

ON THE UNIQUENESS OF CLASSICAL SOLUTIONS OF CAUCHY PROBLEMS

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ABSTRACT. Given that the terminal condition is of at most linear growth, it is well known that a Cauchy problem admits a unique classical solution when the coefficient multiplying the second derivative is also a function of at most linear growth. In this paper, we give a condition on the volatility that is necessary and sufficient for a Cauchy problem to admit a unique solution.

1. MAIN RESULT

Given a terminal-boundary condition $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ with $g(x) \leq C(1+x)$ for some constant $C > 0$, we consider the following Cauchy problem:

$$(1) \quad \begin{aligned} u_t + \frac{1}{2}\sigma^2(x)u_{xx} &= 0, & (x, t) \in (0, \infty) \times [0, T), \\ u(0, t) &= g(0), & t \leq T, \\ u(x, T) &= g(x), \end{aligned}$$

where $\sigma \neq 0$ on $(0, \infty)$, $\sigma^{-2} \in L^1_{loc}(0, \infty)$ (i.e., $\int_a^b \sigma^{-2}(x)dx < \infty$ for any $[a, b] \subset (0, \infty)$), and $\sigma = 0$ on $(-\infty, 0]$.

A solution $u : \mathbb{R}_+ \times [0, T] \mapsto \mathbb{R}$ of (1) is said to be a *classical solution* if $u \in C^{2,1}((0, \infty) \times [0, T))$. A function $f : \mathbb{R}_+ \times [0, T] \mapsto \mathbb{R}$ is said to be of *at most linear growth* if there exists a constant $C > 0$ such that $|f(x, t)| \leq C(1+x)$ for any $(x, t) \in \mathbb{R}_+ \times [0, T]$.

A well-known sufficient condition for (1) to have a unique classical solution among the functions with at most linear growth is that σ itself is of at most linear growth; see, e.g., Chapter 6 of [8] and Theorem 7.6 on page 366 of [10]. On the other hand, consider the SDE

$$(2) \quad dX_s^{t,x} = \sigma(X_s^{t,x})dW_s, \quad X_t^{t,x} = x > 0.$$

The assumptions on σ we made below (1) ensure that (2) has a unique weak solution which is absorbed at zero. (See [6].) The solution $X^{t,x}$ is clearly a local martingale.

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Delbaen and Shirakawa show in [4] that $X^{t,x}$ is a martingale if and only if

$$(3) \quad \int_c^\infty \frac{x}{\sigma^2(x)} dx = \infty, \quad \text{for some } c > 0.$$

(See also [2].)

Below, in Theorem 2, we prove that (3) is also necessary and sufficient for the existence of a unique classical solution of (1). First, in the next theorem, we show that (3), which is weaker than the linear growth condition on σ , is a sufficient condition for the uniqueness.

Theorem 1. *The Cauchy problem (1) has a unique classical solution (if any) in the class of functions with at most linear growth if (3) is satisfied.*

Proof. It suffices to show that (1) with $g \equiv 0$ has a unique solution $u \equiv 0$. Let us define a sequence of stopping times $\tau_n \triangleq \inf\{s \geq t : X_s^{t,x} \geq n \text{ or } X_s^{t,x} \leq 1/n\} \wedge T$ for each $n \in \mathbb{N}$ and $\tau_0 \triangleq \{s \geq t : X_s^{t,x} = 0\} \wedge T$. Then, as in the proof of Theorem 1.6 in [4], we can show that the function defined by

$$\Psi(x) = \begin{cases} x, & x \leq 1, \\ x + \int_1^x \frac{u}{\sigma^2(u)}(x-u)du, & x \geq 1, \end{cases}$$

satisfies $\mathbb{E}[\Psi(X_{\tau_n}^{t,x})] \leq \Psi(x) + xT/2$. Since Ψ is convex, (3) implies that $\lim_{x \rightarrow \infty} \Psi(x)/x = \infty$. Then the criterion of de la Vallée Poussin implies that $\{X_{\tau_n}^{t,x} : n \in \mathbb{N}\}$ is a uniformly integrable family.

Suppose \tilde{u} is another classical solution of at most linear growth. Applying Itô's lemma, we obtain

$$\begin{aligned} \tilde{u}(X_{\tau_n}^{t,x}, \tau_n) &= \tilde{u}(x, t) + \int_t^{\tau_n} \left[\tilde{u}_s(X_s^{t,x}, s) + \frac{1}{2} \sigma^2(X_s^{t,x}) \tilde{u}_{xx}(X_s^{t,x}, s) \right] ds \\ &\quad + \int_t^{\tau_n} \tilde{u}_x(X_s^{t,x}, s) \sigma(X_s^{t,x}) dW_s. \end{aligned}$$

Note that $X^{t,x}$ is a martingale on $[t, \tau_n]$ and we have $\tilde{u}_x(X_s^{t,x}, s)$ bounded for $s \leq \tau_n$. Hence the stochastic integral in the above identity has expectation zero. Taking the expectation of both sides of the above identity, we get $\tilde{u}(x, t) = \mathbb{E}[\tilde{u}(X_{\tau_n}^{t,x}, \tau_n)]$ for each $n \in \mathbb{N}$.

On the other hand, since \tilde{u} is of at most linear growth, there exists a constant C such that $|\tilde{u}(x, t)| \leq C(1+x)$. Therefore $\{\tilde{u}(X_{\tau_n}^{t,x}, \tau_n) : n \in \mathbb{N}\}$ is a uniformly integrable family. This is because it is bounded from above by the uniformly integrable family $\{C(1+X_{\tau_n}^{t,x}) : n \in \mathbb{N}\}$. As a result,

$$\begin{aligned} \tilde{u}(x, t) &= \lim_{n \rightarrow \infty} \mathbb{E}[\tilde{u}(X_{\tau_n}^{t,x}, \tau_n)] = \mathbb{E}\left[\lim_{n \rightarrow \infty} \tilde{u}(X_{\tau_n}^{t,x}, \tau_n)\right] = \mathbb{E}[\tilde{u}(X_{\tau_0}^{t,x}, \tau_0)] \\ &= \mathbb{E}[g(X_T^{t,x}) 1_{\{\tau_0=T\}}] + \mathbb{E}[\tilde{u}(X_{\tau_0}^{t,x}, \tau_0) 1_{\{\tau_0 < T\}}] \\ &= \mathbb{E}[g(X_T^{t,x}) 1_{\{\tau_0=T\}}] + \mathbb{E}[g(0) 1_{\{\tau_0 < T\}}] = 0. \end{aligned}$$

Here the third equality holds since $X^{t,x}$ does not explode (i.e., $\inf\{s \geq t : X_s^{t,x} = +\infty\} = \infty$; see Problem 5.3 on page 332 of [11]), and the one before the last equality follows since $X_{\tau_0}^{t,x} = 0$ on the set $\{\tau_0 < T\}$. \square

Theorem 2. *If we further assume that $\sigma : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is locally Hölder continuous with exponent $1/2$ and g is of linear growth, then the Cauchy problem in (1) has a unique classical solution if and only if (3) is satisfied.*

Proof. First, let us prove the existence of a solution. Let $u(x, t) \triangleq \mathbb{E} g(X_T^{t,x})$ (the value of a call-type European option). Thanks to the Hölder continuity of σ , it follows from Theorem 3.2 in [5] that u is a classical solution of (1). Moreover, it is of at most linear growth due to the assumption that g is of at most linear growth.

Proof of sufficiency. This follows from Theorem 1.

Proof of necessity. If (3) is violated, then $t \rightarrow X_t$ is a strict local martingale on each time interval $(0, T)$, $T \in (0, \infty)$; see [4], [2] and especially the remark after Corollary 4.3 in [12]. Using Theorem 3.2 in [5], it can be seen that $u^*(x, t) \triangleq x - \mathbb{E}[X_T^{x,t}] > 0$ is a classical solution of (1) with zero boundary and terminal conditions. (Note that the Hölder continuity assumption on σ is used in this step as well.) This function clearly has at most linear growth. Therefore $u + \lambda u^*$, for any $\lambda \in \mathbb{R}$, is also a classical solution of (1) which is of at most linear growth. \square

Remark 1. Observe that Theorems 1 and 2 are analogous to Theorem 7.6 and Remark 7.8 on pages 366 and 368 of [10], respectively. Our contribution is the weakening of the linear growth assumption on σ .

A related result is given by Theorem 4.3 of [5] on put-type European options: When g is of strictly sublinear growth (i.e., $\lim_{x \rightarrow \infty} g(x)/x = 0$), then (1) has a unique solution among the functions with strictly sublinear growth (without assuming (3)).

Our result in Theorem 2 complements Theorem 3.2 of [5], which shows that the call-type European option price is a classical solution of (1) of at most linear growth. We prove that Delbaen and Shirakawa's condition (3) is necessary and sufficient to guarantee that the European option price is the only classical solution (of at most linear growth) to this Cauchy problem. [3] and [9] have already observed that the Cauchy problem corresponding to European call options have multiple solutions. (Also see [7] and [1], which consider super hedging prices of call-type options when there are no equivalent local martingale measures.) However, a necessary and sufficient condition under which there is uniqueness/nonuniqueness remains unknown.

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