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MEASURABLE SELECTION THEOREMS FOR OPTIMIZATION PROBLEMS

Ulrich Rieder

In this paper we attempt a unification of several selection theorems in the literature by introducing the notion of a selection class and give sufficient conditions for the existence of measurable ε-maximizers. Various special cases are discussed. Finally, as an application of the selection theorem of Kuratowski and Ryll-Nardzewski [12] a Baire classification of ε-maximizers is determined.

1. Introduction and summary

Let $(X,\mathcal{A})$ and $(Y,\mathcal{B})$ be two measurable spaces and let u be a bounded real-valued function on $X\times Y$. Then for any $\varepsilon\geq 0$, a measurable map $f:X\to Y$ is called ε-maximizer (of u) if for all $x\in X$

$$u(x,f(x)) \geq \sup_{Y} u(x,y) - \varepsilon.$$

The question of the existence of ε-maximizers arises naturally, e.g., in dynamic programming problems and in statistical decision theory, both in the case of Bayes procedures and maximum likelihood procedures.

In recent years there have been several papers (see the references and section 9 of the survey article of Wagner [18]) proving the existence of ε-maximizers under various conditions and various methods. The aim of this paper is to unify these results. The present analysis is based on so-called selection classes. It then becomes possible to derive various well-known selection theorems by simply noting that certain

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subsets belong to a selection class. In particular, it turns out that most of all the selection theorems can be deduced from the fundamental selection theorem of Kuratowski and Ryll-Nardzewski. Moreover, the proofs of the main results also appear to be of interest, because they are relatively elementary.

Section 2 deals with selection classes relative to two measurable spaces. Some examples and properties of selection classes are listed. Section 3 contains the statements and proofs of the main results. Several well-known and some new results are derived from our main theorems in section 4. The results are closely related to the fundamental measurable selection theorem of Kuratowski and Ryll-Nardzewski [12]. They may be regarded as slight generalizations and refinements of results given, e.g., by Schäl [14],[15],[16], Blackwell, Freedman and Orkin [1] and Freedman [5] (cf. also Wagner [18] Theorems 9.1 and 9.2). In the final section 5 a Baire classification for ε-maximizers is determined, which provides a refinement of some measurable selection theorems of section 4, e.g. of the theorem of Whitt [19].

2. Selection classes

Throughout the paper let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be any measurable spaces. If C is a subset of $X\times Y$, then pC denotes the projection of C onto X and for $x \in X$, the x-section of C is denoted by $C(x)$. Then we have $p(C\cap X\times B) = \{x \in pC: C(x)\cap B \neq \emptyset\}$ for all subsets B in Y.

A key tool of the present analysis is the concept of a selection class for $(\mathcal{A}, \mathcal{B})$. Let $\mathcal{L}$ be a family of subsets of $X\times Y$. Then $\mathcal{L} = \mathcal{L}(\mathcal{A}, \mathcal{B})$ will be called a <u>selection class for $(\mathcal{A}, \mathcal{B})$</u>, if

(2.1) $C \in \mathcal{L}$ implies $pC \in \mathcal{A}$,

(2.2) Every non-empty set $C \in \mathcal{L}$ admits a measurable selection, i.e. there is a measurable map f (depending on C)

$f:pC \to Y$ with $(x,f(x)) \in C$ for all $x \in pC$.

We do not assume that $\mathcal{L} \subset \mathcal{A} \otimes \mathcal{B}$ (cf. Example 2.5). In general, it is to be noticed that (2.2) does not imply (2.1). Each subclass of a selection class is a selection class. Since every constant map is measurable, the family $\{A \times B: A \in \mathcal{A}, B \subset Y\}$ is always a selection class for $(\mathcal{A}, \mathcal{B})$. Two measurable spaces $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ may have different selection classes (cf. below).

LEMMA 2.1 *If $\mathcal{L}$ is a selection class for $(\mathcal{A}, \mathcal{B})$, then $\mathcal{L}_o := \{ \bigcup C_n : C_n \in \mathcal{L}, n \in \mathbb{N}\}$ is also a selection class for $(\mathcal{A}, \mathcal{B})$.*

Proof. Since $p(\bigcup C_n) = \bigcup p(C_n)$, (2.1) is true. For $n \in \mathbb{N}$ there exists a measurable selection f_n of C_n.

Define $\quad f(x) = f_n(x) \quad$ on $pC_n - (\bigcup_1^{n-1} pC_k)$, $n \in \mathbb{N}$.

Then f is a measurable selection of $\bigcup C_n$. □

Now we list some examples of selection classes. Other examples may be found e.g. in Wagner [18] and the references there or in Hoffmann-Jørgensen [10] chapter III.9.

If X is a topological space, then $\mathcal{B}(X)$ denotes the σ-algebra of Borel subsets of X. A topological Hausdorff space X is called analytic if it is a continuous image of some Polish (complete, separable metric) space. A subset A of X is analytic if it is analytic in the relative topology (cf. [10]). X is called a Borel space if X is a non-empty Borel subset of a Polish space.

EXAMPLE 2.2 If X is a countable space (with the discrete topology), then the family of all subsets of $X \times Y$ is a selection class for $(\mathcal{B}(X), \mathcal{B})$.

EXAMPLE 2.3 (cf. Sainte-Beuve [13], Theorems 3 and 4) Let $(X, \mathcal{A})$ be a measurable space having the property that $\mathcal{A}$ equals its universal completion. Let Y be an analytic space. Then $\mathcal{A} \otimes \mathcal{B}(Y)$ is a selection class for $(\mathcal{A}, \mathcal{B}(Y))$.

The following example is based on the so-called "Fundamental Measurable Selection Theorem".

EXAMPLE 2.4 (cf. Kuratowski and Ryll-Nardzewski [12],p. 398, Himmelberg [7] Theorem 5.1 and Wagner [18] Theorem 4.2(e)) Let Y be a separable metric space. Then the class

$$\{C \in \mathcal{A}\otimes\mathcal{B}(Y):\ C(x) \text{ is complete for } x \in pC \text{ and } p(C \cap X\times K) \in \mathcal{A} \text{ for all compact sets } K \text{ in } Y\}$$

is a selection class for $(\mathcal{A}, \mathcal{B}(Y))$.

EXAMPLE 2.5 (cf. Hoffmann-Jørgensen [10] Theorems III.7.1 and III.9.6) Let X and Y be analytic spaces. Then the class of all analytic subsets of $X\times Y$ is a selection class for $(\sigma(\mathcal{A}(X)), \mathcal{B}(Y))$, where $\sigma(\mathcal{A}(X))$ is the σ-algebra in X that is generated by the analytic subsets $\mathcal{A}(X)$ of X.

EXAMPLE 2.6 (cf. Brown and Purves [2] Theorem 1) If X and Y are Borel spaces, then

$$\{C \in \mathcal{B}(X) \otimes \mathcal{B}(Y):\ C(x) \text{ is } \sigma\text{-compact for } x \in pC\}$$

is a selection class for $(\mathcal{B}(X), \mathcal{B}(Y))$.

Brown and Purves assume that X and Y are Polish spaces. The stated result follows from that theorem by embedding X and Y in their metric completions.

We remark that Example 2.5 and 2.6 can also be derived directly from Example 2.4. For lack of space the proof will here be omitted.

EXAMPLE 2.7 Let X be a topological space and Y be a separable metric space. Then $\{C \in \mathcal{B}(X) \otimes \mathcal{B}(Y): C$ is closed and $C(x)$ is complete for $x \in pC\}$ is a selection class for $(\mathcal{B}(X), \mathcal{B}(Y))$.

Proof. We will apply Example 2.4. We have to show that $p(C \cap X\times K) \in \mathcal{B}(X)$ for all compact sets K in Y. Since the projection from $X\times K$ onto X is a closed map (cf. Kuratowski [11],p.14), the proof is complete. □

EXAMPLE 2.8 (cf. Schäl [14]) Let Y be a separable metric space. Then $\{C \in \mathcal{A}\otimes\mathcal{B}(Y)$: There exists a countable dense subset Y' (depending on C) of Y such that $Y' \cap C(x)$ is dense in $C(x)$, $x \in pC\}$ is a selection class for $(\mathcal{A}, \mathcal{B}(Y))$.

Proof. If ρ denotes the metric in Y, then we have

$$\{x \in X: \rho(C(x),y)<\alpha\} = \bigcup_{y' \in Y', \rho(y',y)<\alpha} \{x \in X: y' \in C(x)\}$$

where $y \in Y$, $\alpha>0$ and $\rho(C(x),y):=\infty$ if $C(x)=\emptyset$. Thus, $x \to \rho(C(x),y)$ is a measurable function on X for all $y \in Y$. By virtue of Himmelberg [7] Theorem 3.3 we get $pC \in \mathcal{A}$. (2.2) follows from Schäl [14] Lemma 3. □

From Example 2.8 one obtains the following special cases.

EXAMPLE 2.9 If Y is a countable space, then $\mathcal{A}\otimes\mathcal{B}(Y)$ is a selection class for $(\mathcal{A}, \mathcal{B}(Y))$.

EXAMPLE 2.10 Let Y be a separable metric space. Then

$$\{C \in \mathcal{A}\otimes\mathcal{B}(Y): C(x) \text{ is open in } Y \text{ for } x \in pC\}$$

is a selection class for $(\mathcal{A}, \mathcal{B}(Y))$. In particular, if X is a topological space and Y is a separable metric space, then the family of all open subsets of X×Y is a selection class for $(\mathcal{B}(X), \mathcal{B}(Y))$.

Proof. If Y' is a countable dense subset of Y, then it is easily verified that $Y' \cap C(x)$ is dense in $C(x)$ for $x \in pC$. □

3. Main results

Throughout the paper let D be a subset of X×Y and $u: D \to \overline{\mathbb{R}}$. Let $v: pD \to \overline{\mathbb{R}}$ be defined by

$$v(x) := \sup_{y \in D(x)} u(x,y), \quad x \in pD.$$

Let $\varepsilon>0$. A measurable map $f: pD \to Y$ is called ε-maximizer (of u) if for all $x \in pD$

(3.1) $$f(x) \in D(x)$$

(3.2) $$u(x,f(x)) \geq v(x) - \varepsilon, \text{ if } v(x) < \infty$$
$$\geq 1/\varepsilon \quad , \text{ if } v(x) = \infty.$$

A measurable map $f: pD \to Y$ is called <u>maximizer</u> (of u) if

(3.3) $$f(x) \in D(x) \quad \text{and} \quad u(x,f(x)) = v(x), \; x \in pD.$$

Throughout this section let $\mathcal{L}$ be any selection class.

<u>THEOREM 3.1</u> *Suppose* (i) $D \in \mathcal{L}$ *and*
(ii) $\{(x,y) \in D: u(x,y) > c\} \in \mathcal{L}$, $c \in \mathbb{R}$.
Then v *is measurable, and for every* $\varepsilon > 0$ *there exists an* ε-*maximizer*.

<u>Proof.</u> For $c \in \mathbb{R}$, we have

(3.4) $$\{x \in pD: v(x) > c\} = p(\{(x,y) \in D: u(x,y) > c\}).$$

In view of (ii) the last set belongs to $\mathcal{A}$, hence v is measurable. Let $\varepsilon > 0$ be given. For all integers k let $C_k := \{(x,y) \in D: u(x,y) > k\varepsilon\}$. C_k belongs to $\mathcal{L}$ by (ii), and $C_k \supset C_{k+1} \ldots$. Let $A_k := pC_k$. Then $A_k \in \mathcal{A}$ by (2.1) and $A_k \supset A_{k+1} \ldots$. By the first part, the sets

$$A_\infty := \{x \in pD: v(x) = \infty\} \text{ and } A_{-\infty} := \{x \in pD: v(x) = -\infty\}$$

belong also to $\mathcal{A}$. Since $C_k \in \mathcal{L}$, there exists a measurable selection f_k of C_k, and by (i) there is a measurable selection g of D. Let k_o be an integer such that $k_o \geq 1/\varepsilon^2$.
Define

$$f(x) = \begin{cases} f_k(x) & \text{for } x \in A_k - A_{k+1}, \; k \in \mathbb{Z}^{1} \\ f_{k_o}(x) & \text{for } x \in A_\infty \\ g(x) & \text{for } x \in A_{-\infty} \end{cases}$$

The map $f: pD \to Y$ is measurable and $(x,f(x)) \in D$ for all $x \in pD$. This map is also an ε-maximizer of u. Fix $k \in \mathbb{Z}$ and $x \in A_k - A_{k+1}$. Then $u(x,y) \leq (k+1)\varepsilon$ for all $y \in D(x)$, because $x \notin A_{k+1}$, and therefore $v(x) \leq (k+1)\varepsilon$. This implies

$$u(x,f(x)) = u(x,f_k(x)) > k\varepsilon \geq v(x) - \varepsilon.$$

If $x \in A_{-\infty}$, then $u(x,y) = -\infty$ for all $y \in D(x)$ and $u(x,f(x)) = v(x)$. Finally, if $x \in A_\infty$, then $u(x,f(x)) = u(x,f_{k_o}(x)) > k_o\varepsilon \geq 1/\varepsilon$. □

<u>COROLLARY 3.2</u> *Suppose* (i) $D \in \mathcal{L}$ *and*
(ii) $\{(x,y) \in D: u(x,y) \geq c\} \in \mathcal{L}$, $c \in \mathbb{R}$.
Then v *is measurable, and for every* $\varepsilon > 0$ *there exists an* ε-*maximizer*.

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[1] Let $\mathbb{Z}$ denote the set of integers.

Proof. From Lemma 2.1 we know that

$\mathcal{L}_o := \{\bigcup C_n : C_n \in \mathcal{L},\ n \in \mathbb{N}\}$ is also a selection class for $(\mathcal{A}, \mathcal{B})$. Since $D \in \mathcal{L}_o$ and

$$\{(x,y) \in D: u(x,y) > c\} = \bigcup \{(x,y) \in D: u(x,y) \geq c+1/n\}$$

belongs to $\mathcal{L}_o$, the assertions follow from Theorem 3.1. □

If v is finite, then the second result of Theorem 3.1 can be slightly extended by replacing the positive real number ε by a function from pD into $(0,\infty)$, e.g. one obtains

THEOREM 3.3 *Let* $\varepsilon: pD \to (0,\infty)$ *be any function. Suppose* (i) $D \in \mathcal{L}$ *and* (ii) $\{(x,y) \in D: u(x,y) > c\varepsilon(x)\} \in \mathcal{L}$, $c \in \mathbb{R}$. *If* $v < \infty$ on pD *then there exists a measurable selection* f *of* D *such that*

$$u(x,f(x)) \geq v(x) - \varepsilon(x),\ x \in pD.$$

The proof follows along the same lines as the proof of Theorem 3.1 by defining the sets C_k by $\{(x,y) \in D: u(x,y) > k\varepsilon(x)\}$. The set A_∞ is empty. □

COROLLARY 3.4 *Theorem* 3.3 *remains true if* (ii) *is replaced by* (ii)' $\{(x,y) \in D: u(x,y) \geq c\varepsilon(x)\} \in \mathcal{L}$, $c \in \mathbb{R}$.

Now we are interested in maximizers of u.

THEOREM 3.5 *Suppose*

(i) Y *is a separable metric space,* $\mathcal{B} = \mathcal{B}(Y)$,

(ii) $D \in \mathcal{L}$,

(iii) $\{(x,y) \in D: u(x,y) > c\} \in \mathcal{L}$, $c \in \mathbb{R}$,

(iv) $\{y \in D(x): u(x,y) \geq c\}$ *is compact,* $c \in \mathbb{R}$, $x \in pD$.

Then v *is measurable and there exists a maximizer.*

Proof. By (ii) there is a measurable selection g of D. For $n \in \mathbb{N}$ let

$$M_n(x) = \begin{cases} \{y \in D(x): u(x,y) \geq v(x) - 1/n\}, & \text{if } v(x) \in \mathbb{R} \\ \{y \in D(x): u(x,y) \geq n\}, & \text{if } v(x) = \infty \\ \{g(x)\}, & \text{if } v(x) = -\infty \end{cases}$$

From the proof of Theorem 3.1 we get the existence of measurable maps $f_n : pD \to Y$ with $f_n(x) \in M_n(x)$ for all $x \in pD, n \in \mathbb{N}$. We conclude from Lemma 4 in Schäl [14] together with (iv) that there exists a measurable map $f : pD \to Y$ such that

$$f(x) \in \bigcap M_n(x), \; x \in pD$$

which completes the proof. □

Theorem 3.6 is a slight generalization of Theorem 3.5.

<u>THEOREM 3.6</u> *Suppose*

(i) Y *is a separable metric space,* $\mathcal{B} = \mathcal{B}(Y)$,

(ii) *There are sets* D_n *such that* $D = \bigcup D_n$ *and* $D_n \in \mathcal{L}$,

(iii) $\{(x,y) \in D_n : u(x,y) > c\} \in \mathcal{L}$, $c \in \mathbb{R}$, $n \in \mathbb{N}$,

(iv) $\{y \in D_n(x) : u(x,y) \geq c\}$ *is compact,* $c \in \mathbb{R}$, $x \in pD$, $n \in \mathbb{N}$,

(v) $u(x,\cdot)$ *attains its supremum on* $D(x)$, $x \in pD$,

Then v *is measurable and there exists a maximizer.*

<u>Proof.</u> For $n \in \mathbb{N}$ let $v_n(x) = \sup_{y \in D_n(x)} u(x,y)$, $x \in pD_n$.

By Theorem 3.5 there exists a measurable map $f_n : pD_n \to Y$ such that for all $x \in pD_n$, $(x, f(x)) \in D_n$ and $u(x, f_n(x)) = v_n(x)$. By assumption (ii), D belongs to $\mathcal{L}_o$ (see Lemma 2.1) and by (iii), $\{(x,y) \in D : u(x,y) > c\} \in \mathcal{L}_o$ for all $c \in \mathbb{R}$. In view of Lemma 2.1 we get from Theorem 3.1 that v is measurable on pD. Define $A_n := \{x \in pD_n : v_n(x) = v(x)\}$. Then $A_n \in \mathcal{A}$, and by assumption (v), $\bigcup A_n = pD$. Define

$$f(x) = f_n(x) \qquad \text{on } A_n - (A_1 \cup \ldots \cup A_{n-1})$$

for $n \in \mathbb{N}$. Then f is a maximizer of u. □

<u>COROLLARY 3.7</u> *Theorem* 3.5 *and Theorem* 3.6 *remain true if condition* (iii) *is replaced by* $\{(x,y) \in D : u(x,y) \geq c\} \in \mathcal{L}$, *and* $\{(x,y) \in D_n : u(x,y) \geq c\} \in \mathcal{L}$, $c \in \mathbb{R}$, $n \in \mathbb{N}$, *respectively.*

4. Special cases

In this section several well-known and some new selection theorems are derived from our main results.

For $c \in \mathbb{R}$, define $U_c := \{(x,y) \in D\colon u(x,y) \geq c\}$ and $U_{-\infty} := D$.

Theorem 4.1 is a direct extension of the "Fundamental Measurable Selection Theorem" (cf. Kuratowski and Ryll-Nardzewski [12], p. 398, Himmelberg [7] Theorem 5.1 and Wagner [18] Theorem 4.1).

THEOREM 4.1 *Suppose*
(i) Y *is a separable metric space,* $\mathcal{B} = \mathcal{B}(Y)$,
(ii) *For all* $c \in [-\infty, \infty)$, U_c *belongs to* $\mathcal{A} \otimes \mathcal{B}(Y)$, $U_c(x)$ *is complete for* $x \in pD$ *and* $p(U_c \cap X \times K) \in \mathcal{A}$ *for all compact sets* K *in* Y.

Then: (a) v *is measurable and for every* $\varepsilon > 0$ *there exists an* ε*-maximizer.*

(b) *If* $U_c(x)$ *is compact for* $x \in pD$ *and* $c \in \mathbb{R}$, *then there exists a maximizer.*

Proof. Apply Example 2.4 and Corollaries 3.2 and 3.7. □

Remark. It can be shown (cf. the proof of Theorem 5.2) that there exists a set $C_\varepsilon \subset D$ such that $C_\varepsilon(x)$ is complete for $x \in pD$,

$$C_\varepsilon(x) \subset \{y \in D(x)\colon u(x,y) \geq v(x) - \varepsilon\}, \text{ if } v(x) < \infty$$
$$C_\varepsilon(x) \subset \{y \in D(x)\colon u(x,y) \geq 1/\varepsilon\} \quad , \text{ if } v(x) = \infty$$

and $p(C_\varepsilon \cap X \times G) \in \mathcal{A}$ for all open sets G in Y. Then, a direct application of the "Fundamental Measurable Selection Theorem" would also yield the existence of an ε-maximizer.

Since Example 2.5 and Example 2.6 can be deduced from Example 2.4, we get the following Corollaries of Theorem 4.1. Corollary 4.2 is a slight generalization of Theorem (43) in Blackwell, Freedman and Orkin [1] and Theorem 2.5 in Shreve [17].

COROLLARY 4.2 *Suppose*

(i) X *and* Y *are analytic spaces,* $\mathcal{A}:=\sigma(\mathcal{A}(X))$, $\mathcal{B}=\mathcal{B}(Y)$,

(ii) *For all* $c \in [-\infty,\infty)$, U_c *is an analytic subset of* $X\times Y$.

Then the statements of Theorem 4.1(a) *are valid. If also* Y *is metric, then the statement of Theorem* 4.1(b) *is valid.*

COROLLARY 4.3 *Suppose*

(i) X *and* Y *are Borel spaces,* $\mathcal{A}=\mathcal{B}(X)$, $\mathcal{B}=\mathcal{B}(Y)$,

(ii) *For all* $c \in [-\infty,\infty)$, $U_c \in \mathcal{B}(X)\otimes\mathcal{B}(Y)$ *and* $U_c(x)$ *is* σ*-compact for* $x \in pD$.

Then the statements of Theorem 4.1 *are valid.*

Assumption (ii) of Corollary 4.3 is satisfied, e.g., if $D \in \mathcal{B}(X)\otimes\mathcal{B}(Y)$, $D(x)$ is σ-compact for $x \in pD$, u is measurable on D and $u(x,\cdot)$ is upper semi-continuous on $D(x)$ for $x \in pD$ (cf. Brown and Purves [2] Corollary 1, Himmelberg, Parthasarathy and van Vleck [8] Theorem 2).

The next result is due to Schäl ([14] Theorem 2, [15] Theorem 12.1). Here we will give another proof.

COROLLARY 4.4 *Suppose*

(i) Y *is a separable metric space,* $\mathcal{B}=\mathcal{B}(Y)$,

(ii) $D \in \mathcal{A}\otimes\mathcal{B}(Y)$, $D(x)$ *is compact for* $x \in pD$ *and* $p(D \cap X\times G) \in \mathcal{A}$ *for all open sets* G *in* Y,

(iii) u *is the limit of a decreasing sequence of Carathéodory maps.*

Then there exists a maximizer.

Proof. Let $c \in \mathbb{R}$. Let (u_n) be the decreasing sequence of Carathéodory maps, i.e. u_n is measurable on D and $u_n(x,\cdot)$ is continuous on $D(x)$ for $x \in pD$, and $u_n \downarrow u$. We may assume that $u_n \neq u$ for all $n \in \mathbb{N}$. Define for $n \in \mathbb{N}$

$M_n(x) := \mathrm{Cl}\{y \in D(x) : u_n(x,y) > c\}$ [2] and $M_n := \{(x,y) \in D : y \in M_n(x)\}$

Then $M_n \supset M_{n+1}$ and $U_c = \bigcap M_n$. It is clear that $U_c(x)$ is compact for $x \in pD$. In view of Theorem 4.1(b) we have to show that $p(U_c \cap X\times K) \in \mathcal{A}$ for all compact sets K in Y. Let F be a closed

[2] ClB is the closure of B.

subset of Y. If F' is a countable dense subset of F, then

$$p(\{(x,y) \in D: u_n(x,y)>c\} \cap X{\times}F) =$$
$$= \{x \in pD: u_n(x,y)>c \text{ for some } y \in F\}$$
$$= \{x \in pD: u_n(x,y)>c \text{ for some } y \in F'\}$$
$$= \bigcup_{y \in F'} \{x \in pD: u_n(x,y)>c\} \in \mathcal{A}, \quad n \in \mathbb{N}.$$

By Himmelberg [7] Proposition 2.6 and Theorem 3.1, we get $p(M_n \cap X{\times}F) \in \mathcal{A}$ for $n \in \mathbb{N}$, and hence $p(U_c \cap X{\times}F) \in \mathcal{A}$ by Himmelberg [7] Theorem 4.1. The proof is complete. □

Theorem 4.5 generalizes Lemma 3 in Evstigneev [4].

<u>THEOREM 4.5</u> *Suppose*

(i) $(X, \mathcal{A})$ *is a measurable space with the property that* $\mathcal{A}$ *equals its universal completion,*

(ii) Y *is an analytic space,* $\mathcal{B} = \mathcal{B}(Y)$,

(iii) $D \in \mathcal{A} \otimes \mathcal{B}(Y)$,

(iv) u *is measurable on* D.

Then: (a) v *is measurable and for every* $\varepsilon>0$ *there exists an* ε*-maximizer.*

(b) *If also* Y *is metric and* $U_c(x)$ *is compact for* $x \in pD$ *and* $c \in \mathbb{R}$, *then there exists a maximizer.*

<u>Proof.</u> Apply Example 2.3 and Theorems 3.1 and 3.5. □

If Y is a σ-compact metric space, then it suffices to assume that the x-sections of U_c are closed.

<u>THEOREM 4.6</u> *Suppose*

(i) Y *is a* σ*-compact metric space,* $\mathcal{B} = \mathcal{B}(Y)$,

(ii) *For all* $c \in [-\infty,\infty)$, U_c *belongs to* $\mathcal{A} \otimes \mathcal{B}(Y)$, $U_c(x)$ *is closed for* $x \in pD$ *and* $p(U_c \cap X{\times}K) \in \mathcal{A}$ *for all compact sets* K *in* Y.

Then: (a) v *is measurable and for every* $\varepsilon>0$ *there exists an* ε*-maximizer.*

(b) *If* $u(x,\cdot)$ *attains its supremum on* $D(x)$ *for* $x \in pD$, *then there exists a maximizer.*

Proof. Let $Y=\bigcup Y_n$ where each Y_n is compact. Define

$$U_{nc}:=U_c \cap X\times Y_n \ , \ n\in\mathbb{N}, \ c\in[-\infty,\infty).$$

By Himmelberg [7] Theorem 4.1, $p(U_{nc}\cap X\times G)\in\mathcal{A}$ for all open sets G in Y and $n\in\mathbb{N}$. If $\mathcal{L}$ denotes the selection class in Example 2.4, then for all $c\in[-\infty,\infty)$, U_c belongs to the selection class $\mathcal{L}_o:=\{\bigcup C_n: C_n\in\mathcal{L}, n\in\mathbb{N}\}$. An appeal to Corollary 3.2 and Corollary 3.7 completes the proof. □

Assumption (ii) of Theorem 4.6 is satisfied, e.g., if $D\in\mathcal{A}\otimes\mathcal{B}(Y)$, $D(x)$ is closed for $x\in pD$ and $p(D\cap X\times K)\in\mathcal{A}$ for all compact subsets K of Y and u is the limit of a decreasing sequence of Carathéodory maps (cf. the proof of Corollary 4.4 and Schäl [14] Theorem 3).

COROLLARY 4.7 *Suppose*

(i) *X is a topological space,* $\mathcal{A}=\mathcal{B}(X)$,

(ii) *Y is a σ-compact metric space,* $\mathcal{B}=\mathcal{B}(Y)$,

(iii) U_c *is closed for* $c\in[-\infty,\infty)$.

Then the statements of Theorem 4.6 are valid.

Theorem 4.8 is an extension of Corollary (15) in [5].

THEOREM 4.8 *Suppose*

(i) *X is a topological space,* $\mathcal{A}=\mathcal{B}(X)$,

(ii) *Y is a Polish space,* $\mathcal{B}=\mathcal{B}(Y)$,

(iii) *D and* $\{(x,y)\in D: u(x,y)>c\}$ *are* F_σ*-sets in* $X\times Y$, $c\in\mathbb{R}$.

Then: (a) $\{x\in pD: v(x)>c\}$ *is an* F_σ*-set in X, if Y is compact.*

(b) *For each* $\varepsilon>0$ *there exists an* ε*-maximizer.*

(c) *If* $U_c(x)$ *is compact for* $x\in pD$ *and* $c\in\mathbb{R}$, *then there exists a maximizer.*

Proof. The family $\{C\in\mathcal{B}(X)\otimes\mathcal{B}(Y): C \text{ is an } F_\sigma\text{-set}\}$ is a selection class for $(\mathcal{B}(X),\mathcal{B}(Y))$, by virtue of Example 2.7 and Lemma 2.1. Then the statements follow from (3.4), Theorem 3.1 and Theorem 3.5. □

If assumption (ii) is replaced by

(ii)' Y is a σ-compact metric space, $\mathcal{B}=\mathcal{B}(Y)$

then Theorem 4.8 remains true. The proof is analogous to the proof of Theorem 4.8.

Theorem 4.9 is also easily derived from Example 2.7. It is due to Dubins and Savage [3] (cf. Hinderer [9] Theorem 17.9, Schäl [14] Corollary 4).

THEOREM 4.9 *Suppose*

(i) X *is a topological space*, $\mathcal{A} = \mathcal{B}(X)$,

(ii) Y *is a separable metric space*, $\mathcal{B} = \mathcal{B}(Y)$,

(iii) *For* $c \in [-\infty,\infty)$, U_c *is closed and* $U_c(x)$ *is compact for* $x \in pD$.

Then there exists a maximizer.

Theorem 4.10 is a variation of Theorem 1 in Schäl [14], but the proof is new.

THEOREM 4.10 *Suppose*

(i) Y *is a separable metric space*, $\mathcal{B} = \mathcal{B}(Y)$,

(ii) $D \in \mathcal{A} \otimes \mathcal{B}(Y)$ *and there exists a countable dense subset* Y' *of* Y *such that* $Y' \cap D(x)$ *is dense in* $D(x)$, $x \in pD$,

(iii) u *is measurable on* D *and* $u(x,\cdot)$ *is lower semi-continuous on* $D(x)$ *for* $x \in pD$.

Then v *is measurable and for every* $\varepsilon>0$ *there exists an* ε*-maximizer.*

Proof. Let $c \in \mathbb{R}$. The set $\{(x,y) \in D: u(x,y)>c\}$ is measurable and for $x \in pD$, $Y' \cap \{y \in D(x): u(x,y)>c\}$ is dense in $\{y \in D(x): u(x,y)>c\}$, since the last set is open in $D(x)$ by (iii). Then the statements follow from Example 2.8 and Theorem 3.1. $\square$

COROLLARY 4.11 *Suppose*

(i) Y *is a separable metric space*, $\mathcal{B} = \mathcal{B}(Y)$,

(ii) $D \in \mathcal{A} \otimes \mathcal{B}(Y)$ *and* $D(x)$ *is open in* Y *for* $x \in pD$,

(iii) u *is measurable on* D *and* $u(x,\cdot)$ *is lower semi-continuous on* $D(x)$ *for* $x \in pD$.

Then v *is measurable and for every* $\varepsilon>0$ *there exists an* ε*-maximizer.*

5. Baire classification of ε-maximizers

In this section a Baire classification for ε-maximizers is determined which provides a refinement of Theorem 4.1.

Let X and Y be metric spaces. According to a well-known definition, a map $f: X \to Y$ is called of Baire class α (where $\alpha < \Omega$), if for each open set G in Y the set $f^{-1}(G)$ is a Borel set of additive class α [3] in Y (cf. Kuratowski and Ryll-Nardzewski [12]).

For the proof of Theorem 5.2 we will use the following trivial modification of the selection theorem on p. 401 of Kuratowski and Ryll-Nardzewski [12]. Theorem 5.1 follows from that theorem by embedding Y in a complete space.

THEOREM 5.1 (cf. Kuratowski and Ryll-Nardzewski [12] p.401) *Let* X *be a metric and* Y *be a separable metric space. If* C *is a subset of* $X \times Y$ *such that* $C(x)$ *is complete for* $x \in pC$ *and* $p(C \cap X \times G)$ *is a Borel set of additive class* α *(where* $\alpha > 0$*) for all open sets* G *in* Y*, then there exists a selector* f *of* C *and* f *is of Baire class* α.

THEOREM 5.2 *Suppose*

(i) X *is a metric space,* $\mathcal{A} = \mathcal{B}(X)$,

(ii) Y *is a separable metric space,* $\mathcal{B} = \mathcal{B}(Y)$,

(iii) *For all* $c \in [-\infty, \infty)$, U_c *belongs to* $\mathcal{B}(X) \otimes \mathcal{B}(Y)$, $U_c(x)$ *is complete for* $x \in pD$ *and* $p(U_c \cap X \times F)$ *is a Borel set of multiplicative class* $\alpha \geq 0$ *for all closed sets* F *in* Y,

(iv) v *is finite.*

Then: (a) *For each* $\varepsilon > 0$ *there exists a set* $C_\varepsilon \subset D$ *such that* $C_\varepsilon(x)$ *is complete for* $x \in pD$,

(5.1) $$C_\varepsilon(x) \subset \{y \in D(x) : u(x,y) \geq v(x) - \varepsilon\}, \quad x \in pD,$$

and $p(C_\varepsilon \cap X \times G)$ *is a Borel set of additive class* $\alpha+1$ *for all open sets* G *in* Y.

---

[3] Let us recall that the open (closed) sets are Borel sets of additive (multiplicative) class 0; F_σ-sets (G_δ-sets) are of additive (multiplicative) class 1, etc..

(b) *For every* $\varepsilon>0$ *there exists an* ε*-maximizer of Baire class* $\alpha+1$.

(c) *If* $U_c(x)$ *is compact for* $x \in pD$ *and* $c \in \mathbb{R}$, *then there exists a maximizer of Baire class* $\alpha+3$.

Proof. (a) For all integers k let $A_k := pU_{k\varepsilon}$. Then A_k is a Borel set of multiplicative class α by (iii), and $A_k \supset A_{k+1}$. Since X is a metric space, $A_k - A_{k+1}$ is a Borel set of additive class $\alpha+1$. This follows from the fact that each subset of X which is a Borel set of either additive or multiplicative class α, is both of additive and multiplicative class $\alpha+1$ (cf. Hausdorff [6], p. 86-87).

Define $C_\varepsilon(x) := U_{k\varepsilon}(x)$ for $x \in A_k - A_{k+1}$, $k \in \mathbb{Z}$, and
$C_\varepsilon := \{(x,y) \in D\colon\ y \in C_\varepsilon(x)\}$.

Then $C_\varepsilon(x)$ is complete and $C_\varepsilon(x)$ satisfies (5.1) for $x \in pD$. Since

$$p(C_\varepsilon \cap X{\times}G) = \bigcup_k (A_k - A_{k+1}) \cap p(U_{k\varepsilon} \cap X{\times}G) \tag{5.2}$$

it follows that $p(C_\varepsilon \cap X{\times}G)$ is a Borel set of additive class $\alpha+1$ for all open sets G in Y. Thus C_ε satisfies the conditions of (a).

(b) The statement follows from (a) and Theorem 5.1.

(c) The sequence of sets $C_{1/n}$ decreases monotonically to the set

$$\bigcap C_{1/n} = \{(x,y) \in D\colon\ u(x,y) = v(x)\} =: M .$$

By assumption, M is not empty and M(x) is compact for $x \in pD$. We claim that

$$p(M \cap X{\times}F) = \bigcap p(C_{1/n} \cap X{\times}F) \quad \text{for all closed } F. \tag{5.3}$$

Let $x \in \bigcap p(C_{1/n} \cap X{\times}F)$. Then, for every n $C_{1/n}(x) \cap F$ is compact in $C_1(x)$ and not empty. Hence $\bigcap C_{1/n}(x) \cap F$ is compact and not empty, i.e. $x \in p(M \cap X{\times}F)$.

The converse is obvious because of (iv).

If F is closed in Y, then $p(C_{1/n} \cap X{\times}F)$ is a Borel set of multiplicative class $\alpha+2$ (cf. the equality (5.2)) and hence by (5.3) $p(M \cap X{\times}F)$ belongs to this class. Consequently, $p(M \cap X{\times}G)$ is a Borel set of additive class $\alpha+3$ for all open sets G in Y. Now the assertion follows from Theorem 5.1. $\square$

Remarks. (1) Assumption (iii) of Theorem 5.2 is satisfied, e.g., if for all $c \in [-\infty,\infty)$, U_c is closed and Y is compact. This case (and α=0) has been investigated by Whitt [19].

(2) It is easily shown that in Theorem 5.2(c) the set $\{x \in pD: v(x) \geq c\}$ is a Borel set of multiplicative class α for $c \in \mathbb{R}$.

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Ulrich Rieder
Institut für Mathematische Statistik
Universität Karlsruhe
Kaiserstraße 12
D-7500 Karlsruhe
Federal Republik of Germany

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