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LINEAR QUADRATIC DIFFERENTIAL GAMES: CLOSED LOOP SADDLE POINTS*

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Abstract. The object of this paper is to revisit the results of Bernhard [*J. Optim. Theory Appl.*, 27 (1979), pp. 51–69] on two-person zero-sum linear quadratic differential games and generalize them to utility functions without positivity assumptions on the matrices acting on the state variable and to linear dynamics with bounded measurable data matrices. Our paper specializes to *state feedback* via Lebesgue measurable *affine closed loop strategies* with possible non-L^2-integrable singularities. After sharpening the recent results of Delfour [*SIAM J. Control Optim.*, 46 (2007), pp. 750–774] on the characterization of the open loop lower and upper values of the game, it first deals with L^2-integrable closed loop strategies and then with the larger family of strategies that may have non-L^2-integrable singularities. A new conceptually meaningful and mathematically precise definition of a closed loop saddle point is introduced to simultaneously handle state feedbacks of the L^2 type and smooth locally bounded ones, except at most in the neighborhood of finitely many instants of time. A necessary and sufficient condition is that the free end problem be *normalizable almost everywhere*. This relaxation of the classical notion allows singularities in the feedback law at an infinite number of instants, including accumulation points that are not isolated. A complete classification of closed loop saddle points is given in terms of the convexity/concavity properties of the utility function, and connections are given with the open loop lower value, upper value, and value of the game.

Key words. linear quadratic differential game, two person, zero sum, saddle point, value of a game, Riccati differential equation, open loop and closed loop strategies, integrable singularities

AMS subject classifications. 91A05, 91A23, 49N70, 91A25

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1. Introduction. The object of this paper is to revisit the pioneering work of Bernhard [2, 3] on two-person zero-sum linear quadratic differential games and generalize it to utility functions without positivity assumptions on the matrices acting on the state variable and to linear dynamics with bounded measurable data matrices. Our paper specializes to *state feedback* via Lebesgue measurable *affine closed loop strategies* with possible non-L^2-integrable singularities. After sharpening the recent results of Delfour [5] on the characterization of the open loop lower and upper values of the game in section 2, it first deals with L^2-integrable closed loop strategies and then with the larger family of strategies that may have non-L^2-integrable singularities.

In section 3 several equivalent necessary and sufficient conditions are given for the existence of a closed loop saddle point with respect to L^2-integrable affine closed loop strategies, for instance, the *normality* of the problem; the existence of an $H^1(0,T)$ solution to the associated matrix Riccati differential equation. It was shown in [5] that the existence of a solution to the coupled state-adjoint state system is a necessary condition for the existence of a finite open loop lower value, upper value, or value of the game, and that the difference essentially depends on the convexity of the utility function with respect to the control of the minimizing player and on its concavity with respect to the control of the maximizing player. This condition is also necessary

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for the existence of a closed loop saddle point. It leads to a complete classification in terms of the convexity/concavity properties of the utility function.

Section 4 deals with two delicate issues. The first is the very definition of a closed loop saddle point in the presence of closed loop strategies with non-L^2-integrable singularities. As was pointed out in [2, p. 68 and Remark 5.1] such strategies may lead to conflicting terms that simultaneously blow up in the utility function. Under the positivity assumptions, one may possibly get around this problem by setting the utility function equal to $\pm\infty$, but we do not have them here. So we had to introduce a new conceptually meaningful and mathematically precise definition (cf. Definition 4.5). It states that the original problem can be transformed via feedback in such a way that the resulting problem has an open loop saddle point at $(0, 0)$. The second related issue is to specify the class of affine closed loop strategies (cf. Definition 4.3) in such a way that we can simultaneously handle, in the same framework, L^2-integrable closed loop strategies and smooth locally bounded ones, except at most in the neighborhood of finitely many instants of time as in [2].

It turns out that the classical definition of a closed loop saddle point (cf. Definition 3.2) can be an *undeterminate* or a *degenerate* one when either the open loop lower or upper value of the game is not finite (cf. Theorems 4.4 and 4.5). For instance, Berkovitz's equivalence [1] (see our Lemma 3.2) may not apply, as shown in Example 4.2. The proper point of view is that of Definition 4.1, which states that the two closed loop strategies cannot be chosen independently. They must be linked through the *admissibility condition* of Definition 4.3. This subtle difference fundamentally changes the nature of the problem and makes it different from the classical theory of saddle points with respect to two independent sets. We show that the slight relaxation of the definition of *normalizability* of the free end problem in the sense of [2, Definition 3.2] from isolated instants to a set of instants of zero measure is a necessary and sufficient condition for the existence of a closed loop saddle point. This relaxation of the classical notion allows singularities in the feedback law at an infinite number of instants, including accumulation points that are not isolated. This condition is also used to make sense of solutions with singularities to the matrix Riccati differential equation.

In section 4.7, we show that under the convexity-concavity condition, Definitions 3.2 and 4.5 of closed loop saddle points coincide and that closed loop strategies with non-L^2-integrable singularities are useless. These singularities naturally occur when either the open loop lower or upper value of the game is not finite. We complete the classification of closed loop saddle points in section 4.8 along with conditions expressed in terms of the convexity/concavity properties of the utility function. We conclude in section 4.9 with an example of a nonnormalizable problem with finite open loop lower value that can be achieved by state feedback via a solution of the matrix Riccati differential equation.

2. Definitions, notation, and main results.

2.1. System, utility function, values of the game. Given a finite dimensional Euclidean space $\mathbf{R}^d$ of dimension $d \geq 1$, the *norm* and *inner product* will be denoted by $|x|$ and $x \cdot y$, respectively, and irrespective of the dimension d of the space. Given $T > 0$, the norm and inner product in $L^2(0, T; \mathbf{R}^n)$ will be denoted $\|f\|$ and (f, g). The norm in the Sobolev space $H^1(0, T; \mathbf{R}^n)$ will be written $\|f\|_{H^1}$.

Consider the following two-player zero-sum game over the finite time interval

$[0,T]$ characterized by the quadratic *utility function*

$$C_{x_0}(u,v) \overset{\text{def}}{=} Fx(T)\cdot x(T) + \int_0^T Q(t)x(t)\cdot x(t) + |u(t)|^2 - |v(t)|^2\,dt, \tag{2.1}$$

where x is the solution of the linear differential system

$$x'(t) = A(t)x(t) + B_1(t)u(t) + B_2(t)v(t) \quad \text{a.e. in } [0,T], \quad x(0) = x_0, \tag{2.2}$$

$x_0 \in \mathbf{R}^n$ is the *initial state* at time $t = 0$, $u \in L^2(0,T;\mathbf{R}^m)$, $m \geq 1$, is the strategy of the first player, and $v \in L^2(0,T;\mathbf{R}^k)$, $k \geq 1$, is the strategy of the second player. We assume that F is an $n \times n$-matrix and that A, B_1, B_2, and Q are matrix-functions of appropriate orders that are measurable and bounded almost everywhere in $[0,T]$. Moreover, $Q(t)$ and F are symmetrical. It will be convenient to use the following compact notation and drop the a.e. in $[0,T]$ wherever no confusion arises:

$$C_{x_0}(u,v) = Fx(T)\cdot x(T) + \int_0^T Qx\cdot x + |u|^2 - |v|^2\,dt, \tag{2.3}$$

$$x' = Ax + B_1u + B_2v \quad \text{in } [0,T], \quad x(0) = x_0. \tag{2.4}$$

The above assumptions on F, A, B_1, B_2, and Q will be used throughout this paper. The transpose of a matrix M will be denoted $M^\top$, the inverse of its transpose $M^{-\top}$, and $R(t)$ will denote the matrix $B_1(t)B_1(t)^\top - B_2(t)B_2(t)^\top$.

DEFINITION 2.1. *Let x_0 be an initial state in $\mathbf{R}^n$ at time $t = 0$.*

(i) *The game is said to achieve its* open loop lower value *(resp.,* upper value*) if*

$$v^-(x_0) \overset{\text{def}}{=} \sup_{v\in L^2(0,T;\mathbf{R}^k)} \inf_{u\in L^2(0,T;\mathbf{R}^m)} C_{x_0}(u,v) \tag{2.5}$$

$$\left(\textit{resp., } v^+(x_0) \overset{\text{def}}{=} \inf_{u\in L^2(0,T;\mathbf{R}^m)} \sup_{v\in L^2(0,T;\mathbf{R}^k)} C_{x_0}(u,v)\right) \tag{2.6}$$

is finite. By definition, $v^-(x_0) \leq v^+(x_0)$.

(ii) *The game is said to achieve its* open loop value *if its open loop lower value $v^-(x_0)$ and upper value $v^+(x_0)$ are achieved and $v^-(x_0) = v^+(x_0)$. The* open loop value *of the game will be denoted by $v(x_0)$.*

(iii) *A pair $(\bar{u},\bar{v})$ in $L^2(0,T;\mathbf{R}^m) \times L^2(0,T;\mathbf{R}^k)$ is an* open loop saddle point *of $C_{x_0}(u,v)$ in $L^2(0,T;\mathbf{R}^m) \times L^2(0,T;\mathbf{R}^k)$ if for all u in $L^2(0,T;\mathbf{R}^m)$ and all v in $L^2(0,T;\mathbf{R}^k)$,*

$$C_{x_0}(\bar{u},v) \leq C_{x_0}(\bar{u},\bar{v}) \leq C_{x_0}(u,\bar{v}). \tag{2.7}$$

DEFINITION 2.2. *Associate with $x_0 \in \mathbf{R}^n$ the sets*

$$V(x_0) \overset{\text{def}}{=} \left\{ v \in L^2(0,T;\mathbf{R}^k) : \inf_{u\in L^2(0,T;\mathbf{R}^m)} C_{x_0}(u,v) > -\infty \right\}, \tag{2.8}$$

$$U(x_0) \overset{\text{def}}{=} \left\{ u \in L^2(0,T;\mathbf{R}^m) : \sup_{v\in L^2(0,T;\mathbf{R}^k)} C_{x_0}(u,v) < +\infty \right\}. \tag{2.9}$$

2.2. Properties of the utility function. Recall from [5] that the utility function $C_{x_0}(u,v)$ is infinitely differentiable and that its Hessian of second order derivatives is independent of (u,v). Indeed,[1]

$$\frac{1}{2}dC_{x_0}(u,v;\bar{u},\bar{v}) = Fx(T)\cdot\bar{y}(T) + (Qx,\bar{y}) + (u,\bar{u}) - (v,\bar{v}), \tag{2.10}$$

where x is the solution of (2.4) and $\bar{y}$ is the solution of

$$\bar{y}' = A\bar{y} + B_1\bar{u} + B_2\bar{v}, \quad \bar{y}(0) = 0. \tag{2.11}$$

It is customary to introduce the *adjoint system*

$$p' + A^\top p + Qx = 0, \quad p(T) = Fx(T) \tag{2.12}$$

and rewrite expression (2.10) for the gradient in the form

$$\frac{1}{2}dC_{x_0}(u,v;\bar{u},\bar{v}) = (B_1^\top p + u,\bar{u}) + (B_2^\top p - v,\bar{v}). \tag{2.13}$$

Hence $dC_{x_0}(\hat{u},\hat{v};\bar{u},\bar{v}) = 0$ for all $\bar{u}$ and $\bar{v}$ if and only if the *coupled system*

$$\begin{cases} \hat{x}' = A\hat{x} - R\hat{p}, & \hat{x}(0) = x_0, \\ \hat{p}' + A^\top\hat{p} + Q\hat{x} = 0, & \hat{p}(T) = F\hat{x}(T) \end{cases} \tag{2.14}$$

has a solution $(\hat{x},\hat{p})$ in $H^1(0,T;\mathbf{R}^n)^2$ with $(\hat{u},\hat{v}) = (-B_1^\top\hat{p}, B_2^\top\hat{p})$.

As expected, the Hessian is independent of (u,v),

$$\frac{1}{2}d^2C_{x_0}(u,v;\bar{u},\bar{v};\tilde{u},\tilde{v}) = F\tilde{y}(T)\cdot\bar{y}(T) + (Q\tilde{y},\bar{y}) + (\tilde{u},\bar{u}) - (\tilde{v},\bar{v}), \tag{2.15}$$

where $\bar{y}$ is the solution of (2.11) and $\tilde{y}$ is the solution of

$$\tilde{y}' = A\tilde{y} + B_1\tilde{u} + B_2\tilde{v}, \quad \tilde{y}(0) = 0. \tag{2.16}$$

In particular, for all x_0, u, v, $\bar{u}$, and $\bar{v}$, $d^2C_{x_0}(u,v;\bar{u},\bar{v};\bar{u},\bar{v}) = 2\,C_0(\bar{u},\bar{v})$.

2.3. Games with finite open loop lower or upper values. We recall and sharpen the results of [5, Thms. 2.2, 2.3, and 2.4] when the open loop lower or upper value of the game is finite for a given initial state x_0. In each case, the global assumption of finiteness for *all* initial state $x_0 \in \mathbf{R}^n$ yields the *uniqueness* of solution (x,p) of the coupled system (2.14) (cf. [5, Thms. 2.6, 2.7, and 2.8]).

THEOREM 2.1. *The following conditions are equivalent.*

(i) *There exist $\hat{u}$ in $L^2(0,T;\mathbf{R}^m)$ and $\hat{v}$ in $L^2(0,T;\mathbf{R}^k)$ such that*

$$C_{x_0}(\hat{u},\hat{v}) = \inf_{u\in L^2(0,T;\mathbf{R}^m)} C_{x_0}(u,\hat{v}) = \sup_{v\in L^2(0,T;\mathbf{R}^k)} \inf_{u\in L^2(0,T;\mathbf{R}^m)} C_{x_0}(u,v). \tag{2.17}$$

(ii) *The open loop lower value $v^-(x_0)$ of the game is finite.*

[1]Given a real function f defined on a Banach space B, the *first directional semiderivative* at x in the direction v (when it exists) is defined as $df(x;v) = \lim_{t\searrow 0}(f(x+tv)-f(x))/t$. When the map $v \mapsto df(x;v) : B \to \mathbf{R}$ is linear and continuous, it defines the *gradient* $\nabla f(x)$ as an element of the dual B^* of B. The *second order bidirectional derivative* at x in the directions (v,w) (when it exists) is defined as $d^2f(x;v,w) = \lim_{t\searrow 0}(df(x+tw;v)-df(x;v))/t$. When the map $(v,w) \mapsto d^2f(x;v,w) : B\times B \to \mathbf{R}$ is bilinear and continuous, it defines the *Hessian operator* $Hf(x)$ as a continuous linear operator from B to B^*.

(iii) *There exists a solution in* $H^1(0,T;\mathbf{R}^n)^2$ *of the* coupled system

$$\begin{cases} x' = Ax - Rp, & x(0) = x_0, \\ p' + A^\top p + Qx = 0, & p(T) = Fx(T), \end{cases} \tag{2.18}$$

and the following identities are verified:

$$\sup_{v\in V(0)} \inf_{u\in L^2(0,T;\mathbf{R}^m)} C_0(u,v) = \inf_{u\in L^2(0,T;\mathbf{R}^m)} C_0(u,0) = C_0(0,0). \tag{2.19}$$

Under such conditions, the optimal controls and the open loop lower value are given by the expressions

$$\hat u = -B_1^\top p, \quad \hat v = B_2^\top p, \quad \text{and} \quad C_{x_0}(\hat u, \hat v) = p(0)\cdot x_0. \tag{2.20}$$

Remark 2.1. The additional condition $B_2^\top p \in V(x_0)$ that appeared in [5, Thms. 2.2 and 2.6] is redundant. To see that, recall that the last identity (2.19) is equivalent to the convexity of the mapping $u \mapsto C_{x_0}(u,v)$. By [5, Thm. 3.1] the convexity plus a solution of the coupled system (2.18) yields that

$$\inf_{u\in L^2(0,T;\mathbf{R}^m)} C_{x_0}(u, B_2^\top p) = C_{x_0}(-B_1^\top p, B_2^\top p) > -\infty \quad \Rightarrow\ B_2^\top p \in V(x_0).$$

Similarly, the additional condition $-B_1^\top p \in U(x_0)$ that appeared in [5, Thms. 2.3 and 2.7] is also redundant.

Theorem 2.2. *The following conditions are equivalent.*

(i) *There exist* $\hat u$ *in* $L^2(0,T;\mathbf{R}^m)$ *and* $\hat v$ *in* $L^2(0,T;\mathbf{R}^k)$ *such that*

$$C_{x_0}(\hat u,\hat v) = \sup_{v\in L^2(0,T;\mathbf{R}^k)} C_{x_0}(\hat u, v) = \inf_{u\in L^2(0,T;\mathbf{R}^m)} \sup_{v\in L^2(0,T;\mathbf{R}^k)} C_{x_0}(u,v). \tag{2.21}$$

(ii) *The open loop upper value* $v^+(x_0)$ *of the game is finite.*

(iii) *There exists a solution* $(x,p) \in H^1(0,T;\mathbf{R}^n)^2$ *of the* coupled system (2.18)*, and the following identities are verified:*

$$\inf_{u\in U(0)} \sup_{v\in L^2(0,T;\mathbf{R}^k)} C_0(u,v) = \sup_{v\in L^2(0,T;\mathbf{R}^k)} C_0(0,v) = C_0(0,0). \tag{2.22}$$

Under such conditions, the optimal controls and the open loop upper value are given by expressions (2.20)*.*

For the case when $v^-(x_0)$ and $v^+(x_0)$ are both finite, Zhang [9] proved that they are equal. So by combining this with the previous theorems, we get the complete picture.

Theorem 2.3. *The following conditions are equivalent.*

(i) *There exist* $\hat u$ *in* $L^2(0,T;\mathbf{R}^m)$ *and* $\hat v$ *in* $L^2(0,T;\mathbf{R}^k)$ *such that*

$$\begin{aligned} C_{x_0}(\hat u,\hat v) &= \inf_{u\in L^2(0,T;\mathbf{R}^m)} \sup_{v\in L^2(0,T;\mathbf{R}^k)} C_{x_0}(u,v) \\ &= \sup_{v\in L^2(0,T;\mathbf{R}^k)} \inf_{u\in L^2(0,T;\mathbf{R}^m)} C_{x_0}(u,v) \end{aligned} \tag{2.23}$$

(that is, $C_{x_0}(u,v)$ *has a saddle point).*

(ii) *The open loop value* $v(x_0)$ *of the game is finite.*

(iii) *There exists a solution* $(x,p) \in H^1(0,T;\mathbf{R}^n)^2$ *of the* coupled system (2.18), *and the following identities are verified:*

$$\inf_{u\in L^2(0,T;\mathbf{R}^m)} C_0(u,0) = C_0(0,0) = \sup_{v\in L^2(0,T;\mathbf{R}^k)} C_0(0,v). \tag{2.24}$$

(iv) *The lower and upper open loop values,* $v^-(x_0)$ *and* $v^+(x_0)$*, are finite.*
Under such conditions, the optimal controls and the value are given by expressions (2.20) *and* $v(x_0) = v^-(x_0) = v^+(x_0)$.

There are six possible cases according to the fact that $v^-(x_0)$ and $v^+(x_0)$ are finite, $-\infty$, or $+\infty$. If either $v^-(x_0)$ or $v^+(x_0)$ is finite, there is a solution $(x,p) \in H^1(0,T;\mathbf{R}^n)^2$ of the *coupled system* (2.18), and we have the following three cases:

$$\text{(a)}\quad \left|\begin{array}{l} v^-(x_0)\text{ finite}\\ v^+(x_0)\text{ finite}\end{array}\right.\quad \left|\begin{array}{l} \inf_u C_0(u,0) = C_0(0,0)\\ \sup_v C_0(0,v) = C_0(0,0)\end{array}\right.\quad \left|\begin{array}{l} u\mapsto C_0(u,0)\text{ convex}\\ v\mapsto C_0(0,v)\text{ concave}\end{array}\right.$$

$$\text{(b)}\quad \left|\begin{array}{l} v^-(x_0)\text{ finite}\\ v^+(x_0) = +\infty\end{array}\right.\quad \left|\begin{array}{l} \inf_u C_0(u,0) = C_0(0,0)\\ \sup_v \inf_u C_0(u,v) = C_0(0,0)\\ \sup_v C_0(0,v) > C_0(0,0)\end{array}\right.\quad \left|\begin{array}{l} u\mapsto C_0(u,0)\text{ convex}\\ v\mapsto \inf_u C_0(u,v)\text{ concave}\\ v\mapsto C_0(0,v)\text{ not concave}\end{array}\right.$$

$$\text{(c)}\quad \left|\begin{array}{l} v^-(x_0) = -\infty\\ v^+(x_0)\text{ finite}\end{array}\right.\quad \left|\begin{array}{l} \inf_u C_0(u,0) < C_0(0,0)\\ \inf_u \sup_v C_0(u,v) = C_0(0,0)\\ \sup_v C_0(0,v) = C_0(0,0)\end{array}\right.\quad \left|\begin{array}{l} u\mapsto C_0(u,0)\text{ not convex}\\ u\mapsto \sup_v C_0(u,v)\text{ convex}\\ v\mapsto C_0(0,v)\text{ concave}\end{array}\right.$$

There are three more cases as follows that can occur even if the coupled system (2.18) has a solution $(x,p) \in H^1(0,T;\mathbf{R}^n)^2$ and neither $v^-(x_0)$ nor $v^+(x_0)$ is finite:

$$\text{(d)}\quad \left|\begin{array}{l} v^-(x_0) = -\infty\\ v^+(x_0) = +\infty\end{array}\right.\quad \left|\begin{array}{l} \inf_u C_0(u,0) < C_0(0,0)\\ \sup_v C_0(0,v) > C_0(0,0)\end{array}\right.\quad \left|\begin{array}{l} u\mapsto C_0(u,0)\text{ not convex}\\ v\mapsto C_0(0,v)\text{ not concave}\end{array}\right.$$

$$\text{(e)}\quad \left|\begin{array}{l} v^-(x_0) = +\infty\\ v^+(x_0) = +\infty\end{array}\right.\quad \left|\begin{array}{l} \inf_u C_0(u,0) = C_0(0,0)\\ \sup_v \inf_u C_0(u,v) > C_0(0,0)\\ \sup_v C_0(0,v) > C_0(0,0)\end{array}\right.\quad \left|\begin{array}{l} u\mapsto C_0(u,0)\text{ convex}\\ v\mapsto \inf_u C_0(u,v)\text{ not concave}\\ v\mapsto C_0(0,v)\text{ not concave}\end{array}\right.$$

$$\text{(f)}\quad \left|\begin{array}{l} v^-(x_0) = -\infty\\ v^+(x_0) = -\infty\end{array}\right.\quad \left|\begin{array}{l} \inf_u \sup_v C_0(u,v) < C_0(0,0)\\ \inf_u C_0(u,0) < C_0(0,0)\\ \sup_v C_0(0,v) = C_0(0,0)\end{array}\right.\quad \left|\begin{array}{l} u\mapsto C_0(u,0)\text{ not convex}\\ u\mapsto \sup_v C_0(u,v)\text{ not convex}\\ v\mapsto C_0(0,v)\text{ concave}\end{array}\right.$$

Case (d) can occur by combining a system of type (b) with a system of type (c) and a utility function equal to the sum of the two utility functions. Case (e) occurs for the following system and utility function:

$$\begin{aligned} &x'(t) = tu(t) + t^3 v(t) \text{ in } [0,2], \quad x(0) = x_0,\\ &C_{x_0}(u,v) = \frac{3}{8}x(2)\cdot x(2) + \int_0^2 |u(t)|^2 - |v(t)|^2\,dt. \end{aligned} \tag{2.25}$$

Finally, by duality, case (f) can also occur.

3. L^2-integrable closed loop strategies. We generalize classical results to L^2-integrable affine closed loop feedback strategies for general F and $Q(t)$ under the assumptions of section 2.1 on the matrix functions A, B_1, B_2, Q, and F. We also give a classification of the possible cases in terms of the open loop properties of lower value, upper value, and value of the game and the convexity/concavity of the utility function.

3.1. Definitions and main results.

DEFINITION 3.1 (L^2-integrable affine closed loop strategies).

$$\Phi \overset{\text{def}}{=} \left\{ \phi : [0,T] \times \mathbf{R}^n \to \mathbf{R}^m \;\middle|\; \begin{array}{l} \textit{such that } x \mapsto \phi(t,x) \textit{ is affine and} \\ t \mapsto \phi(t,x) \textit{ belongs to } L^2(0,T;\mathbf{R}^m) \end{array} \right\}.$$

$$\Psi \overset{\text{def}}{=} \left\{ \psi : [0,T] \times \mathbf{R}^n \to \mathbf{R}^k \;\middle|\; \begin{array}{l} \textit{such that } x \mapsto \psi(t,x) \textit{ is affine and} \\ t \mapsto \psi(t,x) \textit{ belongs to } L^2(0,T;\mathbf{R}^k) \end{array} \right\}.$$

We say that ϕ or ψ is linear if $\phi(t,x)$ or $\psi(t,x)$ is linear in x.

Remark 3.1. To each $\phi \in \Phi$ (resp., $\psi \in \Psi$) we can associate an $L^2(0,T;\mathbf{R}^m)$-vector function u and an $m \times n$-matrix L^2-function U such that $\phi(t,x) = u(t) + U(t)x$ (resp., an $L^2(0,T;\mathbf{R}^k)$-vector function v and a $k \times n$-matrix L^2-function V such that $\psi(t,x) = v(t) + V(t)x$). The matrix functions U and V may have singularities, but they are globally L^2-integrable. As a result, the fundamental matrix associated with the L^2-matrix function $A + B_1U + B_2V$ will be invertible everywhere in $[0,T]$. Therefore for all $\phi \in \Phi$ and $\psi \in \Psi$, the closed loop system

$$\begin{aligned} &x' = Ax + B_1\phi(x) + B_2\psi(x), \quad x(0) = x_0, \\ &x' = (A + B_1U + B_2V)x + B_1u + B_2v, \quad x(0) = x_0 \end{aligned} \tag{3.1}$$

has a unique solution in $H^1(0,T;\mathbf{R}^n)$ for all $x_0 \in \mathbf{R}^n$. This means that all pairs $(\phi,\psi) \in \Phi \times \Psi$ are *admissible*, and, a fortiori, all pairs of the form (ϕ, v) or (u,ψ) are admissible for all $u \in L^2(0,T;\mathbf{R}^m)$ and $v \in L^2(0,T;\mathbf{R}^k)$.

DEFINITION 3.2.

(i) *Given $x_0 \in \mathbf{R}^n$, we say that $(\phi^*,\psi^*) \in \Phi \times \Psi$ is* an L^2-integrable *closed loop saddle point of $C_{x_0}(\phi,\psi)$ in $\Phi \times \Psi$ if for all $\phi \in \Phi$ and $\psi \in \Psi$,*

$$C_{x_0}(\phi^*,\psi) \le C_{x_0}(\phi^*,\psi^*) \le C_{x_0}(\phi,\psi^*). \tag{3.2}$$

(ii) *We say that $(\phi^*,\psi^*) \in \Phi \times \Psi$ is* an L^2-integrable global closed loop saddle point *of $C_{x_0}(\phi,\psi)$ in $\Phi \times \Psi$ if for all $x_0 \in \mathbf{R}^n$ and for all $\phi \in \Phi$ and $\psi \in \Psi$ the inequalities* (3.2) *are verified.*

By definition, $C_{x_0}(\phi^*,\psi^*)$ is finite. Thus the saddle point is "nondegenerate" in the sense of [2]. The "global version" is better adapted to closed loop strategies. The interest in a closed loop strategy associated with a single initial state is rather limited.

Given any two pairs (ϕ_1,ψ_1) and (ϕ_2,ψ_2) achieving an L^2-integrable closed saddle point, the mixed pairs (ϕ_1,ψ_2) and (ϕ_2,ψ_1) are admissible and also achieve an L^2-integrable closed loop saddle point. Hence the value of the closed loop saddle point is unique (cf. [1]).

LEMMA 3.1. *Given $x_0 \in \mathbf{R}^n$, for all pairs $(\phi_1^*,\psi_1^*) \in \Phi \times \Psi$ and $(\phi_2^*,\psi_2^*) \in \Phi \times \Psi$ verifying* (3.2)*, $C_{x_0}(\phi_1^*,\psi_1^*) = C_{x_0}(\phi_2^*,\psi_2^*)$.*

We quote Berkovitz's equivalence lemma [1].

LEMMA 3.2. *Given $x_0 \in \mathbf{R}^n$, the following statements are equivalent:*

(i) $(\phi^*, \psi^*) \in \Phi \times \Psi$ *is* an L^2-integrable *closed loop saddle point of* $C_{x_0}(\phi, \psi)$*;*
(ii) *there exists a pair* $(\phi^*, \psi^*) \in \Phi \times \Psi$ *such that for all* $u \in L^2(0, T; \mathbf{R}^m)$ *and all* $v \in L^2(0, T; \mathbf{R}^k)$*,*

$$C_{x_0}(\phi^*, v) \le C_{x_0}(\phi^*, \psi^*) \le C_{x_0}(u, \psi^*). \tag{3.3}$$

THEOREM 3.1. *The following statements are equivalent.*

(i) $(\phi^*, \psi^*) \in \Phi \times \Psi$ *is* an L^2-integrable global closed loop saddle point *of* $C_{x_0}(\phi, \psi)$.
(ii) *There exists* $(\phi^*, \psi^*) \in \Phi \times \Psi$ *such that for all* $x_0 \in \mathbf{R}^n$*,* $u \in L^2(0, T; \mathbf{R}^m)$*, and* $v \in L^2(0, T; \mathbf{R}^k)$*,*

$$C_{x_0}(\phi^*, \psi^* + v) \le C_{x_0}(\phi^*, \psi^*) \le C_{x_0}(\phi^* + u, \psi^*). \tag{3.4}$$

(iii) *For all* $x_0 \in \mathbf{R}^n$ *there exist a unique solution* $(\hat{x}, \hat{p}) \in H^1(0, T; \mathbf{R}^n)^2$ *of*

$$\begin{cases} \hat{x}' = A\hat{x} - R\hat{p}, & \hat{x}(0) = x_0, \\ \hat{p}' + A^\top \hat{p} + Q\hat{x} = 0, & \hat{p}(T) = F\hat{x}(T) \end{cases} \tag{3.5}$$

and L^2*-integrable matrices* U_* *and* V_* *of appropriate orders such that for all* $x_0 \in \mathbf{R}^n$*,*

$$\hat{u} = -B_1^\top \hat{p} = U_* \hat{x}, \quad \hat{v} = B_2^\top \hat{p} = V_* \hat{x}. \tag{3.6}$$

(iv) (Invariant embedding). *For all initial times* $s \in [0, T[$*, there exists a unique* $H^1(s, T)$ *solution of the coupled matrix system*

$$\begin{cases} \widehat{X}_s' = A\widehat{X}_s - R\widehat{\Lambda}_s, & \widehat{X}_s(s) = I, \\ \widehat{\Lambda}_s' + A^\top \widehat{\Lambda}_s + Q\widehat{X}_s = 0, & \widehat{\Lambda}_s(T) = F\widehat{X}_s(T). \end{cases} \tag{3.7}$$

By convention, set $\widehat{X}_T(T) = I$ *and* $\widehat{\Lambda}_T(T) = F$*.*

(v) (Normality). $\det X(t) \neq 0$ *everywhere in* $[0, T]$*, where* (X, Λ) *is the* $H^1(0, T)$ *solution of the matrix differential system*

$$\begin{cases} X' = AX - R\Lambda, & X(T) = I, \\ \Lambda' + A^\top \Lambda + QX = 0, & \Lambda(T) = F. \end{cases} \tag{3.8}$$

(vi) *There exists a symmetrical solution* P *with elements in* $H^1(0, T)$ *to the matrix Riccati differential equation*

$$P' + PA + A^\top P - PRP + Q = 0, \quad P(T) = F. \tag{3.9}$$

In particular $C_{x_0}(\phi^*, \psi^*) = P(0)x_0 \cdot x_0$*, and the closed loop strategies are given by*

$$\phi^*(t, x) = -B_1^\top(t)P(t)x = U_*(t)x \text{ and } \psi^*(t, x) = B_2^\top(t)P(t)x = V_*(t)x. \tag{3.10}$$

Proof. (i) $\Rightarrow$ (ii). Let $\hat{x}$ be the trajectory corresponding to the pair (ϕ^*, ψ^*), and denote by $(\hat{u}, \hat{v}) = (\phi^*(x), \psi^*(x))$ the corresponding control pair. Let $U_*(t)$ and $V_*(t)$ be the respective matrices and $u_*(t)$ and $v_*(t)$ be the respective vectors such that $\phi^*(t, x) = U_*(t)x + u_*(t)$ and $\psi^*(t, x) = V_*(t)x + v_*(t)$. Then

$$\hat{x}' = (A + B_1 U_* + B_2 V_*)\hat{x} + B_1 u_* + B_2 v_*, \quad \hat{x}(0) = x_0. \tag{3.11}$$

For all $u \in L^2(0,T;\mathbf{R}^m)$ and $v \in L^2(0,T;\mathbf{R}^k)$, the pair $(\phi^* + u, \psi^* + v) \in \Phi \times \Psi$ and

$$C_{x_0}(\phi^*, \psi^* + v) \le C_{x_0}(\phi^*, \psi^*) \le C_{x_0}(\phi^* + u, \psi^*). \tag{3.12}$$

(ii) $\Rightarrow$ (iii). Introduce the notation $c_{x_0}(u,v)$ for the utility function $C_{x_0}(\phi^* + u, \psi^* + v)$:

$$c_{x_0}(u,v) \overset{\text{def}}{=} Fx(T)\cdot x(T) + \int_0^T Qx\cdot x + |U_*x + u_* + u|^2 - |V_*x + v_* + v|^2\, dt,$$

and denote by x the solution of the corresponding state equation

$$x' = (A + B_1U_* + B_2V_*)x + B_1(u_* + u) + B_2(v_* + v), \quad x(0) = x_0. \tag{3.13}$$

Then the closed loop saddle point inequalities (3.12) become open loop saddle point inequalities for system (3.13) and the new quadratic utility function $c_{x_0}(u,v)$ satisfies the saddle point condition

$$\forall u \in L^2(0,T;\mathbf{R}^m) \text{ and } v \in L^2(0,T;\mathbf{R}^k), \quad c_{x_0}(0,v) \le c_{x_0}(0,0) \le c_{x_0}(u,0), \tag{3.14}$$

and the pair $(0,0)$ achieves that saddle point. By [5, Lemma 3.1] $c_{x_0}(u,v)$ is convex-concave and $dc_{x_0}(0,0;u,v) = 0$ for all u and v. In particular, the coupled system

$$\begin{gathered}
\hat{x}' = (A + B_1U_* + B_2V_*)\hat{x} + B_1u_* + B_2v_*, \quad \hat{x}(0) = x_0,\\
\begin{cases} \hat{p}' + (A + B_1U_* + B_2V_*)^\top \hat{p} + Q\hat{x} + U_*^\top(U_*\hat{x} + u_*) - V_*^\top(V_*\hat{x} + v_*) = 0,\\ \hat{p}(T) = F\hat{x}(T), \end{cases}\\
0 = -B_1^\top \hat{p} - (U_*\hat{x} + u_*) \text{ and } 0 = B_2^\top \hat{p} - (V_*\hat{x} + v_*)
\end{gathered} \tag{3.15}$$

has a solution $(\hat{x}, \hat{p})$ in $H^1(0,T;\mathbf{R}^n)^2$. After substitution, it can be rewritten as

$$\begin{aligned}
&\hat{x}' = A\hat{x} - R\hat{p}, \quad \hat{x}(0) = x_0,\\
&\hat{p}' + A^\top \hat{p} + Q\hat{x} = 0, \quad \hat{p}(T) = F\hat{x}(T).
\end{aligned} \tag{3.16}$$

By assumption this is true for all $x_0 \in \mathbf{R}^n$. But, when system (3.16) has a solution for all x_0, its solution is unique (cf, [5, section 2.6, pp. 760–761]). As a result, for $x_0 = 0$, we have $(\hat{x}, \hat{p}) = (0,0)$, and from identities (3.15),

$$0 = -B_1^\top \hat{p} - (U_*\hat{x} + u_*) \text{ and } 0 = B_2^\top \hat{p} - (V_*\hat{x} + v_*) \quad \Rightarrow u_* = 0 \text{ and } v_* = 0.$$

Hence the feedback controls are of the form $\hat{u} = U_*\hat{x} = -B_1^\top \hat{p}$ and $\hat{v} = V_*\hat{x} = B_2^\top \hat{p}$.

(iii) $\Rightarrow$ (iv). By assumption for all $x_0 \in \mathbf{R}^n$, the coupled system (3.16) has a unique solution $(\hat{x}, \hat{p})$. By linearity of the solution of system (3.5) with respect to x_0, there exist $H^1(0,T)$-matrices $(\widehat{X}, \widehat{\Lambda})$ solution of the matrix system

$$\begin{aligned}
&\widehat{X}' = A\widehat{X} - R\widehat{\Lambda}, \quad \widehat{X}(0) = I,\\
&\widehat{\Lambda}' + A^\top \widehat{\Lambda} + Q\widehat{X} = 0, \quad \widehat{\Lambda}(T) = F\widehat{X}(T).
\end{aligned} \tag{3.17}$$

But the conditions $U_*\hat{x} = -B_1^\top \hat{p}$ and $V_*\hat{x} = B_2^\top \hat{p}$ for all x_0 imply that $U_*\widehat{X} = -B_1^\top \widehat{\Lambda}$ and $V_*\widehat{X} = B_2^\top \widehat{\Lambda}$, and $\widehat{X}$ is also the unique solution of the equation

$$\widehat{X}' = (A + B_1U_* + B_2V_*)\widehat{X}, \quad \widehat{X}(0) = I. \tag{3.18}$$

Since the elements of the matrix function $A + B_1U_* + B_2V_*$ are L^2-functions, the associated fundamental matrix solution $\Phi(t,s)$ is invertible, $\widehat{X}(t)x_0 = \hat{x}(t) = \Phi(t,0)x_0$, and, a fortiori, $\widehat{X}(t) = \Phi(t,0)$ is invertible in $[0,T]$. In particular, for all $s \in [0,T[$, $(\widehat{X}_s(t), \widehat{\Lambda}_s(t)) = (\widehat{X}(t)\widehat{X}(s)^{-1}, \widehat{\Lambda}(t)\widehat{X}(s)^{-1})$ is a solution of system (3.7).

(iv) $\Rightarrow$ (v). First, observe that $\widehat{X}_s(T)$ is invertible for all $s \in [0,T]$. For $s < T$, let $h \neq 0$ be such that $\widehat{X}_s(T)h = 0$. The pair $(x_s(t), p_s(t)) = (\widehat{X}_s(t)h, \widehat{\Lambda}_s(t)h)$ is a solution of the system $x_s' = Ax_s - Rp_s$, $x_s(s) = h$, and $p_s' + A^\top p_s + Qx_s = 0$, $p_s(T) = Fx_s(T)$, with $(x_s(T), p_s(T)) = (0,0)$. Hence $(x_s, p_s) = (0,0)$ and $h = x_s(s) = 0$, a contradiction. For all $0 \le s \le t \le T$, $\widehat{X}_s(T) = \widehat{X}_t(T)\widehat{X}_s(t)$ and $\det \widehat{X}_s(t) \neq 0$. Defining the matrix functions $(X(t), \Lambda(t)) = (\widehat{X}_0(t)\widehat{X}_0(T)^{-1}, \widehat{\Lambda}_0(t)\widehat{X}_0(T)^{-1})$, they are a solution of the matrix differential system (3.7), and necessarily, $\det X(t) \neq 0$ everywhere in $[0,T]$.

(v) $\Rightarrow$ (vi). Since $X(t)$ is invertible for all $t \in [0,T]$, then $P(t) = \Lambda(t)X(t)^{-1}$ is a matrix of $H^1(0,T)$ functions. Moreover, P is symmetrical since $\Lambda^\top X$ is by computing the derivative of $\Lambda^\top X - X^\top \Lambda$. Finally, P is a solution of the Riccati matrix differential equation (3.9). By uniqueness of the solution (x,p) of (3.5), $(x(t), p(t)) = (\widehat{X}_0(t)x_0, \widehat{\Lambda}_0(t)x_0) = (X(t)\widehat{X}_0(T)x_0, \Lambda(t)\widehat{X}_0(T)x_0)$. Hence $p(t) = P(t)x(t)$, $\hat{u}(t) = -B_1^\top p(t) = -B_1^\top P(t)x(t)$, and $\hat{v}(t) = B_2^\top p(t) = B_2^\top P(t)x(t)$.

(vi) $\Rightarrow$ (i). Let $x \in H^1(0,T;\mathbf{R}^n)$ be the solution of

$$x' = Ax + B_1u + B_2v, \quad x(0) = x_0, \tag{3.19}$$

and let P be an $H^1(0,T)$ solution of the matrix Riccati differential equation (3.9). By the classical argument of Bernhard [2], we get

$$C_{x_0}(u,v) = P(0)x_0 \cdot x_0 + \int_0^T \left|u + B_1^\top Px\right|^2 - \left|v - B_2^\top Px\right|^2 \, dt.$$

Choose the closed loop linear strategies $\phi^*(t,x) = -B_1^\top(t)P(t)x$ and $\psi^*(t,x) = B_2^\top(t)P(t)x$. Then for all $v \in L^2(0,T;\mathbf{R}^k)$ and all $u \in L^2(0,T;\mathbf{R}^m)$,

$$\begin{aligned} C_{x_0}(\phi^*, \psi^*) &= P(0)x_0 \cdot x_0, \\ C_{x_0}(u, \psi^*) &= P(0)x_0 \cdot x_0 + \int_0^T \left|u + B_1^\top Px\right|^2 dt \ge P(0)x_0 \cdot x_0 = C_{x_0}(\phi^*, \psi^*), \\ C_{x_0}(\phi^*, v) &= P(0)x_0 \cdot x_0 - \int_0^T \left|v - B_2^\top Px\right|^2 dt \le P(0)x_0 \cdot x_0 = C_{x_0}(\phi^*, \psi^*). \end{aligned}$$

By Lemma 3.2(ii), the linear pair (ϕ^*, ψ^*) is a global closed loop saddle point. Finally, the pair of closed loop strategies $\phi^*(t,x) = -B_1^\top(t)P(t)x = U_*(t)x$ and $\psi^*(t,x) = B_2^\top(t)P(t)x = V_*(t)x$ yields the global closed loop saddle point $P(0)x_0 \cdot x_0$. ∎

3.2. Classification of L^2-integrable closed loop saddle points. One of the necessary conditions for the existence of a closed loop saddle point is the existence of a solution to the coupled system in $(\hat{x}, \hat{p})$ that turns out to also be a necessary condition for the finiteness of the open loop lower value, upper value, or value of the game. The difference essentially depends on the convexity of the utility function with respect to u and on its concavity with respect to v that yields to the following classification.

THEOREM 3.2. *Assume that $(\phi^*, \psi^*) \in \Phi \times \Psi$ is an L^2-integrable closed loop saddle point of $C_{x_0}(\phi, \psi)$.*

(a) *$v(x_0)$ is finite if and only if $C_{x_0}(u,v)$ is convex in u and concave in v.*

(b) *$v^-(x_0)$ is finite and $v^+(x_0) = +\infty$ if and only if $C_{x_0}(u, v)$ is convex in u and not concave in v.*
(c) *$v^+(x_0)$ is finite and $v^-(x_0) = -\infty$ if and only if $C_{x_0}(u, v)$ is concave in v and not convex in u.*
(d) *$v^-(x_0) = -\infty$ and $v^+(x_0) = +\infty$ if and only if $C_{x_0}(u, v)$ is not convex in u and not concave in v.*
(e) *$v^-(x_0) = v^+(x_0) = +\infty$ cannot occur.*
(f) *$v^-(x_0) = v^+(x_0) = -\infty$ cannot occur.*

In the first three cases, $C_{x_0}(\phi^*, \psi^*)$ is equal to $v(x_0)$, $v^-(x_0)$, and $v^+(x_0)$, respectively.

Remark 3.2. Case (d) can occur. An example can be constructed by using a first system of type (b) and a second system of type (c) without interconnection, with utility function equal to the sum of the two utility functions.

Remark 3.3. In Bernhard [2], the utility function was convex in u since $F \geq 0$ and $Q(t) \geq 0$. Only cases (a) and (b) can occur, and case (e) is a degenerate one.

We need the following lemma.

LEMMA 3.3.
(i) *For all $v \in L^2(0, T; \mathbf{R}^k)$,*

$$\inf_{\phi \in \Phi} C_{x_0}(\phi, v) = \inf_{u \in L^2(0,T;\mathbf{R}^m)} C_{x_0}(u, v), \tag{3.20}$$

$$\sup_{\psi \in \Psi} \inf_{\phi \in \Phi} C_{x_0}(\phi, \psi) \geq \sup_{v \in L^2(0,T;\mathbf{R}^k)} \inf_{u \in L^2(0,T;\mathbf{R}^m)} C_{x_0}(u, v). \tag{3.21}$$

(ii) *For all $u \in L^2(0, T; \mathbf{R}^m)$,*

$$\sup_{\psi \in \Psi} C_{x_0}(u, \psi) = \sup_{v \in L^2(0,T;\mathbf{R}^k)} C_{x_0}(u, v), \tag{3.22}$$

$$\inf_{\phi \in \Phi} \sup_{\psi \in \Psi} C_{x_0}(\phi, \psi) \leq \inf_{u \in L^2(0,T;\mathbf{R}^m)} \sup_{v \in L^2(0,T;\mathbf{R}^k)} C_{x_0}(u, v). \tag{3.23}$$

Proof of Lemma 3.3. We need only prove (i). Since $L^2(0, T; \mathbf{R}^m) \subset \Phi$,

$$\inf_{\phi \in \Phi} C_{x_0}(\phi, v) \leq \inf_{u \in L^2(0,T;\mathbf{R}^m)} C_{x_0}(u, v).$$

Conversely, given the pair (ϕ, v), let $x \in H^1(0, T; \mathbf{R}^n)$ be the solution of the system

$$x' = Ax + B_1\phi(x) + B_2 v, \quad x(0) = x_0,$$

and let $u = \phi(x) \in L^2(0, T; \mathbf{R}^m)$. This implies that

$$\begin{aligned} &C_{x_0}(\phi, v) = C_{x_0}(u, v) \geq \inf_{u \in L^2(0,T;\mathbf{R}^m)} C_{x_0}(u, v) \\ &\Rightarrow \inf_{\phi \in \Phi} C_{x_0}(\phi, v) \geq \inf_{u \in L^2(0,T;\mathbf{R}^m)} C_{x_0}(u, v) \Rightarrow \inf_{\phi \in \Phi} C_{x_0}(\phi, v) = \inf_{u \in L^2(0,T;\mathbf{R}^m)} C_{x_0}(u, v). \end{aligned}$$

The second inequality follows from the fact that $L^2(0, T; \mathbf{R}^k) \subset \Psi$. $\square$

Proof of Theorem 3.2. From inequality (3.21), $v^-(x_0) \leq C_{x_0}(\phi^*, \psi^*) < +\infty$ and case (e) cannot occur. Similarly, from inequality (3.23), $v^+(x_0) \geq C_{x_0}(\phi^*, \psi^*) > -\infty$ and case (f) cannot occur. Therefore we are left with the first four cases.

(b) From the first part of the proof of Theorem 3.1, system (3.16) has a solution, and identities (3.15) are verified:

$$\begin{cases} \hat{x}' = A\hat{x} - R\hat{p}, & \hat{x}(0) = x_0, \\ \hat{p}' + A^\top \hat{p} + Q\hat{x} = 0, & \hat{p}(T) = F\hat{x}(T), \end{cases} \tag{3.24}$$

$$0 = -B_1^\top \hat{p} - (U_*\hat{x} + u_*) \text{ and } 0 = B_2^\top \hat{p} - (V_*\hat{x} + v_*). \tag{3.25}$$

Using the controls $(\hat{u}, \hat{v}) = (U_*\hat{x} + u_*, V_*\hat{x} + v_*) = (-B_1^\top \hat{p}, B_2^\top \hat{p})$, the above system can be rewritten as

$$\begin{cases} \hat{x}' = A\hat{x} - B_1 B_1^\top \hat{p} + B_2 \hat{v}, & \hat{x}(0) = x_0, & \hat{u} = -B_1^\top \hat{p}, \\ \hat{p}' + A^\top \hat{p} + Q\hat{x} = 0, & \hat{p}(T) = F\hat{x}(T). & \end{cases}$$

If $C_{x_0}(u, v)$ is convex in u, this implies that $\hat{u}$ is a minimizer of $C_{x_0}(u, \hat{v})$ over u (cf., for instance, [5, Thm 3.1]). Therefore

$$\sup_{v \in L^2(0,T;\mathbf{R}^k)} \inf_{u \in L^2(0,T;\mathbf{R}^m)} C_{x_0}(u, v) \geq \inf_{u \in L^2(0,T;\mathbf{R}^m)} C_{x_0}(u, \hat{v}) = C_{x_0}(\hat{u}, \hat{v}).$$

But, by construction of $(\hat{u}, \hat{v})$, $C_{x_0}(\phi^*, \psi^*) = C_{x_0}(\hat{u}, \hat{v})$. Combining the above inequality with inequality (3.21) in Lemma 3.3, we get

$$v^-(x_0) = \sup_{v \in L^2(0,T;\mathbf{R}^k)} \inf_{u \in L^2(0,T;\mathbf{R}^m)} C_{x_0}(u, v) = \inf_{u \in L^2(0,T;\mathbf{R}^m)} C_{x_0}(u, \hat{v}) = C_{x_0}(\hat{u}, \hat{v})$$

and hence the finiteness of $v^-(x_0)$. If, in addition, $C_{x_0}(u, v)$ was concave in v, then by [5, Thms. 2.5 and 2.4] the value of the game, and hence $v^+(x_0)$, would be finite, and this contradicts our assumption. Conversely, if $v^-(x_0)$ is finite, then the mapping $u \mapsto C_{x_0}(u, v)$ is convex ([5, Thm. 2.2(iii), last part of identity (2.35) and Remark 2.2]). If $v^+(x_0)$ was also finite, then by [5, Thms. 2.5 and 2.4(iii)] $C_{x_0}(u, v)$ would be concave in v in contradiction with our assumption.

The proof of (c) is dual to the proof of (b). The proof of (a) is similar to the proof of (b) and (c). Case (d) is the complement of all the other cases, so it can only occur when $C_{x_0}(u, v)$ is neither convex in u nor concave in v. $\square$

Remark 3.4. If the problem is normal and $F \geq 0$ and $Q(t) \geq 0$, then $v^-(x_0)$ is finite and $v^-(x_0) \geq 0$ for all $x_0 \in \mathbf{R}^n$, and necessarily $P(0)x_0 \cdot x_0 \geq 0$ for all $x_0 \in \mathbf{R}^n$.

4. Not necessarily L^2-integrable affine closed loop strategies.

4.1. The curse of singularities. We now extend the definitions and results of the previous section to Lebesgue measurable feedback matrices with singularities that are not necessarily L^2-integrable in any of its neighborhood. In order to accommodate such strategies, we first enlarge the sets of strategies Φ and Ψ.

DEFINITION 4.1. *The* class of affine closed loop strategies *is defined as follows:*

$$\widetilde{\Phi} \stackrel{\text{def}}{=} \left\{ \phi : [0,T] \times \mathbf{R}^n \to \mathbf{R}^m \;\middle|\; \begin{array}{l} \textit{such that } x \mapsto \phi(t,x) \textit{ is affine,} \\ t \mapsto \phi(t,x) \textit{ is Lebesgue measurable, and} \\ t \mapsto \phi(t,0) \textit{ belongs to } L^2(0,T;\mathbf{R}^m) \end{array} \right\},$$

$$\widetilde{\Psi} \stackrel{\text{def}}{=} \left\{ \psi : [0,T] \times \mathbf{R}^n \to \mathbf{R}^k \;\middle|\; \begin{array}{l} \textit{such that } x \mapsto \psi(t,x) \textit{ is affine,} \\ t \mapsto \psi(t,x) \textit{ is Lebesgue measurable, and} \\ t \mapsto \psi(t,0) \textit{ belongs to } L^2(0,T;\mathbf{R}^k) \end{array} \right\}.$$

We say that ϕ (resp., ψ) is a linear closed loop strategy *if ϕ (resp., ψ) is linear in x.*

To any $(\phi, \psi) \in \widetilde{\Phi} \times \widetilde{\Psi}$ are associated measurable matrix functions $U(t)$ and $V(t)$ and L^2-vector functions u and v such that $\phi(t, x) = U(t)x + u(t)$ and $\psi(t, x) = V(t)x + v(t)$. At that level of generality, an *admissibility condition* on the pair $(\phi, \psi) \in \widetilde{\Phi} \times \widetilde{\Psi}$ is required to make sense of a solution of the underlying differential equation. How should the admissible affine closed loop strategies be chosen in order to preserve the assumption of *normalizability* of [2] that makes sense of a non-$H^1(0, T)$-solution P to the matrix Riccati differential equation? It is clear that the choice of the space of solutions of the matrix Riccati differential equation and the specification of the families Φ and Ψ are closely related.

It has been known that the solution of the scalar Riccati differential equation can exhibit singularities that are not *movable branch points*, at least when the coefficients are smooth functions of t (cf. Ince [6, section 12.51, p. 293]). Another interesting property is that "the general solution of the Riccati equation is expressible rationally in terms of any three distinct particular solutions, and also that the anharmonic ratio of any four solutions is constant. It also shows that the general solution is a rational function of the constant of integration (cf. Ince [6, section 2.15, p. 23 and section 12.51, p. 294])."

This result was extended to the $n \times n$ ($n \geq 2$) solution of the matrix Riccati differential equation by Sorine and Winternitz [8], but with five particular solutions in the general case and four in the *symplectic case* that corresponds to our assumptions on the data matrices. They also give some thought to the space of solutions:[2] "For smooth coefficients A, B, C, and D in the MRE (6) the solution space consists of meromorphic matrices: the matrix elements may have first-order poles, the positions of which depends on the initial conditions. In other words, the MRE (6) has the Painlevé property [6]: the solutions have no moving critical points, i.e., no branch points or essential singularities, the positions of which depend on the initial conditions (cf. [8, pp. 271–272])."

4.2. Bernhard's conditions [2] in the free end case. In the free end case with $F \geq 0$ and $Q(t) \geq 0$, the necessary and sufficient condition of Bernhard [2, Thm. 3.1] for the existence of a nondegenerate closed loop saddle point in the sense of [2, Definition 2.3 and Remark 5.1] reduces to the following three properties:

(ii) $X(t)$ is invertible except possibly at isolated points in $[0, T]$, where (X, Λ) is the unique $H^1(0, T)$ matrix solution of

$$\begin{cases} X' = AX - R\Lambda, \quad X(T) = I, \\ \Lambda' + A^\top \Lambda + QX = 0, \quad \Lambda(T) = F; \end{cases} \tag{4.1}$$

(iii) $x_0 \in \operatorname{Im} X(0)$;

(iv) for all $t \in [0, T]$, $P(t) \geq 0$,

where P is defined in terms of Λ and the pseudoinverse of X as follows:

$$P(t) = \Lambda(t)\, X(t)^\dagger, \quad X(t)^\dagger \stackrel{\text{def}}{=} \begin{cases} \left[X(t)^\top X(t)\right]^{-1} X(t)^\top & \text{if } X(t) \neq 0, \\ \text{arbitrary} & \text{if } X(t) = 0, \end{cases} \tag{4.2}$$

and $\left[X(t)^\top X(t)\right]^{-1}$ is the inverse of $X(t)^\top X(t)$ as a matrix from $\operatorname{Im} X(t)^\top$ onto itself.

[2]Sorine and Winternitz [8] consider solutions W of the general matrix Riccati differential equation $W' = A + WB + CW + WDW$, $W(T) = W_0$.

Condition (ii) defines the matrix function $P(t)$ almost everywhere in $[0, T]$ and gives meaning to a solution of the Riccati differential equation via the solution (Λ, X) of system (4.1). The positivity of F and $Q(t)$ makes the utility function $C_{x_0}(u, v)$ convex in u and this leads to the positivity of $P(t)$ (cf. Remark 3.4). Our objective is to relax those positivity assumptions as in Theorem 3.1. Without them some of the competitive terms in the utility function may simultaneously blow up, making it difficult to set the utility function equal to $\pm\infty$ (cf. [2, p. 68 and Remark 5.1]). Moreover, non-L^2-integrable singularities in the closed loop strategies invalidate the equivalence (ii) of Lemma 3.2 when either the open loop lower or upper value of the game is not finite. So the very definition of a closed loop saddle point has to be properly revisited, and the family of pairs of admissible strategies is no longer $\Phi \times \Psi$ but a subspace S of an enlarged product space $\widetilde{\Phi} \times \widetilde{\Psi}$ containing $\Phi \times \Psi$.

4.3. Normalizability and its consequences. Given the matrices A, B_1, B_2, Q, and F verifying the conditions of section 2.1, system (4.1) always has a unique solution (X, Λ) with elements in $H^1(0, T)$. Introduce the notation

$$Z \stackrel{\text{def}}{=} \{s \in [0, T] : \det X(s) = 0\}. \tag{4.3}$$

By definition, $T \notin Z$ since $X(T) = I$. We now extend the definition of [2] (property (ii) in section 4.2) to a larger class of systems with bounded measurable coefficients.

DEFINITION 4.2. *The problem* (2.1)–(2.2) *is* normalizable *if* $\det X(t) \neq 0$ *almost everywhere in* $[0, T]$*, where* (X, Λ) *is the* $H^1(0, T)$*-solution of the matrix differential system* (4.1).

Here Z is possibly infinite with accumulation points that are not isolated, as can be seen from the following example.

Example 4.1. Consider an extension of the example from [2, Example 5.1, p. 67]:

$$x'(t) = B_1(t)\, u(t) + B_2(t)\, v(t) \text{ a.e. in } [0, 2], \quad x(0) = x_0, \tag{4.4}$$

$$C_{x_0}(u, v) = \frac{1}{2}|x(2)|^2 + \int_0^2 |u(t)|^2 - |v(t)|^2\, dt, \tag{4.5}$$

where

$$B_1(t) \stackrel{\text{def}}{=} \begin{cases} 2 - t, & 1 < t \le 2, \\ 2^{\frac{n}{2}+1}\left(\dfrac{1}{2^n} - t\right), & \dfrac{1}{2^{n+1}} < t \le \dfrac{1}{2^n}, \quad n \ge 0, \end{cases} \tag{4.6}$$

$$B_2(t) \stackrel{\text{def}}{=} \begin{cases} t, & 1 < t \le 2, \\ 2^{\frac{n}{2}+1}\left(t - \dfrac{1}{2^{n+1}}\right), & \dfrac{1}{2^{n+1}} < t \le \dfrac{1}{2^n}, \quad n \ge 0. \end{cases} \tag{4.7}$$

It is readily seen that both B_1 and B_2 are measurable and bounded. Here $A = 0$, $F = 1/2$, $Q = 0$, and $R = B_1 B_1^* - B_2 B_2^*$,

$$R(t) = \begin{cases} 4(1 - t), & 1 < t \le 2, \\ \left(\dfrac{3}{2^n} - 4t\right), & \dfrac{1}{2^{n+1}} < t \le \dfrac{1}{2^n}, \quad n \ge 0. \end{cases} \tag{4.8}$$

The solution of system (4.1) is given by the expressions

$$
\begin{aligned}
X(t) &= \begin{cases} (1-t)^2, & 1 < t \le 2, \\ \left(t - \dfrac{1}{2^n}\right)\left(t - \dfrac{1}{2^{n+1}}\right), & \dfrac{1}{2^{n+1}} < t \le \dfrac{1}{2^n}, \quad n \ge 0, \\ 0, & t = 0, \end{cases} \\
\Lambda(t) &= \frac{1}{2}.
\end{aligned} \tag{4.9}
$$

Here $Z = \{1/2^n : n \ge 0\} \cup \{0\}$ has an infinite number of isolated points plus the accumulation point 0 that is not isolated since $1/2^n \to 0$.

Thus Definition 4.2 is a natural extension of the normalizability in the sense of [2] to systems with bounded measurable coefficients.

LEMMA 4.1. *If problem* (2.1)–(2.2) *is* normalizable *in the sense of Bernhard* [2], *then Z contains a finite number of instants in* $[0, T[$.

Proof. If the compact set Z has an infinite number of points, then there exists a sequence of distinct points $\{t_n\}$ in Z and an accumulation point $t_0 \in Z$, $t_n \neq t_0$, such that $t_n \to t_0$. But this is impossible since each point of Z is an isolated point in $[0,T]$: there is an open interval (a,b) such that $t_0 \in (a,b)$ and $(a,b) \cap [0,T] \backslash \{t_0\} \subset [0,T] \backslash Z$. ∎

Remark 4.1. Since Z has zero measure we know that it does not contain nontrivial intervals. One interesting issue that was raised by the referee is whether the set Z is countable or not. It is countable in our example, but can it be uncountable like the Cantor set? However, all the results of this paper are independent of that open issue.

The normalizability property relies on the fact that the state equation can be solved backward in finite dimension. In general, this would not be true for infinite dimensional evolution systems. Yet, in finite dimension, normalizability turns out to be equivalent to a weaker form of *invariant embedding with respect to the initial time* that would be more natural in infinite dimension. Denote by Z' the set of all initial times $s \in [0, T[$ such that the matrix differential system

$$
\begin{cases} \widehat{X}_s' = A\widehat{X}_s - R\widehat{\Lambda}_s, \quad \widehat{X}_s(s) = I, \\ \widehat{\Lambda}_s' + A^\top \widehat{\Lambda}_s + Q\widehat{X}_s = 0, \quad \widehat{\Lambda}_s(T) = F\widehat{X}_s(T) \end{cases} \tag{4.10}
$$

has a solution $(\widehat{X}_s, \widehat{\Lambda}_s)$ with elements in $H^1(s,T)$.

LEMMA 4.2.

(i) $Z = Z'$.

(ii) *For $s \in [0,T[\backslash Z'$, the matrix differential system* (4.10) *has a unique solution $(\widehat{X}_s, \widehat{\Lambda}_s)$ with elements in $H^1(s,T)$ for all $t \in [s, T[\backslash Z'$, $\det \widehat{X}_s(t) \neq 0$, and $P(s) = \Lambda(s)X(s)^{-1} = \widehat{\Lambda}_s(s)$.*

By convention we set $\widehat{X}_T(T) = I$ and $\widehat{\Lambda}_T(T) = F$.

Remark 4.2. From part (i) of Lemma 4.2, normalizabilty is equivalent to invariant embedding with respect to almost all initial times. This equivalence should be compared with (iv) in Theorem 3.1. It says that the decoupling operator $P(s)$ can be defined almost everywhere as $\widehat{\Lambda}_s(s)$ and that the invariant embedding with respect to almost all initial times can still be done as in [5]. This property was observed for the Riccati differential equation associated with Helmholtz equation[3] of waveguides.

[3]The author would like to thank Jacques Henry (INRIA) for bringing this work to his attention.

Due to a resonance phenomenon, the invariant embedding cannot be done at an at most countable set of initial times. This material can be found in the Ph.D. thesis of Champagne [4], where a sup-inf formulation is also introduced.

Proof. (i) For $s \in [0,T[\backslash Z$, it is easy to check that the pair of matrices $(\widehat{X}_s(t), \widehat{\Lambda}_s(t)) = (X(t)X(s)^{-1}, \Lambda(t)X(s)^{-1})$ is an $H^1(s,T)$-solution of system (4.10). Hence, $[0,T[\backslash Z \subset [0,T[\backslash Z'$ and $Z' \subset Z$. Conversely, for all $s \in [0,T[\backslash Z'$, $\widehat{X}_s(T)$ is invertible. Indeed, if there exists $h \neq 0$ such that $\widehat{X}_s(T)h = 0$, then the pair $(x,p) = (\widehat{X}_s h, \widehat{\Lambda}_s h)$ would be a solution of the system $x' = Ax - Rp$, $x(T) = 0$, $p' + A^\top p + Qx = 0$, $p(T) = Fx(T) = 0$. Hence $(x,p) = (0,0)$ and $0 = x(s) = \widehat{X}_s(s)h = h \neq 0$, a contradiction. Define for $s \in [0,T]\backslash Z'$ the new matrices $(X_s(t), \Lambda_s(t)) = (\widehat{X}_s(t)\widehat{X}_s(T)^{-1}, \widehat{\Lambda}_s(t)\widehat{X}_s(T)^{-1})$. They are a solution of system (4.1) in $[s,T]$. Hence X_s is the restriction of X to $[s,T]$, $X(t) = \widehat{X}_s(t)\widehat{X}_s(T)^{-1}$ on $[s,T]$, $X(s) = \widehat{X}_s(T)^{-1}$. Since $\widehat{X}_s(T)$ is invertible, $X(s)$ is invertible, $s \in [0,T[\,Z$, $[0,T[\,Z' \subset [0,T[\,Z$, and $Z \subset Z'$. Therefore $Z' = Z$.

(ii) To prove that the solution of system (4.10) is unique for each $s \in [0,T[\backslash Z'$, consider an arbitrary solution $(\widehat{X}_s, \widehat{\Lambda}_s)$. Define $(\overline{X}_s(t), \overline{\Lambda}_s(t)) = (\widehat{X}_s(t)X(s), \widehat{\Lambda}_s(t)X(s))$. It is readily seen that they are a solution of the system

$$\begin{cases} \overline{X}_s' = A\overline{X}_s - R\overline{\Lambda}_s, & \overline{X}_s(T) = \widehat{X}_s(T)X(s), \\ \overline{\Lambda}_s' + A^\top\overline{\Lambda}_s + Q\overline{X}_s = 0, & \overline{\Lambda}_s(T) = F\widehat{X}_s(T)X(s). \end{cases}$$

This matrix linear system with final conditions has the unique H^1-solution $\overline{X}_s(t) = X(t)\widehat{X}_s(T)X(s)$ and $\overline{\Lambda}_s(t) = \Lambda(t)\widehat{X}_s(T)X(s)$. But we have shown in part (i) that $\widehat{X}_s(T)$ is invertible and that $X(s) = \widehat{X}_s(T)^{-1}$. Therefore

$$(\widehat{X}_s(t), \widehat{\Lambda}_s(t)) = (X(t)X(s)^{-1}, \Lambda(t)X(s)^{-1})$$

is unique, $P(s) = \Lambda(s)X(s)^{-1} = \widehat{\Lambda}_s(s)$, and $\widehat{X}_s(t)$ is invertible for all $t \in [s,T]\backslash Z'$. ◻

Starting with Definition 4.2 of normalizability, we now proceed in a constructive way to identify the appropriate definition of a closed loop saddle point in the presence of non-L^2-integrable singularities in the closed loop strategies. The normalizability property gives a precise meaning to the *closed loop system* and to a solution of the matrix Riccati differential equation.

LEMMA 4.3. *Assume that the problem* (2.1)–(2.2) *is normalizable. Then*

(i) $P(s) = \Lambda(s)X(s)^{-1}$ *is uniquely defined and symmetrical for all* $s \in [0,T]\backslash Z$, P *verifies the matrix Riccati differential equation*

$$P' + PA + A^\top P - PRP + Q = 0, \quad P(T) = F,$$

in $[0,T]\backslash Z$, *and* PX *is the unique* $H^1(0,T)$*-solution of the matrix equation*

$$(PX)' + A^\top(PX) + QX = 0, \quad (PX)(T) = F.$$

(ii) X *is an* $H^1(0,T)$ *solution of the closed loop matrix equation*

$$X' = (A - RP)X, \quad X(T) = I, \tag{4.11}$$

such that $\det X(t) \neq 0$ *in* $[0,T]\backslash Z$, *and* $-B_1^\top PX = -B_1^\top\Lambda$ *and* $B_2^\top PX = B_2^\top\Lambda$ *belong to* $L^2(0,T)$.

(iii) *for all* $s \in [0,T]\backslash Z$, $\widehat{X}_s$ *is an* $H^1(s,T)$*-solution of the closed loop matrix differential equation*

$$\widehat{X}_s' = (A - RP)\widehat{X}_s, \quad \widehat{X}_s(s) = I, \tag{4.12}$$

such that $\det \widehat{X}_s(t) \neq 0$ *in* $[s,T]\backslash Z$, *and* $P\widehat{X}_s$ *is the unique* $H^1(s,T)$*-solution of the matrix equation*

$$(P\widehat{X}_s)' + A^\top(P\widehat{X}_s) + Q\widehat{X}_s = 0, \quad (P\widehat{X}_s)(T) = F\widehat{X}_s(T).$$

Proof. (i) It is easy to verify that the derivative of the matrix function $X^\top\Lambda - \Lambda^\top X$ is null and that $(X^\top\Lambda - \Lambda^\top X)(T) = 0$. Hence $X^\top\Lambda = \Lambda^\top X$ and $P = \Lambda X^{-1} = (\Lambda X^{-1})^\top = X^{-\top}\Lambda^\top = P^\top$ almost everywhere in $[0,T]$. The second part follows by definition of P and the identity $\Lambda = PX$.

(ii) The second statement follows from the fact that, by definition of P, $X' = AX - R\Lambda = AX - R\Lambda X^{-1}X = (A - RP)X$.

(iii) Equation (4.12) for $\widehat{X}_s$ follows from identity $\widehat{\Lambda}_s(t) = \widehat{\Lambda}_t(t)\widehat{X}_s(t) = P(t)\,\widehat{X}_s(t)$. ∎

4.4. Admissible closed loop affine strategies. The closed loop strategies $(\phi,\psi) \in \widetilde{\Phi} \times \widetilde{\Psi}$ of Definition 4.1 with possible non-L^2-integrable singularities need to be linked through an admissibility condition $(\phi,\psi) \in S$ to make sense of a solution of the underlying differential equation. It fundamentally changes the nature of the problem. This subtle difference prevents the use of the nice classical results of the theory of saddle points with respect to two fixed independent sets Φ and Ψ. For instance, two pairs $(\phi_1,\psi_1) \in S$ and $(\phi_2,\psi_2) \in S$ cannot be mixed: (ϕ_1,ψ_2) and (ϕ_2,ψ_1) need not belong to S as shown in Example 4.2 for the pairs $(\phi^*,\psi^*) \in S$ and $(0,0) \in S$. In particular, property (ii) of Berkovitz's equivalence, Lemma 3.2, is not verified for $u = 0$ or $v = 0$.

Example 4.2. Consider the example from [2, Example 5.1, p. 67]:

$$x'(t) = (2-t)\,u(t) + t\,v(t) \text{ a.e. in } [0,2], \quad x(0) = x_0, \tag{4.13}$$

$$C_{x_0}(u,v) = \frac{1}{2}|x(2)|^2 + \int_0^2 |u(t)|^2 - |v(t)|^2\,dt. \tag{4.14}$$

Here $A = 0$, $B_1(t) = 2-t$, $B_2(t) = t$, $F = 1/2$, $Q = 0$, and $R = B_1B_1^* - B_2B_2^* = 4(1-t)$. The Riccati equation reduces to

$$P' - 4(1-t)P^2 = 0, \quad P(2) = \frac{1}{2} \quad \Rightarrow P(t) = \frac{1}{2(t-1)^2}.$$

Its solution is positive and blows up at $t = 1$. The open loop lower value of the game is $v^-(x_0) = (x_0)^2/2$ and the open loop upper value of the game is $v^+(x_0) = +\infty$ for all $x_0 \in \mathbf{R}$. The closed loop strategies have a singularity in $t = 1$:

$$\phi^*(t,x) = -\frac{2-t}{2(t-1)^2}x \quad \text{and} \quad \psi^*(t,x) = \frac{t}{2(t-1)^2}x. \tag{4.15}$$

Yet the state x is an $H^1(0,2)$-function and the controls $\hat{u}$ and $\hat{v}$ are L^2-functions:

$$x(t) = x_0(t-1)^2, \quad \hat{u}(t) = -(2-t)\frac{1}{2}x_0, \quad \text{and} \quad \hat{v}(t) = t\,\frac{1}{2}x_0. \tag{4.16}$$

Moreover, $X(t) = (t-1)^2$.

In general for the pair (ϕ^*, v), both the resulting L^2-norms of the state x and the control $u = \phi^*(x)$ will blow up even when $v = 0$:

$$x'(t) = -\frac{(2-t)^2}{2(t-1)^2}x(t) + t\,v(t) \text{ in } [0,2], \quad x(0) = x_0, \tag{4.17}$$

$$\text{for } v = 0, \quad x(t) = e^{\frac{1}{2}[\frac{1}{t-1}-(t-1)]}|t-1|\,x_0, \tag{4.18}$$

where $x(1^-) = 0$ and $x(1^+) = \infty$. So the equivalent condition (ii) of Berkovitz's Lemma 3.2 is not verified.

DEFINITION 4.3. *We say that the pair* $(\phi,\psi) \in \widetilde{\Phi} \times \widetilde{\Psi}$ *belongs to* S *or simply that* (ϕ,ψ) *is an* admissible pair *if the associated matrix differential equation*

$$X' = (A + B_1U + B_2V)X, \quad X(T) = I, \tag{4.19}$$

has an $H^1(0,T)$*-solution such that* $\det X(t) \neq 0$ *almost everywhere in* $[0,T]$*,* UX *and* VX *are* $L^2(0,T)$ *matrices,* $|X^{-1}|u \in L^2(0,T;\mathbf{R}^m)$*, and* $|X^{-1}|v \in L^2(0,T;\mathbf{R}^k)$*.*

Remark 4.3. Since A, B_1, and B_2 are L^∞-matrix functions, it implies that the feedback matrix functions U and V will have properties similar to the ones of $X'X^{-1}$. As a consequence, given an admissible pair $(\phi,\psi) \in S$ and $y_0 \in \mathbf{R}^n$, $x(t) = X(t)y_0$ is a solution in $H^1(0,T;\mathbf{R}^n)$ of the equation

$$x' = [A + B_1U + B_2V]x, \quad x(0) = X(0)y_0, \tag{4.20}$$

(or $x(T) = y_0$) such that

$$u = Ux = UXy_0 \in L^2(0,T;\mathbf{R}^m) \quad \text{and} \quad v = Vx = VXy_0 \in L^2(0,T;\mathbf{R}^k).$$

However, this solution of system (4.20) is not unique. Indeed, if x is a solution of system (4.20), $t_i \in Z$, and x_i in $H^1(0,T;\mathbf{R}^n)$ is given by

$$x_i(t) = \begin{cases} 0, & 0 \le t < t_i \\ X(t)z_i, & t_i \le t \le T \end{cases} \quad \forall z_i \in \ker X(t_i),$$

then $X(t)y_0 + x_i(t)$ is also a solution of system (4.20).

The admissibility condition amounts to a change in the state variable via the associated transformation $X(t)$.

LEMMA 4.4. *Assume that* $(\phi,\psi) \in S$*,* $\phi(t,x) = U(t)x + u_0(t)$*, and* $\psi(t,x) = V(t)x + v_0(t)$*. Let* X *be a solution of system* (4.19) *such that* $\det X(t) \neq 0$ *almost everywhere in* $[0,T]$*.*

(i) *For all* $y_0 \in \mathbf{R}^n$*,* $u \in L^2(0,T;\mathbf{R}^m)$ *such that* $|X^{-1}|u \in L^2(0,T;\mathbf{R}^m)$*, and* $v \in L^2(0,T;\mathbf{R}^k)$ *such that* $|X^{-1}|v \in L^2(0,T;\mathbf{R}^k)$*, the system*

$$y' = X^{-1}(B_1u + B_2v), \quad y(0) = y_0, \tag{4.21}$$

has a unique solution in $H^1(0,T;\mathbf{R}^n)$*,* $x \overset{\text{def}}{=} Xy$ *is the unique solution in*

$$H^1_X(0,T;\mathbf{R}^n) \overset{\text{def}}{=} \left\{ x \in H^1(0,T;\mathbf{R}^n) : \begin{array}{l} \exists y \in H^1(0,T;\mathbf{R}^n) \\ \textit{such that } x = Xy \end{array} \right\} \tag{4.22}$$

of the system

$$x' = [A + B_1 U + B_2 V]x + B_1 u + B_2 v, \quad x(0) = X(0)y_0, \tag{4.23}$$

up to a function of the form $X(t)z_0$ *for some* $z_0 \in \ker X(0)$*, and all the solutions of* (4.23) *in* $H^1_X(0,T;\mathbf{R}^n)$ *are given by the expression*

$$x(t) = X(t)\left[y_0 + z_0 + \int_0^t X^{-1}(B_1 u + B_2 v)\, ds\right] \quad \forall z_0 \in \ker X(0). \tag{4.24}$$

(ii) *The subspaces* $\mathcal{U} = \{u \in L^2(0,T;\mathbf{R}^m) : |X^{-1}|u \in L^2(0,T;\mathbf{R}^m)\}$ *and* $\mathcal{V} = \{v \in L^2(0,T;\mathbf{R}^k) : |X^{-1}|v \in L^2(0,T;\mathbf{R}^k)\}$ *are dense in* $L^2(0,T;\mathbf{R}^m)$ *and* $L^2(0,T;\mathbf{R}^k)$*, respectively.*

Proof. (i) By assumption on u and v, the right-hand side of (4.21) belongs to $L^2(0,T;\mathbf{R}^n)$, and its unique solution y belongs to $H^1(0,T;\mathbf{R}^n)$. By direct computation of the derivative of $x = Xy$, it is easy to check that x is a solution of system (4.23). Consider two solutions x_1 and x_2 of system (4.23). The difference $z = x_2 - x_1$ is a solution in $H^1_X(0,T;\mathbf{R}^n)$ of

$$z' = [A + B_1 U + B_2 V]z, \quad z(0) = 0.$$

Since $z \in H^1_X(0,T;\mathbf{R}^n)$, there exists $y \in H^1(0,T;\mathbf{R}^n)$ such that $z = Xy$. The function $y = X^{-1}z$ is the solution in $H^1(0,T)$ of

$$y' = 0 \text{ in } (0,T), \quad y(T) = z(T).$$

Therefore $y(t) = z(T)$ on $[0,T]$. This implies that $z(t) = X(t)z(T)$ in $[0,T]$, $0 = z(0) = X(0)z(T)$, and $z(T) \in \ker X(0)$. Therefore two solutions of system (4.23) can differ only by $X(t)z_0$ for some $z_0 \in \ker X(0)$.

(ii) Define $w_n(t) = 1$ if $|X(t)| \geq 1/n$ and 0 if $|X(t)| < 1/n$. For any $u \in L^2(0,T;\mathbf{R}^n)$, the sequence $\{u_n = u\, w_n\} \subset \mathcal{U}$ converges to u in $L^2(0,T;\mathbf{R}^n)$ by the Lebesgue dominated convergence theorem. ∎

As for normalizability, Definition 4.3 is equivalent to the following definition.

Definition 4.4. *We say that the pair* $(\phi,\psi) \in \widetilde{\Phi} \times \widetilde{\Psi}$ *belongs to* S *or simply that* (ϕ,ψ) *is an* admissible pair *if, for almost all* $s \in [0,T[$*, the matrix differential equation*

$$X_s' = (A + B_1 U + B_2 V)X_s, \quad X_s(s) = I, \tag{4.25}$$

has an $H^1(s,T)$*-solution,* UX_s *and* VX_s *are* $L^2(s,T)$ *matrices,* $|X_s^{-1}|u \in L^2(s,T;\mathbf{R}^m)$*, and* $|X_s^{-1}|v \in L^2(s,T;\mathbf{R}^k)$*.*

4.5. Necessary and sufficient conditions for normalizability.

Lemma 4.5. *Assume that problem* (2.1)–(2.2) *is normalizable. Let* (X,Λ) *be the solution of system* (4.1)*,* $Z = \{t \in [0,T] : \det X(t) = 0\}$*, and let* P *be defined by* (4.2)*.*

(i) X *is a solution in* $H^1(0,T)$ *of the matrix equation* (4.11)*,*

$$X' = (A - RP)X, \quad X(T) = I,$$

such that $\det X(t) \neq 0$ *in* $[0,T]\backslash Z$*,*

$$\begin{aligned} U_* X &= -B_1^\top P X = -B_1^\top \Lambda \in L^2(0,T;\mathbf{R}^n)^n, \\ V_* X &= B_2^\top P X = B_2^\top \Lambda \in L^2(0,T;\mathbf{R}^n)^n \end{aligned} \tag{4.26}$$

for the matrices

$$U_*(t) \stackrel{\text{def}}{=} -B_1^\top(t)P(t) \text{ and } V_*(t) \stackrel{\text{def}}{=} B_2^\top(t)P(t) \tag{4.27}$$

and the pair $(\phi^*, \psi^*) \in S$, *where* $(\phi^*(t,x), \psi^*(t,x)) = (U_*(t)x, V_*(t)x)$.

(ii) *Given* $x_0 \in \operatorname{Im} X(0)$, $u \in L^2(0,T;\mathbf{R}^m)$ *such that* $|X^{-1}|u \in L^2(0,T;\mathbf{R}^m)$, *and* $v \in L^2(0,T;\mathbf{R}^k)$ *such that* $|X^{-1}|v \in L^2(0,T;\mathbf{R}^k)$, *the system*

$$x' = [A - RP]x + B_1 u + B_2 v, \quad x(0) = x_0, \tag{4.28}$$

has a solution $x \in H^1_X(0,T;\mathbf{R}^n)$ *unique up to an element* $X(t)z_0$ *for some* $z_0 \in \ker X(0)$. *Moreover, for any solution* $x \in H^1_X(0,T;\mathbf{R}^n)^n$ *of* (4.28),

$$C_{x_0}(\phi^*(x) + u, \psi^*(x) + v) = \Lambda(0)y_0 \cdot x_0 + \int_0^T |u|^2 - |v|^2 \, dt, \tag{4.29}$$

$C_{x_0}(\phi^*, \psi^*) = \Lambda(0)y_0 \cdot x_0$ *is independent of the choice of* y_0 *such that* $X(0)y_0 = x_0$, *and* $C_{x_0}(\phi^*(x)+u, \psi^*(x)+v)$ *is independent of the choice of the solution* x *in* $H^1_X(0,T;\mathbf{R}^n)$ *to system* (4.28).

(iii) *For all* $x_0 \in \operatorname{Im} X(0)$, $u \in L^2(0,T;\mathbf{R}^m)$ *such that* $|X^{-1}|u \in L^2(0,T;\mathbf{R}^m)$, *and* $v \in L^2(0,T;\mathbf{R}^k)$ *such that* $|X^{-1}|v \in L^2(0,T;\mathbf{R}^k)$,

$$C_{x_0}(\phi^*, \psi^* + v) \le C_{x_0}(\phi^*, \psi^*) \le C_{x_0}(\phi^* + u, \psi^*). \tag{4.30}$$

Remark 4.4. In view of Lemma 4.4(ii), inequalities (4.30) are verified on dense subsets of $L^2(0,T;\mathbf{R}^m)$ and $L^2(0,T;\mathbf{R}^k)$. They define an open loop saddle point in $(0,0) \in \mathcal{U} \times \mathcal{V}$ after a change of the state variable via the transformation $X(t)$.

Proof. (i) This follows from Lemma 4.3(ii).

(ii) Let x be a solution of system (4.28) in $H^1_X(0,T;\mathbf{R}^n)$ and y_0 be such that $X(0)y_0 = x_0$. By Lemma 4.4(i), the solutions x of system (4.28) in $H^1_X(0,T;\mathbf{R}^n)$ are given by (4.24),

$$x(t) = X(t)y(t), \quad y(t) = y_0 + z_0 + \int_0^t X^{-1}(B_1 u + B_2 v)\, ds \quad \forall z_0 \in \ker X(0),$$

and, by definition of $P(t)$, $Px = \Lambda y$. Hence Px is an $H^1(0,T;\mathbf{R}^n)$-function. Differentiate the inner product $\Lambda y \cdot x$ as follows:

$$\begin{aligned}
&\frac{d}{dt}\Lambda y \cdot x = \Lambda' y \cdot x + \Lambda y' \cdot x + \Lambda y \cdot x' \\
&= -[A^\top \Lambda + QX]y \cdot x + \Lambda X^{-1}(B_1 u + B_2 v) \cdot x + \Lambda y \cdot [(A - RP)x + B_1 u + B_2 v] \\
&= -\left[Qx \cdot x + |-B_1^\top Px + u|^2 + |B_2^\top Px + v|^2\right] + |u|^2 - |v|^2
\end{aligned}$$

and

$$\begin{aligned}
C_{x_0}(\phi^* + u, \psi^* + v) &= \Lambda(0)(y_0 + z_0) \cdot x_0 + \int_0^T |u|^2 - |v|^2 \, dt \\
&\Rightarrow C_{x_0}(\phi^*, \psi^*) = \Lambda(0)(y_0 + z_0) \cdot x_0.
\end{aligned}$$

But this last expression is dependent only on x_0. Assume that there exist y_0^1 and y_0^2 such that $X(0)y_0^1 = x_0 = X(0)y_0^2$. Let z_0^1 and z_0^2 be the respective elements of $\ker X(0)$ associated with y_0^1 and y_0^2. Then $y_0^2 + z_0^2 - (y_0^1 - z_0^1) \in \ker X(0)$. By the symmetry in

Lemma 4.3, $X^\top(0)\Lambda(0)(y_0^2+z_0^2-(y_0^1-z_0^1)) = \Lambda(0)^\top X(0)(y_0^2+z_0^2-(y_0^1-z_0^1)) = 0$. Hence

$$\begin{aligned}&\Lambda(0)(y_0^2+z_0^2)\cdot x_0 - \Lambda(0)(y_0^1+z_0^1)\cdot x_0\\ &\quad = \Lambda(0)(y_0^2+z_0^2-(y_0^1-z_0^1))\cdot x_0 = X(0)^\top\Lambda(0)(y_0^2+z_0^2-(y_0^1-z_0^1))\cdot y_0^1 = 0.\end{aligned}$$

The value of $C_{x_0}(\phi^*,\psi^*)$ depends only on x_0. $C_{x_0}(\phi^*+u,\psi^*+v)$ depends only on x_0, u, and v and is independent of the choice of a solution x in $H^1_X(0,T;\mathbf{R}^n)$ to system (4.28).

(iii) It is readily seen from part (ii) that

$$\forall u\in\mathcal{U},\ \forall v\in\mathcal{V},\quad C_{x_0}(\phi^*,\psi^*+v)\le C_{x_0}(\phi^*,\psi^*)\le C_{x_0}(\phi^*+u,\psi^*),$$

and $(\hat u,\hat v)=(0,0)$ is an open loop saddle point. ◻

We are now ready to prove the following result that sheds light on the choice of a definition of the closed loop saddle point in the presence of closed loop strategies with non-L^2-integrable singularities.

THEOREM 4.1. *The following statements are equivalent.*

(i) *Problem* (2.1)–(2.2) *is* normalizable *in the sense of Definition* 4.2.

(ii) *There exists a pair of closed loop strategies* $(\phi^*,\psi^*)\in S$ *that for all* $x_0\in \operatorname{Im}X(0)$*, all* $u\in L^2(0,T;\mathbf{R}^m)$ *such that* $|X^{-1}|u\in L^2(0,T;\mathbf{R}^m)$*, and all* $v\in L^2(0,T;\mathbf{R}^k)$ *such that* $|X^{-1}|v\in L^2(0,T;\mathbf{R}^k)$*,*

$$C_{x_0}(\phi^*,\psi^*+v)\le C_{x_0}(\phi^*,\psi^*)\le C_{x_0}(\phi^*+u,\psi^*). \tag{4.31}$$

(iii) *There exists a pair of linear closed loop strategies* $(\phi^*,\psi^*)\in S$ *(that is,* $\psi^*(t,x)=V_*(t)x$ *and* $\phi^*(t,x)=U_*(t)x$*) such that for all* $x_0\in\operatorname{Im}X(0)$*, there exists a solution* $(\hat x,\hat p)\in H^1_X(0,T;\mathbf{R}^n)\times H^1(0,T;\mathbf{R}^n)$ *of*

$$\begin{cases}\hat x' = A\hat x - R\hat p, & \hat x(0)=x_0,\\ \hat p' + A^\top\hat p + Q\hat x = 0, & \hat p(T) = F\hat x(T),\end{cases} \tag{4.32}$$

and

$$-B_1^\top\hat p = U_*\hat x,\quad B_2^\top\hat p = V_*\hat x. \tag{4.33}$$

For all $x_0\in\operatorname{Im}X(0)$ *the feedback strategies associated with* C_{x_0} *are given by*

$$\phi^*(t,x) = -B_1^\top(t)P(t)x \quad\text{and}\quad \psi^*(t,x) = B_2^\top(t)P(t)x, \tag{4.34}$$

where P *is defined by* (4.2)*, and the value of the closed loop saddle point by*

$$C_{x_0}(\phi^*,\psi^*) = \Lambda(0)y_0\cdot x_0 \tag{4.35}$$

for some $y_0\in\mathbf{R}^n$ *such that* $x_0 = X(0)y_0$*, and this value is independent of the choice of* y_0 *such that* $x_0=X(0)y_0$*.*

Proof. (i) ⇒ (ii). This follows by Lemma 4.5.

(ii) ⇒ (iii). Denote by U_* and V_* and u_* and v_* the matrix and vector functions associated with the pair $(\phi^*,\psi^*)\in S$. By Definition 4.3 and Lemma 4.4, there exists a solution $\overline{X}$ in $H^1_X(0,T)$ to the matrix differential equation (4.19),

$$\overline{X}' = (A+B_1U_*+B_2V_*)\overline{X},\quad \overline{X}(T)=I, \tag{4.36}$$

such that $\det \overline{X}(t) \neq 0$ almost everywhere in $[0,T]$, $U_* \overline{X}$, and $V_* \overline{X}$ in $L^2(0,T)$; $u_* \in L^2(0,T;\mathbf{R}^m)$ such that $|\overline{X}^{-1}| u_* \in L^2(0,T;\mathbf{R}^k)$, and $v_* \in L^2(0,T;\mathbf{R}^m)$ such that $|\overline{X}^{-1}| v_* \in L^2(0,T;\mathbf{R}^k)$.

We know that for all $x_0 \in \operatorname{Im} \overline{X}(0)$, $(\hat{u},\hat{v}) = (0,0)$ is an open loop saddle point of $C_{x_0}(\phi^*(x)+u, \psi^*(x)+v)$, where x is a solution in $H^1_{\overline{X}}(0,T;\mathbf{R}^n)$ of the state equation

$$x' = (A + B_1 U_* + B_2 V_*)x + B_1(u_* + u) + B_2(v_* + v), \quad x(0) = x_0.$$

To get around the nonuniqueness of solution, we start from the other state equation,

$$y' = \overline{X}^{-1}(B_1(u_* + u) + B_2(v_* + v)), \quad y(0) = y_0,$$

for all $y_0 \in \mathbf{R}^n$ and $u \in L^2(0,T;\mathbf{R}^m)$ such that $|\overline{X}^{-1}| u \in L^2(0,T;\mathbf{R}^k)$, and $v \in L^2(0,T;\mathbf{R}^m)$ such that $|\overline{X}^{-1}| v \in L^2(0,T;\mathbf{R}^k)$. By construction, $x(t) = \overline{X}(t) y(t)$, and we just substitute in the utility function to get it in terms of y rather than x: $c_{y_0}(u,v) = C_{\overline{X}(0)y_0}(\phi^*(\overline{X}\, y)+u, \psi^*(\overline{X}\, y)+v)$. This will take care of all $x_0 \in \operatorname{Im} \overline{X}(0)$. From the closed loop saddle point inequalities,

$$c_{y_0}(0,v) \leq c_{y_0}(0,0) \leq c_{y_0}(u,0) \tag{4.37}$$

for all $u \in L^2(0,T;\mathbf{R}^m)$ such that $|\overline{X}^{-1}| u \in L^2(0,T;\mathbf{R}^m)$ and $v \in L^2(0,T;\mathbf{R}^k)$ such that $|\overline{X}^{-1}| v \in L^2(0,T;\mathbf{R}^k)$, the pair $(0,0)$ achieves an open loop saddle point. By [5, Lemma 3.1], $c_{y_0}(u,v)$ is convex-concave and $dc_{y_0}(0,0;u,v) = 0$ for all u and v. A direct computation yields

$$\begin{aligned}\frac{1}{2} dc_{y_0}(0,0;u,v) = & F\hat{y}(T)\cdot z(T) + \int_0^T Q\overline{X}\,\hat{y}\cdot \overline{X}\,z + (U_*\overline{X}\,\hat{y} + u_*)\cdot(U_*\overline{X}\,z + u) \\ & \qquad - (V_*\overline{X}\,\hat{y} + v_*)\cdot(V_*\overline{X}\,z + v)\,dt,\end{aligned}$$

$$\hat{y}' = \overline{X}^{-1}(B_1 u_* + B_2 v_*), \quad \hat{y}(0) = y_0, \quad z' = \overline{X}^{-1}(B_1 u + B_2 v), \quad z(0) = 0. \tag{4.38}$$

By introducing the solution $\pi \in H^1(0,T;\mathbf{R}^n)$ of the adjoint equation

$$\begin{aligned}&\pi' + \overline{X}^{\top} Q\overline{X}\,\hat{y} + \overline{X}^{\top} U_*^{\top}(U_*\overline{X}\,\hat{y} + u_*) - \overline{X}^{\top} V_*^{\top}(V_*\overline{X}\,\hat{y} + v_*) = 0,\\ &\pi(T) = F\hat{y}(T),\end{aligned} \tag{4.39}$$

we get

$$0 = \frac{1}{2} dc_{y_0}(0,0;u,v) = \int_0^T \begin{aligned}[t]&(B_1^{\top}\overline{X}^{-\top}\pi + U_*\overline{X}\,\hat{y} + u_*)\cdot u \\ &+ (B_2^{\top}\overline{X}^{-\top}\pi - V_*\overline{X}\,\hat{y} - v_*)\cdot v\,dt.\end{aligned}$$

By the density of $\mathcal{U}$ in $L^2(0,T;\mathbf{R}^m)$ and $\mathcal{V}$ in $L^2(0,T;\mathbf{R}^k)$ in Lemma 4.4(ii),

$$\begin{aligned}&B_1^{\top}\overline{X}^{-\top}\pi + U_*\overline{X}\,\hat{y} + u_* = 0 \text{ and } B_2^{\top}\overline{X}^{-\top}\pi - V_*\overline{X}\,\hat{y} - v_* = 0\\ &\qquad \Rightarrow\ -B_1^{\top}\overline{X}^{-\top}\pi \in L^2(0,T;\mathbf{R}^m) \text{ and } B_2^{\top}\overline{X}^{-\top}\pi \in L^2(0,T;\mathbf{R}^k).\end{aligned} \tag{4.40}$$

Using the above identities (4.40) to eliminate u^* and v^*, we see that the pair $(\hat{y},\pi)$ is a solution in $H^1(0,T;\mathbf{R}^n) \times H^1(0,T;\mathbf{R}^n)$ of the linear system

$$\begin{cases}\pi' + \overline{X}^{\top} Q\overline{X}\,\hat{y} - \overline{X}^{\top}(U_*^{\top}B_1^{\top} + V_*^{\top}B_2^{\top})\overline{X}^{-\top}\pi = 0, \quad \pi(T) = F\hat{y}(T),\\ \hat{y}' = \overline{X}^{-1}\left[-R\overline{X}^{-\top}\pi - (B_1U_* + B_2V_*)\overline{X}\,\hat{y}\right], \quad \hat{y}(0) = y_0.\end{cases} \tag{4.41}$$

However, the solution of the equation for π is not unique as an element of $H^1(0,T;\mathbf{R}^n)$. Let $\hat{p} \in H^1(0,T;\mathbf{R}^n)$ be the solution of the equation

$$\hat{p}' + A^\top \hat{p} + Q\bar{X}\hat{y} = 0, \quad \hat{p}(T) = F\hat{y}(T), \tag{4.42}$$

and consider the $H^1(0,T;\mathbf{R}^n)$-function $\overline{X}^\top \hat{p}$. A direct computation using equation (4.36) for $\overline{X}$ shows that $\overline{X}^\top \hat{p}$ is also a solution of the first equation of (4.41). So we can choose a solution π in the space

$$H^1_{\overline{X}^\top}(0,T;\mathbf{R}^n) \stackrel{\text{def}}{=} \left\{ q \in H^1(0,T;\mathbf{R}^n) : \begin{array}{l} \exists r \in H^1(0,T;\mathbf{R}^n) \\ \text{such that } q = \overline{X}^\top r \end{array} \right\}. \tag{4.43}$$

Moreover, by using $\pi = \overline{X}^\top \hat{p}$ and by introducing the variable $\hat{x} \stackrel{\text{def}}{=} \overline{X}\,\hat{y}$ in $H^1_{\overline{X}}(0,T;\mathbf{R}^n)$, we finally get that, for all $y_0 \in \mathbf{R}^n$, the pair $(\hat{x},\hat{p})$ is a solution in $H^1_{\overline{X}}(0,T;\mathbf{R}^n)\times H^1(0,T;\mathbf{R}^n)$ of the system

$$\begin{cases} \hat{x}' = A\hat{x} - R\hat{p}, & \hat{x}(0) = X(0)y_0, \\ \hat{p}' + A^\top \hat{p} + Q\hat{x} = 0, & \hat{p}(T) = F\hat{x}(T), \end{cases} \tag{4.44}$$

and from identity (4.40),

$$B_1^\top \hat{p} + U_*\hat{x} + u_* = 0, \quad B_2^\top \hat{p} - V_*\hat{x} - v_* = 0. \tag{4.45}$$

By linearity of system (4.44), the pair $(2\hat{x}, 2\hat{p})$ is also a solution for the initial condition $2y_0$, and necessarily,

$$2(B_1^\top \hat{p} + U_*\hat{x}) + u_* = 0, \quad 2(B_2^\top \hat{p} - V_*\hat{x}) - v_* = 0. \tag{4.46}$$

The last two sets of identities yield $u_* = 0$, $v_* = 0$, and

$$-B_1^\top \hat{p} = U_*\hat{x}, \quad B_2^\top \hat{p} = V_*\hat{x}. \tag{4.47}$$

From this we conclude that the strategies $(\phi^*,\psi^*) \in S$ are linear since $\psi^*(t,x) = V_*(t)x$ and $\phi^*(t,x) = U_*(t)x$.

(iii) $\Rightarrow$ (i). Let $\overline{X}$ be an $H^1(0,T)$-solution of the matrix equation $\overline{X}' = (A + B_1U_* + B_2V_*)\overline{X}$, $\overline{X}(T) = I$. By substituting identities (4.33) into the first equation of (4.32), we get for all $x_0 \in \operatorname{Im}\overline{X}(0)$, $\hat{x}' = (A + B_1U_* + B_2V_*)\hat{x}$, $\hat{x}(0) = x_0$. By Lemma 4.4(i), the solutions of this equation in $H^1_{\overline{X}}(0,T;\mathbf{R}^n)$ are $\overline{X}(t)y_0$ for all $y_0 \in \mathbf{R}^n$ such that $x_0 = X(0)y_0$ and, in particular, $\hat{x}(T) = y_0$. As a consequence, for all $y_0 \in \mathbf{R}^n$ the coupled system

$$\begin{aligned} \hat{x}' = A\hat{x} - R\hat{p}, \quad \hat{x}(T) = y_0, \\ \hat{p}' + A^\top \hat{p} + Q\hat{x} = 0, \quad \hat{p}(T) = Fy_0 \end{aligned} \tag{4.48}$$

has a solution $(\hat{x},\hat{p}) \in H^1_{\overline{X}}(0,T;\mathbf{R}^n) \times H^1(0,T;\mathbf{R}^n)$ such that $\hat{x}(t) = \overline{X}(t)y_0$. But the solution of system (4.48) with final conditions is unique. By linearity, there exists a unique $H^1(0,T)$-solution (X,Λ) to the matrix system

$$\begin{aligned} X' = AX - R\Lambda, \quad X(T) = I, \\ \Lambda' + A^\top \Lambda + QX = 0, \quad \Lambda(T) = F, \end{aligned} \tag{4.49}$$

such that $\hat{x} = Xy_0$ and $\hat{p} = \Lambda y_0$. By uniqueness, for all $y_0 \in \mathbf{R}^n$, $\hat{x}(t) = \overline{X}(t)y_0$. Hence $X(t) = \overline{X}(t)$ and $\det X(t) \neq 0$ for almost all t in $[0,T]$. This shows that the system is normalizable and completes the proof. $\square$

4.6. Definition of a closed loop saddle point. In view of Theorem 4.1, we adopt the following definition that says that the original problem can be changed via feedback in such a way that the resulting problem has an open loop saddle point at $(0,0)$. It will still be referred to as a *closed loop saddle point*, yet it is more a notion of *structural saddle point*.

DEFINITION 4.5.

(i) *Given* $x_0 \in \mathbf{R}^n$, $(\phi^*, \psi^*) \in S$ *is* a closed loop saddle point *of* C_{x_0} *if there exists a solution in* $H^1_X(0,T;\mathbf{R}^n)$ *to the state equation*

$$\hat{x}' = (A + B_1U_* + B_2V_*)\hat{x} + B_1u_* + B_2v_*, \quad \hat{x}(0) = x_0, \tag{4.50}$$

and for all solutions $\hat{x} \in H^1_X(0,T;\mathbf{R}^n)$ *of* (4.50)*, all* $u \in L^2(0,T;\mathbf{R}^m)$ *such that* $|X^{-1}|u \in L^2(0,T;\mathbf{R}^m)$*, all* $v \in L^2(0,T;\mathbf{R}^k)$ *such that* $|X^{-1}|v \in L^2(0,T;\mathbf{R}^k)$*, and all solutions* $x_u, x_v \in H^1_X(0,T;\mathbf{R}^n)$ *of the state equations*

$$x_u' = (A + B_1U_* + B_2V_*)x_u + B_1(u_* + u) + B_2v_*, \quad x_u(0) = x_0, \tag{4.51}$$

$$x_v' = (A + B_1U_* + B_2V_*)x_v + B_1u_* + B_2(v_* + v), \quad x_v(0) = x_0, \tag{4.52}$$

the following inequalities are verified:

$$C_{x_0}(\phi^*(x_v), \psi^*(x_v) + v) \leq C_{x_0}(\phi^*(\hat{x}), \psi^*(\hat{x})) \leq C_{x_0}(\phi^*(x_u) + u, \psi^*(x_u)). \tag{4.53}$$

(ii) *We say that* $(\phi^*, \psi^*) \in S$ *is an* $X(0)$-global closed loop saddle point *of* C_{x_0} *if for all* $x_0 \in \operatorname{Im} X(0)$ *the inequalities* (4.53) *are verified.*

Remark 4.5. By definition of $H^1_X(0,T;\mathbf{R}^n)$, there exists $y \in H^1(0,T;\mathbf{R}^n)$ such that $\hat{x} = Xy$, $x_0 = \hat{x}(0) = X(0)y(0)$, and the existence of a solution to the state equation (4.50) implies that $x_0 \in \operatorname{Im} X(0)$.

Remark 4.6. If $(\phi^*, \psi^*) \in \Phi \times \Psi$ is an L^2-integrable global closed loop saddle point in the sense of Definition 3.2(ii), it is an $X(0)$-global closed loop saddle point by the equivalence of parts (i) and (ii) of Theorem 3.1.

4.7. Closed loop saddle points when $v(x_0)$ is finite. We first study closed loop saddle points when $v(x_0)$ is finite.

THEOREM 4.2. *Given* $x_0 \in \mathbf{R}^n$*, the following statements are equivalent.*

(i) C_{x_0} *has a closed loop saddle point* $(\phi^*, \psi^*) \in S$*, and* $C_{x_0}(u,v)$ *is convex in* u *and concave in* v.

(ii) C_{x_0} *has a closed loop saddle point* $(\phi^*, \psi^*) \in \Phi \times \Psi$*, and* $C_{x_0}(u,v)$ *is convex in* u *and concave in* v (*case* (a) *of Theorem* 3.2).

(iii) C_{x_0} *has an open loop saddle point.*

Proof. (i) $\Rightarrow$ (iii). By Remark 4.5, $x_0 \in \operatorname{Im} X(0)$: there exists y_0 such that $x_0 = X(0)y_0$. From part (ii) of the proof of Theorem 4.1, there exists a solution $(\hat{x}, \hat{p})$ in $H^1(0,T;\mathbf{R}^n)^2$ to the coupled system (4.48) with $\hat{x}(0) = X(0)y_0$. Since $C_{x_0}(u,v)$ is convex in u and concave in v, C_{x_0} has an open loop saddle point by [5, Thm. 2.4].

(iii) $\Rightarrow$ (ii). Denote by $(\hat{u}, \hat{v})$ the pair achieving the open loop saddle point. By [5, Thm. 2.4], $C_{x_0}(u,v)$ is convex in u and concave in v. By definition of $\Phi \times \Psi$, $(\hat{u}, \hat{v}) \in \Phi \times \Psi$. By inequalities (3.20) and (3.21) in Lemma 3.3,

$$\inf_{\phi \in \Phi} C_{x_0}(\phi, \hat{v}) = \inf_{u \in L^2(0,T;\mathbf{R}^m)} C_{x_0}(u, \hat{v}) = C_{x_0}(\hat{u}, \hat{v}),$$

$$\sup_{\psi \in \Psi} \inf_{\phi \in \Phi} C_{x_0}(\phi, \psi) \geq v^-(x_0) \geq \inf_{\phi \in \Phi} C_{x_0}(\phi, \hat{v}) = C_{x_0}(\hat{u}, \hat{v}).$$

By inequalities (3.22) and (3.23) in Lemma 3.3,

$$\sup_{\psi \in \Psi} C_{x_0}(\hat{u}, \psi) = \sup_{v \in L^2(0,T;\mathbf{R}^k)} C_{x_0}(\hat{u}, v) = C_{x_0}(\hat{u}, \hat{v}),$$
$$\inf_{\phi \in \Phi} \sup_{\psi \in \Psi} C_{x_0}(\phi, \psi) \leq v^+(x_0) \leq \sup_{\psi \in \Psi} C_{x_0}(\hat{u}, \psi) = C_{x_0}(\hat{u}, \hat{v}).$$

Hence, there exists $(\hat{u}, \hat{v}) \in \Phi \times \Psi$ such that for all $\phi \in \Phi$ and all $\psi \in \Psi$,

$$C_{x_0}(\hat{u}, \psi) \leq C_{x_0}(\hat{u}, \hat{v}) \leq C_{x_0}(\phi, \hat{v}).$$

By Lemma 3.2, $(\hat{u}, \hat{v})$ is a closed loop saddle point of C_{x_0} in $\Phi \times \Psi$.

(ii) $\Rightarrow$ (i). By assumption, C_{x_0} is convex-concave. As in Remark 4.6, if $(\phi^*, \psi^*) \in \Phi \times \Psi$ is an L^2-integrable closed loop saddle point in the sense of Definition 3.2(i), it is a closed loop saddle point in the sense of Definition 4.5(i). Hence this proof follows from the proof of (i) $\Rightarrow$ (ii) of Theorem 3.1. ◻

We also have a global version of the previous theorem (case (a)).

THEOREM 4.3. *The following statements are equivalent.*

(i) *C_{x_0} has an $X(0)$-global closed loop saddle point $(\phi^*, \psi^*) \in S$, $X(0) = \mathbf{R}^n$, and $C_{x_0}(u, v)$ is convex in u and concave in v.*

(ii) *C_{x_0} has a global closed loop saddle point $(\phi^*, \psi^*) \in \Phi \times \Psi$ and $C_{x_0}(u, v)$ is convex in u and concave in v.*

(iii) *For all $x_0 \in \mathbf{R}^n$, C_{x_0} has an open loop saddle point.*

Proof. (i) $\Rightarrow$ (iii). This follows from the equivalence of (i) and (iii) in Theorem 4.2.

(iii) $\Rightarrow$ (ii). By [5, Thm. 2.4], $C_{x_0}(u, v)$ is convex in u and concave in v. By [5, Thm. 2.9], there exists a unique symmetrical solution to the Riccati equation with elements in $H^1(0, T)$. By Theorem 3.1, C_{x_0} has a global closed loop saddle point $(\phi^*, \psi^*) \in \Phi \times \Psi$.

(ii) $\Rightarrow$ (i). By Theorem 3.1(v), $X(t)$ is invertible for all $[0, T]$, $\operatorname{Im} X(0) = \mathbf{R}^n$, and the problem is normalizable. By Theorem 4.1, C_{x_0} has an $X(0)$-global closed loop saddle point in S and $X(0) = \mathbf{R}^n$. ◻

4.8. Closed loop saddle points when either $v^-(x_0)$ or $v^+(x_0)$ is not finite. From Theorem 4.2, when the value of the game $v(x_0)$ is finite, the closed loop strategy in S can be trivially chosen as $(\phi^*(t, x), \psi^*(t, x)) = (\hat{u}(t), \hat{v}(t))$; from Theorem 4.3, when the value of the game $v(x_0)$ is finite for all $x_0 \in \mathbf{R}^n$, the global closed loop strategy is equal to the L^2-integrable closed loop strategy $(\phi^*, \psi^*) = (-B_1^\top P\hat{x}, B_2^\top P\hat{x}) \in \Phi \times \Psi$, where P is the $H^1(0, T)$ solution of the Riccati differential equation. The conclusion is that *closed loop strategies with non-L^2-integrable singularities will only occur when either $v^-(x_0)$ or $v^+(x_0)$ is not finite.*

The new Definition 4.5 of a closed loop saddle point was introduced to accommodate closed loop strategies with non-L^2-integrable singularities, but its relation to the open loop values is not as straightforward as in the L^2-integrable case of Theorem 3.2. Yet, a complete classification is obtained for the six cases along the lines of Theorem 3.2 in terms of the u-convexity and v-concavity of the utility function.

THEOREM 4.4. *Assume that C_{x_0} has a closed loop saddle point $(\phi^*, \psi^*) \in S$. Denote by $(\hat{u}, \hat{v}) = (\phi^*, \psi^*)$ the associated controls.*

(i) (case (b)). *$v^-(x_0)$ finite and $v^+(x_0) = +\infty$ if and only if $C_{x_0}(u, v)$ is convex in u and not concave in v, and $v \mapsto \inf_v C_{x_0}(u, v)$ is concave. In that case,*

$$v^-(x_0) = \inf_{u \in L^2(0,T;\mathbf{R}^n)} C_{x_0}(u, \hat{v}) = C_{x_0}(\hat{u}, \hat{v}) = C_{x_0}(\phi^*, \psi^*). \tag{4.54}$$

(ii) (case (e)). $v^-(x_0) = v^+(x_0) = +\infty$ *if and only if* $C_{x_0}(u,v)$ *is convex in* u *and not concave in* v*, and* $v \mapsto \inf_v C_{x_0}(u,v)$ *is not concave. In that case,*

$$\sup_{\psi\in\Psi}\inf_{\phi\in\Phi} C_{x_0}(\phi,\psi) = \inf_{\phi\in\Phi}\sup_{\psi\in\Psi} C_{x_0}(\phi,\psi) = +\infty.$$

Proof. From the proof of Theorem 4.1, there exists a solution $(\hat{x},\hat{p})$ in $H^1(0,T;\mathbf{R}^n)^2$ to the coupled system (4.48) ($x_0 \in \operatorname{Im} X(0)$). Since $C_{x_0}(u,v)$ is convex in u, we have

$$v^-(x_0) \geq \inf_{u\in L^2(0,T;\mathbf{R}^m)} C_{x_0}(u,\hat{v}) = C_{x_0}(\hat{u},\hat{v}) > -\infty$$

for the pair $(\hat{u},\hat{v}) = (-B_1^\top \hat{p}, B_2^\top \hat{p})$. Moreover, since $C_{x_0}(u,v)$ is not concave in v, then $v^+(x_0) = +\infty$. As a result, only two cases can occur: $v^-(x_0)$ finite and $v^+(x_0) = +\infty$ or $v^-(x_0) = v^+(x_0) = +\infty$. The property in (ii) follows from Lemma 3.3. ∎

Remark 4.7. Contrarily to the L^2-integrable closed loop saddle point, case (e) can occur, as can be seen from the system and utility function (2.25).

Since the cases (c) and (f) are dual of cases (b) and (e), we have the dual result.

THEOREM 4.5. *Assume that* C_{x_0} *has a closed loop saddle point* $(\phi^*,\psi^*) \in S$*. Denote by* $(\hat{u},\hat{v}) = (\phi^*,\psi^*)$ *the associated controls.*

(i) (case (c)). $v^+(x_0)$ *finite and* $v^-(x_0) = -\infty$ *if and only if* $C_{x_0}(u,v)$ *is concave in* v *and not convex in* u *and* $u \mapsto \sup_v C_{x_0}(u,v)$ *is convex. In that case*

$$v^+(x_0) = \sup_{v\in L^2(0,T;\mathbf{R}^k)} C_{x_0}(\hat{u},v) = C_{x_0}(\hat{u},\hat{v}) = C_{x_0}(\phi^*,\psi^*). \tag{4.55}$$

(ii) (case (f)). $v^-(x_0) = v^+(x_0) = -\infty$ *if and only if* $C_{x_0}(u,v)$ *is concave in* v *and not convex in* u *and* $u \mapsto \sup_v C_{x_0}(u,v)$ *is not convex. In that case*

$$\sup_{\psi\in\Psi}\inf_{\phi\in\Phi} C_{x_0}(\phi,\psi) = \inf_{\phi\in\Phi}\sup_{\psi\in\Psi} C_{x_0}(\phi,\psi) = -\infty.$$

Remark 4.8. Part (ii) of Theorems 4.4 and 4.5 justifies the terminology "degenerate" for cases (e) and (f), since what would be the candidate for a closed loop saddle point identity in $\Phi\times\Psi$ is, respectively, equal to $+\infty$ and $-\infty$. This is to be compared with [2, Definition 2.3 and Theorem 3.1]. If the problem is normalizable, then the conclusions of Theorems 4.4 and 4.5 hold for all $x_0 \in \operatorname{Im} X(0)$ (cf. Theorem 4.1).

Finally, by complementarity, we have the last case (d) of Theorem 3.2.

THEOREM 4.6 (case (d)). *Assume that* C_{x_0} *has a closed loop saddle point* $(\phi^*,\psi^*) \in S$*. Then* $v^-(x_0) = -\infty$ *and* $v^+(x_0) = +\infty$ *if and only if* $C_{x_0}(u,v)$ *is not convex in* u *and not concave in* v*.*

Remark 4.9. Case (d) can definitely occur, as all the other cases. Again, in the work of Bernhard [2], the utility function was convex in u since $F \geq 0$ and $Q(t) \geq 0$. Hence, only cases (a) and (b) could occur, and case (e) is still a degenerate one in the sense of his definition.

4.9. An example of a problem that is not normalizable. We conclude with an example where $\det X(t) = 0$ on an interval of nonzero length. Yet there are solutions to the matrix Riccati differential equation, and the open loop lower value of the game is finite for all initial conditions. The associated strategies can be obtained by feedback. So, the condition $\det X(t) \neq 0$ almost everywhere in $[0,T]$ might not be the most general one, and Definitions 4.3 and 4.5 might be further relaxed or generalized.

Example 4.3. Consider the dynamics and utility function in the time interval $[0, 3]$,

$$x'(t) = b_1(t)\, u(t) + b_2(t)\, v(t), \text{ a.e. in } [0, 3], \quad x(0) = x_0, \tag{4.56}$$

$$C_{x_0}(u, v) = \frac{1}{2}|x(3)|^2 + \int_0^3 |u(t)|^2 - |v(t)|^2\, dt, \tag{4.57}$$

where

$$b_1(t) = \begin{cases} 2 - t, & 0 \le t < 1, \\ 0, & 1 \le t < 2, \\ 3 - t, & 2 \le t \le 3, \end{cases} \quad b_2(t) = \begin{cases} t, & 0 \le t < 1, \\ 0, & 1 \le t < 2, \\ t - 1, & 2 \le t \le 3. \end{cases} \tag{4.58}$$

Here $A = 0$, $B_1(t) = b_1(t)$, $B_2(t) = b_2(t)$, $F = 1/2$, $Q = 0$, and

$$R(t) = b_1(t)^2 - b_2(t)^2 = \begin{cases} 4(1 - t), & 0 \le t < 1, \\ 0, & 1 \le t < 2, \\ 4(2 - t), & 2 \le t \le 3. \end{cases} \tag{4.59}$$

We show that $v^-(x_0) = (x_0)^2/2$ and $v^+(x_0) = +\infty$. For the open loop lower value of the game, the minimization with respect to u has a unique solution for all (x_0, v) since the utility function $u \mapsto C_{x_0}(u, v)$ is convex and bounded below by $-\|v\|^2_{L^2}$. The minimizer is completely characterized by the coupled system

$$\begin{cases} x' = -b_1^2\, p + b_2\, v \text{ a.e. in } [0, 3], & x(0) = x_0, \qquad \hat{u} = -b_1\, p, \\ p' = 0 \text{ a.e. in } [0, 3], & p(3) = \dfrac{1}{2}x(3). \end{cases}$$

From this,

$$J^-_{x_0}(v) \overset{\text{def}}{=} \inf_{u \in L^2(0,2;\mathbf{R})} C_{x_0}(u, v) = C_{x_0}(\hat{u}, v) = \frac{1}{4}\left[x_0 + \int_0^3 b_2\, v\, ds\right]^2 - \int_0^3 |v|^2\, dt.$$

It is readily seen that $J^-_{x_0}$ is concave in v and that the supremum with respect to v of $J^-_{x_0}(v)$ exists. Indeed, from the first order condition, for all v,

$$\frac{1}{2}dJ^-_{x_0}(\hat{v}; v) = \frac{1}{4}\left[x_0 + \int_0^3 b_2\, \hat{v}\, ds\right] \int_0^3 b_2\, v\, ds - \int_0^3 \hat{v}\, v(t)\, dt = 0,$$

there is a unique stationary point $\hat{v}(t) = b_2(t)\, x_0/2$, the Hessian is negative,

$$\frac{1}{2}d^2 J^-_{x_0}(\hat{v}; v; v) = \frac{1}{4}\left[\int_0^3 b_2\, v\, ds\right]^2 - \int_0^3 |v|^2\, dt \le -\frac{1}{3}\int_0^3 |v|^2\, dt,$$

and the open loop lower value of the game is $v^-(x_0) = J^-_{x_0}(\hat{v}) = (x_0)^2/2$. Moreover,

$$\hat{u}(t) = \begin{cases} -(2 - t)\frac{1}{2}x_0, & 0 \le t < 1, \\ 0, & 1 \le t < 2, \\ -(3 - t)\frac{1}{2}x_0, & 2 \le t \le 3, \end{cases} \quad \hat{v}(t) = \begin{cases} t\frac{1}{2}x_0, & 0 \le t < 1, \\ 0, & 1 \le t < 2, \\ (t - 1)\frac{1}{2}x_0, & 2 \le t \le 3. \end{cases} \tag{4.60}$$

However, the open loop upper value of the game is $v^+(x_0) = +\infty$ for all $x_0 \in \mathbf{R}$. Indeed, pick the sequence of controls $\{v_n\}$, $n \geq 1$, $v_n(t) = 0$ in $[0, 2]$ and $v_n(t) = n$ in $[2, 3]$. The corresponding sequence of states at time $t = 3$ is

$$x_n(3) = x_0 + \int_0^3 b_1\, u\, dt + n \int_2^3 (t-1)\, dt = \left[x_0 + \int_0^3 b_1\, u\, dt \right] + \frac{3}{2} n.$$

Denote by X the square bracket that does not depend on n. Then

$$\begin{aligned} C_{x_0}(u, v_n) &= \frac{1}{2} \left| X + \frac{3}{2} n \right|^2 + \int_0^3 |u|^2\, dt - \int_2^3 n^2\, dt \\ &= \frac{1}{8} n^2 + \frac{3}{2} n\, X + \frac{X^2}{2} + \int_0^3 |u|^2\, dt \to +\infty \text{ as } n \to +\infty. \end{aligned}$$

Thus for all $x_0 \in \mathbf{R}$ and $u \in L^2(0, T; \mathbf{R})$,

$$\sup_{v \in L^2(0,T;\mathbf{R})} C_{x_0}(u, v) = +\infty \quad \Rightarrow v^+(x_0) = +\infty.$$

Therefore, $C_{x_0}(u, v)$ has no open loop saddle point. For all x_0, the coupled system

$$\hat{x}' = -R\hat{p}, \quad \hat{x}(0) = x_0 \text{ and } \hat{p} = 0, \quad \hat{p} = \frac{1}{2}\hat{x}(3), \quad \hat{u} = -b_1 \hat{p}, \text{ and } \hat{v} = b_2 \hat{p} \tag{4.61}$$

has a unique solution in $H^1(0, 3)$. The unique solution of system (4.1),

$$\begin{cases} X' = AX - R\Lambda, & X(T) = I, \\ \Lambda' + A^\top \Lambda + QX = 0, & \Lambda(T) = F, \end{cases} \tag{4.62}$$

is given by

(4.63)

$$X(t) = \left\{ \begin{array}{ll} (t-1)^2, & 0 \leq t < 1 \\ 0, & 1 \leq t < 2 \\ (t-2)^2, & 2 \leq t \leq 3 \end{array} \right\}, \quad P_c(t) = \left\{ \begin{array}{ll} \dfrac{1}{2(t-1)^2}, & 0 \leq t < 1 \\ c \text{ (arbitrary)}, & 1 \leq t < 2 \\ \dfrac{1}{2(t-2)^2}, & 2 \leq t \leq 3 \end{array} \right\}.$$
$$\Lambda(t) = 1/2,$$

The problem is not normalizable since $X(t) = 0$ in $[1, 2]$. Yet, the associated optimal strategies are feedback strategies of the usual forms, $\hat{u} = -b_1 P_c x$ and $\hat{v} = b_2 P_c x$, and the linear closed loop strategies are

$$\phi^*(t, x) = U_c(t)x = -b_1(t) P_c(t) x \quad \text{and} \quad \psi^*(t, x) = V_c(t) x = b_2(t) P_c(t) x.$$

The function X is also a solution of (4.19) in Definition 4.3,

$$X' = (b_1 U_c + b_2 V_c) X, \quad X(3) = I.$$

If we adopt the convention that a function $u \in L^2(0, 3)$ such that $|X^{-1}| u \in L^2(0, 3)$ implies that $u = 0$ on $Z = \{t \in [0, 3] : X(t) = 0\} = [1, 2]$ and adopt the same for the function v, it can be shown that $dc_{x_0}(0, 0, : u, v) = 0$ for all u and v. However, we have not been able to prove the convexity-concavity of $c_{x_0}(u, v)$ to conclude that the problem has a closed loop saddle point in the sense of Definitions 4.3 and 4.5.

Another issue is the meaning of a solution to the Riccati equation $P' - R\,P^2 = 0$, with final value $P(3) = 1/2$ and a discontinuous function R. What is the effect of a discontinuity in $R(t)$? For instance the following solutions are continuous in $t = 1$:

$$P(t) = \left\{ \begin{array}{ll} \bar{c}/(1 + 2\bar{c}\,(t-1)^2), & 0 \le t < 1 \\ \bar{c}, & 1 \le t < 2 \\ 1/(2(t-2)^2), & 2 \le t \le 3 \end{array} \right\}$$

for some arbitrary constant $\bar{c} \in \mathbf{R}$. The singularity at $t_2 = 2$ is independent of $\bar{c}$. For $\bar{c} > -1/2$ it is the only singularity. For $\bar{c} \le -1/2$, there is a second singularity at $t_1 = 1 - \sqrt{1/(-2\bar{c})}$ in the interval $[0, 1)$. We also have the solutions P_c of (4.63) with a singularity in $t_1 = 1$. Therefore the solution of the Riccati equation is not unique.

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