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LINEAR QUADRATIC DIFFERENTIAL GAMES: SADDLE POINT AND RICCATI DIFFERENTIAL EQUATION*

MICHEL C. DELFOUR†

Abstract. Zhang [*SIAM J. Control Optim.*, 43 (2005), pp. 2157–2165] recently established the equivalence between the finiteness of the *open loop value* of a two-player zero-sum linear quadratic (LQ) game and the finiteness of its open loop lower and upper values. In this paper we complete and sharpen the results of Zhang for the finiteness of the lower value of the game by providing a set of necessary and sufficient conditions that emphasizes the *feasibility condition*: $(0, 0)$ is a solution of the open loop lower value of the game for the zero initial state. Then we show that, under the assumption of an open loop saddle point in the time horizon $[0, T]$ for all initial states, there is an open loop saddle point in the time horizon $[s, T]$ for all initial times s, $0 \leq s < T$, and all initial states at time s. From this we get an *optimality principle* and adapt the *invariant embedding approach* to construct the decoupling symmetrical matrix function $P(s)$ and show that it is an $H^1(0, T)$ solution of the matrix Riccati differential equation. Thence an open loop saddle point in $[0, T]$ yields closed loop optimal strategies for both players. Furthermore, a necessary and sufficient set of conditions for the existence of an open loop saddle point in $[0, T]$ for all initial states is the convexity-concavity of the utility function and the existence of an $H^1(0, T)$ symmetrical solution to the matrix Riccati differential equation. As an illustration of the cases where the open loop lower/upper value of the game is $-\infty/+\infty$, we work out two informative examples of solutions to the Riccati differential equation with and without blow-up time.

Key words. linear quadratic differential game, saddle point, value of a game, Riccati differential equation, open loop and closed loop strategies, conjugate point, blow-up time

AMS subject classifications. 91A05, 91A23, 49N70, 91A25

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1. Introduction.

We consider the *two-player zero-sum game* with *linear dynamics* and a *quadratic utility function* over a *finite time horizon*. The min sup problem was studied in 1969 by [5]. The fundamental theory of closed loop linear quadratic (LQ) games was given in 1979 by Bernhard [4] followed by the seminal book of Başar and Bernhard [1] in 1991 and 1995. A very nice paper by Zhang [10] in 2005 established the equivalence between the finiteness of the open loop value of a two-player zero-sum LQ game and the finiteness of its open loop lower and upper values.

In this paper we complete and sharpen the results of [10] for the finiteness of the lower value of the game by providing a set of necessary and sufficient conditions (Theorem 2.2) that emphasizes the *feasibility condition*: $(0, 0)$ is a solution of the open loop lower value of the game for the zero initial state. A similar feasibility condition holds for the finiteness of the open loop upper value and value of the game. It also recasts the results in the more intuitive state-adjoint state framework.

Then we show that, under the assumption of an open loop saddle point in the time horizon $[0, T]$ for all initial states, there is a unique open loop saddle point in

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†Centre de recherches mathématiques et Département de mathématiques et de statistique, Université de Montréal, C. P. 6128, succ. Centre-ville, Montréal H3C 3J7, QC, Canada (delfour@crm.UMontreal.ca).

the time horizon $[s, T]$ for all initial times s, $0 \leq s < T$, and all initial states at time s (Theorem 5.2(iii)). From this we get an *optimality principle* and adapt the *invariant embedding approach* of Bellman in the style of Lions [9] to construct the decoupling symmetrical matrix function $P(s)$ (Theorem 2.9) and show that it is an $H^1(0,T)$ solution of the matrix Riccati differential equation. Thence an open loop saddle point in $[0,T]$ yields closed loop optimal strategies for both players who achieve a closed loop-closed loop saddle point in the sense of Bernhard [4]. Furthermore, a necessary and sufficient set of conditions for the existence of an open loop saddle point in $[0,T]$ for all initial states is the convexity-concavity of the utility function and the existence of a symmetrical $H^1(0,T)$ solution to the matrix Riccati differential equation (Theorem 2.10). As an illustration of the cases where the open loop lower/upper value of the game is $-\infty/+\infty$, we work out two informative examples of solutions to the Riccati differential equation with and without blow-up time.

2. Definitions, notation, and main results.

2.1. System, utility function, values of the game. Given a finite dimensional Euclidean space $\mathbf{R}^d$ of dimension $d \geq 1$, the *norm* and *inner product* will be denoted by $|x|$ and $x \cdot y$, respectively, irrespective of the dimension d of the space. Given $T > 0$, the norm and inner product in $L^2(0,T;\mathbf{R}^n)$ will be denoted $\|f\|$ and (f,g). The norm in the Sobolev space $H^1(0,T;\mathbf{R}^n)$ will be written $\|f\|_{H^1}$.

Consider the following two-player zero-sum game over the finite time interval $[0,T]$ characterized by the quadratic *utility function*

$$C_{x_0}(u,v) \stackrel{\text{def}}{=} Fx(T) \cdot x(T) + \int_0^T Q(t)x(t) \cdot x(t) + |u(t)|^2 - |v(t)|^2 \, dt, \tag{2.1}$$

where x is the solution of the linear differential system

$$\frac{dx}{dt}(t) = A(t)x(t) + B_1(t)u(t) + B_2(t)v(t) \quad \text{a.e. in } [0,T], \quad x(0) = x_0, \tag{2.2}$$

x_0 is the *initial state* at time $t = 0$, $u \in L^2(0,T;\mathbf{R}^m)$, $m \geq 1$, is the strategy of the first player, and $v \in L^2(0,T;\mathbf{R}^k)$, $k \geq 1$, is the strategy of the second player. We assume that F is an $n \times n$ matrix and that A, B_1, B_2, and Q are matrix functions of appropriate order that are measurable and bounded a.e. in $[0,T]$. Moreover, $Q(t)$ and F are symmetrical. It will be convenient to use the following compact notation and drop the "a.e. in $[0,T]$" wherever no confusion arises:

$$C_{x_0}(u,v) = Fx(T) \cdot x(T) + \int_0^T Qx \cdot x + |u|^2 - |v|^2 \, dt, \tag{2.3}$$

$$x' = Ax + B_1 u + B_2 v \quad \text{in } [0,T], \quad x(0) = x_0. \tag{2.4}$$

The above assumptions on F, A, B_1, B_2, and Q will be used throughout this paper.

Remark 2.1. The more general quadratic structure involving cross terms and different quadratic weights $N_1 u \cdot u$ and $N_2 v \cdot v$ on u and v (cf., for instance, Bernhard [4, section 2, p. 53]),

$$\int_0^T (x,u,v) \cdot \begin{bmatrix} Q & S & T \\ S^* & N_1 & 0 \\ T^* & 0 & -N_2 \end{bmatrix} \begin{bmatrix} x \\ u \\ v \end{bmatrix} dt,$$

can be brought back to the simpler form (2.1)–(2.2) by the following change of variables:

$$\begin{bmatrix} x \\ u \\ v \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ -N_1^{-1}S^* & N_1^{-1/2} & 0 \\ N_2^{-1}T^* & 0 & N_2^{-1/2} \end{bmatrix} \begin{bmatrix} x \\ \bar{u} \\ \bar{v} \end{bmatrix},$$

where $N_1(t)$ and $N_2(t)$ are symmetrical positive definite matrices such that

$$\begin{aligned} &\exists \nu_1 > 0 \text{ such that } \forall u \in \mathbf{R}^m \text{ and almost all } t, \quad N_1(t)u \cdot u \geq \nu_1 \, |u|^2, \\ &\exists \nu_2 > 0 \text{ such that } \forall v \in \mathbf{R}^k \text{ and almost all } t, \quad N_2(t)v \cdot v \geq \nu_2 \, |v|^2. \end{aligned} \tag{2.5}$$

This yields the simpler initial structure with the system and the utility function

$$x' = \overline{A}x + \overline{B}_1\bar{u} + \overline{B}_2\bar{v}, \qquad \int_0^T \overline{Q}x \cdot x + |\bar{u}|^2 - |\bar{v}|^2 \, dt$$

by introducing the new matrices

$$\begin{aligned} &\overline{A} = A - B_1N_1^{-1}S^* + B_2N_2^{-1}T^*, \quad \overline{B}_1 = B_1N_1^{-1/2}, \quad \overline{B}_2 = B_2N_2^{-1/2}, \\ &\overline{Q} = Q - SN_1^{-1}S^* + TN_2^{-1}T^*. \end{aligned}$$

The matrix functions N_1, N_2, S, and T are all assumed to be measurable and bounded.

DEFINITION 2.1. *Let x_0 be an initial state in $\mathbf{R}^n$ at time $t = 0$.*

(i) *The game is said to achieve its* open loop lower value *(resp.,* upper value*) if*

$$v^-(x_0) \overset{\text{def}}{=} \sup_{v \in L^2(0,T;\mathbf{R}^k)} \inf_{u \in L^2(0,T;\mathbf{R}^m)} C_{x_0}(u,v) \tag{2.6}$$

$$(\textit{resp., } v^+(x_0) \overset{\text{def}}{=} \inf_{u \in L^2(0,T;\mathbf{R}^m)} \sup_{v \in L^2(0,T;\mathbf{R}^k)} C_{x_0}(u,v)) \tag{2.7}$$

is finite. By definition $v^-(x_0) \leq v^+(x_0)$.

(ii) *The game is said to achieve its* open loop value *if its open loop lower value $v^-(x_0)$ and upper value $v^+(x_0)$ are achieved and $v^-(x_0) = v^+(x_0)$. The* open loop value *of the game will be denoted by $v(x_0)$.*

(iii) *A pair $(\bar{u}, \bar{v})$ in $L^2(0,T;\mathbf{R}^m) \times L^2(0,T;\mathbf{R}^k)$ is an* open loop saddle point *of $C_{x_0}(u,v)$ in $L^2(0,T;\mathbf{R}^m) \times L^2(0,T;\mathbf{R}^k)$ if for all u in $L^2(0,T;\mathbf{R}^m)$ and all v in $L^2(0,T;\mathbf{R}^k)$*

$$C_{x_0}(\bar{u}, v) \leq C_{x_0}(\bar{u}, \bar{v}) \leq C_{x_0}(u, \bar{v}). \tag{2.8}$$

In general, (ii) *does not necessarily imply* (iii)*, but we shall see that it does for LQ games.*

DEFINITION 2.2. *Associate with $x_0 \in \mathbf{R}^n$ the sets and the functions*

$$V(x_0) \overset{\text{def}}{=} \left\{ v \in L^2(0,T;\mathbf{R}^k) : \inf_{u \in L^2(0,T;\mathbf{R}^m)} C_{x_0}(u,v) > -\infty \right\}, \tag{2.9}$$

$$U(x_0) \overset{\text{def}}{=} \left\{ u \in L^2(0,T;\mathbf{R}^m) : \sup_{v \in L^2(0,T;\mathbf{R}^k)} C_{x_0}(u,v) < +\infty \right\}, \tag{2.10}$$

$$J^-_{x_0}(v) \overset{\text{def}}{=} \inf_{u \in L^2(0,T;\mathbf{R}^m)} C_{x_0}(u,v), \quad J^+_{x_0}(u) \overset{\text{def}}{=} \sup_{v \in L^2(0,T;\mathbf{R}^k)} C_{x_0}(u,v). \tag{2.11}$$

By definition, $V(x_0) \neq \varnothing$ if and only if $v^-(x_0) > -\infty$, and $U(x_0) \neq \varnothing$ if and only if $v^+(x_0) < +\infty$.

2.2. Saddle points of the game and solution of the Riccati differential equation. In the literature, an important issue is the connection between the existence of a symmetrical solution to the *matrix Riccati differential equation*

$$P' + PA + A^*P - PRP + Q = 0 \text{ a.e. in } [0,T], \quad P(T) = F, \tag{2.12}$$

where $R = B_1B_1^* - B_2B_2^*$, and the existence of either an open or closed loop[1] lower value, upper value, or saddle point of the game. For instance, in the closed loop case, quoting Bernhard [4] in his introduction,

> "It has long been known that, for the two-person, zero-sum differential game with linear dynamics, quadratic payoff, fixed end-time, and free end-state (*standard LQ game*), the existence of a solution to a Riccati equation is a sufficient condition for the existence of a saddle point within the class of instantaneous state feedback strategies (cf. [8], [7]), and therefore within any wider class (cf. [3])."

Similarly, we quote Zhang [10] in his introduction,

> "(a) if the Riccati differential equation admits a solution, then, the game admits a closed loop-closed loop saddle point,"

where he refers to [4].

In the open loop case, the above statements are incomplete (cf. Example 2.2), even under the assumptions

$$F \geq 0 \text{ and } Q(t) \geq 0 \text{ a.e. in } [0,T]$$

used in [4] that necessarily imply the convexity of $C_{x_0}(u,v)$ with respect to u and $V(x_0) = L^2(0,T;\mathbf{R}^k)$ for all $x_0 \in \mathbf{R}^n$. Even when the solution of the Riccati differential equation (2.12) is $H^1(0,T)$ or bounded (Remark 2.5), it is also *necessary* that the utility function be convex in u and concave in v (Theorem 2.10) to get an open loop-open loop saddle point.

This leaves the cases where either the open loop lower or upper value of the game explodes. In such cases the solution of the Riccati differential equation might have a blow-up time as illustrated in Example 2.1 (cf. Bernhard [4, Example 5.1, p. 67]:

> "The following game has a saddle point that *survives* a conjugate point,"

where he means a closed loop-closed loop saddle point). The conjugate point corresponds to a *blow-up time* of the solution of the Riccati equation (2.12), where the solution is not of the $H^1(0,T)$ type. Finally, an open loop saddle point yields closed loop optimal strategies that achieve a closed loop-closed loop saddle point (Theorem 2.9), but the *converse is not necessarily true*. It is informative to first detail the example of Bernhard.

Example 2.1. Consider the dynamics and utility function in the time interval $[0,2]$:

$$x'(t) = (2-t)\,u(t) + t\,v(t) \text{ a.e. in } [0,2], \quad x(0) = x_0, \tag{2.13}$$

$$C_{x_0}(u,v) = \frac{1}{2}|x(2)|^2 + \int_0^2 |u(t)|^2 - |v(t)|^2\,dt. \tag{2.14}$$

Here $A = 0$, $B_1(t) = 2-t$, $B_2(t) = t$, $F = 1/2$, $Q = 0$, and $R = B_1B_1^* - B_2B_2^* = 4(1-t)$. It is shown in [4] that the Riccati equation reduces to

$$P' - 4(1-t)P^2 = 0, \quad P(2) = 1/2 \quad \Rightarrow P(t) = \frac{1}{2(t-1)^2}.$$

[1] The reader is referred to Bernhard [4] for the *closed loop* definitions.

Its solution is positive and blows up at $t = 1$. It is not an element of $H^1(0, 2)$. We now show that there is no open loop saddle point in the time interval $[0, 2]$. For the open loop lower value of the game, the minimization with respect to u has a unique solution for all (x_0, v) since the utility function $u \mapsto C_{x_0}(u, v)$ is convex and bounded below by $-\|v\|^2_{L^2}$. The minimizer is completely characterized by the coupled system

$$\begin{cases} x'(t) = (2 - t)\,\hat{u}(t) + t\,v(t) \text{ a.e. in } [0, 2], & x(0) = x_0, \\ p'(t) = 0 \text{ a.e. in } [0, 2], & p(2) = \dfrac{1}{2}x(2), \end{cases}$$
$$\hat{u}(t) = -(2 - t)\,p(t).$$

From this

$$x(2) = \frac{3}{7}\left[x_0 + \int_0^2 s\,v(s)\,ds\right] \quad \text{and} \quad p(t) = \frac{1}{2}x(2)$$

and

$$\begin{aligned} J^-_{x_0}(v) &\stackrel{\text{def}}{=} \inf_{u \in L^2(0,2;\mathbf{R})} C_{x_0}(u, v) \\ &= C_{x_0}(\hat{u}, v) = \frac{1}{2}x(2)^2 + \frac{1}{4}\,x(2)^2 \int_0^2 (2 - t)^2\,dt - \int_0^2 |v(t)|^2\,dt \\ &= \frac{7}{6}x(2)^2 - \int_0^2 |v(t)|^2\,dt = \frac{3}{14}\left[x_0 + \int_0^2 s\,v(s)\,ds\right]^2 - \int_0^2 |v(t)|^2\,dt. \end{aligned}$$

It is readily seen that $J^-_{x_0}$ is concave in v and that the supremum with respect to v of $J^-_{x_0}(v)$ exists. Indeed, from the first order condition,[2]

$$\forall v,\ \frac{1}{2}dJ^-_{x_0}(\hat{v}; v) = \frac{3}{14}\left[x_0 + \int_0^2 s\,\hat{v}(s)\,ds\right] \int_0^2 s\,v(s)\,ds - \int_0^2 \hat{v}(t)\,v(t)\,dt = 0,$$

there is a unique stationary point $\hat{v}(t) = t\,x_0/2$, the expression of the Hessian

$$\begin{aligned} \frac{1}{2}d^2 J^-_{x_0}(\hat{v}; v; v) &= \frac{3}{14}\left[\int_0^2 s\,v(s)\,ds\right]^2 - \int_0^2 |v(t)|^2\,dt \\ &\le \frac{3}{14}\left[\int_0^2 s^2\,ds\right]\left[\int_0^2 |v(s)|^2\,ds\right] - \int_0^2 |v(t)|^2\,dt \\ &\le \left[\frac{3}{14}\,\frac{2^3}{3} - 1\right] \int_0^2 |v(t)|^2\,dt = -\frac{3}{7}\int_0^2 |v(t)|^2\,dt \le 0 \end{aligned}$$

is negative, and the open loop lower value of the game is $v^-(x_0) = J^-_{x_0}(\hat{v}) = (x_0)^2/2$.

However, the open loop upper value of the game is $v^+(x_0) = +\infty$ for all $x_0 \in \mathbf{R}$. Indeed pick the sequence of controls $\{v_n\}$, $n \ge 1$, $v_n(t) = 0$ in $[0, 1]$, and $v_n(t) = n$ in

[2]Given a real function f defined on a Banach space B, the *first directional semiderivative* at x in the direction v (when it exists) is defined as $df(x; v) \stackrel{\text{def}}{=} \lim_{t \searrow 0} \frac{f(x+tv)-f(x)}{t}$. When the map $v \mapsto df(x; v) : B \to \mathbf{R}$ is linear and continuous, it defines the *gradient* $\nabla f(x)$ as an element of the dual B^* of B. The *second order bidirectional derivative* at x in the directions (v, w) (when it exists) is defined as $d^2 f(x; v, w) \stackrel{\text{def}}{=} \lim_{t \searrow 0} \frac{df(x+tw;v)-df(x;v)}{t}$. When the map $(v, w) \mapsto d^2 f(x; v, w) : B \times B \to \mathbf{R}$ is bilinear and continuous, it defines the *Hessian operator* $Hf(x)$ as a continuous linear operator from B to B^*.

$[1,2]$. The corresponding sequence of states at time $t=2$ is

$$x_n(2) = x_0 + \int_0^2 (2-t)\,u(t)\,dt + n\int_1^2 t\,dt = \left[x_0 + \int_0^2 (2-t)\,u(t)\,dt\right] + \frac{3}{2}n.$$

Denote by X the square bracket that does not depend on n. Then

$$\begin{aligned} C_{x_0}(u,v_n) &= \frac{1}{2}\left|X + \frac{3}{2}n\right|^2 + \int_0^2 |u(t)|^2\,dt - \int_1^2 n^2\,dt \\ &= \frac{1}{8}n^2 + \frac{3}{2}n\,X + \frac{X^2}{2} + \int_0^2 |u(t)|^2\,dt \to +\infty \text{ as } n\to+\infty. \end{aligned}$$

Thus for all $x_0 \in \mathbf{R}$ and $u \in L^2(0,T;\mathbf{R})$

$$\sup_{v\in L^2(0,T;\mathbf{R})} C_{x_0}(u,v) = +\infty \quad \Rightarrow v^+(x_0) = +\infty \quad \text{and} \quad U(x_0) = \varnothing.$$

Therefore, whatever the initial state x_0 is, $C_{x_0}(u,v)$ has no open loop saddle point.

We now consider the example of Zhang [10] of a game without open loop saddle point. We show that the solution of the Riccati differential equation (2.12) is unique, strictly positive, and *infinitely differentiable*.

Example 2.2. Consider the utility function and linear dynamics

$$C_{x_0}(u,v) = \int_0^1 2x^2 + u^2 - v^2\,dt, \quad x' = x + u + v, \quad x(0) = x_0 \tag{2.15}$$

given by Zhang [10]. Here $A = B_1 = B_2 = 1$, $F = 0$, and $Q = 2$. Now $R = B_1B_1^* - B_2B_2^* = 0$, and the associated Riccati differential equation (2.12) reduces to

$$P' + 2P + 2 = 0 \text{ in } [0,1], \quad P(1) = 0.$$

It has a unique infinitely differentiable solution $P(t) = e^{2(1-t)} - 1$ that is strictly positive in $[0,1)$.

We now extend the result of Zhang [10] on the nonexistence of an open loop saddle point from the initial state $x_0 = 0$ to any initial state. For all $x_0 \in \mathbf{R}$ the open loop lower value $v^-(x_0)$ of the game is finite, but the open loop upper value $v^+(x_0)$ is $+\infty$. Indeed for each $v \in L^2(0,T;\mathbf{R})$

$$\begin{aligned} &\inf_{u\in L^2(0,T;\mathbf{R})} C_{x_0}(u,v) \le C_{x_0}(-v,v) = \int_0^1 2\left(x_0\,e^t\right)^2\,dt = \left(e^2-1\right)(x_0)^2 \\ &\quad\Rightarrow v^-(x_0) = \sup_{v\in L^2(0,T;\mathbf{R})}\ \inf_{u\in L^2(0,T;\mathbf{R})} C_{x_0}(u,v) \le \left(e^2-1\right)(x_0)^2. \end{aligned}$$

By definition of the sup,

$$\begin{aligned} v^-(x_0) &= \sup_{v\in L^2(0,T;\mathbf{R})}\ \inf_{u\in L^2(0,T;\mathbf{R})} C_{x_0}(u,v) \\ &\ge \inf_{u\in L^2(0,T;\mathbf{R})} C_{x_0}(u,0) = \inf_{u\in L^2(0,T;\mathbf{R})} \int_0^1 2x^2 + u^2\,dt \ge 0 \\ &\quad\Rightarrow \forall x_0 \in \mathbf{R}, \quad 0 \le v^-(x_0) \le \left(e^2-1\right)(x_0)^2. \end{aligned}$$

For the open loop upper value, associate with each $u \in L^2(0,T;\mathbf{R})$ the sequence of functions $v_n(t) = -u(t) + n$, $n \geq 1$. The corresponding sequence of states is

$$
\begin{aligned}
x_n(t) &= e^t x_0 + n \int_0^t e^{t-s}\, ds = e^t x_0 + n(e^t - 1),\\
C_{x_0}(u, v_n) &= n^2 \int_0^1 2(e^t - 1)^2 - 1\, dt + 2n \int_0^1 u(t)\, dt\\
&\quad + \int_0^1 (e^t x_0)^2\, dt + 2n\, x_0 \int_0^1 e^t\, (e^t - 1)\, dt\\
&= n^2 \int_0^1 1 + 2e^{2t} - 4e^t\, dt + 2n \int_0^1 u(t)\, dt\\
&\quad + \int_0^1 (e^t x_0)^2\, dt + 2n\, x_0\, (e-1)^2\\
&\geq (e-2)^2 n^2 + 2n \left[x_0\, (e-1)^2 + \int_0^1 u(t)\, dt \right] + \int_0^1 (e^t x_0)^2\, dt\\
&\Rightarrow \sup_{v \in L^2(0,T;\mathbf{R})} C_{x_0}(u, v) \geq C_{x_0}(u, v_n) \to +\infty
\end{aligned}
$$

as n goes to infinity. Therefore for all $x_0 \in \mathbf{R}^n$ and all $u \in L^2(0,T;\mathbf{R})$,

$$
\sup_{v \in L^2(0,T;\mathbf{R})} C_{x_0}(u, v) = +\infty \quad \Rightarrow v^+(x_0) = +\infty \quad \text{and} \quad U(x_0) = \varnothing,
$$

and there is no open loop saddle point.

2.3. Properties of the utility function, convexity, concavity, and saddle points. We use a state-adjoint state equation approach to characterize the existence of the open loop upper and lower values as well as the open loop saddle point of the quadratic utility function.

The utility function $C_{x_0}(u, v)$ is infinitely differentiable and, since it is quadratic, its Hessian of second order derivatives is independent of the point (u, v). Indeed,

$$
\frac{1}{2} dC_{x_0}(u, v; \bar{u}, \bar{v}) = Fx(T) \cdot \bar{y}(T) + (Qx, \bar{y}) + (u, \bar{u}) - (v, \bar{v}), \tag{2.16}
$$

where x is the solution of (2.4) and $\bar{y}$ is the solution of

$$
\bar{y}' = A\bar{y} + B_1 \bar{u} + B_2 \bar{v}, \quad \bar{y}(0) = 0. \tag{2.17}
$$

It is customary to introduce the adjoint system

$$
p' + A^* p + Qx = 0, \quad p(T) = Fx(T) \tag{2.18}
$$

and rewrite expression (2.16) for the gradient in the following form:

$$
\frac{1}{2} dC_{x_0}(u, v; \bar{u}, \bar{v}) = (B_1^* p + u, \bar{u}) + (B_2^* p - v, \bar{v}). \tag{2.19}
$$

As predicted, the Hessian is independent of (u, v):

$$
\frac{1}{2} d^2 C_{x_0}(u, v; \bar{u}, \bar{v}; \tilde{u}, \tilde{v}) = F\tilde{y}(T) \cdot \bar{y}(T) + (Q\tilde{y}, \bar{y}) + (\tilde{u}, \bar{u}) - (\tilde{v}, \bar{v}), \tag{2.20}
$$

where $\bar{y}$ is the solution of (2.17) and $\tilde{y}$ is the solution of

$$\tilde{y}' = A\tilde{y} + B_1\tilde{u} + B_2\tilde{v}, \quad \tilde{y}(0) = 0. \tag{2.21}$$

In particular, for all x_0, u, v, $\bar{u}$, and $\bar{v}$

$$d^2 C_{x_0}(u, v; \bar{u}, \bar{v}; \bar{u}, \bar{v}) = 2\, C_0(\bar{u}, \bar{v}), \tag{2.22}$$

and this yields the following characterizations of the u-convexity, v-concavity, and (u, v)-convexity-concavity under the assumptions of section 2.1 on the matrix F and the matrix functions A, B_1, B_2, and Q. Note that the matrices F and $Q(t)$ are symmetrical, but they are not necessarily positive semidefinite.

LEMMA 2.1. *Let F be an $n \times n$ matrix and A, B_1, B_2, and Q be bounded measurable matrix functions of appropriate dimensions, and assume that F and $Q(t)$ are symmetrical for almost all t. Then the following statements are equivalent.*

(i) *The map $u \mapsto C_0(u, 0) : L^2(0, T; \mathbf{R}^m) \to \mathbf{R}$ is convex.*
(ii) *For all $u \in L^2(0, T; \mathbf{R}^m)$, $C_0(u, 0) \geq 0$.*
(iii) $\inf_{u \in L^2(0,T;\mathbf{R}^m)} C_0(u, 0) = C_0(0, 0)$.
(iv) *For all v and x_0 the map $u \mapsto C_{x_0}(u, v) : L^2(0, T; \mathbf{R}^m) \to \mathbf{R}$ is convex.*

COROLLARY 2.1. *The following statements are equivalent.*

(i) *The map $v \mapsto C_0(0, v) : L^2(0, T; \mathbf{R}^k) \to \mathbf{R}$ is concave.*
(ii) *For all $v \in L^2(0, T; \mathbf{R}^k)$, $C_0(0, v) \leq 0$.*
(iii) $\sup_{v \in L^2(0,T;\mathbf{R}^k)} C_0(0, v) = C_0(0, 0)$.
(iv) *For all u and x_0, the map $v \mapsto C_{x_0}(u, v) : L^2(0, T; \mathbf{R}^k) \to \mathbf{R}$ is concave.*

COROLLARY 2.2. *The following statements are equivalent.*

(i) *The map, $(u, v) \mapsto C_0(u, v) : L^2(0, T; \mathbf{R}^m) \times L^2(0, T; \mathbf{R}^k) \to \mathbf{R}$ is (u, v)-convex-concave; that is,*

$$\begin{aligned} &\forall v \in L^2(0, T; \mathbf{R}^k), \quad u \mapsto C_0(u, v) \text{ is convex, and} \\ &\forall u \in L^2(0, T; \mathbf{R}^m), \quad v \mapsto C_0(u, v) \text{ is concave.} \end{aligned} \tag{2.23}$$

(ii) *The pair $(0, 0)$ is a saddle point of $C_0(u, v)$:*

$$\sup_{v \in L^2(0,T;\mathbf{R}^k)} C_0(0, v) = C_0(0, 0) = \inf_{u \in L^2(0,T;\mathbf{R}^m)} C_0(u, 0). \tag{2.24}$$

(iii) $\sup_{v \in L^2(0,T;\mathbf{R}^k)} C_0(0, v) = C_0(0, 0) = \inf_{u \in L^2(0,T;\mathbf{R}^m)} C_0(u, 0)$.
(iv) *For all x_0 the map $(u, v) \mapsto C_{x_0}(u, v) : L^2(0, T; \mathbf{R}^m) \times L^2(0, T; \mathbf{R}^k) \to \mathbf{R}$ is (u, v)-convex-concave; that is,*

$$\begin{aligned} &\forall v \in L^2(0, T; \mathbf{R}^k), \quad u \mapsto C_{x_0}(u, v) \text{ is convex, and} \\ &\forall u \in L^2(0, T; \mathbf{R}^m), \quad v \mapsto C_{x_0}(u, v) \text{ is concave.} \end{aligned} \tag{2.25}$$

2.4. Saddle point and coupled state-adjoint state system. We first obtain necessary and sufficient conditions for the existence of a saddle point of the game and introduce the *coupled (state-adjoint state) system* (cf. Notation 2.1 on page 758) that will also arise in the characterization of the open loop lower and upper values of the game in section 2.5. Theorem 2.4 in section 2.5 will later complete this theorem with the equivalent condition that the value $v(x_0)$ of the game is finite.

THEOREM 2.1. *The following conditions are equivalent.*

(i) *There exists an open loop* saddle point *of $C_{x_0}(u, v)$.*

(ii) *There exists a solution* $(\hat{u},\hat{v})$ *in* $L^2(0,T;\mathbf{R}^m)\times L^2(0,T;\mathbf{R}^k)$ *of the system*

$$\forall u\in L^2(0,T;\mathbf{R}^m),\ \forall v\in L^2(0,T;\mathbf{R}^k),\quad dC_{x_0}(\hat{u},\hat{v};u,v)=0, \tag{2.26}$$

and C_{x_0} *is convex-concave in the sense of* (2.25).

(iii) *There exists a solution* $(x,p)\in H^1(0,T;\mathbf{R}^n)^2$ *of the* coupled system

$$\begin{cases} x' = Ax - B_1B_1^*p + B_2B_2^*p, & x(0)=x_0,\\ p' + A^*p + Qx = 0, & p(T)=Fx(T),\end{cases} \tag{2.27}$$

$$\hat{u} = -B_1^*p,\quad \hat{v} = B_2^*p, \tag{2.28}$$

and

$$\sup_{v\in L^2(0,T;\mathbf{R}^k)} C_0(0,v) = C_0(0,0) = \inf_{u\in L^2(0,T;\mathbf{R}^m)} C_0(u,0). \tag{2.29}$$

Under any one of the above conditions, the value of the game is given by

$$v(x_0) = C_{x_0}(\hat{u},\hat{v}) = p(0)\cdot x_0. \tag{2.30}$$

Proof. (i) $\Rightarrow$ (ii). Let $(\bar{u},\bar{v})$ in $L^2(0,T;\mathbf{R}^m)\times L^2(0,T;\mathbf{R}^k)$ be an open loop saddle point of $C_{x_0}(u,v)$ in $L^2(0,T;\mathbf{R}^m)\times L^2(0,T;\mathbf{R}^k)$. Then by Definition 2.1

$$\sup_{L^2(0,T;\mathbf{R}^k)} C_{x_0}(\bar{u},v) = C_{x_0}(\bar{u},\bar{v}) = \inf_{L^2(0,T;\mathbf{R}^m)} C_{x_0}(u,\bar{v}). \tag{2.31}$$

Since $C_{x_0}(u,v)$ is infinitely differentiable, the minimizing point $\bar{u}$ of $C_{x_0}(u,\bar{v})$ with respect to u is characterized by the first order condition $dC_{x_0}(\bar{u},\bar{v};u,0)=0$ for all u and the second order condition $d^2C_{x_0}(\bar{u},\bar{v};u,0;u,0)\geq 0$ for all u. Since $d^2C_{x_0}(\bar{u},\bar{v};u,0;u,0)$ is independent of $(\bar{u},\bar{v})$ and x_0, $C_{x_0}(u,v)$ is convex in u for all x_0 and all v. A similar argument for the maximum yields $dC_{x_0}(\bar{u},\bar{v};,w)=0$ and $d^2C_{x_0}(\bar{u},\bar{v};0,w;0,w)\leq 0$ for all w and the concavity of $C_{x_0}(u,v)$ in v.

(ii) $\Rightarrow$ (i). By assumption $C_{x_0}(\hat{u},\hat{v})$ is convex-concave and infinitely differentiable and there is a solution to the two first order conditions. By [6, Proposition 1.6], there exists a saddle point.

(ii) $\Leftrightarrow$ (iii). This follows from the previous computations of the gradient and Corollary 2.2.

Finally, we compute the value

$$\begin{aligned} C_{x_0}(\hat{u},\hat{v}) &= Fx(T)\cdot x(T) + (Qx,x) + \|B_1^*p\|^2 - \|B_2^*p\|^2\\ &= p(T)\cdot x(T) - (p'+A^*p,x) + ([B_1B_1^* - B_2B_2^*]p,p)\\ &= p(0)\cdot x(0) + (p, x' - Ax + ([B_1B_1^* - B_2B_2^*]p) = p(0)\cdot x_0. \end{aligned}$$ ◻

NOTATION 2.1. *It will be useful to introduce the set* $\mathcal{N}_{x,p}$ *of all solutions* (y,q) *of the homogeneous coupled system*

$$\begin{cases} y' = Ay - B_1B_1^*q + B_2B_2^*q, & y(0)=0,\\ q' + A^*q + Qy = 0, & q(T)=Fy(T).\end{cases} \tag{2.32}$$

Thus the coupled system has a solution up to an additive pair of $\mathcal{N}_{x,p}$.

2.5. Necessary and sufficient conditions for games with finite values. The quadratic character of the problem yields surprising equivalences that reduce the complexity of its solution. We start with the open loop lower value of the game.

THEOREM 2.2. *The following conditions are equivalent.*

(i) *There exist $\hat{u}$ in $L^2(0,T;\mathbf{R}^m)$ and $\hat{v}$ in $L^2(0,T;\mathbf{R}^k)$ such that*

$$C_{x_0}(\hat{u},\hat{v}) = \inf_{u\in L^2(0,T;\mathbf{R}^m)} C_{x_0}(u,\hat{v}) = \sup_{v\in L^2(0,T;\mathbf{R}^k)} \inf_{u\in L^2(0,T;\mathbf{R}^m)} C_{x_0}(u,v). \tag{2.33}$$

(ii) *The open loop lower value $v^-(x_0)$ of the game is finite.*

(iii) *There exists a solution $(x,p)\in H^1(0,T;\mathbf{R}^n)\times H^1(0,T;\mathbf{R}^n)$ of the* coupled system (2.27) *such that $B_2^*p\in V(x_0)$, the solution pairs $(\hat{u},\hat{v})$ and the open loop lower value are given by the expressions*

$$\hat{u} = -B_1^*p, \quad \hat{v} = B_2^*p, \quad \text{and} \quad v^-(x_0) = C_{x_0}(\hat{u},\hat{v}) = p(0)\cdot x_0, \tag{2.34}$$

and

$$\sup_{v\in V(0)} \inf_{u\in L^2(0,T;\mathbf{R}^m)} C_0(u,v) = \inf_{u\in L^2(0,T;\mathbf{R}^m)} C_0(u,0) = C_0(0,0). \tag{2.35}$$

Proof. The proof of this main theorem will be given in sections 3 and 3.3. ∎

Remark 2.2. The above necessary and sufficient conditions for the finiteness of the open loop value of the game complete the results of Zhang [10] by introducing the new *feasibility condition* (2.35) that is equivalent to saying that the open loop lower value of the game is zero and that $(0,0)$ is a solution for the zero initial state. It also recasts the results in the more intuitive state-adjoint state framework. Condition (2.35) is equivalent to the convexity of $C_{x_0}(u,v)$ with respect to u and the concavity of $J^-_{x_0}(v) = \inf_{u\in L^2(0,T;\mathbf{R}^m)} C_{x_0}(u,v)$ with respect to $v\in V(x_0)$.

Theorem 2.2 has a counterpart for the upper value $v^+(x_0)$ of the game.

THEOREM 2.3. *The following conditions are equivalent.*

(i) *There exist $\hat{u}$ in $L^2(0,T;\mathbf{R}^m)$ and $\hat{v}$ in $L^2(0,T;\mathbf{R}^k)$ such that*

$$C_{x_0}(\hat{u},\hat{v}) = \sup_{v\in L^2(0,T;\mathbf{R}^k)} C_{x_0}(\hat{u},v) = \inf_{u\in L^2(0,T;\mathbf{R}^m)} \sup_{v\in L^2(0,T;\mathbf{R}^k)} C_{x_0}(u,v). \tag{2.36}$$

(ii) *The open loop upper value $v^+(x_0)$ of the game is finite.*

(iii) *There exists a solution $(x,p)\in H^1(0,T;\mathbf{R}^n)\times H^1(0,T;\mathbf{R}^n)$ of the* coupled system (2.27) *such that $-B_1^*p\in U(x_0)$, the solution pairs $(\hat{u},\hat{v})$ and the open loop upper value are given by the expressions*

$$\hat{u} = -B_1^*p, \quad \hat{v} = B_2^*p, \quad \text{and} \quad v^+(x_0) = C_{x_0}(\hat{u},\hat{v}) = p(0)\cdot x_0, \tag{2.37}$$

and

$$\inf_{u\in U(0)} \sup_{v\in L^2(0,T;\mathbf{R}^k)} C_0(u,v) = \sup_{v\in L^2(0,T;\mathbf{R}^k)} C_0(0,v) = C_0(0,0). \tag{2.38}$$

Condition (2.38) says that $C_{x_0}(u,v)$ is concave with respect to v and that $J^+_{x_0}(u) = \sup_{v\in L^2(0,T;\mathbf{R}^k)} C_{x_0}(u,v)$ is convex with respect to $u\in U(x_0)$.

Finally, the necessary and sufficient condition for the finiteness of the value $v(x_0)$ of the game can now be obtained from the above two theorems and Theorem 2.1(iii).

THEOREM 2.4. *The following conditions are equivalent.*

(i) *There exists an open loop* saddle point *of* $C_{x_0}(u,v)$.
(ii) *The open loop value* $v(x_0)$ *of the game is finite.*
(iii) *There exists a solution* $(x,p) \in H^1(0,T;\mathbf{R}^n) \times H^1(0,T;\mathbf{R}^n)$ *of the* coupled system (2.27)*, the solution pair* $(\hat{u},\hat{v})$ *is given by the expressions* (2.28)*, and the convexity-concavity* (2.29) *is verified.*

Under any one of the above conditions, the open loop value is given by expression (2.30).

Proof. (i) $\Rightarrow$ (ii). Since the utility function has a saddle point, the value of the game is finite. (ii) $\Rightarrow$ (iii). From Theorems 2.2 and 2.3 there exists a solution to the coupled system (2.27), and the convexity-concavity condition (2.29) readily follows from (2.38) and (2.35). (iii) $\Rightarrow$ (i). This follows from Theorem 2.1. □

Remark 2.3. The common necessary condition for the finiteness of the lower value $v^-(x_0)$, value $v(x_0)$, and upper value $v^+(x_0)$ of the game is the existence of a solution of the coupled system (2.27). The difference is in the respective *feasibility conditions* (2.35), (2.29), and (2.38): $v^-(0) = 0$, $v(0) = 0$, and $v^+(0) = 0$.

We conclude with the enlightening result proved by Zhang [10, Thm. 4.1] that has shed new light on the characterization of a game with finite value. One of the consequences is that only three cases can occur: (i) $v^+(x_0)$ finite and $v^-(x_0) = -\infty$, (ii) $v^+(x_0) = +\infty$ and $v^-(x_0)$ finite, or (iii) $v(x_0)$ finite.

Theorem 2.5. *Given* $x_0 \in \mathbf{R}^n$*, the following statements are equivalent.*
(i) *There exists an open loop saddle point of* $C_{x_0}(u,v)$.
(ii) *The open loop value of the game of* $C_{x_0}(u,v)$ *is finite.*
(iii) *Both the open loop lower and upper values of* $C_{x_0}(u,v)$ *are finite.*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are obvious. It remains to prove that (iii) $\Rightarrow$ (i). From condition (2.35) of Theorem 2.2 and condition (2.38) of Theorem 2.3 we get condition (2.29) of Theorem 2.4. Finally, both Theorems 2.2 and 2.3 give the existence of a pair $(x,p) \in H^1(0,T;\mathbf{R}^n) \times H^1(0,T;\mathbf{R}^n)$ solution of the coupled system (2.27). Therefore by Theorem 2.4 the utility function has a saddle point. □

2.6. Games with finite values for each initial state. In this section we sharpen the results of the previous section when the open loop lower value, value, or upper value of the game is finite for *all initial states* $x_0 \in \mathbf{R}^n$. In each case this global assumption yields the *uniqueness of solution.*

Theorem 2.6. *The following conditions are equivalent.*
(i) *For each* $x_0 \in \mathbf{R}^n$*, there exist* $\hat{u}$ *in* $L^2(0,T;\mathbf{R}^m)$ *and* $\hat{v}$ *in* $L^2(0,T;\mathbf{R}^k)$ *such that*

$$C_{x_0}(\hat{u},\hat{v}) = \inf_{u \in L^2(0,T;\mathbf{R}^m)} C_{x_0}(u,\hat{v}) = \sup_{v \in L^2(0,T;\mathbf{R}^k)} \inf_{u \in L^2(0,T;\mathbf{R}^m)} C_{x_0}(u,v). \tag{2.39}$$

(ii) *For each* $x_0 \in \mathbf{R}^n$*, the open loop lower value* $v^-(x_0)$ *of the game is finite.*
(iii) *For each* $x_0 \in \mathbf{R}^n$*, there exists a unique pair* $(x,p) \in H^1(0,T;\mathbf{R}^n)^2$ *solution of the* coupled system (2.27) *such that* $B_2^* p \in V(x_0)$*, there exists a unique pair* $(\hat{u},\hat{v})$ *that verifies* (2.28)*, and*

$$\sup_{v \in V(0)} \inf_{u \in L^2(0,T;\mathbf{R}^m)} C_0(u,v) = \inf_{u \in L^2(0,T;\mathbf{R}^m)} C_0(u,0) = C_0(0,0). \tag{2.40}$$

Remark 2.4. The uniqueness under condition (i) was originally given by Zhang et al. in [11] by a different argument. Our short and transparent proof seems to be new. The same proof can readily be used in the context of optimal control [9].

Proof. (i) $\Rightarrow$ (ii) is obvious. (ii) $\Rightarrow$ (iii). This follows from Theorem 2.2 where condition (2.40) is condition (2.35). We need only show the uniqueness of the solution to the coupled system (2.27). By linearity, this amounts to proving that the solution (y,q) of the homogeneous system (2.32) such that $B_2^* q \in V(0)$ is $(0,0)$. Given an arbitrary x_0, consider the expression

$$\begin{aligned} q(0)\cdot x_0 &= q(T)\cdot x(T) - \int_0^T q'\cdot x + q\cdot x'\,dt \\ &= Fx(T)\cdot y(T) + \int_0^T Qx\cdot y + B_1^*p\cdot B_1^*q - B_2^*p\cdot B_2^*q\,dt \\ &= \frac{1}{2} dC_{x_0}(\hat{u},\hat{v};-B_1^*q, B_2^*q) = 0 \end{aligned}$$

from (2.16), (2.27), (2.34), and the fact that $B_2^*q \in V(0)$. Since this identity is true for all $x_0 \in \mathbf{R}^n$, $q(0)=0$. But now we can look at the coupled system (2.32) as a linear differential system in (x,p) with zero initial condition $(y(0),q(0))=(0,0)$ whose unique solution is $(y,q)=(0,0)$. This proves uniqueness. (iii) $\Rightarrow$ (i). This follows, again from Theorem 2.2, since the conditions are verified for each $x_0 \in \mathbf{R}^n$. □

We readily have the dual result.

THEOREM 2.7. *The following conditions are equivalent.*

(i) *For each $x_0 \in \mathbf{R}^n$, there exist $\hat{u}$ in $L^2(0,T;\mathbf{R}^m)$ and $\hat{v}$ in $L^2(0,T;\mathbf{R}^k)$ such that*

$$C_{x_0}(\hat{u},\hat{v}) = \sup_{v\in L^2(0,T;\mathbf{R}^k)} C_{x_0}(\hat{u},v) = \inf_{u\in L^2(0,T;\mathbf{R}^m)} \sup_{v\in L^2(0,T;\mathbf{R}^k)} C_{x_0}(u,v). \tag{2.41}$$

(ii) *For each $x_0 \in \mathbf{R}^n$, the open loop upper value $v^+(x_0)$ of the game is finite.*

(iii) *For each $x_0 \in \mathbf{R}^n$, there exists a unique pair $(x,p) \in H^1(0,T;\mathbf{R}^n)^2$ solution of the* coupled system (2.27) *such that $-B_1^*p \in U(x_0)$, there exists a unique pair $(\hat{u},\hat{v})$ that verifies* (2.28)*, and*

$$\inf_{u\in U(0)} \sup_{v\in L^2(0,T;\mathbf{R}^k)} C_0(u,v) = \sup_{v\in L^2(0,T;\mathbf{R}^k)} C_0(0,v) = C_0(0,0). \tag{2.42}$$

Finally, by combining the last two theorems, we get the saddle point case.

THEOREM 2.8. *The following conditions are equivalent.*

(i) *For each $x_0 \in \mathbf{R}^n$, there exists an open loop* saddle point *of $C_{x_0}(u,v)$.*

(ii) *For each $x_0 \in \mathbf{R}^n$, the open loop value $v(x_0)$ of the game is finite.*

(iii) *For each $x_0 \in \mathbf{R}^n$, there exists a unique pair $(x,p) \in H^1(0,T;\mathbf{R}^n)^2$ solution of the* coupled system (2.27)*, there exists a unique pair $(\hat{u},\hat{v})$ that verifies* (2.28)*, and the convexity-concavity condition* (2.29) *is verified.*

2.7. Open loop saddle point and Riccati differential equation. Under the assumption of the finiteness of the open loop value of the game in $[0,T]$ for each initial state, we can *unexpectedly* use *invariant embedding* and introduce a *decoupling symmetrical matrix* solution of the *matrix Riccati differential equation* (2.12).

THEOREM 2.9. *Assume that the open loop value $v(x_0)$ is finite for all $x_0 \in \mathbf{R}^n$.*

(i) *There exists a unique symmetrical solution with elements in $H^1(0,T)$ of the matrix Riccati differential equation*

$$P' + PA + A^*P - PRP + Q = 0, \quad P(T) = F, \tag{2.43}$$

where $R = B_1B_1^* - B_2B_2^*$. *Moreover,*

$$\hat{p}(t) = P(t)\,\hat{x}(t),\ 0 \le t \le T,\ \text{and } C_{x_0}(\hat{u},\hat{v}) = P(0)x_0 \cdot x_0, \tag{2.44}$$

where $(\hat{x},\hat{p}) \in H^1(0,T;\mathbf{R}^n)^2$ *is the unique solution of the coupled system* (2.27).

(ii) *The optimal strategies of the two players are closed loop*

$$\hat{u} = -B_1^* P\hat{x} \quad \text{and} \quad \hat{v} = B_2^* P\hat{x}, \tag{2.45}$$

and they achieve a closed loop-closed loop saddle point in the sense of [4].

(iii) *For all* $x_0 \in \mathbf{R}^n$ *the function* $C_{x_0}(u,v)$ *is convex-concave.*

Proof. For the proof, see section 5.3. □

The existence of a symmetrical solution to the matrix Riccati differential equation (2.43) implies that, for all $x_0 \in \mathbf{R}^n$, there exists a solution $(\hat{x},\hat{p}) \in H^1(0,T;\mathbf{R}^n) \times H^1(0,T;\mathbf{R}^n)$ of the coupled system (2.27). However, as we have seen in Example 2.2, this is not sufficient to get an open loop saddle point of the utility function $C_{x_0}(u,v)$.

THEOREM 2.10. *A set of necessary and sufficient conditions for the existence of an open loop saddle point of the utility function* $C_{x_0}(u,v)$ *for all* $x_0 \in \mathbf{R}^n$ *is*

(a) *the utility function* $C_{x_0}(u,v)$ *is convex in* u *and concave in* v *for some* x_0*, and*

(b) *there exists a (unique) symmetrical solution in* $H^1(0,T)$ *to the matrix Riccati differential equation* (2.43).

Proof. For the proof, see section 5.4. □

Remark 2.5. The method of *completion of the squares* (cf., for instance, Başar and Bernhard [1, Chap. 9, Thm. 9.4]) can also be used here to obtain

$$\sup_{v\in L^2(0,T;\mathbf{R}^k)} \inf_{u\in L^2(0,T;\mathbf{R}^m)} C_{x_0}(u,v) \le P(0)x_0 \cdot x_0 \le \inf_{u\in L^2(0,T;\mathbf{R}^m)} \sup_{v\in L^2(0,T;\mathbf{R}^k)} C_{x_0}(u,v).$$

So it would be tempting to conclude that there is a saddle point *without* condition (a). But, as illustrated in Example 2.2 where we show that $U(x_0) = \varnothing$ for all x_0, condition (a) is really necessary. In order to get a saddle point, both $v^-(x_0)$ and $v^+(x_0)$ must be finite. Therefore the open loop lower value of the game will be finite if (b) is verified and $V(x_0) \ne \varnothing$; the open loop upper value of the game will be finite if (b) is verified and $U(x_0) \ne \varnothing$.

3. Open loop lower value of the game. We review the three steps: existence and characterization of a minimizer for $v \in V(x_0)$, formulation of the resulting maximization problem with respect to v, and, finally, existence and characterization of the pair that achieves the finite open loop lower value of the game.

3.1. Existence and characterization of the minimizers.

THEOREM 3.1. *Given* $x_0 \in \mathbf{R}^n$ *and* $v \in L^2(0,T;\mathbf{R}^k)$*, the following statements are equivalent.*

(i) *There exists* $\hat{u} \in L^2(0,T;\mathbf{R}^m)$ *such that*

$$C_{x_0}(\hat{u},v) = J^-_{x_0}(v) = \inf_{u\in L^2(0,T;\mathbf{R}^m)} C_{x_0}(u,v). \tag{3.1}$$

(ii) $\inf_{u\in L^2(0,T;\mathbf{R}^m)} C_{x_0}(u,v) > -\infty$ *(that is,* $v \in V(x_0)$*).*

(iii) *There exists a pair* $(x, p) \in H^1(0, T; \mathbf{R}^n) \times H^1(0, T; \mathbf{R}^n)$ *solution of the system*

$$\begin{cases} x' = Ax - B_1 B_1^* p + B_2 v, & x(0) = x_0, \\ p' + A^* p + Qx = 0, & p(T) = Fx(T), \end{cases} \tag{3.2}$$

$$\hat{u}(t) = -B_1^*(t)p(t), \quad J_{x_0}^-(v) = p(0) \cdot x_0 + \int_0^T B_2^* p \cdot v - |v|^2 \, dt, \tag{3.3}$$

and

$$\inf_{u \in L^2(0,T;\mathbf{R}^m)} C_0(u, 0) \geq 0. \tag{3.4}$$

(iv) *The convexity inequality* (3.4) *is verified and*

$$\forall q \in N_p, \quad x_0 \cdot q(0) + \int_0^T v \cdot B_2^* q \, dt = 0, \tag{3.5}$$

where

$$N_p \stackrel{\text{def}}{=} \left\{ q \in H^1(0, T; \mathbf{R}^n) : \forall (y, q) \in N_{x,p} \right\} \tag{3.6}$$

and $N_{x,p}$ *denotes the set of all solutions* (y, q) *of the homogeneous system*

$$\begin{cases} y' = Ay - B_1 \, B_1^* q, & y(0) = 0, \\ q' + A^* q + Qy = 0, & q(T) = Fy(T). \end{cases} \tag{3.7}$$

Proof. The proof follows from the following lemma, the computation of first and second order derivatives (2.19) and (2.22) in section 2.3, and the equivalent condition of Lemma 2.1(ii) for the u-convexity of $C_0(u, v)$.

LEMMA 3.1. *Let* U *be a Hilbert space,* $M : U \to U$ *a continuous linear self-adjoint compact operator,* $f \in U$*,* c *a constant, and* $j(u) = c + 2(f, u) + ([I + M]u, u)$.

(i) *Then the following conditions are equivalent.*

(a)

$$\exists \hat{u} \in U, \quad j(\hat{u}) = \inf_{u \in U} j(u), \tag{3.8}$$

(b)

$$\inf_{u \in U} j(u) > -\infty, \tag{3.9}$$

(c)

$$\exists \hat{u} \in U \text{ such that } [I + M]\hat{u} + f = 0, \text{ and} \tag{3.10}$$

$$\forall u \in U, \quad ([I + M]u, u) \geq 0. \tag{3.11}$$

(ii) *Condition* (3.10) *is equivalent to*

$$\forall w \in \ker[I + M], \quad (f, w) = 0. \tag{3.12}$$

(iii) *Condition* (3.11) *is equivalent to the convexity of* j.

We omit the proof of the lemma. □

NOTATION 3.1. *Given* $x_0 \in \mathbf{R}^n$ *such that* $V(x_0) \neq \varnothing$ *and* $v \in V(x_0)$*, denote by* $\mathcal{P}(v, x_0)$ *the set of all solutions* (x, p) *of system* (3.2)*. It is readily checked that for all* $p \in \mathcal{P}(v, x_0)$*,* $\mathcal{P}(v, x_0) = p + N_p$.

3.2. Some intermediary results.

THEOREM 3.2.

(i) *The sets* $N_{x,p}$, N_p, *and* $B_2^* N_p$ *are finite dimensional linear subspaces of* $H^1(0,T;\mathbf{R}^n)^2$, $H^1(0,T;\mathbf{R}^n)$, *and* $L^2(0,T;\mathbf{R}^k)$, *respectively.* $\mathcal{P}(v,x_0)$ *is a finite dimensional affine subspace of* $H^1(0,T;\mathbf{R}^n)$.

(ii) *If* $V(x_0) \neq \varnothing$ *for some* $x_0 \in \mathbf{R}^n$, *then* $V(x_0)$ *is a closed affine subspace of* $L^2(0,T;\mathbf{R}^k)$, $V(0)$ *is a nonempty closed linear subspace of* $L^2(0,T;\mathbf{R}^k)$,

$$V(0) = (B_2^* N_p)^\perp, \tag{3.13}$$

$$\forall v \in V(x_0), \quad V(x_0) = v + V(0). \tag{3.14}$$

(iii) *Given* $v \in V(x_0)$ *and* $p \in \mathcal{P}(v,x_0)$, *define*

$$v^* \stackrel{\text{def}}{=} v + P_{V(0)}(B_2^* p - v), \tag{3.15}$$

where $P_{V(0)}$ *is the orthogonal projection onto* $V(0)$ *in* $L^2(0,T;\mathbf{R}^k)$. *Then* v^* *is independent of the choice of* p, v^* *is unique in* $V(x_0) \cap B_2^* \mathcal{P}(v,x_0)$, *and there exists* $p^* \in \mathcal{P}(v,x_0)$ *such that* $v^* = B_2^* p^*$. *If, in addition,* $B_2^* p - v \in V(0)^\perp$, *then* $v = v^* = B_2^* p^*$.

Analogues of Theorems 3.1 and 3.2 hold for the open loop upper value.

Remark 3.1. This theorem due to Zhang [10] is a key result in the proof of the existence of a maximizer of the inf problem. We have added part (i) to show that the subspace $B_2^* N_p$ is finite dimensional and hence closed. This is critical in the proof of part (ii). The proof essentially uses the arguments of [10].

Proof of Theorem 3.2. (i) From system (3.7), $N_{x,p}$ is a closed linear subspace of $H^1(0,T;\mathbf{R}^n)^2$ as the kernel of the continuous linear map

$$\begin{aligned}(x,p) &\mapsto \mathcal{A}(x,p) \stackrel{\text{def}}{=} (-x' + Ax - B_1\, B_1^* p, -x(0), p' + A^* p + Qx, Fx(T) - p(T)) \\ &: H^1(0,T;\mathbf{R}^n)^2 \to \left(L^2(0,T;\mathbf{R}^n) \times \mathbf{R}^n\right)^2.\end{aligned}$$

We now use the fact that a topological vector space is finite dimensional if and only if every closed bounded set is compact. Indeed, let K be a closed bounded subset of points (y,q) in $N_{x,p}$ for the $L^2(0,T;\mathbf{R}^n)^2$-topology. Since all the matrices in system (3.7) are bounded, the right-hand sides are bounded and the derivatives (y',q') are also bounded in $L^2(0,T;\mathbf{R}^n)^2$ and, a fortiori, in $H^1(0,T;\mathbf{R}^n)^2$. Since the injection of $H^1(0,T;\mathbf{R}^n)^2$ into $L^2(0,T;\mathbf{R}^n)^2$ is compact, then the closure of K in $L^2(0,T;\mathbf{R}^n)^2$ is compact. But, by assumption, we already know that K is closed. Thence K is compact in $L^2(0,T;\mathbf{R}^n)^2$ and $N_{x,p}$ is finite dimensional.

(ii) Since $V(x_0) \neq \varnothing$, then, by definition, for all v_1, v_2 in $V(x_0)$, condition (ii) of Theorem 3.1 is verified and condition (iii) is also verified for some pairs (x_1,p_1) and (x_2,p_2) verifying the system (3.2). Therefore, for any $\alpha \in \mathbf{R}$, the pair $(x_\alpha, p_\alpha) = (\alpha x_1 + (1-\alpha)x_2, \alpha p_1 + (1-\alpha)p_2)$ is also a solution of system (3.2) for x_0 and $v_\alpha = \alpha v_1 + (1-\alpha) v_2 \in V(x_0)$. Identity (3.14) follows from the fact that $V(x_0)$ is an affine subspace. Moreover, from (3.14), $V(x_0) \neq \varnothing$ necessarily implies that $V(0) \neq \varnothing$. Finally, from condition (3.5) with $x_0 = 0$

$$v \in V(0) \quad \Leftrightarrow \quad \forall q \in N_p, \quad \int_0^T v \cdot B_2^* q \, dt = 0 \quad \Leftrightarrow \; v \in (B_2^* N_p)^\perp,$$

and $V(0) = (B_2^* N_p)^\perp$, a nonempty closed linear subspace.

(iii) Given p_1, p_2 in $\mathcal{P}(v,x_0)$, $p_2-p_1\in N_p$ and

$$v+P_{V(0)}(B_2^*p_2-v)-(v+P_{V(0)}(B_2^*p_1-v))=P_{V(0)}(B_2^*(p_2-p_1))=0,$$

since $B_2^*N_p=V(0)^\perp$. So v^* is independent of the choice of $p\in\mathcal{P}(v,x_0)$. Since $V(x_0)$ is affine, then for all $v\in V(x_0)$,

$$\begin{aligned}
v^* &= v+P_{V(0)}(B_2^*p-v)\in v+V(0)=V(x_0), \qquad (3.16)\\
v^*-B_2^*p &= v-B_2^*p-P_{V(0)}(v-B_2^*p)\in V(0)^\perp=B_2^*N_p\\
&\Rightarrow \exists q\in N_p \text{ such that } v^*-B_2^*p=B_2^*q \quad \Rightarrow v^*=B_2^*(p+q)\in B_2^*\mathcal{P}(v,x_0),
\end{aligned}$$

and $v^*\in V(x_0)\cap B_2^*\mathcal{P}(v,x_0)$. This element is unique since for v_1^* and v_2^* in $V(x_0)\cap B_2^*\mathcal{P}(v,x_0)$, $v_2^*-v_1^*\in V(0)\cap B_2^*N_p=V(0)\cap V(0)^\perp=\{0\}$. Finally, if $B_2^*p-v\in V(0)^\perp$, then from (3.16) we get $v=v^*$. □

3.3. Existence and characterization of maximizers. Assume that $v^-(x_0)$ is finite. By definition of $V(x_0)$, it is not empty and

$$v^-(x_0)=\sup_{v\in L^2(0,T;\mathbf{R}^k)}\inf_{u\in L^2(0,T;\mathbf{R}^m)}C_{x_0}(u,v)=\sup_{v\in V(x_0)}J_{x_0}^-(v), \tag{3.17}$$

where $V(x_0)$ is a closed affine subspace of $L^2(0,T;\mathbf{R}^k)$ and by (3.3) and condition (3.5)

$$J_{x_0}^-(v)=\inf_{u\in L^2(0,T;\mathbf{R}^m)}C_{x_0}(u,v)=p(0)\cdot x_0+\int_0^T B_2^*p\cdot v-|v|^2\,dt, \tag{3.18}$$

or, equivalently,

$$J_{x_0}^-(v)=Fx(T)\cdot x(T)+\int_0^T Q(t)x(t)\cdot x(t)+|B_1^*(t)p(t)|^2-|v(t)|^2\,dt \tag{3.19}$$

for all solutions (x,p) of system (3.2). Define the equivalence class $[(x,p)]=(x,p)+N_{x,p}$. Then for each pair $v\in V(x_0)$, $[(x,p)]$ is the unique solution in $H^1(0,T;\mathbf{R}^n)\times H^1(0,T;\mathbf{R}^n)/N_{x,p}$ of system (3.2). Thus the map

$$v\mapsto[(x,p)]:V(x_0)\to\frac{H^1(0,T;\mathbf{R}^n)\times H^1(0,T;\mathbf{R}^n)}{N_{x,p}} \tag{3.20}$$

is affine and continuous, and the map

$$\begin{aligned}
&(x,p)\mapsto(x(T),x,p)\\
&:H^1(0,T;\mathbf{R}^n)\times H^1(0,T;\mathbf{R}^n)\to\mathbf{R}^n\times L^2(0,T;\mathbf{R}^n)\times L^2(0,T;\mathbf{R}^n)
\end{aligned} \tag{3.21}$$

is continuous and compact.

So we are back to a continuous linear quadratic function $J_{x_0}^-(v)$ that is to be maximized over the closed affine subspace $V(x_0)$. The state is now the pair (x,p) solution of (3.2), but the structure is the same. Lemma 3.1 readily extends to the case of a sup over a closed affine subspace and the following conditions are equivalent:

(a)

$$\exists\hat v\in V(x_0),\quad J_{x_0}^-(\hat v)=\sup_{v\in V(x_0)}J_{x_0}^-(v), \tag{3.22}$$

(b)

$$\sup_{v\in V(x_0)} J^-_{x_0}(v) < +\infty, \tag{3.23}$$

(c)

$$\exists \hat v \in V(x_0) \text{ such that } [I+M]\hat v + f \in V(0)^\perp, \text{ and} \tag{3.24}$$

$$\forall w \in V(0), \quad ([I+M]w, w) \le 0 \tag{3.25}$$

for the new compact operator M corresponding to the new state (x, p).

It remains to compute the directional derivative of $J^-_{x_0}(v)$ at $v \in V(x_0)$ in the direction $w \in V(0)$. By direct computation from formula (2.19)

$$\frac{1}{2} dC_{x_0}(-B_1^* p, v; 0, w) = \int_0^T (B_2^* p - v)\cdot w\, dt, \tag{3.26}$$

which is independent of $p \in \mathcal{P}(v, x_0)$ for all $w \in V(0)$ by Theorem 3.1(iv). Hence

$$dJ^-_{x_0}(v; w) = dC_{x_0}(-B_1^* p, v; 0, w) = 2\int_0^T (B_2^* p - v)\cdot w\, dt, \quad \forall p \in \mathcal{P}(v, x_0). \tag{3.27}$$

As for the second order derivative,

$$\begin{aligned} &\frac{1}{2} d^2 C_{x_0}(-B_1^* p, v; 0, w; 0, w') \\ &= F y_{w'}(T)\cdot y_w(T) + \int_0^T Q y_{w'}\cdot y_w + B_1^* q_{w'}\cdot B_1^* q_w - w'\cdot w\, dt \end{aligned} \tag{3.28}$$

$$\begin{aligned} \Rightarrow\ \frac{1}{2} d^2 J^-_{x_0}(v; w; w) &= \frac{1}{2} d^2 C_{x_0}(-B_1^* p, v; 0, w; 0, w) \\ &= \frac{1}{2} J_0^-(w) = \frac{1}{2} \inf_{u\in L^2(0.T;\mathbf{R}^m)} C_0(u, w), \end{aligned} \tag{3.29}$$

where the last term must be negative or zero for all $w \in V(0)$. But, from Theorem 3.1(iii), $C_0(u, 0)$ is convex in u. By using the equivalent condition of Lemma 2.1(ii) for the u-convexity of $C_0(u, 0)$, we finally get the two-part condition

$$\sup_{v\in V(0)} \inf_{u\in L^2(0,T;\mathbf{R}^m)} C_0(u, v) \le 0 \le \inf_{u\in L^2(0,T;\mathbf{R}^m)} C_0(u, 0).$$

This condition is equivalent to condition (2.35) since $C_0(0, 0) = 0$.

Proof of Theorem 2.2. (i) $\Rightarrow$ (ii) is obvious. (ii) $\Rightarrow$ (iii). From the previous discussion, the finiteness of $v^-(x_0)$ is equivalent to

$$\begin{aligned} \exists \hat v \in V(x_0) \text{ such that } dJ^-_{x_0}(\hat v; w) = 2\int_0^T (B_2^* \hat p - \hat v)\cdot w\, dt = 0, \quad \forall w \in V(0), \\ d^2 J^-_{x_0}(\hat v; w; w) = 2 \inf_{u\in L^2(0.T;\mathbf{R}^m)} C_0(u, w) \le 0, \quad \forall w \in V(0). \end{aligned}$$

The second order condition combined with the fact that $V(x_0) \neq \varnothing$ (Theorem 3.1(iii)) yields condition (2.35). The first order condition says that $B_2^* \hat p - \hat v \in V(0)^\perp$. By Theorem 3.2(iii) there exists $\hat p^* \in \mathcal{P}(\hat v, x_0)$ such that $\hat v = B_2^* \hat p^*$, where $(\hat x^*, \hat p^*)$ is a solution of (3.2). Since $\hat v = B_2^* \hat p^*$, system (2.27) has a solution unique up to an element of $\mathcal{N}_{x,p}$. After substitution of $\hat{v} = B_2^* \hat{p}^*$ in (3.2), $(\hat{x}^*, \hat{p}^*)$ becomes a solution of the coupled system (2.27). This also yields the identities (2.34). (iii) $\Rightarrow$ (i). By assumption $\hat{v} = B_2^* p \in V(x_0)$. The existence of a solution (x, p) to system (2.27) yields the existence of a solution to system (3.2) of Theorem 3.1(iii) with $\hat{u} = -B_1^* p$ as a minimizer. For all $v \in V(x_0)$,

$$J_{x_0}^-(v) = J_{x_0}^-(B_2^* p) + dJ_{x_0}^-(B_2^* p; v - B_2^* p) + \frac{1}{2} d^2 J_{x_0}^-(B_2^* p; v - B_2^* p; v - B_2^* p).$$

The second order term is negative by condition (2.35) since, by assumption, $B_2^* p \in V(x_0)$ and hence $v - B_2^* p \in V(0)$ for all $v \in V(x_0)$. As for the first order term, recall that, in view of (2.34), for all $w \in V(0)$

$$dJ_{x_0}^-(B_2^* p; w) = \int_0^T (B_2^* p - v) \cdot w \, dt = 0.$$

Thus $dJ_{x_0}^-(B_2^* p; v - B_2^* p) = 0$ since $v - B_2^* p \in V(0)$: $B_2^* p$ is a maximizer of $J_{x_0}^-$. $\square$

4. Invariant embedding and convexity-concavity. Consider the LQ game on the time interval $[s, T]$, $0 \le s < T$, with initial state $h \in \mathbf{R}^n$ at time s:

$$C_h^s(u, v) \overset{\text{def}}{=} Fx(T) \cdot x(T) + \int_s^T Qx \cdot x + |u|^2 - |v|^2 \, dt, \tag{4.1}$$

$$x' = Ax + B_1 u + B_2 v \quad \text{a.e. in } [s, T], \quad x(s) = h. \tag{4.2}$$

DEFINITION 4.1. *Let $h \in \mathbf{R}^n$ be an initial state at time s, $0 \le s < T$.*

(i) *The game is said to achieve its* open loop lower value *(resp.,* upper value*) if*

$$v_s^-(h) \overset{\text{def}}{=} \sup_{v \in L^2(s.T; \mathbf{R}^k)} \inf_{u \in L^2(s.T; \mathbf{R}^m)} C_h^s(u, v) \tag{4.3}$$

$$\text{(resp., } v_s^+(h) \overset{\text{def}}{=} \inf_{u \in L^2(s.T; \mathbf{R}^m)} \sup_{v \in L^2(s,T; \mathbf{R}^k)} C_h^s(u, v)\text{)} \tag{4.4}$$

is finite.

(ii) *The game is said to achieve its* open loop value *if its open loop lower value $v_s^-(h)$ and upper value $v_s^+(h)$ are achieved and $v_s^-(h) = v_s^+(h)$. The open loop value of the game will be denoted by $v_s(h)$.*

(iii) *A pair $(\bar{u}, \bar{v})$ in $L^2(s, T; \mathbf{R}^m) \times L^2(s, T; \mathbf{R}^k)$ is an* open loop saddle point *of $C_h^s(u, v)$ if for all u in $L^2(s, T; \mathbf{R}^m)$ and all v in $L^2(s, T; \mathbf{R}^k)$*

$$C_h^s(\bar{u}, v) \le C_h^s(\bar{u}, \bar{v}) \le C_h^s(u, \bar{v}). \tag{4.5}$$

The first result is that, if the $C_{x_0}(u, v)$ is convex, concave, or convex-concave for some x_0, so is $C_h^s(u, v)$ for all $h \in \mathbf{R}^n$ and all s, $0 \le s < T$.

THEOREM 4.1.

(i) *If, for all $(x_0, v) \in \mathbf{R}^n \times L^2(0, T; \mathbf{R}^k)$, the map $u \mapsto C_{x_0}(u, v)$ is convex, then for all s, $0 \le s < T$, and all $(h, v) \in \mathbf{R}^n \times L^2(s, T; \mathbf{R}^k)$ the map $u \mapsto C_h^s(u, v)$ is convex.*

(ii) *If, for all $(x_0, u) \in \mathbf{R}^n \times L^2(0, T; \mathbf{R}^m)$, the map $v \mapsto C_{x_0}(u, v)$ is concave, then for all s, $0 \le s < T$, and all $(h, u) \in \mathbf{R}^n \times L^2(s, T; \mathbf{R}^m)$ the map $v \mapsto C_h^s(u, v)$ is concave.*

Proof. We prove only (i). From (2.20)–(2.22) for all $(u,v) \in L^2(0,T;\mathbf{R}^m) \times L^2(0,T;\mathbf{R}^k)$,

$$\forall \bar u \in L^2(0,T;\mathbf{R}^m), \quad d^2C_{x_0}(u,v;\bar u,0;\bar u,0) \\ = F\bar y(T)\cdot \bar y(T) + (Q\bar y,\bar y) + (\bar u,\bar u) \geq 0, \tag{4.6}$$

where $\bar y$ is the solution of

$$\bar y' = A\bar y + B_1\bar u, \quad \bar y(0) = 0. \tag{4.7}$$

To prove the same result on $[s,T]$, associate with each $\bar u \in L^2(s,T;\mathbf{R}^m)$ its extension by zero $\tilde{\bar u}$ from $[s,T]$ to $[0,T]$. Therefore

$$\forall \bar u \in L^2(s,T;\mathbf{R}^m), \quad F\bar y(T)\cdot \bar y(T) + \int_0^T Q\bar y\cdot \bar y + \tilde{\bar u}\cdot \tilde{\bar u}\, dt \geq 0, \tag{4.8}$$

where $\bar y$ is the solution of

$$\bar y' = A\bar y + B_1\tilde{\bar u}, \quad \bar y(0) = 0. \tag{4.9}$$

Notice that, since $\tilde{\bar u}$ is zero in $[0,s]$, $\bar y = 0$ in $[0,s]$ and $\bar y$ is also the solution of

$$\bar y' = A\bar y + B_1\bar u, \quad \bar y(s) = 0 \tag{4.10}$$

$$\Rightarrow \forall \bar u \in L^2(s,T;\mathbf{R}^m), \quad F\bar y(T)\cdot \bar y(T) + \int_s^T Q\bar y\cdot \bar y + \bar u\cdot \bar u\, dt \geq 0. \tag{4.11}$$

Hence for all $h \in \mathbf{R}^n$, all $(u,v) \in L^2(s,T;\mathbf{R}^m)\times L^2(s,T;\mathbf{R}^k)$, and all $\bar u \in L^2(s,T;\mathbf{R}^m)$,

$$d^2C^s_h(u,v;\bar u,0;\bar u,0) = F\bar y(T)\cdot \bar y(T) + \int_s^T Q\bar y\cdot \bar y + \bar u\cdot \bar u\, dt \\ = d^2C_{x_0}(0,0;\tilde{\bar u},0;\tilde{\bar u},0) \geq 0.$$

Thus for all s and all (h,v), the map $u \mapsto C^s_h(u,v)$ is convex. ∎

5. Decoupling and Riccati differential equation in the saddle point case.

5.1. Open loop saddle point optimality principle. At this juncture, it is important to notice that the necessary conditions (2.35) and (2.38) associated with the respective finiteness of the lower and upper values of the game on $[0,T]$ do not generally survive on $[s,T]$. However the convexity-concavity condition (2.29) does.

THEOREM 5.1. *Assume that $v(x_0)$ is finite for some $x_0 \in \mathbf{R}^n$, let $x(\cdot;x_0), p(\cdot;x_0)$ be a solution of the coupled system* (2.27) *in $[0,T]$, and let s, $0 \le s < T$.*

(i) *The value $v_s(x(s;x_0))$ of the game is finite.*

(ii) *The restriction of (x,p) to $[s,T]$ is a solution of the coupled system*

$$\begin{cases} x_s' = Ax_s - B_1B_1^*p_s + B_2B_2^*p_s \text{ a.e. in } [s,T], & x_s(s) = x(s;x_0), \\ p_s' + A^*p_s + Qx_s = 0, & p_s(T) = Fx_s(T), \end{cases} \tag{5.1}$$

the restrictions $(u_s,v_s) = (u|_{[s,T]}, v_{[s,T]})$ of the controls (u,v) on $[0,T]$ to $[s,T]$ verify

$$u_s = -B_1^*p_s \text{ and } v_s = B_2^*p_s, \quad v_s(x(s;x_0)) = p_s(s)\cdot x(s;x_0), \tag{5.2}$$

$$v(x_0) = v_s(x(s;x_0)) + \int_0^s Qx\cdot x + |u|^2 - |v|^2\, dt, \tag{5.3}$$

and

$$\sup_{v \in L^2(s,T;\mathbf{R}^k)} C_0^s(0,v) = C_0^s(0,0) = \inf_{u \in L^2(s,T;\mathbf{R}^m)} C_0^s(u,0). \tag{5.4}$$

Proof. From Theorem 2.4 on $[s,T]$, part (i) is equivalent to part (ii), and thus it is sufficient to prove part (ii). From Theorem 4.1, the convexity-concavity conditions on $[0,T]$ survive on $[s,T]$, and we get (5.4). Moreover, if $(x(\cdot;x_0), p(\cdot;x_0))$ is a solution of the coupled system (2.27) in $[0,T]$ with initial state x_0 at time 0 and the controls (u,v) verify identities (2.28), then the restrictions $(x_s,p_s) = (x|_{[s,T]}, p|_{[s,T]})$ are a solution to the coupled system (5.1), and the restrictions $(u_s,v_s) = (u|_{[s,T]}, v_{[s,T]})$ of the controls on $[0,T]$ verify (5.2). Thus, by the analogue of Theorem 2.4, we get the finiteness of the value of the game on $[s,T]$. $\square$

THEOREM 5.2. *Assume that $v(x_0)$ is finite for all $x_0 \in \mathbf{R}^n$.*

(i) *The solution (x_s,p_s) of the coupled system* (5.1) *and the controls (u_s,v_s) on $[s,T]$ in* (5.2) *are unique.*

(ii) *The map*

$$x_0 \mapsto X(s)x_0 \stackrel{\text{def}}{=} x(s;x_0) : \mathbf{R}^n \to \mathbf{R}^n \tag{5.5}$$

is a linear bijection, where $x(\cdot;x_0), p(\cdot;x_0)$ is the unique solution of the coupled system (2.27) *in $[0,T]$.*

(iii) *For all $h \in \mathbf{R}^n$, the utility function $C_h^s(u,v)$ has a unique open loop saddle point $(\hat{u}_s,\hat{v}_s) \in L^2(s,T;\mathbf{R}^m) \times L^2(s,T;\mathbf{R}^k)$, and there exists a unique solution $(\hat{x}_s,\hat{p}_s)$ of the coupled system*

$$\begin{cases} \hat{x}_s' = A\hat{x}_s - B_1B_1^*\hat{p}_s + B_2B_2^*\hat{p}_s \text{ a.e. in } [s,T], & \hat{x}_s(s) = h, \\ \hat{p}_s' + A^*\hat{p}_s + Q\hat{x}_s = 0 \text{ a.e. in } [s,T], & \hat{p}_s(T) = F\hat{x}_s(T), \end{cases} \tag{5.6}$$

$$\text{such that } \hat{u}_s = -B_1^*\hat{p}_s \text{ and } \hat{v}_s = B_2^*\hat{p}_s. \tag{5.7}$$

Proof. (i) Assume that the pair $(\hat{u}_s,\hat{v}_s)$ is a saddle point of $C^s_{\hat{x}(s)}$ on the time interval $[s,T]$. Denote by $(\hat{x}_s,\hat{p}_s)$ the corresponding solution to the coupled system (5.1). Consider the new pair on the interval $[0,T]$,

$$\tilde{u} \stackrel{\text{def}}{=} \begin{cases} \hat{u} & \text{in } [0,s], \\ \hat{u}_s & \text{in } [s,T], \end{cases} \quad \tilde{v} \stackrel{\text{def}}{=} \begin{cases} \hat{v} & \text{in } [0,s], \\ \hat{v}_s & \text{in } [s,T], \end{cases} \tag{5.8}$$

and the corresponding solution $(\tilde{x},\tilde{p})$ to the state-adjoint state system (2.4)–(2.18). If it can be shown that the pair $(\tilde{u},\tilde{v})$ is a saddle point of $C_{x_0}(u,v)$ on $[0,T]$, then by uniqueness of the saddle point on $[0,T]$ we can conclude that $(\tilde{u},\tilde{v}) = (\hat{u},\hat{v})$ and hence $(\hat{u}_s,\hat{v}_s) = (\hat{u}|_{[s,T]}, \hat{v}|_{[s,T]})$. From this we get the uniqueness of the saddle point of $C^s_{\hat{x}(s)}$ on $[s,T]$ and the uniqueness of solution to the coupled system (5.1). The first remark is that $\tilde{x}(s) = \hat{x}(s)$ and from (5.3)

$$\begin{aligned} C_{x_0}(\hat{u},\hat{v}) = v(x_0) = \int_0^s Q\hat{x}\cdot\hat{x} + |\hat{u}|^2 - |\hat{v}|^2\,dt + v_s(\hat{x}(s)) \\ = \int_0^s Q\hat{x}\cdot\hat{x} + |\hat{u}|^2 - |\hat{v}|^2\,dt + F\hat{x}_s(T)\cdot\hat{x}_s(T) + \int_s^T Q\hat{x}_s\cdot\hat{x}_s + |\hat{u}_s|^2 - |\hat{v}_s|^2\,dt \\ \Rightarrow C_{x_0}(\hat{u},\hat{v}) = C_{x_0}(\tilde{u},\tilde{v}). \end{aligned}$$

Yet, this is not sufficient to conclude that $(\tilde{u}, \tilde{v})$ is a saddle point of $C_{x_0}(u,v)$. We must show that

$$\sup_{v\in L^2(0,T;\mathbf{R}^k)} C_{x_0}(\tilde{u}, v) = C_{x_0}(\tilde{u}, \tilde{v}) = \inf_{u\in L^2(0,T;\mathbf{R}^m)} C_{x_0}(u, \tilde{v}). \tag{5.9}$$

The second remark is that, since $(\tilde{u}-\hat{u}, \tilde{v}-\hat{v})$ is equal to $(0,0)$ on $[0,s]$, $(\hat{u}_s-\hat{u}, \hat{v}_s-\hat{v})$ is a saddle point of $C_0^s(u_s, v_s)$. Combining this with the fact that, by (5.4), $(0,0)$ is also a saddle point of $C_0^s(u_s, v_s)$, the pairs $(\hat{u}_s-\hat{u}, 0)$ and $(0, \hat{v}_s-\hat{v})$ are also saddle points of $C_0^s(u_s, v_s)$ and $C_0^s(\hat{u}_s-\hat{u}, 0) = C_0^s(0, \hat{v}_s-\hat{v}) = 0$. The third remark is that

$$C_{x_0}(\hat{u}, \tilde{v}) = C_{x_0}(\hat{u}, \hat{v}) + dC_{x_0}(\hat{u}, \hat{v}; 0, \tilde{v}-\hat{v}) + C_0(0, \tilde{v}-\hat{v}) = C_{x_0}(\hat{u}, \hat{v}) + C_0(0, \tilde{v}-\hat{v}).$$

But, since $\tilde{v}-\hat{v}$ is equal to 0 on $[0,s]$,

$$C_0(0, \tilde{v}-\hat{v}) = C_0^s(0, \hat{v}_s-\hat{v}) = 0 \quad \Rightarrow C_{x_0}(\hat{u}, \tilde{v}) = C_{x_0}(\hat{u}, \hat{v}) = C_{x_0}(\tilde{u}, \tilde{v}).$$

We now prove the second part of identity (5.9):

$$C_{x_0}(u, \tilde{v}) = C_{x_0}(\hat{u}, \tilde{v}) + dC_{x_0}(\hat{u}, \tilde{v}; u-\hat{u}, 0) + C_0(u-\hat{u}, 0). \tag{5.10}$$

Since $(0,0)$ is a saddle point of $C_0(u,v)$,

$$\inf_{u\in L^2(0,T;\mathbf{R}^m)} C_0(u-\hat{u}, 0) = \inf_{u\in L^2(0,T;\mathbf{R}^m)} C_0(u, 0) = 0.$$

It remains to prove that for all $u \in L^2(0,T;\mathbf{R}^m)$, $dC_{x_0}(\hat{u}, \tilde{v}; u-\hat{u}, 0) = 0$. First observe that

$$\begin{aligned} dC_{x_0}(\hat{u}, \tilde{v}; u-\hat{u}, 0) &= dC_{x_0}(\hat{u}, \hat{v}; u-\hat{u}, 0) + dC_0(0, \tilde{v}-\hat{v}; u-\hat{u}, 0) \\ &= dC_0(0, \tilde{v}-\hat{v}; u-\hat{u}, 0). \end{aligned}$$

Since $(0, \hat{v}_s-\hat{v})$ is a saddle point of C_0^s on $[s,T]$, there exists a pair (ξ, π) solution of the coupled system

$$\begin{cases} \xi' = A\xi - B_1B_1^*\pi + B_2B_2^*\pi \text{ a.e. in } [s,T], & \xi(s) = 0, \\ \pi' + A^*\pi + Q\xi = 0, & \pi(T) = F\xi(T), \end{cases} \tag{5.11}$$

$$0 = -B_1^*\pi, \quad \hat{v}_s - \hat{v} = B_2^*\pi. \tag{5.12}$$

The first equation can also be written

$$\xi' = A\xi + B_2(\hat{v}_s-\hat{v}) \text{ a.e. in } [s,T], \quad \xi(s) = 0.$$

Denote by $\tilde{\xi}$ the solution of the state equation (2.4) on $[0,T]$ corresponding to the initial state 0 and the control pair $(0, \tilde{v}-\hat{v})$:

$$\tilde{\xi}' = A\tilde{\xi} + B_2(\tilde{v}-\hat{v}) \text{ a.e. in } [0,T], \quad \tilde{\xi}(0) = 0.$$

Then observe that, since the restriction of $\tilde{v}-\hat{v}$ to $[0,s]$ is 0, $\tilde{\xi} = 0$ on $[0,s]$ and $\tilde{\xi} = \xi$ on $[s,T]$. Denoting by y the solution of

$$y' = Ay + B_1(u-\hat{u}) \text{ a.e. in } [0,T], \quad y(0) = 0,$$

we get the expression (cf. (2.16) and (2.19) for the directional derivative)

$$\begin{aligned}
dC_0(0, \tilde{v} - \hat{v}; u - \hat{u}, 0) &= F\tilde{\xi}(T) \cdot y(T) + \int_0^T Q\tilde{\xi} \cdot y + 0 \cdot (u - \hat{u}) + (\tilde{v} - \hat{v}) \cdot 0 \, dt \\
&= F\tilde{\xi}(T) \cdot y(T) + \int_0^T Q\tilde{\xi} \cdot y \, dt = F\tilde{\xi}(T) \cdot y(T) + \int_s^T Q\tilde{\xi} \cdot y \, dt \\
&= F\xi(T) \cdot y(T) + \int_s^T Q\xi \cdot y \, dt = \int_s^T B_1^* \pi \cdot (u - \hat{u}) \, dt = 0,
\end{aligned}$$

since $B_1^* \pi = 0$ on $[s, T]$ from (5.12). This establishes the second part of expression (5.9). The proof of the first part is dual to the proof of the second part. This yields the uniqueness and completes the proof of part (i).

(ii) The map (5.5) is clearly linear (and continuous). Assume that it is not bijective; then there exists some $x_0 \in \mathbf{R}^n$, $x_0 \neq 0$, such that $\hat{x}(s) = 0$. The restriction of $(\hat{x}, \hat{p})$ to the interval $[s, T]$ is a solution of the system

$$\tag{5.13} \begin{cases} \xi' = A\xi - B_1 B_1^* \pi + B_2 B_2^* \pi \text{ a.e. in } [s, T], & \xi(s) = 0 = \hat{x}(s), \\ \pi' + A^* \pi + Q\xi = 0 \text{ a.e. in } [s, T], & \pi(T) = F\xi(T). \end{cases}$$

But from part (i) the unique solution of system (5.13) is $(0, 0)$. Hence

$$\begin{aligned}
&(\hat{x}, \hat{p}) = (0, 0) \text{ in } [s, T] \quad \Rightarrow (\hat{x}(s), \hat{p}(s)) = (0, 0) \\
\Rightarrow &\begin{cases} \hat{x}' = A\hat{x} - B_1 B_1^* \hat{p} + B_2 B_2^* \hat{p} \text{ a.e. in } [0, s], & \hat{x}(s) = 0, \\ \hat{p}' + A^* \hat{p} + Q\hat{x} = 0 \text{ a.e. in } [0, s], & \hat{p}(s) = 0, \end{cases} \\
&\Rightarrow (\hat{x}, \hat{p}) = (0, 0) \text{ in } [0, s] \quad \Rightarrow x_0 = \hat{x}(0) = 0.
\end{aligned}$$

This contradicts our initial conjecture that $x_0 \neq 0$, and we conclude that the linear map (5.5) is injective and, a fortiori, bijective.

(iii) From part (i) for each $h \in \mathbf{R}^n$ and each s, $0 \leq s < T$, there exists a unique $h_0 \in \mathbf{R}^n$ such that $h = X(s)h_0$. But $C_{h_0}(u, v)$ has a unique open loop saddle point in $[0, T]$. From part (i), $C^s_{X(s)h_0}(u, v)$ has a unique open loop saddle point in $[s, T]$. The result now follows from the fact that $h = X(s)h_0$. The equations and the identities follow from Theorem 5.1(ii). $\square$

Remark 5.1. The proof of part (i) is not trivial. It is one of the key elements needed to get the result of part (iii) that says that $C^s_h(u, v)$ has a saddle point for all initial state h and all initial times s.

5.2. Decoupling of the coupled system. We need the following lemma.

LEMMA 5.1. *Assume that the open loop saddle point value $v(x_0)$ is finite for all $x_0 \in \mathbf{R}^n$. Let s, $0 \leq s < T$, and $(\hat{x}_s, \hat{p}_s)$ be the unique solution of the coupled system* (5.6) *with initial state h at time s. Then the map $P(s)$*

$$\tag{5.14} h \mapsto P(s)h \overset{\text{def}}{=} \hat{p}_s(s) : \mathbf{R}^n \to \mathbf{R}^n$$

is linear, continuous, and symmetrical.

Proof. By definition, $P(s)$ is linear and continuous. For the symmetry, let (x, p) and $(\bar{x}, \bar{p})$ be the solutions of the coupled system (5.6) for the respective initial states

h and $\bar{h}$ at time s. By symmetry of F, $Q(t)$, and $B_1(t)B_1^*(t) - B_2(t)B_2^*(t)$,

$$\begin{aligned} P(s)h\cdot\bar{h} &= p(s)\cdot\bar{x}(s) = p(T)\cdot\bar{x}(T) - \int_s^T p'\cdot\bar{x} + p\cdot\bar{x}'\,dt \\ &= Fx(T)\cdot\bar{x}(T) - \int_s^T -(A^*p+Qx)\cdot\bar{x} + p\cdot(A\bar{x} - B_1B_1^*\bar{p} + B_2B_2^*\bar{p})\,dt \\ &= Fx(T)\cdot\bar{x}(T) + \int_s^T Qx\cdot\bar{x} + p\cdot(B_1B_1^* - B_2B_2^*)\bar{p}\,dt = P(s)\bar{h}\cdot h, \end{aligned}$$

and $P(s)^* = P(s)$. $\square$

Remark 5.2. At this juncture the matrix Riccati differential equation can be readily obtained from Lemma 3.1 in [4] since, from Theorem 5.2(ii), the matrix function $X(s)$ is invertible for all s. However, in view of Lemma 5.1, we use invariant embedding to get more a priori information on the decoupling matrix function $P(s)$.

THEOREM 5.3. *Assume that $v(x_0)$ is finite for all $x_0 \in \mathbf{R}^n$.*

(i) *Given the solution of the coupled system (2.27) in $[0,T]$ for $x_0 \in \mathbf{R}^n$,*

$$\hat{p}(s) = P(s)\hat{x}(s), \quad 0 \le s \le T. \tag{5.15}$$

(ii) *The elements of the matrix function P are $H^1(0,T)$-functions, the elements of the matrix functions*

$$A_P \stackrel{\text{def}}{=} A - RP, \quad R \stackrel{\text{def}}{=} B_1B_1^* - B_2B_2^* \tag{5.16}$$

belong to $L^\infty(0,T)$, and the closed loop system

$$\hat{x}' = [A - (B_1B_1^* - B_2B_2^*)\,P]\,\hat{x} \text{ a.e. in } [0,T], \quad \hat{x}(0) = x_0, \tag{5.17}$$

has a unique solution in $H^1(0,T;\mathbf{R}^n)$. For all (t,s), $0 \le s \le t \le T$, the fundamental matrix solution $\Phi_P(t,s)$ associated with the closed loop system (5.17) and its inverse $\Phi_P(t,s)^{-1}$ are continuous in $\{(t,s) : 0 \le s \le t \le T\}$. For all pairs $0 \le s \le t \le T$

$$\frac{\partial}{\partial s}\Phi_P(t,s) + \Phi_P(t,s)A_P(s) = 0 \text{ a.e. in } [0,t], \quad \Phi_P(t,t) = I. \tag{5.18}$$

(iii) *For all h and $\bar{h}$ in $\mathbf{R}^n$*

$$\begin{aligned} h\cdot P(s)\,\bar{h} &= \Phi_P(T,s)h\cdot F\Phi_P(T,s)\bar{h} \\ &\quad + \int_s^T \Phi_P(t,s)h\cdot[Q(t) + P(t)R(t)P(t)]\,\Phi_P(t,s)\bar{h}\,dt. \end{aligned} \tag{5.19}$$

Proof. (i) From Theorems 5.1 and 5.2(i)

$$\hat{x}_s = \hat{x}|_{[s,T]}, \quad \hat{p}_s = \hat{p}|_{[s,T]} \quad \Rightarrow \hat{p}(s) = \hat{p}_s(s) = P(s)\hat{x}_s(s) = P(s)\hat{x}(s),$$

and we get (5.15). The closed loop system is obtained by direct substitution of the identity (5.15) for $\hat{p}$ into the first equation of the coupled system (2.27) in $[0,T]$.

(ii) Associate with the solution of the coupled system (2.27) in $[0,T]$ the matrix function

$$\Lambda(s)x_0 \stackrel{\text{def}}{=} \hat{p}(s;x_0), \quad \forall x_0 \in \mathbf{R}^n,\ 0 \le s \le T. \tag{5.20}$$

From (5.15) in part (i) and the invertibility of $X(s)$

$$\Lambda(s)x_0 \stackrel{\text{def}}{=} P(s)X(s)x_0, \quad \forall x_0 \in \mathbf{R}^n,\ 0 \le s \le T$$
$$\Rightarrow \Lambda(s) = P(s)X(s), \Rightarrow\ P(s) = \Lambda(s)X(s)^{-1}, \quad 0 \le s \le T.$$

Since $X(s)$ is invertible and the elements of the matrices X and Λ are $H^1(0,T)$-functions,

$$P'(s) = \Lambda(s)'X(s)^{-1} - \Lambda(s)X(s)^{-1}X(s)'X(s)^{-1}. \tag{5.21}$$

In particular the elements of the matrix function P are $H^1(0,T)$-functions. Then the matrix function $A_P(t)$ in (5.16) belongs to $L^\infty(0,T)$, and the closed loop system (5.17) has a unique solution in $H^1(0,T;\mathbf{R}^n)$. From this Φ_P has the usual properties of a fundamental matrix solution Φ_P in $\{(t,s) : 0 \le s \le t \le T\}$, $\Phi_P(t,0) = \Phi_P(t,s)\Phi_P(s,0)$, and

$$\frac{\partial \Phi_P}{\partial s}(t,s) + \Phi_P(t,s)A_P(s) \text{ a.e. in } [0,T], \quad \Phi_P(t,t) = I. \tag{5.22}$$

(iii) Let (ψ,φ) (resp., $(\overline{\psi},\overline{\varphi})$) be the solution of the coupled system (5.1) for the initial state h (resp., $\overline{h}$). Then by direct computation

$$h \cdot \overline{\psi}(s) = \varphi(T) \cdot F\overline{\varphi}(T) + \int_s^T \varphi(t) \cdot Q(t)\,\overline{\varphi}(t) + \psi(t) \cdot R(t)\,\overline{\psi}(t)\,dt,$$
$$h \cdot P(s)\,\overline{h} = \Phi_P(T,s)h \cdot F\Phi_P(T,s)\bar{h} + \int_s^T \Phi_P(t,s)h \cdot [Q(t) + P(t)R(t)P(t)]\,\Phi_P(t,s)\bar{h}\,dt. \tag{5.23}$$
□

5.3. Proof of Theorem 2.9. (i) From identity (5.21) in the proof of part (ii) of Theorem 5.3 a straightforward computation yields that the matrix function P is a solution of the matrix Riccati differential equation (2.43). This solution is unique. Indeed if $\bar{P}$ is another solution of the Riccati equation, the closed loop system with $\bar{P}$ has a unique solution $\bar{x}$ and it is easy to check that $\bar{p} = \bar{P}\bar{x}$ is a solution of the associated adjoint equation. But there is a unique solution to the coupled system. By definition of P via invariant embedding we get that $\bar{P} = P$. (ii) and (iii) The proof follows from identities (2.34) and (2.35) in Theorem 2.2.

5.4. Proof of Theorem 2.10. From Theorem 2.9 we get (a) and (b). Conversely, from (a) if P is a solution of the Riccati differential equation, the closed loop system has a unique solution x_P and $p_P = Px_P$ is the solution of the adjoint system. It is then easy to check that the pair (x_P, p_P) is indeed a solution of the coupled system (2.27) in $[0,T]$. Finally from the convexity-concavity property (b) we get the existence of the open loop saddle point.

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