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THE RADON-NIKODYM PROPERTY AND THE KREIN-MILMAN PROPERTY ARE EQUIVALENT FOR STRONGLY REGULAR SETS

WALTER SCHACHERMAYER

ABSTRACT. The result announced in the title is proved. As corollaries we obtain that RNP and KMP are equivalent for subsets of spaces with an unconditional basis and for K-convex Banach spaces. We also obtain a sharpening of a result of R. Huff and P. Morris: A dual space has the RNP iff all separable subspaces have the KMP.

1. Introduction. In the past 15 years a lot of effort has been made to decide the (still open) problem of whether the Radon-Nikodym and the Krein-Milman property (abbreviated RNP and KMP) are equivalent. Several partial results have been obtained (see **[H-M, B-T, H, S1]**). In the present paper we show that these two properties coincide if one assumes strong regularity (a concept used implicitly by J. Bourgain **[B1, B2]** and developed recently by N. Ghoussoub, G. Godefroy and B. Maurey **[G-G-M]**). This property is a close relative of the so-called "convex point-of-continuity property"—abbreviated (CPCP)—which was introduced by J. Bourgain **[B1]** under the name "property $(*)$": A closed convex bounded subset C of a Banach space has (CPCP) if for every closed convex subset D of C the identity map from (D, weak) to $(D, \|\cdot\|)$ has a point of continuity.

The proof of Bourgain's theorem **[B1]**, that a Banach space X has (RNP) iff each subspace with a Schauder-f.d.d. has (RNP) consists of two completely different parts, namely the cases whether X has or fails (CPCP).

We hope that such a distinction of cases may also be useful for the problem, whether (RNP) and (KMP) are equivalent. This hope is stressed by the proof of the two Corollaries 2.10 and 2.11, where we use this distinction of two cases. Corollary 2.10 is a sharpening of a theorem of R. Huff and P. Morris: For dual Banach spaces RNP and KMP are equivalent and separably determined. Corollary 2.11 establishes the equivalence of (RNP) and (KMP) for subsets of spaces with an unconditional basis.

Recall that (RNP) is characterized by the existence of "denting points", which are extreme points and points of weak-to-norm-continuity. Hence the assertion that (KMP) and (CPCP) imply (RNP) (which is an immediate consequence of our theorem) may be viewed in the following way: If there are "many" extreme points and "many" points of weak-to-norm-continuity, then there are in fact "many" denting points.

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Also note that our theorem reduces the problem whether (RNP) and (KMP) are equivalent to the problem whether (KMP) implies strong regularity.

Another immediate consequence of Theorem 2.1 is (cf. 2.15 below) that RNP and KMP are equivalent for K-convex Banach spaces (i.e., spaces not containing l_n^1's uniformly).

The techniques used in the present paper rely heavily on the ideas developed by J. Bourgain [**B1**]. The author has also been influenced by the work of J. Bourgain and H. Rosenthal [**B-R**] and that of N. Ghoussoub and B. Maurey (resp. N. Ghoussoub, G. Godefroy and B. Maurey) in a series of papers on the point-of-continuity property (in particular [**G-M**] and [**G-G-M**]). The concept of a complemented bush which was introduced independently by R. James and A. Ho [**H**], respectively, by H. Rosenthal (personal communication) was also important in developing the ideas underlying the present proof.

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1.1. NOTATION. For the definition of (RNP) and (KMP) we refer to [**D-U**]. If D is a bounded, convex subset of a real Banach space X, a *slice* will be a set of the form

$$S = S(x^*, \alpha) = \left\{ x \in D \colon \left\langle x^*, x \right\rangle > M_{x^*} - \alpha \right\},$$

where x^* is an element of the unit sphere of X^*, $\alpha > 0$, and

$$M_{x^*} = \sup\left\{ \left\langle x^*, x \right\rangle \colon x \in D \right\}.$$

1.2. DEFINITION [**G-G-M**]. A bounded, convex set $D \subset X$ is called *strongly regular* if for every convex C contained in D and $\varepsilon > 0$ there are slices $S_1, \ldots, S_m$ of C such that

$$\operatorname{diam}\left(m^{-1} \sum_{j=1}^{m} S_j \right) < \varepsilon$$

where diam denotes the diameter of a set.

2. The main result.

2.1. THEOREM. *If a convex, bounded, closed subset $D \subset X$ is strongly regular and fails to be an RN-set, then there is a closed, bounded, convex and separable subset C of D which does not have an extreme point. Loosely speaking*

$$\textit{Strong regularity} + \text{KMP} \Rightarrow \text{RNP}. \quad \square$$

To prove the theorem we develop some machinery which will be elaborated on further in a forthcoming paper [**S2**].

2.2 DEFINITIONS AND NOTATIONS. m will denote Lebesgue measure on $[0,1]$ and F will be the positive face of the unit-ball of $L^1(m)$, i.e.

$$F = \left\{ f \in L^1 \colon f \geqslant 0 \text{ and } \|f\|_1 = 1 \right\}.$$

For $\mu \in \bar{F}^*$ (i.e. in the weak-star closure of F in the bidual of $L^1(m)$), $\varepsilon \geqslant 0$ and $P = (A_1, \ldots, A_k)$ a partition of $[0,1]$ into sets of strictly positive measure, let

$$V_{P,\varepsilon}^*(\mu) = \left\{ \nu \in \bar{F}^* \colon \sum_{j=1}^{k} \left| \mu(A_j) - \nu(A_j) \right| \leqslant \varepsilon \right\}$$

where $\mu(A_j)$ denotes $\langle \mu, \chi_{A_j} \rangle$. We also put

$$V_{P,\varepsilon}(\mu) = \left\{ f \in F \colon \sum_{j=1}^{k} \left| \mu(A_j) - \int_{A_j} f\, dm \right| \leqslant \varepsilon \right\}.$$

Note that, for $\varepsilon > 0$, $V^*_{P,\varepsilon}(\mu)$ is a relative weak-star neighborhood of μ in $\bar{F}^*$ and the family of such sets forms a basis of the relative weak-star neighborhoods (when P runs through all partitions into sets of strictly positive measure and ε runs through $]0, 1]$).

$V_{P,\varepsilon}(\mu)$ is the intersection of $V^*_{P,\varepsilon}(\mu)$ with F and is weak-star dense in $V^*_{P,\varepsilon}(\mu)$.

Let $T \colon L^1(m) \to X$ be a (continuous, linear) operator. For $\mu \in \bar{F}^*$ put

$$\rho_T(\mu) = \inf\{\operatorname{diam}(T(V_{P,\varepsilon}(\mu))) \colon \varepsilon > 0 \text{ and } P = (A_1, \ldots, A_n) \text{ a partition of } [0,1]\}$$

and

$$d_T(\mu) = \operatorname{dist}(T^{**}(\mu), X).$$

The notion of $\rho_T(\mu)$ which will be essential in the sequel has been inspired by the use of the "modulus of equi-integrability" $\delta(A)$ in **[G-M]**, Lemma (1).

2.3. Proposition. *With the above notation we have*

(1) $$\rho_T(\mu) \geqslant d_T(\mu).$$

Proof. Since $T(V_{P,\varepsilon}(\mu))$ is weak-star dense in $T^{**}(V^*_{P,\varepsilon}(\mu))$, this is immediate from the definition and the fact that

$$\operatorname{diam}(T(V_{P,\varepsilon}(\mu))) = \operatorname{diam}(T^{**}(V^*_{P,\varepsilon}(\mu))). \quad \square$$

The next result will be crucial for the proof of the theorem.

2.4. Proposition. *Let $T \colon L^1(m) \to X$ be an operator and put*

$$C = \overline{T(F)}^{\|\cdot\|_X}$$

so that

$$\bar{C}^* = \overline{T(F)}^{\sigma(X^{**}, X^*)} = T^{**}(\bar{F}^*).$$

Suppose that every extreme point of $\bar{C}^*$ is in $X^{**} \setminus X$. If x is an extreme point of C then there is $\mu \in \bar{F}^*$ with $T^{**}(\mu) = x$ and

(2) $$\rho_T(\mu) > 0.$$

Proof. Since by assumption, x is not extreme in $\bar{C}^*$, there are x_1^{**} and x_2^{**} in $\bar{C}^*$ with $x_1^{**} \neq x$ such that $x = (x_1^{**} + x_2^{**})/2$. By the extremality of x in C we conclude that x_1^{**} and x_2^{**} are in $X^{**} \setminus X$.

Find $\mu_1, \mu_2 \in \bar{F}^*$ such that, for $i = 1, 2$, $T^{**}(\mu_i) = x_i^{**}$ and let $\mu = (\mu_1 + \mu_2)/2$. Clearly $T^{**}(\mu) = x$. From Proposition 2.3 we conclude that, for $i = 1, 2$,

$$\rho_T(\mu_i) \geqslant d_T(\mu_i) > 0.$$

The subsequent lemma therefore implies that

$$\rho_T(\mu) \geqslant \max(\rho_T(\mu_1), \rho_T(\mu_2))/2 > 0. \quad \square$$

2.5. LEMMA. *Let* $P = (A_1, \ldots, A_k)$ *be a partition of* $[0,1]$ *into sets of strictly positive measure*, $\varepsilon \geqslant 0$, $\mu_1, \mu_2 \in \bar{F}^*$ *and* λ_1, λ_2 *positive scalars with* $\lambda_1 + \lambda_2 = 1$. *Then*

$$\lambda_1 V^*_{P,\varepsilon}(\mu_1) + \lambda_2 V^*_{P,\varepsilon}(\mu_2) = V^*_{P,\varepsilon}(\lambda_1\mu_1 + \lambda_2\mu_2). \tag{3}$$

In particular

$$\rho_T(\lambda_1\mu_1 + \lambda_2\mu_2) \geqslant \max(\lambda_1\rho_T(\mu_1), \lambda_2\rho_T(\mu_2)) \tag{4}$$

PROOF. Suppose first that $\varepsilon = 0$ (in this case formula (3) becomes particularly transparent). An element ν_1 of $\bar{F}^*$ belongs to $V_{P,0}(\mu_1)$ iff for every $j = 1, \ldots, k$, $\nu_1(A_j) = \mu_1(A_j)$.

From this observation it is obvious that if $\lambda_1\nu_1 + \lambda_2\nu_2$ belongs to the left-hand side of (3) then it belongs to the right-hand side. Conversely, if ν belongs to the right-hand side then

$$\nu_1 = \sum_{j=1}^{k} \frac{\mu_1(A_j)}{\nu(A_j)} \cdot (\nu \cdot \chi_{A_j})$$

and

$$\nu_2 = \sum_{j=1}^{k} \frac{\mu_2(A_j)}{\nu(A_j)} \cdot (\nu \cdot \chi_{A_j})$$

furnishes the desired decomposition $\nu = \lambda_1\nu_1 + \lambda_2\nu_2$ as an element of the left-hand side set. (Of course, $\nu \cdot \chi_{A_j}$ denotes the restriction of ν to the set A_j.)

This was the case $\varepsilon = 0$. The case $\varepsilon > 0$ may be treated analogously. However, it is easier to verify the following formula (which we leave to the reader).

$$V^*_{P,\varepsilon}(\mu) = \left[V^*_{P,O}(\mu) + \varepsilon \cdot \text{unit ball}\left(L^1(m)^{**}\right)\right] \cap \bar{F}^*. \tag{5}$$

Formula (3) is then easily seen to hold for any $\varepsilon \geqslant 0$.

Finally, (4) is a consequence of (3) and the (trivial) inequality

$$\operatorname{diam}(A + B) \geqslant \max(\operatorname{diam}(A), \operatorname{diam}(B))$$

for subsets A, B of X. $\square$

After these preparatory results we can formulate a more detailed version of Theorem 2.1. (It follows from Proposition 2.4 that the subsequent theorem implies Theorem 2.1.)

2.6. PROPOSITION. *Let D be a convex, closed, bounded subset of X which is strongly regular but not an RN-set. Then there is an operator* T: $L^1(m) \to X$ *and* $\alpha > 0$ *such that* (*with the notation of* 2.2),

(i) *C is contained in D*,

(ii) *the distance of each extreme point of $\bar{C}^*$ from X is at least α*,

(iii) *for every* $\mu \in \bar{F}^*$, $d_T(\mu) \geqslant \rho_T(\mu)/2$.

In order to prove Theorem 2.6 we still need some lemmata. Their proofs use arguments which are similar to some of those used in **[B1]**.

2.7. LEMMA. *Let D be a convex subset of the unit ball of X, $f \in X^*$, $\|f\| = 1$ and $1 > \varepsilon > 0$. Suppose that there are $\alpha > 0$ and $c \in [-1, +1]$, s.t.*

$$\sup\{\langle f, x\rangle : x \in S(x^*, \alpha\varepsilon/3)\} > c.$$

Then there is a slice T of D with $T \subset S(x^*, \alpha)$ and such that for every $x \in T$, $\langle f, x\rangle > c - \varepsilon$.

PROOF. (We want to thank the referee for suggesting the subsequent simplification of our original argument.)

Define $g = \alpha f + (1 - c + \varepsilon)x^*$. We have

$$\begin{aligned} M_g &= \sup\{g(x) : x \in D\} \\ &\geqslant \sup\{g(x) : x \in S(x^*, \alpha\varepsilon/3)\} \\ &\geqslant \alpha \cdot c + (1 - c + \varepsilon)(M_{x^*} - \alpha\varepsilon/3) \\ &= \alpha(c - (1 - c + \varepsilon) \cdot \varepsilon/3) + (1 - c + \varepsilon)M_{x^*} \\ &> \alpha(c - \varepsilon) + (1 - c + \varepsilon)M_{x^*} \\ &= \alpha + (1 - c + \varepsilon)(M_{x^*} - \alpha) = \beta. \end{aligned}$$

Hence

$$T = \{x \in D : \langle g, x\rangle > \beta\}$$

is a nonempty slice of D. For $x \in T$ we obtain $\langle x^*, x\rangle > M_{x^*} - \alpha$, i.e. $T \subset S(x^*, \alpha)$. Indeed, $\langle x^*, x\rangle \leqslant M_{x^*} - \alpha$ would imply $\langle g, x\rangle \leqslant \alpha + (1 - c + \varepsilon)(M_{x^*} - \alpha) = \beta$, a contradiction.

On the other hand for $x \in T$, we also get $\langle f, x\rangle > c - \varepsilon$. Indeed, otherwise $\langle g, x\rangle \leqslant \alpha(c - \varepsilon) + (1 - c + \varepsilon)M_x^* = \beta$ which again is a contradiction.

The lemma is proved. $\square$

The subsequent lemma contains the basic tool for the proof of Proposition 2.6.

2.8. LEMMA. *Let D be a bounded, convex subset of a Banach space X and $(x_i)_{i=1}^\infty$ a sequence in X. Let $S_1, \ldots, S_m$ be slices of D and $\mu = (\mu_1, \ldots, \mu_m)$ a vector of positive reals with $\sum \mu_i = 1$. Define*

$$\tau(\mu) = \inf\left\{\operatorname{diam}\left(\sum_{j=1}^{m} \mu_j T_j\right) : T_j \text{ are slices contained in } S_j\right\}.$$

Then for $n \in \mathbf{N}$ and $1 > \varepsilon > 0$, there are slices T_j contained in S_j such that

(i) $\operatorname{diam}(\sum_{j=1}^m \mu_j T_j) < \tau(\mu) + \varepsilon$ *and*

(ii) *for* $x \in \sum_{j=1}^m \mu_j T_j$ $\operatorname{dist}(x, E_n) > \tau(\mu)/2 - \varepsilon$ *where E_n denotes the set* $\{x_1, \ldots, x_n\}$.

PROOF. We may assume D is contained in the unit ball of X. By choosing slices $S_j^1 \subset S_j$ we may assume that

$$\operatorname{diam}\left(\sum_{j=1}^{k} \mu_j S_j^1\right) < \tau(\mu) + \varepsilon.$$

If S_j^1 is of the form $S_j^1 = S(x_j^*, \alpha_j)$ let $R_j^1 = S(x_j^*, \alpha_j \varepsilon/6)$. By assumption we may find $f_1 \in X^*$, $\|f_1\| = 1$, s.t.

$$\operatorname{osc}\left(f_1 \mid \sum_{j=1}^{m} \mu_j R_j^1 \right) > \tau(\mu) - \varepsilon.$$

By changing sign of f_1, if necessary, we can ensure that

$$\begin{aligned}\sup\left(f_1 \mid \sum_{j=1}^{m} \mu_j R_j^1 \right) &= \sum_{j=1}^{m} \mu_j \sup\left(f_1 \mid R_j^1 \right) \\ &> \langle f_1, x_1 \rangle + (\tau(\mu) - \varepsilon)/2.\end{aligned}$$

Let $c_j^1 = \sup(f_1 \mid R_j^1)$. We may apply Lemma 2.7 to obtain slices $T_j^1 \subset S_j^1$ such that for $\xi_j \in T_j^1$,

$$\langle f_1, \xi_j \rangle > c_j - \varepsilon/2.$$

Hence for $x \in \sum_{j=1}^m \mu_j T_j^1$

$$\begin{aligned}\langle f_1, x \rangle &= \sum_{j=1}^{m} \mu_j \langle f_1, \xi_j \rangle \\ &> \sum_{j=1}^{m} \mu_j \left(\sup\left(f_1 \mid R_j^1 \right) - \frac{\varepsilon}{2} \right) \\ &> \langle f_1, x_1 \rangle + \tau(\mu)/2 - \varepsilon.\end{aligned}$$

This readily shows that, for $x \in \sum_{j=1}^m \mu_j T_j^1$, we have $\|x - x_1\| > \tau(\mu)/2 - \varepsilon$. Let $S_j^2 = T_j^1$ and repeat the construction to find $T_j^2 \subset S_j^2$ such that for $x \in \sum_{j=1}^m \mu_j T_j^2$, we have $\|x - x_2\| > \tau(\mu)/2 - \varepsilon$. Repeating the construction n times we have proved Lemma 2.8. $\square$

We still need one more easy lemma.

2.9. Lemma. *Let D be a closed, bounded, convex strongly regular set, S a slice of D and $\varepsilon > 0$. Then there are slices $S_1, \ldots, S_m$ of D such that $\bar{S}_j \subset S$ and* $\operatorname{diam}(m^{-1}\sum_{j=1}^m S_j) \leqslant \varepsilon$.

Proof. The set of strongly regular points of D (i.e. those which are contained in the closure of convex combinations of slices of arbitrarily small diameter) is convex and norm-dense in D [**G-G-M**, Proposition III.1]. If S is of the form $S(x^*, \alpha)$ there is therefore a strongly regular point $x_0 \in S(x^*, \alpha/4)$. We may assume that $\varepsilon \leqslant \alpha$. Find slices $R_1, \ldots, R_n$ such that $n^{-1}\sum_{i=1}^n R_i$ is contained in the ball of radius $\varepsilon/4$ around x_0. One easily checks that

$$\text{cardinality}\left\{ i \colon \bar{R}_i \subset S \right\} \geqslant n/2.$$

Relabel the R_i belonging to the above set by $S_1, \ldots, S_m$ and note that from

$$\operatorname{diam}\left(n^{-1} \sum_{i=1}^{n} R_i \right) \leqslant \varepsilon/2$$

it follows that $\operatorname{diam}(m^{-1}\sum_{j=1}^m S_j) \leqslant \varepsilon$. $\square$

PROOF OF PROPOSITION 2.6. We may assume $D \subset$ ball (X). As D is not an RN-set there exists a closed, convex, separable $D_1 \subset D$ and an $\alpha > 0$ such that every slice of D_1 has diameter greater than 2α. To simplify the notation write $D = D_1$. Let $(x_i)_{i=1}^{\infty}$ be a dense sequence in the space spanned by D and denote $E_n = \{x_1, \dots, x_n\}$.

We shall construct an index set $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$ where

$$\Omega_n = \{1, \dots, m_1\} \times \{1, \dots, m_2\} \times \cdots \times \{1, \dots, m_n\}$$

and a system of slices $\{T_\omega : \omega \in \Omega\}$ of D such that for each $\omega \in \Omega_n$,

$$\operatorname{diam}\left[m_{n+1}^{-1} \sum_{j=1}^{m_{n+1}} T_{\omega,j}\right] < 2^{-(n+1)} \tag{6}$$

and

$$\overline{T_{\omega,j}} \subset T_\omega, \qquad j = 1, \dots, m_{n+1}. \tag{7}$$

We proceed by induction on n: For $n = 1$ find slices $S_1, \dots, S_{m_1}$ such that

$$\operatorname{diam}\left[m_1^{-1} \sum_{j=1}^{m_1} S_j\right] < 2^{-1}.$$

Let $\Omega_1 = \{1, \dots, m_1\}$ and $M_1 = m_1$ be the cardinality of Ω_1. Denote by F_{M_1} the positive face of $l_{M_1}^1$, i.e.

$$F_{M_1} = \left\{\mu = (\mu_1, \dots, \mu_{M_1}) : \mu_\omega \geqslant 0 \text{ and } \sum_{\omega \in \Omega_1} \mu_\omega = 1\right\}.$$

Let $\mu^1, \dots, \mu^{p_1}$ be a $1/2$-net of F_{M_1}. We now make a subinduction on $q = 1, \dots, p_1$: For $q = 1$ consider

$$\tau(\mu^1) = \inf\left\{\operatorname{diam}\left(\sum_{\omega \in \Omega_1} \mu_\omega^1 R_\omega\right) : R_\omega \text{ are slices contained in } S_\omega\right\}$$

and apply Lemma 2.8. to find slices $S_\omega^1 \subset S_\omega$ such that for every $x \in \sum_{\omega \in \Omega_1} \mu_\omega^1 S_\omega^1$,

(α) $\operatorname{dist}(x, E_1) > \tau(\mu^1)/2 - 2^{-1}$ and

(β) $\operatorname{diam}(\sum_{\omega \in \Omega_1} \mu_\omega^1 S_\omega^1) < \tau(\mu^1) + 2^{-1}$.

For $q = 2$ consider

$$\tau(\mu^2) = \inf\left\{\operatorname{diam}\left(\sum_{\omega \in \Omega_1} \mu_\omega^2 R_\omega\right) : R_\omega \text{ are slices contained in } S_\omega^1\right\}$$

and find $S_\omega^2 \subset S_\omega^1$ such that for $x \in \sum_{\omega \in \Omega_1} \mu_\omega^2 S_\omega^2$

(α) $\operatorname{dist}(x, E_1) > \tau(\mu^2)/2 - 2^{-1}$ and

(β) $\operatorname{diam}(\sum_{\omega \in \Omega_1} \mu_\omega^2 S_\omega^2) < \tau(\mu^2) + 2^{-1}$.

Continue in the obvious way until $q = p_1$.

For $\omega \in \Omega_1$ let $T_\omega = S_\omega^{p_1}$. This finishes the induction step for $n = 1$.

Now let $n > 1$ and suppose that the construction has been performed up to the $(n-1)$th step. Choose m_n such that for every $\omega \in \Omega_{n-1}$ we may find (by Lemma 2.9) slices $(S_{\omega,j})_{j=1}^{m_n}$ with

$$\operatorname{diam}\left[m_n^{-1} \sum_{j=1}^{m_n} S_{\omega,j}\right] < 2^{-n}$$

and $S_{\omega,j} \subset S_\omega$, $j = 1,\dots,m_n$. Let $\Omega_n = \Omega_{n-1} \times \{1,\dots,m_n\}$ and $M_n = \operatorname{card}(\Omega_n)$. Denote by F_{M_n} the positive face of the M_n-dimensional space $l^1_{M_n}$ and let $\mu^1,\dots,\mu^{p_n}$ be a finite 2^{-n}-net of F_{M_n}. Again we make a subinduction on $q = 1,\dots,p_n$: For $q = 1$ consider

$$\tau(\mu^1) = \inf\left\{\operatorname{diam}\left(\sum_{\omega\in\Omega_n} \mu^1_\omega R_\omega\right) \colon R_\omega \text{ slices contained in } S_\omega\right\}$$

and find, for $\omega \in \Omega_n$, slices $S^1_\omega \subset S_\omega$ such that, for $x \in \sum_{\omega\in\Omega_n} \mu^1_\omega S^1_\omega$

(α) $\operatorname{dist}(x, E_n) > \tau(\mu^1)/2 - 2^{-n}$, and

(β) $\operatorname{diam}(\sum_{\omega\in\Omega_n} \mu^1_\omega S^1_\omega) < \tau(\mu^1) + 2^{-n}$.

Continue in the obvious way for $q = 1,\dots,p_n$ to find, for $\omega \in \Omega_n$, slices $S_\omega \supset S^1_\omega \supset S^2_\omega \supset \cdots \supset S^{p_n}_\omega$ such that for every $q = 1,\dots,p_n$ and $x \in \sum_{\omega\in\Omega_n} \mu^q_\omega S^q_\omega$,

(8) (α) $$\operatorname{dist}(x, E_n) > \tau(\mu^q)/2 - 2^{-n}$$

and

(9) (β) $$\operatorname{diam}\left(\sum_{\omega\in\Omega_n} \mu^q_\omega S^q_\omega\right) < \tau(\mu^q) + 2^{-n}.$$

Finally let $T_\omega = S^{p_n}_\omega$ ($\omega \in \Omega_n$), which finishes the inductive construction.

We are now ready to define the operator T from $L^1(m)$ into X. It will be convenient to take as measure space

$$\Delta = \prod_{n=1}^{\infty} \{1,\dots,m_n\}$$

equipped with the canonical product-measure m_Δ. Clearly $L^1(m_\Delta)$ is lattice-isometric to the Lebesgue space $L^1(m)$.

For $\omega = (k_1,\dots,k_n) \in \Omega_n$ put

$$\Delta_\omega = \left\{\varepsilon = (\varepsilon_j)_{j=1}^{\infty} \in \Delta \colon \varepsilon_j = k_j \text{ for } j = 1,\dots,n\right\}.$$

By definition $m_\Delta(\Delta_\omega) = (m_1 \cdot m_2 \cdots m_n)^{-1} = M_n^{-1}$ and clearly the indicator-functions of Δ_ω span a dense subspace of $L^1(m_\Delta)$.

To define T on this subspace choose first, for each $\omega \in \Omega$, an arbitrary element x_ω of T_ω. For $\omega \in \Omega_n$ define

$$T(M_n \cdot \chi_{\Delta_\omega}) = \lim_{m\to\infty}\left\{(M^{-1}_{n+m} \cdot M_n) \cdot \sum_{\psi\in\Omega^\omega_{n+m}} x_\psi\right\},$$

where Ω^ω_{n+m} denotes those elements of Ω_{n+m} whose first n coordinates coincide with those of ω.

It follows from (6) that T is well defined and it is routine to check that T extends to a continuous linear operator T from $L^1(m_\Delta)$ into X with $\|T\| \leqslant 1$. Formula (7) implies that

(10) $$T(f) \in T_\omega$$

if $\omega \in \Omega_n$ and $f \in F$ is such that f is supported by Δ_ω.

We shall show that T vertifies (i), (ii) and (iii) of Proposition 2.6. Condition (i) is obviously satisfied. We next verify (iii): Fix μ in $\bar{F}^*$ and $n \in \mathbf{N}$ and consider the partition P of Δ into $\{\Delta_\omega: \omega \in \Omega_n\}$. By the construction of the nth induction step, there is $1 \leqslant q \leqslant p_n$ and μ^q in F_{M_n} such that

$$\sum_{\omega \in \Omega_n} |\mu(\Delta_\omega) - \mu_\omega^q| < 2^{-n}.$$

Note that, in view of (10)

$$T(V_{P,0}(\mu)) \subset \sum_{\omega \in \Omega_n} \mu(\Delta_\omega) \cdot T_\omega \subset \sum_{\omega \in \Omega_n} \mu(\Delta_\omega) \cdot S_\omega^q.$$

From the formula

$$V_{P,\varepsilon}(\mu) = \left[V_{P,0}(\mu) + \varepsilon \cdot \operatorname{ball} L^1(m_\Delta)\right] \cap F$$

(which is analogous to formula (5) in the proof of Lemma 2.5) we infer that

$$\begin{aligned} (11) \qquad T(V_{P,2^{-n}}(\mu)) &\subset \left[\sum_{\omega \in \Omega_n} \mu(\Delta_\omega) \cdot S_\omega^q\right] + 2^{-n} \cdot \operatorname{ball}(X) \\ &\subset \left[\sum_{\omega \in \Omega_n} \mu_\omega^q \cdot S_\omega^q\right] + 2 \cdot 2^{-n} \operatorname{ball}(X). \end{aligned}$$

Hence (9) implies that

$$\operatorname{diam}\left[T(V_{P,2^{-n}}(\mu))\right] \leqslant \tau(\mu^q) + 3 \cdot 2^{-n}$$

and

$$\rho_T(\mu) \leqslant \tau(\mu^q) + 3.2^{-n}.$$

On the other hand, it follows from (8) and (11) that for any $x \in T(V_{P,2^{-n}}(\mu))$

$$\operatorname{dist}(x, E_n) > \tau(\mu^q)/2 - 3.2^{-n}.$$

As $n \in \mathbf{N}$ is arbitrary and $(E_n)_{n=1}^\infty$ is dense in the span of D we obtain (iii), i.e.

$$d_T(\mu) \geqslant \rho_T(\mu)/2.$$

We still have to verify condition (ii) of Proposition 2.6. Let x^{**} be an extreme point of $\bar{C}^* = T^{**}(\bar{F}^*)$ and take $n \in \mathbf{N}$. We claim that there is a $\mu \in \bar{F}^*$ such that $T^{**}(\mu) = x^{**}$ and there is one (and only one) $\omega_0 \in \Omega_n$ such that $\mu(\Delta_{\omega_0}) = 1$. Indeed the sets

$$\bar{F}_{\Delta_\omega}^* = \left\{\mu \in \bar{F}^*: \mu(\Delta_\omega) = 1\right\}, \qquad \omega \in \Omega_n,$$

are weak-star compact convex subsets of $\bar{F}^*$ such that

$$\bar{F}^* = \operatorname{conv}\left\{\bar{F}_{\Delta_\omega}^*: \omega \in \Omega_n\right\}.$$

Hence

$$T^{**}(\bar{F}^*) = \operatorname{conv}\left\{T^{**}\left(\bar{F}_{\Delta_\omega}^*\right): \omega \in \Omega_n\right\}.$$

Since x^{**} is extreme in $\bar{C}^* = T^{**}(\bar{F}^*)$, there are $\omega_0 \in \Omega_n$ and $\mu \in \bar{F}_{\Delta_{\omega_0}}^*$ with $T^{**}(\mu) = x^{**}$ which proves the claim. Once again by the n'th induction step, there is a $q \leqslant p_n$ with

$$\sum_{\omega \in \Omega_n} |\mu(\Delta_\omega) - \mu_\omega^q| < 2^{-n}.$$

Hence, in particular, $\mu^q_{\omega_0} > 1 - 2^{-n}$. Since every slice of D has diameter greater than or equal to 2α, we conclude that

$$\tau(\mu^q) \geqslant (1 - 2^{-n})2\alpha.$$

As in the proof of assertion (iii), we may infer that for every $x \in T(V_{P,2^{-n}}(\mu))$,

$$\operatorname{dist}(x, E_n) > (1 - 2^{-n})\alpha - 3.2^{-n}.$$

The $n \in \mathbf{N}$ being arbitrary, we conclude that

$$\operatorname{dist}(x^{**}, X) \geqslant \alpha.$$

This completes the proof of Proposition 2.6. $\square$

We can now harvest the corollaries of Theorem 2.1. The first one gives a more precise version of a theorem due to R. Huff and P. Morris [**H-M**].

2.10. Corollary. *If a weak-star compact, convex subset K of a dual Banach space X^* fails RNP then there is a $\|\cdot\|$-closed, convex, separable subset $C \subset K$ without extreme points.*

Proof. We have to distinguish two cases:

(a) There is a bounded sequence $(x_n)_{n=1}^\infty$ in X such that no subsequence $(x_{n_k})_{k=1}^\infty$ converges pointwise on K.

(b) (a) does not hold.

Let us first deal with case (b): It is a theorem due to J. Bourgain (see [**B2**, Lemma 3.7] or [**G-G-M**], Corollary 4.12) that in this case K is strongly regular. Hence we may apply Theorem 2.1 to prove the corollary for this case.

Case (a) is essentially known: It follows from Rosenthal's theorem [**R**] and (a variant of) a theorem of Pełczyński [**P**] that there is an isomorphic embedding S: $L^1(m) \to X^*$ such that $S(F) \subset K$. Clearly $S(F)$ is a closed, convex, separable set without extreme points.

For the sake of completeness we shall write this up explicitly. It may be deduced from Rosenthal's theorem (see [**Ta**, Theorem 7-3-1 and supplement 7-3-5]) that there is a subsequence $(x_{n_k})_{kn=1}^\infty$ and $\alpha < \beta$ such that for finite disjoint sets I, J in $\mathbf{N}$

$$K \cap \bigcap_{k \in I} \{x_{n_k} \leqslant \alpha\} \cap \bigcap_{k \in J} \{x_{n_k} \geqslant \beta\} \neq \varnothing.$$

If i: $l^1 \to X$ denotes the operator sending the k'th unit vector e_k to x_{n_k} this means, that $i^*(K)$ contains the subset $[\alpha, \beta]^{\mathbf{N}}$ of l^∞. It is an easy exercise to construct an embedding T of $L^1[0,1]$ into l^∞ such that $\|T\| \leqslant \max(|\beta|, |\alpha|)$ and $\|T(h)\| \geqslant (\beta - \alpha)/2$ for $h \in L^1$, $\|h\| = 1$, and $T(F) \subset [\alpha, \beta]^{\mathbf{N}}$. We now lift this operator in a well-known way (see, e.g., [**R-S-U**, p. 532]): For $m \in \mathbf{N}$ and $1 \leqslant j \leqslant 2^m$ find $x^*_{m,k} \in K$ such that

$$i^*(x^*_{m,k}) = T(2^m \chi_{[(j-1)2^{-k}, j2^{-k}]})$$

and let S_m be the operator from $L^1[0,1]$ to X^* given by

$$S_m = \sum_{j=1}^{2^m} \chi_{[(j-1)2^{-k}, j2^{-k}]} \otimes x^*_{m,k}.$$

Let S be any cluster point of the sequence S_m in the weak-star operator-topology. Then S is an isomorphic embedding of $L^1[0,1]$ into X^* such that $S(F) \subset K$. □

We now pass to the second result which we may deduce from Theorem 2.1.

2.11. COROLLARY. *If D is a closed, convex, bounded subset of a space X with an unconditional basis and D fails RNP, then there is a closed, convex $C \subset D$ with no extreme points.*

PROOF. Recall that D has the "convex point-of-continuity property" (CPCP) if for every convex subset $C \subset D$ and $\varepsilon > 0$ there is a relatively weakly open subset $U \subset C$ with $\mathrm{diam}(U) < \varepsilon$ (see [**G-G-M**, III.5]). It was noticed by J. Bourgain that a set D having (CPCP) is strongly regular (see [**G-G-M**, Lemma II.1]).

For the proof of the corollary we split again into two cases:

Case (a). D has (CPCP).

Case (b). D fails (CPCP).

Case (a) is taken care of by Theorem 2.1 and the preceding remark. Case (b) needs a somewhat long argument. Let us note, however, that Case (b) may be settled in a relatively straightforward way if one assumes that D is the unit ball of a closed subspace of X (i.e. for the "global" version of the problem whether RNP and KMP are equivalent). Indeed, as has been pointed out to us by N. Ghoussoub, one may construct in a fairly straightforward way a sequence $(x_n)_{n=1}^\infty$ in D, $\|x_n\| \geqslant \alpha > 0$ and x_n supported by disjoint blocks of the unconditional basis (up to a small perturbation) and such that $x_1 + \cdots + x_n \in D$. By the unconditionality we deduce that there is a constant M such that for all $n \in \mathbf{N}$ and all choices of signs $\varepsilon_1, \ldots, \varepsilon_n$ $\|\varepsilon_1 x_1 + \cdots + \varepsilon_n x_n\| \leqslant M$; hence—under the assumption that D is the unit-ball of a closed subspace of X—the sequence $\{x_n/M\}_{n=1}^\infty$ is such that for all choices of signs $\varepsilon_1 x_1/M + \cdots + \varepsilon_n x_n/M$ is in D. So—by a well-known theorem of Pełczyński (see, e.g., [**D-U**, Theorem 6.15]—we may find an isomorphic embedding T of c_0 into X sending the unit ball of c_0 into D. Of course T (ball(c_0)) then is a closed, convex subset of D with no extreme points.

Let us now pass to the general Case (b), i.e. suppose that D is a closed, convex, bounded subset of X, failing (CPCP) and that X has an unconditional basis $(e_i)_{i=1}^\infty$.

Without loss of generality we may assume that the unconditional basis constant of $(e_i)_{i=1}^\infty$ is 1. We can also find a closed convex $D_1 \subset D$ and $\varepsilon > 0$ such that every relatively weakly open subset $U \subset D_1$ has diameter greater than 2ε. For implicity write $D = D_1$.

It follows that for each $x_0 \in U$

$$x_0 \in \overline{\mathrm{co}}\left((D \setminus B(x_0, \varepsilon)) \cap U\right) \tag{12}$$

where $B(x_0, \varepsilon)$ is the open ball around x_0 of radius ε. Indeed, if this were not the case we could separate x_0 from $(D \setminus B(x_0, \varepsilon)) \cap U$ strictly by an element $x^* \in X^*$, i.e.

$$\langle x^*, x_0 \rangle - \sup\{\langle x^*, x\rangle : x \in (C \setminus B(x_0, \varepsilon)) \cap U\} = \gamma > 0.$$

The set

$$V = U \cap \{x \in C : \langle x^*, x - x_0 \rangle < \gamma/2\}$$

is a relatively weakly open subset of D contained in $B(x_0, \varepsilon)$, a contradiction proving (12).

In particular, if P_n denotes the canonical projection onto the span of $(e_i)_{i=1}^n$ we get for $x_0 \in C$ and $\delta > 0$

$$x_0 \in \overline{\mathrm{co}}\left((D \setminus B(x_0, \varepsilon)) \cap \{x : \|P_n(x - x_0)\| < \delta\}\right). \tag{13}$$

We now construct (somewhat as in **[B1**, Lemma 20]) a bush by induction. For each $n \in \mathbf{N}$ we define a finite subset Ω_n of $\mathbf{N}^n$ and for each $\omega \in \Omega_n$ points $x_\omega \in D$, scalars $1 > \lambda_\omega > 0$, integers $i(\omega)$, $j(\omega)$ and subspaces $\mathscr{C}_\omega$ of Y, where

$$\mathscr{C}_\omega = \mathrm{span}\{e_i : j(\omega) < i \leqslant i(\omega)\}$$

such that

(1) Ω_n is the projection of Ω_{n+1} on the first n coordinates.
(2) $\sum_i \lambda_{\omega,i} = 1$ $(n \in \mathbf{N}, \omega \in \Omega_n)$,
(3) $\|x_\omega - \sum_i \lambda_{\omega,i} x_{\omega,i}\| < \varepsilon/2^{n+2}$ $(n \in \mathbf{N}, \omega \in \Omega_n)$,
(4) $\mathrm{dist}(x_{\omega,i} - x_\omega, \mathscr{C}_\omega) < \varepsilon/2^{n+2}$ $(n \in \mathbf{N}, \omega \in \Omega_n)$,
(5) $\|x_{\omega,i} - x_\omega\| \geqslant \varepsilon$ $(n \in \mathbf{N}, \omega \in \Omega_n)$.

Start by taking $n = 1$, $\Omega_1 = \{1\}$ and x_1 any point of D and $\lambda_1 = 1$. Find $i(0)$ such that

$$\mathrm{dist}(x_1, \mathrm{span}\{e_i : i \leqslant i(0)\}) < \varepsilon/16.$$

For $n = 2$ let $j(1)$ equal $i(0)$ and

$$E_1 = \mathrm{span}\{e_i : i > j(1)\} \cap X$$

and apply (13) to $n = j(1)$ and $\delta = \varepsilon/32$ to find a finite set $\Omega_2 \subset \mathbf{N}^2$ and points x_ω in D and scalars λ_ω $(\omega \in \Omega_2)$ such that (1), (2), (3) and (5) are satisfied. Find $i(1)$ such that

$$\omega \in \Omega_2, \qquad \mathrm{dist}(x_\omega, C_1) < \varepsilon/32,$$

which will take care of (4).

Now assume that, for each $k \leqslant n$, Ω_k and $\{x_\omega, \lambda_\omega : \omega \in \Omega_k\}$ and $\{i(\omega), j(\omega), C_\omega : \omega \in \Omega_{k-1}\}$ are defined. We order Ω_n lexicographically and proceed by subinduction on this finite totally ordered set.

For the first $\omega \in \Omega_n$ let $j(\omega)$ be the maximum of $\{i(\omega) : \omega \in \Omega_{n-1}\}$. Applying (13) to $n = j(\omega)$ and $\delta = \varepsilon/2^{n+3}$ we may find $m(\omega)$ and points $x_{\omega,i}$ in D and scalars $\lambda_{\omega,i}$, $1 \leqslant i \leqslant m(\omega)$, such that (2), (3) and (5) are satisfied. Finally find $i(\omega)$ so large that

$$\mathrm{dist}(x_{\omega,i}, \mathscr{C}_\omega) < \varepsilon/2^{n+3}, \qquad 1 \leqslant i \leqslant m(\omega),$$

to satisfy (4).

If $\omega \in \Omega_n$ has a predecessor $\omega' \in \Omega_n$ then let $j(\omega) = i(\omega')$ and proceed as in the preceding paragraph. When one has thus gone through all of Ω_n it is clear how to define Ω_{n+1}:

$$\Omega_{n+1} = \{(\omega, i) \colon \omega \in \Omega_n, 1 \leqslant i \leqslant m(\omega)\}.$$

This finishes the induction step on n.

We now have constructed on approximate bush $\{x_\omega \colon \omega \in \Omega\}$—with errors controlled by (3)—and similarly as in the proof of Proposition 1.6 we construct a compact set Δ equipped with a probability measure m_Δ corresponding to the index set $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$ and an operator $T \colon L^1(m_\Delta) \to X$.

With the notation of the proof of 1.6 let $n \in \mathbf{N}$, $\omega \in \Omega_n$ and $\mu \in \bar{F}^*$ such that μ is supported by Δ_ω (i.e. $\langle \mu, \Delta_\omega \rangle = 1$). Note that it follows from (4) that $T^{**}(\mu)$ is in X iff $P_\omega^{**}(T^{**}(\mu))$ is in X where $P_\omega \colon X \to X$ is the canonical projection onto the space

$$E_\omega = \overline{\operatorname{span}} \{C_\psi \colon \psi \geqslant \omega\}.$$

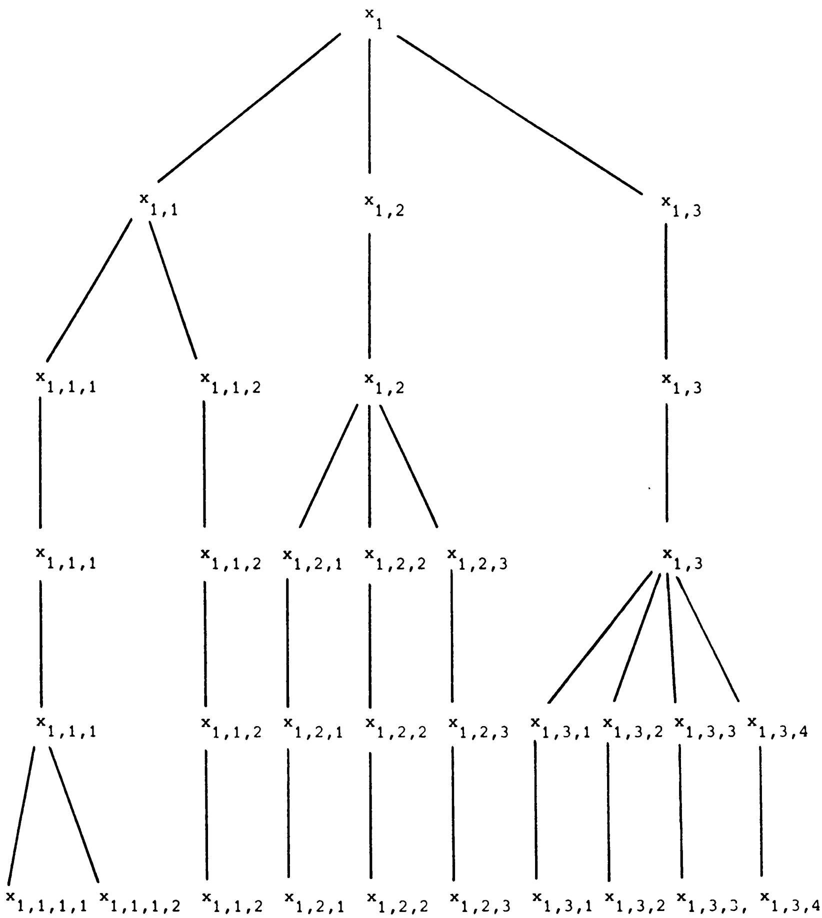
If $\omega' \in \Omega_n$ with $\omega' \neq \omega$ note that E_ω and $E_{\omega'}$ are spaces spanned by disjoint subsets of the unconditional basis (e_i). Hence for $\nu \in \bar{F}^*$, ν supported by $\Delta_{\omega'}$, and λ_1, $\lambda_2 \in [0,1]$, $\lambda_1 + \lambda_2 = 1$ we infer that $T^{**}(\lambda_1\mu + \lambda_2\nu)$ is in X iff $T^{**}(\mu)$ and $T^{**}(\nu)$ are in X. Hence the arguments of [**S1**]—in tandem with the easily verified fact that the extreme points of $T^{**}(\bar{F}^*)$ are in $X^{**} \setminus X$—allows us to conclude that $\overline{T(F)}$ is a closed, convex, bounded subset of X without extreme points. □

2.12 Remark. We can regard the bush constructed above also as a martingale. Having in mind that one interpretation of a martingale is the strategy of a gambler who is playing games with expectation zero we could characterize the above construction as the "strategy of a lazy gambler". Of course, this idea is well known in martingale theory— as was pointed out to us by D. Burkholder—and may also be found implicitly in [**H**, Example 4].

Let us give an intuitive interpretation: A gambler who has the possibility of participating in a sequence of games, each of which has only a finite number of possible results (valued in a Banach space!) and expectation zero, chooses the following "lazy" strategy: At level 1 he plays the first game which has—say—3 possible results, which he enumerates (1,1), (1,2) and (1,3) (this corresponds to Ω_2 above). In the second game the gambler plays only if the result of the first game was (1,1); in the other cases he *simply passes*. Similarly in the third game he only plays if the result of the first game was (1,2), and in the fourth game he only plays if the result of the first game was (1,3).

Games 2 to 4 constitute the second level.

Suppose now that—say—the second game had 2, the third game 3 and the fourth game 4 possible results. Enumerate these by (1,1,1), (1,1,2), (1,2,1), ..., (1,2,3), (1,3,1), ..., (1,3,4). In the fifth game the gambler only plays if the result at the second level was (1,1,1)—in all other cases he passes—etc. Until in the 13th game the gambler only plays if the result at the second level was (1,3,4). This constitutes level 3 and the strategy is further developed in an obvious way.



etc.

A gambler who sticks to this strategy will pass very often—hence the description "lazy". Note that if two gamblers play parallel according to this same strategy, then from that moment on, when they have had a different result in a game, at most one of the gamblers will play a given game—the other will pass. This is the idea of the above construction, which forces the above wedges to lie essentially in subspaces of Y spanned by disjoint subsets of the unconditional basis (e_i).

2.13. Corollary 2.15 below establishes a link between the problem of equivalence of RNP and KMP and the local theory of Banach spaces. We thank E. Odell for pointing out to us the—essentially known—Proposition 2.14 and N. Ghoussoub for suggesting the formulation of Corollary 2.15.

2.14. Proposition. *If a Banach space X fails to be strongly regular, then X^{**} contains a subspace isomorphic to $l^1(I)$, where* $\mathrm{card}(I) = \mathrm{card}([0,1])$. *In particular X contains l_n^1's uniformly.*

PROOF. A Banach space Y does not contain an isomorphic copy of l^1 iff Y^* is "weak-star strongly regular" (Bourgain [**B2**]) iff Y^* is strongly regular [**G-G-M-S**, Corollary VI. 18].

If X fails to be strongly regular then X^{**} fails to be so too, whence X^* contains an isomorphic copy of l^1. Hence X^{**} contains an isomorphic copy of $l^1[0, 1]$.

The last part follows from the principle of local reflexivity. □

2.15. COROLLARY. *If X does not contain l_n^1's uniformly then X has RNP iff X has KMP.* □

Finally we want to stress that we do not know whether for subspaces (or closed, convex, bounded subsets) of L^1 the properties RNP and KMP coincide.

2.16. PROBLEM. Are RNP and KMP equivalent for subspaces of $L^1[0, 1]$?

REFERENCES

[**B1**] J. Bourgain, *Dentability and finite-dimensional decompositions*, Studia Math. **67** (1980), 135–148.

[**B2**] ______, *La propriété de Radon-Nikodym*, Publ. Univ. Pierre et Marie Curie No. 36 (1979).

[**B-R**] J. Bourgain and H. P. Rosenthal, *Geometrical implications of certain finite-dimensional decompositions*, Bull. Soc. Math. Belg. **32** (1980), 57–82.

[**B-T**] J. Bourgain and M. Talagrand, *Dans un espace de Banach réticulé solide, la propriété de Radon-Nikodým et celle de Krein-Milman sont équivalentes*, Proc. Amer. Math. Soc. **81** (1981), 93–96.

[**D-U**] J. Diestel and J. J. Uhl, *Vector measures*, Math. Surveys, no. 15, Amer. Math. Soc., Providence, R.I., 1977.

[**E-W**] G. A. Edgar and R. Wheeler, *Topological properties of Banach spaces*, Pacific J. Math. **115** (1984), 317–350.

[**G-G-M**] N. Ghoussoub, G. Godefroy and B. Maurey, *First class functions around a subset of a compact space and geometrically regular Banach spaces*, preprint.

[**G-G-M-S**] N. Ghoussoub, G. Godefroy, B. Maurey and W. Schachermayer, *Some topological and geometrical structures in Banach spaces* (essentially including [**G-G-M**] and [**S2**]), Mem. Amer. Math. Soc. (submitted).

[**G-M**] N. Ghoussoub and B. Maurey, *The Radon-Nikodym property in function spaces*, Lecture Notes in Math., vol. 1166, Springer-Verlag, Berlin and New York, 1985.

[**G-M-S**] N. Ghoussoub, B. Maurey and W. Schachermayer, *A counterexample to a problem on points of continuity in Banach spaces*, Proc. Amer. Math. Soc. **99** (1987), 278–282.

[**H**] A. Ho, *The Krein-Milman property and complemented bushes in Banach spaces*, Pacific J. Math. **98** (1982), 347–363.

[**H-M**] R. E. Huff and P. D. Morris, *Dual spaces with the Krein-Milman property have the Radon-Nikodym property*, Proc. Amer. Math. Soc. **49** (1975), 104–108.

[**J**] R. C. James, *Uniformly non-square Banach spaces*, Ann. of Math. (2) **80** (1964), 542–550.

[**P**] A. Pełczyńksi, *On Banach spaces containing $L^1(\mu)$*, Studia Math. **30** (1968), 231–246.

[**R-S-U**] L. Riddle, E. Saab and J. Uhl, *Sets with the weak Radon-Nikodym property in dual Banach spaces*, Indiana Univ. Math. J. **32** (1983), 527–541.

[**R**] H. P. Rosenthal, *A characterisation of Banach spaces containing l^1*, Proc. Nat. Acad. Sci. U.S.A. **71** (1974), 2411–2413.

[**S1**] W. Schachermayer, *For a Banach space isomorphic to its square the Radon-Nikodym property and the Krein-Milman property are equivalent*, Studia Math. **81** (1985), 329–339.

[**S2**] ______, *Operators from L^1 into Banach spaces and subsets of L^∞*, preprint.

[**St**] C. Stegall, *The Radon-Nikodym property in Banach spaces*, Part I, Vorlesungen aus dem Fachbereich Mathematik der Universität Essen, Heft 10, Essen, 1983.

[**Ta**] M. Talagrand, *Pettis integral and measure theory*, Mem. Amer. Math. Soc. no. 307, 1984.

[**Th**] E. Thomas, *A converse to Edgar's theorem*, Measure Theory, Oberwolfach 1979 (D. Kölzow, ed.), Lecture Notes in Math., vol. 794, Springer-Verlag, Berlin and New York, 1980, pp. 497–512.

JOHANNES KEPLER UNIVERSITÄT LINZ, INSTITUT FÜR MATHEMATIK, A - 4040 LINZ, AUSTRIA