Mean Field Games Models—A Brief Survey

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2 authors:

Diogo Gomes
King Abdullah University of Science and Technology

Joao Saude
Carnegie Mellon University

Mean field games models - a brief survey

Diogo A. Gomes[*] and João Saúde [†]

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Abstract

The mean-field framework was developed to study systems with an infinite number of rational agents in competition which arise naturally in many applications. The systematic study of these problems was started, in the mathematical community by Lasry and Lions, and independently and around the same time in the engineering community by P. Caines, Minyi Huang and Roland Malhamé. Since these seminal contributions, the research in mean-field games has grown exponentially, and in this paper we present a brief survey of mean-field models as well as recent results and techniques.

In the first part of this paper we study reduced mean-field games, that is, mean-field games which are written as a system of a Hamilton-Jacobi equation and a transport or Fokker-Planck equation. We start by the derivation of the models and by describing some of the existence results available in the literature. Then we discuss the uniqueness of solution and propose a definition of relaxed solution for mean-field games that allows to establish uniqueness under minimal regularity hypothesis. A special class of mean-field games that we discuss in some detail is equivalent to the Euler-Lagrange equation of suitable functionals. We present in detail various additional examples, including extensions to population dynamics models. This section ends with a brief overview of the random variables point of view as well as some applications to extended mean-field games models. These extended models arise in problems where the costs incurred by the agents depend not only on the distribution of the other agents but also on their actions.

The second part of the paper concerns mean-field games in master form. These mean-field games can be modeled as a partial differential equation in an infinite dimensional space. We discuss both deterministic models as well as problems where the agents are correlated. We end the paper with a mean-field model for price impact.

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1 Introduction

Mean field games is a recent area of research developed in the engineering community by Peter Caines, Minyi Huang and Roland Malhamé [HMC06, HCM07], and independently and about the same time by Pierre Louis Lions and Jean Michel Lasry [LL06a, LL06b, LL07a, LL07b] which attempts to understand the limiting behavior of systems involving very large numbers of rational agents which play differential games under partial information and symmetry assumptions. Inspired by ideas in statistical physics, these authors introduced a class of models in which each individual player's contribution is encoded in a mean field that contains only the statistical properties about the ensemble.

[*]Center for Mathematical Analysis, Geometry, and Dynamical Systems - LARSYS and Departamento de Matemática, Instituto Superior Técnico, 1049-001 Lisboa, Portugal and King Abdullah University of Science and Technology (KAUST), CEMSE Division , Thuwal 23955-6900. Saudi Arabia, and KAUST Strategic Research Initiative, Uncertainty Quantification Center.

[†]LARSYS, Instituto Superior Técnico, 1049-001 Lisboa, Portugal

The literature on mean field games and its applications is growing fast, for a recent survey see [LLG10b] and reference therein, as well as the excellent lecture notes [Car11] and the upcoming book [BFY13].

Applications of mean field games arise in the study of growth theory in economics [LLG10a] or environmental policy [LST], for instance, and it is likely that in the future they will play an important rôle in economics and population models. There is also a growing interest in numerical methods for these problems [LST], [ACD10], [ACCD12], [CS12], [AP12], [CS13]. For a survey of numerical methods see [Ach13]. One of the authors and his collaborators have also considered the discrete time, finite state problem [GMS10], and the continuous time finite state problem [GMS13], [FG13]. Such problems have also been addressed in [Gue11b] and [Gue11a]. Various applications and additional models have been worked out in detail in [Gue09a], [Gue09b], [BF13a], [BT13], [GR13], [NCMH13], [TZB13], [T13] [LLG11], [LLLL13], [LM13], [San12] (see also the special edition on Netw. Heterog. Media 7 (2012), no. 2, dedicated to mean-field games). Problems motivated by applications with mixed populations or with a major player were studied in [Hua10, Hua12]. Mean field games have also been analyzed using backwards-forwards stochastic differential equations, see, for instance [NH12], and [CD13b, CD13a, CL13, CDL13]. Linear quadratic problems have been considered from distinct points of view, for instance, in [HCM07], [Bar12], [HCM10], [NH12], [BP13], [BSYY13], and [LZ08].

The rigorous derivation of mean-field models was considered in some models in the original papers by Lions and Lasry. Further developments, using the theory of nonlinear Markov processes were obtained in [KLY11], [KY13b], and [KY13a] (see also the monograph [Kol10]), and using PDE methods in [BF13b]. For finite state problems, the N player problem was studied in [GMS13] where a convergence result was established. For earlier works in the context of statistical physics and interacting particle systems see [Szn91].

The objective of this paper is to give a brief survey of mean-field games models and present some elementary techniques and applications. Many of the results here have been discussed by other authors, most notably by P. L. Lions in his lectures in Collège de France [Lio11], as well as in the references above. Of course in an area growing as fast as mean-field games, it is impossible to present a coherent collection of all available results. As such we have decided to cover two main classes of models: reduced mean-field games models and mean-field games in master form. Due to space constraints we have chosen not to discuss the rigorous justification of mean-field models starting with a finite number of agents N and then taking the limit $N \to \infty$, as well further applications and numerical methods.

This paper is divided into two main parts: reduced mean-field models and mean-field games in master form, which are discussed, respectively in section 2 and section 3. Reduced mean-field models can be formulated as systems of a Hamilton-Jacobi-Bellman equation coupled with a Fokker-Planck or transport equation. We start by discussing the derivation of those models. Then we discuss various existence results both for first and second order equations as well as for problems with local dependence on the measure. Then we address uniqueness questions and propose a definition of relaxed solution that allows to establish uniqueness under minimal regularity hypothesis. A special class of mean-field games can be regarded as the Euler-Lagrange equation of suitable functionals. We present in detail various examples, including extensions to population dynamics models. This section ends with a brief overview of the random variables point of view and some applications to extended mean-field games models. These extended models arise in problems where the costs incurred by the agents depend not only on the distribution of the other agents but also on their actions. We then continue the discussion on mean-field games by considering mean-field games in master form. These were introduced by Lions in [Lio11]. Such master form is particularly useful for the study of problems where agents share a common noise. We present various of these models as well as an application to price formation problems. As with any survey paper, most of the results are not original, but we have tried to provide accurate references and give adequate credit. We apologize in advance for any omissions. A few of the results however have not appeared elsewhere and certain merit additional research. In particular we single out the theory of relaxed solutions and its applications to uniqueness and stability of mean-field games, some of the extended models, population dynamics problems and the price formation model discussed at the end of the paper.

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2 Reduced mean-field models

In this section we consider reduced mean-field models. The models originally studied by Lasry and Lions [LL06a, LL06b, LL07a, LL07b] which consist in a system of a Hamilton-Jacobi type equation and an associated transport or Fokker-Planck equation. We present the derivation of such models and discuss various methods to prove existence and uniqueness of solutions. Stationary models are then briefly discussed. These are quite interesting in their own right but also, under appropriate conditions, encode the long-time asymptotic for mean-field games, as shown in [CLLP12], [CLLP13] (see also [GMS10] and [GMS13] for discrete models). Following [GPV13], we consider also stationary extended models in which the cost for a reference player depends not only in the other players distribution but also on their actions. Then we look at certain variational structures that some of these problems enjoy, and the connections between mean-field models and other now classical problems such as optimal transport and Aubry-Mather theory. We then describe the random variables point of view. This formulation is very close to the one in [HMC06] (although many of the problems considered in this survey are deterministic), but the presentation here reflects also the ideas and methods from the lectures of P. L. Lions in Collège de France [Lio11]. We will show that mean-field games can be set up as a system of Hamilton-Jacobi equation coupled with an ODE in a space of random variables. In this part we discuss only deterministic control problems. This allows us to avoid using backwards stochastic differential equations (see for instance [NH12], and [CD13b, CD13a]) and therefore keeping the presentation elementary. Mean-field models with correlations will be considered in section 3. The random variable point of view makes it easy to consider models where the costs incurred by players depend not only on the distribution of other players but also on their actions. Such models were first studied in [GV13] and are also briefly considered here.

2.1 Derivation of reduced models

Let $U \subset \mathbb{R}^m$ be a convex closed set. As it is usual in stochastic optimal control problems (see [FS06], for instance), consider a vector field $f : \mathbb{R}^d \times U \to \mathbb{R}^d$, and a diffusion matrix $\sigma : \mathbb{R}^d \times U \to \mathcal{M}_{\mathbb{R}}^{d\times m}$, where $\mathcal{M}_{\mathbb{R}}^{d\times m}$ is the set of $d \times m$ real matrices. We suppose that both f and σ are globally Lipschitz in the first coordinate, that is, for all $v \in U$

$$|f(x, v) - f(y, v)|, |\sigma(x, v) - \sigma(y, v)| \le C|x - y|,$$

where the constant C is independent on the control variable v. We also assume the following growth condition

$$|f(x, v)|, |\sigma(x, v)| \le C(1 + |x| + |v|).$$

Let $(\Omega, \mathcal{F}, P)$ be a probability space, where Ω is a set, $\mathcal{F}$ a σ-algebra on Ω and P a probability measure. Let W_t be a Brownian motion on Ω and $\mathcal{F}_t$ the associated filtration. Fix an initial time $t_0 \in [0, T]$. Let $\mathcal{B}_r$ be the Borel σ-algebra on $[t_0, r]$. A control process $\mathbf{v} : [t_0, T] \times \Omega \to U$ is called $\{\mathcal{F}_r\}$-*progressively measurable* if the map $(s, \omega) \to \mathbf{v}(s, \omega)$ from $[t_0, r] \times \Omega$ into U is $\mathcal{B}_r \times \{\mathcal{F}_r\}$-measurable. We denote by $\mathcal{U}$ the set of all progressively measurable control processes.

We consider a population of agents where each agent is allowed to choose a progressively measurable control $\mathbf{v} \in \mathcal{U}$. This control determines the agent's dynamics through the stochastic differential equation

$$d\mathbf{x} = f(\mathbf{x}, \mathbf{v})dt + \sigma(\mathbf{x}, \mathbf{v})dW_t. \tag{1}$$

We will assume that each agent dynamics' is driven by an independent Brownian motion in (1).

Let $\mathcal{P}(\mathbb{R}^d)$ be the set of Borel probability measures in $\mathbb{R}^d$. Let $\theta : [t_0, T] \to \mathcal{P}(\mathbb{R}^d)$ be, for each time t, a probability distribution of agents in $\mathbb{R}^d$. Assume for the moment the trajectory of each agent is determined by an independent copy of (1) where the control $\mathbf{v}$ is given as a (non-time homogeneous) feedback Markovian control, that is

$$\mathbf{v}(t) = \upsilon(\mathbf{x}, t),$$

for some function $\upsilon : \mathbb{R}^d \times [t_0, T] \to U$. Thus each agent of this population will follow the diffusion

$$d\mathbf{x} = f(\mathbf{x}, \upsilon(\mathbf{x}, t))dt + \sigma(\mathbf{x}, \upsilon(\mathbf{x}, t))dW_t. \tag{2}$$

Because the Brownian motion driving each agent is independent from the remaining ones, the population distribution will evolve according to the following Fokker-Planck equation

$$\theta_t + \operatorname{div}(b(x, t)\theta) = \partial^2_{ij}(a_{ij}(x, t)\theta),$$

where

$$b(x, t) = f(x, \upsilon(x, t)), \qquad a_{ij} = \frac{1}{2}\sum_{k=1}^{m} \sigma_{ik}(x, \upsilon(x, t))\sigma_{jk}(x, \upsilon(x, t)),$$

and the initial condition $\theta(x, t)$ is given.

We consider a Lagrangian $L : \mathbb{R}^d \times U \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$, and a terminal cost $\Psi : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$. Suppose L, and Ψ are continuous, bounded by below, and satisfy the following quadratic growth condition

$$|L(x, v, \theta)| \leq C(1 + |x|^2 + |v|^2), \quad |\Psi(x, \theta)| \leq C(1 + |x|^2),$$

for positive constants C independent of (x, v, θ). Assume further, if U is unbounded, that

$$\lim_{|v|\to\infty} \frac{L(x, v, \theta)}{|v|} \to \infty.$$

Fix an agent, which knows the strategy υ used by the other players and whose objective is to find a progressively measurable control $\mathbf{v}$ which minimizes the following cost

$$J(x, t; \mathbf{v}) = E \int_t^T L(\mathbf{x}, \mathbf{v}, \theta)ds + \psi(\mathbf{x}(T), \theta(T)).$$

From the point of view of this agent, its value function is

$$V(x, t) = \inf_{\mathbf{v}\in\mathcal{U}} J(x, t; \mathbf{v}).$$

It is well known that V is then a viscosity solution to the Hamilton-Jacobi equation

$$-V_t + H(x, D_x V, D^2_{xx} V, \theta) = 0, \tag{3}$$

where $H : \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{M}^{d\times d}_{\mathbb{R}} \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ is given by

$$H(x, p, M, \theta) = \sup_{v\in U}\left[-f(x, v)\cdot p - \frac{1}{2}\sigma(x, v)\sigma^T(x, v) : M - L(x, v, \theta)\right].$$

Furthermore, V satisfies the terminal condition

$$V(x, T) = \psi(x, \theta(T)).$$

Suppose now V is a smooth enough solution to (3), and that H is differentiable. Assume further that there exists a function $\bar{v} : \mathbb{R}^d \times [t, T] \to U$ such that

$$H(x, D_x V, D_{xx}^2 V, \theta) = -f(x, \bar{v}) \cdot D_x V - \frac{1}{2}\sigma(x, \bar{v})\sigma^T(x, \bar{v}) : D_{xx}^2 V - L(x, \bar{v}, \theta).$$

Then a simple argument shows that

$$f(x, \bar{v}) = -D_p H(x, D_x V, D_{xx}^2 V, \theta), \qquad \frac{1}{2}\sigma(x, \bar{v})\sigma^T(x, \bar{v}) = -D_M H(x, D_x V, D_{xx}^2 V, \theta),$$

and that the control $\bar{v}$ is optimal.

We assume now that all players have access to the same information and therefore will use the same control the control $\bar{v}$ in (2). This then gives rise to the system

$$\begin{cases} -V_t + H(x, D_x V, D^2 V, \theta) = 0 \\ \theta_t - \operatorname{div}(D_p H \theta) - \partial_{ij}(D_{M_{ij}} H \theta) = 0, \end{cases} \tag{4}$$

coupled with the initial-terminal conditions

$$\begin{cases} V(x, T) = \psi(x, \theta(T)) \\ \theta(x, 0) = \theta_0. \end{cases} \tag{5}$$

The boundary conditions in this problem are non-standard in the sense that part of the unknowns are subject to initial conditions and the rest of them are subject to terminal conditions. Therefore existence of solutions is not obvious and requires some justification, as will be discussed in section 2.2 for three model problems.

In addition to the initial-terminal conditions it is also interesting from the point of view of applications to consider the planning problem, see [ACCD12], and [Por13a]. In this problem we are given two probability measures θ_0 and θ_1 and one looks for a pair (V, θ) solving (4) under the boundary conditions

$$\theta(x, 0) = \theta_0, \qquad \theta(x, T) = \theta_1. \tag{6}$$

In [Por13a] the existence of weak solutions for the planning problem for the second order case was established.

At this stage the key points to address are existence and uniqueness for solutions to (4). This will be done in the following sections.

2.2 Existence of solutions

We now discuss the existence of solutions for the initial-terminal value problem for mean-field games. Rather than considering the most general problem, we consider three model cases. The first two concern first and second order Hamilton-Jacobi equations with smooth dependence on the measure. The third case concerns local dependence on the measure.

We will follow closely in the first two parts of this section the lecture notes by P. Cardaliaguet [Car11]. As such, we will not detail the more technical arguments that can be found in that reference. The case of local potentials will be addressed by establishing various a-priori estimates, using the techniques in [LL06a, LL06b, GSM11, GPSM12, GPSM13a, GPSM13b]. Similarly, we will only outline the main arguments.

2.2.1 First order case

We start by considering first order reduced Mean Field Games,

$$\begin{cases} -V_t + H(x, D_x V, \theta) = 0 & \text{in } \mathbb{R}^d \times [0, T) \\ \theta_t - \operatorname{div}(D_p H \theta) = 0 & \text{in } \mathbb{R}^d \times (0, T], \end{cases} \tag{7}$$

with initial-terminal condition

$$\begin{cases} V(x,T) = \psi(x, \theta(\cdot, T)) \\ \theta(x,0) = \theta_0(x). \end{cases} \tag{8}$$

We will study the particular case where the Hamiltonian is given by

$$H(x, D_x V, \theta) = \frac{1}{2}|D_x V(x,t)|^2 - F(x, \theta(t)), \tag{9}$$

where F is a (nonlocal) operator on probability measures. A solition to (7) is a pair (V, θ), where where V is a bounded locally Lipschitz continuous solution to the Hamilton-Jacobi equation and θ is a weak solution to the transport equation. We denote by $\mathcal{P}_1(\mathbb{R}^d)$ the set of Borel probability measures in $\mathbb{R}^d$ with finite first moments endowed with the $1-$Wasserstein distance. We recall (see [Vil03]) that the $1-$Wasserstein distance between two probability measures θ_1 and θ_2 is defined as

$$d_1(\theta_1, \theta_2) = \inf_{\pi} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y| d\pi(x,y),$$

where the infimum is taken over the set $\Pi(\theta_1, \theta_2)$ of all probability measures π in $\mathbb{R}^d \times \mathbb{R}^d$ whose first marginal is θ_1 and the second marginal is θ_2. We define the norm $\| \cdot \|_{C^2}$ as

$$\|g\|_{C^2} = \sup_{x \in \mathbb{R}^d} \left(|g(x)| + |D_x g(x)| + |D^2_{xx} g(x)| \right),$$

for any $g \in C^2(\mathbb{R}^d)$.

Theorem 1. *Suppose that F, and ψ in (8) are continuous on $\mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d)$, θ_0 is absolutely continuous with respect to the Lebesgue measure, and that there exists a constant $C > 0$ such that*

$$\|F(\cdot, \theta)\|_{C^2}, \|\psi(\cdot, \theta)\|_{C^2} \leq C,$$

uniformly for $\theta \in \mathcal{P}_1$. Then the system (7), for the Hamiltonian (9), and under initial-terminal conditions (8) admits a solution.

Proof. We outline in what follows the proof by a fixed point argument from [Lio11], as detailed in [Car11].

Semiconcavity estimates We recall that a function $\psi : \mathbb{R}^d \to \mathbb{R}$ is semiconcave if there exists a constant C such that $\psi - C|x|^2$ is a concave function. The first step on the fixed point argument consists in proving that the solution to the equation

$$- V_t + H(x, D_x V, \theta) = 0, \tag{10}$$

for a fixed $\theta : [0,T] \to \mathcal{P}_1$ is semiconcave in x, with semiconcavity modulus uniform in θ. This follows from standard viscosity solution techniques, see for instance [BCD97].

Optimal trajectory synthesis Though viscosity solutions may fail to be differentiable, by semiconcavity they are differentiable almost everywhere. Furthermore, if $x \in \mathbb{R}^d$ is a point of differentiability of $V(x,0)$ then the trajectory

$$\begin{cases} \dot{\mathbf{x}} = -D_p H(\mathbf{x}, \mathbf{p}, \theta) \\ \dot{\mathbf{p}} = D_x H(\mathbf{x}, \mathbf{p}, \theta), \end{cases}$$

with

$$\mathbf{x}(0) = x, \qquad \mathbf{p}(0) = D_x V(x, 0),$$

is an optimal trajectory for the optimal control associated with (10), and V is differentiable at $(\mathbf{x}(t), t)$, with $\mathbf{p}(t) = D_x V(\mathbf{x}(t), t)$, for $0 < t < T$.

Transport equation As in [Car11], we can define, using a measurable selection argument a flow $\Phi(x,t,s)$ satisfying

$$\Phi_s(x,t,s) = -D_pH(\Phi(x,t,s), D_xV(\Phi(x,t,s),\theta), \qquad \Phi(x,t,t) = x. \tag{11}$$

Furthermore Φ satisfies the following properties

1. $$|\Phi(x,t,s') - \Phi(x,t,s)| \leq C|s-s'|$$

2. $$|x-y| \leq C|\Phi(x,t,s) - \Phi(y,t,s)|.$$

We then define $\zeta(t) = \Phi(\cdot,0,t)\sharp\theta_0$. It is not hard to check that $\zeta : [0,T] \to \mathcal{P}_1$ is continuous and it is a weak solution to

$$\partial_t\zeta - \operatorname{div}(D_pH(x, D_xV, \theta)\zeta) = 0.$$

Additionally, since θ_0 is absolutely continuous, so is ζ due to the properties of the flow. The key issue is uniqueness. If the vector field $b(x,t) = -D_pH(x, D_xV, \theta)$ were Lipschitz in the x variable, the uniqueness of solution of the conservative transport equation would follow by standard methods. Unfortunately the above vector field may be discontinuous. Consequently, to establish uniqueness one needs to use a approach due to Ambrosio [Amb04], [Amb08], as explained in detail in [Car11].

Stability and fixed point argument The last step of the proof consists in a fixed point argument which depends on the following stability result: for $m \in C([0,T],\mathcal{P}_1)$ denote by $U[m]$ the solution to

$$-V_t + H(x, D_xV, m) = 0$$

with $V(x,T) = \Psi(x, m(T))$. Denote by $\Phi[m]$ the flow induced by $U[m]$ through (11), and $\Theta[m] = \Phi[m]\sharp\theta_0$. Then by stability of viscosity solutions if $m_n \to m$ then $U[m_n] \to U[m]$. By the semiconcavity estimates, we have almost everywhere convergence of $D_xU[m_n]$ to $D_xU[m]$. In addition, $\Theta[m_n]$ is (uniformly) absolutely continuous and so any sublimit will be absolutely continuous. But then $\lim_{n\to\infty}\Theta[m_n]$ is a solution to the transport equation for $D_xU[m]$ and by uniqueness $\Theta[m] = \lim_{n\to\infty}\Theta[m_n]$. This then shows that the map $m \mapsto \Theta[m]$ is continuous. It is also easy to see that it is compact since the properties of the flow imply Lipschitz continuity of $\Theta[m]$ as a map from $[0,T] \to \mathcal{P}_1$. Therefore this map admits a fixed point to which corresponds a solution to (7), as claimed. □

2.2.2 Second order case

Now we consider the second order reduced mean field model (4) with initial-terminal conditions (5). In order to simplify the presentation, and to focus in the main arguments, we assume that the Hamiltonian has the following structure

$$H(x, D_xV, D_x^2V, \theta) = -\Delta V + \frac{1}{2}|D_xV|^2 - F(x,\theta).$$

We assume further

A. F and ψ are uniformly bounded over $\mathbb{R}^d \times \mathcal{P}_1$, and also Lipschitz continuous,

B. θ_0 is absolutely continuous with a continuous density function with finite second moment:

$$\int_{\mathbb{R}^d} |x|^2\theta_0(x)dx < +\infty.$$

Once more we follow the argument proposed in [Lio11] and detailed in [Car11] to prove the existence of solutions for the mean-field equations.

Theorem 2. *Assume that conditions A. and B. hold. Then the reduced mean field game*

$$\begin{cases} -V_t - \Delta V + \frac{1}{2}|D_x V|^2 = F(x,\theta(t)) & \mathbb{R}^d \times [0,T) \\ \theta_t - \Delta\theta - \operatorname{div}(D_x V\theta) = 0 & \mathbb{R}^d \times (0,T] \end{cases}$$

with initial-terminal conditions

$$\begin{cases} V(x,T) = \Psi(x,\theta(T)) \\ \theta(x,0) = \theta_0(x), \end{cases}$$

has a solution (V,θ).

Proof. The proof, as in the previous section is based upon a fixed point argument, of which we outline the main steps.

Fokker-Planck equation The first step consists in studying weak solutions, $\theta \in L^1([0,T],\mathcal{P}_1)$, of the Fokker-Planck equation

$$\begin{cases} \theta_t - \Delta\theta - \operatorname{div}(B\theta) = 0 \\ \theta(x,0) = \theta_0(x), \end{cases} \tag{12}$$

where $B : \mathbb{R}^d \times [0,T] \to \mathbb{R}^d$ is a vector field assumed to be continuous, bounded, and uniformly Lipschitz continuous in x. To do so, consider the stochastic differential equation

$$d\mathbf{x} = B(\mathbf{x},t)dt + \sqrt{2}dW_t,$$

where W_t is a d-dimensional Brownian motion. Suppose further that $\mathcal{L}(\mathbf{x}_0) = \theta_0$. Then it is well known that $\theta(t) = \mathcal{L}(\mathbf{x})$ is a weak solution to the Fokker-Planck equation (12).

Furthermore, there exists a constant depending only on the terminal time T, $C_0 = C_0(T)$ such that

$$d_1(\theta(t),\theta(s)) \le C_0 \left(\|f\|_\infty + \|\sigma\|_\infty\right)|t-s|^{\frac{1}{2}}. \tag{13}$$

Indeed, let $s < t$, and consider the random variables $\mathbf{x}_t$, $\mathbf{x}_s$ with law $\mathcal{L}(\mathbf{x}_t) = \theta(t)$, and $\mathcal{L}(\mathbf{x}_s) = \theta(s)$. Using the definition of the Kantorovitch-Rubinstein distance, and observing that the joint law $\gamma \in \Pi(\theta(t),\theta(s))$ we have

$$d_1(\theta(t),\theta(s)) \le \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|d\gamma(x,y) = E[|\mathbf{x}_t - \mathbf{x}_s|].$$

Since both $\mathbf{x}_t, \mathbf{x}_s$ satisfy (1), with the same initial condition, we get

$$E|\mathbf{x}_t - \mathbf{x}_s| \le E\left[\int_s^t |f(\mathbf{x}_r,r)|dr + \left|\int_s^t \sigma(\mathbf{x}_r,r)dW_t\right|\right] \le K(|f\|_\infty + \|\sigma\|_\infty)\sqrt{t-s}.$$

Additionally, elementary computations show that there exists a constant $C_0 = C_0(T)$ such that

$$\int_{\mathbb{R}^d} |x|^2 d\theta(t,x) \le K\left(\int_{\mathbb{R}^d} |x|^2 d\theta_0(x) + \|f\|_\infty^2 + \|\sigma\|_\infty^2\right). \tag{14}$$

The idea in [Car11] is to consider the set $\mathcal{K}$ given by

$$\mathcal{K} = \left\{ m \in C^0([0,T],\mathcal{P}) : \sup_{s\neq t} \frac{d_1(m(s),m(t))}{|t-s|^{\frac{1}{2}}} \le C_0,\ \sup_{t\in[0,T]} \int_{\mathbb{R}^d} |x|^2 dm(t,x) \le C_0 \right\},$$

for C_0 large enough. This set is a convex compact subset of $C^0([0,T],\mathcal{P})$, which is essential to apply a fixed point argument.

Second-order Hamilton-Jacobi equations The second step consists in looking at the Hamilton-Jacobi equation

$$\begin{cases} -V_t + -\Delta V + \frac{1}{2}|D_x V|^2 = F(x, m(t)) & \mathbb{R}^d \times [0,T] \\ V(x,T) = \Psi(x, m(T)) & \mathbb{R}^d, \end{cases}$$

where m is a given density measure in $\mathcal{K}$. Since F and Ψ satisfy A. and B., and $m \in \mathcal{C}^0([0,T], \mathcal{K})$, using the Cole-Hopf transform is is possible to show that the Hamilton Jacobi equation has unique solution V with $D_x V$ Lipschitz.

Fixed point argument For the last step of the argument, define the map

$$\begin{aligned} \Upsilon : \mathcal{K} &\to \mathcal{K} \\ m &\mapsto \theta = \Upsilon(m), \end{aligned}$$

where $\mathcal{K}$ is the convex, compact set defined before. Note that θ solves the Fokker-Planck equation and is continuous. Since $V \in \mathcal{C}^{2+\frac{1}{2}}$ we have uniqueness of solutions for the Fokker-Planck equation and also $\theta \in \mathcal{C}^{2+\frac{1}{2}}$. From (13) and (14) we conclude that $\theta \in \mathcal{K}$.

We now prove that Υ is continuous. Take a sequence (θ_n) in $\mathcal{K}$ convergent to θ. Then using the local uniformly convergence of Ψ, and F we obtain that (V_n) is also uniformly convergent, say, to V. Using interior regularity estimates we prove that $(D_x V_n)$ is locally uniformly Holder continuous so it converges local uniformly to $D_x V$. Further we have that (θ_n) converges to θ. So now we have all the results we need to state and prove the existence result.

Use the above defined map Υ, between the convex set $\mathcal{K}$ and apply the Schauder fixed point theorem. The solution to the reduced mean field game is precisely the fixed point (V, θ). □

2.2.3 A-priori estimates methods

We will adress a different dependence on the mean-field game on the measure θ. Rather than a smoothing one, we consider a local dependence on the measure. Again, to simplify the presentation we consider the following equation

$$\begin{cases} -V_t + \frac{|D_x V|^2}{2} = \Delta V + g(\theta) \\ \theta_t - \operatorname{div}(D_x V \theta) = \Delta \theta, \end{cases} \tag{15}$$

under initial-terminal data

$$V(x,T) = \psi(x) \qquad \theta(x,0) = \theta_0(x), \tag{16}$$

periodic boundary conditions in the spatial variable, that is $x \in \mathbb{T}^d$, and $g(\theta) = G'(\theta)$, where G is a convex increasing function. We should note that for quadratic Hamiltonians such as (15) smooth solutions are known to exist, see [CLLP12]. However the proof depends on the Hopf-Cole transformation and does not generalize in any obvious way for Hamiltonians which satisfy, for instance, quadratic-type growth conditions. The techniques present here can be generalized easily, and so more general Hamiltonians can be studied with similar techniques, as well as somewhat more general dependence on the measure of the form $g(x, \theta)$.

Lasry-Lions estimates One can easily obtain the following estimate, see [LL06b, LL07a], by multiplying the first equation in (15) by θ and the second equation by V, subtracting these two and integrating by parts:

$$\int_0^T \int_{\mathbb{T}^d} \frac{|D_x V|^2}{2} \theta + G(\theta) \leq C. \tag{17}$$

This estimate is related with the optimality of certain mean-field games as described in section 2.5.

In the case $g \geq 0$ one can obtain the additional estimate

$$\int_0^T \int_{\mathbb{T}^d} |D_x V|^2 dxdt \leq C. \tag{18}$$

By combining the estimates (17) and (18), Lions and Lasry in [LL06b, LL07a] obtained existence of weak solutions for various mean-field games, see also the forthcoming paper [Por13b] and [Car13b].

In [GPSM13a, GPSM13b] the following estimate for mean-field games, also in the case $g \geq 0$ was obtained

$$\int_0^T \int_{\mathbb{T}^d} g'(\theta)|D\theta|^2 + |D^2 V|\theta \leq C, \tag{19}$$

which extends a similar estimate for the stationary case in [Eva09], as well as in [GSM11], for second order problems (a similar estimate was also obtained by P. L. Lions). This result can be established by applying the Laplacian operator to the first equation of (15) and integrating with respect to θ.

Fokker-Planck equation The previous estimates were obtained by looking at the first equation, the Hamilton-Jacobi equation, in (15). However, the second equation also has regularizing properties. By combining the previous results with iterative methods for parabolic equations we obtained in [GPSM13a, GPSM13b] the following result, which only depends on $g \geq 0$:

Theorem 3. *$\theta \in L^\infty((0,T), L^r(\mathbb{T}^d))$, for all $0 < r < \frac{2^*}{2}$, where $2^* = \frac{2d}{d-2}$ is the Sobolev conjugated exponent to 2.*

The proof of this theorem uses an iterative procedure. First we know a-priori that $\theta \in L^\infty((0,T), L^1)$, since the Fokker-Planck equation conserves mass. The idea is to construct a sequence β_n such that at each step one has $\theta \in L^\infty((0,T), L^{1+\beta_n})$. One first obtains the identity

$$\begin{aligned}\int_{\mathbb{T}^d} \theta^{\beta+1}(x,\tau)dx + \frac{4\beta}{\beta+1}\int_0^\tau \int_{\mathbb{T}^d} |D\theta^{\frac{\beta+1}{2}}|^2 dx\, dt \\ = \int_{\mathbb{T}^d} \theta^{\beta+1}(x,0)dx + \beta \int_0^\tau \int_{\mathbb{T}^d} \mathrm{div}(DV)\theta^{\beta+1} dxdt.\end{aligned} \tag{20}$$

The last term can be controlled by

$$\int_0^\tau \int_{\mathbb{T}^d} \Delta V \theta^{\beta+1} \leq C \int_0^\tau \int_{\mathbb{T}^d} (\Delta V)^2\theta + \delta \int_0^\tau \int_{\mathbb{T}^d} \theta^{2\beta+1} dxdt$$

The first term in the right hand side can be estimated by the inequality (19) and the second one is handled by using a combination of Sobolev inequalities and Hölder inequality. This allows to establish an estimate for the $L^\infty((0,T), L^{1+\beta_{n+1}})$ norm of θ in terms of the $L^\infty((0,T), L^{1+\beta_n})$ norm of θ. For the details we refer the reader to [GPSM13a, GPSM13b].

Regularity for Hamilton-Jacobi equation From the integrability properties for θ, we can now look back at the Hamilton-Jacobi equation. We consider the reference case $g(\theta) = \theta^\alpha$. For quadratic Hamiltonians the existence of smooth solutions was established in [CLLP12]. The proof in that paper relies on the Hopf-Cole transformation and depends strongly on the specific quadratic form of the Hamiltonian and does not extend (except perhaps in very specific perturbation regimes) to general Hamiltonians. For Hamiltonians with sub-quadratic growth P.L. Lions, established in [Lio12], the following result:

Theorem 4. *Consider the Hamiltonian*

$$H(p,x) = (1+|p|^2)^{\frac{\gamma}{2}} + V(x)$$

with $1 \leq \gamma < 2$. *If* $\alpha < \frac{2}{d-2}$ *or* $1 \leq \gamma < 1 + \frac{1}{d+1}$ *and* $\alpha > 0$, *then* $D_t V, D^2_{xx} V \in L^p([0,T]\times\mathbb{T}^d)$ *for any* p, *and* $\theta \in L^\infty([0,T], L^p)$.

Once this regularity is obtained then further regularity results can also be established by bootstrapping and standard methods. In [GPSM13a] this result was improved in the subquadratic case and, through a completely different proof, in [GPSM13b] we were also able to study also the superquadratic case. In particular the following result was proved in those papers:

Theorem 5. *Consider the Hamiltonian*

$$H(p,x) = (1+|p|^2)^{\frac{\gamma}{2}} + V(x).$$

Then if $1 + \frac{1}{d+1} < \gamma < 2$ *there exists* $\alpha_{\gamma,d} > \frac{2}{d-2}$ *and for* $2 \leq \gamma < 3$ *for* $\alpha_{\gamma,d} = \frac{2}{d\gamma-2}$, *the solutions of the corresponding mean-field game*

$$-V_t + H(x, DV) = \theta^\alpha + \Delta V, \qquad \theta_t - \operatorname{div}(D_p H\theta) = \Delta\theta,$$

for $\alpha < \alpha_{\gamma,d}$, *with smooth initial-terminal data and* $\theta(x,0)$ *bounded away from zero satisfy* $D_t V, D^2_{xx} V \in L^p([0,T]\times\mathbb{T}^d)$ *for any* p, *and* $\theta \in L^\infty([0,T], L^p)$.

As remarked previously, from this regularity it follows the existence of smooth solutions. Several additional improvements are possible and are discussed in [GPSM13a, GPSM13b].

2.3 Uniqueness by the Lions-Lasry monotonicity method

We now address the uniqueness of classical solutions for the initial-terminal value problem for mean-field games. We start by reviewing the Lions-Lasry monotonicity method. Then we present a definition of weak solution which allows for an improved uniqueness result.

2.3.1 Monotonicity method

We discuss here uniqueness for classical solutions in the second order case (the first order case is just a special case of the second order case) using the technique by Lions and Lasry. This proof yields uniqueness for classical solutions. In the next section we show how to modify the proof so that one can prove uniqueness for viscosity solutions without any regularity hypothesis on the solutions. We consider mean-field games in $\mathbb{R}^d$, but the argument extends to the periodic case without any difficulty.

Theorem 6. *Consider a smooth Hamiltonian of the form* $H(x,p,M,\theta) = H_0(x,p,M) - F(x,\theta)$, *where* $F : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$. *Assume further that* H_0 *is jointly convex in* p *and* M, *that* F *is strictly monotone in* θ, *that is, if* $\theta_1, \theta_2 \in \mathcal{P}_1$, $\theta_1 \neq \theta_2$ *then*

$$\int_0^T \int_{\mathbb{R}^d} \big(F(x,\theta_1) - F(x,\theta_2)\big)(\theta_1-\theta_2)(x) > 0, \tag{21}$$

and

$$\int_{\mathbb{R}^d} (\psi(x,\theta_1) - \psi(x,\theta_2))\,(\theta_1-\theta_2) \geq 0, \quad \forall \theta_1,\theta_2 \in \mathcal{P}_1. \tag{22}$$

Then the initial-terminal value problem for the mean-field game given by

$$\begin{cases} -V_t + H_0(x, D_x V, D_x^2 V) = F(x,\theta) \\ \theta_t - \operatorname{div}(D_p H_0 \theta) - \partial^2_{ij}(D_{M_{ij}} H_0 \theta) = 0, \end{cases} \tag{23}$$

together with conditions (5), *has at most a classical solution* (V,θ).

Proof. We now follow the Lions-Lasry's strategy to prove uniqueness. Suppose, by contradiction that there exist two solutions, (V_1, θ_1), and (V_2, θ_2) of the above mean-field game. We have

$$\int_0^T \int_{\mathbb{R}^d} \frac{d}{dt}(V_1 - V_2)(\theta_1 - \theta_2) = 0.$$

In fact, for the uniqueness proof it suffice to have ≥ 0 in the previous expression. The expression for the left hand side can be obtained by considering the equations for $\bar{V} = V_1 - V_2$ and $\bar{\theta} = \theta_1 - \theta_2$ and multiply them by $\bar{\theta}$ and $\bar{V}$, respectively. After subtracting the later from the former, rearranging various terms we obtain

$$\begin{aligned} 0 = &\int_{\mathbb{R}^d} \bar{V}(x,0)\bar{\theta}_0(x) - (\psi(x,\theta_1(T)) - \psi(x,\theta_2(T)))\,\bar{\theta}(x,T) \\ &- \int_0^T \int_{\mathbb{R}^d} \left(D_p H_0(x, D_x V_1, D^2_{xx} V_1)\theta_1 - D_p H_0(x, D_x V_2, D^2_{xx} V_2)\theta_2\right) \nabla \bar{V} \\ &+ \int_0^T \int_{\mathbb{R}^d} \left(-D_{M_{ij}} H_0(x, D_x V_1, D^2_{xx} V_1)\theta_1 + D_{M_{ij}} H_0(x, D_x V_2, D^2_{xx} V_2)\theta_2\right) \partial^2_{ij} \bar{V} \\ &+ \int_0^T \int_{\mathbb{R}^d} \left(H_0(x, D_x V_1, D^2_{xx} V_1) - H_0(x, D_x V_2, D^2_{xx} V_2)\right) \bar{\theta} + \int_0^T \int_{\mathbb{R}^d} \left(F(x,\theta_2) - F(x,\theta_1)\right) \bar{\theta} \end{aligned} \tag{24}$$

where we have assumed enough regularity and decay to integrate by parts. Using the condition (22), and the convexity of H_0, which implies

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^d} \Big(H_0(x, D_x V_2, D^2_{xx} V_2) - H_0(x, D_x V_1, D^2_{xx} V_1) - D_p H_0(x, D_x V_1, D^2_{xx} V_1)(\nabla V_2 - \nabla V_1) \\ &\qquad\qquad - D_{M_{ij}} H_0(x, D_x V_1, D^2_{xx} V_1)(\partial^2_{ij} V_2 - \partial^2_{ij} V_1)\Big)\theta_1 \\ &+ \int_0^T \int_{\mathbb{R}^d} \Big(H_0(x, D_x V_1, D^2_{xx} V_1) - H_0(x, D_x V_2, D^2_{xx} V_2) - D_p H_0(x, D_x V_2, D^2_{xx} V_2)(\nabla V_1 - \nabla V_2) \\ &\qquad\qquad - D_{M_{ij}} H_0(x, D_x V_2, D^2_{xx} V_2)(\partial^2_{ij} V_1 - \partial^2_{ij} V_2)\Big)\theta_2 \geq 0, \end{aligned}$$

we conclude that

$$\int_0^T \int_{\mathbb{R}^d} \big(F(x,\theta_1) - F(x,\theta_2)\big)(\theta_1 - \theta_2)(x) \leq 0,$$

which contradicts (21). This yields $\theta_1 = \theta_2$. Then uniqueness for viscosity solutions implies $V_1 = V_2$, therefore the solution is unique. □

In the local case one can also use a similar argument to obtain uniqueness. As discussed in P.L. Lions course [Lio11], as described in [Gue11c], take $H(x,p,z) : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^+_0 \to \mathbb{R}$. Then uniqueness for the mean-field game

$$\begin{cases} -V_t - \Delta V + H(x, D_x V, \theta) = 0 \\ \theta_t - \Delta\theta - \operatorname{div}(D_p H \theta) = 0 \end{cases}$$

holds if the H satisfies

$$\begin{bmatrix} z D^2_{pp} H & \frac{1}{2} z D^2_{pz} H \\ \frac{1}{2} z D^2_{zp} H & -D_z H \end{bmatrix} > 0$$

for any $(x,p,z) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^+$.

2.3.2 Relaxed solutions and uniqueness

We now introduce a notion of relaxed solutions for mean-field games that allows to prove uniqueness under minimal regularity assumptions. To simplify the discussion we consider first order mean-field games. Let $H : \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R}$ be a continuous function. Let $L(x, \cdot, \theta)$ be the Legendre transform of $H(x, \cdot, \theta)$, for all $x \in \mathbb{R}^d$ and $\theta \in \mathcal{P}_1(\mathbb{R}^d)$, which to simplify we assume bounded by below.

A relaxed solution for the mean-field game

$$\begin{cases} -u_t + H(x, Du, \theta) = 0 \\ \theta_t - \operatorname{div}(D_p H \theta) = 0. \end{cases} \tag{25}$$

is a triplet (u, θ, J) where $u : \mathbb{R}^d \times [0, T] \to \mathbb{R}$, $\theta \in C([0, T], \mathcal{P}(\mathbb{R}^d))$ and J is a vector valued measure in $[0, T] \times \mathbb{R}^d$ absolutely continuous with respect to θ, satisfying the following properties:

1. $u \in C(\mathbb{R}^d)$ is a viscosity solution of $-u_t + H(x, Du, \theta) = 0$;
2. for $0 \le t \le T$, $u(\cdot, t) \in L^1(d\theta(\cdot, t))$;
3. as a distribution
$$\theta_t + \operatorname{div}(J) = 0;$$
4. since J is absolutely continuous with respect to θ denote by $v(x, t)$ its Radon-Nykodym derivative. Then we require
$$\int u(x, 0) d\theta(x, 0) \ge \int_0^T \int_{\mathbb{R}^d} L(x, v(x, t), \theta) d\theta + \int_{\mathbb{R}^d} u(x, T) d\theta(x, T).$$

Any classical solution to the mean-field game is in fact a relaxed solution, for $J = -D_p H \theta$. Also we observe that (under very mild standard assumptions) from the optimal control representation for viscosity solutions of Hamilton-Jacobi equations, for any pair $(\tilde\theta, \tilde J)$ with $\tilde\theta \in C([0, T], \mathcal{P}(\mathbb{R}^d))$ $\tilde J$ absolutely continuous with respect to $\tilde\theta$ such that $\tilde J = \tilde v \theta$ we have

$$\int u(x, 0) d\tilde\theta(x, 0) \le \int_0^T \int_{\mathbb{R}^d} L(x, \tilde v(x, t), \theta) d\tilde\theta + \int_{\mathbb{R}^d} u(x, T) d\tilde\theta(x, T). \tag{26}$$

Indeed, under mild standard regularity hypothesis it is possible to build a sequence of C^1 functions, such that $u^n \to u$ uniformly, and

$$-u_t^n + H(x, D_x u^n, \theta) \le o(1),$$

as $n \to \infty$. Then

$$-u_t^n - \tilde v D_x u^n \le o(1) - H(x, D_x u^n, \theta) - -\tilde v D_x u^n \le o(1) + L(x, \tilde v, \theta).$$

Integrating and passing to the limit we obtain (26).

One advantage of this notion of relaxed solution is that uniqueness can be proved without any regularity or differentiability assumptions on u.

Theorem 7. *Suppose L satisfies*

$$\int_0^T \int_{\mathbb{R}^d} L(x, \tilde v(x, t), \theta) d\tilde\theta + L(x, v(x, t), \tilde\theta) d\theta - L(x, v(x, t), \theta) d\theta - L(x, \tilde v(x, t), \tilde\theta) d\tilde\theta < 0, \tag{27}$$

whenever $(\theta, J) \ne (\tilde\theta, \tilde J)$, where $J = v\theta$ and $\tilde J = \tilde v \tilde\theta$.

Then the initial-terminal value problem for (25) has at most one relaxed solution.

Proof. Let (u, θ, J) and $(\tilde{u}, \tilde{\theta}, \tilde{J})$ be relaxed solutions to (25). Then

$$\int u(x,0)d\theta(x,0) = \int_0^T \int_{\mathbb{R}^d} L(x, v(x,t), \theta)d\theta + \int_{\mathbb{R}^d} u(x,T)d\theta(x,T),$$

$$\int \tilde{u}(x,0)d\tilde{\theta}(x,0) = \int_0^T \int_{\mathbb{R}^d} L(x, \tilde{v}(x,t), \tilde{\theta})d\tilde{\theta} + \int_{\mathbb{R}^d} \tilde{u}(x,T)d\tilde{\theta}(x,T),$$

$$\int \tilde{u}(x,0)d\tilde{\theta}(x,0) \le \int_0^T \int_{\mathbb{R}^d} L(x, \tilde{v}(x,t), \theta)d\tilde{\theta} + \int_{\mathbb{R}^d} \tilde{u}(x,T)d\tilde{\theta}(x,T),$$

and

$$\int u(x,0)d\theta(x,0) \le \int_0^T \int_{\mathbb{R}^d} L(x, v(x,t), \tilde{\theta})d\theta + \int_{\mathbb{R}^d} u(x,T)d\theta(x,T).$$

Adding the last two inequalities and subtracting the first two equalities yields:

$$0 \le \int_0^T \int_{\mathbb{R}^d} L(x, \tilde{v}(x,t), \theta)\tilde{\theta} + L(x, v(x,t), \tilde{\theta})\theta - L(x, v(x,t), \theta)\theta - L(x, \tilde{v}(x,t), \tilde{\theta})\tilde{\theta},$$

which contradicts (27) unless $(\theta, J) = (\tilde{\theta}, \tilde{J})$. Then $u = \tilde{u}$ by uniqueness for viscosity solutions. □

An example where the previous theorem applies is the separated case

$$L = \frac{|v|^2}{2} + g(\theta),$$

where g is a monotone function (not necessarily local) in the sense that

$$\int_{\mathbb{R}^d} (g(\theta) - g(\tilde{\theta}))(\theta - \tilde{\theta}) > 0,$$

if $\theta \ne \tilde{\theta}$.

2.4 Stationary problems

2.4.1 Stationary problems

In addition to the terminal-initial value or planning problems discussed in the previous section, stationary problems play an important role in many applications. The stationary problem corresponding to (4) consists in finding a triplet $(u, \theta, \bar{H})$, where $u : \mathbb{R}^d \to \mathbb{R}$, $\theta \in \mathcal{P}(\mathbb{R}^d)$ and $\overline{H} \in \mathbb{R}$ which solves

$$\begin{cases} H(D_x u, x, \theta) = \Delta u + \overline{H} \\ -\operatorname{div}(D_p H \theta) - \Delta\theta = 0. \end{cases} \tag{28}$$

The constant $\overline{H}$ is called the effective Hamiltonian as it arises in related problems in homogenization theory, as well as in Aubry-Mather theory (see Section 2.5.4).

There are three natural questions that arise immediately when considering (28). First, of course, is existence (and regularity) of solutions, secondly uniqueness of the constant $\overline{H}$ and of solutions, and finally to what extend one can expect time-dependent mean-field games to converge to stationary solutions.

Concerning existence, one can use similar proofs to the ones in the time dependent case. In particular it is possible to prove suitable a-priori bounds for (28), see [LL06a], [GSM11, GPSM12], for instance. In certain cases uniqueness can be established by monotonicity methods using a procedure similar to the one in Section 2.3. An important class of stationary mean-field games admits a variational formulation. In those cases (28) is the Euler-Lagrange of a (possibly non-coercive) convex functional. For these variational cases, once existence is established, uniqueness follows by standard convexity arguments in the calculus of variations see [GSM11] for instance. We will discuss some variational structures for mean-field games in Section 2.5. The last question, the trend to equilibrium, will be addressed in what follows. We end this section by presenting some results on extended mean-field games.

2.4.2 Trend to equilibrium

A natural question in the initial-terminal value problem the following: suppose one is given an initial probability measure $m^T(x,0) = m_0(x)$ and a terminal cost $u^T(x,T) = u_0(x)$ and then lets $T \to \infty$ - is it true that $m^T(x,t)$ and $u^T(x,t)$ converge to a stationary solution? In some sense this would mean that by taking an initial probability distribution far in the past and a terminal cost far in the future, the present behaves like an equilibrium. The first positive answer to this problem in discrete state and time was given in [GMS10] and then further extended to the continuous time setting in [GMS13]. The key idea in these papers is to adapt the uniqueness proof by monotonicity to extract the convergence result. A similar idea was also independently used in [CLLP12] and in [CLLP13] where the authors studied the continuous time problem both for local coupling and non-local coupling. More recently, the first order case was addressed in [Car13a]. Further results on the finite state problem were established in [FG13] using Γ convergence through a different set of ideas. For Hamilton-Jacobi equations, a new class of ideas using the adjoint method is discussed in [CGMT13].

The long time convergence results in [CLLP12] hold for fairly general Hamiltonians as long as smooth enough solutions are known to exist. In particular the results apply to the following mean-field game:

$$\begin{cases} -u_t - \Delta u + \frac{1}{2}|Du|^2 = F(x,\theta) \\ \theta_t - \Delta\theta - \operatorname{div}(\theta Du) = 0, \end{cases} \tag{29}$$

coupled with initial-terminal conditions

$$\begin{cases} \theta(x,0) = \theta_0(x) \\ u(x,T) = u_0(x), \end{cases}$$

as well as periodic boundary conditions in the spatial variable, that is $x \in \mathbb{T}^d$. Let (u^T, θ^T) be a solution of (29) satisfying, the above, initial-terminal conditions. The existence and uniqueness of solution (u^T, θ^T) of (29) was already discussed in sections 2.2.2 and 2.3. In order to state the convergence results in [CLLP12] we need to consider the following stationary problem (in fact in [CLLP12] more general initial-terminal conditions are considered):

$$\begin{cases} \overline{H} - \Delta\bar{u} + \frac{1}{2}|D\bar{u}|^2 = F(x,\bar{\theta}) \\ -\Delta\bar{\theta} - \operatorname{div}(\bar{\theta} D\bar{u}) = 0 \end{cases} \tag{30}$$

where $\bar{u}$ and $\bar{m}$ satisfy the normalization conditions:

$$\begin{cases} \int_{\mathbb{T}^d} \bar{u} dx = 0 \\ \int_{\mathbb{T}^d} \bar{\theta} dx = 1. \end{cases}$$

We denote by $(\overline{H}\bar{u}, \bar{\theta})$ the solution to the above problem.

A convergence result In the above mentioned paper, the convergence is proved assuming the following conditions:

A1. $F : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$, is C^1, and $\mathbb{Z}^d - periodic$ in the space variable x and increasing in m,

A2. $u_0 : \mathbb{R}^d \to \mathbb{R}$ is $\mathbb{Z}^d$-periodic and of class C^2, Also, for convenience, one considers a rescaled version of (29), $v^T(x,t) = v(x,Tt$, and $\theta^T(x,t) = \theta(x,Tt)$, which satisfies:

$$\begin{cases} -\frac{1}{T}v_t^T - \Delta v^T + \frac{1}{2}|Dv^T|^2 = F(x,\theta^T) \\ \frac{1}{T}\theta_t^T - \Delta\theta^T - \mathrm{div}(\theta^T Dv^T) = 0, \end{cases} \tag{31}$$

with $\theta^T(x,0) = \theta_0(x)$, $v^T(x,1) = u_0(x)$.

Provided the above conditions hold, the convergence results obtained in [CLLP12] are the following:

Theorem 8. *Let (v^T, θ^T) be a solution to (31). Then*

B1. $v^T(t,\cdot)/T$ converges uniformly in $L^1(\mathbb{T}^d)$ to $(1-t)\overline{H}$, for $t \in [0,1]$,

B2. v^T converges uniformly to $\tilde{u}$ in $L^2((0,1)\times\mathbb{T}^d)$, where $\tilde{u} = (1-t)\overline{H}$,

B3. $v^T - \int_{\mathbb{T}^d} v^T(t,y)dy$ converges to $\bar{u}$ in $L^2((0,1)\times\mathbb{T}^d)$,

B4. $\theta^T \to \bar{\theta}$ in $L^p((0,1)\times\mathbb{T}^d)$ for $p < \frac{d+2}{d}$ provided $d > 2$, and for $p < 2$ if $d = 2$,

B5. $F(\cdot,\theta^T) \to F(\cdot,\bar{\theta})$ and $F(\cdot,\theta^T)\theta^T \to F(\cdot,\bar{\theta})\bar{\theta}$ both in $L^1((0,1)\times\mathbb{T}^d)$.

The proof of this theorem relies on the following: first by applying the same technique as in the uniqueness proof one obtains the following identity:

$$\int_0^1 \int_{\mathbb{T}^d} \frac{\theta^T + \bar{\theta}}{2}|Dv^T - D\bar{u}|^2 + (F(x,\theta^T) - F(x,\bar{\theta}))(\theta^T - \bar{\theta})dxdt = -\frac{1}{T}\left[\int_{\mathbb{T}^d}(v^T - \bar{u})(\theta^T - \bar{\theta})\right]_0^1.$$

The second key step consists in obtaining estimates to show that the right hand side is in fact controlled and therefore implies convergence. The proof presented in that paper uses the specific form of the Hamiltonian through the use of the Hopf-Cole transform to obtain various estimates.

Convergence rate The convergence rate's result proved in [CLLP12] assumes, in addition to conditions A1.-A.2, the following one:

A3. The growth rate of F is bounded by below. Let $s \geq t$, so there is a $\gamma > 0$ such that

$$F(x,s) - F(x,t) \geq \gamma(s-t), \quad \forall x \in \mathbb{T}^d.$$

Let $C > 0$ be a constant, and define $\tilde{u}^T(x,t) = u^T(x,t) - \int_{\mathbb{T}^d} u^T(x,t)$, where u^T is a solution to (29).

Theorem 9. *Under the assumptions A1.-A.3 then for $\forall t \in (0,T)$, the following holds:*

$$\|\tilde{u}^T(t) - \bar{u}\|_{L^1(\mathbb{T}^d)} \leq \frac{C}{T-t}\left(e^{-C(T-t)} + e^{-Ct}\right),$$

$$\|\theta^T(t) - \bar{\theta}\|_{L^1(\mathbb{T}^d)} \leq \frac{C}{t}\left(e^{-C(T-t)} + e^{-Ct}\right),$$

and for all $t \in (0, T-1)$

$$\left\|\frac{u^T(t)}{T} - \overline{H}\left(1 - \frac{t}{T}\right)\right\|_{L^1(\mathbb{T}^d)} \leq \frac{C}{T}.$$

where $(\overline{H}, \bar{u}, \bar{\theta})$ is a solution to the associated ergodic problem (30).

2.4.3 Extended stationary models

In many applications it is important to consider mean-field games where the running cost or the dynamics of the players depend not only on the distribution of players but also on their actions. This leads to the class of extended mean-field games considered in [GV13] and [GPV13]. In this section we describe briefly the stationary models and refer the reader to [GPV13] for details and additional results.

We denote by $\chi(\mathbb{T}^d)$ the set of continuous vector fields on $\mathbb{T}^d$, and by $\mathcal{P}^{ac}$ the set of absolutely continuous probability measures in $\mathbb{T}^d$. Let

$$H\colon \mathbb{T}^d \times \mathbb{R}^d \times \mathcal{P}^{ac}(\mathbb{T}^d) \times \chi(\mathbb{T}^d) \to \mathbb{R}.$$

Then one can consider the system:

$$\begin{cases} -\Delta u(x) + H(x, Du(x), \theta, V) = \overline{H} \\ -\Delta\theta(x) + \operatorname{div}(V(x)\theta(x)) = 0 \\ V(x) = -D_pH(x, Du(x), \theta, V). \end{cases} \tag{32}$$

The unknowns for this problems are $u\colon \mathbb{T}^d \to \mathbb{R}$, identified with a $\mathbb{Z}^d$-periodic function on $\mathbb{R}^d$ whenever convenient, a probability measure $m \in \mathcal{P}(\mathbb{T}^d)$, the effective Hamiltonian $\overline{H} \in \mathbb{R}$ and the effective velocity field $V \in \chi(\mathbb{T}^d)$. Among the problems considered in [GPV13] the following example was investigated:

$$H(x, p, \theta, V) = h(x, p) + \delta p \cdot \int_{\mathbb{T}^d} V d\theta - g(\theta) \tag{33}$$

where h is a coercive and satisfies quadratic growth-type conditions and $g : \mathbb{R}^+ \to \mathbb{R}$ is either $g(z) = z^\alpha$ or $g(z) = \ln z$ (other possible functions can also be handled with similar methods but these two are representative of the main techniques and difficulties). In the above reference, we proved:

Theorem 10. *The system* (32) *where H is given by* (33) *admits a unique classical solution, for δ small enough, for $g = \ln m$ in any dimension, and for $g(z) = z^\alpha$, if $d \leq 4$ for any exponent α and if $d \geq 5$ for $\alpha < \frac{1}{d-4}$.*

The proof of this theorem, which rather is lengthy and clearly beyond the scope of this paper, depends upon establishing careful a-priori estimates for the solutions and applying a continuation argument.

2.5 Potential mean-field models

Certain mean-field games admit variational formulations which allows the use of duality and calculus of variations techniques in their study. Some of these structures were already discussed in the papers [LL06a, LL06b, LL07a, LL07b], and used to study the planning problem in [ACCD12] or the long-time behavior in [Car13a]. Also, existence of weak solutions for first order problems was addressed by variational methods in [Car13b]. These will be addressed in section 2.5.1 and consist in optimization problems in the space of measures whose optimality conditions are equivalent to mean-field games. Another related class of variational structures, which are written in terms of integral variational problems, was discovered in the study of the stochastic Evans-Aronsson problem in [GSM11] (see also [GISMY10]). These will be discussed in section 2.5.2 together with some applications and extensions. Then, in section 2.5.3, we investigate, through duality, the connection between these problems defined through multiple integrals and optimization problems in the space of measures.

Rather than developing here a complete theory we present several examples and applications. More general problems can be handled by adapting the ideas outlined in the present paper.

Throughout this section we will work on the periodic setting, that is the state space is $\mathbb{T}^d$, the d-dimensional torus, identified with $[0,1]^d$. The main reason is to avoid problems that could arise by computing integral functionals on non-compact domains. By similar methods, one can consider boundary value problems of various types.

2.5.1 Optimal control in the space of measures

We discuss in this section a class of planning problems for mean-field games which can be seen as optimal control problems in the space of measures.

Let $F : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ be a convex function. Suppose that F is differentiable with gradient with respect to the L^2 inner product $\nabla F(\rho)$. We will work in the setting of section 2.1 under the following simplifying assumptions $U = \mathbb{R}^d$, $f(x,v) = v$, and $L : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$. Consider the problem of minimizing over all (smooth enough) vector fields $b : \mathbb{T}^d \times [0,T] \to \mathbb{R}^d$ and measures ρ in $\mathbb{T}^d \times [0,T]$ the functional

$$\int_0^T \int_{\mathbb{T}^d} F(\rho) + L(x, b(x,t))\rho \tag{34}$$

under the constraint

$$\rho_t + \operatorname{div}(b\rho) = \Delta\rho, \tag{35}$$

with $\rho(x,0) = \theta_0$ and $\rho(x,T) = \theta_1$.

In order to study this problem and obtain optimality conditions we will introduce the dual problem through the minimax principle. Our discussion will be mostly informal as our main objective in this section is to obtain optimality conditions. However, a rigorous discussion of duality in this setting can be found in [ACCD12].

Let $\phi : \mathbb{T}^d \times [0,T] \to \mathbb{R}$ be a smooth function that will act as the Lagrange multiplier for (35). The problem of minimizing (34) under the constraint (35) is equivalent to

$$\min_{b,\rho} \max_{\phi} \int_0^T \int_{\mathbb{T}^d} F(\rho) + L(x, b(x,t))\rho - \phi(\rho_t + \operatorname{div}(b\rho) - \Delta\rho). \tag{36}$$

By definition, the dual problem is the variational problem obtained by switching the minimum with the maximum. In general, the value for the dual problem is a lower bound for the original problem. In many cases, it is possible to show that their values coincide using the Legendre-Fenchel-Rockafellar theorem, see [ACCD12]. Note that the dual problem is simply

$$\max_{\phi} \min_{b,\rho} \int_0^T \int_{\mathbb{T}^d} F(\rho) + L(x, b(x,t))\rho - \phi(\rho_t + \operatorname{div}(b\rho) - \Delta\rho).$$

By integrating by parts and performing the minimization over the vector fields b we obtain that the dual problem is simply

$$\max_{\phi} \min_{\rho} \int_0^T \int_{\mathbb{T}^d} F(\rho) + (\Delta\phi - H(x, D_x\phi) + \phi_t)\,\rho + \int_{\mathbb{T}^d} \phi(x,0)\theta_0 - \int_{\mathbb{T}^d} \phi(x,T)\theta_1, \tag{37}$$

where

$$H(x,p) = \sup_{v \in \mathbb{R}^d} \left[-v \cdot p - L(x,v)\right].$$

Proposition 1. *Let (V, θ) be a solution to*

$$\begin{cases} -V_t - \Delta V + H(x, D_x V) = \nabla F(\theta) \\ \theta_t - \Delta\theta - \operatorname{div}(D_p H(x, D_x V)\theta) = 0 \end{cases}$$

satisfying $\theta(x,0) = \theta_0$, $\theta(x,T) = \theta_T$. Then V is optimal for (37), $(\rho, b) = (\theta, -D_p H(x, D_x V))$ is optimal for (34). Furthermore there is no duality gap, that is, the value of the primal agrees with the one of the dual.

Proof. Denote by P the value in (36), and Q the value of (37). We always have $P \geq Q$.

Clearly, by choosing $\phi = V$ in (37) we obtain the following lower bound:

$$
\begin{aligned}
Q &\geq \min_{\rho} \int_0^T \int_{\mathbb{T}^d} F(\rho) + (\Delta V - H(x, D_x u) + V_t)\,\rho + \int_{\mathbb{T}^d} V(x,0)\theta_0 - \int_{\mathbb{T}^d} V(x,T)\theta_1 \\
&= \min_{\rho} \int_0^T \int_{\mathbb{T}^d} F(\rho) - \nabla F(\theta)\rho + \int_{\mathbb{T}^d} V(x,0)\theta_0 - \int_{\mathbb{T}^d} V(x,T)\theta_1.
\end{aligned}
$$

By convexity we have therefore

$$
Q + \int_{\mathbb{T}^d} V(x,T)\theta_1 - \int_{\mathbb{T}^d} V(x,0)\theta_0 \geq \int_0^T \int_{\mathbb{T}^d} F(\rho) - \nabla F(\theta)\rho \geq \int_0^T \int_{\mathbb{T}^d} F(\theta) - \nabla F(\theta)\theta.
$$

Choosing in (34) $\rho = \theta$ and $b = -D_p H(x, D_x V)$, which by definition satisfy (35) we have the following upper bound:

$$
P \leq \int_0^T \int_{\mathbb{T}^d} F(\theta) + L(x, -D_p H(x, D_x V))\theta.
$$

Using the identity

$$
L(x, -D_p H(x, D_x V)) - D_x V \cdot D_p H(x, D_x V) = -H(x, D_x V),
$$

we have that

$$
P \leq \int_0^T \int_{\mathbb{T}^d} F(\theta) + (V_t - H(x, D_x V) + \Delta V)\,\theta + (-V_t - \Delta V + D_p H(x, D_x V) D_x V)\,\theta.
$$

From this we get

$$
P + \int_{\mathbb{T}^d} V(x,T)\theta_1 - \int_{\mathbb{T}^d} V(x,0)\theta_0 \leq \int_0^T \int_{\mathbb{T}^d} F(\theta) - \nabla F(\theta)\theta,
$$

that is $P \leq Q$. This shows that there is no duality gap and the optimality of $(\theta, -D_p H(x, D_x u))$ and u. $\square$

It is important to observe, however, that not every mean-field game will have such a variational structure. Additionally, it is also not true, in general, that variational problems which involve general costs $L(x, b, \rho)$ rather than a sum of a linear functional in ρ and a nonlinear function of ρ have optimality conditions equivalent to mean-field games.

2.5.2 Calculus of variations with convex non-linear integrands

We consider now a class of variational problems than gives rise to mean-field games through the minimization of functionals defined by multiple integrals. The discussion here is based upon the ideas first developed in [GSM11], [GPSM12], and [GISMY10]. The connection by duality theory between these problems and the ones considered in the previous section will be discussed in section 2.5.3.

In this setting it is more convenient to start in a somewhat more general setting which includes various important examples, such as initial-terminal, planning and stationary problems. Let W be a compact set, in most examples either $W = \mathbb{T}^d \times [0,T]$ or $W = \mathbb{T}^d$. Consider a nonlinear operator denoted by $\mathcal{N} : C^\infty(W) \to C^\infty(W)$. Important examples of such operators are

$$
\mathcal{N}(V) = -V_t + H(x, D_x V) - \Delta V, \tag{38}
$$

with $W = \mathbb{T}^d \times [0,T]$ and

$$\mathcal{N}(V) = H(x, D_x V) - \Delta V, \tag{39}$$

for $W = \mathbb{T}^d$. To simplify we will assume H smooth. Note that many other variations, including first order (simply by omitting the Laplacian), general fully non-linear elliptic operators or even non-local operators can also be considered by the same methods. We will assume $\mathcal{N}$ to be differentiable with respect to u, that is, for any $v \in C_c(W)$ the following directional derivative exists and defines a linear operator $\mathcal{L}_u : C^\infty(W) \to C^\infty(W)$

$$\frac{d}{d\epsilon}\mathcal{N}(u + \epsilon v)\Big|_{\epsilon=0} = \mathcal{L}_u v.$$

The linear operators corresponding to examples (38), and (39) are

$$\mathcal{L}_V v = -v_t + D_p H(x, D_x V) D_x v - \Delta v,$$

and

$$\mathcal{L}_V v = D_p H(x, D_x V) D_x v - \Delta v.$$

Let $G : \mathbb{R} \to \mathbb{R}$ be a convex increasing function. Consider the integral functional

$$\int_W G(\mathcal{N}(V)) dx. \tag{40}$$

Let V be a minimizer of (40). Then, assuming enough regularity, an elementary computation shows that the Euler-Lagrange equation in weak form is

$$\int_W G'(\mathcal{N}(V)) \mathcal{L}_V v dx = 0, \text{ for all } \ v \in C_c^\infty(W).$$

If we define

$$\theta = G'(\mathcal{N}(V)),$$

the above Euler-Lagrange equation can be written in strong form as

$$\begin{cases} \mathcal{N}(V) = (G')^{-1}(\theta) \\ \mathcal{L}_V^* \theta = 0, \end{cases} \tag{41}$$

where $\mathcal{L}_V^*$ is the adjoint of $\mathcal{L}_V$ with respect to the L^2 inner product.

For illustration purposes, we consider now some possible choices of G and $\mathcal{N}$. First set $G(z) = z^\alpha$, and $W = [0,T] \times \mathbb{T}^d$ in (40). Then, for $\alpha \neq 1$, (41) is simply

$$\begin{cases} -V_t + H(x, D_x V) - \Delta V = \left(\frac{\theta}{\alpha}\right)^{\frac{1}{\alpha-1}} \\ \theta_t - \operatorname{div}(D_p H(x, D_x V)\theta) - \Delta\theta = 0. \end{cases}$$

Consider also the case $G(z) = e^z$, then (41) becomes

$$\begin{cases} -V_t + H(x, D_x V) - \Delta V = \ln\theta \\ \theta_t - \operatorname{div}(D_p H(x, D_x V)\theta) - \Delta\theta = 0. \end{cases}$$

This variational interpretation of mean-field games is quite remarkable as it shows that various problems which have been researched intensely in the last few years are closely related to mean-field games. Take for instance

$$\mathcal{N}(V) = |DV|, \qquad G(z) = z^p.$$

The Euler-Lagrange equation for this functional is simply the p-Laplace equation.

Also certain mean-field games have surprising regularizing properties. Take for instance

$$\mathcal{N}(V) = \frac{|DV|^2}{2} + W(x), \qquad G(z) = e^z.$$

This corresponds to the mean-field game

$$\begin{cases} \frac{|DV|^2}{2} + W(x) = \ln \theta \\ -\operatorname{div}(DV\theta) = 0. \end{cases}$$

Though in general first order equations have only Lipschitz or semiconcave solutions, this mean-field game in fact has smooth classical solutions. This was proved in the periodic setting in [Eva09]. In [GSM11] the second order case, which is associated with the non-coercive, convex functional

$$\int_{\mathbb{T}^d} e^{-\Delta V + \frac{|DV|^2}{2} + W(x)},$$

was also studied and shown to admit smooth solutions which are minimizers of the above functional. For further results, see also [GPV13].

In certain applications, it may be necessary to modify the structure of the mean-field game equations. For instance, there may be a source f of agents, or they may die at a rate γ. In this case it is natural to consider equations of the form

$$\begin{cases} \gamma V + H(x, DV) = \Delta u + \ln \theta \\ \gamma\theta - \operatorname{div}(D_p H(x, DV)\theta) = \Delta\theta + f(x). \end{cases}$$

The previous equation is also a Euler-Lagrange equation of the functional

$$\int e^{\gamma V + H(x,DV) - \Delta V} - fV.$$

Various other modifications can also be considered to study optimal switching and obstacle type problems, see [GP13]. Mean-field games with a non-linear Fokker-Planck equation were considered in [GR13].

2.5.3 Duality revisited

We now apply duality theory to the problems discussed in the previous section. We will work a specific example but it should be clear how to apply these ideas in different settings.

Consider the problem

$$\min_{\mathcal{C}} \int_{\mathbb{T}^d} G(-q(x) + H(p(x), x)),$$

where $\mathcal{C}$ is the set of smooth functions (ϕ, p, q), $\phi : \mathbb{T}^d \to \mathbb{R}$, $p : \mathbb{T}^d \to \mathbb{R}^d$, and $q : \mathbb{T}^d \to \mathbb{R}$, which satisfy the constraints

$$p = D_x\phi, \quad q = \Delta\phi.$$

We introduce two Lagrange multipliers $J : \mathbb{T}^d \to \mathbb{R}^d$ and $\theta : \mathbb{T}^d \to \mathbb{R}$. Proceeding as before, we look at the functional

$$\int_{\mathbb{T}^d} G(-q + H(p, x)) + J(p - D_x\phi) + \theta(q - \Delta\phi).$$

The Euler-Lagrange equation for the previous functional can be written as

$$\theta = G'(-q + H(p, x)), \qquad J = \theta D_p H(p, x), \tag{42}$$

and

$$\operatorname{div}(J) - \Delta\theta = 0. \tag{43}$$

Now note that if θ and J are given by (42) then

$$G(-q + H(p,x)) + Jp + \theta q = \inf_{p,q} [G(-q + H(p,x)) + Jp + \theta q] = Z(J,\theta).$$

That is objective functional of dual problem is then

$$\int_{\mathbb{T}^d} Z(J,\theta)dx,$$

together with the constraint (43).

2.5.4 Some very special mean-field games and related problems

Benamou-Brenier reformulation of optimal transport The Monge optimal mass transport problem (see for instance [Vil03] and references therein) consists in moving a certain amount of mass with the least possible cost. More precisely, one is given two measures μ^+ and μ^- in $\mathbb{R}^d$ with finite q moment, that is

$$\int_{\mathbb{R}^d} |x|^q d\mu^\pm(x) < \infty,$$

which satisfy the mass balance condition

$$\int_{\mathbb{R}^d} d\mu^+ = \int_{\mathbb{R}^d} d\mu^-.$$

The Monge problem consists in finding a map $s : \mathbb{R}^d \to \mathbb{R}^d$ which transports μ^+ into μ^-, that is,

$$\int_{\mathbb{R}^d} \varphi(s(x))d\mu^+ = \int_{\mathbb{R}^d} \varphi(y)d\mu^-,$$

for all $\varphi \in C_c^\infty(\mathbb{R}^d)$, in other words $s^\#\mu^+ = \mu^-$, and minimizes the transportation cost

$$\int_{\mathbb{R}^d} |x - s(x)|^q d\mu^+(x).$$

Given a map s for which $s^\#\mu^+ = \mu^-$, one can define a measure μ in $\mathbb{R}^{2d}$ by

$$\int_{\mathbb{R}^{2d}} \phi(x,y)d\mu = \int_{\mathbb{R}^d} \phi(x,s(x))d\mu^+.$$

Furthermore, the marginals $\mu|_x = \mu^+$ and $\mu|_y = \mu^-$. The Monge Kantorowich problem is a relaxed version of Monge's problem, and consists in determining

$$W_p^p(\mu^-,\mu^+) = \min_{\mu\in\Pi(\mu^-,\mu^+)} \int_{\mathbb{R}^{2d}} |x-y|^q d\mu,$$

where $\Pi(\mu^-,\mu^+)$ is the set of all measures which satisfy

$$\int_{\mathbb{R}^{2d}} \varphi(x)d\mu = \int_{\mathbb{R}^d} \varphi(x)d\mu^+,$$

and

$$\int_{\mathbb{R}^{2d}} \varphi(y)d\mu = \int_{\mathbb{R}^d} \varphi(y)d\mu^-.$$

Whereas an optimal transport map s may fail to exist it is always possible to find a solution μ to the relaxed problem. The function W_q is called the q-Wasserstein distance as it is in fact a distance in the space of probability measures with finite q moment.

Benamou and Brenier [BB00] studied a PDE formulation of the mass transport problem which can be regarded as a very special mean-field game. In fact, given two measures $\mu^{\pm}$ with finite second moments one can look at all possible vector fields $b : \mathbb{R}^d \times [0,1] \to \mathbb{R}$ for which there is a solution of

$$\rho_t + \operatorname{div}(b\rho) = 0$$

satisfying $\rho(x,0) = \mu^-$, $\rho(x,1) = \mu^+$. The first observation is that

$$W_2^2(\mu^-, \mu^+) = \inf_b \int_0^1 \int_{\mathbb{R}^d} |b|^2 \rho(x,t) dx dt.$$

Furthermore, the optimality conditions for this problem can be written as $b(x,t) = -Du(x,t)$ where u is a solution of

$$-u_t + \frac{|Du|^2}{2} = 0.$$

Therefore we can rephrase the optimal transport problem as a mean-field game where the Hamilton-Jacobi equation does not depend on the measure:

$$\begin{cases} \rho_t - \operatorname{div}(\rho Du) = 0 \\ -u_t + \frac{|Du|^2}{2} = 0, \end{cases}$$

together with the planning initial-terminal conditions $\rho(x,0) = \mu^-$, $\rho(x,1) = \mu^+$.

Then elementary computations show that we can write the 2-Wasserstein distance as

$$W_2^2(\mu^-, \mu^+) = 2 \int u(x,0) d\mu^- - 2 \int u(x,T) d\mu^+.$$

Aubry-Mather problem The trajectories $\mathbf{x}(\cdot)$ of a system in classical mechanic are determined through a variational principle which asserts that they are critical points of the action:

$$\int_0^T L(\mathbf{x}, \dot{\mathbf{x}}) dt,$$

where L is the Lagrangian, which is the difference between kinetic and potential energy. A case of interest is the one in which $L(x,v) : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ is smooth, strictly convex in v, and $L \geq 0$.

An important issue is to understand the limit as $T \to \infty$. To study this problem, is useful to consider certain measures which encode the asymptotic behavior of minimizing curves. In fact, if $\mathbf{x} : \mathbb{R} \to \mathbb{T}^d$ is globally Lipschitz, one can associate to $\mathbf{x}$ a probability measure μ by

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} \phi(x,v) d\mu = \lim_{T \to \infty} \frac{1}{T} \int_0^T \phi(\mathbf{x}, \dot{\mathbf{x}}) dt,$$

where the limit is taken through an appropriate subsequence.

Since for all $\varphi(x) : \mathbb{T}^d \to \mathbb{R}$ we have

$$\int_0^T \dot{\mathbf{x}} D_x \varphi(\mathbf{x}) dt = O(1),$$

this implies

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} v D_x \varphi(x) d\mu = 0.$$

The **Mather's problem** is the minimization problem

$$\min \int_{\mathbb{T}^d \times \mathbb{R}^d} L d\mu, \tag{44}$$

where the minimum is taken over all positive probability measures μ in $\mathbb{T}^d \times \mathbb{R}^d$ that satisfy the holonomy constraint:

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} v D_x \varphi(x) d\mu = 0,$$

for all $\varphi(x) : \mathbb{T}^d \to \mathbb{R}$. Both the function to be minimized and the constraints are linear in μ.

It turns out that the value of the Mather problem can be characterized by the unique number $\overline{H}$ for which the Hamilton-Jacobi equation

$$H(x, Du) = \overline{H}$$

admits a periodic viscosity solution. In fact the minimum in (44) is $-\overline{H}$. Furthermore, at least formally, see for instance [EG01] or [BG10] for more precise statements, if one takes a probability measure solving, in the weak sense,

$$\operatorname{div}(\theta D_p H(x, Du)) = 0,$$

then the measure μ given by

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} \psi(x, v) d\mu = \int_{\mathbb{T}^d} \psi(x, -D_p H(x, Du)) d\theta$$

is a minimal measure for (44).

The previous observations suggest then that certain aspects of Mather theory can be studied by looking at the very special stationary mean-field game

$$\begin{cases} H(x, Du) = \overline{H} \\ \operatorname{div}(\theta D_p H) = 0. \end{cases}$$

A non-linear version of the Mather problem was studied in [GV07], and [GLM11]. This non-linear version is obtained through the addition of an entropy term to (44). A generalization of the Mather problem for stochastic processes was considered in [Gom02] and further extended in [Gom05].

2.5.5 Population models

Certain population models related to the Hughes model, see [DFMPW11, BDFMW13], can be written as a system of equations of the form

$$\begin{cases} -u_t + f(\theta)|Du|^2 = \Delta u + g(\theta) \\ \theta_t - \operatorname{div}(h(\theta) Du) = \Delta \theta, \end{cases} \tag{45}$$

for suitable functions $f, g, h : \mathbb{R}^+ \to \mathbb{R}$, together with initial-terminal conditions

$$\begin{cases} u(x, T) = u_0(x) \\ \theta(x, 0) = \theta_0(x). \end{cases} \tag{46}$$

By applying ideas similar to the ones in [GPSM13a, GPSM13b] one can obtain various a-priori estimates for these problems. Though these equations do not have the standard mean-field structure of a Hamilton-Jacobi equation coupled with the adjoint of its linearization, many mean-field games ideas and techniques can still be applied, as we sketch next.

Estimates for the Hamilton-Jacobi equation Suppose that g is bounded. Then by maximum principle we conclude that u is bounded by constants that depend only on the bounds for g and the terminal data. In particular, do not depend on f being positive, as in certain applications it is natural to consider the case in which f changes sign.

Let (u, θ) solve (45). By multiplying the first equation in (45) by θ and the second equation by u, subtracting and integrating by parts yields

$$-\frac{d}{dt}\int_{\mathbb{T}^d} u\theta + \int_{\mathbb{T}^d} (\theta f(\theta) - h(\theta))|Du|^2 = \int_{\mathbb{T}^d} \theta g(\theta). \tag{47}$$

From this we have

$$\int_0^T \int_{\mathbb{T}^d} (h(\theta) - \theta f(\theta))|Du|^2 \leq C. \tag{48}$$

Of course the previous inequality will only give an interesting estimate if

$$\kappa(\theta) \equiv h(\theta) - \theta f(\theta) \geq 0.$$

This is indeed the case in the application we have in mind.

Estimates for the transport equation We choose now a convex function $\varphi : \mathbb{R}^+ \to \mathbb{R}$ and multiply the second equation in (45) by $\varphi'(\theta)$. Integrating in $\mathbb{T}^d$ gives rise to the identity

$$\frac{d}{dt}\int_{\mathbb{T}^d} \varphi(\theta) + \int_{\mathbb{T}^d} \varphi''(\theta)h(\theta)D\theta \cdot Du = -\int_{\mathbb{T}^d} \varphi''(\theta)|D\theta|^2.$$

This implies the estimate

$$\frac{d}{dt}\int_{\mathbb{T}^d} \varphi(\theta) \leq \frac{C}{\lambda}\int_{\mathbb{T}^d} \kappa(\theta)|Du|^2 + \int_{\mathbb{T}^d} \left[\lambda \frac{(\varphi''(\theta)h(\theta))^2}{\kappa(\theta)} - \varphi''(\theta)\right] |D\theta|^2.$$

Hence

$$\int_0^T \int_{\mathbb{T}^d} \left[\varphi''(\theta) - \lambda \frac{(\varphi''(\theta)h(\theta))^2}{\kappa(\theta)}\right] |D\theta|^2 \leq \int_{\mathbb{T}^d} \varphi(\theta(y, 0)) - \varphi(\theta(y, T))dy + \frac{C}{\lambda}. \tag{49}$$

Again, this will yield a useful estimate provided for some choice of φ and $\lambda > 0$ we have

$$\zeta_\lambda(\theta) = \varphi''(\theta) - \lambda \frac{(\varphi''(\theta)h(\theta))^2}{\kappa(\theta)} \geq 0.$$

Application to crowded particles models In this model, see [BDFMW13], we have $f(\theta) = \frac{1}{2}(1 - \theta)(1 - 3\theta)$, $g = 1$, and $h = \theta(1 - \theta)^2$. For classical solutions, by maximum principle, if $0 < \theta < 1$ at $t = 0$ then the same is true for all $t > 0$. We have

$$\kappa(\theta) = \frac{\theta}{2}(1 - \theta^2),$$

which is non negative for $0 \leq \theta \leq 1$. Thus from (48) we conclude that

$$\int_0^T \int_{\mathbb{T}^d} \frac{\theta}{2}(1 - \theta^2)|Du|^2 \leq C.$$

For $\varphi(\theta) = \frac{\theta^2}{2}$ we have that

$$\frac{h^2(\theta)}{\kappa(\theta)} \leq C,$$

for $0 < \theta < 1$. Therefore, for λ small enough, taking into account that $0 < \theta < 1$ we obtain the following a-priori estimate

$$\int_0^T \int_{\mathbb{T}^d} |D\theta|^2 dxdt \leq C.$$

2.6 Random variables point of view

We discuss in this section the random variables point of view for deterministic mean-field games. This allows us, in the first order case, to reformulate (4) as a system of a Hamilton-Jacobi equation and a ordinary differential equation in a space of random variables. This formulation is very close to the one originally considered in [HMC06, HCM07]. The presentation here reflects also ideas discussed by P. L. Lions in [Lio11]. The random variables point of view is also convenient to the study extended mean-field games, where the costs incurred by a player depend not only on the positions of the other players but also on their actions [GV13]. A further application of this framework is the stochastic case where the players have a common noise. The latter problem will be briefly discussed in section 3. See also the recent papers [NH12], [CD13b, CD13a, CL13, CDL13] where these problems are addressed using backward-forward stochastic differential equations. This is approach is also natural to address the limit as the number of players N tends to infinity using the interaction particle framework. Such limit is a fundamental problem also in statistical physics, see for instance [Szn91] and references therein.

2.6.1 Random variables

Let $(\Omega, \mathcal{F}, P)$ be a probability space, where Ω is a non-empty set, $\mathcal{F}$ a σ-algebra on Ω and P a probability measure. As usual in probability theory we denote integration with respect to P by the expected value, that is, for any P-integrable real valued function φ we set

$$E\varphi \equiv \int_\Omega \varphi dP.$$

A $\mathbb{R}^d$ valued random variable X is a measurable map $X : \Omega \to \mathbb{R}^d$. For definiteness, we will consider random variables which are L^p integrable. The law of a $\mathbb{R}^d$ valued random variable is the probability measure $\mathcal{L}(X)$ in $\mathbb{R}^d$ defined by

$$\int_{\mathbb{R}^d} \phi d\mathcal{L}(X) = E\phi(X),$$

for any bounded continuous function $\phi : \mathbb{R}^d \to \mathbb{R}$. We say that a function $\Psi : L^p(\Omega) \to \mathbb{R}$ depends only on the law of a random variable if for any pair of random variables $X, Y \in L^p(\Omega)$ such that $\mathcal{L}(X) = \mathcal{L}(Y)$ we have $\Psi(X) = \Psi(Y)$.

Let $\eta : \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$, $\eta(\theta)$. We define a function, $\tilde{\eta} : L^p(\Omega; \mathbb{R}^d) \to \mathbb{R}$ which depends only on the law of the argument, by

$$\tilde{\eta}(X) = \eta(\mathcal{L}(X)).$$

This allows us to identify functions in $\mathcal{P}(\Omega)$ with functions in $L^p(\Omega)$ which depend only on the law.

2.6.2 Dynamics

We regard the set Ω as the collection of all players. We will consider a time dependent family of random variables $\mathbf{X} : \Omega \times [t, T] \to \mathbb{R}^d$. If $\omega \in \Omega$, we interpret $\mathbf{X}_s(\omega)$ as the position of the player ω at time s. For the moment we suppose we are given a vector field determined by a function

$$B : \mathbb{R}^d \times L^p(\Omega) \times [t, T] \to \mathbb{R}^d,$$

depending only on the law on the second coordinate. We suppose the players in Ω follow the deterministic trajectory

$$\dot{\mathbf{X}}_s(\omega) = B(\mathbf{X}_s(\omega), \mathbf{X}_s, s).$$

We observe that for our purposes it is important to distinguish between the dependence of B on the position of a player ω at $\mathbf{X}_s(\omega)$ and the law of the random variable $\mathbf{X}_s$. We consider now a reference player, which has a dynamic

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{v}),$$

where f is as in Section 2.1, and $\mathbf{v} : [t, T] \to U$ is the control of this reference player. As before, we denote by $\mathcal{U}$ the set of bounded controls on $[t, T]$ with values in U.

In this new setting, the Lagrangian is a function $L : \mathbb{R}^d \times L^p(\Omega) \times U \to \mathbb{R}$, that we denote by $L(x, X, v)$, which in the second coordinate depends only on the law. An example of such a Lagrangian is

$$L(x, X, v) = \frac{|v|^2}{2} - EW(x, X),$$

where $W : \mathbb{R}^d \times \mathbb{R}^p(\Omega) \to \mathbb{R}$ is, for instance, a bounded Lipschitz function. The terminal cost is given by a function $\psi : \mathbb{R}^d \times L^p(\Omega) \to \mathbb{R}$ which depends only on the law of the second coordinate.

The objective of the reference player is to minimize

$$V(x, t) = \inf_{\mathbf{v} \in \mathcal{U}} \int_t^T L(\mathbf{x}, \mathbf{X}_s, \mathbf{v}) ds + \psi(\mathbf{x}(T), \mathbf{X}_T).$$

As before, for $(x, X, p) \in \mathbb{R}^d \times L^p(\Omega) \times \mathbb{R}^d$, the Hamiltonian is given by

$$H(x, X, p) = \sup_{v \in U} \left[-f(x, v) \cdot p - L(x, X, v) \right].$$

The Hamiltonian $H : \mathbb{R}^d \times L^p(\Omega) \times \mathbb{R}^d \to \mathbb{R}$, depends only on the law of the second coordinate.

Then, from standard viscosity solution methods, V is the unique viscosity solution of the Hamilton-Jacobi equation

$$-V_t(x, t) + H(x, \mathbf{X}_t, D_x V(x, t)) = 0$$

with the terminal condition $V(x, T) = \psi(x, \mathbf{X}_T)$. As before, the optimal feedback strategy for the reference player yields the dynamics

$$\dot{\mathbf{x}} = -D_p H(\mathbf{x}, \mathbf{X}_t, D_x V(\mathbf{x}, t)).$$

We assume at this stage that each of the players is faced with the same optimization problem. Thus they all have a similar strategy and consequently

$$B(x, X, t) = -D_p H(x, X, D_x V(x, t)).$$

Hence, for $\omega \in \Omega$

$$\dot{\mathbf{X}}_s(\omega) = -D_p H(\mathbf{X}_s(\omega), \mathbf{X}_s, D_x V(\mathbf{X}_s(\omega), s)).$$

Therefore the mean-field equations can be written as

$$\begin{cases} -V_t + H(x, \mathbf{X}, D_x V) = 0 \\ \dot{\mathbf{X}}_s(\omega) = -D_p H(\mathbf{X}_s(\omega), \mathbf{X}_s, D_x V(\mathbf{X}_s(\omega), s)), \end{cases}$$

with the initial-terminal condition

$$\begin{cases} V(x, T) = \psi(x, \mathbf{X}_T) \\ \mathbf{X}_0 = X_0. \end{cases}$$

2.7 Extended mean-field models

We now consider extended mean-field models. These differ from the ones discussed in the previous section because the cost function depends not only on the state of the other players, but also on their strategies. Here, as before, we will formulate the problem using the random variables point of view as in [GV13].

2.7.1 Model set up

We consider the same set up as in the previous subsection, except that the Lagrangian now depends also on the other players actions. More precisely we consider a Lagrangian $L : \mathbb{R}^d \times \mathbb{R}^d \times L^q(\Omega) \times L^q(\Omega) \to \mathbb{R}$. We further assume that the $L(x, v, X, Z)$ depends only on the joint law of $(X, Z) \in L^q(\Omega) \times L^q(\Omega)$. The players positions are determined by a random variable X, and its velocities by the random variable Z.

We consider a reference player and assume as before that the dynamics of the remaining players is described by a differentiable trajectory $\mathbf{X} : [t, T] \to L^q(\Omega)$, $\dot{\mathbf{X}}_s(\omega) = B(\mathbf{X}_s(\omega), \mathbf{X}_s, s)$, with B fixed for now and known by the reference player. This player, which we assume to be at time t in the state $x \in \mathbb{R}^d$, faces the following optimal control problem:

$$V(x, t) = \inf_{\mathbf{v} \in \mathcal{U}} \int_t^T L(\mathbf{x}, \mathbf{v}, \mathbf{X}_s, B) ds + \Psi(\mathbf{x}(T), \mathbf{X}(T)).$$

The Hamiltonian $H : \mathbb{R}^d \times L^q(\Omega) \times \mathbb{R}^d \times L^q(\Omega) \to \mathbb{R}$, is given by

$$H(x, X, p, Z) = \sup_{v \in U} \left[-f(x, v) \cdot p - L(x, X, v, Z) \right],$$

and depends only on the joint law of $(X, Z) \in L^q(\Omega) \times L^q(\Omega)$. Assuming enough regularity on the value function, V is the unique viscosity solution of the following Hamilton-Jacobi equation

$$-V_t(x, t) + H(x, \mathbf{X}_t, D_x V(x, t), \dot{\mathbf{X}}) = 0.$$

As before, all players act rationally, therefore follow optimal trajectories. Then the dynamics, for all players $\omega \in \Omega$, is

$$\dot{\mathbf{X}}_s(\omega) = -D_p H(\mathbf{X}_s(\omega), \dot{\mathbf{X}}_s, D_x V(\mathbf{X}_s(\omega), s), \dot{\mathbf{X}}_s).$$

Henceforth the mean-field equations are

$$\begin{cases} -V_t(x, t) + H(x, \mathbf{X}, D_x V(x, t), \dot{\mathbf{X}}) = 0 \\ \dot{\mathbf{X}}_s(\omega) = -D_p H(\mathbf{X}_s(\omega), \mathbf{X}_s, D_x V(\mathbf{X}_s(\omega), s), \dot{\mathbf{X}}_s), \end{cases} \tag{50}$$

with the following initial-terminal condition

$$\begin{cases} V(x, T) = \psi(x, \mathbf{X}(T)) \\ \mathbf{X}(0) = X_0. \end{cases} \tag{51}$$

2.7.2 Existence

Following [GV13], we address now existence and uniqueness of solutions of the extended mean-field games, (50) with initial-terminal conditions (51). Consider that the following conditions, for $1 \leq q < \infty$ and a Lipschitz bounded function ψ, hold

1. For all $x \in \mathbb{R}^d$, $X, Z \in L^q(\Omega)$, the Lagrangian $L(x, v, X, Z)$ is strictly convex in v and satisfies the coercivity condition

$$\lim_{|v| \to \infty} \frac{L(x, v, X, Z)}{|v|} \to \infty,$$

uniformly in x.

2. $L(x, v, X, Z) > -c_0 E[|X|^q + |Z|^q + 1]$.

3. For all $X, Z \in L^q(\Omega)$ there exists a continuous function $v_0 : L^q(\Omega) \times L^q(\Omega) \to \mathbb{R}^d$ such that $L(x, v_0(X, Z), X, Z) \leq c_1$.
4. $|D_v L|, |D^2_{vv}|L \leq (c_2 L + c_3)E[|X|^q + |Z|^q + 1]$, and $|D_x L|, |D^2_{x,v} L|, |D^2_{xx}| \leq c_2 L + c_3$.
5. $D_x H$ is Lipschitz in $\mathbb{R}^d \times \mathbb{R}^d \times L^q(\Omega) \times L^q(\Omega)$, where $H = L^*$.
6. For any $X, Y, P \in L^q(\Omega)$ the equation $Z = -D_p H(X, P, Y, Z)$ can be solved with respect to Z as
$$Z = G(X, P, Y),$$
where $G : L^q(\Omega) \times L^q(\Omega) \times L^q(\Omega) \to L^q(\Omega)$ is a Lipschitz map.
7. The Hamiltonian H is continuous with respect to X, Z, and locally uniformly in x, p.
8. For any $R > 0$ there exists a constant $C(R)$ such that
$$|H(x, p, X, Z) - H(y, q, Y, W)| \leq C(R)\left(|x - y| + |p - q| + E[X - Y] + E[Z - W]\right),$$
for $|x|, |y|, |p|, |q|, \|X\|_q, \|Y\|_q, \|Z\|_q, \|W\|_q \leq R$.

Theorem 11. *Let the above conditions on L and ψ hold. Suppose X_0 has an absolutely continuous law. Then there exists a solution $(V, X) \in \mathbb{R}^d \times C^{1,1}([0,T] \times L^q(\Omega))$ of the extended mean field game* (50) *with initial-terminal condition* (51). *Furthermore V is a semiconcave and Lipschitz continuous function.*

Proof. The proof in [GV13] uses a fixed point argument that we sketch now, and is divided into the following main steps.

Expanded dynamics Let u be a Lipschitz function. Using the assumption 6. we consider the following system of ODEs in $L^q(\Omega)$,
$$\begin{cases} \dot{X}_s(\omega) = G(P_s(\omega), X_s(\omega), X_s) \\ \dot{P}_s(\omega) = D_x H(P_s(\omega), X_s(\omega), X_s, \dot{X}_s) \\ X_0 = X_0, \quad P_0 = D_x u(X_0). \end{cases} \tag{52}$$
By the assumptions on the Lagrangian L, and on the terminal cost Ψ, the value function u is Lipschitz continuous. Therefore by the Rademacher's theorem $D_x u$ exists a.e.. Furthermore since $\mathcal{L}(X_0)$ is supposed to be absolutely continuous P_0 is well defined. By standard arguments the Lipschitz condition on G and $D_x H$ implies uniqueness of solutions (X, P) for the above system (52).

Optimal control problem Given a solution to (52), we consider the following optimal control problem,
$$\tilde{V}(x,t) = \inf_{\mathbf{x}} \int_t^T L(\mathbf{x}, \dot{\mathbf{x}}, \mathbf{X}_s, \dot{\mathbf{X}}_s) ds + \psi(\mathbf{x}(T), \mathbf{X}_T) \tag{53}$$
where the take the infimum over all absolutely continuous trajectories $\dot{\mathbf{x}}(s)$ with $\dot{\mathbf{x}}(t) = x$.

Lemma 1. *The value function $\tilde{V}(x,t)$ is Lipschitz continuous and semi-concave. Therefore the following conditions hold*

1. $\tilde{V} \leq (T - t)c_1 + \|\psi\|_\infty \quad \forall x \in \mathbb{R}^d, 0 \leq t \leq T.$
2. $|\tilde{V}(x + y, t) - \tilde{V}(x, t)| \leq c_6 |y| \quad \forall x, y, \in \mathbb{R}^d, 0 \leq t \leq T.$
3. $\tilde{V}(x + y, t) - \tilde{V}(x - y, t) - 2\tilde{u}(x, t) \leq c_7 |y|^2 \quad \forall x, y \in \mathbb{R}^d, 0 \leq t \leq T.$

With constants c_6, c_7 depending only on L, ψ and T.

Fixed point We consider the following map, for a Lipschitz function u we associate a trajectory $\mathbf{X}$ by solving (52). Then, we compute the solution $\tilde{V}$ to (53). Denote by Ψ the map $\Psi(u)(x) = \tilde{V}(x, 0)$.

Lemma 2. *Let $\mathcal{A}$ be the set of functions $u \in C^0(\mathbb{R}^d)$ with $|u| \leq c_8$, $Lip(u) \leq c_6$, and semi-concave with constant c_7. Then the mapping Ψ is a continuous and compact mapping from $\mathcal{A}$ into itself.*

Once this lemma is established the existence of fixed point of Ψ follows from Browder's fixed point theorem. Then the argument ends by proving that this fixed point satisfies the mean-field equations (50).

□

2.7.3 Uniqueness

We now address the uniqueness of solutions to extended mean-field games (50). The key technique to prove uniqueness is based upon the Lions-Lasry monotonicity method. In the setting of random variables these monotonicity conditions can be formulated either in terms of the Hamiltonian, as in the original Lions-Lasry argument, or in terms of the Lagrangian (as considered before in section 2.3.2). We discuss both approaches here following [GV13].

Lasry-Lions monotonicity argument Recall that to prove uniqueness of the mean field game (4) the strategy is the following: suppose (V, θ) and $(\bar{V}, \bar{\theta})$ are solutions satisfying (5). Then uniqueness, as explained in section 2.3, follows from considering the quantity

$$\frac{d}{dt} \int (V - \bar{V})(\theta - \bar{\theta}), \tag{54}$$

which together with appropriate monotonicity assumptions on H and the terminal condition yields a contradiction.

The analog idea in the random variable setting is the following: suppose that (V, X), and $(\bar{V}, \bar{X})$ solve the extended mean-field equations (50) with initial-terminal condition (51). Then (54), in this setting, becomes

$$\frac{d}{dt} E\left[V(X,t) - \bar{V}(X,t) + \bar{V}(\bar{X},t) - V(\bar{X},t)\right].$$

In order to illustrate this technique we give a simple example which is also presented in [GV13]. Let the following conditions hold:

1. The Hamiltonian is given by

$$H(x, p, X, Z) = H_0(x, p + \beta E[Z]) + F(x, X),$$

   where H_0 is convex in p, and $\beta \geq 0$.

2. The following monotonicity condition holds true

$$E\left[F(X,X) - F(X,\bar{X}) + F(\bar{X},\bar{X}) - F(\bar{X},X)\right] < 0,$$

   for $X \neq \bar{X}$.

3. The terminal condition ψ satisfies

$$E\left[\psi(X,X) - \psi(X,\bar{X}) + \psi(\bar{X},\bar{X}) - \psi(\bar{X},X)\right] \geq 0.$$

Then we have the following result:

Theorem 12. *Let the Hamiltonian in* (50) *be given by 1. above. Assume further that the conditions 2. and 3. hold. Then uniqueness of (classical) solutions to the extended mean-field game* (50) *with initial-terminal condition* (51) *holds.*

Proof. Suppose that (V, X) and $(\bar{V}, \bar{X})$ are two distinct solutions of (50) satisfying the initial-terminal condition (51). By using the various assumptions we obtain

$$\frac{d}{dt}E\left[(V-\bar{V})(X,t)+(\bar{V}-V)(\bar{X},t)\right]<0. \tag{55}$$

However, by routine computations and using the monotonicity hypothesis we obtain the opposite inequality. Therefore (55) must be a contradiction, henceforth proving the uniqueness result. $\square$

A Lagrangian approach We present now a uniqueness result in terms of monotonicity conditions for the Lagrangian, as presented in [GV13]. Suppose that the Lagrangian function satisfy the following monotonicity condition

$$E\left[L(X,Z,X,Z)-L(\tilde{X},\tilde{Z},X,Z)+L(\tilde{X},\tilde{Z},\tilde{X},\tilde{Z})-L(X,Z,\tilde{X},\tilde{Z})\right]>0, \tag{56}$$

provided that $X\neq\tilde{X}$ or $Z\neq\tilde{Z}$, where $X,\tilde{X},Z,\tilde{Z}\in L^q(\Omega)$.

Remark 1. *As shown in [GV13], provided that L is strictly convex the above monotonicity condition is equivalent to the following differential one*

$$E\left[Z^TD^2_{vZ}LZ+X^TD^2_{xX}LX+Z^TD^2_{vX}LY+Y^TD^2_{xZ}LZ\right]>0,$$

where the the Lagrangian is evaluated at an arbitrary point $(X_1,Z_1,X_2,Z_2)\in(L^q(\Omega))^4$.

We suppose further that the terminal cost function satisfies

$$E\left[\psi(X,X)-\psi(X,\tilde{X})+\psi(\tilde{X},\tilde{X})-\psi(\tilde{X},X)\right]\geq 0. \tag{57}$$

Theorem 13. *Assume that* (56) *and* (57) *hold, then there exists a unique solution to* (50).

Proof. Suppose that (X,V) and $(\tilde{X},\tilde{V})$ are two solutions of (50). Henceforth, for a.e. $\omega\in\Omega$ we have that X, and $\tilde{X}$ are minimizers of optimal control problems for which the value functions are, respectively, given by

$$V(X(0),0)=\int_0^T L(X(s),\dot{X}(s),X(s),\dot{X}(s))ds+\psi(X(T),X(T)), \tag{58}$$

and

$$\tilde{V}(\tilde{X}(0),0)=\int_0^T L(\tilde{X}(s),\dot{\tilde{X}}(s),\tilde{X}(s),\dot{\tilde{X}}(s))ds+\psi(\tilde{X}(T),\tilde{X}(T)). \tag{59}$$

Furthermore we have

$$V(\tilde{X}(0),0)\leq\int_0^T L(\tilde{X}(s),\dot{\tilde{X}}(s),X(s),\dot{X}(s))ds+\psi(\tilde{X}(T),X(T)), \tag{60}$$

$$\tilde{V}(X(0),0)\leq\int_0^T L(X(s),\dot{X}(s),\tilde{X}(s),\dot{\tilde{X}}(s))ds+\psi(X(T),\tilde{X}(T)). \tag{61}$$

Combining the previous inequalities we easily obtain a contradiction to our assumptions (56) and (57). This implies that $X=\tilde{X}$. The identity $V=\tilde{V}$ then follows from the uniqueness of viscosity solutions. $\square$

3 Mean-field models in master form

In this section we discuss a more general formulation for mean-field games, called the master equation. These ideas were introduced by Lions in [Lio11]. Here we focus particularly in the random variables point of view and address both the deterministic and stochastic correlated cases, where the players are subject to a common Brownian motion.

3.1 Deterministic models

We now consider deterministic mean-field games and we derive the master form setting. To do so, we will use the notation and hypotheses from section 2.6.

We start by looking at the optimal control problem

$$V(x, X, t) = \inf_{\mathbf{v}} \left[\int_t^T L(\mathbf{v}(s), \mathbf{x}(s), \mathbf{X}_s) ds + \psi(\mathbf{x}(T), \mathbf{X}_T) \right], \tag{62}$$

where $\mathbf{x}$ is the trajectory of a player which starts at time t at point $\mathbf{x}(t) = x$, and is controlled by $\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{v})$, and $\mathbf{X}_s(\omega)$ is the trajectory of the population of the players which move along a vector field $B : \mathbb{R}^d \times L^q(\Omega; \mathbb{R}^d) \times [0, T] \to L^q(\Omega; \mathbb{R}^d)$,

$$\dot{\mathbf{X}}_s = B(\mathbf{X}_s(\omega), \mathbf{X}_s, s), \qquad \mathbf{X}(t) = X,$$

as previously. The key difference here is that we are considering the value function as a function of both x and X. Then it, at least formally, that the value function V for (62) is a viscosity solution of the equation (see for instance [GN13a, GN13b, GN12], where infinite dimensional optimal control in the space of random variables are considered and the viscosity solution property is established rigorously for related problems).

$$-V_t(x, X, t) - D_X V(x, X, t) \cdot B(X, X, t) + H(x, X, D_x V(x, X, t)) = 0, \tag{63}$$

where H is as before, and $D_X V$ denotes the Frechét derivative with respect to the random variable X.

Furthermore, if V is a smooth enough solution to (63) then any optimal control $\bar{v}$ for (62) satisfies $f(x, \bar{v}(x, X, t)) = -D_p H(x, X, D_x V(x, X, t))$. Since we assume all players act rationally they will all follow the optimal flow. This then yields $B(x, X, t) = -D_p H(x, X, D_x V(x, X, t))$. Thus we arrive to the equation

$$-V_t(x, X, t) + D_X V(x, X, t) \cdot D_p H(X, X, D_x V(X, X, t)) + H(x, X, D_x V(x, X, t)) = 0, \tag{64}$$

with terminal condition $V(x, X, T) = \psi(x, X)$. Equation (64) is called the master equation.

Though a general theory for this class of equations is still lacking, it is possible to establish some basic a-priori estimates for this equation. Namely, suppose we assume

1. ψ is bounded and is Lipschitz in x:

$$|\psi(x_1, X) - \psi(x_2, X)| \leq C|x_1 - x_2|, \quad \forall x_1, x_2 \in \mathbb{R}^d.$$

2. There exist constants $c_0, c_1 > 0$ such that $L(v, x, X) \geq -c_0$ and $L(0, x, X) \leq c_1$ for all $x, v \in \mathbb{R}^d, X \in L^2(\Omega; \mathbb{R}^d)$.

3. L is twice differentiable in x, v and we have the following bounds

$$|D_x L(v, x, X)|, |D^2_{xx} L(v, x, X)|, |D^2_{xv} L(v, x, X)|, |D^2_{vv} L(v, x, X)| \leq C.$$

for all $x, v \in \mathbb{R}^d, X \in L^2(\Omega; \mathbb{R}^d)$.

We have then the following result from [GV13]:

Theorem 14. *Assume that 1-3 hold. Then the function V defined in* (62) *for a fixed vector field B is finite, bounded, Lipschitz and semiconcave in x:*

1. $$|V(x,Y,t)| \leq C, \quad \forall x,h \in \mathbb{R}^d, Y \in L^2(\Omega;\mathbb{R}^d).$$

2. $$|V(x+h,Y,t) - V(x,Y,t)| \leq C|h|, \quad \forall x,h \in \mathbb{R}^d, Y \in L^2(\Omega;\mathbb{R}^d).$$

3. $$V(x+h,Y,t) + V(x-h,Y,t) - 2V(x,Y,t) \leq C|h|^2, \quad \forall x,h \in \mathbb{R}^d, Y \in L^2(\Omega;\mathbb{R}^d).$$

3.2 Correlations

One important applications of master form framework concerns the case where agents are subject to a common noise such as being driven by a common Brownian Motion. Let $(\Omega\times\Omega', \mathcal{F}\times\mathcal{F}', \mathbb{P}\times\mathbb{P}')$ be a probability space, where Ω is the events space, Ω' is the set of all players, $\mathcal{F}\times\mathcal{F}'$ the product σ-algebra on $\Omega\times\Omega'$, and $\mathbb{P}$, and $\mathbb{P}'$ probability measures. As in Subsection 2.6.2 we consider a $L^p(\Omega\times\Omega')$-integrable, time dependent family of random variables $\mathbf{X}: \Omega\times\Omega'\times[0,T] \to \mathbb{R}^d$. We interpret $\mathbf{X}_t(\omega,\omega')$ as the ω realization of the position of player ω' at time t. Let $W_t: [0,T]\times\Omega \to \mathbb{R}^d$ be a Brownian motion. In order to simplify the notation we may write, in the sequel, $\mathbf{X}$, or $\mathbf{X}(\omega')$ instead of $\mathbf{X}_t(\omega,\omega')$. Given a vector field $B: \mathbb{R}^d\times L^p(\Omega') \to \mathbb{R}^d$, and supposing the players follow the random trajectories

$$d\mathbf{X}(\omega') = B(\mathbf{X}(\omega'),\mathbf{X})dt + \sigma(\mathbf{X}(\omega'),\mathbf{v})dW_t, \quad \forall \omega' \in \Omega', \tag{65}$$

where W_t is a Brownian motion, and σ is as defined in the Subsection 2.1. As before, we consider a reference player with dynamics given by

$$d\mathbf{x} = f(\mathbf{x},\mathbf{v})dt + \sigma(\mathbf{x},\mathbf{v})dW_t. \tag{66}$$

We now consider the Lagrangian $L: \mathbb{R}^d\times\mathbb{R}^m\times L^p(\Omega') \to \mathbb{R}$, along with a terminal cost $\psi: \mathbb{R}^d\times L^p(\Omega') \to \mathbb{R}$. Each player aims to minimize

$$V(x,X,t) = \inf_{\mathbf{v}\in\mathcal{U}} E\left[\int_t^T L(\mathbf{x},\mathbf{v},\mathbf{X})ds + \psi(\mathbf{x}_T,\mathbf{X}_T)\right],$$

where the expectation is taken with respect to the probability measure $\mathbb{P}$ in Ω. We define the Hamiltonian $H: \mathbb{R}^d\times L^p(\Omega')\times\mathbb{R}^d \to \mathbb{R}$ as

$$H(x,p,X) = \sup_{v\in U}\left[-f(x,v)\cdot p - L(x,v,X)\right].$$

We we now define certain directional derivatives of functions of random variables. These are the directional derivatives along constant directions on $\mathbb{R}^d$ and play an important role in problems with correlations. Let e_i be the i-th standard coordinate unit vector in $\mathbb{R}^d$. We define the directional first derivative operator as

$$\delta_i V(x,X,t) = \lim_{\varepsilon\to 0}\frac{V(x,X+\varepsilon e_i,t) - V(x,X,t)}{\varepsilon},$$

and the second derivative operator as

$$\delta_i^2 V(x,X,t) = \frac{d^2}{d\varepsilon^2}V(x,X+\varepsilon e_i,t)\Big|_{\varepsilon=0}.$$

Assuming the Dynamic Programming Principle, that $V \in C^{2,2,1}(\mathbb{R}^d \times \mathbb{R}^d \times [0,T])$, and that σ is a constant scalar, we find that V satisfies the following PDE,

$$-V_t(x,X,t) + H\left(x, D_xV(x,X,t), X\right) - B(X(\omega'),X)D_XV(x,X,t) - \frac{\sigma^2}{2}\sum_i^d \delta_i^2 V(x,X,t)$$
$$-\sigma^2\sum_i^d \delta_i D_{x_i}V(x,X,t) - \frac{\sigma^2}{2}\Delta_x V(x,X,t) = 0,$$

with a terminal condition $V(\mathbf{x}_T,\mathbf{X}_T,T) = \psi(\mathbf{x}_T,\mathbf{X}_T)$. Now we assume that all players are rational, which means that when faced with the problem of minimizing their cost functions, all they will use the same optimal strategy. Therefore the feedback strategy followed by the players is

$$B(\mathbf{X}_t(\omega'),\mathbf{X}_t) = -D_pH\left(\mathbf{X}_t(\omega'), D_XV(\mathbf{X}_t(\omega'),\mathbf{X}_t,t),\mathbf{X}_t\right).$$

So the dynamics of the players will be given by

$$d\mathbf{X}_s(\omega') = -D_pH\left(\mathbf{X}_t(\omega'), D_XV(\mathbf{X}_t(\omega'),\mathbf{X}_t,t),\mathbf{X}_t\right)dt + \sigma dW_t.$$

Consequently, the mean-field equations are given by

$$\begin{cases} -V_t + H(x,D_xV,X) - B(X,X)D_XV - \frac{\sigma^2}{2}\sum_i^d\left(\delta_i^2V + 2\delta_iD_{x_i}V + D^2_{x_ix_i}V\right) = 0 \\ B(x,X) = -D_pH\left(x, D_xV(x,X,t),X\right), \end{cases} \tag{67}$$

where the value function V is evaluated at (x,X,t). Furthermore we have the following terminal condition

$$V(x,X,T) = \psi(x,X).$$

The first equation in (67) is called *Master Equation*. As before, a solution of (67) is understood to be a viscosity solution of the first equation, with B fixed, coupled with the second equation, which determines B. Again this is a fixed point problem rather than a single PDE. As in the deterministic case one can prove various partial a-priori regularity results as in Theorem 14 (see [GV13]).

3.3 Extended models

In this section we discuss an extended version of the mean-field games with correlations. Here we look at the case where the Lagrangian depends not only on the state of other players but also on the actions they take. We then present a price-formation model using this set up.

As before we suppose that the players follow the dynamics given by (65). And we consider a reference player which follows (66) and aims to minimizing a cost function. In this extended setting we suppose that the Lagrangian function, $L : \mathbb{R}^d \times \mathbb{R}^m \times L^p(\Omega') \times L^p(\Omega') \to \mathbb{R}$, depends also on the actions taken by the other players. Therefore the value function for the reference player is given by

$$V(x,X,t) = \inf_{\mathbf{v}} E\left[\int_t^T L(\mathbf{x},\mathbf{v},\mathbf{X},B)ds + \psi(\mathbf{x}_T,\mathbf{X}_T)\right],$$

where, as before, $\psi : \mathbb{R}^d \times L^p(\Omega') \to \mathbb{R}$ is a terminal cost, and the expectation is taken with respect to the probability measure $\mathbb{P}$ in Ω. The Hamiltonian, $H : \mathbb{R}^d \times \mathbb{R}^m \times L^p(\Omega) \times L^p(\Omega) \to \mathbb{R}$ is now given by

$$H(x,p,X,Z) = \sup_{v\in U}\left[-f(x,v)\cdot p - L(x,v,X,Z)\right].$$

Note that both the Lagrangian and the Hamiltonian are functions depending only on the joint law of $X, Z \in L^p(\Omega)$.

By standard arguments, the value function V is a viscosity solution of the following PDE

$$-V_t(x,X,t) + H\left(x, D_xV(x,X,t), X, B(X,X,t)\right) - D_XV(x,X,t)\cdot B(X,X,t) - \frac{\sigma^2}{2}\sum_i \delta_i^2 V(x,X,t) \\ -\sigma^2 \sum_i \delta_i D_{x_i}V(x,X,t) - \frac{\sigma^2}{2}\Delta_x V(x,X,t) = 0. \tag{68}$$

Provided V is smooth enough, and for a fixed B, then the optimal control v^* satisfies

$$f(x,v^*) = -D_pH(x, D_xV(x,X,t), X, B).$$

By our assumptions of indistinguishability and rationality of players, every player $\omega' \in \Omega'$ will follow the optimal flow given by

$$B(X(\omega'),X,t) = -D_pH\left(X(\omega'), X, D_xV(X(\omega'),X,t), X, B(X(\omega'),X,t)\right).$$

So plugging the previous optimal flow in the equation (68), we obtain the *Master Equation*

$$\begin{aligned}-V_t(x,X,t) &+ H\left(x, D_xV(x,X,t), X, B(x,X,t)\right) \\ &- D_XV(x,X,t)\cdot D_pH\left(x, D_xV(x,X,t), X, B(x,X,t)\right) \\ &- \frac{\sigma^2}{2}\sum_i \delta_i^2 V(x,X,t) - \sigma^2\sum_i \delta_i D_{x_i}V(x,X,t) - \frac{\sigma^2}{2}\Delta_x V(x,X,t) = 0,\end{aligned}$$

where V satisfies the terminal condition $V(x,X,T) = \psi(x,X)$, where D_XV is the Fréchet derivative of V and δ_iV and δ_i^2V are defined as previously in subsection 3.2.

3.3.1 A price impact model

As an application of the extended formulation we present a modified Merton's portfolio problem where we consider that assets' transactions influence their prices. We formulate the problem for a large number of traders, each one aiming to maximize it's own reward function, while taking the point of view of a reference player. This formulation fits in the previously considered master form of mean-field games. We will continue using the random variables point of view.

Merton's portfolio problem We consider a financial market with two assets, a risk-free asset, *bond* B_t, and a risky asset, *stock* S_t. The dynamics of these variables is given by

$$\begin{cases} dB_t = rB_t dt \\ dS_t = \mu S_t dt + \sigma S_t dW_t, \end{cases}$$

where W_t is the Brownian motion, $r \in (0,\infty)$ is the interest rate, $\mu \in \mathbb{R}$ the drift, and $\sigma \in \mathbb{R}$ the volatility of the stock.

The Merton's problem is to decide what portion of the wealth should be allocated to bonds or stocks, in order to maximize a given reward function. We now consider the Merton's problem for a large number of players. We formulate the problem using the mean-field models in master form discussed in the previous sections. The various players states and actions are encoded by random variables $\mathbf{X}, \mathbf{Y}, \mathbf{C}, \mathbf{L} : \Omega \times \Omega' \times [0,T] \to \mathbb{R}$. At time t the players have an amount of money $\mathbf{X}(t)$ invested in bonds, and an amount $\mathbf{Y}(t)$ in stocks. The players are allowed to consume their wealth, which amounts to withdraw a $\mathbf{C}(t)$ amount from the money invested in bonds. They also can re-allocate their investments by selling a money amount $\mathbf{L}(t)$ of stocks in order to buy bonds. As before we also consider a reference player which allocates an amount $\mathbf{x}(t)$ of his wealth in bonds, and $\mathbf{y}(t)$ in stocks, at a given moment t. This player is allowed to consume an amount $\mathbf{c}(t) \geq 0$ and to change the amount investing in stocks by selling an amount $\mathbf{l}(t)$ (either positive or negative) of stocks in order to buying bonds.

Dynamics of the reference player The reference player has the following dynamics

$$\begin{cases} d\mathbf{x} & = r\mathbf{x}dt + \mathbf{l}dt - \alpha \mathbf{l}E'[\mathbf{L}]dt - \mathbf{c}dt \\ d\mathbf{y} & = \mu \mathbf{y}dt + \sigma \mathbf{y}dW_t - \mathbf{l}dt. \end{cases}$$

where $r, \mu \in \mathbb{R}$ are, respectively, the interest rate and the drift values as before, and $\alpha \geq 0$ is an impact factor of the selling/buying process, and the expectation E' is taken with respect to the probability measure $\mathbb{P}'$ in Ω', that is

$$E'(Z) = \int_{\Omega'} Z(\omega, \omega')d\mathbb{P}'(\omega').$$

The term $-\alpha \mathbf{l}E'[\mathbf{L}]dt$ encodes the price impact cause by a non-equilibrium situation when the sellers are not matched by buyers. In the case, when $\mathbf{l}(t) > 0$, if the expected value of other players' actions is also positive $E[\mathbf{L}'] > 0$ this means that as a whole there are more shares being sold than bought. Therefore it this will adversely affect a player trying to sell. So a player sells what before was valued as a $\mathbf{l}(t)$ amount, and gets instead $\mathbf{l}(t) - \alpha\mathbf{l}(t)E'[\mathbf{L}]$.. In the case where a player $\omega' \in \Omega'$ acts in an opposite direction as the the population of players' average, the impact on the wealth is positive. So, for instance if the player buys $\mathbf{l}(t) < 0$ amount worth of stocks and while the players on average are selling the stock $E'[\mathbf{L}] > 0$, then there will be a positive impact price on the wealth, since $-\mathbf{l}(t)E'[\mathbf{L}(t)] > 0$. Note that, one should have in principle $E'[\mathbf{L}] = 0$. In this model this is not imposed as a constraint but it is natural to expect, as $\alpha \to \infty$, this constraint to be asymptotically satisfied.

Dynamics of the mean-field We assume all players have the same dynamics. Therefore the mean-field variables $\mathbf{X}$ and $\mathbf{Y}$ satisfy

$$\begin{cases} d\mathbf{X} & = r\mathbf{X}dt + \mathbf{L}dt - \alpha \mathbf{L}E'[\mathbf{L}]dt - \mathbf{C}dt \\ d\mathbf{Y} & = \mu \mathbf{Y}dt + \sigma \mathbf{Y}dW_t - \mathbf{L}dt, \end{cases}$$

where we assume for now that $\mathbf{L}$ and $\mathbf{C}$ are known and given in feedback form

$$\mathbf{L} = \Theta(X, Y, X, Y), \qquad \mathbf{C} = \Pi(X, Y, X, Y).$$

Optimization problem for the reference player In order to simplify the expressions that follow we set up some notation first: we write $\mathbf{z} = (\mathbf{x}, \mathbf{y})$, $\mathbf{Z} = (\mathbf{X}, \mathbf{Y})$, $\mathbf{m} = (\mathbf{l}, \mathbf{c})$, and $\mathbf{M} = (\mathbf{L}, \mathbf{C})$. Each player aims to maximize its reward function, which from the point of view of a reference player amounts to:

$$V(z, Z) = \max_{\mathbf{l},\mathbf{c}} E\left[\int_0^\infty e^{-\beta t}\mathcal{L}(\mathbf{z}, \mathbf{m}, \mathbf{Z}, \mathbf{M})dt \,\middle|\, (\mathbf{z}, \mathbf{Z})(0) = (\mathbf{z}_0, \mathbf{Z}_0)\right], \tag{69}$$

where the controls $(\mathbf{l}, \mathbf{c})$ are taken in $L^p(\Omega, \mathbb{R}) \times L^p(\Omega, \mathbb{R}_0^+)$, and $\mathcal{L} : \mathbb{R}^d \times \mathbb{R}^d \times L^p(\Omega') \times L^p(\Omega') \to \mathbb{R}$.

Master equation We define define the Hamiltonian function as

$$H(z, p, q, Z, M) = \inf_{l,c}\left[-(rx + l - \alpha l E'[L] - c) \cdot p - (\mu y - l(z, Z)) \cdot q - \mathcal{L}(z, p, q, Z, M)\right],$$

where $z = (x, y)$, and $Z = (X, Y)$. Now assuming enough regularity, on the value function (69), such that we can use the Itô's formula, and such that the dynamic programming principle applies we have that V satisfies the following PDE:

$$\begin{aligned}
&\beta V(z,Z) + H\Big(z, D_x V(z,Z), D_y V(z,Z), Z, L(Z,Z), C(Z,Z)\Big) \\
&\quad - \big(rX + L(Z,Z) - \alpha L(Z,Z)E'[L(Z,Z)] - C(Z,Z)\big) D_X V(z,Z) \\
&\quad - \big(\mu Y - L(Z,Z)\big) D_Y V(z,Z) \\
&\quad - \frac{1}{2}\sigma^2 \big(y^2 D_{yy} V(z,Z) + Y^2 \delta_i^2 V(z,Z) + y\delta_i D_y V(z,Z)\big) = 0.
\end{aligned} \tag{70}$$

So, provided V is a smooth enough solution to the previous equation, then an optimal control pair (c, l) satisfies

$$\begin{cases} rx + l(z,Z) - \alpha l(z,Z)E'[L(z,Z)] - c(z,Z) = -D_p H(z, D_x V, D_y V, Z, L, C) \\ \mu y - l(z,Z) = -D_q H(z, D_x V, D_y V, Z, L, C). \end{cases}$$

Furthermore we assume that all players are indistinguishable and act rationally henceforth playing optimal strategies. Then this tells us that

$$\begin{cases} rx + \Theta(z,Z)(1 - \alpha E'[\Theta(z,Z)]) - \Pi(z,Z) \\ \qquad = -D_p H(z, D_x V(z,Z,t), D_y V(z,Z,t), Z, \Theta(z,Z), \Pi(z,Z)) \\ \mu y - \Theta(z,Z) \\ \qquad = -D_q H(z, D_x V(z,Z,t), D_y V(z,Z,t), Z, \Theta(z,Z), \Pi(z,Z)), \end{cases} \tag{71}$$

Plugging this controls into the above PDE gives rise to the master equation

$$\begin{aligned}
&\beta V(z,Z) + H\Big(z, D_x V(z,Z), D_y V(z,Z), Z, L(z,Z), C(z,Z)\Big) \\
&\quad + D_p H\big(Z, D_X V, D_Y V, Z, \Theta, \Pi\big) D_X V(z,Z) + D_q H\big(Z, D_X V, D_Y V, Z, \Theta, \Pi\big) D_Y V(z,Z) \\
&\quad - \frac{1}{2}\sigma^2 \big(y^2 D_{yy} V(z,Z) + Y^2 \delta_i^2 V(z,Z) + y\delta_i D_y V(z,Z)\big) = 0.
\end{aligned} \tag{72}$$

Open questions This price formation model illustrates various of the open questions on this area of research. First, it is not clear at all the existence or regularity of solutions. The natural definition of solution is the following: for fixed controls for the mean-field, (in the price formation model Θ and Π) the function V is a viscosity solution of the Hamilton-Jacobi equation (in this case (70)), then the controls for the mean-field are determined by the optimality conditions (such as (71)). This is thus a fixed point problem. In order to study it new techniques to understand the regularity of viscosity solutions of Hamilton-Jacobi equations in infinite dimensions must be developed. Uniqueness, as far as we know is also open, though it may be possible to adapt some of monotonicity techniques developed by Lions in [Lio11] to this setting. From the application point of view it would be extremely important to develop effective numerical methods. It is clear at this stage that any naive attempt to address this would suffer from the curse of dimensionality problem and therefore new ideas are necessary to address this class of problems. Finally, singular perturbation problems such as the one that arises by sending $\alpha \to \infty$ are important, natural, and should certainly be investigated in depth.

References

[ACCD12] Y. Achdou, F. Camilli, and I. Capuzzo-Dolcetta. Mean field games: numerical methods for the planning problem. *SIAM J. Control Optim.*, 50(1):77–109, 2012.

[ACD10] Y. Achdou and I. Capuzzo-Dolcetta. Mean field games: numerical methods. *SIAM J. Numer. Anal.*, 48(3):1136–1162, 2010.

[Ach13] Y. Achdou. Finite difference methods for mean field games. In *Hamilton-Jacobi Equations: Approximations, Numerical Analysis and Applications*, pages 1–47. Springer, 2013.

[Amb04] L. Ambrosio. Transport equation and Cauchy problem for *BV* vector fields. *Invent. Math.*, 158(2):227–260, 2004.

[Amb08] L. Ambrosio. Transport equation and Cauchy problem for non-smooth vector fields. In *Calculus of variations and nonlinear partial differential equations*, volume 1927 of *Lecture Notes in Math.*, pages 1–41. Springer, Berlin, 2008.

[AP12] Y. Achdou and V. Perez. Iterative strategies for solving linearized discrete mean field games systems. *Netw. Heterog. Media*, 7(2):197–217, 2012.

[Bar12] M. Bardi. Explicit solutions of some linear-quadratic mean field games. *Netw. Heterog. Media*, 7(2):243–261, 2012.

[BB00] J.-D. Benamou and Y. Brenier. A computational fluid mechanics solution to the Monge-Kantorovich mass transfer problem. *Numer. Math.*, 84(3):375–393, 2000.

[BCD97] M. Bardi and I. Capuzzo-Dolcetta. *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations.* Birkhäuser Boston Inc., Boston, MA, 1997. With appendices by Maurizio Falcone and Pierpaolo Soravia.

[BDFMW13] M. Burger, M. Di Francesco, P. A. Markowich, and M.-T. Wolfram. On a mean field game optimal control approach modeling fast exit scenarios in human crowds. *preprint*, 2013.

[BF13a] M. Bardi and E. Feleqi. The derivation of ergodic mean field game equations for several populations of players. *Dynamic Games and Applications, to appear*, 2013.

[BF13b] M. Bardi and E. Feleqi. Nonlinear elliptic systems and mean field games. *52nd IEEE Conference on Decision and Control (Florence, December 2013)*, 2013.

[BFY13] A. Bensoussan, J. Frehse, and P. Yam. *Mean field games and mean field type control.* Springer, 2013.

[BG10] A. Biryuk and D. A. Gomes. An introduction to the Aubry-Mather theory. *São Paulo J. Math. Sci.*, 4(1):17–63, 2010.

[BP13] M. Bardi and F. Priuli. LQG mean-field games with ergodic cost. *preprint*, 2013.

[BSYY13] A. Bensoussan, K. Sung, S. Yam, and S. Yung. Linear-quadratic mean field games. *preprint*, 2013.

[BT13] M. Balandat and C. Tomlin. On efficiency in mean field differential games. *2013 American Control Conference (ACC)*, 2013.

[Car11] P. Cardaliaguet. *Notes on mean-field games.* 2011.

[Car13a] P. Cardaliaguet. Long time average of first order mean-field games and weak KAM theory. *preprint*, 2013.

[Car13b] P. Cardaliaguet. Weak solutions for first order mean-field games with local coupling. *preprint*, 2013.

[CD13a] R. Carmona and F. Delarue. Mean field forward-backward stochastic differential equations. *preprint*, 2013.

[CD13b] R. Carmona and F. Delarue. Probabilistic analysis of mean-field games. *preprint*, 2013.

[CDL13] R. Carmona, F. Delarue, and A. Lachapelle. Control of McKean-Vlasov dynamics versus mean field games. *Math. Financ. Econ.*, 7(2):131–166, 2013.

[CGMT13] F. Cagnetti, D. Gomes, H. Mitake, and H. Tran. A new method for large time behavior of convex hamilton-jacobi equations: degenerate equations and weakly coupled systems. *Preprint*, 2013.

[CL13] R. Carmona and D. Lacker. A probabilistic weak formulation of mean field games and applications. *Preprint*, 2013.

[CLLP12] P. Cardaliaguet, J.-M. Lasry, P.-L. Lions, and A. Porretta. Long time average of mean field games. *Netw. Heterog. Media*, 7(2):279–301, 2012.

[CLLP13] P. Cardaliaguet, J.-M. Lasry, P.-L. Lions, and A. Porretta. Long time average of mean field games with a nonlocal coupling. *Siam J. Control Opt. , to appear*, 2013.

[CS12] F. Camilli and F. Silva. A semi-discrete approximation for a first order mean field game problem. *Netw. Heterog. Media*, 7(2):263–277, 2012.

[CS13] E. Carlini and F. J. Silva. A fully-discrete semi-lagrangian scheme for a first order mean field game problem. *Preprint*, 2013.

[DFMPW11] M. Di Francesco, P. A. Markowich, J.-F. Pietschmann, and M.-T. Wolfram. On the Hughes' model for pedestrian flow: the one-dimensional case. *J. Differential Equations*, 250(3):1334–1362, 2011.

[EG01] L. C. Evans and D. Gomes. Effective Hamiltonians and averaging for Hamiltonian dynamics. I. *Arch. Ration. Mech. Anal.*, 157(1):1–33, 2001.

[Eva09] L. C. Evans. Further PDE methods for weak KAM theory. *Calc. Var. Partial Differential Equations*, 35(4):435–462, 2009.

[FG13] R. Ferreira and D. Gomes. On the convergence of finite state mean-field games through Γ-convergence. *preprint*, 2013.

[FS06] W. H. Fleming and H. M. Soner. *Controlled Markov processes and viscosity solutions*, volume 25 of *Stochastic Modelling and Applied Probability.* Springer-Verlag, New York, 2006.

[GISMY10] D. Gomes, R. Iturriaga, H. Sánchez-Morgado, and Y. Yu. Mather measures selected by an approximation scheme. *Proc. Amer. Math. Soc.*, 138(10):3591–3601, 2010.

[GLM11] D. Gomes, A. Lopes, and J. Mohr. The Mather measure and a large deviation principle for the entropy penalized method. *Commun. Contemp. Math.*, 13(2):235–268, 2011.

[GMS10] D. Gomes, J. Mohr, and R. R. Souza. Discrete time, finite state space mean field games. *Journal de Mathématiques Pures et Appliquées*, 93(2):308–328, 2010.

[GMS13] Diogo A. Gomes, Joana Mohr, and Rafael Rigão Souza. Continuous time finite state mean field games. *Appl. Math. Optim.*, 68(1):99–143, 2013.

[GN12] D. Gomes and L. Nurbekyan. Lagrangian dynamics and a weak KAM theorem on the d-infinite dimensional torus. *Dokl. Nats. Akad. Nauk Armen.*, 112(2):152–159, 2012.

[GN13a] D. Gomes and L. Nurbekyan. On the minimizers of calculus of variations problems in Hilbert spaces. *Preprint*, 2013.

[GN13b] D. Gomes and L. Nurbekyan. Weak KAM theory in the d-infinite dimensional torus. *Preprint*, 2013.

[Gom02] D. Gomes. A stochastic analogue of Aubry-Mather theory. *Nonlinearity*, 15(3):581–603, 2002.

[Gom05] D. Gomes. Duality principles for fully nonlinear elliptic equations. In *Trends in partial differential equations of mathematical physics*, volume 61 of *Progr. Nonlinear Differential Equations Appl.*, pages 125–136. Birkhäuser, Basel, 2005.

[GP13] D. Gomes and S. Patrizi. Obstacle and weakly coupled systems problem in mean field games. *preprint*, 2013.

[GPSM12] D. A. Gomes, G. E. Pires, and H. Sánchez-Morgado. A-priori estimates for stationary mean-field games. *Netw. Heterog. Media*, 7(2):303–314, 2012.

[GPSM13a] D. Gomes, E. Pimentel, and H Sanchez-Morgado. Time dependent mean-field games - subquadratic Hamiltonians. *preprint*, 2013.

[GPSM13b] D. Gomes, E. Pimentel, and H Sanchez-Morgado. Time dependent mean-field games - superquadratic Hamiltonians. *preprint*, 2013.

[GPV13] D. Gomes, S. Patrizi, and V. Voskanyan. On the existence of classical solutions for stationary extended mean field games. *preprint*, 2013.

[GR13] D. Gomes and R. Ribeiro. Mean field games with logistic population dynamics. *52nd IEEE Conference on Decision and Control (Florence, December 2013)*, 2013.

[GSM11] D. Gomes and H. Sanchez-Morgado. On the stochastic Evans-Aronsson problem. *preprint*, 2011.

[Gue09a] O. Gueant. *Mean Field Games and Applications to Economics.* Ph.D. Thesis. Université Paris Dauphine, Paris, 2009.

[Gue09b] O. Gueant. A reference case for mean field games models. *J. Math. Pures Appl. (9)*, 92(3):276–294, 2009.

[Gue11a] O. Gueant. An existence and uniqueness result for mean field games with congestion effect on graphs. *preprint*, 2011.

[Gue11b] O. Gueant. From infinity to one: The reduction of some mean field games to a global control problem. *preprint*, 2011.

[Gue11c] O. Gueant. A uniqueness result for mean field games. *Classnotes*, 2011.

[GV07] D. Gomes and E. Valdinoci. Entropy penalization methods for Hamilton-Jacobi equations. *Adv. Math.*, 215(1):94–152, 2007.

[GV13] D. Gomes and V. Voskanyan. Extended mean-field games - formulation, existence, uniqueness and examples. *preprint*, 2013.

[HCM07] M. Huang, P. E. Caines, and R. P. Malhamé. Large-population cost-coupled LQG problems with nonuniform agents: individual-mass behavior and decentralized ϵ-Nash equilibria. *IEEE Trans. Automat. Control*, 52(9):1560–1571, 2007.

[HCM10] M. Huang, P. Caines, and R. P. Malhamé. . The NCE (mean field) principle with locality dependent cost interactions. *IEEE Trans. Automat. Control*, 55(12):2799–2805, 2010.

[HMC06] M. Huang, R. P. Malhamé, and P. E. Caines. Large population stochastic dynamic games: closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle. *Commun. Inf. Syst.*, 6(3):221–251, 2006.

[Hua10] M. Huang. Large-population LQG games involving a major player: the Nash certainty equivalence principle. *SIAM J. Control Optim.*, 48(5):3318–3353, 2009/10.

[Hua12] M. Huang. Mean field stochastic games with discrete states and mixed players. *Proc. GameNets, Vancouver*, 2012.

[KLY11] V. N. Kolokoltsov, J. Li, and W. Yang. Mean field games and nonlinear Markov processes. *preprint*, 2011.

[Kol10] Vassili N. Kolokoltsov. *Nonlinear Markov processes and kinetic equations*, volume 182 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2010.

[KY13a] K. Kolokoltsov and W. Yang. Existence of solutions to path-dependent kinetic equations and related forward-backward systems. *Open Journal of Optimization*, 2:39–44, 2013.

[KY13b] K. Kolokoltsov and W. Yang. Sensitivity analysis for HJB equations with an application to a coupled backward-forward system. *Optimization*, to appear, 2013.

[Lio11] P.-L. Lions. College de france course on mean-field games. 2007-2011.

[Lio12] P.-L. Lions. IMA, University of Minessota. Course on mean-field games. Video. http://www.ima.umn.edu/2012-2013/sw11.12-13.12/. 2012.

[LL06a] J.-M. Lasry and P.-L. Lions. Jeux à champ moyen. I. Le cas stationnaire. *C. R. Math. Acad. Sci. Paris*, 343(9):619–625, 2006.

[LL06b] J.-M. Lasry and P.-L. Lions. Jeux à champ moyen. II. Horizon fini et contrôle optimal. *C. R. Math. Acad. Sci. Paris*, 343(10):679–684, 2006.

[LL07a] J.-M. Lasry and P.-L. Lions. Mean field games. *Jpn. J. Math.*, 2(1):229–260, 2007.

[LL07b] J.-M. Lasry and P.-L. Lions. Mean field games. *Cahiers de la Chaire Finance et Développement Durable*, 2007.

[LLG10a] J.-M. Lasry, P.-L. Lions, and O. Gueant. Application of mean field games to growth theory. *preprint*, 2010.

[LLG10b] J.-M. Lasry, P.-L. Lions, and O. Gueant. Mean field games and applications. *Paris-Princeton lectures on Mathematical Finance*, 2010.

[LLG11] J.-M. Lasry, P.-L. Lions, and O. Gueant. Mean field games and oil production. *preprint*, 2011.

[LLLL13] A. Lachapelle, J.-M. Lasry, C.-A. Lehalle, and P.-L. Lions. Efficiency of the price formation process in presence of high frequency participants: a mean field game analysis. *preprint*, 2013.

[LM13] R. E. Lucas and B Moll. Knowledge growth and the allocation of time. *Journal of Political Economy.*, 2013.

[LST] A. Lachapelle, J. Salomon, and G. Turinici. Computation of mean field equilibria in economics.

[LZ08] Tao Li and Ji-Feng Zhang. Asymptotically optimal decentralized control for large population stochastic multiagent systems. *IEEE Trans. Automat. Control*, 53(7):1643–1660, 2008.

[NCMH13] M. Nourian, P. Caines, R. P. Malhamé, and Minyi Huang. Nash, social and centralized solutions to consensus problems via mean field control theory. *IEEE Trans. Automat. Control*, 58(3):639–653, 2013.

[NH12] S. L. Nguyen and M. Huang. Linear-quadratic-gaussian mixed games with continuum-parametrized minor players. *SIAM J. Control Optim.*, 50(5):2907–2937, 2012.

[Por13a] A. Porretta. On the planning problem for the mean-field games system. *Dyn. Games Appl.*, 2013.

[Por13b] A. Porretta. Weak solutions to Fokker-Planck equations and mean field games. *preprint*, 2013.

[San12] F. Santambrogio. A modest proposal for MFG with density constraints. *Netw. Heterog. Media*, 7(2):337–347, 2012.

[Szn91] Alain-Sol Sznitman. Topics in propagation of chaos. In *École d'Été de Probabilités de Saint-Flour XIX—1989*, volume 1464 of *Lecture Notes in Math.*, pages 165–251. Springer, Berlin, 1991.

[TZB13] H. Tembine, Q. Zhu, and T. Basar. Risk-sensitive mean field games. *Preprint*, 2013.

[T13] H. Tembine. Energy-constrained mean field games in wireless networks. *Journal of Strategic Behavior and the Environment*, accepted 2013.

[Vil03] C. Villani. *Topics in optimal transportation*, volume 58 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2003.