Convex Hedging in Incomplete Markets

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Abstract

In incomplete financial markets not every contingent claim can be replicated by a self-financing strategy. The risk of the resulting shortfall can be measured by convex risk measures, recently introduced by Föllmer and Schied (2002). The dynamic optimization problem of finding a self-financing strategy that minimizes the convex risk of the shortfall can be split into a static optimization problem and a representation problem. It follows that the optimal strategy consists in superhedging the modified claim $\widetilde{\varphi}H$, where H is the payoff of the claim and $\widetilde{\varphi}$ is the solution of the static optimization problem, the optimal randomized test. In this paper, we will deduce necessary and sufficient optimality conditions for the static problem using convex duality methods. The solution of the static optimization problem turns out to be a randomized test with a typical 0-1-structure.

Keywords and phrases: hedging, shortfall risk, convex risk measures, convex duality, generalized Neyman-Pearson lemma

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1 Introduction

In an incomplete financial market not every contingent claim is attainable and the equivalent martingale measure is no longer unique. Thus, a perfect hedge as in the Black-Scholes-Merton model is not possible any longer. Therefore, we are faced with the problem of searching strategies which reduce the risk of the resulting shortfall as much as possible.

One can still stay on the safe side using a 'superhedging' strategy. But from a practical

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point of view, the cost of superhedging is often too high.
For this reason, we consider the possibility of investing less capital than the superhedging price of the liability. This leads to a shortfall, the risk of which, measured by a suitable risk measure, should be minimized. This problem has been studied using the probability (Föllmer and Leukert, 1999), the expectation of a loss function (Föllmer and Leukert, 2000) and a coherent risk measure (Nakano, 2003, 2004; Rudloff, 2005) to quantify the shortfall risk. In our approach, we use convex risk measures, a generalization of coherent risk measures. In analogy to the problems mentioned above, the dynamic optimization problem of finding an admissible strategy that minimizes the convex shortfall risk can be split into a static optimization problem and a representation problem. The optimal strategy consists in superhedging a modified claim $\widetilde{\varphi}H$, where H is the payoff of the claim and $\widetilde{\varphi}$ is a solution of the static optimization problem, an optimal randomized test. In this paper, we show the existence of a solution to the static problem. We deduce with methods of Fenchel duality a necessary and sufficient condition for an optimal randomized test that gives a result about the structure of the solution. The results are new and even improve, when restricted to coherent risk measures, the results obtained by Nakano (2003, 2004) (see Rudloff, 2006, Section 4.1.3).
This paper is organized as follows: In Section 2, we state the formulation of the shortfall problem and review the definition of convex risk measures. Then, we prove the possibility to decompose the dynamic optimization problem into a static and a representation problem. In Section 4, we analyze the static optimization problem. We show the existence of a solution, formulate the dual problem and show that strong duality holds. Thus, the optimal solution is a saddle point of a functional, specified in Section 4.2. To solve the problem, we first consider the inner problem in Section 4.3 and deduce an extended Neyman-Pearson lemma. Then, we solve the saddle point problem in Section 4.4. The optimal solution of the static optimization problem is a randomized test with the typical 0-1-structure.

2 Formulation of the Problem

The discounted price process of the d underlying assets is described as an $\mathbb{R}^d$-valued semimartingale $S = (S_t)_{t\in[0,T]}$ on a complete probability space $(\Omega, \mathcal{F}, P)$ with filtration $(\mathcal{F}_t)_{t\in[0,T]}$. The filtration is supposed to satisfy the usual conditions (see e.g. Karatzas and Shreve (1998)). We write L^1 and L^∞ for $L^1(\Omega, \mathcal{F}, P)$ and $L^\infty(\Omega, \mathcal{F}, P)$, respectively. We endow L^1 and L^∞ with the norm topology. Then, the topological dual space of L^1 can be identified with L^∞ and the topological dual space of L^∞ can be identified with $ba(\Omega, \mathcal{F}, P)$, the space of finitely additive set functions on $(\Omega, \mathcal{F})$ with bounded variation, absolutely continuous to P (see Yosida, 1980, Chapter IV, 9, Example 5). Let $\langle X, Y\rangle$ be the bilinear form between L^∞ and $ba(\Omega, \mathcal{F}, P)$: $\langle X, Y\rangle = \int_\Omega Y dX$ for all $Y \in L^\infty$, $X \in ba(\Omega, \mathcal{F}, P)$. For $X \in L^1$ it reduces to $\langle X, Y\rangle = E[XY]$, where E denotes the mathematical expectation with respect to P.

Let $\widehat{\mathcal{Q}}$ be the set of all probability measures on $(\Omega, \mathcal{F})$ absolutely continuous with respect to P. For $Q \in \widehat{\mathcal{Q}}$ we denote the expectation with respect to Q by E^Q and the Radon-Nikodym derivative dQ/dP by Z_Q.
An $\mathbb{R}^d$-valued semimartingale $S = (S_t)_{t\in[0,T]}$ is called a sigma-martingale if there exists an $\mathbb{R}^d$-valued martingale M and an M-integrable predictable $\mathbb{R}_+$-valued process ξ such that $S_t = \int_0^t \xi_s dM_s$, $t \in [0,T]$ (see Delbaen and Schachermayer (2006), Section 14.2). Let $\mathcal{P}_\sigma$ denote the set of probability measures P^* equivalent to P such that S is a sigma-martingale with respect to P^*. Since we want to exclude arbitrage opportunities, we assume that $\mathcal{P}_\sigma \neq \emptyset$. To be more concrete: S satisfies the condition of 'no free lunch with vanishing risk' if and only if $\mathcal{P}_\sigma \neq \emptyset$ (Theorem 14.1.1, Delbaen and Schachermayer (2006)). The concept of 'no free lunch with vanishing risk' is a mild strengthening of the concept of 'no arbitrage' that has to be used in general semimartingale models. In the case of a finite probability space Ω the above assertion holds true if one replaces the term 'no free lunch with vanishing risk' by the term 'no arbitrage' and the set $\mathcal{P}_\sigma$ by the set of equivalent martingale measures $\mathcal{P}$. In the case of an $\mathbb{R}^d$-valued (locally) bounded semimartingale S one may replace the set $\mathcal{P}_\sigma$ by the set of equivalent (local) martingale measures $\mathcal{P}_{(loc)}$ (see Delbaen and Schachermayer (2006), Chapter 9).
Equations and inequalities between random variables are always understood as $P-a.s.$
A self-financing strategy is given by an initial capital $V_0 \geq 0$ and a predictable process ξ such that the resulting value process

$$V_t = V_0 + \int_0^t \xi_s dS_s, \quad t \in [0,T],$$

is well defined. Such a strategy (V_0, ξ) is called admissible if the corresponding value process V_t satisfies $V_t \geq 0$ for all $t \in [0,T]$.
Consider a contingent claim. Its payoff is given by an $\mathcal{F}_T$ -measurable, nonnegative random variable $H \in L^1$. We assume

$$U_0 = \sup_{P^* \in \mathcal{P}_\sigma} E^{P^*}[H] < +\infty. \tag{1}$$

The above equation is the dual characterization of the superhedging price U_0, the smallest amount V_0 such that there exists an admissible strategy (V_0, ξ) with value process V_t satisfying $V_T \geq H$ (see Delbaen and Schachermayer (2006), Theorem 14.5.20). The corresponding strategy is called the superhedging strategy of the claim H. Again, in the case where S is an $\mathbb{R}^d$-valued (locally) bounded semimartingale, one may replace the set $\mathcal{P}_\sigma$ by the set of equivalent (local) martingale measures $\mathcal{P}_{(loc)}$. In the complete case, where the equivalent sigma-martingale measure P^* is unique, $U_0 = E^{P^*}[H]$ is the unique arbitrage-free price of the contingent claim.
Since superhedging can be quite expensive in the incomplete market (see e.g. Gushchin and Mordecki (2002)), we search for the best hedge an investor can achieve with a smaller amount $\widetilde{V}_0 < U_0$. In other words, we look for an admissible strategy (V_0, ξ) with $0 < V_0 \leq \widetilde{V}_0$ that minimizes the risk of losses due to the shortfall $\{\omega : V_T(\omega) < H(\omega)\}$, this means we want to minimize the risk of $-(H - V_T)^+$. The risk will be measured

by a convex risk measure ρ, recently introduced by Föllmer and Schied (2002). Thus, we consider the dynamic optimization problem of finding an admissible strategy that solves

$$\min_{(V_0,\xi)} \rho\left(-\left(H-V_T\right)^+\right) \tag{2}$$

under the capital constraint of investing less capital than the superhedging price

$$0 < V_0 \leq \widetilde{V}_0 < U_0. \tag{3}$$

For the convenience of the reader we recall the definition and some properties of convex risk measures. In contrast to Föllmer and Schied (2002), where $\rho: L^\infty \to \mathbb{R}$, we consider convex risk measures defined on L^1 that can also attain the value $+\infty$ for investments that are not acceptable in any way.

Definition 2.1 (convex risk measure)**.** *A function $\rho: L^1 \to \mathbb{R} \cup \{+\infty\}$ with $\rho(0) = 0$ is a convex risk measure if it satisfies for all $X_1, X_2 \in L^1$:*

(i) monotonicity: $X_1 \geq X_2 \Rightarrow \rho(X_1) \leq \rho(X_2)$,

(ii) translation property: $c \in \mathbb{R} \Rightarrow \rho(X_1 + c\mathbf{1}) = \rho(X_1) - c$,

(iii) convexity: $\lambda \in [0,1] \Rightarrow \rho(\lambda X_1 + (1-\lambda)X_2) \leq \lambda\rho(X_1) + (1-\lambda)\rho(X_2)$.

The random variable equal to 1 almost surely is denoted by $\mathbf{1}$ in (ii). The assumption $\rho(0) = 0$ is reasonable and ensures that $\rho(X)$ can be interpreted as risk adjusted capital requirement. The set $\mathcal{A} := \{X \in L^1 : \rho(X) \leq 0\}$ is called the acceptance set of the risk measure ρ. It is well known that each lower semicontinuous convex risk measures admits a dual representation (see Föllmer and Schied (2002) for risk measures on L^∞ and for those on general L^p spaces, see for instance Ruszczynski and Shapiro (2006), Hamel (2006), Rudloff (2006)). A function $\rho: L^1 \to \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous, convex risk measure if and only if there exists a representation of the form

$$\rho(X) = \sup_{Q \in \mathcal{Q}} \{E^Q[-X] - \sup_{\widetilde{X} \in \mathcal{A}} E^Q[-\widetilde{X}]\}, \tag{4}$$

where $\mathcal{Q} := \{Q \in \widehat{\mathcal{Q}} : Z_Q \in L^\infty\}$. The conjugate function ρ^* of ρ is nonnegative, convex, proper, weakly* lower semicontinuous,

$$\operatorname{dom} \rho^* \subseteq \{-Z_Q : Q \in \mathcal{Q}\}$$

and ρ^* satisfies for all $Y \in L^\infty$ with $E[Y] = -1$

$$\rho^*(Y) = \sup_{X \in \mathcal{A}} E[XY]. \tag{5}$$

Remark 2.2. Let $\alpha(Q) : \mathcal{Q} \to \mathbb{R} \cup \{+\infty\}$ be a functional with $\inf_{Q\in\mathcal{Q}} \alpha(Q) = 0$. Then,

$$\rho(X) := \sup_{Q\in\mathcal{Q}} \{E^Q[-X] - \alpha(Q)\}$$

is a convex risk measure. The functional α is called a penalty function and $\rho^*(-Z_Q) =: \alpha_{min}(Q)$ for $Q \in \mathcal{Q}$ is the minimal penalty function on $\mathcal{Q}$ that represents ρ (see Föllmer and Schied (2004)).
The penalty function α describes how seriously the probabilistic model $Q \in \mathcal{Q}$ is taken. The value of the convex risk measure $\rho(X)$ is the worst case of the expected loss $E^Q[-X]$ reduced by $\alpha(Q)$, taken over all models $Q \in \mathcal{Q}$ (Föllmer and Schied (2004), Section 3.4).

Coherent risk measures, as introduced in Artzner et al. (1999) and Delbaen (2002), are convex risk measures that are additionally positive homogeneous. For coherent risk measures the dual representation (4) reduces to: ρ is a lower semicontinuous coherent risk measure if and only if there exists a non-empty subset of probability measures $\widetilde{\mathcal{Q}}$ of $\mathcal{Q}$ with $\{Z_Q : Q \in \widetilde{\mathcal{Q}}\}$ convex and weakly* closed in L^∞, such that

$$\rho(X) = \sup_{Q\in\widetilde{\mathcal{Q}}} E^Q[-X]. \tag{6}$$

This conditions is satisfied if ρ admits (4) with $\mathcal{A}$ being a cone, which implies that $\sup_{X\in\mathcal{A}} E^Q[-X] = \mathcal{I}_{\widetilde{\mathcal{Q}}}(Q)$ is an indicator function equal to zero for $Q \in \widetilde{\mathcal{Q}}$ and $+\infty$ otherwise.

3 Decomposition of the Dynamic Problem

The dynamic optimization problem (2), (3) can be split into the following two problems:

1. Static optimization problem: Find an optimal modified claim $\widetilde{\varphi}H$, where $\widetilde{\varphi}$ is a randomized test solving

$$\min_{\varphi\in R_0} \rho\left((\varphi - 1)H\right), \tag{7}$$

$$R_0 = \{\varphi : \ \Omega \to [0,1], \ \mathcal{F}_T - \text{measurable}, \ \sup_{P^*\in\mathcal{P}_\sigma} E^{P^*}[\varphi H] \le \widetilde{V}_0\} \tag{8}$$

2. Representation problem: Find a superhedging strategy for the modified claim $\widetilde{\varphi}H$.

This idea was introduced by Föllmer and Leukert (1999, 2000) using the expectation of a loss function as risk measure and was used for coherent risk measures in Nakano (2003, 2004); Rudloff (2005, 2006) analogously. We obtain the following theorem for convex risk measures:

Theorem 3.1. *Let $\widetilde{\varphi}$ be a solution of the minimization problem (7) and let $(\widetilde{V}_0, \widetilde{\xi})$ be the admissible strategy, where $\widetilde{\xi}$ is the superhedging strategy of the claim $\widetilde{\varphi}H$. Then the strategy $(\widetilde{V}_0, \widetilde{\xi})$ solves the optimization problem (2), (3) and it holds*

$$\min_{(V_0,\xi)} \rho(-(H - V_T)^+) = \min_{\varphi \in R_0} \rho\left((\varphi - 1)H\right). \tag{9}$$

Proof. Let (V_0, ξ) with $V_0 \leq \widetilde{V}_0$ be an admissible strategy. We define the corresponding success ratio $\varphi = \varphi_{(V_0,\xi)}$ as

$$\varphi_{(V_0,\xi)} := I_{\{V_T \geq H\}} + \frac{V_T}{H} I_{\{V_T < H\}},$$

where $I_A(\omega)$ is the stochastic indicator function equal to one for $\omega \in A$ and zero otherwise. Thus, $-(H - V_T)^+ = (\varphi - 1)H$. Since V_t is a $\mathcal{P}_\sigma$-supermartingale (Delbaen and Schachermayer (2006), Theorem 14.5.5) and $\varphi H \leq V_T$:

$$\forall P^* \in \mathcal{P}_\sigma : \quad E^{P^*}[\varphi H] \leq E^{P^*}[V_T] \leq V_0 \leq \widetilde{V}_0,$$

hence, $\varphi \in R_0$. Thus,

$$\rho(-(H - V_T)^+) = \rho((\varphi - 1)H) \geq \rho((\widetilde{\varphi} - 1)H), \tag{10}$$

where $\widetilde{\varphi}$ is the solution to the static optimization problem (7). Consider the admissible strategy $(\overline{V}_0, \widetilde{\xi})$, where $\widetilde{\xi}$ is the superhedging strategy for the modified claim $\widetilde{\varphi}H$ and $\overline{V}_0 \in [\widetilde{U}_0, \widetilde{V}_0]$, where $\widetilde{U}_0 = \sup_{P^* \in \mathcal{P}_\sigma} E^{P^*}[\widetilde{\varphi}H]$ is the superhedging price of the modified claim $\widetilde{\varphi}H$ (see Theorem 14.5.20, Delbaen and Schachermayer (2006)). Inequality (10) is especially satisfied for the success ratio of the admissible strategy $(\overline{V}_0, \widetilde{\xi})$. Thus,

$$\rho((\varphi_{(\overline{V}_0,\widetilde{\xi})} - 1)H) \geq \rho((\widetilde{\varphi} - 1)H). \tag{11}$$

To show the revers inequality, let us consider $\varphi_{(\overline{V}_0,\widetilde{\xi})}H = \min(\widetilde{V}_T, H)$, where $\widetilde{V}_T = \overline{V}_0 + \int_0^T \widetilde{\xi}_s dS_s$. Because of $\widetilde{U}_0 + \int_0^T \widetilde{\xi}_s dS_s \geq \widetilde{\varphi}H$ (superhedging) and $\overline{V}_0 \in [\widetilde{U}_0, \widetilde{V}_0]$, it holds

$$\widetilde{V}_T = \overline{V}_0 + \int_0^T \widetilde{\xi}_s dS_s \geq \widetilde{\varphi}H + \overline{V}_0 - \widetilde{U}_0 \geq \widetilde{\varphi}H.$$

Thus, $\varphi_{(\overline{V}_0,\widetilde{\xi})}H \geq \widetilde{\varphi}H$. Since the convex risk measure ρ is monotone, we obtain

$$\rho((\varphi_{(\overline{V}_0,\widetilde{\xi})} - 1)H) \leq \rho((\widetilde{\varphi} - 1)H).$$

Together with (11), we see that $\varphi_{(\overline{V}_0,\widetilde{\xi})}$ attains the minimum of the static optimization problem (7). Due to (10), we now have

$$\min_{(V_0,\xi)} \rho(-(H - V_T)^+) \geq \rho(-(H - \widetilde{V}_T)^+).$$

Hence, $(\overline{V}_0, \widetilde{\xi})$ with $\overline{V}_0 \in [\widetilde{U}_0, \widetilde{V}_0]$ is the strategy that attains the minimum in the dynamic optimization problem (2), (3) and equation (9) holds true. □

Remark 3.2. In the case of risk measures that allow the construction of $\widetilde{\varphi}$ via the Neyman-Pearson lemma directly (cf. Föllmer and Leukert (1999) and some special cases of Föllmer and Leukert (2000)), one can see that $\widetilde{U}_0 = \widetilde{V}_0$ since the optimal test $\widetilde{\varphi}$ attains the bound $\widetilde{V}_0$ in (8).
In Theorem 4.9, equation (30) of this paper we will show that in the case of convex hedging the bound $\widetilde{V}_0$ is as well attained by the optimal test. Thus, the optimal strategy is $(\widetilde{V}_0, \widetilde{\xi})$.

In the following section, we will consider the static optimization problem (7). To solve it we improve the method used in Rudloff (2006).

4 The Static Optimization Problem

Now, we will show that there exists a solution $\widetilde{\varphi}$ of the static optimization problem (7) and derive necessary and sufficient optimality conditions. Therefore, we will construct the dual problem of (7), deduce a result about the structure of a solution for the inner problem of the dual problem and then solve the whole problem.

4.1 The Primal Problem and the Plan of its Solution

We impose the following assumption that has to be satisfied throughout the remaining part of this paper:

Assumption 4.1. *Let $\rho: L^1 \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous convex risk measure that is continuous and finite in some $(\varphi_0 - 1)H$ with $\varphi_0 \in R_0$.*

Remark 4.2. A convex risk measure $\rho: L^1 \to \mathbb{R} \cup \{+\infty\}$ is (without assuming lower semicontinuity) continuous in the interior of its domain (extended Namioka Theorem, see Ruszczynski and Shapiro (2006), Proposition 3.1 or Frittelli and Biagini (2006), Theorem 2). Especially, if $\rho(X) < +\infty$ for all $X \in L^1$, a convex risk measure admits the representation (4) and is continuous. But for extended real valued convex risk measures we still need the assumption of lower semicontinuity to obtain representation (4).

Let us consider the measurable space $(\mathcal{P}_\sigma, \mathcal{S})$, where $\mathcal{S}$ is the σ-algebra generated by all subsets of $\mathcal{P}_\sigma$. We denote by Λ_+ the set of finite measures on $(\mathcal{P}_\sigma, \mathcal{S})$.
We give an overview over the procedure to solve the static optimization problem:

(i) Prove the existence of a solution $\widetilde{\varphi}$ to the primal problem (7) (Theorem 4.3)

$$p = \min_{\varphi \in R_0} \rho\left((\varphi - 1)H\right) = \min_{\varphi \in R_0} \{\sup_{Q \in \mathcal{Q}} \{E^Q[(1-\varphi)H] - \sup_{X \in \mathcal{A}} E^Q[-X]\}\}.$$

(ii) Deduce the dual problem to (7) by Fenchel duality:

$$d = \sup_{Q\in\mathcal{Q}}\{\inf_{\varphi\in R_0}\{E^Q[(1-\varphi)H] - \sup_{X\in\mathcal{A}} E^Q[-X]\}\} \tag{12}$$

and prove the validity of strong duality $p = d$ (Theorem 4.5). We obtain the existence of a dual solution and can show that the problem is a saddle point problem.

(iii) Consider the inner problem of the dual problem (12) for an arbitrary $Q \in \mathcal{Q}$:

$$p^i(Q) := \max_{\varphi\in R_0} E^Q[\varphi H]. \tag{13}$$

Prove the existence of a solution $\widetilde{\varphi}_Q$ to (13) (Lemma 4.6). Deduce the dual problem of (13) by Fenchel duality:

$$d^i(Q) = \inf_{\lambda\in\Lambda_+}\Big\{\int_\Omega [HZ_Q - H\int_{\mathcal{P}_\sigma} Z_{P^*}d\lambda]^+ dP + \widetilde{V}_0\lambda(\mathcal{P}_\sigma)\Big\}.$$

Prove the validity of strong duality $p^i(Q) = d^i(Q)$ and deduce the necessary and sufficient structure of a solution $\widetilde{\varphi}_Q$ to the inner problem (13) (Theorem 4.8).

(iv) Apply Theorem 4.5 and 4.8 to the primal problem (7) and deduce the necessary and sufficient structure of a solution $\widetilde{\varphi}$ to (7) (Theorem 4.9).

The existence of a solution $\widetilde{\varphi}$ to the static optimization problem (7) can be shown analogously to Nakano (2004, Proposition 1.3), where coherent risk measures were considered.

Theorem 4.3. *There exists a $\widetilde{\varphi} \in R_0$ solving the static optimization problem (7) and $\rho((\widetilde{\varphi}-1)H)$ is finite.*

Proof. The set of randomized tests $R = \{\varphi : \Omega \to [0,1],\ \mathcal{F}_T - \text{measurable}\}$ is weakly* compact as a weakly* closed subset of the weakly* compact unit sphere in L^∞ (Dunford and Schwartz (1988), Theorem V.4.2, V.4.3). Since the map $\varphi \mapsto \sup_{P^*\in\mathcal{P}_\sigma} E^{P^*}[\varphi H]$ is lower semicontinuous in the weak* topology, the constrained set R_0 is weakly* closed, hence weakly* compact. Because of the lower semicontinuity of $\varphi \mapsto \sup_{Q\in\mathcal{Q}}\{E^Q[(1-\varphi)H] - \sup_{X\in\mathcal{A}} E^Q[-X]\}$ in the weak* topology, there exists a $\widetilde{\varphi} \in R_0$ solving (7). $\rho((\widetilde{\varphi}-1)H)$ is finite since ρ is assumed to be finite in some $(\varphi_0 - 1)H$ with $\varphi_0 \in R_0$ (Assumption 4.1). □

Remark 4.4. For measures of risk that are strictly convex one can additionally show that any two solutions coincide $P - a.s.$ on $\{\omega : H > 0\}$ (see Föllmer and Leukert, 2000, Proposition 3.1). A convex risk measure cannot be strictly convex since the translation property of ρ (Definition 2.1 (ii)) and $\rho(0) = 0$ imply the linearity of ρ on the one dimensional subspace of L^1 generated by the random variable equal to 1 a.s. (see Hamel (2006) for further properties of translative functions). This means that for convex risk measures one can only show the existence, not the essential uniqueness of the solution.

4.2 The Dual Problem

In this subsection, we will construct the dual problem of (7) and prove the validity of strong duality. We obtain the existence of a dual solution and show that the problem is a saddle point problem.

Theorem 4.5. *Strong duality holds: The values of the primal problem (7) and its dual problem are equal ($p = d$), where the dual problem of (7) is the following with value d*

$$d = \sup_{Q\in\mathcal{Q}}\{\inf_{\varphi\in R_0}\{E^Q[(1-\varphi)H] - \sup_{X\in\mathcal{A}} E^Q[-X]\}\}. \tag{14}$$

$(\widetilde{Z}_Q, \widetilde{\varphi})$ is a saddle point of the functional $E^Q[(1-\varphi)H] - \sup_{X\in\mathcal{A}} E^Q[-X]$, where $\widetilde{\varphi}$ is the solution of (7) and $\widetilde{Z}_Q = \frac{d\widetilde{Q}}{dP}$ is the solution of (14). Thus,

$$\min_{\varphi\in R_0}\{\max_{Q\in\mathcal{Q}}\{E^Q[(1-\varphi)H] - \sup_{X\in\mathcal{A}} E^Q[-X]\}\} = \max_{Q\in\mathcal{Q}}\{\min_{\varphi\in R_0}\{E^Q[(1-\varphi)H] - \sup_{X\in\mathcal{A}} E^Q[-X]\}\}.$$

Proof. Problem (7) can be rewritten as

$$p = \min_{\varphi\in L^\infty}\{\rho\left((\varphi-1)H\right) + \mathcal{I}_{R_0}(\varphi)\}.$$

We denote $f(\varphi) := \mathcal{I}_{R_0}(\varphi)$ and $g(A\varphi) := \rho\left(A\varphi - H\right) = \rho\left((\varphi-1)H\right)$, where the linear and continuous operator $A : L^\infty \to L^1$ is defined by $A\varphi := H\varphi$. The Fenchel dual problem is (see Ekeland and Temam (1976), Chapter III, equation (4.18))

$$d = \sup_{Y\in L^\infty}\{-f^*(A^*Y) - g^*(-Y)\}, \tag{15}$$

where A^* is the adjoined operator of A and f^*, g^* are the conjugate functions of f and g, respectively. The value p of the primal problem is finite (Theorem 4.3). The function $f : L^\infty \to \mathbb{R}\cup\{+\infty\}$ is convex because of the convexity of R_0. The function $g : L^1 \to \mathbb{R}\cup\{+\infty\}$ is convex since ρ is convex. Since ρ is assumed to be continuous and finite in some $(\varphi_0 - 1)H$ with $\varphi_0 \in R_0$ (Assumption 4.1) we have strong duality $p = d$ (Theorem III.4.1 and Remark III.4.2 in Ekeland and Temam (1976)).
The adjoined operator $A^* : L^\infty \to ba(\Omega, \mathcal{F}, P)$ of A has to satisfy by definition the following equations:

$$\forall Y \in L^\infty, \forall\varphi \in L^\infty : \quad \langle A^*Y, \varphi\rangle = \langle Y, A\varphi\rangle = E[\varphi HY]. \tag{16}$$

To establish the dual problem, we calculate the conjugate functions f^* and g^*. With (16), we obtain

$$f^*(A^*Y) = \sup_{\varphi\in L^\infty}\{\langle A^*Y, \varphi\rangle - f(\varphi)\} = \sup_{\varphi\in R_0} E[\varphi HY].$$

The function g is defined by $g(X) = \rho(X - H)$. Its conjugate function $g^* : L^\infty \to \mathbb{R}\cup\{+\infty\}$ is (Zălinescu, 2002, Theorem 2.3.1 (vi)):

$$g^*(Y) \quad = \quad \rho^*(Y) + \langle Y, H\rangle.$$

Since $\operatorname{dom}\rho^* \subseteq \{-Z_Q : Q \in \mathcal{Q}\}$ and $\rho^*(Y) = \sup_{X\in\mathcal{A}} E[XY]$ for $Y \in L^\infty$ with $E[Y] = -1$ (see equation (5)), the dual problem (15) with value d is

$$d = \sup_{Q\in\mathcal{Q}}\{\inf_{\varphi\in R_0}\{E^Q[(1-\varphi)H] - \sup_{X\in\mathcal{A}} E^Q[-X]\}\}. \tag{17}$$

The existence of a solution $\widetilde{Z}_Q$ to the dual problem follows from the validity of strong duality (see Theorem III.4.1, Ekeland and Temam, 1976). Let $\widetilde{\varphi}$ be the solution to the primal problem (7) (see Theorem 4.3). Since

$$\begin{aligned} p &= \sup_{Q\in\mathcal{Q}}\{E^Q[(1-\widetilde{\varphi})H] - \sup_{X\in\mathcal{A}} E^Q[-X]\} \geq E^{\widetilde{Q}}[(1-\widetilde{\varphi})H] - \sup_{X\in\mathcal{A}} E^{\widetilde{Q}}[-X], \\ d &= \inf_{\varphi\in R_0}\{E^{\widetilde{Q}}[(1-\varphi)H] - \sup_{X\in\mathcal{A}} E^{\widetilde{Q}}[-X]\} \leq E^{\widetilde{Q}}[(1-\widetilde{\varphi})H] - \sup_{X\in\mathcal{A}} E^{\widetilde{Q}}[-X] \end{aligned}$$

and because of strong duality, we have

$$E^{\widetilde{Q}}[(1-\widetilde{\varphi})H] - \sup_{X\in\mathcal{A}} E^{\widetilde{Q}}[-X] \leq p = d \leq E^{\widetilde{Q}}[(1-\widetilde{\varphi})H] - \sup_{X\in\mathcal{A}} E^{\widetilde{Q}}[-X].$$

Hence,

$$\min_{\varphi\in R_0}\{\max_{Q\in\mathcal{Q}}\{E^Q[(1-\varphi)H] - \sup_{X\in\mathcal{A}} E^Q[-X]\}\} = \max_{Q\in\mathcal{Q}}\{\min_{\varphi\in R_0}\{E^Q[(1-\varphi)H] - \sup_{X\in\mathcal{A}} E^Q[-X]\}\}.$$

Thus, $(\widetilde{Z}_Q, \widetilde{\varphi})$ is a saddle point of the function $E^Q[(1-\varphi)H] - \sup_{X\in\mathcal{A}} E^Q[-X]$. □

4.3 The Inner Problem of the Dual Problem

In this subsection, we consider the inner problem of the dual problem (14) for an arbitrary, but fixed $Q \in \mathcal{Q}$. We give a result about the structure of a solution. This makes it possible to deduce a result about a saddle point of Theorem 4.5 in our main theorem in the next subsection. That means, we obtain a result about the structure of a solution of the static optimization problem (7).
First let us consider the inner problem of the dual problem (14) for a $Q \in \mathcal{Q}$ and let us denote with $p^i(Q)$ its optimal value:

$$p^i(Q) := \max_{\varphi\in R_0} E^Q[\varphi H]. \tag{18}$$

Lemma 4.6. *There exists a solution $\widetilde{\varphi}_Q$ to problem (18) and $p^i(Q)$ is finite.*

Proof. The assertion follows since R_0 is weakly* compact (see proof of Theorem 4.3) and $\varphi \mapsto E^Q[\varphi H]$ is continuous in the weak* topology for all $Q \in \mathcal{Q}$. □

Remark 4.7. Problem (18) can be identified as a problem of test theory. Let $R = \{\varphi : \Omega \to [0,1],\ \mathcal{F}_T - \text{measurable}\}$ be the set of randomized tests and let us define the measures O and $O^* = O^*(P^*)$ by $\frac{dO}{dQ} = H$ and $\frac{dO^*}{dP^*} = H$ for $P^* \in \mathcal{P}_\sigma$. Problem (18) turns into

$$\max_{\varphi \in R} E^O[\varphi]$$

subject to

$$\forall P^* \in \mathcal{P}_\sigma : \quad E^{O^*}[\varphi] \leq \widetilde{V}_0 =: \alpha.$$

This is equivalent of looking for an optimal test $\widetilde{\varphi}_Q$ when testing the compound hypothesis $H_0 = \{O^*(P^*) : P^* \in \mathcal{P}_\sigma\}$, parameterized by the class of equivalent sigma-martingale measures, against the simple alternative hypothesis $H_1 = \{O\}$ in a generalized sense. In the generalized test problem (Witting, 1985, Theorem 2.79), O and O^* are not necessarily probability measures, but measures and the significance level α is generalized to be a positive continuous function $\alpha(P^*)$. Witting (1985) deduced a sufficient optimality condition for the optimal test $\widetilde{\varphi}_Q$ and verified the validity of weak duality.

We want to show that strong duality is satisfied. In this case, the typical 0-1-structure of $\widetilde{\varphi}_Q$ is sufficient and necessary for optimality.
We assign to (18) the following Fenchel dual problem and denote by $d^i(Q)$ its optimal value

$$d^i(Q) = \inf_{\lambda \in \Lambda_+} \Big\{ \int_\Omega [HZ_Q - H \int_{\mathcal{P}_\sigma} Z_{P^*} d\lambda]^+ dP + \widetilde{V}_0 \lambda(\mathcal{P}_\sigma) \Big\}. \tag{19}$$

The following strong duality theorem holds true.

Theorem 4.8. *Strong duality holds true for problems (18) and (19), i.e.,*

$$\forall Q \in \mathcal{Q} : \quad d^i(Q) = p^i(Q).$$

Moreover, for each $Q \in \mathcal{Q}$ there exists a solution $\widetilde{\lambda}_Q$ to (19). The optimal randomized test $\widetilde{\varphi}_Q$ of (18) has the following structure:

$$\widetilde{\varphi}_Q(\omega) = \begin{cases} 1 & : \quad HZ_Q > H \int_{\mathcal{P}_\sigma} Z_{P^*} d\widetilde{\lambda}_Q(P^*) \\ 0 & : \quad HZ_Q < H \int_{\mathcal{P}_\sigma} Z_{P^*} d\widetilde{\lambda}_Q(P^*) \end{cases} \qquad P - a.s. \tag{20}$$

and

$$E^{P^*}[\widetilde{\varphi}_Q H] = \widetilde{V}_0 \qquad \widetilde{\lambda}_Q - a.s. \tag{21}$$

Proof. Let $\mathcal{L}$ be the linear space of all bounded and measurable real functions on $(\mathcal{P}_\sigma, \mathcal{S})$ with pointwise addition, multiplication with real numbers and pointwise partial order $l_1 \leq l_2 \Leftrightarrow l_2 - l_1 \in \mathcal{L}_+ := \{l \in \mathcal{L} : \forall P^* \in \mathcal{P}_\sigma : l(P^*) \geq 0\}$. We recall that $\mathcal{S}$ is the σ-algebra generated by all subsets of $\mathcal{P}_\sigma$.
Let Λ be the space of all σ-additive signed measures on $(\mathcal{P}_\sigma, \mathcal{S})$ of bounded variation. We regard $\mathcal{L}$ and Λ as the duality pair associated with the bilinear form $\langle l, \lambda \rangle = \int_{\mathcal{P}_\sigma} l d\lambda$ for $l \in \mathcal{L}$ and $\lambda \in \Lambda$, see Aliprantis and Border (1999, Theorem 13.5). We endow the

space $\mathcal{L}$ with the Mackey topology $\tau(\mathcal{L},\Lambda)$, which ensures that the topological dual of $(\mathcal{L},\tau(\mathcal{L},\Lambda))$ is Λ and that $\mathcal{L}$ is a barrelled space (Husain and Khaleelulla (1978), Corollary II.2, II.4).

We define a linear and continuous operator $B : (L^\infty, \|\cdot\|_{L^\infty}) \to (\mathcal{L},\tau(\mathcal{L},\Lambda))$ by $(B\varphi)(P^*) := -E^{P^*}[H\varphi]$ for $P^* \in \mathcal{P}_\sigma$. B is continuous since for every sequence $\varphi_n \to \varphi$ in $(L^\infty, \|\cdot\|_{L^\infty})$, it holds that $B\varphi_n \to B\varphi$ in $(\mathcal{L}, \|\cdot\|_{\mathcal{L}})$, where $\|l\|_{\mathcal{L}} := \sup_{P^*\in\mathcal{P}_\sigma} |l(P^*)|$, since

$$\sup_{P^*\in\mathcal{P}_\sigma} |B(\varphi_n-\varphi)(P^*)| \le \|\varphi_n-\varphi\|_{L^\infty} U_0$$

and $U_0 < +\infty$ (inequality (1)). Thus, $B\varphi_n$ converges also in the weaker topology $\tau(\mathcal{L},\Lambda)$. We define the functions $\mathbf{1},\mathbf{0} \in \mathcal{L}$ by

$$\forall P^* \in \mathcal{P}_\sigma : \mathbf{1}(P^*) = 1 \in \mathbb{R},\ \mathbf{0}(P^*) = 0 \in \mathbb{R}.$$

Problem (18) is

$$\max_{\varphi\in R} E^Q[\varphi H],$$
$$\forall P^* \in \mathcal{P}_\sigma : \quad E^{P^*}[\varphi H] \le \widetilde{V}_0. \tag{22}$$

The constraint (22) can be rewritten as

$$\widetilde{V}_0\mathbf{1} + B\varphi \ge \mathbf{0} \Leftrightarrow B\varphi \in \mathcal{L}_+ - \widetilde{V}_0\mathbf{1}.$$

Then, we can write problem (18) equivalently as

$$-p^i(Q) = \min_{\varphi\in L^\infty} \Big\{ - E^Q[\varphi H] + \mathcal{I}_R(\varphi) + \mathcal{I}_{\mathcal{L}_+ - \widetilde{V}_0\mathbf{1}}(B\varphi) \Big\}. \tag{23}$$

Let us define the functions $f(\varphi) := -E^Q[\varphi H] + \mathcal{I}_R(\varphi)$ and $g(B\varphi) := \mathcal{I}_{\mathcal{L}_+ - \widetilde{V}_0\mathbf{1}}(B\varphi)$ in (23). We want to establish the dual problem of (23) as in Ekeland and Temam (1976) (Chapter III, equation (4.18)):

$$-d^i(Q) = \sup_{\lambda\in\Lambda} \Big\{ - f^*(B^*\lambda) - g^*(-\lambda) \Big\}. \tag{24}$$

The conjugate function of g is

$$\begin{aligned} g^*(\lambda) &= \sup_{\widetilde{l}\in\mathcal{L}} \Big\{ \langle \widetilde{l},\lambda\rangle - \mathcal{I}_{\mathcal{L}_+ - \widetilde{V}_0\mathbf{1}}(\widetilde{l}) \Big\} = \sup_{\widetilde{l}\in\mathcal{L}_+ - \widetilde{V}_0\mathbf{1}} \langle \widetilde{l},\lambda\rangle = \sup_{l\in\mathcal{L}_+} \langle l - \widetilde{V}_0\mathbf{1},\lambda\rangle \\ &= \sup_{l\in\mathcal{L}_+} \langle l,\lambda\rangle - \widetilde{V}_0 \int_{\mathcal{P}_\sigma} d\lambda = \mathcal{I}_{\mathcal{L}_+^*}(\lambda) - \widetilde{V}_0 \lambda(\mathcal{P}_\sigma), \end{aligned}$$

where $\mathcal{L}_+^*$ is the negative dual cone of $\mathcal{L}_+$. To establish the conjugate function of f

$$f^*(B^*\lambda) = \sup_{\varphi\in L^\infty} \Big\{ \langle B^*\lambda,\varphi\rangle + E^Q[\varphi H] - \mathcal{I}_R(\varphi) \Big\},$$

we have to calculate $\langle B^*\lambda, \varphi\rangle$, where $B^*: \Lambda \to ba(\Omega, \mathcal{F}, P)$ is the adjoined operator of B. By definition of B^*, the equation $\langle B^*\lambda, \varphi\rangle = \langle \lambda, B\varphi\rangle$ has to be satisfied for all $\varphi \in L^\infty, \lambda \in \Lambda$ (see Aliprantis and Border (1999), Definition 6.51). Thus,

$$\forall \varphi \in L^\infty, \forall \lambda \in \Lambda: \langle B^*\lambda, \varphi\rangle = \int_{\mathcal{P}_\sigma} -E^{P^*}[\varphi H] d\lambda.$$

Hence the conjugate function of f is

$$f^*(B^*\lambda) = \sup_{\varphi \in R} \Big\{ -\int_{\mathcal{P}_\sigma} E^{P^*}[\varphi H] d\lambda + E^Q[\varphi H] \Big\}.$$

The dual problem (24) becomes

$$\begin{aligned} -d^i(Q) &= \sup_{\lambda \in \Lambda} \Big\{ -\sup_{\varphi \in R} \Big\{ -\int_{\mathcal{P}_\sigma} E^{P^*}[\varphi H] d\lambda + E^Q[\varphi H] \Big\} - \mathcal{I}_{-\mathcal{L}_+^*}(\lambda) - \widetilde{V}_0 \lambda(\mathcal{P}_\sigma) \Big\}, \\ d^i(Q) &= \inf_{\lambda \in -\mathcal{L}_+^*} \Big\{ \sup_{\varphi \in R} \Big\{ -\int_{\mathcal{P}_\sigma} E^{P^*}[\varphi H] d\lambda + E^Q[\varphi H] \Big\} + \widetilde{V}_0 \lambda(\mathcal{P}_\sigma) \Big\}, \end{aligned}$$

where $-\mathcal{L}_+^* = \{\lambda \in \Lambda : \forall l \in \mathcal{L}_+ : \langle l, \lambda\rangle \geq 0\}$. It can be seen easily that $-\mathcal{L}_+^* = \Lambda_+$ is the set of finite measures on $(\mathcal{P}_\sigma, \mathcal{S})$. Thus,

$$d^i(Q) = \inf_{\lambda \in \Lambda_+} \Big\{ \sup_{\varphi \in R} \Big\{ -\int_{\mathcal{P}_\sigma} E^{P^*}[\varphi H] d\lambda + E^Q[\varphi H] \Big\} + \widetilde{V}_0 \lambda(\mathcal{P}_\sigma) \Big\}. \tag{25}$$

The spaces $(\Omega, \mathcal{F}, P)$ and $(\mathcal{P}_\sigma, \mathcal{S}, \lambda)$ for $\lambda \in \Lambda_+$ are positive, finite measure spaces. Furthermore, the function $f(\omega, P^*) = H(\omega) Z_{P^*}(\omega) \varphi(\omega)$ is measurable for all $\varphi \in R$ and it holds that for all $\lambda \in \Lambda_+$ and for all $\varphi \in R$

$$\int\limits_{\mathcal{P}_\sigma} \int\limits_{\Omega} |H Z_{P^*} \varphi| dP d\lambda \overset{\|\varphi\|_{L^\infty} \leq 1}{\leq} \sup_{P^* \in \mathcal{P}_\sigma} \|H Z_{P^*}\|_{L^1} \lambda(\mathcal{P}_\sigma) \overset{(1)}{<} +\infty.$$

Thus, we can apply Tonelli's Theorem (Dunford and Schwartz, 1988, Corollary III.11.15) and obtain that the order of integration can be changed, i.e., for all $\lambda \in \Lambda_+$ and for all $\varphi \in R$

$$\int\limits_{\mathcal{P}_\sigma} \int\limits_{\Omega} H Z_{P^*} \varphi dP d\lambda = \int\limits_{\Omega} \int\limits_{\mathcal{P}_\sigma} H Z_{P^*} \varphi d\lambda dP < +\infty.$$

Since in (25) only elements $\lambda \in \Lambda_+$ and $\varphi \in R$ have to be considered, we can change the order of integration and obtain

$$d^i(Q) = \inf_{\lambda \in \Lambda_+} \Big\{ \sup_{\varphi \in R} E[\varphi(H Z_Q - H \int_{\mathcal{P}_\sigma} Z_{P^*} d\lambda)] + \widetilde{V}_0 \lambda(\mathcal{P}_\sigma) \Big\}. \tag{26}$$

Since $\varphi \in R$ is a randomized test, the supremum over all $\varphi \in R$ in (26) is attained by

$$\overline{\varphi}(\omega) = \begin{cases} 1 & : \quad H Z_Q > H \int_{\mathcal{P}_\sigma} Z_{P^*} d\lambda \\ 0 & : \quad H Z_Q < H \int_{\mathcal{P}_\sigma} Z_{P^*} d\lambda \end{cases} \qquad P - a.s.$$

If we denote $HZ_Q - H \int\limits_{\mathcal{P}_\sigma} Z_{P^*} d\lambda =: \nu_\lambda(\omega)$, the value of the dual problem is

$$d^i(Q) = \inf_{\lambda \in \Lambda_+} \Big\{ \int\limits_\Omega \nu_\lambda^+(\omega) dP + \widetilde{V}_0 \lambda(\mathcal{P}_\sigma) \Big\}. \tag{27}$$

This is equation (19) of Theorem 4.8. To verify the validity of strong duality we have to use a weaker regularization condition than used in Theorem 4.5. In Borwein and Zhu (2006, Theorem 5) it is shown that strong duality holds if f and g are convex and lower semicontinuous and if there exists some $\varphi_0 \in \operatorname{dom} f$ such that $B\varphi_0 \in \operatorname{core}(\mathcal{L}_+ - \widetilde{V}_0 \mathbf{1})$, where $\operatorname{core}(M)$ is the algebraic interior of a set M. If we take $\varphi_0 \equiv 0,\ 0 \in \operatorname{dom} f$, we have to show that $B\varphi_0 = \mathbf{0} \in \operatorname{core}(\mathcal{L}_+ - \widetilde{V}_0 \mathbf{1})$ for $\widetilde{V}_0 > 0$. This holds true since for all $l \in \mathcal{L}$ there exists a $s = s(l) > 0$, such that for all $0 \leq t \leq s(l)$ it holds $\mathbf{0} + tl \in (\mathcal{L}_+ - \widetilde{V}_0 \mathbf{1})$. We can choose $s(l) = \widetilde{V}_0 / \|l\|_{\mathcal{L}} > 0$ for $l \neq \mathbf{0}$ and $s(l) = c > 0$ for $l = \mathbf{0}$, $c > 0$ arbitrary, where $\|l\|_{\mathcal{L}} := \sup_{P^* \in \mathcal{P}_\sigma} |l(P^*)|$.
The function f is convex and lower semicontinuous since the set R is convex and closed. The function g is convex since the set $(\mathcal{L}_+ - \widetilde{V}_0 \mathbf{1})$ is convex and it is lower semicontinuous w.r.t. the Mackey topology $\tau(\mathcal{L}, \Lambda)$ if and only if the set $(\mathcal{L}_+ - \widetilde{V}_0 \mathbf{1})$ is closed w.r.t. the Mackey topology $\tau(\mathcal{L}, \Lambda)$. To show this, we use that a convex set is closed w.r.t. the Mackey topology $\tau(\mathcal{L}, \Lambda)$ if and only if it is closed w.r.t. the weak topology $\sigma(\mathcal{L}, \Lambda)$. Take a net l_α in $(\mathcal{L}_+ - \widetilde{V}_0 \mathbf{1})$ that converges weakly to l. Thus, for all $\lambda \in \Lambda$ it holds $\int_{\mathcal{P}_\sigma} l_\alpha d\lambda \to \int_{\mathcal{P}_\sigma} l d\lambda$ and for all $P^* \in \mathcal{P}_\sigma$ and all α it holds $l_\alpha(P^*) + \widetilde{V}_0 \geq 0$. Suppose there exists a $\overline{\mathcal{P}} \in \mathcal{S}$ and a $\lambda \in \Lambda_+$ with $\lambda(\overline{\mathcal{P}}) > 0$ and $\lambda(\mathcal{P}_\sigma \backslash \overline{\mathcal{P}}) = 0$ such that $l(P^*) + \widetilde{V}_0 < 0$ for all $P^* \in \overline{\mathcal{P}}$. Then, $\int_{\mathcal{P}_\sigma} (l(P^*) + \widetilde{V}_0) d\lambda < 0$, which is a contradiction to $\int_{\mathcal{P}_\sigma} (l(P^*) + \widetilde{V}_0) d\lambda = \lim_\alpha \int_{\mathcal{P}_\sigma} (l_\alpha(P^*) + \widetilde{V}_0) d\lambda \geq 0$ for all $\lambda \in \Lambda_+$. Since $\mathcal{S}$ contains also the one-point subsets of $\mathcal{P}_\sigma$, it follows that $l \in (\mathcal{L}_+ - \widetilde{V}_0 \mathbf{1})$. Thus, strong duality holds true.
To demonstrate the dependence from the selected measure $Q \in \mathcal{Q}$ we use the notation $\widetilde{\varphi}_Q$ and $\widetilde{\lambda}_Q$ for the primal and dual solution, respectively. The existence of a solution $\widetilde{\varphi}_Q \in R_0$ of the primal problem was proved in Lemma 4.6. Now with strong duality the existence of a dual solution $\widetilde{\lambda}_Q$ follows and the values of the primal and dual objective function at $\widetilde{\varphi}_Q$, respectively $\widetilde{\lambda}_Q$, coincide. This leads to a necessary and sufficient optimality condition.
We consider the primal objective function

$$\begin{aligned} E[\varphi H Z_Q] &= \int\limits_\Omega \varphi H Z_Q dP \\ &= \int\limits_\Omega \varphi \Big[H Z_Q - H \int\limits_{\mathcal{P}_\sigma} Z_{P^*} d\lambda \Big] dP + \int\limits_{\mathcal{P}_\sigma} \int\limits_\Omega \varphi H Z_{P^*} dP d\lambda \\ &= \int\limits_\Omega \varphi \nu_\lambda^+(\omega) dP - \int\limits_\Omega \varphi \nu_\lambda^-(\omega) dP + \int\limits_{\mathcal{P}_\sigma} \int\limits_\Omega \varphi H Z_{P^*} dP d\lambda \end{aligned}$$

and subtract it from the dual objective function. Because of strong duality the difference has to be zero at $\widetilde{\varphi}_Q$, respectively $\widetilde{\lambda}_Q$:

$$\int\limits_{\Omega}\Big[1-\widetilde{\varphi}_Q\Big]\nu^+_{\widetilde{\lambda}_Q}(\omega)dP+\int\limits_{\Omega}\widetilde{\varphi}_Q\nu^-_{\widetilde{\lambda}_Q}(\omega)dP+\int\limits_{\mathcal{P}_\sigma}\Big[\widetilde{V}_0-\int\limits_{\Omega}\widetilde{\varphi}_Q HZ_{P^*}dP\Big]d\widetilde{\lambda}_Q=0.$$

The sum of these three nonnegative integrals is zero if and only if $\widetilde{\varphi}_Q \in R_0$ satisfies condition (20) and (21) of Theorem 4.8. To stress that λ is a measure on $\mathcal{P}_\sigma$, we use in Theorem 4.8 the notation $\lambda(P^*)$. $\square$

For each $Q \in \mathcal{Q}$ there exist a primal and a dual solution $\widetilde{\varphi}_Q$, $\widetilde{\lambda}_Q$, respectively. If $Q=\widetilde{Q}$ is the solution of the outer problem of (14), $\widetilde{\varphi}_{\widetilde{Q}}$ is the solution of the static optimization problem (7).

4.4 The Saddle Point

Now, let us consider the saddle point problem described in Theorem 4.5. With Theorem 4.8 it follows that

$$\begin{aligned}
&\max_{Q\in\mathcal{Q}}\min_{\varphi\in R_0}\{E^Q[(1-\varphi)H]-\sup_{X\in\mathcal{A}}E^Q[-X]\}=\max_{Q\in\mathcal{Q}}\{E^Q[H]-p^i(Q)-\sup_{X\in\mathcal{A}}E^Q[-X]\}\\
&=\max_{Q\in\mathcal{Q}}\{E^Q[H]-d^i(Q)-\sup_{X\in\mathcal{A}}E^Q[-X]\}\\
&=\max_{Q\in\mathcal{Q}}\Big\{\max_{\lambda\in\Lambda_+}\Big\{-E^P[(HZ_Q-H\int\limits_{\mathcal{P}_\sigma}Z_{P^*}d\lambda)^+]-\widetilde{V}_0\lambda(\mathcal{P}_\sigma)\Big\}+E^Q[H]-\sup_{X\in\mathcal{A}}E^Q[-X]\Big\}\\
&=\max_{Q\in\mathcal{Q},\lambda\in\Lambda_+}\Big\{E^P[HZ_Q\wedge H\int\limits_{\mathcal{P}_\sigma}Z_{P^*}d\lambda]-\widetilde{V}_0\lambda(\mathcal{P}_\sigma)-\sup_{X\in\mathcal{A}}E^Q[-X]\Big\},
\end{aligned}$$

where $x\wedge y=min(x,y)$. With Theorem 4.5 it follows that $\widetilde{Q}$ attains the maximum w.r.t. $Q\in\mathcal{Q}$. Theorem 4.8 shows the existence of a $\widetilde{\lambda}=\widetilde{\lambda}_{\widetilde{Q}}$ that attains the maximum w.r.t. $\lambda\in\Lambda_+$. Thus, there exists a pair $(\widetilde{Q},\widetilde{\lambda})$ solving

$$\max_{Q\in\mathcal{Q},\lambda\in\Lambda_+}\Big\{E^P[HZ_Q\wedge H\int\limits_{\mathcal{P}_\sigma}Z_{P^*}d\lambda]-\widetilde{V}_0\lambda(\mathcal{P}_\sigma)-\sup_{X\in\mathcal{A}}E^Q[-X]\Big\}. \tag{28}$$

Now, our main theorem follows.

Theorem 4.9. *Let $(\widetilde{Q},\widetilde{\lambda})$ be the optimal pair in (28).*

- *The solution of the static optimization problem (7) is*

$$\widetilde{\varphi}(\omega)=\begin{cases}1 & : \quad H\widetilde{Z}_Q>H\int_{\mathcal{P}_\sigma}Z_{P^*}d\widetilde{\lambda}(P^*)\\ 0 & : \quad H\widetilde{Z}_Q<H\int_{\mathcal{P}_\sigma}Z_{P^*}d\widetilde{\lambda}(P^*)\end{cases}\qquad P-a.s. \tag{29}$$

  with

$$E^{P^*}[\widetilde{\varphi}H]=\widetilde{V}_0\qquad \widetilde{\lambda}-a.s. \tag{30}$$

- $(\widetilde{\varphi}, \widetilde{Z}_Q)$ *is the saddle point of Theorem 4.5.*
- $(\widetilde{V}_0, \widetilde{\xi})$ *solves the dynamic convex hedging problem (2), (3), where* $\widetilde{\xi}$ *is the superhedging strategy of the modified claim* $\widetilde{\varphi}H$.

Proof. The results follow from Theorem 4.5, 4.8 and 3.1. □

Remark 4.10. It follows that there exists a $[0,1]$-valued random variable δ such that $\widetilde{\varphi}$ as in Theorem 4.9 satisfies

$$\widetilde{\varphi}(\omega) = I_{\{H\widetilde{Z}_Q > H\int_{\mathcal{P}_\sigma} Z_{P^*} d\widetilde{\lambda}(P^*)\}}(\omega) + \delta(\omega) I_{\{H\widetilde{Z}_Q = H\int_{\mathcal{P}_\sigma} Z_{P^*} d\widetilde{\lambda}(P^*)\}}(\omega), \tag{31}$$

where $I_A(\omega)$ is the stochastic indicator function equal to one for $\omega \in A$ and zero otherwise. δ has to be chosen such that $\widetilde{\varphi}$ satisfies (30).
We do not have uniqueness of a solution $\widetilde{\varphi}$ (see Remark 4.4). If we for instance choose δ to be constant, equations (30) and (31) lead to one particular δ and thus to one particular solution $\widetilde{\varphi}$.

Remark 4.11. From equation (30) it follows, that (except in the case where $\widetilde{\lambda}$ takes only the value zero) the superhedging price of the modified claim $\widetilde{\varphi}H$ is equal to the capital boundary $\widetilde{V}_0$. Then, $\widetilde{V}_0$ is the minimal amount of capital that is necessary to solve together with $\widetilde{\xi}$ the dynamic problem (2), (3).

To summarize, the admissible strategy that minimizes the convex shortfall risk consists in superhedging a modified claim $\widetilde{\varphi}H$ that has the form of a knock-out option.

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