = U(s,X_s,[X_s]))_{s\in[t,T]}$ and $(Y_s' = U'(s,X_s,[X_s]))_{s\in[t,T]}$, the drift of X being square-integrable and σ being bounded. Letting $(Z_s = \partial_x^\sigma U(s,X_s,[X_s]))_{s\in[t,T]}$, $(Z_s' = \partial_x U'(s,X_s,[X_s])\sigma(X_s,Y_s',[X_s,Y_s']))_{s\in[t,T]}$, $(\theta_s = (X_s,Y_s,Z_s))_{s\in[t,T]}$, $(\theta_s' = (X_s,Y_s',Z_s'))_{s\in[t,T]}$, $(\theta_s^{(0)} = (X_s,Y_s))_{s\in[t,T]}$ and $(\theta_s^{(0)\prime} = (X_s,Y_s'))_{s\in[t,T]}$, we deduce from the master PDE (2.12) that

$$\begin{aligned}
Y_s - Y_s' &= \int_s^T \big(f(\theta_r,[\theta_r^{(0)}]) - f(\theta_r,[\theta_r^{(0)\prime}])\big)\mathrm{d}r \\
&\quad + \int_s^T \Big\{\partial_x U'(r,X_r,[X_r])\Big(b\big(\theta_r,[\theta_r^{(0)}]\big) - b\big(\theta_r',[\theta_r^{(0)\prime}]\big)\Big) \\
&\qquad + \hat{\mathbb{E}}\Big[\partial_\mu U'(r,X_r,[X_r])(\langle X_r\rangle)\Big(b\big(\langle\theta_r\rangle,[\theta_r^{(0)}]\big) - b\big(\langle\theta_r'\rangle,[\theta_r^{(0)\prime}]\big)\Big)\Big]\Big\}\mathrm{d}r \\
&\quad + \frac{1}{2}\int_s^T \Big\{\mathrm{Tr}\Big[\partial_{xx}^2 U'(r,X_r,[X_r])\Big(\big(\sigma\sigma^\dagger\big)\big(\theta_r,[\theta_r^{(0)}]\big) - \big(\sigma\sigma^\dagger\big)\big(\theta_r',[\theta_r^{(0)\prime}]\big)\Big)\Big] \\
&\qquad + \hat{\mathbb{E}}\Big[\mathrm{Tr}\Big[\partial_v\big[\partial_\mu U'\big](r,X_r,[X_r])\big(\langle X_r\rangle\big) \\
&\qquad\qquad \times \Big(\big(\sigma\sigma^\dagger\big)\big(\langle\theta_r^{(0)}\rangle,[\theta_r^{(0)}]\big) - \big(\sigma\sigma^\dagger\big)\big(\langle\theta_r^{(0)\prime}\rangle,[\theta_r^{(0)\prime}]\big)\Big)\Big]\Big]\Big\}\mathrm{d}r \\
&\quad - \int_s^T \big(Z_r - \partial_x U'(r,X_r,[X_r])\sigma(\theta_r^{(0)\prime},[\theta_r^{(0)\prime}])\big)\mathrm{d}W_r.
\end{aligned}$$

By using Assumptions $(\boldsymbol{H0})$(i) and $(\boldsymbol{H\sigma})$ on the coefficients and Assumptions $(\boldsymbol{H1})$ and $(\boldsymbol{H2})$ that enter in the definition of $\mathcal{D}_\beta$, we deduce from stability estimates for BSDEs, in the spirit of [31], that

$$\begin{aligned}
&\mathbb{E}\big[|Y_s - Y_s'|^2\big] + \mathbb{E}\int_s^T |Z_r - \partial_x U'(r, X_r, [X_r])\sigma(\theta_r^{(0)\prime}, [\theta_r^{(0)\prime}])|^2 \mathrm{d}r \\
&\leqslant C\mathbb{E}\int_s^T |Y_r - Y_r'|^2 \mathrm{d}r + \frac{1}{2}\mathbb{E}\int_s^T |Z_r - Z_r'|^2 \mathrm{d}r,
\end{aligned}$$

from which we get, by the boundedness of $\partial_x U'$, that

$$\mathbb{E}\big[|Y_s - Y_s'|^2\big] + \mathbb{E}\int_s^T |Z_r - Z_r'|^2 \mathrm{d}r \leqslant C\mathbb{E}\int_s^T |Y_r - Y_r'|^2 \mathrm{d}r + \frac{1}{2}\mathbb{E}\int_s^T |Z_r - Z_r'|^2 \mathrm{d}r.$$

We deduce that $Y_s = Y_s'$ for any $s \in [t,T]$, that is $U(t,\xi,[\xi]) = U'(t,\xi,[\xi])$ almost surely. When $[\xi]$ has full support over $\mathbb{R}^d$, continuity of U and U' yield $U(t,x,[\xi]) = U'(t,x,[\xi])$ for all $x \in \mathbb{R}^d$. When the support of $[\xi]$ is strictly included in $\mathbb{R}^d$, we can approximate ξ by a sequence $(\xi_n)_{n\geqslant 1}$ that converges to ξ in L^2 such that, for each $n \geqslant 1$, ξ_n has full support over $\mathbb{R}^d$. Passing to the limit in the relationship $U(t,x,[\xi_n]) = U'(t,x,[\xi_n])$, we complete the proof. □

4. Smoothness for small time horizons – proof of Theorem 2.7

The purpose of this section is to prove that the mapping U given in Definition 2.1 satisfies the required smoothness property for applying the chain rule. Generally speaking, this is proved by showing the smoothness of the corresponding stochastic flows defined in (2.3) and (2.4). More precisely, we prove that the stochastic flows defined in (2.3) and (2.4) are differentiable with respect to ξ, x and μ in the sense discussed in Section 3. This is not a straightforward generalization of the method used by Pardoux and Peng in [31] in order to prove the smoothness of the flow generated by the solution of a classical backward stochastic differential equation as we are facing here two additional difficulties: First, the initial conditions live in non-Euclidean spaces, which requires some special care; second, the backward equation is fully coupled to the forward equation. In order to handle the full coupling between the forward and backward components, we shall assume that T is small. In particular, throughout the whole section, T is less than 1. In the following section, we shall give sufficient conditions for extending the results from small to arbitrary large time horizons.

Below, Assumption $(\boldsymbol{H2})$ is in force. We shall use quite intensively the following lemma, which is an adaptation of the stability estimates in [11]:

Lemma 4.1. *For any $p \geqslant 1$, there exist two constants $c_p := c_p(L) > 0$ and $C_p \geqslant 0$ such that, for $T \leqslant c_p$, for any $t \in [0,T]$, $x \in \mathbb{R}^d$ and $\xi \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)$,*

$$\begin{aligned}
&\|X^{t,\xi}\|_{\mathcal{S}^p,t} + \|Y^{t,\xi}\|_{\mathcal{S}^p,t} + \|Z^{t,\xi}\|_{\mathcal{H}^p,t} \leqslant C_p\big(1 + |\xi| + \|\xi'\|_2\big), \\
&\|X^{t,x,[\xi]}\|_{\mathcal{S}^p} + \|Y^{t,x,[\xi]}\|_{\mathcal{S}^p} + \|Z^{t,x,[\xi]}\|_{\mathcal{H}^p} \leqslant C_p\big(1 + |x| + \|\xi\|_2\big),
\end{aligned} \tag{4.1}$$

and, for any $x' \in \mathbb{R}^d$ and $\xi' \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)$,

$$\begin{aligned}
&\|X^{t,\xi} - X^{t,\xi'}\|_{\mathcal{S}^p,t} + \|Y^{t,\xi} - Y^{t,\xi'}\|_{\mathcal{S}^p,t} + \|Z^{t,\xi} - Z^{t,\xi'}\|_{\mathcal{H}^p,t} \\
&\quad \leqslant C_p\big[|\xi - \xi'| + W_2\big([\xi],[\xi']\big)\big], \\
&\|X^{t,x,[\xi]} - X^{t,x',[\xi']}\|_{\mathcal{S}^p} + \|Y^{t,x,[\xi]} - Y^{t,x',[\xi']}\|_{\mathcal{S}^p} + \|Z^{t,x,\xi} - Z^{t,x',[\xi']}\|_{\mathcal{H}^p} \\
&\quad \leqslant C_p\big[|x - x'| + W_2\big([\xi],[\xi']\big)\big].
\end{aligned} \tag{4.2}$$

In the statement above, the notation $c_p := c_p(L)$ emphasizes the fact that c_p only depends on the Lipschitz constant L introduced in $(\boldsymbol{H0}) - (\boldsymbol{H1})$. The constant C_p is allowed to depend on the other parameters appearing in $(\boldsymbol{H0}) - (\boldsymbol{H2})$, but there is no need to keep track of them for our purpose.

4.1. **Stability estimate for McKean-Vlasov linear FBSDEs.** The strategy for investigating the derivatives of the solutions to (2.3) and (2.4) is standard. We identify the derivatives with the solutions of linearized systems, obtained by formal differentiation of the coefficients. For that reason, the analysis of the differentiability relies on some preliminary stability estimates for linear FBSDEs. Unfortunately, because of the McKean-Vlasov structure of the coefficients, we cannot borrow any estimate from the literature. We thus have to use a tailor-made version, which is the precise purpose of this subsection.

4.1.1. *General set-up.* Generally speaking, we are dealing with a linear FBSDE of the form

$$\begin{aligned}
\mathcal{X}_s &= \eta + \int_t^s B\big(r, \theta_r, \langle \hat{\theta}_r \rangle\big)\big(\vartheta_r, \langle \hat{\vartheta}_r^{(0)} \rangle\big) \mathrm{d}r + \int_t^s \Sigma\big(r, \theta_r^{(0)}, \langle \hat{\theta}_r^{0)} \rangle\big)\big(\vartheta_r^{(0)}, \langle \hat{\vartheta}_r^{0)} \rangle\big) \mathrm{d}W_r, \\
\mathcal{Y}_s &= G\big(X_T, \langle \hat{X}_T \rangle\big)\big(\mathcal{X}_T, \langle \hat{\mathcal{X}}_T \rangle\big) + \int_s^T F\big(r, \theta_r, \langle \hat{\theta}_r^{(0)} \rangle\big)\big(\vartheta_r, \langle \hat{\vartheta}_r^{(0)} \rangle\big) \mathrm{d}r - \int_s^T \mathcal{Z}_r \mathrm{d}W_r,
\end{aligned} \tag{4.3}$$

where η is an initial condition in $L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)$, $\theta = (X, Y, Z)$ and $\hat{\theta} = (\hat{X}, \hat{Y}, \hat{Z})$ are solutions of (2.3) or (2.4), $\vartheta = (\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ denotes the unknowns in the above equation and $\hat{\vartheta} = (\hat{\mathcal{X}}, \hat{\mathcal{Y}}, \hat{\mathcal{Z}})$ is an auxiliary process, which may be ϑ itself (in which case it is unknown). The exponent (0) denotes the restriction of the processes to the two first coordinates, as in (2.3) and (2.4). The processes X, $\hat{X}$, $\mathcal{X}$ and $\hat{\mathcal{X}}$ have the same dimension, the same being true for the processes Y, $\hat{Y}$, $\mathcal{Y}$ and $\hat{\mathcal{Y}}$ and for the processes Z, $\hat{Z}$, $\mathcal{Z}$ and $\hat{\mathcal{Z}}$. In particular, the mappings B, Σ, F and G take values in Euclidean spaces of according dimensions. The symbol $\langle \cdot \rangle$ is used to denote the copy of the underlying random variable onto the probability space $(\hat{\Omega}, \hat{\mathcal{A}}, \hat{\mathbb{P}})$. Although the role of the copy is rather vague at this stage of the paper, it indicates that the coefficients may depend in a non-Markovian way of the various stochastic processes involved. Here is an example:

Example 4.2. *As a typical example for the coefficients B, Σ, F and G, we may think of the derivatives, with respect to some parameter λ, of the original coefficients b, f, σ and g when computed along some triplet $(\theta^\lambda = (X^\lambda, Y^\lambda, Z^\lambda))$ solving (2.1). As a typical example for the parametrization by λ, we may think of the parametrization with respect to the initial condition which is applied to the entire system.*

The shape of the coefficients B, Σ, F and G can then be derived by replacing b, f, σ and g by a generic continuously differentiable Lipschitz function $h : (\mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{m \times d}) \times \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^m) \to \mathbb{R}$. Given such a generic h, we can indeed consider a process of the form

$$\Big(h\big(\theta_r^\lambda, [\theta_r^{\lambda,(0)}]\big)\Big)_{r \in [t,T]}$$

where $\mathbb{R} \ni \lambda \mapsto (\theta_r^\lambda)_{r \in [t,T]} \in \mathcal{S}^2([t,T]; \mathbb{R}^d) \times \mathcal{S}^2([t,T]; \mathbb{R}^m) \times \mathcal{H}^2([t,T]; \mathbb{R}^{m \times d})$ is differentiable with respect to λ, with (derivatives being taken in the aforementioned space)

$$\theta^\lambda_{r\,|\lambda=0} = \theta_r, \quad \frac{\mathrm{d}}{\mathrm{d}\lambda_{|\lambda=0}} \theta_r^\lambda = \vartheta_r, \quad r \in [t,T],$$

the process $(\vartheta_r)_{r\in[t,T]}$ taking its values in $\mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$ (and, for the moment, having nothing to do with the solution of (4.3)*). Then, it is easy to check that the mapping $\mathbb{R} \ni \lambda \mapsto (h(\theta_r^\lambda, [\theta_r^{\lambda,(0)}])_{r\in[t,T]} \in \mathcal{H}^2([t,T];\mathbb{R})$ is differentiable and that the derivative reads as follows*

$$H^{(1)}(r,\theta_r,\langle\theta_r^{(0)}\rangle)(\vartheta_r,\langle\vartheta_r^{(0)}\rangle) := \partial_w h(\theta_r,[\theta_r^{(0)}])\vartheta_r + \hat{\mathbb{E}}\big[\partial_\mu h(\theta_r,[\theta_r^{(0)}])(\langle\theta_r^{(0)}\rangle)\langle\vartheta_r^{(0)}\rangle\big]. \quad (4.4)$$

Of course, if h only acts on $((\theta_r^{(0)},[\theta_r^{(0)}]))_{r\in[t,T]}$ instead of $((\theta_r,[\theta_r^{(0)}]))_{r\in[t,T]}$, then differentiability holds in $\mathcal{S}^2([t,T];\mathbb{R})$.

In Example 4.2, the coefficients B, Σ, F and G are obtained by replacing h by b, σ, f and g and by computing $B^{(1)}$, $\Sigma^{(1)}$, $F^{(1)}$ and $G^{(1)}$ accordingly. Leaving Example 4.2 and going back to the general case, we apply the same procedure: In order to specify the shape of B, Σ, F and G (together with the assumptions they satisfy), it suffices to make explicit the generic form of a function H that may be B, Σ, F or G and to detail the assumptions it satisfies. Given square-integrable processes $(V_r)_{r\in[t,T]}$ and $(\hat{V}_r)_{r\in[t,T]}$, $(V_r)_{r\in[t,T]}$ possibly matching $(X_r)_{r\in[t,T]}$, $(\theta_r^{(0)})_{r\in[t,T]}$ or $(\theta_r)_{r\in[t,T]}$, and similarly for $(\hat{V}_r)_{r\in[t,T]}$, together with other square-integrable processes $(\mathcal{V}_r)_{r\in[t,T]}$ and $(\hat{\mathcal{V}}_r)_{r\in[t,T]}$, $(\mathcal{V}_r)_{r\in[t,T]}$ possibly matching $(\mathcal{X}_r)_{r\in[t,T]}$, $(\vartheta_r^{(0)})_{r\in[t,T]}$ or $(\vartheta_r)_{r\in[t,T]}$, and similarly for $(\hat{\mathcal{V}}_r)_{r\in[t,T]}$, we thus assume that $H(r,V_r,\langle\hat{V}_r^{(0)}\rangle)$ acts on $(\mathcal{V}_r,\hat{\mathcal{V}}_r^{(0)})$ in the following way:

$$H(r,V_r,\langle\hat{V}_r^{(0)}\rangle)(\mathcal{V}_r,\hat{\mathcal{V}}_r^{(0)}) = H_\ell(V_r,\langle\hat{V}_r^{(0)}\rangle)(\mathcal{V}_r,\hat{\mathcal{V}}_r^{(0)}) + H_a(r), \quad (4.5)$$

where $H_a(r)$ is square-integrable and $H_\ell(V_r,\langle\hat{V}_r^{(0)}\rangle)$ acts linearly on $(\mathcal{V}_r,\hat{\mathcal{V}}_r^{(0)})$ in the following sense

$$H_\ell(V_r,\langle\hat{V}_r^{(0)}\rangle)(\mathcal{V}_r,\hat{\mathcal{V}}_r^{(0)}) = h_\ell(V_r,\langle\hat{V}_r^{(0)}\rangle)\mathcal{V}_r + \hat{\mathbb{E}}\big[\hat{H}_\ell(V_r,\langle\hat{V}_r^{(0)}\rangle)\langle\hat{\mathcal{V}}_r^{(0)}\rangle\big]. \quad (4.6)$$

Here h_ℓ and $\hat{H}_\ell$ are maps from $\mathbb{R}^k \times L^2(\hat{\Omega},\hat{\mathcal{A}},\hat{\mathbb{P}};\mathbb{R}^l)$ into $\mathbb{R}^{l'}$ and from $\mathbb{R}^k \times L^2(\hat{\Omega},\hat{\mathcal{A}},\hat{\mathbb{P}};\mathbb{R}^l)$ into $L^2(\hat{\Omega},\hat{\mathcal{A}},\hat{\mathbb{P}};\mathbb{R}^{l''})$ respectively, for suitable k, l, l' and l''. Moreover, there exist three constants $C,K,\alpha \geqslant 0$ and a function $\Phi_\alpha : [L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^l)]^2 \to \mathbb{R}_+$, continuous at any point of the diagonal, such that, for $w,w' \in \mathbb{R}^k$ and $\hat{V}^{(0)},\hat{V}^{(0)\prime} \in L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d)$,

$$\big|h_\ell(w,\langle\hat{V}^{(0)}\rangle)\big| + \hat{\mathbb{E}}\big[|\hat{H}_\ell(w,\langle\hat{V}^{(0)}\rangle)|^2\big]^{1/2} \leqslant K, \quad (4.7)$$

$$|\hat{H}_\ell(w,\langle\hat{V}^{(0)}\rangle)| \leqslant C\big(1 + |\langle\hat{V}^{(0)}\rangle|^{\alpha+1} + \|\hat{V}^{(0)}\|_2^{\alpha+1}\big), \quad (4.8)$$

$$\begin{aligned}&\big|h_\ell(w,\langle\hat{V}^{(0)}\rangle) - h_\ell(w',\langle\hat{V}^{(0)\prime}\rangle)\big|^2 + \hat{\mathbb{E}}\big[|\hat{H}_\ell(w,\langle\hat{V}^{(0)}\rangle) - \hat{H}_\ell(w',\langle\hat{V}^{(0)\prime}\rangle)|^2\big]\\ &\quad\leqslant C\Big\{|w-w'|^2 + \Phi_\alpha^2(\hat{V}^{(0)},\hat{V}^{(0)\prime})\Big\},\end{aligned} \quad (4.9)$$

with the condition that, when $\hat{V}^{(0)} \sim \hat{V}^{(0)\prime}$,

$$\Phi_\alpha(\hat{V}^{(0)},\hat{V}^{(0)\prime}) \leqslant C\mathbb{E}\big[\big(1 + |\hat{V}^{(0)}|^{2\alpha} + |\hat{V}^{(0)\prime}|^{2\alpha} + \|\hat{V}^{(0)}\|_2^{2\alpha}\big)|\hat{V}^{(0)} - \hat{V}^{(0)\prime}|^2\big]^{1/2}. \quad (4.10)$$

We shall also require the additional assumption:

$$\text{For any } \hat{V}^{(0)}, \text{ the family } \big(|\hat{H}_\ell(w,\langle\hat{V}^{(0)}\rangle)|^2\big)_{w\in\mathbb{R}^k} \text{is uniformly integrable.} \quad (4.11)$$

Conditions (4.7), (4.9), (4.10) and (4.11) must be compared with (***H*1**), the constant K in (4.7) playing the role of L in (***H*1**). It is worth mentioning that the constant K has a major role in the sequel as it dictates the size of the time interval on which all the estimates derived in this section hold true.

The comparison between (4.7)–(4.8)–(4.9)–(4.10)–(4.11) and (***H*1**) may be made more explicit within the framework of Example 4.2:

Example 4.3. *(Continuing Example 4.2)*
Assumptions (4.7), (4.8), (4.9) *and* (4.11) *read in the following way when, in the decomposition* (4.4), $h_\ell(V_r, \langle \hat{V}_r^{(0)} \rangle) \equiv \partial_w h(\theta_r, [\theta_r^{(0)}])$ *and* $\hat{H}_\ell(V_r, \langle \hat{V}_r^{(0)} \rangle) \equiv \partial_\mu h(\theta_r, [\theta_r^{(0)}])$*:*

(1) Equation (4.7) *expresses the fact that* h *is Lipschitz continuous with respect to* (w, μ)*, so that the derivatives are bounded, in* L^∞ *in the direction* w *and in* L^2 *in the direction* μ*. Importantly (and as already suggested), the constant* K *corresponds to* L *in* $(\boldsymbol{H1})$*.*
(2) Equation (4.8) *expresses the fact that, for any* (w, μ)*,* $v \mapsto \partial_\mu h(w, \mu)(v)$ *admits a version that is at most of polynomial growth (in* v*) under* $(\boldsymbol{H1})$ *(see the proof right below).*
(3) Equation (4.9) *says that the derivatives in the direction* w *and in the direction of the measure are continuous (in a suitable sense). Except when* $\alpha = 0$*, derivatives may not be Lipschitz continuous in the direction of the measure, which is a crucial relaxation for our purpose.*
(4) Condition (4.11) *expresses the fact that, for* $\chi \in L^2$*, the family* $(\partial_\mu h(w, \mu)(\chi))_{w \in \mathbb{R}^k}$ *must be uniformly square-integrable.*

The existence of a version of $v \mapsto \partial_\mu h(w, \mu)(v)$ *that is at most of polynomial growth can be proved as follows. When* h *is understood as one of the coefficients* b*,* σ*,* f *or* g*, we know that, under* $(\boldsymbol{H1})$*,* $\partial_\mu h$ *(which might be identified with a Fréchet derivative) satisfies, for two random variables* χ *and* χ'*, with the same distribution* μ*,*

$$\begin{aligned} &\mathbb{E}\big[\big|\partial_\mu h(w, \mu)(\chi) - \partial_\mu h(w, \mu)(\chi')\big|^2\big]^{1/2} \\ &\quad \leqslant C\mathbb{E}\big[\big(1 + |\chi|^{2\alpha} + |\chi'|^{2\alpha} + \|\chi\|_2^{2\alpha}\big)|\chi - \chi'|^2\big]^{1/2}, \end{aligned} \tag{4.12}$$

which implies that the mapping $v \mapsto \partial_\mu h(w, \mu)(v)$ *is locally Lipschitz continuous, see Proposition 3.8. More precisely, for a random variable* χ *with* μ *as distribution,*

$$\big|\partial_\mu h(w, \mu)(v) - \partial_\mu h(w, \mu)(v')\big| \leqslant C\big(1 + |v|^\alpha + |v'|^\alpha + \|\chi\|_2^\alpha\big)|v - v'|.$$

Now, we know that,

$$\mathbb{E}\big[|\partial_\mu h(w, \mu)(\chi)|^2\big]^{1/2} \leqslant C. \tag{4.13}$$

Therefore, by the same method as in (3.33)*, we deduce that*

$$\inf_{|v| \leqslant 2\|\chi\|_2} |\partial_\mu h(w, \mu)(v)| \leqslant C,$$

which, together with local Lipschitz property, says that, for any $w \in \mathbb{R}^k$*,*

$$\begin{aligned} \big|\partial_\mu h(w, \mu)(v)\big| &\leqslant C + C\big(1 + |v|^\alpha + \|\chi\|_2^\alpha\big)\big(|v| + \|\chi\|_2\big) \\ &\leqslant C\big(1 + |v|^{\alpha+1} + \|\chi\|_2^{\alpha+1}\big), \end{aligned} \tag{4.14}$$

which completes the proof of the polynomial growth property.

Remark 4.4. *The reader may wonder about the sharpness of the bound* (4.14)*. Indeed, when specialized to the case* $\alpha = 0$ *and* h *independent of* w*,* (4.14) *provides just a linear growth bound for the derivative* $\mathbb{R}^l \ni v \mapsto \partial_\mu h(\mu)(v)$ *of a Lipschitz-continuous function* $h : \mathcal{P}_2(\mathbb{R}^l) \ni \mu \mapsto h(\mu)$ *(the Lipschitz continuity of* h *follows from* (4.13)*). This might seem rather weak and it might be tempting to expect an* L^∞ *bound instead of a linear growth bound. As shown by the example in Remark 2.5, there is no way of guaranteeing that the derivative* $\partial_\mu h$ *of the Lipschitz-continuous function* h *is bounded in* L^∞*, even when* $\alpha = 0$ *(which is the strongest case) . Boundedness of the derivative only holds in* L^2*, as is written in* (4.13)*.*

This important feature explains why the space of boundary conditions we consider in the paper is not limited to functions with derivatives that are globally Lipschitz with respect to the measure argument. Because of the gap in the growth of the derivatives, we would fail to prove that the derivatives of the solution of the master equation (or equivalently of the decoupling field of the FBSDEs (2.3)*) are also globally Lipschitz with respect to the measure argument. Due to this lack of stability, we would not be able to extend the results from short to long time horizons.*

4.1.2. *Estimate of the solution.* Part of our analysis relies on stability estimates for systems of a more general form than (4.3), namely

$$\begin{aligned}
\mathcal{X}_s &= \eta + \int_t^s B\big(r, \bar{\theta}_r, \langle \check{\theta}_r^{(0)} \rangle\big)\big(\bar{\vartheta}_r, \langle \check{\vartheta}_r^{(0)} \rangle\big) \mathrm{d}r + \int_t^s \Sigma\big(r, \bar{\theta}_r^{(0)}, \langle \check{\theta}_r^{(0)} \rangle\big)\big(\bar{\vartheta}_r^{(0)}, \langle \check{\vartheta}_r^{(0)} \rangle\big) \mathrm{d}W_r, \\
\mathcal{Y}_s &= G\big(X_T, \langle \hat{X}_T \rangle\big)\big(\mathcal{X}_T, \langle \hat{\mathcal{X}}_T \rangle\big) + \int_s^T F\big(r, \bar{\theta}_r, \langle \check{\theta}_r \rangle\big)\big(\bar{\vartheta}_r, \langle \check{\vartheta}_r^{(0)} \rangle\big) \mathrm{d}r - \int_s^T \mathcal{Z}_r \mathrm{d}W_r,
\end{aligned} \tag{4.15}$$

the difference between (4.15) and (4.3) being that the coefficients (except the terminal boundary condition) may depend on other triplets $\bar{\theta}$, $\check{\theta}$, $\bar{\vartheta}$ and $\check{\vartheta}$. We shall make use of the following definition, directly inspired from (4.7):

Definition 4.5. *Given triplets $(\theta_r = (X_r, Y_r, Z_r))_{r \in [t,T]}$ and $(\hat{\theta}_r = (\hat{X}_r, \hat{Y}_r, \hat{Z}_r))_{r \in [t,T]}$ of the same form as above, we say that a subset $\mathcal{J}$ of $L^2(\Omega \times \hat{\Omega}, \mathcal{A} \otimes \hat{\mathcal{A}}, \mathbb{P} \otimes \hat{\mathbb{P}}; \mathbb{R}_+)$ is admissible for $(\theta, \hat{\theta})$ if*

(i) for any $r \in [t,T]$, for H matching B, Σ, F or G and $(V_r, \hat{V}_r^{(0)})$ matching $(X_r, \hat{X}_r)$, $(\theta_r, \hat{\theta}_r^{(0)})$ or $(\theta_r^{(0)}, \hat{\theta}_r^{(0)})$, there exists $\Lambda \in \mathcal{J}$ such that $\hat{\mathbb{E}}[|\hat{H}_\ell(r, V_r, \langle \hat{V}_r^{(0)} \rangle)|^2]^{1/2} \leqslant \Lambda$;

(ii) any Λ in $\mathcal{J}$ satisfies $\mathbb{P}(\hat{\mathbb{E}}(\Lambda^2)^{1/2} \leqslant K) = 1$.

Notations. Throughout §4.1.2, $\mathcal{J}$ is an admissible class for both $(\theta, \hat{\theta})$ and $(\bar{\theta}, \check{\theta})$. For a real $\gamma \geqslant T$, an integer $p \geqslant 1$, a real $C > 0$, a triplet $\vartheta = (\mathcal{X}_s, \mathcal{Y}_s, \mathcal{Z}_s)_{s \in [t,T]}$ with values in $\mathcal{S}^2([t,T]; \mathbb{R}^d) \times \mathcal{S}^2([t,T]; \mathbb{R}^m) \times \mathcal{H}^2([t,T]; \mathbb{R}^{m \times d})$ and a pair of random variables (X, χ) with values in a Euclidean space, we let

$$\begin{aligned}
&\mathcal{M}_{\mathbb{M}}^p(\vartheta) := \mathbb{M}\Bigg[\sup_{s \in [t,T]} \big(|\mathcal{X}_s|^p + \gamma^{1/2} |\mathcal{Y}_s|^p\big) + \gamma^{1/2} \bigg(\int_t^T |\mathcal{Z}_s|^2 \mathrm{d}s\bigg)^{p/2}\Bigg], \\
&\mathcal{N}_{\mathbb{M}}^{p,C}(X, \chi) \\
&\quad := \mathrm{esssup}_{\Lambda \in \mathcal{J}} \mathbb{M}\Bigg[\hat{\mathbb{E}}\bigg[\Big\{\Lambda \wedge \Big[C\big(1 + \hat{\mathbb{E}}_t\big[|\langle X \rangle|^{2\alpha+2}\big]^{1/2} + \|X\|_2^{\alpha+1}\big)\Big]\Big\} \hat{\mathbb{E}}_t\big[|\langle \chi \rangle|^2\big]^{1/2}\bigg]^p\Bigg],
\end{aligned} \tag{4.16}$$

with the convention that $\mathbb{M}$ can be $\mathbb{E}_t$ or $\mathbb{E}$ (in the latter case esssup is just a sup). Note that $\mathcal{M}_{\mathbb{M}}^p(\vartheta)$ depends on γ, t and T. We shall omit this dependence in the notation $\mathcal{M}_{\mathbb{M}}^p(\vartheta)$. With these notations, we shall write $\mathcal{M}_{\mathbb{M}}^p(\vartheta^{(0)})$ for $\mathcal{M}_{\mathbb{M}}^p(\mathcal{X}, \mathcal{Y}, 0)$. Similarly, we shall not specify the dependence upon t in the notation $\mathcal{N}_{\mathbb{M}}^{p,C}(X, \chi)$. Regarding the structure of the coefficients, B, Σ, F and G, we also let

$$\mathcal{R}_a^p := \mathbb{E}_t\Bigg[\gamma^{1/2} \big|G_a(T)\big|^p + \bigg(\int_t^T \big|(B_a, F_a)(s)\big| \mathrm{d}s\bigg)^p + \bigg(\int_t^T \big|\Sigma_a(s)\big|^2 \mathrm{d}s\bigg)^{p/2}\Bigg]. \tag{4.17}$$

From Cauchy-Schwarz' inequality and *(ii)* in Definition 4.5, we get that:

Lemma 4.6. *For any pair (X, χ) and any $p \geqslant 1$, $\mathcal{N}_{\mathbb{M}}^{p,C}(X, \chi) \leqslant K^p \|\chi\|_2^p$.*

We deduce that

Lemma 4.7. *For any $p \geqslant 1$, there exist two constants $\Gamma_p := \Gamma_p(K) \geqslant 1$ and $C > 0$ (C independent of p), such that, for $T \leqslant \gamma \leqslant 1/\Gamma_p$ and for any solution ϑ to a system of the same type as* (4.15)*, it holds*

$$\begin{aligned}\mathcal{M}^{2p}_{\mathbb{E}_t}(\vartheta) \leqslant \Gamma_p\bigg[&|\eta|^{2p} + \gamma^{1/2}\mathcal{M}^{2p}_{\mathbb{E}_t}(\bar{\vartheta}) + \mathcal{R}_a^{2p} \\ &+ \gamma^{1/2}\Big\{\mathcal{N}^{2p,C}_{\mathbb{E}_t}\big(\hat{X}_T, \hat{\mathcal{X}}_T\big) + \sup_{s\in[t,T]} \mathcal{N}^{2p,C}_{\mathbb{E}_t}\Big(\check{\theta}^{(0)}_s, \big(\mathcal{M}^2_{\mathbb{E}_t}(\check{\vartheta}^{(0)})\big)^{1/2}\Big)\Big\}\bigg].\end{aligned} \tag{4.18}$$

In particular (redefining the value of Γ_p if necessary),

$$\begin{aligned}\mathcal{M}^{2p}_{\mathbb{E}_t}(\vartheta) \leqslant \Gamma_p\Big[&\big(|\eta| + \|\eta\|_2\big)^{2p} + \mathcal{R}_a^{2p} + \mathbb{E}\big[\mathcal{R}_a^2\big]^p \\ &+ \gamma^{1/2}\Big(\mathcal{M}^{2p}_{\mathbb{E}_t}(\bar{\vartheta}) + \big[\mathcal{M}^2_{\mathbb{E}}(\hat{\vartheta}^{(0)})\big]^p + \big[\mathcal{M}^2_{\mathbb{E}}(\check{\vartheta}^{(0)})\big]^p\Big)\Big].\end{aligned} \tag{4.19}$$

Proof. We make use of standard results for solutions of an FBSDE. We can indeed start with the trivial case when the coefficients B_ℓ, Σ_ℓ and F_ℓ are null and $\hat{G}_\ell$ is also null (see (4.5) and (4.6) for the notations). Then, (4.15) reads as a system driven by the linear part g_ℓ – that appears in the decomposition (4.6) of G – plus a remainder involving B_a, Σ_a, F_a and G_a. Without any McKean-Vlasov interaction, (4.18) follows from stability estimates for standard linear FBSDEs. For instance, following Delarue [11], we get that, for any $p \geqslant 1$, we can find $\Gamma_p := \Gamma_p(K) > 0$ (the value of which is allowed to increase from line to line), such that for $\gamma \leqslant 1/\Gamma_p$, (4.18) holds, but with a simpler right-hand side just consisting of $\Gamma_p[|\eta|^{2p} + \mathcal{R}_a^{2p}]$.

In the case when B_ℓ, Σ_ℓ, F_ℓ are non-zero, we view them, when taken along the values of $(\bar{\theta}, \check{\theta}^{(0)}, \bar{\vartheta}, \check{\vartheta}^{(0)})$ as parts of B_a, Σ_a and F_a. Similarly, we can see $\hat{G}_\ell$, when taken along the values of $(X_T, \langle\hat{X}_T\rangle)$, as a part of G_a. We are thus led back to the previous case, but with a generalized version of the remainder term $\mathcal{R}_a$. In order to complete the proof, it suffices to bound this remainder in L^{2p}. The analysis of the remainder may be split into three pieces: One first term involves b_ℓ, σ_ℓ and f_ℓ; another one involves $\hat{B}_\ell$, $\hat{\Sigma}_\ell$, $\hat{F}_\ell$ and $\hat{G}_\ell$; the last one involves B_a, Σ_a, F_a and G_a and corresponds to the original $\mathcal{R}_a$. As a final bound, we get

$$\mathcal{M}^{2p}_{\mathbb{E}_t}(\vartheta) \leqslant \Gamma_p|\eta|^{2p} \tag{4.20}$$

$$+ \Gamma_p\mathbb{E}_t\bigg[\bigg(\int_t^T \big|(b_\ell, f_\ell)(\bar{\theta}_s, \langle\check{\theta}^{(0)}_s\rangle)\bar{\vartheta}_s\big|\mathrm{d}s\bigg)^{2p} + \bigg(\int_t^T \big|\sigma_\ell(\bar{\theta}^{(0)}_s, \langle\check{\theta}^{(0)}_s\rangle)\bar{\vartheta}^0_s\big|^2\mathrm{d}s\bigg)^{p}\bigg] \tag{4.21}$$

$$\begin{aligned}&+ \Gamma_p\gamma^{1/2}\bigg[\mathbb{E}_t\Big[\big|\hat{\mathbb{E}}\big[\hat{G}_\ell(X_T, \langle\hat{X}_T\rangle)\langle\hat{\mathcal{X}}_T\rangle\big]\big|^{2p}\Big] \\ &\qquad + \gamma^{p/2}\operatorname*{ess\,sup}_{s\in[t,T]} \mathbb{E}_t\Big[\big|\hat{\mathbb{E}}\big[(\hat{B}_\ell, \hat{F}_\ell, \hat{\Sigma}_\ell)(\bar{\theta}_s, \langle\check{\theta}^{(0)}_s\rangle)\langle\check{\vartheta}^{(0)}_s\rangle\big]\big|^{2p}\Big]\bigg]\end{aligned} \tag{4.22}$$

$$+ \Gamma_p\mathbb{E}_t\bigg[\gamma^{1/2}\big|G_a(T)\big|^{2p} + \bigg(\int_t^T \big|(B_a, F_a)(s)\big|\mathrm{d}s\bigg)^{2p} + \bigg(\int_t^T \big|\Sigma_a(s)\big|^2\mathrm{d}s\bigg)^{p}\bigg].$$

Observe that, in (4.22), we used the supremum to get T^p, which we bounded by $\gamma^{1/2}$ times $\gamma^{p/2}$.

Making use of (4.7), we easily handle the term (4.21). In (4.18), it gives the contribution of the form $\gamma^{1/2}\mathcal{M}^{2p}_{\mathbb{E}_t}(\bar{\vartheta})$, the $\gamma^{1/2}$ in front of $\mathcal{M}_{\mathbb{E}_t}$ and the $\gamma^{1/2}$ in the definition of $\mathcal{M}^{2p}_{\mathbb{E}_t}(\bar{\vartheta})$ arising as follows. When handling (b_ℓ, f_ℓ), we can let a power 2 enter inside the time integral. This introduces the $\mathcal{H}^2$-norm of $\bar{\mathcal{Z}}$ times an additional T less than γ, which can be split into $\gamma^{1/2}$ and $\gamma^{1/2}$.

Next we discuss the second term in (4.22). For this we use (4.7) and (4.8). With the shortened notation $H = (B, F, \Sigma)$, we can indeed either say that $\hat{H}_\ell(\bar{\theta}_s, \langle \check{\theta}_s^{(0)} \rangle)$ is bounded in L^2 or use the polynomial growth assumption. We get

$$\Big|\hat{\mathbb{E}}\Big[\hat{H}_\ell(\bar{\theta}_s, \langle \check{\theta}_s^{(0)} \rangle)\langle \check{\vartheta}_s^{(0)} \rangle\Big]\Big| \leqslant \hat{\mathbb{E}}\Big[\Big\{\Lambda_s \wedge \Big(C + C|\langle \check{\theta}_s^{(0)} \rangle|^{\alpha+1} + C\|\check{\theta}_s^{(0)}\|_2^{\alpha+1}\Big)\Big\}|\langle \check{\vartheta}_s^{(0)} \rangle|\Big],$$

where Λ_s is a shortened notation for $|\hat{H}_\ell(\bar{\theta}_s, \langle \check{\theta}_s^{(0)} \rangle)|$. Now, using the conditional Cauchy-Schwarz inequality and the obvious bound $\mathbb{E}_t[S_1 \wedge S_2] \leqslant \mathbb{E}_t[S_1] \wedge \mathbb{E}_t[S_2]$ for two non-negative random variables S_1 and S_2, we obtain:

$$\begin{aligned}&\Big|\hat{\mathbb{E}}\Big[\hat{H}_\ell(\bar{\theta}_s, \langle \check{\theta}_s^{(0)} \rangle)\langle \check{\vartheta}_s^{(0)} \rangle\Big]\Big| \\ &\quad \leqslant \hat{\mathbb{E}}\Big[\Big\{\hat{\mathbb{E}}_t\big[\Lambda_s^2\big]^{1/2} \wedge \Big(C + C\hat{\mathbb{E}}_t\big[|\langle \check{\theta}_s^{(0)} \rangle|^{2\alpha+2}\big]^{1/2} + C\|\check{\theta}_s^{(0)}\|_2^{\alpha+1}\Big)\Big\}\hat{\mathbb{E}}_t\Big[|\langle \check{\vartheta}_s^{(0)} \rangle|^2\Big]^{1/2}\Big].\end{aligned}$$

Taking the power $2p$ and the conditional expectation $\mathbb{E}_t$, we get a term which is less than $\mathcal{N}_{\mathbb{E}_t}^{2p,C}(\check{\theta}_s^{(0)}, \check{\vartheta}_s^{(0)})$. Multiplying by $\gamma^{p/2}$ (see the prefactor in (4.22)), we get that it is less than $\mathcal{N}_{\mathbb{E}_t}^{2p,C}(\check{\theta}_s^{(0)}, [|\check{\mathcal{X}}_s|^2 + \gamma^{1/2}|\check{\mathcal{Y}}_s|^2]^{1/2})$ which is less than $\mathcal{N}_{\mathbb{E}_t}^{2p,C}(\check{\theta}_s^{(0)}, (\mathcal{M}_{\mathbb{E}_t}^2(\check{\vartheta}^{(0)}))^{1/2})$. Of course, we can use the same kind of argument for the first term in (4.22) and get $\mathcal{N}_{\mathbb{E}_t}^{2p,C}(\hat{X}_T, \hat{\mathcal{X}}_T)$ as resulting bound.

The second claim follows from Lemma 4.6. $\square$

In particular, we have the following useful result for systems of the form (4.3) obtained by considering $\vartheta \equiv \bar{\vartheta}$ and $\hat{\vartheta} \equiv \check{\vartheta}$ in (4.19) and setting γ small enough.

Corollary 4.8. *For any $p \geqslant 1$, there exists a constant $\Gamma_p := \Gamma_p(K) \geqslant 1$ such that, for $T \leqslant \gamma \leqslant 1/\Gamma_p$ and for any solution ϑ to a system of the same type as* (4.3), *it holds*

$$\mathcal{M}_{\mathbb{E}_t}^{2p}(\vartheta) \leqslant \Gamma_p\Big[\big(\eta + \|\eta\|_2\big)^{2p} + \mathcal{R}_a^{2p} + \mathbb{E}\big[\mathcal{R}_a^2\big]^p + \gamma^{1/2}\big[\mathcal{M}_{\mathbb{E}}^2(\hat{\vartheta}^{(0)})\big]^p\Big]. \tag{4.23}$$

When $\vartheta \equiv \hat{\vartheta}$, we have (modifying the constant Γ_p if necessary):

$$\mathcal{M}_{\mathbb{E}_t}^{2p}(\vartheta) \leqslant \Gamma_p\Big[\big(\eta + \|\eta\|_2\big)^{2p} + \mathcal{R}_a^{2p} + \mathbb{E}\big[\mathcal{R}_a^2\big]^p\Big]. \tag{4.24}$$

Proof. Inequality (4.23) directly follows from (4.19). To get (4.24), we choose $p = 1$ and then take the expectation. For γ small enough, we obtain $\mathcal{M}_{\mathbb{E}}^2(\vartheta) \leqslant \Gamma_1(\|\eta\|_2^2 + \mathbb{E}[\mathcal{R}_a^2])$ (up to a new value for Γ_1). Plugging the bound into (4.23), we deduce that (4.24) holds. $\square$

4.1.3. *Stability estimates.* The next step is to compare two solutions of (4.15) ϑ and ϑ' driven by two different sets of inputs $(\hat{\theta}, \bar{\theta}, \check{\theta}, \hat{\vartheta}, \bar{\vartheta}, \check{\vartheta})$ and $(\hat{\theta}', \bar{\theta}', \check{\theta}', \hat{\vartheta}', \bar{\vartheta}', \check{\vartheta}')$ but with the same starting point η. Throughout §4.1.3, $\mathcal{J}$ is an admissible class for $(\theta, \hat{\theta})$ and $(\bar{\theta}, \check{\theta})$.

Given an integer $p \geqslant 1$, define similar notations to (4.10) and (4.16) (but without $\gamma^{1/2}$ in front of the terms in $\mathcal{Y}$):

$$\begin{aligned}&\Phi_\alpha\big(\hat{\vartheta}^{(0)}, \hat{\vartheta}^{(0)\prime}\big) := \sup_{s \in [t,T]}\big\{\Phi_\alpha\big(\hat{\vartheta}_s^{(0)}, \hat{\vartheta}_s^{(0)\prime}\big)\big\} \\ &\bar{\mathcal{M}}^{2p}(\vartheta, \hat{\vartheta}) := \sup_{s \in [t,T]}\Big\{\mathbb{E}_t\big[|\mathcal{X}_s|^{2p} + |\mathcal{Y}_s|^{2p}\big] + \|\hat{\mathcal{X}}_s\|_2^{2p} + \|\hat{\mathcal{Y}}_s\|_2^{2p}\Big\} + \|\mathcal{Z}\|_{\mathcal{H}^{2p},t}^{2p}, \\ &\bar{\mathcal{M}}^{2p}\big((\vartheta, \hat{\vartheta}), (\vartheta', \hat{\vartheta}')\big) := \bar{\mathcal{M}}^{2p}\big(\vartheta - \vartheta', \hat{\vartheta} - \hat{\vartheta}'\big) + \Phi_\alpha^{2p}\big(\hat{\vartheta}^{(0)}, \hat{\vartheta}^{(0)\prime}\big), \\ &\bar{\mathcal{M}}^{2p}[\![\vartheta]\!] := \bar{\mathcal{M}}^{2p}\big(\vartheta, \vartheta\big), \quad \bar{\mathcal{M}}^{2p}[\![\vartheta, \vartheta']\!] := \bar{\mathcal{M}}^{2p}\big((\vartheta, \vartheta), (\vartheta', \vartheta')\big),\end{aligned} \tag{4.25}$$

and denote by $\Delta\mathcal{R}_a^{2p}$ the quantity (recall (4.17) for the definition of $\mathcal{R}_a^{2p}$):

$$\begin{aligned}\Delta\mathcal{R}_a^{2p} := \mathbb{E}_t\Big[&\gamma^{1/2}\big|G_a(T)-G_a'(T)\big|^{2p}\\&+\Big(\int_t^T\big|(B_a-B_a',F_a-F_a')(s)\big|\mathrm{d}s\Big)^{2p}+\Big(\int_t^T\big|(\Sigma_a-\Sigma_a')(s)\big|^2\mathrm{d}s\Big)^{p}\Big].\end{aligned}\tag{4.26}$$

(The notations B_a', F_a', Σ_a' and G_a' refer to the fact, along the processes labelled with a 'prime', the remainders in the decomposition of the coefficients may be different.) Then, we have

Lemma 4.9. *For any $p \geqslant 1$, there exist three constants C (independent of p), $\Gamma_p := \Gamma_p(K) \geqslant 1$ and $C_p > 0$, such that, for $T \leqslant \gamma \leqslant 1/\Gamma_p$,*

$$\begin{aligned}\mathcal{M}_{\mathbb{E}_t}^{2p}\big(\vartheta-\vartheta'\big) \leqslant \Gamma_p\gamma^{1/2}\Big\{&\mathcal{M}_{\mathbb{E}_t}^{2p}\big(\bar\vartheta-\bar\vartheta'\big)+\mathcal{N}_{\mathbb{E}_t}^{2p,C}\big(\hat X_T,\hat{\mathcal{X}}_T-\hat{\mathcal{X}}_T'\big)\\&+\sup_{s\in[t,T]}\mathcal{N}_{\mathbb{E}_t}^{2p,C}\Big(\check\theta_s^{(0)},\big(\mathcal{M}_{\mathbb{E}_t}^2(\check\vartheta^{(0)}-\check\vartheta^{(0)\prime})\big)^{1/2}\Big)\Big\}\\&+C_p\Big[\Big(\bar{\mathcal{M}}^{4p}(\vartheta',\hat\vartheta')+\bar{\mathcal{M}}^{4p}(\bar\vartheta',\check\vartheta')\Big)^{1/2}\\&\quad\times\Big\{1\wedge\Big(\bar{\mathcal{M}}^{4p}\big((\theta,\hat\theta),(\theta',\hat\theta')\big)+\bar{\mathcal{M}}^{4p}\big((\bar\theta,\check\theta),(\bar\theta',\check\theta')\big)\Big)\Big\}^{1/2}+\Delta\mathcal{R}_a^{2p}\Big].\end{aligned}\tag{4.27}$$

In particular, choosing $p = 1$ and taking expectation, we have, for some constant $\Gamma' := \Gamma'(K)$ such that $T \leqslant \gamma \leqslant 1/\Gamma'$ and for some $C' > 0$,

$$\begin{aligned}\mathcal{M}_{\mathbb{E}}^{2}\big(\vartheta-\vartheta'\big) \leqslant \Gamma'\gamma^{1/2}\Big\{&\mathcal{M}_{\mathbb{E}}^{2}\big(\bar\vartheta-\bar\vartheta'\big)+\mathcal{M}_{\mathbb{E}}^{2}\big(\hat\vartheta^{(0)}-\hat\vartheta^{(0)\prime}\big)+\mathcal{M}_{\mathbb{E}}^{2}\big(\check\vartheta^{(0)}-\check\vartheta^{(0)\prime}\big)\Big\}\\&+C'\mathbb{E}\Big[\Big(\bar{\mathcal{M}}^{4}(\vartheta',\hat\vartheta')+\bar{\mathcal{M}}^{4}(\bar\vartheta',\check\vartheta')\Big)^{1/2}\\&\quad\times\Big\{1\wedge\Big(\bar{\mathcal{M}}^{4}\big((\theta,\hat\theta),(\theta',\hat\theta')\big)+\bar{\mathcal{M}}^{4}\big((\bar\theta,\check\theta),(\bar\theta',\check\theta')\big)\Big)\Big\}^{1/2}+\Delta\mathcal{R}_a^{2}\Big].\end{aligned}\tag{4.28}$$

Remark 4.10. *Specialized to the case when $\theta \equiv \bar\theta \equiv \theta' \equiv \bar\theta'$, $\hat\theta \equiv \check\theta \equiv \hat\theta' \equiv \check\theta'$, $\vartheta \equiv \bar\vartheta \equiv \hat\vartheta \equiv \check\vartheta$, $\vartheta' \equiv \bar\vartheta' \equiv \hat\vartheta' \equiv \check\vartheta'$ and $\Delta\mathcal{R}_a^2 \equiv 0$, Lemma 4.28 reads as a uniqueness result to* (4.3) *in short time when $\vartheta \equiv \hat\vartheta$ therein.*

Proof. We start with the proof of (4.27). We take benefit of the linearity to make the difference of the two systems of the form (4.15) satisfied by ϑ and ϑ'. The resulting system is linear in $\Delta\vartheta := \vartheta-\vartheta'$, $\Delta\hat\vartheta := \hat\vartheta-\hat\vartheta'$, $\Delta\bar\vartheta := \bar\vartheta-\bar\vartheta'$ and $\Delta\check\vartheta := \check\vartheta-\check\vartheta'$, but contains some remainders. We denote these remainders by ΔB_a, ΔF_a, $\Delta\Sigma_a$ and ΔG_a. Using the notations introduced in (4.5) and (4.6), they may be expanded as:

$$\begin{aligned}\Delta H_a(s) &= \big(h_\ell(\bar\theta_s,\langle\check\theta_s^{(0)}\rangle)-h_\ell(\bar\theta_s',\langle\check\theta_s^{(0)\prime}\rangle)\big)\bar\vartheta_s'\\&\quad+\hat{\mathbb{E}}\big[\big(\hat H_\ell(\bar\theta_s,\langle\check\theta_s^{(0)}\rangle)-\hat H_\ell(\bar\theta_s',\langle\check\theta_s^{(0)\prime}\rangle)\big)\langle\check\vartheta_s^{(0)\prime}\rangle\big]+H_a(s)-H_a'(s),\\\Delta G_a(T) &= \big(g_\ell(X_T,\langle\hat X_T\rangle)-g_\ell(X_T',\langle\hat X_T'\rangle)\big)\mathcal{X}_T'\\&\quad+\hat{\mathbb{E}}\big[\big(\hat G_\ell(X_T,\langle\hat X_T\rangle)-\hat G_\ell(X_T',\langle\hat X_T'\rangle)\big)\langle\hat{\mathcal{X}}_T'\rangle\big]+G_a(T)-G_a'(T),\end{aligned}\tag{4.29}$$

where H may stand for B, F or Σ, with a corresponding meaning for h_ℓ, $\hat H_\ell$ and H_a: h_ℓ may be b_ℓ, f_ℓ, σ_ℓ; $\hat H_\ell$ may be $\hat B_\ell$, $\hat F_\ell$, or $\hat\Sigma_\ell$; H_a may be B_a, F_a or Σ_a; and H_a' may be B_a', F_a' or Σ_a'. With these notations in hand, the terms $\Delta H_a(s)$ and $\Delta G_a(T)$ come from (recall (4.5)):

$$\begin{aligned}
&H\big(r,\bar\theta_r,\langle\check\theta_r^{(0)}\rangle\big)\big(\bar\vartheta_r,\langle\check\vartheta_r^{(0)}\rangle\big) - H\big(r,\bar\theta_r',\langle\check\theta_r^{(0)\prime}\rangle\big)\big(\bar\vartheta_r',\langle\check\vartheta_r^{(0)\prime}\rangle\big)\\
&\quad = H_\ell\big(\bar\theta_r,\langle\check\theta_r^{(0)}\rangle\big)\big(\Delta\bar\vartheta_r,\langle\Delta\check\vartheta_r^{(0)}\rangle\big) + \Delta H_a(r)\,,\\
&G\big(X_T,\langle\hat X_T\rangle\big)\big(\mathcal{X}_T,\langle\hat{\mathcal{X}}_T\rangle\big) - G\big(X_T',\langle\hat X_T'\rangle\big)\big(\mathcal{X}_T',\langle\hat{\mathcal{X}}_T'\rangle\big)\\
&\quad = G_\ell\big(X_T,\langle\hat X_T\rangle\big)\big(\Delta\mathcal{X}_T,\langle\Delta\hat{\mathcal{X}}_T\rangle\big) + \Delta G_a(T)\,.
\end{aligned} \tag{4.30}$$

We will apply Lemma 4.7. In the statement of the Lemma, we see from (4.30) that ϑ must be understood as $\Delta\vartheta$, $\bar\vartheta$ as $\Delta\bar\vartheta$ and similarly for the processes labelled with 'hat' and 'check'. Moreover, the remainder (B_a,F_a,Σ_a,G_a) in the statement must be understood as $(\Delta B_a,\Delta F_a,\Delta\Sigma_a,\Delta G_a)$.

We estimate the remainder terms in (4.18), recalling (4.17) for the meaning we give to the remainder in the stability estimate. By (4.29), the remainder can be split into three pieces according to h_ℓ, $\hat H_\ell$ and H_a.

First step. Upper bound for the terms involving (b_ℓ,f_ℓ), σ_ℓ and g_ℓ. We make use of the assumption (4.9) and of the conditional Cauchy-Schwarz inequality. Getting rid of the constant $\gamma^{1/2}$ in front of $|G_a(T)|^{2p}$ in (4.17), we let

$$\begin{aligned}
\Delta r_\ell^{2p} := \mathbb{E}_t\bigg[&\big|\big(g_\ell(X_T,\langle\hat X_T\rangle) - g_\ell(X_T',\langle\hat X_T'\rangle)\big)\mathcal{X}_T'\big|^{2p}\\
&+\bigg(\int_t^T \big|\big((b_\ell,f_\ell)(\bar\theta_s,\langle\check\theta_s^{(0)}\rangle) - (b_\ell,f_\ell)(\bar\theta_s',\langle\check\theta_s^{(0)\prime}\rangle)\big)\bar\vartheta_s'\big|\mathrm{d}s\bigg)^{2p}\\
&+\bigg(\int_t^T \big|\big(\sigma_\ell(\bar\theta_s^{(0)},\langle\check\theta_s^{(0)}\rangle) - \sigma_\ell(\bar\theta_s^{(0)\prime},\langle\check\theta_s^{(0)\prime}\rangle)\big)\bar\vartheta_s^{(0)\prime}\big|^2\mathrm{d}s\bigg)^{p}\bigg].
\end{aligned}$$

Recalling the Lipschitz property (4.9), we know that, for a generic function h_ℓ, which may be b_ℓ, f_ℓ or σ_ℓ,

$$\big|\big(h_\ell(\bar\theta_s,\langle\check\theta_s^{(0)}\rangle) - h_\ell(\bar\theta_s',\langle\check\theta_s^{(0)\prime}\rangle)\big)\bar\vartheta_s'\big|^2 \leqslant C\big(|\bar\theta_s-\bar\theta_s'|^2 + \Phi_\alpha^2(\check\theta^{(0)},\check\theta^{(0)\prime})\big)|\bar\vartheta_s'|^2. \tag{4.31}$$

Therefore, we get (for a constant C' possibly depending on p and varying from line to line)

$$\begin{aligned}
&\bigg(\int_t^T \big|\big((b_\ell,f_\ell)(\bar\theta_s,\langle\check\theta_s^{0)}\rangle) - (b_\ell,f_\ell)(\bar\theta_s',\langle\check\theta_s^{(0)\prime}\rangle)\big)\bar\vartheta_s'\big|\mathrm{d}s\bigg)^{2p}\\
&\leqslant C'\bigg[\bigg(\int_t^T |\bar\theta_s-\bar\theta_s'|^2\mathrm{d}s\bigg)^p + \Phi_\alpha^{2p}\big(\check\theta^{(0)},\check\theta^{(0)\prime}\big)\bigg]\bigg(\int_t^T |\bar\vartheta_s'|^2\mathrm{d}s\bigg)^p,
\end{aligned}$$

and by conditional Cauchy-Schwarz inequality, we deduce that (with the notation introduced in (4.25)):

$$\begin{aligned}
&\mathbb{E}_t\bigg[\bigg(\int_t^T \big|\big((b_\ell,f_\ell)(\bar\theta_s,\langle\check\theta_s^{(0)}\rangle) - (b_\ell,f_\ell)(\bar\theta_s',\langle\check\theta_s^{(0)\prime}\rangle)\big)\bar\vartheta_s'\big|\mathrm{d}s\bigg)^{2p}\bigg]\\
&\quad\leqslant C'\big\{\bar{\mathcal{M}}^{4p}\big((\bar\theta,\check\theta),(\bar\theta',\check\theta')\big)\big\}^{1/2}\big\{\bar{\mathcal{M}}^{4p}\big(\bar\vartheta',\check\vartheta'\big)\big\}^{1/2}.
\end{aligned}$$

It is pretty clear that we can get a similar bound when replacing (b_ℓ,f_ℓ) by σ_ℓ (using the supremum norm to handle the fact that there is already a square inside the integral).

Finally, the term involving g_ℓ can be also handled in a similar way, paying attention that the 'bar' process has to be replaced by the 'non-bar' process and the 'check' process by the 'hat' process. We thus get

$$\begin{aligned}&\Delta r_\ell^{2p}\\&\leqslant C'\big\{\bar{\mathcal{M}}^{4p}\big((\theta,\hat{\theta}),(\theta',\hat{\theta}')\big)+\bar{\mathcal{M}}^{4p}\big((\bar{\theta},\check{\theta}),(\bar{\theta}',\check{\theta}')\big)\big\}^{1/2}\big\{\mathcal{M}^{4p}\big(\vartheta',\hat{\vartheta}'\big)+\bar{\mathcal{M}}^{4p}\big(\bar{\vartheta}',\check{\vartheta}'\big)\big\}^{1/2}.\end{aligned}$$

Using (4.7), we get another bound for the same quantity, just by taking advantage of the fact that (b_ℓ,f_ℓ), σ_ℓ and g_ℓ are bounded:

$$\Delta r_\ell^{2p}\leqslant C'\bigg\{\mathbb{E}_t\big[|\mathcal{X}_T'|^{2p}\big]+\sup_{s\in[t,T]}\mathbb{E}_t\big[|\bar{\vartheta}_s^{(0)\prime}|^{2p}\big]+\mathbb{E}_t\bigg[\bigg(\int_t^T|\bar{\vartheta}_s'|^2\mathrm{d}s\bigg)^p\bigg]\bigg\},$$

so that

$$\begin{aligned}\Delta r_\ell^{2p}\leqslant C'\Big[1\wedge\big\{\bar{\mathcal{M}}^{4p}\big((\theta,\hat{\theta}),(\theta',\hat{\theta}')\big)+\bar{\mathcal{M}}^{4p}\big((\bar{\theta},\check{\theta}),(\bar{\theta}',\check{\theta}')\big)\big\}^{1/2}\Big]\\\times\big\{\mathcal{M}^{4p}\big(\vartheta',\hat{\vartheta}'\big)+\bar{\mathcal{M}}^{4p}\big(\bar{\vartheta}',\check{\vartheta}'\big)\big\}^{1/2}.\end{aligned}$$

Second step. Upper bound for the terms involving $\hat{B}_\ell$, $\hat{F}_\ell$, $\hat{\Sigma}_\ell$ or $\hat{G}_\ell$. We can make use of the Lipschitz property (4.9) or of the L^2 bound (4.7). For a generic function $\hat{H}_\ell$, which may be $\hat{B}_\ell$, $\hat{F}_\ell$ or $\hat{\Sigma}_\ell$, we get

$$\begin{aligned}&\big|\hat{\mathbb{E}}\big[\big(\hat{H}_\ell(\bar{\theta}_s,\langle\check{\theta}_s^{(0)}\rangle)-\hat{H}_\ell(\bar{\theta}_s',\langle\check{\theta}_s^{(0)\prime})\rangle)\big)\langle\check{\vartheta}_s^{(0)\prime}\rangle\big]\big|^2\\&\qquad\leqslant C\big[1\wedge\big(|\bar{\theta}_s-\bar{\theta}_s'|^2+\Phi_\alpha^2(\check{\theta}^{(0)},\check{\theta}^{(0)\prime}\big)\big]\|\check{\vartheta}_s^{(0)\prime}\|_2^2.\end{aligned}\tag{4.32}$$

Therefore, recalling the bound $\int(1\wedge h)\mathrm{d}\nu\leqslant 1\wedge\int h\mathrm{d}\nu$ that holds for a general measure ν with mass less than 1 and a general measurable nonnegative function h, we get

$$\begin{aligned}&\mathbb{E}_t\bigg[\bigg(\int_t^T\big|\hat{\mathbb{E}}\big[\big(\hat{H}_\ell(\bar{\theta}_s,\langle\check{\theta}_s^{(0)}\rangle)-\hat{H}_\ell(\bar{\theta}_s',\langle\check{\theta}_s^{(0)\prime}\rangle)\big)\langle\check{\vartheta}_s^{(0)\prime}\rangle\big]\big|\mathrm{d}s\bigg)^{2p}\bigg]\\&\leqslant\mathbb{E}_t\bigg[\bigg(\int_t^T\big|\hat{\mathbb{E}}\big[\big(\hat{H}_\ell(\bar{\theta}_s,\langle\check{\theta}_s^{(0)}\rangle)-\hat{H}_\ell(\bar{\theta}_s',\langle\check{\theta}_s^{(0)\prime}\rangle)\big)\langle\check{\vartheta}_s^{(0)\prime}\rangle\big]\big|^2\mathrm{d}s\bigg)^{p}\bigg]\\&\leqslant C'\sup_{s\in[t,T]}\|\check{\vartheta}_s^{(0)\prime}\|_2^{2p}\bigg\{1\wedge\bigg(\mathbb{E}_t\bigg[\bigg(\int_0^T|\bar{\theta}_s-\bar{\theta}_s'|^2\mathrm{d}s\bigg)^p\bigg]+\Phi_\alpha^{2p}\big(\check{\theta}^{(0)},\check{\theta}^{(0)\prime}\big)\bigg)\bigg\},\end{aligned}\tag{4.33}$$

which satisfies the same bound as Δr_ℓ^{2p}. Above the passage from the first to the third line may be applied with H equal to F or B and the passage from the second to the third line may be applied with H equal to Σ. We have a similar bound for the term involving $\hat{G}_\ell$:

$$\begin{aligned}&\mathbb{E}_t\Big[\big|\hat{\mathbb{E}}\big[\big(\hat{G}_\ell(X_T,\langle\hat{X}_T\rangle)-\hat{G}_\ell(X_T',\langle\hat{X}_T'\rangle)\big)\langle\hat{\mathcal{X}}_T'\rangle\big]\big|^{2p}\Big]\\&\leqslant C'\sup_{s\in[t,T]}\|\hat{\vartheta}_s^{(0)\prime}\|_2^{2p}\Big[1\wedge\Big(\sup_{s\in[t,T]}\mathbb{E}_t\big[|\theta_s^{(0)}-\theta_s^{(0),\prime}|^{2p}\big]+\Phi_\alpha^{2p}\big(\hat{\theta}^{(0)},\hat{\theta}^{(0)\prime}\big)\Big)\Big].\end{aligned}\tag{4.34}$$

Conclusion. In order to complete the proof of the first part, notice that the terms labelled by a directly give the remainder $\Delta\mathcal{R}_a^{2p}$ in (4.27). The second part of the statement easily follows from Lemma 4.6.

Remark 4.11. *As the reader may guess, terms of the form $\bar{\mathcal{M}}^{4p}(\vartheta,\hat{\vartheta})$ and $\bar{\mathcal{M}}^{4p}(\bar{\vartheta},\check{\vartheta})$ in* (4.27) *will be handled by means of Corollary 4.8. However, we note that, in comparison with $\bar{\mathcal{M}}^{4p}$, the 'conditional' norm $\mathcal{M}^{4p}$ that is used in Corollary 4.8 incorporates an additional pre-factor $\gamma^{1/2}$, see* (4.16). *Roughly speaking, $\bar{\mathcal{M}}^{4p}(\vartheta,\hat{\vartheta})$ and $\mathcal{M}_{\mathbb{E}_t}^{4p}(\vartheta)+(\mathcal{M}_{\mathbb{E}}^2(\hat{\vartheta}^{(0)})^{2p}$*

are 'equivalent' provided γ is not too small. In the sequel, we often choose γ exactly equal to $1/\Gamma_p$, so that $\bar{\mathcal{M}}^{4p}(\vartheta,\hat{\vartheta})$ and $\mathcal{M}^{4p}_{\mathbb{E}_t}(\vartheta) + (\mathcal{M}^2_{\mathbb{E}}(\hat{\vartheta}^{(0)}))^{2p}$ can be indeed compared.

Corollary 4.12. *Consider a family of progressively-measurable random paths $((\theta^\xi,\hat{\theta}^\xi) : [t,T] \ni s \mapsto (\theta^\xi_s,\hat{\theta}^\xi_s))_\xi$ parametrized by $\xi \in L^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^d)$. Assume that, for any $p \geqslant 1$, there exists a constant C_p such that, for all ξ and ξ' (with the same notation as in (4.25) but with Φ_α defined on $[L^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^d)]^2$ instead of $[L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d)]^2$):*

$$\begin{aligned} &\big(\bar{\mathcal{M}}^{2p}(\theta^\xi,\hat{\theta}^\xi)\big)^{1/2p} \leqslant C_p\big[1 + |\xi| + \|\xi\|_2\big], \\ &\big(\bar{\mathcal{M}}^{2p}\big((\theta^\xi,\hat{\theta}^\xi),(\theta^{\xi'},\hat{\theta}^{\xi'})\big)\big)^{1/2p} \leqslant C_p\big[|\xi-\xi'| + \Phi_\alpha\big(\xi,\xi'\big)\big], \end{aligned} \tag{4.35}$$

Assume also that we can find a Borel subset $\mathcal{O}$ of a Euclidean space, a continuous functional Ψ from $\mathcal{O} \times L^2(\hat{\Omega},\hat{\mathcal{A}},\hat{\mathbb{P}};\mathbb{R}^d)$ into $L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}_+)$ and, for any $\xi \in L^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^d)$, an admissible class $\mathcal{J}^\xi$ for $(\theta^\xi,\hat{\theta}^\xi)$ such that, for any Λ in $\mathcal{J}^\xi$, there exists a random variable $\lambda : (\Omega,\mathcal{A},\mathbb{P}) \to \mathcal{O}$ satisfying $\Lambda(\omega,\cdot) \leqslant \Psi(\lambda(\omega),\langle\xi\rangle)$, where $\Lambda(\omega,\cdot)$ denotes the random variable $\hat{\Omega} \ni \hat{\omega} \mapsto \Lambda(\omega,\hat{\omega})$ on $(\hat{\Omega},\hat{\mathcal{A}},\hat{\mathbb{P}})$.

With C as in Lemma 4.9, we then let, for $\varsigma \in \mathcal{O}$ and $\xi \in L^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^d)$,

$$\bar{\Psi}(\varsigma,\xi)(\omega) = \big(\Psi(\varsigma,\xi)(\omega)\big) \wedge \Big\{C\big(1 + |\xi(\omega)|^{\alpha+1} + \|\xi\|_2^{\alpha+1}\big)\Big\}, \quad \omega \in \Omega, \tag{4.36}$$

where $\Psi(\varsigma,\xi)$ is an abuse of notation for denoting the copy of the variable $\Psi(\varsigma,\langle\xi\rangle)$ on the space Ω instead of $\hat{\Omega}$. (We may indeed assume that $L^2(\hat{\Omega},\hat{\mathcal{A}},\hat{\mathbb{P}};\mathbb{R}^d)$ is a copy of $L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d)$, in which case we can transfer (canonically) $\Psi(\varsigma,\cdot)$ from one space to another.)

Then, for any $p \geqslant 1$, there exist two constants $\Gamma_p := \Gamma_p(K) \geqslant 1$ and $C'_p > 0$, such that, for $T \leqslant \gamma \leqslant 1/\Gamma_p$, choosing $(\bar{\theta},\bar{\vartheta}) \equiv (\theta,\vartheta)$, $(\check{\theta},\check{\vartheta}) \equiv (\hat{\theta},\hat{\vartheta})$, $(\bar{\theta}',\bar{\vartheta}') \equiv (\theta',\vartheta')$ and $(\check{\theta}',\check{\vartheta}') \equiv (\hat{\theta}',\hat{\vartheta}')$ in Lemma 4.9, with $(\theta,\hat{\theta}) :\equiv (\theta^\xi,\hat{\theta}^\xi)$ and $(\theta',\hat{\theta}') :\equiv (\theta^{\xi'},\hat{\theta}^{\xi'})$, it holds that:

$$\begin{aligned} &\big[\mathcal{M}^{2p}_{\mathbb{E}_t}\big(\vartheta-\vartheta'\big)\big]^{1/2p} \\ &\leqslant C'_p\Big\{\big[1 \wedge \big(|\xi-\xi'| + \Phi_\alpha(\xi,\xi')\big)\big] \\ &\qquad\qquad \times \Big(|\eta| + \|\eta\|_2 + \big(\mathcal{R}_a^{4p}\big)^{1/4p} + \mathbb{E}\big(\mathcal{R}_a^2\big)^{1/2} + \big(\mathcal{M}^2_{\mathbb{E}}(\hat{\vartheta}^{(0)\prime})\big)^{1/2}\Big) \\ &\qquad + \big(\Delta\mathcal{R}_a^{2p}\big)^{1/2p} + \sup_{\varsigma\in\mathcal{O}} \sup_{\|\Lambda_0\|_2 \leqslant K} \Big\{\mathbb{E}\Big[\big(\Lambda_0 \wedge \bar{\Psi}(\varsigma,\xi)\big)\big(\mathcal{M}^2_{\mathbb{E}_t}(\hat{\vartheta}^{(0)} - \hat{\vartheta}^{(0)\prime})\big)^{1/2}\Big]\Big\}. \end{aligned} \tag{4.37}$$

When $\vartheta \equiv \hat{\vartheta}$ and $\vartheta' \equiv \hat{\vartheta}'$, we have (modifying the value of Γ_p if necessary):

$$\begin{aligned} &\big[\mathcal{M}^{2p}_{\mathbb{E}_t}\big(\vartheta-\vartheta'\big)\big]^{1/2p} \\ &\leqslant C'_p\Big\{\big[1 \wedge \big(|\xi-\xi'| + \Phi_\alpha(\xi,\xi')\big)\big]\Big(|\eta| + \|\eta\|_2 + \big(\mathcal{R}_a^{4p}\big)^{1/4p} + \mathbb{E}\big[\mathcal{R}_a^2\big]^{1/2}\Big) + \big(\Delta\mathcal{R}_a^{2p}\big)^{1/2p}\Big\} \\ &\quad + C'_p\Big\{\sup_{\varsigma\in\mathcal{O}} \sup_{\|\Lambda_0\|_2 \leqslant K} \mathbb{E}\Big[\big(\Lambda_0 \wedge \bar{\Psi}(\varsigma,\xi)\big)\big[1 \wedge \big(|\xi-\xi'| + \Phi_\alpha(\xi,\xi')\big)\big] \\ &\qquad\qquad\qquad \times \Big(|\eta| + \|\eta\|_2 + \big(\mathcal{R}_a^4\big)^{1/4} + \mathbb{E}\big(\mathcal{R}_a^2\big)^{1/2}\Big)\Big]\Big\} \\ &\quad + C'_p\Big\{\sup_{\varsigma\in\mathcal{O}} \sup_{\|\Lambda_0\|_2 \leqslant K} \mathbb{E}\Big[\big(\Lambda_0 \wedge \bar{\Psi}(\varsigma,\xi)\big)\big(\Delta\mathcal{R}_a^2\big)^{1/2}\Big]\Big\}, \end{aligned} \tag{4.38}$$

the variable Λ_0 in the supremum being in $L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}_+)$ and the function Φ_α differing from the original one in (4.9) and (4.10) but satisfying the same properties on $[L^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^d)]^2$ instead of $[L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d)]^2$.

Remark 4.13. *Before we proceed with the proof of Corollary 4.12, we discuss what the assumptions we made on the structure of $\mathcal{J}^\xi$ permit to say on the term $\mathcal{N}^{2p,C}_{\mathbb{E}_t}(X,\chi)$ in (4.16). Recall indeed that*

$$\mathcal{N}^{p,C}_{\mathbb{E}_t}(X,\chi) = \sup_{\Lambda\in\mathcal{J}^\xi} \mathbb{E}_t\Big[\hat{\mathbb{E}}\Big[\Big\{\Lambda \wedge \Big[C\big(1+\hat{\mathbb{E}}_t\big[|\langle X\rangle|^{2\alpha+2}\big]^{1/2} + \|X\|_2^{\alpha+1}\big)\Big]\Big\}\hat{\mathbb{E}}_t\big[|\langle \mathcal{X}\rangle|^2\big]^{1/2}\Big]^p\Big].$$

Simplifying the notations, the term inside the conditional expectation may be rewritten as $\hat{\mathbb{E}}[(\Lambda\wedge\langle W\rangle)\langle\mathcal{W}\rangle]$, for some random variables $\langle W\rangle$ and $\langle\mathcal{W}\rangle$ in $L^2(\hat\Omega,\hat{\mathcal{A}},\hat{\mathbb{P}};\mathbb{R}_+)$ and for $\Lambda\in\mathcal{J}^\xi$. Allowing the constant C_p in the assumption to increase from line to line, the following bound is proved right below:

$$\mathbb{E}_t\Big[\hat{\mathbb{E}}\Big[\big(\Lambda\wedge\langle W\rangle\big)\langle\mathcal{W}\rangle\Big]^p\Big]^{1/p} \leqslant \sup_{\varsigma\in\mathcal{O}}\sup_{\|\Lambda_0\|_2\leqslant K}\Big\{\mathbb{E}\Big[\big(\Lambda_0\wedge\Psi(\varsigma,\xi)\wedge W\big)\mathcal{W}\Big]\Big\}. \tag{4.39}$$

where, in the above expectation, $\Lambda_0\in L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}_+)$, W and $\mathcal{W}$ are the copies of $\langle W\rangle$ and $\langle\mathcal{W}\rangle$ on the space Ω instead of Ω'.

We first prove the remark:

Proof. [Remark 4.13.] By assumption on the structure of $\mathcal{J}^\xi$, we can find λ such that

$$\begin{aligned}\hat{\mathbb{E}}\Big[\big(\Lambda\wedge\langle W\rangle\big)\langle\mathcal{W}\rangle\Big] &= \hat{\mathbb{E}}\Big[\big(\Lambda\wedge\Psi(\lambda,\langle\xi\rangle)\wedge\langle W\rangle\big)\langle\mathcal{W}\rangle\Big]\\ &\leqslant \sup_{\varsigma\in\mathcal{O}}\hat{\mathbb{E}}\Big[\big(\Lambda\wedge\Psi(\varsigma,\langle\xi\rangle)\wedge\langle W\rangle\big)\langle\mathcal{W}\rangle\Big].\end{aligned}\tag{4.40}$$

Recalling that Λ is a random variable $\Lambda:\Omega\times\hat\Omega\ni(\omega,\hat\omega)\mapsto\Lambda(\omega,\hat\omega)$ on the product space $(\Omega\times\hat\Omega,\mathcal{A}\otimes\hat{\mathcal{A}},\mathbb{P}\otimes\hat{\mathbb{P}})$ such that, for almost every $\omega\in\Omega$, $\Lambda(\omega,\cdot)\in L^2(\hat\Omega,\hat{\mathcal{A}},\hat{\mathbb{P}};\mathbb{R}_+)$ with $\hat{\mathbb{E}}[\Lambda^2(\omega,\cdot)]\leqslant K^2$, we can bound the above right-hand side by

$$\begin{aligned}&\hat{\mathbb{E}}\Big[\big(\Lambda\wedge\Psi(\varsigma,\langle\xi\rangle)\wedge\langle W\rangle\big)\langle\mathcal{W}\rangle\Big]\\ &\leqslant \sup\Big\{\hat{\mathbb{E}}\Big[\big(\langle\Lambda_0\rangle\wedge\Psi(\varsigma,\langle\xi\rangle)\wedge\langle W\rangle\big)\langle\mathcal{W}\rangle\Big];\ \langle\Lambda_0\rangle\in L^2(\hat\Omega,\hat{\mathcal{A}},\hat{\mathbb{P}};\mathbb{R}_+):\hat{\mathbb{E}}\big[\langle\Lambda_0\rangle^2\big]^{1/2}\leqslant K\Big\}.\end{aligned}$$

Transferring the expectation appearing in the supremum into an expectation on Ω, we get (4.39). □

We now turn to:

Proof. [Corollary 4.12.] The strategy is to make use of Lemma 4.9 and to estimate the various terms in (4.27). We use two values for the parameter γ in the definition (4.16) of $\mathcal{M}^p_{\mathbb{M}}$. As suggested in Remark 4.11, we first use $\gamma=1/\Gamma_p$. Since we consider the case $(\bar\theta',\bar\vartheta')\equiv(\theta',\vartheta')$ and $(\check\theta',\check\vartheta')\equiv(\hat\theta',\hat\vartheta')$, we deduce from (4.23) in Corollary 4.8 that there exists a constant C_p' such that

$$\big(\mathcal{M}^{2p}_{\mathbb{E}_t}(\vartheta')\big)^{1/2p} \leqslant C_p'\Big[|\eta|+\|\eta\|_2+\big(\mathcal{R}_a^{2p}\big)^{1/2p}+\mathbb{E}\big(\mathcal{R}_a^2\big)^{1/2}+\big(\mathcal{M}^2_{\mathbb{E}}(\hat\vartheta^{(0)\prime})\big)^{1/2}\Big]. \tag{4.41}$$

Recalling again Remark 4.11 to compare $\mathcal{M}^{4p}_{\mathbb{E}_t}$ and $\bar{\mathcal{M}}^{4p}$ and using in addition (4.35), the last term in (4.27), when put to the power $1/2p$, gives the contribution:

$$\begin{aligned}C_p'\Big[&\big[1\wedge\big(|\xi-\xi'|+\Phi_\alpha(\xi,\xi')\big)\big]\Big(|\eta|+\|\eta\|_2+\big(\mathcal{R}_a^{4p}\big)^{1/4p}+\mathbb{E}\big(\mathcal{R}_a^2\big)^{1/2}+\big(\mathcal{M}^2_{\mathbb{E}}(\hat\vartheta^{(0)\prime})\big)^{1/2}\Big)\\ &+\big(\Delta\mathcal{R}_a^{2p}\big)^{1/2p}\Big].\end{aligned}$$

We now discuss the other terms in (4.27). In this perspective, we use another value for γ, namely $\gamma' \leqslant 1/\Gamma_p$. Note that there is no conflict with the previous choice for γ, which just permitted to handle the terms of the form $\bar{\mathcal{M}}$ in (4.27). We thus turn to the two terms $\mathcal{N}^{2p,C}_{\mathbb{E}_t}$ in (4.27). Taking them to the power $1/2p$ and making use of the first line in (4.35), this brings us with a term of the same form as in the left-hand side of (4.39), with $W = C(1 + |\xi|^{2\alpha+1} + \|\xi\|_2^{2\alpha+1})$ and $\mathcal{W} = [\mathcal{M}^2_{\mathbb{E}_t}(\hat{\vartheta}^{(0)} - \hat{\vartheta}^{(0)\prime})]^{1/2}$. By (4.39), we get the following contribution:

$$\sup_{\varsigma\in\mathcal{O}} \sup_{\|\Lambda_0\|_2\leqslant K} \Big\{\mathbb{E}\Big[\big(\Lambda_0 \wedge \bar{\Psi}(\varsigma,\xi)\big)\big(\mathcal{M}^2_{\mathbb{E}_t}(\hat{\vartheta}^{(0)} - \hat{\vartheta}^{(0)\prime})\big)^{1/2}\Big]\Big\}.$$

We obtain (modifying the constant Γ_p in (4.27) in order to take into account the additional exponent $1/2p$):

$$\begin{aligned}
&\big[\mathcal{M}^{2p}_{\mathbb{E}_t}(\vartheta - \vartheta')\big]^{1/2p} \\
&\leqslant C'_p\Big\{\big[1 \wedge \big(|\xi - \xi'| + \Phi_\alpha(\xi,\xi')\big)\big] \\
&\qquad \times \Big(|\eta| + \|\eta\|_2 + \big(\mathcal{R}^{4p}_a\big)^{1/4p} + \mathbb{E}\big(\mathcal{R}^2_a\big)^{1/2} + \big(\mathcal{M}^2_{\mathbb{E}}(\hat{\vartheta}^{(0)\prime})\big)^{1/2}\Big) + \big(\Delta\mathcal{R}^{2p}_a\big)^{1/2p}\Big\} \\
&\quad + \Gamma_p(\gamma')^{1/4p} \sup_{\varsigma\in\mathcal{O}} \sup_{\|\Lambda_0\|_2\leqslant K} \Big\{\mathbb{E}\Big[\big(\Lambda_0 \wedge \bar{\Psi}(\varsigma,\xi)\big)\big(\mathcal{M}^2_{\mathbb{E}_t}(\hat{\vartheta}^{(0)} - \hat{\vartheta}^{(0)\prime})\big)^{1/2}\Big]\Big\},
\end{aligned} \tag{4.42}$$

which gives (4.37).

We now prove (4.38) when $\vartheta \equiv \hat{\vartheta}$ and $\vartheta' \equiv \hat{\vartheta}'$. We go back to (4.41). Applying (4.24) in Corollary 4.8 with $p = 1$ and taking expectation, we get, for γ small enough,

$$\big(\mathcal{M}^{2p}_{\mathbb{E}_t}(\vartheta')\big)^{1/2p} \leqslant C'_p\Big[|\eta| + \|\eta\|_2 + \big(\mathcal{R}^{2p}_a\big)^{1/2p} + \mathbb{E}\big(\mathcal{R}^2_a\big)^{1/2}\Big],$$

which means that, in (4.42), we can get rid of the term $\mathcal{M}^{2p}_{\mathbb{E}_t}(\hat{\vartheta}^{(0)\prime})$ in the right-hand side.

Let now $p = 1$ in (4.42). Multiply both sides by $\Lambda_0 \wedge \bar{\Psi}(\varsigma,\xi)$ for an $\mathbb{R}_+$-valued random variable Λ_0 such that $\|\Lambda_0\|_2 \leqslant K$ and take the expectation and then the supremum over Λ_0 and ς. For γ' small enough, we get that

$$\begin{aligned}
&\sup_{\varsigma\in\mathcal{O}} \sup_{\|\Lambda_0\|_2\leqslant K} \Big\{\mathbb{E}\Big[\big(\Lambda_0 \wedge \bar{\Psi}(\varsigma,\xi)\big)\big(\mathcal{M}^2_{\mathbb{E}_t}(\vartheta^{(0)} - \vartheta^{(0)\prime})\big)^{1/2}\Big]\Big\} \\
&\leqslant C'\Big\{\sup_{\varsigma\in\mathcal{O}} \sup_{\|\Lambda_0\|_2\leqslant K} \mathbb{E}\Big[\big(\Lambda_0 \wedge \bar{\Psi}(\varsigma,\xi)\big)\big[1 \wedge \big(|\xi - \xi'| + \Phi_\alpha(\xi,\xi')\big)\big] \\
&\qquad\qquad \times \Big(|\eta| + \|\eta\|_2 + \big(\mathcal{R}^4_a\big)^{1/4} + \mathbb{E}\big(\mathcal{R}^2_a\big)^{1/2}\Big)\Big]\Big\} \\
&\quad + C'\Big\{\sup_{\varsigma\in\mathcal{O}} \sup_{\|\Lambda_0\|_2\leqslant K} \mathbb{E}\Big[\big(\Lambda_0 \wedge \bar{\Psi}(\varsigma,\xi)\big)\big(\Delta\mathcal{R}^2_a\big)^{1/2}\Big]\Big\}.
\end{aligned}$$

Plugging the above estimate into (4.42), we complete the proof. □

Here is a very useful condition to check (4.35):

Lemma 4.14. *Consider a family of progressively-measurable random paths* $((\theta^\xi, \hat{\theta}^\xi) : [t,T] \ni s \mapsto (\theta^\xi_s, \hat{\theta}^\xi_s))_\xi$ *parametrized by* $\xi \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)$, *with the property that the paths* $(\hat{\theta}^{\xi,(0)} : [t,T] \ni s \mapsto \hat{\theta}^{\xi,(0)}_s)_\xi$ *are continuous, and that* $(\hat{\theta}^{\xi,(0)}_s)_{s\in[t,T]}$ *and* $(\hat{\theta}^{\xi',(0)}_s)_{s\in[t,T]}$ *have the same distribution when* $\xi \sim \xi'$.

Assume that, for any $p \geqslant 1$, there exists a constant C_p such that, for all ξ and ξ',

$$\begin{aligned}
&\|\theta^{\xi,(0)}\|_{\mathcal{S}^p,t} + \|\hat{\theta}^{\xi,(0)}\|_{\mathcal{S}^p,t} + \|\theta^{\xi}\|_{\mathcal{H}^p,t} \leqslant C_p\big(1 + |\xi| + \|\xi\|_2\big),\\
&\|\theta^{\xi,(0)} - \theta^{\xi',(0)}\|_{\mathcal{S}^p,t} + \|\hat{\theta}^{\xi,(0)} - \hat{\theta}^{\xi',(0)}\|_{\mathcal{S}^p,t} + \|\theta^{\xi} - \theta^{\xi'}\|_{\mathcal{H}^p,t}\\
&\qquad\qquad\qquad\qquad \leqslant C_p\big[|\xi - \xi'| + W_2\big([\xi],[\xi']\big)\big],
\end{aligned} \tag{4.43}$$

then, we can find constants C'_p such that, for all ξ and ξ' (with the notation (4.25)),

$$\begin{aligned}
&\big(\bar{\mathcal{M}}^{2p}(\theta^{\xi},\hat{\theta}^{\xi})\big)^{1/2p} \leqslant C'_p\big[1 + |\xi| + \|\xi\|_2\big],\\
&\big(\bar{\mathcal{M}}^{2p}\big((\theta^{\xi},\hat{\theta}^{\xi}),(\theta^{\xi'},\hat{\theta}^{\xi'})\big)\big)^{1/2p} \leqslant C'_p\big[|\xi - \xi'| + \tilde{\Phi}_\alpha(\xi,\xi')\big],
\end{aligned}$$

where

$$\tilde{\Phi}_\alpha(\xi,\xi') = \mathbb{E}\big[|\xi - \xi'|^2\big]^{1/2} + \sup_{s\in[t,T]} \Phi_\alpha(\hat{\theta}_s^{\xi,(0)},\hat{\theta}_s^{\xi',(0)}), \quad \xi,\xi' \in L^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^d). \tag{4.44}$$

The functional $\tilde{\Phi}_\alpha$ is continuous at any point of the diagonal of $[L^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^d)]^2$ and satisfies (4.10) *(up to a modification of the constant C therein).*

Proof. The bound for $(\bar{\mathcal{M}}^{2p}(\theta^{\xi},\hat{\theta}^{\xi}))^{1/2p}$ is a straightforward consequence of the first line in (4.43). The bound for $(\bar{\mathcal{M}}^{2p}((\theta^{\xi},\hat{\theta}^{\xi}),(\theta^{\xi'},\hat{\theta}^{\xi'}))^{1/2p}$ follows from the second line in (4.43) and from the definition of $\bar{\mathcal{M}}^{2p}$ in (4.25).

The main issue is to check that $\tilde{\Phi}_\alpha$ satisfies the same condition as Φ_α. By (4.43), the map $L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d) \ni \xi \mapsto (\hat{\theta}_s^{\xi,(0)})_{s\in[t,T]} \in \mathcal{S}^2([t,T];\mathbb{R}^l)$ (with the appropriate l) is continuous and, for any $\xi \in L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d)$, the map $[t,T] \ni s \mapsto \hat{\theta}_s^{\xi,(0)} \in L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^l)$ is also continuous, proving that, for any sequence $(\xi_n)_{n\geqslant 1}$ converging to ξ in L^2, the family of random variables $(\hat{\theta}_s^{\xi_n,(0)})_{s\in[t,T],n\geqslant 1}$ is relatively compact. Since, for any compact subset $\mathcal{K} \subset L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d)$, $\sup\{\Phi_\alpha(\chi,\chi'),\chi,\chi' \in \mathcal{K},\|\chi - \chi'\|_2 \leqslant \delta\}$ tends to 0 with δ, continuity of $\tilde{\Phi}_\alpha$ at any point of the diagonal easily follows.

Now, we check that $\tilde{\Phi}_\alpha$ satisfies (4.10) when ξ and ξ' have the same distribution. Since $\hat{\theta}_s^{\xi,(0)}$ and $\hat{\theta}_s^{\xi',(0)}$ have the same distribution, we deduce from (4.43) that

$$\begin{aligned}
&\mathbb{E}\big[\big(1 + |\hat{\theta}_s^{\xi,(0)}|^{2\alpha} + |\hat{\theta}_s^{\xi',(0)}|^{2\alpha} + \|\hat{\theta}_s^{\xi,(0)}\|_2^{2\alpha}\big)|\hat{\theta}_s^{\xi,(0)} - \hat{\theta}_s^{\xi',(0)}|^2\big]^{1/2}\\
&\leqslant \mathbb{E}\Big[\mathbb{E}_t\big[\big(1 + |\hat{\theta}_s^{\xi,(0)}|^{4\alpha} + |\hat{\theta}_s^{\xi',(0)}|^{4\alpha} + \|\hat{\theta}_s^{\xi,(0)}\|_2^{4\alpha}\big)\big]^{1/2}\mathbb{E}_t\big[|\hat{\theta}_s^{\xi,(0)} - \hat{\theta}_s^{\xi',(0)}|^4\big]^{1/2}\Big]^{1/2}\\
&\leqslant C\mathbb{E}\big[\big(1 + |\xi|^{2\alpha} + |\xi'|^{2\alpha} + \|\xi\|_2^{2\alpha}\big)|\xi - \xi'|^2\big]^{1/2}.
\end{aligned}$$

□

Example 4.15. *We illustrate the meaning of* (4.38) *in the simplest (but crucial) case when $\mathcal{R}_a \equiv \Delta\mathcal{R}_a \equiv 0$. Clearly, the most challenging term is*

$$\sup_{\varsigma\in\mathcal{O}} \sup_{\|\Lambda_0\|_2\leqslant K} \mathbb{E}\Big[\big(\Lambda_0 \wedge \bar{\Psi}(\varsigma,\xi)\big)\big[1 \wedge \big(|\xi - \xi'| + \Phi_\alpha(\xi,\xi')\big)\big]\big(|\eta| + \|\eta\|_2\big)\Big],$$

which is less than

$$\sup_{\varsigma\in\mathcal{O}} \sup_{\|\Lambda_0\|_2\leqslant K} \mathbb{E}\Big[\big(\Lambda_0 \wedge \bar{\Psi}(\varsigma,\xi)\big)^2\big[1 \wedge \big(|\xi - \xi'| + \Phi_\alpha(\xi,\xi')\big)\big]^2\Big]^{1/2}\|\eta\|_2 \leqslant \bar{\Phi}(\xi,\xi')\|\eta\|_2,$$

with

$$\bar{\Phi}(\xi,\xi') = \sup_{\varsigma\in\mathcal{O}} \sup_{\|\Lambda_0\|_2\leqslant K} \mathbb{E}\Big[\big(\Lambda_0 \wedge \bar{\Psi}(\varsigma,\xi)\big)^2\big[1 \wedge |\xi - \xi'|^2\big]\Big]^{1/2} + K\Phi_\alpha(\xi,\xi'). \tag{4.45}$$

Recalling the bound $0 \leqslant \bar{\Psi}(\varsigma,\xi) \leqslant C(1+|\xi(\omega)|^{\alpha+1}+\|\xi\|_2^{\alpha+1})$*, there exists a constant* C' *such that, whenever* ξ *and* ξ' *have the same distribution,*

$$\bar{\Phi}(\xi,\xi') \leqslant C'\mathbb{E}\big[\big(1+|\xi|^{2\alpha+2}+|\xi'|^{2\alpha+2}\big)|\xi-\xi'|^2\big]^{1/2},$$

which fits (4.10)*, with* $\alpha+1$ *instead of* α*, up to another multiplicative constant. The functional* $\bar{\Phi}$ *is thus a candidate for being a function of the same type as* $\Phi_{\alpha+1}$*, according to the notation used in the assumptions* (4.7)–(4.11)*. Still, in order to guarantee that* $\bar{\Phi}$ *indeed satisfies the same assumptions as* $\Phi_{\alpha+1}$*, it is necessary to prove that it is continuous at any point of the diagonal. We claim that it is the case under the two additional conditions (the proof is given right below):*

(i) for each $\xi \in L^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^d)$*, the family* $(\Psi^2(\varsigma,\xi))_{\varsigma\in\mathcal{O}}$ *is uniformly integrable,*

(ii) the mappings $(L^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^d) \ni \xi \mapsto \Psi(\varsigma,\xi) \in L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d))_{\varsigma\in\mathcal{O}}$ *are equicontinuous.*

As an example of a family $(\theta^\xi,\hat{\theta}^\xi)_\xi$ *and a functional* $\Psi : \mathcal{O}\times L^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^d) \ni (\varsigma,\xi) \mapsto \Psi(\varsigma,\xi)$ *that satisfy the prescription in Corollary 4.12 together with* (i) *and* (ii)*, we can consider (again, the proof is given right below)* $(\theta^\xi :\equiv \theta^{t,\xi},\hat{\theta}^\xi :\equiv \hat{\theta}^{t,\xi})_\xi$ *or* $(\theta^\xi :\equiv \theta^{t,x,[\xi]},\hat{\theta}^\xi :\equiv \hat{\theta}^{t,\xi})_\xi$ *and*

$$\Psi\big(\varsigma=(w,s),\xi\big) = \sup_{H=B,\Sigma,F,G} \mathbb{E}_t\Big[\big|\hat{H}_\ell\big(w,\hat{\theta}_s^{t,\xi,(0)}\big)\big|^2\Big]^{1/2}, \tag{4.46}$$

for $w \in \mathbb{R}^d\times\mathbb{R}^m\times\mathbb{R}^{m\times d}$ *and* $s\in[t,T]$*. (The definition of* $\Psi(\varsigma,\xi)$ *for a random variable* ξ *that is not* $\mathcal{F}_t$*-measurable is useless here, since* ξ *is exclusively thought as an initial condition of the system* (2.3) *at time* t*.)*

Proof. *First step.* We first check that, under *(i)* and *(ii)*, $\bar{\Phi}$ is continuous at any point of the diagonal. Given two sequences $(\xi_n)_{n\geqslant 0}$ and $(\xi_n')_{n\geqslant 0}$ converging in $L^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^d)$ towards some ξ, we already know that $(\Phi_\alpha(\xi_n,\xi_n'))_{n\geqslant 0}$ converges to 0. Therefore, it suffices to focus on the first term in the right-hand side of (4.45). We have

$$\begin{aligned}
&\Big|\sup_{\varsigma\in\mathcal{O}}\sup_{\|\Lambda_0\|_2\leqslant K}\mathbb{E}\Big[\big(\Lambda_0\wedge\bar{\Psi}(\varsigma,\xi_n)\big)^2\big[1\wedge|\xi_n-\xi_n'|^2\big]\Big] \\
&\quad -\sup_{\varsigma\in\mathcal{O}}\sup_{\|\Lambda_0\|_2\leqslant K}\mathbb{E}\Big[\big(\Lambda_0\wedge\bar{\Psi}(\varsigma,\xi)\big)^2\big[1\wedge|\xi-\xi'|^2\big]\Big]\Big| \\
&\leqslant \sup_{\varsigma\in\mathcal{O}}\sup_{\|\Lambda_0\|_2\leqslant K}\mathbb{E}\Big[\big(\Lambda_0\wedge\bar{\Psi}(\varsigma,\xi)\big)^2\big|1\wedge|\xi-\xi'|^2-1\wedge|\xi_n-\xi_n'|^2\big|\Big] \\
&\quad +\sup_{\varsigma\in\mathcal{O}}\sup_{\|\Lambda_0\|_2\leqslant K}\mathbb{E}\Big[\Big|\big(\Lambda_0\wedge\bar{\Psi}(\varsigma,\xi)\big)^2-\big(\Lambda_0\wedge\bar{\Psi}(\varsigma,\xi_n)\big)^2\Big|\Big].
\end{aligned} \tag{4.47}$$

Recalling the bound $\bar{\Psi}(\varsigma,\xi)\leqslant\Psi(\varsigma,\xi)$, the first term in the right-hand side is less than

$$\sup_{\varsigma\in\mathcal{O}}\mathbb{E}\Big[\Psi^2(\varsigma,\xi)\big|1\wedge|\xi-\xi'|^2-1\wedge|\xi_n-\xi_n'|^2\big|\Big],$$

which tends to 0 by uniform integrability of the family $(\Psi(\varsigma,\xi))_{\varsigma\in\mathcal{O}}$.

Consider now the second term in the right-hand side of (4.47). We have

$$\begin{aligned}
&\sup_{\varsigma\in\mathcal{O}}\sup_{\|\Lambda_0\|_2\leqslant K}\mathbb{E}\Big[\Big|\big(\Lambda_0\wedge\bar{\Psi}(\varsigma,\xi)\big)^2-\big(\Lambda_0\wedge\bar{\Psi}(\varsigma,\xi_n)\big)^2\Big|\Big] \\
&\leqslant 2K\sup_{\varsigma\in\mathcal{O}}\sup_{\|\Lambda_0\|_2\leqslant K}\mathbb{E}\Big[\Big|\Lambda_0\wedge\bar{\Psi}(\varsigma,\xi)-\Lambda_0\wedge\bar{\Psi}(\varsigma,\xi_n)\Big|^2\Big]^{1/2}.
\end{aligned}$$

Recalling from (4.36) that $\bar{\Psi}(\varsigma,\xi) = \Psi(\varsigma,\xi) \wedge \varphi(\xi)$, with $\varphi(\xi) = [C(1+|\xi|^{\alpha+1} + \|\xi\|_2^{\alpha+1}]$, writing $|\Lambda_0 \wedge \bar{\Psi}(\varsigma,\xi) - \Lambda_0 \wedge \bar{\Psi}(\varsigma,\xi_n)| \leqslant |\Lambda_0 \wedge \Psi(\varsigma,\xi) \wedge \varphi(\xi) - \Lambda_0 \wedge \Psi(\varsigma,\xi) \wedge \varphi(\xi_n)| + |\Lambda_0 \wedge \Psi(\varsigma,\xi) \wedge \varphi(\xi_n) - \Lambda_0 \wedge \Psi(\varsigma,\xi_n) \wedge \varphi(\xi_n)|$ and using the Lipschitz property of the map $\mathbb{R} \ni x \mapsto a \wedge x$, for any $a \in \mathbb{R}$, we deduce that

$$\sup_{\varsigma\in\mathcal{O}} \sup_{\|\Lambda_0\|_2 \leqslant K} \mathbb{E}\Big[\Big|\big(\Lambda_0 \wedge \bar{\Psi}(\varsigma,\xi)\big)^2 - \big(\Lambda_0 \wedge \bar{\Psi}(\varsigma,\xi_n)\big)^2\Big|\Big]$$

$$\leqslant 2K\left[\sup_{\varsigma\in\mathcal{O}} \mathbb{E}\Big[\Big|\Psi(\varsigma,\xi) - \Psi(\varsigma,\xi_n)\Big|^2\Big]^{1/2} + \sup_{\varsigma\in\mathcal{O}} \mathbb{E}\Big[\Big|\Psi(\varsigma,\xi) \wedge \varphi(\xi) - \Psi(\varsigma,\xi) \wedge \varphi(\xi_n)\Big|^2\Big]\right]^{1/2}.$$

By uniform continuity of the mappings $(\Psi(\varsigma,\cdot))_{\varsigma\in\mathcal{O}}$, the first term in the right-hand side tends to 0. By uniform integrability of the family $(\Psi^2(\varsigma,\xi))_{\varsigma\in\mathcal{O}}$, the second one also tends to 0.

Second step. We now check the example. By Lemmas 4.1 and 4.14, (4.35) is satisfied with $(\theta^\xi,\hat{\theta}^\xi) :\equiv (\theta^{t,\xi},\hat{\theta}^{t,\xi})$ or $(\theta^\xi,\hat{\theta}^\xi) :\equiv (\theta^{t,x,[\xi]},\hat{\theta}^{t,\xi})$. We prove that Ψ in (4.46) satisfies *(i)* and *(ii)*. We check first the uniform integrability property *(i)*. It suffices to check it for H equal to B, Σ, F or G (if uniform integrability holds for H equal to B, Σ, F or G, then the supremum over H equal to B, Σ, F or G also satisfies *(i)*). Given $\xi \in L^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^d)$, it suffices to prove that the family $(\sup_{r\in[t,T]} |\hat{H}_\ell(w,\hat{\theta}_r^{t,\xi,(0)})|^2)_{w\in\mathbb{R}^k}$ (for the appropriate k) is uniformly integrable. Consider a positive constant $\varepsilon > 0$. Since the path $[t,T] \ni r \mapsto \hat{\theta}_r^{t,\xi,(0)}$ is continuous and Φ_α is continuous at any point of the diagonal, we can find a constant $\delta > 0$ such that

$$\sup_{(r,s)\in[t,T]^2:|s-r|\leqslant\delta} \Phi_\alpha\big(\hat{\theta}_r^{t,\xi,(0)},\hat{\theta}_s^{t,\xi,(0)}\big) \leqslant \varepsilon. \tag{4.48}$$

Then, for $(r,s) \in [t,T]^2$, Cauchy Schwarz' inequality yields

$$\begin{aligned}\Big|\mathbb{E}\big[|\hat{H}_\ell(w,\hat{\theta}_s^{t,\xi,(0)})|^2\big] - \mathbb{E}\big[|\hat{H}_\ell(w,\hat{\theta}_r^{t,\xi,(0)})|^2\big]\Big| &\leqslant 2K\mathbb{E}\Big[\big|\hat{H}_\ell(w,\hat{\theta}_s^{t,\xi,(0)}) - \hat{H}_\ell(w,\hat{\theta}_r^{t,\xi,(0)})\big|^2\Big]^{1/2} \\ &\leqslant 2K\varepsilon^{1/2}.\end{aligned}$$

Therefore, denoting by $(t = s_0 < s_1 < \cdots < s_N = T)$ a subdivision of $[t,T]$ with stepsize less than δ, we deduce that, for any event $A \in \mathcal{A}$,

$$\sup_{w\in\mathbb{R}^k} \sup_{t\leqslant r\leqslant T} \mathbb{E}\Big[\big|\hat{H}_\ell\big(w,\hat{\theta}_r^{t,\xi,(0)}\big)\big|^2 \mathbf{1}_A\Big] \leqslant \sup_{w\in\mathbb{R}^k} \sup_{i=0,\dots,N} \mathbb{E}\Big[\big|\hat{H}_\ell\big(w,\hat{\theta}_{s_i}^{t,\xi,(0)}\big)\big|^2 \mathbf{1}_A\Big] + 2K\varepsilon^{1/2}.$$

By the uniform integrability of each of the family $(|\hat{H}_\ell(w,\hat{\theta}_{s_i}^{t,\xi,(0)})|^2)_{w\in\mathbb{R}^k}$, for $i = 0,\dots,N$, see (***H1***), we deduce that the left-hand side is indeed less than $4K\varepsilon^{1/2}$ for δ small enough.

We check uniform continuity of the mappings $(\xi \mapsto \hat{H}_\ell(w,\hat{\theta}_s^{t,\xi,(0)}))_{w\in\mathbb{R}^k,s\in[t,T]}$:

$$\sup_{w\in\mathbb{R}^k} \sup_{s\in[t,T]} \mathbb{E}\Big[\big|\hat{H}_\ell\big(w,\hat{\theta}_s^{t,\xi,(0)}\big) - \hat{H}_\ell\big(w,\hat{\theta}_s^{t,\xi',(0)}\big)\big|^2\Big]^{1/2} \leqslant C \sup_{s\in[t,T]} \Phi_\alpha\big(\hat{\theta}_s^{t,\xi,(0)},\hat{\theta}_s^{t,\xi',(0)}\big),$$

which tends to 0 as $\xi' - \xi$ tends to 0, by the same argument as in Lemma 4.14. □

We complete the subsection with a very important observation:

Remark 4.16. *Example 4.15 ensures that Corollary 4.12 may be applied with $(\theta^\xi,\hat{\theta}^\xi) :\equiv (\theta^{t,\xi},\theta^{t,\xi})$ or $(\theta^\xi,\hat{\theta}^\xi) :\equiv (\theta^{t,x,[\xi]},\theta^{t,\xi})$, in which case* (4.35) *holds for a suitable function Φ_α (defined on $[L^2(\mathcal{A},\mathcal{F}_t,\mathbb{P};\mathbb{R}^d)]^2$) and the the second term in the right-hand side of* (4.38) *may be bounded by a function of the type $\Phi_{\alpha+1}$ (also defined on $[L^2(\mathcal{A},\mathcal{F}_t,\mathbb{P};\mathbb{R}^d)]^2$).*

It is worth mentioning that, with the construction that is suggested, both Φ_α *and* $\Phi_{\alpha+1}$ *may depend on* t*, which is clear from* (4.44) *and* (4.46)*.*

Below, we want to use versions of both that are independent of t*. This requires first to restrict the domain of definition of both functionals to* $[L^2(\mathcal{A},\mathcal{F}_0,\mathbb{P};\mathbb{R}^d)]^2$*. Second, this requires a suitable adaptation of* (4.44) *and* (4.46)*.*

When $\xi \in L^2(\Omega,\mathcal{F}_0,\mathbb{P};\mathbb{R}^d)$*, we may extend* $(\theta_s^{t,\xi})_{s\in[t,T]}$ *to the interval* $[0,T]$ *by letting* $X_s^{t,\xi} = \xi$*,* $Y_s^{t,\xi} = Y_t^{t,\xi}$ *and* $Z_s^{t,\xi} = 0$ *for* $s \in [0,t]$*. Then, for* $\xi,\xi' \in L^2(\Omega,\mathcal{F}_0,\mathbb{P};\mathbb{R}^d)$*, instead of* (4.44)*, we may let*

$$\tilde{\Phi}_\alpha(\xi,\xi') = \mathbb{E}\big[|\xi-\xi'|^2\big]^{1/2} + \sup_{t\in[0,T]}\sup_{s\in[0,T]}\Phi_\alpha\big(\hat{\theta}_s^{t,\xi,(0)},\hat{\theta}_s^{t,\xi',(0)}\big),$$

(that is we also take the supremum in t*), and, instead of* (4.46)*, we may let*

$$\Psi\big(\varsigma=(w,t,s),\xi\big) = \sup_{H=B,\Sigma,F,G}\mathbb{E}_t\Big[\big|\hat{H}_\ell\big(w,\hat{\theta}_s^{t,\xi,(0)}\big)\big|^2\Big]^{1/2},$$

(that is we include t *in the variable* ς*).*

Then, the resulting new functionals Φ_α *and* $\Phi_{\alpha+1}$ *are independent of* t*, are continuous at any point of the diagonal of* $[L^2(\Omega,\mathcal{F}_0,\mathbb{P};\mathbb{R}^d)]^2$ *and satisfy* (4.10) *with respect to* α *and* $\alpha+1$*. The proof works exactly as in Lemma 4.14 and in Example 4.15, noticing that that the mapping* $L^2(\Omega,\mathcal{F}_0,\mathbb{P};\mathbb{R}^d)\times[0,T]\times[0,T] \ni (\xi,s,t)\mapsto\theta_s^{t,\xi}\in L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d)$ *is continuous (which is the main ingredient to make the argument work).*

4.2. **Analysis of the first-order derivatives.**

4.2.1. *First-order derivatives of the McKean-Vlasov system.* As we already explained in Examples 4.2 and 4.3, the shape of the system (4.3) has been specifically designed in order to investigate the derivative of the system of the original FBSDE in the direction of the measure. Thus, we shall make use of the results from Subsection 4.1, the constant L in $(\boldsymbol{H0})$(i)-$(\boldsymbol{H1})$ now playing the role of the constant K in the above statements. In order to stress the fact that this subsection is devoted to the application of the general results proved above to the specific question of the differentiability of the flow, we shall use constants $c(L)$ or $c_p(L)$ instead of $1/\Gamma(K)$ or $1/\Gamma_p(K)$ for quantifying small time constraints of the type $T \leqslant c(L)$ or $T\leqslant c_p(L)$.

To make things clear, we also recall the identification of h_ℓ, $\hat{H}_\ell$ and H_a in (4.4):

$$h_\ell(w,\langle\hat{V}^{(0)}\rangle) = \partial_w h(w,[\hat{V}^{(0)}]),\ \hat{H}_\ell(w,\langle\hat{V}^{(0)}\rangle) = \partial_\mu h(w,[\hat{V}^{(0)}])(\langle\hat{V}^{(0)}\rangle),\ H_a\equiv 0. \quad (4.49)$$

The next results state the first order differentiability of the McKean-Vlasov system.

Lemma 4.17. *Given a continuously differentiable path of initial conditions* $\mathbb{R}\ni\lambda\mapsto\xi^\lambda\in L^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^d)$*,* t *standing for the initial time in* $[0,T]$*, we can find a constant* $c := c(L) > 0$ *such that, for* $T\leqslant c$*, the path* $\mathbb{R}\ni\lambda\mapsto\theta^\lambda = (X^\lambda,Y^\lambda,Z^\lambda) := \theta^{t,\xi^\lambda}\in\mathcal{S}^2([t,T];\mathbb{R}^d)\times\mathcal{S}^2([t,T],\mathbb{R}^m)\times\mathcal{H}^2([t,T];\mathbb{R}^{m\times d})$ *is continuously differentiable.*

Proof. Under $(\boldsymbol{H0})$(i), existence and uniqueness of a solution to (2.1) may be proved for a small time horizon T by a contraction argument. As in [11], for T small enough, we can approximate $(X^\lambda,Y^\lambda,Z^\lambda)$ as the limit of a Picard sequence $\theta^{n,\lambda} := (X^{n,\lambda},Y^{n,\lambda},Z^{n,\lambda})$, defined by

$$
\begin{aligned}
X_s^{n+1,\lambda} &= \xi^\lambda + \int_t^s b(\theta_r^{n,\lambda}, [\theta_r^{n,\lambda,(0)}])\mathrm{d}r + \int_t^s \sigma(\theta_r^{n,\lambda,(0)}, [\theta_r^{n,\lambda,(0)}])\mathrm{d}W_r \\
Y_s^{n+1,\lambda} &= g(X_T^{n+1,\lambda}, [X_T^{n+1,\lambda}]) + \int_s^T f(\theta_r^{n,\lambda}, [\theta_r^{n,\lambda,(0)}])\mathrm{d}r - \int_s^T Z_r^{n+1,\lambda}\mathrm{d}W_r,
\end{aligned}
$$

where we have used the notation $\theta_s^{n,\lambda,(0)} = (X_s^{n,\lambda}, Y_s^{n,\lambda})$, with the initialization $\theta^{0,\lambda} \equiv 0$. By the standard theory of Itô processes and backward equations (see in particular [31]), we can prove by induction that, for any $n \geqslant 0$, the mapping $\mathbb{R} \ni \lambda \mapsto \theta^\lambda = (X^{n,\lambda}, Y^{n,\lambda}, Z^{n,\lambda}) \in \mathcal{S}^2([t,T];\mathbb{R}^d) \times \mathcal{S}^2([t,T],\mathbb{R}^m) \times \mathcal{H}^2([t,T];\mathbb{R}^{m\times d})$ is continuously differentiable. We give just a sketch of proof. For the forward component, this follows from the fact that given a continuously differentiable path $\mathbb{R} \ni \lambda \mapsto h^\lambda \in \mathcal{H}^2([t,T],\mathbb{R})$, the paths $\mathbb{R} \ni \lambda \mapsto (\int_t^s h_r^\lambda \mathrm{d}r)_{s\in[t,T]}$ and $\mathbb{R} \ni \lambda \mapsto (\int_t^s h_r^\lambda \mathrm{d}W_r)_{s\in[t,T]}$, with values in $\mathcal{S}^2([t,T],\mathbb{R}^l)$ for a suitable dimension l, are continuously differentiable, which is obviously true. To handle the backward component, it suffices to prove first that the path $\mathbb{R} \ni \lambda \mapsto (\mathbb{E}_s[h_T^\lambda])_{s\in[t,T]}$, with values in $\mathcal{S}^2([t,T],\mathbb{R})$, is continuously differentiable, which is straightforward by means of Doob's inequality. This is enough to handle the terminal condition and also the driver since we can split the integral from s to T into an integral from t to s (to which we can apply the result used for the forward component) and an integral from t to T (which can be seen as a new h_T). In this way, we can prove that $\mathbb{R} \ni \lambda \mapsto Y^{n+1,\lambda}$ is continuously differentiable from $\mathbb{R}$ to $\mathcal{S}^2([t,T],\mathbb{R}^m)$. This shows that $\mathbb{R} \ni \lambda \mapsto (\int_t^s Z_r^{n+1,\lambda}\mathrm{d}W_r)_{s\in[t,T]}$ is also continuously differentiable from $\mathbb{R}$ to $\mathcal{S}^2([t,T],\mathbb{R}^m)$. By Itô's isometry, this finally proves that $\mathbb{R} \ni \lambda \mapsto (Z_s^{n+1,\lambda})_{s\in[t,T]}$ is continuously differentiable from $\mathbb{R}$ to $\mathcal{H}^2([t,T],\mathbb{R}^{m\times d})$, the derivative of $Z^{n+1,\lambda}$ writing as the martingale representation term of the derivative of $\int_t^T Z_r^{n+1,\lambda}\mathrm{d}W_r$.

The derivatives, denoted by $(\mathcal{X}^{n,\lambda}, \mathcal{Y}^{n,\lambda}, \mathcal{Z}^{n,\lambda})$, satisfy the system

$$
\begin{aligned}
\mathcal{X}_s^{n+1,\lambda} &= \chi^\lambda + \int_t^s B^{(1)}\big(r, \theta_r^{n,\lambda}, \langle \theta_r^{n,\lambda,(0)}\rangle\big)\big(\vartheta_r^{n,\lambda}, \langle \vartheta_r^{n,\lambda,(0)}\rangle\big)\mathrm{d}r \\
&\quad + \int_t^s \Sigma^{(1)}\big(r, \theta_r^{n,\lambda,(0)}, \langle \theta_r^{n,\lambda,(0)}\rangle\big)\big(\vartheta_r^{n,\lambda,(0)}, \langle \vartheta_r^{n,\lambda,(0)}\rangle\big)\mathrm{d}W_r \\
\mathcal{Y}_s^{n+1,\lambda} &= G^{(1)}\big(X_T^{n+1,\lambda}, \langle X_T^{n+1,\lambda}\rangle\big)\big(\mathcal{X}_T^{n+1,\lambda}, \langle \mathcal{X}_T^{n+1,\lambda}\rangle\big) \\
&\quad + \int_s^T F^{(1)}\big(r, \theta_r^{n,\lambda}, \langle \theta_r^{n,\lambda,(0)}\rangle\big)\big(\vartheta_r^{n,\lambda}, \langle \vartheta_r^{n,\lambda,(0)}\rangle\big)\mathrm{d}r - \int_s^T \mathcal{Z}_r^{n+1,\lambda}\mathrm{d}W_r,
\end{aligned}
\tag{4.50}
$$

where we have used the notations $\chi^\lambda = [\mathrm{d}/\mathrm{d}\lambda]\xi^\lambda$, $\vartheta^{n,\lambda} = (\mathcal{X}^{n,\lambda}, \mathcal{Y}^{n,\lambda}, \mathcal{Z}^{n,\lambda})$ and $\vartheta^{n,\lambda,(0)} = (\mathcal{X}^{n,\lambda}, \mathcal{Y}^{n,\lambda})$ and where B, Σ, F and G are defined according to (4.49) and are denoted by $B^{(1)}$, $\Sigma^{(1)}$, $F^{(1)}$ and $G^{(1)}$ as in (4.4), the superscript (1) stressing the fact that we are dealing with *first-order* derivatives. We thus obtain a system of the form (4.15) with $\theta \equiv \hat{\theta} \equiv \theta^{n+1,\lambda}$, $\bar{\theta} \equiv \check{\theta} \equiv \theta^{n,\lambda}$, $\vartheta \equiv \hat{\vartheta} \equiv \vartheta^{n+1,\lambda}$ and $\bar{\vartheta} \equiv \check{\vartheta} \equiv \vartheta^{n,\lambda}$ and χ^λ playing the role of η. We now apply Lemma 4.7, noticing that the remainder $\mathcal{R}_a$ therein is zero, see (4.49).

First, we set $p = 1$ in (4.19) and choose $\gamma = 1/\Gamma_1(L)$ in (4.16), in agreement with Remark 4.11. We then take expectation on both sides. We get that, for T small, the sequence $(\mathcal{M}_{\mathbb{E}}^2(\vartheta^{n,\lambda}))_{n\geqslant 1}$ is at most of arithmetico-geometric type, with a geometric rate strictly less than 1. By induction, we deduce that there exist two constants $c := c(L) > 0$ and $C \geqslant 0$ (the values of which are allowed to increase from one line to another), such that, for $T \leqslant c$, $\sup_{n \geqslant 0} \mathcal{M}^2_{\mathbb{E}}(\vartheta^{n,\lambda}) \leqslant C\|\chi^\lambda\|_2^2$. Inserting this estimate into (4.19) (with $\gamma = 1/\Gamma_2(L)$ therein), we can prove, in the same way, that, for possibly new values of c and C,

$$\sup_{n \geqslant 1}\Big[\mathcal{M}^4_{\mathbb{E}_t}\big(\vartheta^{n,\lambda}\big)\Big]^{1/2} \leqslant C\big[|\chi^\lambda|^2 + \|\chi^\lambda\|_2^2\big]. \tag{4.51}$$

Exploiting Remark 4.11, we deduce that $[\bar{\mathcal{M}}^4[\![\vartheta^{n,\lambda}]\!]]^{1/2}$ and $[\bar{\mathcal{M}}^4[\![\vartheta^{n+1,\lambda}]\!]]^{1/2}$ in (4.25) are less than $C(|\chi^\lambda|^2 + \|\chi^\lambda\|_2^2)$.

We now make use of (4.28) in Lemma 4.9, with $p = 1$, in order to compare $\vartheta^{n,\lambda}$ and $\vartheta^{n+1,\lambda}$. Clearly, the remainder $\Delta\mathcal{R}_a^2$ in (4.26) is zero since the $\mathcal{R}_a$ terms are here equal to zero, recall (4.49). By the above argument, $[\bar{\mathcal{M}}^4[\![\vartheta^{n,\lambda}]\!]]^{1/2}$ and $[\bar{\mathcal{M}}^4[\![\vartheta^{n+1,\lambda}]\!]]^{1/2}$ in (4.25) are less than $C(|\chi^\lambda|^2 + \|\chi^\lambda\|_2^2)$. In order to apply Lemma 4.9, we also have to estimate $[\bar{\mathcal{M}}^4[\![\theta^{n+1,\lambda}, \theta^{n,\lambda}]\!]]^{1/2}$. Since T is small enough, the Picard scheme for solving (2.3) is geometrically convergent in L^2 and in any L^p, $p \geqslant 2$, conditional on $\mathcal{F}_t$, the geometric rate being independent of the initial conditions. To be precise, there exist $\rho \in (0,1)$ and $C' \geqslant 0$ such that, almost surely, $[\bar{\mathcal{M}}^4[\![\theta^{n+1,\lambda} - \theta^{n,\lambda}]\!]]^{1/2} \leqslant C'(1 + |\xi^\lambda|^2 + \|\xi^\lambda\|_2^2)\rho^n$. By continuity of the map $\mathbb{R} \ni \lambda \mapsto \xi^\lambda \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)$, this shows that $\mathbb{E}([\bar{\mathcal{M}}^4[\![\theta^{n+1,\lambda} - \theta^{n,\lambda}]\!]]^{1/2})$ converges to 0, uniformly in λ in compact subsets. Now, following the proof of Lemma 4.14 and using the fact that the map $\mathbb{R} \ni \lambda \mapsto \xi^\lambda \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)$ is continuous, the family of random variables $(\theta_s^{n,(0),\lambda})_{n \geqslant 0, s \in [t,T], \lambda \in \mathcal{K}}$ is relatively compact in $L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^d \times \mathbb{R}^m)$ for any compact subset $\mathcal{K} \subset \mathbb{R}$. Therefore, $\Phi_\alpha(\theta^{n+1,(0),\lambda}, \theta^{n,(0),\lambda})$ converges to 0, uniformly in λ in compact subsets. We deduce that $\mathbb{E}([\bar{\mathcal{M}}^4[\![\theta^{n+1,\lambda}, \theta^{n,\lambda}]\!]]^{1/2})$ converges to 0, uniformly in λ in compact subsets.

By (4.28) with $\gamma^{1/2}\Gamma'(L) = 1/4$, we deduce that, for $T \leqslant c$ (allowing the value of c to decrease from line to line),

$$\mathcal{M}^2_{\mathbb{E}}\big(\vartheta^{n+1,\lambda} - \vartheta^{n,\lambda}\big) \leqslant \tfrac{1}{2}\mathcal{M}^2_{\mathbb{E}}\big(\vartheta^{n,\lambda} - \vartheta^{n-1,\lambda}\big) + C\mathbb{E}\Big[\big(|\chi^\lambda|^2 + \|\chi^\lambda\|_2^2)\psi_n(\lambda)\big]\big)\Big], \tag{4.52}$$

where $(\psi_n(\lambda))_{n \geqslant 0}$ is a sequence of random variables that are bounded by 1 and that converges in probability to 0 as n tends to ∞, uniformly in λ in compact subsets. By a standard uniform integrability argument, we deduce from the bound $\psi_n(\lambda) \leqslant 1$ and from the continuity property of the mapping $\mathbb{R} \ni \lambda \mapsto \chi^\lambda \in L^2$ that $\mathbb{E}[\big(|\chi^\lambda|^2 + \|\chi^\lambda\|_2^2\big)\psi_n(\lambda)]$ tends to 0 as n tends to ∞, uniformly in λ in compact subsets. Therefore, the left-hand side in (4.52) converges to 0, the convergence being geometric, uniformly in λ in compact subsets. By a Cauchy argument, the proof is completed. □

We emphasize that the derivative process $[\mathrm{d}/\mathrm{d}\lambda]_{|\lambda=0}\theta^\lambda$ given by Lemma 4.17 satisfies (4.3) with $\eta := \chi$, with $\theta \equiv \hat{\theta} :\equiv \theta^0$ and $\vartheta \equiv \hat{\vartheta} :\equiv [\mathrm{d}/\mathrm{d}\lambda]_{|\lambda=0}\theta^\lambda$ and with the coefficients given in (4.49). In particular, for T small enough, the uniqueness of the solution to (4.3) (see Remark 4.10) ensures that the derivative process at $\lambda = 0$ depends only on the family $(\xi^\lambda)_{\lambda \in \mathbb{R}}$ through ξ^0 and $[\mathrm{d}/\mathrm{d}\lambda]_{|\lambda=0}\xi^\lambda$. Thus, when $\xi^0 := \xi$ and $[\mathrm{d}/\mathrm{d}\lambda]_{|\lambda=0}\xi^\lambda := \chi$, we may denote by $\partial_\chi\theta^{t,\xi} = (\partial_\chi X^{t,\xi}, \partial_\chi Y^{t,\xi}, \partial_\chi Z^{t,\xi})$ the tangent process at ξ in the direction χ. By linearity of (4.3), $\partial_\chi\theta^{t,\xi}$ is linear in χ. By a direct application of Corollary 4.8 – recall $H_a \equiv 0$ in the current case –, we have

Lemma 4.18. *For any $p \geqslant 1$, there exist two constants $c_p := c_p(L) > 0$ and C_p, such that, for $T \leqslant c_p$ and with $\gamma = c_p$ in (4.16),*

$$\big[\mathcal{M}^{2p}_{\mathbb{E}_t}\big(\partial_\chi\theta^{t,\xi}\big)\big]^{1/(2p)} \leqslant C_p\big(|\chi| + \|\chi\|_2\big).$$

Choosing $p = 1$ and taking the expectation, we get that the mapping $L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d) \ni \chi \mapsto \partial_\chi \theta^{t,\xi} \in \mathcal{S}^2([t,T];\mathbb{R}^d) \times \mathcal{S}^2([t,T];\mathbb{R}^m) \times \mathcal{H}^2([t,T];\mathbb{R}^{m\times d})$ is continuous, which proves that $L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d) \ni \xi \mapsto \theta^{t,\xi} \in \mathcal{S}^2([t,T];\mathbb{R}^d) \times \mathcal{S}^2([t,T];\mathbb{R}^m) \times \mathcal{H}^2([t,T];\mathbb{R}^{m\times d})$ is Gâteaux differentiable. The next lemma shows that the Gâteaux derivative is continuous:

Lemma 4.19. *For any $p \geqslant 1$, there exist two constants $c_p := c_p(L) > 0$ and C_p, such that, for $T \leqslant c_p$ and with $\gamma = c_p$ in* (4.16),

$$\Big[\mathcal{M}^{2p}_{\mathbb{E}_t}\big(\partial_\chi \theta^{t,\xi} - \partial_\chi \theta^{t,\xi'}\big)\Big]^{1/2p} \leqslant C_p\Big(1 \wedge \big\{|\xi - \xi'| + \Phi_{\alpha+1}(t,\xi,\xi')\big\}\Big)\big(|\chi| + \|\chi\|_2\big),$$

where $\Phi_{\alpha+1}(t,\cdot) : [L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)]^2 \to \mathbb{R}_+$ is continuous at any point of the diagonal, does not depend on p and satisfies (4.10) *with α replaced by $\alpha + 1$. The restriction of $\Phi_{\alpha+1}(t,\cdot,\cdot)$ to $[L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^d)]^2$ may be assumed to be independent of $t \in [0,T]$.*

Proof. The proof is a consequence of Corollary 4.12, with $\mathcal{R}^{2p}_a \equiv \Delta\mathcal{R}^{2p}_a \equiv 0$. Example 4.15 (see in particular (4.46)) guarantees that the conditions of Corollary 4.12 are satisfied. We then deduce that (4.38) holds true. Existence of a function $\Phi_{\alpha+1}(t,\cdot,\cdot)$ satisfying the prescription described in the statement then follows from Example 4.15. By Remark 4.16, we can assume that the restriction to $[L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^d)]^2$ is independent of t. ☐

Remark 4.20. *It is easy to derive from Lemma 4.19 that*

$$\begin{aligned}&\|\partial_\chi X^{t,\xi} - \partial_\chi X^{t,\xi'}_s\|_{\mathcal{S}^1} + \|\partial_\chi Y^{t,\xi} - \partial_\chi Y^{t,\xi'}_s\|_{\mathcal{S}^1} + \|\partial_\chi Z^{t,\xi}_s - \partial_\chi Z^{t,\xi'}_s\|_{\mathcal{H}^1}\\ &\quad\leqslant C\Big(\mathbb{E}\big[\big(1 \wedge |\xi - \xi'|^2\big)\big]^{1/2} + \Phi_{\alpha+1}(\xi,\xi')\Big)\|\chi\|_2.\end{aligned}$$

By [2, Proposition A.3]*, the map $L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d) \ni \xi \mapsto (X^{t,\xi}, Y^{t,\xi}, Z^{t,\xi}) \in \mathcal{S}^1([t,T];\mathbb{R}^d) \times \mathcal{S}^1([t,T];\mathbb{R}^m) \times \mathcal{H}^1([t,T];\mathbb{R}^{m\times d})$ is continuously Fréchet differentiable.*

4.2.2. *First-order derivatives of the non McKean-Vlasov system with respect to the measure argument.* We reproduce the same analysis as above, but with the process $\theta^{t,x,[\xi]}$ instead of $\theta^{t,\xi}$ by taking advantage of the fact that the dependence of the coefficients of the system (2.4) upon the law is already known to be smooth. This permits to discuss the differentiability of $\theta^{t,x,[\xi]}$ in a straightforward manner.

We mimic the strategy of the previous subsection. Considering a continuously differentiable mapping $\lambda \mapsto \xi^\lambda \in L^2(\Omega, \mathcal{F}_t, \mathbb{P})$, we are to prove that $\lambda \mapsto \theta^{t,x,[\xi^\lambda]}$ is continuously differentiable. The specific feature is that, for any λ, the coefficients of the FBSDE (2.4) satisfied by $\theta^{t,x,[\xi^\lambda]}$ depend in a smooth way upon the solution θ^{t,ξ^λ} of the FBSDE (2.3). Since we have already established the continuous differentiability of the mapping $\lambda \mapsto \theta^{t,\xi\lambda}$, it suffices now to prove that the solution of a standard FBSDE depending on a parameter λ in a continuously differentiable way is also continuously differentiable with respect to λ. We shall not perform the proof, as it consists of a simple adaptation of the proof used to prove the differentiability of the flow of a standard FBSDE, see [11]. When $\xi^0 = \xi$ and $[\mathrm{d}/\mathrm{d}\lambda]_{\lambda=0}\xi^\lambda = \chi$, we shall denote the directional derivative at ξ along χ by

$$\big(\partial_\chi X^{t,x,[\xi]}_s, \partial_\chi Y^{t,x,[\xi]}_s, \partial_\chi Z^{t,x,[\xi]}_s\big)_{s\in[t,T]},$$

seen as an element of the space $\mathcal{S}^2([t,T];\mathbb{R}^d) \times \mathcal{S}^2([t,T];\mathbb{R}^m) \times \mathcal{H}^2([t,T];\mathbb{R}^{m\times d})$. By the same argument as above, it only depends on the family $(\xi^\lambda)_{\lambda\in\mathbb{R}}$ through the values of ξ and χ, $(\partial_\chi X^{t,x,[\xi]}, \partial_\chi Y^{t,x,[\xi]}, \partial_\chi Z^{t,x,[\xi]})$ satisfying a 'differentiated' system, of the type (4.3), for which uniqueness holds. In (4.3), $\eta = 0$ (since $[\mathrm{d}/\mathrm{d}\lambda]X^{t,x,[\xi]} = 0$), $\theta \equiv \theta^{t,x,[\xi]}$,

$\hat{\theta} \equiv \theta^{t,[\xi]}$, $\vartheta \equiv \partial_\chi \theta^{t,x,[\xi]}$ and $\hat{\vartheta} \equiv \partial_\chi \theta^{t,\xi}$, the tangent process $\partial_\chi \theta^{t,\xi}$ being given by Lemma 4.17. The coefficients are of the general shape (4.5) and (4.6). When h stands for one of the functions b, f, σ or g and V for $\theta^{t,x,[\xi]}$, $\theta^{t,x,[\xi],(0)}$ or $X^{t,x,[\xi]}$ and $\hat{V}$ for $\theta^{t,\xi}$, $\theta^{t,\xi,(0)}$ or $X^{t,\xi}$, according to the cases, it holds, as in (4.49),

$$h_\ell(V, \langle \hat{V}^{(0)} \rangle) = \partial_x h(V, [\hat{V}^{(0)}]), \ \hat{H}_\ell(V, \langle \hat{V}^{(0)} \rangle) = \partial_\mu h(V, [\hat{V}^{(0)}])(\langle \hat{V}^{(0)} \rangle), \ H_a \equiv 0. \quad (4.53)$$

Lemma 4.21. *For any $p \geqslant 1$, there exist two constants $c_p := c_p(L) > 0$ and C_p, such that, for $T \leqslant c_p$ and with $\gamma = c_p$ in (4.16),*

$$\big[\mathcal{M}_{\mathbb{E}}^{2p}\big(\partial_\chi \theta^{t,x,[\xi]}\big)\big]^{1/2p} \leqslant C_p \|\chi\|_2, \quad (4.54)$$

and

$$\big[\mathcal{M}_{\mathbb{E}}^{2p}\big(\partial_\chi \theta^{t,x,[\xi]} - \partial_\chi \theta^{t,x,[\xi']}\big)\big]^{1/2p} \leqslant C_p \Phi_{\alpha+1}(t,\xi,\xi') \|\chi\|_2, \quad (4.55)$$

where $\Phi_{\alpha+1}(t,\cdot,\cdot) : [L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)]^2 \to \mathbb{R}_+$ is continuous at any point of the diagonal, does not depend on p and satisfies (4.10), with α replaced by $\alpha+1$. The restriction of $\Phi_{\alpha+1}(t,\cdot,\cdot)$ to $[L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^d)]^2$ may be assumed to be independent of $t \in [0,T]$.

Remark 4.22. *Note that there is no conditional expectation on $\mathcal{F}_t$ in the above bounds as the initial condition of $\partial_\chi X^{t,x,[\xi]}$ is zero, which means that the filtration that is used for solving the linear equation can be assumed to be almost-surely trivial at time t. For that reason, the right hand side reduces to $\|\chi\|_2$. We stress the fact that it is not $\|\chi\|_{2p}$ but $\|\chi\|_2$, as the dependence upon χ comes through the McKean-Vlasov interaction terms, which is estimated in L^2 norm.*

Proof. Equation (4.54) is a direct consequence of (4.23) in Corollary 4.8, with $\eta = 0$, $\mathcal{R}_a \equiv 0$ and $\hat{\vartheta} = \partial_\chi \theta^{t,\xi}$, combined with Lemma 4.18 (to control the term $\hat{\vartheta}^{(0)} \equiv \partial_\chi \theta^{t,\xi,(0)}$).

We now turn to (4.55). It follows from (4.37) in Corollary 4.12, with $\eta = 0$, $\mathcal{R}_a \equiv \Delta \mathcal{R}_a \equiv 0$, $\theta^\xi \equiv \theta^{t,x,[\xi]}$, $\vartheta^\xi \equiv \theta^{t,x,[\xi]}$, $\hat{\theta}^\xi \equiv \theta^{t,\xi}$ and $\hat{\vartheta}^\xi \equiv \vartheta^{t,\xi}$ (and the same with ξ' instead of ξ). By Lemma 4.19, we can indeed bound the last term in (4.37) by

$$\sup_{\varsigma \in \mathcal{O}} \sup_{\|\Lambda_0\|_2 \leqslant K} \bigg\{ \mathbb{E}\Big[\big(\Lambda_0 \wedge \bar{\Psi}(\varsigma,\xi)\big) \Big(1 \wedge \big\{|\xi - \xi'| + \Phi_{\alpha+1}(t,\xi,\xi')\big\}\Big) \big(|\chi| + \|\chi\|_2\big) \Big] \bigg\}.$$

with$\bar{\Psi}$ defined in (4.36) and Ψ given in this definition by (4.46). Following Example 4.15 (see in particular (4.45)), we deduce that (4.55) holds true (modifying $\Phi_{\alpha+1}(t,\cdot,\cdot)$ if necessary). By Remark 4.16, we can assume that the restriction of $\Phi_{\alpha+1}(t,\cdot,\cdot)$ to $[L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^d)]^2$ is independent of t. $\square$

Following Remark 4.20 to pass from Gâteaux to Fréchet, we deduce:

Lemma 4.23. *For $T \leqslant c$ with $c := c(L) > 0$, $t \in [0,T]$ and $x \in \mathbb{R}^d$, the function $L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d) \ni \xi \mapsto \mathcal{U}(t,x,\xi) = Y_t^{t,x,[\xi]}$ is Fréchet continuously differentiable. In particular, the function $\mathcal{P}^2(\mathbb{R}^d) \ni \mu \mapsto U(t,x,\mu)$ is differentiable in Lions' sense. Moreover, for all $x \in \mathbb{R}^d$, for all $\xi, \xi' \in L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^d)$, we have, with $\mu = [\xi]$ and $\mu' = [\xi']$,*

$$\|\partial_\mu U(t,x,\mu)(\xi)\|_2 \leqslant C, \quad \|\partial_\mu U(t,x,\mu)(\xi) - \partial_\mu U(t,x,\mu')(\xi')\|_2 \leqslant C \Phi_{\alpha+1}(\xi,\xi'), \quad (4.56)$$

where $\Phi_{\alpha+1} : [L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^d)]^2 \to \mathbb{R}_+$ is continuous at any point of the diagonal and satisfies (4.10), with α replaced by $\alpha+1$.

Proof. Fréchet differentiability is a consequence of the continuity Lemma 4.21 that permits to pass from Gâteaux to Fréchet on the model of Remark 4.20. We then have $\partial_\chi Y_t^{t,x,[\xi]} = \mathbb{E}[\partial_\mu U(t,x,[\xi])(\xi)\chi]$. Combined with Lemma 4.21, this gives (4.56), but with $\Phi_{\alpha+1}$ defined on $[L^2(\Omega,\mathcal{F}_0,\mathbb{P};\mathbb{R}^d)]^2$, which requires that ξ and ξ' belong to $L^2(\Omega,\mathcal{F}_0,\mathbb{P};\mathbb{R}^d)$. The main issue is to prove that $\Phi_{\alpha+1}$ may be defined on the whole $[L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d)]^2$. It is then worth mentioning that $\|\partial_\mu U(t,x,\mu)(\xi)-\partial_\mu U(t,x,\mu')(\xi')\|_2$ only depends on the law of (ξ,ξ'). Given $(\xi,\xi')\in[L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d)]^2$, we can always find a pair $(\tilde\xi,\tilde\xi')\in[L^2(\Omega,\mathcal{F}_0,\mathbb{P};\mathbb{R}^d)]^2$ with the same distribution (provided that $(\Omega,\mathcal{F}_0,\mathbb{P})$ is rich enough). This says that, with the $\Phi_{\alpha+1}$ given by Lemma 4.21, for all $\xi,\xi'\in L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d)$,

$$\begin{aligned}&\big\|\partial_\mu U(t,x,\mu)(\xi)-\partial_\mu U(t,x,\mu')(\xi')\big\|_2\leqslant\tilde\Phi_{\alpha+1}(\xi,\xi'),\\&\quad\text{with }\tilde\Phi_{\alpha+1}(\xi,\xi'):=\inf\big\{\Phi_{\alpha+1}(\tilde\xi,\tilde\xi'),\quad(\tilde\xi,\tilde\xi')\in L^2(\Omega,\mathcal{F}_0,\mathbb{P};\mathbb{R}^d),\quad(\tilde\xi,\tilde\xi')\sim(\xi,\xi')\big\}.\end{aligned}$$

Clearly, $\tilde\Phi_{\alpha+1}$ is defined on the whole $[L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d)]^2$. It satisfies (4.10). Continuity at any point of the diagonal may be proved as follows. Given a sequence $(\xi_n,\xi_n')_{n\geqslant1}$ that converges to some (ξ,ξ) in $[L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d)]^2$, we may find a pair $(\tilde\xi,\tilde\xi)\in[L^2(\Omega,\mathcal{F}_0,\mathbb{P};\mathbb{R}^d)]^2$ with the same law as (ξ,ξ). Now, for any $n\geqslant1$, we can construct $(\tilde\xi_n,\tilde\xi_n')$ in $[L^2(\Omega,\mathcal{F}_0,\mathbb{P};\mathbb{R}^d)]^2$ such that the 4-tuple $(\tilde\xi_n,\tilde\xi_n',\tilde\xi,\tilde\xi)$ has the same law as (ξ_n,ξ_n',ξ,ξ) (it suffices to use the conditional law of (ξ_n,ξ_n') given (ξ,ξ)). Then, $(\tilde\xi_n,\tilde\xi_n')_{n\geqslant1}$ converges to $(\tilde\xi,\tilde\xi)$ in L^2. From the inequality $\tilde\Phi_{\alpha+1}(\xi_n,\xi_n')\leqslant\Phi_{\alpha+1}(\tilde\xi_n,\tilde\xi_n')$, $\tilde\Phi_{\alpha+1}(\xi_n,\xi_n')$ tends to 0. □

We now discuss the Lipschitz property in x of $\partial_\mu U(t,x,\mu)$:

Lemma 4.24. *For $T\leqslant c$, with $c:=c(L)>0$, we can find a constant C such that, for ξ with μ as distribution,*

$$\forall x,x'\in\mathbb{R}^d,\quad\|\partial_\mu U(t,x,\mu)(\xi)-\partial_\mu U(t,x',\mu)(\xi)\|_2\leqslant C|x-x'|.$$

Proof. Thanks to the relationship $\partial_\chi Y_t^{t,x,[\xi]}=\mathbb{E}[\partial_\mu U(t,x,[\xi])(\xi)\chi]$, it suffices to discuss the Lipschitz property (in x) of the tangent process $(\partial_\chi X_s^{t,x,\xi},\partial_\chi Y_s^{t,x,\xi},\partial_\chi Z_s^{t,x,\xi})_{s\in[t,T]}$, seen as an element of $\mathcal{S}^2([t,T];\mathbb{R}^d)\times\mathcal{S}^2([t,T];\mathbb{R}^m)\times\mathcal{H}^2([t,T];\mathbb{R}^{m\times d})$, ξ and χ denoting elements of $L^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^d)$.

Basically, the strategy is the same as in the proofs of Lemmas 4.19 and 4.21. It is based on a tailored-made version of Corollary 4.12, obtained by applying Lemma 4.1 and Lemma 4.9 with $\theta\equiv\bar\theta:=\theta^{t,x,[\xi]}$, $\theta'\equiv\bar\theta':=\theta^{t,x',[\xi]}$, $\hat\theta\equiv\hat\theta'\equiv\check\theta\equiv\check\theta':=\theta^{t,\xi}$ and $\hat\vartheta\equiv\hat\vartheta'\equiv\check\vartheta\equiv\check\vartheta':=\partial_\chi\theta^{t,\xi}$. Informally, it consists in choosing $\eta=0$ and in replacing $|\xi-\xi'|$ by $|x-x'|$ and $\Phi_\alpha(\xi,\xi')$ by 0 in the statement of Corollary 4.12. We end up with $|\partial_\chi Y_t^{t,x,[\xi]}-\partial_\chi Y_t^{t,x',[\xi]}|\leqslant C|x'-x|\|\chi\|_2$. □

4.2.3. *Derivatives with respect to the space argument.* We now discuss the derivatives of U with respect to the variable x. Since the process $\theta^{t,x,[\xi]}=(X^{t,x,[\xi]},Y^{t,x,[\xi]},Z^{t,x,[\xi]})$ may be seen as the solution of a standard FBSDE parametrized by the law of ξ, we can apply the results in [11] on the smoothness of the flow of a classical FBSDE in short time. Given $t\in[0,T]$ and $\xi\in L^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^d)$, we deduce that the function $\mathbb{R}^d\ni x\mapsto\theta^{t,x,[\xi]}=(X^{t,x,[\xi]},Y^{t,x,[\xi]},Z^{t,x,[\xi]})\in\mathcal{S}^2([t,T];\mathbb{R}^d)\times\mathcal{S}^2([t,T];\mathbb{R}^m)\times\mathcal{H}^2([t,T];\mathbb{R}^{m\times d})$ is continuously differentiable, the derivative process at point $x\in\mathbb{R}^d$ being denoted by $\partial_x\theta^{t,x,[\xi]}=(\partial_x X^{t,x,[\xi]},\partial_x Y^{t,x,[\xi]},\partial_x Z^{t,x,[\xi]})$. To be self-contained, notice that the same result could be obtained by applying the results of Subsection 4.1, with the following version of $H(r,\cdot)$:

$$H(r,V_r^{t,x,[\xi]})(\mathcal{V}_r^{t,x,[\xi]})=\partial_x h(V_r^{t,x,[\xi]},[V_r^{t,\xi}])\mathcal{V}_r^{t,x,[\xi]}.\tag{4.57}$$

As a consequence, we easily get, for $T \leqslant c_p$, $c_p := c_p(L)$ and with $\gamma = c_p$ in (4.16), $[\mathcal{M}_{\mathbb{E}}^{2p}(\partial_x \theta^{t,x,[\xi]})]^{1/2p} \leqslant C_p$. Recalling the identity $U(t,x,[\xi]) = \theta_t^{t,x,[\xi]}$, we recover the fact that $\mathbb{R}^d \ni x \mapsto U(t,x,[\xi])$ is continuously differentiable and that $\|\partial_x U\|_\infty \leqslant C$, see also [11]. On the same model (for instance by adapting Lemmas 4.19 or 4.21 to investigate the difference $\partial_x \theta^{t,x,[\xi]} - \partial_x \theta^{t,x',[\xi]}$ for two different $x, x' \in \mathbb{R}^d$ or by taking benefit from the results proved in [11]), it can be checked that, for any $t \in [0,T]$, any $\xi \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)$, the mapping $\mathbb{R}^d \ni x \mapsto \partial_x U(t,x,[\xi])$ is C-Lipschitz continuous. Intuitively, such a bound is much simpler to get than the bound for the continuity of $\partial_\mu U$ because of the very simple structure of $H(r,\cdot)$ in (4.57), the function $\partial_x h$ being Lipschitz-continuous with respect to the first argument.

To get the smoothness of $\partial_x U$ in the direction μ, we may investigate the difference $\partial_x \theta^{t,x,[\xi]} - \partial_x \theta^{t,x,[\xi']}$ for two different $\xi, \xi' \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)$. Reapplying Corollary 4.12, exactly in the same way as in the proof of Lemma 4.21, we deduce

$$\forall x \in \mathbb{R}^d,\ \xi, \xi' \in L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^d),\ \big|\partial_x U(t,x,[\xi]) - \partial_x U(t,x,[\xi'])\big| \leqslant \Phi_{\alpha+1}(\xi,\xi'). \tag{4.58}$$

Actually, the above bound could be improved. Indeed, it also holds with $\Phi_{\alpha+1}(\xi,\xi')$ replaced by $\Phi_\alpha(\xi,\xi')$. The reason is that, in the analysis of $\partial_x \theta^{t,x,[\xi]} - \partial_x \theta^{t,x,[\xi']}$, there are no derivatives in the direction of the measure, whereas these are precisely these terms that make $\Phi_{\alpha+1}(\xi,\xi')$ appear in the proof of Lemma 4.23 (or equivalently of Lemma 4.21). In order to keep some homogeneity between the various estimates we have on the derivatives of U, we feel it is more convenient to keep $\Phi_{\alpha+1}(\xi,\xi')$ in the above right-hand side.

4.2.4. *Final statement.* The following is the complete statement about the first-order differentiability:

Theorem 4.25. *For $T \leqslant c$, with $c := c(L) > 0$ and $t \in [0,T]$, the function $\mathbb{R}^d \times L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^d) \ni \xi \mapsto U(t,x,[\xi]) = \mathcal{U}(t,x,\xi)$ is continuously differentiable and there exists a constant $C \geqslant 0$, such that for all $x, x' \in \mathbb{R}^d$, for all $\xi, \xi' \in L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^d)$,*

$$\begin{aligned}
&|U(t,x,\mu) - U(t,x',\mu')| \leqslant C\big(|x-x'| + W_2(\mu,\mu')\big) \\
&|\partial_x U(t,x,\mu) - \partial_x U(t,x',\mu')| + \|\partial_\mu U(t,x,\mu)(\xi) - \partial_\mu U(t,x',\mu')(\xi')\|_2 \\
&\quad \leqslant C\big(|x-x'| + \Phi_{\alpha+1}(\xi,\xi')\big),
\end{aligned} \tag{4.59}$$

where $\Phi_{\alpha+1} : [L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^d)]^2 \to \mathbb{R}_+$ is continuous at any point of the diagonal and satisfies (4.10), with α replaced by $\alpha+1$. In particular, for any $x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, we can find a locally Lipschitz continuous version of the mapping $\mathbb{R}^d \ni v \mapsto \partial_\mu U(t,x,\mu)(v)$.

Moreover, the functions $[0,T] \times \mathbb{R}^d \times L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^d) \ni (t,x,\xi) \mapsto \partial_x U(t,x,[\xi]) \in \mathbb{R}^d$ and $[0,T] \times \mathbb{R}^d \times L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^d) \ni (t,x,\xi) \mapsto \partial_\mu U(t,x,[\xi])(\xi) \in L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^d)$ are continuous.

Finally, for any $t \in [0,T]$, $\xi \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)$ and $C' \geqslant 0$,

$$\lim_{\mathbb{P}(A) \to 0, A \in \mathcal{A}} \sup_{(t,x) \in [0,T] \times \mathbb{R}^d} \sup_{\Lambda \in L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^d) : \|\Lambda\|_2 \leqslant C'} \big|\mathbb{E}\big[\partial_\mu U(t,x,[\xi])(\xi)\Lambda \mathbf{1}_A\big]\big| = 0, \tag{4.60}$$

which is the analogue of the uniform integrability property described in ($\boldsymbol{H}\mathbf{1}$) for the original coefficients b, σ, f and g.

Proof. The Lipschitz property of U is a direct consequence of the bounds we have for $\partial_x U$ and $\partial_\mu U$ (or equivalently of Lemma 4.1). The joint continuous differentiability is a consequence of the partial continuous differentiability and of the joint continuity properties of the derivatives. The extension of $\Phi_{\alpha+1}$ to the whole $[L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d)]^2$ is achieved as in the proof of Lemma 4.23.

The local Lipschitz property of $\mathbb{R}^d \ni v \mapsto \partial_\mu U(t,x,\mu)(v)$ follows from Proposition 3.8.

We now discuss the continuity of $[0,T] \ni t \mapsto \partial_\mu U(t,x,[\xi])(\xi) \in L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d)$. Clearly, there is no loss of generality in assuming that $\xi \in L^2(\Omega,\mathcal{F}_0,\mathbb{P};\mathbb{R}^d)$. Given $\xi,\chi \in L^2(\Omega,\mathcal{F}_0,\mathbb{P};\mathbb{R}^d)$ and $0 \leqslant t \leqslant s \leqslant T$, it suffices to bound the time increment $\mathbb{E}[(\partial_\mu U(t,x,[\xi])(\xi) - \partial_\mu U(s,x,[\xi])(\xi))\chi]$ by $C(t,s)\|\chi\|_2$, the constant $C(t,s)$ being independent of χ and converging to 0 as $s-t$ tends to 0. We have

$$\begin{aligned}
&\mathbb{E}\big[\big(\partial_\mu U(t,x,[\xi])(\xi) - \partial_\mu U(s,x,[\xi])(\xi)\big)\chi\big] \\
&= \hat{\mathbb{E}}\big[\big(\partial_\mu U(t,x,[\xi])(\langle\xi\rangle) - \partial_\mu U(s,x,[\xi])(\langle\xi\rangle)\big)\langle\chi\rangle\big] \\
&= \mathbb{E}\hat{\mathbb{E}}\big[\big(\partial_\mu U\big(s,X_s^{t,x,[\xi]},\big[X_s^{t,\xi}\big]\big)\big(\big\langle X_s^{t,\xi}\big\rangle\big) - \partial_\mu U(s,x,[\xi])\big(\langle\xi\rangle\big)\big)\langle\chi\rangle\big] \\
&\quad + \mathbb{E}\hat{\mathbb{E}}\big[\big(\partial_\mu U(t,x,[\xi])\big(\langle\xi\rangle\big) - \partial_\mu U\big(s,X_s^{t,x,[\xi]},\big[X_s^{t,\xi}\big]\big)\big(\big\langle X_s^{t,\xi}\big\rangle\big)\big)\langle\chi\rangle\big].
\end{aligned} \tag{4.61}$$

By the smoothness property of $\partial_\mu U(s,\cdot)$, the first term in the right-hand side is bounded by $C(\mathbb{E}[|X_s^{t,x,[\xi]} - x|^2]^{1/2} + \Phi_{\alpha+1}(X_s^{t,\xi},\xi))\|\chi\|_2$, the constant C being allowed to increase from line to line. The coefficients of (2.3) and (2.4) being at most of linear growth, we deduce from (4.1) that $\mathbb{E}[|X_s^{t,\xi} - \xi|^2]^{1/2}$ and $\mathbb{E}[|X_s^{t,x,[\xi]} - x|^2]^{1/2}$ are less than $C(1+\|\xi\|_2)(s-t)^{1/2}$ and $C(1+|x|+\|\xi\|_2)(s-t)^{1/2}$ respectively. Therefore, the first term in the last line of (4.61) is bounded by

$$C\Big[\big(1+|x|+\|\xi\|_2\big)(s-t)^{1/2} + \sup_{\xi':\|\xi'-\xi\|_2 \leqslant C(1+\|\xi\|_2)(s-t)^{1/2}} \Phi_{\alpha+1}(\xi',\xi)\Big]\|\chi\|_2. \tag{4.62}$$

Clearly, the term in brackets goes to 0 with $s-t$.

We now handle the second term in the last line of (4.61). Differentiating (with respect to ξ in the direction χ) the relationships $U(t,x,[\xi]) = Y_t^{t,x,[\xi]}$ and $\mathbb{E}[U(t,X_s^{t,x,[\xi]},[X_s^{t,\xi}])] = \mathbb{E}[Y_s^{t,x,[\xi]}]$, we obtain

$$\begin{aligned}
&\mathbb{E}\hat{\mathbb{E}}\big[\big(\partial_\mu U(t,x,[\xi])\big(\langle\xi\rangle\big) - \partial_\mu U\big(s,X_s^{t,x,[\xi]},\big[X_s^{t,\xi}\big]\big)\big(\big\langle X_s^{t,\xi}\big\rangle\big)\big)\langle\chi\rangle\big] \\
&= \mathbb{E}\big[\partial_\chi Y_t^{t,x,[\xi]} - \partial_\chi Y_s^{t,x,[\xi]}\big] + \mathbb{E}\big[\partial_x U\big(s,X_s^{t,x,[\xi]},[X_s^{t,\xi}]\big)\partial_\chi X_s^{t,x,[\xi]}\big] \\
&+ \mathbb{E}\Big[\hat{\mathbb{E}}\Big[\partial_\mu U\big(s,X_s^{t,x,[\xi]},\big[X_s^{t,\xi}\big]\big)\big(\langle X_s^{t,\xi}\rangle\big)\langle\partial_\chi X_s^{t,\xi} - \chi\rangle\Big]\Big].
\end{aligned}$$

The first term is equal to $\mathbb{E}\int_t^s F^{(1)}(r,\theta_r^{t,x,[\xi]},\langle\theta_r^{t,\xi,(0)}\rangle)(\partial_\chi\theta_r^{t,x,[\xi]},\langle\partial_\chi\theta_r^{t,\xi,(0)}\rangle)\mathrm{d}r$ (with the notations of (4.4)). By $(\boldsymbol{H1})$ and Lemmas 4.18 and 4.21 (with $p=1$), it is bounded by $C(s-t)^{1/2}\|\chi\|_2$. Since $\partial_x U$ is bounded, the second term is less than $C\mathbb{E}[|\partial_\chi X_s^{t,x,[\xi]}|] = C\mathbb{E}[|\partial_\chi X_s^{t,x,[\xi]} - \partial_\chi X_t^{t,x,[\xi]}|]$. By $(\boldsymbol{H1})$ and Lemmas 4.18 and 4.21 again, it is less than $C(s-t)^{1/2}\|\chi\|_2$. For the third term, we first apply Cauchy-Schwarz inequality to get that it is less than $C\mathbb{E}[|\partial_\chi X_s^{t,\xi} - \chi|^2]^{1/2}$, recall (4.56). Then, by $(\boldsymbol{H1})$ and Lemma 4.18, it is bounded by $C(s-t)^{1/2}\|\chi\|_2$. Continuity of $[0,T] \ni t \mapsto \partial_\mu U(t,x,[\xi])(\xi) \in L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d)$ easily follows. Continuity of $[0,T] \ni t \mapsto \partial_x U(t,x,[\xi]) \in \mathbb{R}^d$ may be proved in the same way. Together with the uniform continuity estimates (4.59), we deduce that the functions $[0,T]\times\mathbb{R}^d\times L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d) \ni (t,x,\xi) \mapsto \partial_x U(t,x,[\xi]) \in \mathbb{R}^d$ and $[0,T]\times\mathbb{R}^d\times L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d) \ni (t,x,\xi) \mapsto \partial_\mu U(t,x,[\xi])(\xi) \in L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d)$ are continuous.

We now prove (4.60). For $A \in \mathcal{A}$ and $\Lambda \in L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d)$ with $\|\Lambda\|_2 \leqslant C'$, we have

$$\big|\mathbb{E}\big[\partial_\mu U(t,x,[\xi])(\xi)\Lambda\mathbf{1}_A\big]\big| = \partial_\chi Y_t^{t,x,[\xi]}, \quad \text{with } \chi = \Lambda\mathbf{1}_A. \tag{4.63}$$

We now apply (4.18) in Lemma 4.7 with $\theta \equiv \bar{\theta} :\equiv \theta^{t,x,[\xi]}$, $\vartheta \equiv \bar{\vartheta} :\equiv \partial_\chi \theta^{t,x,[\xi]}$, $\hat{\theta} \equiv \check{\theta} :\equiv \theta^{t,\xi}$, $\hat{\vartheta} \equiv \check{\vartheta} :\equiv \partial_\chi \theta^{t,\xi}$ and $\eta = 0$. The coefficients driving (4.3) are given by (4.53). By (4.18), we get that, for $T \leqslant \gamma$ with γ in (4.16) given by $\gamma^{1/2}\Gamma_2(L) = \min[(1/8L^2), 1/2]$,

$$[\partial_\chi Y_t^{t,x,[\xi]}]^2 \leqslant \frac{1}{4L^2} \sup_{s\in[t,T]} \mathcal{N}^{2,C}_{\mathbb{E}_t}\Big(\theta_s^{t,\xi,(0)}, \big(\mathcal{M}^2_{\mathbb{E}_t}(\partial_\chi \theta^{t,\xi,(0)})\big)^{1/2}\Big). \tag{4.64}$$

We use (4.39) (with Ψ given by (4.46)) to bound the above term. For any $\varepsilon > 0$,

$$\begin{aligned}
&\sup_{x\in\mathbb{R}^d, s\in[t,T]} \mathcal{N}^{2,C}_{\mathbb{E}_t}\Big(\theta_s^{t,\xi,(0)}, \big(\mathcal{M}^2_{\mathbb{E}_t}(\partial_\chi \theta^{t,\xi,(0)})\big)^{1/2}\Big) \\
&\leqslant \sup_{(w,s)\in\mathbb{R}^k\times[t,T]} \sup_{\|\Lambda_0\|_2\leqslant L} \Big\{\mathbb{E}\Big[\big(\Lambda_0 \wedge \Psi((w,s),\xi)\big)\big(\mathcal{M}^2_{\mathbb{E}_t}(\partial_\chi \theta^{t,\xi,(0)})\big)^{1/2}\Big]^2\Big\} \qquad (4.65)\\
&\leqslant L^2\varepsilon \\
&\quad + \sup_{(w,s)\in\mathbb{R}^k\times[t,T]} \sup_{\|\Lambda_0\|_2\leqslant L} \Big\{\mathbb{E}\Big[\big(\Lambda_0 \wedge \Psi((w,s),\xi)\big)^2 \mathbf{1}_{\{\mathcal{M}^2_{\mathbb{E}_t}(\partial_\chi\theta^{t,\xi,(0)})\geqslant\varepsilon\}}\Big]^2 \mathcal{M}^2_{\mathbb{E}}(\partial_\chi \theta^{t,\xi,(0)})\Big\},
\end{aligned}$$

where we denoted $\mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$ by $\mathbb{R}^k$ and we used Cauchy-Schwarz' inequality in the last line. Recall that in the above suprema, Λ_0 takes values in $\mathbb{R}_+$.

By Lemma 4.18, $[\mathcal{M}^2_{\mathbb{E}}(\partial_\chi \theta^{t,\xi,(0)})]^{1/2} \leqslant C\|\chi\|_2 \leqslant CC'$. By uniform integrability of the family $(\Psi^2((w,s),\xi))_{w\in\mathbb{R}^k, s\in[t,T]}$, it thus suffices to prove that

$$\lim_{\mathbb{P}(A)\to 0} \mathbb{P}\big(\mathcal{M}^2_{\mathbb{E}_t}(\partial_\chi \theta^{t,\xi,(0)}) \geqslant \varepsilon\big) = 0 \tag{4.66}$$

in order to prove (4.60) (recall that the above probability depends on A through $\chi = \Lambda\mathbf{1}_A$). We reapply (4.18) in Lemma 4.7, but with $\theta \equiv \bar{\theta} \equiv \hat{\theta} \equiv \check{\theta} :\equiv \theta^{t,\xi}$, $\vartheta \equiv \bar{\vartheta} \equiv \hat{\vartheta} \equiv \check{\vartheta} :\equiv \partial_\chi \theta^{t,\xi}$ and $\eta = \Lambda\mathbf{1}_A$. Following (4.64) and (4.65), we get

$$\begin{aligned}
&\mathcal{M}^2_{\mathbb{E}_t}(\partial_\chi \theta^{t,\xi,(0)}) \qquad (4.67)\\
&\leqslant C\Lambda^2\mathbf{1}_A + \frac{1}{4L^2} \sup_{(w,s)\in\mathbb{R}^k\times[t,T]} \sup_{\|\Lambda_0\|_2\leqslant L} \Big\{\mathbb{E}\Big[\big(\Lambda_0 \wedge \Psi((w,s),\xi)\big)\big(\mathcal{M}^2_{\mathbb{E}_t}(\partial_\chi \theta^{t,\xi,(0)})\big)^{1/2}\Big]^2\Big\}.
\end{aligned}$$

Multiplying by $\mathbf{1}_{A^\complement}$ and taking the expectation, we deduce that

$$\begin{aligned}
&\mathbb{E}\big[\mathbf{1}_{A^\complement}\mathcal{M}^2_{\mathbb{E}_t}(\partial_\chi \theta^{t,\xi,(0)})\big] \\
&\leqslant \frac{1}{2L^2} \sup_{(w,s)\in\mathbb{R}^k\times[t,T]} \sup_{\|\Lambda_0\|_2\leqslant L} \Big\{\mathbb{E}\Big[\big(\Lambda_0 \wedge \Psi((w,s),\xi)\big)\big(\mathcal{M}^2_{\mathbb{E}_t}(\partial_\chi \theta^{t,\xi,(0)})\big)^{1/2}\mathbf{1}_{A^\complement}\Big]^2\Big\} \\
&\quad + \frac{1}{2L^2} \sup_{(w,s)\in\mathbb{R}^k\times[t,T]} \sup_{\|\Lambda_0\|_2\leqslant L} \Big\{\mathbb{E}\Big[\big(\Lambda_0 \wedge \Psi((w,s),\xi)\big)\big(\mathcal{M}^2_{\mathbb{E}_t}(\partial_\chi \theta^{t,\xi,(0)})\big)^{1/2}\mathbf{1}_{A}\Big]^2\Big\} \\
&\leqslant \frac{1}{2}\mathbb{E}\big[\mathbf{1}_{A^\complement}\mathcal{M}^2_{\mathbb{E}_t}(\partial_\chi \theta^{t,\xi,(0)})\big] + C \sup_{(w,s)\in\mathbb{R}^d\times[t,T]} \sup_{\|\Lambda_0\|_2\leqslant L} \Big\{\mathbb{E}\Big[\big(\Lambda_0 \wedge \Psi((w,s),\xi)\big)^2\mathbf{1}_A\Big]\Big\},
\end{aligned}$$

where we used Cauchy-Schwarz' inequality twice to get the last line. By uniform integrability of the family $(\Psi^2((w,s),\xi))_{x\in\mathbb{R}^k, s\in[t,T]}$, the second term in the last line tends to 0 with $\mathbb{P}(A)$. Therefore, $\mathbb{E}[\mathbf{1}_{A^\complement}\mathcal{M}^2_{\mathbb{E}_t}(\partial_\chi \theta^{t,\xi,(0)})]$ also tends to 0 with $\mathbb{P}(A)$.

Going back to (4.67), taking the root and then the expectation and splitting the expectation in the right-hand side according to the indicator functions of $A^\complement$ and A, we get in the same way

$$\mathbb{E}\big[\big(\mathcal{M}^2_{\mathbb{E}_t}(\partial_\chi \theta^{t,\xi,(0)})\big)^{1/2}\big] \leqslant C\big(\mathbb{P}(A)\big)^{1/2} + C\mathbb{E}\big[\mathbf{1}_{A^{\complement}}\mathcal{M}^2_{\mathbb{E}_t}(\partial_\chi \theta^{t,\xi,(0)})\big]^{1/2}$$
$$+ C \sup_{(x,s)\in\mathbb{R}^d\times[t,T]} \sup_{\|\Lambda_0\|_2\leqslant L} \Big\{\mathbb{E}\Big[\big(\Lambda_0 \wedge \Psi((x,s),\xi)\big)^2 \mathbf{1}_A\Big]^{1/2}\Big\}.$$

The right-hand side tends to 0 with $\mathbb{P}(A)$, which proves (4.66). □

4.3. **Study of the second-order differentiability.** The goal is now to discuss the second-order differentiability of U.

4.3.1. *Path property of $Z^{t,x,[\xi]}$ in $\mathcal{S}^2([t,T];\mathbb{R}^{m\times d})$.* We start with the following remark. In the previous subsection, we proved that the function $\partial_x U$ was Lipschitz continuous with respect to the variables x and μ. Recalling the standard representation formula

$$Z^{t,x,[\xi]}_s = \partial_x U\big(s, X^{t,x,[\xi]}_s, [X^{t,\xi}_s]\big)\sigma\big(X^{t,x,[\xi]}_s, Y^{t,x,[\xi]}_s, [X^{t,\xi}_s, Y^{t,\xi}_s]\big), \quad s \in [t,T], \tag{4.68}$$

see (2.7), we may derive bounds for $Z^{t,x,[\xi]}$ in the space $\mathcal{S}^2([t,T],\mathbb{R}^{m\times d})$ instead of $\mathcal{H}^2([t,T],\mathbb{R}^{m\times d})$ (and similarly for $Z^{t,\xi}$ by replacing x by ξ in the above formula). Under assumption (***H2***), which contains (***Hσ***), σ is known to be bounded, so that $Z^{t,x,[\xi]}$ and $Z^{t,\xi,[\xi]}$ are indeed bounded (in L^∞), independently of ξ. Moreover, for any $p \geqslant 1$, for $T \leqslant c_p$ with $c_p := c_p(L)$, we can find $C_p \geqslant 0$ such that, for $\xi,\xi' \in L^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^d)$,

$$\begin{aligned}
&\mathbb{E}_t\Big[\sup_{r\in[t,T]}\big|Z^{t,x,[\xi]}_r - Z^{t,x',[\xi']}_r\big|^{2p}\Big]^{1/2p} \leqslant C_p\Big(1 \wedge \big\{|x-x'| + \Phi_{\alpha+1}(t,\xi,\xi')\big\}\Big),\\
&\mathbb{E}_t\Big[\sup_{r\in[t,T]}\big|Z^{t,\xi}_r - Z^{t,\xi'}_r\big|^{2p}\Big]^{1/2p} \leqslant C_p\Big(1 \wedge \big\{|\xi-\xi'| + \Phi_{\alpha+1}(t,\xi,\xi')\big\}\Big).
\end{aligned} \tag{4.69}$$

Note that the term $\Phi_{\alpha+1}(t,\xi,\xi')$ comes from the fact that, when handling the difference $\partial_x U(s, X^{t,x,[\xi]}_s, [X^{t,\xi}_s]) - \partial_x U(s, X^{t,x',[\xi']}_s, [X^{t,\xi'}_s])$, we get $C[|X^{t,x,[\xi]}_s - X^{t,x',[\xi']}_s| + \Phi_{\alpha+1}(X^{t,\xi}_s, X^{t,\xi'}_s)]$ as bound. We then apply (4.44) in Lemma 4.14 (with $\alpha+1$ instead of α) to handle $\Phi_{\alpha+1}(X^{t,\xi}_s, X^{t,\xi'}_s)$. The part involving σ in the definition of $Z^{t,x,[\xi]}_s$ can be treated by means of Lemma 4.1 using the fact that σ is Lipschitz continuous. Following Remark 4.16 and as in the statement of Lemma 4.19, the restriction of $\Phi_{\alpha+1}(t,\cdot,\cdot)$ to the space $[L^2(\Omega,\mathcal{F}_0,\mathbb{P};\mathbb{R}^d)]^2$ may be assumed to be independent of $t \in [0,T]$.

4.3.2. *Partial smoothness of $\partial_\mu U$. Overview.* By making use of (4.69), we first discuss the existence and the smoothness of the second-order derivatives of U in the measure argument. The first remark is that we only need to discuss partial $\mathcal{C}^2$ differentiability in order to prove the chain rule. This says that, when investigating the second-order derivatives, there is no need to prove that the function U has a twice Fréchet differentiable lifted version. Roughly speaking, the only thing we need is the differentiability of the mapping $\mathbb{R}^d \times \mathbb{R}^d \ni (x,v) \mapsto \partial_\mu U(t,x,\mu)(v)$ (at least when v is restricted to the support of μ), together with the continuity (in (t,x,μ,v)) of the derivatives (again, at least when v is restricted to the support of μ). In order to differentiate in the direction v without differentiating in the direction μ, we shall make use of Theorem 3.6, which has been specifically designed for that purpose. Basically, we are to differentiate the lifted version of $\partial_\mu U(t,x,\mu)$ along trajectories $(\xi^\lambda)_{\lambda\in\mathbb{R}}$ that are continuously differentiable in

$L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)$, with the constraint that all the $(\xi^\lambda)_{\lambda\in\mathbb{R}}$ have the same distribution and the assumption that

$$\forall \lambda \in \mathbb{R}, \quad \Big\| \frac{\mathrm{d}}{\mathrm{d}\lambda} \xi^\lambda \Big\|_\infty \leqslant 1, \quad \text{with the additional notation } \zeta := \frac{\mathrm{d}}{\mathrm{d}\lambda}_{|\lambda=0} \xi^\lambda. \tag{4.70}$$

In this framework, we will make use of the following technical lemma:

Lemma 4.26. *Consider a function $h : \mathbb{R}^k \times \mathcal{P}_2(\mathbb{R}^l) \to \mathbb{R}$ as in ($\boldsymbol{H2}$), a continuously differentiable mapping $\mathbb{R} \ni \lambda \mapsto \chi^\lambda \in L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^l)$ with the property that all the χ^λ have the same distribution, and a random variable $\varpi \in L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^l)$ such that, for any bounded interval $[a, b] \subset \mathbb{R}$, the family*

$$\Big(\frac{\mathrm{d}\chi^\lambda}{\mathrm{d}\lambda} \otimes \varpi \Big)_{\lambda\in[a,b]}$$

is uniformly square integrable (the tensorial product acting on $\mathbb{R}^l$). Then, the function

$$\mathbb{R}^k \times \mathbb{R} \ni (w, \lambda) \mapsto \mathbb{E}\big[\partial_\mu h(w, [\chi^\lambda])(\chi^\lambda)\varpi\big] = \mathbb{E}\big[\partial_\mu h(w, [\chi^0])(\chi^\lambda)\varpi\big]$$

is continuously differentiable, with

$$\mathbb{R}^k \times \mathbb{R} \ni (w, \lambda) \mapsto \mathbb{E}\Big[\partial_w\big[\partial_\mu h(w, [\chi^0])\big](\chi^\lambda)\varpi\Big]$$

$$\mathbb{R}^k \times \mathbb{R} \ni (w, \lambda) \mapsto \mathbb{E}\Big[\partial_v\big[\partial_\mu h(w, [\chi^0])\big](\chi^\lambda)\frac{\mathrm{d}\chi^\lambda}{\mathrm{d}\lambda} \otimes \varpi\Big]$$

as respective partial derivatives in w and λ.

Proof. For $w, w' \in \mathbb{R}^k$ and $\lambda, \lambda' \in \mathbb{R}$, we write (thanks to ($\boldsymbol{H2}$)):

$$\begin{aligned}
&\partial_\mu h(w', [\chi^0])(\chi^{\lambda'}) - \partial_\mu h(w, [\chi^0])(\chi^\lambda) \\
&= \partial_\mu h(w', [\chi^0])(\chi^{\lambda'}) - \partial_\mu h(w', [\chi^0])(\chi^\lambda) + \partial_\mu h(w', [\chi^0])(\chi^\lambda) - \partial_\mu h(w, [\chi^0])(\chi^\lambda) \\
&= \Big(\int_0^1 \partial_v\big[\partial_\mu h(w', [\chi^0])\big]\big(s\chi^{\lambda'} + (1-s)\chi^\lambda\big)\mathrm{d}s \Big)\big(\chi^{\lambda'} - \chi^\lambda\big) \\
&\quad + \Big(\int_0^1 \partial_w\big[\partial_\mu h\big(sw' + (1-s)w, [\chi^0]\big)\big]\big(\chi^\lambda\big)\mathrm{d}s \Big)(w' - w).
\end{aligned} \tag{4.71}$$

Thanks to the L^2 bounds on $\partial_w[\partial_\mu h]$ and $\partial_v[\partial_\mu h]$ in ($\boldsymbol{H2}$), we deduce that, as $(w', \lambda') \to (w, \lambda)$,

$$\begin{aligned}
&\mathbb{E}\Big[\Big| \int_0^1 \partial_v\big[\partial_\mu h(w', [\chi^0])\big]\big(s\chi^{\lambda'} + (1-s)\chi^\lambda\big)\mathrm{d}s - \partial_v\big[\partial_\mu h(w, [\chi^0])\big]\big(\chi^\lambda\big) \Big|^2\Big] \to 0, \\
&\mathbb{E}\Big[\Big| \int_0^1 \partial_w\big[\partial_\mu h\big(sw' + (1-s)w, [\chi^0]\big)\big]\big(\chi^\lambda\big)\mathrm{d}s - \partial_w\big[\partial_\mu h(w, [\chi^0])\big]\big(\chi^\lambda\big) \Big|^2\Big] \to 0.
\end{aligned} \tag{4.72}$$

Notice now from the uniform integrability property in the assumption that, as $\lambda' \to \lambda$ (with $\lambda' \neq \lambda$),

$$\mathbb{E}\Big[\big|\big(\frac{\chi^{\lambda'} - \chi^\lambda}{\lambda' - \lambda} - \frac{\mathrm{d}\chi^\lambda}{\mathrm{d}\lambda}\big) \otimes \varpi\big|^2\Big] \to 0. \tag{4.73}$$

Plugging (4.72) and (4.73) into (4.71), we easily deduce that the mapping (4.26) is differentiable. Continuity of the partial derivatives is proved in the same way. $\square$

4.3.3. *Partial smoothness of $\partial_\mu U$. Strategy.* Generally speaking, the strategy is the same as for proving the first-order continuous differentiability and consists in discussing the continuous differentiability of the derivative processes $\partial_\chi \theta^{t,\xi^\lambda} = (\partial_\chi X^{t,\xi^\lambda}, \partial_\chi Y^{t,\xi^\lambda}, \partial_\chi Z^{t,\xi^\lambda})$ and $\partial_\chi \theta^{t,x,[\xi^\lambda]} = (\partial_\chi X^{t,x,[\xi^\lambda]}, \partial_\chi Y^{t,x,[\xi^\lambda]}, \partial_\chi Z^{t,x,[\xi^\lambda]})$ with respect to λ when the family $(\xi^\lambda)_{\lambda\in\mathbb{R}}$ satisfies the aforementioned prescriptions and χ is in $L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)$. Together with the relationship $\partial_\chi Y_t^{t,x,[\xi^\lambda]} = D\mathcal{U}(t,x,\xi^\lambda)\cdot\chi$, this will permit to apply Theorem 3.6 (compare in particular with (3.31)).

Intuitively, one has in mind to consider first the partial second order tangent process of the McKean-Vlasov FBSDE (2.3) in the direction χ and ζ, which we shall denote by $\partial^2_{\zeta,\chi}\theta^{t,\xi} = (\partial^2_{\zeta,\chi} X^{t,\xi}, \partial^2_{\zeta,\chi} Y^{t,\xi}, \partial^2_{\zeta,\chi} Z^{t,\xi}) := [\mathrm{d}/\mathrm{d}\lambda]_{|\lambda=0}\partial_\chi\theta^{t,\xi^\lambda}$. Informally, this process should satisfy a system of the form (4.3), with coefficients of the generic form (4.5). Precisely, the coefficients H_ℓ should have the same decomposition as in the first order case, see (4.49), V and $\hat{V}$ also standing for θ, $\theta^{(0)}$ or X but $\mathcal{V}$ and $\hat{\mathcal{V}}$ now standing for $\partial^2_{\zeta,\chi}\theta$, $\partial^2_{\zeta,\chi}\theta^{(0)}$ or $\partial^2_{\zeta,\chi}X$ (with the usual convention that the symbol (0) in $V^{(0)}$ and $\mathcal{V}^{(0)}$ indicates the restriction to the two first coordinates). Terms B_a, Σ_a, F_a and G_a in (4.5) should not be zero anymore and should be defined as follows for a generic coefficient h that may be b, σ, f or g:

$$
\begin{aligned}
H_a(r) &= \partial^2_{ww} h\big(\theta_r^{t,\xi}, [\theta_r^{t,\xi,(0)}]\big)\partial_\chi\theta_r^{t,\xi}\otimes\partial_\zeta\theta_r^{t,\xi} \\
&\quad + \hat{\mathbb{E}}\big[\partial_w\big[\partial_\mu h\big(\theta_r^{t,\xi}, [\theta_r^{t,\xi,(0)}]\big)\big]\big(\langle\theta_r^{t,\xi,(0)}\rangle\big)\langle\partial_\chi\theta_r^{t,\xi,(0)}\rangle\otimes\partial_\zeta\theta_r^{t,\xi}\big] \\
&\quad + \hat{\mathbb{E}}\big[\partial_v\big[\partial_\mu h\big(\theta_r^{t,\xi}, [\theta_r^{t,\xi,(0)}]\big)\big]\big(\langle\theta_r^{t,\xi,(0)}\rangle\big)\langle\partial_\chi\theta_r^{t,\xi,(0)}\rangle\otimes\langle\partial_\zeta\theta_r^{t,\xi,(0)}\rangle\big] \\
&=: H_a^{(2)}\big(\theta_r^{t,\xi}, \langle\theta_r^{t,\xi,(0)}\rangle, \partial_\chi\theta_r^{t,\xi}, \partial_\zeta\theta_r^{t,\xi}, \langle\partial_\chi\theta_r^{t,\xi,(0)}\rangle, \langle\partial_\zeta\theta_r^{t,\xi,(0)}\rangle\big) \\
&=: H_a^{ww}(r) + H_a^{w\mu}(r) + H_a^{v\mu}(r),
\end{aligned}
\tag{4.74}
$$

where $H_a^{(2)}$ could be expressed (in an obvious way) as a function of general arguments θ_r, $\langle\hat{\theta}_r^{(0)}\rangle$, ϑ_r^1, ϑ_r^2, $\langle\hat{\vartheta}_r^{1,(0)}\rangle$ and $\langle\hat{\vartheta}_r^{2,(0)}\rangle$ instead of $\theta_r^{t,\xi}$, $\langle\theta_r^{t,\xi,(0)}\rangle$, $\partial_\chi\theta_r^{t,\xi}$, $\partial_\zeta\theta_r^{t,\xi}$, $\langle\partial_\chi\theta_r^{t,\xi,(0)}\rangle$ and $\langle\partial_\zeta\theta_r^{t,\xi,(0)}\rangle$. By analogy with (4.4), we can let

$$
\begin{aligned}
&H^{(2)}\big(r, \theta_r, \langle\hat{\theta}_r^{(0)}\rangle, \vartheta_r^1, \vartheta_r^2, \langle\hat{\vartheta}_r^{1,(0)}\rangle, \langle\hat{\vartheta}_r^{2,(0)}\rangle\big)\big(\vartheta_r, \langle\hat{\vartheta}_r^{(0)}\rangle\big) \\
&:= \partial_w h(\theta_r, [\hat{\theta}_r^{(0)}])\vartheta_r + \hat{\mathbb{E}}\big[\partial_\mu h(\theta_r, [\hat{\theta}_r^{(0)}])(\langle\hat{\theta}_r^{(0)}\rangle)\langle\hat{\vartheta}_r^{(0)}\rangle\big] \\
&\quad + H_a^{(2)}\big(\theta_r, \langle\hat{\theta}_r^{(0)}\rangle, \vartheta_r^1, \vartheta_r^2, \langle\hat{\vartheta}_r^{1,(0)}\rangle, \langle\hat{\vartheta}_r^{2,(0)}\rangle\big), \quad r\in[t,T].
\end{aligned}
\tag{4.75}
$$

Pay attention that there is no 'second-order derivatives' in the direction of the measure (*i.e.* '$\partial^2_{\mu\mu}h$') in (4.74). Indeed, the fact that the initial conditions $(\xi^\lambda)_\lambda$ have the same distribution forces the solutions $(\theta^\lambda)_\lambda$ to be identically distributed as well. For the same reason, there is no crossed derivative of the form '$\partial_\mu[\partial_w h]$'. On the opposite, notice that $(\partial_\chi\theta^\lambda)$ (resp. $(\partial_\zeta\theta^\lambda)$) are not identically distributed since the coupling between χ (resp. ζ) and ξ^λ may vary. In particular, when differentiating with respect to λ an expression of the form $\hat{\mathbb{E}}[\partial_\mu h(\theta^\lambda, [\theta^{\lambda,(0)}])(\langle\theta^{\lambda,(0)}\rangle)\langle\partial_\chi\theta^{\lambda,(0)}\rangle]$ for a function h as above, the input $[\theta^{\lambda,(0)}]$ has a zero derivative as it is constant in λ, but the two last inputs, namely $\langle\theta^{\lambda,(0)}\rangle$ and $\langle\partial_\chi\theta^{\lambda,(0)}\rangle$, may give a non-trivial contribution.

On the model of (4.7), (4.8) and (4.9), we shall use the following assumptions on the coefficients (compare also with $(\boldsymbol{H2})$):

$$\begin{aligned}
&|\partial^2_{ww} h(w,[\hat{V}^{(0)}])| + \widehat{\mathbb{E}}\Big[\big|\partial_w\big[\partial_\mu h(w,[\hat{V}^{(0)}])\big](\langle\hat{V}^{(0)}\rangle)\big|^2\Big]^{1/2} \\
&\quad + \widehat{\mathbb{E}}\Big[\big|\partial_v\big[\partial_\mu h(w,[\hat{V}^{(0)}])\big](\langle\hat{V}^{(0)}\rangle)\big|^2\Big]^{1/2} \leqslant C,
\end{aligned} \tag{4.76}$$

and

$$\begin{aligned}
&|\partial^2_{ww} h(w,[\hat{V}^{(0)}]) - \partial^2_{ww} h(w',[\hat{V}^{(0)\prime}])| \\
&\quad + \widehat{\mathbb{E}}\Big[\big|\partial_w\big[\partial_\mu h(w,[\hat{V}^{(0)}])\big](\langle\hat{V}^{(0)}\rangle) - \partial_w\big[\partial_\mu h(w',[\hat{V}^{(0)\prime}])\big](\langle\hat{V}^{(0)\prime}\rangle)\big|^2\Big]^{\frac12} \\
&\quad + \widehat{\mathbb{E}}\Big[\big|\partial_v\big[\partial_\mu h(w,[\hat{V}^{(0)}])\big](\langle\hat{V}^{(0)}\rangle) - \partial_v\big[\partial_\mu h(w',[\hat{V}^{(0)\prime}])\big](\langle\hat{V}^{(0)\prime}\rangle)\big|^2\Big]^{\frac12} \\
&\leqslant C\big(|w-w'| + \Phi_\alpha(\hat{V}^{(0)},\hat{V}^{(0)\prime})\big)\ .
\end{aligned} \tag{4.77}$$

4.3.4. *Preliminary estimates.* We start with the following bound of the remainder term $H_a^{(2)}$ in (4.75):

Lemma 4.27. *Given generic processes θ, $\hat{\theta}$, ϑ^1, ϑ^2, $\hat{\vartheta}^1$ and $\hat{\vartheta}^2$, denote by $H_a^{(2)}(r)$ the term $H_a^{(2)}(\theta_r,\langle\hat{\theta}_r^{(0)}\rangle,\vartheta_r^1,\vartheta_r^2,\langle\hat{\vartheta}_r^{1,(0)}\rangle,\langle\hat{\vartheta}_r^{2,(0)}\rangle)$ in (4.75), H matching B, Σ, F or G. For any $p \geqslant 1$, we can find a constant C_p (independent of the processes) such that (using the notation $\bar{\mathcal{M}}$ from (4.25))*

$$\begin{aligned}
&\mathbb{E}_t\bigg[|G_a^{(2)}(T)|^{2p} + \bigg(\int_t^T \big(|B_a^{(2)}(s)| + |F_a^{(2)}(s)|\big)\mathrm{d}s\bigg)^{2p} + \bigg(\int_t^T |\Sigma_a^{(2)}(s)|^2\mathrm{d}s\bigg)^p\bigg]^{1/2p} \\
&\leqslant C_p\bigg[\Big(\bar{\mathcal{M}}^{4p}(\vartheta^1,\hat{\vartheta}^1)\Big)^{1/4p}\Big(\bar{\mathcal{M}}^{4p}(\vartheta^2,\hat{\vartheta}^2)\Big)^{1/4p} + \mathbb{E}\Big[\|\hat{\vartheta}^{1,(0)}\|^2_{\mathcal{S}^4,t}\|\hat{\vartheta}^{2,(0)}\|^2_{\mathcal{S}^4,t}\Big]^{1/2}\bigg].
\end{aligned} \tag{4.78}$$

Proof. We start with the case $H = B$ (resp. F), or equivalently $h = b$ (resp. f). We use a decomposition of the same type as (4.74) (with the same notations). By conditional Cauchy-Schwartz inequality and (4.76), we can find a constant C_p such that

$$\mathbb{E}_t\bigg[\bigg(\int_t^T |H_a^{ww}(s)|\mathrm{d}s\bigg)^{2p}\bigg] \leqslant C_p\|\vartheta^1\|^{2p}_{\mathcal{H}^{4p},t}\|\vartheta^2\|^{2p}_{\mathcal{H}^{4p},t}. \tag{4.79}$$

We now aim to obtain similar upper bound for the other terms in (4.74). We therefore observe, using (4.76), $|H_a^{w\mu}(s)| \leqslant C\|\hat{\vartheta}_s^{1,(0)}\|_2|\vartheta_s^2|$, so that

$$\mathbb{E}_t\bigg[\bigg(\int_t^T |H_a^{w\mu}(s)|\mathrm{d}s\bigg)^{2p}\bigg] \leqslant C_p\|\vartheta^2\|^{2p}_{\mathcal{H}^{2p},t}\|\hat{\vartheta}^{1,(0)}\|^{2p}_{\mathcal{S}^2}. \tag{4.80}$$

We now handle $H_a^{v\mu}$. By conditional Hölder inequality, we observe that, under condition (4.76), $|H_a^{v\mu}(s)| \leqslant C\mathbb{E}[\|\hat{\vartheta}^{1,(0)}\|^2_{\mathcal{S}^4,t}\|\hat{\vartheta}^{2,(0)}\|^2_{\mathcal{S}^4,t}]^{1/2}$, from which we get

$$\mathbb{E}_t\bigg[\bigg(\int_t^T |H_a^{v\mu}(s)|\mathrm{d}s\bigg)^{2p}\bigg] \leqslant C_p\mathbb{E}\Big[\|\hat{\vartheta}^{1,(0)}\|^2_{\mathcal{S}^4,t}\|\hat{\vartheta}^{2,(0)}\|^2_{\mathcal{S}^4,t}\Big]^p. \tag{4.81}$$

By (4.79), (4.80) and (4.81) and with the notation (4.25), we get (4.78). The cases when $H = \Sigma$ or G may be handled in the same way. $\square$

Lemma 4.28. *Given processes θ, θ', $\hat\theta$, $\hat\theta'$, ϑ^1, $\vartheta^{1\prime}$, ϑ^2, $\vartheta^{2\prime}$, $\hat\vartheta^1$, $\hat\vartheta^{1\prime}$, $\hat\vartheta^2$ and $\hat\vartheta^{2\prime}$, denote the terms $H_a^{(2)}(\theta_r,\langle\hat\theta_r^{(0)}\rangle,\vartheta_r^1,\vartheta_r^2,\langle\hat\vartheta_r^{1,(0)}\rangle,\langle\hat\vartheta_r^{2,(0)}\rangle)$ in (4.75) by $H_a^{(2)}(r)$ and use a similar definition for $H_a^{(2)\prime}(r)$. For any $p\geqslant 1$, we can find a constant C_p (independent of the processes) such that, for any random variable ε with values in $\mathbb{R}_+$ (with the notation $\bar{\mathcal{M}}$ from (4.25)),*

$$\begin{aligned}
&\mathbb{E}_t\biggl[|G_a^{(2)}(T)-G_a^{(2)\prime}(T)|^{2p}+\biggl(\int_t^T\bigl(|B_a^{(2)}(s)-B_a^{(2)\prime}(s)|+|F_a^{(2)}(s)-F_a^{(2)\prime}(s)|\bigr)\mathrm{d}s\biggr)^{2p}\\
&\qquad+\biggl(\int_t^T|\Sigma_a^{(2)}(s)-\Sigma_a^{(2)\prime}(s)|^2\mathrm{d}s\biggr)^{p}\biggr]^{1/2p}\\
&\leqslant C_p\Bigl\{\Bigl(1\wedge\Bigl[\mathbb{E}_t\bigl(\varepsilon^{4p}\bigr)^{1/4p}+\Bigl(\bar{\mathcal{M}}^{4p}\bigl(\theta-\theta',\hat\theta-\hat\theta'\bigr)\Bigr)^{1/4p}+\Phi_\alpha\bigl(\hat\theta^{(0)},\hat\theta^{(0)\prime}\bigr)\Bigr]\Bigr)\\
&\qquad\times\Bigl[\Bigl(\bar{\mathcal{M}}^{8p}\bigl(\vartheta^1,\hat\vartheta^1\bigr)\Bigr)^{1/8p}\Bigl(\bar{\mathcal{M}}^{8p}\bigl(\vartheta^2,\hat\vartheta^2\bigr)\Bigr)^{1/8p}+\mathbb{E}\Bigl[\|\hat\vartheta^{1,(0)}\|^2_{\mathcal{S}^4,t}\|\hat\vartheta^{2,(0)}\|^2_{\mathcal{S}^4,t}\Bigr]^{1/2}\Bigr]\Bigr\}\\
&\quad+C_p\Bigl\{\Bigl(\bar{\mathcal{M}}^{4p}\bigl(\vartheta^1-\vartheta^{1\prime},\hat\vartheta^1-\hat\vartheta^{1\prime}\bigr)\Bigr)^{1/4p}\Bigl(\bar{\mathcal{M}}^{4p}\bigl(\vartheta^2,\hat\vartheta^2\bigr)\Bigr)^{1/4p}\\
&\qquad+\Bigl(\bar{\mathcal{M}}^{4p}\bigl(\vartheta^{1\prime},\hat\vartheta^{1\prime}\bigr)\Bigr)^{1/4p}\Bigl(\bar{\mathcal{M}}^{4p}\bigl(\vartheta^2-\vartheta^{2\prime},\hat\vartheta^2-\hat\vartheta^{2\prime}\bigr)\Bigr)^{1/4p}\Bigr\}\\
&\quad+C_p\Bigl\{\mathbb{E}\Bigl[\|\hat\vartheta^{1,(0)\prime}\|^2_{\mathcal{S}^4,t}\|\hat\vartheta^{2,(0)}-\hat\vartheta^{2,(0)\prime}\|^2_{\mathcal{S}^4,t}\Bigr]^{1/2}+\mathbb{E}\Bigl[\|\hat\vartheta^{1,(0)}-\hat\vartheta^{1,(0)\prime}\|^2_{\mathcal{S}^4,t}\|\hat\vartheta^{2,(0)}\|^2_{\mathcal{S}^4,t}\Bigr]^{1/2}\Bigr\}\\
&\quad+\mathbb{E}_t\biggl[\biggl(\int_t^T\mathbf{1}_{\{|\theta_s-\theta'_s|>\varepsilon\}}|\vartheta_s^1|\,|\vartheta_s^2|\mathrm{d}s\biggr)^{2p}\biggr]^{1/2p}.
\end{aligned}\tag{4.82}$$

Proof. We start with the case when $H=B,F$. As in the proof of Lemma 4.27, we make use of the decomposition (4.74). Denoting by H_2^{ww} and $H_2^{ww\prime}$ the related terms in (4.74), we compute:

$$\begin{aligned}
&\mathbb{E}_t\biggl[\biggl(\int_t^T|H_a^{ww}(s)-H_a^{ww\prime}(s)|\mathrm{d}s\biggr)^{2p}\biggr]^{1/2p}\\
&\leqslant\mathbb{E}_t\biggl[\biggl(\int_t^T\Bigl|\Bigl\{\partial^2_{ww}h\bigl(\theta_s,[\hat\theta_s^{(0)}]\bigr)-\partial^2_{ww}h\bigl(\theta'_s,[\hat\theta_s^{(0)\prime}]\bigr)\Bigr\}\vartheta_s^1\otimes\vartheta_s^2\Bigr|\,\mathrm{d}s\biggr)^{2p}\biggr]^{1/2p}\\
&\quad+\mathbb{E}_t\biggl[\biggl(\int_t^T\Bigl|\partial^2_{ww}h\bigl(\theta'_s,[\hat\theta_s^{(0)\prime}]\bigr)\,\{\vartheta_s^1-\vartheta_s^{1\prime}\}\otimes\vartheta_s^2\Bigr|\,\mathrm{d}s\biggr)^{2p}\biggr]^{1/2p}\\
&\quad+\mathbb{E}_t\biggl[\biggl(\int_t^T\Bigl|\partial^2_{ww}h\bigl(\theta'_s,[\hat\theta_s^{(0)\prime}]\bigr)\vartheta_s^{1\prime}\otimes\{\vartheta_s^2-\vartheta_s^{2\prime}\}\Bigr|\,\mathrm{d}s\biggr)^{2p}\biggr]^{1/2p}\\
&:=A_1+A_2+A_3.
\end{aligned}\tag{4.83}$$

Bounding the difference of the terms in $\partial^2_{ww} h$ by a constant or by the increment of the underlying variables, we thus obtain, for any random variable ε with values in $\mathbb{R}_+$,

$$\begin{aligned}
A_1 &\leqslant C\mathbb{E}_t\Bigl[\Bigl\{1 \wedge \Bigl(\varepsilon + \sup_{s\in[t,T]} \Phi_\alpha\bigl(\hat\theta_s^{(0)}, \hat\theta_s^{(0)\prime}\bigr)\Bigr)^{2p}\Bigr\}\Bigl(\int_t^T |\vartheta_s^1|\,|\vartheta_s^2| \mathrm{d}s\Bigr)^{2p}\Bigr]^{1/2p} \\
&\quad + \mathbb{E}_t\Bigl[\Bigl(\int_t^T \mathbf{1}_{\{|\theta_s-\theta_s'|>\varepsilon\}}|\vartheta_s^1|\,|\vartheta_s^2|\mathrm{d}s\Bigr)^{2p}\Bigr]^{1/2p} \\
&\leqslant C_p\Bigl(1 \wedge \Bigl[\mathbb{E}_t\bigl(\varepsilon^{4p}\bigr)^{1/4p} + \sup_{s\in[t,T]} \Phi_\alpha\bigl(\hat\theta_s^{(0)}, \hat\theta_s^{(0)\prime}\bigr)\Bigr]\Bigr)\|\vartheta^1\|_{\mathcal{H}^{8p},t}\|\vartheta^2\|_{\mathcal{H}^{8p},t} \\
&\quad + \mathbb{E}_t\Bigl[\Bigl(\int_t^T \mathbf{1}_{\{|\theta_s-\theta_s'|>\varepsilon\}}|\vartheta_s^1|\,|\vartheta_s^2|\mathrm{d}s\Bigr)^{2p}\Bigr].
\end{aligned}$$

We also have

$$A_2 + A_3 \leqslant C_p\Bigl(\|\vartheta^1 - \vartheta^{1\prime}\|_{\mathcal{H}^{4p},t}\|\vartheta^2\|_{\mathcal{H}^{4p},t} + \|\vartheta^{1\prime}\|_{\mathcal{H}^{4p},t}\|\vartheta^2 - \vartheta^{2\prime}\|_{\mathcal{H}^{4p},t}\Bigr).$$

Next, using a similar decomposition to (4.83), we compute

$$\begin{aligned}
&\mathbb{E}\Bigl[\Bigl(\int_t^T |H_a^{w\mu}(s) - H_a^{w\mu\prime}(s)|\mathrm{d}s\Bigr)^{2p}\Bigr]^{1/2p} \\
&\leqslant C_p\Bigl\{\Bigl(1 \wedge \bigl\{\|\theta-\theta'\|_{\mathcal{H}^{4p},t} + \sup_{s\in[t,T]} \Phi_\alpha(\hat\theta_s^{(0)}, \hat\theta_s^{(0)\prime})\bigr\}\Bigr)\|\hat\vartheta^{1,(0)}\|_{\mathcal{S}^2}\|\vartheta^2\|_{\mathcal{H}^{4p},t} \\
&\quad + \|\hat\vartheta^{1,(0)} - \hat\vartheta^{1,(0)\prime}\|_{\mathcal{S}^2}\|\vartheta^2\|_{\mathcal{H}^{2p},t} + \|\hat\vartheta^{1,(0)\prime}\|_{\mathcal{S}^2}\|\vartheta^2 - \vartheta^{2\prime}\|_{\mathcal{H}^{2p},t}\Bigr\}.
\end{aligned}$$

We also get

$$\begin{aligned}
&\mathbb{E}_t\Bigl[\Bigl(\int_t^T |H_a^{v\mu}(s) - H_a^{v\mu\prime}(s)|\mathrm{d}s\Bigr)^{2p}\Bigr]^{1/2p} \\
&\leqslant C_p\Bigl\{\Bigl(1 \wedge \bigl\{\|\theta-\theta'\|_{\mathcal{H}^{2p},t} + \sup_{s\in[t,T]} \Phi_\alpha(\hat\theta_s^{(0)}, \hat\theta_s^{(0)\prime})\bigr\}\Bigr)\mathbb{E}\Bigl[\|\hat\vartheta^{1,(0)}\|^2_{\mathcal{S}^4,t}\|\hat\vartheta^{2,(0)}\|^2_{\mathcal{S}^4,t}\Bigr]^{1/2} \\
&\quad + \mathbb{E}\Bigl[\|\hat\vartheta^{1,(0)\prime}\|^2_{\mathcal{S}^4,t}\|\hat\vartheta^{2,(0)} - \hat\vartheta^{2,(0)\prime}\|^2_{\mathcal{S}^4,t}\Bigr]^{1/2} + \mathbb{E}\Bigl[\|\hat\vartheta^{1,(0)} - \hat\vartheta^{1,(0)\prime}\|^2_{\mathcal{S}^4,t}\|\hat\vartheta^{2,(0)}\|^2_{\mathcal{S}^4,t}\Bigr]^{1/2}\Bigr\}.
\end{aligned}$$

Collecting the various inequalities, we get (4.82).

The proof is quite similar when $H = \Sigma$ or G, but there are two main differences. The first one is that, in the analysis of $\Sigma_a^{(2)}$ and $G_a^{(2)}$, processes are estimated with $\mathcal{S}$ instead of $\mathcal{H}$ norms. Obviously, this does not affect (4.82) since $\Sigma_a^{(2)}$ and $G_a^{(2)}$ only involve the two first coordinates of θ, ϑ^1 and ϑ^2. The second main difference comes from A_1. Since neither σ nor g depend on the component Z, we can replace $|\theta_s - \theta_s'|$ by $|\theta_s^{(0)} - \theta_s^{(0)\prime}|$ in the analysis of the term corresponding to A_1. Choosing $\varepsilon = \sup_{s\in[t,T]} |\theta_s^{(0)} - \theta_s^{(0)\prime}|$, we get rid of the remaining term containing the indicator function of the event $\{|\theta_s^{(0)} - \theta_s^{(0)\prime}| > \varepsilon\}$. Then, $\mathbb{E}_t[\varepsilon^{4p}]^{1/4p}$ is exactly equal to $\|\theta^{(0)} - \theta^{(0)\prime}\|_{\mathcal{S}^{4p},t}$, which is less than $(\bar{\mathcal{M}}^{4p}(\theta-\theta', \hat\theta - \hat\theta'))^{1/4p}$. □

4.3.5. *Proof of the differentiability of the McKean-Vlasov system.* We claim:

Lemma 4.29. *There exists $c := c(L) > 0$ such that, for $T \leqslant c$, for χ and $(\xi^\lambda)_\lambda$ as in §4.3.2, the mapping*

$$\mathbb{R} \ni \lambda \mapsto \partial_\chi \theta^{t,\xi^\lambda} = \bigl(\partial_\chi X^{t,\xi^\lambda}, \partial_\chi Y^{t,\xi^\lambda}, \partial_\chi Z^{t,\xi^\lambda}\bigr),$$

with values in $\mathcal{S}^2([t,T];\mathbb{R}^d) \times \mathcal{S}^2([t,T];\mathbb{R}^m) \times \mathcal{H}^2([t,T];\mathbb{R}^{m\times d})$, is continuously differentiable. The derivative at $\lambda = 0$ only depends upon the family $(\xi^\lambda)_{\lambda\in\mathbb{R}}$ through the value of $\xi := \xi^0$ and $\zeta := [\mathrm{d}/\mathrm{d}\lambda]_{\lambda=0}\xi^\lambda$ (see footnote 10 on page 26 for a precise meaning). It is denoted by $\partial^2_{\zeta,\chi}\theta^{t,\xi}$.

Proof. We adapt the proof of Lemma 4.17. To do so, we use the Picard sequence $((\theta^{n,\lambda},\partial_\chi\theta^{n,\lambda}))_{n\geqslant 1}$ solving (4.50), with $X_t^{n,\lambda} = \xi^\lambda$ and $\chi^\lambda = \chi$ for any $\lambda \in \mathbb{R}$. The sequence converges in $[\mathcal{S}^2([t,T];\mathbb{R}^d) \times \mathcal{S}^2([t,T],\mathbb{R}^m) \times \mathcal{H}^2([t,T];\mathbb{R}^{m\times d})]^2$ towards $(\theta^{t,\xi^\lambda},\partial_\chi\theta^{t,\xi^\lambda})$, uniformly in λ in compact subsets. Pay attention that, in (4.50), the choice $\chi^\lambda = \chi$, for all $\lambda \in \mathbb{R}$, fits the framework of Theorem 3.6 in which χ is kept frozen, independently of λ.

Similarly, we denote by $((\theta^{n,\lambda},\partial_\zeta\theta^{n,\lambda}))_{n\geqslant 1}$ the Picard sequence solving (4.50), with $X_t^{n,\lambda} = \xi^\lambda$ and $\chi^\lambda = [\mathrm{d}/\mathrm{d}\lambda]\xi^\lambda$ for any $\lambda \in \mathbb{R}$. The sequence converges in $[\mathcal{S}^2([t,T];\mathbb{R}^d) \times \mathcal{S}^2([t,T],\mathbb{R}^m) \times \mathcal{H}^2([t,T];\mathbb{R}^{m\times d})]^2$ towards $(\theta^{t,\xi^\lambda},\partial_\zeta\theta^{t,\xi^\lambda})$, uniformly in λ in compact subsets. In (4.50), the choice $\chi^\lambda = [\mathrm{d}/\mathrm{d}\lambda]\xi^\lambda$, for any $\lambda \in \mathbb{R}$, fits the framework of Theorem 3.6 in which $\zeta = [\mathrm{d}/\mathrm{d}\lambda]_{|\lambda=0}X^\lambda$, with X^λ therein playing the role of ξ^λ.

Notice that $(\theta^{n,\lambda})_{n\geqslant 1}$, which appears in each of the two Picard sequences, denotes the same process. The difference between $(\partial_\chi\theta^{n,\lambda})_{n\geqslant 1}$ and $(\partial_\zeta\theta^{n,\lambda})_{n\geqslant 1}$ is that $[\mathrm{d}/\mathrm{d}\lambda]\theta^{n,\lambda} = \partial_\zeta\theta^{n,\lambda}$ but $[\mathrm{d}/\mathrm{d}\lambda]\theta^{n,\lambda} \neq \partial_\chi\theta^{n,\lambda}$. The motivation for considering $\partial_\chi\theta^{n,\lambda}$ is that $\partial_\chi Y_t^{n,\lambda}$ converges to $\mathbb{E}[\partial_\mu U(t,\xi^\lambda,[\xi^\lambda])\chi]$, which is precisely the quantity that we aim at differentiating with respect to λ.

First step. The first point is to prove that, for any $n \geqslant 0$, the map $\lambda \mapsto (\partial_\chi\theta_s^{n,\lambda})_{s\in[t,T]}$ is continuously differentiable from $\mathbb{R}$ to $\mathcal{S}^2([t,T];\mathbb{R}^d)\times\mathcal{S}^2([t,T];\mathbb{R}^m)\times\mathcal{H}^2([t,T];\mathbb{R}^{m\times d})$. To do so, we recall the system (4.50):

$$
\begin{aligned}
\partial_\chi X_s^{n+1,\lambda} &= \chi + \int_t^s B^{(1)}\big(r,\theta_r^{n,\lambda},\langle\theta_r^{n,\lambda,(0)}\rangle\big)\big(\partial_\chi\theta_r^{n,\lambda},\langle\partial_\chi\theta_r^{n,\lambda,(0)}\rangle\big)\mathrm{d}r \\
&\quad + \int_t^s \Sigma^{(1)}\big(r,\theta_r^{n,\lambda,(0)},\langle\theta_r^{n,\lambda,(0)}\rangle\big)\big(\partial_\chi\theta_r^{n,\lambda,(0)},\langle\partial_\chi\theta_r^{n,\lambda,(0)}\rangle\big)\mathrm{d}W_r \\
\partial_\chi Y_s^{n+1,\lambda} &= G^{(1)}\big(X_T^{n+1,\lambda},\langle X_T^{n+1,\lambda}\rangle\big)\big(\partial_\chi X_T^{n+1,\lambda},\langle\partial_\chi X_T^{n+1,\lambda}\rangle\big) \\
&\quad + \int_s^T F^{(1)}\big(r,\theta_r^{n,\lambda},\langle\theta_r^{n,\lambda,(0)}\rangle\big)\big(\partial_\chi\theta_r^{n,\lambda},\langle\partial_\chi\theta_r^{n,\lambda,(0)}\rangle\big)\mathrm{d}r - \int_s^T \partial_\chi\mathcal{Z}_r^{n+1,\lambda}\mathrm{d}W_r,
\end{aligned}
\tag{4.84}
$$

with $\partial_\chi\theta^{0,\lambda} \equiv (0,0,0)$ as initialization, with a similar system for $\partial_\zeta\theta^{n,\lambda}$, replacing χ by $[\mathrm{d}/\mathrm{d}\lambda]X^\lambda$.

Generally speaking, the proof is the same as that of Lemma 4.17: We argue by induction, assuming at each step $n \geqslant 1$ that $\lambda \mapsto (\partial_\chi\theta_s^{n,\lambda})_{s\in[t,T]}$ is continuously differentiable (the derivative being denoted by $(\partial^2_{\zeta,\chi}\theta_s^{n,\lambda})_{s\in[t,T]}$); we prove first the differentiability of the forward component and then the differentiability of the backward one in (4.84).

In comparison with the proof of Lemma 4.17, we must pay attention to the two following points. The first question is to justify the differentiability under the various expectation symbols that appear in the definitions of $B^{(1)}$, $\Sigma^{(1)}$, $F^{(1)}$ and $G^{(1)}$. Thanks to (4.51) and from the fact that the sequence $[\mathrm{d}/\mathrm{d}\lambda](\xi^\lambda)$ is bounded in L^∞ (see 4.70), we know that

$$
\sup_{n\geqslant 1}\Big[\mathcal{M}^4_{\mathbb{E}_t}\big(\partial_\chi\theta^{n,\lambda}\big)\Big]^{1/4} \leqslant C\big[|\chi| + \|\chi\|_2\big], \quad \sup_{n\geqslant 1}\Big[\mathcal{M}^{2p}_{\mathbb{E}_t}\big(\partial_\zeta\theta^{n,\lambda}\big)\Big]^{1/4} \leqslant C, \tag{4.85}
$$

so that Lemma 4.26 applies with $[\mathrm{d}/\mathrm{d}\lambda]\chi^\lambda = \partial_\zeta \theta^{n,\lambda}$ and $\varpi = \partial_\chi \theta^{n,\lambda}$ and permits to guarantee the differentiability of the terms driven by an expectation.

Another problem is that the coefficients now involve the product of two terms that are differentiable in $\mathcal{H}^2([t,T];\mathbb{R}^k)$ (or $\mathcal{S}^2([t,T];\mathbb{R}^k)$ in some cases), for a suitable $k \geqslant 1$, so that the product is differentiable in $\mathcal{H}^1([t,T];\mathbb{R}^k)$ (or $\mathcal{S}^1([t,T];\mathbb{R}^k)$) only (for another k). We make this clear for $(\int_t^s B^{(1)}(r,\theta_r^{n,\lambda},\langle\theta_r^{n,\lambda,(0)}\rangle)(\partial_\chi\theta_r^{n,\lambda},\langle\partial_\chi\theta_r^{n,\lambda,(0)}\rangle)\mathrm{d}r)_{t\leqslant s\leqslant T}$, the other terms being handled in a similar fashion. Repeating the analysis of Lemma 4.17, it is differentiable from $\mathbb{R}$ to $\mathcal{S}^1([t,T];\mathbb{R}^d)$, the derivative process writing

$$\begin{aligned}&\int_t^s B^{(2)}\big(r,\Theta_r^{n,\lambda}\big)\big(\partial^2_{\zeta,\chi}\theta_r^{n,\lambda},\langle\partial^2_{\zeta,\chi}\theta_r^{n,\lambda,(0)}\rangle\big)\mathrm{d}r,\\ &\qquad\text{with } \Theta_r^{n,\lambda} = \big(\theta_r^{n,\lambda},\langle\theta_r^{n,\lambda,(0)}\rangle,\partial_\chi\theta_r^{n,\lambda},\partial_\zeta\theta_r^{n,\lambda},\langle\partial_\chi\theta_r^{n,\lambda,(0)}\rangle,\langle\partial_\zeta\theta_r^{n,\lambda,(0)}\rangle\big)\end{aligned}\tag{4.86}$$

with $s \in [t,T]$. As explained in (4.75) and on the model of Lemma 4.26, we here used the crucial assumption that all the $(\xi^\lambda)_\lambda$ are identically distributed to get the shape of $B^{(2)}$.

In order to prove differentiability in $\mathcal{S}^2([t,T];\mathbb{R}^d)$, a uniform integrability argument is needed. Assume indeed that a path $\mathbb{R} \ni \lambda \mapsto \vartheta^\lambda = (\vartheta_s^\lambda)_{s\in[t,T]} \in \mathcal{S}^1([t,T],\mathbb{R}^k)$, for some $k \geqslant 1$, is continuously differentiable and that, for any finite interval I, the family $(\sup_{s\in[t,T]}|[\mathrm{d}/\mathrm{d}\lambda]\vartheta_s^\lambda|^2)_{\lambda\in I}$ is uniformly integrable. Then, $\mathbb{R}\ni\lambda\mapsto\vartheta^\lambda\in\mathcal{S}^2([t,T];\mathbb{R}^k)$ is continuously differentiable.

In our framework, the form of $([\mathrm{d}/\mathrm{d}\lambda]\vartheta_s^\lambda)_{s\in[t,T]}$ is explicitly given by (4.86). The coefficient $B^{(2)}$ may be expanded by means of (4.75). Clearly, it involves a linear term in $(\partial^2_{\zeta,\chi}\theta_r^{n,\lambda},\langle\partial^2_{\zeta,\chi}\theta_r^{n,\lambda,(0)}\rangle)$ and the remainder $B_a^{(2)}(\Theta_r^{n,\lambda})$. By $(\boldsymbol{H1})$ and $(\boldsymbol{H2})$, we get that

$$\sup_{s\in[t,T]}\Big|\frac{\mathrm{d}}{\mathrm{d}\lambda}\vartheta_s^\lambda\Big| \leqslant C\bigg[\bigg(\int_t^T\big[|\partial^2_{\zeta,\chi}\theta_r^{n,\lambda}|^2+\|\partial^2_{\zeta,\chi}\theta_r^{n,\lambda}\|_2^2\big]\mathrm{d}r\bigg)^{1/2}+\mathbb{E}\int_t^T|B_a^{(2)}(\Theta_r^{n,\lambda})|\mathrm{d}r\bigg].\tag{4.87}$$

By continuity of $\mathbb{R}\ni\lambda\mapsto\partial^2_{\zeta,\chi}\theta^{n,\lambda}\in\mathcal{H}^2([t,T];\mathbb{R}^k)$, the first term in the right-hand side is uniformly square integrable. We thus discuss the term in $B_a^{(2)}$. Recalling Lemma 4.18 and the similar version (4.51) for the Picard scheme in Lemma 4.17 (which is given for $p=1$ only but which could be generalized), we have the more general version of (4.85):

$$\sup_{n\geqslant1}\Big[\mathcal{M}^{2p}_{\mathbb{E}_t}\big(\partial_\chi\theta^{n,\lambda}\big)\Big]^{1/2p}\leqslant C_p\big[|\chi|+\|\chi\|_2\big],\quad \sup_{n\geqslant1}\Big[\mathcal{M}^{2p}_{\mathbb{E}_t}\big(\partial_\zeta\theta^{n,\lambda}\big)\Big]^{1/2p}\leqslant C_p,\tag{4.88}$$

so that, by Lemma 4.27 (with $\theta\equiv\hat\theta:\equiv\theta^{n,\lambda}$, $\vartheta^1\equiv\hat\vartheta^1:\equiv\partial_\chi\theta^{n,\lambda}$ and $\vartheta^2\equiv\hat\vartheta^2:\equiv\partial_\zeta\theta^{n,\lambda}$, and, as usual, for $T\leqslant c$)

$$\mathbb{E}_t\bigg[\bigg(\int_t^T|B_a^{(2)}(\Theta_r^{n,\lambda})\mathrm{d}r\bigg)^{2p}\bigg]^{1/2p}\leqslant C_p\big(|\chi|+\|\chi\|_2\big).\tag{4.89}$$

Now, choosing $p=2$, we get that, for any event $A\in\mathcal{A}$,

$$\begin{aligned}&\mathbb{E}\bigg[\mathbf{1}_A\bigg(\int_t^T|B_a^{(2)}(\Theta_r^{n,\lambda})\mathrm{d}r\bigg)^2\bigg]\\ &\qquad\leqslant C\mathbb{E}\big[\mathbb{E}_t[\mathbf{1}_A]^{\frac12}\,\big(|\chi|+\|\chi\|_2\big)^2\big] = C\mathbb{E}\Big[\mathbb{E}_t\big[\mathbf{1}_A(|\chi|+\|\chi\|_2)^2\big]^{\frac12}\,\big(|\chi|+\|\chi\|_2\big)\Big]\\ &\qquad\leqslant C\|\chi\|_2\mathbb{E}\big[\mathbf{1}_A\big(|\chi|+\|\chi\|_2\big)^2\big]^{\frac12},\end{aligned}$$

where we have used the fact that χ is $\mathcal{F}_t$ measurable. The above bound permits to establish the required uniform integrability argument, a similar argument holding true for the terms driven by $F_a^{(2)}$, $\Sigma_a^{(2)}$ and $G_a^{(2)}$. Inductively, this permits to prove that the map $\lambda \mapsto \partial_\chi \theta^{n,\lambda}$ is continuously differentiable from $\mathbb{R}$ to $\mathcal{S}^2([t,T],\mathbb{R}^d) \times \mathcal{S}^2([t,T],\mathbb{R}^m) \times \mathcal{H}^2([t,T],\mathbb{R}^{m\times d})$. With the same notation as in (4.86), we have, for any $n \geqslant 0$,

$$\begin{aligned} \partial^2_{\zeta,\chi} X_s^{n+1,\lambda} &= \int_t^s B^{(2)}\big(r,\Theta_r^{n,\lambda}\big)\big(\partial^2_{\zeta,\chi}\theta_r^{n,\lambda}, \langle \partial^2_{\zeta,\chi}\theta_r^{n,\lambda,(0)}\rangle\big)\mathrm{d}r \\ &\quad + \int_t^s \Sigma^{(2)}\big(r,\Theta_r^{n,\lambda,(0)}\big)\big(\partial^2_{\zeta,\chi}\theta_r^{n,\lambda,(0)}, \langle \partial^2_{\zeta,\chi}\theta_r^{n,\lambda,(0)}\rangle\big)\mathrm{d}W_r, \end{aligned} \tag{4.90}$$

and

$$\begin{aligned} \partial^2_{\zeta,\chi} Y_s^{n+1,\lambda} &= G^{(2)}(\Xi_T^{n+1,\lambda})\big(\partial^2_{\zeta,\chi} X_T^{n+1,\lambda}, \langle \partial^2_{\zeta,\chi} X_T^{n+1,\lambda}\rangle\big) \\ &\quad + \int_s^T F^{(2)}\big(r,\Theta_r^{n,\lambda}\big)\big(\partial^2_{\zeta,\chi}\theta_r^{n,\lambda}, \langle \partial^2_{\zeta,\chi}\theta_r^{n,\lambda,(0)}\rangle\big)\mathrm{d}r - \int_s^T \partial^2_{\zeta,\chi} Z_r^{n+1,\lambda}\mathrm{d}W_r, \end{aligned} \tag{4.91}$$

where we have let:

$$\begin{aligned} \Theta_r^{n,\lambda,(0)} &= \big(\theta_r^{n,\lambda,(0)}, \langle \theta_r^{n,\lambda,(0)}\rangle, \partial_\chi \theta_r^{n,\lambda,(0)}, \partial_\zeta \theta_r^{n,\lambda,(0)}, \langle \partial_\chi \theta_r^{n,\lambda,(0)}\rangle, \langle \partial_\zeta \theta_r^{n,\lambda,(0)}\rangle\big), \\ \Xi_T^{n,\lambda} &= \big(X_T^{n,\lambda}, \langle X_T^{n,\lambda}\rangle, \partial_\chi X_T^{n,\lambda}, \partial_\zeta X_T^{n,\lambda}, \langle \partial_\chi X_T^{n,\lambda}\rangle, \langle \partial_\zeta X_T^{n,\lambda}\rangle\big). \end{aligned}$$

Second step. Convergence of the sequence $(\partial^2_{\zeta,\chi}\theta^{n,\lambda})_{n\geqslant 0}$ in the space $\mathcal{S}^2([t,T],\mathbb{R}^d) \times \mathcal{S}^2([t,T],\mathbb{R}^m) \times \mathcal{H}^2([t,T],\mathbb{R}^{m\times d})$ is then shown as in the proof of Lemma 4.17. Generally speaking, the point is to compare approximations at steps n and $n+1$ and then to prove that the norm of the difference decays geometrically fast as n tends to ∞. As in the first step, some precaution is needed as the system differs from the one involved in the proof of Lemma 4.17, the difference coming from the remainder term $H_a^{(2)}$ in (4.74). Precisely, the proof of Lemma 4.17 relies on Lemmas 4.7 and 4.9, with 0 as remainder term $\mathcal{R}_a$, but, in the current framework, the remainder term is equal to $(H_a^{(2)}(\Theta_s^{n,\lambda}))_{s\in[t,T]}$ when $H = B$, Σ or F and $G_a^{(2)}(\Xi_T^{n+1,\lambda})$ when $H = G$, and is thus non-zero. The analysis thus imitates the proof of Lemma 4.17, but with a non-zero remainder term $\Delta\mathcal{R}_a$ in (4.28) that corresponds to the difference of the remainders $\mathcal{R}_a$ at steps n and $n+1$. In short, it is enough to prove that $\mathbb{E}[\Delta\mathcal{R}_a^2]$ tends to 0 as n tends to ∞ (to simplify, we omit to specify the index n in $\Delta\mathcal{R}_a^2$). By convergence of $(\theta_s^{n,\lambda}, \partial_\chi\theta_s^{n,\lambda}, \partial_\zeta\theta_s^{n,\lambda})_{s\in[t,T]}$ to $(\theta_s^{t,\xi^\lambda}, \partial_\chi\theta_s^{\xi\lambda}, \partial_\zeta\theta_s^{t,\xi^\lambda})_{s\in[t,T]}$, we can deduce from Lemma 4.28 (with $\theta \equiv \hat\theta := \theta^{n,\lambda}$, $\vartheta^1 \equiv \hat\vartheta^1 := \partial_\chi\theta^{n,\lambda}$, $\vartheta^2 \equiv \hat\vartheta^2 := \partial_\zeta\theta^{n,\lambda}$ and $\theta' \equiv \hat\theta' := \theta^{n+1,\lambda}$, $\vartheta^{1,\prime} \equiv \hat\vartheta^{1,\prime} := \partial_\chi\theta^{n+1,\lambda}$, $\vartheta^{2,\prime} \equiv \hat\vartheta^{2,\prime} := \partial_\zeta\theta^{n+1,\lambda}$) that $\Delta\mathcal{R}_a^2$ tends to 0 in probability as n tends to ∞, the convergence being uniform with respect to λ in compact subsets: In (4.82), we can check that all the terms not containing the variable ε tend 0; choosing ε as a small deterministic real, it is standard to prove that the expectation of the last term in (4.82) tends to 0. The latter property follows from the following fact: For any compact $I \subset \mathbb{R}$, the sequence $(\partial_\chi\theta^{n,\lambda})_{n\geqslant 1}$ and $(\partial_\zeta\theta^{n,\lambda})_{n\geqslant 1}$ are convergent in the L^2 sense on $\Omega\times[t,T]$, so that the families $(\partial_\chi\theta^{n,\lambda})_{n\geqslant 1,\lambda\in I}$ and $(\partial_\zeta\theta^{n,\lambda})_{n\geqslant 1,\lambda\in I}$ are uniformly square integrable on $\Omega\times[t,T]$.

The convergence of $\Delta\mathcal{R}_a^2$ to 0 actually holds in the L^1 sense on Ω, since the bound (4.89) (with similar bounds for F, Σ and G) allows to apply another argument of uniform integrability. The convergence is uniform with respect to λ in compact sets. This proves the continuous differentiability of $\mathbb{R} \ni \lambda \mapsto \partial_\chi\theta^{t,\xi^\lambda} \in \mathcal{S}^2([t,T];\mathbb{R}^d) \times \mathcal{S}^2([t,T];\mathbb{R}^m) \times \mathcal{H}^2([t,T];\mathbb{R}^{m\times d})$. The derivative at $\lambda = 0$ satisfies a system of the form (4.3) (obtained by an obvious adaptation of (4.90) and (4.91)), which is uniquely solvable in short time. This proves that the derivative at $\lambda = 0$ only depends on the family $(X^\lambda)_{\lambda\in\mathbb{R}}$ through X^0 and ζ.

We complete the analysis as in the proof of Lemma 4.17. □

4.3.6. *Estimates of the directional derivatives of the McKean-Vlasov system.* We claim:

Lemma 4.30. *Recall the notations* (4.16)*. For any $p \geqslant 1$, there exist two constants $c := c_p(L) > 0$ and C_p, such that, for $T \leqslant c_p$ (and with $\gamma = c_p$ in* (4.16)*),*

$$\big[\mathcal{M}^{2p}_{\mathbb{E}_t}\big(\partial^2_{\zeta,\chi}\theta^{t,\xi}\big)\big]^{1/(2p)} \leqslant C_p\big(|\chi| + \|\chi\|_2\big).$$

Proof. The result follows from Corollary 4.8 with $\eta = 0$, $\theta \equiv \hat{\theta} := \theta^{t,\xi}$, $\vartheta \equiv \hat{\vartheta} := \partial^2_{\zeta,\chi}\theta^{t,\xi}$, H given by (4.74), and, in particular, with remainders $\mathcal{R}^{2p}_a$ and $\mathcal{R}^2_a$ coming from $H^{(2)}_a$ in (4.74). Recalling Lemma 4.18 and the assumption $\|\zeta\|_\infty \leqslant 1$, the remainders may be estimated by means of Lemma 4.27, with $\theta \equiv \hat{\theta} := \theta^{t,\xi}$, $\vartheta^1 \equiv \hat{\vartheta}^1 := \partial_\chi\theta^{t,\xi}$ and $\vartheta^2 \equiv \hat{\vartheta}^2 := \partial_\zeta\theta^{t,\xi}$. □

We now discuss the continuity with respect to ξ. We claim:

Lemma 4.31. *For any $p \geqslant 1$, there exist two constants $c_p := c_p(L) > 0$ and C_p such that, for $T \leqslant c_p$ (and with $\gamma = c_p$ in* (4.16)*),*

$$\Big[\mathcal{M}^{2p}_{\mathbb{E}_t}\big(\partial^2_{\zeta,\chi}\theta^{t,\xi} - \partial^2_{\zeta,\chi}\theta^{t,\xi'}\big)\Big]^{1/2p} \leqslant C_p\Big(1 \wedge \big\{|\xi - \xi'| + \Phi_{\alpha+1}(t,\xi,\xi')\big\}\Big)\big(|\chi| + \|\chi\|_2\big), \quad (4.92)$$

where $\Phi_{\alpha+1}(t,\cdot) : [L^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^d)]^2 \to \mathbb{R}_+$ is continuous at any point of the diagonal, does not depend on p and satisfies (4.10) *with α replaced by $\alpha + 1$. The restriction of $\Phi_{\alpha+1}(t,\cdot,\cdot)$ to $[L^2(\Omega,\mathcal{F}_0,\mathbb{P};\mathbb{R}^d)]^2$ may be assumed to be independent of $t \in [0,T]$.*

Proof. Generally speaking, the strategy is to apply Corollary 4.12, with $\eta = 0$, $\theta^\xi \equiv \hat{\theta}^\xi :\equiv \theta^{t,\xi}$, $\vartheta \equiv \hat{\vartheta} := \partial^2_{\zeta,\chi}\theta^{t,\xi}$ (and the same for ξ'), H given by (4.74) and, in particular, with remainders $\mathcal{R}^{2p}_a$ and $\mathcal{R}^2_a$ coming from $H^{(2)}_a$ in (4.74) (and the same for the remainders labelled with 'prime'). As in the proof of the previous Lemma 4.30, we can bound the remainders $(\mathcal{R}^{2p}_a)^{1/2p}$ by $C_p(|\chi| + \|\chi\|_2)$.

In order to estimate $(\Delta\mathcal{R}^{2p}_a)^{1/2p}$, we apply Lemma 4.28. A crucial fact is that we have (4.69). This says that, instead of working in conditional norm $[\bar{\mathcal{M}}^{2p}[\![\cdot]\!]]^{1/2p}$ for estimating the distance between $\theta^{t,\xi}$ and $\theta^{t,\xi'}$, we can directly work with the conditional norm $\|\cdot\|_{\mathcal{S}^{2p},t} + \|\cdot\|_{\mathcal{S}^2}$. As a byproduct, we can choose $\varepsilon = \sup_{s\in[t,T]}|\theta^{t,\xi}_s - \theta^{t,\xi'}_s|$ in (4.82). By (4.69), we thus get $C_p(1 \wedge \{|\xi - \xi'| + \Phi_{\alpha+1}(t,\xi,\xi')\})(|\chi| + \|\chi\|_2)$ as a bound for the terms containing the symbol ε in (4.82) ($\Phi_{\alpha+1}$ being independent of t when ξ and ξ' are $\mathcal{F}_0$-measurable). By Lemmas 4.18 and 4.19, all the terms involving an $\bar{\mathcal{M}}$ may be bounded in the same way. By (4.44) in Lemma 4.14, the same is true for the term involving Φ_α. By Cauchy-Schwarz inequality and once again by Lemmas 4.18 and 4.19, the same bound holds for the terms integrated under $\mathbb{E}$. In the end, the whole right-hand side in (4.82) may be bounded by $C_p(1 \wedge \{|\xi - \xi'| + \Phi_{\alpha+1}(t,\xi,\xi')\})(|\chi| + \|\chi\|_2)$ (without the t when $\xi,\xi' \in L^2(\Omega,\mathcal{F}_0,\mathbb{P};\mathbb{R}^d)$). In (4.38), this brings us to the case when the remainders are zero, but Φ_α is replaced by $\Phi_{\alpha+1}$. Applying (4.45) in Example 4.15, we complete the proof of (4.92). The last part of the statement (choice of a version of $\Phi_{\alpha+1}$ which is independent of t) follows from Remark 4.16. □

4.3.7. *Study of the Non McKean-Vlasov system.* We now repeat the same analysis but for the process $(\theta^{t,x,[\xi]}, \partial_\chi\theta^{t,x,[\xi]})$ (instead of $(\theta^{t,\xi}, \partial_\chi\theta^{t,\xi})$). Considering a continuously differentiable path $\lambda \mapsto \xi^\lambda$ from $\mathbb{R}$ into $L^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^d)$ such that $|[\mathrm{d}/\mathrm{d}\lambda]\xi^\lambda| \leqslant 1$, we are first to prove that the mapping $\mathbb{R} \ni \lambda \mapsto \partial_\chi\theta^{t,x,[\xi^\lambda]} \in \mathcal{S}^2([t,T];\mathbb{R}^d) \times \mathcal{S}^2([t,T];\mathbb{R}^m) \times$

$\mathcal{H}^2([t,T];\mathbb{R}^{m\times d})$ is continuously differentiable. Before we discuss the proof, we must say a word about the notation itself, which is slightly ambiguous. Since the law of ξ^λ is independent of λ, we could be indeed tempted to say that $\partial_\chi \theta^{t,x,[\xi^\lambda]}$ is independent of λ, which is obviously false. The reason is that, in the coefficients driving the system satisfied by $\partial_\chi \theta^{t,x,[\xi^\lambda]}$, there are terms of the form $\hat{\mathbb{E}}[\partial_\mu H(\theta^{t,x,[\xi^\lambda]},[\theta^{t,\xi^\lambda,(0)}])(\langle \theta^{t,\xi^\lambda,(0)}\rangle)\langle \partial_\chi \theta^{t,\xi^\lambda,(0)}\rangle]$, see (4.53), which explicitly depend upon the joint law of χ and ξ^λ. Clearly, there is no reason for the joint law to be independent of λ.

Recalling (4.53), we know that $\partial_\chi \theta^{t,x,[\xi^\lambda]}$ satisfies a standard linear FBSDE with $\hat{\mathbb{E}}[\partial_\mu H(\theta^{t,x,[\xi^\lambda]},[\theta^{t,\xi^\lambda,(0)}])(\langle \theta^{t,\xi^\lambda,(0)}\rangle)\langle \partial_\chi \theta^{t,\xi^\lambda,(0)}\rangle]$ as affine part. The coefficients of the FBSDE read as coefficients parametrized by λ through the values of $(\theta^{t,x,[\xi^\lambda]},\theta^{t,\xi^\lambda},\partial_\chi \theta^{t,\xi^\lambda})$. Now that the continuous differentiability of $\mathbb{R}\ni\lambda\mapsto(\theta^{t,x,[\xi^\lambda]},\theta^{t,\xi^\lambda},\partial_\chi\theta^{t,\xi^\lambda})$ has been proved, we can repeat the arguments used in the proof of Lemma 4.29 to show that $\mathbb{R}\ni\lambda\mapsto\partial_\chi\theta^{t,x,[\xi^\lambda]}\in\mathcal{S}^2([t,T];\mathbb{R}^d)\times\mathcal{S}^2([t,T];\mathbb{R}^m)\times\mathcal{H}^2([t,T];\mathbb{R}^{m\times d})$ is also continuously differentiable. (The complete proof is left to the reader.)

With the notation $\zeta := [\mathrm{d}/\mathrm{d}\lambda]_{|\lambda=0}\xi^\lambda$, we denote the second-order tangent process by $\partial^2_{\zeta,\chi}\theta^{t,x,[\xi]} := [\mathrm{d}/\mathrm{d}\lambda]_{|\lambda=0}\partial_\chi\theta^{t,x,[\xi^\lambda]}$. It satisfies a system of the form (4.3) with $\theta\equiv\theta^{t,x,[\xi]}$, $\hat\theta\equiv\theta^{t,\xi}$, $\vartheta\equiv\partial^2_{\zeta,\chi}\theta^{t,x,[\xi]}$ and $\hat\vartheta\equiv\partial^2_{\zeta,\chi}\theta^{t,\xi}$ and with generic coefficients H given by (compare if needed with (4.53)):

$$h_\ell(V,\langle\hat V^{(0)}\rangle)=\partial_x h(V,[\hat V^{(0)}]),\ \hat H_\ell(V,\langle\hat V^{(0)}\rangle)=\partial_\mu h(V,[\hat V^{(0)}])(\langle\hat V^{(0)}\rangle),\ H_a\equiv\tilde H^{(2)}_a,$$

where $\tilde H^{(2)}_a(r)$ is a variant of $H^{(2)}_a$ in (4.74) and reads:

$$\tilde H^{(2)}_a(r) := H^{(2)}_a\big(\theta^{t,x,[\xi]}_r,\langle\theta^{t,\xi,(0)}_r\rangle,\partial_\chi\theta^{t,x,[\xi]}_r,\partial_\zeta\theta^{t,x,[\xi]}_r,\langle\partial_\chi\theta^{t,\xi,(0)}_r\rangle,\langle\partial_\zeta\theta^{t,\xi,(0)}_r\rangle\big). \qquad (4.93)$$

On the model of Lemmas 4.30 and 4.31, we claim (compare with Lemma 4.21):

Lemma 4.32. *For any $p\geqslant 1$, there exist two constants $c_p := c_p(L) > 0$ and C_p such that, for $T\leqslant c_p$ and with $\gamma = c_p$ in* (4.16),

$$\big[\mathcal{M}^{2p}_{\mathbb{E}}\big(\partial^2_{\zeta,\chi}\theta^{t,x,[\xi]}\big)\big]^{1/2p}\leqslant C_p\|\chi\|_2,$$

and

$$\Big[\mathcal{M}^{2p}_{\mathbb{E}}\big(\partial^2_{\zeta,\chi}\theta^{t,x,[\xi]}-\partial^2_{\zeta,\chi}\theta^{t,x',[\xi']}\big)\Big]^{1/2p}\leqslant C_p\big(|x-x'|+\Phi_{\alpha+1}(t,\xi,\xi')\big)\|\chi\|_2,$$

where $\Phi_{\alpha+1}(t,\cdot):[L^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^d)]^2\to\mathbb{R}_+$ is continuous at any point of the diagonal, does not depend on p and satisfies (4.10) *with α replaced by $\alpha+1$. The restriction of $\Phi_{\alpha+1}(t,\cdot,\cdot)$ to $[L^2(\Omega,\mathcal{F}_0,\mathbb{P};\mathbb{R}^d)]^2$ may be assumed to be independent of $t\in[0,T]$.*

Proof. Loosely speaking, the result is similar to Lemmas 4.30 and 4.31, but with the realizations of ξ and ξ' therein replaced by x and x'. Actually, the main difference with the computations made for the McKean-Vlasov system comes from the shape of the remainder $\mathcal{R}_a$ that is implemented in the stability Corollary 4.12. In the proofs of Lemmas 4.30 and 4.31, the definition of the remainder $\mathcal{R}_a$ is based on the formula (4.74). In the current framework, it is based on the formula (4.93), which is slightly different. It can be estimated by means of Lemma 4.28. The proof is then completed as that one of Lemma 4.21. $\square$

4.3.8. *Final statement.* We finally claim:

Theorem 4.33. *There exists a constant $c := c(L) > 0$ such that, for $T \leqslant c$:*

- *for any $t \in [0,T]$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, the function $\mathbb{R}^d \ni x \mapsto U(t,x,\mu)$ is $\mathcal{C}^2$ and the functions $[0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \ni (t,x,\mu) \mapsto U(t,x,\mu)$, $[0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \ni (t,x,\mu) \mapsto \partial_x U(t,x,\mu)$ and $[0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \ni (t,x,\mu) \mapsto \partial^2_{xx} U(t,x,\mu)$ are continuous,*
- *for any $(t,x) \in [0,T] \times \mathbb{R}^d$, the function $\mathcal{P}^2(\mathbb{R}) \ni \mu \mapsto U(t,x,\mu)$ is partially $\mathcal{C}^2$; for any $(t,\mu) \in [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$, there exists a version of $\mathbb{R}^d \ni v \mapsto \partial_\mu U(t,x,\mu)(v) \in \mathbb{R}^d$ such that $\mathbb{R}^d \times \mathbb{R}^d \ni (x,v) \mapsto \partial_\mu U(t,x,\mu)(v) \in \mathbb{R}^d$ is differentiable at any (x,v) such that $v \in \mathrm{Supp}(\mu)$, the partial derivative $\mathbb{R}^d \times \mathbb{R}^d \ni (x,v) \mapsto \partial_v[\partial_\mu U(t,x,\mu)](v)$ being continuous at any (w,v) such that $v \in \mathrm{Supp}(\mu)$ and the partial derivative $\mathbb{R}^d \times \mathrm{Supp}(\mu) \ni (x,v) \mapsto \partial_x[\partial_\mu U(t,x,,\mu)](v)$ being continuous in (x,v).*

Moreover, we can find a constant C such that, for all $x \in \mathbb{R}^d$, for all $\xi \in L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d)$,

$$\big|\partial^2_{xx} U(t,x,[\xi])\big| + \mathbb{E}\big[\big|\partial_x\big[\partial_\mu U(t,x,[\xi])\big](\xi)\big|^2\big]^{1/2} + \mathbb{E}\big[\big|\partial_v\big[\partial_\mu U(t,x,[\xi])\big](\xi)\big|^2\big]^{1/2} \leqslant C,$$

and, for all $x,x' \in \mathbb{R}^d$, for all $\xi,\xi' \in L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d)$,

$$\begin{aligned}
&\big|\partial^2_{xx} U(t,x,[\xi]) - \partial^2_{xx} U(t,x',[\xi'])\big| \\
&\quad + \mathbb{E}\big[\big|\partial_x\big[\partial_\mu U(t,x,[\xi])\big](\xi) - \partial_x\big[\partial_\mu U(t,x',[\xi'])\big](\xi')\big|^2\big]^{1/2} \\
&\quad + \mathbb{E}\big[\big|\partial_v\big[\partial_\mu U(t,x,[\xi])\big](\xi) - \partial_v\big[\partial_\mu U(t,x',[\xi'])\big](\xi')\big|^2\big]^{1/2} \\
&\leqslant C\big\{|x-x'| + \Phi_{\alpha+1}(\xi,\xi')\big\},
\end{aligned}$$

where $\Phi_{\alpha+1} : [L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d)]^2 \to \mathbb{R}_+$ satisfies (4.10), with α replaced by $\alpha+1$. In particular, for any $x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, we can find a locally Lipschitz continuous version of the mappings $\mathbb{R}^d \ni v \mapsto \partial_x[\partial_\mu U(t,x,\mu)](v)$ and $\mathbb{R}^d \ni v \mapsto \partial_v[\partial_\mu U(t,x,\mu)](v)$.

The functions $[0,T] \times \mathbb{R}^d \times L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d) \ni (t,x,\xi) \mapsto \partial^2_{xx} U(t,x,[\xi]) \in \mathbb{R}^d$, $[0,T] \times \mathbb{R}^d \times L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d) \ni (t,x,\xi) \mapsto \partial_x[\partial_\mu U(t,x,[\xi])](\xi) \in L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d)$ and $[0,T] \times \mathbb{R}^d \times L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d) \ni (t,x,\xi) \mapsto \partial_v[\partial_\mu U(t,x,[\xi])](\xi) \in L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d)$ are continuous.

Proof. We first apply Theorem 3.6 in order to prove the $\mathcal{C}^2$-partial property of $\mu \mapsto U(t,x,\mu)$. By Theorem 4.25, we already know that the lifted version $L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d) \ni \xi \mapsto \mathcal{U}(t,x,\xi) = U(t,x,[\xi])$ is continuously differentiable in the sense of Fréchet. Recalling the identity

$$\partial_\chi Y_t^{t,x,[\xi]} = \mathbb{E}\big[D\mathcal{U}(t,x,[\xi])(\xi)\chi\big],$$

we deduce from Lemmas 4.18 and 4.19 that the gradient $D\mathcal{U}(t,x,\cdot)$ satisfies *(i)* and *(ii)* in the statement of Theorem 3.6. Now, using the same sequence $(\xi^\lambda)_{\lambda\in\mathbb{R}}$ as in §4.3.2, we notice that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}_{|\lambda=0} \mathbb{E}\big[D\mathcal{U}(t,x,\xi^\lambda)\chi\big] = \partial^2_{\zeta,\chi} Y_t^{t,x,[\xi]},$$

which satisfies *(i)* and *(ii)* in the statement of Theorem 3.6 thanks to Lemmas 4.30 and 4.31 (with ξ^λ playing the role of X^λ in the statement of Theorem 3.6). We deduce that, for any $(t,x) \in [0,T] \times \mathbb{R}^d$, the map $\mathcal{P}_2(\mathbb{R}^d) \ni \mu \mapsto U(t,x,\mu)$ is partially $\mathcal{C}^2$. In particular, for any $(t,x,\mu) \in [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$, we can find a version of $\mathbb{R}^d \ni v \mapsto \partial_\mu U(t,x,\mu)(v)$ that is continuously differentiable, such a version being uniquely defined on the support of μ. Moreover, by (3.42), we have the relationship

$$\partial^2_{\mathrm{sign}(Z')e,\mathrm{sign}(Z')\chi} Y_t^{t,x,[\xi]} = \mathbb{E}\big[\mathrm{Tr}\big\{\big(\partial_v[\partial_\mu U(t,x,\mu)](\xi)\big)\big(\chi \otimes e\big)\big\}\big],$$

which holds true for any $e \in \mathbb{R}^d$ and any $\xi, \chi \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)$, with $\xi \sim \mu$, and for a prescribed random variable Z' independent of (ξ, χ). From Lemma 4.32, we deduce that

$$\mathbb{E}[|\partial_v[\partial_\mu U(t,x,[\xi])](\xi)|^2]^{1/2} \leqslant C,$$
$$\mathbb{E}[|\partial_v[\partial_\mu U(t,x,[\xi])](\xi) - \partial_v[\partial_\mu U(t,x',[\xi'])](\xi')|^2]^{1/2} \leqslant C[|x-x'| + \Phi_{\alpha+1}(\xi,\xi')],$$

the extension of $\Phi_{\alpha+1}$ to the whole $[L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^d)]^2$ being achieved as in the proof of Lemma 4.23.

By means of Proposition 3.8, we deduce that, for given $t \in [0,T]$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, we can choose, for any $x \in \mathbb{R}^d$, a version of $\mathbb{R}^d \ni v \mapsto \partial_\mu U(t,x,\mu)(v)$ such that the derivative mapping $\mathbb{R}^d \ni v \mapsto \partial_v[\partial_\mu U(t,x,\mu)](v)$ is continuous on compact subsets of $\mathbb{R}^d$, uniformly in $x \in \mathbb{R}^d$. Using the same trick as in (3.33), we deduce that the family $(\mathbb{R}^d \ni v \mapsto \partial_v[\partial_\mu U(t,x,\mu)](v))_{x \in \mathbb{R}^d}$ is relatively compact for the topology of uniform convergence on compact subsets. Considering a sequence $(x_n)_{n \geqslant 1}$ that converges to x, we already know that the sequence of functions $(\mathbb{R}^d \ni v \mapsto \partial_v[\partial_\mu U(t,x_n,\mu)](v) \in \mathbb{R}^{d \times d})_{n \geqslant 1}$ converges in $L^2(\mathbb{R}^d, \mu; \mathbb{R}^{d \times d})$ to $\mathbb{R}^d \ni v \mapsto \partial_v[\partial_\mu U(t,x,\mu)](v) \in \mathbb{R}^{d \times d}$. Since $\partial_v[\partial_\mu U(t,x,\mu)]$ is uniquely defined on the support of μ, the limit of any converging subsequence (for the topology of uniform convergence on compact subsets of $\mathbb{R}^d$) of $(\partial_v[\partial_\mu U(t,x_n,\mu)](\cdot))_{n \geqslant 1}$ coincides with $\partial_v[\partial_\mu U(t,x,\mu)](\cdot)$ on the support of μ. We deduce that the function $\mathbb{R}^d \times \mathbb{R}^d \ni (x,v) \mapsto \partial_v[\partial_\mu U(t,x,\mu)](v) \in \mathbb{R}^{d \times d}$ is continuous at any (x,v) such that $v \in \mathrm{Supp}(\mu)$.

Proving a similar version of Lemma 4.32, but for $\partial^2_{xx}\theta^{t,x,[\xi]}$, we can show in the same way that U is twice differentiable in x and satisfies

$$\big|\partial^2_{xx} U(t,x,[\xi])\big|, \quad \big|\partial^2_{xx} U(t,x,[\xi]) - \partial^2_{xx} U(t,x',[\xi'])\big| \leqslant C\big[|x-x'| + \Phi_{\alpha+1}(\xi,\xi')\big],$$

We notice indeed that, for $\xi \sim \mu$, $\partial^2_{xx} Y_t^{t,x,[\xi]}$ coincides with $\partial^2_{xx} U(t,x,\mu)$.

Similarly, we can investigate $\partial_x[\partial_\chi \theta^{t,x,[\xi]}]$. By means of Lemma 6.1 in Appendix, we can prove that, once a continuous version of $\partial_\mu U(t,x,\mu)$ has be chosen for any $(t,x,\mu) \in [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$, the function $\mathbb{R}^d \ni x \mapsto \partial_\mu U(t,x,\mu)(v)$ is differentiable at any point (x,v) such that $v \in \mathrm{Supp}(\mu)$, the derivative function $\mathbb{R}^d \times \mathrm{Supp}(\mu) \ni (x,v) \mapsto \partial_x[\partial_\mu U(t,x,\mu)](v)$ being continuous. Combining with the continuous differentiability property in v, we deduce that the mapping $\mathbb{R}^d \times \mathbb{R}^d \ni (x,v) \mapsto \partial_\mu U(t,x,\mu)(v)$ is differentiable at any point (x,v) such that $v \in \mathrm{Supp}(\mu)$, with the aforementioned prescribed continuity properties of the partial derivatives.

Then $\partial_x[\partial_\chi Y_t^{t,x,[\xi]}]$ identifies with $\mathbb{E}[\partial_x[\partial_\mu U(t,x,\mu)](\xi)\chi]$. Moreover,

$$\mathbb{E}\big[|\partial_x[\partial_\mu U(t,x,[\xi])](\xi)|^2\big]^{1/2} \leqslant C,$$
$$\mathbb{E}\big[|\partial_x[\partial_\mu U(t,x,[\xi])](\xi) - \partial_x[\partial_\mu U(t,x',[\xi'])](\xi')|^2\big]^{1/2} \leqslant C\big[|x-x'| + \Phi_{\alpha+1}(\xi,\xi')\big].$$

Generally speaking, time continuity of the derivatives can be proved as in Theorem 4.25. Anyhow, some precaution is needed since the drivers of the backward equations that represent all the second-order derivatives involve quadratic terms in $\partial_\chi Z^{t,\xi}$ and $\partial_\chi Z^{t,x,[\xi]}$, see for instance (4.74). The *a priori* difficulty is that, so far, we have exhibited bounds for $\partial_\chi Z^{t,\xi}$ and $\partial_\chi Z^{t,x,[\xi]}$ in $\mathcal{H}$ norm only, which might not suffice for investigating the time regularity. The key point is then to notice that all these terms may be estimated in $\mathcal{S}$ instead of $\mathcal{H}$ norm. The trick is to invoke the representation formula (4.68) for the process $Z^{t,x,[\xi]}$, to differentiate it and then to make use of the bounds we just proved for $\partial^2_{xx} U$ and $\partial_x[\partial_\mu U]$. □

We now turn to

Proof. [Proof of Theorem 2.7] We first prove that U is a classical solution of the PDE (2.12). The main argument follows from (2.10), the idea being to apply the chain rule to $U(t+h,x,\cdot)$, which is licit thanks to Theorem 4.33. Following (2.9), we get

$$\begin{aligned}
&U\big(t+h,x,[X^{t,\xi}_{t+h}]\big) - U\big(t+h,x,[\xi]\big)\\
&= \int_t^{t+h} \widehat{\mathbb{E}}\Big[\partial_\mu U\big(t+h,x,[X^{t,\xi}_r]\big)\big(\langle X^{t,\xi}_r\rangle\big) b\big(\langle \theta^{t,\xi}_r\rangle, [\theta^{t,\xi,(0)}_r]\big)\Big]\,\mathrm{d}r\\
&\quad + \frac{1}{2}\int_t^{t+h} \widehat{\mathbb{E}}\Big[\mathrm{Trace}\big[\partial_v\big[\partial_\mu U\big]\big(t+h,x,[X^{t,\xi}_r]\big)\big(\langle X^{t,\xi}_r\rangle\big)\big(\sigma\sigma^\dagger\big)\big(\langle \theta^{t,\xi,(0)}_r\rangle, [\theta^{t,\xi,(0)}_r]\big)\big]\Big]\,\mathrm{d}r.
\end{aligned}$$

Assumption (**$H0$**) and Theorems 4.25 and 4.33 provide estimates on the smoothness of b, $\sigma\sigma^\dagger$, $\partial_\mu U$ and $\partial_v[\partial_\mu U]$. We deduce that we can find a non-negative functional Φ on $[L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d\times\mathbb{R}^m\times\mathbb{R}^{m\times d})]^2$, continuous at any point of the diagonal, matching 0 on the diagonal, such that

$$\begin{aligned}
&\Big| U\big(t+h,x,[X^{t,\xi}_{t+h}]\big) - U\big(t+h,x,[\xi]\big)\\
&\qquad - h\widehat{\mathbb{E}}\Big[\partial_\mu U\big(t+h,x,[\xi]\big)\big(\langle \xi\rangle\big) b\big(\langle \theta^{t,\xi}_t\rangle, [\theta^{t,\xi,(0)}_t]\big)\Big]\\
&\qquad - \frac{h}{2}\widehat{\mathbb{E}}\Big[\mathrm{Trace}\big[\partial_v\big[\partial_\mu U\big]\big(t+h,x,[\xi]\big)\big(\langle \xi\rangle\big)\big(\sigma\sigma^\dagger\big)\big(\langle \theta^{t,\xi,(0)}_t\rangle, [\theta^{t,\xi,(0)}_t]\big)\big]\Big]\Big|\\
&\leqslant h \sup_{r\in[t,t+h]} \Phi\big(\theta^{t,\xi}_r, \theta^{t,\xi}_t\big).
\end{aligned}$$

Recalling that $\theta^{t,\xi}_r = (X^{t,\xi}_r, Y^{t,\xi}_r, \partial_x U(r, X^{t,\xi}_r, [X^{t,\xi}_r])\sigma(X^{t,\xi}_r, Y^{t,\xi}_r))$, we deduce from Theorem 4.25 (smoothness of $\partial_x U$ both in time and in space) that it converges (in L^2) to $\theta^{t,\xi}_t$ as r tends to t, proving that the supremum above tends to 0 as h tends to 0. Now, using the time continuity of the derivatives $\partial_\mu U$ and $\partial_v[\partial_\mu U]$ (see Theorem 4.33), we deduce that there exists a function $\varepsilon : \mathbb{R} \ni u \mapsto \varepsilon_u \in \mathbb{R}_+$, with $\lim_{u\to 0}\varepsilon_u = 0$, such that

$$\begin{aligned}
&\Big| U\big(t+h,x,[X^{t,\xi}_{t+h}]\big) - U\big(t+h,x,[\xi]\big)\\
&\qquad - h\Big[\widehat{\mathbb{E}}\Big[\partial_\mu U\big(t,x,[\xi]\big)\big(\langle \xi\rangle\big) b\big(\langle \theta^{t,\xi}_t\rangle, [\theta^{t,\xi,(0)}_t]\big)\Big]\\
&\qquad - \frac{h}{2}\widehat{\mathbb{E}}\Big[\mathrm{Trace}\big[\partial_v\big[\partial_\mu U\big]\big(t,x,[\xi]\big)\big(\langle \xi\rangle\big)\big(\sigma\sigma^\dagger\big)\big(\langle \theta^{t,\xi,(0)}_t\rangle, [\theta^{t,\xi,(0)}_t]\big)\big]\Big]\Big]\Big| \leqslant h\varepsilon_h.
\end{aligned} \tag{4.94}$$

Now, we can plug (4.94) into (2.10). Following (2.11), we get that the time increment $[U(t+h,x,[\xi]) - U(t,x,[\xi])]/h$ has a limit as h tends to 0. As in Subsection 2.3, the right derivative in time satisfies (2.12) and is thus continuous in time. Since U is obviously continuous in time, we deduce that the mapping $[0,T] \ni t \mapsto U(t,x,[\xi])$ is differentiable and that the PDE (2.12) holds true. $\square$

5. Large population stochastic control – proof of Theorem 2.9

In this section, we discuss two applications of our previous results to large population stochastic control. The first application is related to mean-field games, whilst the second one is related to the optimal control of McKean-Vlasov equations.

5.1. **The global smoothness of the decoupling field.** So far, smoothness of the decoupling field U has been discussed for small time intervals $[0,T]$; namely for $T \leqslant \delta_0$ where $\delta_0 > 0$ only depends upon the Lipschitz constants of the coefficients b, f, σ and g, denoted by the common letter L in condition $(\boldsymbol{H0})$(i). A natural, though quite challenging, question concerns the possible extension of such a result to the case when T is arbitrarily large.

The principle for extending the result to an arbitrarily large time horizon is discussed in the earlier paper [11]. It consists of a backward recursion starting from the terminal time T. Thanks to the short time result proved in the previous section, the mapping $[T-\delta_0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \ni (t,x,\mu) \mapsto U(t,x,\mu) \in \mathbb{R}^m$ is rigorously defined as the initial value $Y_t^{t,x,\mu}$ of the backward component of the system (2.4), existence and uniqueness of the solution of the forward-backward system following from the condition $T-t \leqslant \delta_0$. By Lemma 4.1, U is Lipschitz continuous in (x,μ), uniformly in $t \in [T-\delta_0,T]$. Up to a modification of the choice of the constant δ_0, δ_0 still depending on the Lipschitz constants of the coefficients only, the results established in Section 4 show that, under the assumptions detailed in Subsection 2.4, U belongs to the class $\bigcup_{\beta \geqslant 0} \mathcal{D}_\beta([T-\delta_0,T])$. As in [11], we proceed by reapplying the short time existence, uniqueness and differentiability result to a new interval of the form $[T-(\delta_0+\delta_1), T-\delta_0]$, with the new terminal condition $U(T-\delta_0,\cdot,\cdot)$ at time $T-\delta_0$ replacing the terminal condition g at time T. A preliminary condition for iterating the short time solvability property is that $U(T-\delta_0,\cdot,\cdot)$ is an admissible boundary condition. Under $(\boldsymbol{H2})$, Theorems 4.25 and 4.33 say that it is indeed the case, up to a deterioration of α into $\alpha+1$, the exponent α driving the local Lipschitz regularity of the derivatives of the coefficients in $(\boldsymbol{H1})$ and $(\boldsymbol{H2})$. This makes possible to reapply the existence and uniqueness result for short time horizons with α be replaced by $\alpha+1$. Fortunately, the length δ_1 of the new interval of existence and uniqueness only depends on the Lipschitz constant of b, f, σ and $U(T-\delta_0,\cdot,\cdot)$. In particular, it does not suffer from the deterioration of the exponent α into $\alpha+1$, which is a crucial fact. As a result we are able to extend the definition of U to $[T-(\delta_0+\delta_1), T-\delta_0] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$. Since the new terminal condition $U(T-\delta_0,\cdot,\cdot)$ has the same properties as g (but possibly with a different Lipschitz constant and a different α), the extended version of U is in the class $\mathcal{D}_{\alpha+1}([T-(\delta_0+\delta_1),T]) \subset \bigcup_{\beta \geqslant 0} \mathcal{D}_\beta([T-(\delta_0+\delta_1),T])$. The argument can be applied recursively on a sequence of small intervals of the form $[T-(\delta_0+\cdots+\delta_{n+1}), T-(\delta_0+\cdots+\delta_n)]$, $n \geqslant 0$. Of course, the issue is that the lengths $(\delta_n)_{n \geqslant 0}$ may be smaller and smaller so that the sum $\sum_{n \geqslant 0} \delta_n$ may not exceed T. This happens if the Lipschitz constant of U at times $(T-(\delta_0+\cdots+\delta_n))_{n \geqslant 1}$ blows up before that the sequence $(\delta_0+\cdots+\delta_n)_{n \geqslant 1}$ exceeds T. Put it differently, the construction of the smooth decoupling field U on $[0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$ can be achieved by means of a backward recursion provided that the Lipschitz constant of $U(t,\cdot,\cdot)$ remain bounded as t runs backward along the induction.

The crux of the matter is thus to get such a Lipschitz estimate. In the following, we present two examples, derived from large population stochastic control, for which the following assumption holds true:

Assumption$((\boldsymbol{H3}))$. *For any $t \in [0,T]$ and any square integrable $\mathcal{F}_t$-measurable random variable ξ, the system (2.3) has a unique solution $(X_s^{t,\xi}, Y_s^{t,\xi}, Z_s^{t,\xi})_{s \in [t,T]}$ and it satisfies, for all $\xi, \xi' \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)$,*

$$\mathbb{E}\big[|Y_t^{t,\xi} - Y_t^{t,\xi'}|^2\big]^{1/2} \leqslant \Lambda \mathbb{E}\big[|\xi-\xi'|^2\big]^{1/2}, \tag{5.1}$$

with Λ a positive constant that does not depend on ξ, ξ' nor on t.

We will show below that, under ($\boldsymbol{H3}$), the decoupling field U constructed along the induction must satisfy at any time t at which it has been defined

$$\forall\, \xi, \xi' \in L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^d), \quad \mathbb{E}\big[|U(t,\xi,[\xi]) - U(t,\xi',[\xi'])|^2\big]^{1/2} \leqslant \Lambda \mathbb{E}\big[|\xi - \xi'|^2\big]^{1/2}. \quad (5.2)$$

Although it is a first step in the control of the Lipschitz constant for U, it remains insufficient for our purposes. The reason is that the control is here stated along the diagonal only. Fortunately, the next Lemma permits to fill the gap and to bound the Lipschitz constant of U, in x and μ, on the entire domain:

Lemma 5.1. *Under ($\boldsymbol{H2}$), assume that U has been constructed on some interval $[T_0, T]$, for $T_0 \in [0,T]$. Assume moreover that it satisfies* (5.2) *for any $t \in [T_0,T]$ and that it is continuously differentiable in the directions x and μ at any time $t \in [T_0,T]$. Then, we can find a constant $\tilde{\Lambda}$, independent of T_0, such that for $t \in [T_0,T]$, $x, x' \in \mathbb{R}^d$ and $\mu, \mu' \in \mathcal{P}_2(\mathbb{R}^d)$:*

$$|U(t,x,\mu) - U(t,x',\mu')| \leqslant \tilde{\Lambda}\big(|x-x'| + W_2(\mu,\mu')\big).$$

Proof.

Step 1. Applying Proposition 3.8 (with $\alpha = 0$) we get that U is Λ-Lipschitz continuous in x, or equivalently that $\|\partial_x U(t,\cdot,\cdot)\|_\infty \leqslant \Lambda$ for $t \in [T_0,T]$.

Step 2a. Now, for $t \in [T_0,T]$, $x \in \mathbb{R}^d$ and $\xi, \xi' \in L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^d)$, we have

$$\begin{aligned}
&|U(t,x,[\xi]) - U(t,x,[\xi'])| \\
&\quad = \left| \int_0^1 \mathbb{E}\big[\partial_\mu U(t,x,[(1-\lambda)\xi + \lambda\xi'])\big((1-\lambda)\xi + \lambda\xi'\big)\big(\xi - \xi'\big)\big] \mathrm{d}\lambda \right| \\
&\quad \leqslant \mathbb{E}\big[|\xi' - \xi|^2\big]^{1/2} \int_0^1 \mathbb{E}\big[\big|\partial_\mu U(t,x,[(1-\lambda)\xi + \lambda\xi'])\big((1-\lambda)\xi + \lambda\xi'\big)\big|^2\big]^{1/2} \mathrm{d}\lambda.
\end{aligned}$$

In particular, in order to complete the proof, it suffices to find a constant C, independent of T_0, such that, for all $(t,x,\mu) \in [T_0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$,

$$\mathbb{E}\big[|\partial_\mu U(t,x,\mu)(\xi)|^2\big]^{1/2} \leqslant C. \quad (5.3)$$

Step 2b. Combining Step 1 and (5.2), we obtain

$$\mathbb{E}\big[|U(t,\xi,[\xi]) - U(t,\xi,[\xi'])|^2\big]^{1/2} \leqslant 2\Lambda \mathbb{E}\big[|\xi - \xi'|^2\big]^{1/2},$$

which at the level of the gradient says (choosing $\xi' - \xi = h\chi$, letting h tend to 0 and applying Fatou's lemma)

$$\forall \chi \in L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^d), \quad \mathbb{E}\Big[\hat{\mathbb{E}}\big[\partial_\mu U(t,\xi,[\xi])(\langle\xi\rangle)\langle\chi\rangle\big]^2\Big]^{1/2} \leqslant 2\Lambda \mathbb{E}[|\chi|^2]^{1/2}. \quad (5.4)$$

This control is weaker than (5.3). In order to get (5.3), the strategy is to apply, on some small interval $[t,S]$, the results proved in Section 4 on the first-order differentiability of U with respect to the measure. Assuming that ξ is $\mathcal{F}_t$ measurable, we make use of Lemma 4.17 but on the interval $[t,S]$ and with g replaced by $U(S,\cdot,\cdot)$, the value of S being specified next. In the backward component of the system of the type (4.3) satisfied by the derivative process $(\partial_\chi X_s^{t,\xi}, \partial_\chi Y_s^{t,\xi}, \partial_\chi Z_s^{t,\xi})_{s \in [t,S]}$, the boundary condition reads as

$$\partial_\chi Y_S^{t,\xi} = \partial_x U\big(S, X_S^{t,\xi}, [X_S^{t,\xi}]\big)\partial_\chi X_S^{t,\xi} + \hat{\mathbb{E}}\big[\partial_\mu U\big(S, X_S^{t,\xi}, [X_S^{t,\xi}]\big)\big(\langle X_S^{t,\xi}\rangle\big)\langle\partial_\chi X_S^{t,\xi}\rangle\big].$$

Now, by the a priori bound (5.4) and Step 1, we get that

$$\mathbb{E}\big[|\partial_\chi Y_S^{t,\xi}|^2\big]^{1/2} \leqslant C\mathbb{E}\big[|\partial_\chi X_S^{t,\xi}|^2\big]^{1/2}. \quad (5.5)$$

Above and in the computations below, the constant C may change from line to line, it depends on the parameters in assumptions and, importantly, is uniform with respect to $0 \leqslant T_0 \leqslant t \leqslant S \leqslant T$. The bound (5.5) reads as a Lipschitz bound (in L^2), with a constant C.

We can make use of (4.24) in Corollary 4.8, with $p = 1$, $\gamma \leqslant 1/\Gamma_1$, $g_\ell \equiv 0$, $\hat{G}_\ell \equiv 0$, $G_a(S) = \partial_\chi Y_S^{t,\xi}$ (which is to say, in rough terms, that we put the whole terminal condition in the remainder) and $[0,T]$ replaced by $[t,S]$. The remainder term $\mathbb{E}[\mathcal{R}_a^2]$ is thus equal to $\gamma^{1/2}\mathbb{E}[|\partial_\chi Y_S^{t,\xi}|^2]$, which is less than $C\gamma^{1/2}\mathbb{E}[|\partial_\chi X_S^{t,\xi}|^2]$. Therefore, choosing $\Gamma_1 C\gamma^{1/2} = 1/2$, we have, for $S - t \leqslant c := c(L)$,

$$\mathbb{E}\biggl[\sup_{s\in[t,S]}\bigl(|\partial_\chi X_s^{t,\xi}|^2 + |\partial_\chi Y_s^{t,\xi}|^2\bigr) + \int_t^S |\partial_\chi Z_s^{t,\xi}|^2 \mathrm{d}s\biggr]^{1/2} \leqslant C\|\chi\|_2. \tag{5.6}$$

Now, consider the derivative process of the non McKean-Vlasov system (2.4). It satisfies a forward-backward system of the type (4.3). The boundary condition in the backward component may be expressed as

$$\partial_\chi Y_S^{t,x,[\xi]} = \partial_x U\bigl(S, X_S^{t,x,[\xi]}, [X_S^{t,\xi}]\bigr)\partial_\chi X_S^{t,x,\xi} + \hat{\mathbb{E}}\bigl[\partial_\mu U\bigl(S, X_S^{t,x,[\xi]}, [X_S^{t,\xi}]\bigr)(\langle X_S^{t,\xi}\rangle)\langle \partial_\chi X_S^{t,\xi}\rangle\bigr].$$

Under the notations (4.5) and (4.6), the above writing reads as the decomposition of the terminal condition in the form $g_\ell(X_S^{t,x,[\xi]}, [X_S^{t,\xi}]) = \partial_x U(S, X_S^{t,x,[\xi]}, [X_S^{t,\xi}])$, $\hat{G}_\ell \equiv 0$ and $G_a(S) = \hat{\mathbb{E}}[\partial_\mu U(S, X_S^{t,x,[\xi]}, [X_S^{t,\xi}])\langle \partial_\chi X_S^{t,\xi}\rangle]$. We can apply once again Corollary 4.8, with $p = 1$, $\vartheta = \partial_\chi \theta^{t,x,[\xi]}$, $\hat{\vartheta} = \partial_\chi \theta^{t,\xi}$ and B, Σ and F given by (4.53). Recalling that $\partial_x U$ is bounded by Λ, we get for $S - t \leqslant \tilde{c} := \tilde{c}(\Lambda \vee L)$,

$$\begin{aligned}|\partial_\chi Y_t^{t,x,[\xi]}| \leqslant C\Bigl(&\mathbb{E}\bigl[\bigl|\hat{\mathbb{E}}[\partial_\mu U(S, X_S^{t,x,[\xi]}, [X_S^{t,\xi}])(\langle X_S^{t,\xi}\rangle)\langle \partial_\chi X_S^{t,\xi}\rangle]\bigr|^2\bigr]^{1/2} \\ &+ \mathbb{E}\bigl[\sup_{s\in[t,S]}\bigl(|\partial_\chi X_s^{t,\xi}|^2 + |\partial_\chi Y_s^{t,\xi}|^2\bigr)\bigr]^{1/2}\Bigr),\end{aligned}$$

the second part coming from the remainder term H_a in (4.53) when $H = B, \Sigma, F$.

Therefore, from the relationship $\partial_\chi Y_t^{t,x,[\xi]} = \hat{\mathbb{E}}[\partial_\mu U(t,x,[\xi])(\langle \xi\rangle)\chi]$ and from (5.6), we get

$$\hat{\mathbb{E}}\bigl[|\partial_\mu U(t,x,[\xi])(\langle\xi\rangle)|^2\bigr]^{1/2} \leqslant C\Bigl(1 + \mathbb{E}\hat{\mathbb{E}}\bigl[|\partial_\mu U\bigl(S, X_S^{t,x,[\xi]}, [X_S^{t,\xi}]\bigr)(\langle X_S^{t,\xi}\rangle)|^2\bigr]^{1/2}\Bigr).$$

We deduce

$$\begin{aligned}&\sup_{x\in\mathbb{R}^d,\xi\in L^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^d)} \hat{\mathbb{E}}\bigl[|\partial_\mu U(t,x,[\xi])(\langle\xi\rangle)|^2\bigr]^{1/2} \\ &\quad\leqslant C\Bigl(1 + \sup_{x\in\mathbb{R}^d,\xi\in L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d)} \hat{\mathbb{E}}\bigl[|\partial_\mu U\bigl(S,x,[\xi]\bigr)(\langle\xi\rangle)|^2\bigr]^{1/2}\Bigr).\end{aligned}$$

Since the terms in the suprema only depend on the law of ξ, we can assume that the supremum in the left-hand side is taken over $\xi \in L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^d)$. Assuming without any loss of generality that $C \geqslant 1$ and iterating the inequality, we get

$$\begin{aligned}1 + &\sup_{x\in\mathbb{R}^d,\xi\in L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d)} \hat{\mathbb{E}}\bigl[|\partial_\mu U(t,x,[\xi])(\langle\xi\rangle)|^2\bigr]^{1/2} \\ &\leqslant 2C\Bigl(1 + \sup_{x\in\mathbb{R}^d,\xi\in L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d)} \hat{\mathbb{E}}\bigl[|\partial_\mu U\bigl(S,x,[\xi]\bigr)(\langle\xi\rangle)|^2\bigr]^{1/2}\Bigr) \\ &\leqslant (2C)^n\Bigl(1 + \sup_{x\in\mathbb{R}^d,\xi\in L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d)} \hat{\mathbb{E}}\bigl[|\partial_\mu g\bigl(x,[\xi]\bigr)(\langle\xi\rangle)|^2\bigr]^{1/2}\Bigr),\end{aligned}$$

with $n = \lceil (T-t)/\tilde{c} \rceil$. Recalling the notation L in $(\boldsymbol{H0})$(i), we deduce that

$$\sup_{x\in\mathbb{R}^d,\xi\in L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d)} \hat{\mathbb{E}}\big[|\partial_\mu U(t,x,[\xi])(\langle\xi\rangle)|^2\big]^{1/2} \leqslant LC^{T/\tilde{c}+1},$$

which proves (5.3) and thus completes the proof. □

Proposition 5.2. *Assume that b, f, σ and g satisfy $(\boldsymbol{H2})$ and that the statement $(\boldsymbol{H3})$ holds true. Then there exists a mapping $U : [0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d) \ni (t,x,\mu) \mapsto U(t,x,\mu) \in \mathbb{R}^m$, Lipschitz continuous in (x,μ), uniformly in $t\in[0,T]$, such that, for all $t\in[0,T]$ and $\xi\in L^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^d)$,*

$$Y_s^{t,\xi} = U\big(s, X_s^{t,\xi}, [X_s^{t,\xi}]\big).$$

Moreover, U belongs to $\bigcup_{\beta\geqslant 0}\mathcal{D}_\beta$ and satisfies the master equation (2.12).

Proof. The proposition is proved by induction. Given a large integer $N \geqslant 1$ (the value of which is fixed below), let $\delta = T/N$. The induction hypothesis reads, for $n\in\{1,\dots,N\}$:

$(\mathcal{I}_n)$: There exists a mapping $U : [T-n\delta,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d) \ni (t,x,\mu) \mapsto U(t,x,\mu) \in \mathbb{R}^m$ that belongs to $\bigcup_{\beta\geqslant 0}\mathcal{D}_\beta([T-n\delta,T])$ such that

(i) for any $t\in[T-n\delta,T]$, the function $U(t,\cdot,\cdot)$ satisfies the same assumption as g in $(\boldsymbol{H0})$(i), $(\boldsymbol{H1})$ $(\boldsymbol{H2})$, but with the constant L replaced by $\tilde{\Lambda}$ coming from Lemma 5.1 ($\tilde{L}$ and α being replaced by some $\tilde{L}_n$ and $\tilde{\alpha}_n$);

(ii) U satisfies the master PDE (2.12) on $[T-n\delta,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)$

(iii) for all $t\in[T-n\delta,T]$ and $\xi\in L^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^d)$, $Y_t^{t,\xi} = U(t,\xi,[\xi])$.

Step 1. In this step, we first specify the value of N and we prove that $(\mathcal{I}_1)$ is satisfied.

First, notice that $\tilde{\Lambda}$ in Lemma 5.1 may be assumed to be larger than L in $(\boldsymbol{H0})$(i), $(\boldsymbol{H1})$ and $(\boldsymbol{H2})$. We then choose N as the smallest integer such that $\delta := T/N \leqslant c(\tilde{\Lambda})$, where c is given by Theorems 4.25 and 4.33 (or more precisely by the minimum of the c's in these two statements). For $T-t\leqslant\delta$, we know that, for any $x\in\mathbb{R}^d$ and $\mu\in\mathcal{P}_2(\mathbb{R}^d)$, the system (2.4) has a unique solution $(X_s^{t,x,\mu}, Y_s^{t,x,\mu}, Z_s^{t,x,\mu})_{s\in[t,T]}$ and, by Theorems 4.25 and 4.33, U belongs to $\bigcup_{\beta\geqslant 0}\mathcal{D}_\beta([T-\delta,T])$ and satisfies the master equation on $[T-\delta,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)$.

Now, by Corollary 1.5 in [11] (which holds true for small time horizons), we can replace x by a square-integrable $\mathcal{F}_t$-measurable random initial condition ξ in (2.4). With obvious notations, it must satisfy $Y_t^{t,\xi,\mu} = U(t,\xi,\mu)$. Choosing ξ with distribution μ, we deduce from uniqueness in small time to the system (2.3) that $(X_s^{t,\xi,\mu}, Y_s^{t,\xi,\mu}, Z_s^{t,\xi,\mu})_{s\in[t,T]}$ coincides with $(X_s^{t,\xi}, Y_s^{t,\xi}, Z_s^{t,\xi})_{s\in[t,T]}$. Indeed, $(X_s^{t,\xi}, Y_s^{t,\xi}, Z_s^{t,\xi})_{s\in[t,T]}$ solves (2.4) with x replaced by ξ and the system (2.4) has a unique solution. Therefore, we deduce that, with probability 1,

$$Y_t^{t,\xi} = U(t,\xi,[\xi]), \text{ for all } \quad t\in[T-\delta,T].$$

By $(\boldsymbol{H3})$, U satisfies (5.2) so that, by Lemma 5.1, $(\mathcal{I}_1)$ is indeed satisfied.

Step 2 Assume that, for some $n\in\{1,\dots,N-1\}$, $(\mathcal{I}_n)$ holds true.

For any $t\in[T-(n+1)\delta,T]$ and $\xi\in L^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^d)$, we consider again the forward-backward system (2.3). By $(\boldsymbol{H3})$, it admits a unique solution. In particular, by the uniqueness property guaranteed by $(\boldsymbol{H3})$, it must hold that

$$Y_{T-n\delta}^{t,\xi} = Y_{T-n\delta}^{T-n\delta, X_{T-n\delta}^{t,\xi}}. \tag{5.7}$$

By the induction hypothesis, $Y^{t,\xi}_{T-n\delta}$ must have the form

$$Y^{t,\xi}_{T-n\delta} = U\big(T-n\delta, X^{t,\xi}_{T-n\delta}, [X^{t,\xi}_{T-n\delta}]\big).$$

Therefore, we now consider (2.3) but on $[T-(n+1)\delta, T-n\delta]$, with $U(T-n\delta,\cdot,\cdot)$ as terminal boundary condition. By the induction hypothesis, we know that $U(T-n\delta,\cdot,\cdot)$ is $\tilde{\Lambda}$-Lipschitz continuous, so that existence and uniqueness to (2.3) with $U(T-n\delta,\cdot,\cdot)$ as terminal boundary condition hold true. This permits to extend the definition of U to $[T-(n+1)\delta, T-n\delta]$. By Theorems 4.25 and 4.33, the extension of U belongs to $\bigcup_{\beta\geqslant 0}\mathcal{D}_\beta([T-(n+1)\delta, T-n\delta])$ and thus to $\bigcup_{\beta\geqslant 0}\mathcal{D}_\beta([T-(n+1)\delta, T])$. Moreover, it satisfies the master equation on $[T-(n+1)\delta, T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)$.

Consider now the restriction of the global solution $(X^{t,\xi}_s, Y^{t,\xi}_s, Z^{t,\xi}_s)_{s\in[t,T]}$ to the small interval $[T-(n+1)\delta, T-n\delta]$. By (5.7), it must coincide with the short time solution constructed on $[t, T-n\delta]$ with $U(T-n\delta,\cdot,\cdot)$ as terminal boundary conditions. By the same arguments as in Step 1, we thus get that

$$Y^{t,\xi}_t = U(t,\xi,[\xi])$$

with probability one. This shows that U satisfies (5.2) and applying Lemma 5.1, we get that $(\mathcal{I}_{n+1})$ is satisfied. □

5.2. **Mean-field games.**

5.2.1. *General set-up.* Mean-field games were introduced simultaneously by Lasry and Lions [23, 24, 25] and by Huang, Caines and Malhamé [20]. Their purpose is to describe asymptotic Nash equilibria within large population of controlled agents interacting with one another through the empirical distribution of the system. When players are driven by similar dynamics and subject to similar cost functionals, asymptotic equilibria are expected to obey some propagation of chaos, limiting the analysis of the whole population to the analysis of one single player and thus reducing the complexity in a drastic way.

The dynamics of one single player read as

$$\mathrm{d}X_t = b(X_t,\mu_t,\alpha_t)\mathrm{d}t + \sigma(X_t,\mu_t)\mathrm{d}W_t, \quad t\in[0,T], \tag{5.8}$$

for some possibly random initial condition X_0, where $(W_t)_{t\in[0,T]}$ is an $\mathbb{R}^d$-valued Brownian motion and $b:\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\times\mathbb{R}^k\to\mathbb{R}^d$ and $\sigma:\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)$ are Lipschitz-continuous on the model of $(\boldsymbol{H0})$(i). Above, $(\alpha_t)_{t\in[0,T]}$ denotes the control process. It takes values in $\mathbb{R}^k$ and is assumed to be progressively-measurable and to satisfy:

$$\mathbb{E}\int_0^T |\alpha_t|^2\mathrm{d}t < +\infty.$$

The family $(\mu_t)_{t\in[0,T]}$ denotes an arbitrary flow of probability measures in $\mathcal{P}_2(\mathbb{R}^d)$. It is intended to describe the statistical equilibrium of the game, the notion of equilibrium being defined according to some cost functional

$$J\big((\alpha_t)_{t\in[0,T]}\big) = \mathbb{E}\bigg[G(X_T,\mu_T) + \int_0^T F(X_t,\mu_t,\alpha_t)\mathrm{d}t\bigg],$$

and being actually given by the solution of a fixed point problem, the description of which is taken from [7]:

(i) Given the family $(\mu_t)_{t\in[0,T]}$, solve the optimization problem

$$\inf_{(\alpha_t)_{t\in[0,T]}} J\big((\alpha_t)_{t\in[0,T]}\big).$$

Assume that the optimal path is uniquely defined and denote it by $(\hat{X}_t^{(\mu_s)_{s\in[0,T]}})_{t\in[0,T]}$.

(ii) Find $(\mu_s)_{s\in[0,T]}$ such that $[\hat{X}_t^{(\mu_s)_{s\in[0,T]}}] = \mu_t$ for all $t \in [0,T]$.

Generally speaking, there are two ways to characterize the optimal paths in (i) by means of an FBSDE. The first one is to represent the value function of the optimization problem (i) as the decoupling field of a forward-backward system, in which case equilibria solving (ii) may be described through a McKean-Vlasov FBSDE along the lines of [10]. Another way is to make use of the stochastic Pontryagin principle to represent directly the optimal path in (i) as the forward component of the solution of a forward-backward system, in which case equilibria solving (ii) may be described through a McKean-Vlasov FBSDE along the lines of [7]. When using the stochastic Pontryagin principle, the decoupling field of the underlying forward-backward system is then understood as the gradient of the value function of the optimization problem (i).

Here we are willing to show that, in both cases, the decoupling field of the McKean-Vlasov FBSDE used to characterize equilibria of the game is indeed a classical solution of a master PDE of the type (2.12) and, then, to make the connection with the so-called *master equation* presented in Lions' lectures at the *Collège de France*. In each case, we exhibit sufficient conditions under which the master PDE is solvable for an arbitrary time horizon T. In short, the two types of representation apply under slightly different assumptions. The direct representation of the value function is well-fitted to cases when σ is uniformly non-degenerate, since standard theory for uniformly parabolic semilinear PDEs then applies. The stochastic Pontryagin principle is more adapted to cases when the underlying Hamiltonian is convex in both the space and control variables, σ being possible degenerate. In both cases, we shall implement the Lasry-Lions monotonicity condition, see $(\boldsymbol{H4})$(iii) below, in order to investigate the Lipschitz property of the solution of the corresponding master PDE in the direction of the measure.

5.2.2. *Use of the Stochastic Pontryagin Principle.* We first explain how things work when using the stochastic Pontryagin principle in order to characterize the optimal paths in (i). Then, following [7], the matching problem (ii) is solved by forcing the forward component of the FBSDE derived from the Pontryagin principle to have $(\mu_t)_{t\in[0,T]}$ as marginal laws. The resulting system becomes ($(Y_s)_{s\in[t,T]}$ being seen as a row vector process)

$$\begin{aligned} \mathrm{d}X_t &= b\big(X_t, [X_t], \hat{\alpha}(X_t, [X_t], Y_t)\big)\mathrm{d}t + \sigma(X_t, [X_t])\mathrm{d}W_t \\ \mathrm{d}Y_t &= -\partial_x H\big(X_t, [X_t], Y_t, \hat{\alpha}(X_t, [X_t], Y_t)\big)\mathrm{d}t + Z_t \mathrm{d}W_t, \end{aligned} \tag{5.9}$$

with the boundary condition $Y_T = \partial_x G(X_T, [X_T])$, where H denotes the so-called *extended Hamiltonian* of the system:

$$H(x,\mu,y,\alpha) = y^\dagger b(x,\mu,\alpha) + F(x,\mu,\alpha), \quad x, y \in \mathbb{R}^d,\ \alpha \in \mathbb{R}^k,\ \mu \in \mathcal{P}_2(\mathbb{R}^d), \tag{5.10}$$

and $\hat{\alpha}(x,\mu,y)$ denotes the minimizer:

$$\hat{\alpha}(x,\mu,y) = \mathrm{argmin}_\alpha H(x,\mu,y,\alpha). \tag{5.11}$$

We shall specify below assumptions under which the minimizer is indeed well-defined. For the moment, we concentrate on the regularity properties we need on the coefficients. As we aim at applying Proposition 5.2, we let:

Assumption ($(\boldsymbol{H4})$(i))**.** *The running cost F may be decomposed as*

$$F(x,\mu,\alpha) = F_0(x,\mu) + F_1(x,\alpha), \quad x \in \mathbb{R}^d,\ \mu \in \mathcal{P}_2(\mathbb{R}^d),\ \alpha \in \mathbb{R}^k, \tag{5.12}$$

the function F_1 being three times differentiable, with bounded and Lipschitz-continuous derivatives of order 2 and 3. The functions F_0 and G are locally Lipschitz continuous in x and μ, the Lipschitz constant being at most of linear growth in $|x|$ and in $(\int_{\mathbb{R}^d} |x'|^2 \mathrm{d}\mu(x'))^{1/2}$. Moreover, F_0 and G are differentiable with respect to x and the coefficients $f_0 = \partial_x F_0$ and $g = \partial_x G$ are Lipschitz in (x,μ) and satisfy ($\boldsymbol{H1}$) and ($\boldsymbol{H2}$) with $h = f_0, g$ and $w = x$.

In particular, there exists a constant C such that, for all $x \in \mathbb{R}^d$, $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, $\alpha \in \mathbb{R}^k$,

$$
\begin{aligned}
&|G(x,\mu)| \leqslant C\Big[1 + |x|^2 + \int_{\mathbb{R}^d} |x'|^2 d\mu(x')\Big],\\
&|F_0(x,\mu)| + |F_1(x,\alpha)| \leqslant C\Big[1 + |x|^2 + \int_{\mathbb{R}^d} |x'|^2 d\mu(x') + |\alpha|^2\Big].
\end{aligned}
\tag{5.13}
$$

Actually, the decomposition (5.12) is motivated by the uniqueness criterion we use below. We introduce it now and not later since it makes the exposition of the regularity assumption much simpler. The growth conditions on the Lipschitz constant of the derivatives are motivated by the typical example when F and G have a quadratic structure in x and α (see [9]).

The reader may notice that nothing is said about the smoothness of b and σ. The reason is the following. Generally speaking, the uniqueness of the minimizer in (5.11) is ensured under strict convexity of the Hamiltonian in the direction α, but, for our purpose, we will use more. We indeed require the *full extended Hamiltonian*

$$
H'(x,\mu,y,z,\alpha) = H(x,\mu,y,\alpha) + \mathrm{Trace}\big(z\sigma(x,\mu)\big),
$$

for $x \in \mathbb{R}^d$, $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, $y \in \mathbb{R}^d$, $z \in \mathbb{R}^{d\times d}$, $\alpha \in \mathbb{R}^k$, to be convex in (x,α), namely

$$
\begin{aligned}
H'(x',\mu,y,z,\alpha') - H'(x,\mu,y,z,\alpha) - \langle x'-x, \partial_x H'(x,\mu,y,z,\alpha)\rangle&\\
- \langle \alpha'-\alpha, \partial_\alpha H'(x,\mu,y,z,\alpha)\rangle &\geqslant \lambda|\alpha'-\alpha|^2,
\end{aligned}
\tag{5.14}
$$

for some $\lambda > 0$. In order to guarantee the convexity of H, we must assume that $b(x,\mu,\alpha)$ is a linear function in (x,α) of the form $b_0(\mu) + b_1 x + b_2\alpha$, for some matrices $b_1 \in \mathbb{R}^{d\times d}$ and $b_2 \in \mathbb{R}^{d\times k}$ and $b_0 : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d$. Moreover, because of the uniqueness criterion we use below, we shall restrict ourselves to the case $b_0 \equiv 0$ so that the drift reduces to the linear combination $b(x,\alpha) = b_1 x + b_2 \alpha$. Similarly, we must assume that $\sigma(x,\mu)$ is a linear function in x, which implies that σ is independent of x as we need it to be bounded (see ($\boldsymbol{H\sigma}$)). Again, because of the uniqueness criterion we use below, we restrict ourselves to the case when σ is also independent of μ, namely $\sigma(x,\mu) = \sigma$ for some constant matrix σ of dimension $d \times d$. Then, the convexity property (5.14) holds provided F satisfies it. In particular, $H'(x',\mu,y,z,\alpha') - H'(x,\mu,y,z,\alpha) = H(x',\mu,y,\alpha') - H(x,\mu,y,\alpha)$ so that the analysis of the *full extended Hamiltonian* H' may be reduced to the analysis of the *extended Hamiltonian* H. We thus require

Assumption (($\boldsymbol{H4}$)(ii))**.** *There exist $b_1 \in \mathbb{R}^{d\times d}$, $b_2 \in \mathbb{R}^{d\times k}$ and $\sigma \in \mathbb{R}^{d\times d}$ such that $b(x,\mu,\alpha) = b_1 x + b_2\alpha$ and $\sigma(x,\mu) = \sigma$, for any $x \in \mathbb{R}^d$, $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ and $\alpha \in \mathbb{R}^k$. Moreover, F satisfies* (5.14) *and the mapping $\mathbb{R}^d \ni x \mapsto G(x,\mu) \in \mathbb{R}$ is convex in the x-variable for any $\mu \in \mathcal{P}_2(\mathbb{R}^d)$.*

We then notice that $\hat{\alpha}(x,y,\mu)$ solves the equation:

$$
y^\dagger b_2 + \partial_\alpha F\big(x,\mu,\hat{\alpha}(x,\mu,y)\big) = 0. \tag{5.15}
$$

Since $\partial_\alpha F = \partial_\alpha F_1$ does not depend upon μ, we deduce that $\hat{\alpha}(x,\mu,y)$ reduces to $\hat{\alpha}(x,y)$. It is then straightforward to prove from the implicit function theorem that the mapping

$(x, y) \mapsto \hat{\alpha}(x, y)$ is twice differentiable with respect to (x, y) with bounded and Lipschitz-continuous derivatives. This says in particular that, in (5.9), there is no McKean-Vlasov interaction in the forward equation. Moreover, we deduce, by composition, that Assumption $(\boldsymbol{H2})$ is satisfied (and thus $(\boldsymbol{H0})$ and $(\boldsymbol{H1})$ as well).

Existence, uniqueness and differentiability of the solution. In [7], it is proved that (5.9) admits a unique solution provided the following assumption is in force (in addition to $(\boldsymbol{H4})$(i) and $(\boldsymbol{H4})$(ii)):

Assumption ($(\boldsymbol{H4})$(iii))**.** *There exists $c > 0$ such that*

(1) For all $x \in \mathbb{R}^d$, $|\partial_\alpha F_1(x, 0)| \leqslant c$,

(2) For all $x \in \mathbb{R}^d$, $\langle x, \partial_x F_0(0, \delta_x)\rangle \geqslant -c(1 + |x|)$, $\langle x, \partial_x G(0, \delta_x)\rangle \geqslant -c(1 + |x|)$,

where δ_x is the Dirac mass at point x. Moreover, the following Lasry-Lions monotonicity condition is in force:

$$\int_{\mathbb{R}^d} \big(F_0(x, \mu) - F_0(x, \mu')\big)\mathrm{d}\big(\mu - \mu'\big)(x) \geqslant 0, \quad \int_{\mathbb{R}^d} \big(G(x, \mu) - G(x, \mu')\big)\mathrm{d}\big(\mu - \mu'\big)(x) \geqslant 0.$$

Actually, not only existence and uniqueness hold, but also the key Lipschitz estimate (5.1) is true, justifying $(\boldsymbol{H3})$. The argument is the same as the one given in [7, Proposition 3.7] for proving uniqueness. The only difference is that initial conditions may be different. More precisely, given $t \in [0, T]$ and two square-integrable $\mathcal{F}_t$-measurable random variables ξ and ξ', the same argument as in [7], combined with (3.6) therein to take into account the fact that the initial conditions are different, shows that

$$2\lambda \mathbb{E} \int_t^T |\hat{\alpha}(X_s^{t,\xi}, Y_s^{t,\xi}) - \hat{\alpha}(X_s^{t,\xi'}, Y_s^{t,\xi'})|^2 \mathrm{d}s \leqslant \mathbb{E}\big[\langle \xi - \xi', Y_t^{t,\xi} - Y_t^{t,\xi'}\rangle\big]. \tag{5.16}$$

(Here $(X^{t,\xi}, Y^{t,\xi}, Z^{t,\xi})$ and $(X^{t,\xi'}, Y^{t,\xi'}, Z^{t,\xi'})$ satisfy (5.9) with $X_t^{t,\xi} = \xi$ and $X_t^{t,\xi'} = \xi'$.) Now, it is quite straightforward to see that

$$\begin{aligned} &\mathbb{E}\big[|Y_t^{t,\xi} - Y_t^{t,\xi'}|^2\big] \\ &\quad \leqslant C\Big(\sup_{s \in [t,T]} \mathbb{E}\big[|X_s^{t,\xi} - X_s^{t,\xi'}|^2\big] + \mathbb{E}\int_t^T |\hat{\alpha}(X_s^{t,\xi}, Y_s^{t,\xi}) - \hat{\alpha}(X_s^{t,\xi'}, Y_s^{t,\xi'})|^2 \mathrm{d}s\Big), \end{aligned} \tag{5.17}$$

and,

$$\begin{aligned} &\sup_{s \in [t,T]} \mathbb{E}\big[|X_s^{t,\xi} - X_s^{t,\xi'}|^2\big] \\ &\quad \leqslant C\Big(\mathbb{E}\big[|\xi - \xi'|^2\big] + \mathbb{E}\int_t^T |\hat{\alpha}(X_s^{t,\xi}, Y_s^{t,\xi}) - \hat{\alpha}(X_s^{t,\xi'}, Y_s^{t,\xi'})|^2 \mathrm{d}s\Big). \end{aligned} \tag{5.18}$$

Therefore, from (5.17) and (5.18),

$$\mathbb{E}\big[|Y_t^{t,\xi} - Y_t^{t,\xi'}|^2\big] \leqslant C\Big(\mathbb{E}\big[|\xi - \xi'|^2\big] + \mathbb{E}\int_t^T |\hat{\alpha}(X_s^{t,\xi}, Y_s^{t,\xi}) - \hat{\alpha}(X_s^{t,\xi'}, Y_s^{t,\xi'})|^2 \mathrm{d}s\Big),$$

Plugging (5.16) into the above equation, we get (5.1).

Master equation. The fact that $(\boldsymbol{H3})$ holds permits us to apply Proposition 5.2. It follows that the decoupling field U of the forward-backward equation (5.9) satisfies the corresponding master PDE (2.12).

We emphasize that the master PDE that we derive is not the standard master equation in mean-field games theory. Loosely speaking, the master equation in mean-field games is the equation satisfied by V, such that U is the gradient of V, which stands for the value function of the game, namely

$$\begin{aligned} V(t,x,\mu) \\ = \mathbb{E}\biggl[G\bigl(X_T^{t,x,\mu},[X_T^{t,\xi}]\bigr) + \int_t^T F\bigl(X_s^{t,x,\mu},[X_s^{t,\xi}],\hat{\alpha}(X_s^{t,x,\mu},Y_s^{t,x,\mu})\bigr)\mathrm{d}s\biggr], \quad \xi \sim \mu, \end{aligned} \tag{5.19}$$

in other words $V(t,x,\mu)$ is the optimal cost when the private player is initialized at x and the equilibrium strategy for the population is initialized at μ. (Here $(X^{t,x,\mu},Y^{t,x,\mu},Z^{t,x,\mu})$ solves (2.4) with the coefficients of (5.9).)

Now that U is known to belong to $\bigcup_{\beta\geqslant 0}\mathcal{D}_\beta$, we can see $X^{t,\xi}$ and $X^{t,x,\mu}$ as solutions of autonomous forward SDEs driven by smooth Lipschitz-continuous coefficients (the drift being just obtained by a composition of b with $\alpha(\cdot,U(\cdot,\cdot,\cdot))$). In particular, $X^{t,\xi}$ and $X^{t,x,\mu}$ must have the same smoothness properties as in the various results of Section 4, but for arbitrary time since the backward constraint has been removed. Another way to understand that claim is to prove regularity inductively, by means of a forward induction, applying successively the results obtained in Section 4 on $[t,T-n\delta]$, $[T-n\delta,T-(n-1)\delta]$, ..., $[T-\delta,T]$, for the same δ as in the proof of Proposition 5.2 and for n such that $t\in[T-(n+1)\delta,T-n\delta)$. The induction is then based on the flow property, which says that, for $s\in[T-k\delta,T-(k-1)\delta]$,

$$X_s^{t,\xi} = X_s^{T-k\delta,X_{T-k\delta}^{t,\xi}} \quad \text{and} \quad X_s^{t,x,[\xi]} = X_s^{T-k\delta,X_{T-k\delta}^{t,x,[\xi]},[X_{T-k\delta}^{t,\xi}]}, \tag{5.20}$$

and, thus, permits the transfer from one interval to another.

Basically, this permits us to prove that V is smooth in x and μ by differentiating under the expectation, provided that G and F_0 are smooth enough in the direction of the measure. Motivated by the fact that the coefficients are required to satisfy the convexity assumption $(\boldsymbol{H4})$(ii), assume for instance that

Assumption $((\boldsymbol{H4})(\mathrm{iv}))$**.** *The functions*

$$\begin{aligned} &\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\ni(x,\mu)\mapsto\frac{F_0(x,\mu)}{\sqrt{1+|x|^2+\int_{\mathbb{R}^d}|v|^2\mathrm{d}\mu(v)}},\\ &\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\ni(x,\mu)\mapsto\frac{G(x,\mu)}{\sqrt{1+|x|^2+\int_{\mathbb{R}^d}|v|^2\mathrm{d}\mu(v)}}, \end{aligned} \tag{5.21}$$

satisfy $(\boldsymbol{H0})$(i)–$(\boldsymbol{H1})$–$(\boldsymbol{H2})$ *(for some values of the parameters therein). In particular,* F_0 *and* G *satisfy the same differentiability property as in* $(\boldsymbol{H0})$(i)–$(\boldsymbol{H1})$–$(\boldsymbol{H2})$ *but the derivatives are locally (instead of globally) controlled.*

Then, we can differentiate the representation formula for V as we differentiated the backward components of (2.3) and (2.4) in Section 4, up to the slight difference that the derivatives of G and $F(\cdot,\cdot,\hat{\alpha}(\cdot,\cdot))$ in x, y and μ may be of linear growth in all the arguments. The key point to circumvent it is to notice from $(\boldsymbol{H4})$(iv) that the random variable

$$\frac{G(X_T^{t,x,\mu},[X_T^{t,\xi}])}{\sqrt{1+|X_T^{t,x,\mu}|^2+\|X_T^{t,\xi}\|_2^2}}$$

satisfies the same first-order and second-order differentiability properties as $\theta_T^{t,x,\xi}$ in Lemmas 4.21 and 4.32. Since all the estimates in Lemmas 4.21 and 4.32 hold in L^2, it is then pretty clear that the mapping

$$\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \ni (x,\mu) \mapsto \mathbb{E}\Big[\sqrt{1+|X_T^{t,x,\mu}|^2+\|X_T^{t,\xi}\|_2^2}\frac{G(X_T^{t,x,\mu},[X_T^{t,\xi}])}{\sqrt{1+|X_T^{t,x,\mu}|^2+\|X_T^{t,\xi}\|_2^2}}\Big]$$
$$= \mathbb{E}\big[G(X_T^{t,x,\mu},[X_T^{t,\xi}])\big], \quad \text{with} \quad \xi \sim \mu,$$

satisfies the same assumption as F_0 and G in $(\boldsymbol{H4})$(ii).

We then may proceed in the same way with $F(\cdot,\cdot,\hat{\alpha}(\cdot,\cdot))$ instead of $G(\cdot,\cdot)$ (recalling that F_1 has bounded derivatives of order 2 and 3, that $\hat{\alpha}$ has bounded derivatives of order 1 and 2 and that $U(t,\cdot,\cdot)$ satisfies $(\boldsymbol{H0})$(i)–$(\boldsymbol{H1})$–$(\boldsymbol{H2})$, the values of the parameters therein being uniform in $t \in [0,T]$).

In the spirit of Theorems 4.25 and 4.33, this permits to show that, for any $t \in [0,T]$, $V(t,\cdot,\cdot)$ satisfies the same assumption as F_0 and G in $(\boldsymbol{H4})$(ii), the parameters that appear in $(\boldsymbol{H0})$(i)–$(\boldsymbol{H1})$–$(\boldsymbol{H2})$ being uniform in $t \in [0,T]$.

It thus remains to identify the shape of the master PDE and, in the same time, to prove the continuity of V and of its derivatives with respect to t. From the same flow property as in (5.20), we notice that, for any $(t,x,\mu) \in [0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)$ and any $s \in [t,T]$,

$$\begin{aligned}
V(t,x,\mu) &= \mathbb{E}\bigg[G\big(X_T^{s,X_s^{t,x,\mu},[X_s^{t,\xi}]},[X_T^{s,[X_s^{t,\xi}]}]\big) \\
&\quad + \int_s^T F\big(X_r^{s,X_s^{t,x,\mu},[X_s^{t,\xi}]},[X_r^{s,[X_s^{t,\xi}]}],\hat{\alpha}(X_r^{s,X_s^{t,x,\mu},[X_s^{t,\xi}]},Y_r^{s,X_s^{t,x,\mu},[X_s^{t,\xi}]})\big)\mathrm{d}r\bigg] \\
&\quad + \mathbb{E}\bigg[\int_t^s F\big(X_r^{t,x,\mu},[X_r^{t,\xi}],\hat{\alpha}(X_r^{t,x,\mu},Y_r^{t,x,\mu})\big)\mathrm{d}r\bigg] \\
&= \mathbb{E}\bigg[V\big(s,X_s^{t,x,\mu},[X_s^{t,\xi}]\big)+\int_t^s F\big(X_r^{t,x,\mu},[X_r^{t,\xi}],\hat{\alpha}(X_r^{t,x,\mu},Y_r^{t,x,\mu})\big)\mathrm{d}r\bigg],
\end{aligned}$$

from which we may repeat the arguments from Theorems 4.25, 4.33 and 2.7 (see also Subsection 2.3). We finally obtain

Theorem 5.3. *Under* $(\boldsymbol{H4})$(i–iv)*, the function* $V(t,\cdot,\cdot)$ *satisfies the same assumption as* F_0 *and* G *in* $(\boldsymbol{H4})$(iv)*, the parameters that appear in* $(\boldsymbol{H0})$(i)–$(\boldsymbol{H1})$–$(\boldsymbol{H2})$ *being uniform in* $t \in [0,T]$*. Moreover, for any* $x \in \mathbb{R}^d$ *and* $\mu \in \mathcal{P}_2(\mathbb{R}^d)$*, the function* $[0,T] \ni t \mapsto V(t,x,\mu)$ *is continuously differentiable, the derivative being continuous in* (t,x,μ)*. For any* $x \in \mathbb{R}^d$ *and* $\xi \in L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d)$*, the functions* $[0,T]\times\mathbb{R}^d\times L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d) \ni (t,x,\xi) \mapsto \partial_\mu V(t,x,[\xi])(\xi) \in L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d)$ *and* $[0,T]\times\mathbb{R}^d\times L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d) \ni (t,x,\xi) \mapsto \partial_v[\partial_\mu V(t,x,[\xi])](\xi) \in L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d)$ *are continuous.*

Finally, V *satisfies the master equation*

$$\begin{aligned}
&\partial_t V(t,x,\mu)+\partial_x V(t,x,\mu)b\big(x,\hat{\alpha}(x,U(t,x,\mu))\big)+F\big(x,\mu,\hat{\alpha}(x,U(t,x,\mu))\big) \\
&\quad + \int_{\mathbb{R}^d}\partial_\mu V(t,x,\mu)(v)b\big(v,\hat{\alpha}(v,U(t,v,\mu))\big)\,\mathrm{d}\mu(v) \\
&\quad + \frac{1}{2}\mathrm{Tr}\bigg[\bigg(\partial^2_{xx}V(t,x,\mu)+\int_{\mathbb{R}^d}\partial_v\big(\partial_\mu V(t,x,\mu)\big)(v)\mathrm{d}\mu(v)\bigg)\sigma\sigma^\dagger\bigg]=0,
\end{aligned} \tag{5.22}$$

with $V(T,x,\mu)=G(x,\mu)$ *as terminal condition and with* U *denoting the decoupling field of* (5.9)*.*

Remark 5.4. *The identification $U(t,x,\mu) = \partial_x V(t,x,\mu)$ can be checked directly as:*

$$\begin{aligned}\partial_x V(t,x,\mu) = \mathbb{E}\bigg[&\partial_x G\big(X_T^{t,x,\mu},[X_T^{t,\xi}]\big)\partial_x X_T^{t,x,\mu}\\ &+\int_t^T \partial_x F\big(X_s^{t,x,\mu},[X_s^{t,\xi}],\hat{\alpha}(X_s^{t,x,\mu},Y_s^{t,x,\mu})\big)\partial_x X_s^{t,x,\mu}\mathrm{d}s\\ &+\int_t^T \partial_\alpha F\big(X_s^{t,x,\mu},[X_s^{t,\xi}],\hat{\alpha}(X_s^{t,x,\mu},Y_s^{t,x,\mu})\big)\partial_x\big(\hat{\alpha}(X_s^{t,x,\mu},Y_s^{t,x,\mu})\big)\mathrm{d}s\bigg], \quad \xi\sim\mu.\end{aligned}$$

Now, (5.15) *says that $\partial_\alpha F(X_s^{t,x,\mu},[X_s^{t,\xi}],\hat{\alpha}(X_s^{t,x,\mu},Y_s^{t,x,\mu})) = -b_2^\dagger Y_s^{t,x,\mu}$, so that*

$$\begin{aligned}\partial_x V(t,x,\mu) = \mathbb{E}\bigg[&Y_T^{t,x,\mu}\partial_x X_T^{t,x,\mu} + \int_t^T \partial_x F\big(X_s^{t,x,\mu},[X_s^{t,\xi}],\hat{\alpha}(X_s^{t,x,\mu},Y_s^{t,x,\mu})\big)\partial_x X_s^{t,x,\mu}\mathrm{d}s\\ &- b_2^\dagger\int_t^T Y_s^{t,x,\mu}\partial_x\big(\hat{\alpha}(X_s^{t,x,\mu},Y_s^{t,x,\mu})\big)\mathrm{d}s\bigg].\end{aligned}$$

Using (5.9) *and Itô's formula to expand $(Y_s^{t,x,\mu}\partial_x X_s^{t,x,\mu})_{t\leqslant s\leqslant T}$, we get that the right-hand side is equal to $Y_t^{t,x,\mu}$. We omit the details of the computation here.*

5.2.3. *Direct approach.* Theorem 5.3 is specifically designed to handle the case when the coefficients may be quadratic in x, provided that the *extended Hamiltonian* has a convex structure in (x,α). When the coefficients are bounded in x and μ and σ is non-degenerate, we can give a direct proof of the solvability of the master equation (5.22) under the weaker assumption that the *extended Hamiltonian* is convex in α. The key point is to represent directly the value function V in (5.19) by means of a McKean-Vlasov FBSDE, and thus to avoid any further reference to the stochastic Pontryagin principle. In particular, contrary to the last paragraph, we shall prove existence and uniqueness to (2.3) without relying on results in [7]. Of course, as previously, we shall need to check that the processes that enter the representation of the value function satisfy (***H*3**), or equivalently, that the key estimate (5.2) holds true. We shall assume:

Assumption ((***H*5**))**.** *The coefficient σ has the form $\sigma:\mathbb{R}^d\ni x\mapsto\sigma(x)\in\mathbb{R}^{d\times d}$, is bounded, twice differentiable, with bounded and Lipschitz-continuous derivatives of order 1 and 2, and, for any $x\in\mathbb{R}^d$, the matrix $\sigma(x)$ is invertible with $\sup_{x\in\mathbb{R}^d}|\sigma^{-1}(x)|<\infty$.*

The parameter k is equal to d and b may decomposed as

$$b(x,\alpha) = b_0(x)+\alpha, \quad x\in\mathbb{R}^d, \quad \alpha\in\mathbb{R}^d,$$

the function b_0 being bounded and twice continuously differentiable with bounded and Lipschitz-continuous derivatives of order 1 and 2.

The running cost F may be decomposed as

$$F(x,\mu,\alpha) = F_0(x,\mu)+F_1(x,\alpha), \quad x\in\mathbb{R}^d,\ \mu\in\mathcal{P}_2(\mathbb{R}^d),\ \alpha\in\mathbb{R}^d,$$

where

- *the functions F_0 and G are bounded and satisfy* (***H*0**)(i)*,* (***H*1**)*,* (***H*2**)*;*
- *the function F_1 is bounded in x and at most of quadratic growth in α, uniformly in x; it is is three times differentiable in (x,α), the derivatives of order 2 and 3 being bounded and Lipschitz-continuous, the derivative of order* 1 *in x being bounded and the derivative of order* 1 *in α being at most of linear growth in α, uniformly in x; there exists $\lambda>0$ such that it satisfies the convexity assumption*

$$F_1(x,\alpha')-F_1(x,\alpha)-\langle\alpha'-\alpha,\partial_\alpha F_1(x,\alpha)\rangle\geqslant\lambda|\alpha'-\alpha|^2,$$

And, the Lasry-Lions monotonicity condition in the last line of $(\boldsymbol{H4})$(iii) *holds true.*

We here prove that

Theorem 5.5. *For a given* $T > 0$ *and under* $(\boldsymbol{H5})$*, the master PDE* (5.22) *has a unique classical solution in the space* $\bigcup_\beta \mathcal{D}_{\beta \geqslant 0}$

Proof. In comparison with the proof of Theorem 5.3, the difficulty here is that we do not have an *a priori* existence and uniqueness result for the McKean-Vlasov FBSDE system representing the master PDE (5.22). In order to proceed, we thus revisit the proof of Proposition 5.2 and show, by induction, that there exist an integer $N \geqslant 1$ and a constant $\tilde{\Lambda} \geqslant 0$ such that, with $\delta = T/N$, the following holds true for any $n \in \{1, \dots, N\}$:

$(\mathcal{I}_n)$: There exists a mapping $V : [T-n\delta, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \ni (t,x,\mu) \mapsto V(t,x,\mu) \in \mathbb{R}$ that belongs to $\bigcup_{\beta \geqslant 0} \mathcal{D}_\beta([T-n\delta, T])$ such that

(i) for any $t \in [T-n\delta, T]$, the function $V(t,\cdot,\cdot)$ satisfies the same assumption as g in $(\boldsymbol{H0})$(i), $(\boldsymbol{H1})$ $(\boldsymbol{H2})$, but with the constant L replaced by $\tilde{\Lambda}$ and $\tilde{L}$ and α being replaced by some $\tilde{L}_n$ and $\tilde{\alpha}_n$;

(ii) V satisfies the master PDE (5.22) on $[T-n\delta, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$.

First step. We start with the following observation. Equation (5.22) is of the type (2.12), with $m = 1$, $b(x,y,z,\nu) = b_0(x) + \hat{\alpha}(x, z\sigma^{-1}(x))$ [12], $\sigma(x,\nu) = \sigma(x)$, $f(x,y,z,\nu) = F_0(x,\mu) + F_1(x, \hat{\alpha}(x, z\sigma^{-1}(x)))$ and $g(x,\mu) = G(x,\mu)$ (recall that the product $z\sigma^{-1}(x)$ makes sense since z reads as an element of $\mathbb{R}^{1\times m}$, that is a row vector). Since b does not rely on y and ν, we shall write $b(x,z)$ for $b(x,y,z,\nu)$. Similarly, since f is independent of the variable y and depends on the variable $\nu \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R})$ through its first marginal $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ only (recall that, formally, ν is understood as the joint marginal law of the process (X,Y)), we shall write $f(x,z,\mu)$ for $f(x,y,z,\nu)$. Now, recalling (5.15), we know that $(x,z) \mapsto \hat{\alpha}(x,z)$ is twice differentiable with respect to (x,z) with bounded and Lipschitz-continuous derivatives of order 1 and 2. In particular, we can find a constant C such that, for all $x, z \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$,

$$|\partial_z f(x,z,\mu)| \leqslant C(1+|z|), \tag{5.23}$$

which plays an important role below.

Of course, the assumption $(\boldsymbol{H0})$(i) is not satisfied because of the quadratic growth of f in the variable z. In order to apply Theorem 2.7, we shall make use of a truncation argument. Considering a smooth function $\varphi_R : \mathbb{R}^d \to \mathbb{R}^d$ that matches the identity on the ball of center 0 and of radius R, that is zero outside the ball of center 0 and of radius $2R$ and that satisfies $|\nabla \varphi_R| \leqslant C$ with C independent of R, we let $b_R(x,z) = b(x, \varphi_R(z))$ and $f_R(x,z,\mu) = f(x, \varphi_R(z), \mu)$.

In particular, for any $(t,x) \in [0,T] \times \mathbb{R}^d$ and any flow of probability measures $(\mu_u)_{u \in [t,T]}$ with values in $\mathcal{P}_2(\mathbb{R}^d)$, we know from [11] that the FBSDE system

$$\left\{ \begin{aligned} &X_s^{t,x,(\mu_r)_{r\in[t,T]}} \\ &\quad = x + \int_t^s b_R\big(X_r^{t,x,(\mu_u)_{u\in[t,T]}}, Z_r^{t,x,(\mu_u)_{u\in[t,T]}}\big)\mathrm{d}r + \int_t^s \sigma\big(X_r^{t,x,(\mu_u)_{u\in[t,T]}}\big)\mathrm{d}W_r, \\ &Y_s^{t,x,(\mu_r)_{r\in[t,T]}} \\ &\quad = g\big(X_T^{t,x,(\mu_u)_{u\in[t,T]}}, \mu_T\big) + \int_s^T f_R\big(X_r^{t,x,(\mu_u)_{u\in[t,T]}}, Z_r^{t,x,(\mu_u)_{u\in[t,T]}}, \mu_r\big)\mathrm{d}r \\ &\qquad\qquad - \int_s^T Z_r^{t,x,(\mu_u)_{u\in[t,T]}}\mathrm{d}W_r, \quad s \in [t,T], \end{aligned} \right. \tag{5.24}$$

[12]Pay attention that the letter b is used both to denote the first-order coefficient in (2.12) and the drift in (5.8). We feel that the reader can easily make the distinction between the two of them.

admits a unique solution. It satisfies $|Z_s^{t,x,(\mu_u)_{u\in[t,T]}}| \leqslant C_R$ $\mathrm{d}s \otimes \mathrm{d}\mathbb{P}$ almost everywhere, for a constant C_R that may depend upon R (but not on $(\mu_u)_{u\in[t,T]}$).

We now prove that C_R may be chosen independently of R. The proof is as follows. We write

$$f(x, \varphi_R(z), \mu) = f(x, 0, \mu) + \left(\int_0^1 \partial_z f(x, \varphi_R(\lambda z), \mu) \nabla \varphi_R(\lambda z) \mathrm{d}\lambda\right) z^\dagger.$$

By a standard Girsanov argument, the above decomposition of f_R says that the FBSDE (5.24) may be written, under a new probability, as a new FBSDE system with $f(X_s^{t,x,(\mu_u)_{u\in[t,T]}}, 0, \mu_r)$ as driver in the backward component and with

$$\begin{aligned} &b_R\big(X_r^{t,x,(\mu_u)_{u\in[t,T]}}, Z_r^{t,x,(\mu_u)_{u\in[t,T]}}\big) \\ &\quad + \left(\int_0^1 \partial_z f\big(X_r^{t,x,(\mu_u)_{u\in[t,T]}}, \varphi_R(\lambda Z_r^{t,x,(\mu_u)_{u\in[t,T]}}), \mu\big) \nabla \varphi_R(\lambda Z_r^{t,x,(\mu_u)_{u\in[t,T]}}) \mathrm{d}\lambda\right)^\dagger \end{aligned}$$

as drift in the forward component: The driver in the backward component is bounded and, by (5.23), the drift in the forward component is bounded in the variable x and at most of linear growth in the variable z. In particular, by [12], there exists a constant Γ, independent of R and $(\mu_u)_{u\in[t,T]}$, such that, we indeed have $|Z_s^{t,x,(\mu_u)_{u\in[t,T]}}| \leqslant \Gamma$.

The coefficients G and F_0 being bounded, we also have $|Y_s^{t,x,(\mu_u)_{u\in[t,T]}}| \leqslant C$, for C independent of R and of $(\mu_u)_{u\in[t,T]}$.

Second step. We now construct $\delta > 0$ such that the master PDE (5.22) admits a solution in $\bigcup_{\beta\geqslant 0} \mathcal{D}_\beta([T-\delta, T])$ on $[T-\delta] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$. With the same Γ as in the previous step, we indeed apply Theorem 2.7 with (b_R, σ, f_R, g) instead of (b, σ, f, g), for some $R > \Gamma \|\sigma^{-1}\|_\infty$. This says that, for some $\delta \in (0, T]$, there exists a function

$$V : [T-\delta, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$$

in $\bigcup_{\beta\geqslant 0} \mathcal{D}_\beta([T-\delta, T])$ that solves (5.22) with b replaced by b_R and f replaced by f_R.

Now, for any $(t, x, \mu) \in [T-\delta, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$, for any $s \in [t, T]$,

$$\partial_x V(s, X_s^{t,x,\mu}, [X_s^{t,\xi}]) = Z_s^{t,x,\mu} \sigma^{-1}(X_s^{t,x,\mu}),$$

where $\xi \sim \mu$ and $(X^{t,\xi}, Y^{t,\xi}, Z^{t,\xi})$ and $(X^{t,x,\mu}, Y^{t,x,\mu}, Z^{t,x,\mu})$ solve (2.3) and (2.4) with (b, f) replaced by (b_R, f_R). In particular, $|\partial_x V(t, x, \mu)| \leqslant \Gamma \|\sigma^{-1}\|_\infty < R$. Therefore, V also solves (5.22). It also satisfies $|V| \leqslant C$, for some C independent of R. Basically, this proves *(ii)* in $(\mathcal{I}_1)$.

Third step. In order to prove *(i)* in $(\mathcal{I}_1)$ and more generally in $(\mathcal{I}_n)$ for any $n = 2, \dots, N$, we must identify the constant $\tilde{\Lambda}$ first. We thus proceed as follows. We assume that there exists a time $t \in [0, T]$ such that, on $[t, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$, the master PDE (5.22) has a solution V in $\bigcup_{\beta\geqslant 0} \mathcal{D}_\beta([t, T])$. We are then willing to provide a bound for $\sup_{x\in\mathbb{R}^d, \xi\in L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d)} \|\partial_\mu V(t, x, [\xi])(\xi)\|_2$, independently of $t \in [0, T]$.

Since $V \in \bigcup_{\beta\geqslant 0} \mathcal{D}_\beta([t, T])$, we can find some $R > 0$ such that $\|\partial_x V(s, \cdot, \mu)\|_\infty \|\sigma\|_\infty < R$ for any $s \in [t, T]$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$. In particular, V also solves the master PDE associated with (b_R, σ, f_R, g) instead of (b, σ, f, g). Since (b_R, σ, f_R, g) satisfies $(\boldsymbol{H0})$(i)–$(\boldsymbol{H1})$–$(\boldsymbol{H2})$, we can imitate the proof of Theorem 2.8 and build a solution to (2.3) for any $\xi \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)$. The forward process is defined as a solution of (3.44). We shall prove right below that it is uniquely defined, so that we can denote it by $(X_s^{t,\xi})_{s\in[t,T]}$.

Uniqueness is a consequence of a more general result of stability, the proof of which is as follows. Given $\xi, \xi' \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)$, we consider two solutions $(X_s^{t,\xi})_{s\in[t,T]}$ and $(X_s^{t,\xi'})_{s\in[t,T]}$ to the SDE (3.44), with ξ and ξ' as respective initial solutions. We then expand, by means of Itô's formula $(V(s, X_s^{t,\xi}, [X_s^{t,\xi}]) - V(s, X_s^{t,\xi'}, [X_s^{t,\xi}]))_{s\in[t,T]}$ (observe that, in both terms, the measure argument is driven by ξ). By Proposition 3.9 (either by generalizing to the case when the process plugged in the spatial argument is not the same as the one plugged in the measure argument or by extending the dimension in order to see $(X_s^{t,\xi}, X_s^{t,\xi'})_{s\in[t,T]}$ as a single process), we get for $s \in [t,T]$ (using the fact that $R > \|\partial_x V\|_\infty \|\sigma\|_\infty$)

$$
\begin{aligned}
\mathrm{d}\big[V\big(s, X_s^{t,\xi}, [X_s^{t,\xi}]\big)\big] &= -f\big(X_s^{t,\xi}, [X_s^{t,\xi}], \hat{\alpha}\big(X_s^{t,\xi}, \partial_x V(s, X_s^{t,\xi}, [X_s^{t,\xi}])\big)\big)\mathrm{d}s \\
&\quad + \partial_x V(s, X_s^{t,\xi}, [X_s^{t,\xi}])\sigma(X_s^{t,\xi})\mathrm{d}W_s,
\end{aligned}
\tag{5.25}
$$

and

$$
\begin{aligned}
\mathrm{d}\big[V\big(s, X_s^{t,\xi'}, [X_s^{t,\xi}]\big)\big] &= \Big[-f\big(X_s^{t,\xi'}, [X_s^{t,\xi}], \hat{\alpha}\big(X_s^{t,\xi'}, \partial_x V(s, X_s^{t,\xi'}, [X_s^{t,\xi}])\big)\big) \\
&\quad + \partial_x V(s, X_s^{t,\xi'}, [X_s^{t,\xi}])\Big(\hat{\alpha}\big(X_s^{t,\xi'}, \partial_x V(s, X_s^{t,\xi'}, [X_s^{t,\xi'}])\big) \\
&\qquad - \hat{\alpha}\big(X_s^{t,\xi'}, \partial_x V(s, X_s^{t,\xi'}, [X_s^{t,\xi}])\big)\Big)\Big]\mathrm{d}s \\
&\quad + \partial_x V(s, X_s^{t,\xi'}, [X_s^{t,\xi}])\sigma(X_s^{t,\xi'})\mathrm{d}W_s.
\end{aligned}
\tag{5.26}
$$

Taking the difference between (5.25) and (5.26) and using the same notation H for the Hamiltonian as in (5.10), we obtain

$$
\begin{aligned}
&\mathrm{d}\big[V\big(s, X_s^{t,\xi}, [X_s^{t,\xi}]\big) - V\big(s, X_s^{t,\xi'}, [X_s^{t,\xi}]\big)\big] \\
&\quad = -\Big[f\big(X_s^{t,\xi}, [X_s^{t,\xi}], \hat{\alpha}\big(X_s^{t,\xi}, \partial_x V(s, X_s^{t,\xi}, [X_s^{t,\xi}])\big)\big) \\
&\qquad\quad - f\big(X_s^{t,\xi'}, [X_s^{t,\xi}], \hat{\alpha}\big(X_s^{t,\xi'}, \partial_x V(s, X_s^{t,\xi'}, [X_s^{t,\xi'}])\big)\big)\Big]\mathrm{d}s \\
&\qquad - \Big[H\Big(X_s^{t,\xi'}, [X_s^{t,\xi}], \partial_x V(s, X_s^{t,\xi'}, [X_s^{t,\xi}]), \hat{\alpha}\big(X_s^{t,\xi'}, \partial_x V(s, X_s^{t,\xi'}, [X_s^{t,\xi'}])\big)\Big) \\
&\qquad\quad - H\Big(X_s^{t,\xi'}, [X_s^{t,\xi}], \partial_x V(s, X_s^{t,\xi'}, [X_s^{t,\xi}]), \hat{\alpha}\big(X_s^{t,\xi'}, \partial_x V(s, X_s^{t,\xi'}, [X_s^{t,\xi}])\big)\Big)\Big]\mathrm{d}s \\
&\qquad + \Big[\partial_x V(s, X_s^{t,\xi}, [X_s^{t,\xi}])\sigma(X_s^{t,\xi}) - \partial_x V(s, X_s^{t,\xi'}, [X_s^{t,\xi}])\sigma(X_s^{t,\xi'})\Big]\mathrm{d}W_s.
\end{aligned}
$$

Therefore, taking the expectation and integrating in s from t to T, we get from the convexity of H in α (that follows from the convexity of F_1 and the linear structure of the drift in α in (5.8)) that

$$
\begin{aligned}
&\mathbb{E}\big[V(t, \xi, [\xi]) - V(t, \xi', [\xi])\big] \\
&\quad - \mathbb{E}\int_t^T \Big[F_1\Big(X_s^{t,\xi}, \hat{\alpha}\big(X_s^{t,\xi}, \partial_x V(s, X_s^{t,\xi}, [X_s^{t,\xi}])\big)\Big) \\
&\qquad\qquad - F_1\Big(X_s^{t,\xi'}, \hat{\alpha}\big(X_s^{t,\xi'}, \partial_x V(s, X_s^{t,\xi'}, [X_s^{t,\xi'}])\big)\Big)\Big]\mathrm{d}s \\
&\geqslant \mathbb{E}\big[G(X_T^{t,\xi}, [X_T^{t,\xi}]) - G(X_T^{t,\xi'}, [X_T^{t,\xi}])\big] + \mathbb{E}\int_t^T \big(F_0(X_s^{t,\xi}, [X_s^{t,\xi}]) - F_0(X_s^{t,\xi'}, [X_s^{t,\xi}])\big)\mathrm{d}s \\
&\quad + \lambda\mathbb{E}\int_t^T \big|\hat{\alpha}\big(X_s^{t,\xi'}, \partial_x V(s, X_s^{t,\xi'}, [X_s^{t,\xi'}])\big) - \hat{\alpha}\big(X_s^{t,\xi'}, \partial_x V(s, X_s^{t,\xi'}, [X_s^{t,\xi}])\big)\big|^2 \mathrm{d}s.
\end{aligned}
$$

By exchanging the roles of ξ and ξ' and then by summing up, we deduce that

$$\begin{aligned}
&\mathbb{E}\big[V(t,\xi,[\xi]) - V(t,\xi',[\xi]) - \big(V(t,\xi,[\xi']) - V(t,\xi',[\xi'])\big)\big] \\
&\geqslant \mathbb{E}\big[G(X_T^{t,\xi},[X_T^{t,\xi}]) - G(X_T^{t,\xi'},[X_T^{t,\xi}]) - \big(G(X_T^{t,\xi},[X_T^{t,\xi'}]) - G(X_T^{t,\xi'},[X_T^{t,\xi'}])\big)\big] \\
&\quad + \mathbb{E}\int_t^T \big[\big(F_0(X_s^{t,\xi},[X_s^{t,\xi}]) - F_0(X_s^{t,\xi'},[X_s^{t,\xi}])\big) \\
&\qquad\qquad - \big(F_0(X_s^{t,\xi},[X_s^{t,\xi'}]) - F_0(X_s^{t,\xi'},[X_s^{t,\xi'}])\big)\big]\mathrm{d}s \\
&\quad + \lambda\mathbb{E}\int_t^T \big|\hat{\alpha}\big(X_s^{t,\xi'},\partial_x V(s,X_s^{t,\xi'},[X_s^{t,\xi'}])\big) - \hat{\alpha}\big(X_s^{t,\xi'},\partial_x V(s,X_s^{t,\xi'},[X_s^{t,\xi}])\big)\big|^2\mathrm{d}s \\
&\quad + \lambda\mathbb{E}\int_t^T \big|\hat{\alpha}\big(X_s^{t,\xi},\partial_x V(s,X_s^{t,\xi},[X_s^{t,\xi}])\big) - \hat{\alpha}\big(X_s^{t,\xi},\partial_x V(s,X_s^{t,\xi},[X_s^{t,\xi'}])\big)\big|^2\mathrm{d}s.
\end{aligned}$$

Finally, rearranging the terms, we deduce from the Lasry-Lions condition that

$$\begin{aligned}
&\mathbb{E}\big[V(t,\xi,[\xi]) - V(t,\xi',[\xi]) - \big(V(t,\xi,[\xi']) - V(t,\xi',[\xi'])\big)\big] \\
&\geqslant \lambda\mathbb{E}\int_t^T \big|\hat{\alpha}\big(X_s^{t,\xi'},\partial_x V(s,X_s^{t,\xi'},[X_s^{t,\xi'}])\big) - \hat{\alpha}\big(X_s^{t,\xi'},\partial_x V(s,X_s^{t,\xi'},[X_s^{t,\xi}])\big)\big|^2\mathrm{d}s \\
&\quad + \lambda\mathbb{E}\int_t^T \big|\hat{\alpha}\big(X_s^{t,\xi},\partial_x V(s,X_s^{t,\xi},[X_s^{t,\xi}])\big) - \hat{\alpha}\big(X_s^{t,\xi},\partial_x V(s,X_s^{t,\xi},[X_s^{t,\xi'}])\big)\big|^2\mathrm{d}s.
\end{aligned} \tag{5.27}$$

When $\xi = \xi'$, the left-hand side is zero. Denoting by X and X' two solutions to the SDE (3.44) with the same initial condition ξ, the above inequality (with the formal identification $X \equiv X^{t,\xi}$ and $X' \equiv X^{t,\xi'}$) says that $\hat{\alpha}(X_s', \partial_x V(s, X_s', [X_s'])) = \hat{\alpha}(X_s', \partial_x V(s, X_s', [X_s]))$. Then, uniqueness to (3.44) follows from the fact that, by assumption, $\partial_x V$ is Lipschitz continuous in x.

Fourth step. Given the flow of probability measures $([X_s^{t,\xi}])_{s\in[t,T]}$ we just constructed, we know from [11] that, for any $x \in \mathbb{R}^d$, the FBSDE (2.4), when driven by (b_R, σ, f_R, g) and by $\mu = [\xi]$, is uniquely solvable. By the first step, the solution must solve (2.4), when driven by (b, σ, f, g). Moreover, it satisfies $|Z_s^{t,x,\mu}| \leqslant \Gamma$ $\mathrm{d}s \otimes \mathrm{d}\mathbb{P}$ almost everywhere. Applying Itô's formula to $(V(s, X_s^{t,x,\mu}, [X_s^{t,\xi}]))_{s\in[t,T]}$, we can check, in the spirit of Theorem 2.8, that $Y_s^{t,x,\mu} = V(s, X_s^{t,x,\mu}, [X_s^{t,\xi}])$ and $Z_s^{t,x,\mu} = \partial_x V(s, X_s^{t,x,\mu}, [X_s^{t,\xi}])\sigma(X_s^{t,x,\mu})$, $s \in [t,T]$, so that, on $[t,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$,

$$\|\partial_x V\|_\infty \leqslant \Gamma \|\sigma^{-1}\|_\infty.$$

Another way to make the connection with (2.4) is to see expansions (5.25) and (5.26) as standard verification arguments, as often used in stochastic control theory. Indeed, we are just using the fact that the mapping $(s,x) \mapsto V(s,x,[X_s^{t,\xi}])$ is a solution of a standard HJB equation, corresponding to the optimization problem (i) in the description of a mean-field game on page 80. We can indeed differentiate in time $V(s,x,\mu_s)$ for a given $x \in \mathbb{R}^d$, where $\mu_s = [X_s^{t,\xi}]$. Applying the chain rule proved in Section 3 and combining with the master PDE (5.22), we then recover the HJB equation:

$$\begin{aligned}
&\partial_s\big[V(s,x,\mu_s)\big] + \partial_x V(s,x,\mu_s)\big(b_0(x) + \hat{\alpha}(x,\partial_x V(s,x,\mu_s))\big) \\
&\quad + \frac{1}{2}\mathrm{Tr}\big[\sigma\sigma^\dagger(x)\partial_{xx}^2 V(s,x,\mu_s)\big] + F\big(x,\mu_s,\hat{\alpha}(x,\partial_x V(s,x,\mu_s))\big) = 0,
\end{aligned} \tag{5.28}$$

for $s \in [t,T]$ and $x \in \mathbb{R}^d$, with $V(T,x,\mu_T) = G(x,\mu_T)$. We know that $\partial_x V$ is bounded by $\Gamma\|\sigma^{-1}\|_\infty$. Therefore, (5.28) reads as a standard semilinear uniformly parabolic equation driven by smooth coefficients in x. Since f is Lipschitz-continuous in the direction of the measure and $[t,T] \ni s \mapsto \mu_s$ is $1/2$-Hölder continuous (the drift of the diffusion $X^{t,\xi}$ being bounded), the coefficients are $1/2$-Hölder continuous in time. By Schauder's theory for semilinear parabolic equation (see [15, Chapter 7]), we can find a bound Γ' for $\partial^2_{xx} V$ that is independent of $t \in [0,T]$.

Now, going back to (3.44), we may use the bound for $\partial^2_{xx} V$ as a Lipschitz bound for $\partial_x V$ in the direction x. It is then pretty standard to deduce, from Gronwall's lemma, that, for any $\xi, \xi' \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)$,

$$
\begin{aligned}
\mathbb{E}\big[\sup_{s\in[t,T]} |X_s^{t,\xi} - X_s^{t,\xi'}|^2\big] &\leqslant C\Big(\mathbb{E}\big[|\xi-\xi'|^2\big] \\
&\quad + \mathbb{E}\int_t^T \big|\hat{\alpha}\big(X_s^{t,\xi}, \partial_x V(s, X_s^{t,\xi}, [X_s^{t,\xi}])\big) - \hat{\alpha}\big(X_s^{t,\xi}, \partial_x V(s, X_s^{t,\xi}, [X_s^{t,\xi'}])\big)\big|^2 \mathrm{d}s\Big),
\end{aligned}
\tag{5.29}
$$

for a constant C that is independent of t, ξ and ξ' and the value of which is allowed to increase from line to line. In particular, using the Lipschitz property of $\hat{\alpha}$ and once again the bound for $\partial^2_{xx} V$, we deduce that

$$
\begin{aligned}
&\mathbb{E}\int_t^T \big|\hat{\alpha}\big(X_s^{t,\xi}, \partial_x V(s, X_s^{t,\xi}, [X_s^{t,\xi}])\big) - \hat{\alpha}\big(X_s^{t,\xi'}, \partial_x V(s, X_s^{t,\xi'}, [X_s^{t,\xi'}])\big)\big|^2 \mathrm{d}s \\
&\leqslant C\Big(\mathbb{E}\big[|\xi-\xi'|^2\big] \\
&\quad + \mathbb{E}\int_t^T \big|\hat{\alpha}\big(X_s^{t,\xi}, \partial_x V(s, X_s^{t,\xi}, [X_s^{t,\xi}])\big) - \hat{\alpha}\big(X_s^{t,\xi}, \partial_x V(s, X_s^{t,\xi}, [X_s^{t,\xi'}])\big)\big|^2 \mathrm{d}s\Big) \\
&\leqslant C\Big(\mathbb{E}\big[|\xi-\xi'|^2\big] \\
&\quad + \mathbb{E}\big[V(t,\xi,[\xi]) - V(t,\xi',[\xi]) - \big(V(t,\xi,[\xi']) - V(t,\xi',[\xi'])\big)\big]\Big),
\end{aligned}
\tag{5.30}
$$

the last line following from (5.27) (paying attention that the last term in the right-hand side is non-negative).

We now make use of Remark 5.4. By differentiating $(Y_s^{t,x,\mu})_{s\in[t,T]}$ with respect to x (which is licit as it reads $(Y_s^{t,x,\mu} = V(s, X_s^{t,x,\mu}, [X_s^{t,\xi}]))_{s\in[t,T]}$ and $(X_s^{t,x,\mu})_{s\in[t,T]}$ solves a standard SDE with smooth coefficients) and then, by applying Itô's formula, we can indeed check that $(\partial_x Y_s^{t,x,\mu}(\partial_x X_s^{t,x,\mu})^{-1})_{s\in[t,T]}$, solves the backward SDE in (5.9), so that

$$
\begin{aligned}
\partial_x V(t,x,\mu) &= \mathbb{E}\Big[\partial_x G\big(X_T^{t,x,\mu}, [X_T^{t,\xi}]\big) \\
&\quad + \int_t^T \partial_x H\big(X_s^{t,x,\mu}, [X_s^{t,\xi}], \partial_x V(s, X_s^{t,x,\mu}, [X_s^{t,\xi}]), \hat{\alpha}(X_s^{t,x,\mu}, \partial_x V(s, X_s^{t,x,\mu}, [X_s^{t,\xi}]))\big)\mathrm{d}s\Big],
\end{aligned}
$$

and thus

$$
\begin{aligned}
\partial_x V(t,\xi,[\xi]) &= \mathbb{E}\Big[\partial_x G\big(X_T^{t,\xi}, [X_T^{t,\xi}]\big) \\
&\quad + \int_t^T \partial_x H\big(X_s^{t,\xi}, [X_s^{t,\xi}], \partial_x V(s, X_s^{t,\xi}, [X_s^{t,\xi}]), \hat{\alpha}(X_s^{t,\xi}, \partial_x V(s, X_s^{t,\xi}, [X_s^{t,\xi}]))\big)\mathrm{d}s | \mathcal{F}_t\Big].
\end{aligned}
$$

Therefore, thanks to (5.29) and (5.30), and by the Lipschitz property of $\partial_x F$ and $\partial_x G$ in the variables x, μ and α, we get that, for any $\xi, \xi' \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)$,

$$
\begin{aligned}
&\mathbb{E}\big[|\partial_x V(t,\xi,[\xi]) - \partial_x V(t,\xi',[\xi'])|^2\big] \\
&\quad \leqslant C\Big(\mathbb{E}\big[|\xi-\xi'|^2\big] + \mathbb{E}\big[V(t,\xi,[\xi]) - V(t,\xi',[\xi]) - \big(V(t,\xi,[\xi']) - V(t,\xi',[\xi'])\big)\big]\Big).
\end{aligned}
$$

Since V is smooth in x and $\partial_x V$ is Γ'-Lipschitz in x, we can write

$$\begin{aligned}
&\mathbb{E}\big[|\partial_x V(t,\xi,[\xi]) - \partial_x V(t,\xi,[\xi'])|^2\big] \\
&\leqslant C\Big(\mathbb{E}\big[|\xi-\xi'|^2\big] \\
&\quad + \int_0^1 \mathbb{E}\big[\big(\partial_x V\big(t,\lambda\xi+(1-\lambda)\xi',[\xi]\big) - \partial_x V\big(t,\lambda\xi+(1-\lambda)\xi',[\xi']\big)\big)\big(\xi-\xi'\big)\big]\mathrm{d}\lambda\Big) \\
&\leqslant C\Big(\mathbb{E}\big[|\xi-\xi'|^2\big] + \mathbb{E}\big[\big(\partial_x V\big(t,\xi,[\xi]\big) - \partial_x V\big(t,\xi,[\xi']\big)\big)\big(\xi-\xi'\big)\big]\Big).
\end{aligned}$$

We finally get that

$$\mathbb{E}\big[|\partial_x V(t,\xi,[\xi]) - \partial_x V(t,\xi,[\xi'])|^2\big] \leqslant C\|\xi'-\xi\|_2^2,$$

the constant C being independent of t, ξ and ξ'. Plugging into (3.44), we can deduce that

$$\sup_{s\in[t,T]} \mathbb{E}\big[|X_s^{t,\xi} - X_s^{t,\xi'}|^2\big] \leqslant C\|\xi'-\xi\|_2^2. \tag{5.31}$$

We now look at the backward equation in (2.3) (driven by (b_R,σ,f_R,g)). Now that we have proven a Lipschitz estimate for the forward component, it is standard to prove a similar estimate for the backward one. We deduce that (5.1) and thus (5.2) hold true. Applying Lemma 5.1, we get the required $\tilde{\Lambda}$ in *(i)* of the induction property $(\mathcal{I}_n)$.

Last step. From the second and fourth steps, it is clear that *(i)* in $(\mathcal{I}_1)$ holds true, which completes the proof of $(\mathcal{I}_1)$.

We then apply Theorem 2.7 iteratively along the lines of the proof of Proposition 5.2. Notice that here there is no need of the assumption *(iii)* in the induction scheme used in the proof of Proposition 5.2. Indeed, by the fourth step above, we have a direct way to establish (5.2), whereas, in the proof of Proposition 5.2, the bound (5.2) is obtained by means of the induction assumption *(iii)*.

Uniqueness follows from Theorem 2.8, observing that the quadratic term in the equation may be truncated (as any solution in the class $\bigcup_{\beta\geqslant 0}\mathcal{D}_\beta$ has a bounded gradient). $\square$

5.3. **Control of McKean-Vlasov equations.**

5.3.1. *General set-up.* Another example taken from large population stochastic control is the optimal control of McKean-Vlasov equations. We refer to [5, 9] for a complete review. The idea here is to minimize the cost functional

$$J\big((\alpha_t)_{t\in[0,T]}\big) = \mathbb{E}\bigg[G(X_T,[X_T]) + \int_0^T F(X_t,[X_t],\alpha_t)dt\bigg],$$

over controlled McKean-Vlasov diffusion processes of the form

$$dX_t = b(X_t,[X_t],\alpha_t)dt + \sigma dW_t, \quad t\in[0,T], \tag{5.32}$$

for some possibly random initial condition X_0. As in (5.8), $(W_t)_{t\in[0,T]}$ is an $\mathbb{R}^d$-valued Brownian motion, $b:\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\times\mathbb{R}^k\to\mathbb{R}^d$ is Lipschitz-continuous on the same model as in $(\boldsymbol{H0})$(i) and $(\alpha_t)_{t\in[0,T]}$ denotes the progressively-measurable square-integrable control process. Note that we shall only consider the case σ constant.

Unlike the mean-field games example, in which the McKean-Vlasov constraint is imposed in step (ii) only, see page 80, the McKean-Vlasov prescription is here given first. In particular, the problem now consists of a true optimization problem.

Below, we make use of the stochastic Pontryagin principle in order to characterize the optimal paths. Although the form of the Pontryagin principle is different from what it is in mean-field games, it imposes, in a rather similar way, restrictive conditions on the structure of the SDE (5.32), among which the fact that σ has to be constant. The Hamiltonian is defined in the same way as before, see (5.10), but the FBSDE derived from the stochastic Pontryagin principle has a more complicated form (see [5]):

$$\begin{aligned} dX_t &= b\big(X_t,[X_t],\hat\alpha(X_t,[X_t],Y_t)\big)dt + \sigma dW_t \\ dY_t &= -\partial_x H\big(X_t,[X_t],Y_t,\hat\alpha(X_t,[X_t],Y_t)\big)dt \\ &\qquad - \hat{\mathbb{E}}\big[\partial_\mu H\big(\langle X_t\rangle,[X_t],\langle Y_t\rangle,\hat\alpha(\langle X_t\rangle,[X_t],\langle Y_t\rangle)\big)(X_t)\big] + Z_t dW_t, \end{aligned} \tag{5.33}$$

with the boundary condition $Y_T = \partial_x G(X_T,[X_T]) + \hat{\mathbb{E}}[\partial_\mu G(\langle X_T\rangle,[X_T])(X_T)]$. The reason is that the state space over which the optimization is performed is the enlarged space $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$. This means that, in the extended Hamiltonian, the state variable is the pair (x,μ) and not x itself. The additional terms in the driver and in the boundary condition deriving from the stochastic Pontryagin principle thus express the sensitivity of the Hamiltonian with respect to the measure argument. We notice that these two terms may be reformulated as

$$\begin{aligned} &\hat{\mathbb{E}}\big[\partial_\mu H\big(\langle X_t\rangle,[X_t],\langle Y_t\rangle,\hat\alpha(\langle X_t\rangle,[X_t],\langle Y_t\rangle)\big)(X_t)\big] = \tilde h\big(X_t,[X_t,Y_t]\big), \\ &\hat{\mathbb{E}}\big[\partial_\mu G\big(\langle X_T\rangle,[X_T]\big)(X_T)\big] = \tilde g\big(X_T,[X_T]\big), \end{aligned}$$

where

$$\begin{aligned} \tilde h(x,\nu) &= \int_{\mathbb{R}^d\times\mathbb{R}^d} \partial_\mu H\big(v,\pi_1\sharp\nu,w,\hat\alpha(v,\pi_1\sharp\nu,w)\big)(x)\mathrm{d}\nu(v,w), \\ \tilde g(x,\mu) &= \int_{\mathbb{R}^d} \partial_\mu G(v,\mu)(x)\mathrm{d}\mu(v), \end{aligned}$$

with $x\in\mathbb{R}^d$, $\nu\in\mathcal{P}_2(\mathbb{R}^d\times\mathbb{R}^d)$, $\mu\in\mathcal{P}_2(\mathbb{R}^d)$ and $\pi_1 : \mathbb{R}^d\times\mathbb{R}^d \ni (x,y)\mapsto x\in\mathbb{R}^d$.

Existence and uniqueness of a solution to (5.33) have been established under the following assumption (see [5]):

Assumption(($\boldsymbol{H6}$)(i)). *The drift b is of the linear form $b(x,\mu,\alpha) = b_0 x + b_1\int_{\mathbb{R}^d} v\mathrm{d}\mu(v) + b_2\alpha$. The cost functions F and G are locally Lipschitz continuous in (x,μ,α), the local Lipschitz constant being at most of linear growth in $|x|$, $(\int_{\mathbb{R}^d}|v|^2\mathrm{d}\mu(v))^{1/2}$ and $|\alpha|$. Moreover, F and G are also $\mathcal{C}^1$ in (x,μ,α), the derivative in (x,α) being Lipschitz continuous in (x,μ,α) and the functions $\partial_\mu F$ and $\partial_\mu G$ satisfying (with $h = F$ and $w = (x,\alpha)$ or $h = g$ and $w = x$)*

$$\mathbb{E}\big[\big|\partial_\mu h(w,[\xi])(\xi) - \partial_\mu h(w',[\xi'])(\xi')\big|^2\big]^{1/2} \leqslant \tilde L\big\{|w-w'| + \mathbb{E}\big[|\xi-\xi'|^2\big]^{1/2}\big\}.$$

Finally, there exists $\lambda > 0$ such that

$$\begin{aligned} &F(x',\mu',\alpha') - F(x,\mu,\alpha) - \langle x'-x,\partial_x F(x,\mu,\alpha)\rangle \\ &\qquad - \langle \alpha'-\alpha,\partial_\alpha F(x,\mu,\alpha)\rangle - \mathbb{E}\big[\langle \xi'-\xi,\partial_\mu F(x,\mu,\alpha)(\xi)\rangle\big] \geqslant \lambda|\alpha'-\alpha|^2, \end{aligned} \tag{5.34}$$

for any pair (ξ,ξ') with μ and μ' as marginal distributions, where $x,x'\in\mathbb{R}^d$, $\mu,\mu'\in\mathcal{P}_2(\mathbb{R}^d)$ and $\alpha,\alpha'\in\mathbb{R}^k$.

In a similar way, the function $(x,\mu)\mapsto G(x,\mu)$ is convex in the joint variable (x,μ).

Of course, the Hamiltonian is convex in α under ($\boldsymbol{H6}$)(i) so that the minimizer (5.11) is well-defined. By (5.15) and by a suitable version of the implicit function theorem, the function $\hat\alpha$ inherits the smoothness of $\partial_\alpha H$. For instance, assume that

Assumption ((***H*6**)(ii))**.** *The function* $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^k \ni (x,\mu,\alpha) \mapsto \partial_\alpha F(x,\mu,\alpha)$ *satisfy* (***H*2**) *(and thus* (***H*0**) *and* (***H*1**) *as well) (with* $w = (x,\alpha)$ *in the notations used in* (***H*1**) *and* (***H*2**)*).*

Then,

Lemma 5.6. *Under* (***H*6**)(i) *and* (***H*6**)(ii)*, the function* $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^k \ni (x,\mu,\alpha) \mapsto \hat{\alpha}(x,\mu,y)$ *satisfies* (***H*0**)*,* (***H*1**) *and* (***H*2**) *(with* $w = (x,y)$ *in the notations used in* (***H*1**) *and* (***H*2**)*).*

Proof. The starting point is (5.15). By (5.34) and by the Lipschitz property of $\partial_\alpha F$, we can reproduce the argument used in [5] to prove that $\hat{\alpha}$ is also Lipschitz continuous. More generally, the smoothness in x, y follows from a standard application of the implicit function theorem.

We now discuss the regularity of $\hat{\alpha}$ in the direction μ. Given $\xi, \chi \in L^2(\Omega, \mathcal{A}, \mathbb{P})$, we deduce from (5.15) that, for any $t \in \mathbb{R}$,

$$y^\dagger b_2 + \partial_\alpha F\big(x, [\xi + t\chi], \hat{\alpha}(x, [\xi + t\chi], y)\big) = 0.$$

By the standard implicit function theorem, we deduce that the function $\mathbb{R} \ni t \mapsto \hat{\alpha}(x, [\xi + t\chi], y) \in \mathbb{R}^k$ is differentiable and that

$$\begin{aligned}&\mathbb{E}\big\{\partial_\mu\big[\partial_\alpha F\big(x,[\xi],\hat{\alpha}(x,[\xi],y)\big)(\xi)\chi\big]\big\}\\&\quad + \partial^2_{\alpha\alpha} F\big(x,[\xi],\hat{\alpha}(x,[\xi],y)\big)\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\big[\hat{\alpha}(x,[\xi+t\chi],y)\big] = 0.\end{aligned}$$

By strict convexity, the matrix $\partial^2_{\alpha\alpha} F(x,[\xi],\hat{\alpha}(x,[\xi],y))$ is invertible. We easily deduce that the mapping $L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d) \ni \xi \mapsto \hat{\alpha}(x,[\xi],y) \in \mathbb{R}^k$ is Fréchet differentiable. In particular, the mapping $\mathcal{P}_2(\mathbb{R}^d) \ni \mu \mapsto \hat{\alpha}(x,\mu,y)$ is differentiable in Lions' sense and

$$\partial_\mu \hat{\alpha}(x,\mu,y)(v) = \big[\partial^2_{\alpha\alpha} F\big(x,\mu,\hat{\alpha}(x,\mu,y)\big)\big]^{-1}\partial_\mu\big[\partial_\alpha F\big(x,\mu,\hat{\alpha}(x,\mu,y)\big)\big](v).$$

The corresponding bounds in (***H*1**) together with the uniform integrability property are easily checked. Now, the smoothness in v follows from that one of $\partial_\mu[\partial_\alpha F]$ and the related bounds in (***H*2**) hold true. The smoothness of $\partial_\mu \hat{\alpha}$ in x, y is satisfied once we have the smoothness of $\hat{\alpha}$ in x, y. □

5.3.2. *Master equation.* The point is now to apply Proposition 5.2 with b as above, σ constant and

$$\begin{aligned}&f(x,y,\nu) = \partial_x H\big(x, \pi_1\sharp\nu, \hat{\alpha}(x,\pi_1\sharp\nu,y)\big) + \tilde{h}(x,\nu), \quad x,y \in \mathbb{R}^d,\ \nu \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d),\\&g(x,\mu) = \partial_x G(x,\mu) + \tilde{g}(x,\mu),\ x \in \mathbb{R}^d,\ \mu \in \mathcal{P}_2(\mathbb{R}^d).\end{aligned}$$

Notice also that (***H*3**) is satisfied, see again [5]. It thus remains to check that (***H*2**) is satisfied.

We thus assume that

Assumption ((***H*6**)(iii))**.** *The functions* $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^k \ni (x,\mu,\alpha) \mapsto \partial_x F(x,\mu,\alpha)$ *and* $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \ni (x,\mu) \mapsto \partial_x G(x,\mu)$ *satisfy* (***H*0**)(i)*,* (***H*1**) *and* (***H*2**) *(with* $w = (x,\alpha)$ *and* $w = x$ *respectively).*

For any $(x,\mu,\alpha) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^k$*, there exist versions of* $\partial_\mu F(x,\mu,\alpha)(\cdot)$ *and of* $\partial_\mu G(x,\mu)(\cdot)$ *such that* $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^k \times \mathbb{R}^d \ni (x,\mu,\alpha,v) \mapsto \partial_\mu F(x,\mu,\alpha)(v)$ *and* $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \ni (x,\mu,v) \mapsto \partial_\mu G(x,\mu)(v)$ *that satisfy* (***H*0**)(i)*,* (***H*1**) *and* (***H*2**) *(with* $w = (x,\alpha,v)$ *and* $w = (x,v)$ *respectively).*

Under $(\boldsymbol{H6})$(i-ii-iii), by Lemma 5.6, the function $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \ni (x, \mu, y) \mapsto \partial_x H(x, \mu, \hat\alpha(x, \mu, y))$ satisfies $(\boldsymbol{H0})$(i), $(\boldsymbol{H1})$ and $(\boldsymbol{H2})$. We now discuss $\tilde h$. By linearity of b, we first observe that (recalling that $\partial_\mu[\int_{\mathbb{R}^d} h(v')\mathrm{d}\mu(v')](v) = \nabla h(v)$, see [5])

$$\tilde h(x,\nu) = \int_{\mathbb{R}^d\times\mathbb{R}^d} w^\dagger b_1 \mathrm{d}\nu(v,w) + \int_{\mathbb{R}^d\times\mathbb{R}^d} \partial_\mu F\big(v, \pi_1\sharp\nu, \hat\alpha(v,\pi_1\sharp\nu,w)\big)(x)\mathrm{d}\nu(v,w),$$

The smoothness of the first term is easily handled, the smoothness of the second one in x as well. The difficulty is to differentiate the second one with respect to ν. We get

$$\begin{aligned}
&\partial_\nu\bigg[\int_{\mathbb{R}^d\times\mathbb{R}^d} \partial_\mu F\big(v', \pi_1\sharp\nu, \hat\alpha(v',\pi_1\sharp\nu,w')\big)(x)\mathrm{d}\nu(v',w')\bigg](v,w)\\
&= \partial_{(v,w)}\big[\partial_\mu F\big(v,\pi_1\sharp\nu,\hat\alpha(v,\pi_1\sharp\nu,w)\big)(x)\big]\\
&\quad + \bigg(\int_{\mathbb{R}^d\times\mathbb{R}^d} \partial_\mu\big[\partial_\mu F\big(v',\pi_1\sharp\nu,\hat\alpha(v',\pi_1\sharp\nu,w')\big)(x)\big](v)\mathrm{d}\nu(v',w'),0\bigg)\\
&\quad + \bigg(\int_{\mathbb{R}^d\times\mathbb{R}^d} \Big[\partial_\alpha\big[\partial_\mu F\big(v',\pi_1\sharp\nu,\hat\alpha(v',\pi_1\sharp\nu,w')\big)(x)\big]\partial_\mu\big[\hat\alpha(v',\pi_1\sharp\nu,w')\big]\Big](v)\mathrm{d}\nu(v',w'),0\bigg),
\end{aligned}$$

where the '0' indicates that the derivative in the direction w is zero. We let the reader check the required conditions for the derivative in the direction ν in $(\boldsymbol{H1})$ and $(\boldsymbol{H2})$ are indeed satisfied. Derivatives in the direction x are easily handled.

We deduce that Proposition 5.2 applies. As for mean-field games, the master PDE satisfied by U is not the 'natural' equation associated with the optimization problem. Following the previous subsection, we thus define

$$V(t,x,\mu) = \mathbb{E}\bigg[G\big(X_T^{t,x,\mu},[X_T^{t,\xi}]\big) + \int_t^T F\big(X_s^{t,x,\mu},[X_s^{t,\xi}],\hat\alpha(X_s^{t,x,\mu},[X_s^{t,\xi}],Y_s^{t,x,\mu})\big)ds\bigg],$$

where $\xi \sim \mu$, $(X_s^{t,\xi})_{s\in[t,T]}$ denotes the forward component in (5.33), under the initial condition $X_t^{t,\xi} = \xi$, and $(X_s^{t,x,\mu})_{s\in[t,T]}$ denotes the corresponding solution of (2.4).

As in the proof of Theorem 5.3, we are willing to apply the results from Section 4 in order to investigate the smoothness of V. Again, this requires some precaution as the coefficients may be of quadratic growth in the space variable and in the measure argument. Proceeding as in the proof of Theorem 5.3, we have [13]

Theorem 5.7. *Under* $(\boldsymbol{H6})$(i–iii)*, the function V satisfies the statement of Theorem 5.3, with the same master equation expect that U inside is the decoupling field of* (5.33)*.*

On the model of Remark 5.4, the identification of $U(t,x,\mu)$ in terms of $V(t,x,\mu)$ now reads

$$U(t,x,\mu) = \partial_x V(t,x,\mu) + \int_{\mathbb{R}^d} \partial_\mu V(t,x',\mu)(x)d\mu(x'), \tag{5.35}$$

which can be proved by differentiating the map $L^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^d) \ni \xi \mapsto \mathbb{E}[V(t,\xi,[\xi])] \in \mathbb{R}$ in the direction $\chi \in L^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^d)$. By the same kind of expansion as in Remark 5.4, we get

$$\mathbb{E}\Big[\partial_x V(t,\xi,[\xi])\chi + \hat{\mathbb{E}}\big[\partial_\mu V(t,\xi,[\xi])(\langle\xi\rangle)\langle\chi\rangle\big]\Big] = \mathbb{E}\big[Y_t^{t,\xi}\chi\big] = \mathbb{E}\big[U(t,\xi,[\xi])\chi\big],$$

[13] Pay attention that there is no need for an analogue of $(\boldsymbol{H4})$(iv), since $(\boldsymbol{H4})$(iv) is necessarily true under $(\boldsymbol{H6})$(i–iii), with $F_0(x,\mu)$ in (5.21) replaced by $F_0(x,\mu,\hat\alpha(x,\mu,U(t,x,\mu)))$, the constants appearing in $(\boldsymbol{H0})$(i)–$(\boldsymbol{H1})$–$(\boldsymbol{H2})$ being uniform in $t \in [0,T]$.

where $Y_t^{t,\xi}$ and $U(t,\xi,[\xi])$ are seen as row vectors. By Fubini's theorem, this identifies $U(t,\xi,[\xi])$ with $\partial_x V(t,\xi,[\xi]) + \hat{\mathbb{E}}[\partial_\mu V(t,\langle\xi\rangle,[\xi])(\xi)]$. This proves (5.35) when the law of ξ has $\mathbb{R}^d$ as support. In the general case, we can approximate ξ by random variables with $\mathbb{R}^d$ as support. Passing to the limit in (5.35), this completes the proof of the identification.

We refer to [8] for additional comments about the differences between the shapes of the master equation in mean-field games and in the control of McKean-Vlasov equations.

6. Appendix

6.1. **Proof of Proposition 3.8.** The proof is a straightforward adaptation of Lemma 3.3 in [5]. Basically, it suffices to prove the result when μ has a smooth positive density denoted by p, and p and its derivatives being at most of exponential decay at the infinity. It is then possible to construct a quantile function $U : (0,1)^d \ni (z_1,\dots,z_d) \mapsto U(z_1,\dots,z_d) \in \mathbb{R}^d$ (this is the notation used in [5], but this has nothing to do with the generic notation U used in the paper for denoting a function of the measure) such that $U(\eta_1,\dots,\eta_d)$ has law μ when $\eta_1,\dots,\eta_d$ are i.i.d. random variables with uniform distribution on $(0,1)$. Moreover, $\partial U_i/\partial z_i \neq 0$ and $\partial U_j/\partial z_i = 0$ if $i < j$.

Going to (69) therein, we see from the assumption imposed on V that the bound becomes

$$\begin{aligned}
&\int_{|r|<h} \big| V_n\big[U\big(z^0 + r - 2r_d e_d\big)\big] - V_n\big(U(z^0+r)\big)\big|^2 \mathrm{d}r \\
&\leqslant C_n^2 \int_{|r|<h} \bigg[1 + |U\big(z^0 + r - 2r_d e_d\big)|^{2\alpha} + |U(z^0+r)|^{2\alpha} + \Big(\int_{\mathbb{R}^d} |x|^2 d\mu(x) \Big)^{2\alpha} \bigg] \\
&\qquad\qquad\qquad\qquad \times \big| U\big(z^0 + r - 2r_d e_d\big) - (U(z^0+r)\big|^2 \mathrm{d}r,
\end{aligned}$$

where V_n is a mollification of V that satisfies (3.32) with respect to a constant C_n that converges to C as n tends to the infinity. Dividing by h^d and following the lines of the original argument, we get, for a given $z^0 \in \mathbb{R}^d$,

$$\Big| \frac{\partial V_n}{\partial x_d}\big(U(z^0)\big) \frac{\partial U_d}{\partial z_d}\big(z^0\big) \Big|^2 \leqslant C_n^2 \bigg[1 + 2|U\big(z^0\big)|^{2\alpha} + \Big(\int_{\mathbb{R}^d} |x|^2 d\mu(x) \Big)^{2\alpha} \bigg] \Big| \frac{\partial U_d}{\partial z_d}\big(U(z^0)\big) \Big|^2.$$

Dividing by $|[\partial U_d/\partial z_d](U(z^0))|$ and letting n tend to the infinity, we complete the proof. □

6.2. **Differentiability lemma.**

Lemma 6.1. *Consider a function $V : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}^d$ such that, for any $\xi, \chi \in L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d)$, the mapping $\mathbb{R}^d \ni x \mapsto \mathbb{E}[\langle V(x,[\xi],\xi),\chi\rangle]$ is differentiable (where $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{R}^d$). Assume moreover that there exist a constant $C \geqslant 0$ and a function Φ_α as in ($\boldsymbol{H1}$) such that, for all $x, x' \in \mathbb{R}^d$ and $\xi, \xi', \chi \in L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d)$:*

$$\begin{aligned}
&\Big| \frac{\mathrm{d}}{\mathrm{d}x} \mathbb{E}\big[\langle V(x,[\xi],\xi),\chi\rangle\big]\Big| \leqslant C\|\chi\|_2, \\
&\Big| \frac{\mathrm{d}}{\mathrm{d}x} \mathbb{E}\big[\langle V(x,[\xi],\xi),\chi\rangle\big] - \frac{\mathrm{d}}{\mathrm{d}x} \mathbb{E}\big[\langle V(x',[\xi'],\xi'),\chi\rangle\big]\Big| \leqslant C\big(|x-x'| + \Phi_\alpha(\xi,\xi')\big)\|\chi\|_2.
\end{aligned}$$

Then, for any $x \in \mathbb{R}^d$ and any $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, we can find a continuous version of $V(x,\mu,\cdot)$, uniquely defined on Supp(μ)*, such that the mapping $\mathbb{R}^d \times \mathrm{Supp}(\mu) \ni (x,v) \mapsto V(x,\mu,v)$ is differentiable with respect to x. Moreover, we can find a mapping $\mathbb{R}^d \times \mathbb{R}^d \ni (x,v) \mapsto \partial V(x,\mu,v)$, continuous in v for any given $x \in \mathbb{R}^d$, jointly continuous at any point (x,v)*

with $v \in \mathrm{Supp}(\mu)$*, such that* $\partial V(x,\mu,v)$ *identifies with* $\partial_x V(x,\mu,v)$ *whenever* $v \in \mathrm{Supp}(\mu)$*. In particular,* $\partial_x V(\cdot,\mu,\cdot)$ *is continuous on* $\mathbb{R}^d \times \mathrm{Supp}(\mu)$*.*

Proof. By Riesz' theorem, for any $i \in \{1,\dots,d\}$, for any $x \in \mathbb{R}^d$ and any $\xi \in L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d)$, we can find an element $V^i_{x,\xi} \in L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d)$ such that

$$\frac{\mathrm{d}}{\mathrm{d}x_i}\mathbb{E}\big[\langle V(x,[\xi],\xi),\chi\rangle\big] = \mathbb{E}\big[\langle V^i_{x,\xi},\chi\rangle\big].$$

Now, for $h \neq 0$, denoting by e_i the ith vector of the canonical basis,

$$\begin{aligned}&\mathbb{E}\Big[\Big\langle h^{-1}\big(V(x+he_i,[\xi],\xi) - V(x,[\xi],\xi),\chi\big) - V^i_{x,\xi},\chi\Big\rangle\Big] \\ &= \int_0^1 \Big(\frac{\mathrm{d}}{\mathrm{d}x_i}\mathbb{E}\big[\langle V(x+she_i,[\xi],\xi),\chi\rangle\big] - \frac{\mathrm{d}}{\mathrm{d}x_i}\mathbb{E}\big[\langle V(x,[\xi],\xi),\chi\rangle\big]\Big)\mathrm{d}s.\end{aligned}$$

By assumption, we thus get that $h^{-1}(V(x+he_i,[\xi],\xi) - V(x,[\xi],\xi)) - V^i_{x,\xi}$ tends to 0 in $L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d)$. Therefore $V^i_{x,\xi}$ is a random variable in $L^2(\Omega,\sigma(\xi),\mathbb{P};\mathbb{R}^d)$ and we can express it as $\partial_i V(x,[\xi],\xi)$ where $\partial_i V(x,[\xi],\cdot)$ is a function in $L^2(\mathbb{R}^d,[\xi];\mathbb{R}^d)$.

We have

$$\mathbb{E}\big[|\partial V(x,[\xi],\xi) - \partial V(x',[\xi'],\xi')|^2\big] \leqslant C\big(|x-x'|^2 + \Phi^2_\alpha(\xi,\xi')\big).$$

Choosing $x = x'$, we deduce from Proposition 3.8 that, for any $x \in \mathbb{R}^d$ and any $\xi \in L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d)$, there exists a version of the mapping $\mathbb{R}^d \ni v \mapsto \partial V(x,[\xi],v) = (\partial_1 V(x,[\xi],v),...,\partial_d V(x,[\xi],v)) \in \mathbb{R}^{d\times d}$ that is continuous on compact subsets of $\mathbb{R}^d$, uniformly in $x \in \mathbb{R}^d$, such a version being uniquely defined on the support of $[\xi]$. By the same method as in (3.33), we deduce that the family $(\mathbb{R}^d \ni v \mapsto \partial V(x,[\xi],v) \in \mathbb{R}^{d\times d})_{x\in\mathbb{R}^d}$ is relatively compact for the topology of uniform convergence on compact subsets. Considering a sequence $(x_n)_{n\geqslant 1}$ that converges to $x \in \mathbb{R}^d$, we already know that the sequence of functions $(\mathbb{R}^d \ni v \mapsto \partial V(x_n,[\xi],v) \in \mathbb{R}^{d\times d})_{n\geqslant 1}$ converges in $L^2(\mathbb{R}^d,[\xi];\mathbb{R}^{d\times d})$ to $\mathbb{R}^d \ni v \mapsto \partial V(x,[\xi],v) \in \mathbb{R}^{d\times d}$. Since $\partial V(x,[\xi],\cdot)$ is uniquely defined on the support of $[\xi]$, the limit of any converging subsequence (for the topology of uniform convergence on compact subsets of $\mathbb{R}^d$) of $(\partial V(x_n,[\xi],\cdot))_{n\geqslant 1}$ coincides with $\partial V(x,[\xi],\cdot)$ on the support of $[\xi]$. We easily deduce that the function $\mathbb{R}^d \times \mathbb{R}^d \ni (x,v) \mapsto \partial V(x,[\xi],v) \in \mathbb{R}^{d\times d}$ is continuous at any point (x,v) such that $v \in \mathrm{Supp}([\xi])$.

Similarly, we deduce from the identity

$$\begin{aligned}&\mathbb{E}\Big[\Big\langle \frac{1}{h}\Big(\big(V(x+he_i,[\xi],\xi) - V(x,[\xi],\xi)\big) - \big(V(x'+he_i,[\xi'],\xi') - V(x',[\xi'],\xi')\big)\Big),\chi\Big\rangle\Big] \\ &= \int_0^1 \mathbb{E}\Big\{\Big\langle\big(\partial_i V(x+sh,[\xi],\xi) - \partial_i V(x'+sh,[\xi'],\xi')\big),\chi\Big\rangle\Big\}\mathrm{d}s,\end{aligned}$$

that $\|h^{-1}[(V(x+he_i,[\xi],\xi) - V(x,[\xi],\xi)) - (V(x'+he_i,[\xi'],\xi') - V(x',[\xi'],\xi'))]\|_2 \leqslant C(|x-x'| + \Phi_\alpha(\xi,\xi'))$, from which we get that, for any $x \in \mathbb{R}^d$, any $h \neq 0$ and any $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, there exists a version of the mapping $\mathbb{R}^d \ni v \mapsto h^{-1}[V(x+he_i,\mu,v) - V(x,\mu,v)]$ that is continuous on compact subsets of $\mathbb{R}^d$, uniformly in $x \in \mathbb{R}^d$ and in $h \neq 0$. As above, we deduce that the family $(\mathbb{R}^d \ni v \mapsto h^{-1}[V(x+he_i,\mu,v) - V(x,\mu,v)])_{x\in\mathbb{R}^d,h\neq 0}$ is relatively compact, for the topology of uniform convergence on compact subsets. Once again, following the same argument as above, this says that, for any $x \in \mathbb{R}^d$, the functions $(\mathrm{Supp}(\mu) \ni v \mapsto h^{-1}[V(x+he_i,\mu,v) - V(x,\mu,v)] \in \mathbb{R}^d)_{h\neq 0}$ converge uniformly on compact subsets as h tends to 0 to some derivative function, which identifies with $\mathrm{Supp}(\mu) \ni v \mapsto \partial_i V(x,\mu,v) \in \mathbb{R}^d$. $\square$

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