

# The Capriciousness of Numerical Methods for Singular Perturbations\*

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*Dedicated to Wolfgang Wasow on the occasion of his 100th birthday*

**Abstract.** A collection of typical examples shows the exotic behavior of numerical methods when applied to singular perturbation problems. While standard meshes are used in the first six examples, even on layer-adapted meshes several surprising phenomena are shown to occur.

**Key words.** singular perturbations, finite differences, finite elements, layer-adapted meshes

**AMS subject classifications.** 35L10, 35L12, 35L60

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**1. Introduction.** The history of singular perturbations began early in the last century with Prandtl's work in hydrodynamics (1904) and Birkhoff's contributions to the asymptotic integration of linear ordinary differential equations (1908). Wolfgang Wasow, born in 1909, stimulated considerable mathematical activity in singularly perturbed boundary value problems with his 1965 book *Asymptotic Expansions for Ordinary Differential Equations*.

The history of numerical methods for singular perturbation problems began in 1969 with two significant Russian papers, but its development has been based on the fundamental books on asymptotic expansions by Wasow and by Vasil'eva and Butuzov (1973) (see [23] for a short historical overview).

Wolfgang Wasow worked on a variety of asymptotic problems for forty years and, as a result of his influence, singular perturbations are well understood today and are a standard part of graduate students' training in applied mathematics. His paper "The Capriciousness of Singular Perturbations" [27] inspired the title of the present paper. In the introduction to his paper, Wasow explained the capriciousness of singular perturbations:

The designation 'singular' is appropriate in more than one sense. When one deals with regular perturbations, intuition is usually a reliable guide as to what really happens for small  $\varepsilon$ . The theory of singular perturbations, however, is full of peculiar and unexpected features. The purpose of this article is to illustrate this statement by a number of typical examples. We will describe some results and try to make them plausible. For proofs and

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additional material the reader will have to turn to the original research papers.

We follow Wasow's approach, but the features of the numerical methods for singular perturbations that we present are largely unexpected.

For simplicity, we restrict ourselves to one-dimensional (1D) convection-diffusion problems (the “most ubiquitous and challenging task in the numerical approximation of differential equations” [13]), but also give some references to the literature for problems in several space dimensions.

**2. Equidistant Meshes.** Consider the boundary value problems

$$(2.1a) \quad -\varepsilon u'' + b(x)u' + c(x, u) = f(x) \quad \text{in } (\alpha, \beta),$$

$$(2.1b) \quad u(\alpha) = A, \quad u(\beta) = B.$$

Here  $b, c$ , and  $f$  are given smooth functions, with the exception of those in Example 2.6 below. We study mainly linear problems with  $c(x, u) := c(x)u$ . The parameter  $\varepsilon$  is nonnegative but small, and is in general positive (only in Example 2.6 is  $\varepsilon = 0$ ).

The solution of (2.1) is, for small  $\varepsilon$ , characterized by the existence of layers. Usually (depending on the precise boundary conditions) the solution has an exponential boundary layer at  $x = \alpha$  if  $b(\alpha) < 0$  and at  $x = \beta$  if  $b(\beta) > 0$ . Furthermore, interior layers can occur if, for instance,  $b$  changes sign or if the reduced equation

$$b(x)w' + c(x, w) = f(x)$$

has a complicated solution structure, which happens typically if  $c(x, w)$  is nonlinear.

In this section we discuss finite difference and finite element methods on an equidistant mesh with mesh size  $h = (\beta - \alpha)/N$ :

$$(2.2) \quad \alpha = x_0 < x_1 < \cdots < x_{N-1} < x_N = \beta \quad \text{with} \quad x_i = \alpha + ih.$$

*Example 2.1* (stability of numerical schemes). Consider

$$(2.3) \quad -\varepsilon u'' - u' = 1 \quad \text{in } (0, 1), \quad u(0) = u(1) = 0.$$

In the exact solution  $u(x) = (1 - e^{-x/\varepsilon})/(1 - e^{-1/\varepsilon}) - x$ , we recognize the typical exponential layer term  $e^{-x/\varepsilon}$  at  $x = 0$ .

If we discretize (2.3) using the central difference scheme

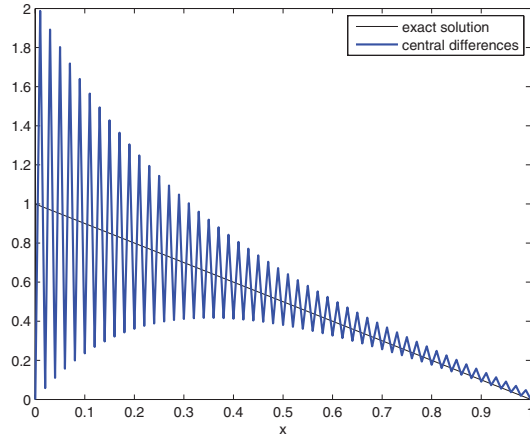
$$(2.4) \quad -\frac{\varepsilon}{h^2}(u_{i-1} - 2u_i + u_{i+1}) - \frac{u_{i+1} - u_{i-1}}{2h} = 1$$

(or using linear finite elements), then for  $\varepsilon < Ch$  we observe oscillatory and inaccurate solutions; see Figure 2.1 for  $\varepsilon = 10^{-4}$  and  $N = 100$ . The reason is the lack of stability in the discrete problem.

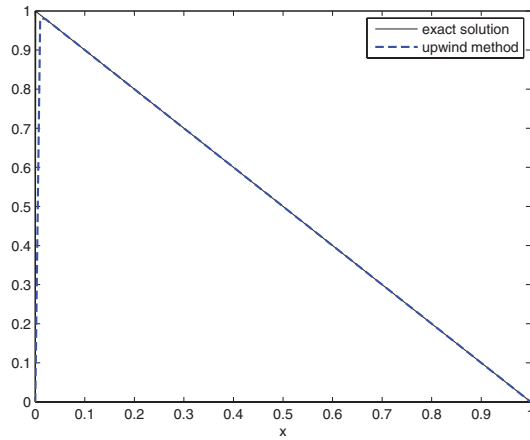
Stability here is related to the M-matrix property of the discrete problem generated. A matrix  $A$  is called an M-matrix if its entries  $a_{ij}$  satisfy  $a_{ij} \leq 0$  for  $i \neq j$  and its inverse  $A^{-1}$  exists with  $A^{-1} \geq 0$ . The stability constant  $K$  in the stability inequality

$$(2.5) \quad \|u_h\|_\infty \leq K \|Au_h\|_\infty$$

can be found using, e.g., the M-criterion; see [17]. In (2.5),  $u_h$  is the numerical solution at the mesh points and  $A$  is the matrix corresponding to the numerical method—such as (2.4).



**Fig. 2.1** Central difference scheme (2.4) with  $\varepsilon = 1e - 4$  and  $N = 100$ .



**Fig. 2.2** Upwind difference scheme (2.6) with  $\varepsilon = 1e - 4$  and  $N = 100$ .

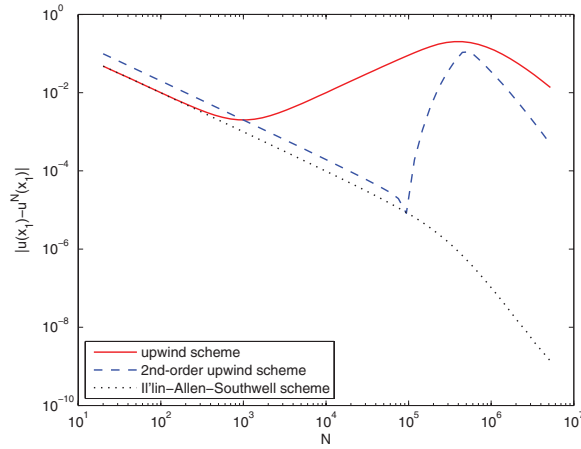
The simplest stable discretization is the upwind scheme

$$(2.6) \quad -\frac{\varepsilon}{h^2}(u_{i-1} - 2u_i + u_{i+1}) - \frac{u_{i+1} - u_i}{h} = 1.$$

We remark that (2.6) is equivalent to

$$-\frac{\varepsilon}{h^2}(u_{i-1} - 2u_i + u_{i+1}) - \frac{u_{i+1} - u_{i-1}}{2h} - \frac{h}{2} \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} = 1.$$

In other words, upwind differencing can be regarded as a stabilization technique for the central difference scheme. Figure 2.2 shows the oscillation-free numerical solution for first-order upwind differencing.



**Fig. 2.3** Error in the first mesh node of numerical solutions for (2.7) with  $\varepsilon = 1e-6$ ; solid line: upwind scheme; dashed line: second-order upwind scheme; dotted line: Il'in-Allen-Southwell scheme.

Today many stabilization techniques exist; see [17]. Stability related to M-matrix properties is restricted to low-order methods. Higher-order methods are often stable with respect to a “stronger” norm (see [17]). For such methods one cannot avoid oscillations completely, but the oscillations are restricted to a small region near the layer.

*Example 2.2* (the error behavior near the layer). For a standard problem one expects that the error of a discretization method decreases as the mesh size  $h$  decreases. But for singularly perturbed problems this is not necessarily the case if  $\varepsilon$  is fixed and  $h$  varies.

Consider the problem

$$(2.7) \quad -\varepsilon u'' - u' = 2x \quad \text{in } (0, 1), \quad u(0) = u(1) = 0,$$

with  $\varepsilon = 10^{-6}$ . We apply three different finite difference methods. The first is again the first-order upwind difference method

$$-\frac{\varepsilon}{h^2}(u_{i-1} - 2u_i + u_{i+1}) - \frac{u_{i+1} - u_i}{h} = 2ih.$$

The second method is the second-order upwind scheme from [18]:

$$-\frac{\varepsilon}{h^2}(u_{i-1} - 2u_i + u_{i+1}) - \frac{u_{i+1} - u_{i-1}}{2h} + \frac{\lambda}{h}(-u_{i-1} + 3u_i - 3u_{i+1} + u_{i+2}) = 2ih$$

with  $\lambda = \frac{1}{2} - \frac{\varepsilon}{h}$ . Finally, the third method is an exponentially fitted scheme, the Il'in-Allen-Southwell method,

$$-\frac{\varepsilon}{h^2}\sigma(q)(u_{i-1} - 2u_i + u_{i+1}) - \frac{u_{i+1} - u_{i-1}}{2h} = 2ih$$

with  $\sigma(q) = q \coth(q)$  and  $q = h/(2\varepsilon)$ .

Figure 2.3 shows the error behavior at the first mesh node for the three methods applied to (2.7). On standard meshes, this behavior is typical for all schemes that are stabilized but not exponentially fitted.

When first-order upwind differencing is applied to a boundary value problem with an exponential layer at  $x = \alpha$ , the typical error behavior at the mesh node  $x_i$  if  $\varepsilon \leq h$  is of the type

$$C_1 h + C_2 \exp\left(-\gamma \frac{x_i - \alpha}{\varepsilon}\right)$$

with positive constants  $C_1$ ,  $C_2$ , and  $\gamma$ . Therefore, the error at the first mesh point is  $\mathcal{O}(1)$  if  $h = \varepsilon$ , and for a certain range of  $h$  (and fixed  $\varepsilon$ ) one finds that the error increases as  $h$  decreases.

Of course, the situation is different for uniformly convergent schemes; these satisfy

$$(2.8) \quad |(u - u_h)(x_i)| \leq Ch^\gamma$$

with some  $C$  independent of  $\varepsilon$  and  $h$ . Then, for fixed  $\varepsilon$ , the error decreases as  $h$  decreases; see the results for the Il'in–Allen–Southwell method (for which  $\gamma = 1$ ) in Figure 2.3.

Unfortunately, the construction of exponentially fitted schemes in two dimensions is difficult and, in the case of characteristic boundary layers, is impossible; see [17]. Consequently, one should use layer-adapted meshes if one is interested in achieving high accuracy in layer regions. We shall discuss such meshes in section 3.

*Example 2.3* (higher-order Galerkin methods). Consider the boundary value problem

$$(2.9) \quad -\varepsilon u'' - u' + u = 1 \quad \text{in } (0, 1), \quad u(0) = u(1) = 0,$$

and its finite element discretization: using some finite element space  $V_h \subset H_0^1(0, 1)$ , we seek  $u_h \in V_h$  such that

$$(2.10) \quad a(u_h, v_h) = (f, v_h) \quad \text{for all } v_h \in V_h,$$

where

$$(2.11) \quad a(w, v) := \varepsilon \int_0^1 w' v' + \int_0^1 (-w' + w)v \quad \text{and} \quad (f, v) := \int_0^1 v.$$

While for linear finite elements we observe oscillations as in central differencing—see Figure 2.1 for a similar problem—higher-order elements behave differently. It turns out that the linear component of the Galerkin solution is satisfactory and the oscillations occur only in the higher-order modes—see Figure 2.4, where we used 40 elements and  $\varepsilon = 10^{-3}$ .

The behavior of the Galerkin method with quadratic elements was first explained in [4] by decomposing the solution into

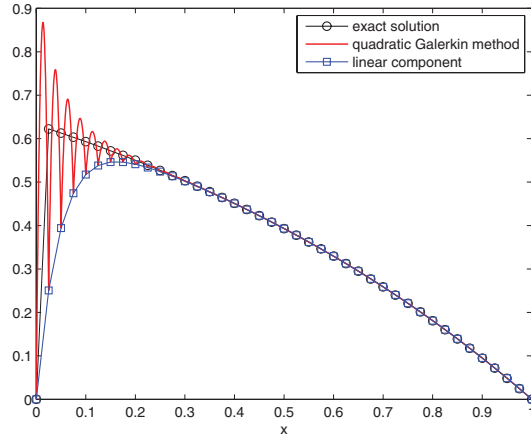
$$u_h = u_L + u_B,$$

where  $u_L$  denotes the linear part of the approximation and  $u_B$  a quadratic bubble part. Elimination of the bubble part leads to a discrete problem of the form

$$(2.12) \quad a(u_L, v_L) + S(u_L, v_L) = (f, v_L).$$

For instance, for the differential operator  $-\varepsilon u'' + \beta u'$  with constant coefficients, the stabilization term  $S(\cdot, \cdot)$ , when written in the form of a difference operator, is

$$\frac{\beta^2 h^2}{12\varepsilon} \frac{u_{i+1} - 2u_i + u_{i-1}}{h}.$$



**Fig. 2.4** Galerkin method (2.10) with quadratic elements, its linear component, and the exact solution.

This stabilization term accounts for the smooth but smeared curve (plain solid line) in Figure 2.4.

For the standard Galerkin method and higher-order elements it turns out that stability is already guaranteed for certain subspaces of the finite element space, and only some high-frequency modes have to be stabilized [10]. Streamline diffusion can be used to stabilize the oscillating solution obtained using a Galerkin method with linear elements. This stabilization technique can be related to the bubble stabilization presented above for higher-order elements, and to local projection stabilization [26].

*Example 2.4* (weak imposition of Dirichlet boundary conditions). We revisit the solution of the boundary value problem (2.9) by Galerkin methods with linear elements. In the standard approach the essential Dirichlet conditions are strongly satisfied for functions  $v_h \in V_h$  from the finite element space, i.e.,  $v_h(0) = v_h(1) = 0$ .

Is there an alternative? Already in 1971 Nitsche [14] proposed that one should satisfy the boundary conditions only weakly. To be precise, let  $V_h$  be the space of linear finite elements on the given mesh without the boundary conditions. Then integration by parts yields

$$(2.13) \quad -\varepsilon \int_0^1 w'' v = -\varepsilon \int_0^1 w' v' - \varepsilon w' v|_0^1$$

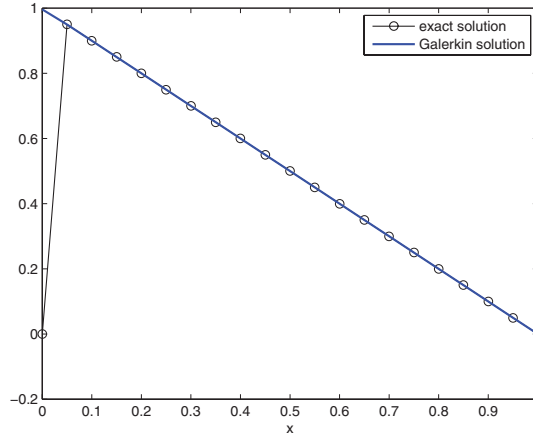
and now the second term does not vanish. Moreover, we have difficulties with coercivity because

$$-\int_0^1 v' v = -\frac{1}{2} v^2|_0^1.$$

We therefore add to the bilinear form  $a(\cdot, \cdot)$  the Nitsche quantity

$$n(w, v) := -\varepsilon w' v|_0^1 - \varepsilon w v'|_0^1 + \frac{\varepsilon \gamma}{h} (w(1)v(1) + v(0)w(0)) + w(1)v(1),$$

where the first term is the missing term in  $a(\cdot, \cdot)$  of (2.13), the second term is its symmetric counterpart that is zero for the exact solution, the third term helps us to



**Fig. 2.5** Galerkin method (2.14) with weakly imposed boundary conditions.

enforce the boundary conditions using a penalty parameter  $\gamma$ , and the last one ensures coercivity. The new weak formulation reads as follows: Find  $u_h \in V_h$  with

$$(2.14) \quad a(u_h, v_h) + n(u_h, v_h) = (f, v_h) \quad \text{for all } v_h \in V_h.$$

This new bilinear form has improved stability properties. It is coercive with respect to a stronger norm,

$$\|v\|_{Ni}^2 := \varepsilon \|v'\|_0^2 + \|v\|_0^2 + \frac{\varepsilon\gamma}{h}(v^2(1) + v^2(0)) + v^2(1),$$

where we have used the  $L_2$ -norm  $\|\cdot\|_0$ .

Surprisingly, with weakly imposed Dirichlet boundary conditions one obtains reasonable results even for linear finite elements. We remark that it is sufficient to impose weak boundary conditions at the boundary layer; see Figure 2.5, where we used  $\gamma = 2$  and  $\varepsilon = 10^{-4}$ . An error estimate in this case [19] uses a splitting

$$u - u_h = (u - S) + (S - S^I) + (S^I - u_h)$$

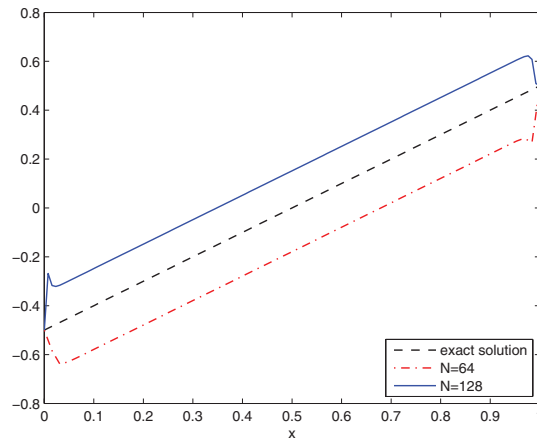
of the error, where  $S$  is the smooth component of the solution decomposition and  $S^I$  its piecewise linear interpolant. Because there are no boundary values incorporated in  $V_h$ , we have the key relation  $S^I \in V_h$  and all three error components can be estimated.

Weak imposition of the boundary conditions also improves high-order Galerkin methods and stabilized methods. Its use in continuous interior penalty stabilization is discussed in [19]. For applications in fluid mechanics, see [3].

*Example 2.5* (upwind method fails). Consider the boundary value problem

$$(2.15) \quad -\varepsilon u'' + \left(x - \frac{1}{2}\right) u' = x - \frac{1}{2}, \quad u(0) = A, u(1) = B.$$

This type of problem is different from the examples studied so far: we have a so-called turning point problem because the coefficient  $(x - 1/2)$  has a zero in the interval studied. The signs of  $(x - 1/2)$  for  $x = 0$  and  $x = 1$  imply that we expect layers



**Fig. 2.6** Upwind scheme for (2.15) with  $N = 64$  and  $N = 128$  and  $\varepsilon = 1e - 3$ .

at  $x = 0$  and  $x = 1$ . However, the solution of the reduced problem (which does not satisfy any boundary condition) is  $u_0(x) = x + C$  with an unknown constant  $C$ . For the determination of  $C$ , see [6, Chapter 7, section 3.4]. This situation is known as “resonance” because an eigenvalue of the differential operator tends to zero as  $\varepsilon \rightarrow 0$ .

Consider (2.15) with  $A = -1/2$ ,  $B = 1/2$ , and the solution  $u(x) = x - 1/2$ , so that no boundary layer is present. One expects the upwind method to work for this problem. But for several combinations of  $N$  and  $\varepsilon$  we obtain discrete solutions that are wrong: the error at the interior mesh points is of order  $\mathcal{O}(1)$  (see Figure 2.6, where  $\varepsilon = 10^{-3}$  and  $N = 64$  and  $128$ ).

Surprisingly, in [24] uniform convergence of the Il’in–Allen–Southwell scheme was proved for a similar problem, namely,

$$-\varepsilon u'' + xu' = 0, \quad u(-1) = A, u(1) = B,$$

which has reduced solution  $u_0 = C = (A + B)/2$  and two boundary layers. But this theoretical result has limited practical value because the conditioning of the discrete scheme degrades as  $\varepsilon$  decreases. For instance, with  $\varepsilon = 0.01$  it was not possible to solve the discrete problem for  $N > 24$  even in double precision; see [24, 25]. Quadruple precision improved the situation somewhat.

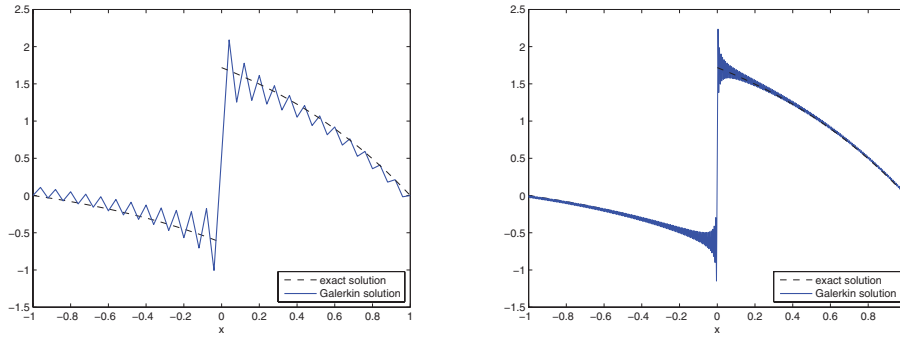
How to handle numerically such ill-conditioned boundary value problems is more or less an open problem.

*Example 2.6* (oscillations in the vicinity of a shock layer: is there a Gibbs phenomenon?). Let us study the pure convection problem

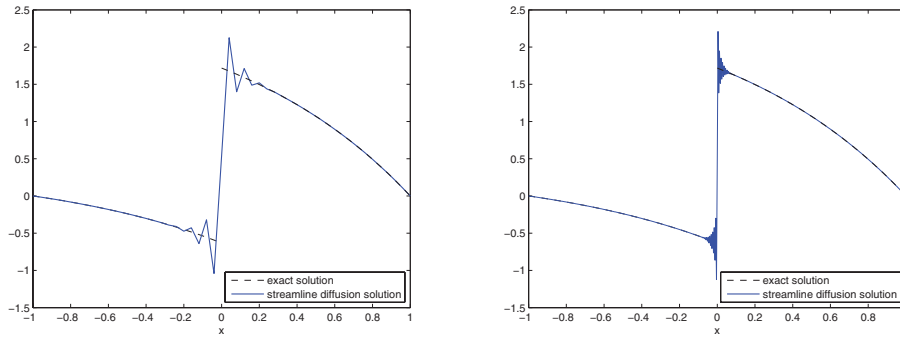
$$(2.16) \quad b(x)u' + c(x)u = f(x) \quad \text{in } (\alpha, \beta)$$

with the condition  $u|_{\Gamma_-} = 0$ . Here  $\Gamma_-$  denotes the inflow boundary:  $\alpha \in \Gamma_-$  if  $b(\beta) < 0$ , while  $\beta \in \Gamma_+$  if  $b(\alpha) > 0$ . If the condition  $c - b'/2 \geq \mu > 0$  is satisfied, the problem (2.16) has a unique solution in  $V = \{v \in L_2 : bv' \in L_2 \text{ and } v|_{\Gamma_-} = 0\}$ .





**Fig. 2.7** Galerkin method for (2.17) with  $N = 50$  (left) and  $N = 500$  (right).



**Fig. 2.8** Streamline diffusion for (2.17) with  $N = 50$  (left) and  $N = 500$  (right).

If we use linear finite elements to solve (2.16), we observe the usual oscillations for convection-diffusion problems. Moreover, it is well known that the standard Galerkin method cannot produce optimal error estimates. As early as 1984, the authors of [9] proposed the use of streamline diffusion or discontinuous Galerkin methods instead of the standard method. If the solution of the given problem is sufficiently smooth, then similar error estimates can be proved for both methods.

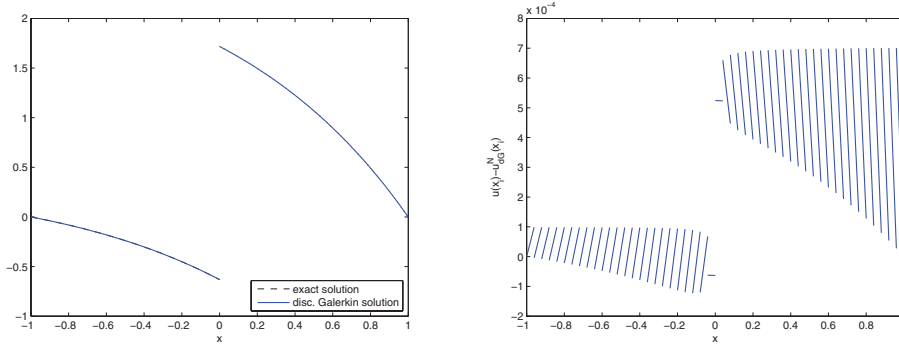
Here we want to compare a Galerkin method using linear elements, with and without streamline diffusion, and a discontinuous Galerkin method for the problem

$$(2.17) \quad -xu' = xe^x \quad \text{in } (-1, 1) \quad \text{with } u(-1) = u(1) = 0.$$

The solution is characterized by a shock at  $x = 0$ .

As is seen in Figures 2.7 and 2.8, for standard Galerkin and streamline diffusion there are oscillations in the discrete solution, while for streamline diffusion the oscillations are restricted to a neighborhood of the shock layer. Nevertheless, the oscillations around the discontinuity are not reduced by increasing  $N$ ; in this sense we have a Gibbs phenomenon for the streamline diffusion method.

For discontinuous Galerkin, on the other hand, the situation is completely different: the oscillations tend to zero as the mesh size tends to zero—see Figure 2.9. Consequently, for problems whose solutions exhibit discontinuities we recommend the use of the discontinuous Galerkin method [15].



**Fig. 2.9** Discontinuous Galerkin for (2.17) with  $N = 50$  (left) and the error (right).

**3. Layer-Adapted Meshes.** Assume that the linear convection-diffusion problem (2.1) is posed in the interval  $(0, 1)$  and has exactly one (boundary) layer at  $x = 0$  that is characterized by the function  $\exp(-\gamma x/\varepsilon)$ . Then it is natural to condense the mesh in the layer region near  $x = 0$ . In 1969 Bakhvalov [2] introduced a mesh that was defined by a mapping onto a scaled boundary layer function and was graded inside the layer region. In the early 1990s Shishkin [20] (see also the recent [21] for finite differences on piecewise equidistant meshes) proposed a much simpler piecewise equidistant mesh. For an overview of layer-adapted meshes we recommend the recent survey [12].

We refer to the fine mesh as “the nodes in the layer region” and the coarse mesh as “the nodes outside the layer region.” We consider both layer-adapted meshes in this section and define them now.

A Bakhvalov-type mesh has mesh points  $x_i = \lambda(i/N)$  for  $i = 0, 1, \dots, N$  defined by the mesh-generating function

$$(3.1) \quad \lambda(t) = \begin{cases} \psi(t) := -A\varepsilon \ln\left(1 - \frac{t}{q}\right) & \text{for } t \in [0, \tau], \\ \pi(t) := \psi(\tau) + (t - \tau)\psi'(\tau) & \text{for } t \in [\tau, 1], \end{cases}$$

with  $q$  roughly the portion of mesh points used to resolve the layer and  $A$  a constant that depends on the underlying problem. For the standard Bakhvalov mesh the point  $\tau$  satisfies

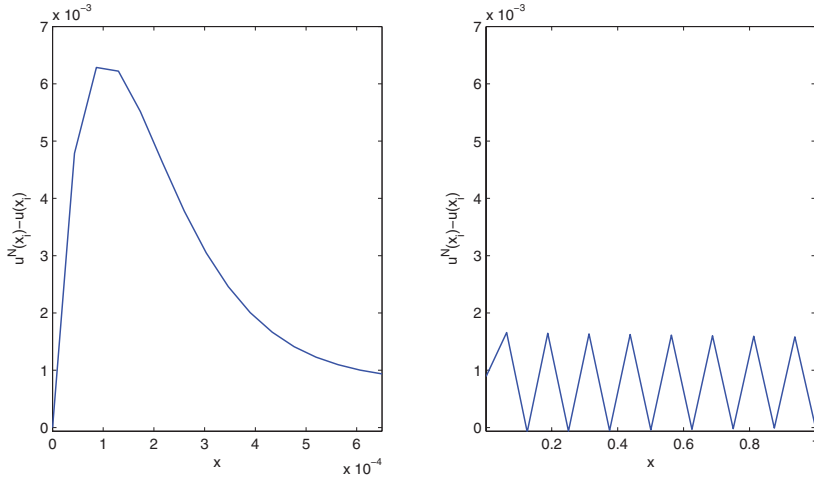
$$\psi'(\tau) = \frac{1 - \psi(\tau)}{1 - \tau}.$$

This nonlinear equation cannot be solved explicitly, but the approximation  $\tau = q - A\varepsilon$  gives Bakhvalov-type meshes with similar properties; see [1]. For the following examples we use this definition of  $\tau$  with  $q = 1/2$  and  $A = 2$ .

Shishkin’s piecewise equidistant meshes are much simpler. They are defined by

$$(3.2) \quad x_i = \begin{cases} ih & \text{for } i = 0, 1, \dots, N/2, \\ 1 - (N - i)H & \text{for } i = N/2 + 1, \dots, N, \end{cases}$$

with  $h = \frac{2\sigma}{N}$ ,  $H = 2(1 - \sigma)/N$ , and  $\sigma = \sigma_0\varepsilon \ln N$ . Typically,  $\sigma_0$  is equal to the formal



**Fig. 3.1** Errors for central differences on a Shishkin mesh: fine mesh (left) and coarse mesh (right).

order of the numerical method or is chosen to accommodate the error analysis. We will use  $\sigma_0 = 2$ .

In one dimension, many convergence results for several discretization methods on these (and other layer-adapted meshes) are known, while in two dimensions we have results mainly for Shishkin and Shishkin-type meshes [12, 17]. For instance, central differencing or linear finite elements satisfy in one dimension the following error estimate, which is uniform with respect to  $\varepsilon$ :

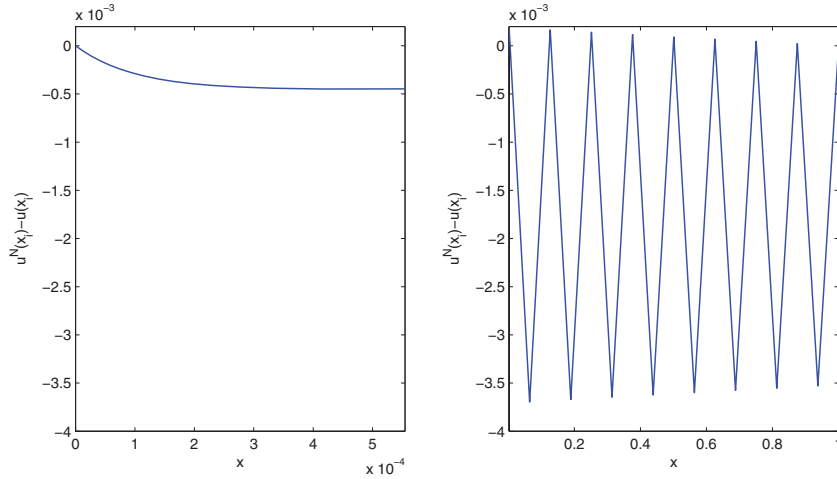
$$(3.3) \quad \max_{i=0,\dots,N} |u(x_i) - u^N(x_i)| \leq C \begin{cases} N^{-2} & \text{for a Bakhvalov-type mesh,} \\ (N^{-1} \ln N)^2 & \text{for a Shishkin mesh.} \end{cases}$$

*Example 3.1* (oscillations on layer-adapted meshes). For central differencing on layer-adapted meshes one loses the M-matrix property of the discrete problem, which is the basis of stability for some upwind techniques. Nevertheless, the use of layer-adapted meshes induces some additional stability. In fact, proofs of (3.3) are based on the  $(l_\infty, l_1)$  or  $(l_\infty, w_{-1,\infty})$  stability of the discrete operator; see [12].

Although graphs of the numerical solutions generated by central differencing on layer-adapted meshes look satisfactory, the computed solutions have small oscillations on the coarse mesh; see Figures 3.1 and 3.2.

While for an S-mesh the error is smaller outside the layer than inside the layer, for a Bakhvalov mesh the maximum error is obtained on the coarse mesh (see Figures 3.1 and 3.2 for problem (2.3) with  $N = 32$  and  $\varepsilon = 10^{-6}$ ).

We remark that an unorthodox way of extracting useful information from the oscillatory central difference solution is described and analyzed in [22]. Using central differencing, solve the discrete problem on an equidistant mesh  $\{x_i\}_0^N$ , then add an extra grid point in the mesh interval  $(0, x_1)$ , and solve the problem again; for each of these two (oscillatory) computed solutions, join the nodal values by straight lines (i.e., construct the piecewise linear finite element solution); finally, find the points  $(\hat{x}_i, \hat{y}_i)$  where these two piecewise linear solutions intersect. Then one has  $x_{i-1} \leq \hat{x}_i \leq x_i$  for all  $i$  and  $\max_{0 \leq i \leq N-1} |u(\hat{x}_i) - \hat{y}_i| \leq CN^{-2}$ .



**Fig. 3.2** Errors for central differences on a Bakhvalov mesh: fine mesh (left) and coarse mesh (right).

Summarizing, on layer-adapted meshes one can use standard discretization techniques, but some small oscillations still appear in the discrete solution. Additional stabilization improves the situation. See [16] for a survey of stabilizations on layer-adapted meshes.

*Example 3.2* (order reduction for S-meshes that are not locally almost equidistant). Now suppose that we have a modified Shishkin mesh for handling a convection-diffusion problem with an exponential boundary layer at  $x = 0$ , namely,

$$(3.4) \quad x_i = \begin{cases} ih & \text{for } i = 0, \dots, N/2, \\ \text{arbitrary} : \sigma_0 \varepsilon \ln N = x_{N/2} < x_{N/2+1} < \dots < x_N = 1, \end{cases}$$

with  $h = 2\sigma_0 \varepsilon \ln N/N$ , where  $\sigma_0 = 2$  is chosen in our experiments.

Does the nonuniformity of the *coarse* mesh influence the convergence rate for linear finite elements?

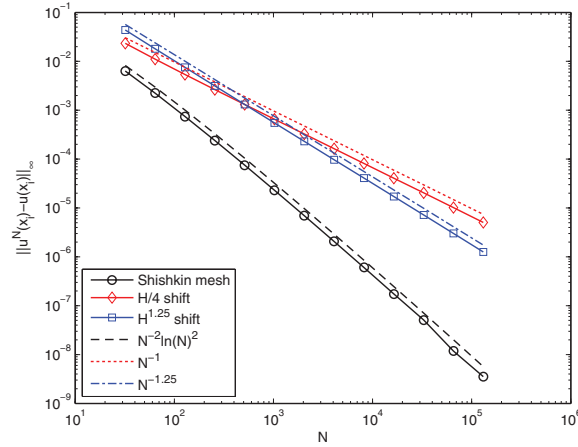
At first sight, error estimates in the energy norm, supercloseness results, and  $L_2$ -norm error estimates give the impression that this is not the case. But a careful study of the pointwise error [5] shows that the convergence rate can be  $N^{-\delta}$  if the coarse mesh sizes  $H_i$  hold,  $|H_i - H_{i+1}| = \mathcal{O}(N^{-\delta})$ . A locally almost equidistant mesh is a mesh for which  $\delta = 2$ . A mesh  $\{\hat{x}_i\}$  with  $\delta = 1$  can be generated by taking a standard Shishkin mesh  $\{x_i\}$  and moving all even-index inner grid points of the coarse mesh a distance  $H/4$  to the right, i.e.,

$$\hat{x}_{2i} := x_{2i} + H/4,$$

and a mesh  $\{\tilde{x}_i\}$  with  $\delta \in (1, 2)$  can be constructed in a similar manner by moving the even-index inner grid points of the coarse part of a Shishkin mesh:

$$\tilde{x}_{2i} := x_{2i} + H^\alpha$$

with  $\alpha = \delta$ .



**Fig. 3.3** Errors for linear finite elements on perturbed Shishkin meshes.

We consider the test problem

$$\begin{aligned} -\varepsilon u'' - u' &= -2\varepsilon - 2x, \\ u(0) &= u(1) = 0, \end{aligned}$$

whose exact solution is  $u = x^2 - 1 + \frac{\exp(-x/\varepsilon) - \exp(-1/\varepsilon)}{1 - \exp(-1/\varepsilon)}$ .

For the numerical experiment we use  $\varepsilon = 10^{-8}$ , the standard Shishkin mesh  $\{x_i\}$ , and the shifted meshes  $\{\hat{x}_i\}$  and  $\{\tilde{x}_i\}$  with  $\alpha = 5/4$  as described above. Figure 3.3 shows the corresponding error curves, and we observe a reduced order of convergence for the perturbed meshes.

For stabilized methods one would expect better behavior. In fact, for streamline diffusion the convergence rate is insensitive to the perturbation of the grid; see [5].

Thus, nonstabilized methods for convection-diffusion problems have at least two drawbacks: the methods are relatively sensitive with respect to perturbations of the grid, and it is difficult to solve efficiently the discrete problems generated.

*Example 3.3* (parasitic and inaccurate solution for a nonlinear problem with an interior layer). For nonlinear problems the reduced equation may have more than one solution. It is then unclear which of these is the correct limit (as  $\varepsilon \rightarrow 0$ ) of the exact solution in a given subinterval and where a transition from one reduced solution to another might take place.

In [7] Howes discusses in detail the analytical behavior of solutions of

$$-\varepsilon^2 u'' + F(x, u, u') = 0, \quad u(\alpha) = A, \quad u(\beta) = B,$$

for the cases  $F(x, u, u') = c(x, u)$  and  $F(x, u, u') = f(x, u)u' + g(x, u)$  and also the quadratic case  $F(x, u, u') = p(x, u)u'^2 + q(x, u)u' + r(x, u)$ .

Here we consider the semilinear reaction-diffusion problem

$$(3.5) \quad -\varepsilon^2 u'' + c(x, u) = 0, \quad u(\alpha) = A, \quad u(\beta) = B.$$

In particular, we are interested in having a shock layer at an interior point  $x^*$  of  $(\alpha, \beta)$ , for which sufficient conditions are given by [7, Theorem 4.4]. Without presenting all

the details, sufficient conditions are characterized by the existence of two solutions  $u_L$  and  $u_R$  of the reduced problem, which are in some sense stable and are such that

$$(3.6) \quad u(x) = \begin{cases} u_L + \chi_L + \mathcal{O}(\varepsilon) & \text{for } \alpha \leq x \leq x^*, \\ u_R + \chi_R + \mathcal{O}(\varepsilon) & \text{for } x^* \leq x \leq \beta; \end{cases}$$

here,  $\chi_L$  and  $\chi_R$  are shock layer functions at  $x = x^*$ , while  $u_L$  satisfies  $u_L(\alpha) = A$  and  $u_R$  satisfies  $u_R(\beta) = B$ . Moreover, assuming  $u_L < u_R$ , the functional

$$J(x) = \int_{u_L(x)}^{u_R(x)} c(x, s) ds$$

has to have certain properties. The important point  $x^*$  is characterized by  $J(x^*) = 0$ .

A standard example from the literature (see, e.g., [11]) is the boundary value problem

$$(3.7) \quad -\varepsilon^2 u'' + u(u-1) \left( u - \left( x + \frac{3}{2} \right) \right) = 0, \quad u(0) = 0, \quad u(1) = \frac{5}{2}.$$

Here,  $u_L \equiv 0$  satisfies  $u_L(0) = 0$  and  $c_u(x, u_L) = x + 3/2 \geq 3/2$ , which guarantees its stability. Furthermore,  $u_R = x + 3/2$  satisfies  $u_R(1) = 5/2$  and  $c_u(x, u_R) = (x + 3/2)(x + 1/2) \geq 3/4$ . The reduced solution  $u \equiv 1$  is unstable. It is easy to see that

$$J(x) = \int_0^{x+3/2} s(s-1) \left( s - \left( x + \frac{3}{2} \right) \right) ds = \left( x + \frac{3}{2} \right)^3 \left( \frac{1}{2} - x \right) / 12$$

has exactly one zero in  $(0, 1)$ :  $x^* = 1/2$ . The shock layer is located here.

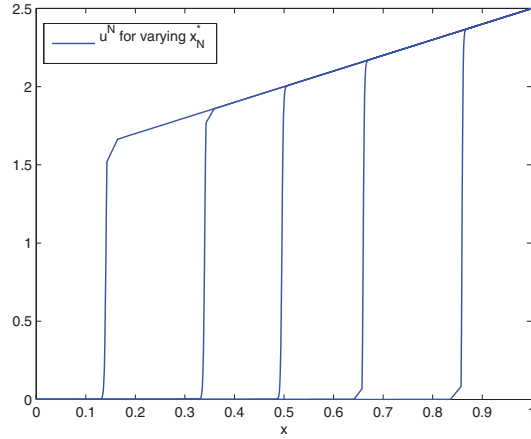
Now we solve (3.7) numerically using the central difference scheme on a Shishkin mesh for the shock layer region. For a given value  $x_N^* \in (0, 1)$  a fine equidistant mesh using  $N/2$  intervals is defined on  $[x_N^* - \lambda, x_N^* + \lambda]$  with  $\lambda = 4.2\varepsilon \ln N$ , following the suggestions in [11]. Outside this interval a coarse equidistant mesh is constructed.

The first surprising fact is that if one centers the fine mesh at any point  $x_N^*$  in  $(0, 1)$ , the computed solution has an interior shock layer around that point, as shown in Figure 3.4. In this sense the position of the fine mesh selects a particular solution from the list of possible solutions because of the nonuniqueness property of these nonlinear problems.

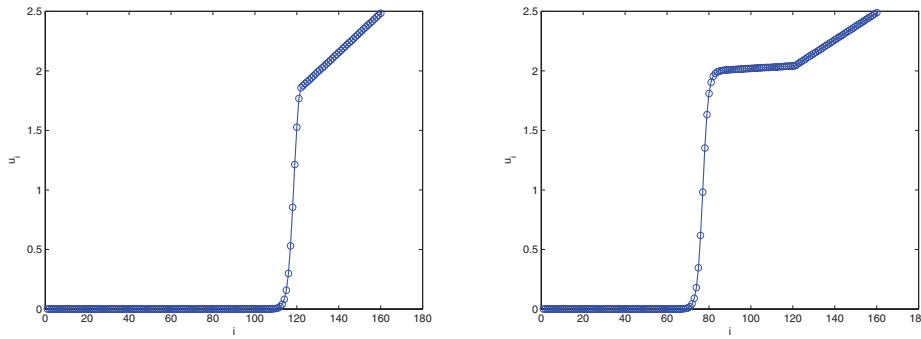
If we stretch the numerical solution, we see a completely different behavior of the numerical solutions for the different choices of  $x_N^*$ ; see Figure 3.5 for  $N = 160$  and  $\varepsilon = 2 \cdot 10^{-3}$ . In these examples the fine mesh starts at  $i = 40$  and ends at  $i = 120$ . For  $x_N^* = 0.3$  the transition between  $u_L$  and  $u_R$  starts at the right end of the fine mesh, at which  $x \approx 0.34$ , but for  $x_N^* = 0.5$  the transition is in the middle of the fine mesh.

In order to solve the discrete nonlinear system we use Newton's method. It turns out that the choice of the initial solution  $u_0$  is extremely important. The use of a linear interpolant of the boundary values yields no convergence. We use instead  $u_0 = u_R$  to the right of the fine mesh and  $u_0 = u_L$  elsewhere.

For  $x_N^* = 0.5$  we observe extremely slow convergence of Newton's method. This differs from the case  $x_N^* \neq 0.5$ .



**Fig. 3.4** Numerical solutions for (3.7) using  $x_N^* = 0.1, 0.3, 0.5, 0.7$ , and  $0.9$  from left to right.



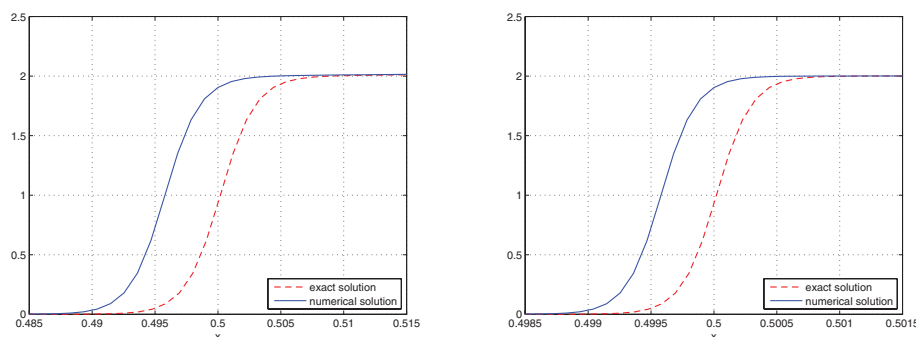
**Fig. 3.5** Stretched numerical solutions for (3.7) using  $x_N^* = 0.3$  (left) and  $x_N^* = 0.5$  (right).

Even if the layer-adapted mesh is correctly located (for our example, centered on  $x_N^* = x^* = 0.5$ ), nevertheless the computed solution can be inaccurate. In [11] it is proved that for a certain range of  $N$  and  $\varepsilon$  there exists a constant  $\tilde{C}$  such that

$$|u_i^N - u(x_i - \tilde{C}\varepsilon)| \leq C(N^{-1} \ln N)^2.$$

Consequently, the error in the layer region can be  $\mathcal{O}(1)$  since the computed solution lies close only to a translation of the exact solution. This observation is the basis for a postprocessing procedure [11] that generates an accurate numerical solution. In our numerical examples,  $\tilde{C} \approx 2.2$  and we can construct an approximation to the exact solution by shifting the numerical solution  $2.2\varepsilon$  to the right; see Figure 3.6 for a close-up of the numerical solutions around  $x_N^* = x^* = 0.5$ .

For nonlinear singularly perturbed problems, at present there are few results in the literature regarding robust numerical solution methods. In particular, interior layers cause enormous difficulties. The examples in [27] show that this is unsurprising: some nonlinear problems exhibit exotic analytical phenomena that are completely different from the properties of linear problems.



**Fig. 3.6** Numerical solutions and shifted ones for (3.7) and  $\varepsilon = 2 \cdot 10^{-3}$  (left),  $\varepsilon = 2 \cdot 10^{-4}$  (right).

**Conclusion.** Singularly perturbed boundary value problems are often used as test problems for checking the quality of numerical methods.

As our examples show, on standard locally uniform meshes it is difficult to guarantee high accuracy near the layers, even for 1D problems. While for some linear 1D problems it is still possible to construct uniformly convergent schemes (such as the Il'in–Allen–Southwell scheme), in two dimensions or for nonlinear problems one should use layer-adapted meshes. In some practical applications it is sufficient to obtain accurate solutions far from the layers—then stabilized methods such as streamline diffusion or discontinuous Galerkin work well.

In the last ten years the analysis of discretization methods on layer-adapted meshes has made some progress insofar as linear problems with straightforward layer structures can be analyzed—see [12, 17]. But many open problems remain. For instance, the interaction of singularities and layers is not well understood at present. Moreover, the layer-adapted meshes used are constructed a priori but not a posteriori; all existing adaptive codes in two dimensions have difficulty in detecting the layers in convection-diffusion problems that have different layers (see the examples in [8]).

We hope that our paper encourages the reader to enter the fascinating world of singularly perturbed problems and to attack in the near future some of the unsolved problems that we have described.

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#### REFERENCES

- [1] V. B. ANDREEV AND N. KOPTEVA, *On the convergence, uniform with respect to a small parameter, of monotone three-point finite-difference approximations*, Differ. Equ., 34 (1998), pp. 921–929.
- [2] N. BAKHVALOV, *Towards optimization of methods for solving boundary value problems in the presence of boundary layers*, Zh. Vychisl. Mat. i Mat. fiz., 9 (1969), pp. 841–859 (in Russian).
- [3] Y. BAZILEVS AND T. HUGHES, *Weak imposition of Dirichlet boundary conditions in fluid mechanics*, Comput. & Fluids, 36 (2007), pp. 12–26.
- [4] F. BREZZI AND A. RUSSO, *Choosing bubbles for advection-diffusion problems*, Math. Models Methods Appl. Sci., 4 (1994), pp. 571–587.



- [5] L. CHEN AND J. XU, *An optimal streamline diffusion finite element method for a singularly perturbed problem*, in Recent Advances in Adaptive Computation, Contemp. Math. 383, AMS, Providence, RI, 2005, pp. 236–246.
- [6] E. M. DE JAGER AND J. F. FURU, *The Theory of Singular Perturbation*, North-Holland Ser. Appl. Math. Mech., Elsevier, Amsterdam, 1996.
- [7] F. HOWES, *Boundary-interior layer interactions in nonlinear singular perturbation theory*, Mem. Amer. Math. Soc., 15 (1978), no. 203.
- [8] V. JOHN, *A computational study of a posteriori error estimators for convection-diffusion problems*, Comput. Methods Appl. Mech. Engrg., 190 (2000), pp. 757–781.
- [9] C. JOHNSON, U. NÄVERT, AND J. PITKÄRANTA, *Finite element methods for linear hyperbolic problems*, Comput. Methods Appl. Mech. Engrg., 45 (1984), pp. 285–312.
- [10] P. KNOBLOCH AND L. TOBISKA, *On the stability of finite-element discretizations of convection-diffusion-reaction equations*, IMA J. Numer. Anal., doi:10.1093/imanum/drp020 (2009).
- [11] N. KOPTEVA AND M. STYNES, *Stabilised approximation of interior-layer solutions in a singularly perturbed semilinear reaction-diffusion problem*, submitted.
- [12] T. LINSS, *Layer-Adapted Meshes for Reaction-Convection-Diffusion Problems*, Lecture Notes in Math. 1985, Springer, Berlin, Heidelberg, 2010.
- [13] K. W. MORTON, *Numerical Solution of Convection-Diffusion Problems*, Chapman & Hall, London, 1996.
- [14] J. NITSCHKE, *Über ein Variationsprinzip zur Lösung von Dirichlet-Problemen bei Verwendung von Teilräumen, die keinen Randbedingungen unterworfen sind*, Abh. Math. Univ. Hamburg, 36 (1971), pp. 9–15.
- [15] B. RIVIÈRE, *Discontinuous Galerkin Methods for Solving Elliptic and Parabolic Equations: Theory and Implementation*, Frontiers Appl. Math. 35, SIAM, Philadelphia, 2008.
- [16] H.-G. ROOS, *Stabilized FEM for convection-diffusion problems on layer-adapted meshes*, J. Comput. Math., 27 (2009), pp. 266–279.
- [17] H.-G. ROOS, M. STYNES, AND L. TOBISKA, *Robust Numerical Methods for Singularly Perturbed Differential Equations*, Springer Ser. Comput. Math., Springer, Berlin, Heidelberg, 2008.
- [18] H.-G. ROOS AND R. VANSELOW, *A comparison of four- and five-point difference approximations for stabilizing the one-dimensional stationary convection-diffusion equation*, Electron. Trans. Numer. Anal., 32 (2008), pp. 63–75.
- [19] F. SCHIEWECK, *On the role of boundary conditions for the CIP stabilization of higher order finite elements*, Electron. Trans. Numer. Anal., 32 (2008), pp. 1–16.
- [20] G. SHISHKIN, *Grid Approximation of Singularly Perturbed Elliptic and Parabolic Equations*, Second doctoral thesis, Keldysh Institute, Moscow, 1990 (in Russian).
- [21] G. SHISHKIN AND L. SHISHKINA, *Difference Methods for Singularly Perturbed Problems*, Chapman and Hall, Boca Raton, FL, 2009.
- [22] Q. SONG, G. YIN, AND Z. ZHANG, *An  $\varepsilon$ -uniform finite element method for singularly perturbed boundary value problems*, Int. J. Numer. Anal. Model., 4 (2007), pp. 128–141.
- [23] M. STYNES, *Numerical methods for convection-diffusion problems or the 30 years war*, in Proceedings of the 20th Biennial Conference on Numerical Analysis (Dundee), Numerical Analysis Report NA/217, D. F. Griffiths and G. A. Watson, eds., University of Dundee, Dundee, Scotland, 2003, pp. 95–103.
- [24] X. SUN, *Numerical analysis of an exponentially ill-conditioned boundary value problem with applications to metastable problems*, IMA J. Numer. Anal., 21 (2001), pp. 817–842.
- [25] X. SUN AND M. WARD, *Numerical analysis of an exponentially ill-conditioned boundary value problem with applications to metastable problems*, Methods Appl. Anal., 8 (2001), pp. 333–347.
- [26] L. TOBISKA, *On the relationship of local projection stabilization to other stabilized methods for one-dimensional advection-diffusion equations*, Comput. Methods Appl. Mech. Engrg., 198 (2009), pp. 831–837.
- [27] W. WASOW, *The capriciousness of singular perturbations*, Nieuw Arch. Wisk., 18 (1970), pp. 190–210.