

CREDIT RISK MODELING AND VALUATION: AN INTRODUCTION

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Abstract

Credit risk is the distribution of financial losses due to unexpected changes in the credit quality of a counterparty in a financial agreement. We review the structural, reduced form and incomplete information approaches to estimating joint default probabilities and prices of credit sensitive securities.

Key words: credit risk; default risk; structural approach; reduced form approach; incomplete information approach; intensity; trend; compensator.

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1 Introduction

Credit risk is the distribution of financial losses due to unexpected changes in the credit quality of a counterparty in a financial agreement. Examples range from agency downgrades to failure to service debt to liquidation. Credit risk pervades virtually all financial transactions.

The distribution of credit losses is complex. At its center is the probability of default, by which we mean any type of failure to honor a financial agreement. To estimate the probability of default, we need to specify

- a model of investor uncertainty;
- a model of the available information and its evolution over time; and
- a model definition of the default event.

However, default probabilities alone are not sufficient to price credit sensitive securities. We need, in addition,

- a model for the riskfree interest rate;
- a model of recovery upon default; and
- a model of the premium investors require as compensation for bearing systematic credit risk.

The credit premium maps actual default probabilities to market-implied probabilities that are embedded in market prices. To price securities that are sensitive to the credit risk of multiple issuers and to measure aggregated portfolio credit risk, we also need to specify

- a model that links defaults of several entities.

There are three main quantitative approaches to analyzing credit. In the *structural* approach, we make explicit assumptions about the dynamics of a firm's assets, its capital structure, and its debt and share holders. A firm defaults if its assets are insufficient according to some measure. In this situation a corporate liability can be characterized as an option on the firm's assets. The *reduced form* approach is silent about why a firm defaults. Instead, the dynamics of default are exogenously given through a default rate, or intensity. In this approach, prices of credit sensitive securities can be calculated as if they were default free using an interest rate that is the riskfree rate adjusted

by the intensity. The *incomplete information* approach combines the structural and reduced form models. While avoiding their difficulties, it picks the best features of both approaches: the economic and intuitive appeal of the structural approach and the tractability and empirical fit of the reduced form approach.

This article reviews these approaches in the context of the multiple facets of credit modeling that are mentioned above. Our goal is to provide a concise overview and a guide to the large and growing literature on credit risk.

2 Structural credit models

The basis of the structural approach, which goes back to Black & Scholes (1973) and Merton (1974), is that corporate liabilities are contingent claims on the assets of a firm. The market value of the firm is the fundamental source of uncertainty driving credit risk.

2.1 Classical approach

Consider a firm with market value V , which represents the expected discounted future cash flows of the firm. The firm is financed by equity and a zero coupon bond with face value K and maturity date T . The firm's contractual obligation is to repay the amount K to the bond investors at time T . Debt covenants grant bond investors absolute priority: if the firm cannot fulfil its payment obligation, then bond holders will immediately take over the firm. Hence the default time τ is a discrete random variable given by

$$\tau = \begin{cases} T & \text{if } V_T < K \\ \infty & \text{if else.} \end{cases} \quad (1)$$

Figure 1 depicts the situation graphically.

To calculate the probability of default, we make assumptions about the distribution of assets at debt maturity under the physical probability P . The standard model for the evolution of asset prices over time is geometric Brownian motion:

$$\frac{dV_t}{V_t} = \mu dt + \sigma dW_t, \quad V_0 > 0, \quad (2)$$

where $\mu \in \mathbb{R}$ is a drift parameter, $\sigma > 0$ is a volatility parameter, and W is a standard Brownian motion. Setting $m = \mu - \frac{1}{2}\sigma^2$, Ito's lemma implies that

$$V_t = V_0 e^{mt + \sigma W_t}.$$

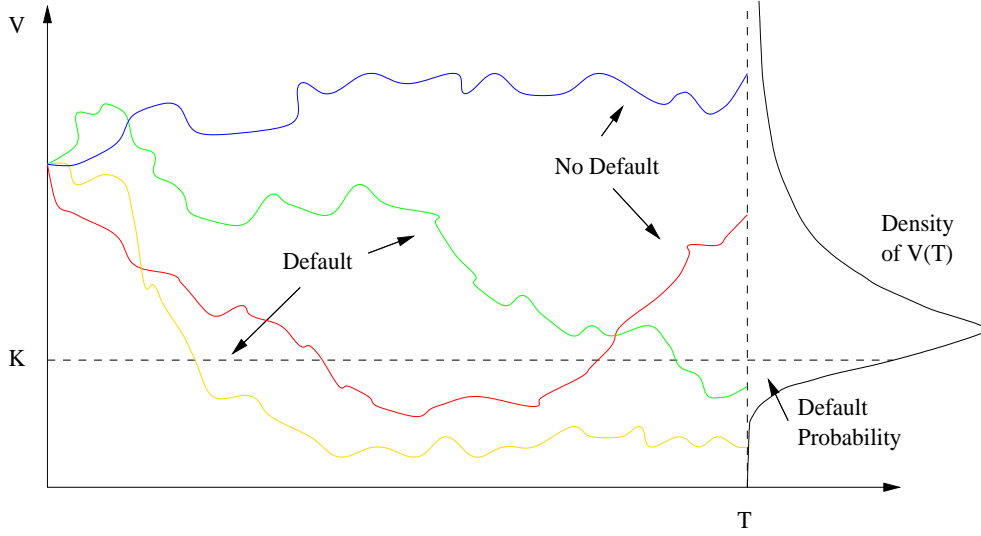


Figure 1: Default in the classical approach.

Since W_T is normally distributed with mean zero and variance T , default probabilities $p(T)$ are given by

$$p(T) = P[V_T < K] = P[\sigma W_T < \log L - mT] = \Phi\left(\frac{\log L - mT}{\sigma\sqrt{T}}\right)$$

where $L = \frac{K}{V_0}$ is the initial leverage ratio and Φ is the standard normal distribution function.

Assuming that the firm can neither repurchase shares nor issue new senior debt, the payoffs to the firm's liabilities at debt maturity T are as summarized in Table 1. If the asset value V_T exceeds or equals the face value K of the bonds, the bond holders will receive their promised payment K and the shareholders will get the remaining $V_T - K$. However, if the value of assets V_T is less than K , the ownership of the firm will be transferred to the bondholders, who lose the amount $K - V_T$. Equity is worthless because of limited liability. Summarizing, the value of the bond issue B_T^T at time T is given by

$$B_T^T = \min(K, V_T) = K - \max(0, K - V_T).$$

This payoff is equivalent to that of a portfolio composed of a default-free loan with face value K maturing at T and a short European put position on the assets of the firm with strike K and maturity T . The value of the equity E_T

	Assets	Bonds	Equity
No Default	$V_T \geq K$	K	$V_T - K$
Default	$V_T < K$	V_T	0

Table 1: Payoffs at maturity in the classical approach.

at time T is given by

$$E_T = \max(0, V_T - K),$$

which is equivalent to the payoff of a European call option on the assets of the firm with strike K and maturity T .

Pricing equity and credit risky debt reduces to pricing European options. We consider the classical Black-Scholes setting. The financial market is frictionless, trading takes place continuously in time, riskfree interest rates $r > 0$ are constant and firm assets X follow geometric Brownian motion (2). Also, the value of the firm is a traded asset. The equity value is given by the Black-Scholes call option formula C :

$$E_0 = C(\sigma, T, K, r, V_0) = V_0 \Phi(d_+) - e^{-rT} K \Phi(d_-) \quad (3)$$

where

$$d_{\pm} = \frac{(r \pm \frac{1}{2}\sigma^2)T - \log L}{\sigma\sqrt{T}}.$$

We note that the equity pricing function is monotone in firm volatility σ : equity holders always benefit from an increase in firm volatility.

While riskfree zero coupon bond prices are just $K \exp(-rT)$ with T being the bond maturity, the value of the corresponding credit-risky bonds is

$$B_0^T = K e^{-rT} - P(\sigma, T, K, r, V_0)$$

where P is the Black-Scholes vanilla put option formula. We note that the value of the put is just equal to the present value of the default loss suffered by bond investors. This is the discount for default risk relative to the riskfree bond, which is valued at $K \exp(-rT)$. This yields

$$B_0^T = V_0 - V_0 \Phi(d_1) + e^{-rT} K \Phi(d_2)$$

which together with (3) proves the market value identity

$$V_0 = E_0 + B_0^T.$$

While clearly both equity and debt values depend on the firm's leverage ratio, this equation shows that their sum does not. This shows that the Modigliani & Miller (1958) theorem holds also in the presence of default. This result asserts that the market value of the firm is independent of its leverage, see Rubinstein (2003) for a discussion. This is not the case for all credit models as we show below.

The *credit spread* is the difference between the yield on a defaultable bond and the yield on an otherwise equivalent default-free zero bond. It gives the excess return demanded by bond investors to bear the potential default losses. Since the yield $y(t, T)$ on a bond with price $b(t, T)$ satisfies $b(t, T) = \exp(-y(t, T)(T - t))$, we have for the credit spread $S(t, T)$ at time t ,

$$S(t, T) = -\frac{1}{T - t} \log \left(\frac{B_t^T}{\bar{B}_t^T} \right), \quad T > t, \quad (4)$$

where \bar{B}_t^T is the price of a default-free bond maturing at T . The term structure of credit spreads is the schedule of $S(t, T)$ against T , holding t fixed. In the Black-Scholes setting, we have $\bar{B}_t^T = K \exp(-r(T - t))$ and we obtain

$$S(0, T) = -\frac{1}{T} \log \left(\Phi(d_-) + \frac{1}{L} e^{rT} \Phi(-d_+) \right), \quad T > 0, \quad (5)$$

which is a function of maturity T , asset volatility σ (the firm's business risk), the initial leverage ratio L , and riskfree rates r . Letting leverage be 80% and riskfree rates be 5%, in Figure 2 we plot the term structure of credit spreads for varying asset volatilities.

2.2 First-passage approach

In the classical approach, firm value can dwindle to almost nothing without triggering default. This is unfavorable to bondholders, as noted by Black & Cox (1976). Bond indenture provisions often include safety covenants that give bond investors the right to reorganize a firm if its value falls below a given barrier.

Suppose the default barrier D is a constant valued in $(0, V_0)$. Then the default time τ is a continuous random variable valued in $(0, \infty]$ given by

$$\tau = \inf\{t > 0 : V_t < D\} \quad (6)$$

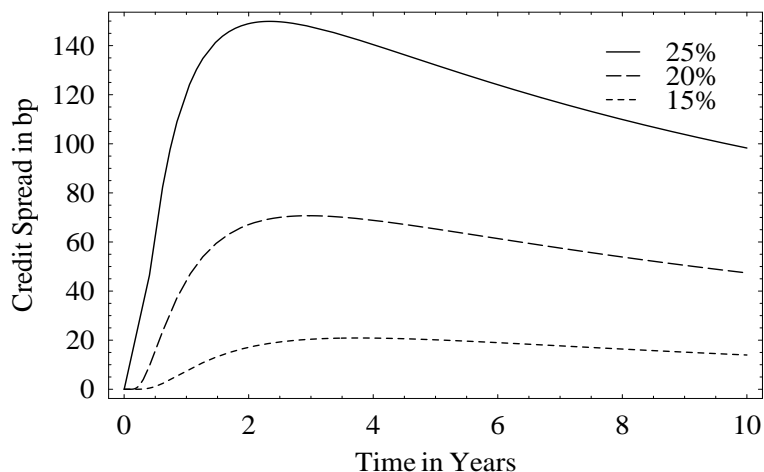


Figure 2: Term structure of credit spreads given by (5) for varying asset volatilities σ , in the classical approach. We set $L = 80\%$ and $r = 5\%$.

Figure 3 depicts the situation graphically. In the Black-Scholes setting with asset dynamics (2), default probabilities are calculated as

$$p(T) = P[M_T < D] = P[\min_{s \leq T}(ms + \sigma W_s) < \log(D/V_0)].$$

where M is the historical low of firm values,

$$M_t = \min_{s \leq t} V_s.$$

Since the distribution of the historical low of an arithmetic Brownian motion is inverse Gaussian,¹ we have

$$p(T) = \Phi\left(\frac{\log(D/V_0) - mT}{\sigma\sqrt{T}}\right) + \left(\frac{D}{V_0}\right)^{\frac{2m}{\sigma^2}} \Phi\left(\frac{\log(D/V_0) + mT}{\sigma\sqrt{T}}\right). \quad (7)$$

We check whether this default definition is consistent with the payoff to investors. We need to consider two scenarios. The first is when $D \geq K$. If the firm value never falls below the barrier D over the term of the bond ($M_T > D$), then bond investors receive the face value $K < V_0$ and the equity holders

¹To find that distribution, one first calculates the joint distribution of the pair $(W_t, \min_{s \leq t} W_s)$ by the reflection principle. Girsanov's theorem is used to extend to the case of Brownian motion with drift.

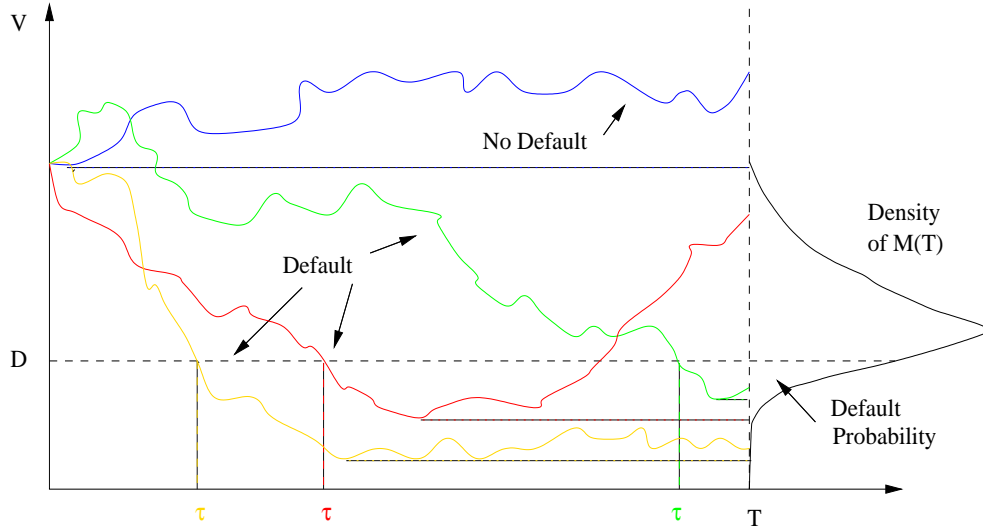


Figure 3: Default in the first-passage approach.

receive the remaining $V_T - K$. However, if the firm value falls below the barrier at some point during the bond's term ($M_T \leq D$), then the firm defaults. In this case the firm stops operating, bond investors take over its assets D and equity investors receive nothing. Bond investors are fully protected: they receive at least the face value K upon default and the bond is not subject to default risk any more.

This anomaly does not occur if we assume $D < K$ so that bond holders are both exposed to some default risk and compensated for bearing that risk. If $M_T > D$ and $V_T \geq K$, then bond investors receive the face value K and the equity holders receive the remaining $V_T - K$. If $M_T > D$ but $V_T < K$, then the firm defaults, since the remaining assets are not sufficient to pay off the debt in full. Bond investors collect the remaining assets V_T and equity becomes worthless. If $M_T \leq D$, then the firm defaults as well. Bond investors receive $D < K$ at default and equity becomes worthless.

Reisz & Perlich (2004) point out that if the barrier is below the bond's face value, then our earlier definition (6) does not reflect economic reality anymore: it does not capture the situation when the firm is in default because $V_T < K$ although $M_T > D$. We discuss two remedies to avoid this inconsistency.

Re-define default. We re-define default as firm value falling below the barrier $D < K$ at any time before maturity *or* firm value falling below face value

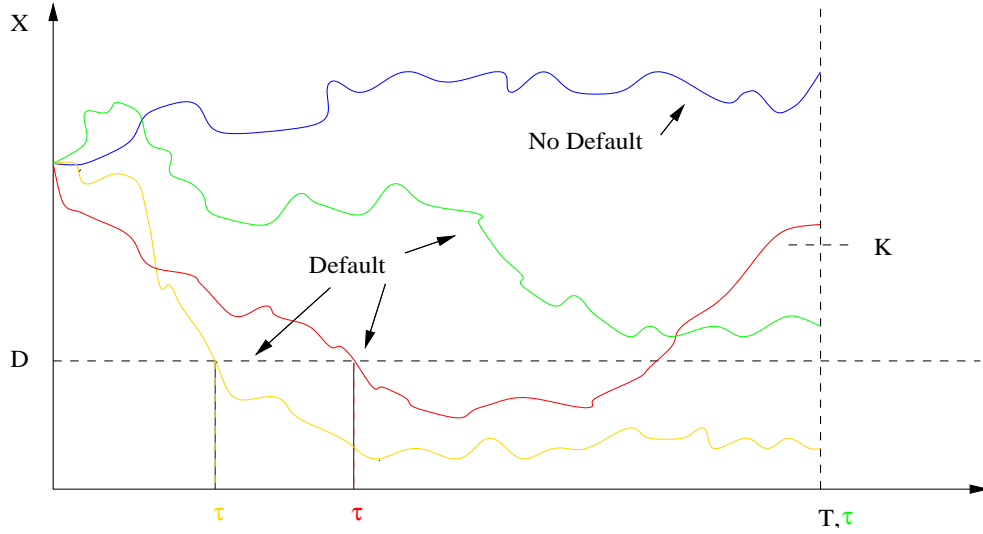


Figure 4: Default at first passage of firm value to the default barrier or at debt maturity T if the corresponding firm value V_T is less than the debt's face value K .

K at maturity. Formally, the default time is now given by

$$\tau = \min(\tau^1, \tau^2), \quad (8)$$

where τ^1 is the first passage time of assets to the barrier D and τ^2 is the maturity time T if assets $V_T < K$ at T and ∞ otherwise. In other words, the default time is defined as the minimum of the first-passage default time (6) and Merton's default time (1). This definition of default is consistent with the payoff to equity and bonds. Even if the firm value does not fall below the barrier, if assets are below the bond's face value at maturity the firm defaults, see Figure 4. We get for the corresponding default probabilities

$$\begin{aligned} p(T) &= 1 - P[\min(\tau^1, \tau^2) > T] \\ &= 1 - P[\tau^1 > T, \tau^2 > T] \\ &= 1 - P[M_T > D, V_T > K] \\ &= 1 - P[\min_{t \leq T}(mt + \sigma W_t) > \log(D/V_0), mT + \sigma W_T > \log L] \end{aligned}$$

Using the joint distribution of an arithmetic Brownian and its running minimum, we get immediately

$$p(T) = \Phi\left(\frac{\log L - mT}{\sigma\sqrt{T}}\right) + \left(\frac{D}{V_0}\right)^{\frac{2m}{\sigma^2}} \Phi\left(\frac{\log(D^2/(KV_0)) + mT}{\sigma\sqrt{T}}\right). \quad (9)$$

This default probability is obviously higher than the corresponding probability in the classical approach, which is obtained as the special case where $D = 0$.

The corresponding payoff to equity investors at maturity is

$$E_T = \max(0, V_T - K)1_{\{M_T \geq D\}} \quad (10)$$

where 1_A is the indicator function of the event A . The equity position is equivalent to a European down-and-out call option position on firm assets V with strike K , barrier $D < K$, and maturity T . Pricing equity reduces to pricing European barrier options. In the Black-Scholes setting with constant interest rates and asset dynamics (2), we find the value

$$E_0 = C(\sigma, T, K, r, V_0) - V_0 \left(\frac{D}{V_0}\right)^{\frac{2r}{\sigma^2}+1} \Phi(h_+) + Ke^{-rT} \left(\frac{D}{V_0}\right)^{\frac{2r}{\sigma^2}-1} \Phi(h_-) \quad (11)$$

where C is the vanilla call value and where

$$h_{\pm} = \frac{(r \pm \frac{1}{2}\sigma^2)T + \log(D^2/(KV_0))}{\sigma\sqrt{T}}.$$

We make two observations. First, the down-and-out call is worth at most as much as the corresponding vanilla call. The barrier call value converges to the vanilla call value as $D \rightarrow 0$. Second, the barrier option value is, unlike the vanilla call value, not monotone in firm volatility σ . Unlike in the classical approach, in the first-passage approach equity investors do not always benefit from an increase in asset volatility. This has important implications for model calibration.

The corresponding payoff to bond investors at maturity is

$$B_T^T = K - (K - V_T)^+ + (V_T - K)^+ 1_{\{M_T < D\}} \quad (12)$$

This position is equivalent to a portfolio composed of a riskfree loan with face value K maturing at T , a short European put on the firm with strike K and maturity T and a long European down-and-in call on the firm with strike K and maturity T . In the first-passage approach bonds are worth at least as

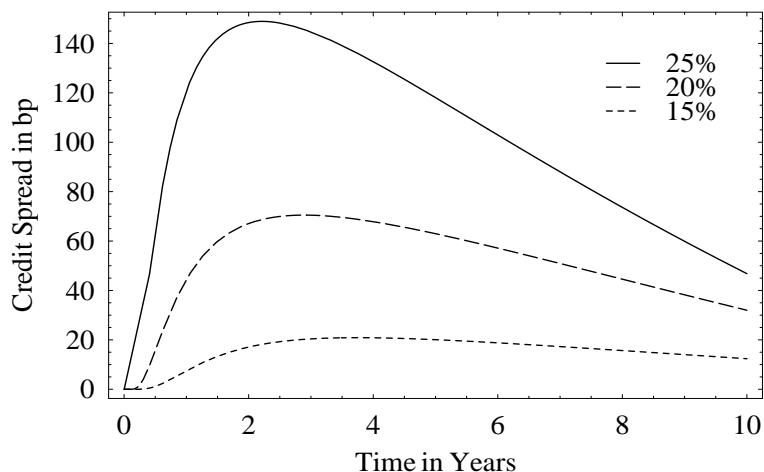


Figure 5: Term structure of credit spreads implied by (11) for varying firm volatilities σ . We set $V_0 = 100$, $K = 75$, $D = 50$ and $r = 5\%$.

much as in the classical approach. In the first-passage model bond investors have additionally a barrier option on the firm that knocks in if the firm defaults before the maturity T . Correspondingly,

$$B_0^T = Ke^{-rT} - P(\sigma, T, K, r, V_0) + DIC(\sigma, T, K, D, r, V_0) \quad (13)$$

where P (resp. DIC) is the vanilla put (resp. down-and-in) option value. The combined value of the option positions gives the present value of the default loss suffered by bond investors. We get

$$B_0^T = V_0 - C(\sigma, T, K, V_0) + V_0 \left(\frac{D}{V_0} \right)^{\frac{2r}{\sigma^2} + 1} \Phi(h_+) + Ke^{-rT} \left(\frac{D}{V_0} \right)^{\frac{2r}{\sigma^2} - 1} \Phi(h_-)$$

which again implies the value identity $V_0 = S_0 + B_0^T$.

In Figure 5, we plot the corresponding term structure of credit spreads $S(0, T)$. With increasing maturity T , the spread asymptotically approaches zero. This is at odds with empirical observation: spreads tend to increase with increasing maturity, reflecting the fact that uncertainty is greater in the distant future than in the near term. This discrepancy follows from two model properties: the firm value grows at a positive (riskfree) rate and the capital structure is constant. We can address this issue by assuming that the total debt grows at a positive rate, or that firms maintain some target leverage ratio as in Collin-Dufresne & Goldstein (2001).

Time-varying default barrier. The second way to avoid the inconsistency discussed above is to introduce a time-varying default barrier $D(t) \leq K$ for all $t \leq T$. For some constant $k > 0$, consider the deterministic function

$$D(t) = Ke^{-k(T-t)} \quad (14)$$

which can be thought of as the face value of the debt, discounted back to time t at a continuously compounding rate k . The firm defaults at

$$\tau = \inf\{t > 0 : V_t < D(t)\}. \quad (15)$$

Observing that

$$\{V_t < D(t)\} = \{(m - k)t + \sigma W_t < \log L - kT\}$$

we have for the default probability

$$p(T) = P[\min_{t \leq T} ((m - k)t + \sigma W_t) < \log L - kT].$$

Now we have reduced the problem to calculating the distribution of the historical low of an arithmetic Brownian motion with drift $m - k$. We get

$$p(T) = \Phi\left(\frac{\log L - mT}{\sigma\sqrt{T}}\right) + (Le^{-kT})^{\frac{2}{\sigma^2}(m-k)} \Phi\left(\frac{\log L + (m - 2k)T}{\sigma\sqrt{T}}\right). \quad (16)$$

The corresponding equity position is a European down-and-out call option on firm assets with strike K , time-varying barrier $D(t)$, and maturity T :

$$E_T = (V_T - K)^+ 1_{\{M_T^k \geq D\}} \quad (17)$$

where $M_t^k = \min_{s \leq t} V_0 \exp((m - k)s + \sigma W_s)$ is the running minimum of the firm value with adjusted arithmetic growth rate $m - k$ and $D = K \exp(-kT)$. Merton (1973) gives a closed-form expression for E_0 . The bond position is, in analogy to (12), given by

$$B_T^T = K - (K - V_T)^+ + (V_T - K)^+ 1_{\{M_T^k < D\}}. \quad (18)$$

Bond values B_0^T can be calculated as the residual $X_0 - E_0$.

2.3 Excursion approach

The first-passage approach assumes that bond investors immediately take over control of the firm when its value falls below a barrier whose value is often prescribed by a debt covenant. In practice, bankruptcy codes often grant firms an extended period of time to reorganize operations after a default. If the restructuring is successful, a firm emerges from bankruptcy and continues operating. If the outcome is negative, bond investors seize control and liquidate the remaining firm assets.

We model the reorganization process following a default by considering the *excursion* of firm value after first passage to the default barrier D . We assume that D is constant. For some non-negative bounded function f on $[0, \infty)^2$, we consider the continuous functional $F(V)$ defined by

$$F(V)_t = \int_0^t f(s, t) 1_{\{V_s \leq D\}} ds. \quad (19)$$

This functional measures the risk of the firm to be liquidated in that the firm's liquidation time τ_L is defined by

$$\tau_L = \inf\{t > 0 : F(V)_t > C\} \quad (20)$$

where C is some non-negative constant. We observe that for strictly positive f and $C = 0$, the liquidation time τ_L becomes the first-passage time of V to D considered in (6).

In order to discuss some examples, we introduce a non-negative weight function w on $[0, \infty)^2$. A standard example is $w(s, t) = \exp(-\int_s^t k_s ds)$ for $s \leq t$ and k a non-negative constant.

Example 2.1. Suppose

$$f(s, t) = w(s, t) 1_{\{L_t \leq s\}} \quad (21)$$

where $L_t = \sup\{s \leq t : V_s = D\}$ is the last time before t when the firm value was equal to the default barrier. Then $F(V)_t$ is the weighted *consecutive* excursion time of firm value below the default barrier at time t . It takes into account only the most recent excursion of V before t . Defaults which do not lead to liquidation are “forgotten.” After a firm emerges from default (V crosses D from below), the firm starts anew without any records on past defaults. \square

Example 2.2. Suppose

$$f(s, t) = w(s, t). \quad (22)$$

Then $F(V)_t$ is the weighted *cumulative* excursion time of firm value below the default barrier at time t . It takes into account all excursions of V before t : defaults are never forgotten. \square

Example 2.3. Suppose

$$f(s, t) = w(s, t)(D - V_s). \quad (23)$$

Then $F(V)_t$ measures the weighted *cumulative shortfall* $\int_0^t w(t, s)(D - V_s)^+ ds$ of the excursion of firm value below the default barrier at time t . Unlike the previous two examples, this specification accounts for the “success of the firm reorganization” by measuring the weighted area of excursions. \square

We define $\tau = \min(\tau_L, \tau_C)$, where

$$\tau_C = \begin{cases} T & \text{if } V_T < K \\ \infty & \text{if else.} \end{cases} \quad (24)$$

is the classical default time. The probability of ultimate bankruptcy $p(T) = P[\tau \leq T]$ is given by $p(T) = 1 - P[\tau_L > T, \tau_C > T]$. We calculate this probability in Example 2.2 with $w(s, t) = 1$ for all s and t . In this case $F(V)_t = \int_0^t 1_{\{V_s \leq D\}} ds$ is the excursion time of V below D . We note that $\{\tau_L > t\} = \{F(V)_t \leq C\}$ since $F(V)$ is non-decreasing so we need the joint distribution of $(V_T, F(V)_T)$ to calculate $p(T)$. See Borodin & Salminen (1996) and Hugonnier (1999).

The payoff to equity investors at debt maturity T is given by

$$S_T = (V_T - K)^+ 1_{\{\tau_L > T\}} \quad (25)$$

In case of example 2.1, this is called a *Parasian* option. In case of example 2.2, this is a *Parisian* option. For the pricing, see Hugonnier (1999), Moraux (2001), and François & Morellec (2002).

2.4 Dependent Defaults

Credit spreads of different issuers are correlated through time. Two patterns are found in time series of spreads. The first is that spreads vary smoothly

with general macro-economic factors in a correlated fashion. This means that firms share a common dependence on the economic environment, which results in *cyclical correlation* between defaults. The second relates to the jumps in spreads: we observe that these are often common to several firms or even entire markets. This suggests that the sudden large variation in the credit risk of one issuer, which causes a spread jump in the first place, can propagate to other issuers as well. The rationale is that economic distress is *contagious* and propagates from firm to firm. A typical channel for these effects are borrowing and lending chains. Here the financial health of a firm also depends on the status of other firms as well.

2.4.1 Cyclical dependence

We assume that firm values of several firms are correlated through time. This corresponds to common factors driving asset returns. We consider the simplest case with two firms and asset dynamics

$$\frac{dV_t^i}{V_t^i} = \mu_i dt + \sigma_i dW_t^i, \quad V_0^i > 0, \quad i = 1, 2,$$

where $\mu_i \in \mathbb{R}$ is a drift parameter, $\sigma_i > 0$ is a volatility parameter, and (W^1, W^2) is a two-dimensional Brownian motion with correlation ρ . That is, $(W_{t_1}^1, W_{t_2}^2) \sim N(\mathbf{0}, \Sigma)$ with covariance matrix

$$\Sigma = \begin{bmatrix} t_1 & \rho\sqrt{t_1 t_2} \\ \rho\sqrt{t_1 t_2} & t_2 \end{bmatrix}$$

Letting $m_i = \mu_i - \frac{1}{2}\sigma_i^2$, by Itô's formula we get $V_t^i = V_0^i \exp(m_i t + \sigma_i W_t^i)$.

Example 2.4. Consider the classical approach (1). We obtain for the joint probability of firm 1 to default at T_1 (the fixed debt maturity) and firm 2 to default at T_2

$$\begin{aligned} p(T_1, T_2) &= P[V_{T_1}^1 < K_1, V_{T_2}^2 < K_2] \\ &= \Phi_2\left(\rho, \frac{\log L_1 - m_1 T_1}{\sigma_1 \sqrt{T_1}}, \frac{\log L_2 - m_2 T_2}{\sigma_2 \sqrt{T_2}}\right) \end{aligned} \quad (26)$$

where $L_i = K_i/V_0^i$ and $\Phi_2(r, \cdot, \cdot)$ is the bivariate standard normal distribution function with linear correlation parameter $|r| < 1$, given by

$$\Phi_2(r, a, b) = \int_{-\infty}^a \int_{-\infty}^b \frac{1}{2\pi\sqrt{1-r^2}} \exp\left(\frac{2rxy - x^2 - y^2}{2(1-r^2)}\right) dx dy \quad (27)$$

□

Example 2.5. Consider the first-passage approach (6). Letting $M_t^i = \min_{s \leq t} V_s^i$ be the running minimum value of firm i at time t , we get for the joint probability of firm 1 to default before T_1 and firm 2 to default before T_2

$$\begin{aligned} p(T_1, T_2) &= P[M_{T_1}^1 \leq D_1, M_{T_2}^2 \leq D_2] \\ &= \Psi_2 \left(\rho; T_1, T_2; \log \frac{D_1}{V_0^1}, \log \frac{D_2}{V_0^2} \right) \end{aligned}$$

where D_i is the constant default barrier of firm i and, holding $x, y \leq 0$ fixed, $\Psi_2(r; \cdot, \cdot; x, y)$ is the bivariate inverse Gaussian distribution function with correlation r . This function is given in closed-form in Iyengar (1985) and Zhou (2001a). A first-order approximation is given in Wise & Bhansali (2004). \square

Default Time Copulas. The joint default probability p provides a comprehensive characterization of the default risk of both firms. It describes simultaneously the individual likelihood of a firm to default and the likelihood that both firms default jointly. In the portfolio context we are often interested in the component of p describing the default dependence structure only. It turns out that we can isolate this dependence structure from p by means of a *copula*. Formally, the copula C of the vector (τ_1, τ_2) is a function that maps the individual default probabilities p_i into the joint default probability p ,

$$p(T_1, T_2) = C(p_1(T_1), p_2(T_2)).$$

There is only one such mapping C if p is continuous. In this case we can also go the other way around, and find C from a given p through

$$C(u, v) = p(p_1^{-1}(u), p_2^{-1}(v))$$

for all u and v in $[0, 1]$. Here

$$p_i^{-1}(u) = \inf\{x \geq 0 : p_i(x) \geq u\} \quad (28)$$

is the generalized inverse of the individual default probability. It is also referred to as the u -quantile $q_u(\tau_i)$ of τ_i . See Nelsen (1999) for an introduction to copulas.

Example 2.6. In the classical approach the default dependence structure is given by the Gaussian copula C_r^{Ga} with correlation r , defined by

$$\begin{aligned} C_r^{Ga}(u, v) &= \Phi_2(r, \Phi^{-1}(u), \Phi^{-1}(v)) \\ &= \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi\sqrt{1-r^2}} \exp\left(\frac{2rxy - x^2 - y^2}{2(1-r^2)}\right) dx dy \quad (29) \end{aligned}$$

To see this, note that the vector $(W_{T_1}^1, W_{T_2}^2)$ is Gaussian with $\text{Cov}(W_{T_1}^1, W_{T_2}^2) = \rho\sqrt{T_1 T_2}$ so that

$$\begin{aligned} p(T_1, T_2) &= P[W_{T_1}^1 < (\log L_1 - m_1 T_1)/(\sigma_1 T_1), W_{T_2}^2 < (\log L_2 - m_2 T_2)/(\sigma_2 T_2)] \\ &= C_\rho^{Ga}(p_1(T_1), p_2(T_2)). \end{aligned} \quad (30)$$

□

Analogously, in the first passage approach, the default dependence structure is given by the inverse Gaussian copula with correlation ρ .

Above we have focused on the joint distribution function p of the default times. Equivalently, we can consider the “survival copula” \bar{C} of the joint survival function s of (τ_1, τ_2) . It satisfies

$$s(T_1, T_2) = P[\tau_1 > T_1, \tau_2 > T_2] = \bar{C}(s_1(T_1), s_2(T_2)),$$

where $s_i(T) = 1 - p_i(T)$ is the marginal survival function of firm i . Using the relationship between p and s , we can calculate \bar{C} from C and vice versa:

$$\bar{C}(u, v) = u + v + C(1 - u, 1 - v) - 1.$$

The survival copula governs the univariate distribution of the *first-to-default* time $\tau = \tau_1 \wedge \tau_2$. We have

$$P[\tau > T] = s(T, T) = \bar{C}(s_1(T), s_2(T)).$$

A copula is a joint distribution function with standard uniform marginals. To see this, we assume that the marginals p_i are continuous and transform the default times by their marginals to obtain

$$C(u, v) = P[p_1(\tau_1) \leq u, p_2(\tau_2) \leq v] = P[U_1 \leq u, U_2 \leq v]$$

where $U_i = p_i(\tau_i)$ is standard uniform. As a joint distribution function, a copula satisfies the Fréchet bound inequality

$$\max(u + v - 1, 0) \leq C(u, v) \leq \min(u, v)$$

for all u and v in $[0, 1]$. Figure 6 shows the two surfaces in the unit cube. It is easy to check that $\min(u, v) = P[U \leq u, U \leq v]$ so that the upper bound copula is the joint distribution function of the vector (U, U) . If C takes on the upper bound, defaults are perfectly positively dependent. This corresponds to

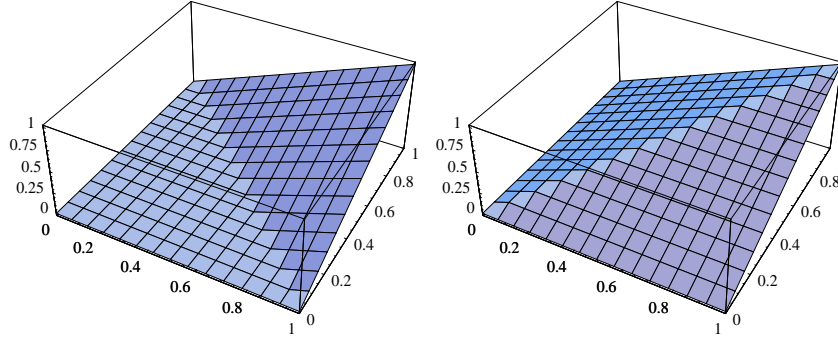


Figure 6: Fréchet lower and upper bound copulas.

an asset correlation of $\rho = 1$ and means that the time of default of one firm is an increasing function of the default time of the other firm:

$$\tau_2 = T(\tau_1), \quad \text{a.s.}, \quad T = p_2^{-1} \circ p_1 \text{ increasing.}$$

In the special case where $p_1 = p_2$, both firms default literally at the same time: $\tau_1 = \tau_2$, almost surely. Since $\max(u + v - 1, 0) = P[U \leq u, 1 - U \leq v]$, the lower bound copula is the joint distribution function of the vector $(U, 1 - U)$. If C takes on the lower bound, defaults are perfectly negatively correlated. This corresponds to an asset correlation of $\rho = -1$ and means that one default time is an decreasing function of the other:

$$\tau_2 = T(\tau_1), \quad \text{a.s.}, \quad T = p_2^{-1} \circ (1 - p_1) \text{ decreasing.}$$

It is easy to check that $C(u, v) = uv$ if and only if defaults are independent.

Measuring default dependence. The default copula measures the complete non-linear dependence between the defaults. It is straightforward to show that the copula is invariant under strictly increasing transformations T_i : the transformed times $T_i(\tau_i)$ have copula C as well. An intuitive bivariate scalar-valued measure of default dependence can easily be constructed. We consider Spearman's rank correlation, cf. Embrechts, McNeil & Straumann (2001). For the default times τ_1 and τ_2 it is given as the linear correlation of the copula C :

$$\rho^\tau = 12 \int_0^1 \int_0^1 (C(u, v) - uv) du dv.$$

This shows that ρ^τ is a scaled version of the volume enclosed by C and the independence copula. Moreover, ρ^τ is a function of the copula only, and is hence invariant under increasing transformations.

The quantity ρ^τ describes the degree of *monotonic* default dependence through a number in $[-1, 1]$, with the left (right) endpoint referring to perfect negative (positive) default dependence. Rank correlation should be contrasted with linear correlation $\rho(\tau_1, \tau_2)$ of the default times and linear correlation of the Bernoulli default indicators $1_{\{\tau_i \leq T\}}$. These measures are often used in the literature; they describe the degree of *linear* default dependence through a number in $[-1, 1]$. Unless the default times/default indicators are jointly elliptically distributed, linear correlation based measures will misrepresent default dependence: they do not cover the non-linear part of the dependence. Rank correlation ρ^τ does not suffer from this defect: it summarizes monotonic dependence.

Tail dependence. Tail dependence refers to the degree of dependence in the lower and upper quadrant tail of a bivariate distribution. We measure this degree by the coefficient of upper and lower tail dependence of a given copula. Suppose the underlying random variables are continuous. A copula C is called *lower tail dependent* if the coefficient of lower tail dependence

$$\lim_{u \downarrow 0} \frac{C(u, u)}{u} \quad (31)$$

exists and takes a value in $(0, 1]$. A lower tail dependent copula exhibits a pronounced tendency of generating low values in all marginals simultaneously. This can be seen when we rewrite the tail dependence condition as

$$\lim_{u \downarrow 0} P[\tau_2 \leq q_u(\tau_2) \mid \tau_1 \leq q_u(\tau_1)] \in (0, 1], \quad (32)$$

where $q_u(\tau_i) = p_i^{-1}(u) = \inf\{x \geq 0 : p_i(x) \geq u\}$ is the u -quantile of the distribution p_i of τ_i , see (28). This is the conditional probability that firm 2 will default very early given firm 1 defaults very early. In other words, tail dependence is an important concept that relates to the likelihood of multiple default scenarios.

A copula C is called *upper tail dependent* if the coefficient of upper tail dependence

$$\lim_{u \uparrow 1} \frac{1 - 2u + C(u, u)}{1 - u} \quad (33)$$

exists and takes a value in $(0, 1]$. An upper tail dependent copula exhibits a pronounced tendency of generating high values in all marginals simultaneously, which can be seen by the equivalent formulation of the condition as

$$\lim_{u \uparrow 1} P[\tau_2 > q_u(\tau_2) \mid \tau_1 > q_u(\tau_1)] \in (0, 1]. \quad (34)$$

This is the conditional probability that firm 2 will default in the indefinite future given firm 1 defaults in the indefinite future.

Copulas C for which the (lower resp. upper) tail coefficient is zero are called asymptotically independent in the (lower resp. upper) tail.

Since tail dependence is a copula property, it is invariant under strictly increasing transformations of the underlying random variables.

2.4.2 Bernoulli mixture models

The Bernoulli mixture framework provides a common perspective on the construction of portfolio loss distributions that account for cyclical default dependence. We fix a horizon, say one year, and consider the default indicator $Y_i = 1_{\{\tau_i \leq 1\}}$. Then $Y = (Y_1, \dots, Y_n)$ is a vector of Bernoulli random variables Y_i with success probability $p_i = P[Y_i = 1]$. The goal is to provide tractable models for the joint default probability, i.e. the distribution of Y .

The general idea. We consider an insightful but very special case first. Suppose firms are independent and equally likely to default: $p_i = p$ for all i . The sequence (Y_i) is a sequence of classical Bernoulli trials. The distribution of the sum $L_n = Y_1 + \dots + Y_n$ for $n \geq 1$ is Binomial with parameter vector (n, p) . That is,

$$P[L_n = k] = \binom{n}{k} p^k (1 - p)^{n-k}, \quad k \leq n, \quad (35)$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the Binomial coefficient. When defaults result in unit losses, the random variable L_n gives the aggregate loss in a portfolio of n firms and default indicators Y_i . (35) gives the corresponding loss distribution.

Of course, firm defaults and thus the Bernoulli variables (Y_i) are not independent. Firms depend on common macro-economic factors. This induces cyclical default dependence as we discussed above: if the economy is in a bad state, default probabilities p are high. If the economy is doing well, p is low. In other words, the default probability p depends on the realization of the state of

the economy and is thus random. Let F be the distribution of p on the interval $[0, 1]$ which describes our uncertainty about p . We average the binomial loss probabilities (35) over F to obtain for the corresponding loss distribution

$$P[L_n = k] = \binom{n}{k} \int_0^1 z^k (1 - z)^{n-k} dF(z), \quad k \leq n. \quad (36)$$

We can interpret this construction as a two step *randomization* procedure. First, default probabilities p are selected according to the distribution F . Given the realization of p , firm defaults are independent. Second, the total number of defaults is selected according to the Binomial distribution with parameter p . Another way to interpret the loss probability (36) is as a *mixture* of Binomial probabilities, with the mixing distribution given by F .

This neat and simple construction of loss probabilities for dependent defaults is quite natural, as de Finetti's theorem shows. This result represents the joint distribution of a sequence (Y_i) of Bernoulli variables. Two assumptions are necessary: the sequence (Y_i) is *infinite* and *exchangeable*. The latter property means that for any $k \in \mathbb{N}$, the vector (Y_1, \dots, Y_k) has the same distribution as the vector $(Y_{\Pi(1)}, \dots, Y_{\Pi(k)})$ for any permutation Π of the indices $\{1, \dots, k\}$. This is the same as saying that all firms have equal default probability and the dependence between the firms is symmetric. An obvious situation where this holds is when firms share the same default probability and are mutually independent, which was the situation we considered at the beginning. De Finetti's theorem asserts that then there always exists a distribution F on $[0, 1]$ such that for all $k \leq n \in \mathbb{N}$ loss probabilities are given by (36). In other words, the marginal distributions of the sequence (Y_i) are always given by a mixture of binomial probabilities.

Homogeneous portfolios. We consider homogeneous portfolios, for which the corresponding sequence (Y_i) of default indicators is exchangeable. What we do is model the random default probabilities p in (36) as conditional default probabilities. Let X be an independent random variable that models the state of the economy. The conditional probability of default given X is defined as

$$p(X) = E[Y_i | X] = P[Y_i = 1 | X]. \quad (37)$$

We note that $p(X)$ is a random variable whose distribution depends on that of X . We call Y a *homogeneous Bernoulli mixture model* with factor vector X , if conditionally on X , the Y_i are independent Bernoulli random variables with

common success probability $p(X)$. Dependence of firms on X induces cyclical default dependence.

In the structural models, default occurs if firm assets are sufficiently low relative to liabilities according to some measure. We therefore set

$$Y_i = 1 \Leftrightarrow A_i < B_i \quad (38)$$

for a random variable A_i and a constant B_i .

Example 2.7. In the classical approach (1), a firm defaults if its value is below the face value of the debt at maturity. Thus

$$A_i = W_1^i = \frac{\log(V_1^i/V_0^i) - m_i}{\sigma_i} \quad \text{and} \quad B_i = \frac{\log L_i - m_i}{\sigma_i}, \quad (39)$$

where A_i is the standardized asset return and B_i is the standardized face value of the debt, which is sometimes called the distance to default. The vector (A_1, \dots, A_1) is Gaussian with mean vector zero and covariance matrix $\Sigma = (\rho_{ij})$ with $\rho_{ij} = \text{Cov}(W_1^i, W_1^j)$ being the asset correlation. \square

Example 2.8. In the first-passage approach (6), a firm defaults if its value falls below the default barrier D_i before maturity. Thus

$$A_i = \min_{s \leq 1} (m_i s + \sigma_i W_s^i) \quad \text{and} \quad B_i = \log(D_i/V_0^i), \quad (40)$$

where A_i is the running minimum log-value of firm i at time 1, and B_i is the standardized default barrier. The vector (A_1, \dots, A_n) is inverse Gaussian with mean vector zero and covariance matrix $\Sigma = (\rho_{ij})$ with $\rho_{ij} = \text{Cov}(W_1^i, W_1^j)$ being the asset correlation. \square

We continue to discuss the classical approach in a homogeneous setting. Since $p_i = P[W_1^i < B_i] = \Phi(B_i) = p$ for all i , the standardized face value of the debt B_i is given by $B = \Phi^{-1}(p)$ for all firms i . The asset correlation matrix Σ is of the form $\rho_{ij} = \rho$ for $i \neq j$ and $\rho_{ij} = 1$ for $i = j$. That is, the asset correlation between any two firms is equal to the constant ρ . In this case the standardized asset return W_1^i can be parameterized by the one-factor linear model

$$W_1^i = \sqrt{\rho}X + \sqrt{1 - \rho}Z_i, \quad (41)$$

where $X \sim N(0, 1)$ is a systematic factor and $Z_i \sim N(0, 1)$ is an independent idiosyncratic factor. Here $\sqrt{\rho}X$ can be interpreted as systematic risk in asset

returns, while $\sqrt{1-\rho}Z_i$ stands for the idiosyncratic risk in asset returns. The weight $\sqrt{\rho}$ describes the sensitivity of a firm with respect to the systematic factor X . The regression coefficient of the linear regression (41) is equal to ρ . We get immediately

$$p(X) = \Phi \left(\frac{\Phi^{-1}(p) - \sqrt{\rho}X}{\sqrt{1-\rho}} \right), \quad (42)$$

which depends on two parameters: the individual default probability p and the asset correlation ρ . We see that $p(X)$ is decreasing in X : positive values of the macro-factor correspond to a “healthy” state of the economy, while negative values correspond to a distressed economy. Since conditionally on the macro-factor X the Y_i are independent Bernoulli variables with success probability $p(X)$, we have for the joint probability of default/survival

$$P[Y_1 = y_1, \dots, Y_n = y_n] = \int_{-\infty}^{\infty} (p(x))^{\sum_i^n y_i} (1-p(x))^{n-\sum_i^n y_i} \phi(x) dx,$$

where $y_i \in \{0, 1\}$ and ϕ is the standard normal density. The joint default probability is given as the special case

$$P[Y_1 = 1, \dots, Y_n = 1] = \int_{-\infty}^{\infty} \Phi \left(\frac{\Phi^{-1}(p) - \sqrt{\rho}x}{\sqrt{1-\rho}} \right)^n \phi(x) dx. \quad (43)$$

We are interested in the distribution of aggregated losses on a portfolio of n firms in model (41). Suppose for simplicity that a default of a firm results in a loss of 1 dollar. Then losses are given by the discrete random variable

$$L_n = \begin{cases} 0 & \text{if } n = 0 \\ Y_1 + \dots + Y_n & \text{if } n = 1, 2, \dots \end{cases}$$

Conditionally on the realization of the macro-factor X , the variable L_n has a Binomial distribution with parameter vector $(n, p(X))$. In agreement with (36), the unconditional loss distribution is therefore given by

$$P[L_n \leq k] = \sum_{i=1}^k \binom{n}{i} \int_{-\infty}^{\infty} (p(x))^i (1-p(x))^{n-i} \phi(x) dx.$$

It is clear how to generalize this to general distributions of the A_i in (38).

Some approximations. We are interested in the distribution of aggregate losses L_n if $n \rightarrow \infty$ i.e. if the portfolio becomes large. This leads to a number of useful approximations to the loss distribution for large credit portfolios. We consider the percentage portfolio loss L_n/n and its conditional moments:

$$E[L_n/n \mid X] = p(X)$$

$$\text{Var} [L_n/n \mid X] = \frac{1}{n}p(X)(1 - p(X)).$$

Chebychev's inequality yields an upper bound on the probability that the loss fraction deviates from $p(X)$ by an amount larger than some $\epsilon > 0$:

$$P \left[\left| \frac{L_n}{n} - p(X) \right| > \epsilon \mid X \right] \leq \frac{\text{Var} [L_n/n \mid X]}{\epsilon^2} = \frac{p(X)(1 - p(X))}{n\epsilon^2}.$$

Taking expectations on both sides, and letting $n \rightarrow \infty$, we get

$$\lim_{n \rightarrow \infty} P \left[\left| \frac{L_n}{n} - p(X) \right| > \epsilon \right] = 0$$

for any $\epsilon > 0$. We also write

$$\lim_{n \rightarrow \infty} \frac{L_n}{n} = p(X) \quad \text{in probability.}$$

This is called a weak *law of large numbers* (LLN). It says that asymptotically (for “large” portfolios) the probability that the percentage loss deviates from $p(X)$ by more than $\epsilon > 0$ in absolute value is zero.

We are interested in a stronger convergence result. Obviously the sequence $L_n(\omega)/n$ does not converge to $p(X)$ for *all* states of the world $\omega \in \Omega$. There are states where the sequence does not converge. An example is the scenario ω where all firms default ($Y_i(\omega) = 1$ for all $i \in \mathbb{N}$) so that $L_n(\omega) = n$. Another example is the scenario where no firm defaults and $L_n(\omega) = 0$. But these states have probability zero so they do not matter! We can thus say that the percentage loss L_n/n converges to $p(X)$ in almost all states of the world:

$$\lim_{n \rightarrow \infty} \frac{L_n}{n} = p(X) \quad \text{almost surely,}$$

i.e. with probability one. This is called the strong LLN. We see that the strong law implies the weak law, but not the other way around. Under an additional condition, the strong law also holds in the more general situation where the

names i in the portfolio have different weights k_i such that $\sum_i k_i = 1$. Redefining the portfolio losses as

$$L_n^k = \begin{cases} 0 & \text{if } n = 0 \\ k_1 Y_1 + \dots + k_n Y_n & \text{if } n = 1, 2, \dots, \end{cases}$$

the loss fraction $L_n^k/n \rightarrow p(X)$ almost surely if and only if $\sum_i k_i^2 \rightarrow 0$, i.e. if there are not too many “large” exposures that dominate all others.

We draw two important conclusions. First, the limit $p(X)$ is a random variable: it depends on the macro-factor X , which is random. Thus, average portfolio losses in very large portfolios are governed by the distribution of X (the mixing distribution). Qualitatively, the higher the volatility of the macro-factor, the higher are the fluctuations of aggregate losses. Second, in the limit all idiosyncratic risk induced by fluctuation of the firm-specific factors Z_i in asset returns diversify away. Only systematic risk induced by the fluctuation of the macro-factor X remains. This systematic risk is not diversifiable, even in infinitely large credit portfolios.

We can use the LLN to derive the limiting loss distribution on a large credit portfolio. Letting L denote the loss fraction on the limiting portfolio and using the fact that $p(x)$ is decreasing in x , we have that

$$F_{p,\rho}(x) = P[L \leq x] = P[p(X) \leq x] = P[X > p^{-1}(x)] = \Phi(-p^{-1}(x)).$$

Using the inverse p^{-1} from (42) we obtain

$$F_{p,\rho}(x) = \Phi\left(\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}\Phi^{-1}(x) - \Phi^{-1}(p)\right)\right) \quad (44)$$

for $x \in [0, 1]$. The limiting loss distribution $F_{p,\rho}$ depends on two parameters: the uniform individual default probability p and the uniform asset correlation ρ . This makes it a parsimonious model for a variety of applications, including the determination of regulatory credit risk capital under the Basel II guidelines for financial institutions. Figure 7 plots the limiting loss density for several $\rho \in \{5\%, 10\%, 15\%\}$. We fix $p = 0.5\%$.

The Value-at-Risk at a confidence level α is given by the α -quantile $q_\alpha(L)$ of the limiting loss distribution $F_{p,\rho}$. We calculate

$$q_\alpha(L) = F_{p,\rho}^{-1}(\alpha) = \Phi\left(\frac{1}{\sqrt{1-\rho}}\left(\sqrt{\rho}q_\alpha(X) + \Phi^{-1}(p)\right)\right)$$

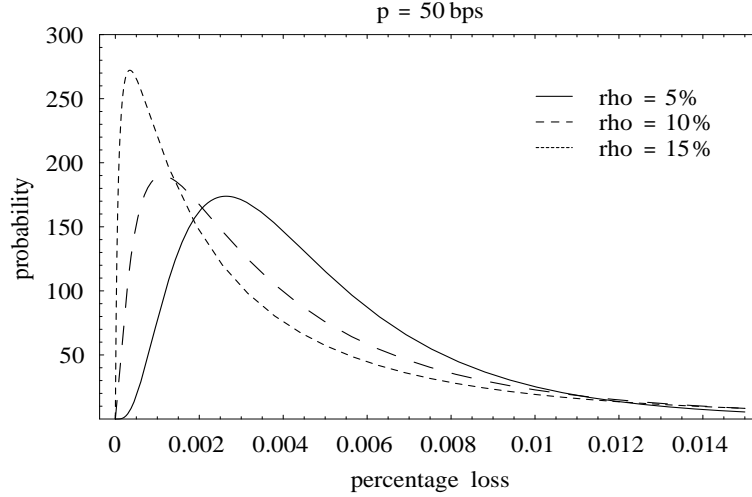


Figure 7: Density of the limiting loss distribution for $p = 0.5\%$, varying asset correlation ρ .

where $q_\alpha(X) = \Phi^{-1}(\alpha)$ is the α -quantile of the standard normal macro-factor X . This shows that asymptotically, the quantile of the macro-factor essentially governs the quantile of the loss distribution. The higher the potential of excessive volatility of the macro-factor, the higher is also the potential of excessive credit default losses. This is closely related to our earlier conclusion that asymptotically, average losses are governed by the distribution of the macro-factor.

The LLN establishes the convergence of percentage losses L_n/n to the conditional default probability $p(X)$ if the portfolio becomes large. We are interested in the probability that L_n deviates from its mean $np(X)$ by an amount of order \sqrt{n} . Such deviations are considered “normal.” Deviations of order n are considered “large;” there is a sophisticated theory that analyzes such large deviations, see Dembo & Zeitouni (1998) for example. It has also been applied to Bernoulli mixture models in Dembo, Deuschel & Duffie (2002). We consider normal deviations by means of the classical *central limit theorem* for Bernoulli sequences. It asserts that given the macro-factor X , the distribution of standardized losses converges to standard normal distribution:

$$\lim_{n \rightarrow \infty} \frac{L_n - E[L_n | X]}{\sqrt{\text{Var}[L_n | X]}} = N(0, 1) \quad \text{in distribution.}$$

Again, this is due to the fact that conditionally on X , the default indicators Y_i are iid Bernoulli variables. We can also write

$$\lim_{n \rightarrow \infty} P \left[\frac{L_n - np(X)}{\sqrt{np(X)(1-p(X))}} \leq x \mid X \right] = \Phi(x).$$

Asymptotically, aggregated losses L_n in large portfolios are conditionally normal with mean $np(X)$ and variance $np(X)(1-p(X))$. Unconditional loss probabilities can be uniformly approximated by the function K_n given by

$$K_n(x) = \int_{-\infty}^{\infty} \Phi \left(\frac{x - np(k)}{\sqrt{np(k)(1-p(k))}} \right) \phi(k) dk, \quad (45)$$

which is a mixture of normal probabilities. We have

$$\sup_{x \geq 0} |P[L_n \leq x] - K_n(x)| \leq \epsilon_n,$$

where $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$. The Berry-Esseen theorem (e.g. Durrett (1996, Theorem 4.9, Chapter 2)) gives an upper bound on the approximation error in dependence of n .

Heterogeneous portfolios. We describe the state of the economy by a random vector $X = (X_1, \dots, X_m)$ with $m \ll n$ and define the conditional default probability

$$p_i(X) = E[Y_i \mid X] = P[Y_i = 1 \mid X]. \quad (46)$$

Then Y is called a *Bernoulli mixture model* with factor vector X , if conditionally on X , the Y_i are independent Bernoulli random variables with success probability $p_i(X)$.

We continue to discuss the classical approach in a heterogeneous setting. We introduce the multi-factor linear model

$$W_1^i = a_i' X + b_i Z_i \quad (47)$$

for asset returns. Here X is normal with zero mean vector and covariance matrix $\bar{\Sigma}$, the Z_i are independent and standard normal, $a_i = (a_{i1}, \dots, a_{im})$ is a vector of constant factor weights, and b_i is a constant as well.

The Bernoulli mixture model (39) with a linear multi-factor model (47) for asset returns is representative for the industry credit portfolio models provided

by Moody's KMV ("PortfolioManager") and RiskMetrics ("CreditMetrics"). In these models we have for conditional default probabilities

$$p_i(X) = P[W_i^i < B_i | X] = \Phi\left(\frac{B_i - a'_i X}{b_i}\right). \quad (48)$$

Using the fact that conditionally on X defaults are independent, we have for joint default probabilities

$$\begin{aligned} P[Y_1 = 1, \dots, Y_n = 1] &= E[P[Y_1 = 1, \dots, Y_n = 1 | X]] \\ &= E\left[\prod_{i=1}^n p_i(X)\right] \\ &= \int_{\mathbb{R}^m} \prod_{i=1}^n \Phi\left(\frac{B_i - a'_i x}{b_i}\right) \phi_m(\bar{\Sigma}; x) dx, \end{aligned} \quad (49)$$

where $\phi_m(\bar{\Sigma}; \cdot)$ denotes the m -variate normal density function with covariance matrix $\bar{\Sigma}$. The joint default probability is a *mixture* of Gaussian probabilities, where the mixing distribution is the distribution of the macro-factor X .

Multivariate normal mixtures. The assumption of multivariate normally distributed asset returns is often violated in practice. We introduce a more general multivariate normal mixture model for the asset returns W , which accommodates a wide range of more realistic distributions, such as the t -distribution. For some independent random variable U , we set

$$W_1^i = c_i(U) + d(U)(a'_i X + b_i Z_i), \quad (50)$$

where $c_i : \mathbb{R} \rightarrow \mathbb{R}$, $d : \mathbb{R} \rightarrow (0, \infty)$ and the other variables are defined as above in (47). The distribution of the vector (W^1, \dots, W^n) depends on the choices for c_i, d and the distribution of U . Conditional on (X, U) , the random variable W^i is independent and normally distributed with mean $c_i(U) + d(U)a'_i X$ and variance $(d(U)b_i)^2$. Thus we get for the conditional default probability

$$P[W_i^i < B_i | X, U] = \Phi\left(\frac{B_i - c_i(U) - d(U)a'_i X}{d(U)b_i}\right).$$

Letting f_U denote the density of U , we get for joint default probabilities

$$P[Y_1 = 1, \dots, Y_n = 1] = \int_{\mathbb{R}^{m+1}} \prod_{i=1}^n \Phi\left(\frac{B_i - c_i(u) - d(u)a'_i x}{d(u)b_i}\right) \phi_m(\bar{\Sigma}; x) f_U(u) dx du.$$

Different factor models are obtained by varying the specifications for the functions c_i, d and the law of U .

Example 2.9. Suppose $c_i(u) = 0$ and $d(u) = 1$ for all u . Then the factor vector (W^1, \dots, W^n) is multivariate normal with mean vector zero and covariance matrix Σ , as in our first model (47). The dependence structure of the asset returns is governed by the Gaussian copula (29). \square

Example 2.10. Suppose $C_i(u) = 0$ and $d(u) = \sqrt{\nu/u}$ for some $\nu > 0$ and $U \sim \chi^2(\nu)$. In this case the factor vector (W^1, \dots, W^n) is multivariate t -distributed with zero mean vector, covariance matrix $\frac{\nu}{\nu-2}\Sigma$ and $\nu > 2$ degrees of freedom. The dependence structure of the asset returns is governed by the t -copula. Letting $t_2(r, \nu, \cdot, \cdot)$ denote the bivariate standard t -distribution function with correlation parameter r and ν degrees of freedom, the corresponding copula is given by

$$\begin{aligned} C_{r,\nu}^t(u, v) &= t_2(r, \nu, t_\nu^{-1}(u), t_\nu^{-1}(v)) \\ &= \int_{-\infty}^{t_\nu^{-1}(u)} \int_{-\infty}^{t_\nu^{-1}(v)} \frac{1}{2\pi\sqrt{1-r^2}} \exp\left(1 + \frac{x^2 - 2rxy + y^2}{\nu(1-r^2)}\right)^{-(\nu+2)/2} dx dy \end{aligned} \quad (51)$$

where t_ν is the standard t -distribution function with ν degrees of freedom. \square

We have seen in the context of the classical structural approach that the Gaussian copula of the asset returns governs the copula of the default times. Consider a more general version of definition (39), given by

$$Y_i(T) = 1 \Leftrightarrow W_T^i < B_i.$$

It follows that the joint default probability satisfies

$$p(T_1, \dots, T_n) = C(p_1(T_1), \dots, p_n(T_n)) = C_{T_1, \dots, T_n}^W(p_1(T_1), \dots, p_n(T_n)).$$

We conclude that the copula C_{T_1, \dots, T_n}^W of the return vector $(W_{T_1}^1, \dots, W_{T_n}^n)$ is equal to the copula C of the default times. Thus, in Example 2.9, the default dependence structure is Gaussian while in Example 2.10, it is the t -copula.

The tail dependence properties of the underlying asset return copula determine the joint default behavior. While the Gaussian copula is asymptotically independent, the t -copula is lower tail dependent. Since marginal default probabilities p_i are typically very small, if compared with the Gaussian model, the t -model exhibits a pronounced tendency of generating low asset returns for all firms simultaneously. In the t -model there is a higher chance of observing multiple defaults relatively early than in the Gaussian model. In other words, for a given covariance matrix Σ , the t -model is more conservative in the sense that

it estimates higher probabilities for scenarios where multiple defaults happen early. From a modeling perspective, a given asset return covariance matrix Σ does not imply a unique model for correlated defaults: we need to choose an appropriate asset return copula additionally.

One factor copulas. The joint default probabilities (43) in the homogeneous one factor model are easy to calculate. Consider the corresponding copula of the default times. It is given by

$$C(u_1, \dots, u_n) = \int_{-\infty}^{\infty} \Phi \left(\frac{\Phi^{-1}(u_1) - \sqrt{\rho}x}{\sqrt{1-\rho}} \right)^n \phi(x) dx. \quad (52)$$

This copula is exchangeable—it expresses symmetric dependence between the defaults. It is sometimes called the *one-factor Gaussian copula*, see Gregory & Laurent (2002). It can be used with arbitrary marginals to obtain a new joint default distribution that can be used in other applications.

The symmetric dependence structure (52) is controlled by a single parameter ρ only. For some applications, this might not be flexible enough to fit calibration instruments. A refinement is to introduce non-symmetric correlation between asset returns by replacing the one-factor homogeneous model (41) with the one-factor version of the general model (47),

$$W_1^i = \rho_i X + \sqrt{1 - \rho_i^2} Z_i \quad (53)$$

so that $\text{Cov}(W_1^i, W_1^j) = \rho_i \rho_j$. The firm-specific parameter ρ_i governs the linear correlation between the systematic factor X and the asset return of firm i . Conditional default probabilities and joint default probabilities are given by the one-factor versions of (48) and (49), respectively. The corresponding copula provides n correlation parameters to characterize the dependence between the default times. It is given by

$$C(u_1, \dots, u_n) = \int_{-\infty}^{\infty} \prod_{i=1}^n \Phi \left(\frac{\Phi^{-1}(u_i) - \rho_i x}{\sqrt{1 - \rho_i^2}} \right) \phi(x) dx.$$

Calibrating the approximations. If we approximate the percentage loss distribution of an actual, heterogeneous finite portfolio with one of the distributions obtained above, we need to choose the parameters p and ρ . It seems natural to set the mean p of the approximating portfolio equal to the mean of the actual portfolio. There are several reasonable ways to calibrate the asset

correlation ρ of the approximating portfolio. One is to choose ρ such that the variance of the two portfolios are equal. Another is to match the (approximate) tail decay rate, as suggested by Glasserman (2003). The benchmark quantities of the actual portfolio can be calculated through a Monte Carlo simulation, or through another calibrated model, for example the heterogeneous Bernoulli mixture model (49).

2.4.3 Credit Contagion

Asset correlation captures the dependence of firms on common economic factors in a natural way. Modeling default contagion effects is much more difficult. A straightforward idea is to consider a jump-diffusion model for firm value. We would stipulate that a downward jump in the value of a given firm triggers subsequent jumps in the firm values of other firms with some probability. This would correspond to the propagation of economic distress. This approach fails however due to the lack of (closed-form) results on the joint distribution of firms' historical asset lows in higher dimensions. This is what we need to calculate the probability of joint default.

Generalized Bernoulli mixture models. A more successful attempt is to introduce interaction effects through the standardized default barriers B_i in the Bernoulli mixture model (48). Giesecke & Weber (2004) suppose the barrier is random and depends on the firm's liquidity state, which in turn depends on the default status of the firm's counterparties. If a firm's liquidity reserves are stressed due to a payment default of a counterparty, it finances the loss by issuing more debt. This increases the default barrier: the firm is now more likely to default, all else being equal. With no counterparty defaults the default barrier remains unaffected. Giesecke & Weber (2004) provide a non-classical CLT-type approximation to the credit portfolio loss distribution, analogous to (45).

2.5 Credit premium

Issuers of credit sensitive securities share a common dependence on the economic environment. It follows that aggregated credit risk cannot be diversified away. This undiversifiable or systematic risk commands a premium, which compensates risk-averse investors for assuming credit risk.

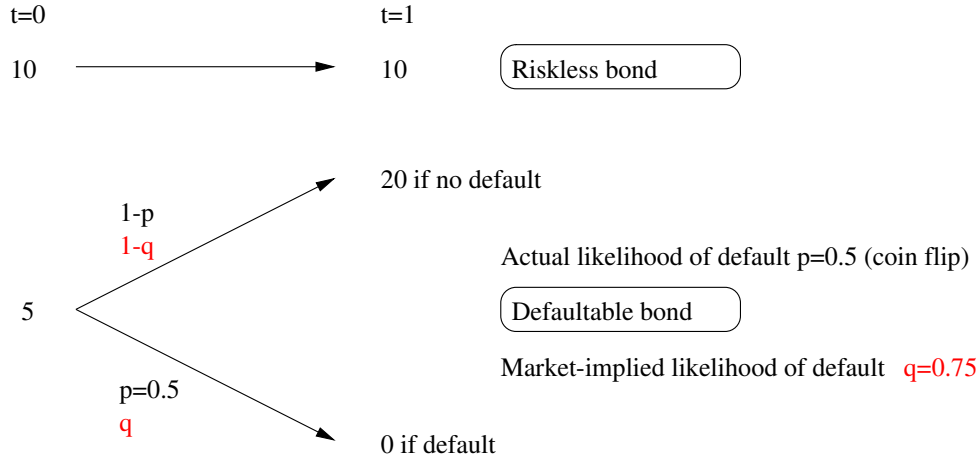


Figure 8: Physical vs. market-implied likelihood of default.

The credit premium is empirically well-documented and theoretically complex. Its importance relates to the uses of a quantitative credit model. As a default probability forecasting tool, a credit model must reflect the historical default experience. As a tool for pricing credit sensitive securities, it must fit observed market prices. To make use of both market data and historical default data in the calibration and application of a credit model, we need to understand the relationship between actual defaults and defaultable security prices. Here the risk premium comes into play: it maps the actual or physical likelihood of default $p(T)$ into the market-implied likelihood of default $q(T)$ that is embedded in security prices.

We examine the difference between the two probabilities using a simple example, see Figure 8. We consider a one-period market with two securities, a riskfree bond paying 10 and trading at 10 (riskfree rates are zero) and a defaultable bond trading at 5, that pays 20 in case of no default and zero in case the issuer defaults by the end of the trading period $T = 1$. Suppose the physical probability of default is $p = p(1) = 0.5$. This is however not the probability the market uses for pricing the bond: it would lead to a price of $20E^P[1_{\{\tau>1\}}] = 20(1 - p) = 10$, which is double the price the bonds is actually trading. At this price, risk-averse investors would rather put their money into the riskfree bond that costs 10 as well, unless they get a discount as compensation for the default risk. The discount makes the risk acceptable to the investors. The market requires a discount of 5, and the corresponding price reflects the market-implied probability of default $q = q(1)$, which satisfies

$5 = 20E^Q[1_{\{\tau>1\}}] = 20(1 - q)$. This yields $q = 0.75$, which is bigger than the physical probability of default $p = 0.5$. To account for risk aversion in calculating the expected payoff of the defaultable bond, the market puts more weight on unfavorable states of the world in which the firm defaults.

This basic insight is the same in our continuous trading market with non-zero interest rates. We consider the pricing of a zero-coupon bond with face value 1 and maturity T . Its payoff is $1_{\{\tau>T\}}$ at T . Calculating the fair value of the bond using physical probabilities gives

$$e^{-rT}E^P[1_{\{\tau>T\}}] = e^{-rT}P[\tau > T] = e^{-rT}(1 - p(T)). \quad (54)$$

This valuation principle is also known as the *actuarial principle*. Although convenient, this principle has significant deficiencies. As we have seen above, the price difference between the riskfree and risky bond covers only expected default losses $p(T)e^{-rT}$. It does not cover a risk premium as compensation for the risk of default.

To account for risk aversion, the actuarial principle must be modified to generate higher compensation to the investor or equivalently, lower bond prices. The standard approach is to retain the form of the principle (54) and to substitute a pricing measure Q for the physical probability P . Events such as “default by time T ” are assigned new probabilities that do not necessarily reflect the actual likelihood of default. Rather, they are consistent with the bond’s market price. The price calculated as expected discounted payoff under the probability Q ,

$$B_0^T = e^{-rT}E^Q[1_{\{\tau>T\}}] = e^{-rT}Q[\tau > T] = e^{-rT}(1 - q(T)). \quad (55)$$

accounts for both the expected default loss and the risk premium. This relation suggests to call Q also a market-implied probability.

A pricing probability Q is characterized by two properties.

(1) Martingale property:

The discounted price process (C_te^{-rt}) of any traded default contingent security with price C must be a martingale with respect to the pricing measure. This implies that

$$C_0 = E^Q[e^{-rT}C_T]. \quad (56)$$

The price of a security is given by its expected discounted future cash flows under Q . In case of the zero bond considered above, this yields

(55). Since a martingale is a “fair” process whose expected loss or gain is zero, after accounting for the time value of money, prices calculated under Q are “fair.”

(2) Equivalence:

The pricing measure and physical measure agree on which events have zero probability. That is, an event has zero probability under P if and only if it has zero probability under Q .

The mathematical conditions determining the set of pricing measures \mathcal{P} arise from a fundamental economic result in Delbaen & Schachermayer (1997) that goes back to Harrison & Kreps (1979) and Harrison & Pliska (1981): Under broad assumptions, \mathcal{P} is non-empty if and only if the security prices generated by the elements in \mathcal{P} do not admit arbitrage opportunities. Further, \mathcal{P} consists of a single measure if and only if markets are complete and every contingent claim can be perfectly hedged. These deep results point to the most serious deficiency of the actuarial pricing principle: it does not guarantee the absence of arbitrage opportunities. In fact, if markets are complete and the risk premium is non-trivial, the actuarial principle implies an arbitrage.

All pricing measures account for default risk and each one accounts for the risk premium in its own way. If the financial market is arbitrage-free but incomplete, there are infinitely many martingale measures and thus, infinitely many risk premia. This is because \mathcal{P} is convex and has more than one element. In this case, the two conditions do not lead to a unique price but to an arbitrage-free price interval $(\inf_{Q \in \mathcal{P}} C_0(Q), \sup_{Q \in \mathcal{P}} C_0(Q))$.

The relationship between P and its equivalent measures Q is characterized through a random variable Z_T given by

$$Z_T = \frac{dQ}{dP}.$$

It is called the Radon-Nikodym density and can also be understood as a likelihood ratio. This intuition is easily confirmed in our simple example shown in Figure 8. There are two states of the world, “no default” (denoted ω_1) and “default” (denoted ω_2). The Radon-Nikodym density is simply the ratio of the path probabilities. For the “no-default” path

$$Z_1(\omega_1) = \frac{Q[\omega_1]}{P[\omega_1]} = \frac{1-q}{1-p} = \frac{1}{2}$$

and for the “default” path

$$Z_1(\omega_2) = \frac{Q[\omega_2]}{P[\omega_2]} = \frac{q}{p} = \frac{3}{2}.$$

Denoting the payoff of the bond by X (so $X(\omega_1) = 20$ and $X(\omega_2) = 0$), we can calculate

$$E^Q[X] = X(\omega_1)Q[\omega_1] = X(\omega_1)Z_1(\omega_1)P[\omega_1] = E^P[XZ_1] = 5$$

and vice versa $E^P[X] = E^Q[X\frac{1}{Z_1}] = 10$. The relations between the Radon-Nikodym density and expectations under P and Q hold also in the general case for suitable random variables X , allowing us to go back and forth between P and Q . Also observe that $E^P[Z_T] = 1$.

Drawing from the analogy with option pricing, the Radon-Nikodym density can be characterized explicitly in the structural models. It is given by

$$Z_T = \exp\left(-\alpha W_T - \frac{1}{2}\alpha^2 T\right) \quad (57)$$

where W is the Brownian motion driving the uncertainty about firm assets and hence credit risk. This is the only source of uncertainty in the model. The constant α is the risk premium for this uncertainty.

The set of equivalent measures is characterized by (57). It is parameterized by the risk premium α . The set of pricing measures \mathcal{P} sits inside the set of equivalent measures. A pricing measure is an equivalent measure that makes the discounted prices of traded securities martingales. In our market, this condition uniquely determines the risk premium α . It is given as the excess return on firm assets over the riskfree return per unit of firm risk, measured in terms of asset volatility:

$$\alpha = \frac{\mu - r}{\sigma}. \quad (58)$$

This is analogous to the risk premium in the standard Black-Scholes model. If the market is risk averse, then α is positive: investors in credit-risky firm assets require a return that is higher than the riskfree return. The excess return on any credit sensitive security is given by its volatility times α . For equity,

$$\mu^E - r = \alpha\sigma^E,$$

where μ^E (σ^E) is the growth rate (volatility) of equity. Given the risk premium and equity volatility we can calculate expected return of the firm’s stock.

The uniqueness of the risk premium implies the uniqueness of the Radon-Nikodym derivative and the uniqueness of the pricing probability Q . The fundamental theorem of asset pricing then asserts that a market in which either firm assets, equity or bonds are traded is complete so that default can be perfectly hedged by dynamic trading in firm assets.

Girsanov's theorem implies that the process defined by $W_t^Q = W_t + \alpha t$ is a Brownian motion under the pricing probability Q . The firm value dynamics under Q are thus given by

$$\frac{dV_t}{V_t} = \mu dt + \sigma(dW_t^Q - \alpha dt) = r dt + \sigma dW_t^Q, \quad X_0 > 0. \quad (59)$$

We emphasize that only the drift is changed if we move from physical asset dynamics to asset dynamics in a pricing world. Under Q , the firm value grows at the riskfree rate r which is smaller than the actual growth rate μ . We lower the firm growth rate to account for risk aversion, i.e. we put more weight on unfavorable states of the world where the firm does not grow as fast. We conclude that we can easily obtain the pricing default probability $q(T)$ from the physical default probability $p(T)$ that we obtained above for various default definitions: we just set the growth rate μ in these formulas equal to r .

2.6 Calibration

The calibration of a quantitative credit model is closely related to its use. To price single-name credit sensitive securities using a structural model, we need to calibrate the following vector of constant parameters:

$$(r, \sigma, V_0, K, D, T),$$

The first three parameters refer to firm value dynamics, whereas the remaining parameters relate to the debt of the firm. The barrier D is relevant only in the first passage approach. To use the model to forecast actual default probabilities, we need to calibrate additionally the growth rate μ of firm assets or, equivalently, the risk premium α . In a multiple firm setting we need to estimate asset correlations in addition to the single-name parameters.

Firm values are not directly observable. The goal is to estimate the parameters of the firm value process based on equity prices, which can be observed for public firms. Riskfree interest rates can be estimated from default-free Treasury bond prices via standard procedures. We bypass estimation of face value and maturity of firm debt from balance sheet data, which is non-trivial given

the complex capital structure of firms. In practice these parameters are often fixed ad-hoc, as some average of short-term and long-term debt, for example. We introduce a more reasonable solution to this problem later.

We consider the classical approach. Given equity prices E_t and equity volatility σ^E , Jones, Mason & Rosenfeld (1984) and many others suggest to back out V_t and σ by numerically solving a system of two equations. The first equation relates the equity price to asset value, time and asset volatility:

$$E_t = f(V_t, t) \quad (60)$$

where $f(x, t)$ is the Black-Scholes pricing function for a European call with strike K and maturity T . The second equation relates the equity price to asset and equity volatility, the Delta of equity, and asset value:

$$E_t = \frac{\sigma}{\sigma^E} f_x(V_t, t) V_t, \quad (61)$$

where a subscript on f refers to a partial derivative. This relation is obtained from applying Itô's formula to (60), yielding

$$\begin{aligned} df(V_t, t) \\ = \left(f_x(V_t, t) \mu V_t + \frac{1}{2} f_{xx}(V_t, t) \sigma^2 V_t^2 + f_t(V_t, t) \right) dt + \sigma f_x(V_t, t) V_t dW_t, \end{aligned} \quad (62)$$

and comparing the diffusion coefficient to that of the equity value dynamics

$$dE_t = \mu^E E_t dt + \sigma^E E_t dW_t,$$

where μ^E is the equity growth rate. Both μ^E and σ^E are random variables that depend on the value of the firm.

Supposing the system (60) and (61) admits a unique solution for a given equity value and equity volatility, we can use it to “translate” a time series of equity values into a time series of asset values and volatilities. As for the equity volatility, we can use the empirical standard deviation of equity returns, or implied volatilities from options on the stock. Given a time series of asset returns, the empirical growth rate yields an estimate of μ and hence the market price of credit risk (58).² Further, given asset return time series of several firms, asset correlation can be estimated. Alternatively, we can introduce a linear

²The standard estimate of the firm growth rate is very poor: it is based on two asset return observations only.

factor model for normally distributed asset returns, which expresses the idea that firms share a common dependence on general economic factors:

$$\log \left(\frac{V_T^i}{V_0^i} \right) = w_i \psi_i + \epsilon_i.$$

Here ψ_i is a normally distributed systematic factor, which can be constructed as a weighted sum of global, country and industry specific factors. The constant w_i is the factor loading, and expresses the linear correlation between asset returns and the systematic factors. The ϵ_i are (mutually) independent normally distributed factors, which capture idiosyncratic risk in asset returns. The asset correlation is now determined through the factors loadings, see Crouhy, Galai & Mark (2000) for details.

An alternative maximum likelihood estimation procedure is based on the framework proposed by Duan (1994). Here we consider equity price data as transformed asset price data, with the equity pricing function defining the transformation. Suppose we observe a time series of daily equity values $(E^i)_{i=1,2,\dots,n}$; we drop the time index for clarity. Letting $\Delta t = 1/240$, the likelihood function can be derived as

$$\mathcal{L}(E^1, \dots, E^n) = \prod_{i=1}^{n-1} \frac{1}{\sqrt{2\pi\Delta t\sigma V^{i+1}} |f_x(V^{i+1}, 0, \sigma)|} e^{-\frac{1}{2\sigma^2\Delta t} [\log(\frac{V^{i+1}}{V^i}) - m\Delta t]^2}$$

where V^i is given as the solution of the equation

$$E^i = f(V^i, 0, \sigma), \tag{63}$$

if a unique solution exists. Here f is the equity pricing function with asset value, time and asset volatility as arguments. It is given through the underlying structural credit approach. The estimates \hat{m} and $\hat{\sigma}$ are the parameter values that maximize $\log \mathcal{L}$ given the equity time series. Note that we obtain an estimate of the firm growth rate, enabling us to obtain an estimate of the market price of credit risk via (58). Given $\hat{\sigma}$, estimates of asset values \hat{V}^i are obtained as the solutions to $E^i = f(\hat{V}^i, 0, \hat{\sigma})$, if such solutions uniquely exist. Duan (1994) shows that these estimators are asymptotically normal. Duan, Gauthier, Simonato & Zaanoun (2003) extend this procedure to a setting with multiple firms in the classical approach, and obtain estimates of asset correlations as well.

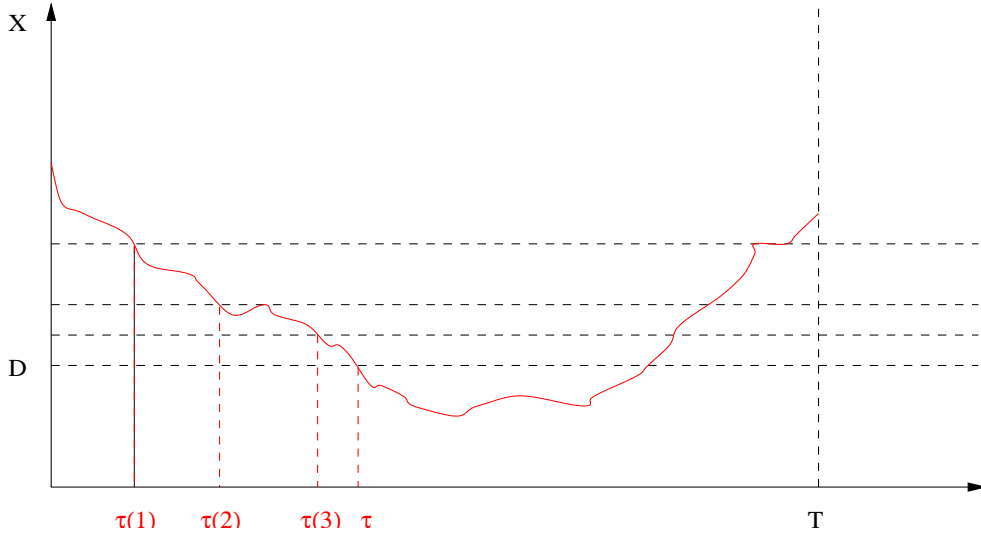


Figure 9: Announcing the default τ by a sequence $(\tau(n))$ of “pre-defaults” in the first-passage approach.

2.7 Can we predict the future?

To a certain extent, users of structural models implicitly assume they can. In structural models, firm value is the single source of uncertainty that drives credit risk. Investors observe the distance of default as it evolves over time. If the firm value has no jumps, this implies that the default event is not a total surprise. There are “pre-default events” which announce the default of a firm. In the first passage approach, for example, we can think of a pre-default event as the first time assets fall dangerously close to the default barrier, see Figure 9. Mathematically, there is an increasing sequence of event times $(\tau(n))$ that converge to the default time τ ; we say the default is *predictable*.

This predictability of default is not just a technical obscurity, but has significant implications for the fitting of structural models to market prices. First, since default can be anticipated, the model price of a credit sensitive security converges continuously to its recovery value. Second, the model credit spread tends to zero with time to maturity going to zero:

$$\lim_{T \downarrow t} S(t, T) = 0 \quad (64)$$

almost surely, see Giesecke (2001). Quite telling in this regard are the credit spreads implied by the classical and first-passage approaches, see Figures 2

and 5. Both properties are at odds with intuition and market reality. Market prices do exhibit surprise downward jumps upon default. Even for very short maturities in the range of weeks, market credit spreads remain positive. This indicates investors do have substantive short-term uncertainty about defaults, in contrast to the predictions of the structural models.

3 Reduced form credit models

Reduced form models go back to Artzner & Delbaen (1995), Jarrow & Turnbull (1995) and Duffie & Singleton (1999). Here we assume that default occurs without warning at an exogenous default rate, or intensity. The dynamics of the intensity are specified under the pricing probability. Instead of asking why the firm defaults, the intensity model is calibrated from market prices.

3.1 Default intensity

The reduced form approach is not based on a model definition of default. The dynamics of default are prescribed exogenously, directly under a pricing probability Q . The problem can be cast in the framework of point processes. Taking as given the random default time τ , we define the default process by

$$N_t = 1_{\{\tau \leq t\}} = \begin{cases} 1 & \text{if } \tau \leq t \\ 0 & \text{if else.} \end{cases}$$

This is a point process with one jump of size one at default.

Since the default process is increasing, it has an upward trend: the conditional probability at time t that the firm defaults by time $s \geq t$ is at least as big as N_t itself. A process with this property is called a submartingale. A process with zero trend is called a martingale. This is a “fair” process in the sense that the expected gain or loss is zero.

The Doob-Meyer decomposition theorem enables us to isolate the upward trend from N . This fundamental result states that there exists an increasing process A^τ starting at zero such that $N - A^\tau$ becomes a martingale, see Delalacherie & Meyer (1982). The unique process A^τ counteracts the upward trend in N ; it is therefore often called compensator.

Interestingly, the analytic properties of the compensator correspond to the probabilistic properties of default. For example, the compensator is continuous if and only if the default time τ is unpredictable. In this case the default comes

without warning; a sequence of announcing pre-default times does not exist. This is a desirable model property since it allows us to fit the model to market credit spreads.

The compensator describes the cumulative, conditional likelihood of default. In the reduced form approach to credit, the compensator is parameterized through a non-negative process λ by setting

$$A_t^\tau = \int_0^{\min(t, \tau)} \lambda_s ds = \int_0^t \lambda_s 1_{\{\tau > s\}} ds. \quad (65)$$

With this assumption, λ_t describes the conditional default rate, or *intensity*: for small Δt and $t < \tau$, the product $\lambda_t \cdot \Delta t$ approximates the pricing probability that default occurs in the interval $(t, t + \Delta t]$. Any given non-negative process λ can be used to parameterize the dynamics of default. No economic model of firm default is needed for this purpose any more!

Example 3.1. Suppose λ is a constant. Then N is a homogeneous Poisson process with intensity λ , stopped at its first jump. Thus τ is exponentially distributed with parameter λ and the pricing probability of default is given by

$$q(T) = 1 - e^{-\lambda T}.$$

Given the default probability, we can calculate the intensity as

$$\lambda = \frac{d(T)}{1 - q(T)}$$

where d is the density of q . In view of this representation, in the statistics literature λ is often called hazard rate. \square

Example 3.2. Suppose $\lambda = \lambda(t)$ is a deterministic function of time t . Then N is an inhomogeneous Poisson process with intensity function λ , stopped at its first jump. The default probability is given by

$$q(T) = 1 - e^{-\int_0^T \lambda(u) du}.$$

A simple but useful parametric intensity model is

$$\lambda(t) = h_i, \quad t \in [T_{i-1}, T_i), \quad i = 1, 2, \dots \quad (66)$$

for constants h_i and T_i , which can be calibrated from market data. \square

Example 3.3. Suppose that $\lambda = (\lambda_t)$ is a stochastic process such that conditional on the realization of the intensity, N is an inhomogeneous Poisson process stopped at its first jump. Then N is called a Cox process, or doubly-stochastic Poisson process. The conditional default probability given the intensity path up to time T is given by $1 - \exp(-\int_0^T \lambda_u du)$. By the law of iterated expectations we find the default probability

$$q(T) = 1 - E^Q[e^{-\int_0^T \lambda_u du}]. \quad (67)$$

To construct a Cox process intensity models we proceed in two steps. We first choose a vector of latent state variables X . These are thought of as “risk factors” which drive the intensity. In a second step we choose a nonnegative function Λ that maps the risk factors into the intensity, such that $\int_0^t \Lambda(X_s) ds$ is finite. We then set $\lambda = \Lambda(X)$. \square

3.2 Rating transition intensity

We have focused so far on the modeling the stochastic structure of the default event by an intensity. Other types of credit events, such as rating transitions, can be modeled in terms of intensities as well.

Lando (1998) considers credit ratings. He proposes the Cox process framework to model default as the first time $\tau = \inf\{t \geq 0 : U_t = Y\}$ a continuous-time Markov credit rating chain U with state space $\{1, \dots, Y\}$ hits the absorbing state Y . State 1 is interpreted as the highest credit rating category, state $Y - 1$ is interpreted as the lowest rating before default, and state Y is the default state. The dynamics of U are described by a generator matrix G with transition intensities of the form $\lambda_t(i, j) = \Lambda_{i,j}(X_t)$, where $\Lambda_{i,j}$ is a (continuous) nonnegative function on \mathbb{R}^d , which maps the risk factors X into the transition intensity. Intuitively speaking, for small Δt the product $\lambda_t(i, j)\Delta t$ is the probability that the firm currently in rating class i will migrate to class j within the time interval Δt . The generator matrix has the form

$$G_t = \begin{pmatrix} -\lambda_t(1) & \lambda_t(1, 2) & \dots & \lambda_t(1, Y) \\ \lambda_t(2, 1) & -\lambda_t(2) & \dots & \lambda_t(2, Y) \\ \vdots & & & \\ \lambda_t(Y-1, 1) & -\lambda_t(Y-1, 2) & \dots & \lambda_t(Y-1, Y) \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

where

$$\lambda_t(i) = \sum_{j=1, j \neq i}^Y \lambda_t(i, j), \quad i = 1, \dots, Y - 1.$$

Intuitively, $\lambda_t(i)\Delta t$ is the probability that there will be any rating change in Δt for a firm currently in class i . This generalizes Jarrow, Lando & Turnbull (1997), where the transition intensities are assumed to be constant. The corresponding default process N is a Cox process with intensity $\lambda_t(U_t, Y)$ at time t , which is represented by the last column in G_t .

3.3 Affine intensity models

After specifying an intensity model, we face the problem of computing default probabilities $q(T) = Q[\tau \leq T]$. While this is easy with deterministic intensities as we have seen above, in all other cases we have to calculate an expectation over the pricing trend. While this might not be easy in general, the affine framework of Duffie & Kan (1996) provides a powerful class of intensity models that admit closed-form solutions for $q(T)$ up to the solution of an ordinary differential equation. The affine framework takes advantage of the Cox process specification of N .

Affine diffusion model. We assume that the risk factor X solves the stochastic differential equation

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t \tag{68}$$

where the coefficients are affine functions of the state variables:

- $\mu(x) = \mu_0 + \mu_1 x$, where $\mu_0 \in \mathbb{R}^d$ is a vector of constants and $\mu_1 \in \mathbb{R}^{d \times d}$ is a matrix of constants,
- $(\sigma(x)\sigma(x)^\top)_{ij} = (\sigma_0)_{ij} + (\sigma_1)_{ij} \cdot x$, where $\sigma_0 \in \mathbb{R}^{d \times d}$ and $\sigma_1 \in \mathbb{R}^{d \times d}$ are matrices of constants, and
- $W \in \mathbb{R}^d$ is a standard Brownian motion under Q .

Additionally, we assume that the function Λ that maps the state variables X into the default intensity is affine as well: $\Lambda(x) = \Lambda_0 + \Lambda_1 \cdot x$ for $\Lambda_0 \in \mathbb{R}$ and

$\Lambda_1 \in \mathbb{R}^d$. Under these assumptions, the default probability is exponentially affine in the initial state X_0 :

$$q(T) = 1 - \exp(a(T) - b(T) \cdot X_0), \quad (69)$$

where the coefficient functions solve a system of ordinary differential equations given in Duffie & Kan (1996), which can be explicitly solved in some cases.

Example 3.4. Suppose that $d = 1$ and $\mu(x) = c\mu - cx$ for constants $\mu \in \mathbb{R}$ and $c > 0$, and $\sigma^2(x) = \sigma^2$ for a constant $\sigma > 0$. Then X follows the Gaussian process of Vasicek (1977):

$$dX_t = c(\mu - X_t)dt + \sigma dW_t, \quad X_0 \in \mathbb{R}. \quad (70)$$

This process exhibits a mean-reverting behavior. If X_t is below μ , then the positive drift $c(\mu - X_t)$ counteracts the process to fall further and forces it back to μ . If X_t is above μ , then the negative drift $c(\mu - X_t)$ counteracts the process to increase further and forces it back to μ . The scaling parameter c can be interpreted as the speed of mean reversion. Ito's formula can be used to prove that

$$X_t = \mu + e^{-ct}(X_0 - \mu) + \sigma \int_0^t e^{c(s-t)} dW_s$$

showing that X_t is normally distributed with moments

$$\begin{aligned} E^Q[X_t] &= \mu + e^{-ct}(X_0 - \mu) \\ \text{Var}^Q[X_t] &= \frac{\sigma^2}{2c}(1 - e^{-2ct}). \end{aligned}$$

This follows from the fact that the stochastic integral $\int_0^t f(s)dW_s$ is normally distributed with zero mean and variance $\int_0^t f^2(s)ds$ for suitable functions f . Letting $t \rightarrow \infty$, we have that $X_t \rightarrow N(\mu, \frac{\sigma^2}{2c})$ in distribution, suggesting to call μ the long-run mean of X . Letting $\Lambda(x) = x$, we have that

$$\begin{aligned} b(T) &= \frac{1}{c}(1 - e^{-cT}) \\ a(T) &= \mu(b(T) - T) + \frac{\sigma^2}{2c^2} \left[T - 2b(T) + \frac{1}{2c}(1 - e^{-2cT}) \right], \end{aligned}$$

providing a closed-form expression for default probabilities $q(T)$ via (69). Note that by choosing $\Lambda(x) = x$ we allow $\lambda < 0$ with strictly positive probability.

This is inconsistent and the price we have to pay for a nice parametric default probability model. In practice, one can try to calibrate the parameters such that $\frac{\sigma^2}{2c}$ is small relative to μ , so that for large t the probability of negative values of λ becomes small. \square

Example 3.5. Suppose that $d = 1$ and $\mu(x) = c\mu - cx$ for constants $\mu \in \mathbb{R}$ and $c > 0$, and $\sigma^2(x) = \sigma^2 x$ for a constant $\sigma > 0$. Then X follows the square-root diffusion of Cox, Ingersoll & Ross (1985):

$$dX_t = c(\mu - X_t)dt + \sigma\sqrt{X_t}dW_t, \quad X_0 \in \mathbb{R}. \quad (71)$$

If we choose $2c\mu > \sigma^2$ and $X_0 > 0$, then the process stays strictly positive almost surely. This is called the Feller-condition. Letting $\Lambda(x) = x$, we find

$$b(T) = \frac{2(e^{\gamma T} - 1)}{(\gamma - c)(e^{\gamma T} - 1) + 2\gamma}$$

$$a(T) = \frac{2c\mu}{\sigma^2} \log \left(\frac{2\gamma e^{(\gamma-c)T/2}}{(\gamma - c)(e^{\gamma T} - 1) + 2\gamma} \right),$$

where $\gamma = \sqrt{c^2 + 2\sigma^2}$. \square

Affine jump diffusion model. We can extend the affine diffusion model to include unexpected jumps, which can model the arrival of news in the economy. We assume that the risk factor X solves the stochastic differential equation

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t + dJ_t \quad (72)$$

where the coefficients μ and $\sigma\sigma^\top$ are affine functions of X as described above, $W \in \mathbb{R}^d$ is a standard Brownian motion under Q , and J is a pure jump process whose arrival intensity $h(X_t)$ at time t is affine in X_t as well: $h(x) = h_0 + h_1 \cdot x$ for $h_0 \in \mathbb{R}$ and $h_1 \in \mathbb{R}^d$. Conditional on the path of X , the jump times of J form an inhomogeneous Poisson process with intensity $h(X)$. If T is such a jump time, then the distribution j of the jump size is independent of the path of X up to T . For more details we refer to Duffie, Filipovic & Schachermayer (2003).

If we additionally assume that the default intensity is affine in the state variables as well, then the default probability is exponentially affine in the initial state X_0 :

$$q(T) = 1 - \exp(a(T) - b(T) \cdot X_0), \quad (73)$$

where the coefficient functions solve a system of ordinary differential equations given in Duffie, Pan & Singleton (2000), which can be explicitly solved in some cases.

Example 3.6. Suppose that $d = 1$ and $\mu(x) = c\mu - cx$ for constants $\mu \in \mathbb{R}$ and $c > 0$, and $\sigma^2(x) = \sigma^2 x$ for a constant $\sigma > 0$. Suppose further that $h(x) = h \geq 0$ and the jump distribution j is exponential with mean ν (that is, J is a compound Poisson process with iid exponential jumps). Then X follows the “basic affine process” of Duffie & Garleanu (2001):

$$dX_t = c(\mu - X_t)dt + \sigma\sqrt{X_t}dW_t + dJ_t, \quad X_0 \in \mathbb{R}. \quad (74)$$

Letting $\Lambda(x) = x$, the coefficient functions a and b are explicitly given in Duffie & Garleanu (2001). \square

3.4 Valuation

The description of the default dynamics through the market-implied default intensity λ leads to tractable valuation formulas. Below, we describe several different specifications of these formulas corresponding to different units for the value recovered by investors at default.

We start by considering a simple example that conveys the intuition of the general results. Consider a zero coupon bond paying 1 at maturity T if there is no default and R at T if the firm defaults before time T . Here the variable $R \in [0, 1]$ specifies the recovery of face value on the bond. With constant interest rates and constant recovery, the bond price is

$$B_0^T = e^{-rT} E^Q[R1_{\{\tau \leq T\}} + 1_{\{\tau > T\}}] = e^{-rT} - e^{-rT}(1 - R)q(T) \quad (75)$$

where $q(T)$ is the market-implied default probability. The value of the bond is the value of an otherwise equivalent riskfree bond minus the present value of the default loss. If the intensity is constant as in Example 3.1 and recovery is zero, we obtain

$$B_0^T = e^{-rT}(1 - q(T)) = e^{-(r+\lambda)T}. \quad (76)$$

This means the value of the defaultable bond is calculated as if the bond were riskfree by using a default-adjusted discount rate. The new discount rate is the sum of the riskfree rate r and the intensity λ . This parallel between pricing formulas for defaultable bonds and otherwise equivalent default free bonds is one of the best features of reduced form models.

The convenient parallel extends to more complicated securities. Consider a general credit sensitive security specified by the triple (T, c_T, R) . It pays the amount c_T at T if no default occurs before T , the maturity of the security. In case of default, investors receive some recovery payment that is modeled with a stochastic process R such that each R_t is bounded. If default occurs at time τ , the recovery payment is R_τ .

For $c_T = 1$ and nontrivial R , this security is a defaultable zero-coupon bond. For $c_T = (S_T - K)^+$ and nontrivial R , this security is a vulnerable call option on S struck at K , which is written by a defaultable counterparty. For c_T nontrivial and $R = 0$, this security represents a single fee payment at time T in a default swap, as we will see below.

We place ourselves into the Cox process framework of Example 3.3 for the intensity. Additionally, we assume that $c_T = c(X_T)$ for some bounded measurable function $c : \mathbb{R}^d \rightarrow \mathbb{R}$. The idea is that X represents a state vector that drives default and payment risk.

The zero-recovery claim $(T, c_T, 0)$. Assuming that interest rates are given by the constant $r > 0$, the price C_0 of the claim at time zero is given by its discounted expected payoff under the pricing probability,

$$C_0 = E^Q[e^{-rT} c_T 1_{\{\tau > T\}}] = e^{-rT} E^Q[c_T 1_{\{\tau > T\}}].$$

If τ and c_T were independent, then of course $C_0 = e^{-rT} E^Q[c_T](1 - q(T))$. In the general case, using the same “conditioning trick” that we used already in Example 3.3, we have that

$$\begin{aligned} C_0 &= e^{-rT} E^Q[E^Q[c_T 1_{\{\tau > T\}} \mid (X_s)_{s \leq T}]] \\ &= e^{-rT} E^Q[c_T Q[\tau > T \mid (X_s)_{s \leq T}]] \\ &= e^{-rT} E^Q[c_T e^{-\int_0^T \lambda_s ds}]. \end{aligned}$$

In the second line we used the fact that c_T is known given X_T , and in the third line we used the fact that conditionally on the path of the state variables, τ is the first jump of an inhomogeneous Poisson process.

Our argument can also be applied when interest rates are stochastic. Indeed, assuming that $r_t = g(X_t)$ for some bounded measurable function

$g : \mathbb{R}^d \rightarrow [0, \infty)$, we have that

$$\begin{aligned} C_0 &= E^Q \left[e^{-\int_0^T r_s ds} c_T 1_{\{\tau > T\}} \right] \\ &= E^Q \left[e^{-\int_0^T r_s ds} c_T Q[\tau > T \mid (X_s)_{s \leq T}] \right] \\ &= E^Q \left[c_T e^{-\int_0^T (r_s + \lambda_s) ds} \right]. \end{aligned}$$

This of course reduces to the formula (76) in case $c_T = 1$ and the intensity and interest rates are both constant.

The general claim (T, c_T, R) . We start right away in the general setup with stochastic interest rates $r_t = g(X_t)$. We have that $C_0 = C_0^F + C_0^R$, where

$$C_0^F = E^Q \left[e^{-\int_0^T r_s ds} c_T 1_{\{\tau > T\}} \right] \quad \text{and} \quad C_0^R = E^Q \left[e^{-\int_0^T r_s ds} R_\tau 1_{\{\tau \leq T\}} \right],$$

where C_0^F is the value of the final payment which we calculated above and C_0^R is the value of the recovery payment. Applying our conditioning trick,

$$\begin{aligned} C_0^R &= E^Q \left[E^Q \left[e^{-\int_0^T r_s ds} R_\tau 1_{\{\tau \leq T\}} \mid (X_s)_{s \leq T} \right] \right] \\ &= E^Q \left[\int_0^\infty e^{-\int_0^u r_s ds} R_u 1_{\{u \leq T\}} k(u) du \right] \\ &= \int_0^T E^Q \left[e^{-\int_0^u r_s ds} R_u k(u) \right] du, \end{aligned}$$

where in the last line we have used Fubini's theorem to interchange the order of integration. Here $k(u)$ is the conditional density of τ at u given the path $(X_s)_{s \leq T}$ for all $0 \leq u \leq T$. In the Cox process framework this density exists and is given by

$$k(u) = \frac{d}{du} Q[\tau \leq u \mid (X_s)_{s \leq T}] = \lambda_u \exp \left(- \int_0^u \lambda_s ds \right).$$

Thus we obtain the convenient expression

$$C_0^R = \int_0^T E^Q \left[e^{-\int_0^u (r_s + \lambda_s) ds} R_u \lambda_u \right] du.$$

Recovery specifications. The recovery payment at default can be measured in different units. In the *recovery of face value* scheme R_t is given as some fraction $\bar{R}_t \in [0, 1]$ of the security's face value. In the *equivalent recovery*

scheme R_t is given as a fraction \bar{R}_t of an equivalent but default-free version of the security. In the *fractional recovery* scheme investors receive a fraction \bar{R} of the security's market value just before default. If default occurs at time t , then this value is $C_{t-} = \lim_{t \uparrow \tau} C_t$. The recovery process can be written as $R_t = \bar{R}_t C_{t-}$. This convention makes only sense if C_τ differs from $C_{\tau-}$, i.e. if there is a surprise jump in the security price at default. Recall that in the structural models $C_\tau = C_{\tau-}$. We calculate

$$C_0 = E^Q \left[C_T e^{-\int_0^T (r_s + (1 - \bar{R}_s) \lambda_s) ds} \right]. \quad (77)$$

This is the value of the claim $(T, c_T, 0)$ when the issuer's default intensity is “thinned” to $\lambda(1 - \bar{R})$, see (76). The intuition behind this is as follows. Suppose the bond defaults with intensity λ . At default, the bond becomes worthless with probability $1 - \bar{R}$, and its value remains unchanged with probability \bar{R} . Clearly, the pre-default value $C_{\tau-}$ of the claim is not changed by this way of looking at default. Consequently, for pricing we can ignore the “harmless” default, which occurs with intensity $\lambda \bar{R}$. We then price the claim as if it had zero recovery and a default intensity $\lambda(1 - \bar{R})$.

3.5 Credit spreads

We take a closer look at the credit spreads implied by reduced form models. Consider a zero-recovery zero coupon bond $(T, 1, 0)$ in the Cox process framework with a right-continuous intensity $\lambda = \Lambda(X)$. For simplicity, we assume that risk-free rates are independent from the default time. From definition (4),

$$\begin{aligned} \lim_{T \downarrow t} S(t, T) &= - \lim_{T \downarrow t} \frac{\partial}{\partial T} E^Q \left[e^{-\int_t^T \lambda_s ds} \mid (X_s)_{s \leq t} \right] \\ &= - \lim_{T \downarrow t} E^Q \left[-\lambda_T e^{-\int_t^T \lambda_s ds} \mid (X_s)_{s \leq t} \right] \\ &= \lambda_t \end{aligned}$$

almost surely, provided that $t < \tau$. This result should be contrasted with the structural models, where the spread goes to zero with time to maturity going to zero, see (64). In the reduced form models the default event is unpredictable, it comes without warning. There is always short-term uncertainty about the default event, for which investors demand a premium. This premium, expressed in terms of yield, is given by the intensity. It corresponds to the Q -expected loss of one dollar over an infinitesimal horizon.

The unpredictability of default has another important consequence. In line with empirical observation, the model price of a credit sensitive security will abruptly drop to its recovery value upon default. This is in direct conflict with the structural models in which the price converges to its default contingent value. There no surprise jumps are possible.

3.6 Dependent defaults

In the reduced form model we can introduce cyclical default correlation by assuming that firms' default intensities are smoothly correlated through time. An effective framework for this is the Cox process model of Example 3.3, extended to the multivariate case. The indicator processes N^1, \dots, N^n of the default times τ_1, \dots, τ_n with respective intensities $\lambda^1, \dots, \lambda^n$ follow a multivariate Cox process if the individual indicators follow independent Cox processes. To construct such a multivariate Cox process, we start with some state process X which models the systematic and idiosyncratic factors driving the credit risk of firms. We model the intensity as $\lambda^i = \Lambda_i(X)$, for some bounded non-negative function Λ_i . We get

$$Q[\tau_i > T \mid (X_s)_{s \leq T}] = \exp \left(- \int_0^T \lambda_s^i ds \right). \quad (78)$$

Example 3.7. A simple but effective multi-factor model is as follows. We take as given a state process $X \in \mathbb{R}_+^{n+1}$ for n firms, where the X^i are mutually independent. Here X^{n+1} represents systematic credit risk among the firms, while X^i represents the idiosyncratic credit risk of firm $i = 1, \dots, n$. We set

$$\lambda^i = X^{n+1} + X^i, \quad i = 1, \dots, n. \quad (79)$$

By the independence of the factors and (78), we have

$$Q[\tau_i > T] = E^Q[e^{-\int_0^T X_s^{n+1} ds}] \cdot E^Q[e^{-\int_0^T X_s^i ds}], \quad (80)$$

which can be calculated in closed-form in the affine framework of Section 3.3. The generalization to multiple systematic factors is obvious. \square

In the Cox process model, the dependence among the defaults of firms comes from the sensitivity of their intensities on common factors in the state vector X . If the uncertainty about X is removed, defaults become independent. The multivariate Cox model hence generates *conditionally independent*

defaults. We consider horizons T_i with $T = \max_i T_i$. Conditional on the realization of X , the N^i are independent Cox processes with conditional survival probability (78). Thus

$$\begin{aligned} Q[\tau_1 > T_1, \dots, \tau_n > T_n \mid (X_s)_{s \leq T}] &= \prod_{i=1}^n Q[\tau_i > T_i \mid (X_s)_{s \leq T}] \\ &= \exp \left(- \sum_{i=1}^n \int_0^{T_i} \lambda_s^i ds \right). \end{aligned}$$

Taking expectations on both sides gives

$$s(T_1, \dots, T_n) = Q[\tau_1 > T_1, \dots, \tau_n > T_n] = E^Q \left[e^{-\sum_{i=1}^n \int_0^{T_i} \lambda_s^i ds} \right]. \quad (81)$$

While for pricing problems it is often more convenient to work with the joint survival probability s , standard arguments can be applied to calculate the joint default probability from s .

First-to-default intensity. We are interested in the intensity λ of the first default time $\tau = \min \tau_i$. Taking $T_i = T$ for all i in (81), we have

$$s(T, \dots, T) = Q[\tau > T] = E^Q \left[e^{-\int_0^T \sum_{i=1}^n \lambda_s^i ds} \right],$$

This suggests that

$$\lambda = \lambda^1 + \dots + \lambda^n. \quad (82)$$

The intuition behind this result is that, unless defaults can happen at literally the same time with strictly positive probability, the sum of the individual local default probabilities λ^i gives the local probability of *any* default arrival. We can use this result immediately to value credit sensitive securities (T, c_T, R) whose payoffs depend on the first-to-default in a pool of names. All the formulas that we derived in Section 3.4 apply without changes.

In view of (82), we can re-interpret the intensity model (79) in Example 3.7 in terms of a shock model. Suppose the economy is subject to systematic and idiosyncratic shocks. We take the non-negative processes X^i , $i = 1, \dots, n+1$, to be the intensities of the shock arrival times δ_i . We think of δ_{n+1} as the arrival time of a systematic shock that affects all firms in the market. Similarly, δ_i is the arrival time of an idiosyncratic shock that affects only firm i . With (82), the intensity model (79) then follows from the definition $\tau_i = \min(\delta_i, \delta_{n+1})$. This definition means in particular that a systematic shock arrival causes the simultaneous default of all firms in the market.

Copulas. To the survival probability s corresponds a copula C given by

$$C(u_1, \dots, u_n) = s(s_1^{-1}(u_1), \dots, s_n^{-1}(u_n)),$$

where $s_i(T) = Q[\tau_i > T] = 1 - q_i(T)$ is the marginal survival probability with inverse s^{-1} . The copula C will depend on the particular choice of the functions Λ_i and the dynamics of X . More generally, we can use any copula to build tractable models for correlated defaults. Together with arbitrary marginals s_i , any copula C specifies a proper joint survival function s via

$$s(T_1, \dots, T_n) = C(s_1(T_1), \dots, s_n(T_n)).$$

A popular choice in practice is to take the Gaussian copula corresponding to the structural joint default probability (26) and a piecewise constant intensity function $\lambda_i(t)$ with marginal survival function

$$s_i(T) = \exp\left(-\int_0^T \lambda_i(s) ds\right).$$

This modeling choice is, strictly speaking, inconsistent given the incompatible assumptions of the models underlying the copula and the marginals. On the other hand, the calibration is straightforward: we bootstrap the intensity λ^i directly from market prices and estimate the asset correlation parameterizing the copula C from a factor model of asset or equity returns. This is often the only way to calibrate the correlation parameters, since the market for multi-name credit sensitive claims is not very liquid yet.

The choice of the copula C is essentially arbitrary—there is no “natural choice.” This introduces model risk, since different copulas lead to quite different joint default characteristics as we observed already in the context of the structural models. In lack of empirical data of correlated defaults, it is hard to say which characteristics are natural. For a discussion of these model risk issues we refer to Frey & McNeil (2003).

Credit contagion. Taking account of contagious default correlation is not an easy exercise. The idea is that there are correlated jumps in firms’ default intensities, corresponding to the correlated jumps we observe in credit spreads. A variant of this assumes that there are market-wide events that can trigger joint defaults, see Duffie & Singleton (1998) and Giesecke (2003). Another variant assumes that the default intensity of a firm depends explicitly on the

default status of related counterparty firms in the market. A parametrization of this idea is

$$\lambda_t^i = h_t^i + \sum_{j \neq i} a^j N_t^j$$

see Jarrow & Yu (2001). Here h^i is the base default intensity and N^j is the default indicator process of firm j . The parameters a^j are chosen such that λ^i is non-negative. To avoid running into a circularity problem, one can suppose that only the default of designated “primary” firms has an effect on other, “secondary” firms.

While Jarrow & Yu (2001) focus on the pricing of credit sensitive securities in the presence of contagion effects, it is difficult to calculate joint default probabilities and portfolio loss distributions within this approach. As Davis & Lo (2001) and Giesecke & Weber (2003) show, one can obtain tractable closed-form characterizations of loss distributions at the cost of more restricting assumptions, which relate to the homogeneity of firms and the symmetry in their counterparty relations.

3.7 Calibration

Reduced form models are typically formulated directly under a pricing probability. This suggests that we calibrate directly from market prices of various credit sensitive securities. One often uses liquid debt prices or credit default swap spreads, although Jarrow (2001) argues that equity is a good candidate as well. Depending on the characteristics of the calibration security, it may be necessary to make parametric assumptions about the recovery process as well. With fractional recovery and zero bonds for example, the problem is to choose the parameters of the adjusted short rate model $r + (1 - R)\lambda$ such that model bond prices best fit observed market prices. Here one can either parameterize the adjusted short rate directly or specify the component processes separately. With a separate specification identification problems may arise, since only the product $(1 - R)\lambda$ enters the pricing formula. In general, in the estimation problem one can draw from the experience related to non-defaultable term structure models, given the close analogy to reduced form defaultable models. We refer to Dai & Singleton (2003) for an overview of available techniques. Standard methods include maximum likelihood and least squares.

4 Incomplete information credit models

The incomplete information framework provides a common perspective on the structural and reduced form approaches to analyzing credit. This perspective enables us to see models of both types as members of a common family. This family contains previously unrecognized structural/reduced form hybrids, some of which incorporate the best features of both traditional approaches. Incomplete information credit models were introduced by Duffie & Lando (2001), Giesecke (2001) and Çetin, Jarrow, Protter & Yildirim (2002). A non-technical discussion of incomplete information models is in Goldberg (2004).

4.1 Default trend

Underlying *all* credit models is the increasing default process N and its compensator A^τ . Thanks to the Doob-Meyer decomposition, the compensator can be isolated from the default process. The difference is a martingale, a fair process whose expected gains or losses are zero. The compensator represents the fair cumulative compensation for the short-term credit risk embedded in the default process. If there is short-term uncertainty about default in any state of the world, then there is a process A that generates the compensator:

$$A_t^\tau = A_{\min(t,\tau)} = \begin{cases} A_t & , \quad t < \tau \\ A_\tau & , \quad \text{else.} \end{cases}$$

The process A is called the *default trend*. It can be used to estimate default probabilities and price credit sensitive securities.

In traditional structural models default can be anticipated. In this case there is no short-term credit risk that would require compensation. Correspondingly, the compensator is trivially given by the default process itself. In reduced form models, it is assumed that default cannot be anticipated, so there is short-term credit risk by assumption. The compensator is directly parameterized through an intensity λ . We simply define the trend as

$$A_t = \int_0^t \lambda_s ds. \tag{83}$$

Hence we can think of the trend as the cumulative default intensity. In this situation the dynamics of model default probabilities and security prices are immediately implied by the exogenous intensity dynamics.

Instead of focusing on the default intensity and making ad-hoc assumptions about its dynamics, incomplete information models seek to specify the trend based on a model definition of default. Here we provide an endogenous characterization of the trend in terms of a firm's assets and liabilities via an underlying structural model. But this works only if we can modify the underlying structural model to admit short-term credit risk.

There are two approaches to introduce short-term uncertainty into structural models. The first is to allow for “surprise” jumps in the firm value, as in Zhou (2001*b*), Hilberink & Rogers (2002) and Kijima & Suzuki (2001). In this situation there is always a chance that the firm value jumps below the default barrier. This cannot be anticipated. However, there is also a chance that the firm just “diffuses” to the barrier, as in the traditional models with continuous value process. Here default can be anticipated. So depending on the state of the world, there may or may not be short-term credit risk.

There is another approach that guarantees default cannot be anticipated so there is short-term credit risk in any state of the world. This approach arises through a re-examination of the informational assumptions underlying the traditional structural models. In these models, it is implicitly assumed that the information we need to calibrate and run the model is publicly available. This information includes the firm value process and its parameters as well as the default barrier. In the incomplete information framework, we address the fact that in reality, our information about these quantities is imperfect. The information we have is much coarser than the idealized traditional structural models suggest, as highlighted by the high profile scandals at Enron, Tyco and WorldCom. Concretely this means that we may not be sure either of the true value of the firm or of the exact condition of the firm that will trigger default. It follows that we are always uncertain about the distance to default. Thus, default is a complete surprise: it cannot be anticipated. The trend, which represents the compensation for the associated short-term credit risk, can be characterized explicitly in terms of firm assets and default barrier.

Example 4.1 (I^2 credit model). Suppose default is described by the first passage model (6). Assume the default barrier D is a random variable that cannot be observed. Let D be independent with continuous distribution function G on $(0, V_0)$. Giesecke (2001) shows that the trend A is given by

$$A_t = -\log G(M_t) \tag{84}$$

where, as in Section 2.2, M_t is the historical low of firm value at time t . In

view of (83), we need only differentiate the trend to get the intensity. Under the assumption that G is differentiable, the derivative of A is however zero almost surely. This means that we cannot write the trend as in (83) in terms of an intensity. \square

Example 4.2. In the first passage model (6), suppose we do not observe the firm value directly but instead receive noisy asset reports from time to time. Let $f(\cdot, t)$ be the conditional density of the log-firm value at time t on $(d, 0)$ where $d = \log(D/V_0)$. Duffie & Lando (2001) show that

$$A_t = \frac{1}{2}\sigma^2 \int_0^t f_x(d, s)ds$$

Here an intensity exists and is given by $\lambda_t = \frac{1}{2}\sigma^2 f_x(d, t)$. \square

The trend is the key to the calculation of default probabilities and prices of credit sensitive securities. Under technical conditions stated in Giesecke (2001), we have the generalized reduced form formula

$$q(T) = 1 - E^Q[e^{-A_T}] \quad (85)$$

where A is the default trend under the pricing probability Q . This formula simplifies to the reduced form formula (67) if the trend admits an intensity. There are closed form expressions for $q(T)$ in some cases.

Example 4.3 (I^2 credit model). Suppose the default barrier D is uniform on $(0, V_0)$ under the pricing probability Q . We have for the default probability

$$q(T) = 1 + \left(\frac{\sigma^2}{2r} - 1\right) \Phi\left(\frac{\nu\sqrt{T}}{\sigma}\right) - e^{rT} \left(1 + \frac{\sigma^2}{2r}\right) \Phi\left(\frac{-\bar{\nu}\sqrt{T}}{\sigma}\right) \quad (86)$$

where $\nu = r - \sigma^2/2$ and $\bar{\nu} = r + \sigma^2/2$. \square

Consider the credit sensitive security (T, c_T, R) with fractional recovery $R = \bar{R}C_-$, see Section 3.4. Under technical conditions stated in Giesecke & Goldberg (2003), we have for the security's pre-default value the generalized reduced form formula

$$e^{-rT} E^Q[c_T e^{-\int_0^T (1-R_s)dA_s}] \quad (87)$$

where A is the default trend under Q . If the trend admits an intensity, (87) simplifies to (77) with r constant.

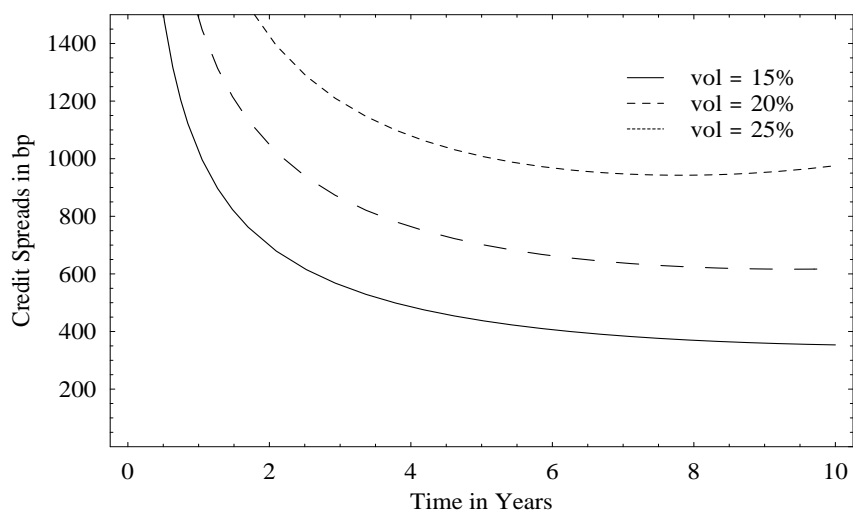


Figure 10: Term structure of credit spreads, varying asset volatility σ in the I^2 model.

The incomplete information models share many of the good properties of both structural and reduced form models while avoiding their difficulties. While built on an intuitive and economically meaningful structural approach, default cannot be anticipated as in the traditional structural models. This has several desirable consequences. First, any incomplete information model admits a non-trivial trend that can be characterized explicitly. The trend can be used to calculate default probabilities and prices of credit sensitive securities through tractable generalized reduced form formulas. In the traditional structural models these convenient reduced form formulas fail. Second, consistent with empirical observations, prices of credit sensitive securities drop abruptly to their recovery values upon default. Third, short-term credit spreads are typically bounded away from zero. To illustrate this, we consider a zero recovery zero bond with face value 1 maturing at T . The bond is priced at $e^{-rT}(1-q(T))$. Letting $r = 6\%$, in Figure 10, we plot the corresponding credit spreads

$$S(0, T) = -\frac{1}{T} \log(1 - q(T)), \quad T > 0,$$

in the I^2 model. Giesecke & Goldberg (2004b) calibrate the I^2 model from market data and further analyze its empirical properties. In particular, the I^2 model output is empirically compared to a traditional first passage model. Two main conclusions can be drawn. The I^2 model reacts more quickly since

it takes direct account of the entire history of public information rather than just current values. This can be seen from the structure of the trend in (84): it depends on the historical low of the firm value. Furthermore, the I^2 model predicts positive short spreads for firms in distress. The traditional first passage model always predicts that short spreads are zero.

4.2 Dependent defaults

Since incomplete information models are based on the structural approach, we can model cyclical default correlation through firm value correlation.

Contagious default correlation arises very naturally with incomplete information. Consider the I^2 model. As discussed in detail in Giesecke (2004), with defaults of firms arriving over time, we learn about the unobserved default barriers of the surviving firms. This means we update the distribution we put on a firm's default barrier with the information we extract from the unanticipated defaults of counterparty firms, and re-assess firms' default probabilities. The situation in which we do not directly observe firm values (Example 4.2) is very similar; it is analyzed in Collin-Dufresne, Goldstein & Helwege (2002). In both scenarios the “contagious” jumps in credit spreads we observe in credit markets are implied by informational asymmetries.

The same way we introduced the trend in the single firm case to estimate default probabilities and prices of securities, we can develop the concept of the trend in a situation with multiple firms under incomplete information. The trend can then be used to estimate prices of securities that depend on the credit risk of multiple firms. It can also be used to construct efficient simulation algorithms for the simulation of correlated default events. This analysis is carried out in Giesecke & Goldberg (2004d).

4.3 Credit premium

The credit risk premium is the mapping between the actual probability P and the pricing probability Q . To understand the structure of the premium, we examine the dynamics of firm value and corporate liabilities in the I^2 model. We argued above that thanks to the unpredictability of default, prices of credit sensitive claims including firm equity and debt drop precipitously at default. Empirical observation shows that equity drops to near zero. This makes sense since equity holders have no stake in the firm after default. The value of the bonds is diminished by bankruptcy costs, which is described by some fractional

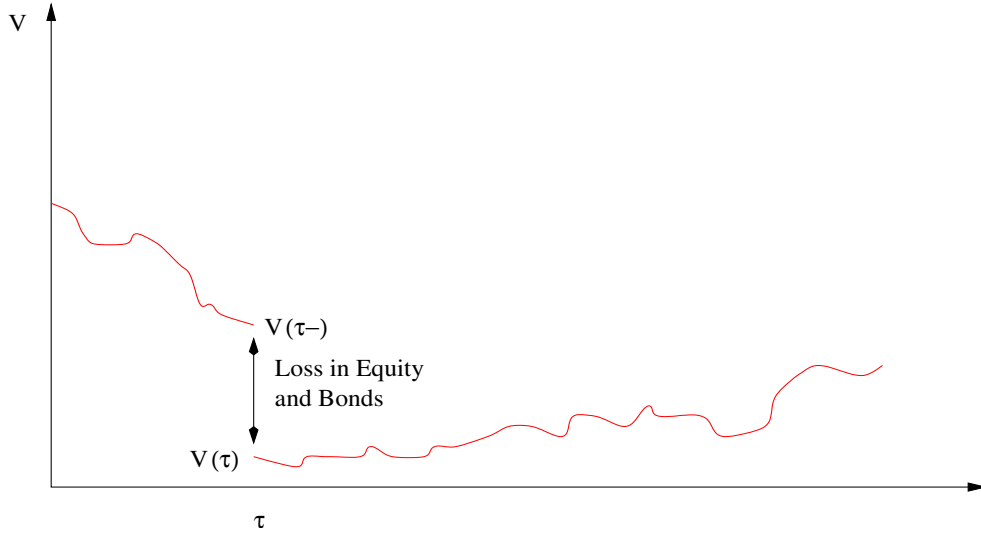


Figure 11: Firm value in the incomplete information model.

recovery process R . Consequently, firm value, which is equal to the sum of equity and debt values, also drops at default. This is shown in Figure 11. If default were to occur at time t , the combined default losses in equity and debt value relative to V are given by

$$J_t = \frac{1}{V_{t-}} (E_{t-} + (1 - R_t) \cdot B_{t-}).$$

Here E denotes the value of equity and B denotes the value of the bonds. If prior to default firm value follows a geometric Brownian motion, then the firm value process can hence be written as the jump diffusion

$$\frac{dV_t}{V_{t-}} = \mu dt + \sigma dW_t - J_t dN_t. \quad (88)$$

This shows that there are two sources of uncertainty related to firm value. The first is the diffusive uncertainty represented by the volatility σ . The second is the uncertainty associated with the downward jump in firm value at default.

The density describing the relation between the probabilities P and Q is now richer than (57) as Giesecke & Goldberg (2003) show in the context of the I^2 model. The density is parameterized by the risk premium. In the incomplete information models the risk premium can be decomposed into two components, which correspond to the two sources of uncertainty. The diffusive risk premium α compensates investors for the diffusive uncertainty in firm

value. As in the traditional structural models, it is realized as a change to the drift term in firm value dynamics: $\mu - r = \alpha_t \sigma$. The default event risk premium β is not present in the traditional structural models. It compensates investors for the jump uncertainty in firm value and is realized as a change to the default probability. Driessen (2002) empirically confirms that this event risk premium is a significant factor in corporate bond returns.

In Giesecke & Goldberg (2003), it is shown that the assumption of no arbitrage is realized in the mathematical relationships among α , β , the recovery rate assumed by the market, and the coefficients of the price processes of traded securities. The price processes depend explicitly on the leverage ratio, so the premia α and β do as well. In this case the density depends on firm leverage. As Giesecke & Goldberg (2004c) discuss, this violates an important condition for the Modigliani & Miller (1958) theorem. The I^2 model is therefore not consistent with the Modigliani-Miller theorem. It provides a new way to measure the deviation of real markets from the idealized markets in which the Modigliani-Miller theorem holds.

The structure of the incomplete information risk premium is analogous to the risk premium in reduced form models considered in El Karoui & Martellini (2001) and Jarrow, Lando & Yu (2003). The diffusive premium related to the firm value process corresponds to a premium for diffusive risk in the default intensity process. The event risk premium is analogous to the default event risk premium in intensity based models. However, in the incomplete information setting it is defined in the general reduced form context where an intensity need not exist. Interestingly, Jarrow et al. (2003) show that in the multi-firm intensity based Cox model of Section 3.6, where defaults are conditionally independent, the default event risk premium asymptotically diversifies away.

4.4 Calibration

There is a lively debate in the literature concerning which data should be used to calibrate credit. Jarrow (2001) points to a division between structural and reduced form modelers on this issue. Traditionally, structural models are fit to equity markets and reduced form models are fit to bond markets. Jarrow (2001) argues that the equity and bond data can be used in aggregate to calibrate a credit model and he gives a recipe for doing this in a reduced form setting.

Giesecke & Goldberg (2004a) apply the reasoning in Jarrow (2001) to calibrate the I^2 model. The estimation procedure makes use of historical default

rates in conjunction with data from equity, bond and credit default swap markets. Huang & Huang (2003) give empirical evidence that structural models yield more plausible results if calibrated to both kinds of data. Importantly, the physical and pricing probabilities are fit simultaneously. The output of the calibration includes estimates of the risk premium, market implied recovery, model security prices and physical probabilities of default.

One issue addressed in Giesecke & Goldberg (2004*a*) is the relationship between model and actual capital structures. In the classical setting, equity is a European option with strike price and date equal to the face value and maturity of a zero bond. This model is internally consistent. However, it fits market data only to the extent that firm debt can be adequately represented as a zero bond. Giesecke & Goldberg (2004*a*) make use of the flexibility imparted by incomplete information to give a more realistic picture of equity. Specifically, equity is a down-and-out call with a stochastic strike price and lower barrier. This approach sidesteps the intractable problem of describing a complex capital structure in terms of a single face value.

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