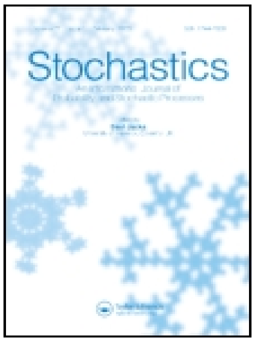

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Optimal stopping of switching diffusions with state dependent switching rates

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ABSTRACT

This paper is concerned with a continuous-time and infinite-horizon optimal stopping problem in switching diffusion models. In contrast to the assumption commonly made in the literature that the regime-switching is modeled by an independent Markov chain, we consider in this paper the case of state-dependent regime-switching. The Hamilton–Jacobi–Bellman (HJB) equation associated with the optimal stopping problem is given by a system of coupled variational inequalities. By means of the dynamic programming (DP) principle, we prove that the value function is the unique viscosity solution of the HJB system. As an interesting application in mathematical finance, we examine the problem of pricing perpetual American put options with state-dependent regime-switching. A numerical procedure is developed based on the DP approach and an efficient discrete tree approximation of the continuous asset price process. Numerical results are reported.

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1. Introduction

Markovian regime-switching models have been well studied in the literature of financial mathematics due to their capability of modeling complex systems with uncertainty. For example, see Liu and Zhao [11] and references therein. There are two random sources in this setup: a Brownian motion is used for the continuous dynamics of the underlying state process, and an exogenous Markov chain with finite state space is used for capturing the random switching across different regimes (e.g. different stages of business cycles in financial market). A fundamental assumption is that the Markov chain is independent of the Brownian motion, implying that the continuous state process depends on the Markovian regime, but not visa versa. In other words, the random switching of regimes changes the behavior of the state process, but the state process does not have any influence on how fast or show the regime switches. This independence assumption is reasonable for some scenarios but may not for others. Take option pricing as an example. For options written on individual stocks, it seems fine to think that the impact of the price change of the particular stock on the overall market is negligible. However, for options written on indexes, the influence of the index on the overall market needs to be included into the model, since the index itself can be an indicator of market condition (e.g. options on S&P500). For this reason, it is necessary to relax the independence assumption in the model

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and allow the regime-switching to be dependent of the underlying continuous state process. This has motivated the study of stochastic systems with state dependent regime-switching.

Stochastic systems with state dependent regime-switching are systematically treated in Yin and Zhu [14] in which both theoretical results and some applications are presented. Optimal stopping has been an important problem in stochastic analysis and can date back to McKean [12] who solved the problem without regime-switching. Guo [4] solved the problem for a Markovian regime-switching model. Guo and Zhang [5] derived a closed-form solution for the perpetual American put options assuming the model has two regimes. Zhang [15], Guo and Zhang [6] studied a problem of finding the optimal selling rules in a Markovian regime-switching model. Recently Pemy [13] studied an optimal stopping problem for a Markovian regime-switching jump diffusion model by using the viscosity solution approach. However, no existing literature deals with the problem in the more general models with state dependent regime-switching. Thus the present paper is the first attempt along this new direction of extension of the optimal stopping problem and application.

In this work we consider an infinite-horizon optimal stopping problem while the underlying state process is a switching diffusion, and the rates of regime switchings depend on the continuous state process. Under fairly general conditions, we show that the value function of the optimal stopping problem is the unique viscosity solution of the HJB equation which is given by a system of coupled variational inequalities. We then examine the problem of pricing perpetual American put options with state-dependent regime-switching. We develop a numerical procedure to determine the optimal exercise levels as well as the approximate option prices. The procedure is based on the dynamic programming (DP) approach and an efficient discrete tree approximation of the continuous stock price. To our best knowledge, this is the first example of considering American options in regime-switching models with state dependent switching rates.

The paper is organized as follows. Section 2 presents the problem formulation and the associated HJB system. Section 3 establishes the viscosity solution property of the value function to the HJB system. Section 4 develops a numerical procedure for pricing perpetual American put options with state-dependent regime-switching, and reports numerical results. Section 5 provides further remarks and concludes the paper.

2. Problem formulation

We follow the setup of switching diffusions as in [14, Chapter 2]. For simplicity of exposition and without loss of generality, in this work we consider an one-dimensional switching diffusion with state-dependent switching rates. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space equipped with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions of right continuity and completeness. Let $\{B_t\}_{t\geq 0}$ be an one-dimensional standard Brownian motion. Let $\{\alpha_t\}_{t\geq 0}$ be a stochastic process with right-continuous sample paths, valued in a finite set $\mathcal{M} := \{1, \ldots, m_0\}$ with $m_0 > 0$ fixed. We allow the intensity matrix (or the generator) of α_t to be state dependent. Specifically, let $Q(\cdot) = (q_{ij}(\cdot))_{m_0\times m_0}$ be the intensity matrix of α_t with the properties: $q_{ij} : \mathbb{R} \to \mathbb{R}$ is continuous and bounded, $1 \leq i, j \leq m_0$. Moveover, q_{ij}s satisfy the q-property: $q_{ij}(x) \geq 0$ if $i \neq j$, $q_{ii}(x) \leq 0$, $q_{ii}(x) = -\sum_{j\neq i} q_{ij}(x)$, $1 \leq i, j \leq m_0$, $x \in \mathbb{R}$.

Consider the two-component process $(X_t, \alpha_t), t \geq 0$, where the continuous state process X_t is governed by

$$dX_t = \mu(X_t, \alpha_t)dt + \sigma(X_t, \alpha_t)dB_t, \quad t \geq 0, \tag{2.1}$$

where $\mu : \mathbb{R} \times \mathcal{M} \to \mathbb{R}$ and $\sigma : \mathbb{R} \times \mathcal{M} \to \mathbb{R}$ are appropriate functions satisfying the Lipschitz condition

$$|\mu(x_1, i) - \mu(x_2, i)| \leq K|x_1 - x_2|, \quad |\sigma(x_1, i) - \sigma(x_2, i)| \leq K|x_1 - x_2|, \tag{2.2}$$

and the linear growth condition,

$$|\mu(x, i)| \leq K(1 + |x|), \quad |\sigma(x, i)| \leq K(1 + |x|), \tag{2.3}$$

for all $x, x_1, x_2 \in \mathbb{R}$ and for each $i \in \mathcal{M}$, where K is a positive constant (In what follows we will use K as a generic positive constant; that is, its values may vary at different places). In addition, we assume $\sigma(x, i) > 0$ for all $x \in \mathbb{R}$ and for each $i \in \mathcal{M}$.

On the other hand, the discrete component α_t evolves according to the probability law:

$$P\{\alpha_{t+\Delta t} = j | \alpha_t = i, X_s, \alpha_s, 0 \leq s \leq t\} = q_{ij}(X_t)\Delta t + o(\Delta t), \ \forall j \neq i. \tag{2.4}$$

Note that neither X_t nor α_t itself satisfies the Markov property. However the joint process $(X_t, \alpha_t), t \geq 0$ is a two-dimensional Markov process. In view of [14, Theorem 2.1], there exists a unique solution (X_t, α_t) to Equation (2.1) with given initial state $(X_0, \alpha_0) = (x, i)$ in which the evolution of the jump component α_t is governed by (2.4).

Now we present the optimal stopping problem associated with the switching process (X_t, α_t). Let $\mathcal{T}$ denote the collection of $\mathcal{F}_t$ stopping times taking values in $[0, \infty]$. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be two functions that satisfy the Lipschitz condition:

$$|f(x_1) - f(x_2)| \leq K|x_1 - x_2|, \quad |g(x_1) - g(x_2)| \leq K|x_1 - x_2|, \tag{2.5}$$

and the linear growth condition,

$$|f(x)| \leq K(1 + |x|), \quad |g(x)| \leq K(1 + |x|), \tag{2.6}$$

for all $x, x_1, x_2 \in \mathbb{R}$.

The value function of the optimal stopping problem is defined by

$$V(x, i) = \sup_{\tau \in \mathcal{T}} E\left[\int_0^\tau e^{-\beta t} f(X_t)dt + e^{-\beta\tau} g(X_\tau)\Big| X_0 = x, \alpha_0 = i\right] \tag{2.7}$$

where $\beta > 0$ is a discount factor. Note that $e^{-\beta\tau(\omega)} = 0$ if $\tau(\omega) = \infty$ for some $\omega \in \Omega$.

Remark 1: It is natural and meaningful to extend the model by allowing the discount factor β, the rewarding functions f and g to depend on the current state of economy. That is, by using regime-dependent discount factor $\beta(\alpha_t)$, and regime-dependent rewarding functions $f(X_t, \alpha_t)$ and $g(X_t, \alpha_t)$, the optimal stopping problem is then given by

$$V(x, i) = \sup_{\tau \in \mathcal{T}} E\left[\int_0^\tau e^{-\int_0^t \beta(\alpha_s)ds} f(X_t, \alpha_t)dt + e^{-\int_0^\tau \beta(\alpha_s)ds} g(X_\tau, \alpha_\tau)\Big| X_0 = x, \alpha_0 = i\right].$$

The viscosity solution method developed in this paper can be applied to this generalized model in a straightforward way and the same results can be established.

The HJB equation associated with the optimal stopping problem (2.7) is a system of m_0 coupled variational inequalities given by:

$$\min\left\{\beta v(x,i) - \mu(x,i)\frac{\mathrm{d}v}{\mathrm{d}x}(x,i) - \frac{1}{2}\sigma^2(x,i)\frac{\mathrm{d}^2v}{\mathrm{d}x^2}(x,i) - f(x) \right.$$
$$\left. - \sum_{j\neq i} q_{ij}(x)[v(x,j) - v(x,i)],\ \ v(x,i) - g(x)\right\} = 0,\ i = 1,\ldots,m_0. \tag{2.8}$$

Let $\left(X_t^{x,i}, \alpha_t^{x,i}\right)$ denote the solution of (2.1) and (2.4) with given initial state $(X_0,\alpha_0) = (x,i)$. Note that in what follows we may still use (X_t,α_t) for notational simplicity. In view of Krylov [7] and Yin and Zhu [14], we have the following estimates:

$$E\left[\sup_{0\le s\le t}|X_s^{x,i}|\right] \le e^{\beta_0 t}(1+|x|),\ t\ge 0,\ x\in\mathbb{R},\ i\in\mathcal{M}, \tag{2.9}$$

$$E\left[\sup_{t\ge 0} e^{-\beta t}|X_t^{x,i}|\right] \le 1+|x|,\ x\in\mathbb{R},\ i\in\mathcal{M},\ \forall\beta>\beta_0, \tag{2.10}$$

where $\beta_0 > 0$ is a constant. Moreover, following [14, Chapter 2], one can show that the value function $V(x,i)$ is continuous in x for each $i\in\mathcal{M}$ and satisfies a linear growth condition.

We present the DP equation for the optimal stopping problem considered in this paper, which plays a crucial role in establishing the viscosity solution property in the next section.

Let $\theta\in\mathcal{T}$ be a stopping time. Then

$$V(x,i) = \sup_{\tau\in\mathcal{T}} E\left[\int_0^{\tau\wedge\theta} e^{-\beta t}f(X_t)\mathrm{d}t + e^{-\beta\tau}g(X_\tau)I_{\{\tau<\theta\}} + e^{-\beta\theta}V(X_\theta,\alpha_\theta)I_{\{\theta\le\tau\}}\right], \tag{2.11}$$

where I_A denotes the indicator function of the event $A\in\mathcal{F}$.

3. Viscosity solution

In this section we show that the value function (2.7) is the unique viscosity solution to the HJB system (2.8). We first give the definition of the viscosity solution to the HJB system (2.8), which is a natural modification of that for second-order partial differential equations introduced in Crandall et al. [1].

Definition 1 (Viscosity Solution). Assume that $w(x,i)$ is continuous in x for each $i\in\mathcal{M}$ and satisfies a polynomial growth condition (i.e. $|w(x,i)| \le K(1+|x|^n)$ for some integer $n\ge 1$).

(1) If for each $i \in \mathcal{M}$,

$$\min\left\{\beta w(x_0,i) - \mu(x_0,i)\frac{\mathrm{d}\psi}{\mathrm{d}x}(x_0) - \frac{1}{2}\sigma^2(x_0,i)\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2}(x_0) - f(x_0)\right.$$
$$\left. - \sum_{j\neq i} q_{ij}(x_0)[w(x_0,j) - w(x_0,i)],\ \ w(x_0,i) - g(x_0)\right\} \geq 0 \tag{3.1}$$

whenever $\psi \in C^2(\mathbb{R})$ such that $w(x,i) - \psi(x)$ has a local minimum at $x_0 \in \mathbb{R}$ and $w(x_0,i) = \psi(x_0)$, then $w(x,i)$, $i \in \mathcal{M}$ is a viscosity super-solution of the HJB system (2.8).

(2) If for each $i \in \mathcal{M}$,

$$\min\left\{\beta w(x_0,i) - \mu(x_0,i)\frac{\mathrm{d}\varphi}{\mathrm{d}x}(x_0) - \frac{1}{2}\sigma^2(x_0,i)\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2}(x_0) - f(x_0)\right.$$
$$\left. - \sum_{j\neq i} q_{ij}(x_0)[w(x_0,j) - w(x_0,i)],\ \ w(x_0,i) - g(x_0)\right\} \leq 0 \tag{3.2}$$

whenever $\varphi \in C^2(\mathbb{R})$ such that $w(x,i) - \varphi(x)$ has a local maximum at $x_0 \in \mathbb{R}$ and $w(x_0,i) = \varphi(x_0)$, then $w(x,i)$, $i \in \mathcal{M}$ is a viscosity sub-solution of the HJB system (2.8).

(3) $w(x,i)$, $i \in \mathcal{M}$ is a viscosity solution of the HJB system (2.8) if it is both a viscosity super-solution and a viscosity sub-solution of the HJB system (2.8).

Theorem 1: *The value function $V(x,i)$, $i \in \mathcal{M}$ is a viscosity solution of the HJB system* (2.8).

Proof: We establish the viscosity super- and sub-solution properties, respectively in the following two steps.

Step 1. $V(x,i)$, $i \in \mathcal{M}$ is a viscosity super-solution of (2.8).

Considering $\tau = 0$ in (2.7), we obtain immediately, $V(x,i) \geq g(x)$.

Next, for a given $i \in \mathcal{M}$, let $\psi \in C^2(\mathbb{R})$ be a testing function, that is, $V(x,i) - \psi(x)$ has a local minimum at a point $x_0 \in \mathbb{R}$ in a bounded neighborhood $N(x_0)$ and $V(x_0,i) = \psi(x_0)$. Define

$$\Psi(x,j) = \begin{cases} \psi(x), & \text{if } j = i, \\ V(x,j), & \text{if } j \neq i, \end{cases} \quad j \in \mathcal{M}. \tag{3.3}$$

Let τ_α be the first jump time of $\alpha_t \left(= \alpha_t^{x_0,i}\right)$, i.e. $\tau_\alpha = \min\{t \geq 0 : \alpha_t \neq i\}$. Then $\tau_\alpha > 0$, *a.s.* Let $\theta_0 \in (0, \tau_\alpha)$ be such that the state $X_t (= X_t^{x_0,i})$ starts at x_0 and stays in $N(x_0)$ for $0 \leq t \leq \theta_0$. Applying the generalized Itô's formula to the switching process $e^{-\beta t}\Psi(X_t, \alpha_t)$ and then taking integral from $t = 0$ to $t = \theta_0 \wedge h$, where $h > 0$ is a positive constant, we have

$$e^{-\beta(\theta_0\wedge h)}\Psi(X_{\theta_0\wedge h}, \alpha_{\theta_0\wedge h})$$
$$= \Psi(x_0,i) + \int_0^{\theta_0\wedge h} e^{-\beta t}\Big[- \beta\Psi(X_t,\alpha_t) + \mu(X_t,\alpha_t)\frac{\mathrm{d}\Psi}{\mathrm{d}x}(X_t,\alpha_t)$$

$$+\frac{1}{2}\sigma^2(X_t,\alpha_t)\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2}(X_t,\alpha_t)+\sum_{j\neq\alpha_t}q_{\alpha_t j}(X_t)[\Psi(X_t,j)-\Psi(X_t,\alpha_t)]\Big]\mathrm{d}t$$
$$+\int_0^{\theta_0\wedge h}e^{-\beta t}\sigma(X_t,\alpha_t)\frac{\mathrm{d}\Psi}{\mathrm{d}x}(X_t,\alpha_t)\mathrm{d}B_t+\int_0^{\theta_0\wedge h}\mathrm{d}M_\Psi(t), \tag{3.4}$$

where M_Ψ is a martingale provided that Ψ is bounded. Note that for $0\le t\le\theta_0$, $X_t\in N(x_0)$, hence $\Psi(X_t,\alpha_t)$ and $\frac{\mathrm{d}\Psi}{\mathrm{d}x}(X_t,\alpha_t)$ are bounded. It follows that

$$E\left[\int_0^{\theta_0\wedge h}e^{-\beta t}\sigma(X_t,\alpha_t)\frac{\mathrm{d}\Psi}{\mathrm{d}x}(X_t,\alpha_t)\mathrm{d}B_t+\int_0^{\theta_0\wedge h}\mathrm{d}M_\Psi(t)\right]=0. \tag{3.5}$$

Also noting that $\alpha_t=i$ for $0\le t\le\theta_0$. Taking expectation in (3.4) and using (3.3) and (3.5) yield

$$\begin{aligned}E\left[e^{-\beta(\theta_0\wedge h)}\psi(X_{\theta_0\wedge h})\right]&=\psi(x_0)+E\Big[\int_0^{\theta_0\wedge h}e^{-\beta t}\Big[-\beta\psi(X_t)+\mu(X_t,i)\frac{\mathrm{d}\psi}{\mathrm{d}x}(X_t)\\&\quad+\frac{1}{2}\sigma^2(X_t,i)\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2}(X_t)+\sum_{j\neq i}q_{ij}(X_t)[V(X_t,j)-\psi(X_t)]\Big]\mathrm{d}t\Big]\\&\ge\psi(x_0)+E\Big[\int_0^{\theta_0\wedge h}e^{-\beta t}\Big[-\beta V(X_t,i)+\mu(X_t,i)\frac{\mathrm{d}\psi}{\mathrm{d}x}(X_t)\\&\quad+\frac{1}{2}\sigma^2(X_t,i)\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2}(X_t)+\sum_{j\neq i}q_{ij}(X_t)[V(X_t,j)-V(X_t,i)]\Big]\mathrm{d}t\Big]\end{aligned} \tag{3.6}$$

where the last inequality is due to the fact that $V(X_t,i)\ge\psi(X_t)$ for $X_t\in N(x_0)$.

Setting $x=x_0$ and $\tau=\theta=\theta_0\wedge h$ in (2.11), we have

$$\begin{aligned}\psi(x_0)=V(x_0,i)&\ge E\left[\int_0^{\theta_0\wedge h}e^{-\beta t}f(X_t)\mathrm{d}t+e^{-\beta(\theta_0\wedge h)}V(X_{\theta_0\wedge h},i)\right]\\&\ge E\left[\int_0^{\theta_0\wedge h}e^{-\beta t}f(X_t)\mathrm{d}t+e^{-\beta(\theta_0\wedge h)}\psi(X_{\theta_0\wedge h})\right].\end{aligned} \tag{3.7}$$

Combining (3.6) and (3.7) and dividing by h, we obtain

$$\begin{aligned}&E\Big[\frac{1}{h}\int_0^{\theta_0\wedge h}e^{-\beta t}\Big[\beta V(X_t,i)-\mu(X_t,i)\frac{\mathrm{d}\psi}{\mathrm{d}x}(X_t)-\frac{1}{2}\sigma^2(X_t,i)\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2}(X_t)\\&\qquad-\sum_{j\neq i}q_{ij}(X_t)[V(X_t,j)-V(X_t,i)]-f(X_t)\Big]\mathrm{d}t\Big]\ge 0.\end{aligned} \tag{3.8}$$

Sending $h \downarrow 0$ and using the dominated convergence theorem, we obtain

$$\begin{aligned}\beta V(x_0,i) - \mu(x_0,i)\frac{\mathrm{d}\psi}{\mathrm{d}x}(x_0) - \frac{1}{2}\sigma^2(x_0,i)\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2}(x_0)\\ - \sum_{j\neq i} q_{ij}(x_0)[V(x_0,j) - V(x_0,i)] - f(x_0) \geq 0.\end{aligned} \tag{3.9}$$

This shows that the value function $V(x,i)$, $i \in \mathcal{M}$ satisfies the viscosity super-solution property (3.1).

Step 2. $V(x,i)$, $i \in \mathcal{M}$ is a viscosity sub-solution of (2.8).

We argue by contradiction. Assume that there exist an $i_0 \in \mathcal{M}$, a point $x_0 \in \mathbb{R}$ and a testing function $\varphi_0 \in C^2(\mathbb{R})$ such that $V(x,i_0) - \varphi_0(x)$ has a local maximum at x_0 in a bounded neighborhood $N(x_0)$, $V(x_0,i_0) = \varphi_0(x_0)$, and

$$\begin{aligned}\min\Bigg\{&\beta V(x_0,i_0) - \mu(x_0,i_0)\frac{\mathrm{d}\varphi_0}{\mathrm{d}x}(x_0) - \frac{1}{2}\sigma^2(x_0,i_0)\frac{\mathrm{d}^2\varphi_0}{\mathrm{d}x^2}(x_0) - f(x_0)\\ &- \sum_{j\neq i_0} q_{i_0j}(x_0)[V(x_0,j) - V(x_0,i_0)],\quad V(x_0,i_0) - g(x_0)\Bigg\} > 0,\end{aligned}$$

i.e.

$$\begin{aligned}\beta V(x_0,i_0) - \mu(x_0,i_0)\frac{\mathrm{d}\varphi_0}{\mathrm{d}x}(x_0) - \frac{1}{2}\sigma^2(x_0,i_0)\frac{\mathrm{d}^2\varphi_0}{\mathrm{d}x^2}(x_0)\\ - \sum_{j\neq i_0} q_{i_0j}(x_0)[V(x_0,j) - V(x_0,i_0)] - f(x_0) > 0\end{aligned} \tag{3.10}$$

and

$$V(x_0,i_0) - g(x_0) > 0. \tag{3.11}$$

By the continuity of all functions involved in (3.10) and (3.11) (V, φ_0, φ_0', φ_0'', μ, σ, q_{ij}, f, g), there exist a $\delta > 0$ and an open interval $B(x_0,\delta) = (x_0 - \delta, x_0 + \delta) \subset N(x_0)$ such that

$$\begin{aligned}\beta V(x,i_0) - \mu(x,i_0)\frac{\mathrm{d}\varphi_0}{\mathrm{d}x}(x) - \frac{1}{2}\sigma^2(x,i_0)\frac{\mathrm{d}^2\varphi_0}{\mathrm{d}x^2}(x)\\ - \sum_{j\neq i_0} q_{i_0j}(x)[V(x,j) - V(x,i_0)] - f(x) > \delta,\quad x \in B(x_0,\delta),\end{aligned} \tag{3.12}$$

and

$$V(x,i_0) - g(x) > \delta,\quad x \in B(x_0,\delta). \tag{3.13}$$

Let $\theta_\delta = \min\{t \geq 0 : X_t \notin B(x_0,\delta)\}$ be the first exit time of $X_t \left(= X_t^{x_0,i_0}\right)$ from $B(x_0,\delta)$. Let $\theta = \theta_\delta \wedge \tau_\alpha$ where τ_α is the first jump time of $\alpha_t \left(= \alpha_t^{x_0,i_0}\right)$. Then $\theta > 0$, *a.s.*. For $0 \leq t \leq \theta$, we have

$$\beta V(X_t,i_0) - \mu(X_t,i_0)\frac{\mathrm{d}\varphi_0}{\mathrm{d}x}(X_t) - \frac{1}{2}\sigma^2(X_t,i_0)\frac{\mathrm{d}^2\varphi_0}{\mathrm{d}x^2}(X_t)$$
$$- \sum_{j\neq i_0} q_{i_0 j}(X_t)[V(X_t,j) - V(X_t,i_0)] - f(X_t) > \delta, \tag{3.14}$$

and

$$V(X_t,i_0) - g(X_t) > \delta. \tag{3.15}$$

Define

$$\Phi(x,j) = \begin{cases} \varphi_0(x), & \text{if } j = i_0, \\ V(x,j), & \text{if } j \neq i_0, \end{cases} \quad j \in \mathcal{M}. \tag{3.16}$$

For any stopping time $\tau \in \mathcal{T}$, applying the Itô's formula to $e^{-\beta t}\Phi(X_t,\alpha_t)$, taking integral from $t = 0$ to $t = (\theta \wedge \tau)-$, and then taking expectation yield

$$E\left[e^{-\beta(\theta\wedge\tau)}\varphi_0(X_{\theta\wedge\tau})\right] - V(x_0,i_0) = E\Bigg[\int_0^{(\theta\wedge\tau)-} e^{-\beta t}\Big[-\beta\varphi_0(X_t) + \mu(X_t,i_0)\frac{\mathrm{d}\varphi_0}{\mathrm{d}x}(X_t)$$
$$+ \frac{1}{2}\sigma^2(X_t,i_0)\frac{\mathrm{d}^2\varphi_0}{\mathrm{d}x^2}(X_t) + \sum_{j\neq i_0} q_{i_0 j}(X_t)[V(X_t,j) - \varphi_0(X_t)]\Big]\mathrm{d}t\Bigg], \tag{3.17}$$

in which we used $E\left[e^{-\beta(\theta\wedge\tau)}\varphi_0(X_{(\theta\wedge\tau)-})\right] = E\left[e^{-\beta(\theta\wedge\tau)}\varphi_0(X_{\theta\wedge\tau})\right]$ due to continuity. Noting that the integrand in the RHS of (3.17) is continuous in t. Using (3.14), (3.15) and that $V(X_t,i_0) \le \varphi_0(X_t)$ in (3.17), it follows

$$V(x_0,i_0) \ge E\Bigg[\int_0^{\theta\wedge\tau} e^{-\beta t}\Big[\beta V(X_t,i_0) - \mu(X_t,i_0)\frac{\mathrm{d}\varphi_0}{\mathrm{d}x}(X_t) - \frac{1}{2}\sigma^2(X_t,i_0)\frac{\mathrm{d}^2\varphi_0}{\mathrm{d}x^2}(X_t)$$
$$- \sum_{j\neq i_0} q_{i_0 j}(X_t)[V(X_t,j) - V(X_t,i_0)]\Big]\mathrm{d}t + e^{-\beta(\theta\wedge\tau)}V(X_{\theta\wedge\tau},i_0)\Bigg]$$
$$\ge E\Bigg[\int_0^{\theta\wedge\tau} e^{-\beta t}[f(X_t)+\delta]\mathrm{d}t + e^{-\beta\tau}V(X_\tau,i_0)I_{\{\tau<\theta\}} + e^{-\beta\theta}V(X_\theta,i_0)I_{\{\theta\le\tau\}}\Bigg]$$
$$\ge E\Bigg[\int_0^{\theta\wedge\tau} e^{-\beta t}[f(X_t)+\delta]\mathrm{d}t + e^{-\beta\tau}[g(X_\tau)+\delta]I_{\{\tau<\theta\}} + e^{-\beta\theta}V(X_\theta,i_0)I_{\{\theta\le\tau\}}\Bigg]$$
$$= E\Bigg[\int_0^{\theta\wedge\tau} e^{-\beta t}f(X_t)\mathrm{d}t + e^{-\beta\tau}g(X_\tau)I_{\{\tau<\theta\}} + e^{-\beta\theta}V(X_\theta,i_0)I_{\{\theta\le\tau\}}\Bigg]$$
$$+ \delta E\left[\int_0^{\theta\wedge\tau} e^{-\beta t}\mathrm{d}t + e^{-\beta\tau}I_{\{\tau<\theta\}}\right]. \tag{3.18}$$

Now we estimate the term $E\left[\int_0^{\theta\wedge\tau} e^{-\beta t}\mathrm{d}t + e^{-\beta\tau}I_{\{\tau<\theta\}}\right]$. Choose a positive constant C_0 such that

$$0 < C_0 \le \min\left\{\left(\beta + \frac{2}{\delta}\sup_{x\in B(x_0,\delta)}|\mu(x,i_0)| + \frac{1}{\delta^2}\sup_{x\in B(x_0,\delta)}\sigma^2(x,i_0)\right)^{-1}, 1\right\}. \tag{3.19}$$

Consider the function $\phi(x) = C_0\left(1 - \frac{(x-x_0)^2}{\delta^2}\right)$. Then, $\phi(x) \le C_0 \le 1$, $\phi(x_0) = C_0 > 0$, $\phi(x_0 \pm \delta) = 0$, and for $x \in B(x_0, \delta)$, we have

$$\begin{aligned}
&\beta\phi(x) - \mu(x, i_0)\frac{\mathrm{d}\phi}{\mathrm{d}x}(x) - \frac{1}{2}\sigma^2(x, i_0)\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}(x) \\
&\quad = \beta C_0\left(1 - \frac{(x-x_0)^2}{\delta^2}\right) + \mu(x, i_0)\frac{2C_0}{\delta^2}(x - x_0) + \sigma^2(x, i_0)\frac{C_0}{\delta^2} \\
&\quad \le \beta C_0 + \sup_{x\in B(x_0,\delta)} |\mu(x, i_0)|\frac{2C_0}{\delta^2}\delta + \sup_{x\in B(x_0,\delta)} \sigma^2(x, i_0)\frac{C_0}{\delta^2} \le 1. \qquad (3.20)
\end{aligned}$$

Applying the Itô's formula to $e^{-\beta t}\phi(X_t)$ between $t = 0$ to $t = \theta \wedge \tau$, we have

$$\begin{aligned}
C_0 = \phi(x_0) &= E\left[\int_0^{\theta\wedge\tau} e^{-\beta t}\left[\beta\phi(X_t) - \mu(X_t, i_0)\frac{\mathrm{d}\phi}{\mathrm{d}x}(X_t) - \frac{1}{2}\sigma^2(X_t, i_0)\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}(X_t)\right]\mathrm{d}t\right] \\
&\quad + E\left[e^{-\beta(\theta\wedge\tau)}\phi(X_{\theta\wedge\tau})\right] \\
&\le E\left[\int_0^{\theta\wedge\tau} e^{-\beta t}\mathrm{d}t + e^{-\beta\tau}I_{\{\tau<\theta\}}\right] + E\left[e^{-\beta\theta}\phi(X_\theta)I_{\{\theta\le\tau\}}\right]. \qquad (3.21)
\end{aligned}$$

Recall that $\theta = \theta_\delta \wedge \tau_\alpha$ where τ_α is the first jump time of α_t and θ_δ is the first exit time of X_t from $B(x_0, \delta)$. Then we have $\phi(X_{\theta_\delta}) = 0$ and $\phi(X_{\tau_\alpha}) \le C_0$. It follows that

$$\begin{aligned}
E\left[e^{-\beta\theta}\phi(X_\theta)I_{\{\theta\le\tau\}}\right] &= E\left[e^{-\beta\theta_\delta}\phi(X_{\theta_\delta})I_{\{\theta_\delta\le\tau_\alpha\le\tau\}}\right] + E\left[e^{-\beta\tau_\alpha}\phi(X_{\tau_\alpha})I_{\{\tau_\alpha<\theta_\delta\le\tau\}}\right] \\
&\le C_0 E\left[e^{-\beta\tau_\alpha}\right]. \qquad (3.22)
\end{aligned}$$

Using (3.22) in (3.21), we obtain

$$E\left[\int_0^{\theta\wedge\tau} e^{-\beta t}\mathrm{d}t + e^{-\beta\tau}I_{\{\tau<\theta\}}\right] \ge C_0\left(1 - E\left[e^{-\beta\tau_\alpha}\right]\right). \qquad (3.23)$$

Using (3.23) in (3.18), we have

$$\begin{aligned}
V(x_0, i_0) &\ge E\left[\int_0^{\theta\wedge\tau} e^{-\beta t}f(X_t)\mathrm{d}t + e^{-\beta\tau}g(X_\tau)I_{\{\tau<\theta\}} + e^{-\beta\theta}V(X_\theta, i_0)I_{\{\theta\le\tau\}}\right] \\
&\quad + \delta C_0\left(1 - E\left[e^{-\beta\tau_\alpha}\right]\right). \qquad (3.24)
\end{aligned}$$

It follows that

$$\begin{aligned}
V(x_0, i_0) &\ge \sup_{\tau\in\mathcal{T}} E\left[\int_0^{\theta\wedge\tau} e^{-\beta t}f(X_t)\mathrm{d}t + e^{-\beta\tau}g(X_\tau)I_{\{\tau<\theta\}} + e^{-\beta\theta}V(X_\theta, \alpha_\theta)I_{\{\theta\le\tau\}}\right] \\
&\quad + \delta C_0\left(1 - E\left[e^{-\beta\tau_\alpha}\right]\right), \qquad (3.25)
\end{aligned}$$

which is a contradiction to the DP principle (2.11) since $E\left[e^{-\beta\tau_\alpha}\right] < 1$. Therefore the value function $V(x, i)$, $i \in \mathcal{M}$ is a viscosity sub-solution of the system (2.8).

This completes the proof of Theorem 1. □

Next, we show the uniqueness of the viscosity solution. To this end, the following equivalent formulation of viscosity solution of the HJB system (2.8) is needed. See [1].

Consider function $h : \mathbb{R} \to \mathbb{R}$. Given $x \in \mathbb{R}$. Define the (second order) superjet of $h(x)$ by

$$\mathcal{P}^{2,+}h(x) = \Big\{(q,M) \in \mathbb{R}\times\mathbb{R} : h(y) \le h(x) + q(y-x) + \frac{1}{2}M(y-x)^2 + o((y-x)^2) \text{ as } y \to x\Big\} \tag{3.26}$$

and its closure by

$$\overline{\mathcal{P}}^{2,+}h(x) = \Big\{(q,M) = \lim_{n\to\infty}(q_n,M_n) \text{ with } (q_n,M_n) \in \mathcal{P}^{2,+}h(x_n) \text{ and } \lim_{n\to\infty}(x_n,h(x_n)) = (x,h(x))\Big\}. \tag{3.27}$$

Similarly, define the (second order) subjet of $h(x)$ by

$$\mathcal{P}^{2,-}h(x) = \Big\{(q,M) \in \mathbb{R}\times\mathbb{R} : h(y) \ge h(x) + q(y-x) + \frac{1}{2}M(y-x)^2 + o((y-x)^2) \text{ as } y \to x\Big\} \tag{3.28}$$

and its closure by

$$\overline{\mathcal{P}}^{2,-}h(x) = \Big\{(q,M) = \lim_{n\to\infty}(q_n,M_n) \text{ with } (q_n,M_n) \in \mathcal{P}^{2,-}h(x_n) \text{ and } \lim_{n\to\infty}(x_n,h(x_n)) = (x,h(x))\Big\}. \tag{3.29}$$

The following relations hold:

$$\mathcal{P}^{2,-}h(x) = -\mathcal{P}^{2,+}(-h)(x), \quad \overline{\mathcal{P}}^{2,-}h(x) = -\overline{\mathcal{P}}^{2,+}(-h)(x). \tag{3.30}$$

Note that for function $w : \mathbb{R}\times\mathcal{M} \to \mathbb{R}$, we can define $\mathcal{P}^{2,+}w(x,i), \mathcal{P}^{2,-}w(x,i), \overline{\mathcal{P}}^{2,+}w(x,i)$ and $\overline{\mathcal{P}}^{2,-}w(x,i)$ for each $i \in \mathcal{M}$ by identifying $h(x) = w(x,i)$ in the above definitions.

The following results have been established. See [1].

Lemma 1: *$\mathcal{P}^{2,+}h(x)$ (resp. $\mathcal{P}^{2,-}h(x)$) consists of the set $\big(\phi'(x), \phi''(x)\big)$ where $\phi(x) \in C^2(\mathbb{R})$ and $h(x) - \phi(x)$ has a global maximum (resp. minimum) at x and $h(x) = \phi(x)$.*

Lemma 2 (Equivalent Formulation of Viscosity Solution): *Assume that $w(x,i)$ is continuous in x for each $i \in \mathcal{M}$ and satisfies a polynomial growth condition.*

(1) *If for each $i \in \mathcal{M}$, each $x_0 \in \mathbb{R}$, and each $(q,M) \in \mathcal{P}^{2,-}w(x_0,i)$,*

$$\min\Big\{\beta w(x_0,i) - \mu(x_0,i)q - \frac{1}{2}\sigma^2(x_0,i)M - f(x_0) - \sum_{j\neq i} q_{ij}(x_0)[w(x_0,j) - w(x_0,i)], \quad w(x_0,i) - g(x_0)\Big\} \ge 0, \tag{3.31}$$

then $w(x,i)$, $i \in \mathcal{M}$ is a viscosity super-solution of the HJB system (2.8).

(2) *If for each $i \in \mathcal{M}$, each $x_0 \in \mathbb{R}$, and each $(q, M) \in \mathcal{P}^{2,+}w(x_0, i)$,*

$$\min\Bigg\{\beta w(x_0, i) - \mu(x_0, i)q - \frac{1}{2}\sigma^2(x_0, i)M - f(x_0) \\ - \sum_{j\neq i} q_{ij}(x_0)[w(x_0, j) - w(x_0, i)],\quad w(x_0, i) - g(x_0)\Bigg\} \leq 0, \tag{3.32}$$

then $w(x, i)$, $i \in \mathcal{M}$ is a viscosity sub-solution of the HJB system (2.8).

Theorem 2 (Comparison Principle): *If $w_1(x, i)$ (resp. $w_2(x, i)$), $i \in \mathcal{M}$ is a viscosity sub-solution (resp. super-solution) of the HJB system (2.8), $w_1(x, i)$ and $w_2(x, i)$ are continuous in x for each $i \in \mathcal{M}$ and have at most a linear growth, then*

$$w_1(x, i) \leq w_2(x, i),\quad \forall x \in \mathbb{R},\ i \in \mathcal{M}. \tag{3.33}$$

The well-known Ishii's Lemma is needed in establishing the comparison principle. We state this lemma in the following Theorem 3 in a suitable way for our application. See [1].

Theorem 3: *Let $\phi_1(x)$, $\phi_2(x)$ be upper (resp. lower) semi-continuous functions on $\mathbb{R}$, $\Phi(x, y) \in C^2(\mathbb{R} \times \mathbb{R})$, $(\bar{x}, \bar{y})$ be a local maximum of $\phi_1(x) - \phi_2(y) - \Phi(x, y)$. Then for each $\varepsilon > 0$, there exist X_ε, $Y_\varepsilon \in \mathbb{R}$ such that*

$$\left(\frac{\partial\Phi}{\partial x}(\bar{x}, \bar{y}), X_\varepsilon\right) \in \overline{\mathcal{P}}^{2,+}\phi_1(\bar{x}),\quad \left(-\frac{\partial\Phi}{\partial y}(\bar{x}, \bar{y}), Y_\varepsilon\right) \in \overline{\mathcal{P}}^{2,-}\phi_2(\bar{x}), \tag{3.34}$$

and

$$\begin{pmatrix} X_\varepsilon & 0 \\ 0 & -Y_\varepsilon \end{pmatrix} \leq D^2\Phi(\bar{x}, \bar{y}) + \varepsilon[D^2\Phi(\bar{x}, \bar{y})]^2, \tag{3.35}$$

where $D^2\Phi$ denotes the Heissian of Φ.

Proof of Theorem 2: For each $0 < \lambda < 1$, we introduce

$$w_2^\lambda(x, i) = w_2(x, i) + \lambda(1 + x^2),\ x \in \mathbb{R},\ i \in \mathcal{M}. \tag{3.36}$$

We first show that $w_2^\lambda(x, i)$, $i \in \mathcal{M}$ is a viscosity super-solution of the HJB system (2.8). To this end, for given $i \in \mathcal{M}$, let $\psi \in C^2(\mathbb{R})$ be a testing function such that $w_2^\lambda(x, i) - \psi(x)$ has a local minimum at some point $x_0 \in \mathbb{R}$ and $w_2^\lambda(x_0, i) = \psi(x_0)$, that is, $w_2(x, i) - [\psi(x) - \lambda(1 + x^2)]$ has a local minimum at $x_0 \in \mathbb{R}$. Hence $\psi(x) - \lambda(1 + x^2)$ is a testing function for the viscosity super-solution $w_2(x, i)$, $i \in \mathcal{M}$. It follows from (3.1) that

$$\beta\left[\psi(x_0) - \lambda\left(1 + x_0^2\right)\right] - \mu(x_0, i)\left[\psi'(x_0) - 2\lambda x_0\right] - \frac{1}{2}\sigma^2(x_0, i)\left[\psi''(x_0) - 2\lambda\right] \\ - \sum_{j\neq i} q_{ij}(x_0)[w_2(x_0, j) - w_2(x_0, i)] - f(x_0) \geq 0 \tag{3.37}$$

and

$$w_2(x_0, i) - g(x_0) \geq 0. \tag{3.38}$$

Note that (3.37) implies that

$$\begin{aligned}\beta\psi(x_0) - \mu(x_0,i)\psi'(x_0) - \frac{1}{2}\sigma^2(x_0,i)\psi''(x_0) - \sum_{j\neq i} q_{ij}(x_0)\left[w_2^\lambda(x_0,j) - w_2^\lambda(x_0,i)\right] - f(x_0) \\ \geq \beta\lambda\left(1+x_0^2\right) - 2\lambda x_0\mu(x_0,i) - \lambda\sigma^2(x_0,i). \qquad (3.39)\end{aligned}$$

In view of the linear growth condition (2.3), we have $|x_0\mu(x_0,i)| \leq 2K(1+x_0^2)$, $\sigma^2(x_0,i) \leq 2K(1+x_0^2)$ where K is the constant specified in (2.3). It then follows that

$$\begin{aligned}\beta\psi(x_0) - \mu(x_0,i)\psi'(x_0) - \frac{1}{2}\sigma^2(x_0,i)\psi''(x_0) - \sum_{j\neq i} q_{ij}(x_0)\left[w_2^\lambda(x_0,j) - w_2^\lambda(x_0,i)\right] - f(x_0) \\ \geq \lambda\left(1+x_0^2\right)(\beta - 6K) \geq 0\end{aligned}$$

provided that $\beta \geq 6K$. On the other hand, (3.38) implies that $w_2^\lambda(x_0,i) - g(x_0) \geq 0$. This establishes the super-solution property of $w_2^\lambda(x,i)$, $i \in \mathcal{M}$.

Next, we show that for all $0 < \lambda < 1$,

$$\max_{i\in\mathcal{M}} \sup_{x\in\mathbb{R}} \left[w_1(x,i) - w_2^\lambda(x,i)\right] \leq 0. \qquad (3.40)$$

Given $\lambda \in (0,1)$. Let $i_0 \in \mathcal{M}$ be such that

$$W := \sup_{x\in\mathbb{R}}\left[w_1(x,i_0) - w_2^\lambda(x,i_0)\right] = \max_{i\in\mathcal{M}} \sup_{x\in\mathbb{R}}\left[w_1(x,i) - w_2^\lambda(x,i)\right].$$

Note that $w_1(x,i) - w_2^\lambda(x,i) = w_1(x,i) - w_2(x,i) - \lambda(1+x^2) \to -\infty$ as $|x| \to \infty$ since $w_1(x,i)$ and $w_2(x,i)$ have at most a linear growth. Thus, the continuity of $w_1(x,i) - w_2^\lambda(x,i)$ implies that W is attained on some open bounded interval (a,b), $-\infty < a < b < \infty$, i.e.

$$W = \sup_{x\in(a,b)}\left[w_1(x,i_0) - w_2^\lambda(x,i_0)\right]. \qquad (3.41)$$

We introduce, for each $0 < \varepsilon < 1$, $\phi_\varepsilon(x,y) = \frac{1}{\varepsilon}(x-y)^2$, $(x,y) \in \mathbb{R}^2$ and

$$\Phi_\varepsilon(x,y,i_0) = w_1(x,i_0) - w_2^\lambda(y,i_0) - \phi_\varepsilon(x,y), \ (x,y) \in \mathbb{R}^2. \qquad (3.42)$$

The continuous function $\Phi_\varepsilon(x,y,i_0)$ takes its maximum W_ε on the closed rectangle $[a,b]\times[a,b]$ at some point $(x_\varepsilon,y_\varepsilon) \in [a,b]\times[a,b]$, i.e.

$$W_\varepsilon = \Phi_\varepsilon(x_\varepsilon,y_\varepsilon,i_0) = \max_{(x,y)\in[a,b]^2} \Phi_\varepsilon(x,y,i_0).$$

Note that the set $(x_\varepsilon,y_\varepsilon)_\varepsilon$ are bounded, so we can extract convergent sequences, still denoted by $(x_\varepsilon,y_\varepsilon)_\varepsilon$ so that $\lim_{\varepsilon\to 0}(x_\varepsilon,y_\varepsilon) = (\hat{x},\hat{y}) \in [a,b]^2$. We now show that as $\varepsilon \to 0$, $W_\varepsilon \to W$ and $\phi_\varepsilon(x_\varepsilon,y_\varepsilon) = \frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon)^2 \to 0$. Note that

$$\begin{aligned}W \leq W_\varepsilon &= \Phi_\varepsilon(x_\varepsilon,y_\varepsilon,i_0) \\ &= w_1(x_\varepsilon,i_0) - w_2^\lambda(y_\varepsilon,i_0) - \phi_\varepsilon(x_\varepsilon,y_\varepsilon) \leq w_1(x_\varepsilon,i_0) - w_2^\lambda(y_\varepsilon,i_0). \qquad (3.43)\end{aligned}$$

Since the sequence $\{w_1(x_\varepsilon, i_0) - w_2^\lambda(y_\varepsilon, i_0)\}_\varepsilon$ is bounded, it follows that the sequence $\{\phi_\varepsilon(x_\varepsilon, y_\varepsilon)\}_\varepsilon$ is bounded too. Hence we must have $\hat{x} = \hat{y}$. Sending $\varepsilon \to 0$ in (3.43), we have

$$W \leq \lim_{\varepsilon\to 0} W_\varepsilon = w_1\left(\hat{x}, i_0\right) - w_2^\lambda\left(\hat{x}, i_0\right) - \lim_{\varepsilon\to 0} \phi_\varepsilon(x_\varepsilon, y_\varepsilon) \leq w_1\left(\hat{x}, i_0\right) - w_2^\lambda\left(\hat{x}, i_0\right) \leq W.$$

Hence, $W = w_1(\hat{x}, i_0) - w_2^\lambda(\hat{x}, i_0)$, $\lim_{\varepsilon\to 0} W_\varepsilon = W$, and $\lim_{\varepsilon\to 0} \phi_\varepsilon(x_\varepsilon, y_\varepsilon) = 0$. Since $(x_\varepsilon, y_\varepsilon)$ is a local maximum of $\Phi_\varepsilon(x, y, i_0)$, using the Ishii's Lemma (Theorem 3), there exist X_ε, $Y_\varepsilon \in \mathbb{R}$ such that

$$\begin{aligned} \left(\frac{\partial\phi_\varepsilon}{\partial x}(x_\varepsilon, y_\varepsilon), X_\varepsilon\right) &= \left(\frac{2(x_\varepsilon - y_\varepsilon)}{\varepsilon}, X_\varepsilon\right) \in \overline{\mathcal{P}}^{2,+} w_1(x_\varepsilon, i_0), \\ \left(-\frac{\partial\phi_\varepsilon}{\partial y}(x_\varepsilon, y_\varepsilon), Y_\varepsilon\right) &= \left(\frac{2(x_\varepsilon - y_\varepsilon)}{\varepsilon}, Y_\varepsilon\right) \in \overline{\mathcal{P}}^{2,-} w_2^\lambda(y_\varepsilon, i_0), \end{aligned} \tag{3.44}$$

and

$$\begin{aligned} \begin{pmatrix} X_\varepsilon & 0 \\ 0 & -Y_\varepsilon \end{pmatrix} &\leq D^2\phi(x_\varepsilon, y_\varepsilon) + \varepsilon\left[D^2\phi(x_\varepsilon, y_\varepsilon)\right]^2 \\ &\leq \frac{2}{\varepsilon}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \varepsilon\left[\frac{2}{\varepsilon}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\right]^2 = \frac{10}{\varepsilon}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \end{aligned} \tag{3.45}$$

It follows that

$$\begin{aligned} \sigma^2(x_\varepsilon, i_0)X_\varepsilon - \sigma^2(y_\varepsilon, i_0)Y_\varepsilon &= (\sigma(x_\varepsilon, i_0)\ \sigma(y_\varepsilon, i_0))\begin{pmatrix} X_\varepsilon & 0 \\ 0 & -Y_\varepsilon \end{pmatrix}\begin{pmatrix} \sigma(x_\varepsilon, i_0) \\ \sigma(y_\varepsilon, i_0) \end{pmatrix} \\ &\leq (\sigma(x_\varepsilon, i_0)\ \sigma(y_\varepsilon, i_0))\frac{10}{\varepsilon}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \sigma(x_\varepsilon, i_0) \\ \sigma(y_\varepsilon, i_0) \end{pmatrix} \\ &= \frac{10}{\varepsilon}\left(\sigma(x_\varepsilon, i_0) - \sigma(y_\varepsilon, i_0)\right)^2. \end{aligned} \tag{3.46}$$

Using the equivalent formulation of viscosity solution (Lemma 2), we have

$$\begin{aligned} \min\Bigg\{&\beta w_1(x_\varepsilon, i_0) - \mu(x_\varepsilon, i_0)\frac{2(x_\varepsilon - y_\varepsilon)}{\varepsilon} - \frac{1}{2}\sigma^2(x_\varepsilon, i_0)X_\varepsilon - f(x_\varepsilon) \\ &- \sum_{j\neq i_0} q_{i_0 j}(x_\varepsilon)[w_1(x_\varepsilon, j) - w_1(x_\varepsilon, i_0)],\ \ w_1(x_\varepsilon, i_0) - g(x_\varepsilon)\Bigg\} \leq 0, \end{aligned} \tag{3.47}$$

and

$$\begin{aligned} \min\Bigg\{&\beta w_2^\lambda(y_\varepsilon, i_0) - \mu(y_\varepsilon, i_0)\frac{2(x_\varepsilon - y_\varepsilon)}{\varepsilon} - \frac{1}{2}\sigma^2(y_\varepsilon, i_0)Y_\varepsilon - f(y_\varepsilon) \\ &- \sum_{j\neq i_0} q_{i_0 j}(y_\varepsilon)\left[w_2^\lambda(y_\varepsilon, j) - w_2^\lambda(y_\varepsilon, i_0)\right],\ \ w_2^\lambda(y_\varepsilon, i_0) - g(y_\varepsilon)\Bigg\} \geq 0. \end{aligned} \tag{3.48}$$

We consider two cases.

Case 1: There exists an $0 < \varepsilon_0 < 1$ such that $w_1(x_\varepsilon, i_0) - g(x_\varepsilon) > 0$ for all $\varepsilon < \varepsilon_0$. In this case, (3.47) implies that

$$\beta w_1(x_\varepsilon, i_0) - \mu(x_\varepsilon, i_0)\frac{2(x_\varepsilon - y_\varepsilon)}{\varepsilon} - \frac{1}{2}\sigma^2(x_\varepsilon, i_0)X_\varepsilon - f(x_\varepsilon) \\ - \sum_{j\neq i_0} q_{i_0 j}(x_\varepsilon)[w_1(x_\varepsilon, j) - w_1(x_\varepsilon, i_0)] \leq 0, \quad \forall \varepsilon < \varepsilon_0. \tag{3.49}$$

On the other hand, (3.48) implies that

$$\beta w_2^\lambda(y_\varepsilon, i_0) - \mu(y_\varepsilon, i_0)\frac{2(x_\varepsilon - y_\varepsilon)}{\varepsilon} - \frac{1}{2}\sigma^2(y_\varepsilon, i_0)Y_\varepsilon - f(y_\varepsilon) \\ - \sum_{j\neq i_0} q_{i_0 j}(y_\varepsilon)\left[w_2^\lambda(y_\varepsilon, j) - w_2^\lambda(y_\varepsilon, i_0)\right] \geq 0, \quad \forall 0 < \varepsilon < 1. \tag{3.50}$$

Subtracting (3.50) from (3.49), using (3.46) and the Lipschitz conditions (2.2) and (2.5) for $\mu(x,i)$, $\sigma(x,i)$ and $f(x)$, we have, for $\varepsilon < \varepsilon_0$,

$$\begin{aligned}
\beta\left[w_1(x_\varepsilon, i_0) - w_2^\lambda(y_\varepsilon, i_0)\right] \leq\ & \frac{2(x_\varepsilon - y_\varepsilon)}{\varepsilon}\left[\mu(x_\varepsilon, i_0) - \mu(y_\varepsilon, i_0)\right] \\
& + \frac{1}{2}\left[\sigma^2(x_\varepsilon, i_0)X_\varepsilon - \sigma^2(y_\varepsilon, i_0)Y_\varepsilon\right] \\
& + [f(x_\varepsilon) - f(y_\varepsilon)] + \sum_{j\neq i_0} q_{i_0 j}(x_\varepsilon)[w_1(x_\varepsilon, j) - w_1(x_\varepsilon, i_0)] \\
& - \sum_{j\neq i_0} q_{i_0 j}(y_\varepsilon)\left[w_2^\lambda(y_\varepsilon, j) - w_2^\lambda(y_\varepsilon, i_0)\right] \\
\leq\ & 2K\phi_\varepsilon(x_\varepsilon, y_\varepsilon) + 5K^2\phi_\varepsilon(x_\varepsilon, y_\varepsilon) + K|x_\varepsilon - y_\varepsilon| \\
& + \sum_{j\neq i_0} q_{i_0 j}(x_\varepsilon)[w_1(x_\varepsilon, j) - w_1(x_\varepsilon, i_0)] \\
& - \sum_{j\neq i_0} q_{i_0 j}(y_\varepsilon)[w_2^\lambda(y_\varepsilon, j) - w_2^\lambda(y_\varepsilon, i_0)].
\end{aligned}$$

Sending $\varepsilon \to 0$, we obtain

$$\begin{aligned}
\beta W &= \beta\left[w_1\left(\hat{x}, i_0\right) - w_2^\lambda\left(\hat{x}, i_0\right)\right] \\
&\leq \sum_{j\neq i_0} q_{i_0 j}(\hat{x})[w_1(\hat{x}, j) - w_1(\hat{x}, i_0)] - \sum_{j\neq i_0} q_{i_0 j}(\hat{x})\left[w_2^\lambda(\hat{x}, j) - w_2^\lambda(\hat{x}, i_0)\right] \\
&= \sum_{j\neq i_0} q_{i_0 j}(\hat{x})\left[(w_1(\hat{x}, j) - w_2^\lambda(\hat{x}, j)) - (w_1(\hat{x}, i_0) - w_2^\lambda(\hat{x}, i_0))\right] \leq 0
\end{aligned} \tag{3.51}$$

since $w_1(\hat{x}, i_0) - w_2^\lambda(\hat{x}, i_0) = \max_{i\in\mathcal{M}}\ \sup_{x\in\mathbb{R}}[w_1(x,i) - w_2^\lambda(x,i)]$. It follows that $W \leq 0$.

Case 2: There exists a subsequence ε_n, $n \geq 1$ such that $\lim_{n\to\infty} \varepsilon_n = 0$ and

$$w_1(x_{\varepsilon_n}, i_0) - g(x_{\varepsilon_n}) \leq 0, \ n \geq 1.$$

From (3.48), we have

$$w_2^\lambda(y_{\varepsilon_n}, i_0) - g(y_{\varepsilon_n}) \geq 0, \ n \geq 1.$$

It follows that

$$w_1(x_{\varepsilon_n}, i_0) - w_2^\lambda(y_{\varepsilon_n}, i_0) \leq g(x_{\varepsilon_n}) - g(y_{\varepsilon_n}), \ n \geq 1.$$

Sending $n \to \infty$, we obtain

$$W = w_1(\hat{x}, i_0) - w_2^\lambda(\hat{x}, i_0) \leq g(\hat{x}) - g(\hat{x}) = 0.$$

For both cases we have shown that $W \leq 0$, which implies that

$$w_1(x, i) \leq w_2^\lambda(x, i), \ x \in \mathbb{R}, \ i \in \mathcal{M}, \ \lambda \in (0, 1).$$

Sending $\lambda \to 0$, we have

$$w_1(x, i) \leq w_2(x, i), \ x \in \mathbb{R}, \ i \in \mathcal{M}.$$

This completes the proof of Theorem 2. □

The following corollary follows from Theorems 1 and 2.

Corollary 1: *The value function $V(x, i)$, $i \in \mathcal{M}$ is the unique viscosity solution of the HJB system* (2.8) *that has at most a linear growth.*

4. Perpetual American put option with state dependent regime-switching

In this section we consider the problem of pricing perpetual American put options written on an asset (a stock or an index, for example) whose price X_t at time $t \geq 0$ is governed by the following switching diffusion:

$$\mathrm{d}X_t = X_t[\mu(\alpha_t)\mathrm{d}t + \sigma(\alpha_t)\mathrm{d}B_t], \quad t \geq 0, \tag{4.1}$$

where $\mu(\alpha_t)$ and $\sigma(\alpha_t)$ are the expected rate of return and volatility, respectively, of the asset price. For the purpose of option pricing, here we assume that the given $(\Omega, \mathcal{F}, \mathcal{P})$ is a properly chosen risk-neutral probability space (a regime-switching random Esscher transform can be employed, following Elliott et al. [3] to obtain a risk-neutral probability measure). The new feature of the option pricing problem is that we allow the jump component α_t to be state dependent, that is, the generator $Q(X_t)$ is a (continuous and bounded) function of the asset price X_t.

Let $g(x) = (E - x)^+$ be the payoff function of the put option where $E > 0$ denotes the exercise price. Let $r > 0$ be the risk-free interest rate. Then the risk-neutral price of the perpetual American put option is given by

$$V(x, i) = \sup_{\tau \in \mathcal{T}} E\left[e^{-r\tau} g(X_\tau) \middle| X_0 = x, \alpha_0 = i\right] = \sup_{\tau \in \mathcal{T}} E\left[e^{-r\tau}(E - X_\tau)^+ \middle| X_0 = x, \alpha_0 = i\right]. \tag{4.2}$$

The HJB system associated with (4.2) is:

$$\min\left\{rv(x,i)-\mu(i)x\frac{\mathrm{d}v}{\mathrm{d}x}(x,i)-\frac{1}{2}\sigma^2(i)x^2\frac{\mathrm{d}^2v}{\mathrm{d}x^2}(x,i)-\sum_{j\neq i}q_{ij}(x)[v(x,j)-v(x,i)],\right.$$
$$\left.v(x,i)-(E-x)^+\right\}=0,\ i=1,\ldots,m_0. \tag{4.3}$$

In view of Section 3, we know that the perpetual American put option price function V defined by (4.2) is the unique viscosity solution of the HJB system (4.3). To numerically solve (4.3), we employ an approximation procedure based on the DP principle and an efficient discrete tree approximation of the continuous-time switching diffusion process $(X_t,\alpha_t), t\geq 0$ where X_t is given by (4.1) and α_t is governed by (2.4).

Liu [9] developed a completely recombined tree for a Markovian regime-switching model that grows linearly as the number of time steps increases. Since the joint process (X_t,α_t) is a two-dimensional Markov process (although α_t itself is not a Markov chain), the idea of [9] can be followed to develop a tree approach for the switching diffusion model (4.1) and (2.4). In what follows we briefly describe the main steps of the tree construction and refer the readers to [9] for further details.

Fix $h>0$ for the time step size. We consider the discrete grid $\{kh, k=0,1,2,\ldots\}$ of the continuous time domain $[0,\infty)$. The discrete grid for the asset price is given by $\{x_0\exp(j\bar{\sigma}\sqrt{h}), j=0,\pm1,\pm2,\ldots\}$ where $\bar{\sigma}>0$ specifies the spatial step size and x_0 is the initial asset price $(X_0=x_0)$. Let (X_k,α_k) denote the discrete approximation of the switching process (X_t,α_t) at time $t=kh, k=0,1,2\ldots$. Assume $(X_k,\alpha_k)=(x,i)$. Then at the next time step, the asset price X_{k+1} can either move up to $x\exp(l_i\bar{\sigma}\sqrt{h})$ with probability $p_{i,u}$, or stay at x (no move) with probability $p_{i,m}$, or move down to $x\exp(-l_i\bar{\sigma}\sqrt{h})$ with probability $p_{i,d}$, where l_i (depending on i) is the number of upward/downward moves (in the unit of $\bar{\sigma}\sqrt{h}$) of the log-price $\ln(X_{k+1})$ (see [9] for discussions of proper choice of l_i). On the other hand, the discrete component α_{k+1} at the $(k+1)$th step may stay at the state i with (conditional) probability $p^{\alpha}_{ii}(x)$ or jump to any other state $j\neq i$ with (conditional) probability $p^{\alpha}_{ij}(x)$, where the switching probabilities $p^{\alpha}_{ij}(x)$ can be approximated [see (2.4)] by

$$p^{\alpha}_{ij}(x)\approx q_{ij}(x)h,\ j\neq i,\quad p^{\alpha}_{ii}(x)\approx 1-\sum_{j\neq i}q_{ij}(x)h=1+q_{ii}(x)h. \tag{4.4}$$

Consequently, conditioned on $(X_k,\alpha_k)=(x,i)$, (X_{k+1},α_{k+1}) can take $3m_0$ possible values with corresponding (conditional) probabilities given as

$$(X_{k+1},\alpha_{k+1})=\begin{cases}(x\exp(l_i\bar{\sigma}\sqrt{h}),j), & \text{with probability } p^{\alpha}_{ij}(x)p_{i,u},\\ (x,j), & \text{with probability } p^{\alpha}_{ij}(x)p_{i,m},\ j=1,\ldots m_0.\\ (x\exp(-l_i\bar{\sigma}\sqrt{h}),j), & \text{with probability } p^{\alpha}_{ij}(x)p_{i,d},\end{cases} \tag{4.5}$$

By matching the conditional mean and variance implied by the trinomial lattice to that implied by the continuous-time process (4.1), the conditional probabilities $p_{i,u}$, $p_{i,m}$ and $p_{i,d}$ can be calculated by

$$
\begin{aligned}
p_{i,u} &= \frac{\sigma^2(i) + a(i)(l_i\bar{\sigma})\sqrt{h} + a^2(i)h}{2(l_i\bar{\sigma})^2}, \\
p_{i,d} &= \frac{\sigma^2(i) - a(i)(l_i\bar{\sigma})\sqrt{h} + a^2(i)h}{2(l_i\bar{\sigma})^2}, \\
p_{i,m} &= 1 - \frac{\sigma^2(i) + a^2(i)h}{(l_i\bar{\sigma})^2},
\end{aligned}
\tag{4.6}
$$

where $a(i) = \mu(i) - \frac{1}{2}\sigma^2(i)$. See [9] for a derivation of (4.6). Following Liu [10] and Yin and Zhu [14], we can show that the so constructed discrete Markov chain approximation $\{(X_k, \alpha_k), k = 0, 1, 2, \ldots\}$ converges weakly to the continuous-time process $\{(X_t, \alpha_t), t \geq 0\}$ as $h \to 0$.

The discrete version of the optimal stopping problem (4.2) is given by

$$
V^h(x, i) = \sup_{\tau} E\left[e^{-rh\tau}(E - X_\tau)^+ \middle| X_0 = x, \alpha_0 = i\right], \tag{4.7}
$$

where $\tau \in \{0, 1, 2, \ldots, \infty\}$ is a discrete stopping time and $V^h(x, i)$ denotes the approximation of the put option price $V(x, i)$ when the grid size h is used for the discretization.

The following iterative procedure, which produces a sequence of approximation option prices, is based on the DP principle of the optimal stopping problem. For $i = 1, 2, \ldots, m_0$, $k = 0, 1, 2, \ldots$,

$$
V_0(x, i) = (E - x)^+, \tag{4.8}
$$

$$
\begin{aligned}
V_{k+1}(x, i) = \max\Bigg\{ & V_k(x, i),\ e^{-rh} \sum_{j=1}^{m_0} p_{ij}^{\alpha}(x)\Big[V_k(x \exp(l_i\bar{\sigma}\sqrt{h}), j)p_{i,u} \\
& + V_k(x \exp(-l_i\bar{\sigma}\sqrt{h}), j)p_{i,d} + V_k(x, j)p_{i,m}\Big]\Bigg\}.
\end{aligned}
\tag{4.9}
$$

Clearly, $V_k(x, i), k \geq 0$ is nondecreasing and bounded and hence convergent. Moreover,

$$
V^h(x, i) = \lim_{k \to \infty} V_k(x, i),\ x > 0,\ i \in \mathcal{M}.
$$

Following the arguments in Kushner and Martins [8], Davis et al. [2], one can show that as $h \to 0$, the discrete approximation $V^h(x, i)$ converges to the unique continuous viscosity solution of the HJB system (4.3), i.e. the option price $V(x, i)$.

A Numerical Example. We consider a model with two regimes, namely $m_0 = 2$. In this case, the generator of the switching component α_t takes the form of

$$
Q(x) = \begin{pmatrix} -q_{12}(x) & q_{12}(x) \\ q_{21}(x) & -q_{21}(x) \end{pmatrix}, \quad q_{12}(x) > 0,\ q_{21}(x) > 0,\ x \in \mathbb{R},
$$

where q_{12} is the jump rate from regime 1 to regime 2 and q_{21} is the jump rate from regime 2 to regime 1. In the numerical experiments, we use $q_{12}(x) = 1+\sin x$ and $q_{21}(x) = 1-\cos x$. Hence the jump rates vary in between 0 and 2, depending on the asset price x.

The other model parameters are specified as following: $\sigma(1) = 0.15$, $\sigma(2) = 0.25$, $\mu(1) = \mu(2) = r = 0.05$, $E = 1$. The time step size is $h = 0.01$ and the spatial step size is

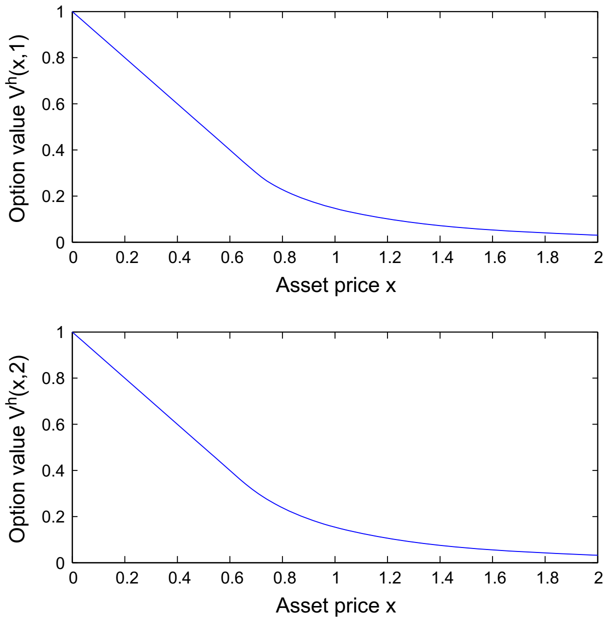

Figure 1. Perpetual American put value functions with two regimes.

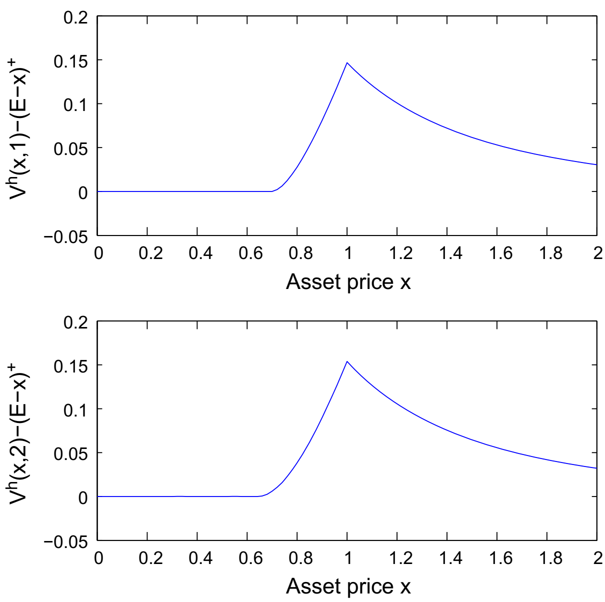

Figure 2. Value function less payoff.

$\bar{\sigma} = 0.2$. The iterative procedure (4.8) is run for 3000 times to obtain a stationary option value. The algorithm produces the optimal exercise levels $x_1^* = 0.717$, $x_2^* = 0.659$, where x_1^* and x_2^* are determined by

$$x_i^* = \min\{x : V^h(x,i) > (E-x)^+\},\ i = 1,2,$$

where V^h denotes the calculated approximation of the option price. That means when the market is in regime 1, the American put option should be exercised whenever the asset price satisfies $x \le x_1^*(= 0.717)$; when the market is in regime 2, the option should be exercised whenever the asset price satisfies $x \le x_2^*(= 0.659)$. Figure 1 shows the two approximated value functions $V^h(x,1)$, $V^h(x,2)$ corresponding to the two regimes, for $0 \le x \le 2$. Figure 2 displays the differences between the approximated value functions and the option payoff, which confirms that $V^h(x,i) \ge (E-x)^+$, $i = 1,2$, a crucial feature for American put options.

5. Concluding remarks

We treat an infinite-horizon optimal stopping problem in switching diffusion models with state dependent switching rates. Using the viscosity solution approach, we prove that the value function is the unique viscosity solution of the associated system of HJB equations. As an application, the problem of pricing perpetual American put options is examined. The results obtained in this paper extend the existing literature on Markovian regime-switching models to the new and more general regime-switching models that can better characterize the dynamics of market prices.

We note that the present paper considers the optimal stopping of a underlying regime-switching diffusion where the rates of switchings are influenced by the same underlying process. A more interesting and extended problem is that the switching rates can depend on a multi-dimensional diffusion including the process that is being optimized and other processes that have impacts on the underlying process. This will be further studied in the future.

Disclosure statement

No potential conflict of interest was reported by the author.

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