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BOUNDEDLY NONHOMOGENEOUS ELLIPTIC AND PARABOLIC EQUATIONS

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ABSTRACT. This paper considers elliptic equations of the form

$$0 = F(u_{x^ix^j}, u_{x^i}, u, 1, x) \qquad (*)$$

and parabolic equations of the form

$$u_t = F(u_{x^ix^j}, u_{x^i}, u, 1, t, x), \qquad (**)$$

where $F(u_{ij}, u_i, u, \beta, x)$ and $F(u_{ij}, u_i, u, \beta, t, x)$ are positive homogeneous functions of the first order of homogeneity with respect to (u_{ij}, u_i, u, β), convex upwards with respect to (u_{ij}) and satisfying a uniform condition of strict ellipticity. Under certain smoothness conditions on F and boundedness from above of the second derivatives of F with respect to (u_{ij}, u_i, u), solvability is established for these equations of a problem over the whole space, of the Dirichlet problem in a domain with a sufficiently regular boundary (for the equation $(*)$), and of the Cauchy problem and the first boundary value problem (for equation $(**)$). Solutions are sought in the classes $C^{2+\alpha}$, and their existence is proved with the aid of internal a priori estimates in $C^{2+\alpha}$.

Bibliography: 29 titles.

Boundedly nonhomogeneous elliptic and parabolic equations were introduced in [1 They are defined by functions $F = F(u_{ij}, u_i, u)$ such that the first derivatives of F wit respect to u_{ij}, u_i and u are bounded and such that the expression

$$\left|F - \left(F_{u_{ij}}u_{ij} + F_{u_i}u_i + F_u u\right)\right|,$$

which is called the *measure of the nonhomogeneity of* F, is also bounded. The meaning (the last condition becomes clear (see [1]) if for $\beta > 0$ we introduce into consideratio instead of F, the function

$$\Phi(u_{ij}, u_i, u, \beta) = \beta F\left(\tfrac{1}{\beta}u_{ij}, \tfrac{1}{\beta}u_i, \tfrac{1}{\beta}u\right).$$

Then Φ becomes a positive homogeneous function of (u_{ij}, u_i, u, β) of the first order (homogeneity, the first derivatives of which with respect to (u_{ij}, u_i, u, β) are bounded. was shown in [1] that all boundedly nonhomogeneous functions may be written in th form $\inf_\alpha \sup_\beta F^{\alpha\beta}$, where the $F^{\alpha\beta}$ are linear (affine) functions of u_{ij}, u_i, and u, and th elliptic and parabolic equations with such functions are connected in a natural way wi game problems of control by means of diffusion processes.

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The theory of solvability of equations in Sobolev classes, formulated in [1], is only suitable for the case in which the number of space variables is equal to two. In this article we construct, by completely different methods, a theory of solvability in Hölder classes under much stronger assumptions than in [1]. The most essential assumption is the requirement of upwards convexity of F with respect to the variables u_{ij} (this requirement is not imposed with respect to the remaining variables).

The most typical representatives of elliptic equations with boundedly nonhomogeneous functions F, convex upwards with respect to (u_{ij}, u_i, u), are the Bellman equations of the form

$$\inf_s (L^s u + f^s) = 0, \tag{1}$$

where the L^s are linear elliptic operators. We recall that for such equations it was proved in [2] and [3], under very weak assumptions of nonsingularity of the L^s, that a solution exists over the whole space with bounded Sobolev derivatives (see also [4]). This was accomplished by means of probability methods, which were subsequently used in [5] and [6] to do away with the nonsingularity assumptions in considering the problem over the whole space. A method was then presented in [7] for reducing the optimal control problem to a sequence of optimal stopping problems with switching from one control to another, and a proof was given for the solvability of the first boundary value problem for parabolic degenerate Bellman equations (a development of the results in [7] is given in [8]; see also [9]). In all these papers, and also [10] and [11], the derivatives are taken in a generalized sense of one kind or another.

Up to about 1979, probability methods predominated in the study of the Bellman equations. There then appeared the papers [12]–[14], and others, in which this study was carried out by methods of the theory of differential equations. In [12] it was shown for the first time that if s takes on only two values the solution of (1) has second Hölder derivatives. In [13] a method was employed which, from the probability theory point of view, is similar to the method employed in [7] and [8]. One of the features of this method, in a probability interpretation, is the introduction of randomized switching from one control to another (treated by the authors as a "penalty" for switching), having the same sense as the randomized stopping in [15]. The basic results of [13] and [14] relate to elliptic equations and, more to the point, they are weaker than the results in [7] since they require fulfillment of the stong condition of nondegeneracy and finiteness of the set of values s. It shoud be noted, however, that in [14] there are no conditions on the magnitude of the coefficient of the unknown function; but this condition would appear automatically if the results of [7] carry over directly (without the addition of methods known at the present time) to elliptic equations.

The methods of this paper are standard methods of the theory of differential equations, different from the methods of [12]–[14], and are based on obtaining an a priori interior estimate of the norm of the solution in $C^{2+\alpha}$. This estimate, which we derive in §2, is based in turn on [16] and the simple consideration that the second derivative of the lower bound of a set of functions does not exceed the upper bound of the second derivatives of these functions. This consideration allows us, for example, in the case of equation (1), to insert the sign of the second derivative under the inf sign, replacing it by sup and replacing the equals sign by an inequality sign. After this we carry the sign of the second derivative under the sign of L^s, adding additional coordinates, and we then make it possible, with the aid of the results from [16] and equation (1) itself, to estimate the

oscillation of the second derivative of the solution in a small ball in terms of the oscillation in a large ball. The additional coordinates, of which we have spoken, have the probability meaning of the derivatives of the solution of a stochastic equation with respect to the initial data (these derivatives appear automatically upon differentiating the probability solution with respect to the initial data) and in §4 they also help us to obtain an estimate of the maxima of the second derivatives. Unfortunately, these ideas are realized in §§2 and 4 much more easily and breifly without explaining them to the last detail and without explaining the true reasons for the appearance of these or different objects.

As this paper was being readied for the printer (at the end of June 1981), the author obtained the preprints [17] and [18], in which, also with the aid of [16], interior estimates were derived in $C^{2+\alpha}$ for solutions of equations of the type (1) and solvability of the Dirichlet problem in $C^{2+\alpha}$ was proved in a bounded smooth domain. It should be said that in a technical sense our paper differs from [17] and [18], although some of the ideas of §2 can, in a first approximation, be regarded as similar to corresponding ideas of [17] and [18]. Moreover, in the basic one of these papers ([18]) elliptic equations only of the form (1) are considered, where the L^s are linear operators and s assumes a finite set of values. We examine both elliptic and parabolic equations of the form (1), where L^s can be a nonlinear operator and s assumes, generally speaking, a denumerable set of values; for these equations in §5 we solve the Cauchy problem, the first boundary value problem in a cylinder, and the Dirichlet problem in a domain. M. V. Safonov has communicated to the author the fact that he has also proved interior estimates in $C^{2+\alpha}$ for the solutions of elliptic equations of the form (1). It is of interest to note that his method differs from both our method and the methods used in [17] and [18].

We note also the papers [19] and [20], in which differential equation methods are used to obtain various characteristics of the continuity of the second derivatives of the solution; this is accomplished, however, on condition that in the principal terms there is little difference from linearity. Finally, speaking of nonlinear equations, we cannot fail to mention the papers [21] and [22] dealing with Monge-Ampère equations and their connection (established in [2]) with equations of the form (1) for an infinite set of values s.

Besides §§2, 4 and 5 already mentioned, this paper contains three additional sections. In §1 we introduce notation and formulate two basic results; in §3 we obtain estimates in $C^{2+\alpha}$ of the solution of the Cauchy problem up to the boundary ($t = 0$) and deduce standard consequences concerning the existence of the solution; in §6 we give three examples of equations to which our results apply.

§1. Notation and some basic results

Let $T \in (0, \infty)$, $\nu \in (0, 1]$, and the integer $d \geqslant 1$ all be fixed. Let $E_d = \{x = (x^1, \dots, x^d)$: $x^i \in (-\infty, \infty)\}$ be a euclidean space; let $xy = x^i y^i$ denote the inner product of $x, y \in E_d$, where (throughout the following) repeated indices denote summation from 1 to d; also, let $|x| = \sqrt{xx}$. For functions u specified on E_d, we denote, as usual, by u_{x^i}, $u_{x^i x^j}$, etc., the partial derivatives of u with respect to x^i, $x^i x^j$; we denote the gradient of u by u_x; and by $u_{(\xi)} = \xi^i u_{x^i}$ and $u_{(\xi)(\xi)} = \xi^i \xi^j u_{x^i x^j}$ we shall mean the first and second derivatives of u with respect to x along the vector $\xi \in E_d$. We employ a similar notation for functions u specified in domains from E_d, and also for functions of the form $u(t, x)$ and $F(u_{ij}, u_i, u, t, x)$; the latter will be differentiated not only with respect to t and x but also with respect to (u_{ij}, u_i, u) along certain vectors $(\tilde{u}_{ij}, \tilde{u}_i, \tilde{u})$ in a euclidean space of corresponding dimensionality.

If $D \subset E_d$, $Q = (0, T) \times D$ and $\bar{Q} = [0, T] \times \bar{D}$, we denote by $\mathfrak{F}_0(\nu, Q)$ the totality of all real functions $F(u_{ij}, u_i, u, \beta, t, x)$ possessing the following properties:

1.1) F is defined for all $(t, x) \in \bar{Q}$, $\beta > 0$, and real u_{ij} $(i, j = 1, \ldots, d)$, u_i $(i = 1, \ldots, d)$ and u.

1.2) The function F is positive homogeneous of the first order of homogeneity with respect to (u_{ij}, u_i, u, β), and for each t is twice continuously differentiable with respect to the variables $(u_{ij}, u_i, u, \beta, x)$.

1.3) $\nu |\xi|^2 \leqslant F_{u_{ij}} \xi^i \xi^j$ for all $\xi \in E_d$.

1.4) F is convex upwards with respect to the set of variables (u_{ij}).

1.5) The second derivative of F with respect to (u_{ij}, u_i, u) along an arbitrary vector $(\tilde{u}_{ij}, \tilde{u}_i, \tilde{u})$ does not exceed

$$\nu^{-1}\beta^{-1}\left(\sum_i |\tilde{u}_i|^2 + |\tilde{u}|^2\right),$$

and the absolute values of $F_{u_{kr}}$, F_{u_k}, F_β, $F_{u_{kr}x^i}$, F_{ux^i}, $F_{\beta x^i}$ do not exceed ν^{-1}, and $|F_{x^i x^j}| \leqslant \nu^{-1} w$, where $w = (\beta^2 + u^2 + u_i u_i + u_j u_{ij})^{1/2}$.

We stress the fact that in the first required estimate in condition 1.5) the concern is only with an estimate from above. Therefore condition 1.4), singled out from systematic considerations, is contained in 1.5).

We note also that

$$F_{x^i} = F_{u_{kr}x^i} u_{kr} + F_{u_k u^i} u_k + F_{ux^i} u + \beta F_{\beta x^i}$$

and, therefore, that $|F_{x^i}| \leqslant N(d, \nu) w$.

Suppose that in addition to conditions 1.1)–1.5) we also have

1.6) F is continuous with respect to $(u_{ij}, u_i, u, \beta, t, x)$ and differentiable with respect to t, and $|F_t| \leqslant \nu^{-1} w$ (in the domain of definition of F).

Then we shall write $F \in \mathfrak{F}(\nu, Q)$.

For those cases in which F does not depend on t and $F \in \mathfrak{F}_0(\nu, Q)$ we write $F \in \mathfrak{F}(\nu, D)$. If some domain $Q \subset (-\infty, \infty) \times E_d$ and $M \geqslant 0$, then we denote by $P(M, Q)$ the totality of all the sets (u_{ij}, u_i, u, t, x) such that $(t, x) \in \bar{Q}$ and

$$\sum_{i,j} |u_{ij}| + \sum_i |u_i| + |u| \leqslant M.$$

For $Q \subset E_d$ we define $P(M, Q)$, omitting t in the previous definition and replacing $(t, x) \in \bar{Q}$ by $x \in \bar{Q}$.

Let $C(\Gamma)$ be the space of functions u, continuous on Γ, with the usual norm $\|u\|_{C(\Gamma)}$. For $\alpha \in (0, 1]$ and $\Gamma \subset E_d$, let $C^\alpha(\Gamma)$ be the subspace of $C(\Gamma)$ consisting of functions u for which the norm

$$\|u\|_{C^\alpha(\Gamma)} = \|u\|_{C(\Gamma)} + \sup_{x_i \in \Gamma, x_1 \neq x_2} \frac{|u(x_1) - u(x_2)|}{|x_1 - x_2|^\alpha} \tag{1.1}$$

is finite.

The second term in (1.1) is called the *Hölder constant* of the function u of order α with respect to the distance $|x_1 - x_2|$. When $\alpha \in (0, \infty]$ and $\Gamma \subset (-\infty, \infty) \times E_d$, we define $C^\alpha(\Gamma)$ in a similar way except that we measure the Hölder contant with respect to the distance $|t_1 - t_2|^{1/2} + |x_1 - x_2|$. In speaking of Hölder constants of the function $F(u_{ij}, u_i, u, t, x)$ on the set $P(M, Q)$ we shall have in mind the distance

$$\sum |u^1_{ij} - u^2_{ij}| + \sum |u^1_i - u^2_i| + |u^1 - u^2| + |x_1 - x_2| + |t_1 - t_2|^{1/2}.$$

Further, if $D \subset E_d$, then $C^2(D)$ is the space of all functions u, twice continuousl differentiable with respect to x in $\bar{D}$, for which the norm

$$\|u\|_{C^2(\bar{D})} = \sum_{i,j} \|u_{x^i x^j}\|_{C(\bar{D})} + \sum_i \|u_{x^i}\|_{C(\bar{D})} + \|u\|_{C(\bar{D})} \tag{1.2}$$

is finite.

If the domain $D \subset (-\infty, \infty) \times E_d$, the definition of $C^2(\bar{D})$ also involves the requir ment that the first derivative of u with respect to t be continuous, and, in the definition c $\|u\|_{C^2(\bar{D})}$, the addition of $\|u_t\|_{C(\bar{D})}$ to the right-hand side of (1.2). When $\alpha \in (0, 1)$ th spaces $C^{2+\alpha}(\bar{D})$ are defined like $C^2(\bar{D})$, except that in the definition of $\|u\|_{C^{2+\alpha}(\bar{D})}$ w take $C^\alpha(\bar{D})$ instead of $C(\bar{D})$. Finally, sets of functions of the type $C^\alpha_{\mathrm{loc}}(D)$ are defined lik $\bigcup C^\alpha(\bar{D}_1)$, where the sum is taken over all bounded domains $D_1 \subset \bar{D}_1 \subset D$.

Also let $\mathrm{meas}_d\, \Gamma$ be the Lebesgue measure in E_d of the set $\Gamma \subset E_d$, and let

$$a \wedge b = \min(a, b), \qquad a \vee b = \max(a, b),$$
$$a_+ = \tfrac{1}{2}(|a| + a), \qquad a_- = \tfrac{1}{2}(|a| - a).$$

To give the reader an idea of the results to be obtained in this paper we state two of th theorems from §5 (in a somewhat weakened form).

THEOREM 1.1. *Let $Q = (0, T) \times E_d$, and let the functions $F^s \in \mathfrak{F}(\nu, Q)$ be defined fc $s = 1, 2, \ldots$. Let $F = \inf_s F^s$ and assume that $\alpha \in (0, 1)$ and $\varphi \in C^{2+\alpha}(E_d)$. Then th Cauchy problem*

$$u_t = F(u_{x^i x^j}, u_{x^i}, u, 1, t, x) \quad in\ Q, \qquad u\big|_{t=0} = \varphi \quad in\ E_d$$

has a solution (moreover, a unique one) belonging to $C^{2+\alpha_1}(\bar{Q})$ for some $\alpha_1 \in (0, 1)$.

THEOREM 1.2. *Let the domain $D \subset E_d$ be such that for some $\varepsilon_0 > 0$, for an arbitrary poir $x \in \partial D$, a closed ball of radius ε_0 can be found lying in $E_d \setminus D$ and containing x (on i boundary). On ∂D let there be given a continuous bounded function φ, and, for $s = 1, 2, \ldots$ let the functions $F^s \in \mathfrak{F}(\nu, E_d)$ be defined such that $F^s_u \leqslant -\nu$. Put $F = \inf_s F^s$. Then th Dirichlet problem*

$$0 = F(u_{x^i x^j}, u_{x^i}, u, 1, x) \quad in\ D, \qquad u\big|_{\partial D} = \varphi$$

has a solution $u \in C(\bar{D}) \cap C^2_{\mathrm{loc}}(D)$, which, moreover, is unique. In addition, for a arbitrary bounded domain $D_1 \subset \bar{D}_1 \subset D$, there is an $\alpha \in (0, 1)$ such that $u \in C^{2+\alpha}(\bar{D}_1)$.

§2. Estimates of the norms of u_t and u_{xx} in C^α

Let the bounded domain $Q \subset (-\infty, \infty) \times E_d$, let $Q_1 \subset Q$, and let the real functio $F = F(u_{ij}, u_i, u, t, x)$ be defined for $(t, x) \in \bar{Q}$ and for real u_{ij}, u_i $(i, j = 1, \ldots, d)$ and u We fix the constants $M, M_1 \geqslant 0$ and $\mu \in (0, 1)$, and we assume that the followin conditions are satisfied on $P(M, Q)$.

2.1) The function F is once continuously differentiable with respect to all of it arguments, and the function and its derivatives do not exceed M_1 in absolute value.

2.2) $\mu |\xi|^2 \leqslant F_{u_{ij}} \xi^i \xi^j \leqslant \mu^{-1} |\xi|^2$ for all $\xi \in E_d$.

2.3) F is convex upwards with respect to the set of variables (u_{ij}) (for fixed values o the other variables).

2.4) F is twice continuously differentiable with respect to the arguments $(u_{ij}, u_i, u, x$ for fixed t; moreover, its second derivative along an arbitrary vector $(\tilde{u}_{ij}, \tilde{u}_i, \tilde{u}, \tilde{x})$ doe not exceed

$$M_1\Big(\sum |\tilde{u}_{ij}| + \sum |\tilde{u}_i| + |\tilde{u}| + |\tilde{x}|\Big)\Big(\sum |\tilde{u}_i| + |\tilde{u}| + |\tilde{x}|\Big).$$

As for the relation between conditions 2.3) and 2.4), the same remarks could be made as were made concerning 1.4) and 1.5).

We assume that on $\bar{Q}$ the function u is defined and that $u, u_t, u_{xx} \in C^2(\bar{Q})$; also on $\bar{Q}$

$$u_t(t, x) = F(u_{x^ix^j}(t, x), u_{x^i}(t, x), u(t, x), t, x), \tag{2.1}$$

$$\sum_{i,j} |u_{x^ix^j}| + \sum |u_{x^i}| + |u| \leqslant M. \tag{2.2}$$

Finally, we let

$$\rho_1 = \tfrac{1}{2} \inf\left\{|t_0 - t_1|^{1/2} + |x_0 - x_1| : (t_0, x_0) \in \partial Q, (t_1, x_1) \in Q_1, t_0 \leqslant t_1\right\},$$

and we assume that $\rho_1 > 0$ and (for convenience) that $\rho_1 \leqslant 1$.

THEOREM 2.1. *There exist $\alpha \in (0, 1)$ and $N \geqslant 0$, depending only on d, M, M_1, and μ, such that the Hölder constants of order α of the functions u_t and u_{xx} in Q_1 do not exceed $N\rho_1^{-\alpha}$.*

For the proof of this theorem we require several lemmas. The following two lemmas are, apparently, well known. We include their proofs for completeness.

LEMMA 2.1. *There exist an integer $n \geqslant 1$, a number $\rho_1 > 0$, and unit vectors $l_1, \ldots, l_n \in E_d$, depending only on μ and d, such that the inequality*

$$\operatorname{tr} au \geqslant \rho_0 \sum_{i=1}^{n} \left(\operatorname{tr} \tilde{l}_i \tilde{l}_i^* u\right)_+ - 3\mu^{-1} \sum_{i=1}^{n} \left(\operatorname{tr} \tilde{l}_i \tilde{l}_i^* u\right)_-$$

is satisfied for all matrices a, all $d \times d$ symmetric matrices u, and all vectors $\tilde{l}_i$ satisfying the conditions $|\tilde{l}_i - l_i| \leqslant 2\rho_0$ and $\mu|\xi|^2 \leqslant \operatorname{tr} \xi\xi^* a \leqslant \mu^{-1}|\xi|^2$, $i = 1, \ldots, n$, $\xi \in E_d$.

PROOF. Considering instead of a the matrix $\frac{1}{2}(a + a^*)$, we reduce the situation to the case in which a is symmetric. Further, the set $\Gamma(\mu)$ of all $d \times d$ symmetric matrices a satisfying the condition $\mu|\xi|^2 \leqslant \operatorname{tr} \xi\xi^* a \leqslant \mu^{-1}|\xi|^2$ for all $\xi \in E_d$ forms a closed convex set in the euclidean space of all $d \times d$ symmetric matrices. We enclose $\Gamma(\mu)$ in the open polyhedron $\pi \subset \Gamma(\frac{1}{2}\mu)$. This is possible since $\Gamma(\mu)$ lies in the interior of $\Gamma(\frac{1}{2}\mu)$. Let $a_1, \ldots, a_{n_1}$ be all the vertices of π, and let $\lambda_i(m)$ be the eigenvalues and $l_i(m)$ the unit eigenvectors of the matrix a_m, $i = 1, \ldots, d$, $m = 1, \ldots, n_1$. Next, we choose $\rho_0 > 0$ so small that for arbitrary vectors $\tilde{l}_i(m)$, satisfying the condition $|\tilde{l}_i(m) - l_i(m)| \leqslant 2\rho_0$, the smallest closed convex polyhedron containing all the matrices $\Sigma_i \lambda_i(m)\tilde{l}_i(m)\tilde{l}_i^*(m)$ ($m = 1, \ldots, n_1$), would also contain the set

$$\Gamma(\mu) - \rho_0 \sum_{m=1}^{n_1} \sum_{i=1}^{d} \tilde{l}_i(m)\tilde{l}_i^*(m).$$

Such a choice of ρ_0 is possible since $\Gamma(\mu)$, together with some neighborhood, lies in π and $a_m = \Sigma_i \lambda_i(m) l_i(m) l_i^*(m)$. For arbitrary $a \in \Gamma(\mu)$ we then have

$$a = \sum_{m,i} p_m \lambda_i(m)\tilde{l}_i(m)\tilde{l}_i^*(m) + \rho_0 \sum_{m,i} \tilde{l}_i(m)\tilde{l}_i^*(m)$$

with some $p_m \geqslant 0$ for which $\Sigma\, p_m = 1$. Our result then follows from this representation of the elements of $\Gamma(\mu)$ with the aid of elementary transformations if we use the notation $n = n_1 d$ and $\{l_1, \ldots, l_n\} = \{l_i(m) : i = 1, \ldots, d, m = 1, \ldots, n_1\}$ and note that $0 \leqslant \lambda_i(m) \leqslant 2\mu^{-1}$. This completes the proof of the lemma.

LEMMA 2.2. *Let $N \geqslant 0$, $\alpha \in (0, 1)$, $\gamma \in (0, 1)$ and $\rho_2 > 0$, and let n be a positive integ For each $\rho \in [0, \rho_2)$ let there be defined the set $A(\rho) \subset \{1, \ldots, n\}$ (possibly empty) and $[0, 2\rho_2)$ let there be defined nonnegative functions $w_i(\rho)$, $i = 1, \ldots, n$, nondecreasing w respect to ρ. Finally, let the following inequalities be satisfied for $\rho \in [0, \rho_2)$.*

$$N\left(\sum_{i \in A(\rho)} w_i(2\rho) + \left(\frac{\rho}{\rho_2}\right)^{\alpha}\right) \geqslant \sum_{i \notin A(\rho)} w_i(\rho), \qquad \gamma \sum_{i \in A(\rho)} w_i(2\rho) \geqslant \sum_{i \in A(\rho)} w_i(\rho).$$

Then there exist constants $N_1 \geqslant 0$ and $\alpha_1 \in (0, 1)$, depending only on N, α and γ, such tl for $\rho \in [0, 2\rho_2)$

$$\sum_{i=1}^{n} w_i(\rho) \leqslant N_1 \left(\frac{\rho}{\rho_2}\right)^{\alpha_1} \left(\sum_{i=1}^{n} w_i(2\rho_2 -) + 1\right).$$

PROOF. We choose $\kappa > 0$ so that

$$(N + 1)\frac{\kappa(1 + \kappa)}{1 - \gamma - \kappa\gamma} \leqslant \frac{1}{2} \qquad (1 + \kappa^2 \leqslant 2^{\alpha}),$$

and we show that for each $\rho \in [0, \rho_2)$ one of the following two inequalities is satisfied:

$$\sum_{i=1}^{n} w_i(2\rho) \geqslant (1 + \kappa) \sum_{i=1}^{n} w_i(\rho) \quad \text{or} \quad \sum_{i=1}^{n} w_i(\rho) \leqslant 2N\left(\frac{\rho}{\rho^2}\right)^{\alpha}. \tag{2}$$

We fix a $\rho \in [0, \rho_2)$ and we assume that the first inequality is not satisfied. We tl have

$$\sum_i w_i(\rho) \leqslant \sum_{i \notin A(\rho)} w_i(2\rho) + \gamma \sum_{i \in A(\rho)} w_i(2\rho),$$

$$\sum_i w_i(2\rho) \leqslant (1 + \kappa)\left(\sum_{i \notin A(\rho)} w_i(2\rho) + \gamma \sum_{i \in A(\rho)} w_i(2\rho)\right),$$

$$(1 - \gamma - \kappa\gamma) \sum_{i \in A(\rho)} w_i(2\rho) \leqslant \kappa \sum_{i \notin A(\rho)} w_i(2\rho) \leqslant \kappa(1 + \kappa)\sum_i w_i(\rho),$$

$$\sum_{i \notin A(\rho)} w_i(\rho) + \sum_{i \in A(\rho)} w_i(\rho) \leqslant N\left(\frac{\rho}{\rho_2}\right)^{\alpha} + (N + 1)\frac{\kappa(1 + \kappa)}{1 - \gamma - \kappa\gamma}\sum_i w_i(\rho).$$

Collecting like terms in the last inequality, we obtain the second inequality in (2.3). \ now put

$$\psi(\rho) = \kappa \sum_i w_i(\rho) + 2N\left(\frac{\rho}{\rho_2}\right)^{\alpha}.$$

It is not difficult to verify, by virtue of (2.3), that when $\rho \in [0, \rho_2)$ we have $(1 + \kappa)\psi($ $\leqslant \psi(2\rho)$. It follows from this, as we know, that constants N_1 and γ_1, depending only on exist such that $\psi(\rho) \leqslant N_1 \rho^{\alpha_1} \rho_2^{-\alpha_1} \psi(2\rho_2 -)$ for $\rho \in [0, 2\rho_2)$. This completes the proof of th lemma.

A part of Theorem 2.1 is proved in the following lemma.

LEMMA 2.3. *exist constants $N \geqslant 0$ and $\alpha_0 \in (0, 1)$, depending only on d, M, M_1, and μ, such that*

$$|u_t(t_0, x_0) - u_t(t_1, x_1)| + |u_{x^i}(t_0, x_0) - u_{x^i}(t_1, x_1)| \leqslant N\rho_1^{-\alpha_0}\rho^{\alpha_0},$$

where $\rho = |t_0 - t_1|^{1/2} + |x_0 - x_1|$, for arbitrary $(t_0, x_0), (t_1, x_1) \in Q_1$.

PROOF. Since $|u_t|$ is a bounded function in Q (see (2.1)), and the same is true for $|u_x|$, we can assume in the proof of the lemma that $\rho < \rho_1$. Assume also that $t_1 \leqslant t_0$. We then have

$$(t_0, x_0), (t_1, x_1) \in \{(t, x): \rho_1^2 < t - t_0 \leqslant 0, |x - x_0| < \rho_1\}$$
$$\subset \{(t, x): 4\rho_1^2 < t - t_0 \leqslant 0, |x - x_0| < 2\rho_1\} \subset Q. \tag{2.4}$$

In addition, from (2.1) we obtain

$$(u_t)_t = F_{u_{ij}}(u_t)_{x^ix^j} + F_{u_i}(u_t)_{x^i} + F_u u_t + F_t. \tag{2.5}$$

From this and from (2.4) the estimate needed for u_t follows according to Theorem 4.2 of [16]. An estimate for u_{x^i} (even when $\alpha_0 = 1$) follows from Lemma 3.1 in Chapter II of [23] (and its proof), since u_t and u_{xx} are bounded in Q. This completes the proof of the lemma.

Next, for $(t, x) \in Q$ and $|\xi| < 2$, $\xi \in E_d$, we define functions $\sigma_{ij}(t, x, \xi)$ in accordance with the formula

$$\sigma_{ij} = \frac{1}{2}M_1\Big(\sum_i\Big|\sum_k u_{x^ix^k}\xi^k\Big| + \Big|\sum_k u_{x^k}\xi^k\Big| + |\xi|\Big)\operatorname{sgn}\Big(\sum_k u_{x^ix^jx^k}\xi^k\Big).$$

Clearly, there exists a constant $N_0 > 0$, depending only on μ, M, M_1, and d, such that for all $\tilde{x}, \tilde{\xi} \in E_d$, $(t, x) \in Q$ and $|\xi| < 2$, we have

$$\mu|\tilde{x}|^2 + \sigma_{ij}\tilde{x}^i\tilde{\xi}^j + N_0|\tilde{\xi}|^2 \geqslant \tfrac{1}{2}\mu\big(|\tilde{x}|^2 + |\tilde{\xi}|^2\big).$$

Consequently, if on sufficiently smooth functions v of $(t, x, \xi) \in Q \times E_d$ we define an operator L according to the formula

$$Lv = F_{u_{ij}}v_{x^ix^j} + F_{u_i}v_{x^i} + \sigma_{ij}v_{x^i\xi^j} + N_0\big(v_{\xi^1\xi^1} + \cdots + v_{\xi^d\xi^d}\big),$$

it follows that L will be a uniformly elliptic operator with respect to the variables (x, ξ) in $Q \times (|\xi| < 2)$. We explain that here, as in (2.5), in the arguments of $F_{u_{ij}}$ we have introduced $(u_{x^ix^j}(t, x), u_{x^i}(t, x), u(t, x))$ in place of (u_{ij}, u_i, u). We also employ this subsitution in what follows in this section.

LEMMA 2.4. *There exists a constant N_1, depending only on μ, M, M_1, and d, such that the function*

$$v(t, x, \xi) \equiv N_1|\xi|^2 + u_{x^ix^j}(t, x)\xi^i\xi^j$$

satisfies the inequality $v_t \leqslant Lv$ in $Q \times (|\xi| < 2)$.

PROOF. We let G denote the second derivative of F along the vector $(\tilde{u}_{ij}, \tilde{u}_i, \tilde{u}, \tilde{x})$ and in it we put

$$(u_{ij}, u_i, u) = (u_{x^ix^j}(t, x), u_{x^i}(t, x), u(t, x)),$$
$$(\tilde{u}_{ij}, \tilde{u}_i, \tilde{u}, \tilde{x}) = \big(\xi^k u_{x^ix^jx^k}(t, x), \xi^k u_{x^ix^k}(t, x), \xi^k u_{x^k}(t, x), \xi\big).$$

A direct calculation shows, by virtue of (2.1), that

$$v_t = Lv + G - 2\sigma_{ij}u_{x^ix^jx^k}\xi^k - 2N_0\Delta u - 2N_0N_1d + F_u u_{x^ix^j}\xi^i\xi^j.$$

In accordance with the construction of σ and condition 2.4),

$$G - 2\sigma_{ij}u_{x^ix^jx^k}\xi^k \leqslant M_1\Big(\sum\Big|\sum u_{x^ix^k}\xi^k\Big| + \Big|\sum u_{x^k}\xi^k\Big| + |\xi|\Big)^2.$$

Since only the first and second derivatives of u enter into Δu and the right-hand side of the last inequality, the required inequality is readily attained, by virtue of (2.2), with sufficiently large choice for N_1. This completes the proof of the lemma.

In the following lemma we state one of the results from [16] in a form convenient to us.

LEMMA 2.5. *Let $\mu_1 \in (0, 1)$, $\beta \in (0, 1)$, $D = [0, 1] \times \{y \in E_{d_1}: |y^i| \leqslant 1 = 1, \ldots, d_1\}$, $v \in W^{1,2}_{d_1+1}(D) \cap C(D)$, and*

$$v_t(t, y) \leqslant \sum_{i,j=1}^{d_1} a^{ij}(t, y)v_{y^iy^j}(t, y) + \sum_{i=1}^{d_1} b^i(t, y)v_{y^i}(t, y)$$

(almost everywhere in D), where the measurable functions a^{ij} and b^i are such that $|b^i| \leqslant \mu_1^{-1}$, the matrix (a^{ij}) is symmetric and all of its eigenvalues lie in $[\mu_1, \mu_1^{-1}]$. Let

$$D_1 = [\tfrac{3}{4}, 1] \times \{y \in E_{d_1}: |y^i| \leqslant \tfrac{1}{2}, i = 1, \ldots, d_1\},$$

and assume that $\xi < \max\{v, D\}$ and

$$\operatorname{meas}_{d_1+1}\{(t, y) \in D: v \leqslant \xi, t \leqslant \tfrac{3}{4}\} \geqslant \beta \operatorname{meas}_{d_1+1}\{(t, y) \in D: t \leqslant \tfrac{3}{4}\}.$$

Then there exists a constant $\gamma \in (0, 1)$, depending only on d_1, ν, and β, such that

$$\max\{v, D_1\} \leqslant (1 - \gamma)\zeta + \gamma\max\{v, D\}.$$

This assertion is obvious for an arbitrary $\gamma \in (0, 1)$ if $\zeta \geqslant \max\{v, D_1\}$. If $\zeta < \max\{v, D_1\} \equiv m$, the assertion may be proved by considering $(v - \zeta)/(m - \zeta)$ in place of v and repeating literally the discussion that follows formula (4.4) of [16], replacing the 1/4 by β.

PROOF OF THEOREM 2.1. 1°. We take ρ_0, n and $l_1, \ldots, l_n$ from Lemma 2.1 and v from Lemma 2.4, and we put

$$Q_i(\rho) = \{(t, x, \xi): -\rho^2 \leqslant t \leqslant 0, |\xi^j - l_i^j| \leqslant \rho, |x^j| \leqslant \rho, j = 1, \ldots, d\}.$$

In addition, with no loss of generality, we consider that $\rho_0 \leqslant 1$ and we let

$$\rho_2 = \tfrac{1}{2}(\rho_1 \wedge \rho_0),$$
$$m_i(\rho, t_0, x_0, \xi_0) = \min\{v: (t, x, \xi) \in Q_i(\rho) + (t_0, x_0, \xi_0)\},$$
$$M_i(\rho, t_0, x_0, \xi_0) = \max\{v: (t, x, \xi) \in Q_i(\rho) + (t_0, x_0, \xi_0)\},$$
$$w_i(\rho, t_0, x_0, \xi_0) = M_i(\rho, t_0, x_0, \xi_0) - m_i(\rho, t_0, x_0, \xi_0).$$

Our problem is to prove the existence of constants $N \geqslant 0$ and $\alpha \in (0, 1)$, depending only on μ, M_1, M, and d, such that

$$\sum_{i=1}^n w_i(\rho, t_0, x_0, \xi_0) \leqslant N\Big(\frac{\rho}{\rho_2}\Big)^\alpha \tag{2.6}$$

for arbitrary $(t_0, x_0, \xi_0) \in Q_1 \times (|\xi| < \rho_0)$ and $\rho < 2\rho_2$.

Since $\rho_2 \geqslant \frac{1}{2}\rho_1\rho_0$, $|u_{xx}| \leqslant M$ and ρ_0 depends only on μ and d, it clearly follows from (2.3) for all $(t_0, x_0), (t_1, x_1) \in Q_1$, $t_1 \leqslant t_0$, and $|\xi_0| \leqslant \rho_0$, that

$$|v(t_0, x_0, \xi_0 + l_1) - v(t_1, x_1, \xi_0 + l_1)| \leqslant N\rho_1^{-\alpha}\big(|t_0 - t_1|^{1/2} + |x_0 - x_1|\big)^\alpha, \tag{2.7}$$

where N depends only on μ, M_1, M, and d. Since $v(t_0, x_0, \xi) - v(t_1, x_1, \xi)$ is a quadratic form in ξ with coefficients $u_{x^ix^j}(t_0, x_0) - u_{x^ix^j}(t_1, x_1)$, we obtain from (2.7) an estimate for them similar to (2.7). This, along with Lemma 2.4, concludes the proof of the theorem.

2°. Thus it remains to prove (2.6). We fix $\rho \in [0, \rho_2)$ and $(t_0, x_0, \xi_0) \in Q_1 \times (|\xi| < \rho_0)$, and, where it will cause no misunderstanding, we omit the arguments ρ, t_0, x_0 and ξ_0. Also, we put

$$\varepsilon = \mu\rho_0/6d, \qquad Q_i' = \left(Q_i(2\rho) + (t_0, x_0, \xi_0)\right) \cap \left\{(t, x, \xi): t - t_0 \leqslant -\rho^2\right\},$$

$\pi_{ix}Q'$ is the projection of Q_i' onto the space of the variables (t, x) (it does not depend on i), $\pi_\xi Q_i'$ is the projection of Q_i' onto the space of the variables ξ, and

$$\Gamma_i = \left\{(t, x, \xi): v(t, x, \xi) \leqslant (1-\varepsilon)M_i(\rho) + \varepsilon m_i(\rho)\right\} \cap Q_i',$$

$$A = \left\{i: \operatorname{meas}_{2d+1}\Gamma_i \leqslant \tfrac{1}{n}\operatorname{meas}_{2d+1}Q_i'\right\}, \qquad B = \left\{i: \operatorname{meas}_{2d+1}\Gamma_i < \tfrac{1}{n}\operatorname{meas}_{2d+1}Q_i'\right\}.$$

We notice that if $i \in B$, the set τ_i of all those $(t, x) \in \pi_{tx}Q'$ for which the inequality $v(t, x, \xi) \leqslant (1-\varepsilon)M_i(\rho) + \varepsilon m_i(\rho)$ is satisfied everywhere on $\pi_\xi Q_i^!$ has by Fubini's theorem a $(d+1)$-dimensional Lebesgue measure, which is less than $(1/n)\operatorname{meas}_{d+1}\pi_{tx}Q'$. Consequently, the $(d+1)$-dimensional Lebesgue measure of the set $\pi_{tx}Q' \setminus \cup_B \tau_i$ is greater than zero and the set itself is nonempty. Therefore, t, x and ξ_i can be found such that

$$(t, x, \xi_i) \in Q_i', \quad v(t, x, \xi_i) \geqslant (1-\varepsilon)M_i(\rho) + \varepsilon m_i(\rho) \quad \forall i \in B. \tag{2.8}$$

We complete the enumeration of the objects we shall require below by introducing the points $(t_i, x_i, \eta_i) \in Q_i(\rho) + (t_0, x_0, \xi_0)$, at which $v = m_i(\rho)$, and the vectors $\tilde{l}_i = \xi_i$ for $i \in B$ and $\tilde{l}_i = \eta_i$ for $i \in A$. Finally, we agree to use the same letter N to denote, generally speaking, various constants depending only on μ, M_1, M and d.

3°. We now prove that

$$\sum_{i\in B} w_i(\rho) \leqslant N\left(\sum_{i \in A} w_i(2\rho) + \left(\frac{\rho}{\rho_2}\right)^{\alpha_0}\right), \tag{2.9}$$

where α_0 is taken from Lemma 2.3. Understandably, we can assume that $B \neq \varnothing$. We apply Lemma 2.1 to the difference of the values of F on the solution u at the point (t, x) and at the point (t_j, x_j) for $j \in B$. Naturally, it is necessary first to represent this difference with the help of Lagrange's formula in the form

$$a_{pq}\left[u_{x^px^q}(t, x) - u_{x^px^q}(t_j, x_j)\right] + b_p\left[u_{x^p}(t, x) - u_{x^p}(t_j, x_j)\right] \\ + c\left[u(t, x) - u(t_j, x_j)\right] + f.$$

In addition, we use the fact that here, in accordance with Lemma 2.3 (applied to $Q_1 = (t_0, x_0) + \{(t, x): -4\rho_2^2 < t < 0, |x| < 2\rho_2\}$), three of the last terms do not exceed $N\rho^{\alpha_0}\rho_2^{-\alpha_0}$ in absolute value, and, obviously,

$$\left(u_{x^px^q}(t, x) - u_{x^px^q}(t_j, x_j)\right)l^pl^q = v(t, x, l_j) - v(t_j, x_j, l).$$

We then obtain from (2.1), in accordance with Lemmas 2.1 and 2.3,

$$N\left(\frac{\rho}{\rho_2}\right)^{\alpha_0} \geqslant \rho_0\left[v(t, x, \tilde{l}_j) - v(t_j, x_j, \tilde{l}_j)\right]_+ \\ -3\mu^{-1}\sum_{i=1}^{n}\left[v(t, x, \tilde{l}_i) - v(t_j, x_j, \tilde{l}_i)\right]_- . \tag{2.10}$$

To estimate the first expression on the right side of (2.10) we note that $j \in B$ and $\tilde{l}_j = \xi_j$, and that by virtue of (2.8)

$$v(t, x, \tilde{l}_j) - v(t_j, x_j, \tilde{l}_j) \geqslant (1-\varepsilon)M_j(\rho) + \varepsilon m_j(\rho) - v(t_j, x_j, \eta_j) - N\rho \\ = (1-\varepsilon)w_j(\rho) - N\rho \geqslant -N\rho.$$

Similarly, for $i \in B$

$$v(t, x, \tilde{l}_i) - v(t_j, x_j, \tilde{l}_i) \geqslant (1-\varepsilon)M_i(\rho) + \varepsilon m_i(\rho) - v(t_j, x_j, \xi_i) \\ \geqslant (1-\varepsilon)M_i(\rho) + \varepsilon m_i(\rho) - M_i(\rho) - N\rho = -\varepsilon w_i(\rho) - N\rho.$$

For $i \in A$ we estimate the corresponding expressions crudely from below in terms of $(-w_i(2\rho))$. As a result we derive from (2.10) the following inequality:

$$N\left(\frac{\rho}{\rho_2}\right)^{\alpha_0} \geqslant \rho_0(1-\varepsilon)w_j(\rho) - 3\mu^{-1}\varepsilon\sum_{i \in B, i\neq j} w_i(\rho) - 3\mu^{-1}\sum_{i\in A} w_i(2\rho) \\ \geqslant \rho_0 w_j(\rho) - 3\mu^{-1}\varepsilon\sum_{i\in B} w_i(\rho) - 3\mu^{-1}\sum_{i \in A} w_i(2\rho).$$

Adding such inequalities for $j \in B$ and recalling that $3\mu^{-1}\varepsilon d = \frac{1}{2}\rho_0$, we obtain (2.9).

4°. As Lemma 2.2 shows, to complete the proof of (2.6) and, hence, the theorem, it is sufficient to establish the existence of a constant $\gamma \in (0, 1)$, depending only on d, μ, M, and M_1, such that $\gamma w_i(2\rho) \geqslant w_i(\rho)$ for $i \in A$.

With the help of the coordinate transformation $t \to \frac{1}{4}\rho^{-2}t$, $x \to \frac{1}{2}\rho^{-1}x$, $\xi \to \frac{1}{2}\rho^{-1}\xi$ and Lemmas 2.4 and 2.5 (with $\beta = 1/n$) we obtain for $i \in A$

$$M_i(\rho) \leqslant (1-\gamma)\left((1-\varepsilon)M_i(\rho) + \varepsilon m_i(\rho)\right) + \gamma M_i(2\rho) \\ = (1-\gamma)M_i(\rho) - \varepsilon(1-\gamma)w_i(\rho) + \gamma M_i(2\rho), \\ \gamma M_i(\rho) + \varepsilon(1-\gamma)w_i(\rho) \leqslant \gamma M_i(2\rho),$$

where γ depends only on d, μ, M, M_1, and $\gamma \in (0, 1)$.

Since we also have $-\gamma m_i(\rho) \leqslant -\gamma m_i(2\rho)$, it follows that $(\gamma + \varepsilon(1-\gamma))w_i(\rho) \leqslant \gamma w_i(2\rho)$. This completes the proof of the theorem.

§3. Estimates of the norms of u_t and u_{xx} in C^α in a closed domain, and their consequences

We fix the bounded domains $D_1 \subset \overline{D}_1 \subset D \subset E_d$ and we put $Q = (0, T) \times D$ and $Q_1 = (0, T) \times D_1$. We consider that the assumptions of the previous section regarding F and u are satisfied. Let $u_0(x) = u(0, x)$, $\bar{\rho}_1 = \frac{1}{2}\operatorname{dist}(\partial D, \partial D_1)$, and

$$\psi(\alpha) = \bar{\rho}_1^{-\alpha} + \|u_0\|_{C^{2+\alpha}(\overline{D})}$$

and, for convenience, assume that $\bar{\rho} \leqslant 1$.

THEOREM 3.1. *There exist constants $\alpha_0 \in (0, 1)$ and $N \geqslant 0$, depending only on d, M, M_1 and μ, such that for $\alpha \in (0, \alpha_0]$*

$$\left|u_t(t_0, x_0) - u_t(t_1, x_1)\right| \leqslant N\psi(\alpha)\left(|t_0 - t_1|^{1/2} + |x_0 - x_1|\right)^{\alpha/2}, \tag{3.1}$$

$$\left|u_{x^ix^j}(t_0, x_0) - u_{x^ix^j}(t_1, x_1)\right| \leqslant N\psi(\alpha)\left(|t_0 - t_1|^{1/2} + |x_0 - x_1|\right)^{\alpha/4} \tag{3.2}$$

for arbitrary (t_0, x_0), $(t_1, x_1) \in Q_1$ and $i, j = 1, \ldots, d$.

We fix (t_0, x_0), $(t_1, x_1) \in Q_1$, we assume that $t_1 \leqslant t_0$, and we let $|t_0 - t_1|^{1/2} + |x_0 - x_1|$. For brevity we also agree to take no note of the fact that constants of the type α_i and N depend only on d, M, M_1, and μ, and we shall denote the various constants, generally speaking, by one and the same letter N. To prove the theorem we require two lemmas. The first is well known and we state it without proof. Its proof is very easily obtained, for example, through use of a probability representation of the solution of a parabolic equation.

LEMMA 3.1. *Let $\alpha \in [0, 1]$ and $v \in C^2(\bar{Q})$, and in Q let v satisfy the inequality*

$$v_t \leqslant a^{ij}v_{x^ix^j} + b^iv_{x^i} + cv + f,$$

where the measurable functions a^{ij}, b^i, c and f are such that $|a^{ij}|, |b^i|, |c|, |f| \leqslant M_1$, and the matrix $(a^{ij}) \geqslant 0$. Then there exists a constant N, depending only on M_1 and d, such that

$$(v(t, x) - v(0, x))_+ \leqslant Nt^{\alpha/2}\left(\|v\|_{C^\alpha(\bar{D})} + 1 + \|v\|_{C(\bar{Q})}\bar{\rho}_1^{-\alpha}\right),$$

if $(t, x) \in Q_1$ and $t \leqslant 2$.

LEMMA 3.2. *Let $v \in C^2(\bar{Q})$, and in Q let v satisfy the equation*

$$v_t = a^{ij}v_{x^ix^j} + b^iv_{x^i} + cv + f,$$

where the measurable functions a^{ij}, b^i, c and f are such that $|a^{ij}|, |b^i|, |c|, |f| \leqslant M_1$ and $\mu|\xi|^2 \leqslant a^{ij}\xi^i\xi^j$ for all $\xi \in E_d$. Then there exist constants $\alpha_1 \in (0, 1)$ and $N \geqslant 0$ such that

$$|v(t_0, x_0) - v(t_1, x_1)| \leqslant N\left(\|v\|_{C^\alpha(\bar{D})} + 1 + \|v\|_{C(Q)}\bar{\rho}_1^{-\alpha}\right)\rho^{\alpha/2} \tag{3.3}$$

for all $\alpha \in [0, \alpha_1]$.

PROOF. Obviously, it is sufficient to prove (3.3) only for $\rho \leqslant \bar{\rho}_1$. We note that then $t_0 \leqslant t_1 + \rho^2 \leqslant t_1 + 1$. We consider separately the three cases: a) $t_1 \geqslant \bar{\rho}_1^2$; b) $\rho \leqslant t_1 \leqslant \bar{\rho}_1^2$; c) $t_1 \leqslant \rho$.

In case a) we apply Theorem 4.2 of [16] in the domain $Q_2 = \{(t, x): |x - x_0| < 2\bar{\rho}_1, t_1 - \bar{\rho}_1^2 < t < t_0\} \subset Q$ and we note that the norm of f in $\mathfrak{L}_{d+1}(Q_2)$ does not exceed $N\bar{\rho}_1^{d/(d+1)}$. We then obtain the existence of constants $\alpha_1 \in (0, d/(d+1)]$ and $N \geqslant 0$ for which

$$|v(t_0, x_0) - v(t_1, x_1)| \leqslant N\left(1 + \|v\|_{C(\bar{Q})}\bar{\rho}_1^{-\alpha}\right)\rho^\alpha$$

for arbitrary $\alpha \in [0, \alpha_1]$.

In case b), in accordance with Theorem 4.2 of [16], applied in the domain $\{(t, x): |x - x_0| < 2\bar{\rho}_1, 0 < t < t_0\}$, for the same α we have

$$|v(t_0, x_0) - v(t_1, x_1)| \leqslant Nt_1^{-\alpha/2}\left(\|v\|_{C(\bar{Q})} + 1\right)\rho^\alpha \leqslant N\left(1 + \|v\|_{C(\bar{Q})}\right)\rho^{\alpha/2}.$$

Finally, in case c), $t_0 \leqslant \rho^2 + t_1 \leqslant 2\rho$, and, according to Lemma 3.1,

$$\begin{aligned}|v(t_0, x_0) - v(t_1, x_1)| &\leqslant Nt_0^{\alpha/2}\left(\|v\|_{C^\alpha(\bar{D})} + 1 + \|v\|_{C(\bar{Q})}\bar{\rho}_1^{-\alpha}\right) + \rho^\alpha\|v\|_{C^\alpha(\bar{D})}\\ &\leqslant N\left(\|v\|_{C^\alpha(\bar{D})} + 1 + \|v\|_{C(\bar{Q})}\bar{\rho}_1^{-\alpha}\right)\rho^{\alpha/2}.\end{aligned}$$

Since inequality (3.3) is satisfied in all the cases, this completes the proof of the lemma.

PROOF OF THE THEOREM. We remark that equation (2.1) is true even when $t = 0$. From it and from (2.5) we find, according to Lemma 3.2, that there exist constants $\alpha_1 \in (0, 1)$ and $N \geqslant 0$ such that

$$|u_t(t_0, x_0) - u_t(t_1, x_1)| \leqslant N\psi(\alpha)\rho^{\alpha/2} \tag{3.4}$$

for $\alpha \in [0, \alpha_1]$. With this we have proved (3.1).

In proving (3.2) we assume, with no loss of generality, that $\rho \leqslant \bar{\rho}_1$, and we consider two cases separately: a) $\rho \leqslant t_1$; b) $t_1 < \rho$. In case a) we apply Theorem 2.1 to the domain $\{(t, x): |x - x_0| < \bar{\rho}_1, t_1 < t < t_0\}$, which we take in place of Q_1 in Theorem 2.1. In addition, we note that

$$\rho \leqslant \bar{\rho}_1, \quad \rho \leqslant \rho^{1/2} \leqslant t_1^{1/2}, \quad \rho \leqslant \bar{\rho}_1 \wedge t_1^{1/2}, \quad \bar{\rho}_1 \wedge t_1^{1/2} \geqslant \bar{\rho}_1\rho^{1/2}.$$

We then find that there exist constants $\alpha_2 \in (0, 1)$ and $N \geqslant 0$ such that

$$\begin{aligned}|u_{x^ix^j}(t_0, x_0) - u_{x^ix^j}(t_1, x_1)| &\leqslant N\rho^{\alpha_2}\left(\bar{\rho}_1 \wedge t_1^{1/2}\right)^{-\alpha_2}\\ &\leqslant N\rho^\alpha\left(\bar{\rho}_1 \wedge t_1^{1/2}\right)^{-\alpha} \leqslant N\rho^{\alpha/2}\bar{\rho}_1^{-\alpha} \leqslant N\rho^{\alpha/4}\psi(\alpha)\end{aligned} \tag{3.5}$$

for arbitrary $\alpha \in [0, \alpha_2]$ and $i, j = 1, \ldots, d$.

In case b) we take ρ_0, n and $l_1, \ldots, l_n$ from Lemma 2.1, and v from Lemma 2.4, $|\xi_0| < \rho_0$, and we put $\tilde{l}_i = l_i + \xi_0$. Since $t_0 \leqslant t_1 + \rho^2 \leqslant 2\rho \leqslant 2\rho^{1/2}$, we have according to Lemma 2.4 and Lemma 3.1 for $\alpha \in [0, 1]$

$$\left(v\left(t_i, x_i, \tilde{l}_j\right) - v\left(0, x_i, \tilde{l}_j\right)\right)_+ \leqslant N\rho^{\alpha/4}\psi(\alpha) \tag{3.6}$$

for $i = 0, 1$ and $j = 1, \ldots, n$. On the other hand, as in part 3° of the proof of Theorem 2.1, we find, upon comparing F on the solution u at the point (t_i, x_i) and at the point $(0, x_i)$, from (2.1) and (3.4) with $\alpha \in [0, \alpha_1]$ that

$$\begin{aligned}N\psi(\alpha)\rho^{\alpha/4} \geqslant \rho_0&\left[v\left(0, x_i, \tilde{l}_1\right) - v\left(t_i, x_i, \tilde{l}_1\right)\right]_+\\ &-3\mu^{-1}\sum_{k=1}^n\left[v\left(0, x_i, \tilde{l}_k\right) - v\left(t_i, x_i, \tilde{l}_k\right)\right]_-.\end{aligned}$$

From this and from (3.6) for $\alpha \in [0, \alpha_1]$ we conclude that

$$\begin{gathered}\left[v\left(0, x_i, \tilde{l}_1\right) - v\left(t_i, x_i, \tilde{l}_1\right)\right]_+ \leqslant N\rho^{\alpha/4}\psi(\alpha),\\ \left|v\left(0, x_i, \tilde{l}_1\right) - v\left(t_i, x_i, \tilde{l}_1\right)\right| \leqslant N\rho^{\alpha/4}\psi(\alpha),\\ \left|v\left(t_0, x_0, \tilde{l}_1\right) - v\left(t_1, x_1, \tilde{l}_1\right)\right| \leqslant N\rho^{\alpha/4}\psi(\alpha).\end{gathered}$$

Since here $\tilde{l}_1$ is an arbitrary vector from $\{\xi: |\xi - l_1| < \rho_0\}$, it follows that in case b)

$$|u_{x^ix^j}(t_0, x_0) - u_{x^ix^j}(t_1, x_1)| \leqslant N\rho^{\alpha/4}\psi(\alpha)$$

for $\alpha \in [0, \alpha_1]$. By virtue of (3.5) this inequality will be true for $\alpha \in [0, \alpha_0]$, and it will be true in case a) if we take $\alpha_0 = \alpha_1 \wedge \alpha_2$. Such an α_0 is arrived at also for (3.1). This completes the proof of the theorem.

A number of consequences, standard in the theory of differential equations, follow from this theorem. Provided that a priori estimates have been established in $C^{2+\alpha}$, the following results are very well known. However, since the author could not find their proofs in the literature, he decided, from systematic considerations, to supply the proofs.

THEOREM 3.2. *Let the assumptions made at the beginning of this section be satisfied; let $\alpha \in (0, 1)$ and $u_0 \in C^{4+\alpha}(\bar{D})$. Assume that the first and second derivatives of F with respect to u_{ij}, u_i, u and x, and their Hölder constants of order α with respect to (u_{ij}, u_i, u, t, x) in $P(M, Q)$ do not exceed M_1. Assume also that the first derivative of F with respect to t and its Hölder constant of order α with respect to (u_{ij}, u_i, u, t, x) in $P(M, Q)$ does not exceed M_1. Then u, u_t, $u_{xx} \in C^{2+\alpha}(\bar{Q}_1)$, and the norms of these functions in $C^{2+\alpha}(\bar{Q}_1)$ may be estimated from above by a constant depending only on $\bar{\rho}_1$, d, M, M_1, μ and $\|u_0\|_{C^{4+\alpha}(\bar{D})}$.*

PROOF. We shall follow closely the proof of Theorem 13 in [24], Chapter III, §5, and, for brevity, we shall take no note of the dependence of the constants on $\bar{\rho}_1$, d, M, M_1, μ and $\|u_0\|_{C^{4+\alpha}(\bar{D})}$. We take smooth domains D_2 and D_3 such that $\bar{D}_1 \subset D_2 \subset \bar{D}_2 \subset D_3 \subset \bar{D}_3 \subset D$, and we let $Q_i = (0, T) \times D_i$. Differentiating (2.1) once with respect to x^i, we obtain a linear parabolic equation for u_{x^i} whose coefficients, by virtue of Theorem 3.1, satisfy in

Q_3 a Hölder condition with an exponent $\varepsilon \in (0, \alpha)$. From the theory of linear equations (see [23] and [24]) it follows that

$$\|u_{x^i}\|_{C^{2+\varepsilon}(\bar{Q}_3)} \leqslant N.$$

In particular, this yields an estimate of $|(u_x)_{xx}|$ and $|(u_x)_t|$ in Q_3. It follows from this, in accordance with Lemma 3.1, of [23], Chapter II, that

$$\|u_{xx}\|_{C^1(\bar{Q}_3)} \leqslant N.$$

From the last inequality we find, in turn, that the coefficients of the equation for u_{x^i} satisfy in Q_3 a Hölder condition with exponent α. Therefore,

$$\|u_{x^i}\|_{C^{2+\alpha}(\bar{Q}_2)} \leqslant N.$$

If we now differentiate (2.1) twice with respect to x, we readily see that

$$\|u_{xx}\|_{C^{2+\alpha}(\bar{Q}_1)} \leqslant N.$$

A single differentiation of (2.1) with respect to t, together with the simply verifiable inclusion

$$u_t\big|_{t=0} = F(u_{0x^ix^j}, u_{0x^i}, u_0, 0, x) \in C^{2+\alpha}(\bar{D})$$

gives

$$\|u_t\|_{C^{2+\alpha}(\bar{Q}_1)} \leqslant N.$$

This completes the proof of the theorem.

REMARK 3.1. It is evident from the proof of this theorem that if in its formulation we take the domains Q and Q_1 from §2, replace $\bar{\rho}_1$ by ρ_1, and omit the condition $u_0 \in C^{4+\alpha}(D)$ and the expression $\|u_0\|_{C^{4+\alpha}(\bar{D})}$, it remains valid. Upon comparing with Theorem 2.1 we find there is the advantage that the exponent α in the Hölder condition for u_t and u_{xx} can be made arbitrary in its dependence on the properties of F. Moreover, it is obviously important, in this connection, that we also have $u_t, x_{xx} \in C^{2+\alpha}(\bar{Q}_1)$.

THEOREM 3.3. *Let $Q = (0, T) \times E_d$, $\alpha \in (0, 1)$, $\varphi \in C^{4+\alpha}(E_d)$ and $\varepsilon > 0$. Assume that in $P(M + \varepsilon, Q)$ the conditions of §2 and the conditions of Theorem 3.2 relating to F are satisfied. For each $\kappa \in [0, 1]$ let (2.2) be satisfied for any function $u \in C^2(\bar{Q})$ that is a solution of the problem*

$$u_t = (1 - \kappa)\Delta u + \kappa F(u_{x^ix^j}, u_{x^i}, u, t, x) \quad \text{in } Q, \qquad u\big|_{t=0} = \varphi \tag{3.7}$$

and is such that $u_t, u_{xx} \in C^2(\bar{Q})$. Then (2.1) has a solution such that

$$u, u_t, u_{xx} \in C^{2+\alpha}(\bar{Q}), \qquad u(0, x) = \varphi(x).$$

For the proof of this theorem we need the following lemma.

LEMMA 3.3. *Assume that $N \geqslant 0$, that the function $\Phi(\xi, t, x)$ is defined for $\xi \in \{\eta \in E_d: |\eta| \leqslant N\}$ and $(t, x) \in \bar{Q}$ and, together with its first derivatives with respect to ξ, is bounded. Assume, for each ξ, that the functions Φ and Φ_ξ satisfy a Hölder condition with respect to (t, x) with exponent α and a constant independent of ξ. Assume also that Φ_ξ is uniformly continuous with respect to (ξ, t, x). Then*

$$\Phi: \xi(t, x) \to \Phi(\xi(t, x), t, x)$$

is a continuous operator in

$$C^\alpha(\bar{Q}) \cap \{\xi: \|\xi\|_{C(\bar{Q})} \leqslant N\}.$$

PROOF. For simplicity of the notation we take $d_1 = 1$. We assume that the lemma is false. Then, as is readily seen, we can find $\delta > 0$, $\xi_n \in C^\alpha(\bar{Q}) \cap \{\xi: \|\xi\|_{C(\bar{Q})} \leqslant N\}$ and $(t'_n, x'_n), (t''_n, x''_n) \in \bar{Q}$, $n = 0, 1, 2, \ldots$, such that

$$\|\xi_n - \xi_0\|_{C^\alpha(\bar{Q})} \to 0, \tag{3.8}$$

$$\begin{aligned}&\big|[\Phi(\xi_n(t'_n, x'_n), t'_n, x'_n) - \Phi(\xi_0(t'_n, x'_n), t'_n, x'_n)]\\ &\qquad - [\Phi(\xi_n(t''_n, x''_n), t''_n, x''_n) - \Phi(\xi_0(t''_n, x''_n), t''_n, x''_n)]\big| \geqslant \delta\rho_n^\alpha,\end{aligned}$$

where $\rho_n = |t'_n - t''_n|^{1/2} + |x'_n - x''_n|$. Obviously, we can assume that $\rho_n \to 0$. In the calculations to follow we shall omit the subscript n. Since

$$\eta \equiv [\xi(t', x') - \xi_0(t', x')] - [\xi(t'', x'') - \xi_0(t'', x'')] = o(\rho^\alpha),$$
$$\zeta \equiv \xi_0(t', x') - \xi_0(t'', x'') = O(\rho^\alpha),$$

it follows from Hadamard's formula that

$$\begin{aligned}&[\Phi(\xi(t', x'), t', x') - \Phi(\xi(t'', x''), t', x')]\\ &\quad - [\Phi(\xi_0(t', x'), t', x') - \Phi(\xi_0(t'', x''), t', x')]\\ &= \eta\int_0^1 \Phi_\xi(\theta\xi(t', x') + (1 - \theta)\xi(t'', x''), t', x')\,d\theta\\ &\quad + \zeta\int_0^1 \big[\Phi_\xi(\theta\xi(t', x') + (1 - \theta)\xi(t'', x''), t', x')\\ &\qquad - \Phi_\xi(\theta\xi_0(t', x') + (1 - \theta)\xi_0(t'', x''), t', x')\big]\,d\theta = o(\rho^\alpha).\end{aligned}$$

Moreover,

$$\begin{aligned}&[\Phi(\xi(t'', x''), t', x') - \Phi(\xi_0(t'', x''), t', x')]\\ &\quad - [\Phi(\xi(t'', x''), t'', x'') - \Phi(\xi_0(t'', x''), t'', x'')]\\ &= (\xi(t'', x'') - \xi_0(t'', x''))\int_0^1 \big[\Phi_\xi((\theta\xi + (1 - \theta)\xi_0), t', x')\\ &\qquad - \Phi_\xi((\theta\xi + (1 - \theta)\xi_0)(t'', x''), t'', x'')\big]\,d\theta = o(\rho^\alpha),\end{aligned}$$

since $\zeta(t'', x'') - \xi_0(t'', x'') \to 0$. Adding the resulting relations, we obtain a contradiction with (3.8). This completes the proof of the lemma.

PROOF OF THEOREM 3.3. We apply a standard method of continuation with respect to a parameter and we repeat several of the arguments from the remarks made in §9 of [25]. Let I be the set of all those $\kappa \in [0, 1]$ for each of which problem (3.7) has a solution u^κ such that $u^\kappa, u^\kappa_t, u^\kappa_{xx} \in C^{2+\alpha}(\bar{Q})$. It is known (see [23] and [24]) that $0 \in I$. We prove that $I = [0, 1]$ and, by doing so, we prove the theorem, if we show that I is closed and open in the relative topology of $[0, 1]$. Closure of I is a direct consequence of Theorem 3.2 and Arzelà's theorem.

To prove that I is open in $[0, 1]$ we take $\kappa_1 \in I$ and we let $u' = u^{\kappa_1}$. Considering w to be a known function for fixed $\kappa \in [0, 1]$, we consider in Q the following equation for v:

$$\begin{aligned}v_t = (1 - \kappa)\Delta v + \kappa\big(a^{ij}v_{x^ix^j} + b^iv_{x^i} + cv\big)&\\ + \kappa\big(F(w_{x^ix^j}, w_{x^i}, w, t, x) - a^{ij}w_{x^ix^j} - b^iw_{x^i} - cw\big)&\end{aligned} \tag{3.9}$$

with initial data $v(0, x) = \varphi(x)$, where

$$(a^{ij}, b^i, c) = \big(F_{u_{ij}}, F_{u_i}, F_u\big)(u'_{x^ix^j}, u'_{x^i}, u', t, x).$$

By virtue of our assumptions we have $a^{ij}, b^i, c \in C^\alpha(\overline{Q})$. Moreover, there exists a $\delta_0 > 0$ so small that for

$$w \in S_{\delta_0} = \{v\colon \|v - u'\|_{C^{2+\alpha}(\overline{Q})} \leqslant \delta_0\}$$

and for all $(t, x) \in \overline{Q}$

$$(w_{x^ix^j}, w_{x^i}, w, t, x) \in P(M + \varepsilon, Q).$$

For such w the last coefficient of κ in (3.9) belongs to $C^\alpha(\overline{Q})$ and, therefore, the Cauchy problem in question has a solution $v \in C^{2+\alpha}(\overline{Q})$, which is, moreover, unique. By the same token, for each $\kappa \in [0, 1]$ there is defined on the ball S_{δ_0} a mapping $\Psi_\kappa\colon w \to v$, acting in $C^{2+\alpha}(\overline{Q})$.

Our most immediate problem is to prove that for some $\delta > 0$, for all κ sufficiently close to κ_1, the mapping Ψ_κ acts from S_δ to S_δ and is a contraction in the metric of $C^{2+\alpha}(\overline{Q})$.

It is not difficult to see, by virtue of (3.9), that

$$\begin{aligned}(v - u')_t = (1 - \kappa)\Delta(v - u') &+ \kappa\big[a^{ij}(v - u')_{x^ix^j} + b^i(v - u')_{x^i} + c(v - u')\big] \\ &+ \kappa\big[F(w_{x^ix^j}, w_{x^i}, w, t, x) - F(u'_{x^ix^j}, u'_{x^i}, u', t, x) \\ &\qquad - a^{ij}(w - u')_{x^ix^j} - b^i(w - u')_{x^i} - c(w - u')\big] \\ &+ (\kappa^1 - \kappa)\big[\Delta u' - F(u'_{x^ix^j}, u_{x^i}, u', t, x)\big].\end{aligned} \tag{3.10}$$

Next, using Hadamard's formula, we transform

$$F(w_{x^ix^j}, w_{x^i}, w, t, x) - F(u'_{x^ix^j}, u'_{x^i}, u', t, x)$$

and make use of Lemma 3.3. We then find that for $w \in S_\delta$ the norm in $C^\alpha(\overline{Q})$ of the last coefficient of κ in (3.10) may be estimated in terms of $\gamma(\delta)\delta$, where $\gamma(\delta) \to 0$ as $\delta \to 0$ and $\gamma(\delta)$ does not depend on w. If follows from this and from (3.10), by virtue of known estimates of the solutions of linear equations, that there is a $\delta \in (0, \delta_0]$ such that for all κ sufficiently close to κ_1 the mapping Ψ_κ acts from S_δ into S_δ. Similarly, it may be proved that Ψ_κ is a contraction.

Thus, for all $\kappa \in [0, 1]$ sufficiently close to κ^1 there is a $w \in C^{2+\alpha}(\overline{Q})$ such that v set equal to w satisfies (3.9) and $v(0, x) = \varphi(x)$. It is obvious that w is a solution of problem (3.7), and it then remains for us to show that not only $w \in C^{2+\alpha}(\overline{Q})$ but also $w_t, w_{xx} \in C^{2+\alpha}(\overline{Q})$. We do this by repeating the proof of Theorem 13 in [24], Chapter III, §5 or the proof of Theorem 3.2, with the only difference in the proof of the latter being that here, as in [24], we cannot at once differentiate the equation for w with respect to x but, instead, form a difference ratio with respect to x and then pass to the limit. We thus find that $w_x, w_{xx} \in C^{2+\alpha}(\overline{Q})$. The relation $w_t \in C^{2+\alpha}(\overline{Q})$ follows directly from what has already been proved and from the expression for w_t yielded by (3.7). This completes the proof of the theorem.

From Theorem 2.1 we have deduced, for a parabolic equation, Theorems 3.1 and 3.2, and, after that, Theorems 3.3. From Theorem 2.1 we can obtain at once an analog of Theorem 3.3 for an elliptic equation. It is merely necessary to note that Theorem 2.1 is valid for the case in which u and F do not depend on t. An almost literal repetition of the proofs of Theorems 3.2 and 3.3 leads to the following result.

THEOREM 3.4. *Let F be independent of t and let $Q = (0, 1) \times E_d$, $\alpha \in (0\ 1)$ and $\varepsilon > 0$. Assume that in $P(M + \varepsilon, Q)$ the conditions of §2 and of Theorem 3.2 are satisfied relative to F. For each $\kappa \in [0, 1]$ and for any function $u \in C^2(E_d)$ that is a solution of the problem*

$$(1 - \kappa)(\Delta u - u) + \kappa F(u_{x^ix^j}, u_{x^i}, u, x) = 0 \quad \text{in } E_d \tag{3.11}$$

and is such that $u_{xx} \in C^2(E_d)$, let inequality (2.2) hold in E_d. Then for $\kappa = 1$ equation (3.1 has a solution, and this solution is such that $u, u_{xx} \in C^{2+\alpha}(E_d)$ (i.e., $u \in C^{4+\alpha}(E_d)$).

§4. Estimates of u_x, u_{xx} and u_t in C

The results obtained in §3 reduce the proof of existence theorems to the proof of priori boundedness of u, u_x and u_{xx}. In this section we obtain the estimates needed for u_x and u_{xx} with the aid of ideas going back to S. N. Bernstein and techniques essential taken from the theory of optimal control of diffusion processes. In order to consider tl elliptic and parabolic cases at the same time, we introduce a parameter χ, which assum one of the two values 0 or 1. The value of χ is assumed to be chosen and fixed througho this section.

We take bounded domains $D_1 \subset D \subset E_d$ such that $\overline{\rho} = \operatorname{dist}(\partial D_1, \partial D) > 0$. Let $Q =$ $(0, T) \times D$ and $Q_1 = (0, T) \times D_1$, and assume that we are given $F \in \mathfrak{F}_0(\nu, Q)$ and function $u \in C^2(\overline{Q})$ such that $u_t, u_{xx} \in C^2(\overline{Q})$ and such that in $\overline{Q}$

$$\chi u_t = F(u_{x^ix^j}, u_{x^i}, u, 1, t, x). \tag{4.1}$$

We put $u_0(x) = u(0, x)$. For brevity we also agree to take no note in our proofs of tl dependence of the constants N on the known quantities, and we shall use one and tl same letter N, generally speaking, to denote various constants.

LEMMA 4.1. *There exists a constant $N = N(d, \nu, \overline{\rho})$ such that*

$$\|u_x\|_{C(\overline{Q}_1)} \leqslant N\big(\|u\|_{C(\overline{Q})} + 1 + \chi\|u_0\|_{C(\overline{D})}\big).$$

PROOF. We take $\zeta \in C_0^\infty(E_d)$ such that $\zeta = 0$ outside of D and $\zeta = 1$ on D_1; we fix constant $\alpha > 0$, the means for choosing which we indicate below, and we consider tl function $\zeta^2 u_{x^i} + \alpha u^2$. Let (t_0, x_0) be one of the points at which this function attains i upper bound. Only three possibilities present themselves:

a) $t_0 > 0$, $x_0 \in D$ or $\chi = 0$, $x_0 \in D$; b) $x_0 \in \partial D$; c) $t_0 = 0$, $\chi = 1$.

We consider case a) first. At the point (t_0, x_0) we have

$$\chi\big(\zeta^2 u_{x^i} u_{x^it} + \alpha u u_t\big) \geqslant 0, \qquad \zeta\zeta_{x^k} u_{x^i} u_{x^i} + \zeta^2 u_{x^ix^k} u_{x^i} + \alpha u u_{x^k} = 0, \tag{4.2}$$

and the matrix with the elements

$$\begin{aligned}\zeta^2 u_{x^ix^kx^r} u_{x^i} &+ \zeta^2 u_{x^ix^k} u_{x^ix^r} + 4\zeta\zeta_{x^r} u_{x^ix^k} u_{x^i} \\ &+ \tfrac{1}{2}(\zeta^2)_{x^kx^r} u_{x^i} u_{x^i} + \alpha u u_{x^kx^r} + \alpha u_{x^k} u_{x^r}\end{aligned} \tag{4.3}$$

is nonpositive definite. Multiplying the relations

$$\chi u_{tx^i} = F_{u_{kr}} u_{x^ix^kx^r} + F_{u_k} u_{x^ix^k} + F_u u_{x^i} + F_{x^i}, \quad \chi u_t = F_{u_{kr}} u_{x^kx^r} + F_{u_k} u_{x^k} + F_u u + F_\beta$$

by $\zeta^2 u_{x^i}$ and αu, respectively, adding and using (4.2), we obtain

$$\begin{aligned}0 \leqslant F_{u_{kr}} \zeta^2 u_{x^ix^kx^r} u_{x^i} &- F_{u_k} \zeta\zeta_{x^k} u_{x^i} u_{x^i} + F_u\big(\zeta^2 u_{x^i} u_{x^i} + \alpha u\big)^2 \\ &+ F_{x^i} \zeta^2 u_{x^i} + F_{u_{kr}} \alpha u u_{x^kx^r} + F_\beta \alpha u.\end{aligned}$$

To obtain an upper estimate for the first term we use the negative definiteness of tl matrix with the elements (4.3). We then obtain

$$\begin{aligned}&\alpha F_{u_{kr}} u_{x^k} u_{x^r} + \zeta^2 F_{u_{kr}} u_{x^ix^k} u_{x^ix^r} \\ &\quad \leqslant -4F_{u_{kr}} \zeta\zeta_{x^r} u_{x^ix^k} u_{x^i} - F_{u_{kr}} \tfrac{1}{2}(\zeta^2)_{x^kx^r} u_{x^i} u_{x^i} - F_{u_k} \zeta\zeta_{x^k} u_{x^i} u_{x^i} \\ &\quad\quad + F_u\big(\zeta u_{x^i} u_{x^i} + \alpha u^2\big) + F_{x^i} \zeta^2 u_{x^i} + F_\beta \alpha u.\end{aligned} \tag{4.4}$$

Next, we use condition 1.3) and inequalities of the type

$$|F_{u_{kr}}\zeta\zeta_{x^r}u_{x^rx^k}u_{x^l}| \leqslant \varepsilon\zeta^2\sum u^2_{x^rx^k} + Nu_{x^l}u_{x^l},$$
$$|F_{x^l}\zeta^2 u_{x^l}| \leqslant Nw\zeta^2\sum|u_{x^l}| \leqslant \varepsilon\zeta^2\sum u^2_{x^rx^k} + N(u_{x^l}u_{x^l} + u^2 + 1).$$

From (4.4) we obtain

$$(\alpha - N)u_{x^k}u_{x^k} \leqslant N(1+\alpha)\left(1 + \|u\|^2_{C(\bar Q)}\right). \tag{4.5}$$

If from the very start we had taken $\alpha = N + 1$ (N is the first of the constants in (4.5)), then at the point (t_0, x_0) we would have obtained $\zeta^2 u_{x^l}u_{x^l} + \alpha u^2 \leqslant N(1 + \|u\|^2_{C(\bar Q)})$, whence the estimate needed in case a) would follow. With this indication (generally, with α arbitrary) the lemma becomes completely obvious in cases b) and c) also. This completes the proof of the lemma.

THEOREM 4.1. *There exists a constant N, depending only on d, ν, $\bar p$, $\|u\|_{C(\bar Q)}$ and $\chi\|u_0\|_{C^2(\bar D)}$, such that $u_{(\xi)(\xi)} \leqslant N$ in Q_1 for all unit vectors $\xi \in E_d$.*

For the proof of this theorem we need a lemma and some notation. In what follows we shall meet objects u_{ijkr}, u_{ijk}, u_{ij}, u_i and u, and certain statements relating to them. We shall always consider only u_{ijkr}, u_{ijk} and u_{ij} that are invariant relative to arbitrary permutations of the subscripts, and we shall say that a statement involving them is satisfied by the solution u if the statement is satisfied for all $(t, x) \in Q$ following the substitution

$$(u_{ijkr}, u_{ijk}, u_{ik}, u_i, u, \beta)$$
$$= (u_{x^ix^jx^kx^r}(t,x), u_{x^ix^jx^k}(t,x), u_{x^ix^k}(t,x), u_{x^i}(t,x), u(t,x), 1).$$

Further, we introduce a function ζ, as in the previous proof, we fix $\alpha > 0$ (the method for choosing α will be indicated below), and we let (t_0, x_0) be one of the points of $\bar Q$ at which the quantity

$$\zeta^2\Big[\max_{|\xi|=1}(u_{(\xi)(\xi)})_+\Big]^2 + \alpha u_{x^i}u_{x^i}$$

attains its upper bound. We remark that such an expression (one, however, without a maximum with respect to ξ) was encountered in [13]. In addition, we let

$$\lambda = \max_{|\xi|=1}\left(u_{(\xi)(\xi)}(t_0, x_0)\right)_+,$$

and we let ξ_0 be a vector such that $|\xi_0|^2 = \lambda\zeta(t_0, x_0)$ and $\lambda\xi_0^i = u_{x^ix^j}(t_0, x_0)\xi_0^j$. Finally, we let

$$A^2u_{ij} = u_{krij}\xi_0^k\xi_0^r, \quad A^2u_i = u_{kri}\xi_0^k\xi_0^r, \quad A^2u = u_{kr}\xi_0^k\xi_0^r,$$
$$Au_{ij} = u_{kij}\xi_0^k, \quad Au_i = u_{ki}\xi_0^k, \quad Au = u_k\xi_0^k.$$

LEMMA 4.2. a) *If $x_0 \in D$, then at the point (t_0, x_0), for arbitrary $\varepsilon > 0$, the matrix*

$$\big[\zeta A^2u_{ij} + \alpha u_ku_{kij} + \alpha u_{ki}u_{kj} + A^2u\zeta_{x^ix^j} + 2A^2u_i\zeta_{x^j}$$
$$+ 2v_\varepsilon^{kr}(\zeta Au_{ki} + Au_k\zeta_{x^i})(\zeta Au_{rj} + Au_r\zeta_{x^j})\big] \leqslant 0, \tag{4.6}$$

on the solution u, where (v_ε^{kr}) is the matrix inverse to $(v_{kr} + \varepsilon\delta_{kr})$, and $v_{kr} = |\xi_0|^2\delta_{kr} + 2\xi_0^k\xi_0^r - \zeta u_{kr}$.

b) *If $x_0 \in D$, $t_0 > 0$ or if $x_0 \in D$, $\chi = 0$, then at the point (t_0, x_0) on the solution u*

$$0 \leqslant F_{u_{ij}}\left(\zeta A^2u_{ij} + \alpha u_ku_{kij}\right) - A^2uF_{u_i}\zeta_{x^i} + F_u\left(\zeta A^2u + \alpha u_iu_i\right)$$
$$+\zeta G + 2\zeta\left[F_{u_{ij}(\xi_0)}Au_{ij} + F_{u_i(\xi_0)}Au_i + F_{u(\xi_0)}Au\right] + \zeta F_{(\xi_0)(\xi_0)} + \alpha u_kF_{x^k},$$

where G is the second derivative of F with respect to (u_{ij}, u_i, u) along the vector $(\tilde u_{ij}$, ι in which we have put $(\tilde u_{ij}, \tilde u_i, \tilde u) = (Au_{ij}, Au_i, Au)$.

PROOF. a) It is easy to see that the expression $2\zeta u_{ij}\xi^i\xi^j + \alpha u_iu_i - |\xi|^4$, which is n from the point of view of diffusion processes, attains its upper bound with resp $(t, x, \xi) \in Q \times E_d$ on the solution u at the point (t_0, x_0, ξ_0). Therefore, on the solu at this point the first derivatives of this expression with respect to x are equal to zer the matrix of the second derivatives with respect to (x, ξ) is nonpositive. In other wo

$$\zeta A^2u_i + \zeta_{x^i}A^2u + \alpha u_{ik}u_k = 0$$

and for arbitrary $\tilde x$, $\tilde\xi \in E_d$

$$\left(\zeta A^2u_{ij} + 2\zeta_{x^i}A^2u_j + \zeta_{x^ix^j}A^2u + \alpha u_{ik}u_{jk} + \alpha u_{ijk}u_k\right)\tilde x^i\tilde x^j$$
$$+4(\zeta_{x^i}Au_j + \zeta Au_{ij})\tilde x^i\tilde\xi^j - 2v_{ij}\tilde\xi^i\tilde\xi^j \leqslant 0.$$

Clearly, the matrix $(v_{ij}) \geqslant 0$; therefore, $(v_{ij} + \varepsilon\delta_{ij}) > 0$ for $\varepsilon > 0$, and if in (4. replace v_{ij} by $v_{ij} + \varepsilon\delta_{ij}$ we obtain a valid inequality. If after this replacement we cal the upper bound of the left side of (4.9) with respect to $\tilde\xi$, we obtain (4.6).

b) For the proof of (4.7) it is sufficient to differentiate (4.1) with respect to x^k, mu by αu_k, and add the result of twofold differentiation of (4.1) with respect to ξ, mult by ζ. Following this, it is necessary to substitute $(t, x, \xi) = (t_0, x_0, \xi_0)$, and take account (4.8) and the fact that when $t_0 > 0$ or $\chi = 0$ the result of the substitution wi be less than zero. This completes the proof of the lemma.

PROOF OF THEOREM 4.1. Passing, if necessary, from D to a smaller domain, w assume, by virtue of Lemma 4.1, that $|u_x|$ has been estimated in D. As in the prc Lemma 4.1 we distinguish cases a), b) and c); we begin by considering the first of ι $t_0 > 0$, $x_0 \in D$ or $\chi = 0$, $x_0 \in D$. Let $w_2 = (u_{ij}u_{ij})^{1/2}$. We note, first of all, that o solution u at the point (t_0, x_0)

$$|\xi_0|^2 = \lambda\xi, \quad A^2u = \lambda^2\zeta \leqslant w_2^2, \quad |Au_i| = \lambda|\xi_0^i| \leqslant \lambda^{3/2}\zeta^{1/2},$$
$$|Au| \leqslant N|\xi_0| \leqslant N(\lambda^{3/2}\zeta^{1/2} + 1),$$
$$A^2u_i = Au_{ki}\xi_0^k = (\zeta Au_{ki} + Au_k\zeta_{x^i})\tfrac{1}{\zeta}\xi_0^k - \tfrac{1}{\zeta}A^2u\zeta_{x^i}, \text{(}^1\text{)}$$
$$|A^2u_i| \leqslant \lambda^{1/2}\zeta^{-1/2}|\zeta Au_{ki} + Au_k\zeta_{x^i}| + Nw_2^2. \qquad ($$

Next, we obtain upper estimates of some of the terms in (4.7). In accordance assumption 1.5) and by virtue of (4.10) we have

$$\zeta G \leqslant \nu^{-1}\left(\sum|Au_i|^2 + |Au|^2\right) \leqslant N\zeta\lambda^3 + N,$$
$$\zeta F_{(\xi_0)(\xi_0)} \leqslant \nu^{-1}|\xi_0|^2w \leqslant Nw_2^2 + N, \quad |\alpha u_kF_{x^k}| \leqslant N\alpha w \leqslant N\alpha(w_2 + 1). \qquad ($$

(1) $c/0 = 0$.

We now replace the first term in (4.7) by its obvious estimate obtained with the aid of (4.6). In addition, we use (4.10) and (4.11). Then for arbitrary $\varepsilon > 0$ we readily obtain

$$\begin{aligned} 0 \leqslant & -\alpha w_2^2 + N w_2^2 + N\lambda^{1/2}\zeta^{-1/2}\sum|\zeta A u_{ki} + A u_k \zeta_{x^i}| \\ & -2v_\varepsilon^{kr}(\zeta A u_{ki} + A u_k \zeta_{x^i})(\zeta A u_{ri} + A u_r \zeta_{x^i}) + N(\alpha+1) \\ & + N\zeta\lambda^3 + N\zeta^{1/2}\lambda^{1/2}\sum|\zeta A u_{ij} + A u_j \zeta_{x^i}| + N\alpha w_2. \end{aligned} \tag{4.12}$$

We also make use of the fact that, for arbitrary $\eta \in E_d$ and $\delta > 0$,

$$|\eta| \leqslant \left(v_\varepsilon^{kr}\eta^k\eta^r\right)^{1/2}\left(\operatorname{tr}(v_{kr} + \varepsilon\delta_{kr})\right)^{1/2},$$

$$\lambda^{1/2}\zeta^{-1/2}|\eta| \leqslant \delta v_\varepsilon^{kr}\eta^k\eta^r + \frac{1}{\delta}\frac{\lambda}{\zeta}\operatorname{tr}(v_{kr} + \varepsilon\delta_{kr})$$

and $\operatorname{tr}(v_{kr}) \leqslant N w_w \zeta$. By virtue of these inequalities we conclude from (4.12) that

$$0 \leqslant w_2^2(N_1 + N_2\zeta\lambda - \alpha) + N\alpha w_2 + N(\alpha+1). \tag{4.13}$$

We emphasize that in (4.13) the constants N do not depend on α. Therefore, if from the very beginning we had taken α such that

$$\frac{1}{2}(\alpha - 1) = N_1 + N_2 \max_{\bar{Q}, |\xi|=1} \zeta(u_{(\xi)(\xi)})_+,$$

from (4.13) we would have obtained $w_2 \leqslant N$ and $\lambda \leqslant N$. Therefore, in case a) in Q we have

$$\begin{aligned} \zeta^2\Big(\max_{|\xi|=1}(u_{(\xi)(\xi)})_+\Big)^2 + \alpha u_{x^i}u_{x^i} &\leqslant (\zeta^2\lambda^2 + \alpha u_{x^i}u_{x^i})(t_0, x_0) \\ &\leqslant N + N\alpha \leqslant N + N\max_{\bar{Q}, |\xi|=1}\zeta(u_{(\xi)(\xi)})_+. \end{aligned} \tag{4.14}$$

An inequality such as this is obvious also in cases b) and c). Since the estimate sought may be obtained from (4.14) the proof of the theorem is complete.

One of the main results of this section is

THEOREM 4.2. *There exists a constant N, depending only on d, ν, $\bar{p}$, $\|u\|_{C(\bar{Q})}$, $\chi\|u_0\|_{C^2(\bar{D})}$ and $\chi\|u_t\|_{C(\bar{Q}_1)}$, such that*

$$\|u_x\|_{C(\bar{Q}_1)} + \|u_{xx}\|_{C(\bar{Q}_1)} \leqslant N.$$

PROOF. By virtue of Lemma 4.1 and Theorem 4.1, it only remains for us to show that $u_{(\xi)(\xi)} \geqslant -N$ on Q_1 for all unit vectors $\xi \in E_d$. In this connection, it is obvious that for fixed $(t, x) \in Q_1$ we need only consider the case in which ξ is an eigenvector of the matrix $(u_{x^ix^j})$ corresponding to a negative eigenvalue, say, μ. We expand the matrix $(u_{x^ix^j})$ at this point in terms of the eigenvectors

$$+u_{x^ix^j} = \mu\xi^i\xi^j + \sum_{k=1}^{k-1}\lambda^k\xi_k^i\xi_k^j.$$

Since $\lambda^k \leqslant N$ in accordance with Theorem 4.1 and $\mu \leqslant 0$, it follows from Lemma 4.1 that

$$\chi u_t = F_{u_{ij}}u_{x^ix^j} + F_{u_i}u_{x^i} + F_u u + F_\beta \leqslant \mu\nu + N.$$

Hence $u_{(\xi)(\xi)}\nu = \mu\nu \geqslant -N$, which is what we wished to prove.

THEOREM 4.3. *Let $\chi = 1$ and let $F \in \mathfrak{F}(\nu, Q)$. Then there exists a constant $N = N(d, \nu,$ such that*

$$\|u_t\|_{C(\bar{Q}_1)} \leqslant N\left(1 + \|u_0\|_{C^2(\bar{D})} + \|u\|_{C(\bar{Q})}\right). \tag{4.1}$$

PROOF. As in the proof of Lemma 4.1 we introduce the function ζ, and for $\alpha > 0$ consider the function $\zeta^2 u_t^2 + \alpha u_{x^i}u_{x^i}$. Let (t_0, x_0) be one of those points of $\bar{Q}$ at whi this function attains its upper bound. We assume that $t_0 > 0$ and $x_0 \in D$. Then at $(t_0, x$ we have

$$\zeta^2 u_t u_{tt} + \alpha u_{tx^i}u_{x^i} \geqslant \zeta\zeta_{x^k}u_t^2 + \zeta^2 u_t u_{tx^k} + \alpha u_{x^ix^k}u_{x^i} = 0,$$

and the matrix with the elements

$$\zeta^2 u_{tx^kx^r} + \zeta^2 u_{tx^k}u_{tx^r} + 4\zeta\zeta_{x^k}u_t u_{tx^r} + \tfrac{1}{2}(\zeta^2)_{x^kx^r}u_t^2 + \alpha u_{x^ix^k}u_{x^ix^r} + \alpha u_{x^ix^kx^r}u_{x^i}$$

is nonpositive definite. With the aid of these facts we arrive at the inequality

$$\begin{aligned} &\alpha F_{u_{kr}}u_{x^kx^i}u_{x^rx^i} + \zeta^2 F_{u_{kr}}u_{tx^k}u_{tx^r} \\ &\quad\leqslant -4F_{u_{kr}}\zeta\zeta_{x^k}u_t u_{tx^r} - \tfrac{1}{2}F_{u_{kr}}(\zeta^2)_{x^kx^r}u_t^2 - F_{u_k}\zeta\zeta_{x^k}u_t^2 \\ &\quad\quad + F_u(\zeta^2 u_t^2 + \alpha u_{x^i}u_{x^i}) + F_t\zeta^2 u_t + \alpha F_{x^i}u_{x^i} \end{aligned} \tag{4.1}$$

upon differentiating (4.1) with respect to t and x^i in the now well-known way.

Next, we use the fact that on the solution u

$$u_t^2 = F^2 \leqslant N w^2, \quad |\zeta\zeta_{x^k}u_t u_{tx^r}| \leqslant \varepsilon\zeta^2(u_{tx^r})^2 + \frac{1}{\varepsilon}|\zeta_{x^k}|^2 u_t^2, \quad |F_t u_t| \leqslant \nu^{-1}w|u_t| \leqslant N w^2.$$

Then from (4.16) we obtain

$$(\alpha\nu - N)u_{x^kx^i}u_{x^kx^i} \leqslant N(1+\alpha)(u_{x^i}u_{x^i} + u^2 + 1).$$

Since, in fact, $u_t^2 \leqslant N(u_{x^kx^i}u_{x^kx^i} + u_{x^i}u_{x^i} + u^2 + 1)$, from these inequalities we obta with a suitable α, at (t_0, x_0),

$$\zeta^2 u_t^2 + \alpha u_{x^i}u_{x^i} \leqslant N\left(\|u_x\|^2_{C(\bar{Q})} + \|u\|^2_{C(\bar{Q})} + \|u_0\|^2_{C^2(\bar{D})} + 1\right).$$

This type of inequality is, in fact, valid even when $t = 0$ or $x_0 \in \partial D$. It remains eliminate the quantity $\|u_x\|_{C(\bar{Q})}$ from it in the well-known way. This completes the prc of the theorem.

We proceed now to a derivation of estimates for u_x, u_{xx} and u_t, which would be routi relative to t. We fix $\varepsilon \in (0, T)$ and we let $Q_2 = (\varepsilon, T) \times D_1$.

THEOREM 4.4. *If in the statements of Lemma 4.1 and Theorems 4.1, 4.2 and 4.3 we replc Q_1 by Q_2 (with retention of the additional assumptions in Theorem 4.3) and solve with constants N depending on ε, then in these statements we can omit $\|u_0\|_{C(\bar{D})}$ and $\|u_0\|_{C^2(}$*

PROOF. Obviously, we can assume that $\chi = 1$. We first take up Lemma 4.1. In its prc we take in place of $\zeta(x)$ a function $\zeta(t, x)$ such that $\zeta \in C_0^\infty(E_{d+1})$, $\zeta = 1$ on Q_2, a $\zeta = 0$ for $t = 0$ or $x \notin D$, and we then follow the same reasoning as in the proof Lemma 4.1. Then in the case of unique existence, $t_0 > 0$, $x_0 \in D$, we arrive at inequal (4.4), on whose right-hand side it is necessary to add the term $\zeta\zeta_t u_{x^i}u_{x^i}$. However, t addition of this term causes no change in the reasoning that follows (4.4).

The situation is similar in connection with the proofs of Theorems 4.1 and 4.3. He with the indicated ζ on the right-hand sides of (4.7) and (4.16) there appear, respective the additional terms $\zeta_t A^2 u$ and $\zeta\zeta_t u_t^2$, which cause no change in the proofs of Theorems 4.1 and 4.3. Finally, the proof of Theorem 4.2 goes through without any changes. This completes the proof of the theorem.

Sometimes the inclusion of u_t in the estimates of u_x and u_{xx}, as in Theorem 4.2, is undesirable. In this case the following theorem is useful.

THEOREM 4.5. *Let $\chi = 1$. Then the norms of u_t and u_{xx} in $\mathfrak{L}_2(Q_2)$ may be estimated from above by a constant depending only on d, ν, $\bar{p}$, ε, T, the diameter of D_1, and $\|u\|_{C(\bar{Q})}$.*

For the proof it is sufficient to use the appropriate reasoning from the proofs of Lemma 2.4 in [26] and Theorem 5.1 in [6], noting that

$$u_t = F_{u_{ij}} u_{x^i x^j} + F_{u_i} u_{x^i} + F_u u + F_\beta, \quad F_{u_{ij}} u_{x^i x^j} = F_{u_{ij}}\big(u_{x^i x^j} - N\delta_{ij}\big) + N\operatorname{tr}\big(F_{u_{ij}}\big),$$

and, consequently, by virtue of Lemma 4.1 and Theorem 4.3, in Q_1

$$N\Delta u - N \leqslant F_{u_{ij}} u_{x^i x^j} \leqslant \nu\Delta u + N, \quad F_{u_{ij}} u_{x^i x^j} \leqslant N, \qquad q N\Delta u - N \leqslant \mu_t \leqslant N,$$

where the constants $N \geqslant 0$ depend only on d, ν, $\bar{p}$, ε and $\|u\|_{C(\bar{Q})}$.

We conclude this section with a generalization of Theorems 4.1–4.4. We shall need this generalization in the following section when we consider F of the form $\min(F^1,\ldots,F^n)$.

For $s = 1,\ldots,n$ let there be defined functions $F^s \in \mathfrak{F}_0(\nu, Q)$. On E_n let there be given a function $\Phi(f^1,\ldots,f^n)$, twice continuously differentiable and convex upwards, such that

$$\Phi_{f^s} \geqslant 0, \quad \nu \leqslant \sum_{s=1}^{n} \Phi_{f^s} \leqslant \nu^{-1}, \quad \big|\Phi - f^s\Phi_{f^s}\big| \leqslant \nu^{-1}. \tag{4.17}$$

Let

$$F(u_{ij}, u_i, u, \beta, t, x) = \beta\Phi\big(\tfrac{1}{\beta}F^1(u_{ij}, u_i, u, \beta, t, x),\ldots,\tfrac{1}{\beta}F^n(u_{ij}, u_i, u, \beta, t, x)\big),$$

and let u be the solution of (4.1) with a function introduced in this way.

THEOREM 4.6. *Let the assumptions made in the preceding two paragraphs be satisfied. Then*: a) *the statements of Lemma* 4.1 *and Theorems* 4.1, 4.2 *and* 4.5, *as well as the statements of Theorem* 4.4 *concerning Lemma* 4.1 *and Theorems* 4.1 *and* 4.2, *continue to hold*; *and* b), *if, in addition, $F^2 \in \mathfrak{F}(\nu, Q)$ (for each s), then the statement of Theorem* 4.3 *and the statement of Theorem* 4.4 *concerning Theorem* 4.3 *remain valid.*

It is important to note that in those estimates obtained with the aid of this theorem, nowhere do n and the characteristics of the second derivatives of Φ enter.

PROOF. The proof of Lemma 4.1 and the derivation of Theorems 4.2 and 4.5 from Theorem 4.1 change very little since, by virtue of (4.17),

$$F_{u_{ij}}\xi^i\xi^j \geqslant \nu^2|\xi|^2, \quad F = F_{u_{ij}}u_{ij} + F_{u_i}u_i + F_u u + \beta F_\beta,$$

where

$$\begin{aligned}|F_\beta| = \Big|&\Phi\big(\tfrac{1}{\beta}F^1,\ldots,\tfrac{1}{\beta}F^n\big) - \tfrac{1}{\beta}F^s\Phi_{f^s}\big(\tfrac{1}{\beta}F^1,\ldots,\tfrac{1}{\beta}F^n\big)\\ &+ F^s_\beta\Phi_{f^s}\big(\tfrac{1}{\beta}F^1,\ldots,\tfrac{1}{\beta}F^n\big)\Big| \leqslant N.\end{aligned} \tag{4.18}$$

It is obvious that to prove Theorem 4.1 it is sufficient to prove inequality (4.7) with F replaced by F^s for some s (the same replacement being made in the definition of G). Let $(4.7)^s$ denote the inequality obtained upon making this replacement in (4.7). If we carry out the operations indicated in the proof of statement b) of Lemma 4.2, then, as is readily

seen, we obtain the sum of the inequalities $(4.7)^s$, multiplied by Φ_{f^s}, on the right si which an expression of the form $\Phi_{f^s f^p}\kappa^s\kappa^p$ has been added. Since the latter expressi nonpositive and the second inequality in (4.17) is valid, it follows that one o inequalities $(4.7)^s$ holds automatically.

The validity of Theorem 4.3 (as well as that of Lemma 4.1) may be established ir case without the upward convexity of Φ by means of an almost word-for-word repet of its proof.

The validity of the statement of Theorem 4.4 may be proved in our case for Lemm and Theorems 4.2 and 4.3 in the same way as in the previous proof of Theorem 4.4. I case of Theorem 4.1, upon carrying out the proof of Theorem 4.3, we obtain, inste (4.7), the sum of the inequalities $(4.7)^s$, multiplied by Φ_{f^s}, on the right side of which is added $\zeta_t A^2 u$ and an expression of the form $\Phi_{f^s f^p}\kappa^s\kappa^p$. If we note that

$$|\zeta_t A^2 u| \leqslant N\sum_{s=1}^{n}\Phi_{f^s}|A^2 u|,$$

it will then follow from the aforementioned inequality that for some s the inequality will be valid; on the right side of this inequality there will be the additional term $N|A$ nothing in the sequel being changed. This completes the proof of the theorem.

§5. Theorems on the existence of a solution

Let $Q = (0, T) \times E_d$.

THEOREM 5.1. *Let $\alpha \in (0, 1)$, $F \in \mathfrak{F}(\nu, Q)$, and for $\beta = 1$, on each set $P(M, Q)$, le first and second derivatives of F with respect to (u_{ij}, u_i, u, x) and the function F_t be bou and have finite Hölder constants of order α with respect to (u_{ij}, u_i, u, t, x). Assume als $\varphi \in C^{4+\alpha}(E_d)$. Then the Cauchy problem*

$$u_t = F(u_{x^i x^j}, u_{x^i}, u, 1, t, x) \quad in\ Q, \quad u\big|_{t=0} = \varphi \qquad in\ E_d$$

has a solution u such that $u_t, u_{xx} \in C^{2+\alpha}(\bar{Q})$; moreover, this solution is unique.

PROOF. Let $F(u_{ij}, u_i, u, t, x) = F(u_{ij}, u_i, u, 1, t, x)$. Theorem 3.3 shows that to p existence of the solution of problem (5.1) it is sufficient to establish the a priori esti $\|u^\kappa\|_{C^2(\bar{Q})} \leqslant N$ for an arbitrary $\kappa \in [0, 1]$ and an arbitrary sufficiently smooth Theorem 3.3) solution u^κ of problem (3.7) with the constant N independent of κ. Theorems 4.2 and 4.3 we take D_1 and D to be concentric balls of radii 1 an respectively, and if we locate their common center at various points, we see that to o this a priori estimate it is sufficient to establish the much weaker estimate $\|u^\kappa\|_{C(\bar{Q})}$ where N does not depend on κ. The latter estimate follows readily from the maxi principle, the boundedness of $F_{u_{ij}}$, F_{u_i}, F_u and F_β, and the fact that u^κ satisfies a li equation of the form

$$u_t = (1-\kappa)\Delta u + \kappa\big(F_{u_{ij}}u_{x^i x^j} + F_{u_i}u_{x^i} + F_u u + F_\beta\big).$$

The uniqueness of the solution of (5.1), as we know, is easily proved with the aid o maximum principle and the Hadamard formula. This completes the proof of the theor

In the case of elliptic equations with bounded coefficients, to obtain an estimate o maximum of a bounded solution over the whole space it is sufficient, as we know, tha coefficient of u be strictly negative. Therefore, analogous to the proof of Theorem 5.1 obtain from Theorem 3.4 and Theorem 4.2

THEOREM 5.2. *Let $\alpha \in (0,1)$, let $F(u_{ij}, u_i, u, \beta, x)$ be independent of t, and let $F \in \mathfrak{F}(\nu, E_d)$. Assume that for $\beta = 1$, on each set $P(M, E_d)$, the first and second derivatives of F with respect to (u_{ij}, u_i, u, x) are bounded and have finite Hölder constants of order α with respect to (u_{ij}, u_i, u, x). Finally, let $F_u \leqslant -\nu$. Then the equation $0 = F(u_{x^i x^j}, u_{x^i}, u, 1, x)$ has in E_d a solution u such that $u, u_{xx} \in C^{2+\alpha}(E_d)$; moreover, this solution is unique.*

We proceed to the proof of existence of solutions under fewer assumptions of smoothness on F and φ. We remark that the statement of Theorem 5.3, even when $F^1 = \cdots = F^n = \cdots$, cannot, it seems to the author, be obtained from Theorem 5.1 without using the results of §§2–4, i.e., without using estimates of the solutions of nonlinear equations. At the same time, Theorem 5.1 can be obtained from Theorem 5.3 only with the aid of estimates of the solutions of linear equations (see the proof of Theorem 13 in [24], Chapter III, §5, or the proof of Theorem 3.2).

THEOREM 5.3. *Let the functions $F^s \in \mathfrak{F}(\nu, Q)$ be defined for $s = 1, 2, \ldots$, and let $\alpha \in (0,1)$ and $\varphi \in C^{2+\alpha}(E_d)$. Let $F = \inf_s F^s$. Then the Cauchy problem (5.1) has a solution $u \in C^2(\bar{Q})$, which, moreover, is unique. In addition, there exists a constant $\alpha_1 \in (0,1)$, depending only on d, ν, $\|u\|_{C(\bar{Q})}$ and $\|\varphi\|_{C^2(E_d)}$, such that $u \in C^{2+\alpha_2}(\bar{Q}) \cap C^{2+\alpha_1}_{\rm loc}(Q)$, where $\alpha_2 = \frac{1}{4}(\alpha_1 \wedge \alpha)$.*

PROOF. 1°. We define F^s for $t < 0$ and for $t > T$ by

$$F^s\big|_{t<0} = F^s\big|_{t=0}, \qquad F^s\big|_{t>T} = F^s\big|_{t=T};$$

we take any nonnegative $\zeta \in C_0^\infty(-\infty, \infty)$ such that $\int \zeta\, dt = 1$, and for $\varepsilon \in (0,1)$ we define $\tilde{F}^{s(\varepsilon)}$, rescaling F^s with respect to the variables u_{ij}, u_i, u, t and x with

$$\zeta_\varepsilon = \left(\prod_{i,j} \frac{1}{\varepsilon}\zeta\left(\frac{1}{\varepsilon}u_{ij}\right)\right)\left(\prod_i \frac{1}{\varepsilon}\zeta\left(\frac{1}{\varepsilon}u_i\right)\right)\frac{1}{\varepsilon^2}\zeta\left(\frac{1}{\varepsilon}u\right)\zeta\left(\frac{1}{\varepsilon}t\right)\prod_i \frac{1}{\varepsilon}\zeta\left(\frac{1}{\varepsilon}x^i\right).$$

For $\delta \in (0,1)$ let $\Phi_n^{(\delta)}$ be the convolution with respect to the variables $f^1, \ldots, f^n$ of the function $\Phi_n = \min(f^1, \ldots, f^n)$ with $\prod_1^n \frac{1}{\delta}\zeta(\frac{1}{\delta} f^s)$. We note at once that Φ_n has the first Sobolev derivatives

$$\Phi_{nf^s} \geqslant 0, \quad \sum_s \Phi_{nf^s} = 1, \quad \Phi_n = f^s \Phi_{nf^s} \quad \text{(almost everywhere)}.$$

Therefore,

$$\Phi_{nf^s}^{(\delta)} \geqslant 0, \qquad \sum_s \Phi_{nf^s}^{(\delta)} = 1,$$

$$\left|\Phi_n^{(\delta)} - f^s\Phi_{nf^s}^{(\delta)}\right| = \left|\delta \int g^s \Phi_{nf^s}(f^i - \delta g^i) \prod_{r=1}^n \zeta(g^r)\, dg^r\right| \leqslant \delta N, \tag{5.3}$$

where N does not depend on n and δ. We now put

$$F^{s(\varepsilon)}(u_{ij}, u_i, u, \beta, t, x) = \beta\tilde{F}^{s(\varepsilon)}\left(\tfrac{1}{\beta}u_{ij}, \tfrac{1}{\beta}u_i, \tfrac{1}{\beta}u, 1, t, x\right),$$

$$F_n^{\varepsilon\delta}(u_{ij}, u_i, u, \beta, t, x)$$
$$= \beta\Phi_n^{(\delta)}\left(\tfrac{1}{\beta}F^{1(\varepsilon)}(u_{ij}, u_i, u, \beta, t, x), \ldots, \tfrac{1}{\beta}F^{n(\varepsilon)}(u_{ij}, u_i, u, \beta, t, x)\right),$$

$$\varphi^{(\varepsilon)}(x) = \varphi(x) * \prod_i \frac{1}{\varepsilon}\zeta\left(\frac{1}{\varepsilon}x^i\right).$$

2°. It is easy to verify that for $s = 1, 2, \ldots$ all the functions $F^{s(\varepsilon)}$ satisfy co
1.1)–1.4) (for all values of the arguments with the same constant ν).

In condition 1.5) the estimate of the second derivative with respect to the dire elementary. Estimates of the moduli of $F_\beta^{s(\varepsilon)}$ and $F_{\beta x^i}^{s(\varepsilon)}$ may be obtained with th inequalities analogous to the last inequality in (5.3). Estimates of the other der appearing in 1.5) are obtained very simply. Thus, we can assert the existence of ar such that for $\varepsilon \in (0, \varepsilon_0)$ the functions $F^{s(\varepsilon)} \in \mathfrak{F}(\nu/2, Q)$ for all s.

3°. Further, inequalities (5.3) and (4.18) show that there exists a $\delta_0 > 0$ su $|F_{n\beta}^{\varepsilon\delta}| \leqslant 4\nu^{-1}$ for all $\varepsilon \in (0, \varepsilon_0)$, $\delta \in (0, \delta_0)$ and $n \geqslant 1$. In addition, it is obvious the first derivatives of $F_n^{\varepsilon\delta}$ with respect to u_{ij}, u_i, u do not exceed $2\nu^{-1}$ in absolu and that

$$\left|F_n^{\varepsilon\delta}\right| = \left|F_{nu_{ij}}^{\varepsilon\delta}u_{ij} + F_{nu_i}^{\varepsilon\delta}u_i + F_{nu}^{\varepsilon\delta}u + F_{n\beta}^{\varepsilon\delta}\beta\right| \leqslant N(d)w,$$
$$\left|F_{nx^i}^{\varepsilon\delta}\right| \leqslant N(d)w, \qquad \left|F_{nt}^{\varepsilon\delta}\right| \leqslant N(d)w.$$

We note also that $F_n^{\varepsilon\delta}$ is convex upwards with respect to the set of variables $(u$ that

$$N^{-1}(d)\nu|\xi|^2 \leqslant F_{nu_{ij}}^{\varepsilon\delta}\xi^i\xi^j \leqslant N(d)\nu^{-1}|\xi|^2$$

for all $\xi \in E_d$. Finally, for $\beta = 1$ the derivatives of arbitrary order of $F_n^{\varepsilon\delta}$ with res (u_{ij}, u_i, u, t, x) are bounded for all $\varepsilon > 0$, $\delta > 0$, and $n \geqslant 1$ in an arbitrary set $P($ and the second derivative of $F_n^{\varepsilon\delta}$ with respect to (u_{ij}, u_i, u, x) along an arbitrary $(\tilde{u}_{ij}, \tilde{u}_i, \tilde{u}, \tilde{x})$ does not exceed

$$2\nu^{-1}\left(u_i\tilde{u}_i + \tilde{u}^2\right) + 2\Phi_{nf^s}^{(\delta)}\left(F_{u_{kr}x^i}^{s(\varepsilon)}\tilde{u}_{kr} + F_{u_k x^i}^{s(\varepsilon)}\tilde{u}_k + F_u^{s(\varepsilon)}\tilde{u}\right)\tilde{x}^i + \Phi_{nf^s}^{(\delta)}F_{(\tilde{x})(\tilde{x})}^{s(\varepsilon)}$$
$$\leqslant N(d)\nu^{-1}w\left(\sum|\tilde{u}_{ij}| + \sum|\tilde{u}_i| + |\tilde{u}| + |\tilde{x}|\right)\left(\sum|\tilde{u}_i| + |\tilde{u}| + |\tilde{x}|\right)$$

for $\varepsilon \in (0, \varepsilon_0)$, $\delta \in (0,1)$ and $n \geqslant 1$.

4°. According to Theorem 5.1, by virtue of the results from 3°, for any $\varepsilon \in ($ $\delta \in (0, \delta_0)$ and $n \geqslant 1$ the Cauchy problem

$$u_t = F_n^{\varepsilon\delta}(u_{x^i x^j}, u_{x^i}, u, 1, t, x) \quad \text{in } Q, \qquad u\big|_{t=0} = \varphi^{(\varepsilon)} \quad \text{in } E_d$$

has a solution $u_n^{\varepsilon\delta} \in C^2(\bar{Q})$ such that $u_{nt}^{\varepsilon\delta}, u_{nxx}^{\varepsilon\delta} \in C^2(\bar{Q})$. Further, we use the symbo denote constants not depending on ε, δ and n.

From a formula of the type (5.2) and the inequalities $|F_{nu}^{\varepsilon\delta}| \leqslant 2\nu^{-1}$ and $|F_{n\beta}^{\varepsilon\delta}| \leqslant$ follows that $\|u_n^{\varepsilon\delta}\|_{C(\bar{Q})} \leqslant N$. From this, in accordance with Theorem 4.6, Lemma 4 Theorems 4.2 and 4.3, we have $\|u_n^{\varepsilon\delta}\|_{C^2(\bar{Q})} \leqslant N$. This, along with the results from 3°, m possible to conclude, according to Theorems 3.1 and 2.1, that there exist con $\alpha_1 \in (0,1)$ and $N \geqslant 0$ such that

$$\left\|u_n^{\varepsilon\delta}\right\|_{C^{2+\alpha_1/4}(\bar{Q})} \leqslant N \quad \text{for } \alpha \geqslant \alpha_1, \qquad \left\|u_n^{\varepsilon\delta}\right\|_{C^{2+\alpha/4}(\bar{Q})} \leqslant N \quad \text{for } \alpha \leqslant \alpha_1,$$
$$\left\|u_n^{\varepsilon\delta}\right\|_{C^{2+\alpha_1}([\gamma,T]\times E_d)} \leqslant \gamma^{-\alpha_1}N \quad \text{for } \gamma \in (0, T).$$

These estimates allow us to pass to the limit in the relations

$$u_{nt}^{\varepsilon\delta} = F_n^{\varepsilon\delta}\left(u_{nx^ix^j}^{\varepsilon\delta}, u_{nx^i}^{\varepsilon\delta}, u_n^{\varepsilon\delta}, 1, t, x\right) \quad \text{in } Q, \qquad u_n^{\varepsilon\delta}\big|_{t=0} = \varphi^{(\varepsilon)}$$

along appropriate subsequences as $\varepsilon \downarrow 0$, $\delta \downarrow 0$ and $n \to \infty$. The first two limi elementary, since F_n and Φ_n are continuous functions and their Sobolev means conve them uniformly on compacta. In the limit as $n \to \infty$ it is useful to note that F is continuous with respect to (u_{ij}, u_i, u, t, x) by virtue of the uniform continuity of F^s with respect to s. Since the continuous functions $\Phi_n(F^1, \dots, F^n)$ converge to F in decreasing fashion, in accordance with Dini's theorem they also converge uniformly on compacta. With this the existence of a solution u of the Cauchy problem (5.1) is established.

5°. We now analyze what the constant α_1 depends on. Let S_1, S_2, S_3 be three concentric balls of radii 1, 2, 3, respectively, and let $Q_i = (0, T) \times S_i$. In (5.4) we replace Q by Q_1. Then, according to Theorems 3.1 and 2.1, with such a replacement we can take the index α_1 in (5.4) to be dependent only on d and on those constants μ, M and M_1 with which the conditions 2.1)–2.4) are satisfied for $F_n^{\varepsilon\delta}$ in $P(\|u_n^{\varepsilon\delta}\|_{C^2(\bar{Q}_2)}, Q_2)$. The investigation carried out in 3° shows that these constants depend only on ν, d, and on an estimate of the norm of $u_n^{\varepsilon\delta}$ in $C^2(\bar{Q}_2)$. According to Theorem 4.6 we can estimate this norm in terms of $\|\varphi\|_{C^2(\bar{S}_2)}$ and $\|u_n^{\varepsilon\delta}\|_{C(\bar{Q}_3)}$, where the last norm, in turn, does not exceed $\|u\|_{C(\bar{Q}_3)} + 1$ for sufficiently small ε, δ, and large n (taken, clearly, from the aforementioned subsequences). Thus, with the replacement of Q by Q_1 in (5.4), the index α_1 can be regarded as depending only on d, ν, $\|\varphi\|_{C^2(E_d)}$ and $\|u\|_{C(Q)}$, but since the center S_1 can be chosen arbitrarily, it follows that the assertion of the theorem relative to α_1 is proved as well.

6°. We concern ourselves now with the question of uniqueness of the solution of (5.1) in $C^2(\bar{Q})$. Let $u^1, u^2 \in C^2(\bar{Q})$ and let them be solutions of the Cauchy problem (5.1). Then

$$\inf_s \left[F^s\left(u^1_{x^ix^j}, u^1_{x^i}, u^1, 1, t, x\right) - F^s\left(u^2_{x^ix^j}, u^2_{x^i}, u^2, 1, t, x\right)\right]$$
$$\leqslant (u^1 - u^2)_t \leqslant \sup_s \left[F^s\left(u^1_{x^ix^j}, u^1_{x^i}, u^1, 1, t, x\right) - F^s\left(u^2_{x^ix^j}, u^2_{x^i}, u^2, 1, t, x\right)\right].$$

It follows from this and from Hadamard's formula that there exist measurable functions a^{ij}, b^i, c, such that

$$a^{ij}\xi^i\xi^j \geqslant 0 \quad (\geqslant \nu|\xi|^2), \quad (u^1 - u^2)_t = a^{ij}(u^1 - u^2)_{x^ix^j} + b^i(u^1 - u^2)_{x^i} + c(u^1 - u^2).$$

In accordance with the maximum principle we conclude that $u^1 = u^2$. This completes the proof of the theorem.

REMARK 5.1. The uniqueness of the solution of problem (5.1) allows us to assert that $u_n^{\varepsilon\delta}$ converges to u for $\varepsilon \downarrow 0$, $\delta \downarrow 0$, $n \to \infty$.

Reasoning similar to that in 5° of the last proof, along with the use of Theorem 4.6 on interior estimates, easily brings us to the following result.

THEOREM 5.4. *Let the assumptions of Theorem* 5.3 *be satisfied and let u be a solution of the Cauchy problem formulated in its proof; let $r > 0$ and $\gamma \in (0, T)$, and let S_1 and S_2 be concentric balls in E_d of radii r and $2r$, respectively; let $Q_i = (\gamma/i, T) \times S_i$. Let $\nu_1 \in (0, 1]$ and $F^s \in \mathfrak{F}(\nu_1, Q_2)$ for all $s \geqslant 1$. Then there exist constants $\alpha_1 \in (0, 1)$ and $N \geqslant 0$, depending only on d, ν_1, γ, r and $\|u\|_{C(\bar{Q}_2)}$, such that*

$$\|u\|_{C^{2+\alpha_1}(\bar{Q}_1)} \leqslant N.$$

To prove theorems concerning the existence of a solution of the Cauchy problem when $\varphi \in C(E_d)$ and a solution of the first boundary value problem, we require some notation and a lemma. Let the domain D in E_d be such that for some $\varepsilon_0 > 0$, and for an arbitrary point $x \in \partial D$, a closed ball of radius ε_0 can be found which lies in $E_d \setminus D$ and contains x (on its boundary). Let $Q' = (0, T) \times D$, $\Lambda_\delta = [0, T] \times \{x \in D\colon \operatorname{dist}(x, \partial D) \leqslant \delta\}$, $\pi_\delta = \Lambda_\delta \cup ([0, \delta] \times \bar{D})$ and $\pi = \pi_0$. Also, let c_0 be a function from $C^2(E_d)$ such that $1 \geqslant c_0 > 0$

on $E_d \setminus \bar{D}$ and $c_0 = 0$ on $\bar{D}$. We note that the case $D = E_d$ is not excluded; then, of co
$c_0 = 0$.

We prove the following lemma at the close of this section.

LEMMA 5.1. *Let $\mu \in (0, 1]$, and let the real functions a^{ij}, b^i $(i, j = 1, \dots, d)$, c, f a satisfy the following conditions in $\bar{Q}$: a) $\mu|\xi|^2 \leqslant a^{ij}\xi^i\xi^j$ for all $\xi \in E_d$; b) $|a^{ij}| + |b^i| + |f| + |\varphi| \leqslant \mu^{-1}$ for all i, j and for φ continuous in $\bar{Q}$. For $m \geqslant 0$ assume that the func $u^m \in C^2_{\mathrm{loc}}(Q) \cap C(\bar{Q})$ and are solutions of the Cauchy problem*

$$u_t = a^{ij}u_{x^ix^j} + b^iu_{x^i} + cu + f - c_0m(u - \varphi) \quad in\ Q, \qquad u\,\big|_{t=0} = \varphi \quad in\ E_d.$$

Then 1) $\|u^m\|_{C(\bar{Q})} \leqslant N(\mu, T)$; *and* 2) *for all $\gamma > 0$ and $R > 0$ there exist $\delta > 0$ and n depending only on d, γ, μ, T, R and the modulus of continuity of φ in $[0, T] \times (|x| \leqslant R$ such that $|\varphi - u^m| \leqslant \gamma$ in $\pi_\delta \cap ([0, T] \times (|x| \geqslant R))$ for $m \geqslant n$.*

This lemma allows us, first of all, to prove the following theorem on the existence solution of the Cauchy problem for the initial condition $\varphi \in C(E_d)$.

THEOREM 5.5. *Let $\varphi \in C(E_d)$ and $\nu \in (0,]$, and for $s = 1, 2, \dots$ let the func $F^s \in \mathfrak{F}(\nu, Q)$ be given. We put $F = \inf_s F^s$. Then the Cauchy problem* (5.1) *has a sol $u \in C^2_{\mathrm{loc}}(Q) \cap C(\bar{Q})$, which, moreover, is unique. In addition, if $\nu_1 \in (0, 1)$, $\gamma \in (0$ $r > 0$, the S_i are concentric balls in E_d of radii ir $(i = 1, 2)$, $Q_i = (\gamma/i, T) \times S_i$ $F^s \in \mathfrak{F}(\nu_1, Q_2)$ for all $s \geqslant 1$, there then exist constants $\alpha \in (0, 1)$ and $N \geqslant 0$, depending on d, ν_1, γ, r and $\|u\|_{C(\bar{Q}_2)}$, such that*

$$\|u\|_{C^{2+\alpha}(\bar{Q}_1)} \leqslant N.$$

PROOF. We take a sequence $\varphi^n \in C^3(E_d)$ such that on each compactum from E functions φ^n will be equicontinuous and $\varphi^n \to \varphi$ as $n \to \infty$. Let $u(n)$ be a solution i $C^2(\bar{Q})$ of the Cauchy problem (5.1) in which we replace φ by φ^n. According to Theo 5.3 such solutions $u(n)$ exist and are unique; by Theorem 5.4 we have uniform int estimates for $u(n)$ in $C^{2+\alpha}$. Therefore, we can find a function $u \in C^2_{\mathrm{loc}}(Q)$, equal to φ $t = 0$ and such that some subsequence $(u(n), u_t(n), u_{x^i}(n), u_{x^ix^j}(n))$ converge (u, u_t, u_x, u_{xx}) on any compactum lying in $(0, T] \times E_d$. Since uniqueness of the solu can be proved just as it was in Theorem 5.3 and the interior estimates for u are e obtained from interior estimates for $u(n)$, it is only necessary for us to prove $u \in C(\bar{Q})$.

The reasoning used in 6° of the proof of Theorem 5.3 shows that for every n we can functions $a^{ij}(n)$, $b^i(n)$, $c(n)$ and $f(n)$ satisfying the conditions of Lemma 5.1 wi constant μ, not depending on n, such that $u_t(n) = a^{ij}(n)u_{x^ix^j}(n) + b^i(n)u_{x^i}(n$ $c(n)u(n) + f(n)$ in Q. We now apply Lemma 5.1 with $D = E_d$. In this case $c_0 = 0$ and parameter m actually does not enter into (5.5). In accordance with Lemma 5.1 we find $|\varphi^n - u(n)| \to 0$ as $t \downarrow 0$, uniformly with respect to n and with x varying over an arbit compactum in E_d. Since $\varphi^n \to \varphi$ and $u(n) \to u$ over some subsequence in $(0, T] \times E$ follows that $|\varphi - u| \to 0$ as $t \downarrow 0$ uniformly on such compacta. This completes the proo the theorem.

THEOREM 5.6. *Let the continuous bounded function φ be given on π, let $\nu \in (0, 1]$, an the functions $F^s \in \mathfrak{F}(\nu, Q)$ be defined for $s = 1, 2, \dots$. Put $F = \inf_s F^2$. Then the proble*

$$u_t = F(u_{x^ix^j}, u_{x^i}, u, 1, t, x) \quad in\ (0, T] \times D, \qquad u\,\big|_\pi = \varphi \qquad ($$

has a solution $u \in C(\bar{Q}') \cap C^2_{\text{loc}}(Q')$, which is, moreover, unique. In addition, if $\nu_1 \in (0,1]$, $\gamma \in (0,T)$, the domain $D_1 \subset \bar{D}_1 \subset D$, and $F^s \in \mathfrak{F}(\nu_1, (\gamma, T) \times D_1)$ for all s, then there exist constants $\alpha \in (0,1)$, $N \geq 0$, depending only on d, ν_1, γ, $\|u\|_{C(\bar{Q}')}$ and the distance between the boundaries D_1 and D, such that

$$\|u\|_{C^{2+\alpha}([\gamma,T]\times\bar{D}_1)} \leq N.$$

PROOF. Uniqueness of the solution of (5.6) is proved in the same way in Theorem 5.3. For the proof of existence we assume that φ is extended onto Q as a continuous and bounded function. For $m \geq 0$ we consider the Cauchy problem

$$\begin{aligned} u_t &= F(u_{x^ix^j}, u_{x^i}, u, 1, t, x) - c_0 m(u - \varphi^m) \quad \text{in } Q, \\ (u - \varphi^m)\big|_{t=0} &= 0 \quad \text{in } E_d, \end{aligned} \tag{5.7}$$

where the functions φ^m are taken as in the previous proof.

Since

$$F - c_0 m(u - \varphi^m\beta) = \inf_s \left[F^s - c_0 m(u - \varphi^m\beta)\right],$$

there exists, in accordance with Theorem 5.5, for each m a solution $u(m)$ of (5.7) from the class $C(\bar{Q}) \cap C^2_{\text{loc}}(Q)$. Moreover, it is obvious that $F^s - c_0 m(u - \varphi^m\beta) \in \mathfrak{F}(\nu, Q)$. Therefore Theorem 5.5 also yields interior estimates of $u(m)$, uniform with respect to m, in $C^{2+\alpha}([\gamma, T] \times \bar{D}_1)$ in terms of $\|u(m)\|_{C(\bar{Q}')}$. From these estimates and relations of the type

$$u_t(m) = a^{ij}(m)u_{x^ix^j}(m) + b^i(m)u_{x^i}(m), \text{and} + c(m)u(m) - c_0 m(u(m) - \varphi^m)$$

we deduce at once, with the aid of Lemma 5.1, the existence of the required solution of (5.6), for which the interior estimates asserted in the theorem are valid. This completes the proof of the theorem.

In the proof of Theorems 5.5 and 5.6 a fundamental role is played by Theorems 5.3 and 5.4 and Lemma 5.1. From the proof of the latter the reader sees that when $c \leq -\gamma$ the constants N, δ and n, which appear in their statements, do not depend on T and, therefore, the natural analog of Lemma 5.1 holds for elliptic equations. Moreover, the interior estimates from §2 are valid, in an obvious way, for functions u which are independent of t, and the proofs of the corresponding analogs of Theorems 5.3 and 5.4 for elliptic equations can be carried through by an almost word-for-word repetition of the proofs for parabolic equations. Therefore, in order to avoid repeating the reasoning already employed, we present Theorem 5.6 for elliptic equations without proof.

THEOREM 5.7. *Let a continuous bounded function φ be given on ∂D, let $\nu \in (0,1]$, and let the functions $F^2 \in \mathfrak{F}(\nu, E_d)$ be defined for $s = 1, 2, \ldots$ in such a way that $F^s_u \leq -\nu$. Put $F = \inf_s F^s$. Then the problem*

$$0 = F(u_{x^ix^j}, u_{x^i}, u, 1, x) \quad \text{in } D, \qquad u\big|_{\partial D} = \varphi$$

has a solution $u \in C(\bar{D}) \cap C^2_{\text{loc}}(D)$, which, moreover, is unique. In addition, if $\nu_1 \in (0,1]$, the bounded domain $D_1 \subset \bar{D}_1 \subset D$, and $F^s \in \mathfrak{F}(\nu_1, D_1)$ for all s, then there exist constants $\alpha \in (0,1)$ and $N \geq 0$, depending only on d, ν_1, $\|u\|_{C(\bar{D})}$ and the distance between the boundaries D_1 and D, such that

$$\|u\|_{C^{2+\alpha}(\bar{D}_1)} \leq N.$$

PROOF OF LEMMA 5.1. 1°. We note, first, considering the function $u\exp(-t/\mu)$ instead of u, that it is easy to reduce the situation to the case $c \leq -\mu$. Therefore, we assume that $c \leq -\mu$. Further, from the maximum principle we obtain $|u^m| \leq \mu^{-2} + \|\varphi\|_{C(\bar{Q})}$, which establishes assertion 1).

2°. We proceed now to the proof of assertion 2). Assume that we have succeeded in constructing functions $\delta = \delta(d, \gamma, \mu) > 0$ and $n = n(d, \gamma, \mu) > 0$ such that, as soon as the conditions of the lemma are satisfied, $m \geq n$, $\gamma > 0$, $R > 0$, then

$$|u^m| \leq \gamma + \|\varphi\|_{C([0,T]\times(|x|\leq R+1))} \tag{5.8}$$

in $\pi_\delta \cap ([0,T] \times (|x| \leq R))$. We now show how assertion 2) is then deduced. We fix $\gamma > 0$ and $R > 0$, and we also fix $\varepsilon > 0$ so that $|\varphi - \varphi^{(\varepsilon)}| \leq \gamma$ in $[0,T] \times (|x| \leq R+1)$ where

$$\varphi^{(\varepsilon)}(t,x) = \varphi(t,x) * \left(\frac{1}{\varepsilon}\zeta\left(\frac{1}{\varepsilon}t\right)\prod_i \frac{1}{\varepsilon}\zeta\left(\frac{1}{\varepsilon}x^i\right)\right)$$

and where, in calculating the latter convolution, we assume that $\varphi\big|_{t<0} = \varphi\big|_{t=0}$ and $\varphi\big|_{t>T} = \varphi\big|_{t=T}$. A suitable ε, as is well known, can be chosen, depending only φ, R, d and the modulus of continuity of φ in $[0,T] \times (|x| \leq R+2)$. Further, we note that

$$\begin{aligned} (u - \varphi^{(\varepsilon)})_t &= a^{ij}(u - \varphi^{(\varepsilon)})_{x^ix^j} + b^i(u - \varphi^{(\varepsilon)})_{x^i} + c(u - \varphi^{(\varepsilon)}) \\ &\quad + \left[f + a^{ij}\varphi^{(\varepsilon)}_{x^ix^j} + b^i\varphi^{(\varepsilon)}_{x^i} + c\varphi^{(\varepsilon)}\right] - c_0 m\left((u - \varphi^{(\varepsilon)}) - (\varphi - \varphi^{(\varepsilon)})\right). \end{aligned}$$

In addition, the absolute value of the expression in the square brackets does not exceed $\mu^{-2}(1 + \varepsilon^{-2})N(d)$. Consequently, in accordance with our assumption, when $\delta = \delta(d, \gamma, \mu^{-2}(1+\varepsilon^{-2})N(d))$ and $m \geq n(d, \gamma, \mu^{-2}(1+\varepsilon^{-2})N(d))$ in the set $\pi_\delta \cap ([0,T] \times (|x| \leq R))$, we have

$$|u^m - \varphi| \leq \gamma + |u^m - \varphi^{(\varepsilon)}| \leq 2\gamma + \|\varphi - \varphi^{(\varepsilon)}\|_{C([0,T]\times(|x|\leq R+1))} \leq 3\gamma.$$

Thus it remains to prove the assertion connected with (5.8). We do this using the standard technique of barriers. We fix $\gamma > 0$ and, with no loss of generality, we assume that $\varepsilon_0 \leq 1/4$. For simplicity we agree also to take no note of the fact that the constants selected below necessarily depend on the initial objects in a particular way.

3°. We show, first of all, that for any $\varepsilon > 0$ we can find an $n_1 = n_1(d, \gamma, \mu, \varepsilon)$ such that for $t_0 \in [0,T]$, $m \geq n_1$ and $\operatorname{dist}(x_0, D) \geq \varepsilon$, we have $|u^m(t_0, x_0)| \leq \gamma + M(x_0)$, where

$$M(x_0) = \max\{|\varphi(t,x)| : t \in [0,T], |x - x_0| \leq \tfrac{1}{4}\}.$$

We fix $\varepsilon > 0$, and we let $(t_0, x_0) \in Q \setminus Q'$, $\operatorname{dist}(x_0, D) \geq \varepsilon$ and $\eta \in (0, \varepsilon \wedge \frac{1}{4})$. On the set $(|x - x_0| \leq \varepsilon)$ the function c_0 is bounded from below by a positive constant. Using this, it is not hard to select a small constant $\kappa > 0$ such that the function $\cosh(\kappa\sqrt{m}\,|x - x_0|)$ on $[0,T] \times (|x - x_0| \leq \varepsilon)$ will satisfy the inequality $a^{ij}v_{x^ix^j} + b^iv_{x^i} + (c - c_0 m)v \leq 0$ for all sufficiently large m. Assuming that η is sufficiently small, it is easy to show that the function $\eta^2 - |x - x_0|^2$ satisfies in $[0,T] \times (|x - x_0| \leq \eta)$ the inequality

$$a^{ij}v_{x^ix^j} + b^iv_{x^i} + (c - c_0 m)v + \mu \leq 0$$

for all m. Comparing u^m with

$$M(x_0) + \mu^{-2}\left(\eta^2 - |x - x_0|^2\right) + \|u^m\|_{C(\bar{Q})}\left(\cosh\left(\kappa\eta\sqrt{m}\right)\right)^{-1}\cosh\left(\kappa\sqrt{m}\,|x - x_0|\right),$$

we obtain, in accordance with the maximum principle,

$$|u^m(t_0, x_0)| \leq M(x_0) + \mu^{-2}\eta^{-2} + 2\mu^{-2}\left(\cosh\left(\kappa\eta\sqrt{m}\right)\right)^{-1}.$$

It then remains to put $\eta = m^{-1/4}$ and to consider sufficiently large m.

4°. We prove yet another intermediate assertion. Simple calculations show that there exists an $\varepsilon_1 \in (\varepsilon_0, 2\varepsilon_0)$, close to ε_0, such that for all $x_0 \in E_d$ and $\varepsilon \in [0, \varepsilon_1 - \varepsilon_0]$ the function

$$\tfrac{1}{4}(\varepsilon_1 - \varepsilon_0 + \varepsilon)^2 - \left(|x - x_0| - \tfrac{1}{2}(\varepsilon_0 + \varepsilon_1 - \varepsilon)\right)^2$$

satisfies the inequality

$$a^{ij}v_{x^ix^j} + b^iv_{x^i} + (c - c_0m)v + \mu \leqslant 0$$

for all $m \geqslant 0$, $t \in [0, T]$ and $|x - x_0| \in [\varepsilon_0 - \varepsilon, \varepsilon_1]$. In addition, we can find a $\kappa < 0$ such that for $x_0 \in E_d$ the function

$$\left(|x - x_0|^{\kappa} - \varepsilon_0^{\kappa}\right)\left(\varepsilon_1^{\kappa} - \varepsilon_0^{\kappa}\right)^{-1}$$

satisfies the inequality

$$a^{ij}v_{x^ix^j} + b^iv_{x^i} + (c - c_0m)v \leqslant 0$$

for the same m, t and x. It follows from this, according to the maximum principle, that

$$\begin{aligned} |u^m(t, x)| \leqslant \left(|x - x_0|^{\kappa} - \varepsilon_0^{\kappa}\right)\left(\varepsilon_1^{\kappa} - \varepsilon_0^{\kappa}\right)^{-1}\|u^m\|_{C(\bar{Q})} \\ + \mu^{-2}\left[\tfrac{1}{4}(\varepsilon_1 - \varepsilon_0 + \varepsilon)^2 - \left(|x - x_0| - \tfrac{1}{2}(\varepsilon_0 - +\varepsilon_1 - \varepsilon)^2\right)\right] + M_m(\varepsilon). \end{aligned} \tag{5.9}$$

where $M_m(\varepsilon)$ is the largest value of $|u^m(t, x)|$ on

$$\left([0, T] \times \left(|x - x_0| = \varepsilon_0 - \varepsilon\right)\right) \cup \left\{(t, x) \colon t = 0, |x - x_0| \in [\varepsilon_0 - \varepsilon, \varepsilon_1]\right\}.$$

5°. Assume now that $x \in D$, $|x| \leqslant R$ and $\operatorname{dist}(x, \partial D) \leqslant \delta \leqslant \varepsilon_1 - \varepsilon_0$. As x_1 we take a point on ∂D such that $|x - x_1| \leqslant \delta$, and as x_0 we take a point such that the ball of radius ε_0 with center at x_0 lies outside of D and contains x_1. It is clear that $|x - x_0| \in [\varepsilon_0, \varepsilon_1]$. Moreover, in accordance with the result in 3°, when $\varepsilon \in (0, \varepsilon_1 - \varepsilon_0]$ and $m \geqslant n_1(d, \gamma, \mu, \varepsilon)$ the last term in (5.9) does not exceed the right side of (5.8). The sum of the first two terms on the right side of (5.9) does not exceed

$$2\mu^{-2}\left((\delta + \varepsilon_0)^{\kappa} - \varepsilon_0^{\kappa}\right)\left(\varepsilon_1^{\kappa} - \varepsilon_0^{\kappa}\right)^{-1} + \tfrac{1}{4}\mu^{-2}\left[(\varepsilon_1 - \varepsilon_0 + \varepsilon)^2 - (\varepsilon_1 - \varepsilon_0 - \varepsilon - 2\delta)^2\right],$$

if $\varepsilon + \delta \leqslant \frac{1}{2}(\varepsilon_0 + \varepsilon_1 - \varepsilon)$. The last expression can be made arbitrarily small by taking ε and δ sufficiently small. With this we have proved (5.8) if we replace γ therein by 2γ on the set $\Lambda_\delta \cap ([0, T] \times (|x| \leqslant R))$.

6°. Finally, for an arbitrary fixed $x_0 \in D$, $|x_0| \leqslant R$, the function $|x - x_0|^2 + Nt$ for a sufficiently large constant N satisfies the inequality

$$v_t \geqslant a^{ij}v_{x^ix^j} + b^iv_{x^i} + (c - c_0m) + \mu^{-1} \quad \text{in } \left(|x - x_0| \leqslant 1\right).$$

Hence, by the maximum principle,

$$|u^m(t, x)| \leqslant \left(\|u^m\|_{C(\bar{Q})} + 1\right)\left(|x - x_0|^2 + Nt\right) + \|\varphi\|_{C([0,T] \times (|x| \leqslant R+1))}.$$

Substituting $x = x_0$ here, we obtain (5.8) in $[0, \delta] \times (|x| \leqslant R)$ for sufficiently small δ. This completes the proof of the lemma.

REMARK 5.1. The requirements on the domain D in Theorems 5.6 and 5.7 have, clearly, been raised. Once these theorems have been proved for domains D with a sufficiently smooth boundary they can be extended to a much wider class of domains, in a standard way, for the boundaries of which there exist appropriate barriers (approximating these domains from within by domains with smooth boundaries).

We note also that Lemma 5.1 is easily extended to a wide class of domains Q' which are curvilinear cylinders with respect to t; also, with the lemma so extended, we can easily deduce the existence of a solution of the first boundary value problem from the solvability of the Cauchy problem.

§6. Examples

1°. Let the domain $D \subset E_d$ satisfy the conditions stated in §5 preceding Lemma 5.1. For $s = 1, \dots, n$ let the functions f^s and the elliptic operators

$$L^s = a_{ij}^s \frac{\partial^2}{\partial x^i \partial x^j} + b_i^s \frac{\partial}{\partial x^i} + c^s$$

be specified on E_d.

We assume that $a_{ij}^s, b_i^s, c^s, f^s \in C^2(E_d)$, $c^s \leqslant -\nu$, and $\nu|\xi|^2 \leqslant a_{ij}^s\xi^i\xi^j$ for all $\xi \in E_d$, s and x, where the constant $\nu > 0$. Let $\varphi \in C(E_d)$. A direct application of Theorem 5.7 in the case $F^s = a_{ij}^s u_{ij} + b_i^s u_i + c^s u + \beta f^s$ leads to an assertion concerning the existence of a unique solution $u \in C(\bar{D})C_{\text{loc}}^2(D)$ of the problem

$$\min_s \left(L^{(s)}u + f^s\right) = 0 \quad \text{in } D, \qquad (u - \varphi)\big|_{\partial D} = 0.$$

Moreover, for an arbitrary bounded domain $D_1 \subset \bar{D}_1 \subset D$ this solution belongs to $C^{2+\alpha}(D_1)$ for some $\alpha \in (0, 1)$ (possibly depending on D_1). This result was obtained earlier in [12] for the case of two operators L^s under, essentially, for fewer assumptions by another method. For the case of a finite number of operators L^s for a smooth boundary of D and $\varphi = 0$ in [18] this result was strengthened by an assertion concerning the boundedness of the second derivatives of u in D. Concerning the estimates of u_{xx} on ∂D, see also [8] and [29]. We remark also that instead of $L^s u$ we could have taken $L^s u + (|u_x|^2 + 1)^{1/2}$.

2°. A less trivial example of an application of our results is furnished by the following Monge-Ampere type equation. Let $\lambda > 0$ and $f \in C^2(E_d)$. In E_d we consider the equation

$$\det\left(u_{x^ix^j} - \lambda u\delta_{ij}\right) = d^{-d}(f_-)^d, \tag{6.1}$$

whose solution we seek in a class of functions u satisfying in E_d the condition

$$\left(u_{x^ix^j} - \lambda u\delta_{ij}\right) \geqslant 0. \tag{6.2}$$

With the aid of optimal control theory the existence and the uniqueness of the solution of (6.1) (understood in the sense of almost everywhere) was proved in [27] and [28] in the class of bounded functions with bounded first and second Sobolev derivatives, the solution satisfying (6.2) (almost everywhere). We show here how this result can be obtained in another way under the added assumption that $f < 0$.

It is known from [27] and [28] that (6.1), in the class of functions satisfying (6.2), is equivalent to the equation

$$\inf_{a = a^* \geqslant 0, \operatorname{tr} a = 1} \left(a_{ij}u_{x^ix^j} - \lambda u + \sqrt[d]{\det af}\right) = 0. \tag{6.3}$$

Along with this equation we consider also $(6.3)_n$, which is obtained if in place of $(-\lambda u)$ in (6.3) we write $(\frac{1}{n}\Delta u - \lambda u)$. In $(6.3)_n$ we can replace the infimum over all a under consideration by the infimum over the denumerable set of a^s. If following this we put

$$F^s = a^s_{ij}u_{ij} + \tfrac{1}{n}\delta_{ij}u_{ij} - \lambda u + \beta\sqrt[d]{\det a^s f},$$

then, in accordance with Theorem 5.7, we find that $(6.3)_n$ has a solution $u^n \in C^2(E_d)$, which, moreover, is unique. It readily follows from the maximum principle that

$$|u^n| \leq \lambda^{-1}\|f\|_{C(E_d)}.$$

We now estimate the second derivatives of u^n. Since the second difference of the lower bound of a set of functions does not exceed the upper bound of the second differences, we then obtain, upon applying to (6.3) the operation of taking the second difference with respect to an arbitrary direction h and then using the maximum principle.

$$u^n(x+h) - 2u^n(x) + u^n(x-h) \leq \tfrac{1}{\lambda}h^2\|f\|_{C^2(E_d)} = N_1 h^2.$$

Therefore, the matrix $(u^n_{x^ix^j}) \leq N_1(\delta_{ij})$. On the other hand, if in $(6.3)_n$ we take $\xi^i\xi^j$, where $|\xi| = 1$, in place of a_{ij}, and if we use the fact that $\Delta u^n \leq N_1 d$, we obtain $(u^n_{x^ix^j}) \geq N(\delta_{ij})$, where N does not depend on n.

Thus, the second derivatives of the u^n are uniformly bounded.

In addition, analogously to what was said concerning the equivalence of (6.1), (6.2) and (6.3), equation (6.3) signifies that

$$\left(u^n_{x^ix^j} + \delta_{ij}\left(\tfrac{1}{n}\Delta u^n - \lambda u^n\right)\right) \geq 0, \quad \det\left(u^n_{x^ix^j} + \delta_{ij}\left(\tfrac{1}{n}\Delta u^n - \lambda u^n\right)\right) = d^{-d}|f|^d. \quad (6.4)$$

From these two relations, the boundedness of the second derivatives of u^n and the inequality $f < 0$, it follows that in each compactum the eigenvalues of the matrix appearing in (6.4) are separated from zero, uniformly with respect to n. It is obvious from this that in $(6.3)_n$ we can take the infimum subject to the additional condition $a \geq \mu(\delta_{ij})$, where $\mu(x) > 0$ is continuous, and that u^n thereby remains a solution of the equation so modified. Following this, in accordance with Theorem 5.7, we obtain, for an arbitrary compactum K, a uniform estimate of the norm of u^n in $C^{2+\alpha}(K)$ with a constant $\alpha \in (0,1)$ (possibly depending on K but not on n). Passing to the limit along a suitable subsequence in (6.4) leads to the existence of a solution of (6.1) belonging to the class $C^{2+\alpha}(K)$ for any compactum K with a constant $\alpha \in (0,1)$.

3°. We consider also an example of a parabolic Monge-Ampere equation, which plays a very important role in the theory of optimal control of diffusion processes, in the derivation of estimates from [16], and, finally, in the theory developed in this paper. Let $D = \{x \in E_d\colon |x| < 1\}$, $T \in (0,\infty)$ and $Q_1 = (0,T) \times D$. Let $f \in C^2(\overline{Q}_1)$, $f(0,x) = 0$. We consider in Q_1 the equation

$$-u_t \det(u_{x^ix^j}) = (d+1)^{-(d+1)}\left(1-|x|^2\right)(f_-)^{d+1} \quad (6.5)$$

with the boundary conditions $u|_{t=0} = u|_{|x|=1} = 0$. As shown in §5 of [11], equation (6.5) (understood in the sense of almost everywhere in Q_1) with these boundary conditions has a solution $v \in C(\overline{Q}_1)$ such that the Sobolev derivatives v_t, v_x and v_{xx} are bounded in Q_1, $v_t \leq 0$ and $(v_{x^ix^j}) \geq 0$ (almost everywhere in Q_1). We show that $v \in C^{2+\alpha}(K)$ for an arbitrary compactum $K \subset Q_1$ in which $f < 0$, where $\alpha \in (0,1)$ (and possibly depends on K).

Let, for example, the smooth domain $D_1 \subset \overline{D}_1 \subset D$, $0 < \gamma_1 < \gamma_2 \leq T$, and assume that $f < 0$ in $\overline{Q}_2$, where $Q_2 = (\gamma_1,\gamma_2) \times D_1$. For $u = v$ and for v_t and $v_{x^ix^j}$ bounded, it follows at once from (6.5) that $(-v_t)$ and the eigenvalues of the matrix $(v_{x^ix^j})$ are separated away from zero from below on Q_2 (almost everywhere) by a constant $\mu > 0$. In addition, v (almost everywhere) satisfies the equation (see [1])

$$\inf_{\substack{r\geq 0,\, a=a^*\geq 0\\ r+\operatorname{tr} a=1}} \left(-ru_t + a_{ij}u_{x^ix^j} + \left(1-|x|^2\right)^{1/(d+1)}(r\det a)^{1/(d+1)}f\right) = 0. \quad (6.6)$$

From these properties of the derivatives of v it follows that if this equation is considered on Q_2 and if the lower bound is taken subject to the added conditions $r \geq \mu_1$ and $a \geq \mu_1(\delta_{ij})$, with some constant $\mu_1 > 0$, then v (almost everywhere in Q_2) will remain its solution. We make use of this, and so solve the modified equation (6.6) for u_t, taking r outside the parentheses. We then find that v (almost everywhere in Q_2) satisfies the equation

$$u_t = \inf_{\substack{r\geq\mu_1,\, a=a^*\geq\mu_1(\delta_{ij})\\ r+\operatorname{tr} a=1}} \left(\frac{1}{r}a_{ij}u_{x^ix^j} + \frac{1}{r}\left(1-|x|^2\right)^{1/(d+1)}(r\det a)^{1/(d+1)}f\right).$$

According to Theorem 5.6, this equation, in Q_2 with the boundary condition $u = v$ on the lower base and the lateral surface of Q_2, has a solution $v_1 \in C(\overline{Q}_2)$, belonging to $C^{2+\alpha}(K)$ for an arbitrary compactum $K \subset Q_2$ with some $\alpha \in (0,1)$ (possibly depending on K). It then remains to use the uniqueness theorems from [4] in order to confirm that $v = v_1$ in $\overline{Q}_2$.

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AN ESTIMATE FOR POLYNOMIALS ON ANALYTIC SETS

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ABSTRACT. Let A be a connected, analytic (in general, not closed) subset of the complex space $\mathbf{C}^n$ and let $K \subset A$ be a compact set which is not pluri-polar in A. In this article it is proved that the extremal function $V(z, K)$ is locally bounded on A if and only if A belongs to some algebraic set of the same dimension as A. Moreover, it is shown that for an algebraic set A in a neighborhood of any ordinary point $z^0 \in A_0$ the function $V(z, K)$ can be represented as the limit of an increasing sequence of maximal functions.

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We consider the following problem: let A be a connected analytic (in general, not closed) subset of the complex space $\mathbf{C}^n$ and $K \subset A$ a compact set which is not pluri-polar in A.(1) It is required to give on A an estimate of the form

$$|P_j(z)|^{1/j} \leqslant \Phi(z) \tag{1}$$

for all polynomials $P_j(z)$, $z = (z_1, \ldots, z_n) \in \mathbf{C}^n$, of degree j with norm $\|P_j\|_K^{1/j} \leqslant M$, where $\Phi(z)$ is some function which depends only on K and M. Without loss of generality we may let $M = 1$. Then for the best estimate in (1) it is necessary for us to take as $\Phi(z)$ the extremal function

$$\Phi(z, K) = \sup\left\{|P_j|^{1/j}\colon P_j \text{ is a polynomial of degree } j,\ \|P_j\|_K \leqslant 1, j = 1, 2, \ldots\right\}. \tag{2}$$

Therefore, the existence of an estimate of type (1) depends on the behavior of the function $\Phi(z, K)$ or, more precisely, on whether $\Phi(z, K)$ is finite at all points $z \in A$.

We note that $\Phi(z, K) \not\equiv \infty$ in $\mathbf{C}^n$ if and only if K is not a pluri-polar set in $\mathbf{C}^n$. Also, if K is a pluri-polar set, then $\Phi(z, K) = \infty$ in $\mathbf{C}^n$ except on some pluri-polar set $E \supset K$. Since in our situation A is itself a pluri-polar set, it follows that the set $\{z \in \mathbf{C}^n\colon \Phi(z, K) < \infty\}$ is pluri-polar.

The goal of this article is to study the properties (mainly the property of local boundedness) of the function $\Phi(z, K)$ on A. We prove that for a compact set $K \subset A$ which is not pluri-polar in A the function $\Phi(z, K)$ is locally bounded in A if and only if A is a piece of an algebraic set, i.e., if and only if A lies in an algebraic set of the same

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(1) The connectedness of A is to be understood in the sense that the set of ordinary points A^0 forms a complex manifold of some dimension m. A set $K \subset A$ is called *pluri-polar* in A if for any point $z^0 \in K \cap A^0$ there exist a neighborhood $U \subset A^0$ of this point and a plurisubharmonic function $u(z) \not\equiv -\infty$ in U such that $u(z) = -\infty$ for all $z \in K \cap U$.

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