A Maximum Principle for Optimal Control of Discrete-Time Stochastic Systems With Multiplicative Noise

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***Abstract*—The maximum principle (MP) for the discrete-time stochastic optimal control problems is proved. It is shown that the adjoint equations of the MP are a pair of backward stochastic difference equations.**

***Index Terms*—Backward stochastic difference equations, discrete-time stochastic systems, maximum principle.**

I. Introduction

In the past decades, the stochastic optimal control problems have been extensively studied, in particular, various results on stochastic maximum principle (MP) have appeared; see [1], [2], [4]–[8], [10]–[12] and the references therein. However, these results are mostly concentrated on the continuous-time stochastic Itô systems. As far as discrete-time stochastic systems with multiplicative noise, most results for MP are analogous to the deterministic systems, which are based on the Lagrange multiplier method [2], [14]. However, as said in [15], from Peng's MP on optimal control of Itô systems [7], we know that the stochastic MP is different from the deterministic systems, which reflects the stochastic nature of this problem. It can be found that, up to now, few results appear on discrete-time version of Peng's continuous-time MP. On the other hand, theoretically, the discrete-time backward stochastic difference equations (BSDEs), which are the counterparts of continuous-time BSDEs applied in [7], do not necessarily have adapted solutions (see Remark 3.3) in general. This is our main motivation to study the stochastic MP for discrete-time systems.

In this paper, we will discuss the discrete-time stochastic MP associated with the following optimal control problem: minimize the cost functional

$$J(v) = \mathbb{E}\sum_{k=0}^{N-1} l(x_k, v_k) + \mathbb{E}h(x_N) \tag{1}$$

subject to

$$\begin{aligned} x_{k+1} &= g(x_k, v_k) + \sigma(x_k, v_k)\omega_k, \\ x_0 &\in \mathbb{R}^n, k = 1, 2, \cdots, N-1 \end{aligned} \tag{2}$$

where, $v = \{v_k\}_{k=0}^{N-1}$ is the admissible control variable with $v_k \in U_k$, $x = \{x_k\}_{k=0}^{N}$ is the state variable and $\{\omega_k = (\omega_k^1, \cdots, \omega_k^d)^T\}_{k=1}^N$ are the given random sequence. If the optimal control $u = \{u_k\}_{k=0}^{N-1}$ is given, in order to obtain the necessary condition for the optimization problem (1), (2), we extend the spike variation method to the discrete-time stochastic systems, and let the perturbed admissible control $\bar{u}_k = (1-\delta_{km})u_k + \delta_{km}v, v \in U_m$. Since, for the discrete-time systems, the time difference is constant and cannot be infinitesimal. So, Peng's method in [7], where the variation of $J(v)$ is based on the time difference, cannot be applied to the discrete-time case directly. In order to overcome this difficulty, we assume that v has the form $v = u_m + \epsilon\Delta v$, where ϵ is a positive constant number and Δv is a bounded random variable. Based on this idea, we obtain the variation of the cost functional $J(v)$ and the MP for the optimal control problem (1), (2). Moreover, we derive the adjoint equations of the discrete-time stochastic MP as a pair of BSDEs.

For convenience, we adopt the following notations: $\mathbb{R}^n$: the set of all real n-dimensional vectors; $\mathbb{R}^{n\times d}$: the set of $n \times d$ real matrices; A^T, x^T: transpose of a matrix A or vector x; $\mathrm{tr}(A)$: trace of a square matrix A; $\langle\cdot,\cdot\rangle$: inner product of two vectors or matrices with $\langle x, y\rangle = \mathrm{tr}(x^T y)$ and the corresponding norm $|x| = \sqrt{\langle x, x\rangle}$; $A \otimes B$: the Kronecker product of two matrices A and B; δ_{ij}: The Kronecker delta, i.e., $\delta_{ij} = 1$ when $i = j$ while $\delta_{ij} = 0$ when $i \neq j$; I_d: $d \times d$ identity matrix.

II. Preliminaries

In a given complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $\{\omega_k\}_{k=0,1,\cdots,N}$ be a sequence of $\mathcal{F}$-measurable, $\mathbb{R}^d$-valued random variables and satisfy the following conditions $(i, j, k = 0, 1, \cdots, N)$:

(i) $\omega_0, \omega_1, \cdots, \omega_N$ are independent, and for every $\omega_k = \left(\omega_k^1, \cdots, \omega_k^d\right)^T, \omega_k^1, \cdots, \omega_k^d$ are independent $\mathbb{R}$-valued random variables.

(ii) $\mathbb{E}\omega_k = 0_{d\times 1}, \mathbb{E}\left[\omega_i\omega_j^T\right] = \delta_{ij} I_d.$ (3)

Let $\mathcal{F}_k \subset \mathcal{F}$ be the σ-field generated by $\omega_0, \omega_1, \cdots, \omega_{k-1}$, i.e.

$$\mathcal{F}_k = \sigma\{\omega_0, \omega_1, \cdots, \omega_k\}, k = 0, 1, \cdots, N.$$

and, $\mathcal{F}_{-1} = \{\emptyset, \Omega\}$($\emptyset$ is the empty set). Set $\mathbb{F} = \{\mathcal{F}_k\}_{k=0}^N$. A random sequence $y = \{y_k\}_{k=0}^N$ is called $\mathbb{F}$-predictable, if for every $k = 0, 1, \cdots, N$, the random variable y_k is $\mathcal{F}_{k-1}$-measurable. Denote $L^2(\Omega, \mathcal{F}_k; \mathbb{R}^n)$ the set of all $\mathbb{R}^n$-valued $\mathcal{F}_k$-measurable random variables X with $\mathbb{E}|X|^2 < \infty$.

Lemma 2.1: (Riesz representation theorem) If $f : \mathcal{H} \to \mathbb{R}$ is a continuous linear functional on the Hilbert space $\mathcal{H}$, then there exists a unique $y_f \in \mathcal{H}$ such that

$$f(x) = \langle y_f, x\rangle_{\mathcal{H}}, \quad \forall x \in \mathcal{H}$$

where $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ denotes the inner product.

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For the given data as

$$\phi=\{\phi_k\}_{k=0}^{N-1}, \psi=\{\psi_k\}_{k=0}^{N-1}, \phi_k\in L^2(\Omega,\mathcal{F}_{k-1};\mathbb{R}^n),$$
$$\psi_k=\left(\psi_k^1,\psi_k^2,\cdots,\psi_k^d\right)\in L^2(\Omega,\mathcal{F}_{k-1};\mathbb{R}^{n\times d})$$

suppose $z(\phi,\psi)=\{z_k\}_{k=0}^N$ satisfies the following discrete-time difference equation:

$$\begin{cases} z_{k+1}=(g_kz_k+\phi_k)+\sum_{j=1}^d(\sigma_k^jz_k+\psi_k^j)\omega_k^j,\\ z_0=0, k=0,1,\cdots,N-1,\end{cases} \tag{4}$$

where, $g_k,\sigma_k^j\in L^2(\Omega,\mathcal{F}_{k-1};\mathbb{R}^{n\times n}), j=1,2,\cdots,d,$ are bounded $\mathbb{R}^{n\times n}$-valued random variables.

For each given $l_k\in L^2(\Omega,\mathcal{F}_{k-1};\mathbb{R}^n)$, we consider the following linear functional:

$$I(\phi,\psi)=\mathbb{E}\sum_{k=0}^N\langle l_k,z_k\rangle. \tag{5}$$

Construct a pair of discrete-time BSDEs as follows:

$$\begin{cases} p_k=\mathbb{E}\left[l_{k+1}+g_{k+1}^Tp_{k+1}+\sum_{j=1}^d{\sigma_{k+1}^j}^TQ_{k+1}^j\middle|\mathcal{F}_{k-1}\right]\\ Q_k=\mathbb{E}\left[\left(l_{k+1}+g_{k+1}^Tp_{k+1}+\sum_{j=1}^d{\sigma_{k+1}^j}^TQ_{k+1}^j\right)\omega_k^T\middle|\mathcal{F}_{k-1}\right],\\ p_{N-1}=\mathbb{E}[l_N|\mathcal{F}_{N-2}],\ Q_{N-1}=\mathbb{E}[l_N\omega_{N-1}|\mathcal{F}_{N-2}],\\ \quad k=0,1,\cdots,N-2.\end{cases} \tag{6}$$

It is easy to see that BSDE (6) has a unique $\mathbb{F}-$predictable solution $(p,Q)=\{(p_k,Q_k)\}_{k=0}^{N-1}$ and

$$(p_k,Q_k)\in L^2(\Omega,\mathcal{F}_{k-1};\mathbb{R}^n)\times L^2(\Omega,\mathcal{F}_{k-1};\mathbb{R}^{n\times d}).$$

Lemma 2.2: Suppose (p,Q) is the solution of (6), then the functional $I(\cdot,\cdot)$ can be uniquely represented as

$$I(\phi,\psi)=\mathbb{E}\sum_{k=0}^{N-1}\left[\langle p_k,\phi_k\rangle+\langle Q_k,\psi_k\rangle\right]. \tag{7}$$

Proof: Denote $A_k=g_k+\sum_{j=1}^d{\sigma_k^j}^T\omega_k^j, k=0,1,\cdots,N-1,$ then the solution of difference (4) can be solved by

$$z_k=\sum_{i=0}^{k-1}A_{i+1}^k\left(\phi_i+\sum_{j=1}^d\psi_i^j\omega_i^j\right), k\ge 1$$

where, $A_i^k, 1\le i\le k\le N$ is defined as $A_i^k=I_n$ when $i=k$, and $A_i^k=A_{k-1}A_{k-2}\cdots A_i$ when $1\le i<k$. Substituting the above z_k into (5), we have

$$I(\phi,\psi)=\mathbb{E}\sum_{k=0}^{N-1}\left[\left\langle\sum_{i=k+1}^N\left(A_{k+1}^i\right)^Tl_i,\phi_k\right\rangle+\left\langle\sum_{i=k+1}^N\left(A_{k+1}^i\right)^Tl_i\omega_k^T,\psi_k\right\rangle\right].$$

By the smoothing property of conditional expectation, we have

$$I(\phi,\psi)=\mathbb{E}\sum_{k=0}^{N-1}\left\{\left\langle\mathbb{E}\left[\sum_{i=k+1}^N\left(A_{k+1}^i\right)^Tl_i|\mathcal{F}_{k-1}\right],\phi_k\right\rangle+\left\langle\mathbb{E}\left[\sum_{i=k+1}^N\left(A_{k+1}^i\right)^Tl_i\omega_k^T|\mathcal{F}_{k-1}\right],\phi_k\right\rangle\right\}.$$

Let

$$p_k=\mathbb{E}\left[\sum_{i=k+1}^N\left(A_{k+1}^i\right)^Tl_i\middle|\mathcal{F}_{k-1}\right],$$
$$Q_k=\mathbb{E}\left[\sum_{i=k+1}^N\left(A_{k+1}^i\right)^Tl_i\omega_k^T\middle|\mathcal{F}_{k-1}\right],$$
$$k=0,\cdots,N-1.$$

Then, for $Q_k=(Q_k^1,Q_k^2,\cdots,Q_k^d)$, we have

$$Q_k^j=\mathbb{E}\left[\sum_{i=k+1}^N\left(A_{k+1}^i\right)^Tl_i\omega_k^j\middle|\mathcal{F}_{k-1}\right], j=1,\cdots,d.$$

Denote the Hilbert space

$$\mathcal{H}_N=\left\{L^2(\Omega,\mathcal{F}_k;\mathbb{R}^n)\times L^2(\Omega,\mathcal{F}_k;\mathbb{R}^{n\times d})\right\}_{k=0}^{N-1}$$

with the inner product as

$$\langle\alpha,\beta\rangle_{\mathcal{H}_N}=\mathbb{E}\sum_{k=0}^{N-1}\left[\langle\alpha_k^1,\beta_k^1\rangle+\left\langle\alpha_k^2,\beta_k^2\right\rangle\right]$$

where

$$\alpha_k^1,\beta_k^1\in L^2(\Omega,\mathcal{F}_k;\mathbb{R}^n), \alpha_k^2,\beta_k^2\in L^2(\Omega,\mathcal{F}_k;\mathbb{R}^{n\times d}),$$
$$\alpha=\left\{\left(\alpha_k^1,\alpha_k^2\right)\right\}_{k=0}^{N-1}, \beta=\left\{\left(\beta_k^1,\beta_k^2\right)\right\}_{k=0}^{N-1}\in\mathcal{H}_N.$$

It is easy to verify that $I(\phi,\psi)$ is a continuous linear functional on $\mathcal{H}_N$, $(\phi,\psi)\in\mathcal{H}_N$ and $(p_k,Q_k), k=0,1,\cdots,N-1$ satisfy the BSDE (6). By the Riesz representation theorem (Lemma 2.1), there exists a unique $\{(p_k,Q_k)\}_{k=0}^{N-1}\in\mathcal{H}_N$ satisfying (7). This proves the uniqueness of the representation of (7). ■

III. Maximum Principle for the Stochastic Discrete-Times Systems

In this section, we suppose that, in the optimal control problem (1), (2), the admissible control set $U=\{U_k\}_{k=0}^{N-1}, U_k=L^2(\Omega,\mathcal{F}_k;\mathbb{R}^{n_v})$ and $\{\omega_k=(\omega_k^1,\cdots,\omega_k^d)^T\}_{k=1}^N$ are the random sequence and satisfy (3). We also assume that

Assumption 1: ϕ is twice continuously differentiable w.r.t. x and v, $\phi_x,\phi_{xx},\phi_v,\phi_{vv},l_{xx}$ are bounded, where $\phi=g,\sigma,h$. ω_k has the fourth moment, $\mathbb{E}|\omega_k|^4<\infty$.

Suppose that $u=\{u_k\}_{k=0}^{N-1}$ is the optimal control of problem (1), (2). For fixed integer $0\le m\le N-1$, construct the perturbed admissible control $\bar{u}_k=(1-\delta_{km})u_k+\delta_{km}v, v\in U_m$. In the rest of this paper, we let v have the form of $v=u_m+\epsilon\Delta v, \epsilon>0$, where Δv is an arbitrary random variable with values in $\mathcal{U}_m\subset L^2(\Omega,\mathcal{F}_m;\mathbb{R}^{n_v})$, such that

$$\sup_{\omega\in\Omega}|\Delta v(\omega)|<\infty. \tag{8}$$

Then $\bar{u}_k=u_k+\delta_{km}\epsilon\Delta v$. Let $\bar{x}=\{\bar{x}_k\}_{k=0}^N$ be the solution of (2) corresponding to the control $\bar{u}$ with the initial value $\bar{x}_0=x_0$. Then, we have the following lemmas.

Lemma 3.1: Under Assumption 1, the following hold:

$$\sup_{0\le k\le N}\mathbb{E}|\bar{x}_k-x_k|^2\le C_1\epsilon^2\mathbb{E}|\Delta v|^2 \tag{9}$$

$$\sup_{0\le k\le N}\mathbb{E}|\bar{x}_k-x_k|^4\le C_1\epsilon^4\mathbb{E}|\Delta v|^4. \tag{10}$$

Let $y = \{y_k\}_{k=0}^N$ be the solution of the following difference equation

$$\begin{cases} y_{k+1} = [g_x(x_k,u_k)y_k + \delta_{km} g_v(x_m,u_m)\epsilon\Delta v] \\ \quad +[\sigma_x(x_k,u_k)y_k + \delta_{km}\sigma_v(x_m,u_m)\epsilon\Delta v]\omega_{k+1} \\ y_0 = 0. \end{cases} \tag{11}$$

Then, we obtain the following result.

Lemma 3.2: Under Assumption 1, we have

$$\sup_{0\le k\le N} \mathbb{E}|\bar{x}_k - x_k - y_k|^2 \le M\epsilon^4\mathbb{E}|\Delta v|^4, \tag{12}$$

with $M > 0$ being a constant.

Proof: Denote $\hat{x}_k = \bar{x}_k - x_k$, then $\hat{x}_0 = 0$ and

$$\hat{x}_{k+1} = g_x(x_k,u_k)\hat{x}_k + \phi_k^1 + \delta_{km}\left[g_v(x_m,u_m)\epsilon\Delta v + \phi_m^2\right] + \sum_{j=1}^d \left\{\sigma_x^j(x_k,u_k)\hat{x}_k + \psi_k^{1,j} + \delta_{km}\left[\sigma_v^j(x_m,u_m)\epsilon\Delta v + \psi_m^{2,j}\right]\right\}\epsilon_k^j$$

where

$$\varphi_x(u,v) = \begin{bmatrix} \varphi_{x_1^1}(u,v) & \varphi_{x_2}^1(u,v) & \cdots & \varphi_{x_n}^1(u,v) \\ \varphi_{x_1}^2(u,v) & \varphi_{x_2^2}(u,v) & \cdots & \varphi_{x_n^2}(u,v) \\ \vdots & \vdots & & \vdots \\ \varphi_{x_1^n}(u,v) & \varphi_{x_2}(u,v)^n & \cdots & \varphi_{x_n}^n(u,v) \end{bmatrix}$$

and φ_v has the similar meaning. Moreover, $\phi_k^1 = (\hat{x}_k^T \otimes I_n)g_{xx} \times (x_k+\theta_k\hat{x}_k,u_k)(\hat{x}_k \otimes 1_n)$, $\psi_k^{1,j} = (\hat{x}_k^T\otimes I_n)\sigma_{xx}^j(x_k+\theta_k^j\hat{x}_k,u_k) \times (\hat{x}_k\otimes 1_n)$ when $k \ge m+1$, and $\phi_k^1 = 0$ when $k \le m$. $\phi_m^2 = \epsilon^2(\Delta v^T\otimes I_n)g_{vv}(x_m,u_m+\tilde{\theta}_m\epsilon\Delta v)(\Delta v\otimes 1_n)$, $\psi_m^{2,j} = \epsilon^2(\Delta v^T\otimes I_n)\sigma_{vv}^j(x_m,u_m+\tilde{\theta}_m^j\epsilon\Delta v)(\Delta v\otimes 1_n)$, where

$$\varphi_{xx}(x_k+\theta_k\hat{x}_k,u_k) = \begin{bmatrix} \varphi_{xx}^1(x_k+\theta_k^1\hat{x}_k,u_k) & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \varphi_{xx}^n(x_k+\theta_k^n\hat{x}_k,u_k) \end{bmatrix}$$

and

$$\varphi_{vv}(x_k,u_k+\theta_k\hat{u}_m) = \begin{bmatrix} \varphi_{vv}^1(x_k,u_k+\tilde{\theta}_k^1\hat{u}_m) & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \varphi_{vv}^n(x_k,u_k+\tilde{\theta}_k^n\hat{u}_m) \end{bmatrix}.$$

In above notations, $1_n = (1,1,\cdots,1)^T \in \mathbb{R}^n$, $\varphi = g,\sigma^j, j = 1,\cdots,d$ and $\varphi = (\varphi^1,\varphi^2,\cdots,\varphi^n)^T$, and all θ, such as $\theta_k^i, \tilde{\theta}_k^i$, satisfy $0 < \theta < 1$, and $\hat{u}_m = \epsilon\Delta v$.

By the boundness of φ_{xx} and Lemma 3.1, we have the following:

$$\begin{aligned} \mathbb{E}|\phi_k^1|^2 &\le C_1\mathbb{E}|\hat{x}_k|^4 \le C_2\epsilon^4\mathbb{E}|\Delta v|^4, \\ \mathbb{E}|\phi_m^2|^2 &\le C_2\epsilon^4\mathbb{E}|\Delta v|^4, \\ \mathbb{E}|\psi_k^{1,j}|^2 &\le C_2\epsilon^4\mathbb{E}|\Delta v|^4, \\ \mathbb{E}|\psi_m^{2,j}|^2 &\le C_2\epsilon^4\mathbb{E}|\Delta v|^4, k \ge m+1, j = 1,\cdots,d. \end{aligned}$$

Denote $Z_k = \bar{x}_k - x_k - y_k$, then $Z_k = 0$ when $k = 0,1,\cdots,m$. For $k \ge m+1$, we have

$$Z_{k+1} = \left[g_x(x_k,u_k)Z_k + \phi_k^1\right] + \sum_{j=1}^d \left[\sigma_x^j(x_k,u_k)Z_k + \psi_k^{1,j}\right]\omega_k,$$

$$Z_{m+1} = \phi_m^2 + \sum_{j=1}^d \psi_m^{2,j}\epsilon_m^j, k = m+1,\cdots,N-1.$$

Since

$$\mathbb{E}|Z_{k+1}|^2 = \mathbb{E}\left|g_x(x_k,u_k)Z_k + \phi_k^1\right|^2 + \sum_{j=1}^d \mathbb{E}\left|\sigma_x^j(x_k,u_k)Z_k + \psi_k^{1,j}\right|^2$$

we have

$$\begin{aligned} \mathbb{E}|Z_{k+1}|^2 &\le C_3\mathbb{E}|Z_k|^2 + C_3\epsilon^4\mathbb{E}|\Delta v|^4, \\ \mathbb{E}|Z_{m+1}|^2 &\le C_3\epsilon^4\mathbb{E}|\Delta v|^4, k = m+1,\cdots,N-1. \end{aligned}$$

By iteration, the inequality (12) can be shown. ■

By Lemma 3.2, we see that y_k is the first-order variation of x_k. So, (11) is called the first-order variational equation of (2).

Similarly, we can verify that the variation of $J(v)$ at u is

$$\mathcal{L}J(u) = \mathbb{E}\sum_{k=0}^{N-1}\left[\langle l_x(x_k,u_k), y_k\rangle + \delta_{km}\langle l_v(x_m,u_m),\epsilon\Delta v\rangle\right] + \mathbb{E}\langle h_x(x_N), y_N\rangle.$$

We take the Hamiltonian function as

$$H(x,v,p,Q) = l(x,v) + \langle p, g(x,v)\rangle + \sum_{j=1}^d \left\langle Q^j, \sigma^j(x,v)\right\rangle,$$

$$x\in\mathbb{R}^n, v\in\mathbb{R}^{n_v}, p\in\mathbb{R}^n, \quad Q = (Q^1,\cdots,Q^d)\in\mathbb{R}^{n\times d}.$$

Construct the discrete-time BSDE as

$$\begin{cases} p_k = \mathbb{E}\Big[l_x(x_{k+1},u_{k+1}) + g_x^T(x_{k+1},u_{k+1})p_{k+1} \\ \qquad + \sum_{j=1}^d {\sigma_x^j}^T(x_{k+1},u_{k+1})Q_{k+1}^j|\mathcal{F}_{k-1}\Big] \\ Q_k = \mathbb{E}\Big\{\Big[l_x(x_{k+1},u_{k+1}) + g_x^T(x_{k+1},u_{k+1})p_{k+1} \\ \qquad + \sum_{j=1}^d {\sigma_x^j}^T(x_{k+1},u_{k+1})Q_{k+1}^j\Big]\omega_k^T|\mathcal{F}_{k-1}\Big\} \\ p_{N-1} = \mathbb{E}[h_x(x_N)|\mathcal{F}_{N-2}] \\ Q_{N-1} = \mathbb{E}[h_x(x_N)\omega_{N-1}|\mathcal{F}_{N-2}] \\ k = 0,1,\cdots,N-2. \end{cases} \tag{13}$$

Now, we can obtain our main result, which is a necessary condition, called the MP of the optimal control problem (1), (2).

Theorem 3.1: Under Assumption 1, if $(\{x_k\}_{k=0}^N, \{u_k\}_{k=0}^{N-1})$ are solutions of the optimal control problem (1), (2), then the solutions

$$\left\{(p_k,Q_k)\in L^2(\Omega,\mathcal{F}_k;\mathbb{R}^n)\times L^2(\Omega,\mathcal{F}_k;\mathbb{R}^n)\right\}_{k=0}^{N-1} \tag{14}$$

to BSDE (13) satisfy the following equality:

$$H_v(x_m,u_m,p_m,Q_m) = 0 \quad a.s. \tag{15}$$

Proof: By Lemma 2.2, we have

$$\mathcal{L}J(u) = \mathbb{E}\sum_{k=0}^{N-1}\left[\langle p_k, \delta_{km}g_v(x_m,u_m)\epsilon\Delta v\rangle + \langle Q_k,\delta_{km}\sigma_v(x_m,u_m)\epsilon\Delta v\rangle + \delta_{km}\langle l_v(x_m,u_m),\epsilon\Delta v\rangle\right].$$

So

$$\mathcal{L}J(u) = \mathbb{E}\langle H_v(x_m,u_m,p_m,Q_m),\epsilon\Delta v\rangle.$$

Since u is the optimal control, it follows

$$\mathcal{L}J(u) = 0.$$

This implies

$$\mathbb{E}\left\langle H_v(x_m, u_m, p_m, Q_m), \Delta v\right\rangle = 0, \qquad \forall \Delta v \in \mathcal{U}. \tag{16}$$

For every integer $K > 0$, let

$$\Delta v = H_v(x_m, u_m, p_m, Q_m) 1_{\{|H_v(x_m,u_m,p_m,Q_m)|<K\}} \in \mathcal{U}.$$

By (16), we have

$$\mathbb{E}\left[|H_v(x_m, u, p_m, Q_m)|^2 1_{\{|H_v(x_m,u_m,p_m,Q_m)|<K\}}\right] = 0.$$

So

$$H_v(x_m, u_m, p_m, Q_m) 1_{\{|H_v(x_m,u_m,p_m,Q_m)|<K\}} = 0 \quad a.s.$$

Let $K \to \infty$, we can obtain (15). ■

Remark 3.1: The discrete-time BSDE (13) can also be described by

$$\begin{cases} p_k = \mathbb{E}\left[H_x(x_{k+1}, u_{k+1}, p_{k+1}, Q_{k+1})|\mathcal{F}_{k-1}\right], \\ Q_k = \mathbb{E}\left[H_x(x_{k+1}, u_{k+1}, p_{k+1}, Q_{k+1})\omega_{k+1}^T|\mathcal{F}_{k-1}\right], \\ p_{N-1} = \mathbb{E}\left[h_x(x_N)|\mathcal{F}_{N-2}\right], \\ Q_{N-1} = \mathbb{E}\left[h_x(x_N)\omega_{N-1}|\mathcal{F}_{N-2}\right], \\ k = 0, 1, \cdots, N-2. \end{cases} \tag{17}$$

Consider the following deterministic optimal control problem: Minimize the cost functional

$$J(v) = \sum_{k=0}^{N-1} l(x_k, v_k) + h(x_N) \tag{18}$$

subject to

$$x_{k+1} = g(x_k, v_k), \qquad x_0 \in \mathbb{R}^n, k = 1, 2, \cdots, N-1. \tag{19}$$

The adjoint (13) becomes

$$\begin{cases} p_k = l_x(x_{k+1}, u_{k+1}) + g_x^T(x_{k+1}, u_{k+1})p_{k+1} \\ p_{N-1} = h_x(x_N), k = 0, 1, \cdots, N-2 \end{cases} \tag{20}$$

in which Q vanishes, and the corresponding Hamilton function becomes

$$H(x, v, p) = l(x, v) + \langle p, g(x, v)\rangle. \tag{21}$$

We have the following corollary.

Corollary 3.1: Suppose $(\{x_k\}_{k=0}^N, \{u_k\}_{k=0}^{N-1})$ are solutions of the optimal control problem (18), (19), then the solution $p = \{p_k\}_{k=0}^{N-1}$ to (20) satisfies the following equality:

$$H_v(x_m, u_m, p_m) = 0 \quad a.s. \tag{22}$$

Proof: Let $\sigma(x, v) \equiv 0$ in Theorem 3.1 and the adjoint (13), the proof is straightforward. ■

Remark 3.2: In particular, in Corollary 3.1, if we further let $g(x, v) = x + f(x, v)$, $l(x, v) \equiv 0$, the adjoint (20) becomes

$$p_k - p_{k+1} = f_x(x_{k+1}, u_{k+1})p_{k+1}$$

which is just the one given by Halkin [3].

Remark 3.3: Generally speaking, the discrete-time BSDEs corresponding to the continuous-time backward stochastic differential equations applied by Peng [7] can be written as the following forms:

$$y_k = f_{k+1}(y_{k+1}, z_{k+1}) - z_k\omega_k. \tag{23}$$

However, the following example shows that BSDEs with the form of (23) do not necessarily have the adapted solutions w.r.t. the filtration $\mathbb{F}$ generated by $\{\omega_k\}$. Considering the one-step BSDE of (23). Let $N = 1$, $f_1(y, z) = y$, $y, z \in \mathbb{R}$, and ω_0 takes values in $\{-\sqrt{3/2}, 0, \sqrt{3/2}\}$ with probability

$$\mathbb{P}\left(\omega_0 = -\sqrt{\frac{3}{2}}\right) = \mathbb{P}(\omega_0 = 0) = \mathbb{P}\left(\omega_0 = \sqrt{\frac{3}{2}}\right) = \frac{1}{3}.$$

Let $y_1 = \omega_0^2$, $z_1 = 0$, suppose there exists $(y_0, z_0) \in L^2(\Omega, \mathcal{F}_{-1}, \mathbb{R}) \times L^2(\Omega, \mathcal{F}_{-1}, \mathbb{R})$ such that

$$y_0 = f_1(y_1, z_1) + z_0\omega_0. \tag{24}$$

Since $(y_0, z_0) \in L^2(\Omega, \mathcal{F}_{-1}, \mathbb{R}) \times L^2(\Omega, \mathcal{F}_{-1}, \mathbb{R})$, we know that y_0, z_0 are two real numbers, and

$$y_0 = 1, \quad z_0 = 0.$$

Substituting y_0 and z_0 into (24), we have

$$\mathbb{P}\left\{f_1(y_1, z_1) = 1\right\} = 1.$$

This contradicts $f_1(y_1, z_1) = 0$ or 3/2, because the random variable $f_1(y_1, z_1)$ has only two possible values 0 or 3/2 with probability

$$\mathbb{P}\left\{f_1(y_1, z_1) = 0\right\} = \frac{1}{3}, \mathbb{P}\left\{f_1(y_1, z_1) = \frac{3}{2}\right\} = \frac{2}{3}.$$

Theorem 3.1 gives a necessary condition of the optimal control problem (1), (2). Now, we discuss the sufficient conditions for problem (1), (2). Let $u = \{u_k\}$ an admissible control, denote by $x(u) = \{x_k(u)\}$ and $(p, Q) = \{(p_k(u), Q_k(u))\}$ the solutions of (2) and (13) w.r.t. u. Denote by $\partial^2\phi$ the Hessian matrix of $\mathbb{R}$-valued function $\phi(x, v)$. Let p_k^i be the i'th component of p_k. If

$$\begin{aligned} &p_k^i \geq 0, \text{ and } \partial^2 f^i \geq 0, \text{ simultaneously,} \\ &\text{or} \\ &p_k^i \leq 0, \text{ and } \partial^2 f^i \leq 0, \text{ simultaneously.} \end{aligned} \tag{25}$$

Similarly, we can define $Q_k^{j^i}$ and $\partial^2\sigma^{j^i}$. We assume $(p_k^i(u), Q_k^{j^i}(u))$ and $(\partial^2 f^i, \partial^2\sigma^{j^i})$ have the same sign, $i = 1, \cdots, n, j = 1, \cdots, d$.

Theorem 3.2: Under Assumption 1 and $h_{xx} \geq 0$, $\partial^2 l \geq 0$, if the admissible control $u = \{u_k\}$ satisfies (15), and $(p_k^i(u), Q_k^{j^i}(u))$ have the same sign with $(\partial^2 f^i, \partial^2\sigma^{j^i})$ for $k = 0, 1, \cdots, N-1, j = 1, \cdots, d, i = 1, \cdots, n$, then $u = \{u_k\}$ is the optimal control of the problem (1), (2).

Proof: For another admissible control $v = \{v_k\}$, denote $\Delta u_k = v_k - u_k$, $\Delta x_k = x_k(v) - x_k(u)$. By Taylor's formula, we can obtain

$$\begin{aligned} J(v) - J(u) \geq\; &\mathbb{E}\left[p_{N-1}(u)\Delta x_{N-1}\right] \\ &+ \mathbb{E}\sum_{k=0}^{N-2}\left[l_x^T(x_k(u), u_k)\Delta x_k + l_v^T(x_k(u), u_k)\Delta v_k\right] \\ &+ \mathbb{E}H_v(x_{N-1}(u), u_{N-1}, p_{N-1}(u), Q_{N-1}(u)). \end{aligned}$$

Since $\Delta x_0 = 0$, by (15), we have

$$J(v) - J(u) \geq \mathbb{E}[p_0(u)\Delta x_0] + \sum_{k=0}^{N-1}\mathbb{E}H_v(x_k(u), u_k, p_k(u), Q_k(u)).$$

Since $\Delta x_0 = 0$, by (15), we have

$$J(v) - J(u) \geq 0$$

for all admissible v. This proves that $u = \{u_k\}$ is the optimal control of problem (1), (2). ■

IV. EXAMPLES

Example 4.1: Consider the following system with $d = 1$ (i.e., ω_k is $\mathbb{R}$-valued):

$$x_{k+1} = Ax_k + A_0 v_k + (Bx_k + B_0 v_k)\omega_k,$$
$$x_0 \in \mathbb{R}^n, k = 0, 1, \cdots, N-1 \tag{26}$$

with the cost functional

$$J(v) = \mathbb{E}\sum_{k=0}^{N-1}\left(|x_k|^2 + |u_k|^2\right) + \mathbb{E}|x_N|^2 \tag{27}$$

where $A, B \in \mathbb{R}^{n\times n}$, $A_0, B_0 \in \mathbb{R}^{n\times n_v}$. By Theorem 4.3 of [13], we can obtain the optimal control for the problem (26), (27) as follows:

$$\begin{aligned} u_k =& -\left(I_d + A_0^T P_{k+1} A_0 + B_0^T P_{k+1} B_0\right)^{-1} \\ &\times \left(A_0^T P_{k+1} A + B_0^T P_{k+1} B\right) x_k, \\ k =& 0, 1, \cdots, N-1 \end{aligned}$$

where $\{x_k\}_{k=0}^N$ are the corresponding states and $\{P_k\}_{k=0}^N \geq 0$ are the corresponding solutions of the generalized difference Riccati (6) of [13].

Let the Hamiltonian function

$$H(x, v, p, Q) = |x|^2 + |v|^2 + \langle p, Ax + A_0 v\rangle + \langle Q, Bx + B_0 v\rangle$$

and the corresponding BSDEs

$$\begin{cases} p_k = \mathbb{E}\left[2x_{k+1} + A^T p_{k+1} + B^T Q_{k+1} | \mathcal{F}_{k-1}\right], \\ Q_k = \mathbb{E}\left\{\left[2x_{k+1} + A^T p_{k+1} + B^T Q_{k+1}\right]\omega_k | \mathcal{F}_{k-1}\right\}, \\ p_{N-1} = \mathbb{E}[2x_N | \mathcal{F}_{N-2}], Q_{N-1} = \mathbb{E}[2x_N \omega_{N-1} | \mathcal{F}_{N-2}], \\ \quad k = 0, 1, \cdots, N-2. \end{cases} \tag{28}$$

In (26), we further assume the state x_k and control v_k to be all 1-dimensional cases. Let $A = 0.3$, $A_0 = 0.1$, $B = 0.12$, $B_0 = 0.2$, $x_0 = 1$ and $\{\omega_k\}_{k=0}^{N-1}$ are independent identically distributed with probability $\mathbb{P}(\omega_k = -1) = \mathbb{P}(\omega_k = 1) = 1/2$, such a model is called a random walking model. The optimal control with $N = 4$ is

$$u_0 = -0.0569x_0, \quad u_1 = -0.0569x_1,$$
$$u_2 = -0.0564x_2, \quad u_3 = -0.0514x_3.$$

The trajectories of $\{p_k\}_{k=0}^{N-1}$, $\{Q_k\}_{k=0}^{N-1}$ and $H_v(x_k, u_k, p_k, Q_k)$ are illustrated in Fig. 1 with $N = 5$.

In Fig. 1, we only illustrate the trajectories of p_k and Q_k for one sample, but for $H_v(x_m, u_m, p_m, Q_m)$, all the possible trajectories are illustrated, which are just consistent with the equality (15) of Theorem 3.1.

Example 4.2: In this example, we apply our MP to study a kind of production and consumption choice optimization problems. We suppose that an investor is able to invest his wealth to produce some production, and he can get profit from the production. Denote by x_t the capital of this investor at time t, and by $c_t \geq 0$ the rate of consumption. In 1928, Ramsey [9] introduced the continuous-time model to describe the law of motion for capital accumulation. Now, we extend it to the discrete-time case with some risk in his investment process

$$\begin{cases} x_{t+1} = f(x_t) + (1-\delta)x_t - c_t + \sigma(x_t)\omega_t, \\ x_0 \in \mathbb{R}^+ \quad \text{is given}, \ t = 0, 1, \cdots, T-1 \end{cases} \tag{29}$$

where $f(x)$ is the income of production, δ is the depreciation rate of the capital, $\sigma(x)$ denotes the effect influenced by the exogenous

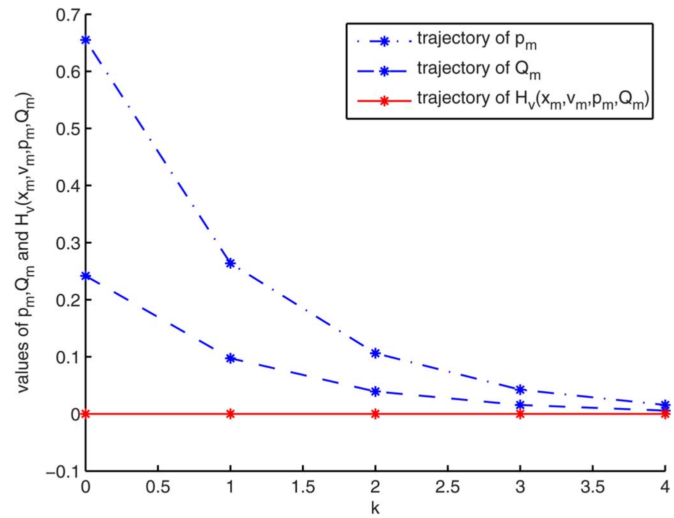

Fig. 1. Trajectories of p, Q and $H_v(x_m, u_m, p_m, Q_m)$.

environment (risk) and $\{\omega_t\}$ are the 1−dimensional white noises. Our objective is to choose the optimal consumption rate $c_t \geq 0$ to maximize the following functional $J(c)$:

$$J(c) = \mathbb{E}x_T + \mathbb{E}\sum_{k=0}^{T-1} u(c_t) \tag{30}$$

where x_T is the capital left over after consumption in the last period T. $u(\cdot)$ is the utility function given by

$$u(c) = \frac{c^{1-\frac{1}{\delta}}}{1-\frac{1}{\delta}}, 0 < \delta < 1.$$

Then the Hamiltonian function is

$$H(x, c, p, Q) = u(c) + p\left(f(x) + (1-\delta)x - c\right) + Q\sigma(x)$$

and the corresponding adjoint BSDE is

$$\begin{cases} p_t = \mathbb{E}\left[\left(f_x(x_{t+1}) + 1 - \delta\right) p_{t+1} + \sigma_x(x_{t+1}) Q_{t+1} | \mathcal{F}_{t-1}\right] \\ Q_t = \mathbb{E}\left\{\left[\left(f_x(x_{t+1}) + 1 - \delta\right) p_{t+1} \right.\right. \\ \qquad \left.\left. + \sigma_x(x_{t+1}) Q_{t+1}\right]\omega_t | \mathcal{F}_{t-1}\right\} \\ p_{T-1} = 1, Q_{T-1} = 0, t = 0, \cdots, T-2. \end{cases} \tag{31}$$

For simplicity, let $f(x) = x, \sigma(x) = 1/2x$, then we have $f''(x) = 0, \sigma''(x) = 0, Q_t = 0$ and

$$p_t = (2-\delta)p_{t+1}, Q_t = 0, p_{T-1} = 1.$$

Since $u'(c) = c^{-(1/\delta)}$, $u''(c) = -(1/\delta)c^{-(1/\delta)-1} \leq 0$. Solving the equation

$$H_c(x_t, c_t, p_t, Q_t) = 0$$

we have

$$c_t = p_t^{-\delta}, t = 0, 1, \cdots, T-1.$$

By Theorem 3.2, we know that $\{c_t\}$ is the optimal consumption rate for the optimization problem (29), (30). Fig. 2 is the trajectory of c_t with $\delta = 0.05$, $T = 6$.

V. CONCLUSION

In this paper, we have studied the MP of discrete-time stochastic optimal control problems. By using the spike variation method, we find

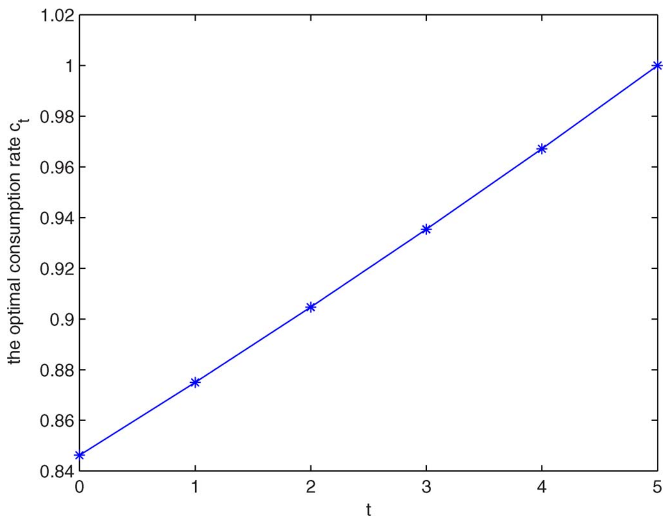

Fig. 2. Trajectories of optimal consumption rate c_t.

that the necessary condition of such problems is associated with the solutions of a pair of BSDEs, based on which, the MP for the optimal control problem (1), (2) is obtained. Moreover, a sufficient condition for the optimization problem (1), (2) is also given.

References

[1] J. M. Bismut, “An introductory approach to duality in optimal stochastic control,” *SIAM Rev.*, vol. 20, pp. 62–78, 1978.

[2] C. A. Burdet and S. P. Sethi, “On the maximum principle for a class of discrete dynamical systems with lags,” *J. Optimiz. Theory and Applic.*, vol. 19, no. 3, pp. 445–454, 1976.

[3] H. Halkin, “A Maximum Principle of the Pontryagin type for systems described by nonlinear difference equations,” *SIAM J. Control and Optimiz.*, vol. 4, pp. 90–111, 1966.

[4] H. J. Kushner and F. C. Schweppe, “A maximum principle for stochastic control systems,” *J. Mathematical Anal. and Applic.*, vol. 8, pp. 287–302, 1964.

[5] H. J. Kushner, “Necessary conditions for continuous parameter stochastic optimization problems,” *SIAM J. Control and Optimiz.*, vol. 10, pp. 550–560, 1972.

[6] J. Li and S. Peng, “Stochastic optimization theory of backward stochastic differential equations with jumps and viscosity solutions of Hamilton-Jacobi-Bellman equations,” *Nonlin. Analy. Theory Methods and Applic.*, vol. 70, pp. 1776–1796, 2009.

[7] S. Peng, “A general stochastic maximum principle for optimal control problems,” *SIAM J. Control and Optimiz.*, vol. 28, no. 4, pp. 966–979, 1990.

[8] S. Peng and Z. Wu, “Fully coupled forward-backward stochastic differential equations and applications to optimal control,” *SIAM J. Control and Optimiz.*, vol. 37, no. 3, pp. 825–843, 1999.

[9] F. P. Ramsey, “A mathematical theory of savings,” *The Economic J.*, vol. 38, pp. 543–559, 1928.

[10] J. Shi and Z. Wu, “Relationship between MP and DPP for the stochastic optimal control problem of jump diffusions,” *Appl. Mathemat. and Optimiz.*, vol. 63, pp. 151–189, 2011.

[11] Z. Wu, “A general maximum principle for optimal control of forward-backward stochastic systems,” *Automatica*, vol. 49, no. 5, pp. 1473–1480, 2013.

[12] G. Wang and Z. Yu, “A pontryagin’s maximum principle for non-zero sum differential games of BSDEs with applications,” *IEEE Trans. Autom. Control*, vol. 55, no. 7, pp. 1742–1747, Jul. 2010.

[13] M. Ait Rami, X. Chen, and X. Y. Zhou, “Discrete-time indefinite LQ control with state and control dependent noises,” *J. Global Optimiz.*, vol. 23, no. 3, pp. 245–265, 2002.

[14] H. Wang, H. Zhang, and X. Wang, “Optimal control for stochastic discrete-time systems with multiple input-delays,” in *Proc. 10th World Congr. Intelligent Control and Automation*, Beijing, China, Jul. 6–8, 2012, pp. 1529–1534.

[15] J. Yong and X. Y. Zhou, *Stochastic Controls: Hamiltonian Systems and HJB Equations*. New York: Springer-Verlag, 1999.