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Towards a deterministic KPZ equation with fractional diffusion: the stationary problem*

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Abstract

In this work, we investigate by analysis the possibility of a solution to the fractional quasilinear problem:

$$\begin{cases} (-\Delta)^s u &= |\nabla u|^q + \lambda f & \text{in } \Omega, \\ u &= 0 & \text{in } \mathbb{R}^N \setminus \Omega, \\ u &> 0 & \text{in } \Omega, \end{cases}$$

where $\Omega \subset I\!R^N$ is a bounded regular domain ($\mathcal{C}^2$ is sufficient), $s \in (\frac{1}{2}, 1)$, $1 < q$ and f is a measurable non-negative function with suitable hypotheses.

The analysis is done separately in three cases: subcritical, $1 < q < 2s$; critical, $q = 2s$; and supercritical, $q > 2s$.

Keywords: fractional diffusion, nonlinear gradient terms, stationary KPZ equation
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1. Introduction

The aim of this paper is to discuss the existence of a weak solution of the following nonlocal elliptic problem with gradient term:

$$\begin{cases} (-\Delta)^s u &= |\nabla u|^q + \lambda f & \text{in } \Omega, \\ u &= 0 & \text{in } \mathbb{R}^N \setminus \Omega, \\ u &> 0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset I\!R^N$ is a bounded regular domain ($\mathcal{C}^2$ is sufficient), $s \in (\frac{1}{2}, 1)$, $1 < q$ and f is a measurable non-negative function. Here, $(-\Delta)^s$ means the classical fractional Laplacian operator of order $2s$ defined by

$$(-\Delta)^s u(x) := a_{N,s} \text{ P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} \, \mathrm{d}y, \; s \in (0, 1), \tag{1.2}$$

where

$$a_{N,s} := 2^{2s-1} \pi^{-\frac{N}{2}} \frac{\Gamma(\frac{N+2s}{2})}{|\Gamma(-s)|},$$

is the normalization constant to have the identity

$$(-\Delta)^s u = \mathcal{F}^{-1}(|\xi|^{2s} \mathcal{F} u), \; \xi \in \mathbb{R}^N, s \in (0, 1),$$

in $\mathcal{S}(\mathbb{R}^N)$, the class of Schwartz functions.

Our goal is to find *natural* conditions on f in order to determine the existence of a positive solution.

The problem (1.1) can be seen as a Kardar–Parisi–Zhang (KPZ) stationary equation with fractional diffusion. See [32] for the derivation of the model in the local case.

In this sense, since the KPZ model assumes the growth of the interface in the direction of its normal, it seems to be natural to assume that $s > \frac{1}{2}$.

The local problem has been extensively studied by many authors—see, for instance, [3, 6, 11, 22, 23, 28, 30] and the references therein. In the above works, the case $q \leqslant 2$ (subcritical and critical growth) is studied in depth. The existence of a solution is determined using *a priori* estimates—that follow from the use of suitable test functions—together with comparison principles. In the last three references, some sharp regularity results are obtained. As a consequence, the authors in [3] were able to prove a characterization of all positive solutions which implies a *wild* non-uniqueness result. We refer also to the recent paper [29], where a fine analysis of the stochastic case is considered.

The supercritical case, $q > 2$, has been studied, for instance, in the references [30, 41] and [40]. The existence of a solution is determined using Potential Theory and certain fixed point arguments. It is clear that, in any case, the existence results are guaranteed under regularity hypotheses on f and a smallness condition on λ.

We refer to [31, 46] for physical motivations and results for the corresponding evolutionary problem.

The aim of this article is the analysis of the nonlocal case. There are significant differences from the local case. First of all, it is necessary to identify the critical growth in the fractional setting. By homogeneity, the critical power seems to be $q = 2s$; and this is, in fact, the threshold to use the comparison techniques when $q < 2s$. It is worth pointing out that, in such a critical growth, there is no known change of variables similar to the Hopf–Cole change in the local case. Moreover, the techniques of nonlinear test functions, in general, are difficult to adapt to the nonlocal problem.

Among the main tools in our analysis are sharp estimates on the Green's function of the *fractional laplacian* obtained in [14–16, 18, 25]. Some interesting results by using such estimates, appear in the papers by Chen and Veron [19, 20]. They consider a nonlinear term $|\nabla u|^q$ with $q < p_* := \frac{N}{N-2s+1}$, and obtain sharp existence results for f a Radon measure.

We will concentrate on covering the range $q \geqslant p_*$; it seems that the argument used in [19, 20] cannot be extended directly to this range, and thus we need to use a different approach. To deal with the subcritical case $q < 2s$, we will prove a new comparison principle in the spirit of [4] and [42]. This comparison result allows us to prove the existence and uniqueness of a *nice* solution to problem (1.1). In the case $q < p_*$, we are able to prove the uniqueness of the solution for every datum in L^1. Techniques based on the comparison principle have a serious limitation in the critical and supercritical cases—that is, for $q \geqslant 2s$. Such difficulties restrict us to a super-solution when we start from an ordered family of approximation problems in order to solve problem (1.1). To overcome this lack of compactness, we will use estimates from potential theory, and apply a fixed point argument inspired by the papers [41] and [40] in the local case.

The paper is organized as follows.

In section 2, we begin with some basic results about the problem, with general datum in L^1 or in the space of Radon measures. As was observed in the local case, the existence of a solution to problem (1.1) is strongly related to the regularity of the solution to problem

$$\begin{cases} (-\Delta)^s v &= f \quad \text{in } \Omega, \\ \qquad v &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.3}$$

In the same section, we will also specify the sense in which the solutions are understood, and establish for $s > \frac{1}{2}$ the regularity of the solution v to problem (1.3) according to the regularity of the datum f. In section 3, we state the comparison principle to be used in the subcritical case. The proof relies on a Harnack type inequality of the fractional operator, perturbed by a first order linear term. Having obtained the comparison principle, we are able to prove existence and uniqueness results for approximation problems, and to obtain the uniqueness result for the problem studied in [20]—that is, for $q < p_*$. Problem (1.1) with $q < 2s$ is treated in section 4. The proof of existence of solution uses the comparison principle and the construction of a suitable supersolution. As in the local framework, the existence of a solution will be guaranteed under additional hypotheses on f and smallness condition on λ. The compactness argument used in this section cannot be used to treat the critical or supercritical cases; for this reason, the analysis of these two cases will be performed in section 5, using suitable estimates from Potential Theory and the classical Schauder fixed point theorem. Finally, in the last section, we collect some open problems that seem interesting for future study.

2. Preliminaries and auxiliary results

In this section, we present some useful results about the problem

$$\begin{cases} (-\Delta)^s v = \nu & \text{in } \Omega, \\ v = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{2.1}$$

where ν is a bounded Radon measure. We give some definitions about the set of Radon measures and the sense in which a solution to problem (2.1) is considered (see [19], [20] and [35]).

Definition 2.1. Let μ be a bounded Radon measure and $\beta > 0$. We say that $\mu \in \mathfrak{M}(\Omega, d^\beta)$ if

$$\int_\Omega \mathrm{d}^\beta(x)\mathrm{d}|\mu| < +\infty,$$

with $d(x) = \text{ dist } (x, \partial\Omega)$.

In the same way, if f is a locally integrable function, then $f \in \mathfrak{M}(\Omega, d^\beta)$ if

$$\int_\Omega |f(x)|\mathrm{d}^\beta(x)\mathrm{d}x < +\infty.$$

It is clear that if $f \in L^1(\Omega)$, then $f \in \mathfrak{M}(\Omega, d^\beta)$ for all $\beta > 0$.

Next, we specify the sense in which solutions are defined for this general class of data.

Definition 2.2. We say that u is a weak solution to problem (2.1) if $u \in L^1(\Omega)$, and for all $\phi \in \mathbb{X}_s$, we have

$$\int_\Omega u(-\Delta)^s\phi\mathrm{d}x = \int_\Omega \phi \mathrm{d}\nu,$$

where

$$\mathbb{X}_s \equiv \Big\{\phi \in \mathcal{C}(I\!R^N) \mid \text{ supp } (\phi) \subset \overline{\Omega},\ (-\Delta)^s\phi(x) \text{ is pointwise defined and } |(-\Delta)^s\phi(x)| < C \text{ in } \Omega\Big\}.$$

Despite that it is a classical result based in the local results by Stampacchia in [43] and developed for instance in [35], we include the uniqueness result of weak solution.

Proposition 2.3. *Let ν be a bounded Radon measure in Ω. Then problem (2.1) has at most a weak solution.*

Proof. If u_1, u_2 are two weak solutions to problem (2.1), then $(u_1 - u_2)$ satisfies

$$\int_\Omega (u_1 - u_2)(-\Delta)^s\phi\mathrm{d}x = 0 \text{ for all } \phi \in \mathbb{X}_s,$$

where

$$\mathbb{X}_s \equiv \Big\{\phi \in \mathcal{C}(I\!R^N) \mid \text{ supp } (\phi) \subset \overline{\Omega},\ (-\Delta)^s\phi(x) \text{ pointwise defined and } |(-\Delta)^s\phi(x)| < C \text{ in } \Omega\Big\}.$$

So, if $g \in \mathcal{C}_0^\infty(\Omega)$ and defining ϕ to be the unique solution to the problem

$$\begin{cases} (-\Delta)^s\phi &= g \quad \text{in } \Omega, \\ \phi &= 0 \quad \text{in } I\!R^N \backslash \Omega, \end{cases} \tag{2.2}$$

it holds that

$$\int_\Omega (u_1 - u_2)g\mathrm{d}x = 0 \text{ for all } g \in \mathcal{C}_0^\infty(\Omega).$$

Hence $u_1 = u_2$ a.e. in Ω. Since $u_1 = u_2 = 0$ in $I\!R^N \backslash \Omega$, we reach that $u_1 = u_2$ a.e. in $I\!R^N$. $\square$

The functional framework to obtain a solution to the truncated problem in the fractional Sobolev space is given in the next definition.

Definition 2.4. For $0 < s < 1$, we define the fractional Sobolev space of order s as

$$H^s(\mathbb{R}^N) := \{u \in L^2(\mathbb{R}^N) \mid \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y < +\infty\}.$$

We now define the space $H_0^s(\Omega)$ as the completion of $\mathcal{C}_0^\infty(\Omega)$ with respect to the norm of $H^s(\boldsymbol{R}^N)$. Note that if $u \in H_0^s(\Omega)$, we have $u = 0$ a.e. in $\boldsymbol{R}^N \setminus \Omega$, and we can write

$$\int_{I\!R^N}\int_{I\!R^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, \mathrm{d}x\, \mathrm{d}y = \iint_{D_\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, \mathrm{d}x\, \mathrm{d}y,$$

where

$$D_\Omega := I\!R^N \times I\!R^N \setminus \big(\mathcal{C}\Omega \times \mathcal{C}\Omega\big).$$

Definition 2.5. For $\sigma \in \mathbb{R}$, we set

$$T_k(\sigma) = \max(-k, \min(k, \sigma)) \quad \text{and} \quad G_k(\sigma) = \sigma - T_k(\sigma).$$

Let us recall the next classical tools from Fourier analysis that will be used systematically in the rest of the paper.

Lemma 2.6. *Let $0 < \lambda < N, 1 \leqslant p < l < \infty$ be such that $\frac{1}{l} + 1 = \frac{1}{p} + \frac{\lambda}{N}$. For $g \in L^p(I\!R^N)$, we define*

$$J_\lambda(g)(x) = \int_{I\!R^N} \frac{g(y)}{|x-y|^\lambda} \mathrm{d}y.$$

(a) *J_λ is well defined, in the sense that the integral converges absolutely for almost all $x \in \mathbb{R}^N$.*
(b) *If $p > 1$, then $\|J_\lambda(g)\|_p \leqslant c_{p,l}\|g\|_l$.*
(c) *If $p = 1$, then $|\{x \in \mathbb{R}^N\, | J_\lambda(g)(x) > \sigma\}| \leqslant \big(\frac{A\|g\|_1}{\sigma}\big)^l$.*

See, for instance, sections 1.2 of 5 in [44] for the proof.

To end this collection of preliminaries, we recall the classical Hardy–Littlewood–Sobolev inequality.

Lemma 2.7. *Let $0 < \lambda < N$, $\theta, \gamma > 1$ with $\frac{1}{\theta} + \frac{1}{\gamma} + \frac{\lambda}{N} = 2$. Assume that $g \in L^\theta(I\!R^N)$ and $k \in L^\gamma(I\!R^N)$; then*

$$\Big|\iint_{I\!R^N \times I\!R^N} \frac{g(x)k(y)}{|x-y|^\lambda} \mathrm{d}x\mathrm{d}y\Big| \leqslant C(N, \lambda, \theta)\|k\|_{L^\gamma(I\!R^N)}\|g\|_{L^\theta}.$$

For the proof, we refer, for instance, to section 4.3 in [37] or to the paper [36].

From [19, 35] and [2] the following result holds.

Theorem 2.8. *Assume that $f \in L^1(\Omega)$; then problem (2.1) has a unique weak solution u that is obtained as the limit of $\{u_n\}_{n\in\mathbb{N}}$, the sequence of the unique solutions to the approximation problems*

$$\begin{cases} (-\Delta)^s u_n &= f_n(x) \quad \text{in } \Omega,\\ u_n &= 0 \quad \text{in } I\!R^N\setminus\Omega, \end{cases} \tag{2.3}$$

with $f_n = T_n(f)$. Further,

$$T_k(u_n) \to T_k(u) \text{ strongly in } H_0^s(\Omega), \quad \forall k>0, \tag{2.4}$$

$$u \in L^\theta(\Omega), \qquad \forall\, \theta \in \Big(1, \frac{N}{N-2s}\Big) \tag{2.5}$$

and

$$\big|(-\Delta)^{\frac{s}{2}} u\big| \in L^r(\Omega), \qquad \forall\, r \in \Big(1, \frac{N}{N-s}\Big). \tag{2.6}$$

Moreover,

$$u_n \to u \text{ strongly in } W_0^{s,q_1}(\Omega) \text{ for all } q_1 < \frac{N}{N-2s+1}. \tag{2.7}$$

Remarks 2.9.

(1) In [19], the authors proved the existence and the uniqueness of a weak solution to problem (2.1), in the sense of definition 2.2 for every bounded Radon measure f. The uniqueness is a consequence of the classical duality argument by Stampacchia in [43]. See [35] for some details in the fractional framework.

(2) In [2], the more general framework of the fractional p-Laplacian operator with non-negative datum is studied. The uniqueness of non-negative solution in the entropy setting is proved. Since, in this work, we are considering the linear case $p=2$, the existence and uniqueness of weak solutions are obtained without any sign condition on the datum.

2.1. The Green's function and the representation formula

Let $\mathcal{G}_s(x,y)$ be the Green's function associated with the fractional laplacian $(-\Delta)^s$; that is, $\mathcal{G}_s(x,y)$ is, by definition, the solution to the problem

$$\begin{cases} (-\Delta)_y^s \mathcal{G}_s(x,y) &= \delta_x \quad \text{if } y \in \Omega,\\ \mathcal{G}_s(x,y) &= 0 \quad \text{if } y \in I\!R^N\setminus\Omega, \end{cases} \tag{2.8}$$

for each fixed $x \in \Omega$, where δ_x is the unit Dirac mass concentrated at x. As consequence of the proposition 2.3 we obtain that the Green's function is unique.

Recall that in $I\!R^N$, by using the Fourier transform, the fundamental solution for the fractional laplacian is, modulo s-harmonic functions, $\mathcal{G}_{s,I\!R^N}(x,y) = \frac{C}{|x-y|^{N-2s}}$. It is clear, moreover, that if $s \in (\frac{1}{2}, 1)$, then the gradient of $\mathcal{G}_{s,I\!R^N}$ exists in the classical sense for all $x \neq y$, and is locally integrable in $I\!R^N$.

For a bounded regular domain Ω, we have $\mathcal{G}_s(x,y) \leqslant \mathcal{G}_{s,I\!R^N}(x,y)$. Indeed, we can write $\mathcal{G}_s(x,y) = \mathcal{G}_{s,I\!R^N}(x,y) + \Phi_s(x,y)$, where $\Phi_s(x,y)$ is the s-harmonic function such that

$\Phi_s(x,y) = -\mathcal{G}_{s,\mathbb{R}^N}(x,y)$ for each $y \in \mathbb{R}^N \setminus \Omega$ and each fixed $x \in \Omega$; specifically, $\Phi_s(x,y)$ is the unique solution to the problem

$$\begin{cases} (-\Delta)^s_y \Phi_s(x,y) &= 0 \qquad \text{if } y \in \Omega, \\ \Phi_s(x,y) &= -\mathcal{G}_{s,\mathbb{R}^N}(x,y) \quad \text{if } y \in \mathbb{R}^N \setminus \Omega, \end{cases} \tag{2.9}$$

for each fixed $x \in \Omega$.

Therefore, $\mathcal{G}_s(x,y)$ has the same singularity as $\mathcal{G}_{s,\mathbb{R}^N}(x,y)$.

The authors in [12] and [14] were able to drive some pointwise properties of the Green's function and its gradient when $s \in (\frac{1}{2}, 1)$. See, too, the survey [45], the references therein and, in particular, formula 2.19 in [26].

We collect these estimates in the following Lemma.

Lemma 2.10. *Assume that $s \in (\frac{1}{2}, 1)$; then*

$$\mathcal{G}_s(x,y) \leqslant C_1 \min\{\frac{1}{|x-y|^{N-2s}}, \frac{d^s(x)}{|x-y|^{N-s}}, \frac{d^s(y)}{|x-y|^{N-s}}\}, \tag{2.10}$$

and

$$|\nabla_x \mathcal{G}_s(x,y)| \leqslant C_2 \mathcal{G}_s(x,y) \max\{\frac{1}{|x-y|}, \frac{1}{d(x)}\}. \tag{2.11}$$

In particular,

$$|\nabla_x \mathcal{G}_s(x,y)| \leqslant \frac{C_3}{|x-y|^{N-2s+1}},$$

where C_1, C_2, C_3 are independent of x and y.

Remarks 2.11. It is worth pointing out that since the fractional laplacian is the infinitesimal generator of stable Lévy processes, it follows—as was proved in [12, 33]—that if f is a regular bounded function, then we have the next representation formula for the unique solution to problem (2.1):

$$u(x) = \int_\Omega \mathcal{G}_s(x,y) f(y) \mathrm{d}y = E^x\left(\int_0^{\tau_\Omega} f(X_\sigma) d\sigma\right), \tag{2.12}$$

where (X_σ) is a $2s$-stable process in $\mathbb{R}^N$, and τ_Ω is the first exit time of X_t from Ω. We refer to [5, 13] for more details about the Lévy process and its connection with the fractional Laplacian.

We consider now the general case $f \in L^1(\Omega)$. As was stated in theorem 2.8, the problem (2.1) has a unique weak solution u in the sense of definition 2.2. Define

$$w(x) = \int_\Omega \mathcal{G}_s(x,y) f(y) \mathrm{d}y. \tag{2.13}$$

Note that the function w is well defined. Indeed, since $f \in L^1(\Omega)$, using the classical estimate,

$$\mathcal{G}_s(x,y) \leqslant \frac{C}{|x-y|^{N-2s}},$$

and lemma 2.6, we conclude the proof of lemma 2.10. By the definition of $\mathcal{G}_s$, we also deduce that $w = 0$ in $\mathbb{R}^N \backslash \Omega$. Moreover, $u = w$. This follows directly because w is a weak solution in the sense of definition (2.2) to the Dirichlet problem, and by the uniqueness of the weak solution.

For the reader's convenience, we will give a direct proof.

Recall that u is obtained as a limit of the sequence $\{u_n\}_n$, where u_n is the unique solution to the approximation problem (2.3) (see for instance [35]). Choosing f_n regular and bounded, then by remark 2.11, the solution u_n is also given by the representation formula, that is,

$$u_n(x) = \int_\Omega \mathcal{G}_s(x, y) f_n(y) \mathrm{d}y. \tag{2.14}$$

Thus,

$$|w(x) - u_n(x)| \leqslant \int_\Omega \mathcal{G}_s(x, y)|f(y) - f_n(y)| \mathrm{d}y \leqslant C \int_\Omega \frac{|f(y) - f_n(y)|}{|x - y|^{N-2s}} \mathrm{d}y.$$

Hence, by lemma 2.6, it holds that

$$|\{x \in \mathbb{R}^N \, |w(x) - u_n(x)| > \sigma\}| \leqslant \Big(\frac{A\|f - f_n\|_{L^1(\Omega)}}{\sigma}\Big)^l, \text{ for all } l < \frac{N}{N - 2s}.$$

Letting $n \to \infty$, we conclude that $u_n \to w$ in the Marcinkiewicz space $M^{\frac{N}{N-2s}}(\Omega)$ and then $u_n \to w$ strongly in the Lebesgue space $L^\theta(\Omega)$ for all $\theta < \frac{N}{N-2s}$. Hence, $w = u$, and thus u, the unique solution to problem (2.1), is also given by

$$u(x) = \int_\Omega \mathcal{G}_s(x, y) f(y) \mathrm{d}y. \tag{2.15}$$

As a consequence of the estimates in lemma 2.10, the authors in [20] obtain the following regularity result.

Theorem 2.12. *Suppose that $s \in (\frac{1}{2}, 1)$, and let $f \in \mathfrak{M}(\Omega)$ be a Radon measure. Then the problem (2.1) has a unique weak solution u in the sense of definition 2.2, such that*

(1) $|\nabla u| \in M^{p_*}(\Omega)$, the Marcinkiewicz space, with $p_* = \frac{N}{N-2s+1}$, and as a consequence, $u \in W_0^{1,\theta}(\Omega)$ for all $\theta < p_*$. Moreover,

$$\|u\|_{W_0^{1,\theta}(\Omega)} \leqslant C(N, q, \Omega)\|f\|_{\mathfrak{M}(\Omega)}. \tag{2.16}$$

(2) For $f \in L^1(\Omega)$, setting $T : L^1(\Omega) \to W_0^{1,\theta}(\Omega)$, with $T(f) = u$, T is a compact operator.

Remark 1. If $f \in L^1(\Omega, d^\beta)$ with $0 \leqslant \beta \leqslant s$, then for $p \in (1, \frac{N}{N+\beta-2s})$ there exists $c_p > 0$ such that

$$\|u\|_{W^{2s-\gamma,p}} \leqslant \|f\|_{L^1(\Omega, d^\beta)},$$

where $\gamma = \beta + \frac{N}{p'}$ if $\beta > 0$, and $\gamma > \frac{N}{p'}$ if $\beta = 0$.

We will use theorems 2.8 and 2.12 as the starting point of our analysis.

2.2. A technical result

In what follows, we will assume that $s \in (\frac{1}{2}, 1)$.

In the local case, $s = 1$, if u is the solution to the corresponding problem (2.1), it is known that $T_k(u) \in W_0^{1,2}(\Omega)$ for all $k > 0$. This follows by using $T_k(u)$ as a test function in (2.1). To prove, in the next subsection, a similar result in the fractional setting, we need the following result.

Lemma 2.13. *Let Ω be a bounded smooth domain. Assume $\frac{1}{2} < s < 1$ and $\alpha \in I\!R$ verifying $1 < \alpha < 2s$. Then the problem*

$$\begin{cases} (-\Delta)^s \rho &= \frac{1}{d^\alpha(x)} & \text{in } \Omega, \\ \rho &= 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{2.17}$$

has a positive solution ρ, such that $\rho \in L^\infty(\Omega)$ and $\rho^{\frac{\beta+1}{2}} \in H_0^s(\Omega)$ for all $\beta > \max\{\frac{\alpha}{2s-\alpha}, 1\}$. In particular, $\rho^\theta \in H_0^s(\Omega)$ for all $\theta > \max\{\frac{s}{2s-\alpha}, 1\}$.

Proof. Recall that Ω is a regular domain; since $s > \frac{1}{2}$, the following Hardy type inequality was obtained in [38],

$$C \int_\Omega \frac{\phi^2}{d^{2s}(x)} \mathrm{d}x \leqslant \|\phi\|^2_{H_0^s(\Omega)} \text{ for all } \phi \in H_0^s(\Omega), \tag{2.18}$$

where $C > 0$ depend only on s, N and Ω.

Define ρ_n to be the unique solution to the approximating problem

$$\begin{cases} (-\Delta)^s \rho_n &= \frac{1}{d^\alpha(x) + \frac{1}{n}} & \text{in } \Omega, \\ \rho_n &= 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{2.19}$$

It is clear that $\rho_n \in L^\infty(\Omega) \cap H_0^s(\Omega)$, $\rho_n > 0$ in Ω and $\rho_n \leqslant \rho_{n+1}$ for all n.

Since $\alpha < 2s$, we can pick up $\beta > 1$ such that $\alpha < \frac{2s\beta}{\beta+1}$.

Taking ρ_n^β as a test function in (2.19), and using the numerical inequality,

$$(a-b)(a^\gamma - b^\gamma) \geqslant c|a^{\frac{1+\gamma}{2}} - b^{\frac{1+\gamma}{2}}|^2 \text{ for all } a, b \in I\!R^+ \text{ and } \gamma, c > 0, \tag{2.20}$$

(see [34]), it follows that

$$C\|\rho_n^{\frac{\beta+1}{2}}\|^2_{H_0^s(\Omega)} \leqslant \int_\Omega \frac{\rho_n^\beta}{d^\alpha(x)} \mathrm{d}x. \tag{2.21}$$

Thus, using the Hölder inequality, the following holds:

$$\int_\Omega \frac{\rho_n^\beta}{d^\alpha(x)} \mathrm{d}x \leqslant \Big(\int_\Omega \frac{\rho_n^{\beta+1}}{d^{2s}(x)} \mathrm{d}x \Big)^{\frac{\beta}{\beta+1}} \Big(\int_\Omega \frac{1}{d^{(\beta+1)(\alpha - \frac{2s\beta}{\beta+1})}(x)} \mathrm{d}x \Big)^{\frac{1}{\beta+1}}.$$

Note that, since $\alpha < \frac{2s\beta}{\beta+1}$, it follows that $(\beta+1)(\alpha - \frac{2s\beta}{\beta+1}) < 0$,

$$\int_\Omega \frac{1}{d^{(\beta+1)(\alpha - \frac{2s\beta}{\beta+1})}(x)} \mathrm{d}x < \infty.$$

Therefore, by Hardy inequality (2.18),

$$\int_\Omega \frac{\rho_n^\beta}{d^\alpha(x)} \mathrm{d}x \leqslant C \Big(\int_\Omega \frac{\rho_n^{\beta+1}}{d^{2s}(x)} \mathrm{d}x \Big)^{\frac{\beta}{\beta+1}} \leqslant C\|\rho_n^{\frac{\beta+1}{2}}\|^{\frac{\beta}{\beta+1}}_{H_0^s(\Omega)}. \tag{2.22}$$

Hence, from (2.21) and (2.22), we have $\|\rho_n^{\frac{\beta+1}{2}}\|_{H_0^s(\Omega)} \leqslant C$ for all $n \in \mathbb{N}$. Thus, there exists ρ such that $\rho^{\frac{\beta+1}{2}} \in H_0^s(\Omega)$, $\rho_n \uparrow \rho$ strongly in $L^{\frac{2_s^*(\beta+1)}{2}}(\Omega)$ and $\rho_n^{\frac{\beta+1}{2}} \rightharpoonup \rho^{\frac{\beta+1}{2}}$ weakly in $H_0^s(\Omega)$.

It is not difficult to show that ρ is a solution to problem (2.17) in the sense of distribution, and that $\frac{\rho_n^{\beta+1}}{d^\alpha(x)} \to \frac{\rho^{\beta+1}}{d^\alpha(x)}$ strongly in $L^1(\Omega)$. In the same way, we can show that $\rho \in L^\sigma(\Omega)$ for all $\sigma > 0$.

To prove that $\rho \in L^\infty(\Omega)$, we follow [35], where the classical Stampacchia argument in [43] is adapted to the fractional setting. For the reader's convenience, we give some details—mainly the estimates involving the Hardy inequality.

Take $G_k^\beta(\rho_n)$, with $k > 0$, as a test function in (2.19). Note that by using (2.20), we have

$$(\rho_n(x) - \rho_n(y))\left(G_k^\beta(\rho_n(x)) - G_k^\beta(\rho_n(y))\right) \geqslant C\left|G_k^{\frac{\beta+1}{2}}(\rho_n(x)) - G_k^{\frac{\beta+1}{2}}(\rho_n(y))\right|^2;$$

thus,

$$\begin{aligned} C\iint_{D_\Omega} \frac{|G_k^{\frac{\beta+1}{2}}(\rho_n(x)) - G_k^{\frac{\beta+1}{2}}(\rho_n(y))|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y &\leqslant \int_\Omega \frac{G_k^\beta(\rho_n(x))}{\mathrm{d}^\alpha(x)}\mathrm{d}x \\ &\leqslant \Big(\int_\Omega \frac{G_k^{\beta+1}(\rho_n(x))}{\mathrm{d}^{2s}(x)}\mathrm{d}x\Big)^{\frac{\beta}{\beta+1}}\Big(\int_{A_k} \mathrm{d}^{(\beta+1)(\frac{2s\beta}{\beta+1}-\alpha)}(x)\mathrm{d}x\Big)^{\frac{1}{\beta+1}}. \end{aligned}$$

Letting $n \to \infty$, it follows that

$$\iint_{D_\Omega} \frac{|G_k^{\frac{\beta+1}{2}}(\rho(x)) - G_k^{\frac{\beta+1}{2}}(\rho(y))|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y \leqslant \Big(\int_\Omega \frac{G_k^{\beta+1}(\rho(x))}{\mathrm{d}^{2s}(x)}\mathrm{d}x\Big)^{\frac{\beta}{\beta+1}}|A_k|^{\frac{1}{\beta+1}},$$

where $A_k = \{x \in \Omega : |u(x)| \geqslant k\}$. Hence, by the Hardy inequality,

$$\|G_k^{\frac{\beta+1}{2}}(\rho)\|_{H_0^s(\Omega)}^{\frac{1}{\beta+1}} \leqslant C|A_k|^{\frac{1}{\beta+1}};$$

and so, by the Sobolev inequality,

$$S\|G_k^{\frac{\beta+1}{2}}(\rho)\|_{L^{2_s^*}(\Omega)} \leqslant C|A_k|.$$

Let $h > k$; since $A_h \subset A_k$, we obtain that

$$(h-k)^{\frac{\beta+1}{2}}|A_h|^{\frac{1}{2_s^*}} \leqslant C|A_k|,$$

that is,

$$|A_h| \leqslant \frac{C|A_k|^{2_s^*}}{(h-k)^{\frac{2_s^*(\beta+1)}{2}}}.$$

Since $2_s^* > 1$, by the classical numerical lemma of Stampacchia in [43], there exists $k_0 > 0$ such that $|A_h| = 0$ for all $h \geqslant k_0$. Thus ρ is bounded. $\square$

2.3. Some regularity results

We start by proving the following regularity result of $T_k(v)$, which will be a useful tool in the subsequent arguments.

Theorem 2.14. *Assume that $f \in L^1(\Omega)$, and define u to be the unique weak solution to problem (2.1). Then $T_k(u) \in W_0^{1,\alpha}(\Omega) \cap H_0^s(\Omega)$ for any $1 < \alpha < 2s$; moreover,*

$$\int_\Omega |\nabla T_k(u)|^\alpha \,\mathrm{d}x \leqslant Ck^{\alpha-1}\|f\|_{L^1(\Omega)}.$$

Proof. Without loss of generality, we can assume that $f \geqslant 0$. By theorem 2.8, $T_k(u) \in H_0^s(\Omega)$. Let $\mathcal{G}_s$ be the Green's kernel of $(-\Delta)^s$, then

$$u(x) = \int_\Omega \mathcal{G}_s(x,y) f(y)\mathrm{d}y. \tag{2.23}$$

Thus,

$$|\nabla T_k(u)| = |\nabla u(x)|\chi_{\{|u(x)|<k\}} \leqslant \Big(\int_\Omega |\nabla_x \mathcal{G}_s(x,y)| f(y)\mathrm{d}y\Big)\chi_{\{|u(x)|<k\}}.$$

Fix $1 < \alpha < 2s$; then, using the Hölder inequality, it follows that

$$\begin{aligned}
|\nabla T_k(u)|^\alpha &\leqslant \Big(\int_\Omega |\nabla_x \mathcal{G}_s(x,y)| f(y)\mathrm{d}y\Big)^\alpha \chi_{\{|u(x)|<k\}}\\
&\leqslant \Big(\int_\Omega \frac{|\nabla_x \mathcal{G}_s(x,y)|}{\mathcal{G}_s(x,y)}\mathcal{G}_s(x,y) f(y)\mathrm{d}y\Big)^\alpha \chi_{\{|u(x)|<k\}}\\
&\leqslant \Big(\int_\Omega \Big[\frac{|\nabla_x \mathcal{G}_s(x,y)|}{\mathcal{G}_s(x,y)}\Big]^\alpha \mathcal{G}_s(x,y) f(y)\mathrm{d}y\Big)\Big(\int_\Omega \mathcal{G}_s(x,y) f(y)\mathrm{d}y\Big)^{\alpha-1} \chi_{\{|u(x)|<k\}}\\
&\leqslant \Big(\int_\Omega \Big[\frac{|\nabla_x \mathcal{G}_s(x,y)|}{\mathcal{G}_s(x,y)}\Big]^\alpha \mathcal{G}_s(x,y) f(y)\mathrm{d}y\Big) u^{\alpha-1}(x)\chi_{\{|u(x)|<k\}}\\
&\leqslant k^{\alpha-1}\Big(\int_\Omega \Big[\frac{|\nabla_x \mathcal{G}_s(x,y)|}{\mathcal{G}_s(x,y)}\Big]^\alpha \mathcal{G}_s(x,y) f(y)\mathrm{d}y\Big)\chi_{\{|u(x)|<k\}}.
\end{aligned}$$

From lemma 2.10, we know that

$$\mathcal{G}_s(x,y) \leqslant C_1 \min\left\{\frac{1}{|x-y|^{N-2s}}, \frac{d^s(x)}{|x-y|^{N-s}}, \frac{d^s(y)}{|x-y|^{N-s}}\right\},$$

and

$$|\nabla_x \mathcal{G}_s(x,y)| \leqslant C_2 \mathcal{G}_s(x,y) \max\left\{\frac{1}{|x-y|}, \frac{1}{d(x)}\right\}.$$

Thus, setting $h(x,y) = \frac{|\nabla_x \mathcal{G}_s(x,y)|}{\mathcal{G}_s(x,y)}$, we reach that

$$\int_\Omega |\nabla T_k(u)|^\alpha \mathrm{d}x \leqslant k^{\alpha-1}\int_\Omega f(y)\Big(\int_\Omega h(x,y)^\alpha \mathcal{G}_s(x,y)\mathrm{d}x\Big)\mathrm{d}y.$$

It is clear that $h(x,y)^{\alpha} \leqslant C\left(\frac{1}{|x-y|^{\alpha}} + \frac{1}{d^{\alpha}(x)}\right)$. Therefore, we conclude that

$$\begin{aligned}\int_{\Omega} |\nabla T_k(u)|^{\alpha} \mathrm{d}x &\leqslant Ck^{\frac{\alpha}{\alpha'}}\left\{\int_{\Omega} f(y)\left(\int_{\Omega} \frac{1}{|x-y|^{\alpha}}\mathcal{G}_s(x,y)\mathrm{d}x\right)\mathrm{d}y + \int_{\Omega} f(y)\left(\int_{\Omega} \frac{1}{d^{\alpha}(x)}\mathcal{G}_s(x,y)\mathrm{d}x\right)\mathrm{d}y\right\}\\ &\leqslant Ck^{\alpha-1}(I_1+I_2).\end{aligned}$$

With respect to I_1, using the fact that $\mathcal{G}_s(x,y) \leqslant \frac{C}{|x-y|^{N-2s}}$, the following holds:

$$I_1 \leqslant C\int_{\Omega} f(y)\int_{\Omega}\left(\frac{1}{|x-y|^{\alpha}}\mathcal{G}_s(x,y)\mathrm{d}x\right)\mathrm{d}y \leqslant C\int_{\Omega} f(y)\left(\int_{\Omega} \frac{1}{|x-y|^{N-2s+\alpha}}\mathrm{d}x\right)\mathrm{d}y.$$

Since $\alpha < 2s$,

$$\int_{\Omega} \frac{1}{|x-y|^{N-2s+\alpha}}\mathrm{d}x \leqslant \int_{B_R(y)} \frac{1}{|x-y|^{N-2s+\alpha}}\mathrm{d}x \leqslant CR^{2s-\alpha}$$

for $R \gg 2\mathrm{diam}\,(\Omega)$. We deal now with I_2. Consider ρ, the unique solution to problem

$$\begin{cases}(-\Delta)^s\rho &= \frac{1}{d^{\alpha}(x)} \quad \text{in } \Omega,\\ \rho &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega\end{cases}$$

obtained in lemma 2.13. Since $\alpha < 2s$, by lemma 2.13, $\rho \in L^{\infty}(\Omega)$. Thus,

$$I_2 \leqslant C\int_{\Omega} f(y)\left(\int_{\Omega} \frac{1}{d^{\alpha}}\mathcal{G}_s(x,y)\mathrm{d}x\right)\mathrm{d}y = C\int_{\Omega} f(y)\rho(y)\mathrm{d}y \leqslant C\|f\|_{L^1(\Omega)}.$$

Combining the above estimates, it follows that

$$\int_{\Omega} |\nabla T_k(u)|^{\alpha}\mathrm{d}x \leqslant Ck^{\alpha-1}\|f\|_{L^1(\Omega)}.$$

Thus, we conclude the proof. □

We next state some results required in order to find the regularity of v when f is assumed to be more regular.

Lemma 2.15. *Suppose that $f \in L^m(\Omega)$ with $m \geqslant 1$ and define v to be the unique solution to problem*

$$\begin{cases}(-\Delta)^s v &= f \quad \text{in } \Omega,\\ v &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega,\end{cases} \tag{2.24}$$

with $s > \frac{1}{2}$. Then for all $p < \frac{mN}{N-m(2s-1)}$, there exists a positive constant $C \equiv C(\Omega, N, s, p)$ such that

$$\|\nabla v\|_{L^p(\Omega)} \leqslant C\|f\|_{L^m(\Omega)}. \tag{2.25}$$

Moreover,

(1) If $m = \frac{N}{2s-1}$, then $|\nabla v| \in L^p(\Omega)$ for all $p < \infty$.
(2) If $m > \frac{N}{2s-1}$, then $v \in \mathcal{C}^{1,\sigma}(\Omega)$ for some $\sigma \in (0,1)$.

To prove the previous lemma, we need the following elementary result.

Lemma 2.16. *Suppose that the same hypotheses as above hold. Let $\phi \in \mathcal{C}_0^1(\Omega)$, and define ψ to be the unique solution to problem*

$$\begin{cases} (-\Delta)^s \psi &= \frac{\partial \phi}{\partial x_i} \quad \text{in } \Omega, \\ \psi &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega; \end{cases} \tag{2.26}$$

then, for $\beta < \frac{N}{2s-1}$ and for all $r < \frac{\beta N}{N-\beta(2s-1)}$, there exists a positive constant $C \equiv C(N, \beta, r, \Omega)$ such that

$$\|\psi\|_{L^r(\Omega)} \leqslant C\|\phi\|_{L^\beta(\Omega)}. \tag{2.27}$$

Proof. Let $\mathcal{G}_s$ be the Green's kernel of $(-\Delta)^s$, then

$$\psi(x) = \int_\Omega \mathcal{G}_s(x,y)\frac{\partial \phi}{\partial y_i}(y)\mathrm{d}y.$$

Hence, using integration by parts,

$$|\psi(x)| \leqslant \int_\Omega |\frac{\partial \mathcal{G}_s(x,y)}{\partial y_i}||\phi(y)|\mathrm{d}y.$$

Using the Hardy–Littlewood–Sobolev inequality in lemma 2.7, and since

$$|\frac{\partial \mathcal{G}_s(x,y)}{\partial y_i}| \leqslant \frac{C}{|x-y|^{N-2s+1}}$$

(see lemma 2.10 above), it follows that

$$\|\psi\|_{L^r(\Omega)} \leqslant C\|\phi\|_{L^\beta(\Omega)},$$

where $\frac{1}{r} = \frac{1}{\beta} + \frac{N-2s+1}{N} - 1$. Thus, we conclude the proof. □

Note that if $\beta \nearrow \frac{N}{2s-1}$, then $r \to \infty$. We are now able to prove lemma 2.15.

Proof of lemma 2.15. As a consequence of theorem 2.12 (see [20]) we have that $v \in W_0^{1,q}(\Omega)$ for all $q < p_*$. Let $\phi \in \mathcal{C}_0^1(\Omega)$; then

$$\langle \frac{\partial v}{\partial x_1}, \phi\rangle = -\int_\Omega v\frac{\partial \phi}{\partial y_1}\mathrm{d}y.$$

Hence, if ψ is the solution of problem (2.26), it follows that

$$|\langle \frac{\partial v}{\partial x_1}, \phi\rangle| = |\int_\Omega v(-\Delta)^s\psi \mathrm{d}y| = |\int_\Omega f\psi \mathrm{d}y|.$$

Thus, using the Hölder inequality,

$$|\langle \frac{\partial v}{\partial x_1}, \phi\rangle| \leqslant \|f\|_{L^m(\Omega)}\|\psi\|_{L^{m'}(\Omega)},$$

and by lemma 2.16, setting $r = m'$, we reach that

$$|\langle \frac{\partial v}{\partial x_1}, \phi \rangle| \leqslant C\|f\|_{L^m(\Omega)}\|\phi\|_{L^\beta(\Omega)} \quad \text{for} \quad \frac{1}{\beta} + \frac{N-2s+1}{N} < 1 + \frac{1}{m'}.$$

Therefore, by duality, $\frac{\partial v}{\partial x_1} \in L^{\beta'}(\Omega)$ and

$$\|\frac{\partial v}{\partial x_1}\|_{L^{\beta'}(\Omega)} \leqslant C\|f\|_{L^\beta(\Omega)}.$$

Since $m' < \frac{\beta N}{N-\beta(2s-1)}$, then $\beta' < \frac{pN}{N-p(2s-1)}$ and the result follows. It is clear that if $m = \frac{N}{2s-1}$, then we can take β' any positive constant and then $|\nabla v| \in L^p(\Omega)$ for all $p < \infty$.

We thereby prove point (2).

Note that if $m > \frac{N}{2s-1}$, then $|\nabla v| \in W^{2s-\gamma-1,l}(\Omega)$, where $0 < \gamma < 2s-1$ and $(2s-\gamma-1)l > N$.

Indeed, we proceed with similar duality arguments to those in the proof of theorem 2.12 in [19]. Let $\phi \in \mathcal{C}_0^\infty(\Omega)$; then

$$|\int_\Omega \phi(-\Delta)^s v \mathrm{d}x| = |\int_\Omega f\phi \mathrm{d}x| \leqslant \|f\|_{L^m(\Omega)}\|\phi\|_{L^{m'}(\Omega)}.$$

Setting $l = \frac{Nm}{N-\gamma m}$ with $\gamma \in [0,1]$, such that $\gamma < \min\{(2s-1), \frac{N}{m}\}$, then $m' = \frac{l'N}{N-l'\gamma}$. It is clear that $\frac{N}{2s-1} < m < l$.

Now, using the definition of q and by the Sobolev inequality, it follows that

$$|\int_\Omega \phi(-\Delta)^s v \mathrm{d}x| \leqslant \|f\|_{L^m(\Omega)}\|\phi\|_{L^{\frac{l'N}{N-l'\gamma}}(\Omega)} \leqslant C\|f\|_{L^m(\Omega)}\|\phi\|_{W_0^{\gamma,l'}(\Omega)}.$$

Thus, $(-\Delta)^s v \in W^{-\gamma,l}(\Omega)$, and

$$\|(-\Delta)^s v\|_{W^{-\gamma,l}(\Omega)} \leqslant C\|f\|_{L^m(\Omega)}.$$

Recalling that $(-\Delta)^s$, realize an isomorphism between $W^{2s-\gamma,l}(\Omega)$ and $W^{-\gamma,l}(\Omega)$; hence, we conclude that $v \in W^{2s-\gamma,l}(\Omega)$ and

$$\|v\|_{W^{2s-\gamma,l}(\Omega)} \leqslant C\|f\|_{L^m}.$$

Since $2s-\gamma > 1$, then $|\nabla v| \in W^{2s-\gamma-1,l}(\Omega)$. Now, using the definition of l, we reach easily that $(2s-\gamma-1)l > N$. Thus, by the fractional Morrey inequality, $|\nabla v| \in \mathcal{C}^{0,\sigma}(\Omega)$, with $\sigma = \frac{N}{(2s-\gamma-1)l}$ and

$$\|v\|_{\mathcal{C}^{1,\sigma}(\Omega)} \leqslant C\|f\|_{L^m(\Omega)}. \qquad \square$$

3. Comparison principle and applications

In this section, we will prove a comparison principle that extends the one proved in [4] in the local case. More precisely, we will prove the following result.

Theorem 3.1 (Comparison principle). *Let $g \in L^1(\Omega)$ be a non-negative function. Assume that for all $\xi_1, \xi_2 \in \mathbb{R}^N$,*

$$H : \Omega \times \mathbb{R}^N \to \mathbb{R}^+ \text{ verifies } |H(x,\xi_1) - H(x,\xi_2)| \leqslant Cb(x)|\xi_1 - \xi_2|,$$

where $b \in L^\sigma(\Omega)$ for some $\sigma > \frac{N}{2s-1}$. Consider w_1, w_2: two positive functions such that $w_1, w_2 \in W^{1,p}(\Omega)$ for all $p < p_*$, $(-\Delta)^s w_1, (-\Delta)^s w_2 \in L^1(\Omega)$, $w_1 \leqslant w_2$ in $I\!R^N \setminus \Omega$ and

$$\begin{cases} (-\Delta)^s w_1 & \leqslant & H(x, \nabla w_1) + g & \text{in } \Omega, \\ (-\Delta)^s w_2 & \geqslant & H(x, \nabla w_2) + g & \text{in } \Omega. \end{cases} \tag{3.1}$$

Then $w_2 \geqslant w_1$ in Ω.

In order to prove theorem 3.1, we will follow the arguments used by Porretta in [42] for differential equations, and that we shall need to prove some results on problems with first order terms.

Let us begin by recalling the following Harnack inequality, proved in [14].

Proposition 3.2 (Harnack inequality). *Assume that $B \in (L^\sigma(\Omega))^N$ with $\sigma > \frac{N}{2s-1}$ and let $w \in \mathcal{C}^{1,\alpha}(\Omega)$ be a non-negative function in $I\!R^N$ such that*

$$(-\Delta)^s w - \langle B(x), \nabla w \rangle = 0 \text{ in } B_R$$

with $B_R \subset\subset \Omega$. Then

$$\sup_{B_R} w \leqslant C \inf_{B_R} w,$$

where $C \equiv C(\Omega, B_R)$.

3.1. A uniqueness result for a related problem

We prove the following uniqueness result.

Lemma 3.3. *Let B be a vector field in Ω. Assume that $B \in (L^\sigma(\Omega))^N$ with $\sigma > \frac{N}{2s-1}$, and let w be a solution to the problem*

$$\begin{cases} (-\Delta)^s w & = & \langle B(x), \nabla w \rangle & \text{in } \Omega, \\ w & = & 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3.2}$$

with $|\nabla w| \in M^{p_*}(\Omega)$, the Marcinkiewicz space; then $w = 0$.

Proof. We claim that $w \in \mathcal{C}^{1,\alpha}(\Omega)$ for some $\alpha \in (0,1)$.

We divide the proof of the claim into two steps:

First step. Suppose that $|\nabla w| \in L^m(\Omega)$ with $m = \frac{N\sigma}{(2s-1)\sigma - N}$; then, setting $h(x) = |\langle B(x), \nabla w \rangle|$, and taking into account the regularity of B, it follows that $h \in L^{\frac{N}{2s-1}}(\Omega)$. Going back to (3.2), and by the first point in lemma 2.15, we conclude that $|\nabla w| \in L^p(\Omega)$ for all $p < \infty$. Thus, $h \in L^\sigma(\Omega)$. Since $\sigma > \frac{N}{2s-1}$, by the second point in lemma 2.15, we conclude that $|\nabla w| \in \mathcal{C}^{0,\alpha}(\Omega)$ for some $\alpha \in (0,1)$, and the result follows.
Second step. We prove the regularity result of the first step: specifically, that $|\nabla w| \in L^{\frac{N\sigma}{(2s-1)\sigma - N}}(\Omega)$.

We will use a bootstrapping argument.

Since $|\nabla w| \in M^{p_*}(\Omega)$, $h \in L^{l_1}(\Omega)$ for $1 < l_1 < \frac{p_*\sigma}{\sigma+p_*}$. Fix l_1 as above; then, using lemma 2.16, it follows that $|\nabla w| \in L^{r_1}(\Omega)$, with $r_1 = \frac{l_1 N}{N-l_1(2s-1)}$. Hence, by the Hölder inequality, we conclude that $h \in L^{l_2}(\Omega)$ with $l_2 = \frac{r_1\sigma}{\sigma+r_2}$. Using again lemma 2.16, it follows that $|\nabla w| \in L^{r_2}(\Omega)$ with $r_2 = \frac{l_2 N}{N-l_2(2s-1)}$.

It is clear that $r_2 > r_1$. Define by iteration the sequence $\{r_n\}_n$ by

$$r_{n+1} = \frac{N\sigma r_n}{N\sigma - r_n(\sigma(2s-1)-N)}.$$

If, for some n_0, $r_{n_0} \geqslant \frac{\sigma N}{(2s-1)\sigma - N}$, then the result follows.

We argue by contradiction. Assume that $r_n < \frac{\sigma N}{(2s-1)\sigma-N}$ for all n. It is easy to show that $\{r_n\}_n$ is an increasing sequence. Hence, there exists $\bar{r}$ such that $r_n \uparrow \bar{r} \leqslant \frac{\sigma N}{(2s-1)\sigma-N}$. Thus, $\bar{r} = \frac{N\sigma\bar{r}}{N\sigma-\bar{r}(\sigma(2s-1)-N)}$; hence, $\bar{r} = 0$, contradicting the fact that $\{r_n\}_n$ is an increasing sequence.

Therefore, there exists $n_0 \in \mathbb{N}$ such that $r_{n_0} \geqslant \frac{\sigma N}{(2s-1)\sigma-N}$, and the claim follows.

Let us prove now that $w \leqslant 0$. If, by contradiction, $C = \max_{x\in\Omega} w(x) > 0$, then there exists $x_0 \in \Omega$ such that $w(x_0) = C$. We set $w_1 = C - w$; then $w_1 \geqslant 0$ in $\mathbb{R}^N$, $w_1(x_0) = 0$ and

$$(-\Delta)^s w_1 - B(x)\nabla w_1 = 0 \text{ in } \Omega.$$

Consider $B_R = B_r(x_0) \subset\subset \Omega$. By applying the Harnack inequality in proposition 3.2 to w_1, we conclude that $\sup_{B_r(x_0)} w_1 \leqslant C(\Omega, B_r) \inf_{B_r(x_0)} w_1 = 0$. Thus $w_1 \equiv 0$ in $B_r(x_0)$. Since Ω is a bounded domain, we apply the Harnack inequality, and prove in a finite number of steps that $w_1 = 0$ in Ω. Thus, $C \leqslant 0$, and so $w \leqslant 0$.

The linearity of the problem makes it possible to apply similar arguments to $-w$ (which is also a solution to (3.2)). Thus $w \equiv 0$, and the result follows. □

3.2. Existence for an auxiliary problem

Let us prove now the following existence result for an auxiliary problem.

Lemma 3.4. *Assume that $B \in (L^{\sigma_1}(\Omega))^N$ with $\sigma_1 > \frac{N}{2s-1}$ and $0 \leqslant f \in L^{\sigma_2}(\Omega)$ with $\sigma_2 > \frac{N}{2s}$, then the problem*

$$\begin{cases} (-\Delta)^s w &= \langle B(x), \nabla w\rangle + f \quad \text{in } \Omega, \\ \qquad w &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3.3}$$

has a unique non-negative bounded solution w such that $|\nabla w| \in L^a(\Omega)$ for all $a < \infty$. In addition, if $\sigma_2 > \frac{N}{2s-1}$, then $\nabla w \in (\mathcal{C}^{0,\alpha}(\Omega))^N$ for some $\alpha \in (0, 1)$.

Proof. It is clear that the regularity of the solution follows using the same iteration argument as in the proof of the regularity result in lemma 3.3. Let us prove the existence part.

Fix $p < p_*$ be such that $p' < \sigma_1$, and define the operator $T : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$ by setting $w = T(u)$, where w solves

$$\begin{cases} (-\Delta)^s w &= \langle B(x), \nabla u_+\rangle + f \quad \text{in } \Omega, \\ w &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{3.4}$$

Since $u \in W_0^{1,p}(\Omega)$, then the existence and uniqueness of w follow by an approximating argument and the results of [20]. It is clear that if w is a fixed point of T, then w is a non-negative solution to (3.3). To show that T has a fixed point, we will use the Schauder fixed point theorem (see Theorem 11.3 in [27]).

From the result of [20], we conclude that T is a compact operator.

We claim that there exists $M > 0$ such that if $u = \lambda T(u)$, with $\lambda \in [0, 1]$, then $\|u\|_{W_0^{1,p}(\Omega)} \leqslant M$.

To prove the claim, we argue by contradiction.

Assume that there exist sequences $\{\lambda_n\} \subset [0, 1]$ and $\{u_n\}_n$, such that $u_n = \lambda_n T(u_n)$ and $\|u\|_{W_0^{1,p}(\Omega)} \to \infty$ as $n \to \infty$. Define $v_n = \frac{u_n}{\|u_n\|_{W_0^{1,p}(\Omega)}}$, then $\|v_n\|_{W_0^{1,p}(\Omega)} = 1$, and v_n solves

$$\begin{cases} (-\Delta)^s v_n &= \langle B(x), \nabla (v_n)_+\rangle + \lambda_n \frac{f}{\|u_n\|_{W_0^{1,p}(\Omega)}} \quad \text{in } \Omega, \\ v_n &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{3.5}$$

It is clear that $v_n \geqslant 0$ and $\|\langle B(x), \nabla(v_n)_+\rangle\|_{L^1(\Omega)} \leqslant C$; thus, for all $l < p_*$, there exists a positive constant C, depending only on the data and independents of the sequence $\{v_n\}_n$, such that

$$\|v_n\|_{W_0^{1,l}(\Omega)} < C,$$

and

$$\|\nabla v_n\|_{W_0^{2s-1-\gamma,l}(\Omega)} \leqslant C \text{ where } \gamma = \frac{N}{l}.$$

Up to a subsequence, we find that $v_n \rightharpoonup v$ weakly in $W_0^{1,l}(\Omega)$ for all $l < p_*$, $v \in W_0^{1,l}(\Omega)$, $v \geqslant 0$, and v solves

$$\begin{cases} (-\Delta)^s v &= \langle B(x), \nabla v\rangle \quad \text{in } \Omega, \\ v &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

From lemma 3.3, we obtain that $v = 0$; and from the compactness result of [20], we obtain that $v_n \to v$ strongly in $W_0^{1,l}(\Omega)$ for all $l < p_*$; in particular, for $l = p$. Then $\|v\|_{W_0^{1,p}(\Omega)} = 1$, which is a contradiction. Hence, the claim follows.

Thus T has a fixed point, and thus problem (3.3) has a non-negative solution.

The uniqueness immediately follows. Indeed, if w_1 and w_2 are two solutions to problem (3.3), then $\tilde{w} = w_1 - w_2$ solves

$$\begin{cases} (-\Delta)^s \tilde{w} &= \langle B(x), \nabla \tilde{w}\rangle \quad \text{in } \Omega, \\ \tilde{w} &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

By lemma 3.3, we know that $\tilde{w} = 0$, thus $w_1 = w_2$. □

Remark 2. We are able to prove the existence result in lemma 3.4 without the positivity condition on f, in fact, consider w_1 and w_2 the solutions to problem (3.3) with data f_+ and f_- respectively. Set $w = w_1 - w_2$; then w solves

$$(-\Delta)^s w = \langle B(x), \nabla w\rangle + f \text{ in } \Omega,$$

with the same regularity.

A remarkable result derives from the following observations.

Since $s > \frac{1}{2}$, we set $E = W^{2s,2}(\Omega) \cap H_0^s(\Omega)$. Then, if $w \in E$, it holds that $|\nabla w| \in W^{2s-1,2}(\Omega)$; hence, $|\nabla w| \in L^{\frac{2N}{N-2(2s-1)}}(\Omega)$. By hypothesis $B \in (L^{\sigma_1}(\Omega))^N$ with $\sigma_1 > \frac{N}{2s-1}$, then $|\langle B(x), \nabla w\rangle| \in L^2(\Omega)$ for all $w \in E$.

Define $L(w) = (-\Delta)^s w - \langle B(x), \nabla w\rangle$; then

$$L : L^2(\Omega) \to L^2(\Omega), \text{ with } Dom(L) = E.$$

By lemma 3.3, we have that $Ker(L) = \{0\}$. Thus, $L^{-1} : L^2(\Omega) \to E$ is a well defined compact operator. Therefore, by the Fredholm alternative theorem we conclude that for all $f \in L^2(\Omega)$, the problem (3.4) has a unique solution $u \in E$.

Now, observe that for $u, v \in E$, we have

$$\begin{aligned}\langle L(u), v\rangle &= \int_\Omega (-\Delta)^s u\, v \mathrm{d}x - \int_\Omega \langle B(x), \nabla u\rangle v \,\mathrm{d}x \\ &= \int_\Omega u(-\Delta)^s v \mathrm{d}x - \int_\Omega \text{ div } (B(x)v) u \mathrm{d}x \\ &= \int_\Omega u\Big((-\Delta)^s v - \text{ div } (B(x)v)\Big) u \mathrm{d}x.\end{aligned}$$

Hence, by defining

$$K(v) = (-\Delta)^s v - \text{ div } (B(x)v),$$

we have

$$\langle K(v), u\rangle = \langle v, Lu\rangle;$$

that is, K is the adjoint operator of L. Since $0 = \text{ dim Ker } (L) = \text{ dim Ker}(K)$, then for all $f \in L^2(\Omega)$, the problem

$$\begin{cases} (-\Delta)^s v - \text{ div } (B(x)v) &= f \quad \text{in } \Omega, \\ \qquad\qquad v &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3.6}$$

has a unique solution $u \in E$. In particular, using the same regularity techniques as in lemmas 2.16 and 3.3, we find the following consequence.

Corollary 3.5. *For all $f \in L^{\sigma_2}(\Omega)$ with $\sigma_2 > \frac{N}{2s}$, there exists a unique solution v to problem (3.6), such that $|\nabla v| \in L^p(\Omega)$ for all $p < \infty$. In addition, if $\sigma_2 > \frac{N}{2s-1}$, then $\nabla v \in (\mathcal{C}^{0,\alpha}(\Omega))^N$ for some $\alpha \in (0, 1)$. If $f \geqslant 0$, then $v \geqslant 0$.*

3.3. Proof of theorem 3.1

We are now in a position to prove the comparison principle in theorem 3.1.

Consider $w = w_1 - w_2$, then $w \in W^{1,p}(\Omega)$ for all $p < p_*$ and $(-\Delta)^s w \in L^1(\Omega)$.

We have just to show that $w^+ = 0$. It is clear that $w \leqslant 0$ in $\mathbb{R}^N \setminus \Omega$. By (3.1), it follows that

$$(-\Delta)^s w \quad \leqslant \quad H(x, \nabla w_1) - H(x, \nabla w_2) \leqslant b(x)|\nabla w|.$$

Now, using Kato's inequality (see, for instance, [35]) we get

$$(-\Delta)^s w_+ \leqslant b(x)|\nabla w_+|, \;\; w_+ = \max\{\mathrm{w}, 0\} \in \mathrm{W}_0^{1,\mathrm{q}}(\Omega) \text{ for all } \mathrm{q} < \mathrm{p}_*. \tag{3.7}$$

Let v be the unique positive bounded solution to problem

$$\begin{cases} (-\Delta)^s v + \operatorname{div}(\mathcal{F}(x)v) &= 1 \quad \text{in } \Omega, \\ v &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3.8}$$

where

$$\mathcal{F}(x) = \begin{cases} b(x)\frac{\nabla w_+(x)}{|\nabla w_+(x)|} & \text{if } |\nabla w_+(x)| \neq 0 \\ 0 & \text{if } |\nabla w_+(x)| = 0. \end{cases}$$

Recall that $b \in L^\sigma(\Omega)$ for some $\sigma > \frac{N}{2s-1}$; thus, $\mathcal{F} \in L^\sigma(\Omega)$. Now, using corollary 3.5 with $f \equiv 1$, it holds that $\nabla v \in (\mathcal{C}^{0,\alpha}(\Omega))^N$ for some $\alpha \in (0, 1)$.

The main idea is to use v as a test function in (3.7). To show that we have to prove that the term $\int v(-\Delta)^s w_+ \mathrm{d}x$ is well defined.

It is well known that $(-\Delta)^s$ realizes an isomorphism between $W^{2s-\gamma,l}(\Omega)$ and $W^{-\gamma,l}(\Omega)$. See, for instance, [21]. Since $2s > 1$, we can choose $\gamma = 1$. Now, taking into account that $w_+ \in W_0^{1,q}(\Omega)$ for all $q < p_*$, it follows that $w_+ \in W_0^{2s-1,q}(\Omega)$. Thus, $(-\Delta)^s w_+ \in W^{-1,q}(\Omega)$. Since $v \in W_0^{1,l}(\Omega)$, we can choose $l = q'$; then

$$\left| \int v(-\Delta)^s w_+ \mathrm{d}x \right| \leqslant C\|w_+\|_{W_0^{1,q}(\Omega)} \|v\|_{W^{1,q'}(\Omega)} < \infty.$$

In the same way, we can show that the term $\int w_+(-\Delta)^s v \mathrm{d}x$ is well defined, and that

$$\left| \int w_+(-\Delta)^s v \,\mathrm{d}x \right| \leqslant \|w_+\|_{W_0^{1,q}(\Omega)} \|v\|_{W^{1,q'}(\Omega)} < \infty.$$

As a consequence, we can use v as a test function in (3.7), and then it follows that

$$\int_\Omega (-\Delta)^s w_+ v \mathrm{d}x \leqslant \int_\Omega b(x)|\nabla w_+(x)| v(x) \mathrm{d}x.$$

On the other hand we have

$$\begin{aligned} \int_\Omega (-\Delta)^s w_+ v \mathrm{d}x &= \int_\Omega w_+(-\Delta)^s v \mathrm{d}x \\ &= \int_\Omega w_+(-\operatorname{div} \mathcal{F}(x)v) \mathrm{d}x + \int_\Omega w_+ \mathrm{d}x \\ &= \int_\Omega b(x)|\nabla w_+(x)| v(x) \mathrm{d}x + \int_\Omega w_+ \mathrm{d}x. \end{aligned}$$

Hence, $\int_\Omega w_+ \leqslant 0$, and so $w \leqslant 0$ in Ω. Thus, we conclude the proof. □

As byproducts of the previous result, we obtain the following uniqueness results.

Corollary 3.6. *Let $g \in L^1(\Omega)$ be a non-negative function. Suppose that $q \geqslant 1$ and $a > 0$, then the problem*

$$\begin{cases} (-\Delta)^s w &= \frac{|\nabla w|^q}{a+|\nabla w|^q} + g & \text{in } \Omega, \\ w &= 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3.9}$$

has a unique non-negative solution w, such that $w \in W_0^{1,\theta}(\Omega)$ for all $\theta < p_*$, and $T_k(w) \in H_0^s(\Omega) \cap W^{1,\alpha}(\Omega)$ for all $\alpha < 2s$.

Corollary 3.7. *Consider the problem*

$$\begin{cases} (-\Delta)^s w &= |\nabla w|^q + \lambda g & \text{in } \Omega, \\ w &= 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3.10}$$

with $1 < q < p_*$ and $g \in L^1(\Omega)$, $g \geqslant 0$. Then there exists λ^* such if $\lambda < \lambda^*$, problem (3.10) has a unique positive solution w, such that $w \in W_0^{1,\theta}(\Omega)$ for all $\theta < p_*$, and $T_k(w) \in H_0^s(\Omega)$ for all $k > 0$.

Proof. The existence and regularity can be seen in [20]. We prove the uniqueness. Indeed, if w_1 and w_2 are two positive solutions to problem (3.10) with the above regularity, defining $\bar{w} = w_1 - w_2$, then $(-\Delta)^s \bar{w} \in L^1(\Omega)$, $\bar{w} \in W_0^{1,q}(\Omega)$ for all $q < p_*$, $T_k(\bar{w}) \in H_0^s(\Omega)$ for all $k > 0$ and

$$(-\Delta)^s \bar{w} \leqslant H(x, \nabla w_1) - H(x, \nabla w_2) \leqslant b(x)|\nabla \bar{w}|,$$

where $b(x) = q\Big(|\nabla w_1| + |\nabla w_2|\Big)^{q-1}$. Since $q < p_*$, $q' > p_*'$; hence, $b \in L^\sigma(\Omega)$ for some $\sigma > \frac{N}{2s-1}$. Thus, using the comparison principle in lemma 3.1, we conclude that $\bar{w}_+ = 0$. In the same way, setting $\widehat{w} = w_2 - w_1$, we obtain that $\widehat{w}_+ = 0$. Thus $w_1 = w_2$. $\square$

In particular, we can state the following comparison principle.

Theorem 3.8. *Assume that $g \in L^1(\Omega)$ is a non-negative function. Let w_1, w_2 be two non-negative functions such that $w_1, w_2 \in W^{1,p}(\Omega)$ for some $1 \leqslant q < p_*$, $(-\Delta)^s w_1, (-\Delta)^s w_2 \in L^1(\Omega)$, $w_1 \leqslant w_2$ in $I\!R^N \setminus \Omega$ and*

$$\begin{cases} (-\Delta)^s w_1 &\leqslant |\nabla w_1|^q + g & \text{in } \Omega, \\ (-\Delta)^s w_2 &\geqslant |\nabla w_2|^q + g & \text{in } \Omega. \end{cases} \tag{3.11}$$

Then $w_2 \geqslant w_1$ in Ω.

As a consequence, if $1 \leqslant q < p_*$ and $g \equiv 0$, then the unique weak solution to problem (3.10) is $w = 0$.

4. The subcritical problem: existence results via comparison arguments

In this section, we consider the problem

$$\begin{cases} (-\Delta)^s u &= |\nabla u|^q + \lambda f & \text{in } \Omega, \\ u &= 0 & \text{in } \mathbb{R}^N \setminus \Omega, \\ u &> 0 & \text{in } \Omega, \end{cases} \tag{4.1}$$

where $s \in (\frac{1}{2}, 1)$, $q < 2s$ and $f \in L^{\sigma}(\Omega)$ for some convenient $\sigma > 1$. The main goal of this section is to show that, under additional conditions on f, we are able to build a suitable supersolution, whence the comparison principle in theorem 3.1 allows us to use a monotony argument to prove the existence of a minimal positive solution.

Remark 3. Notice that in the local case, the existence of a solution is guaranteed under the condition

$$\inf_{\phi \in \mathcal{C}_0^{\infty}(\Omega)} \frac{\int_{\Omega} |\nabla \phi|^{q'} \mathrm{d}x}{\int_{\Omega} f |\phi|^{q'} \mathrm{d}x} > 0. \tag{4.2}$$

By the spectral theory, it is clear that the above condition holds if $f \in L^{\sigma}(\Omega)$ for some $\sigma > \frac{N}{q'}$.

4.1. A radial supersolution

We will start by building a radial supersolution with the required regularity.

Define $w(x) = (1 - |x|^{\alpha})_+$, where $1 < \alpha < 2s$. Since w is a radial function, by closely following the computations in [24], it can be seen that

$$(-\Delta)^s w(x) = r^{-2s} \int_1^{\infty} \left((w(r) - w(\sigma r)) + (w(r) - w(\frac{r}{\sigma}))\sigma^{2s-N} \right) \sigma(\sigma^2 - 1)^{-1-2s} H(\sigma) \mathrm{d}\sigma,$$

where $r = |x|$ and H is a continuous positive function defined on $[1, \infty)$, with $H(\sigma) \simeq \sigma^{2s}$ as $\sigma \to \infty$. Leaving $r_0 < 1$ to be chosen later, for $r < r_0$, we have

$$\begin{aligned}
(-\Delta)^s w(x) &= r^{-2s} \int_1^{\frac{1}{r}} \left(r^{\alpha}(\sigma^{\alpha} - 1) + r^{\alpha}(\frac{1}{\sigma^{\alpha}} - 1)\sigma^{2s-N} \right) \sigma(\sigma^2 - 1)^{-1-2s} H(\sigma) \mathrm{d}\sigma \\
&\quad + r^{-2s} \int_{\frac{1}{r}}^{\infty} \left((1 - r^{\alpha}) + r^{\alpha}(\frac{1}{\sigma^{\alpha}} - 1)\sigma^{2s-N} \right) \sigma(\sigma^2 - 1)^{-1-2s} H(\sigma) \mathrm{d}\sigma \\
&= r^{\alpha-2s} \int_1^{\frac{1}{r}} (\sigma^{\alpha} - 1)(1 - \sigma^{2s-N-\alpha}) \sigma(\sigma^2 - 1)^{-1-2s} H(\sigma) \mathrm{d}\sigma \\
&\quad + r^{\alpha-2s} \int_{\frac{1}{r}}^{\infty} \left((\frac{1}{r^{\alpha}} - 1) - (1 - \frac{1}{\sigma^{\alpha}})\sigma^{2s-N} \right) \sigma(\sigma^2 - 1)^{-1-2s} H(\sigma) \mathrm{d}\sigma \\
&= r^{\alpha-2s} F(r)
\end{aligned}$$

with

$$\begin{aligned}
F(r) = &\int_1^{\frac{1}{r}} (\sigma^{\alpha} - 1)(1 - \sigma^{2s-N-\alpha}) \sigma(\sigma^2 - 1)^{-1-2s} H(\sigma) \mathrm{d}\sigma \\
&+ \int_{\frac{1}{r}}^{\infty} \left((\frac{1}{r^{\alpha}} - 1) - (1 - \frac{1}{\sigma^{\alpha}})\sigma^{2s-N} \right) \sigma(\sigma^2 - 1)^{-1-2s} H(\sigma) \mathrm{d}\sigma.
\end{aligned}$$

We claim that $F(r) \geqslant C(r_0) > 0$ for all $r \in (0, r_0)$. By a direct computation, we find that $F'(r) < 0$; hence, to conclude, we have only to show that $F(r_0) \geqslant C(r_0)$ for suitable $r_0 < 1$.

Note that $(\sigma^\alpha - 1)(1 - \sigma^{2s-N-\alpha})\sigma(\sigma^2 - 1)^{-1-2s}H(\sigma) > 0$ for all $\sigma > 1$. On the other hand, we have $(1 - \frac{1}{\sigma^\alpha})\sigma^{2s-N} \leqslant (1 - \frac{1}{\sigma_0^\alpha})\sigma_0^{2s-N}$, where $\sigma_0 = (\frac{N+\alpha-2s}{N-2s})^{\frac{1}{\alpha}}$. Thus, for $\sigma > \frac{1}{r}$, we have

$$(\frac{1}{r^\alpha} - 1) - (1 - \frac{1}{\sigma^\alpha})\sigma^{2s-N} \geqslant (\frac{1}{r^\alpha} - 1) - (1 - \frac{1}{\sigma_0^\alpha})\sigma_0^{2s-N}.$$

Defining $r_0 \equiv \left(\frac{N+\alpha-2s}{N+\alpha-2s+\sigma_0^{2s-N}}\right)^{\frac{1}{\alpha}} < 1$, for $r \leqslant r_0$ and $\sigma \geqslant \frac{1}{r}$, it holds that

$$(\frac{1}{r^\alpha} - 1) - (1 - \frac{1}{\sigma^\alpha})\sigma^{2s-N} \geqslant 0.$$

Combining the above estimates, we reach that $F(r) \geqslant C(r_0) > 0$ for all $r \leqslant r_0$. Note that $|\nabla w| = \alpha |x|^{\alpha-1}\chi_{\{|x|<1\}}$. Since $1 < \alpha < 2s$, setting $w_1 = Cw$ for some $C > 0$, we obtain that w_1 satisfies

$$\begin{cases} (-\Delta)^s w_1 & \geqslant & |\nabla w_1|^q + \lambda \frac{1}{|x|^{2s-\alpha}} & \text{in } B_r(0), \\ w_1 & \geqslant & 0 & \text{in } \mathbb{R}^N \setminus B_r(0), \\ w_1 & > & 0 & \text{in } B_r(0), \end{cases} \tag{4.3}$$

for some $\lambda > 0$.

It is clear that, modulo a rescaling argument, the above construction holds in any bounded domain.

In the case $p_* < q < 2s$, we can guess a positive supersolution of the form

$$S(x) = A|x|^{-\alpha}, \quad A > 0, \quad 0 < \alpha < N - 2s.$$

By direct calculation, we obtain

$$(-\Delta)^s S(x) = C_{N,s}(\alpha) A |x|^{-\alpha-2s}, \quad C_{N,s}(\alpha) > 0,$$

and $|\nabla S(x)| \leqslant A\alpha |x|^{-(\alpha+1)}$.

Therefore, to have a radial solution in the whole $\mathbb{R}^N$, the following identity must be verified:

$$C_{N,s}(\alpha)|x|^{-\alpha-2s} \geqslant \alpha^q A^{q-1} |x|^{-(\alpha+1)} \text{ for all } x \in \mathbb{R}^N,$$

that is, necessarily $\alpha = \frac{2s-q}{q-1}$, and the condition $q < 2s$ appears in a natural way.

Hence, it is sufficient to pick up A such that $\alpha^q A^{q-1} \leqslant C_{N,s}(\alpha)$.

If the source term $f \in L^\infty(\Omega)$, then we just have to choose $\lambda > 0$ small enough, in order to have a supersolution S in Ω.

Since $q > \frac{N}{N-2s+1} \equiv p_*$, then $S \in W^{1,q}_{loc}(\mathbb{R}^N)$ and any translation $\tilde{S}(x) = S(x - x_0)$ is also a supersolution, thus choosing $x_0 \in I\!R^N \backslash \overline{\Omega}$, $\tilde{S}(x)$ is a bounded supersolution.

4.2. A first result on the existence of a weak solution

We have the next existence result.

Theorem 4.1. *Assume $f \in L^\infty(\Omega)$. Let w be a bounded supersolution to (4.1), such that $w \in W^{1,q}_0(\Omega) \cap L^\infty(\Omega)$. Then problem (4.1) has a solution u such that $u \in W^{1,\alpha}_0(\Omega)$ for all $\alpha < 2s$.*

Proof. Let u_n be the unique solution to the approximation problem

$$\begin{cases} (-\Delta)^s u_n &= \frac{|\nabla u_n|^q}{1+\frac{1}{n}|\nabla u_n|^q} + \lambda f \quad \text{in } \Omega, \\ u_n &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{4.4}$$

By the comparison principle in theorem 3.1, it follows that $u_n \leqslant u_{n+1} \leqslant w$ for all n. Hence, there exists u such that $u_n \uparrow u$ strongly in $L^{q^*}(\Omega)$. Let $g_n(x) = \frac{|\nabla u_n|^q}{1+\frac{1}{n}|\nabla u_n|^q} + \lambda f$ and define ρ to be the unique solution to the problem

$$\begin{cases} (-\Delta)^s \rho &= 1 \quad \text{in } \Omega, \\ \rho &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{4.5}$$

Using ρ as a test function in (4.4), and since $u_n \leqslant w$, it follows that

$$\int_\Omega g_n(x)\rho \leqslant C \text{ for all } n.$$

We claim that the sequence $\{u_n\}_n$ is bounded in $W_0^{1,a}(\Omega)$ for all $a < 2s$. We follow closely the same ideas as in the proof of theorem 2.14. We have that

$$u_n(x) = \int_\Omega \mathcal{G}_s(x,y) g_n(y) \mathrm{d}y.$$

Hence,

$$|\nabla u_n(x)| \leqslant \int_\Omega |\nabla_x \mathcal{G}_s(x,y)| g_n(y) \mathrm{d}y.$$

Fixing $1 < \alpha < 2s$, and defining $h(x,y) = \max\left\{\frac{1}{|x-y|}, \frac{1}{d(x)}\right\}$, it follows that

$$\begin{aligned} |\nabla u_n(x)|^\alpha &\leqslant \Big(\int_\Omega |\nabla_x \mathcal{G}_s(x,y)| g_n(y)\mathrm{d}y\Big)^\alpha \leqslant \Big(\int_\Omega h(x,y)\mathcal{G}_s(x,y) g_n(y)\mathrm{d}y\Big)^\alpha \\ &\leqslant \Big(\int_\Omega (h(x,y))^\alpha \mathcal{G}_s(x,y) g_n(y)\mathrm{d}y\Big)\Big(\int_\Omega \mathcal{G}_s(x,y) g_n(y)\mathrm{d}y\Big)^{\alpha-1} \\ &\leqslant \Big(\int_\Omega (h^\alpha(x,y)\mathcal{G}_s(x,y) g_n(y)\mathrm{d}y\Big) u_n^{\alpha-1}(x) \\ &\leqslant \int_\Omega \Big(h^\alpha(x,y)\mathcal{G}_s(x,y) g_n(y)\mathrm{d}y\Big) w^{\alpha-1}(x) \\ &\leqslant \int_\Omega \Big(h^\alpha(x,y)\mathcal{G}_s(x,y)|\nabla u_n(y)|^q\mathrm{d}y\Big) w^{\alpha-1}(x) + \lambda \int_\Omega \Big(h^\alpha(x,y)\mathcal{G}_s(x,y) f(y)\mathrm{d}y\Big) w^{\alpha-1}(x). \end{aligned}$$

Thus,

$$\begin{aligned} \int_\Omega |\nabla u_n|^\alpha \mathrm{d}x \leqslant & \int_\Omega |\nabla u_n(y)|^q \Big(\int_\Omega h^\alpha(x,y)\mathcal{G}_s(x,y) w^{\alpha-1}(x)\Big)\mathrm{d}y \\ & + \lambda \int_\Omega f(y)\Big(\int_\Omega h^\alpha(x,y)\mathcal{G}_s(x,y) w^{\alpha-1}(x)\mathrm{d}x\Big)\mathrm{d}y \\ \equiv & J_1 + J_2. \end{aligned}$$

Since $w \in L^\infty(\Omega)$ and $\alpha < 2s$, following the same computations as in the proof of theorem 2.14, we reach that

$$J_1 \leqslant C \int_\Omega |\nabla u_n(y)|^q \Big(\int_\Omega h^\alpha(x,y)\mathcal{G}_s(x,y)\mathrm{d}x\Big)\mathrm{d}y \leqslant C \int_\Omega |\nabla u_n(y)|^q \mathrm{d}y$$

and

$$J_2 \leqslant C \int_\Omega f(y)\mathrm{d}y.$$

Therefore, we conclude that

$$\int_\Omega |\nabla u_n(x)|^\alpha \mathrm{d}x \leqslant C_1 \int_\Omega |\nabla u_n(x)|^q \mathrm{d}x + C_2.$$

Choosing $\alpha > q$, by the Hölder inequality, we obtain that

$$\int_\Omega |\nabla u_n(x)|^\alpha \mathrm{d}x \leqslant C \text{ for all } n.$$

As a consequence, we get that $\{g_n\}_n$ is bounded in $L^{1+\varepsilon}(\Omega)$ for some $\varepsilon > 0$. By the compactness result in proposition 2.12, we obtain that, up to a subsequence, $u_n \to u$ strongly in $W_0^{1,r}(\Omega)$ for all $r < p_*$, and $|\nabla u_n| \to |\nabla u|$ a.e. in Ω. Hence, by Vitali's lemma we reach that $u_n \to u$ strongly in $W_0^{1,\alpha}(\Omega)$ for all $\alpha < 2s$; in particular, for $\alpha = q$. Thus u is a solution to (4.1) with $u \in W_0^{1,\alpha}(\Omega)$ for all $a < 2s$. □

As a consequence of the above Theorem and the construction of the supersolution at the beginning of this subsection, we get the following result.

Corollary 4.2. *Assume that $q < 2s$, and that $f \in L^\infty(\Omega)$ with $f \gneqq 0$. Then there exists $\lambda^* > 0$, such that for all $\lambda < \lambda^*$, problem (4.1) has a bounded positive solution u such that $u \in W_0^{1,\alpha}(\Omega)$ for all $\alpha < 2s$.*

4.3. A second existence result

Now, as in the local case, we assume that $f \in L^\gamma(\Omega)$ for some $\gamma > \frac{N}{q'(2s-1)}$, $q' = \frac{q}{q-1}$. In order to obtain a solution, we need some extra condition on the supersolution. We obtain the following result.

Theorem 4.3. *Assume that $f \in L^\gamma(\Omega)$ for some $\gamma > \frac{N}{q'(2s-1)}$. Let w be a non-negative supersolution to (4.1) such that $w \in W^{1,\sigma}(\Omega)$ for some $q < \sigma \leqslant 2s$. Suppose that the following estimate holds:*

$$\sup_{x\in\Omega} \int_\Omega \frac{w^{\sigma-1}(x)\mathcal{G}_s(x,y)}{|x-y|^\sigma}\mathrm{d}x \leqslant C. \tag{4.6}$$

Then problem (4.1) has a solution u such that $u \in W_0^{1,\sigma}(\Omega)$ and $T_k(u) \in H_0^s(\Omega) \cap W_0^{1,\alpha}(\Omega)$ for all $\alpha < 2s$.

Proof. Define ψ to be the solution to problem

$$\begin{cases} (-\Delta)^s\psi &= \frac{w^{\sigma-1}(x)}{d^\sigma(x)} \quad \text{in } \Omega, \\ \psi &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{4.7}$$

Then $\psi \in L^\theta(\Omega)$ for $\theta > \max\{\frac{\alpha}{q-\sigma}, \frac{N}{N-q'(2s-1)}\}$ if $q'(2s-1) < N$, i.e. $q > p_*$, and $\psi \in L^\infty(\Omega)$ if $q \leqslant p_*$.

Let u_n be the unique solution to the approximation problem (4.4), then $u_n \leqslant u_{n+1} \leqslant w$ for all n. Since $w \in W^{1,\sigma}(\Omega)$, we get the existence of u such that $u_n \uparrow u$ strongly in $L^{\sigma^*}(\Omega)$. As in the proof of the theorem 4.1, setting $g_n(x) = \frac{|\nabla u_n|^p}{\frac{1}{n}+|\nabla u_n|^p} + \lambda f$, it follows that

$$u_n(x) = \int_\Omega \mathcal{G}_s(x,y)g_n(y)\mathrm{d}y \quad \text{and then } |\nabla u_n(x)| \leqslant \int_\Omega |\nabla_x \mathcal{G}_s(x,y)|g_n(y)\mathrm{d}y.$$

Therefore,

$$\begin{aligned} |\nabla u_n(x)|^\sigma &\leqslant \Big(\int_\Omega |\nabla_x \mathcal{G}_s(x,y)|g_n(y)\mathrm{d}y\Big)^\sigma \leqslant \Big(\int_\Omega h(x,y)\mathcal{G}_s(x,y)g_n(y)\mathrm{d}y\Big)^\sigma \\ &\leqslant \Big(\int_\Omega (h(x,y))^\alpha \mathcal{G}_s(x,y)g_n(y)\mathrm{d}y\Big)\Big(\int_\Omega \mathcal{G}_s(x,y)g_n(y)\mathrm{d}y\Big)^{\sigma-1} \\ &\leqslant \Big(\int_\Omega (h^\sigma(x,y)\mathcal{G}_s(x,y)g_n(y)\mathrm{d}y\Big)u_n^{\sigma-1}(x) \\ &\leqslant \int_\Omega \Big(h^\sigma(x,y)\mathcal{G}_s(x,y)g_n(y)\mathrm{d}y\Big)w^{\sigma-1}(x) \\ &\leqslant \int_\Omega \Big(h^\sigma(x,y)\mathcal{G}_s(x,y)|\nabla u_n(y)|^q\mathrm{d}y\Big)w^{\sigma-1}(x) + \lambda\int_\Omega \Big(h^\sigma(x,y)\mathcal{G}_s(x,y)f(y)\mathrm{d}y\Big)w^{\sigma-1}(x). \end{aligned}$$

Thus,

$$\begin{aligned} \int_\Omega |\nabla u_n|^\sigma \mathrm{d}x \leqslant & \int_\Omega |\nabla u_n(y)|^q\Big(\int_\Omega h^\sigma(x,y)\mathcal{G}_s(x,y)w^{\sigma-1}(x)\mathrm{d}x\Big)\mathrm{d}y \\ &+ \lambda\int_\Omega f(y)\Big(\int_\Omega h^\sigma(x,y)\mathcal{G}_s(x,y)w^{\sigma-1}(x)\mathrm{d}x\Big)\mathrm{d}y \equiv J_1 + J_2. \end{aligned}$$

Recall that $h(x,y) = \max\{\frac{1}{|x-y|}, \frac{1}{d(x)}\}$, then

$$\begin{aligned} J_1 &\leqslant \int_\Omega |\nabla u_n(y)|^q\Big(\int_\Omega \frac{w^{\sigma-1}(x)G(x,y)}{|x-y|^\sigma}\mathrm{d}x\Big)\mathrm{d}y + \int_\Omega |\nabla u_n(y)|^q\Big(\int_\Omega \frac{w^{\sigma-1}(x)G(x,y)}{d^\sigma(x)}\mathrm{d}x\Big)\mathrm{d}y \\ &\leqslant \int_\Omega |\nabla u_n(y)|^q\Big(\int_\Omega \frac{w^{\sigma-1}(x)G(x,y)}{|x-y|^\sigma}\mathrm{d}x\Big)\mathrm{d}y + \int_\Omega |\nabla u_n(y)|^q\psi(y)\mathrm{d}y. \end{aligned}$$

By the hypothesis on w, we reach that

$$\begin{aligned} J_1 &\leqslant C\int_\Omega |\nabla u_n(y)|^q\mathrm{d}y + \Big(\int_\Omega |\nabla u_n(y)|^\sigma\mathrm{d}y\Big)^{\frac{q}{\sigma}}\Big(\int_\Omega \psi^{\frac{\sigma}{q-\sigma}}\mathrm{d}y\Big)^{\frac{\sigma-q}{\sigma}} \\ &\leqslant C_1\int_\Omega |\nabla u_n(y)|^q\mathrm{d}y + C_2\Big(\int_\Omega |\nabla u_n(y)|^\sigma\mathrm{d}y\Big)^{\frac{q}{\sigma}}. \end{aligned}$$

For J_2, we have

$$J_2 \leqslant \int_\Omega f(y)\Big(\int_\Omega \frac{w^{\sigma-1}(x)G(x,y)}{|x-y|^\sigma}\mathrm{d}x\Big)\mathrm{d}y + \int_\Omega f(y)\psi(y)\mathrm{d}y.$$

Hence,

$$J_2 \leqslant C\int_\Omega f(y)\mathrm{d}y + \Big(\int_\Omega f^{\frac{N}{q'(2s-1)}}(y)\mathrm{d}y\Big)^{\frac{q'(2s-1)}{N}}\Big(\int_\Omega \psi^{\frac{N}{N-q'(2s-1)}}(y)\mathrm{d}y\Big)^{\frac{N-q'(2s-1)}{N}} \leqslant C.$$

Thus,

$$\int_\Omega |\nabla u_n(x)|^\sigma \mathrm{d}x \leqslant C_1\int_\Omega |\nabla u_n(x)|^q\mathrm{d}x + C_2\Big(\int_\Omega |\nabla u_n(y)|^\alpha \mathrm{d}y\Big)^{\frac{q}{\alpha}} + C_3.$$

Since $\sigma > q$, using the Hölder inequality, the following holds:

$$\int_\Omega |\nabla u_n(x)|^\sigma \mathrm{d}x \leqslant C \text{ for all } n.$$

Hence, up to a subsequence, $u_n \rightharpoonup u$ weakly in $W_0^{1,\sigma}(\Omega)$. By the compactness result in proposition 2.12, we obtain that, up to a subsequence, $|\nabla u_n| \to |\nabla u|$ a.e. in Ω. Hence by Vitali's lemma, taking into account that $q < \sigma$, we reach that $u_n \to u$ strongly in $W_0^{1,q}(\Omega)$. Thus, u is a solution to (4.1), with $u \in W_0^{1,\sigma}(\Omega)$. Define $F \equiv |\nabla u|^q + \lambda f$; then $F \in L^1(\Omega)$. Hence, using theorem 2.14, it holds that $T_k(u) \in H_0^s(\Omega) \cap W_0^{1,\sigma}(\Omega)$ for all $\sigma < 2s$. Thus, we conclude the proof. □

As a consequence of theorem 4.3, we get the following application in a concrete case.

Theorem 4.4. *Assume that $p_* < q < 2s$, and let $f(x) = \frac{1}{|x|^\theta}$, with $0 < \theta < q'(2s-1) < N$. Then there exists λ^* such that for all $\lambda < \lambda^*$, problem (4.1) has a solution u with $u \in W_0^{1,\sigma}(\Omega)$ for $q < \sigma < \min\{\frac{N}{\theta-2s+1}, 2s\}$ and $T_k(u) \in H_0^s(\Omega) \cap W_0^{1,\alpha}(\Omega)$, for all $\alpha < 2s$.*

Proof. From theorem 4.3, we have only to build a supersolution w such that $w \in W^{1,\sigma}(\Omega)$ for some $q < \sigma \leqslant 2s$. It is clear that $f \in L^\gamma(\Omega)$ for some $\gamma > \frac{N}{q'(2s-1)}$. Without loss of generality, we can assume that $\theta > 2s$. Define $w_1(x) = \frac{1}{|x|^{\theta-2s}}$; then

$$(-\Delta)^s w_1 = \frac{C}{|x|^\theta} \geqslant C_1(\Omega)|\nabla w_1|^q + C_2 f \text{ in } \Omega.$$

Hence, setting $w = cw_1$, we reach that, for small λ,

$$(-\Delta)^s w \geqslant |\nabla w|^q + \lambda f \text{ in } \Omega.$$

It is clear that $w \in W^{1,\beta}(\Omega)$ for all $\beta < \frac{N}{\theta-2s+1}$. Since $q > p_*$, $q < \frac{\theta}{\theta-2s+1} < \frac{N}{\theta-2s+1}$. Hence, there exists $q < \sigma < \min\{\frac{N}{\theta-2s+1}, 2s\}$ such that $w \in W^{1,\sigma}(\Omega)$. Moreover, condition (4.6) holds. Indeed,

$$\int_\Omega \frac{w^{\sigma-1}(x)G(x,y)}{|x-y|^\sigma}\mathrm{d}x \leqslant C\int_\Omega \frac{1}{|x|^{(\theta-2s)(\sigma-1)}|x-y|^{N-2s+\alpha_0}}\mathrm{d}x.$$

Since $0 < \theta < q'(2s-1)$, then we can choose $q < \sigma < 2s$ such that $\sigma < \frac{\theta}{\theta-2s+1}$. That is, $\frac{1}{|x|^{(\theta-2s)(\sigma-1)}} \in L^{\gamma}(\Omega)$ for some $\gamma > \frac{N}{2s-1}$. Therefore, using Hölder inequality, we obtain that

$$\int_\Omega \frac{w^{\sigma-1}(x)G(x,y)}{|x-y|^\sigma}\mathrm{d}x \leqslant C\Big(\int_\Omega \frac{1}{|x|^{(\theta-2s)(\sigma-1)}})^\sigma \mathrm{d}x\Big)^{\frac{1}{\gamma}}\Big(\int_\Omega \frac{1}{|x-y|^{\sigma'(N-2s+\sigma)}}\mathrm{d}x\Big)^{\frac{1}{\gamma'}}.$$

Since $\gamma'(N-2s+\sigma) < N$,

$$\int_\Omega \frac{w^{\sigma-1}(x)G(x,y)}{|x-y|^\sigma}\mathrm{d}x \leqslant C.$$

Define now ψ to be the unique solution to the problem

$$\begin{cases} (-\Delta)^s\psi = \frac{w^{\sigma-1}(x)}{d^\sigma(x)} = \frac{1}{|x|^{(\theta-2s)(\sigma-1)}d^\sigma(x)} & \text{in } \Omega, \\ \psi = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{4.8}$$

Since $(\theta-2s)(\sigma-1) < 2s$, we prove that $\psi \in L^\infty(\Omega)$. Fixing $B_r(0) \subset\subset \Omega$, and taking ψ_1 and ψ_2 to be the solutions to problems

$$\begin{cases} (-\Delta)^s\psi_1 = \frac{1}{|x|^{(\theta-2s)(\sigma-1)}d^\sigma(x)}\chi_{B_r(0)} & \text{in } \Omega, \\ \psi_1 = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{4.9}$$

and

$$\begin{cases} (-\Delta)^s\psi_2 = \frac{1}{|x|^{(\theta-2s)(\sigma-1)}d^\sigma(x)}\chi_{\{\Omega\setminus B_r(0)\}} & \text{in } \Omega, \\ \psi_2 = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{4.10}$$

it is clear that $\psi = \psi_1 + \psi_2$. Since $(\theta - 2s)(\sigma - 1) < 2s$, there exists $\sigma_1 > \frac{N}{2s}$ such that $\frac{1}{|x|^{(\theta-2s)(\sigma-1)}d^\sigma(x)}\chi_{B_r(0)} \in L^{\sigma_1}(\Omega)$. Therefore, $\psi_1 \in L^\infty(\Omega)$. With respect to ψ_2, it is clear that $\frac{1}{|x|^{(\theta-2s)(\sigma-1)}d^\sigma(x)}\chi_{\{\Omega\setminus B_r(0)\}} \leqslant \frac{C}{d^\sigma(x)}$. Thus, since $\sigma < 2s$, using similar arguments as in the proof of lemma 2.13, we reach that $\psi_2 \in L^\infty(\Omega)$. Thus, $\psi \in L^\infty(\Omega)$, and the claim follows.

Hence, we conclude that all conditions of theorem 4.3 hold, and therefore there exists a solution u to problem (4.1) with $u \in W_0^{1,\sigma}(\Omega)$ and $T_k(u) \in H_0^s(\Omega) \cap W_0^{1,\alpha}(\Omega)$ for all $\alpha < 2s$. $\square$

Remark 4. We do not reach the extremal case $\alpha = 2s$. It is clear that by the previous monotonicity method we cannot reach the case $q \geqslant 2s$. However, in the next section, we will show the existence of a solution if $q \geqslant 2s$, using some arguments from Potential Theory.

5. Existence result using potential theory

In this section, we will complete the above existence results for all $q > p_*$. We will use some techniques from potential theory. The key is to construct a suitable supersolution, using hypotheses on f that allow us to use *potential theory estimates*. In [30] the authors prove the existence of solution under potential type hypothesis on f and for any $q \geqslant 1$ in the local setting. The hypothesis on f is equivalent to the condition (4.2). This type of argument was also used in [17] for $s = 1$, for some potentials, instead of a gradient term.

In what follows, we will assume that $f \in L^m(\Omega)$ with $m > \frac{N}{q'(2s-1)}$; then, following the ideas of [17] and [30], we will build a suitable supersolution to problem (4.1) in the whole space $I\!R^N$ under natural conditions on f.

Consider the Riez potential $J_{(N-\alpha)}$ defined in lemma 2.6. We call $I_\alpha = J_{(N-\alpha)}$, that is,

$$I_\alpha(g)(x) = \int_{I\!R^N} \frac{g(y)}{|x-y|^{N-\alpha}} \mathrm{d}y.$$

Note that if $0 < \alpha < 2$, then $I_\alpha = ((-\Delta)^{\frac{\alpha}{2}})^{-1}$ and $G_\alpha(x,y) \equiv \frac{1}{|x-y|^{N-\alpha}}$ is a constant multiple of the fundamental solution associated to the operator $(-\Delta)^{\frac{\alpha}{2}}$.

For $f \in L^m(\Omega)$, we consider its extension by 0 to the whole of $\mathbb{R}^N$: specifically,

$$f_0(x) = \begin{cases} f(x) & \text{if} \quad x \in \Omega, \\ 0 & \text{if} \quad x \in \mathbb{R}^N \setminus \Omega. \end{cases} \tag{5.1}$$

If q is the exponent in the problem (4.1), we define

$$F_0(x) = \Big(I_{2s-1}(f_0)(x)\Big)^q.$$

Then the key hypothesis on f is that the inequality

$$I_{2s-1}(F_0) \leqslant C_1 I_{2s-1}(f_0) \tag{5.2}$$

holds.

The first result of this Section is the following.

Theorem 5.1. *Assume that the hypothesis (5.2) holds; then problem (4.1) has a positive supersolution $\bar{u}$ such that $\bar{u} \in D^{1,\gamma}(I\!R^N)$ where $\gamma = \frac{mN}{N-m(2s-1)} \geqslant q$. In particular, $\bar{u} \in W^{1,q}_{loc}(I\!R^N)$.*

Proof. For the precise definition of the space $D^{1,\gamma}(I\!R^N)$, the reader can consult section 8.2 in [37]. Assume that (5.2) holds, and define $\bar{u} = u_1 + u_2$, where u_1, u_2 solve the problems

$$(-\Delta)^s u_1 = \lambda f_0 \text{ in } I\!R^N, (-\Delta)^s u_2 = C_2 F_0(x) \text{ in } I\!R^N, \qquad \text{respectively.} \tag{5.3}$$

It is clear that

$$\bar{u}(x) = I_{2s}(\lambda f_0 + C_2 F_0)(x);$$

thus,

$$|\nabla \bar{u}| \leqslant (N-2s)\Big(I_{2s-1}(\lambda f_0 + C_2 F_0)\Big).$$

Hence, using (5.2), we reach that

$$|\nabla \bar{u}|^q \leqslant \Big((N-2s)(\lambda + C_2 C_1)\Big)^q \Big(I_{2s-1}(f_0)\Big)^q \leqslant \Big((N-2s)(\lambda + C_2 C_1)\Big)^q F_0.$$

Therefore, we conclude that

$$(-\Delta)^s \bar{u} = \lambda f_0 + C_2 F_0 \geqslant \lambda f_0 + \frac{C_2 |\nabla \bar{u}|^q}{\Big((N-2s)(\lambda + C_2 C_1)\Big)^q}.$$

Let $\hat{u} = a\bar{u}$, where $a = \dfrac{C_2^{\frac{1}{q-1}}}{\Big((N-2s)(\lambda+C_2C_1)\Big)^{\frac{q}{q-1}}}$; then

$$(-\Delta)^s \hat{u} \geqslant |\nabla \hat{u}|^q + \lambda^* f_0 \text{ with } \lambda^* = a\lambda. \tag{5.4}$$

Since $|\nabla \hat{u}| \leqslant C I_{2s-1}(f_0)$, $f_0 \in L^m(I\!R^N)$ with $m > \frac{N}{q'(2s-1)}$ and $q > p_*$, by using lemma 2.6 and the Dominated Convergence Theorem, it follows that $|\nabla \hat{u}| \in L^\gamma(I\!R^N)$, where $\gamma = \frac{mN}{N-m(2s-1)} \geqslant q$. The result follows. □

In the next proposition, we analyze the regularity of the solution $\hat{u}$ obtained above.

Proposition 1. *Let $\hat{u}$ be the supersolution obtained in theorem 5.1.*

(1) If $m > \frac{N}{2s}$, then $\hat{u} \in L^\infty(I\!R^N)$.
(2) If $\frac{N}{q'(2s-1)} < m \leqslant \frac{N}{2s}$, then $\hat{u} \in L^{\frac{mN}{N-2sm}}(I\!R^N)$; in particular, for any α verifying

$$1 < \alpha < \alpha_0 \equiv \frac{N}{N-m(2s-1)} < 2s,$$

and for any bounded domain Ω, we have

$$\sup_{\{y\in\Omega\}} \int_\Omega \frac{\hat{u}^{\alpha-1}(x)}{|x-y|^{N-2s+\alpha}} \mathrm{d}x \leqslant C. \tag{5.5}$$

Proof. Consider u_1 and u_2 defined in the proof of theorem 5.1.

Assume first that $m > \frac{N}{2s-1}$; then we easily get that $u_1 \in L^\infty(I\!R^N)$. Since

$$F_0(x) = \Big(\int_\Omega \frac{f_0(y)}{|x-y|^{N-2s+1}} \mathrm{d}y\Big)^q,$$

using the Hölder inequality, we reach that $F_0 \in L^\infty(I\!R^N)$. Moreover, as a consequence of lemma 2.6, $F_0 \in L^1(I\!R^N)$. Thus, $F_0 \in L^1(I\!R^N) \cap L^\infty(I\!R^N)$. Hence, $u_2 \in L^\infty(I\!R^N)$, and the result then follows in this case.

If $\frac{N}{2s} < m \leqslant \frac{N}{2s-1}$ then $u_1 \in L^\infty(I\!R^N)$.

With respect to u_2, we have $F_0 \in L^{\gamma_1}(I\!R^N)$ with $\gamma_1 = \frac{mN}{q(N-m(2s-1))}$. A direct computation shows that for $\frac{N}{2s} < m \leqslant \frac{N}{2s-1}$, we have $\gamma_1 > \frac{N}{q}$.

Using again lemma 2.6, it holds that $F_0 \in L^{\frac{2N}{N+2s}}(I\!R^N)$. Thus $F_0 \in L^\sigma(I\!R^N)$ for all $\sigma \in [\frac{2N}{N+2s}, \gamma_1]$. Hence $u_2 \in L^\infty(I\!R^N)$, and thus, $\hat{u} \in L^\infty(I\!R^N)$.

Consider hypothesis (2); specifically, $\frac{N}{q'(2s-1)} < m \leqslant \frac{N}{2s}$.

Since $|\nabla \hat{u}| \in L^\gamma(I\!R^N)$ with $\gamma = \frac{mN}{N-m(2s-1)}$, by the Sobolev inequality, we conclude that $\hat{u} \in L^{\frac{mN}{N-2sm}}(I\!R^N)$.

Let us now prove inequality (5.5). Fix $1 < \alpha < \alpha_0$, and define

$$K_1(y) = \int_\Omega \frac{u_1^{\alpha-1}(x)}{|x-y|^{N-2s+\alpha}} \mathrm{d}x \quad \text{and} \quad K_2(y) = \int_\Omega \frac{u_2^{\alpha-1}(x)}{|x-y|^{N-2s+\alpha}} \mathrm{d}x.$$

Using Hölder's inequality, we obtain

$$K_i(y) \leqslant \left(\int_\Omega u_i^{(\alpha-1)\sigma}(x)\mathrm{d}x\right)^{\frac{1}{\sigma}} \left(\int_\Omega \frac{1}{|x-y|^{(N-2s+\alpha)\sigma'}}\mathrm{d}x\right)^{\frac{1}{\sigma'}} \text{ for } i=1,2.$$

Since Ω is a bounded domain, $\int_\Omega \frac{1}{|x-y|^{(N-2s+\alpha)\sigma'}}\mathrm{d}x \leqslant C(\Omega)$ if $(N-2s+\alpha)\sigma' < N$, which means that $\sigma > \frac{N}{2s-\alpha}$. Since $\alpha < \alpha_0$, we can find $\sigma > \frac{N}{2s-\alpha}$ such that $(\alpha-1)\sigma \leqslant \frac{mN}{N-2sm}$. Hence, $K_1(y) \leqslant C(u_1,\Omega)$ for all $y \in \Omega$.

We deal now with K_2. By lemma 2.6, we get $F_0 \in L^\theta(I\!R^N)$ for $\theta = \frac{mN}{q(N-m(2s-1))}$. Therefore, again using lemma 2.6, we find that $u_2 \in L^{\theta_1}(I\!R^N)$ where $\theta_1 = \frac{mN}{q(N-m(2s-1))-2sm}$. Since Ω is bounded, and $\alpha < \alpha_0$, we can find a σ such that $\sigma > \frac{N}{2s-\alpha}$ and $\theta_1 > (\alpha-1)\sigma$. Hence,

$$\int_\Omega u_2^{(\alpha-1)\sigma}(x)\mathrm{d}x \leqslant C.$$

As $\hat{u} = u_1 + u_2$, this concludes the proof. □

Remark 5.

(1) It is clear that the above computations allow us to get a *regular* supersolution to problem (4.1) for all $q > 1$, given suitable hypotheses of f and λ.

(2) If, moreover, $q < p_* = \frac{N}{N-2s+1}$, then, as was stated in theorem 3.8, a comparison principle holds in the sense that if v is satisfied,

$$\begin{cases} (-\Delta)^s v & \leqslant & |\nabla v|^q + \lambda g & \text{in } \Omega, \\ \quad v & \leqslant & 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{5.6}$$

with $(-\Delta)^s v, |\nabla v|^q \in L^1(\Omega)$ and $g \leqslant f$, then $v \leqslant \hat{u}$ in $I\!R^N$. In the case where $g = 0$, we know from theorem 3.8 that $v \leqslant 0$; thus, trivially, $v \leqslant \hat{u}$.

(3) If $q \geqslant p_*$, then in general, the comparison principle does not hold. In the local case, i.e. $s = 1$, it was proved in [1] that for $q \geqslant \frac{N}{N-1}$, the problem

$$-\Delta u = |\nabla u|^q \text{ in } B_1(0), u \in W_0^{1,q}(B_1(0)),$$

has a non-trivial solution u, and thus, as was proved in [3, 1], there exist infinitely many non-comparable positive solutions.

Now, as in [30], we can prove the following existence result.

Theorem 5.2. *Under the general hypotheses on s and q, assume that $\Omega \equiv \mathbb{R}^N$. There exists a constant C_1 depending on q and N such that if (5.2) holds with constant C_1, then there exists a positive weak solution $u \in W^{1,q}_{loc}(I\!R^N)$ to the equation*

$$(-\Delta)^s u = |\nabla u|^q + \lambda f, \quad f \geqslant 0 \text{ in } \mathbb{R}^N. \tag{5.7}$$

Proof. Take u_1 as the solution to problem

$$(-\Delta)^s u_1 = \lambda f \text{ in } I\!R^N,$$

and define u_{k+1} by recursion, setting

$$u_{k+1} = I_{2s}(|\nabla u_k|^q) + I_{2s}(\lambda f).$$

Then

$$(-\Delta)^s u_{k+1} = |\nabla u_k|^q + \lambda f \text{ in } I\!R^N.$$

We claim that

$$|\nabla u_k| \leqslant C_1 I_{2s-1}(\lambda f), \tag{5.8}$$

$$|\nabla u_{k+1} - \nabla u_k| \leqslant C_2 \delta^k I_{2s-1}(\lambda f) \text{ for some } \delta < 1, \tag{5.9}$$

and the sequence $\{u_k\}_k$ is a Cauchy sequence in the space $D^{1,\gamma}(I\!R^N)$, where $\gamma = \frac{mN}{N-m(2s-1)} \geqslant q$.

We first prove (5.8), arguing by induction. It is clear that (5.8) holds for $k = 1$. Assume that

$$|\nabla u_k| \leqslant a_k I_{2s-1}(f);$$

then we know that

$$\begin{aligned} |\nabla u_{k+1}| &= |\nabla(I_{2s}(|\nabla u_k|^q)) + \nabla(I_{2s}(f))| \\ &\leqslant C\Big(I_{2s-1}(|\nabla u_k|^q) + I_{2s-1}(f)\Big) \\ &\leqslant C\Big(a_k^q I_{2s-1}((f_0)^q) + I_{2s-1}(f)\Big). \end{aligned}$$

Now, using (5.2), we conclude that

$$|\nabla u_{k+1}| \leqslant a_{k+1} I_{2s-1}(f),$$

with

$$a_{k+1} = C(C_1 a_k^q + 1).$$

Thus, if $C_1 \leqslant \frac{q'^{1-q}}{qC^q}$,

$$\lim_{k\to\infty} a_k = a \leqslant Cq',$$

where a is the smaller root of the equation $\tau = C(C_1\tau^q + 1)$. Hence, (5.8) follows.

As a conclusion, using the same computation as in the last part of the proof of theorem 5.1, we reach that the sequence $\{u_k\}_k$ is bounded in the space $D^{1,\gamma}(I\!R^N)$, where $\gamma = \frac{mN}{N-m(2s-1)}$. Now, using estimate (5.8), we reach that $\{u_k\}_k$ is a Cauchy sequence in $D^{1,\gamma}(I\!R^N)$. Hence, we get the existence of $u \in D^{1,\gamma}(I\!R^N)$ such that $u_k \to u$ strongly in $D^{1,\gamma}(I\!R^N)$. Since $q < \gamma$, $u_k \to u$ strongly in $W^{1,q}_{loc}(I\!R^N)$, and thus u solves (4.1), at least in the sense of distributions. □

5.1. The subcritical case $q < 2s$

In this subsection, we consider the case $q < 2s$. We will combine the above ideas in order to show the existence of a *suitable supersolution* under *natural* conditions on *f*. Then using the comparison principle and the representation formula, as in the previous section, we show the existence of a minimal solution. More precisely, we have the following result.

Theorem 5.3. *Assume that $q < 2s$. Suppose that $f \in L^m(\Omega)$, where $m > \frac{N}{q'(2s-1)}$, and the hypothesis (5.2) holds. Then there exists $\lambda^* > 0$ such that for all $0 < \lambda < \lambda^*$, problem (4.1) has a solution u such that $u \in W_0^{1,q}(\Omega)$ and $T_k(u) \in H_0^s(\Omega) \cap W_0^{1,\sigma}(\Omega)$ for all $\sigma < 2s$.*

Proof. We follow closely the argument used in the proof of theorem 4.3. Let u_n be the unique solution to the approximation problem (4.4); then $u_n \leqslant u_{n+1}$. Fix $\lambda < \lambda^*$ defined in (5.4) and let $\hat{u}$ be the supersolution obtained in theorem 5.1. It is clear that $\hat{u}$ is a supersolution to problem (4.4). Hence, by the comparison principle in theorem (3.1), we reach that $u_n \leqslant \hat{u}$ for all n. Hence, following the same computation as in the proof of theorem 4.1, we obtain that

$$\begin{aligned}\int_\Omega |\nabla u_n(x)|^\alpha \mathrm{d}x \quad &\leqslant \int_\Omega |\nabla u_n(y)|^q \Big(\int_\Omega h^\alpha(x,y)\mathcal{G}_s(x,y)\hat{u}^{\alpha-1}(x)\mathrm{d}x \Big)\mathrm{d}y \\ &+\lambda \int_\Omega f(y) \Big(\int_\Omega h^\alpha(x,y)\mathcal{G}_s(x,y)\hat{u}^{\alpha-1}(x)\mathrm{d}x \Big)\mathrm{d}y. \end{aligned} \tag{5.10}$$

Now, we divide the proof into two cases, according to the value of m and the regularity of $\hat{u}$.

5.1.1. The first case. $\frac{N}{2s} < m$. In this case, using proposition 1, we know that $\hat{u} \in L^\infty(\Omega)$. Hence, following again the proof of theorem 4.1, and taking into consideration (5.10), we conclude that $\int_\Omega |\nabla u_n|^\alpha \mathrm{d}x \leqslant C$ for all n provided that $\alpha < 2s$. The rest of the proof now exactly follows the proof of theorem 4.1.

5.1.2. The second case. $\frac{N}{q'(2s-1)} < m \leqslant \frac{N}{2s}$. Since $q \geqslant p_*$, then using lemma 2.6, we obtain that $|\nabla \hat{u}| \in L^\gamma(\Omega)$ where $\gamma = \frac{mN}{N-m(2s-1)} > q$.

We set

$$K(y) = \int_\Omega h^\alpha(x,y)\mathcal{G}_s(x,y)\hat{u}^{\alpha-1}(x)\mathrm{d}x;$$

then by (5.10), we have

$$\int_\Omega |\nabla u_n(x)|^\alpha \mathrm{d}x \leqslant \int_\Omega |\nabla u_n(y)|^q K(y)\mathrm{d}y + \lambda \int_\Omega f(y)K(y)\mathrm{d}y. \tag{5.11}$$

We claim that for $q < \alpha < \alpha_0$, defined in proposition 1, we have $K \in L^\infty(\Omega)$. Note that

$$K(y) \leqslant \int_\Omega \frac{\hat{u}^{\alpha-1}(x)G(x,y)}{|x-y|^\alpha}\mathrm{d}x + \int_\Omega \frac{\hat{u}^{\alpha-1}(x)G(x,y)}{d^\alpha(x)}\mathrm{d}x = J_1 + J_2.$$

It is clear that

$$J_1 \leqslant \int_\Omega \frac{\hat{u}^{\alpha-1}(x)}{|x-y|^{N-2s+\alpha}}\mathrm{d}x.$$

Hence, by the second point in proposition 1, we obtain that $\int_\Omega \frac{\hat{u}^{\alpha-1}(x)}{|x-y|^{N-2s+\alpha}}\mathrm{d}x \leqslant C$. Thus, $J_1(y) \leqslant C$. To analyze J_2, we follow the same computation as in the proof of theorem 4.1. As a consequence, we reach that $J_2 \leqslant C$.

Therefore, we conclude that $K(y) \leqslant C$ for all $y \in \Omega$, and the claim follows.

Following again the last part of the proof of theorem 4.1, we get the existence of a solution u to problem (4.1) such that $u \leqslant \hat{u}$, $u \in W_0^{1,\alpha}(\Omega)$ for all $\alpha < \alpha_0$ and $T_k(u) \in H_0^s(\Omega) \cap W_0^{1,\sigma}(\Omega)$ for all $\sigma < 2s$. It is clear that u is a minimal solution to (4.1). □

We now give a capacity-based condition on f, in order to show that condition (5.2) holds. Recall the following result proved in [39].

Theorem 5.4. *Assume that $f \in L^1(I\!R^N)$ is a non-negative function. For any compact set $E \subset I\!R^N$, we define $|E|_f = \int_E f(y)\mathrm{d}y$.*

Then f satisfies the condition (5.2) if and only if, for any compact set $E \subset I\!R^N$,

$$|E|_f \leqslant C \ \mathrm{cap}_{2s-1,q'}(E), \tag{5.12}$$

where

$$\mathrm{cap}_{2s-1,q'}(E) \equiv \inf\left\{ \int_{I\!R^N}\int_{I\!R^N} \frac{|\phi(x)-\phi(y)|^{q'}}{|x-y|^{N+q'(2s-1)}}\mathrm{d}x\mathrm{d}y \text{ with } \phi \in \mathcal{C}_0^\infty(I\!R^N) \text{ and } \phi \geqslant \chi_E \right\}.$$

As a consequence we have the following result.

Theorem 5.5. *Let $f \in L^m(\Omega)$, with $m \geqslant \frac{N}{q'(2s-1)}$, and define f_0 as in (5.1). Then the condition (5.2) holds for f_0.*

Proof. We have only to show that the condition (5.12) holds. Let E be a compact set and consider $\phi \in \mathcal{C}_0^\infty(I\!R^N)$ be such that $\phi \geqslant \chi_E$. Using the Hölder inequality,

$$|E|_{f_0} \leqslant \int_E f_0|\phi|^{q'}\mathrm{d}x \leqslant \left(\int_{I\!R^N} f_0^{\frac{p^*_{s_1}}{p^*_{s_1}-p}}(x)\mathrm{d}x\right)^{\frac{p^*_{s_1}-p}{p^*_{s_1}}} \left(\int_{I\!R^N} |\phi(x)|^{p^*_{s_1}}\mathrm{d}x\right)^{\frac{p}{p^*_{s_1}}}$$

where $p = q'$, $s_1 = 2s-1$ and $p^*_{s_1} = \frac{pN}{N-ps_1}$. It is clear that $\frac{p^*_{s_1}}{p^*_{s_1}-p} = \frac{N}{q'(2s-1)}$. Since $m \geqslant \frac{N}{q'(2s-1)}$, using the Sobolev inequality, it holds that

$$|E|_{f_0} \leqslant C(\Omega)\|f\|_{L^m(\Omega)} \ \mathrm{cap}_{2s-1,q'}(E).$$

Thus, we conclude the proof. □

5.2. The critical case q = 2s

In this case, there are difficulties in using the comparison arguments. It is possible to find a supersolution, but it is not clear how to pass to the limit in the gradient term when dealing with the family of approximation problems.

In the local case this difficulty is overpassed by using convenient nonlinear test functions and a suitable change of variable. In the nonlocal framework it seems to be necessary to change this point of view and to adapt a different approach. We will use in a convenient way the Schauder fixed point theorem following the strategy used recently in [41] and [40] for the local case.

Theorem 5.6. *Suppose that Ω is a bounded regular domain and that $f \in L^m(\Omega)$ where $m > \frac{N}{2s}$. Then there exists $\lambda^*(f) > 0$ such that for all $\lambda < \lambda^*$, problem (4.1) has a solution $u \in W_0^{1,2s}(\Omega)$.*

Proof. Suppose that $f \in L^m(\Omega)$, where $m > \frac{N}{2s}$. We know by lemma 2.15 that if v is a solution to problem (2.24), then for all $p < \frac{mN}{N-m(2s-1)}$, there exists a positive constant C_0 such that

$$\|\nabla v\|_{L^p(\Omega)} \leqslant C_0 \|f\|_{L^m(\Omega)}. \tag{5.13}$$

Since $m > \frac{N}{2s}$, $\sigma_0 \equiv 2sm < \frac{mN}{N-m(2s-1)}$. Now, taking into account that $2s > 1$, we can choose $\lambda^* > 0$ such that for some $l > 0$, we have

$$C_0(l + \lambda^* \|f\|_{L^m(\Omega)}) = l^{\frac{1}{2s}}.$$

Fix $\delta > 0$ small enough to be chosen later; $\lambda < \lambda^*$ and $l > 0$, as above. We define the set

$$E = \{v \in W_0^{1,1}(\Omega) : v \in W_0^{1,2s(1+\delta)}(\Omega) \text{ and } \|\nabla v\|_{L^{2sm}} \leqslant l^{\frac{1}{2s}}\}. \tag{5.14}$$

It is easy to check that E is a closed convex set of $W_0^{1,1}(\Omega)$. Consider the operator

$$\begin{array}{rcl} T : E & \to & W_0^{1,1}(\Omega) \\ v & \to & T(v) = u, \end{array}$$

where u is the unique solution to problem

$$\begin{cases} (-\Delta)^s u &=& |\nabla v|^{2s} + \lambda f & \text{in } \Omega, \\ u &=& 0 & \text{in } \mathbb{R}^N \setminus \Omega, \\ u &>& 0 & \text{in } \Omega. \end{cases} \tag{5.15}$$

Since $|\nabla v|^{2s} + \lambda f \in L^1(\Omega)$, the existence of u is a consequence of theorem 2.12. Moreover, $|\nabla u| \in L^q(\Omega)$ for all $q < p_* = \frac{N}{N-2s+1}$. Hence, T is well defined.

We claim that:

(1) For small enough $\delta > 0$, $T(E) \subset E$,
(2) T is a continuous and compact operator on E.

Proof of (1). Since $\sigma_0 = 2sm < \frac{mN}{N-m(2s-1)}$, by using lemma 2.15, it follows that

$$\|\nabla u\|_{L^{\sigma_0}(\Omega)} \leqslant C_0 \left\| |\nabla v|^{2s} + \lambda f \right\|_{L^m(\Omega)}.$$

Thus,

$$\begin{aligned} \|\nabla u\|_{L^{\sigma_0}(\Omega)} &\leqslant C_0 \left(\left\| |\nabla v|^{2s} \right\|_{L^m(\Omega)} + \lambda \|f\|_{L^m(\Omega)} \right) \\ &\leqslant C_0 \left(\|\nabla v\|^{2s}_{L^{2sm}(\Omega)} + \lambda \|f\|_{L^m(\Omega)} \right) \\ &\leqslant C_0 \left(l + \lambda^* \|f\|_{L^m(\Omega)} \right) = l^{\frac{1}{2s}}. \end{aligned}$$

It is clear that for small δ, $2s(1+\delta) < 2sm$. Hence $u \in E$.

Proof of (2). To show the continuity of T with respect to the topology of $W_0^{1,1}(\Omega)$, we consider $\{v_n\}_n \subset E$ such that $v_n \to v$ strongly in $W_0^{1,1}(\Omega)$. Define $u_n = T(v_n), u = T(v)$.

We have to show that $u_n \to u$ strongly in $W_0^{1,1}(\Omega)$; to do this, we prove that

$$\|\nabla v_n - \nabla v\|_{L^{2s}(\Omega)} \to 0 \text{ as } n \to \infty.$$

Recall that $\{v_n\}_n \subset E$, and $\|v_n - v\|_{W_0^{1,1}(\Omega)} \to 0$ as $n \to \infty$; thus, $\nabla v_n \to \nabla v$ strongly in $(L^1(\Omega))^N$, and $\|\nabla v_n\|_{L^{2sm}(\Omega)} \leqslant l^{\frac{1}{2s}}$.

Since $2sm > 1$, setting $a = \frac{2s(m-1)}{2sm-1} < 1$, it follows that $\frac{2s-a}{1-a} = 2sm$. Hence, by the Hölder inequality, we conclude that

$$\begin{aligned}\|\nabla v_n - \nabla v\|_{L^{2s}(\Omega)} &\leqslant \|\nabla v_n - \nabla v\|_{L^1(\Omega)}^{\frac{a}{2s}} \|\nabla v_n - \nabla v\|_{L^{\frac{2s-a}{1-a}}(\Omega)}^{\frac{2s-a}{2s}} \\ &\leqslant C\|\nabla v_n - \nabla v\|_{L^1(\Omega)}^{\frac{a}{2s}} \to 0 \text{ as } n \to \infty.\end{aligned}$$

Now, using the definition of u_n and u, it can be seen that $u_n \to u$ strongly in $W_0^{1,1}(\Omega)$. Thus, T is continuous.

To finish, we have just to show that T is compact with respect to the topology of $W_0^{1,1}(\Omega)$.

Let $\{v_n\}_n \subset E$ be such that $\|v_n\|_{W_0^{1,1}(\Omega)} \leqslant C$. Since $\{v_n\}_n \subset E$, then $\|\nabla v_n\|_{L^{2s(1+\delta)}(\Omega)} \leqslant C$, and therefore, up to a subsequence, $v_{n_k} \rightharpoonup v$ weakly in $W_0^{1,2s(1+\delta)}(\Omega)$.

Define

$$F_n = |\nabla v_n|^{2s} + \lambda f, F = |\nabla v|^{2s} + \lambda f;$$

it is clear that F_n is bounded in $L^{1+\delta}(\Omega)$, and $F_n \rightharpoonup F$ weakly in $L^{1+\delta}(\Omega)$. Using the compactness result of [19], we conclude that, up to a subsequence, $u_{n_k} \to u$ strongly in $W_0^{1,1}(\Omega)$; hence, the claim follows.

In conclusion, using the Schauder fixed point theorem, there exists $u \in E$ such that $T(u) = u$, then $u \in W_0^{1,2s}(\Omega)$ and u solves (4.1). □

Remark 6.

(1) The solution obtained above is the unique solution in E. Indeed, we may assume u_1 and $u_2 \in E$ to be solutions to problem (4.1). Therefore, in particular, $\|\nabla u_1\|_{L^{2sm}(\Omega)} < \infty$ and $\|\nabla u_2\|_{L^{2sm}(\Omega)} < \infty$. Define $w = u_1 - u_2$; then $\|\nabla w\|_{L^{2sm}(\Omega)} < \infty$, and w solves the problem

$$\begin{cases}(-\Delta)^s w = |\nabla u_1|^{2s} - |\nabla u_2|^{2s} & \text{in } \Omega, \\ w = 0 & \text{in } \mathbb{R}^N \setminus \Omega.\end{cases}$$

Setting $b(x) = |\nabla u_1|^{2s-1} + |\nabla u_2|^{2s-1}$, the following inequality holds:

$$(-\Delta)^s w \leqslant 2sb(x)|\nabla w| \text{ in } \Omega.$$

Since $m > \frac{N}{2s}$, $b \in L^\sigma(\Omega)$ for $\sigma > \frac{N}{2s-1}$. By the comparison principle in theorem 3.1, it follows that $w_+ = 0$. Thus $u_1 \leqslant u_2$. In a similar way, we get $u_2 \leqslant u_1$. Hence, $u_1 = u_2$.

(2) The solution $u \in E$ is the minimal solution to problem (4.1). Assume that v is an other solution to (4.1), then $v \in W_0^{1,q}(\Omega)$ and $(-\Delta)^s v \in L^1(\Omega)$. As above, setting $w = u - v$ and using the fact that for all $\xi_1, \xi_2 \in I\!R^N$, for all $\alpha > 1$, we have

$$|\xi_1|^\alpha - |\xi_2|^\alpha \leqslant \alpha |\xi_1|^{\alpha-2} \langle \xi_1, \xi_1 - \xi_2 \rangle;$$

it follows that

$$\begin{cases} (-\Delta)^s w &= |\nabla u|^{2s} - |\nabla v|^{2s} \leqslant 2s|\nabla u|^{2s-1}|\nabla w| & \text{in } \Omega, \\ w &= 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Setting $b(x) = 2s|\nabla u|^{2s-1}$ and using the fact that $u \in E$, there results that $b \in L^\sigma(\Omega)$ for $\sigma = \frac{2sm}{2s-1} > \frac{N}{2s-1}$. As above, using the comparison principle in theorem 3.1, we conclude that $w_+ = 0$. Hence, $u \leqslant v$.

5.3. The supercritical case $q > 2s$

In a similar way as in the critical case, we can handle the supercritical case, $q > 2s$, and prove the following result.

Theorem 5.7. *Suppose that Ω is a bounded regular domain, and that $f \in L^m(\Omega)$, where $m > \frac{N}{q'(2s-1)}$. Then there exists $\lambda^*(f) > 0$ such that for all $\lambda < \lambda^*$, problem (4.1) has a solution $u \in W_0^{1,q}(\Omega)$.*

Proof. In this case, we choose $l > 0$ and σ_0 such that

$$\sigma_0 \equiv 2sm < \frac{mN}{N - m(2s-1)} \text{ and } C_0(l + \lambda^* \|f\|_{L^m(\Omega)}) = l^{\frac{1}{q}}.$$

Define the set

$$E = \{v \in W_0^{1,1}(\Omega) : v \in W_0^{1,q(1+\delta)}(\Omega) \text{ and } \|\nabla v\|_{L^{qm}(\Omega)} \leqslant l^{\frac{1}{q}}\}. \tag{5.16}$$

As in the proof of theorem 5.6, we consider $T_q : E \to W_0^{1,1}(\Omega)$ defined by $u = T_q(v)$, where u is the unique solution to problem

$$\begin{cases} (-\Delta)^s u &= |\nabla v|^q + \lambda f & \text{in } \Omega, \\ u &= 0 & \text{in } \mathbb{R}^N \setminus \Omega, \\ u &> 0 & \text{in } \Omega. \end{cases} \tag{5.17}$$

By similar computations as in the proof of theorem 5.6, and for fixed $\lambda < \lambda^*(f)$, we can prove that T_q has a fixed point in E_q; thus, problem (4.1) has a solution $u \in E_q$. $\square$

Remark 7. As in the case $q = 2s$, since $m > \frac{N}{q'(2s-1)}$, by the same kind of arguments as in remark 6, it holds that problem (4.1) has a unique positive solution in the convex set E_q that is the minimal solution of (4.1).

6. Some open problems

The regularity result proved in lemma 2.15 is the key to showing the existence results, and it is worth pointing out that it depends directly on the representation formula given in (2.23), and on the pointwise estimates on the Green's function G_s.

With the previous remark in mind, we can formulate the following open problems, which should be interesting to study.

(1) Let consider the operator L_k defined by

$$L_k(u) = P.V \int_{I\!R^N} (u(x) - u(y))k(x, y)\,\mathrm{d}y,$$

where k is a suitable symmetric function. Consider the problem

$$\begin{cases} L_k(u) &= |\nabla u|^p + \lambda f \quad \text{in } \Omega,\\ u &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega,\\ u &> 0 \quad \text{in } \Omega, \end{cases}$$

It seems interesting to seek conditions on L_k in order to find the same kind of regularity results—for instance, getting estimates without the explicit representation formula.

(2) In the local case $s = 1$, and for the critical exponent $q = 2$, an exponential regularity is obtained for any solution to problem (4.1)—see [3]. Specifically, the result is that any positive solution satisfies $\mathrm{e}^{\alpha u} - 1 \in W_0^{1,2}(\Omega)$ for all $\alpha < \frac{1}{2}$. It seems to be natural to ask for the optimal regularity in the fractional case.

(3) Consider the nonlinear operator

$$(-\Delta_p^s)\, u(x) := P.V \int_{I\!R^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+ps}}\,\mathrm{d}y$$

with $1 < p < N$ and $s \in (0, 1)$; then, as was proved in [2], the problem

$$\begin{cases} (-\Delta_p^s)u &= f(x) \quad \text{in } \Omega,\\ u &= 0 \quad \text{in } I\!R^N \setminus \Omega, \end{cases}$$

has a unique entropy solution for any non-negative datum. It seems interesting to show the regularity of $|\nabla u|$ if $sp' > \frac{(2-p)N}{p-1} + 1$, and to consider the nonlinear, nonlocal version of problem (1.1).

(4) A problem with nonlocal diffusion and nonlocal growth term could be formulated for all $s \in (0, 1)$, and it should be interesting to analyze it in detail.

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References

[1] Abdellaoui B 2008 Multiplicity result for quasilinear elliptic problems with general growth in the gradient *Adv. Nonlinear Stud.* **8** 289–301

[2] Abdellaoui B, Attar A and Bentifour R 2016 On the fractional p-laplacian equations with weights and general datum *Adv. Nonlinear Anal.* (https://doi.org/10.1515/anona-2016-0072)

[3] Abdellaoui B, Dall'Aglio A and Peral I 2006 Some remarks on elliptic problems with critical growth in the gradient *J. Differ. Equ.* **222** 21–62
[4] Alaa N E and Pierre M 1993 Weak solutions of some quasilinear elliptic equations with data measures *SIAM J. Math. Anal.* **24** 23–35
[5] Applebaum D 2009 *Lévy Processes and Stochastic Calculus* (*Cambridge Studies in Advanced Mathematics* vol 116) (Cambridge: Cambridge University Press)
[6] Boccardo L, Murat F and Puel J-P 1982 Existence des solutions non bornées pour certains équations quasi-linéaires *Port. Math.* **41** 507–34
[7] Boccardo L, Murat F and Puel J-P 1983 Existence de solutions faibles pour des équations elliptiques quasi-linéaires à croissance quadratique *Nonlinear Partial Differential Equations and their Applications, Collège de France Seminar, Vol.* IV (*Research Notes in Mathematics* vol 84) ed J-L Lions and H Brezis (Boston, MA: Pitman) pp 19–73
[8] Boccardo L, Murat F and Puel J-P 1984 Resultats d'existence pour certains problèmes elliptiques quasi-linéaires *Ann. Scuola Norm. Super. Pisa* **11** 213–35
[9] Boccardo L, Murat F and Puel J-P 1988 Existence of bounded solutions for nonlinear elliptic unilateral problems *Ann. Math. Pura Appl.* **152** 183–96
[10] Boccardo L, Murat F and Puel J-P 1992 L^{∞} estimates for some nonlinear elliptic partial differential equations and application to an existence result *SIAM J. Math. Anal.* **2** 326–33
[11] Boccardo L, Gallouët T and Orsina L 1997 Existence and nonexistence of solutions for some nonlinear elliptic equations *J. Anal. Math.* **73** 203–23
[12] Bogdan K and Byczkowski T 1999 Potential theory for the a-stable Schrödinger operator on bounded Lipschitz domain *Stud. Math.* **133** 53–92
[13] Bogdan K, Byczkowski T, Kulczycki T, Ryznar M, Song R and Vondra Z 2009 *Potential Analysis of Stable Processes and its Extensions* (*Lecture Notes in Mathematics* vol 1980) (Berlin: Springer)
[14] Bogdan K and Jakubowski T 2012 Estimates of the green function for the fractional Laplacian perturbed by gradient *Potential Anal.* **36** 455–81
[15] Bogdan K, Kulczycki T and Nowak A 2002 Gradient estimates for harmonic and $q-$ harmonic funcitons of Symmetric stable processes *Illinois J. Math.* **46** 541–56
[16] Bogdan K, Kulczycki T and Kwaśnicki M 2008 Estimates and structure of a-harmonic functions *Probab. Theory Relat. Fields* **140** 345–81
[17] Brezis H and Cabré X 1998 Some simple nonlinear PDE's without solution *Boll. Unione. Mat. Ital. Sez.* B **8** 223–62
[18] Chen Z and Song R 1998 Estimates on Green's functions and Poisson kernels for symmetric stable process *Math. Ann.* **312** 465–501
[19] Chen H and Veron L 2014 Semilinear fractional elliptic equations involving measures *J. Differ. Equ.* **257** 1457–86
[20] Chen H and Veron L 2014 Semilinear fractional elliptic equations with gradient nonlinearity involving measures *J. Funct. Anal.* **266** 5467–92
[21] Di Nezza E, Palatucci G and Valdinoci E 2012 Hitchhiker's guide to the fractional Sobolev spaces *Bull. Sci. Math.* **136** 521–73
[22] Ferone V and Murat F 1998 Quasilinear problems having quadratic growth in the gradient: an existence result when the source term is small *Equations aux Dérivées Partielles et Applications* ed M R Gauthier-Villars (Amsterdam: Elsevier) pp 497–515
[23] Ferone V and Murat F 2000 Nonlinear problems having natural growth in the gradient: an existence result when the source terms are small *Nonlinear Anal. T.M.A.* **42** 1309–26
[24] Ferrari F and Verbitsky I 2012 Radial fractional Laplace operators and Hessian inequalities *J. Differ. Equ.* **253** 244–72
[25] Frank R L, Lieb E H and Seiringer R 2008 Hardy–Lieb–Thirring inequalities for fractional Schrödinger operators *J. Am. Math. Soc.* **21** 925–50
[26] Frazier M W and Verbitsky I E 2010 Global green's function estimates *Around the Research of Vladimir Maz'ya.* III (*Int. Mathematical Series* vol 13) (New York: Springer) pp 105–52
[27] Gilbarg D and Trudinger N 1998 *Elliptic Partial Differential Equations of Second Order* (*Classics in Mathematics*) (Berlin: Springer) (reprint of the 1998 edn)
[28] Grenon N, Murat F and Porretta A 2006 Existence and *a priori* estimate for elliptic problems with subquadratic gradient dependent terms *C. R. Acad. Sci. Paris* I **342** 23–8
[29] Hairer M 2013 Solving the KPZ equation *Ann. Math.* **178** 559–664
[30] Hansson K, Maz'ya V G and Verbitsky I E 1999 Criteria of solvability for multidimensional Riccati equations *Ark. Mat.* **37** 87–120

[31] Karch G and Woyczyński W A 2008 Fractal Hamilton–Jacobi-KPZ equations *Trans. Am. Math. Soc.* **360** 2423–42
[32] Kardar M, Parisi G and Zhang Y C 1986 Dynamic scaling of growing interfaces *Phys. Rev. Lett.* **56** 889–92
[33] Kulczycki T 2013 Gradinet estimates of $q-$ harmonic functions of fractional schrodinger operator *Potential Anal.* **39** 69–98
[34] Kassmann M 2009 *A priori* estimates for integro-differential operators with measurable kernels *Calc. Var.* **34** 1–21
[35] Leonori T, Peral I, Primo A and Soria F 2015 Basic estimates for solution of elliptic and parabolic equations for a class of nonlocal operators *Discrete Contin. Dyn. Syst.* A **35** 6031–68
[36] Lieb E H 1983 Sharp constants in the Hardy–Littlewood–Sobolev and related inequalities *Ann. Math.* **118** 349–74
[37] Lieb E H and Loss M 2001 *Analysis* (*Graduate Studies in Mathematics* vol 14) (Providence, RI: American Mathematical Society)
[38] Loss M and Sloane C 2010 Hardy inequalities for fractional integrals on general domains *J. Funct. Anal.* **259** 1369–79
[39] Maz'ya V G and Verbitsky I E 1995 Capacitary estimates for fractional integrals, with applications to partial differential equations and Sobolev multipliers *Ark. Mat.* **33** 81–115
[40] Mengesha T and Cong P N 2016 Quasilinear Riccati type equations with distributional data in Morrey space framework *J. Differ. Equ.* **260** 5421–49
[41] Cong P N 2014 Morrey global bounds and qusilinear Riccarti type equation bellow the natural exponent *J. Math. Pures Appl.* **102** 99–123
[42] Porretta A 2009 *Elliptic equations with first order terms (Lectures notes of the course at Alexandria, Ecole Cimpa)* (Egypt) (http://archive.schools.cimpa.info/anciensite/NotesCours/PDF/2009/Alexandrie_Porretta.pdf)
[43] Stampacchia G 1965 Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus *Ann. Inst. Fourier* **15** 189–258
[44] Stein E M 1970 *Singular Integrals and Differentiability Properties of Functions* (Princeton, NJ: Princeton University Press)
[45] Verbitsky I E 2010 *Green's Function Estimates for Some Linear and Nonlinear Elliptic Problems* (*Contemporary Mathematics* vol 528) (Providence, RI: American Mathematical Society) pp 59–69
[46] Woyczyński W A 1998 *Burgers-KPZ Turbulence: Göttingen Lectures* (*Lecture Notes in Mathematics* vol 1700) (Berlin: Springer)