Calc. Var. 14, 345–357 (2002) **Calculus of Variations**

DOI (Digital Object Identifier) 10.1007/s005260100106

Diogo Aguiar Gomes

Viscosity solutions of Hamilton-Jacobi equations, and asymptotics for Hamiltonian systems

Received: 16 October 2000 / Accepted: 23 February 2001 /
Published online: 12 October 2001 – © Springer-Verlag 2002

Abstract. In this paper we apply the theory of viscosity solutions of Hamilton-Jacobi equations to understand the structure of certain Hamiltonian flows. In particular, we describe the asymptotic behavior of minimizing orbits, and prove analogs of the classical Hamilton-Jacobi integrability theory that hold under very general conditions. Then, combining partial differential equations techniques with dynamical systems ideas (Mather measures, ergodicity) we study solutions of time-independent Hamilton-Jacobi equation, namely, uniform continuity, difference quotients and non-uniqueness.

1. Introduction

Consider the Hamiltonian differential equations

$$\dot{x} = -D_p H(p,x) \qquad \dot{p} = D_x H(p,x), \tag{1}$$

where $H(p,x) : \mathbb{R}^{2n} \to \mathbb{R}$ is smooth function. The first objective of this paper is to understand the asymptotic behavior of the trajectories of (1) that are characteristics of viscosity solutions of Hamilton-Jacobi equations,

$$H(P + D_x u, x) = \overline{H}(P). \tag{2}$$

For such trajectories we will prove that analogs of Hamilton-Jacobi classical integrability theory still hold under general conditions – even when (1) is non-integrable. The second objective is to make clear the connections between Mather measures [14] (invariant measures under the flow of (1) with certain minimizing properties) and viscosity solutions of Hamilton-Jacobi equations (2). The Mather measures will be obtained as weak limits of measures supported on characteristics of viscosity solutions of Hamilton-Jacobi equations. The third and last objective is to study the regularity properties of viscosity solutions of (2). Such regularity properties are

D. A. Gomes: Institute of Advanced Study, Princeton University, Princeton, NJ 08540, USA
(e-mail: dgomes@math.ist.utl.pt)

Universidade Tecnica Lisboa, Instituto Superior Tecnico (IST),
Departamento de Matematica, Avenida Rovisco Pais, 1049-001 Lisboa, Portugal

Supported in part by FCT (Portugal) through programs POCIT, POCIT/32931/MAT/2000, BPD 1531/2000 and NSF grant DMS 97-29992.

not only important in themselves but also have applications in proving asymptotic estimates on the trajectories of (1).

We make the assumption that H is strictly convex in p, and $\mathbb{Z}^n$ periodic in x, i.e.

$$H(p, x + k) = H(p, x)$$

for all $(p, x) \in \mathbb{R}^{2n}$ and $k \in \mathbb{Z}^n$. This hypothesis is satisfied in many important applications, for instance, the motion of particles in a lattice potential or perturbations of Hamiltonian systems in action-angle coordinates.

It is well known [1] that by solving a Hamilton-Jacobi PDE (2) it is possible to construct a change of variables that simplifies the dynamics of (1). Suppose for each $P \in \mathbb{R}^n$ there are smooth functions $\overline{H}(P)$ and $u(x, P)$ solving (2). Furthermore assume that the equations

$$p = P + D_x u(x, P) \qquad X = x + D_P u(x, P) \tag{3}$$

define a smooth change of coordinates $(x, p) \to (X, P)$. Then, in the new coordinates (X, P), (1) is

$$\dot{X} = -D_P \overline{H}(P) \qquad \dot{P} = 0. \tag{4}$$

Therefore, since (4) is trivial to solve, we would have solved (1), up to changes of coordinates.

Unfortunately there are several points where this method can fail. Firstly (2) may not have any classical solution. Secondly, for fixed P there is not uniqueness and therefore u may not be differentiable in P. Finally, in the very special situation where u is smooth both in P and in x, (4) may not be solvable or may not define a gobal smooth change of coordinates.

Ignoring the previous remarks, and assuming that there are no problems, we point out the following facts:

– Since $\dot{P} = 0$, for each P the graph $p = P + D_x u$ is an invariant set.
– In this set the trajectories are straight lines (up to a change of coordinates), because $\dot{X} = -D_P \overline{H}(P)$.
– Since $D_P u$ is bounded, solutions with initial conditions on the invariant set have the asymptotic property

$$\lim_{t \to \infty} \frac{x(t)}{t} = -D_P \overline{H}.$$

It turns out that analogs of these statements are still true as long as classical solutions are replaced by viscosity solutions.

Recall that u is a viscosity solution of (2), provided that whenever ϕ is a smooth function such that $u - \phi$ has a local maximum at a point x_0 (resp. minimum) then $H(P + D_x \phi(x_0), x_0) \leq \overline{H}(P)$ (resp. $\geq$). A classical-yet-unpublished theorem from Lions, Papanicolaou, and Varadhan [13] (see also [3], [5], [4], or [2]) guarantees the existence of a viscosity solution of (2). More precisely, for each P there exists a unique $\overline{H}(P)$ and a function $u(x, P)$ (possibly not unique), solving (2) in the viscosity sense. Furthermore $\overline{H}$ is convex in P and $u(x, P)$ Lipschitz in x.

If u is a viscosity solution of (2) then [12]

$$u(x,P) = \inf_{x(0)=x} \int_0^t \left[L(x,\dot{x}) + P\dot{x} + \overline{H}(P)\right] ds + u(x(t),P), \tag{5}$$

where the infimum is taken over all Lipschitz trajectories $x(\cdot)$ with initial condition x, and $L(x,v) = \sup_p \left[-v \cdot p - H(p,x)\right]$ is the Legendre transform of H. Furthermore there exists an optimal trajectory $x^*(s)$, $0 \leq s \leq t$. Let

$$p^* = P + D_v L(x^*, \dot{x}^*).$$

Then (x^*, p^*) is a solution of (1). For $0 < s < t$,

$$p^*(s) = P + D_x u(x^*(s), P),$$

in particular, u is differentiable along the optimal trajectory and if u is differentiable at x, $p^*(0) = P + D_x u(x,P)$.

The results by A. Fathi [8], [9], [10], [11], and W. E [6] make clear the connection between viscosity solutions and Hamiltonian dynamics. The main idea is that if $u(x,P)$ is a viscosity solution of (2) then there exists an invariant set $\mathcal{I}$ contained on the graph

$$\{(x, P + D_x u(x,P))\}.$$

Furthermore, $\mathcal{I}$ is a subset of a Lipschitz graph, i.e., $D_x u(x,P)$ is a Lipschitz function on $\pi(\mathcal{I})$, where $\pi(x,p) = x$. If $\overline{H}$ is differentiable at P, then any solution $(x(t), p(t))$ of (1) with initial conditions on $\mathcal{I}$ satisfies

$$\lim_{t\to\infty} \frac{|x(t) + D_P \overline{H} t|}{t} = 0. \tag{6}$$

In Sect. 2 we discuss several improved versions of (6) using viscosity solutions methods (theorem 1). These versions will need additional regularity properties of $\overline{H}$ and u. Since $\overline{H}(P)$ is convex, additional regularity for $\overline{H}(P)$ (twice differentiability) is not a problem for generic values of P. Increased regularity properties for u (uniform continuity in P or L^2 difference quotients) are more delicate and are studied in Sects. 4 and 5. Other regularity issues for u are also discussed in [7]. To study such problems we construct certain probability measures (Mather measures) that encode information about the Hamiltonian dynamics contained in viscosity solution of (2). In Sect. 3 we prove that there is a one-to-one correspondence between Mather measures and viscosity solutions (see also [7]). This, for instance, explains why viscosity solutions of (2) may not be unique. In Sect. 4 we give conditions under which the solution $u(x,P)$ is uniformly continuous in P. The proof relies in understanding how viscosity solutions encode information about Mather measures. Finally, in the last section, we study difference quotients and prove that change of coordinates (3) satisfies a weak non-degeneracy condition if $\overline{H}$ is uniformly convex.

I am deeply indebt to Professor Lawrence C. Evans for his suggestions and patience in reading and discussing with me several drafts of this paper. I would like to thank M. Bardi, L. Barreira, and W. Oliva discussing this paper with me and for many references and suggestions.

2. Improved asymptotics

Suppose that u is a periodic viscosity solution of (2). Define

$$\mathcal{G} = \{(x, P + D_x u) : u \text{ is differentiable at } x\}.$$

Let Ξ_t be the flow corresponding to (1). Observe that, for all $t > 0$, $\Xi_t(\mathcal{G}) \subset \mathcal{G}$. Define $\mathcal{G}_t = \Xi_t(\overline{\mathcal{G}})$ and let

$$\mathcal{I} = \cap_{t>0}\mathcal{G}_t.$$

Then [6] $\mathcal{I}$ is a nonempty closed invariant set with respect to the Hamiltonian flow. Furthermore, if $\overline{H}(P)$ is differentiable at P, the trajectories $(x(t), p(t))$ of the Hamiltonian flow with initial conditions on the invariant set $\mathcal{I}(P)$ satisfy

$$\lim_{t\to\infty} \frac{x(t)}{t} = -D_P\overline{H}(P).$$

The next theorem improves the previous asymptotic estimate.

Theorem 1. *Suppose $(x(t), p(t))$ is a solution of (1) with initial conditions on the invariant set $\mathcal{I}$. Assume $\overline{H}$ is twice differentiable at P (since $\overline{H}$ is convex, it is twice differentiable for almost every P). Let*

$$\|x - y\| \equiv \min_{k\in\mathbb{Z}^n} |x - y + k|,$$

i.e., the "periodic distance" between x and y. Then there exists a constat C such that

$$|x(t) - x(0) + D_P\overline{H}t| \le C\sqrt{\|x(t) - x(0)\|t}.$$

Furthermore if there exists a continuous function ω, with $\omega(0) = 0$, such that

$$|u(x, P) - u(x, P')| \le \omega(|P - P'|).$$

Then

$$|x(t) - x(0) + D_P\overline{H}t| \le \min_\delta \frac{\|x(t) - x(0)\| \wedge \omega(\delta)}{\delta} + Ct\delta. \tag{7}$$

Finally, if in $\mathcal{I}$ u is uniformly differentiable in P,

$$x(t) + D_P u(x(t), P) - x(0) - D_P u(x(0), P) + D_P\overline{H}t = 0.$$

Proof. Let $u(x, P)$ be a viscosity solution of (2). For some C^1 function $x^*(s)$ with $x^*(0) = x$

$$u(x, P) = u(y, P) + \int_0^t \left[L(x^*, \dot{x}^*) + P\cdot\dot{x}^* + \overline{H}(P)\right] ds,$$

where $y = x^*(t)$. for any other P'

$$u(x, P') \le u(y, P') + \int_0^t \left[L(x^*, \dot{x}^*) + P'\cdot\dot{x}^* + \overline{H}(P')\right] ds.$$

Thus

$$u(x,P)-u(x,P') \geq u(y,P)-u(y,P')+ \\ +\int_0^t \left[(P-P')\cdot \dot{x}^*+\overline{H}(P)-\overline{H}(P')\right] ds.$$

If $\overline{H}$ is twice differentiable (or at least $C^{1,1}$) at P then

$$\overline{H}(P') \leq \overline{H}(P)+\eta\cdot(P'-P)+C|P'-P|^2,$$

where $\eta = D_P\overline{H}(P)$. Thus

$$u(x,P)-u(x,P')+u(y,P')-u(y,P) \geq \\ (P-P')\cdot\int_0^t [\dot{x}^*+\eta]-Ct|P'-P|^2.$$

The left hand side can be estimated by

$$u(x,P)-u(x,P')+u(y,P')-u(y,P) \leq C\|x-y\|.$$

Choose $|P-P'| = \sqrt{\frac{\|x-y\|}{t}}$ then

$$|\int_0^t [\dot{x}^*+\eta]\,| \leq C\sqrt{\|x-y\|t}.$$

If there exists a continuous function ω, with $\omega(0)=0$, such that

$$|u(x,P)-u(x,P')| \leq \omega(|P-P'|)$$

then

$$u(x,P)-u(x,P')+u(y,P')-u(y,P) \leq \|x-y\| \wedge \omega(|P-P'|),$$

thus, by choosing $|P-P'|$ appropriately, we obtain (7).

Finally, if u is uniformly differentiable in P, we get

$$x(t)+D_Pu(x(t),P)-x(0)-D_Pu(x(0),P)+D_P\overline{H}t=0,$$

by using the same techniques. This last equality shows that if the expression

$$X = x+D_Pu$$

is well defined in the invariant set then $\dot{X} = D_P\overline{H}$. □

In Sect. 4 we investigate sufficient conditions for the existence of a modulus of continuity $\omega(\delta)$ for u. Such conditions in conjunction with estimate (7) yield a sharper asymptotic estimate.

3. Mather measures

A general approach to study dynamical systems is to consider invariant probability measures and then try to understand their properties: for instance determine the supports or ergodic properties.

In this section we construct certain invariant measures (Mather measures) that will have supports in the invariant sets. These measures admit a characterization as "action minimizing" measures [14], i.e., μ is a Mather measure if it minimizes

$$\int L + P \cdot v d\eta,$$

over all probability measures η that are invariant under the flow Ξ_t (here $v = -D_p H(p,x)$). Alternatively, consider a trajectory $(x(t), p(t))$ of (1) with initial conditions on the invariant set $\mathcal{I}$. For $E \subset T^n \times \mathbb{R}^n$ define the measure

$$\mu_T(E) = \frac{1}{T}\int_0^T 1_E(x(t), p(t)).$$

μ_T is a probability measure and since $p(t)$ is bounded we can extract a weakly converging subsequence

$$\mu_T \rightharpoonup \mu \tag{8}$$

to some measure μ. Since $\mathcal{I}$ is closed this measure will be supported on $\mathcal{I}$. We will prove that these measures are indeed Mather measures, and conversely, that any Mather measure can be obtained from a procedure like this.

Theorem 2. *Let μ, as in (8), be a invariant probability measure associated with a periodic viscosity solution of*

$$H(P + D_x u, x) = \overline{H}(P).$$

Then μ minimizes

$$\int L + P \cdot v d\eta,$$

over all probability measures η that are invariant under the flow Ξ_t, and therefore μ is a Mather measure.

Proof. If the claim were false, there would be an invariant probability measure ν such that

$$-\overline{H} = \int L + P \cdot v d\mu > \int L + P \cdot v d\nu = -\lambda.$$

We may assume that ν is ergodic, otherwise we could choose an ergodic component of ν for which the previous inequality holds. Take a generic point of (x,p) in the support of ν and consider its orbit $x(s)$. Then

$$u(x(0), P) - \overline{H}(P)t \leq \int_0^t L(x(s), \dot{x}(s)) + P \cdot \dot{x}(s) ds + u(x(t), P).$$

As $t \to \infty$

$$\frac{1}{t}\int_0^t L(x(s), \dot{x}(s)) + P\dot{x}(s)ds \to -\lambda,$$

by the ergodic theorem. Hence

$$-\overline{H} \le -\lambda,$$

which is a contradiction. □

Next we prove that any Mather measure is "embedded" in a viscosity solution of a Hamilton-Jacobi equation. Recall that any Mather measure μ is supported on a Lipschitz graph $(x, p(x))$ [14]. Let $\pi(p, x) = x$ be the vertical projection and $\pi_*\mu$ the vertical projection of μ. Suppose μ is a ergodic Mather measure. Then [15] there exists a Lipschitz function $W : \mathrm{supp}(\pi_*\mu) \to \mathbb{R}$ and a constant $\overline{H}(P)$ such that

$$-L - P \cdot v = \overline{H}(P) + D_x W v.$$

By taking W as terminal condition for a time-dependent Hamilton-Jacobi equation we can construct a periodic viscosity solution of (2) and then recover μ through a weak limit as in (8). More precisely:

Theorem 3. *Suppose μ is an ergodic Mather measure. Then there exists a viscosity solution u of (2) such that $u = W$ on $\mathrm{supp}(\pi_*\mu)$. Furthermore, for almost every $x \in \mathrm{supp}(\pi_*\mu)$, the measure μ can be recovered by taking minimizing trajectories that pass trough x and taking a weak limit as in (8).*

Proof. Consider the terminal value problem $V(x, 0) = W(x)$ if $x \in \mathrm{supp}(\pi_*\mu)$ and $V(x, 0) = +\infty$ elsewhere, with

$$-D_t V + H(P + D_x V, x) = \overline{H}(P),$$

for $t < 0$. Then, for $x \in \mathrm{supp}(\pi_*\mu)$

$$V(x, t) = W(x).$$

Also if $x \notin \mathrm{supp}(\pi_*\mu)$ then

$$V(x, t) \le V(x, s),$$

if $t < s < 0$. Hence, as $t \to -\infty$ the function $V(x, t)$ decreases pointwise. Since V is bounded and uniformly Lipschitz in x it must converge uniformly (because V is periodic) to some function u. Then u will be a viscosity solution of

$$H(P + D_x u, x) = \overline{H}(P).$$

Since $u = W$ on the support of $\pi_*\mu$, the second part of the theorem is a consequence of the ergodic theorem. □

Finally, to complete the picture, recall a theorem from [7] that states that any Mather measure is supported on the graph $p = P + D_x u$, for any u viscosity solution of (2). This theorem shows that any viscosity solution of (2) encodes all the information about all possible Mather measures.

In case in which, for the same P, there are distinct ergodic Mather measures $\mu_1 \dots \mu_k$ with disjoint supports, we could use the functions $W_1 \dots W_k$ as initial condition (with $+\infty$ outside the union of $\operatorname{supp} \pi_* \mu_i$) to construct a viscosity solution of (2). By adding arbitrary constants to W_i we may change the solution, and therefore proving non-uniqueness - this is will be discussed in a forthcoming paper.

4. Uniform continuity of viscosity solutions

From Sect. 2 we know that results about the uniform continuity in P of solutions of (2) would yield improved asymptotic estimates for trajactories in the invariant set. In this section we adress the question whether viscosity solutions of (2) are uniformly continuous in P. Obviously, adding an arbitrary function of P to u produces another viscosity solution. We could think that by defining a new family of solutions $v = u + f(P)$, with an appropriate choice for f (for instance such that $v(0, P) = 0$) we would get a continuous family of solutions v. However, the non-uniqueness observation from the previous section as well as simple examples imply that such results are not to be expected in general. However, as we prove bellow, when there is a unique Mather measure μ (unique ergodicity) then, u is uniformly continuous in P on the support of $\pi_* \mu$.

Proposition 1. *Suppose μ is a Mather measure as in the previous section. Let $P_n \to P$. Then there exists a point x in the support of $\pi_* \mu$ such that for any T*

$$\sup_{0 \le t \le T} |u(x^*(t), P) - u(x^*(t), P_n)| \to 0,$$

as $n \to \infty$, provided $u(x, P_n) = u(x, P)$.

Proof. We start by proving an auxiliary lemma

Lemma 1. *There exist a point (x, p) in the support of μ, and sequences $x_n, \tilde{x}_n \to x$, $p_n, \tilde{p}_n \to p$, with $(x_n, p_n) \in \operatorname{supp} \mu$ optimal pair for P, and $(\tilde{x}_n, \tilde{p}_n)$ optimal pairs for P_n.*

Remark. The non-trivial point of the lemma is that the limits of p_n and $\tilde{p}_n$ are the same.

Proof. Take a generic point (x_0, p_0) in the support of μ. Let $x^*(t)$ be the optimal trajectory for P with initial condition (x_0, p_0). Then for all $t > 0$

$$H(P + D_x u(x^*(t), P), x^*(t)) = \overline{H}(P).$$

Also, for almost every y

$$H(P + D_x u(x^*(t) + y, P_n), x^*(t)) = \overline{H}(P_n) + O(|y|),$$

for almost every t. Choose y_n with $|y_n| \le |P - P_n|$ such that the previous identity holds. By strict convexity of H in p

$$\dot{x}^*(t)\xi + \theta|\xi|^2 \le C|P_n - P|,$$

where

$$\xi = [P - P_n + D_x u(x^*(t), P) - D_x u(x^*(t) + y_n, P_n)],$$

$$\dot{x}^*(t) = -D_p H(P + D_x u(x^*(t), P), x^*(t)),$$

and $\theta > 0$. Note that

$$\left|\frac{1}{T}\int_0^T \dot{x}^*(t)\xi\right| \leq |P - P_n| + \frac{|u(x^*(0), P) - u(x^*(T), P)|}{T} + \\ + \frac{|u(x^*(0) + y_n, P_n) - u(x^*(T) + y_n, P_n)|}{T}.$$

Therefore we may choose T (depending on n) such that

$$\left|\frac{1}{T}\int_0^T \dot{x}^*(t)\xi\right| \leq 2|P - P_n|.$$

Thus

$$\frac{1}{T}\int_0^T |P + D_x u(x^*(t), P) - P_n - D_x u(x^*(t) + y_n, P_n)|^2 \leq \\ \leq C|P - P_n|.$$

Choose $0 \leq t_n \leq T$ for which

$$|P + D_x u(x^*(t_n), P) - P_n - D_x u(x^*(t_n) + y_n, P_n)|^2 \leq C|P - P_n|.$$

Let $x_n = x^*(t_n)$, $\tilde{x}_n = x^*(t_n) + y_n$, and

$$p_n = P + D_x u(x^*(t_n), P) \qquad \tilde{p}_n = P_n + D_x u(x^*(t_n) + y_n, P_n).$$

By extracting a subsequence, if necessary, we may assume $x_n \to x$, $\tilde{x}_n \to x$, etc. ■

To see that the lemma implies the proposition, let $x_n^*(t)$ be the optimal trajectory for P with initial conditions (x_n, p_n). Similarly, let $\tilde{x}_n^*(t)$ be the optimal trajectory for P_n with initial conditions $(\tilde{x}_n, \tilde{p}_n)$. Then

$$u(x_n, P) = \int_0^t L(x_n^*, \dot{x}_n^*) + P \cdot \dot{x}_n^* + \overline{H}(P) ds + u(x_n^*(t), P),$$

and

$$u(\tilde{x}_n, P_n) = \int_0^t L(\tilde{x}_n^*, \dot{\tilde{x}}_n^*) + P_n \cdot \dot{\tilde{x}}_n^* + \overline{H}(P_n) ds + u(\tilde{x}_n^*(t), P_n).$$

On $0 \leq t \leq T$ both x_n^* and $\tilde{x}_n^*$ converge uniformly, and, since by hypothesis,

$$u(x_n, P), u(\tilde{x}_n, P_n) \to u(x, P),$$

we conclude that

$$u(\tilde{x}_n^*(t), P_n) - u(x_n^*(t), P) \to 0$$

uniformly on $0 \leq t \leq T$. Therefore

$$u(x^*(t), P_n) - u(x^*(t), P) \to 0$$

uniformly on $0 \leq t \leq T$. □

Given a viscosity solution u of (2) we define the restricted flow corresponding to the differential equation

$$\dot{x} = -D_p H(P + D_x u, x), \tag{9}$$

for all $x \in \pi(\mathcal{I})$ since outside the invariant set this differential equation only defines a semi-flow. In particular, given an ergodic Mather measure μ associated with u, (9) restricted to $\operatorname{supp}(\pi_*\mu)$ defines a flow. We say that the flow (9) is uniquely ergodic if there is only one invariant probability measure.

Theorem 4. *Suppose μ is an ergodic Mather measure associated with a viscosity solution u of (2). Assume that the flow (9) restricted to $\operatorname{supp}(\pi_*\mu)$ is uniquely ergodic. Let $P_n \to P$. Then*

$$u(x, P_n) \to u(x, P),$$

uniformly on the support of $\pi_*\mu$, provided that an appropriate constant $C(P_n)$ is added to $u(x, P_n)$.

Proof. Fix $\epsilon > 0$. We need to show that if n is sufficiently large then

$$\sup_{x \in \operatorname{supp}(\pi_*\mu)} |u(x, P_n) - u(x, P)| < \epsilon.$$

Choose M such that $\|D_x u(x, P)\|, \|D_x u(x, P_n)\| \leq M$. Let $\delta = \frac{\epsilon}{8M}$. Cover $\operatorname{supp} \pi_*\mu$ with finitely many balls B_i with radius $\leq \delta$. Choose (x, p) as in the previous proposition. Let $(x^*(t), p^*(t))$ be the optimal trajectory for P with initial condition (x, p). Then there exists T_δ and $0 \leq t_i \leq T_\delta$ such that $x_i = x^*(t_i) \in B_i$. Choose n sufficiently large such that

$$\sup_{0 \leq t \leq T_\delta} |u(x^*(t), P) - u(x^*(t), P_n)| \leq \frac{\epsilon}{2}.$$

Then, on each y in B_i

$$\begin{aligned} |u(y, P) - u(y, P_n)| \leq |u(y, P) - u(y_i, P)| + |u(y_i, P) - u(y_i, P_n)| + \\ + |u(y_i, P_n) - u(y, P_n)| \leq 4M\delta + \frac{\epsilon}{2} \leq \epsilon. \end{aligned}$$

□

Actually, the unique ergodicity hypothesis is not too restrictive since by Mane's results [15] "most" Mather measures are uniquely ergodic (in the sense that after small generic perturbations to the Lagrangian the restricted flow (9) is uniquely ergodic).

5. Non-degeneracy

In order to the classical change of coordinates (3) to be well defined we need the non-degeneracy condition $\det(I + D^2_{xP}u) \neq 0$. In this section we will show that under the hypothesis that $\overline{H}$ is strictly convex the change of coordinates (3) satisfies a week non-degeneracy condition.

As motivation for our computations consider the following proposition

Proposition 2. *Suppose both $\overline{H}(P)$ and $u(x,P)$ are smooth functions and $\overline{H}(P)$ is strictly convex at P. Suppose μ is an ergodic Mather measure. Then for any vector ξ*

$$c|\xi|^2 \leq \int \left| \left[I + D^2_{xP}u(x,P) \right] \xi \right|^2 d\mu \leq C|\xi|^2, \tag{10}$$

with $0 < c \leq C$. In particular, $0 < \left| \det \left[I + D^2_{xP}u(x(t),P) \right] \right| < \infty$ μ–a.e..

Proof. Let $D_\xi u = D_P u \xi$. Applying $D^2_{\xi\xi}$ to equation (2) we get

$$c \left| I + D^2_{x\xi}u \right|^2 + D_p H D^3_{x\xi\xi}u = D^2_{\xi\xi}\overline{H},$$

since by uniform convexity $D^2_{pp}H > c$. Integrating along a generic trajectory $(x(t), p(t))$ in the support of μ we conclude

$$\int_0^T D_p H D^3_{x\xi\xi}u = O(1),$$

uniformly in T. Thus

$$c|\xi|^2 \leq \lim_{T\to\infty} \frac{1}{T} \int_0^T \left| \left[I + D^2_{xP}u(x(t),P) \right] \xi \right|^2.$$

The proof of the other inequality is similar, using the second derivative bound $D^2_{pp}H \leq C$. ☐

With the help of difference quotients we can make the previous proposition precise in the case where u is a viscosity solution. An analog of the inequality

$$\int \left| \left[I + D^2_{xP}u(x(t),P) \right] \xi \right|^2 d\mu \leq C|\xi|^2$$

was proved in [7]; the next theorem is a slightly different version such estimate.

Theorem 5. *Suppose $\overline{H}(P)$ is twice differentiable at P. Then for almost every y sufficently small (for instance, we may take $|y| \leq |P - P'|^4$)*

$$\limsup_{T\to\infty} \frac{1}{T} \int_0^T |P' + D_x u(x(t) + y, P') - P - D_x u(x(t), P)|^2 dt \leq$$
$$\leq C|P' - P|^2,$$

where $(x(t), p(t))$ is a solution of (1) with initial conditions on $\mathcal{I}$.

Remark. The idea of considering difference quotients in P with "slightly" shifted functions in x has to do with the fact that $u(x(t), P')$ may not be differentiable along $x(t)$. However for almost every $y \in \mathbb{R}^n$ $u(x(t)+y, P')$ will be differentiable for almost every t.

Proof. Note that

$$H(P + D_x u(x(t), P), x(t)) = \overline{H}(P) \tag{11}$$

and for almost every y sufficiently small,

$$H(P' + D_x u(x(t)+y, P'), x(t)) = \overline{H}(P') + O(|P-P'|^2). \tag{12}$$

Subtracting (12) from (11) we obtain the inequality

$$D_p H(P + D_x u(x(t), P), P)\zeta + \theta|\zeta|^2 \leq D_P \overline{H}(P'-P) + C|P-P'|^2,$$

where

$$\zeta = P' + D_x u(x(t)+y, P') - P + D_x u(x(t), P), \tag{13}$$

using the strict convexity of H and twice differentiablity of $\overline{H}$. Observe that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T D_p H(P + D_x u(x(t), P), P)\zeta = D_P \overline{H}(P'-P),$$

since $\dot{x} = D_p H$. Thus

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T |\zeta|^2 \leq C|P-P'|^2,$$

as we claim. □

The converse result is

Theorem 6. *Suppose $\overline{H}(P)$ is strictly convex at P. Then for almost every y sufficently small (for instance $|y| \leq |P-P'|^2$ will do)*

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T |P' + D_x u(x(t)+y, P') - P - D_x u(x(t), P)|^2 dt \geq$$
$$\geq c|P'-P|^2.$$

Proof. Using the notation from the previous theorem and the hypothesis that $\overline{H}$ is strictly convex at P we obtain the inequality

$$D_p H(P + D_x u(x(t), P), P)\zeta + \theta|\zeta|^2 \geq D_P \overline{H}(P'-P) + c|P-P'|^2.$$

Thus, by integration, we conclude

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T |\zeta|^2 \geq c|P-P'|^2,$$

as we claim. □

This last theorem shows that strict convexity of $\overline{H}$ implies that at least in a measure-theoretic sense, the graphs $(x, P + D_x u(x, P))$ and $(x, P' + D_x u(x, P'))$ are distinct on the support of the Mather measures. Therefore the change of coordinates (3) is in a weak sense non-degenerate.

References

1. V. I. Arnold. Mathematical methods of classical mechanics. New York: Springer, 1989. Translated from the 1974 original by K. Vogtmann and A. Weinstein.
2. A. Braides and A. Defranceschi. Homogenization of Multiple Integrals. Oxford Univ. Press, 1998.
3. M. Concordel. Periodic homogenization of Hamilton-Jacobi equations. PhD thesis, UC Berkeley, 1995.
4. Marie C. Concordel. Periodic homogenization of Hamilton-Jacobi equations: additive eigenvalues und variational formula. Indiana Univ. Math. J., **45**(4): 1095–1117, 1996.
5. Marie C. Concordel. Periodic homogenization of Hamilton-Jacobi equations. II. Eikonal equations. Proc. Roy. Soc. Edinburgh Sect. A, **127**(4): 665–689, 1997.
6. E. Weinan. Aubry-Mather theory and periodic solutions of the forced Burgers equation. Comm. Pure Appl. Math., **52**(7): 811–828, 1999.
7. L. C. Evans and D. Gomes. Effective Hamiltonians and averaging for Hamiltonian dynamicsI. Preprint, 1999.
8. Albert Fathi. Solutions KAM faibles conjuguées et barrières de Peierls. C. R. Acad. Sci. Paris Sér. I. Math., **325**(6): 649–652, 1997.
9. Albert Fathi. Théorème KAM faible et théorie de Mather sur les systèmes lagrangiens. C. R. Acad. Sci. Paris Sér. I. Math., **324**(9): 1043–1046, 1997.
10. Albert Fathi. Orbite hétéroclines et ensemble de Peierls. C. R. Acad. Sci. Paris Sér. I. Math., **326**: 1213–1216, 1998.
11. Albert Fathi. Sur la convergence du semi-groupe de Lax-Oleinik. C. R. Acad. Sci. Paris Sér. I. Math., **327**: 267–270, 1998.
12. Wendell H. Fleming and H. Mete Soner. Controlled Markov processes and viscosity solutions. New York: Springer, 1993.
13. P. L. Lions, G. Papanicolaou, and S. R. S. Varadhan. Homogenization of Hamilton-Jacobi equations. Preliminary Version, 1988.
14. John N. Mather. Action minimizing invariant measures for positive definite Lagrangian systems. Math. Z., **207**(2): 169–207, 1991.
15. Ricardo Mañé. Generic properties and problems of minimizing measures of Lagrangian systems. Nonlinearity, **9**(2): 273–310, 1996.