A Brief Introduction to Brownian Motion on a Riemannian Manifold

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Contents

Introductory Remarks

These lecture notes constitute a brief introduction to stochastic analysis on manifolds in general, and Brownian motion on Riemannian manifolds in particular. Instead of going into detailed proofs and not accomplish much, I will outline main ideas and refer the interested reader to the literature for more thorough discussion. This is especially true for the last lecture, in which I only discuss the flat space case. Therefore it should be only served as a guide to what one should expect for the path and loop spaces over a Riemannian manifold.

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Lecture 1. Brownian Motion on a Riemannian Manifold

1.1. Brownian motion on euclidean space

Brownian motion on euclidean space is the most basic continuous time Markov process with continuous sample paths. By general theory of Markov processes, its probabilistic behavior is uniquely determined by its initial distribution and its transition mechanism. The latter can be specified by either its transition density function or its infinitesimal generator. For Brownian motion on ^{n}, its transition density function is the Gaussian heat kernel

$$p(t,x,y) = \left(\frac{1}{2\pi t}\right)^{n/2} e^{-|x-y|^2/2t}, \tag{1.1.1}$$

and its infinitesimal generator is half of the Laplace operator:

$$\frac{1}{2}\Delta = \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}.$$

The law $\mathbb{P}_x$ of Brownian motion starting from x is therefore a probability measure on the euclidean path space $C(\mathbb{R}_+, \mathbb{R}^n)$. If we use $\mathbb{E}_x$ to denote the integration with respect to $\mathbb{P}_x$, then we have Dynkin's formula

$$\mathbb{E}_x f(X_t) = f(x) + \frac{1}{2}\int_0^t \Delta f(X_s)\, ds.$$

Here X stands for the so-called coordinate process on $C(\mathbb{R}_+, \mathbb{R}^n)$:

$$X(\omega)_t = X_t(\omega) = \omega_t, \quad \omega \in C(\mathbb{R}_+, \mathbb{R}^n).$$

Dynkin's formula is the starting point of applications of Brownian motion to analysis. If we want to do *stochastic analysis* (as opposed to analysis), then we need Itô's formula:

$$f(X_t) = f(X_0) + \int_0^t \langle \nabla f(X_s), dX_s\rangle + \frac{1}{2}\int_0^t \Delta f(X_s)\, ds.$$

This formula can be regarded as a microscopic formulation (a refinement) of Dynkin's formula. Lying between the two is the martingale characterization of Brownian motion:

$$f(X_t) = f(X_0) + \text{martingale} + \frac{1}{2}\int_0^t \Delta f(X_s)\, ds.$$

All these formulations (Dynkin's formula, Itô's formula, martingale characterization) are equivalent in the sense that each one of them defines uniquely the probability measure $\mathbb{P}_x$ under which the coordinate process is a Markov process with the Gaussian transition density function (1.1.1).

All of the above formulations of Brownian motion will find its counterpart when $\mathbb{R}^n$ is replaced by a Riemannian manifold.

As a way of thinking, it is useful to regard paths of Brownian motion as the characteristic lines of the Laplace operator Δ. Since Δ is an elliptic operator, we know that it has no real characteristic lines in the classical sense. It is a basic fact in the theory of partial differential equations that the solution of the initial value problem is a weighted average of the data on the characteristic lines emanating from the point at which we are seeking solution. Take, for example, the initial value problem for the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad u(0,x) = f(x), \quad \frac{\partial u}{\partial x}(0,x) = 0.$$

The characteristic lines of the wave operator are $x \pm t = \text{const}$. The solution of the problem is

$$u(t,x) = \frac{f(x+t) + f(x-t)}{2},$$

which is the expected value of the initial value $f(x+t)$ and $f(x-t)$ on the characteristic lines from (x,t) as if we assign each characteristic line the probability $1/2$. Now consider the Dirichlet problem on the smooth domain D in $\mathbb{R}^n$:

$$\begin{cases} \Delta u = 0 & \text{on } D, \\ u = f & \text{on } \partial D. \end{cases}$$

Each curve X starting from x meets the boundary at X_{τ_D}, where τ_D is the first exit time of D:

$$\tau_D = \inf \{ t \geq 0 : X_t \notin D \}.$$

The value of the data for this "characteristic line" is $f(X_{\tau_D})$. The probability is assigned to these "characteristic lines" according to the law of Brownian motion $\mathbb{P}_x$. In analogy with the case of the wave equation, we arrive heuristically the formula

$$u_f(x) = \mathbb{E}_x f(X_{\tau_D}), \quad x \in D,$$

which is Doob's representation of the solution of the Dirichlet problem.

1.2. Laplace-Beltrami operator and the heat kernel

As we have seen in SECTION 1.1, the Laplace operator and the Gaussian transition density function (heat kernel) are the basic analytic objects associated with Brownian motion on $\mathbb{R}^n$. Of the two, the Laplace operator is the more fundamental one and the heat kernel can be obtained as (minimal) fundamental solution of the heat equation associated with the Laplace operator, namely, the function $p(t,x,y)$ is the smallest positive solution of the following initial value problem:

$$\frac{\partial p}{\partial t} = \frac{1}{2}\Delta u, \quad \lim_{t\downarrow 0} p(t,x,y) = \delta_x(y).$$

The counterpart of the Laplace operator on a Riemannian manifold M is the Laplace-Beltrami operator Δ_M, which will serves as the infinitesimal generator for Brown motion on M. This operator can be briefly described as follows. We denote the Riemannian metric on M $\langle\cdot,\cdot\rangle_x$. The gradient grad$f$ of a function f on M is a vector field on M defined uniquely by

$$\langle \mathrm{grand} f, X\rangle = X(f), \quad X \in \mathit{\Gamma}(M).$$

[Here $\mathit{\Gamma}(M)$ is the space of smooth vector field on M.] The divergence div$X$ of a vector field X is characterized by

$$\int_M X(f)d\mu = -\int_M f \mathrm{div} X\, d\mu.$$

Here μ is the Riemannian volume measure. The Laplace-Beltrami operator is

$$\Delta_M f = \mathrm{div}(\mathrm{grad} f).$$

In local coordinates $x^1, \cdots, x^n$, the Riemannian metric is written in the form

$$ds^2 = g_{ij}dx^i dx^j.$$

The Laplace-Beltrami operator is given by

$$\Delta_M f = \frac{1}{\sqrt{G}}\frac{\partial}{\partial x^j}\left(\sqrt{G} g^{ij}\frac{\partial f}{\partial x^i}\right).$$

Here G is the determinant of the matrix $\{g_{ij}\}$ and $\{g^{ij}\}$ is its inverse. We have

$$\Delta_M f = g^{ij}\frac{\partial^2 f}{\partial x^i \partial x^j} + b^i \frac{\partial f}{\partial x^i},$$

where

$$b^i = \frac{1}{\sqrt{G}}\frac{\partial(\sqrt{G}g^{ij})}{\partial x^j} = g^{jk}\mathit{\Gamma}^i jk,$$

where $\mathit{\Gamma}^i jk$ are the Christoffel symbols of the metric ds^2 in the local coordinates. Therefore Δ_M is a second order, strictly elliptic operator.

The construction of the heat kernel (minimal fundamental solution) $p(t,x,y)$ associated with the Laplace-Beltrami operator is not a trivial task and belongs to the fields of partial differential equations and differential geometry. It is carried out in great detail in Chavel[**1**]. We will see later that the theory of stochastic differential equations allows in some sense avoid this construction.

A general stochastic differential equation in Stratonovich formulation has the form

$$dX_t = V_\alpha(X)_t \circ dW_t^\alpha + V_0(X_t)\, dt.$$

It generates a diffusion process (i.e., Markov process with continuous sample paths) whose infinitesimal generator is

$$L = \frac{1}{2}\sum_{i=1}^{l} V_\alpha^2 + V_0.$$

Thus it has the form of a "sum of squares." Unfortunately, on a general Riemannian manifold, there is no natural way of writing the Laplace-Beltrami operator as a sum of squares. Such a representation of Δ_M is possible if M is embedded isometrically in some euclidean space Δ^l. For a point $x \in M$, let P_α denote the projection of the unit coordinate vector ξ_α on the tangent space T_xM. We obtain in this way l vector fields P_α on M. It can be shown that ([**11**], 77–78)

$$\Delta_M = \sum_{\alpha=1}^{l} P_\alpha^2.$$

According to Nash's theorem, one can always embedd a Riemannian manifold isometrically in some euclidean space. This fact and the above expression for Δ_M can be used to give an extrinsic construction of Brownian motion on M.

1.3. Brownian motion on a Riemannian manifold

We define Brownian motion on M to be a Markov process whose transition density function is $p(t, x, y)$, the heat kernel associated with the Laplace-Beltrami operator. General theory of Markov processes shows how such a process can be constructed, see Chung[**4**]. It turns out to be a diffusion process, i.e., a strong Markov process with continuous sample paths.

On a general Riemannian manifold it may happen that

$$\int_M p(t, x, y)dy < 1.$$

Probabilistically this means that Brownian motion may not run for all time. More precisely, there is a finite stopping time e, called the lifetime of Brownian motion such that

$$\lim_{t\uparrow\uparrow e} X_t = \infty_M.$$

where $\widehat{M} = M \cup \{\infty_M\}$ is the one-point compactification of M and the above limit is understood in the topology of $\widehat{M}$. Intuitively speaking, Brownian motion may go off the manifold in a finite amount of time. Naturally this case does not happen if M is compact. We have

$$\mathbb{P}_x\{e \geq t\} = \int_M p(t, x, y)\, dy.$$

When $\mathbb{P}_x\{e < \infty\} = 1$ for some x and t (hence for all x and t), we say that the manifold M is stochastically complete. We will address the question when a Riemannian manifold is stochastically complete.

Once Brownian motion is constructed as a diffusion processes with transition density function $p(t,x,y)$, the fact that $p(t,x,y)$ is the minimal fundamental solution of the heat equation for the Laplace-Beltrami operator give immediately Dynkin's formula

$$\mathbb{E}_x f(X_t) = f(x) + \mathbb{E}_x \int_0^t \Delta_M f(X_s)\, ds$$

for all reasonable functions on M, say smooth with compact support. Without little extra effort, this can be expended to read

$$f(X_t) = f(X_0) + M_t^f + \frac{1}{2}\int_0^t \Delta f(X_s)\, ds, \qquad 0 \le t < e. \tag{1.3.1}$$

Here M^f is a (local) martingale depending on f. The quadratic variation process of M^f can be identified.

Proposition 1.3.1. *We have*

$$\langle M^f\rangle_t = \int_0^t |\nabla f(X_s)|^2 ds.$$

Proof. We decompose $f(X_t)^2$ in two ways. First, we use Itô's formula and (1.3.1) to obtain

$$\begin{aligned} f(X_t)^2 &= f(x)^2 + 2\int_0^t f(X_s)\, df(X_s) + \langle M^f\rangle_t \\ &= f(x)^2 + 2\int_0^t f(X_s) dM_s^f + \int_0^t f(X_s)\Delta_M f(X_s)\, ds + \langle M^f\rangle_t. \end{aligned}$$

Second we use (1.3.1) with f replaced by f^2 and obtain

$$f(X_t)^2 = f(x)^2 + M_t^{f^2} + \frac{1}{2}\int_0^t \Delta_M(f^2)(X_s)\, ds.$$

Equating the bounded variation parts of the two expressions, we have

$$\langle M^f\rangle_t = \frac{1}{2}\int_0^t \left\{\Delta_M(f^2)(X_s) - 2f(X_s)\Delta_M f(X_s)\right\} ds.$$

Finally,

$$\Delta_M(f^2) - 2f\Delta_M f = 2|\nabla f|^2.$$

□

We thus have established the following fact about Brownian motion on M:

$$f(X_t) = f(X_0) + M_t^f + \frac{1}{2}\int_0^t \Delta_M f(X_s)\, ds, \qquad 0 \le t < e,$$

with where M^f is a local martingale with

$$\langle M^f\rangle_t = \int_0^t |\nabla f(X_s)|^2 ds.$$

This is property of Brownian motion is sufficient for most applications in analysis and geometry, for in these applications we rarely need to know the joint distribution of the martingale M^f and the stochastic integral $\int_0^t \Delta_M f(X_s)\,ds$. For more delicate stochastic analysis, we need to have an explicit description of the martingale M^f. If $M = \mathbb{R}^n$, we have

$$M_t^f = \int_0^t \langle \nabla f(X_s), dX_t\rangle.$$

In the present formulation of Brownian motion, there is not a direct way of writing down M^f.

1.4. Brownian motion by embedding

If M is a submanifold of a euclidean space $\mathbb{R}^l$, Brownian motion on M can be obtained by solving a stochastic differential equation on M. In SECTION 1.2 we mentioned that the Laplace-Beltrami operator Δ_M can be written in the form of a sum of squares:

$$\Delta_M = \sum_{\alpha=1}^{l} P_\alpha^2,$$

where P_α is the projection of the αth coordinate unit vector ξ_α on the tangent space T_xM. Each P_α is a vector field on M. Consider the following Stratonovich stochastic differential equation on M driven by a l-dimensional euclidean Brownian motion W:

$$dX_t = P_\alpha(X_t)\circ dW_t^\alpha, \qquad X_0 \in M. \tag{1.4.1}$$

[The summation convention is used.] This is a stochastic differential equation on M because P_α are vector fields on M. Extending P_α arbitrarily to the whole ambient space we can solve this equation as if it is an equation on $\mathbb{R}^l$ by the usual Picard's iteration. It can be verified that if the initial value X_0 lies on the manifold M, then the solution lies on M for all time:

$$\mathbb{P}_x\left\{X_t \in M \quad \text{for all } t < e | X_0 \in M\right\} = 1,$$

see [**11**], 22–23. Furthermore, the solution is a diffusion process generated by $(1/2)\sum_{\alpha=1}^l P_\alpha^2 = (1/2)\Delta_M$. Using Itô's formula we have

$$f(X_t) = f(X_0) + M_t^f + \frac{1}{2}\int_0^t \Delta_M f(X_s)\,ds, \qquad 0 \le t < e,$$

where

$$M_t^f = \int_0^t \langle P_\alpha f(X_s), dW_s^\alpha\rangle.$$

EXAMPLE 1.4.1. (Brownian motion on a sphere) Consider the unit sphere $\mathbb{S}^n$ canonically embedded in $\mathbb{R}^{n+1}$. The projection to the tangent sphere at x is given by

$$P(x)\,\xi = \xi - \langle \xi, x\rangle x, \qquad x \in \mathbb{S}^n, \quad \xi \in \mathbb{R}^{n+1}.$$

Hence the matrix $P = \{P_1, \cdots, P_{n+1}\}$ is

$$P(x)_{ij} = \delta_{ij} - x_i x_j.$$

Brownian motion on $\mathbb{S}^n$ is the solution of the stochastic differential equation

$$X_t^i = X_0^i + \int_0^t (\delta_{ij} - X_s^i X_s^j) \circ dW_s^j, \quad X_0 \in \mathbb{S}^n.$$

This is Stroock's representation of spherical Brownian motion.

The representation 1.4.1 is a good way to visualize Brownian motion on M as a pathwise object (rather than a measure on the path space $C(\mathbb{R}_+, M)$). It is an extrinsic representation because it depends on the embedding of M into some euclidean space $\mathbb{R}^l$. It has the drawback that the equation (1.4.1) is driven by a Brownian motion W whose dimension l is in general larger than the dimension n of the manifold M, whereas in some sense Brownian motion on M should still be an n-dimensional object. Full strength of Brownian motion on M can only be revealed after we write it faithfully as an n-dimensional object, i.e., as the solution of a stochastic differential equation driven by an n-dimensional euclidean Brownian motion.

1.5. Brownian motion in local coordinates

As we have mentioned in SECTION 1.2, the Laplace-Beltrami operator can be written as

$$\Delta_M f = \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^j} \left(\sqrt{G} g^{ij} \frac{\partial f}{\partial x^i} \right).$$

This gives a way of constructing Brownian motion valid up to the first time it exits from the local coordinate chart. Let $\sigma = \left\{ \sigma_i^j \right\}$ be the unique symmetric square root of $g^{-1} = \left\{ g^{ij} \right\}$. Consider the solution of the stochastic differential equation for a process $X_t = \left\{ X_t^1, \ldots, X_t^n \right\}$:

$$dX_t^i = \sigma_j^i(X_t)\, dB_t^j + \frac{1}{2} b^i(X_t)\, dt.$$

Then it is easy to verify by Itô's formula that X is a diffusion process generated by $(1/2)\Delta_M$, i.e., X is a Brownian motion on M. Brownian motion can be studied this way by choosing an appropriate local coordinate system in which the Laplace-Beltrami operator Δ_M takes special a special form.

Lecture 2. Brownian Motion and Geometry

We study the effect of curvature on the behavior of Brownian motion and hope that it will lead to interesting results about the manifold itself. We will concentrate some problems which can be studied through the distance function $r(x) = d(x, o)$, where o is a fixed point on the manifold. In this respect, the radial process $r_t = r(X_t)$ is a natural object of investigation. We often compare this process with the same process on a radially symmetric manifold satisfying certain curvature conditions. On such a manifold, problems often becomes one-dimensional and can be solved explicitly.

2.1. Radially symmetric manifolds

A radially symmetric manifold M has a distinguished point o, call the pole of M. In the polar coordinates (r, θ) induced by the exponential map $\exp_o : \mathbb{R}^n \to M$ based at o, the metric has the following form

$$ds^2 = dr^2 + G(r)^2 d\theta^2.$$

Here $d\theta^2$ denotes the standard Riemannian metric on the $n-1$-sphere $\mathbb{S}^{n-1}$, and G is a smooth function on an interval $[0, D)$ satisfying

$$G(0) = 0, \ G'(0) = 1, \quad 0 \le r < D.$$

In terms of these coordinates, the Laplace-Beltrami operator has the form The Laplace-Beltrami operator has the form

$$\Delta_M = L_r + \frac{1}{G(r)^2} \Delta_{\mathbb{S}^{n-1}}, \tag{2.1.1}$$

where L_r is the radial Laplacian

$$L_r = \left(\frac{\partial}{\partial r}\right)^2 + (n-1)\frac{G'(r)}{G(r)}\frac{\partial}{\partial r},$$

and $\Delta_{\mathbb{S}^{n-1}}$ is the Laplace-Beltrami operator on $\mathbb{S}^{n-1}$. The main feature of this case is that the radial component is completely decoupled from the angular component and the angular component is a scaling of $\Delta_{ms^{n-1}}$ with the scale $1/G(r)^2$ depending on the radial component. In the terminology of differential geometry, the metric ds^2 has the form of a warped product.

Let $X_t = (r_t, \theta_t)$ be a Brownian motion on a radially symmetric manifold M written in polar coordinates. Using polar coordinates, we find that the radial component is the solution of the stochastic differential equation:

$$r_t = r_0 + W_t + \frac{n-1}{2}\int_0^t \frac{G'(r_s)}{G(r_s)}ds, \tag{2.1.2}$$

where W is a 1-dimensional Brownian motion. The angular component can also be described easily. Let $Y = \{Y_t\}$ be a Brownian motion on $\mathbb{S}^{n-1}$ independent of W (hence also independent of $\{r_t\}$). Define a new time scale

$$l_t = \int_0^t \frac{ds}{G(r_s)^2}, \tag{2.1.3}$$

and let $\theta_t = Y_{l_t}$ be the time change of the spherical Brownian motion Y. Then $X_t = (r_t, \theta_t)$ constructed this way is a Brownian motion on M (see [**11**], 84–85). From this description of Brownian motion it is clear that the behavior of Brownian motion on a radially symmetric manifold is controlled by and large by its radial process. The radially process is a one-dimensional diffusion process, which has been well studied in probability theory.

Let's study a special case more closely. A complete, simply connected manifolds of negative sectional curvature is called a Cartan-Hadamard manifold. Suppose that M is such a manifold with constant negative curvature $-K^2$ (space form). Then it is a radially symmetric manifold with the metric $ds^2 = dr^2 + G(r)^2 d\theta^2$ is given by

$$G(r) = \frac{\sinh Kr}{K}.$$

The behavior of Brownian motion on this manifold is typical for Brownian motion on Cartan-Hadamard manfiold whose curvature is pinned between two negative constant.

The radial process is the solution of

$$dr_t = dW_t + \frac{n-1}{2} K \coth Kr_t \, dt.$$

We write the equation in the form

$$r_t = W_t + \frac{(n-1)}{2}\int_0^t K \coth Kr_s \, ds.$$

We have $\coth Kr_t \geq 1$ for all t because $r_t \geq 0$. Hence

$$r_t \geq r_0 + W_t + \frac{(n-1)}{2} K\, t. \tag{2.1.4}$$

According to the law of the iterated logarithm,

$$\limsup_{t\to\infty} \frac{W_t}{\sqrt{2t \ln\ln t}} = 1,$$

Thus the term $r_0 + W_t$ on the right side of (2.1.4) is of lower order than the last term and we have (with probability one) $r_t \to \infty$. Now that $\coth Kr_t \to 1$ as $t \to \infty$, from

$$\frac{r_t}{t} = \frac{r_0 + W_t}{t} + \frac{(n-1)}{2}\frac{K}{t}\int_0^t \coth Kr_s \, ds$$

we find the asymptotic behavior of the radial process

$$\lim_{t\to\infty} \frac{r_t}{t} = \frac{(n-1)K}{2}. \tag{2.1.5}$$

We now examine at the angular process. From the above discussion, we know that it is a time-change of a Brownian motion on the sphere $\mathbb{S}^{n-1}$. In the present case, the time change is

$$l_t = \int_0^t \left(\frac{K}{\sinh Kr_s}\right)^2 ds.$$

From (2.1.5) the integral converges as $t \uparrow \infty$. It follows that

$$\lim_{t\to\infty} \theta_t = Y_{l_\infty}. \tag{2.1.6}$$

(2.1.5) and (2.1.6) give a fairly good picture of the asymptotic behavior of Brownian motion on a complete, simply connected manifold of constant negative curvature.

See [**12**] for more recent work on angular convergence of Brownian motion and its relation with the Dirichlet problem at infinity for Cartan-Hadamard manifolds.

2.2. Radial process

The concept of a radial process can be introduced for Brownian motion on a general Riemannian manifold M. Fix a reference point $o \in M$, and let $r(x) = d(x, o)$ be the Riemannian distance between x and o. We define the radial process $r_t = r(X_t)$. It is natural to try to use Itô's formula to decompose this into a martingale part and a bounded variation part. The function $r : M \to \mathbb{R}_+$ has a well behaved singularity at the origin. In particular,

$$\Delta_M r \sim \frac{n-1}{r} \qquad \text{near } r = 0.$$

This singularity will not cause any problem for us because, except for the trivial one-dimensional case, Brownian motion X never hits o for $t > 0$. However, $x \mapsto r(x)$ is not a smooth function on $M\backslash \{o\}$. Differential geometry ([**8**] tells us exactly where it is smooth.

For simplicity we assume that M is geodesically complete. Every geodesic segment can be extended in both directions indefinitely and every pair of points can be connected by a distance-minimizing geodesic. For each unit vector $e \in T_oM$, there is a unique geodesic $C_e : [0, \infty) \to M$ such that $\dot{C}_e(0) = e$. The exponential map $\exp : T_oM \to M$ is

$$\exp te = C_e(t).$$

If we identify T_oM with $\mathbb{R}^n$ by an orthonormal frame, the exponential map becomes a map from $\mathbb{R}^n$ onto M. For small t, the geodesic $C_e[0, t]$ is the unique distance-minimizing geodesic between its endpoints. Let $t(e)$ be the largest t such that the geodesic $C_e[0,t]$ is distance-minimizing from $C_e(0)$ to $C_e(t)$. Define

$$\widetilde{C}_o = \{t(e)e : e \in T_oM,\ |e| = 1\}.$$

Then the cutlocus of o is the set $C_o = \exp \widetilde{C}_o$. Sometimes we also call $\widetilde{C}_o$ the cutlocus of o. The set within the cutlocus is the star-shaped domain

$$\widehat{E}_o = \{te \in T_oM :\ e \in T_oM,\ 0 \le t < t(e),\ |e| = 1\}.$$

On M the set within cutlocus is $E_o = \exp \widetilde{E}_o$. We have the following basic results from differential geometry (see [**8**] [**2**] [**3**]).

Theorem 2.2.1. *(i) The map* $\exp : \widetilde{E}_o \to E_o$ *is a diffeomorphism.*
(ii) The cutlocus C_o is a closet subset of measure zero.
(iii) If $x \in C_y$, then $y \in C_x$.
(iv) E_o and C_o are disjoint and $M = E_0 \cup C_o$. □

According to the above theorem the polar coordinates (r, θ) are well behaved on the region $M \backslash C_o$ within the cutlocus. The set they do not cover is the cutlocus C_o, a set of measure zero. The radial function $r(x) = d(x, o)$ is smooth on $M \backslash C_o$ and Lipschitz on all of M. Furthermore, $|\nabla r| = 1$ everywhere on $M \backslash C_o$.

If X is a Brownian motion on M starting within E_o, then, before it hits the cutlocus C_o,

$$r(X_t) = r(X_0) + W_t + \frac{1}{2}\int_0^t \Delta_M r(X_s)\, ds, \quad t < T_{C_o}, \tag{2.2.1}$$

where T_{C_o} is the first hitting time X of the cutlocus C_o and W is a martingale. Its quadratic variation is

$$\langle W \rangle_t = \int_0^t \Gamma(r, r)(X_s)\, ds = \int_0^t |\nabla r(X_s)|^2\, ds = t.$$

Hence by Lévy's criterion, W is a Brownian motion. (2.2.1) shows that the behavior of the radial process is largely controlled by the Laplacian of the distance function $\Delta_M r$. In practice we try to bound $\Delta_M r$ by a known function of r and then control $r(X_t)$ by comparing it with a one-dimensional diffusion process.

(2.2.1) is good enough if the cutlocus C_o is empty, e.g., if M is a Cartan-Hadamard manifold. What happens to the when Brownian motion crosses the cutlocus? Very complicated. A very detailed study of this problem can be found in [**6**]. In most cases, the following result due to W. Kendall is sufficient.

Theorem 2.2.2. *Suppose that X is a Brownian motion on Riemannian manifold M. Let $r(x) = d(x, o)$ be the distance function from a fixed point $o \in M$. Then there exist a one-dimensional euclidean Brownian motion*

W and a nondecreasing process L which increases only when $X_t \in C_o$ (the cutlocus of o) such that

$$r(X_t) = r_0 + W_t + \frac{1}{2}\int_0^t \Delta_M r(X_s)\, ds - L_t, \quad t < e.$$

According this theorem, we always have a lower bound

$$r_t \le r_0 + W_t + \frac{1}{2}\int_0^t \Delta_M r(X_s)\, ds, \quad t < e.$$

If we need to bound the radial process from above, we have to assume that the cutlocus is empty. In this case,

$$r(X_t) = r_0 + W_t + \frac{1}{2}\int_0^t \Delta_M r(X_s)\, ds, \quad t < e.$$

2.3. Comparison theorems

The next step is to study the Laplacian $\Delta_M r$ of the distance function. The goal is to bound $\Delta_M r$ by a simple function of r. There is a host of Laplacian (more generally, Hessian) comparison theorems of this type one can draw from differential geometry (see [**3**] [**15**]). We cite two simple ones which compare an arbitrary Riemannian manifold with manifolds of constant curvature (see [**15**]).

THEOREM 2.3.1. *Let $K_M(x)$ denote any sectional curvature at $x \in M$ and assume that*

$$-K_1^2 \le K_M(x) \le K_2^2.$$

Then we have at any smooth point of the distance function $r(x)$,

$$(n-1)K_2 \cot K_2 r(x) \le \Delta_M r(x) \le (n-1)K_1 \coth K_1 r(x).$$

This result can be used to control the radial process, or more precisely, to compare the radial process on M with those on manifolds of constant curvatures K_1^2 and $-K_2^2$. Let's first bound the radial process from below. We have

$$r_t \le r_0 + W_t + \frac{1}{2}\int_0^t \Delta_M r(X_s)\, ds.$$

Next, consider the equation

$$r_t^1 = r_0 + W_t + \frac{n-1}{2}\int_0^t K_1 \coth K_1 r_s^1\, ds.$$

r_t^1 is the radial process of a Brownian motion on the space form of constant curvature $-K_1^2$. Note that it is driven by the same Brownian motion W. Since we have $\Delta_M r(x) \le (n-1)K_1 \coth K_1 r(x)$, the drift of r_t is smaller that the drift of r_t^1. By the standard comparison theorem for one-dimensional processes (see [**13**]), we have $r_t \le r_t^1$ for all $t \ge 0$.

Next, if M does not have cutlocus, or if $t \le T_{C_o}$, we have

$$r_t = r_0 + W_t + \frac{1}{2}\int_0^t \Delta_M r(X_s)\, ds.$$

This time we compare with the process r_t^2 determined by

$$r_t^2 = r_0 + W_t + \frac{n-1}{2}\int_0^t K_2 \cot K_2 r_s^2\, ds.$$

r_t^2 is the radial process on the $(n-1)$-dimensional sphere of radius $1/K_2$. By the same argument as before, we have $r_t \ge r_t^2$.

Let's see two applications. The next result is due to S. T. Yau.

Theorem 2.3.2. *Let M be a complete manifold whose sectional curvature is bounded from below by a constant. Then it is stochastically complete, i.e.,*

$$\int_M p(t,x,y)\, dy = 1$$

for all $x \in M, t > 0$.

Proof. If M is not stochastically complete, that Brownian motion blows up with positive probability, i.e., it goes to infinity in finite amount of time. Suppose that $K_M(x) \ge -K_1^2$. Then we have shown that $r_t \le r_t^2$. If r_t goes to infinity in finite amount of time, certain r_t^2 will do the same. But we have shown that $r_t \sim (n-1)Kt/2$ as $t \to \infty$, which means that r_t^2 does not blow up. Nor will r_t. □

We say that Brownian motion on M is transient if for some $x \in M$ (hence for all $x \in M$),

$$\mathbb{P}_x \left\{ \lim_{t \uparrow\uparrow e} X_t = \infty_M \right\} = 1.$$

Otherwise, we say Brownian motion is recurrent on M. There is a simple analytic criterion for recurrence and transience. Let

$$G(x,y) = \int_0^\infty p(t,x,y)\, dt$$

be Green's function of M. Then Brownian motion on M is transient if and only if $G(x,y) < \infty$ for some pair of points $x \ne y$ (hence for all such pairs of points). It is well known that euclidean Brownian motion of dimension 1 and 2 is recurrent, and of dimension 3 or higher is transient.

Theorem 2.3.3. *Suppose that M is a Cartan-Hadamard manifold of dimension greater than 2. Then Brownian motion on M is transient.*

Proof. M does not have cutlocus and $K_M(x) \le 0$. Therefore $r_t \ge r_t^2$, where r_t^2 is the radial process of euclidean Brownian motion of dimension n. Since $r_t^2 \to \infty$ as $t \uparrow \infty$, we must have $r_t \to \infty$ as $t \uparrow\uparrow e$, which means Brownian motion on M must be transient. □

More refined results along these lines can be found in [**11**].

Lecture 3. Stochastic Calculus on manifolds

From a theoretical point of view, the most satisfactory construction of Brownian motion on a manifold is that of Eells-Elworthy-Malliavin.

3.1. Orthonormal frame bundle

Let $\mathscr{O}_x(M)$ be the set of orthonormal frames of the tangent space T_xM. The orthonormal frame bundle

$$\mathscr{O}(M) = \bigcup_{x \in M} \mathscr{O}_x(M)$$

has a natural structure of a smooth manifold of dimension $n(n+1)/2$. Let $\pi : \mathscr{O}(M) \to M$ be the canonical projection. Each element $u \in \mathscr{O}(M)$ is therefore an isometry

$$u : \mathbb{R}^n \to T_{\pi u}M.$$

Let $u \in \mathscr{O}(M)$ and $\pi u = x$. The fibre $\pi^{-1}x = \mathscr{O}_x M$ is naturally a smooth submanifold of dimension $n(n-1)/2$. Its tangent space $V_u\mathscr{O}(M)$ is a subspace of the same dimension of the full tangent space $T_u\mathscr{O}(M)$. A curve $\{u_t\}$ in $\mathscr{O}(M)$ is horizontal if u_t is the parallel transport of u_0 along the projection curve $\{\pi u_t\}$. The set of tangent vectors of horizontal curves passing through a fixed point $u \in \mathscr{O}(M)$ is the horizontal subspace $H_u\mathscr{O}(M)$ of dimension n of $T_u\mathscr{O}(M)$ and we have the relation

$$T_u\mathscr{O}(M) = H_u\mathscr{O}(M) \oplus V_u\mathscr{O}(M),$$

and the projection $\pi : \mathscr{O}(M) \to M$ induces an isomorphism $\pi_* : H_u\mathscr{O}(M) \to T_xM$. On the orthonormal frame bundle, we have n well defined horizontal vector field H_i. At each $u \in \mathscr{O}(M)$, $H_i(u)$ is the unique horizontal vector in $H_u\mathscr{O}(M)$ whose projection is the ith unit vector ue_i of the orthonormal frame; i.e.,

$$\pi_* H_i(u) = ue_i, \qquad H_i(u) \in H_u\mathscr{O}(M).$$

The operator

$$\Delta_{\mathscr{O}(M)} = \sum_{i=1}^n H_i^2$$

is called Bochner's horizontal Laplacian on $\mathscr{O}(M)$. The Eells-Elworthy-Malliavin construction is based on the following relation.

Proposition 3.1.1. *For any smooth function f on M, we have*

$$\Delta_M f(x) = \Delta_{\mathscr{O}(M)}(f \circ \pi)(u)$$

for any $u \in \mathscr{O}(M)$ such that $\pi u = x$.

3.2. Eells-Elworhthy-Malliavin construction of Brownian motion

We note that $\Delta_{\mathscr{O}(M)}$ is in the form of the sum of n squares, where n is the dimension of the manifold M. Of course the price to pay is that it is an operator on a much larger space $\mathscr{O}(M)$ instead of on the manifold M itself. Consider the following stochastic differential equation on $\mathscr{O}(M)$:

$$dU_t = \sum_{i=1}^{n} H_i(U_t) \circ dW_t^i.$$

It is driven by an n-dimensional Brownian motion W. A solution of this equation is called a horizontal Brownian motion (on $\mathscr{O}(M)$). It is a diffusion process generated by $\Delta_{\mathscr{O}(M)}$. Itô's formula takes the following form:

$$F(U_t) = F(U_0) + \sum_{i=1}^{n} \int_0^t H_i F(U_s)\, dW_s^i + \frac{1}{2} \int_0^t \Delta_{\mathscr{O}(M)} F(U_s)\, ds,$$

where F is a smooth function on $\mathscr{O}(M)$. Now, if we apply this to a function of the form $F = f \circ \pi$, the lift of a smooth function f on M, then by Proposition 3.1.1,

$$f(X_t) = f(X_0) + \sum_{i=1}^{n} \int_0^t H_i(f \circ \pi)(U_s)\, dW_s^i + \frac{1}{2} \int_0^t \Delta_M f(X_s)\, ds.$$

Here $X_t = \pi U_t$ is the projection of the horizontal Brownian motion U_t on the manifold M. It follows that X_t is a Brownian motion on M starting from $X_0 = \pi U_0$.

Suppose that we want to construct a Brownian motion starting from x. We fix an orthonormal frame $u \in \mathscr{O}(M)$ over x, i.e., $\pi u = x$. There is a unique horizontal Brownian motion U_t starting from the frame u. The projection $X_t = \pi U_t$ is a Brownian motion from x. This Brownian motion is not uniquely determined by the driving Brownian motion W because the initial frame u can be chosen arbitrarily. Of course, the law of X is completely determined by the initial point x and does not depend on the choice of either u or W. Once a frame u is fixed, $W \mapsto X$ establishes a measure preserving map

$$J : (P_o(\mathbb{R}^n), \mu) \to (P_x(M), \nu),$$

where μ is the Wiener measure on the euclidean path space $P_o(\mathbb{R}^n) = C_o(\mathbb{R}_+, \mathbb{R}^n)$ (the law of euclidean Brownian motion) and ν is the law of Brownian motion on M starting from x. The map J is usually called the Itô map for the reason that it is obtained through solving an Itô type stochastic differential equations. We will see later that this map is invertible. The image $X = JW$ is called a stochastic development of W.

3.3. Stochastic horizontal lift

In differential geometry, a smooth curve on M can be lifted to a horizontal curve with the help of the Riemannian connection (or equivalently, the concept of parallel transport). If $c : I \to M$ is such a curve, we choose a frame $u(0)$ over $c(0$ and simply let $u(t)$ be the parallel transport of $u(0)$ along $c[0, t]$, which is accomplished by solving an ordinary differential equation in local charts. The lift $\{u(t),\, t \in I\}$ in turn defines a smooth curve w in $\mathbb{R}^n$ by

$$w(t) = u(t)^{-1}\dot{c}(t).$$

The curve w is called an anti-development of c. The standard reference for this part of differential geometry is [**14**].

A similar procedure can be carried out if the smooth curve c is replaced by a Brownian motion, or more generally, a semimartingale X on M. We expect that the horizontal lift U of X is obtained by solving a stochastic differential equation driven by X. Unlike a smooth curve on M, a semimartingale on M is not a local object. A construction of U using local charts is possible but technically unwieldy. If we assume that M is embedded in some euclidean space, then a relatively clean construction is possible.

Before we proceed further, let us give the definition of a semimartingale on M. Let $(\Omega, \mathscr{F}_*, \mathbb{P})$ be a probability space with filtration $\mathscr{F}_* = \{\mathscr{F}_t, t \geq 0\}$. A semimartingale $X = \{X_t,\, t \geq 0\}$ on M is an M-valued, $\mathscr{F}_*$-adapted process such that $\{f(X_t),\, t \geq 0\}$ is a real-valued semimartingale for all smooth functions f on M.

Let M be a submanifold of $\mathbb{R}^l$ and recall that $P_\alpha(x)$ is the projection of the αth coordinate unit vector ξ_α to the tangent space T_xM at $x \in M$. Suppose that X is a semimartingale on M. Since M is a submanifold of $\mathbb{R}^l$, X can be regarded as a semimartingale on $\mathbb{R}^l$, i.e., $X = \left\{X^1, \ldots, X^l\right\}$. Let $P^*_\alpha(u)$ be the horizontal lift of $P_\alpha(\pi u)$ to $u \in \mathscr{O}(M)$. Then we obtain l horizontal vector field. Consider the following stochastic differential equation on $\mathscr{O}(M)$ driven by X:

$$dU_t = \sum_{\alpha=1}^{l} P^*_\alpha(U_t) \circ dX^\alpha_t. \tag{3.3.1}$$

It has a unique solution once an initial frame U_0 is given.

Theorem 3.3.1. *The solution of (3.3.1) is a horizontal lift of X to $\mathscr{O}(M)$.*

Proof. We sketch a proof, see [**11**] for details.

The proof is based on the following identity which holds for any semimartingale X on M:

$$X_t = X_0 + \sum_{\alpha=1}^{l} \int_0^t P_\alpha(X_s) \circ dX_s^\alpha. \tag{3.3.2}$$

This can be regarded as a stochastic differential equation for X on M. Laurent Schwartz observed once that every semimartingale on a manifold M is a solution of a stochastic differential equation on M.

Let $f : \mathscr{O}(M) \to M \subseteq \mathbb{R}^l$ be the projection $\pi : \mathscr{O}(M) \to M$ regarded as an $\mathbb{R}^l$-valued function on $\mathscr{O}(M)$. Let $Y_t = f(U_t) = \pi U_t$. We have to show that $Y_t = X_t$. Apply Itô's formula to $f(U_t)$, we obtain

$$Y_t = Y_0 + \sum_{\alpha=1}^{l} P_\alpha^* f(U_s) \circ dX_s^\alpha.$$

A not so difficult calculation shows that

$$P_\alpha^* f(u) = P_\alpha(\pi u).$$

Hence

$$Y_t = Y_0 + \sum_{\alpha=1}^{l} P_\alpha(Y_s) \circ dX_s^\alpha.$$

Therefore Y satisfies the same stochastic differential equation as X (see (3.3.2)). By the uniqueness of solutions we must have $X = Y = \pi U$; namely, U is a horizontal lift of X. □

Once we have found the horizontal lift U of X, it is not hard to write down the anti-development of X:

$$W_t = \int_0^t U_s^{-1} P_\alpha(U_s) \circ dX_s^\alpha.$$

It can be verified easily that this $\mathbb{R}^l$-valued semimartingale drives an equation for U, namely,

$$dU_t = \sum_{i=1}^{n} H_i(U_t) \circ dW_t^i.$$

Furthermore, if X is a Brownian motion on M, then W is a Brownian motion on $\mathbb{R}^n$. This can be verified using Lévy's criterion.

The correspondences

$$W \longleftrightarrow U \longleftrightarrow X$$

are very useful because it converts a manifold-valued process X into a euclidean space valued process W, which is much easier to handle. We emphasize two points: (1) these correspondences are valid for any M-valued semimartingale X; (2) they depend on the connection we have used to define horizontal lift for vectors. Here we used the Riemannian connection, but the whole construction can be carried out for any affine connection ∇. In this case the orthonormal frame bundle $\mathscr{O}(M)$ should be replaced by the general linear frame bundle $F(M)$, otherwise everything remains the same.

3.4. Stochastic integrals on a manifold

As applications of the concepts of stochastic horizontal lift and anti-development, we define some stochastic integrals on a manifold.

Let X be a semimartingale and U and X its horizontal lift and anti-development, respectively. Let θ a 1-form on M. The stochastic line integral of θ along $X[0,t]$ is defined by

$$\int_{X[0,t]} \theta = \sum_{i=1}^{n} \int_0^t \theta(U_s e_i) \circ dW_s^i.$$

It can be verified that this definition is independent of the choice of the connection. It is possible to write some other definitions in which the connection does not show up. For example, if M is a submanifold of $\mathbb{R}^l$, then we have

$$\int_{X[0,t]} = \sum_{\alpha=1}^{l} \int_0^t \theta(P_\alpha)(X) \circ dX_t^\alpha.$$

If $\theta = df$ is an exact 1-form, then

$$\int_{X[0,t]} \theta = f(X_t) - f(X_0).$$

Another interesting fact is that the stochastic anti-development W itself is a stochastic line integral. Let Θ be the $\mathbb{R}^n$-valued solder form on $\mathscr{O}(M)$, i.e.,

$$\Theta(Z)(u) = u^{-1}\pi_* Z, \qquad Z \in \Gamma(\mathscr{O}(M)).$$

We have

$$W_t = \int_{U[0,t]} \Theta.$$

Let h be a (0,2)-tensor on M. The h-quadratic variation of X is defined by

$$\int_0^t h(dX_s, dX_s) = \int_0^t h(U_s e_i, U_s e_j) d\langle W^i, W^j\rangle_s.$$

Again we have

$$\int_0^t h(dX_s, dX_s) = \int_0^t h(P_\alpha, P_\beta)(X_s)\, d\langle X^i, X^j\rangle_s,$$

so the definition is independent of the choice of the connection.

If we take $h = df_1 \otimes df_2$ for some smooth functions f_1, f_2 on M, then

$$\int_0^t (df_1 \otimes df_2)(dX_s, dX_s) == \langle f_1(X), f_2(X)\rangle_t.$$

These concepts are useful in the study of manifold-valued martingales.

Lecture 4. Analysis on Path and Loop Spaces

Let $P_o(M) = C_o([0,1], M)$ be the space of continuous functions from $[0,1$ to M starting from a fixed point $o \in M$. The loop space is $L_o(M) = \{\gamma \in P_o(M) : \gamma(1) = o\}$. These are typical examples of infinite-dimensional space. We want to do analysis on these space. The measure we use for $P_o(M)$ is the Wiener measure ν, the law of Brownian motion on M starting from o. For the loop space $L_o(M)$, we use the law ν_o of Brownian bridge based at o. To do analysis, we need the concept of a gradient operator. Due to time limit, we will only discuss the case of the flat path space $P_o(\mathbb{R}^n)$. For generalization of the results discuss here to a general Riemannian manifold see [**11**].

4.1. Quasi-invariance of the Wiener measure

If an $h \in P_o(\mathbb{R}^n)$ is absolutely continuous and $\dot{h} \in L^2(I;\mathbb{R}^n)$ we define

$$|h|_{\mathscr{H}} = \sqrt{\int_0^1 |\dot{h}_s|^2 ds}\,;$$

otherwise we set $|h|_{\mathscr{H}} = \infty$. The ($\mathbb{R}^n$-valued) Cameron-Martin space is

$$\mathscr{H} = \{h \in P_o(\mathbb{R}^n) : |h|_{\mathscr{H}} < \infty\}\,.$$

THEOREM 4.1.1. *(Cameron-Martin-Maruyama) Let $h \in \mathscr{H}$ and*

$$\xi_h \omega = \omega + h, \qquad \omega \in P_o(\mathbb{R}^n)$$

a Cameron-Martin shift on the path space. Then the shifted Wiener measure $\mu^h = \mu \circ (\xi_h)^{-1}$ is absolutely continuous with respect to μ and

$$\frac{d\mu^h}{d\mu}(\omega) = \exp\left[\langle h_s, \omega\rangle_{\mathscr{H}} - \frac{1}{2}|h|^2_{\mathscr{H}}\right]. \tag{4.1.1}$$

Here

$$\langle h, \omega\rangle_{\mathscr{H}} = \int_0^1 \langle \dot{h}_s, d\omega_s\rangle.$$

Cameron-Martin shifts are the only shift which preserves the measure class of the Wiener measure. More precisely we have the following converse of the above theorem.

THEOREM 4.1.2. *Let $h \in P_o(\mathbb{R}^n)$, and let*

$$\xi_h \omega = \omega + h, \qquad \omega \in P_o(\mathbb{R}^n)$$

be the shift on the path space by h. Let μ be the Wiener measure on $P_o(\mathbb{R}^d)$. If the shifted Wiener measure $\mu^h = \mu \circ (\xi_h)^{-1}$ is absolutely continuous with respect to μ, then $h \in \mathscr{H}$.

PROOF. We show that if $h \notin \mathscr{H}$, then the measures μ and μ^h are mutually singluar, i.e., there is a set A such that $\mu A = 1$ and $\mu^h A = 0$. Let

$$\langle f, g \rangle_{\mathscr{H}} = \int_0^1 \dot{f}_s dg_s,$$

whenever the integral is well defined. If $f \in \mathscr{H}$ such that $\dot{f}$ is a step function on $[0, 1]$:

$$\dot{f} = \sum_{i=0}^{l-1} f_i I_{[s_i, s_{i+1})},$$

where $f_i \in \mathbb{R}^n$ and $0 = s_0 < s_1 < \cdots < s_l = 1$, then

$$\langle f, h \rangle_{\mathscr{H}} = \sum_{i=0}^{l-1} f_i \left(h_{s_{i+1}} - h_{s_i} \right)$$

is well defined. It is an easy exercise to show that if there is a constant C such that

$$\langle f, h \rangle_{\mathscr{H}} \le C |f|_{\mathscr{H}}$$

for all step functions $\dot{f}$, then h is absolutely continuous and $\dot{h}$ is square-integrable, namely, $h \in \mathscr{H}$.

Suppose that $h \notin \mathscr{H}$. Then there is a sequence $\{f_l\}$ such that

$$|f_l|_{\mathscr{H}} = 1 \quad \text{and} \quad \langle h, f_l \rangle_{\mathscr{H}} \ge 2l.$$

Let W be the coordinate process on $P_o(\mathbb{R}^d)$. Then it is a Brownian motion under μ and the stochastic integral

$$\langle f_l, W \rangle_{\mathscr{H}} = \int_0^1 \left\langle \dot{f}_{l,s}, dW_s \right\rangle$$

is well defined. Let

$$A_l = \{\langle f_l, W \rangle_{\mathscr{H}} \le l\}$$

and $A = \limsup_{l \to \infty} A_l$. Since $|f_l|_{\mathscr{H}} = 1$, the random variable $\langle f_l, W \rangle_{\mathscr{H}}$ is standard Gaussian under μ; hence

$$\mu A_l \ge 1 - e^{-l^2/2}.$$

This shows that $\mu A = 1$. On the other hand,

$$\mu^h A_l = \mu \{\langle f_l, W + h \rangle_{\mathscr{H}} \le l\} \le \mu \{\langle f_l, W \rangle_{\mathscr{H}} \le -l\}.$$

Hence $\mu^h A_l \le e^{-l^2/2}$ and $\mu^h A = 0$. Therefore μ and μ^h are mutually singular. $\square$

The above quasi-invariance result can be carried over to the flat loop space. Let

$$\mathscr{H}_o = \{h \in \mathscr{H} : h(1) = 0\}.$$

We will show that the Wiener measure μ_o on $L_o(\mathbb{R}^n)$ is quasi-invariant under the Cameron-Martin shift $\xi^h : L_o(\mathbb{R}^n) \to L_o(\mathbb{R}^n)$ for $h \in \mathscr{H}_o$.

Let $\{\omega_s\}$ be the coordinate process on the space $P_o(\mathbb{R}^n)$ and consider the stochastic differential equation for Brownian bridge

$$d\gamma_s = d\omega_s - \frac{\gamma_s ds}{1-s}, \quad \gamma_0 = o.$$

The assignment $J\omega = \gamma$ defines a measurable map $J : P_o(\mathbb{R}^n) \to L_o(\mathbb{R}^n)$. The map J can also be viewed as an $L_o(\mathbb{R}^n)$-valued random variable. Suppose that $h \in \mathscr{H}_o$. A simple computation shows that

$$d\{\gamma_s + h_s\} = d\{\omega_s + k_s\} - \frac{\gamma_s + h_s}{1-s} ds,$$

where

$$k_s = h_s + \int_0^s \frac{h_\tau}{1-\tau} d\tau.$$

This shows that through the map J, the shift $\gamma \mapsto \gamma + h$ in the loop space $L_o(\mathbb{R}^n)$ is equivalent to a shift $\omega \mapsto \omega + k$ in the path space $P_o(\mathbb{R}^n)$. The following lemma shows that the latter is a Cameron-Martin shift.

LEMMA 4.1.3. *(Hardy's inequality)*

$$\int_0^1 \left|\frac{h_s}{1-s}\right|^2 ds \le 4 \int_0^1 |\dot h_s|^2 ds.$$

PROOF. We have for any $t \in (0,1)$,

$$\begin{aligned}\int_0^t \left|\frac{h_s}{1-s}\right|^2 ds =& \int_0^t |h_s|^2 d\left[\frac{1}{1-s}\right] \\ =& 2\int_0^t \frac{h_s \cdot \dot h_s}{1-s} ds + \frac{|h_t|^2}{1-t} \\ \le& \frac{1}{2}\int_0^t \left|\frac{h_s}{1-s}\right|^2 ds + 2\int_0^t |\dot h_s|^2 ds + \frac{|h_t|^2}{1-t}.\end{aligned}$$

In the last step we have used inequality

$$2ab \le \frac{1}{2}a^2 + 2b^2.$$

Therefore

$$\int_0^t \left|\frac{h_s}{1-s}\right|^2 ds \le 4\int_0^t |\dot h_s|^2 ds + \frac{2|h_t|^2}{1-t}.$$

The desired inequality follows by letting $t \to 1$ in the above inequality because

$$\frac{|h_t|^2}{1-t} = \frac{1}{1-t}\left|\int_t^1 \dot h_s ds\right|^2 \le \int_t^1 |\dot h_s|^2 ds \to 0.$$

□

The lemma implies that $k \in \mathscr{H}$. Define the exponential martingale

$$e_s = \exp\left[\int_0^s \langle \dot{k}_\tau, d\omega_\tau\rangle - \frac{1}{2}\int_0^s |\dot{k}_\tau|^2 d\tau\right].$$

Let μ^k be the probability measure on $P_o(\mathbb{R}^n)$ defined by

$$\frac{d\mu^k}{d\mu} = e_1. \tag{4.1.2}$$

By the Cameron-Martin-Maruyama theorem, μ^k is the law of $\omega + k$. Since it is absolutely continuous with respect to μ, the random variable $\omega \mapsto J(\omega+k)$ is well-defined and

$$J(\omega + k) = \gamma + h. \tag{4.1.3}$$

Let μ_o^h be the law of the shifted Brownian bridge $\gamma + h$. Then

$$\begin{aligned}\mu_o^h(C) =& \mu_o(C-h) = \mu(J^{-1}C - k)\\ =& \mu^k(J^{-1}C) = \mu(e_1; J^{-1}C)\\ =& \mu_o(e_1 \circ J; C),\end{aligned}$$

where we have used (4.1.3) and (4.1.2) in the second and the fourth steps, respectively. Now it is clear that μ_o^h and μ_o are mutually equivalent on $L_o(\mathbb{R}^n)$ and

$$\frac{d\mu_o^h}{d\mu_o} = e_1 \circ J. \tag{4.1.4}$$

Finally it is easy to verify that

$$\int_0^1 \langle \dot{k}_s, d\omega_s\rangle = \int_0^1 \langle \dot{h}_s, d\gamma_s\rangle, \qquad \int_0^1 |\dot{k}_s|^2 ds = \int_0^1 |\dot{h}_s|^2 ds. \tag{4.1.5}$$

This means that

$$e_1(J\gamma) = e_1(\omega) = \exp\left[\int_0^1 \langle \dot{h}_s, d\gamma_s\rangle - \frac{1}{2}\int_0^1 |\dot{h}_s|^2 ds\right].$$

We have proved the following result.

Theorem 4.1.4. *Let $h \in \mathscr{H}_o$ and $\xi^h\gamma = \gamma + h$ for $\gamma \in L_o(\mathbb{R}^n)$. Then the shifted Wiener measure $\mu_o^h = \mu_o \circ (\xi^h)^{-1}$ on the loop space $L_o(\mathbb{R}^n)$ is equivalent to μ_o and*

$$\frac{d\mu_o^h}{d\mu_o} = \exp\left[\int_0^1 \langle \dot{h}_s, d\gamma_s\rangle - \frac{1}{2}\int_0^1 |\dot{h}_s|^2 ds\right].$$

4.2. Gradient operator

We now define the gradient operator in the path space $P_o(\mathbb{R}^n)$. By analogy with finite dimensional space, each element $h \in P_o(\mathbb{R}^n)$ represents a direction along which one can differentiate a nice functions F on $P_o(\mathbb{R}^n)$. Naturally the directional derivative of F along h should be defined by the formula

$$D_h F(\omega) = \lim_{t\to 0} \frac{F(\omega + th) - F(\omega)}{t} \tag{4.2.1}$$

if the limit exists in some sense. The preliminary class of functions on $P_o(M)$ for which the above definition of $D_h F$ makes immediate sense is that of cylinder functions.

DEFINITION 4.2.1. *Let $\mathbb{E}$ be a Banach space. A function $F : P_o(\mathbb{R}^n) \to \mathbb{E}$ is called an $\mathbb{E}$-valued cylinder function if it has the form*

$$F(\omega) = f(\omega_{s_1}, \cdots, \omega_{s_l}), \tag{4.2.2}$$

where $0 < s_1 < \cdots < s_l \le 1$ and f is an $\mathbb{E}$-valued smooth function on $(\mathbb{R}^n)^l$ such that all its derivatives have at most polynomial growth. The set of $\mathbb{E}$-valued cylinder functions is denoted by $\mathcal{C}(\mathbb{E})$. Typically $\mathbb{E} = \mathbb{R}^1, \mathbb{R}^n$, or $\mathscr{H}$. We denote $\mathcal{C}(\mathbb{R}^1)$ simply by $\mathcal{C}$.

If $F \in \mathcal{C}$ is given by (4.2.2), then it is clear that the limit (4.2.1) exists everywhere and we have

$$D_h F(\omega) = \sum_{i=1}^{l} \langle \nabla^i F(\omega), h_{s_i} \rangle_{\mathbb{R}^n}, \tag{4.2.3}$$

where

$$\nabla^i F(\omega) = \nabla^i f(\omega_{s_1}, \cdots, \omega_{s_l}).$$

Here $\nabla^i f$ denotes the gradient of f with respect to the ith variable.

It is natural to define the gradient DF of a function $F \in \mathcal{C}$ to be an $\mathscr{H}$-valued functions on $P_o(\mathbb{R}^n)$ such that

$$\langle DF(\omega), h \rangle_{\mathscr{H}} = D_h F(\omega).$$

A simple calculation shows that

$$DF(\omega)_s = \sum_{i=1}^{l} \min(s, s_i) \nabla^i F(\omega) \tag{4.2.4}$$

and

$$|DF(\omega)|^2_{\mathscr{H}} = \sum_{i=1}^{l} (s_i - s_{i-1}) \left| \sum_{j=i}^{l} \nabla^j F(\omega) \right|^2. \tag{4.2.5}$$

4.3. Integration by parts

We will use the notation

$$(F,G) = \int_{P_o(\mathbb{R}^n)} F(\omega)G(\omega)\mu(d\omega).$$

THEOREM 4.3.1. *Let $F, G \in \mathcal{C}$ and $h \in \mathscr{H}$. Then*

$$(D_h F, G) = (F, D_h^* G), \tag{4.3.1}$$

where

$$D_h^* = -D_h + \int_0^1 \langle \dot{h}_s, d\omega_s \rangle.$$

PROOF. Let $\xi^{th}\omega = \omega + th$ and $\mu^{th} = \mu \circ \left(\xi^{th}\right)^{-1}$. Then μ^{th} and μ are mutually absolutely continuous. We have

$$\int (F \circ \xi^{th}) G d\mu = \int F(G \circ \xi^{-th}) d\mu^{th} = \int F(G \circ \xi^{-th}) \frac{d\mu^{th}}{d\mu} d\mu.$$

We differentiate with respect to t and set $t = 0$. Using the formula for the Radon-Nikodym derivative (4.1.1) in the Cameron-Martin-Maruyama theorem we have at $t = 0$

$$\frac{d}{dt}\left\{\frac{d\mu^{th}}{d\mu}\right\} = \int_0^1 \langle \dot{h}_s, d\omega_s \rangle = \langle h, \omega \rangle_{\mathscr{H}}.$$

The formula follows immediately. $\square$

To understand the integration by parts formula better, let's look at its finite dimensional analog and find out the proper replacement for the stochastic integral in D_h^*. Let $h \in \mathbb{R}^N$ and consider the differential operator

$$D_h = \sum_{i=1}^N h^i \frac{d}{dx^i}.$$

Let μ be the Gaussian measure on $\mathbb{R}^N$, i.e.,

$$\frac{d\mu}{dx} = \left(\frac{1}{2\pi}\right)^{N/2} e^{-|x|^2/2}.$$

[dx is the Lebesgue measure.] For smooth functions F, G on $\mathbb{R}^N$ with compact support we have by the usual integration by parts for the Lebesgue measure

$$(D_h F, G) = (F, D_h^* G), \tag{4.3.2}$$

where $D_h^* = -D_h + \langle h, x \rangle$ at $x \in \mathbb{R}^N$.

REMARK 4.3.2. Since D_h is a derivation we have

$$D_h^* G = -D_h G + (D_h^* 1) G.$$

Therefore to find the formal adjoint of D_h it is enough compute $D_h^* 1$, which is denoted by $\operatorname{div}(D_h)$ or $\delta(h)$ by various authors. We have

$$D_h^* 1(\omega) = \int_0^1 \langle \dot{h}_s, d\omega_s \rangle.$$

An integration by parts formula for the gradient operator D can be obtained as follows. Fix an orthonormal basis $\{h^j\}$ for $\mathscr{H}$. Denote by $\mathcal{C}_0(\mathscr{H})$ the set of $\mathscr{H}$-valued functions G of the form $G = \sum_j G_j h^j$, where each $G_j \in \mathcal{C}$ and almost all of them are equal to zero. It is easy to check that $\mathcal{C}_0(\mathscr{H})$ is dense in $L^p(\mu; \mathscr{H})$ for all $p \in [1, \infty)$.

Since $DF = \sum_j (D_{h^j} F) h^j$ in $L^2(\mu; \mathscr{H})$, we have

$$(DF, G) = \sum_j (D_{h^j} F, G_j) = \sum_j (F, D_{h^j}^* G_j).$$

The assumption that $G \in \mathcal{C}_o(\mathscr{H})$ means that the sums are finite. Let

$$D^* G = \sum_{j=0}^{\infty} D_{h^j}^* G_j = -\sum_{j=0}^{\infty} D_{h^j} G_j + \sum_{j=0}^{\infty} G_j \int_0^1 \langle \dot{h}_s^j, d\omega_s \rangle.$$

We rewrite this formula in a more compact form. If

$$J = \sum_{j,k} J_{jk} h^j \otimes h^k$$

is an $\mathscr{H} \otimes_{\mathbb{R}} \mathscr{H}$-valued function, we write

$$\operatorname{Trace} J = \sum_{j=0}^{\infty} J_{jj}.$$

For $G = \sum_k G_k h^j$ we define its gradient to be

$$DG = \sum_{j,k} (D_{h^j} G_k) h^j \otimes h^k.$$

Then it is clear that

$$\operatorname{Trace} DG = \sum_j D_{h^j} G_j.$$

For $G = \sum_j G_j h^j$ we define

$$\int_0^1 \langle \dot{G}_s, d\omega_s \rangle = \sum_j G_j \int_0^1 \langle \dot{h}_s^j, d\omega_s \rangle.$$

[This is a term-by-term integration with respect to a specific basis for $\mathscr{H}$, not anticipative stochastic integral!] Then we can write

$$D^* G = -\operatorname{Trace} DG + \int_0^1 \langle \dot{G}_s, d\omega_s \rangle. \tag{4.3.3}$$

THEOREM 4.3.3. *Let* $F \in \mathcal{C}$ *and* $G \in \mathcal{C}_0(\mathscr{H})$. *Then*

$$(DF, G) = (F, D^*G),$$

where D^*G *is given by* (4.3.3).

The Gradient operator can also be defined on the loop space $L_o(\mathbb{R}^n)$. Its integration by parts formula takes the same form as in the path space.

4.4. Ornstein Uhlenbeck operator

Let $\mathrm{Dom}(\mathcal{E}) = \mathrm{Dom}(D)$ and define the positive symmetric quadratic form $\mathcal{E} : \mathrm{Dom}(\mathcal{E}) \times \mathrm{Dom}(\mathcal{E}) \to \mathbb{R}$ by

$$\mathcal{E}(F, F) = (DF, DF)_{L^2(\mu;\mathscr{H})} = E|DF|^2_{\mathscr{H}}.$$

Then $(\mathcal{E}, D(\mathcal{E}))$ is a closed quadratic form, i.e., $\mathrm{Dom}(\mathcal{E})$ is complete with respect to the inner product

$$\mathcal{E}_1(F, F) = \mathcal{E}(F, F) + (F, F).$$

Furthermore the pair $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ is a Dirichlet form; see [**9**].

By general theory of closed symmetric forms (Fukushima[**9**], 17-19), there exists a non-positive self-adjoint operator L such that $\mathrm{Dom}(\mathcal{E}) = \mathrm{Dom}(\sqrt{-L})$ and

$$\mathcal{E}(F, F) = (\sqrt{-L}F, \sqrt{-L}F). \tag{4.4.1}$$

L is called the Ornstein-Uhlenbeck operator on the path space $P_o(\mathbb{R}^n)$. In fact we have $L = -D^*D$, where D is the gradient operator and D^* its adjoint.

The Ornstein-Uhlenbeck operator is an infinite-dimensional generalization of the usual Ornstein-Uhlenbeck operator on $\mathbb{R}^1$:

$$L = -D^*D = \frac{d^2}{dx^2} - x\frac{d}{dx},$$

where

$$D = \frac{d}{dx}, \qquad D^* = -\frac{d}{dx}.$$

D^* is the adjoint of D with respect to the standard Gaussian measure μ on $\mathbb{R}^1$:

$$\frac{d\mu}{dx} = \frac{e^{-x^2/2}}{\sqrt{2\pi}}.$$

It is a classical result (see [**5**]) that the Hermite polynomials

$$H_N(x) = \frac{(-1)^N}{\sqrt{N!}}\, e^{x^2/2}\frac{d^N}{dx^N}\, e^{-x^2/2}, \qquad N \in \mathbb{Z}_+$$

form a complete set of L^2-eigenfunctions for the self-adjoint operator L on $L^2(\mathbb{R}^1, \mu)$:

$$LH_N = -NH_N, \qquad N \in \mathbb{Z}_+.$$

Returning to the path space, for simplicity we will assume in the following that the base space has dimension $n = 1$. Let $\{h^i\}$ be an orthonormal basis for $\mathscr{H}$. Then the $\{\langle h^i, \omega\rangle_{\mathbb{H}}\}$ is i.i.d. sequence with standard Gaussian distribution. Hence the map $T : \omega \mapsto \{\langle h^i, \omega\rangle_{\mathbb{H}}\}$ is an isometry (measure-preserving map) between the two measure spaces ($\tilde{\mu}$ is the Gaussian measure on $\mathbb{R}^{\mathbb{Z}_+}$). With this isometry in mind, the following construction is in order.

Let $\mathcal{I}$ denote the set of indices $I = \{n_i\}$ such that $n_i \in \mathbb{Z}_+$ and almost all of them are equal to zero. Denote $|I| = n_1 + n_2 + \cdots$. For $I \in \mathcal{I}$ define

$$H_I(\omega) = \prod_i H_{n_i}(\langle h^i, \omega\rangle_{\mathscr{H}}).$$

Then the fact that the Hermite polynomials $\{H_N, N \in \mathbb{Z}_+\}$ form an orthonormal basis for $L^2(\mathbb{R}, \mu)$ implies immediately that $\{H_I, I \in \mathcal{I}\}$ is an orthonormal basis for $L^2(P_o(\mathbb{R}), \mu)$. Moreover, the eigenspace of L for the eigenvalue N is

$$C_N = \text{the linear span of } \{H_I : |I| = N\}$$

and

$$L^2(P_o(\mathbb{R}), \mu) = C_0 \oplus C_1 \oplus C_2 \oplus \cdots .$$

This is the Wiener chaos decomposition. The following theorem completely describes the spectrum $\mathrm{Spec}(-L)$ of $-L$.

THEOREM 4.4.1. $\mathrm{Spec}(-L) = \mathbb{Z}_+$ *and* C_N *is the eigenspace for the eigenvalue* N*. Let* $P_N : L^2(P_o(\mathbb{R}), \mu) \to C_N$ *be the orthogonal projection to* C_N*. Then*

$$LF = -\sum_{N=0}^{\infty} N P_N F.$$

Note that all eigenspaces are infinite dimensional except for $C_0 = \mathbb{R}$*.*

Let $\mathscr{P}_t = e^{tL/2}$ in the sense of spectral theory. The Ornstein-Uhlenbeck semigroup $\{\mathscr{P}_t\}$ is the strongly continuous L^2-semigroup generated by the Ornstein-Uhlenbeck operator. Clearly,

$$\mathscr{P}_t F = e^{-Nt/2} F, \qquad F \in C_N.$$

Each $\mathscr{P}_t$ is a conservative, L^2-contraction, i.e., $P_t 1 = 1$ and $\|P_t F\|_2 \le \|F\|_2$.

PROPOSITION 4.4.2. *The semigroup* $\{\mathscr{P}_t\}$ *is positive, namely* $\mathscr{P}_t F \ge 0$ *if* $F \ge 0$*. For each* $p \in [1, \infty]$*, the Ornstein-Uhlenbeck semigroup* $\{\mathscr{P}_t\}$ *is a positive, conservative, and contractive* L^p*-semigroup.*

PROOF. The positivity follows from the fact that the semigroup comes from a Dirichlet form, see [**9**], 22-24. The L^p-contraction follows from the positivity and L^2-contraction. □

4.5. Logarithmic Sobolev inequality

Infinite dimensional analysis, Sobolev inequalities in general do not hold. In their stead, under certain conditions, we can prove a weaker inequality called logarithmic Sobolev inequality. In its form presented here the inequality is due to E. Nelson.

THEOREM 4.5.1. *Let μ be the standard Gaussian measure on $\mathbb{R}^N$ and ∇ the usual gradient operator. Suppose that f is a smooth function on $\mathbb{R}^N$ such that both f and ∇f have at most polynomial growth. Then we have*

$$\int_{\mathbb{R}^N} |f|^2 \log |f| d\mu \le \int_{\mathbb{R}^N} |\nabla f|^2 d\mu + \|f\|_2^2 \log \|f\|_2. \tag{4.5.1}$$

Here $\|f\|_2$ is the norm of f in $L^2(\mathbb{R}^N, \mu)$.

PROOF. For a positive s let μ_s be the Gaussian measure

$$\mu_s(dx) = \left(\frac{1}{2\pi s}\right)^{l/2} e^{-|x|^2/2s} dx,$$

where dx denotes the Lebesgue measure. Then $\mu = \mu_1$. Let $g = f^2$ and

$$P_s g(x) = \int_{\mathbb{R}^l} g(x-y)\mu_s(dy).$$

Consider the function $H_s = P_s\phi(P_{1-s}g)$, where $\phi(t) = 2^{-1}t\log t$. Differentiating with respect to s and noting that Δ commutes with P_s we have

$$\begin{aligned}
\frac{dH_s}{ds} &= \frac{1}{2}P_s\Delta\phi(P_{1-s}g) - \frac{1}{2}P_s\left\{\phi'(P_{1-s}g)\Delta P_{1-s}g\right\} \\
&= \frac{1}{2}P_s\left\{\phi'(P_{1-s}g)\Delta P_{1-s}g + \phi''(P_{1-s}g)\left|\nabla P_{1-s}g\right|^2\right\} \\
&\quad -\frac{1}{2}P_s\left\{\phi'(P_{1-s}g)\Delta P_{1-s}g\right\} \\
&= \frac{1}{2}P_s\left\{\phi''(P_{1-s}g)\left|\nabla P_{1-s}g\right|^2\right\} \\
&\le \frac{1}{4}P_s\left\{\frac{(P_{1-s}|\nabla g|)^2}{P_{1-s}g}\right\} \\
&\le P_s\left\{P_{1-s}|\nabla f|^2\right\} \\
&= P_1|\nabla f|^2.
\end{aligned}$$

Here we have used the fact that $|\nabla P_{1-s}g| \le P_{1-s}|\nabla g|$ in the fourth step and the inequality

$$(P_{s-r}|\nabla g|)^2 \le 4P_{s-r}gP_{s-r}|\nabla f|^2$$

in the fifth step, the latter being a consequence of the Cauchy-Schwarz inequality. Now integrating from 0 to 1 we obtain the desired result immediately. □

Translating the finite dimensional logarithmic Sobolev inequality in Theorem 4.5.1 to the path space $P_o(\mathbb{R}^n)$ we obtain the following result.

THEOREM 4.5.2. *If $F \in Dom(D)$, then we have*

$$\int_{P_o(\mathbb{R}^n)} |F|^2 \log |F| d\mu \le \int_{P_o(\mathbb{R}^n)} |DF|_{\mathscr{H}}^2 d\mu + \|F\|_2^2 \log \|F\|_2. \tag{4.5.2}$$

PROOF. We assume $n = 1$ for simplicity. The set of functions of the form

$$F(\omega) = f(\langle h_1, \omega\rangle_{\mathscr{H}}, \cdots, \langle h_0, \omega\rangle_{\mathscr{H}})$$

is dense in Dom(D). For such an F, we have

$$DF(\omega) = \sum_{i=0}^{l} f_{x_i}(\omega) h^i.$$

It is then clear that

$$|DF(\omega)|_{\mathscr{H}} = |\nabla f(\langle h^0, \omega\rangle_{\mathscr{H}}, \cdots, \langle h^l, \omega\rangle_{\mathscr{H}})|.$$

On the other hand, the distribution of $\left\{\langle h^0, \omega\rangle_{\mathscr{H}}, \cdots, \langle h^l, \omega\rangle_{\mathscr{H}}\right\}$ is the standard Gaussian measure on $\mathbb{R}^{l+1}$. Therefore (4.5.2) reduces to (4.5.1). □

There is a general result due to L. Gross which says that a logarithmic Sobolev inequality is equivalent to an hypercontractivity property of the corresponding semigroup.

THEOREM 4.5.3. *Let $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ be a Dirichlet form on a probability space $(X, \mathscr{B}, \mu)$ and $\{\mathscr{P}_t\}$ be the associated semigroup. The following statements are equivalent for a positive constant C .*

(I) Hypercontractivity for $\{\mathscr{P}_t\}$: $\|\mathscr{P}_t\|_{q,p} = 1$ for all (t, p, q) such that $t > 0, 1 < p < q$ and

$$e^{t/C} \geq \frac{q-1}{p-1}.$$

(II) The logarithmic Sobolev inequality for $\mathcal{E}$:

$$\mathbb{E}\left(F^2 \ln F^2\right) \leq 2C\mathcal{E}(F, F) + \mathbb{E}\, F^2\, \ln \mathbb{E}\, F^2.$$

PROOF. See [**7**] or [**11**]. □

THEOREM 4.5.4. *The Ornstein-Uhlenbeck semigroup $\{\mathscr{P}_t\}$ is hypercontractive. More precisely $\|\mathscr{P}_t\|_{q,p} = 1$ for all (t, p, q) such that $t > 0, 1 < p < q$ and*

$$e^t \geq \frac{q-1}{p-1}.$$

4.6. Concluding remarks

For a compact Riemannian manifold, a logarithmic Sobolev inequality for the gradient operator on the path space is known [**11**]. In the proof of the logarithmic Sobolev inequality presented here, we have taken advantage of the Gaussian structure of the underlying linear Gaussian structure. For a general manifold a completely different approach is needed. In general a logarithmic inequality implies the existence of a positive spectrum gap. In the flat case we simply verify this fact by direct computation. A parallel (or maybe not so parallel) theory for a manifold with boundary is a current area of active research.

For the flat loop space $L_o(\mathbb{R}^n)$ the theory is the same as the flat path space because from the Gaussian point of view, the path and loop spaces are isometric. For a general loop space $L_o(M)$, where M is a compact Riemannian manifold, S. Aida proved that the 0-eigenspace of the Ornstein-Uhlenbeck operator L is simple for each homotopy class. Even for simply connected M, A. Erbele proved that there is in general no positive spectral gap. It is believed that this anomaly is due to the presence of negative curvature on M. Therefore the current effort is aiming at proving the existence of a logarithmic Sobolev inequality for a compact, simply connected manifold of positive curvature.

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Lectures
Probability Summer School
Kyushu, August 19 to 22, 2003

Spatial and Harness processes,
random trees and Poisson approximation

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First lecture: Spatial processes. Construction and perfect simulation. We present a construction that is at the same time a perfect simulation algorithm for measures that are absolutely continuous with respect to some Poisson process and can be obtained as invariant measures of birth-and-death processes. Examples include area- and perimeter-interacting point processes, invariant measures of loss networks, and the Ising contour and random cluster models. It directly provides perfect samples of finite windows of the *infinite-volume* measure and it is based on a two-step procedure: (i) the construction of the (finite and random) relevant portion of a (space-time) marked Poisson processes (free birth-and-death process), and (ii) a "cleaning" algorithm that trims out this process according to the interaction rules of the target process. The first step involves the generation of "ancestors" of a given object, that is of predecessors that may have an influence on the birth-rate under the target process. The second step, and hence the whole procedure, is feasible if these "ancestors" form a finite set with probability one. We present a sufficiency criteria for this condition, based on the absence of infinite clusters for an associated (backwards) oriented percolation model.

This lecture is based on the papers math.PR/9806131 and math.PR/9911162 in collaboration with Fernandez and Garcia.

Second lecture: Harness processes and Gaussian massless random fields The state space of Hammersley harness process is R^{Z^d} and the time is continuous. At rate 1 the height at each site x of Z^d is substituted by an average of the heights of the neighboring sites plus a centered random variable. The average is taken with some given translation invariant stochastic matrix $P = (p(x, y))$ with $p(x, x) \equiv 0$. When the noise is

Gaussian with variance $1/\beta$, the dynamics turns out to be the Glauber dynamics for the Gaussian massless random field related to the Hamiltonian $H(\eta) = \sum p(x,y)(\eta(x) - \eta(y))^2$ at inverse temperature β. We propose a Harris graphical construction of the Harness process that allows to construct the equilibrium state as an almost sure limit of proceses "from the past", gives speed of convergence to equilibrium and almost sure thermodynamical limit for the Gaussian field. Some of the results can be exported to the non Gaussian case where the invariant measure is not explicitely known.

This lecture is based on joint work with Beat Niederhauser. There is no still paper available, but a draft of the slides can be found in here.

Third lecture: Poisson trees and Brownian webs We construct graphs whose vertices are the points of a homogeneous Poisson process. Under our construction in $d = 1, 2$ the resulting graph is a unique tree with finite branches. The result is then used to enumerate the points of the Poisson process in an "origin independent" way. In four or more dimensions our construction produces infinitely many trees. We also describe the convergence of the trees to the so called Brownian web, when the intensity of the Poisson process and the construction of the trees are suitable rescaled.

This lecture is based on the paper math.PR/0209395 in collaboration with Landim and Thorisson.

Fourth lecture: Poissonian approximation of a tagged particle The famous Burke's theorem of queuing theory says that Poisson arrivals to a stationary $M/M/1$ queue produces Poisson departures. We present an extension of the Burke's theorem to a family of zero range processes and then apply it to the problem of a tagged particle in asymmetric nearest neighbors simple exclusion process, as follows. We consider the position of a tagged particle in the one dimensional asymmetric nearest neighbors simple exclusion process. Each particle attempts to jump to the site to its right at rate p and to the site to its left at rate q. The jump is realized if the destination site is empty. We assume $p > q$. The initial distribution is the product measure with density λ, conditioned to have a particle at the origin. We call X_t the position at time t of this particle and construct in the same space X_t, N_t, a Poisson process of parameter $(p-q)(1-\lambda)$ and B_t, a stationary process satisfying $E\exp(\theta|B_t|) < \infty$ for all $\theta > 0$ satisfying

$$X_t = N_t - B_t + B_0$$

for all $t \geq 0$. As a corollary we obtain that —properly centered and rescaled— the process $\{X_t\}$ converges to Brownian motion.

This lecture is based on the papers ff1 and ff2 in collaboration with Fontes.

Fermion 測度とその周辺

白井 朋之 (金沢大・理学部)

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1 はじめに

１９５０年代に原子核（多体系）のスペクトルの統計的な研究のために Wigner がランダム行列を導入し，Wigner の半円則を証明した [45] のをきっかけにランダム行列の研究は物理と数学の両分野で多方面に拡がっている．数あるランダム行列の中でも特に GUE は深い数学的対象であることがわかってきている．GUE とは $N \times N$-Hermite 行列全体 $\mathcal{H}_N$ にガウス測度

$$P_N(dX) \propto \exp(-\operatorname{Tr}(X^2))dX \tag{1.1}$$

を入れたランダム行列である．Hermite 行列 X の行列成分のうち，上三角の非対角成分の実部と虚部 $\operatorname{Re} X_{ij}, \operatorname{Im} X_{ij}$ $(1 \le i < j \le N)$，対角成分 X_{ii} $(1 \le i \le N)$ の N^2 個が独立な変数なので，$\mathcal{H}_N$ は自然に $\mathbf{R}^{N^2}$ と見なせ，dX はそう見たときの $\mathcal{H}_N$ 上の Lebesgue 測度

$$dX = \prod_{i=1}^{N} dX_{ii} \times \prod_{1 \le i < j \le N} d(\operatorname{Re} X_{ij}) d(\operatorname{Im} X_{ij})$$

である．$\operatorname{Tr}(X^2)$ が各成分の２乗和であることに注意すれば，(1.1) は，各成分 $\operatorname{Re} X_{ij}, \operatorname{Im} X_{ij}$ $(1 \le i < j \le N)$，X_{ii} $(1 \le i \le N)$ が独立な 1 次元の Gauss 分布に従っていることを意味する．

$\mathcal{H}_N$ の元の N 個の実固有値 $x = (x_1, \ldots, x_N)$ の $\mathbf{R}^N$ における（対称化した）分布密度は，

$$\begin{aligned}
\mu_N(x) &= \mu_N(x_1, \ldots, x_N) \\
&= Z_N^{-1} \prod_{1 \le i < j \le N} (x_i - x_j)^2 \exp\left(-\sum_{i=1}^{N} x_i^2\right) \\
&= \det\left(K^{(N)}(x_i, x_j)\right)_{i,j=1}^{N}
\end{aligned}$$

という非常に特別な形になることが知られている [29]．ただし，

$$
\begin{aligned}
K^{(N)}(x,y) &= \sum_{k=0}^{N-1} \varphi_k(x)\varphi_k(y) \\
&= \left(\frac{N}{2}\right)^{1/2} \frac{\varphi_N(x)\varphi_{N-1}(y) - \varphi_{N-1}(x)\varphi_N(y)}{x-y}
\end{aligned}
$$

と定義される積分核である．また $\varphi_k(x)$ は正規化された Hermite 関数で，

$$
\varphi_k(x) = \frac{1}{(\sqrt{\pi}k!2^k)^{1/2}} H_k(x) e^{-x^2/2}
$$

によって与えられる．$H_k(x)$ は Hermite 多項式

$$
H_k(x) = (-1)^k e^{x^2} \frac{d^k}{dx^k} e^{-x^2}
$$

で重み $e^{-x^2}dx$ に関する直交多項式である．$K^{(N)}(x,y)$ についての二つ目の等式は直交多項式論で有名な Christoffel-Darboux の公式による [39]．

ここに与えられた確率密度関数の $N-n$ 個の変数を積分することにより n 点相関関数が得られる．n 点相関関数とは以下のようなものである．

$$
\rho_n^{(N)}(x_1,\ldots,x_n) = \frac{N!}{(N-n)!} \int_{\mathbf{R}^{N-n}} p_N(x_1,\ldots,x_N) dx_{n+1}\ldots dx_N \tag{1.2}
$$

によって定義する．この特別な形の確率密度関数 (相関関数) が後で見るように Fermion 測度の特徴であり，GUE の固有値分布は Fermion 測度となることがわかる．

Hermite 関数の漸近形は詳しく研究されており (cf. [20])，その結果を用いると $K^{(N)}(x,y)$ のスケーリング極限が計算される．

$\alpha_N = \frac{\pi}{\sqrt{2N}}$ とおくと，

$$
\lim_{N\to\infty} \alpha_N K^{(N)}(\alpha_N x, \alpha_N y) = \frac{\sin\pi(x-y)}{\pi(x-y)} =: K^{sine}(x,y)
$$

となり，$K^{sine}(x,y)$ は正弦核と呼ばれる．

また $\beta_N = \frac{1}{\sqrt{2}N^{1/6}}$ とおくと，

$$
\begin{aligned}
\lim_{N\to\infty} \beta_N K^{(N)}(\sqrt{2N}+\beta_N x, \sqrt{2N}+\beta_N y) &= \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x-y} \\
&= : K^{Airy}(x,y)
\end{aligned}
$$

となり，$K^{Airy}(x,y)$ は Airy 核と呼ばれる ([11, 42])．ただし，

$$
K^{Airy}(x,x) = \mathrm{Ai}'(x)^2 - \mathrm{Ai}''(x)\mathrm{Ai}(x) = \mathrm{Ai}'(x)^2 - x\mathrm{Ai}(x)^2.
$$

Γ 分布の行列版を考えよう．非負定値 Hermite 行列は必ず X^*X の形に書けることに注意して，$\alpha > -1$ とするとき非負定値行列の空間に

$$
\exp(-\mathrm{Tr}(X^*X))\det(X^*X)^\alpha dX \tag{1.3}
$$

に比例する確率分布を考えたものは Laguarre Ensemble とよばれる．ただし，dX は複素行列全体上の Lebesgue 測度．このとき X^*X の固有値は非負となりその分布は GUE と同様に行列式の形をもつ．

$$
\begin{aligned}
\mu_N(x) &= \mu_N(x_1,\ldots,x_N) \\
&= Z_N^{-1} \prod_{1\le i<j\le N} (x_i - x_j)^2 \prod_{i=1}^N x_i^\alpha \exp\left(-\sum_{i=1}^N x_i\right) \\
&= \det\left(K^{(N)}(x_i,x_j)\right)_{i,j=1}^N
\end{aligned}
$$

ここで積分核は $K^{(N)}(x,y)=\sum_{k=0}^{N-1}\psi_k^{(\alpha)}(x)\psi_k^{(\alpha)}(y)$ である．ただし，$\psi_k^{(\alpha)}(x)=L_k^{(\alpha)}(x)x^\alpha e^{-x}$ で $L_k^{(\alpha)}(x)$ は Laguarre 多項式である．つまり，$(0,\infty)$ における重み $x^\alpha e^{-x}dx$ に関する直交多項式で

$$L_k^{(\alpha)}(x)=\frac{x^{-\alpha}e^x}{k!}\frac{d^k}{dx^k}x^\alpha e^{-x}$$

で与えられる．$\gamma_N=\frac{1}{4N}$ とおくと

$$\begin{aligned}\lim_{N\to\infty}\gamma_N K^{(N)}(\gamma_N x,\gamma_N y) &= \frac{J_\alpha(\sqrt{x})\sqrt{y}J'_\alpha(\sqrt{y})-\sqrt{x}J'_\alpha(\sqrt{x})J_\alpha(\sqrt{y})}{2(x-y)}\\ &= :K^{Bessel}(x,y)\end{aligned}$$

となる．ここで J_α は Bessel 関数である．$K^{Bessel}(x,y)$ は Bessel 核と呼ばれる (cf. [42])．

注意 *1.1.* これらの積分核 $K^{(N)},K_{sine},K_{Airy},K_{Bessel}$ などから定まる積分作用素はすべて射影作用素になっていることに注意する．

本講演ではまず Fermion 測度の性質について (特に離散の場合に) 詳しく述べた後，その類似物である α-Boson 測度について考えたい．またこの他 Wishart 過程や表現論と α-Boson 測度との関係や，Tracy-Widom 分布と Airy 過程などについても考えたい．

2 離散 Fermion 測度 (Determinantal 測度)

2.1 アイディア

まず Fermion 測度のアイディアを簡単に説明しよう．

二つの行列

$$K=\begin{pmatrix}a & b\\ c & d\end{pmatrix},\quad I-K=\begin{pmatrix}1-a & -b\\ -c & 1-d\end{pmatrix}$$

を用意して，$\{0,1\}^2$ 上の関数 $\mu(x_1x_2)$ を以下のように定める：

(1) x_1x_2 の値によって $K,I-K$ からあらたな行列 $K^{(x_1x_2)}$ をつくる．$K^{(x_1x_2)}$ の 1 行目は $x_1=1,0$ に従って $K,I-K$ の 1 行目とする．$K^{(x_1x_2)}$ の 2 行目は $x_2=1,0$ に従って $K,I-K$ の 2 行目とする．

(2) $\mu(x_1x_2)=\det(K^{(x_1x_2)})$ と定める．具体的には行列 $K^{(x_1x_2)}$ は

$$K^{(11)}=\begin{pmatrix}a & b\\ c & d\end{pmatrix},\quad K^{(10)}=\begin{pmatrix}a & b\\ -c & 1-d\end{pmatrix},\quad K^{(01)}=\begin{pmatrix}1-a & -b\\ c & d\end{pmatrix},\quad K^{(00)}=\begin{pmatrix}1-a & -b\\ -c & 1-d\end{pmatrix}$$

となり，

$$\begin{aligned}&\mu(11)=\det K^{(11)}=ad-bc,\quad \mu(10)=\det K^{(10)}=a(1-d)+bc,\\ &\mu(01)=\det K^{(01)}=(1-a)d+bc,\quad \mu(00)=\det K^{(00)}=(1-a)(1-d)-bc.\end{aligned}$$

と定める．行列式の多重線型性により $\mu(11)+\mu(10)=\begin{pmatrix}a & b\\ 0 & 1\end{pmatrix}=a$, $\mu(01)+\mu(00)=\begin{pmatrix}1-a & -b\\ 0 & 1\end{pmatrix}=1-a$ であるから

$$\mu(11)+\mu(10)+\mu(01)+\mu(00)=1$$

であることがわかる．またすべての $x_1x_2\in\{0,1\}^2$ に対して $\mu(x_1x_2)\geq 0$ となるための必要十分条件は，

$$\begin{cases}0\leq a,d\leq 1,\ bc\in\mathbf{R},\\ -\min((1-a)d,a(1-d))\leq bc\leq\min(ad,(1-a)(1-d))\end{cases}$$

である．つまり，K が上の条件を満たすとき, μ は集合 $\{0,1\}^2$ 上の確率分布を定める．このアイディアを一般の場合に拡張しよう．

2.2 離散 Fermion 測度

以下，R を高々可算な集合とし，配置空間を $Q=Q(R)=\{\xi : R\to\{0,1\}\}$ とし直積位相を入れて位相空間とみなす．ξ の台と R の部分集合との対応でしばしばべき集合 2^R と可算直積空間 Q とを同一視する．R を添字集合とする (無限次) 複素行列 $K=(K(x,y))_{x,y\in R}$ が以下の仮定を満たすとする．

仮定 2.1. 任意の互いに素な R の有限部分集合 Λ_0,Λ_1 に対して，

$$\det(P_{\Lambda_0}(I_\Lambda-K_\Lambda)+P_{\Lambda_1}K_\Lambda)=\det\begin{pmatrix} I_{\Lambda_0}-K_{\Lambda_0} & -K_{\Lambda_0\Lambda_1}\\ K_{\Lambda_1\Lambda_0} & K_{\Lambda_1}\end{pmatrix}\geq 0$$

が成り立つ．ただし，

$$\Lambda=\Lambda_0\sqcup\Lambda_1,\quad K_{\Lambda\Lambda'}:=P_\Lambda KP_{\Lambda'},\quad K_\Lambda:=K_{\Lambda\Lambda}$$

とする．

Λ_0,Λ_1 を互いに素な R の有限部分集合とする．$\xi : Q\to\{0,1\}$ に対して Λ_0 上で 0, Λ_1 上で 1 となるような関数全体からなる筒集合 $0^{\Lambda_0}1^{\Lambda_1}$ を

$$0^{\Lambda_0}1^{\Lambda_1}=\{\xi\in Q\,;\,\xi(x)=i\ \text{ if } x\in\Lambda_i,\ i=0,1\}$$

とする．また Q 上の筒集合全体を $\mathcal{C}$ とする．

$\mathcal{C}$ 上の非負集合関数を

$$\begin{aligned}\mu(0^{\Lambda_0}1^{\Lambda_1}) &= \det(P_{\Lambda_0}(I_\Lambda-K_\Lambda)+P_{\Lambda_1}K_\Lambda),\\ \mu(Q) &= 1\end{aligned}$$

によって定義する．ただし，$\Lambda=\Lambda_0\sqcup\Lambda_1$．このとき次のことがわかる．

補題 2.2. $\{\mu(0^{\Lambda_0}1^{\Lambda_1})\,;\,\Lambda_0,\Lambda_1\subset R,\Lambda_0\cap\Lambda_1=\emptyset\}$ は次の意味で無矛盾：任意の互いに素な有限部分集合 $\Lambda_0,\Lambda_1\subset R$ と $\Lambda:=\Lambda_0\sqcup\Lambda_1$ に含まれない任意の $x\in R$ に対して，

$$\mu(0^{\Lambda_0\cup\{x\}}1^{\Lambda_1})+\mu(0^{\Lambda_0}1^{\Lambda_1\cup\{x\}})=\mu(0^{\Lambda_0}1^{\Lambda_1}).$$

さらに，

$$\sum_{\Lambda_0\sqcup\Lambda_1=\Lambda}\mu(0^{\Lambda_0}1^{\Lambda_1})=1. \tag{2.1}$$

証明. 証明は $N=|\Lambda|$ による帰納法．$N=0$ のときは $(1-K(x,x))+K(x,x)=1$ より明らか．また $N=|\Lambda|$ まで正しいと仮定する．$\tilde\Lambda=\Lambda\cup\{x\}$ として

$$K_{\tilde\Lambda}=\begin{pmatrix} K & b\\ {}^tc & k\end{pmatrix}$$

とかく．ただし，$K=K_\Lambda,\ k=K(x,x)$ としている．このとき行列式の多重線型性より

$$\begin{aligned}&\mu(0^{\Lambda_0\cup\{x\}}1^{\Lambda_1})+\mu(0^{\Lambda_0}1^{\Lambda_1\cup\{x\}})\\ &= \det\begin{pmatrix} P_{\Lambda_1}K+P_{\Lambda_0}(I-K) & P_{\Lambda_1}b-P_{\Lambda_0}b\\ -{}^tc & 1-k\end{pmatrix}+\det\begin{pmatrix} P_{\Lambda_1}K+P_{\Lambda_0}(I-K) & P_{\Lambda_1}b-P_{\Lambda_0}b\\ {}^tc & k\end{pmatrix}\\ &= \det\begin{pmatrix} P_{\Lambda_1}K+P_{\Lambda_0}(I-K) & P_{\Lambda_1}b-P_{\Lambda_0}b\\ {}^t0 & 1\end{pmatrix}=\det\left(P_{\Lambda_1}K+P_{\Lambda_0}(I-K)\right)\\ &= \mu(0^{\Lambda_0}1^{\Lambda_1}).\end{aligned}$$

よって第一の主張は証明された．二つ目の主張はこのことを繰り返して用いればよい． □

注意 *2.3.* 上の補題 2.2 の式 (2.1) は次の事実の特別な場合 ($A=K,\ B=I-K$) となっている：任意の $|\Lambda|\times|\Lambda|$ 次行列 A,B に対して，

$$\sum_{\Lambda_0\sqcup\Lambda_1=\Lambda}\det\left(P_{\Lambda_1}A+P_{\Lambda_0}B\right)=\det(A+B)$$

が成り立つ．

補題 2.2 と Kolmogorov の拡張定理から次の結果を得る．

定理 2.4. *K は 仮定 2.1 を満たすとする．このとき，Q 上の Borel 確率測度 μ_K で有限次元分布が*

$$\mu_K(0^{\Lambda_0}1^{\Lambda_1})=\det(P_{\Lambda_0}(I_\Lambda-K_\Lambda)+P_{\Lambda_1}K_\Lambda)$$

によって与えられるものがただ一つ存在する．

この定理により得られる $Q=Q(R)$ 上の Borel 確率測度 μ_K を K に付随する離散 Fermion 測度とよぶことにする．

以下のことに注意しよう．

注意 *2.5.* K が 仮定 2.1 を満たせば $I-K$ も満たす．このとき μ_{I-K} は写像 $Q\ni\xi\mapsto 1-\xi\in Q$ による μ_K の像測度になる．この事実は R が離散であることによっている．

さて筒集合の測度は別の形でも計算できる．

補題 2.6. *筒集合の測度は以下のような別表示をもつ．*

$$\begin{aligned}
\mu(0^{\Lambda_0}1^{\Lambda_1}) &= \det(P_{\Lambda_0}(I_\Lambda-K_\Lambda)+P_{\Lambda_1}K_\Lambda)\\
&= \begin{cases}\det(J[\Lambda]_{\Lambda_1})\det(I_\Lambda-K_\Lambda), & I_\Lambda-K_\Lambda \text{ が可逆のとき}\\ \det((J[\Lambda]^{-1})_{\Lambda_0})\det K_\Lambda, & K_\Lambda \text{ が可逆のとき}\end{cases}\\
&= (-1)^{|\Lambda_0|}\det(K_\Lambda-P_{\Lambda_0})\\
&= (-1)^{|\Lambda_1|}\det(I_\Lambda-K_\Lambda-P_{\Lambda_1}).
\end{aligned}$$

ただし，$J[\Lambda]=K_\Lambda(I_\Lambda-K_\Lambda)^{-1}$, $J[\Lambda]^{-1}=(I_\Lambda-K_\Lambda)K_\Lambda^{-1}$.

証明. $I_\Lambda-K_\Lambda$ は可逆であるとする．

$$\begin{aligned}
\det(P_{\Lambda_0}(I_\Lambda-K_\Lambda)+P_{\Lambda_1}K_\Lambda) &= \det(P_{\Lambda_0}+P_{\Lambda_1}J[\Lambda])\det(I_\Lambda-K_\Lambda)\\
&= \det(P_{\Lambda_0}+P_{\Lambda_1}J[\Lambda]P_{\Lambda_1}]\det(I_\Lambda-K_\Lambda)\\
&= \det(J[\Lambda]_{\Lambda_1})\det(I_\Lambda-K_\Lambda).
\end{aligned}$$

K_Λ が可逆の場合もまったく同様．また $(P_{\Lambda_1}-P_{\Lambda_0})^2=I_\Lambda$ であることに注意して，

$$\begin{aligned}
\det(P_{\Lambda_0}(I_\Lambda-K_\Lambda)+P_{\Lambda_1}K_\Lambda) &= \det(P_{\Lambda_0}-P_{\Lambda_1})^2(P_{\Lambda_0}(I_\Lambda-K_\Lambda)+P_{\Lambda_1}K_\Lambda)\\
&= \det(P_{\Lambda_1}-P_{\Lambda_0})\det(-P_{\Lambda_0}(I_\Lambda-K_\Lambda)+P_{\Lambda_1}K_\Lambda)\\
&= (-1)^{|\Lambda_0|}\det(K_\Lambda-P_{\Lambda_0}).
\end{aligned}$$

□

例 2.7. (1) K は $\ell^2(R)$ 上の Hermite 作用素で，$O\le K\le I$ を満たすとする．このとき，補題 2.6 に注意すれば K は 仮定 2.1 を満たすことがわかる．

(2) $R = \mathbf{Z}^1$ とする．K は $\|K\| \le 1$ で，さらに totally positive であるとする．つまり，任意の $n \in \mathbf{N}$ と任意の $x_1 < x_2 < \cdots < x_n$ と $y_1 < y_2 < \cdots < y_n$ に対して

$$\det(K(x_i, y_j))_{i,j=1}^n \ge 0$$

とする．このとき K は 仮定 2.1 を満たす.
(3) (1) の特別な場合である．$R = \mathbf{Z}^d$ とする．$\widehat{k} : \mathbf{T}^d \to [0,1]$ に対して，

$$k(x) = \left(\frac{1}{2\pi}\right)^d \int_{\mathbf{T}^d} \widehat{k}(\theta) e^{ix\theta} d\theta, \quad x \in \mathbf{Z}^d$$

によって定義し，Toeplitz 作用素 $K : \ell^2(\mathbf{Z}^d) \to \ell^2(\mathbf{Z}^d)$ を

$$Kf(x) = \sum_{y \in \mathbf{Z}^d} k(x-y) f(y)$$

とすると仮定 2.1 の条件を満たすので，確率測度 $\mu_{\widehat{k}}$ が存在する．構成の仕方より $\mathbf{Z}^d$ の平行移動に関して不変な確率測度となる．特に $\widehat{k} \equiv \alpha$ $(0 \le \alpha \le 1)$ とすると $K = \alpha I$ となり，対応する μ_K は $(\alpha, 1-\alpha)$-Bernoulli 測度となる.

例 2.8. (cf.[24]) 連結有限グラフ $G = (V, E)$ が与えられときその部分グラフですべての点を含む木を全域木という．グラフ G が与えられたとき

$$M(x, e) = \begin{cases} 1, & x = o(e), \\ -1, & x = t(e), \\ 0, & otherwise \end{cases}$$

によって定まる $|V| \times |E|$-行列 M を接続行列という．行列 M の階数は $|V| - 1$ に等しい．行列 M の 1 行から $|V| - 1$ 行目までの行ベクトルで張られる線型空間 V は $\mathbf{R}^{|E|} \cong \ell^2(E)$ の部分空間をなす．$\ell^2(E)$ から V への直交射影に対する Fermion 測度はグラフ G の全域木上の一様分布とみなせる．ちなみに G の全域木の個数は行列 M の 1 行から $|V| - 1$ 行目までをとりだして得られる $(|V| - 1) \times |E|$-部分行列 N をもちいると $\det NN^*$ によって与えられる.

さて，ξ の Laplace 変換を計算しよう.

定理 2.9. K は 仮定 *2.1* を満たすとする.

$$\int_Q e^{-\langle \xi, f \rangle} \mu(d\xi) = \det(I - K(1 - e^{-f(x)})), \tag{2.2}$$

ただし，$\operatorname{supp} f$ は有限集合で $\langle \xi, f \rangle = \sum_{x \in R} \xi(x) f(x)$ である．また $(1 - e^{-f(x)})$ はかけ算作用素.

証明. $\operatorname{supp} f = \Lambda$ とする．注意 2.3 より

$$\begin{aligned} \int_Q e^{-\langle \xi, f \rangle} \mu(d\xi) &= \sum_{\Lambda_0 \sqcup \Lambda_1 = \Lambda} e^{-\sum_{x \in \Lambda_1} f(x)} \mu(0^{\Lambda_0} 1^{\Lambda_1}) \\ &= \sum_{\Lambda_0 \sqcup \Lambda_1 = \Lambda} e^{-\sum_{x \in \Lambda_1} f(x)} \det(P_{\Lambda_0}(I_\Lambda - K_\Lambda) + P_{\Lambda_1} K_\Lambda) \\ &= \sum_{\Lambda_0 \sqcup \Lambda_1 = \Lambda} \det(P_{\Lambda_0}(I_\Lambda - K_\Lambda) + P_{\Lambda_1} e^{-f} K_\Lambda) \\ &= \det(I_\Lambda - K_\Lambda + e^{-f} K_\Lambda). \end{aligned}$$

□

この Laplace 変換の式 (2.2) よりモーメントが求まる．例えば，

系 2.10. $\langle \xi, f\rangle$ の平均は

$$\int_Q \mu(d\xi)\langle \xi, f\rangle = \sum_{x\in R} K(x,x)f(x).$$

特に Λ にある 1 の個数の平均は $\mathrm{Tr}(K_\Lambda) = \sum_{x\in\Lambda} K(x,x)$ で与えられる．

証明. 式 (2.2) において f の代わりに tf として，$t=0$ で微分すればよい． □

行列 K_Λ の階数は Λ 内の粒子の個数 (1 の個数) に対応し，行列 $I_\Lambda - K_\Lambda$ の階数は Λ 内の 0 の個数に対応している．

命題 2.11. *(1)* $\mathrm{rank}\, K_\Lambda = n$ とすると $\mu(\xi(\Lambda) \le n) = 1$．また $\mathrm{rank}(I_\Lambda - K_\Lambda) \le m$ ならば $\mu(\xi(\Lambda) \ge N-m) = 1$. ただし $\xi(\Lambda) = \langle \xi, 1_\Lambda\rangle$ で Λ 内の 1 の個数をあらわす.
(2) 特に K が $\mathrm{rank}\, K = n$ の射影であるときは，$\mu(\xi(R) = n) = 1$ となる．

証明. $\mathrm{rank}\, K_\Lambda = n$ とする．$|\Lambda_1| \ge n+1$ のとき行列 $P_{\Lambda_0}(I_\Lambda - K_\Lambda) + P_{\Lambda_1}K_\Lambda$ の $|\Lambda_1| \times |\Lambda|$ 小行列 $P_{\Lambda_1}K_\Lambda$ の中にかならず一次従属なベクトルが存在する．よって行列式は 0 となり，$\mu(0^{\Lambda_0}1^{\Lambda_1}) = 0$ となる．つまり $\mu(\xi(\Lambda) \ge n+1) = 0$ であるから主張を得る．後半もまったく同様. □

上の命題は次の事実からも従う.

命題 2.12. 任意の $\Lambda \subset R(|\Lambda| = n)$ と $k \ge 0$ に対して，

$$\begin{aligned}\mu(\xi(\Lambda) = k) &= \det(I_\Lambda - K_\Lambda)\,\mathrm{Tr}(\wedge^k J[\Lambda]) \\ &= \sum_{J\subset\{1,2,\ldots,n\}} \prod_{j\in J}\lambda_j \prod_{j\in J^c}(1-\lambda_j)\end{aligned}$$

ここで $\lambda_1, \ldots, \lambda_n$ は K_Λ の固有値である．$I_\Lambda - K_\Lambda$ が可逆でないときは 2 行目の意味で理解する．特に

$$\mu(\xi(\Lambda) = 0) = \det(I_\Lambda - K_\Lambda).$$

統計物理でよく知られているように Fermion は Pauli の排他律にしたがい，二つの粒子が同じ状態に入ることは禁じられる．このことから Fermion 測度はその条件付き確率がまた Fermion 測度となるという著しい性質をもつ.

定理 2.13. A は R の有限部分集合とする．$\mu(1^A) > 0$ とする．

$$\mu^A(\cdot) = \mu(\cdot \mid 1^A) = \mu(\cdot \mid \xi \equiv 1 \ on\ A)$$

は以下に定義する行列 K^A に対する *Fermion* 測度となる．$A = \{x_1, \ldots, x_n\}$ とすると

$$\begin{aligned}K^A(x,y) &= \left(\det(K(x_i,x_j))_{i,j=1}^n\right)^{-1} \\ &\quad \times \det\begin{pmatrix} K(x,y) & K(x,x_1) & \cdots & K(x,x_n) \\ K(x_1,y) & K(x_1,x_1) & \cdots & K(x_1,x_n) \\ \vdots & \vdots & \ddots & \vdots \\ K(x_n,y) & K(x_n,x_1) & \cdots & K(x_n,x_n)\end{pmatrix} \\ &= K(x,y) - \langle K(x,\cdot), K_A^{-1}K(\cdot,y)\rangle.\end{aligned}$$

ただし，$K(x,\cdot) = (K(x,x_i))_{i=1}^n$，$K(\cdot,y) = (K(x_i,y))_{i=1}^n$ である．

補題 2.14. $\{0,1\}^R$ 上の確率測度 μ に対して，ある行列 K が存在して任意の R の有限部分集合 $\Lambda \subset R$ に対して

$$\rho(\Lambda) := \mu(1^\Lambda) = \det(K(x,y))_{x,y\in\Lambda}$$

が成り立つとする．このとき，μ は行列 K に対する *Fermion* 測度となる．

証明．補題 2.2 の証明と同様に関係

$$\det\begin{pmatrix} A & b \\ c & d \end{pmatrix} + \det\begin{pmatrix} A & b \\ -c & 1-d \end{pmatrix} = \det A$$

を繰り返せばすべての筒集合の確率が矛盾なく定義できる． □

定理 *2.13* の証明．$A = \{a\}$ の場合だけ示す．A と互いに素な Λ に対して，

$$\mu(1^{\Lambda\cup\{a\}}) = \det(K(x,y))_{x,y\in\Lambda\cup\{a\}}$$

である．このとき，$\mu(1^A) = K(a,a) > 0$ であることに注意して第 a 行に $K(x,a)/K(a,a)$ をかけて第 x 行から引くと，行列の (x,y) 成分は $K^a(x,y)$ となり，(x,a) 成分は (a,a) 成分以外すべて 0 になる．よって

$$\mu(1^{\Lambda\cup\{a\}}) = \det(K^a(x,y))_{x,y\in\Lambda}\cdot K(a,a) = \det(K^a(x,y))_{x,y\in\Lambda}\cdot\mu(1^{\{a\}})$$

であるから補題 2.14 を用いると結論を得る． □

2.3 Fermion 測度に関する相関不等式

以下では K は Hermite 作用素であることを仮定する．筒集合の測度に関して以下のような相関不等式が成り立つ．

命題 2.15. *(1)* A, B は互いに素な有限集合とすると

$$\mu(1^{A\cup B}) \le \mu(1^A)\cdot\mu(1^B).$$

もっと一般に A, B は任意の有限集合とすると

$$\mu(1^{A\cup B})\mu(1^{A\cap B}) \le \mu(1^A)\cdot\mu(1^B).$$

(2) 以下の不等式がなりたつ．

$$\mu(0^{\Lambda_0})\mu(1^{\Lambda_1}) \le \mu(0^{\Lambda_0}1^{\Lambda_1}) \le \{\mu(0^{\Lambda_0})\mu(1^{\Lambda_1})\}^{1/2}.$$

証明．命題の不等式の証明には以下のような行列式に関する不等式を用いればよい：行列 A, C は非負定値であるとすると

$$\det\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \le \det A \det C, \quad \det\begin{pmatrix} A & B \\ -B^* & C \end{pmatrix} \ge \det A \det C$$

が成り立つ． □

注意 *2.16.* K が Toeplitz 行列で定まるときはシフトのエントロピーが

$$\lim_{\Lambda\to R} -\frac{1}{|\Lambda|}\sum_{\Lambda_0\sqcup\Lambda_1=\Lambda}\mu(0^{\Lambda_0}1^{\Lambda_1})\log\mu(0^{\Lambda_0}1^{\Lambda_1})$$

によって定義されるが，命題 2.15(2) の不等式から $K \ne O, I$ ならば正のエントロピーをもつことが示される [24, 25, 36].

ところで Q の元は R 上の $\{0,1\}$-値関数であったから自然な順序で Q は半順序集合になる．$A \in \mathcal{B}(Q)$ に対して，$\xi \in A$ かつ $\xi \leq \eta$ ならば $\eta \in A$ となるとき，A は increasing であるという．また μ, ν は Q 上の確率測度とする．任意の increasing な A に対して，$\mu(A) \leq \nu(A)$ となるとき，$\mu \leq \nu$ とかく．

命題 2.17 ([38]). μ, ν は Q 上の確率測度とする．$\mu \leq \nu$ であるための必要十分条件は，μ と ν の単調なカップリングが存在することである．ただし，カップリング P が単調であるとは $\operatorname{supp} P \subset \{(\xi,\eta) \in Q \times Q\,;\, \xi \leq \eta\}$ となるときをいう．

さて例 2.7 の Toeplitz 行列の場合を考えよう．$\widehat{k} : \mathbf{T}^d \to [0,1]$ から定まる Fermion 測度を $\mu_{\widehat{k}}$ とかく．このとき以下のことが知られている [24].

命題 2.18. $0 \leq \widehat{k}_1 \leq \widehat{k}_2 \leq 1$ ならば $\mu_{\widehat{k}_1} \leq \mu_{\widehat{k}_2}$.

このことから $0 \leq \widehat{k}_1 \leq \widehat{k}_2 \leq 1$ ならば $\mu_{\widehat{k}_1}$ と $\mu_{\widehat{k}_2}$ との単調なカップリングが存在する．例えば，$0 \leq p \leq 1$ とし $\widehat{k}_1 = p\widehat{k}_2$ とする．このとき以下のようにしてカップリングを構成できる．$\xi, \eta : \Omega \to Q$ を Q 上の分布がそれぞれ $(p, 1-p)$-ベルヌイと $\mu_{\widehat{k}}$ である確率変数とする．このとき，$\min(\xi,\eta) = \xi \cdot \eta$ (各点での min をとる) の分布は $\mu_{p\widehat{k}}$ に等しい．

問題 2.19. $0 \leq \widehat{k}_1 \leq \widehat{k}_2 \leq 1$ のとき，一般に $\mu_{\widehat{k}_1}$ と $\mu_{\widehat{k}_2}$ との単調なカップリングを構成できるか？

3 Fermion 測度と Boson 測度とその一般化

3.1 一般の空間における Fermion 測度

R を可算基を持つ局所コンパクトハウスドルフ空間とし，その上の Radon 測度を $\lambda(dx)$ として，以下固定する．R 上の非負整数値 Radon 測度を R 上の局所有限な配置といい，その全体を $Q = Q(R)$ と表わす．Q には漠位相をいれ $\mathcal{B}(Q)$ は位相的 Borel 集合体とする．Q の要素 ξ は $\xi = \sum_i \delta_{x_i}$ の形に書けることに注意しておく.（多重点がある場合は違う点だとみなして，繰り返し和に加える.）μ を $(Q, \mathcal{B}(Q))$ 上の確率測度とするとき，$(Q, \mathcal{B}(Q), \mu)$ を点過程またはランダム場という

$\xi \in Q$ と台がコンパクトな連続関数 $f \in C_c(R)$ に対して，

$$\langle \xi, f \rangle = \int_R f(x)\xi(dx) = \sum_{x \in \xi} f(x)$$

とおく．また一般に $\xi \in Q$ と台がコンパクトな連続関数 $f_n \in C_c(R^n)$ に対して，

$$\langle \xi_n, f_n \rangle = \sum_{x_1, x_2, \ldots x_n \in \xi : \text{互いに異なる}} f_n(x_1, \ldots, x_n)$$

とおく．ξ_1 は ξ とみなす.

定義 3.1. 任意の $f_n \in C_c(R^n)$ に対して，

$$\int_Q \mu(d\xi)\langle \xi_n, f_n \rangle = \int_{R^n} \lambda_n(dx_1 \cdots dx_n) f_n(x_1, \ldots, x_n)$$

を満たす R^n 上の測度 $\lambda_n(dx_1 \cdots dx_n)$ が存在すれば，これを ξ の n 次相関測度と呼ぶ．特に Λ_1 は平均測度ともいう．さらに，λ_n が $\lambda^{\otimes n}$ に関して絶対連続であるとき，

$$\rho_n(x_1, \ldots, x_n) = \frac{d\lambda_n}{d\lambda^{\otimes n}}(x_1, \ldots, x_n)$$

を n 次相関関数と呼ぶ．シンボリックには

$$\lambda_n = \int_Q \mu(d\xi)\xi_n$$

である．ここで，定義した相関関数は (1.2) で定義したものと一致する．

点過程は Laplace 変換

$$L_\mu(f) = \int_Q \mu(d\xi) \exp(-\langle \xi, f \rangle) \quad f \in C_c^+(R)$$

によって一意に決定される．よく知られているように平均測度 ν のポアソン点過程 Π_ν の Laplace 変換は

$$\int_Q \Pi_\nu(d\xi) \exp(-\langle \xi, f \rangle) = \exp\left(-\int_R (1 - e^{-f(x)}) \nu(dx) \right) \tag{3.1}$$

さて定理 2.9 の Laplace 変換の形を考慮して以下のような結果を得る．

定理 3.2 ([34, 35, 37]). $L^2(R, d\lambda)$ 上の積分作用素 K は局所トレース族かつ自己共役で $O \le K \le I$ を満たすものとする[1]．このとき *Laplace* 変換が

$$\int_Q \mu_K(d\xi) \exp(-\langle \xi, f \rangle) = \mathrm{Det}(I - K_\varphi) \tag{3.2}$$

となる $Q = Q(R)$ 上の確率測度 μ_K が唯一存在する．ただし，$f \in C_c^+(R)$ はコンパクトな台をもつ任意の非負連続関数，$\varphi = 1 - \exp(-f)$, $K_\varphi = \sqrt{\varphi} K \sqrt{\varphi}$．また右辺は積分作用素 K_φ の *Fredholm* 行列式．さらに，n 次の相関関数は以下で与えられる．

$$\begin{aligned} \rho_n(x_1, \ldots, x_n) &= \det(K(x_i, x_j))_{i,j=1}^n \\ &= \sum_{\sigma \in \mathcal{S}_n} \mathrm{sgn}(\sigma) K(x_1, x_{\sigma(1)}) \cdots K(x_n, x_{\sigma(n)}). \end{aligned}$$

注意 *3.3.* ここで得られた確率測度を [26, 27] に従って Fermion 点過程もしくは Fermion 測度と呼ぶ[2]．もちろん離散 Fermion 測度はこの定理の特別な場合である．

証明は以下のようにして Kolmogorov の拡張定理による (cf.[21])．まず有界集合 Λ 上の配置空間を $Q(\Lambda)$ とすると $Q(\Lambda)$ は $\cup_{n=0}^\infty \Lambda^n / \sim$ と同一視される．ここで $\sim$ は座標の置換に関する同値関係をあらわす．さて $\bigcup_{n=0}^\infty \Lambda^n$ 上の対称関数 $\sigma_{\Lambda,K}$ を

$$\begin{aligned} \sigma_{\Lambda,K}(x_1, \ldots, x_n) &= \mathrm{Det}(I - K_\Lambda) \det(J[\Lambda](x_i, x_j))_{i,j=1}^n \text{ on } \Lambda^n, \\ \sigma_{\Lambda,K}(\emptyset) &= \mathrm{Det}(I - K_\Lambda) \text{ on } \Lambda^0 = \{\emptyset\}, \end{aligned}$$

によって定義する．ただし，$J[\Lambda] = K_\Lambda (I - K_\Lambda)^{-1}$ とする．このとき，$Q(\Lambda)$ 上の確率測度 $\mu_{\Lambda,K}$ を

$$\begin{aligned} &\int_{Q(\Lambda)} \mu_{\Lambda,K}(d\xi) \exp(-\langle \xi, f \rangle) \\ &= \sum_{n=0}^\infty \frac{1}{n!} \int_{\Lambda^n} \sigma_{\Lambda,K}(x_1, \ldots, x_n) \exp\left(-\sum_{k=1}^n f(x_k) \right) \lambda^{\otimes n}(dx_1 \cdots dx_n) \end{aligned}$$

とする．$J[\Lambda]$ が非負定値であることから $\sigma_{\Lambda,K} \ge 0$ は明らかである．このとき以下の補題が成り立つ．

補題 3.4. f が R 上のコンパクト台をもつ非負連続関数とする．$\mathrm{supp}\, f \subset \Lambda$ とすると

$$\int_{Q(\Lambda)} \mu_{\Lambda,K}(d\xi) \exp(-\langle \xi, f \rangle) = \mathrm{Det}(I - K_\varphi). \tag{3.3}$$

ただし，$\varphi = 1 - e^{-f}$.

[1] K が局所トレース族であるとは，任意のコンパクト集合 $\Lambda \subset R$ に対して，$K_\Lambda = 1_\Lambda K 1_\Lambda$ がトレース族作用素であることと定義する．

[2] Determinantal 測度と呼んでいる文献も多い．

証明．$\operatorname{supp} f \subset \Lambda$ とすると．

$$
\begin{aligned}
\operatorname{Det}(I-K_\varphi) &= \operatorname{Det}(I-K_\Lambda)\operatorname{Det}(I+(J[\Lambda])_{e^{-f}}) \\
&= \operatorname{Det}(I-K_\Lambda) \\
&\quad \times\left\{1+\sum_{n=1}^{\infty}\frac{1}{n!}\int_{\Lambda^n}\det\left((J[\Lambda])_{e^{-f}}(x_i,x_j)\right)_{i,j=1}^{n}\lambda^{\otimes n}(dx_1\cdots dx_n)\right\} \\
&= \operatorname{Det}(I-K_\Lambda) \\
&\quad \times\left\{1+\sum_{n=1}^{\infty}\frac{1}{n!}\int_{\Lambda^n}\det(J[\Lambda](x_i,x_j))_{i,j=1}^{n}\exp\left(-\sum_{k=1}^{n}f(x_k)\right)\lambda^{\otimes n}(dx_1\cdots dx_n)\right\} \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\int_{\Lambda^n}\sigma_{\Lambda,K}(x_1,\ldots,x_n)\exp\left(-\sum_{k=1}^{n}f(x_k)\right)\lambda^{\otimes n}(dx_1\cdots dx_n) \\
&= \int_{Q(\Lambda)}\mu_{\Lambda,K}(d\xi)\exp(-\langle\xi,f\rangle).
\end{aligned}
$$

□

任意のコンパクト台をもつ非負連続関数 f に対して $\operatorname{supp} f \subset \Lambda \subset \Lambda'$ ならば，

$$
\int_{Q(\Lambda')}\mu_{\Lambda',K}(d\xi)\exp(-\langle\xi,f\rangle)=\int_{Q(\Lambda)}\mu_{\Lambda,K}(d\xi)\exp(-\langle\xi,f\rangle)=\operatorname{Det}(I-K_\varphi)
$$

となることより，確率測度の系 $\{\mu_{\Lambda,K}\}$ は無矛盾である．よって Kolmogorov の拡張定理により定理を得る (cf.[21])．以上の証明は K_Λ が固有値 1 をもたないことを仮定しているが，例えば sK_Λ を考えて $s\uparrow 1$ の極限を考えればよい．

例 3.5. 1 節であげた GUE と Laguarre ensemble の行列サイズ N が無限大で表われる 3 つの積分核はそれぞれ $\mathbf{R}$ 上の Fermion 点過程を定義し重要である．$K_{sine}(x,y)$ の場合は平行移動不変な点過程になる．さらに $K_{Airy}(x,y)$ の場合は確率 1 で最右端の粒子が存在し，GUE の最大固有値に対応するものである．さらに $K_{Bessel}(x,y)$ の場合は $(-\infty,0)$ に粒子は存在しない．

例 3.6. $R=\sqcup_{i=1}^{N}E_i$ とする．簡単のために $E_i\cong E$ とする．K は $L^2(R)\cong\bigoplus_{i=1}^{N}L^2(E)$ 上の局所トレース族の積分作用素とする．積分核は行列形 $K(x,y)=(K_{rs}(x,y))_{1\le r,s\le N}, x,y\in E$ になり，相関関数は

$$
\begin{aligned}
\rho_n(x_1^{(1)},\ldots,x_{k_1}^{(1)},\ldots,x_1^{(N)},\ldots,x_{k_N}^{(N)}) &= \rho_{k_1,\ldots,k_N}(x_1^{(1)},\ldots,x_{k_1}^{(1)},\ldots,x_1^{(N)},\ldots,x_{k_N}^{(N)}) \\
&= \det(K_{rs}(x_{i_r}^{(r)},x_{j_s}^{(s)})_{1\le r,s\le N,1\le i_r\le k_r,1\le j_s\le k_s}
\end{aligned}
$$

によって与えられる．ただし，$k_1+k_2+\cdots+k_N=n$.

問題 3.7. 微小な独立確率変数の和が Poisson 分布に収束するという Poisson の少数法則と同様の定理を相関のある場合に K_{sine} に対応する Fermion 測度への極限定理として定式化できるか？

3.2 Boson 測度と α-Boson 測度

Laplace 変換の形に注目して Fermion 測度は以下のような形で Boson 測度に拡張される．

定理 3.8 ([35]). $L^2(R,d\lambda)$ 上の積分作用素 K は局所トレース族かつ自己共役で $K\ge O$ を満たすものとする．コンパクトな台をもつ任意の非負連続関数 $f\in C_c^+(R)$ に対して *Laplace* 変換が

$$
\int_Q \mu(d\xi)\exp(-\langle\xi,f\rangle)=\operatorname{Det}(I+K_\varphi)^{-1} \tag{3.4}
$$

となる $Q=Q(R)$ 上の確率測度が唯一存在する．さらに，n 次の相関関数は以下で与えられる．

$$\begin{aligned}\rho_n(x_1,\ldots,x_n) &= \operatorname{per}(K(x_i,x_j))_{i,j=1}^n \\ &= \sum_{\sigma\in\mathcal{S}_n} K(x_1,x_{\sigma(1)})\cdots K(x_n,x_{\sigma(n)}).\end{aligned}$$

Laplace 変換の形から自然に次のように問題を一般化することが考えられる [35].

問題 3.9. $L^2(R,d\lambda)$ 上の積分作用素 K は局所トレース族かつ自己共役で $K\geq O$ を満たすものとする．このとき次の形の Laplace 変換

$$\int_Q \mu_{\alpha,K}(d\xi)\exp(-\langle\xi,f\rangle) = \det(I+\alpha K_\varphi)^{-1/\alpha}. \tag{3.5}$$

を持つ $Q=Q(R)$ 上の確率測度は存在するか？もし存在すれば n 次の相関関数は以下で与えられる．

$$\begin{aligned}\rho_n(x_1,\ldots,x_n) &= \det{}_\alpha(K(x_i,x_j))_{i,j=1}^n \\ &= \sum_{\sigma\in\mathcal{S}_n} \alpha^{d(\sigma)}K(x_1,x_{\sigma(1)})\cdots K(x_n,x_{\sigma(n)}).\end{aligned}$$

ただし，$d(\sigma)$ は置換 σ を互換の積としてあらわすのに必要な互換の最小の個数[3]．特に

$$\det{}_{-1}A=\det A,\quad \det{}_0 A=\prod_{i=1}^n a_{ii},\quad \det{}_1 A=\operatorname{per}A.$$

もし $\mu_{\alpha,K}$ が存在するときには α-Boson 測度とよぶことにする．もちろん，$\alpha=\pm1$ のときは Fermion 測度と Boson 測度に対応して，$\alpha=0$ のときは極限をとったものとみなせば Poisson 測度に対応する．

R が一点からなる場合は (3.5) の右辺は一般化された二項分布の Laplace 変換を与える．よって，(3.5) の右辺が確率分布の Laplace 変換になるための必要十分条件は，$\alpha\in\{-1/m\,;\,m\in\mathbf{N}\}\cup[0,\infty)$ である．以降はおもに $\alpha>0$ について考えることにする．

さて $J_\alpha[\Lambda]=K_\Lambda(I+\alpha K_\Lambda)^{-1/\alpha}$ とし，

$$\begin{aligned}\sigma_{\Lambda,\alpha,K}(x_1,\ldots,x_n) &= \operatorname{Det}(I+\alpha K_\Lambda)^{-1/\alpha}\det{}_\alpha(J_\alpha[\Lambda](x_i,x_j))_{i,j=1}^n \text{ on } \Lambda^n, \\ \sigma_{\Lambda,\alpha,K}(\emptyset) &= \operatorname{Det}(I+\alpha K_\Lambda)^{-1/\alpha} \text{ on } \Lambda^0=\{\emptyset\}.\end{aligned}$$

によって定義する．$Q(\Lambda)$ 上の測度 $\mu_{\Lambda,\alpha,K}$ を

$$\begin{aligned}&\int_{Q(\Lambda)} \mu_{\Lambda,\alpha,K}(d\xi)\exp(-\langle\xi,f\rangle) \\ &= \sum_{n=0}^\infty \frac{1}{n!}\int_{\Lambda^n}\sigma_{\Lambda,\alpha,K}(x_1,\ldots,x_n)\exp\left(-\sum_{k=1}^n f(x_k)\right)\lambda^{\otimes n}(dx_1\cdots dx_n).\end{aligned}$$

とする．

Fredholm 行列式について以下の展開公式がなりたつ．これは $(1-x)^{-\alpha}$ の展開公式の無限次元への一般化である．

定理 3.10. J は $L^2(R,\lambda)$ 上のトレース族の積分作用素とする．$\|\alpha J\|<1$ ならば

$$\operatorname{Det}(I-\alpha J)^{-1/\alpha} = \sum_{n=0}^\infty \frac{1}{n!}\int_{R^n}\det{}_\alpha\left(J(x_i,x_j)\right)_{i,j=1}^n \lambda^{\otimes n}(dx_1\cdots dx_n), \tag{3.6}$$

$\alpha\in\{-1/m\ ;\ m\in\mathbf{N}\}$ のときは $\|\alpha J\|<1$ の条件なしで *(3.6)* は成り立つ．

[3] $d(\sigma,\eta)=d(\sigma^{-1}\eta)$ とおくと $\mathcal{S}_n$ 上の距離となる．

このことに注意すると，

$$\int_{Q(\Lambda)} \mu_{\Lambda,\alpha,K}(d\xi)\exp(-\langle \xi, f\rangle) = \mathrm{Det}(I+\alpha K_\varphi)^{-1/\alpha}$$

となる．つまり，Fermion 測度の場合と同様にして $\sigma_{\Lambda,\alpha,K}(x_1,\ldots,x_n)$ の非負性が保証されれば Q 上の確率測度 $\mu_{\alpha,K}$ が存在することがわかる．例えば $\det_\alpha$ の定義より $J_\alpha[\Lambda]$ が非負行列であれば上の非負性はあきらかである．

上の関係を考慮すると以下のような問題が考えられる．

問題 3.11. $\alpha > 0$ とする．以下の同値な 2 条件が成立するような α の範囲は？

$\mu_{\alpha,K}$ が任意の $K \geq O$ に対して存在する $\Longleftrightarrow$ $\det_\alpha A \geq 0$ が任意の $A \geq O$ に対して成り立つ

$\alpha = \pm 1$ のときは線型代数的な性質 ($\mathrm{per}\, A \geq \det A \geq 0,\ \forall A \geq O$) から確率測度 $\mu_{\pm 1,K}$ の存在がわかり，Laplace 変換の形とその性質から $\alpha \in \{\pm 1/m, ; m \in \mathbf{N}\}$ のときは $\mu_{\pm 1,K}$ の畳み込みとして確率測度 $\mu_{\pm 1/m,K}$ が構成されて線型代数の問題が肯定的に解ける．

注意 *3.12.* 上の考察と次節の結果をあわせると現在のところ $\alpha \in \{-1/m\,;\, m \in \mathbf{N}\} \cup \{0\} \cup \{2/m\,;\, m \in \mathbf{N}\}$ の場合について問題 3.11 が肯定的に解決されている．負の場合については $\alpha \in \{-1/m\,;\, m \in \mathbf{N}\}$ 以外は否定的に結論される．

3.3 Gauss 場と Boson 測度

さて $\alpha = 2$ の場合を考えよう．この場合は Gauss 場と密接な関係がある [9, 35].

定理 3.13. $\{X(x)\}_{x \in R}$ は平均 0, 共分散 K の *Gauss* 場とすると

$$E[\Pi_{X^2}(d\xi)] = \mu_{2,K}(d\xi) \tag{3.7}$$

となる．ただし，Π_{X^2} は *intensity* $X(x)^2 \cdot \lambda(dx)$ の Q 上の *Poisson* 測度であり，E は *Gauss* 場 $X(x)$ による平均をあらわす.

証明. (3.1) に注意して Laplace 変換を計算すると以下のようになることから定理は示される．

$$\begin{aligned} E\left[\int_Q \Pi_{X^2}(d\xi)\exp\left(-\langle \xi, f\rangle\right)\right] &= E\left[\exp - \int_R (1-e^{-f(x)})X(x)^2\lambda(dx)\right] \\ &= \mathrm{Det}(I + 2(1-e^{-f})K)^{-1/2}. \end{aligned}$$

□

系 3.14. n 次非負定値行列 A に対して平均 0 で共分散行列が A である *Gauss* 確率変数を $(Z_1,\ldots,Z_n)$ とすると

$$\det_2 A = E[Z_1^2 \cdots Z_n^2] \geq 0$$

である[4].

[4]記号がよくないがここでの $\det_2$ は regularized determinant ではなく，$\det_\alpha$ の $\alpha = 2$ の場合である．

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