and combining with (95) yields, correct to fourth order:

$$\text{(6)} \qquad \operatorname{Cov}\{e(k/k-1)\} = A^2(k)\sigma_{k-1}^2 + 2B^2(k)\sigma_{k-1}^4 + 6A(k)C(k)\sigma_{k-1}^4.$$

This compares to the authors' equations (95) and the incorrect equation (107) combined to give:

$$\text{(7)} \qquad \operatorname{Cov}\{e(k/k-1)\} = A^2(k)\sigma_{k-1}^2 - B^2(k)\sigma_{k-1}^4.$$

While the difference in the original incorrect form (7) and the correct form (6) is of second order, the iteration of the incorrect equations thousands of times as in the suggested maneuvering reentry vehicle tracking application can, and indeed does, render the Cov $\{e(k/k-1)\}$ nonpositive definite through repeated subtraction of the small quantity $B^2(k)\sigma_{k-1}^4$, yielding incorrect results and numerical instability in the filter equations.

Approximating of (6) by its second order approximation:

$$\text{(8)} \qquad \operatorname{Cov}\{e(k/k-1)\} = A^2(k)\sigma_{k-1}^2$$

as used in linearized Kalman filtering, or justifying deleting the term $6A(k)C(k)\sigma_{k-1}^4$ to eliminate computation of the matrices of third partials, merits further study for specific applications. The approximation represented by the original (107) combined with (95) is, however, invalid for the reasons demonstrated by the example.

REFERENCES

[1] J. R. KRASNAKEVITCH AND R. A. HADDAD, *Generalized prediction-correction estimation*, this Journal, 7 (1969), pp. 496–511.

[2] ———, *The Kindler formulation for unbiased BRV/MaRV tracking*, Rep. 7612-1, Kindler Assoc., Cambridge, MA, revised Feb. 1976.

[3] ———, *Filter equations—The Kindler formulation for unbiased BRV/MaRV tracking*, Rep. 7612-2, Kindler Assoc., Cambridge, MA, Feb. 1976.

SIAM J. CONTROL AND OPTIMIZATION
Vol. 15, No. 5, August 1977

SURVEY OF MEASURABLE SELECTION THEOREMS*

DANIEL H. WAGNER†

Abstract. Suppose $(T, \mathcal{M})$ is a measurable space, X is a topological space, and $\varnothing \neq F(t) \subset X$ for $t \in T$. Denote $\operatorname{Gr} F = \{(t, x): x \in F(t)\}$. The problem surveyed (reviewing work of others) is that of existence of $f: T \to X$ such that $f(t) \in F(t)$ for $t \in T$ and $f^{-1}(U) \in \mathcal{M}$ for open $U \subset X$. The principal conditions that yield such f are (i) X is Polish, each $F(t)$ is closed, and $\{t: F(t) \cap U \neq \varnothing\} \in \mathcal{M}$ whenever $U \subset X$ is open (Kuratowski and Ryll-Nardzewski and, under stronger assumption, Castaing), or (ii) T is a Hausdorff space, $\operatorname{Gr} F$ is a continuous image of a Polish space, and $\mathcal{M}$ is the σ-algebra of sets measurable with respect to an outer measure, among which are the open sets of T (primarily von Neumann). The latter result follows from the former by lifting F in a natural way to a map into the closed sets of a Polish space. This procedure leads to the theory of set-valued functions of Suslin type (Leese), which extends the result (i) to comprehend a considerable portion of the results on the problem surveyed. Among the topics addressed, measurable implicit functions and the case where X is a linear space and each $F(t)$ is convex and compact are particularly important to control theory, for example. With $T = X = [0, 1]$ and $\operatorname{Gr} F$ Borel, an elegant partition of $\operatorname{Gr} F$ into Lebesgue measurable maps from T to X, parameterized by Borel functions, has been found (Wesley) via Cohen forcing methods. Other topics discussed include pointwise optimal selections, selections of partitions, uniformization, non-σ-algebras in place of $\mathcal{M}$, Lusin measurability, and set-valued measures. Substantial historical comments and an extensive bibliography are included. (See addenda (i)–(iii).)

1. Introduction. This paper surveys the subject of existence of a measurable function which is a selection of a given set-valued function mapping a measurable space into subsets of a topological space. The subject has undergone considerable development in the past decade. We attempt to review the principal results currently available and to give a history of prior work, dating primarily from[0] a 1949 lemma of von Neumann [NE] and from precursors on the subject by Lusin [LS], Novikov [NO1], and others of the 1930 era. (See addenda (i), (ii).)

Measurable selection problems arise in a variety of ways in control theory, mathematical economics, probability theory, statistics, and operator theory, among other fields. For example, Aumann's influential 1965 paper [AU1] was motivated by economics. Numerous applications are given in the referenced papers. Although we will have little to say about applications, let us note two examples.

Suppose $d: R^2 \to R^n$ and $D(q) = \int_R d(t, q(t))\,dt$ for all $q: R \to R$ for which the (Lebesgue) integral is finite. Suppose it is known that $\lambda \in R^n$ and q^* have the property that

$$\text{(1.1)} \qquad \lambda \cdot D(q^*) \geqq \lambda \cdot D(q) \quad \text{for all admissible } q,$$

the dot being ordinary inner product. Do we then have

$$\text{(1.2)} \qquad \lambda \cdot d(t, q^*(t)) \geqq \lambda \cdot d(t, y) \quad \text{for } y \in R, \quad \text{a.e. } t \in R?$$

In other words, does satisfaction of a functional multiplier rule imply satisfaction of a pointwise multiplier rule? The answer was shown by Aumann and Perles

* Received by the editors June 21, 1976, and in revised form December 14, 1976.

† Daniel H. Wagner, Associates, Paoli, Pennsylvania 19301. This work was supported in part by the Office of Naval Research under Contract N 00014-70-C-0232.

[0] Additional credit to Jankov [JN], Novikov [NO2], and Rokhlin [RK2], and other recent information are given in addenda in proof at the end of the paper.

[AP], and in more generality by Wagner and Stone [WS], to be affirmative providing d is a Borel function. An example in [WS] also shows that it does not suffice for d to be Lebesgue measurable. In this application, one defines the set-valued function F by

$$F(t)=\{x : x \in R^n \text{ and } \lambda \cdot d(t,x) > \lambda \cdot d(t, q^*(t))\} \quad \text{for } t \in R.$$

If (1.2) fails, one finds a Borel $T \subset R$ of positive measure such that $F(t) \neq \varnothing$ for $t \in T$. Since d is a Borel function, von Neumann's theorem mentioned above (e.g., Corollary 5.2 below) assures the existence of a Lebesgue measurable function f on T such that $f(t) \in F(t)$ for $t \in T$, i.e., f is a measurable selection of $F|T$, from which a contradiction to (1.1) is easily deduced.

As the second example, consider the problem of generalizing the LaSalle bang–bang principle of control theory: Any output attainable via an admissible control function is attainable by a control which utilizes only extreme points of each instantaneous (compact convex) set of possible choices. Proving such statements usually involves recognizing that each such choice is a convex combination of extreme points, and one needs to find such a representation in a measurable way (see § 8 below, [WG1], [AU1], [CA5], [HV5], or [VA3], for example).

The preliminaries in § 2 include some instructive counterexamples due to Dauer and Van Vleck and to Kaniewski. Early history is discussed in § 3, primarily work of Lusin, Novikov, and Saks.

The main fundamentals of measurable selections are given in § 4 on closed-valued functions, § 5 on set-valued functions with measurable graph, § 6 on set-valued functions of Suslin type, and § 7 on implicit functions. Of foremost importance is pioneering work of von Neumann, Kuratowski and Ryll-Nardzewski, and Castaing. Prominent in these sections is the prolific work of Castaing, Himmelberg and Van Vleck, and Leese. Section 6, based on work of Leese, unifies much of the developments in § 4, § 5, and § 7. Numerous additional authors have contributed to these developments. In particular, the papers on graph-conditioned theorems by Aumann and Sainte-Beuve are quite interesting and some expositions by Rockafellar and Himmelberg are especially helpful. The initial contribution to the implicit function topic of § 7 was Filippov's.

Convex-valued functions are discussed in § 8, notably Valadier's work on scalarly measurable selections and results of Himmelberg and Van Vleck and of Leese on extreme point selections. We note applications of selection theory for convex-valued functions to bang–bang problems and, primarily by Rockafellar, to optimization of convex integral functionals by duality methods. In § 9 we review results on pointwise optimal measurable selections, initiated by Dubins and Savage.

In § 10 we discuss decomposition of the graph of a set-valued function into measurable selections, notably an elegant result of Wesley which appears to be the most profound result to date in measurable selection theory, judging by its proof via Cohen forcing methods.

Regarding a partition of a set as a set-valued function, in § 11 we have an alternative approach to selection problems, used in early results by Mackey and Dixmier and later more extensively by Hoffman-Jørgensen, Kuratowski, Maitra, and Rao, among others. The subject of uniformization, discussed in § 12, usually

treats selections with emphasis on their properties as subsets of a product space; this subject is older than that of emphasizing *function* properties of selections, and our coverage here is less complete than that of most topics discussed. It includes theorems relating these two types of properties. Replacing the σ-algebra of the measurable space with other structures is discussed in § 13. Lusin measurability is reviewed in § 14, including generalizations of Lusin's theorem involving set-valued functions. In § 15, we discuss work on set-valued measures, led by Godet-Thobie and Artstein. A few works which do not come directly under our other topic headings are noted in § 16; the final work discussed is the very recent "measurable fields" approach by Delode, Arino, and Penot, which appears to be quite promising.

A sequence of recommended initial reading is given in § 17—some readers may wish to turn to this first.

Numerous results come under more than one of these topic headings. We have tried to give or discuss each in the section where its greatest interest appears to lie.

A significant special topic that we do not discuss is that of differential equations involving set-valued functions, in particular orientor fields. Our only discussion of *continuous* selections, an important topic related to measurable selections, is to cite a few general references in § 13.

An extensive bibliography is provided, categorized as described in its introduction.

An acknowledgement to several sources of help is given at the end. Regarding accreditation, let us emphasize the well-known fact that "superseded" results have usually contributed to the development of the subject by their earlier appearance. By including considerable historical comments, we have tried to do some justice to this point, but certainly very inadequately. Indeed even with fairly recent literature, the heavy volume of results on the subject has required that much excellent work be reviewed in only a superficial way, e.g., our discussion of set-valued measures in § 15.

2. Preliminaries. For every set S we define $\mathscr{P}(S)=\{A : \varnothing \neq A \subset S\}$. When $\mathscr{L}$ is a set of sets, by $\mathscr{L}_\sigma$ we mean the set of countable unions of members of $\mathscr{L}$. If S is topologized, by $\mathscr{B}(S)$ we mean the σ-algebra of Borel sets of S, i.e., that generated by the open sets of S, and for $A \subset S$, by cl A we mean the closure of A. If $\mathscr{A}$ and $\mathscr{D}$ are σ-algebras, by $\mathscr{A} \otimes \mathscr{D}$ we mean the smallest σ-algebra containing $\{A \times D : A \in \mathscr{A} \text{ and } D \in \mathscr{D}\}$. We denote the set of real numbers by R and Euclidean n-space by R^n.

We make considerable use of fixed notations denoting fundamental objects in the structure of the problem. Definitions stated with respect to T, μ, $\mathscr{M}$, X, or F as fixed below apply in obvious ways to counterpart other objects.

We fix $T \neq \varnothing$ as a set, not necessarily topologized, and μ as a nonnegative (possibly infinite) measure over T. *Measurability always refers to μ unless stated otherwise.* Often we specify that μ is an outer measure, meaning that $\mu(S)$ is defined for *each* $S \subset T$ and μ is countably sub-additive; then measurability of $S \subset T$ is defined by Carathéodory's criterion [FE, § 2.12] and the set of measurable sets is a σ-algebra. At other times μ is merely defined on a given σ-algebra, i.e., the family of measurable sets, and is countably additive. Often it will not matter which of these measure concepts is used. *In all cases we fix $\mathcal{M}$ as the* σ-algebra of measurable sets. If Z is a topological space and $f: T \to Z$, we say f is a *measurable function* if $f^{-1}(U) \in \mathcal{M}$ whenever $U \subset Z$ is open. (Many of the results reviewed here are taken from papers based on the measure foundations of Bourbaki [BO2] who defines an integral as a linear functional and the measure of a set as the integral of its characteristic function.)

It should be recognized that our measure conventions include the case where no measure is present, i.e., when one is dealing with a *measurable* space $(T, \mathcal{M})$, meaning $\mathcal{M}$ is an arbitrary σ-algebra of subsets of T and $T \in \mathcal{M}$; one may let μ be the trivial measure given by $\mu(S) = 0$ for $S \in \mathcal{M}$ to bring this case into our framework. For theorem statements which do not mention properties of μ, in fact, one may just as well consider that μ is not present.

We fix X as a topological space (except in Theorems 5.8 and 12.1), and we reserve F to mean $F: T \to \mathcal{P}(X)$, i.e., F is a *set-valued function* (also called multifunction, multivalued function, in French, multiapplication, or in German, Multiabbildung).

We say F is (adjective)-valued if $F(t)$ is (adjective) for $t \in T$, and we apply operations on sets to operations on set-valued functions in an obvious fashion, e.g., if $G: T \to \mathcal{P}(X)$, then $(F \cap G)(t) = F(t) \cap G(t)$ for $t \in T$.

A *selection* (also called selector, section, uniformization, or, in German, Schnitt) of F is a function $f: T \to X$ such that $f(t) \in F(t)$ for $t \in T$. We denote

$$\mathcal{S}(F) = \{f: f \text{ is a measurable selection of } F\}.$$

We say f is an *a.e. measurable selection* of F if for some $S \in \mathcal{M}$, $\mu(T \backslash S) = 0$ and f is a measurable selection of $F|S$. The problems considered here are: when does one have $\mathcal{S}(F) \neq \varnothing$ (i.e., there exists a measurable selection of F) or when does there exist at least an a.e. measurable selection of F? Of course, from the axiom of choice, every set-valued function has a selection.

Following [RC6], we say $\{f_1, f_2, \cdots\}$ is a *Castaing representation* of F if $f_i \in \mathcal{S}(F)$ for $i = 1, 2, \cdots$, and $\{f_1(t), f_2(t), \cdots\}$ is dense in $F(t)$ for $t \in T$. Under weak conditions (see Theorem 4.2 below), existence of a Castaing representation, which is an additional problem of interest, is equivalent to measurability of F as defined next; this fact lends itself to manipulation of closed-valued functions in ways which help to solve our primary problem of proving $\mathcal{S}(F) \neq \varnothing$, as shown particularly well by Rockafellar [RC2, 6]. (See addenda (ii), (v).)

For $A \subset X$, we define $F^{-}(A) = T \cap \{t: F(t) \cap A \neq \varnothing\}$. We say F is *measurable*, as a set-valued function, if $F^{-}(K) \in \mathcal{M}$ whenever $K \subset X$ is closed, and *weakly measurable* if $F^{-}(U) \in \mathcal{M}$ whenever $U \subset X$ is open. Early uses of variations on these concepts of measurability were made by Rokhlin [RK2], Berge [BG], Pliś [PL1], Debreu [DE], and Kuratowski and Ryll-Nardzewski [KRN]. The definition of a measurable set-valued function was formalized and exploited by Castaing in his thesis [CA4, 5]. The term "weak measurability" (although not the concept) was introduced by Himmelberg, Jacobs, and Van Vleck [HJV].

We define the *graph* of F, denoted Gr F, by

$$\operatorname{Gr} F = (T \times X) \cap \{(t, x): x \in F(t)\}.$$

For a function $f: T \to X$ we do not refer to the graph of f (which we regard as the same as f). It should be clear when we are referring to properties of f as a subset of $T \times X$ (such as being a Borel set) or as a map on T to X (such as being a Borel function). If Y and Z are topological spaces, we say $f: T \times Y \to Z$ is a *Carathéodory map* if $f(t, \cdot)$ is continuous for $t \in T$ and $f(\cdot, x)$ is measurable for $x \in Y$. We denote $\pi_T(t, x) = t$ and $\pi_X(t, x) = x$ for $t \in T$, $x \in X$.

If T is topologized and F is closed-valued, we say F is *upper*{*lower*} *semi-continuous*, abbreviated usc{lsc}, as a set-valued function, if for each closed{open} $A \subset X$, $F^{-}(A)$ is closed{open} (see [KU1, Chap. 1, § 18]); F is usc implies Gr F is closed. One says F is *continuous* if F is usc and lsc. The abbreviations usc and lsc are also applied to $f: T \to R$.

When F is compact-valued and X is separable metric, measurability and continuity of F as a set-valued function are respectively equivalent to measurability and continuity as a "point valued" function with respect to the Hausdorff metric on the set of compact subsets of X. This is applied, e.g., by Castaing [CA4, 5, Chap. 4] and in earlier work by Debreu [DE].

An excellent source for measurability properties of set-valued functions is Himmelberg [HM2]; see also Rockafellar [RC6, 2, 3] and Castaing [CA5]. Several references in the bibliography are additional sources; those marked with a single prime are included because, in this respect, they augment the unprimed references (sources on existence of measurable selections), in some cases peripherally. Debreu's [DE] (1965) was a pioneering paper on measurability properties of F, without going into selection questions.

If every closed subset of X is a G_δ (e.g., if X is metrizable), then measurability of F implies weak measurability; the converse fails as shown by Example 2.4 below. We cannot omit the condition on X: Let $T = X = R$, $\mathcal{M} = \{\varnothing, T\}$, the open sets of X be the open right half-lines, and $F(t) = \{x: x \leqq t\}$ for $t \in T$. In most theorems below where F is weakly measurable, we also have X metrizable, so measurability may be substituted for weak measurability in those cases.

If $F_i: T \to \mathcal{P}(X)$ is {weakly} measurable for $i = 1, 2, \cdots$, then so is $\bigcup_{i=1}^{\infty} F_i$. Unfortunately the same cannot be said for intersections—see Example 2.3 below. However, if each F_i is weakly measurable and closed-valued, and *either* (i) X is σ-compact and metric, (ii) X is separable metric and for $t \in T$, for some i, $F_i(t)$ is compact, (iii) X has a countable base and for some i, F_i is measurable and compact-valued, or (iv) $X = R^n$, then $\bigcap_{i=1}^{\infty} F_i$ is measurable [HM2], [LE3], [RC6].

The principal additional properties of closed-valued F are summarized in Theorem 4.2 below.

By a *Polish space* is meant a (not necessarily complete) homeomorph of a complete separable metric space. We say that S is a *Suslin*{*Lusin*} *space* if S is topologized as a Hausdorff space and there exist a Polish space P and a continuous surjective {bijective} $\varphi: P \to S$. We define a *weakly* Suslin space in the same way without the Hausdorff requirement. A {*weakly*} *Suslin set* in a topological space is a subset which is a {weakly} Suslin space. Suslin sets play important roles in measurable selection theory. Probably the most thorough treatment of them is [HJ]. Other excellent references include [FE, § 2], [KU1, Chap. III, § 38], [BO1, Chap. IX, § 6], and [CH]. A subset of a Hausdorff space which is a Lusin space is a Borel subset, and in a Polish space the converse holds [FE, § 2.2.10]. A Borel subset of a Suslin space is a Suslin set [HV3, Lem.]. If μ is an outer measure, T is Hausdorff and $\mathcal{M}$ contains the open sets of T, then $\mathcal{M}$ contains the Suslin sets of T [FE, § 2.2.12]. Any Suslin space is a continuous image of the set of irrational numbers.

The definition of Suslin set given here is more general than that given in [KU1] (there called analytic set) and [BO1] and less general than that given in [FE]. It is important to note that [BO1] requires Suslin spaces and Lusin spaces to be metrizable by definition, but we are advised that a forthcoming edition of [BO1] will use the definitions employed here. Note that Castaing and his colleagues at l'Université du Languedoc, Montpellier, have consistently considered Suslin spaces to be Hausdorff, not necessarily metrizable, although that has not been explicit in their earlier publications. What we term Suslin and Lusin spaces are respectively called analytic and standard spaces in [HJ], [CH], and [MG]. Not much can be said about properties of weakly Suslin sets ([LE5] calls them "classical analytic")—that definition merely affords a weaker hypothesis which suffices for some theorems in non-Hausdorff spaces. Still weaker hypotheses, related variations under the term "analytic," are used in, e.g., [LE3, 5] and [SN] (see below: Theorem 4.11, remarks before Theorem 5.6, and Theorem 12.3).

One says μ is *complete* if $S' \subset S \in \mathcal{M}$ and $\mu(S) = 0$ imply $S' \in \mathcal{M}$ (always true if μ is an outer measure). We say $S \subset T$ is *universally measurable* (w.r.t. $\mathcal{M}$ and without reference to μ) if S is measurable for each bounded (equivalently, σ-finite) outer (equivalently, complete) measure whose set of measurable sets contains $\mathcal{M}$. If $\mathcal{M}$ contains all of its universally measurable sets, it is said to be *complete*. Of course, if μ is σ-finite and complete, then $\mathcal{M}$ is complete.

We shall frequently employ an assumption that is weaker than $\mathcal{M}$ being complete, viz., that $\mathcal{M}$ is a Suslin family, defined next. This definition employs the Suslin operation, which has been central to the classical development of the theory of Suslin sets. We make little use of the Suslin operation other than to define "Suslin family"; however, it is used in several papers to prove results cited below.

We fix $\mathscr{V}$ and $\mathscr{V}^*$ as the respective sets of infinite and finite sequences of positive integers. Let $\mathscr{F}$ be a family of sets, and $A: \mathscr{V}^* \to \mathscr{F}$. For $\sigma \in \mathscr{V}$, denote $(\sigma_1, \cdots, \sigma_n)$ by $\sigma|n$, following [RG]. Then

$$\bigcup_{\sigma \in \mathscr{V}} \bigcap_{n=1}^{\infty} A_{\sigma|n}$$

is said to be *obtained from $\mathscr{F}$ by the Suslin operation*. If every set obtained from $\mathscr{F}$ in this way is also in $\mathscr{F}$, we say $\mathscr{F}$ is a *Suslin family* ([RB] and [LE2–5], for example, say $\mathscr{F}$ admits the Suslin operation, and [DL] and [DAP2] say $\mathscr{F}$ is "souslinienne"). We always have $\{D: D$ is obtained from $\mathscr{F}$ by the Suslin operation$\}$ is a Suslin family ("generated by $\mathscr{F}$") [HF, § 19]. In a Hausdorff space, each Suslin set is in the Suslin family generated by the set of closed sets [RGW, Theorem 2]; in a Suslin space the converse holds (adapt the proof in [KU1, § 39, II]).

If μ is an outer measure, then $\mathcal{M}$ is a Suslin family, e.g., [SK, p. 50]. Consequently, if $\mathcal{M}$ is complete, $\mathcal{M}$ is a Suslin family. Also, as noted in [LE2], if μ is a Radon measure on a locally compact Hausdorff space, then it follows from [KU1, p. 95] that $\mathcal{M}$ is a Suslin family. These observations obviate most of the

complication which appears to be introduced by considering the Suslin operation, in contrast to consideration of continuous images of Polish spaces.

Let us consider some cases where measurable selections do not exist, i.e., $\mathscr{S}(F) = \varnothing$. The most elementary example is the case where $f: T \to X$ is not a measurable function and $F(t) = \{f(t)\}$ for $t \in T$. Then f is obviously the only selection of F. Suppose in particular that $T = X = [0, 1]$, μ is outer Lebesgue measure over T, $T \supset S \notin \mathcal{M}$, and f is the characteristic function of S. Now $\mathrm{Gr}\, F$ is measurable with respect to 2-dimensional (outer) Lebesgue measure, since it has measure 0 (but $\mathrm{Gr}\, F$ is not Borel). Thus, one can have $\mathscr{S}(F) = \varnothing$ even when $\mathrm{Gr}\, F$ has fairly nice measurability.

We shall see in Theorem 5.3, for example, that if $\mathrm{Gr}\, F$ is a Borel, or even Suslin, subset of R^2, then F has a selection which is a Lebesgue measurable function, and which will also (by Lusin's theorem, Theorem 14.1 below) be a.e. equal to a Borel function. However, with $\mathrm{Gr}\, F$ Borel in R^2 there need not exist a selection of F which is a Borel function on *all* of T, i.e., if $\mathcal{M} = \mathscr{B}(T)$, we may have $\mathscr{S}(F) = \varnothing$. This was shown by an example given in Novikov's [NO1] and in [LS] (see § 3) and a later example by Blackwell [BL].

Where assumptions on $\mathrm{Gr}\, F$ are not made, it helps for F to be measurable and to have closed values. The following three examples of Dauer and Van Vleck [DV] illustrate some bad behavior of measurable set-valued functions which are not closed valued. For Examples 2.1, 2.2, and 2.3, we let $T = X = [0, 1]$, μ be outer Lebesgue measure and $S \subset T$ have inner measure 0 and outer measure 1.

Example 2.1 [DV]. Let Q, Q' be disjoint dense countable subsets of $[0, 1]$. Let $F(t) = Q$ for $t \in S$ and $F(t) = Q'$ for $t \in T \backslash S$. Then F is not measurable, since $F^{-}(\{a\}) = S$ for $a \in Q$. However, F is weakly measurable, since $F^{-}(U)$ is $\varnothing$ or T for open $U \subset X$. Also, F is countable-valued. In [DV] it is shown that $\mathscr{S}(F) = \varnothing$.

Example 2.2 [DV]. Let $F(t) = X \backslash \{t\}$ if $t \in S$ and $F(t) = X$ if[1] $t \in T \backslash S$. Then F is measurable but $\mathrm{Gr}\, F \notin \mathcal{M} \otimes \mathscr{B}(X)$.

Example 2.3 [DV]. Let F be as in Example 2.2 and let $G(t) = \{t, 1\}$ for $t \in T$. Then[1] $(F \cap G)(t) = \{1\}$ for $t \in S$ and $(F \cap G)(t) = \{t, 1\}$ for $t \in T \backslash S$. While F and G are measurable, $F \cap G$ is not even weakly measurable.

The following example of Kaniewski (privately communicated via Kuratowski and Himmelberg) shows that a weakly measurable *closed-valued* function need not be measurable, even when T and X are Polish. Leese [LE3, p. 73, Example (vi)] had independently shown that this is true (with the same T and $\mathcal{M}$) without exhibiting an example.

Example 2.4 (Kaniewski). Let $T = [0, 1]$, Z be the set of irrationals, $X = T \times Z$, $p(t, n) = t$ for $(t, n) \in X$, $F = p^{-1}$, and $\mathcal{M} = \mathscr{B}(T)$. Since p is an open mapping, F is weakly measurable, in fact lsc. That F is not measurable is seen by taking a closed $K \subset X$ such that $p(K)$ is not Borel.

3. Pre-1949 history. A reasonable starting point for an historical discussion of measurable selections appears to be Lusin's 1930 book [LS, Chap. IV] and Novikov's 1931 paper [NO1]. Reference [LS] is a classic treatment of the theory of Suslin sets in R^n, the early development of which is primarily due to Suslin,

[1] Typographical error in [DV].

Sierpinski, and Lusin. Both [LS] and [NO1] make nonspecific reference to the other author's work, but neither cites these references. Both treat implicit functions in a way which constitutes a setting for the subject of Borel function selections. They consider a Borel $f: R^m \times R^p \to R^q$ from which one may define (we are rendering usages)

$$F(t) = R^p \cap \{y: f(t, y) = 0\} \quad \text{for } t \in R^m,$$

$$E = R^m \cap \{t: F(t) \neq \varnothing\}.$$

Both showed the following:

(i) If each $F(t)$ is countable, then E is a Borel set and $F|E$ has a Borel function selection.

(ii) Without the requirement that F be countable-valued, E need not be a Borel set and $F|E$ need not have a Borel function selection.

In giving (i), Lusin showed more, by way of decomposing Gr F—see § 10. Note that the assumption that F is countable-valued is a severe restriction. Achievement of (ii) centered on showing that there exist disjoint complementary Suslin (i.e., CA) sets which cannot be separated by Borel sets, the original demonstration of which Lusin credits to Novikov. Incidentally, since Gr F is Borel, we now know that F has a Lebesgue measurable selection (Corollary 5.2 below), which by other work of Lusin (Theorem 14.1 below) agrees a.e. with a Borel function.

Lusin also addressed the question: Given $g: E \to R^p$ such that $f(t, g(t)) = 0$ for $t \in E$, does there exist a Borel $h: R^m \to R^p$ such that $g = h|E$? This is a case of the extension problem noted in § 16 below and discussed in [HM2].

The usages "uniforme" (i.e., single-valued) and "multiforme" (i.e., multi-valued) functions in [LS], [NO1], and earlier works appear to have given rise to the term "uniformization," which is conceptually the same as "selection," but with a different emphasis on properties of the selections. This topic affords additional early history—see § 12.

Another early result is the following of Saks [SK, Lem. 7.1, p. 282] (first edition was 1933): If $X \subset R$ is compact, and $f: X \to T$ is continuous, then there exists a Borel $A \subset X$ such that $f(A) = f(X)$ and $f|A$ is one-to-one. Then $[f|A]^{-1}$ is a Borel function selection of $F = f^{-1}$—see Kuratowski [KU1, Chap. III, § 39, V, Thm. 3] (first edition was 1933). Also X could be any compact metric space, since such is a continuous image of a Cantor set. Saks' lemma was generalized to Lebesgue measurable f by Federer and Morse [FM] (1943), but in a way which does not appear to generalize the measurable selection consequence just stated. Mackey [MC1, Lem. 1.13] (1952) applied [FM] as noted in § 11 below. Baker [BA, Lem. 3] (1965) adapted Mackey's argument with [FM] to generalize Saks' lemma to the case where T and X each have a topology with countable base and are "almost Hausdorff" as defined in Theorem 12.4 below; the same Borel selection consequence follows.

The earliest result on existence of measurable selections without assuming countability or compactness of the values of F is von Neumann's in 1949, which we will come to in § 5. (See addenda (i), (ii).)

4. Closed-valued functions. In this section we survey selection results when F is closed-valued, generally without assumptions on Gr F. We remind the reader that (see § 2) μ is a measure over T for which $\mathcal{M}$ is the σ-algebra of measurable

sets, X is topologized, $\varnothing \neq F(t) \subset X$ for $t \in T$, and $\mathcal{S}(F)$ is the set of measurable selections of F.

The assumption that F is closed-valued is not as restrictive as might first appear. For example, if T is topologized as a T_1 space, $f: X \to T$ is continuous, and $F = f^{-1}$, then F is automatically closed-valued. We shall see in the next section how this observation may be used to derive graph-conditioned selection results from results of the sort given in this section.

Probably the most important result to date in the entire theory of measurable selections is the following theorem. Its hypotheses are sufficiently weak that it suffices for most applications, and numerous measurable selection results have been derived from it, including the earlier result of von Neumann [NE], Theorem 5.1 below. It has also been generalized somewhat.

THEOREM 4.1. *If F is weakly measurable and closed-valued and X is Polish, then $\mathcal{S}(F) \neq \varnothing$.*

Because Theorem 1 is so important, we discuss its origin in detail. This result was given in 1965, by Kuratowski and Ryll-Nardzewski in stronger form as Theorem 1 of [KRN] (see also [KU2, p. 74]), and independently by Castaing in more restricted form as Théorème 3 of [CA1]. In [KRN], $\mathcal{M}$ is permitted to be $\mathcal{L}_\sigma$, where $\mathcal{L}$ is a field (i.e., Boolean algebra) of subsets of T; this hypothesis is weaker than the requirement that $\mathcal{M}$ be a σ-algebra. Castaing's statement in [CA1] is an announcement, with proof deferred to Théorème 3 of [CA2] (1966) and Théorème 5.2 of his thesis [CA4, 5] (1967). In [CA1, 2, 4, 5] the assumption is made and utilized in proof that F is measurable, not just weakly measurable. Characteristic of Bourbaki measure foundations, it is also hypothesized that μ is a Radon measure on T, a locally compact space (in [CA1, 2] a compact space), but the method of proof (which uses a sifting, i.e., "criblage," of X) requires neither a topology nor a measure on T. The proofs in [KRN] and [CA2, 4, 5] construct in different ways a Cauchy sequence of functions which converges uniformly to a selection. Castaing was the first to show, by Théorème 5.4 of [CA4, 5] (same hypothesis as Théorème 5.2), that one can in fact obtain what has been termed a Castaing representation of F—see Theorem 4.2 below. (See addendum (ii).)

Subsequent to the appearance of [KRN] and [CA5], workers in the field became aware of the existence of Rokhlin's 1949 statement [RK2, § 2.9, Lem. 2] which was similar to Theorem 4.1 except that $\mathcal{M}$ was specialized to be isomorphic to the σ-algebra generated by the Lebesgue measurable subsets of [0, 1] and a countable family of atoms. In recent years Rokhlin has often been credited with the origination of, in effect, Theorem 4.1. However, although the statement in [RK2] is correct, the proof is not—the recursive construction does not satisfy (10n).[2] (See addendum (iii).)

A special variant of Theorem 4.1 was given in 1962 by Dixmier [DI, Lem. 2]: assuming also $T = X$, $\mathcal{M} = \mathcal{B}(T)$, and $\{F(t): t \in T\}$ is a partition of T, he obtained a selection f of F with range f Borel (Corollary 11.2(ii) below). Now there is a fairly easy metric argument in [HM2, Theorem 3.3] showing that where X is separable metric, F is weakly measurable only if Gr cl $F \in \mathcal{M} \otimes \mathcal{B}(X)$. This argument may be used (i) to deduce from [DI], Theorem 4.1 with the added conditions that T is

[2] For confirming our finding on this point we are indebted to Roman Pol and Pawel Szeptycki, who reviewed the original Russian version, and to Fred Van Vleck who reviewed the English translation.

Polish and $\mathcal{M}=\mathcal{B}(T)$ (by applying [DI] to the partition $\{\{t\}\times F(t): t\in T\}$ of Gr F), and (ii) to deduce this special case of Theorem 4.1 from von Neumann's earlier theorem, discussed in § 5 below. Dixmier used sifting methods. Plausibly [DI, Lem. 1] could be used with Castaing's argument to prove [CA4, 5, Thm. 5.2] with F weakly measurable rather than measurable.

An additional source for the proof of Kuratowski and Ryll-Nardzewski of Theorem 4.1 is [PR1], the first text on measurable selections.

That a closed-valued F is well-behaved is seen in the following theorem, which summarizes properties of such F given by Castaing [CA9], Rockafellar [RC3], Himmelberg [HM2], Himmelberg and Van Vleck [HV6], Leese [LE3], and Delode, Arino, and Penot [DAP2].

THEOREM 4.2. *Suppose F is closed-valued. Consider the following:*

(i) $F^{-}(B)\in\mathcal{M}$ *for* $B\in\mathcal{B}(X)$;
(ii) $F^{-}(K)\in\mathcal{M}$ *for closed* $K\subset X$, *i.e.*, F *is measurable*;
(iii) $F^{-}(U)\in\mathcal{M}$ *for open* $U\subset X$, *i.e.*, F *is weakly measurable*;
(iv) *for some metric d on X, $d(x, F(\cdot))$ is a measurable function for $x\in X$*;
(v) Gr $F\in\mathcal{M}\otimes\mathcal{B}(X)$;
(vi) Gr F *is the Suslin family generated by* $\mathcal{M}\otimes\mathcal{B}(X)$;
(vii) $\pi_T(A\cap \mathrm{Gr}\, F)\in\mathcal{M}$ *for* $A\in\mathcal{M}\otimes\mathcal{B}(X)$;
(viii) $\pi_T(A\cap \mathrm{Gr}\, F)\in\mathcal{M}$ *for A in the Suslin family generated by* $\mathcal{M}\otimes\mathcal{B}(X)$;
(ix) *F has a Castaing representation*;
(x) *there exists a measurable $f_i: T\to X$ for $i=1,2,\cdots$, such that $\{f_1(t), f_2(t),\cdots\}\cap F(t)$ is dense in $F(t)$ for $t\in T$ and $T\cap\{t: f_i(t)\in F(t)\}$ is measurable for $i=1,2,\cdots$*;
(xi) $F^{-}(C)\in\mathcal{M}$ *for compact* $C\subset X$.

We then have the following:

(a) (ix)$\Leftrightarrow$(x).
(b) *If X has a countable base, then* (iii)$\Rightarrow$(v).
(c) *If X is regular and a continuous image of a space with a countable base, then* (ii)$\Rightarrow$(v).
(d) *If X is separable metric, then* (ii)$\Rightarrow$(iii)$\Leftrightarrow$(iv)$\Rightarrow$(xi), (iii)$\Rightarrow$(v), *and* (ix)$\Rightarrow$(xi). *If also X is σ-compact, then* (ii)$\Leftrightarrow$(iii)$\Leftrightarrow$(ix)$\Leftrightarrow$(xi).
(e) *If X is separable metric and F is complete-valued, then* (iii)$\Leftrightarrow$(ix)$\Leftrightarrow$(xi).
(f) *If $\mathcal{M}$ is a Suslin family and X is regular and a weakly Suslin space, then* (ii)$\Leftrightarrow$(v)$\Rightarrow$(ix).
(g) *If $\mathcal{M}$ is a Suslin family and X is metric Suslin, then* (i) *through* (x) *are equivalent.*

Proof. The proof of [RC6, Thm. 1B] proves (a); (b) and (c) are given as [LE3, Thms. 3.6 and 3.7]; (d) and (e) come from [HM2, Thms. 3.5 and 5.6] and [HV6, Thm. 1′]; (f) follows from Theorem 6.1 below and [LE3, Thm. 3.9], observing that F is of Suslin type.

It remains to prove (g). From what has been proved and obvious observations, (viii)$\Rightarrow$(vii)$\Rightarrow$(i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(iv)$\Rightarrow$(v)$\Rightarrow$(vi) and (ix)$\Leftrightarrow$(x). The proof of [RC6, Thm. 1B] shows (ix)$\Rightarrow$(ii). Leese (personal communication) has deduced (ix) and (viii) from (vi) as follows. One shows that $\mathcal{M}\otimes\mathcal{B}(X)$ and hence the Suslin family generated by $\mathcal{M}\otimes\mathcal{B}(X)$ are contained in the Suslin family generated by $\{S\times K: S\in\mathcal{M}$ and $K\subset X$ is closed$\}$. Hence (vi) implies (viii) by [LE5, Thm. 5.5]. It also follows that (vi) implies that F is of Suslin type by [LE2, Thm. 6], from which (ix) follows by Theorem 6.1 below; thus (vi)$\Rightarrow$(ix). □

The most comprehensive conclusion in Theorem 4.2 is (g). For σ-finite complete μ and Polish X, Castaing [CA9, Lem. 1] gave (i)$\Leftrightarrow$(iii)$\Leftrightarrow$(iv)$\Leftrightarrow$(v)$\Leftrightarrow$(ix) (and (iv)$\Rightarrow$(ii)$\Rightarrow$(iii) is elementary). Rockafellar [RC3, Thm. 1] added that these are equivalent to (x). Very recently, Delode, Arino, and Penot [DAP2] added (vi), (vii), and (viii) to the equivalence and weakened the requirement on μ to $\mathcal{M}$ being a Suslin family. Leese (and subsequently Delode [DL, 2.9]) observed that Polish X could be weakened to metric Suslin X. Under separable metric X, Himmelberg [HM2] has given (b), (c), (d), (e), and various related facts.

The equivalence (iii)$\Leftrightarrow$(ix) in Theorem 4.2(e) is useful in both directions. For example, Rockafellar [RC2] applies this equivalence with $X=R^n$ to show measurability of the intersection and the closed vector sum of measurable closed-valued functions. He gives a more comprehensive treatment of related manipulations in [RC6]. (An even more powerful manipulative tool is Leese's theory of Suslin type—see § 6.) The set values involved in Rockafellar's [RC1–6] are primarily epigraphs of convex real-valued functions on a separable reflexive Banach space, parameterized on a complete σ-finite measure space. Measurability of the epigraph-valued function is a key criterion for a convex "integrand" to be "normal" (see [RC3, 6]). The above (iii)$\Leftrightarrow$(ix) is used in showing, for instance, that a convex integrand which is a finite Carathéodory map is normal, and that conjugation of convex normal integrands is reflexive. These facts are, in turn, useful to optimization of convex integral functionals by duality methods. Following is an application of (iii)$\Leftrightarrow$(x). (See addendum (v).)

COROLLARY 4.3 (Rockafellar [RC6, Cor. 1D]). *Suppose $\mathcal{M}$ is a Suslin family, X is Polish, and for a.e. $t\in T$, $F(t)=$* cl interior *$F(t)$ (as is true, for instance, if $X=R^n$ and $F(t)$ is an n-dimensional closed convex set). Then F is measurable iff $F^{-}(\{x\})$ is measurable for $x\in X$.*

In the remainder of this section we give a chronological review of additional results with closed-valued F.

Himmelberg and Van Vleck [HV2, Thm. 5] observed that the completeness requirement in Theorem 4.1 could be put on the values of F (as in Theorem 4.2(e)) rather than on a homeomorph of X. They obtained a precursor to [HV3] (see § 5) with X a metric Lusin space, and various results pertaining to (xi) in Theorem 4.2 and to $\mathcal{M}$ being merely a σ-ring. Reference [HV2] supersedes [HV1].

In extending Scorza-Dragoni's generalization of Lusin's theorem (see § 14), Castaing [CA13, Thm. 5] gave a result to the effect that if T is compact, μ is Radon, X is metric, and F is "approximately lower semi-continuous" and complete-valued, then F has an a.e. measurable selection. Jacobs [JC1] and Himmelberg, Jacobs, and Van Vleck [HJV] gave related results.

In the following theorem, Castaing has substituted existence of a suitable sifting [BO1, Chap. IX, § 6.5] of X for some of the assumptions in Theorem 4.2, motivated by his proof of [CA5, Thm. 5.2].

THEOREM 4.4 [CA15, 16, Thm. 1]. *Suppose X is a Hausdorff space, $((C_1, p_1, \varphi_1), (C_2, p_2, \varphi_2),\cdots)$ is a sifting of X, $F^{-}(\varphi_n(c))\in\mathcal{M}$ for $c\in C_n$ and $n=1,2,\cdots$, and F is closed-valued. Then there exists a selection of F which is a pointwise limit of measurable functions on T to X with countable range.*

Following is a corollary to this theorem.

Mägerl [MG, Kapitel IV, Korollar 2.4] independently obtained $\mathcal{S}(F)\neq\varnothing$ under the bracketed hypothesis of Corollary 4.5. Valadier [VA4, 5, Lem. 1] obtained the conclusion of Theorem 4.4 in Castaing representation form assuming F is closed-valued, $\mathrm{Gr}\, F\in\mathcal{M}\otimes\mathcal{B}(X)$, X is Suslin, and μ is complete and σ-finite.

COROLLARY 4.5 [CA16, Cor. 6] *Suppose X is a Suslin {Lusin} space, F is closed-valued, and $F^-(A)\in\mathcal{M}$ for every Suslin {Borel} $A\subset X$. Then F has a Castaing representation.*

Following is a novel theorem of Robertson [RB] using a "left set," i.e., the set A. Theorem 2 of [RB] is an antecedent to the "Suslin type" development of his student S. J. Leese (see § 6).

THEOREM 4.6 [RB, Thm. 4]. *Suppose F is measurable and closed-valued and X is a continuous image of a set $A\subset R$ with the property that $\inf D\in A$ for $\varnothing\neq D\subset A$. Then $\mathcal{S}(F)\neq\varnothing$.*

The next theorem is a generalization by Leese (personal communication) of Kuratowski's [KU4, Thm. 5.2]. The latter has T and X metric Suslin and concludes $\mathcal{S}(F)\neq\varnothing$.

THEOREM 4.7. *Suppose T is topologized and let $\mathcal{L}$ be the Suslin family generated by the closed sets of T. Suppose $\mathcal{M}\supset\mathcal{L}$, X is regular and weakly Suslin, F is closed-valued, and $F^-(A)\in\mathcal{L}$ for closed $A\subset X$. Then F is of Suslin type (see § 6) and hence F has a Castaing representation.*

Proof. Note § 6 and follow the proof of [RB, Lemma 1]. □

Maitra and Rao have weakened the separability of X in Theorem 4.1, adding other restrictions, as follows.

THEOREM 4.8 [MR1, Cor. 4]. *Assume the Zermelo–Frankel axioms, the axiom of choice, and Martin's axiom. Suppose $T=R$, $\mathcal{M}$ is the set of Lebesgue measurable subsets of R or the set of subsets of R having the Baire property, X is complete metric with base of cardinality less than $2^{\aleph_0}$, and F is closed-valued and weakly measurable. Then $\mathcal{S}(F)\neq\varnothing$.*

Artstein [AR2, Prop. 4.12] has shown that under conditions resembling those of Theorem 4.9 given next, if for $i=1,2,\cdots,(F_1,F_2,\cdots)$ "converges weakly" to F, and $f\in\mathcal{S}(F)$, then there exists $f_i\in\mathcal{S}(F_i)$ for $i=1,2,\cdots$, such that $(f_1,f_2,\cdots)$ converges weakly to f.

THEOREM 4.9 [AR2, Thm. 2.7]. *Suppose $T=[0,1]$, μ is Lebesgue measure, $X=R^n$, and F is closed-valued. Then there exists a closed-valued $G\colon T\to\mathcal{P}(X)$ such that $\mathrm{Gr}\, G$ is Borel, $G(t)\subset F(t)$ for a.e. $t\in T$, and the a.e. measurable selections of F coincide with those of G.*

Many selection results make the strong assumption that F is compact-valued. Following is such a result by Leese which has weak assumptions in other respects. Combined with Theorem 4.11, we have a rather general selection result for closed-valued F. Theorem 4.10 is given in [LE5] under kinds of generalizations mentioned in § 13 below. As observed by Leese, 4.10(i) implies [RB, Thm. 1], which assumes X is a Hausdorff continuous image of a separable metric space.

THEOREM 4.10 [LE5, Thms. 4.1 and 4.2]. *Suppose F is compact-valued and measurable. Then $\mathcal{S}(F)\neq\varnothing$ providing one of the following holds:*

(i) *there exist closed $K_1,K_2,\cdots\subset X$ such that for each distinct pair of points in X, some K_n contains one and not both (Leese's Condition (S)); or*

(ii) *$\mathcal{B}(X)$ is generated by a family of closed sets whose cardinality is at most the first uncountable cardinal (Leese's Condition (B)).*

THEOREM 4.11 [LE3, Thm. 8.6]. *Suppose $\mathcal{M}$ is a Suslin family, X is regular and analytic in the sense of being a continuous image of a countable intersection of countable unions of closed compact subsets of some topological space, and F is measurable and closed-valued. Then there exists a measurable compact-valued $G\colon T\to\mathcal{P}(X)$ such that $G(t)\subset F(t)$ for $t\in T$.*

5. Graph-conditioned theorems. In this section we recount the development of selection theorems based on properties of $\mathrm{Gr}\, F$ rather than on conditions on the values of F. The two topics are linked, as shown in Theorem 4.2, in the proof of Theorem 5.3, and, more extensively, in § 6. We again remind the reader (for the last time) that T, $\mathcal{M}$, μ, X, F and $\mathcal{S}(F)$ are fixed in § 2. (See addendum (i).)

The present topic begins with the 1949 selection result of von Neumann (also given with same proof in [PR1]).

THEOREM 5.1 [NE, Lem. 5]. *Suppose $T=R$, X is a Suslin subset of a Polish space, $f\colon X\to T$ is continuous and surjective, $F=f^{-1}$, and μ arises from a non-decreasing right-continuous bounded $g\colon R\to R$. Then $\mathcal{S}(F)\neq\varnothing$.*

Proof (outline). Represent X as a continuous image of ω^ω, topologized homeomorphic to the irrationals, where $\omega=\{1,2,\cdots\}$. For each $t\in T$, select the lexicographic minimum in ω^ω of the counterimage of $F(t)$ and map this back to $F(t)$. □

This in effect is what von Neumann stated. His proof is still valid if the conditions on T and μ are replaced by the condition that T be Hausdorff and $\mathcal{M}$ contain the Suslin sets of T. In this generality we note a corollary of a form (Suslin graph) in which von Neumann's theorem is often given. Recall that when T is Hausdorff, $\mathcal{M}$ contains the Suslin sets of T if, in particular, $\mathcal{M}\supset\mathcal{B}(T)$ and μ is an outer measure.

COROLLARY 5.2. *Suppose T and X are Polish, $\mathrm{Gr}\, F$ is Suslin, and $\mathcal{M}$ contains the Suslin sets of T. Then $\mathcal{S}(F)\neq\varnothing$.*

Proof. Let $f=\pi_T$ and, replacing X by $T\times X$, apply Theorem 5.1 (generalized as noted). □

Von Neumann's result seems to have been little known until around 1965 when it surfaced separately in mathematical economics, notably in Aumann's [AU1, 2], and in control theory, although it was referenced and used by Mackey [MC2] in 1957, for example. (We know of three leaders in measure theory who were unaware of it in 1971.) Ironically, one suspects that its recognition suffered from submergence under the prolific output of a giant.

We now depart from chronology to note how graph-conditioned theorems, including 5.1 and 5.2 just given, can be derived from a closed-valued result such as Theorem 4.1. Castaing [CA4, p. 123] was the first to do this—one lifts a set-valued function with Suslin graph to a measurable closed-valued function into a Polish space. This idea was used by Himmelberg and Van Vleck in [HV3] to prove a version of Theorem 5.3 in a more direct way than in [CA4]. It has been exploited more extensively by Leese in his "Suslin type" approach—see § 6.

The following theorem and proof are largely given by Leese [LE5, Thm. 7.4]. Both Theorem 5.3 and Corollary 5.4 were for the most part contained in a personal communication we received from Castaing in 1972 under the stronger assumption that T and X are Suslin spaces (see also [SB3, Thm. 2] and [HJ, Thm. III.9.6]). Here it is assumed what seems just enough to make the proof of Theorem 5.3 work—the method is closer to that of [HV3] than to Castaing's.

THEOREM 5.3. *Suppose T is topologized as a T_1 space, Gr F is weakly Suslin, and $\mathcal{M}$ contains each weakly Suslin subset of T. Then F has a Castaing representation.*

Proof. Take a Polish space P and a continuous surjective $\varphi: P \to \operatorname{Gr} F$. Let $G = (\pi_T \circ \varphi)^{-1}$. Since $\pi_T \circ \varphi$ is continuous and T is a T_1 space, G is closed-valued. Also, G is measurable, because for closed $A \subset P$, $G^-(A) = \pi_T(\varphi(A))$ so $G^-(A)$ is a continuous image of the Polish space A, whence $G^-(A) \in \mathcal{M}$. By Theorem 4.2(e), G has a Castaing representation $\{g_1, g_2, \cdots\}$. Then for $i = 1, 2, \cdots$, $f_i \equiv \pi_X \circ \varphi \circ g_i \in \mathcal{S}(F)$, and $\{f_1(t), f_2(t), \cdots\}$ is dense in $F(t)$ for $t \in T$. □

COROLLARY 5.4. *Suppose T and X are Suslin spaces, $\mathcal{M}$ contains each Suslin subset of T, $f: X \to T$ is continuous and surjective, and $F = f^{-1}$. Then F has a Castaing representation.*

Proof. Since f is continuous, Gr F is closed in $T \times X$, and hence is Suslin [HV3, Lem.]. Thus, Theorem 5.3 applies. □

Christensen and Jayne have shown [CH, Thm. 4.3] that a continuous map on a Polish space onto a compact metric space need not have a Borel function inverse; of course, when $\mathcal{M} = \mathcal{B}(T)$, $\mathcal{M}$ will not ordinarily contain all Suslin sets of T.

Hoffman-Jørgensen has obtained a Borel function inverse of a Borel function, as follows (a somewhat related result in [CH, Thm. 4.3] is a specialization of Theorem 6.1 below).

THEOREM 5.5 [HJ Thms. III.11.B. 8–11]. *Suppose T and X are Suslin spaces, $f: X \to T$ is a Borel function, $F = f^{-1}$, and $\mathcal{M} = \mathcal{B}(T)$. Then $\mathcal{S}(F) \neq \varnothing$ providing one of the following holds.*

(i) *F is weakly measurable and either F is compact-valued or* Gr *F is Polish;*
(ii) Gr F is σ-compact;
(iii) Gr *F is Lusin and F is countable-valued.*

Mägerl's [MG, Kapitel III, Satze 2.6, 2.7] follow from Theorem 5.3 by letting T be Hausdorff and μ be an outer measure for Satz 2.6 and T be locally compact Hausdorff and μ be Radon for Satz 2.7.

Returning to chronology, the first generalization of von Neumann's result was the following by Sion in 1960. Sion's paper has been well known in uniformization theory, but belatedly known in measurable selection theory; it does not reference [NE]. Note that his condition on X is satisfied when X is Polish. He made a weaker assumption on Gr F than that given here, viz., that Gr F is "analytic" in $T \times X$, by which he means a continuous image of a countable intersection of countable unions of compact subsets of a Hausdorff space. In [SN, Cor. 4.4], the assumption on μ is omitted, but $\mathcal{M}$ is generated by the "analytic" subsets of T.

THEOREM 5.6 [SN, Cor. 4.5]. *Suppose T is Hausdorff, μ is an outer measure, $\mathcal{M} \supset \mathcal{B}(T)$, X is a regular Hausdorff Lindelöf space with a base of cardinality no greater than $\aleph_1$, and* Gr *F is Suslin. Then $\mathcal{S}(F) \neq \varnothing$.*

The next graph-conditioned theorem to appear was the following of Blackwell and Ryll-Nardzewski in 1962. It is unusual in imposing measure-theoretic

conditions on X. Its motivation was to prove that if μ is a probability measure, f is a real random variable on T, and range f is not Borel, then there does not exist an everywhere proper conditional distribution given f. It is applied again in [FU] and [BD].

THEOREM 5.7 [BRN, Thm. 2]. *Suppose T and X are Borel subsets of Polish spaces, $\mathcal{M} \subset \mathcal{B}(T)$, $\mathcal{M}$ is countably generated, and* Gr *$F \in \mathcal{M} \otimes \mathcal{B}(X)$. Suppose also that there exists $g: T \times \mathcal{B}(X) \to R$ such that $g(t, \cdot)$ is a probability measure on $\mathcal{B}(X)$ for $t \in T$, $g(\cdot, B)$ is a measurable function for $B \in \mathcal{B}(X)$, and $g(t, F(t)) > 0$ for $t \in T$. Then $\mathcal{S}(F) \neq \varnothing$.*

Aumann [AU3] in 1967 made a significant advance with the following graph-conditioned theorem which involves no topological assumption, although as observed by Sainte-Beuve in [SB3], one may just as well assume that X is a Lusin space and $\mathcal{A} = \mathcal{B}(X)$. Following [AU3], we say $(X, \mathcal{A})$ is a *standard space* if $\mathcal{A}$ is a σ-algebra of subsets of X and there is a one-one correspondence between X (not necessarily topologized) and R which induces a one-one correspondence between $\mathcal{A}$ and $\mathcal{B}(R)$.

THEOREM 5.8 [AU3]. *Suppose μ is σ-finite, $(X, \mathcal{A})$ is a standard space, and* Gr *$F \in \mathcal{M} \otimes \mathcal{A}$. Then there exist $S \in \mathcal{M}$ and a selection f of $F|S$ such that $\mu(T \backslash S) = 0$ and $f^{-1}(A) \in \mathcal{M}$ for $A \in \mathcal{A}$.*

An example in [AU3] due to Lindenstrauss shows that one may not let $\mathcal{A}$ be an arbitrary σ-algebra on X and Aumann shows that σ-finiteness may not be omitted. Also given in [AU3] is an interesting discussion of the question of whether a theorem such as 5.2 holds if Gr F is complementary Suslin.

Complementary Suslin sets also arise in the following result of Castaing.

THEOREM 5.9 [CA8, Prop. 1]. *Suppose T is a Suslin space, X is a metric Suslin space, F is complete-valued, $T \backslash F^-(A)$ is Suslin for every closed $A \subset X$, and $\mathcal{M} = \mathcal{B}(T)$. Then* Gr *$F$ is a Suslin space iff F has a Castaing representation.*

Sainte-Beuve generalized Theorem 5.8 in [SB1, 2, 3]. She assumed $\mathcal{M}$ is complete (no assumption on μ) and X is Suslin instead of Lusin and obtained $\mathcal{S}(F) \neq \varnothing$. Following is a further generalization by Leese [LE2, Cor. to Thm. 7] yielded by Theorem 6.1 below.

THEOREM 5.10. *Suppose $\mathcal{M}$ is a Suslin family, X is a weakly Suslin space, and* Gr *$F \in \mathcal{M} \otimes \mathcal{B}(X)$. Then F has a Castaing representation.*

To see that this, and similarly [SB1, 2, 3], generalize Theorem 5.8, extend the μ of Theorem 5.8 to an outer measure μ^* so that each μ^* measurable set differs from a μ measurable set by a set contained in a set of μ measure zero; the σ-algebra of μ^* measurable sets is a Suslin family. Apply Theorem 5.10 and check the counterimages of a countable base of X.

Dauer and Van Vleck have shown how measurable selections of cl F can be approximated by those of F. A metric on X induces the essential supremum pseudometric in $\mathcal{S}(F)$ and in Theorem 5.11 we topologize $\mathcal{S}(F)$ with this. A similar conclusion is in [LE3, Thm. 8.7], assuming F is weakly measurable instead of Gr F is Suslin.

THEOREM 5.11 [DV, Thms. 1, 2]. *Suppose T is locally compact separable metric, μ is Radon, X is metric Suslin and* Gr *F is Suslin. Then $\mathcal{S}(\operatorname{cl} F) = \operatorname{cl} \mathcal{S}(F)$. If instead the hypothesis of Theorem 5.8 is satisfied, then this conclusion holds for a.e. selections.*

When μ is Borel regular and σ-finite, von Neumann's theorem yields an a.e. selection of F which is a Borel function, by application of Lusin's theorem. The following version proved by Federer [WS, Thm. 4.1] obtains a Borel selection on most of T without assuming Borel regularity of μ. This version may also be proved by von Neumann's argument in Theorem 5.1 or, as Castaing has pointed out, by observing that every Suslin space is a Radon space [BO2, Chap. IX, § 3.3] and applying Theorem 5.3.

THEOREM 5.12. *Suppose T is Hausdorff, $\mathcal{M} \supset \mathcal{B}(T)$, μ is a bounded outer measure, X is a Suslin subset of a Polish space, $h: X \to T$ is continuous and surjective, $F = h^{-1}$, and $\varepsilon > 0$. Then there exist a compact $C \subset T$ and $f \in \mathcal{S}(F|C)$ such that $\mu(T \backslash C) < \varepsilon$ and f is a Borel function.*

We close this section with what seem to be the main graph-conditioned results of Leese's [LE5]. He generalizes in the following directions not shown here: (a) a "partial uniformization," i.e., existence of a well-behaved compact-valued subfunction of F, as in Theorem 4.11, is given, (b) properties are given of the T-projection of Gr F (F not necessarily defined on all of T), (c) X weakly Suslin as defined here, is sometimes weakened to X "analytic" and (d) non-σ-algebras are sometimes employed in place of $\mathcal{M}$ (see § 13).

THEOREM 5.13 [LE5, Thm. 5.5]. *Suppose $\mathcal{M}$ is a Suslin family, Gr F is in the Suslin family generated by $\{S \times K: S \in M$ and $K \subset X$ is closed$\}$, and X is weakly Suslin. Then $\mathcal{S}(F) \neq \varnothing$.*

THEOREM 5.14 [LE5, Thm. 6.2 or 6.3]. *Suppose T is topologized, $\mathcal{M}$ contains the Suslin family generated by the closed sets of T, X is weakly Suslin and Gr F is in the Suslin family generated by the closed sets of $T \times X$. Then $\mathcal{S}(F) \neq \varnothing$.*

6. Set-valued functions of Suslin type. We summarize in this section Leese's Suslin type approach, given in [LE2]. Theorem 6.1 is a succinct statement from which a great deal of the above results and of those in § 7 may be readily deduced. The theme of lifting F to a well-behaved map into the subsets of a Polish space, which underlies this development, has antecedents in work of Castaing [CA4], Himmelberg and Van Vleck [HV3], and Robertson [RB], as noted in § 4 and § 5.

We follow Leese [LE2], and add the bracketed "weak" version in the definition and associated theorems, in saying *F is of {weak} Suslin type* if there exist a Polish space P, a continuous $\varphi: P \to X$, and a {weakly} measurable closed-valued $G: T \to \mathcal{P}(P)$ such that $F(t) = \varphi(G(t))$ for $t \in T$. (Note that the significance of the word "weak" here differs from its significance in the definition of weak Suslin space.) The F in Kaniewski's Example 2.4 is of weak Suslin type but not of Suslin type. When F is of Suslin type, it is of weak Suslin type.

By Theorem 4.2(e) and the proof of Theorem 5.3, one easily obtains the following.

THEOREM 6.1 [LE2, Thm. 7]. *If F is of weak Suslin type, then F has a Castaing representation, so $\mathcal{S}(F) \neq \varnothing$.*

By itself, Theorem 6.1 adds little to prior knowledge. The usefulness of Leese's [LE2], which is considerable, lies in showing that many kinds of F are of Suslin type, and in giving additional properties of such F; this development holds for weak Suslin type also.

The following result of Robertson shows that Suslin type and weak Suslin type are the same thing in an important case.

THEOREM 6.2 [RB, Thm. 3]. *Suppose $\mathcal{M}$ is a Suslin family, X is a metrizable Suslin space, and F is closed-valued. Then F is measurable iff F is weakly measurable.*

COROLLARY 6.3. *If $\mathcal{M}$ is a Suslin family, then F is of Suslin type iff F is of weak Suslin type.*

Proof. Apply Theorem 6.2 to the G of the above definition. □

If X is Hausdorff and F is of weak Suslin type, then Gr F is in the Suslin family generated by $\{A \times B: A \in \mathcal{M}$ and $B \subset X$ is closed$\}$. The proof in [LE2, Thm. 5] must be modified with "weak" by using an open sifting and, as Leese has pointed out, by adding details to show $x = \varphi(y)$ on page 405.

Suppose $\mathcal{M}$ is a Suslin family and X is Hausdorff. Then the class of set-valued functions of Suslin type is a Suslin family (operating pointwise on T), in particular it is closed under countable union and intersection. It is also closed under countable Cartesian product. If also X is a topological vector space, then this class is closed under vector addition, multiplication by a measurable scalar function, and formation of closed convex hulls. These properties are in [LE2].

In each of the following cases F is of {weak} Suslin type (largely in [LE2]–[LE4] corrects the proof of [LE2, Thm. 6]):

(i) X is Polish and F is closed-valued and {weakly} measurable;
(ii) $G: T \to \mathcal{P}(X)$ is compact-valued and {weakly} measurable, X is separable metric with completion $\hat{X}$, i embeds X in $\hat{X}$, and $F = i \circ G$;
(iii) $\mathcal{M}$ is a Suslin family, X is a regular space and a Suslin space, and F is {weakly} measurable and closed-valued;
(iv) $\mathcal{M}$ is a Suslin family, X is a weakly Suslin space, and Gr $F \in \mathcal{M} \otimes \mathcal{B}(X)$;
(v) T is a T_1 space, $\mathcal{M}$ contains each weakly Suslin subset of T, and Gr F is weakly Suslin.

With these observations and other hints in [LE2], one may readily deduce from Theorem 6.1 all of 4.1 (the primary basis of 6.1), 4.2(f)(g) [(ii)⇒(ix)], 4.5, 4.7, 5.1, 5.2, 5.3, 5.4, 5.6, 5.8, 5.10, and also 7.1 and 7.2 of the next section.

7. Measurable implicit functions. In this section we fix a topological space Y, a function $g:$ Gr $F \to Y$, and a measurable function $h: T \to Y$ such that $h(t) \in g(\{t\} \times F(t))$ for $t \in T$. We are concerned with whether there exists $f \in \mathcal{S}(F)$ such that $h = g(\cdot, f(\cdot))$; such an f is a measurable implicit function pertaining to this structure. If we define

$$G(t) = X \cap \{x: g(t, x) = h(t)\} \quad \text{for } t \in T, \tag{7.1}$$

this becomes the question of whether $\mathcal{S}(F \cap G) \neq \varnothing$. Results on this question have been quite numerous, apparently because many applications, notably in control theory, arise naturally in this form. They are sometimes called Filippov type theorems, recalling the lemma of [FI], which was the first selection result of this kind.

Theorems 7.1 and 7.2, due to Leese, 7.3, due to Hoffman-Jørgensen, and 7.4, due largely to Castaing and Himmelberg, are rather general theorems of the sort sought. (Theorem 7.4(i) was given by Castaing [CA9, Corollaire] under Polish X and σ-finite complete μ.) They treat the respective cases where g is $\mathcal{M} \otimes \mathcal{B}(X)$ measurable (i.e., $g^{-1}(U) \in \mathcal{M} \otimes \mathcal{B}(X)$ for open $U \subset X$), continuous, Borel, and a Carathéodory map. Under the latter condition, Lemma 7.5 (which generalizes

[HM2, Thm. 6.1 and KU1, p. 378]) further facilitates application of Theorem 7.1. In Theorems 7.1 and 7.2, $\mathcal{M}$ must be a Suslin family (§ 2), which is weak enough for most applications. Set-valued functions of Suslin type are defined in § 6. Lusin measurability of a function is defined in § 14; it implies ordinary measurability, and when the range space is separable metric and μ is σ-finite, the converse holds.

The statement of Theorem 7.2 in [LE2, Thm. 9] also assumes that X is regular, but Leese has shown [LE3, p. 82] that this assumption may be omitted.

In [HM2, Thm. 7.1] separability of Y is omitted. However, Leese has pointed out, and Himmelberg concurs (both in personal correspondence), that the *argument* fails with this omission; if Y is not separable, $p: T \to Y$ and $q: T \to Y$ are measurable, and $r(t) = (p(t), q(t))$ for $t \in T$, then r need not be measurable. The same difficulty arises in [HM2, Thms. 7.2 and 7.4]. The validity of these three theorems of [HM2] without separability of Y is an open question.

THEOREM 7.1 [LE2, Thm. 8].[3] *Suppose $\mathcal{M}$ is a Suslin family, X is Hausdorff, Y is separable metric, F is of Suslin type, and g is $\mathcal{M}\otimes\mathcal{B}(X)$ measurable. Then there exists $f \in \mathcal{S}(F)$ such that $h = g(\cdot, f(\cdot))$.*

THEOREM 7.2 [LE2, Thm. 9] and [LE3, p. 82]. *Suppose T is locally compact Hausdorff, μ is Radon, X and Y are Hausdorff, F is of Suslin type, g is continuous, and h is Lusin measurable* (see § 14). *Then there exists $f \in \mathcal{S}(F)$ such that $h = g(\cdot, f(\cdot))$.*

THEOREM 7.3 [HJ, Thm. III.16.10].[3] *Suppose T, X, Y, and* Gr *F are Suslin, g is a Borel function, and either* (i) *$\mathcal{M}$ is generated by the Suslin subsets of T, or* (ii) *$\mathcal{M} \supset \mathcal{B}(T)$ and μ is σ-finite and complete. Then there exists $f \in \mathcal{S}(F)$ such that $h = g(\cdot, f(\cdot))$.*

THEOREM 7.4. *Suppose Y is separable metric, and g is a Carathéodory map. Then there exists $f \in \mathcal{S}(F)$ such that $h = g(\cdot, f(\cdot))$, providing one of the following holds:*

(i) *$\mathcal{M}$ is a Suslin family, X is weakly Suslin, and* Gr *$F \in \mathcal{M}\otimes\mathcal{B}(X)$; or*

(ii) *$\mathcal{M}$ is a Suslin family, X is Hausdorff, and F is of Suslin type; or*

(iii) [HM2, Thm. 7.1] *X is separable metric, F is measurable, and either F is compact-valued or F is closed-valued and X is σ-compact.*

Proof. Under (ii), the definition of Suslin type lets us confine to a Suslin subspace of X. By Lemma 7.5 given next, g is $\mathcal{M}\otimes\mathcal{B}(X)$ measurable. Somewhat as in [HM2, Thm. 7.4], let $\psi(t, x) = (g(t, x), h(t))$ for $t \in T$, $x \in X$. Since Y is separable metric, $\mathcal{B}(Y \times Y) = \mathcal{B}(Y)\otimes\mathcal{B}(Y)$. Hence ψ is $\mathcal{M}\otimes\mathcal{B}(X)$ measurable. With G as in (7.1), Gr $G = \psi^{-1}(\{(y, y): y \in Y\}) \in \mathcal{M}\otimes\mathcal{B}(X)$, so G is of Suslin type (§ 6), hence so is $F \cap G$. By Theorem 6.1, $(F \cap G) \neq \varnothing$. The proof under (i) is similar, using (iv) at the end of § 6. □

[3] Leese has observed that "Y is separable metric" in Theorem 7.1 may be weakened to "Y satisfies Condition (S)" (see Theorem 4.10): Then

$$(T \times X)\backslash \mathrm{Gr}\, G = \bigcup_{n=1}^{\infty} [g^{-1}(K_n) \cap (h^{-1}(Y\backslash K_n) \times X)] \in \mathcal{M}\otimes\mathcal{B}(X),$$

where $\{K_1, K_2, \cdots\}$ is the separating family and G is as in (7.1), whence $F \cap G$ is of Suslin type. From this, Theorem 7.3(ii) follows from Theorem 7.1.

LEMMA 7.5 [LE3, Lem. 14.1]. *If X has a countable base, Y is perfectly normal, i.e., if Y is normal and each open set of Y is an F_σ, and g is a Carathéodory map, then g is $\mathcal{M}\otimes\mathcal{B}(X)$ measurable.*

If in implicit function results of this form we specialize g so that each $g(\cdot, x)$ is constant, replacing it with $k: X \to Y$, we obtain a lifting theorem, i.e., assurance of existence of $f \in \mathcal{S}(F)$ such that $h = k \circ f$. Theorems 7.1–7.4 yield fairly general statements of this nature. An additional lifting result is the following, suggested by Leese.

THEOREM 7.6. *Suppose F is of weak Suslin type, $\mathcal{M}$ is a Suslin family, Y is a Hausdorff space, $k: X \to Y$ is continuous, and $h(t) \in k(F(t))$ for $t \in T$. Then there exists $f \in \mathcal{S}(F)$ such that $h = k \circ f$.*

Himmelberg and Van Vleck [HV2] give lifting results with measurability of F, h, and f defined to mean that inverse images of compact sets are measurable ((xi) of Theorem 4.2) and also with $\mathcal{M}$ being a σ-ring rather than a σ-algebra. McShane and Warfield [MW] (see also [YO]) gave early results in lifting form; these are generalized by [HV2].

Hoffman-Jørgensen [HJ, III. 11] has given such lifting results, not involving F, and also results on the symmetric problem: Given $p: Z \to X$ and $q: Z \to T$, find a "nice" $f: T \to X$ such that $f \circ q = p$.

Under measurability of inverse images of compact sets, Himmelberg and Van Vleck have given an implicit function result as [HV6, Thm. 4(ii)]. Part (i) of that theorem follows from Theorem 7.2, above.

We now review various other results on measurable implicit functions, all of which may be readily deduced from the foregoing, most from Theorem 7.1.

In Filippov's highly influential 1959 lemma [FI], T, X, and Y are in Euclidean spaces, g is continuous, and F is compact-valued and usc, among other restrictions. Another early result is Wazewski's [WZ] (1961), heavily conditioned by compactness. Aronszayn in 1964 permitted F to be G_δ-valued, but constant, reported in [SV]. Olech [OL] in 1965 had g a Carathéodory map with X compact—he obtained a selection by lexicographic minimization, which has componentwise recursiveness in common with Filippov's approach. All of these were directly motivated by control theory applications.

Castaing [CA1] in 1965 (proof in [CA2]) was somewhat more general with X Polish, T compact metric, Y Hausdorff, and F closed-valued with Suslin graph, but with g continuous and μ Radon. Generalizations in similar vein were given in [CA4, 5, § 5] (with weaker assumptions on T) and by Jacobs [JC1, Thm. 2.2; JC2, Thms. 2.5, 2.5′]. Himmelberg, Jacobs, and Van Vleck [HJV, Theorems 3, 3′] put completeness on the values of F instead of on X.

In [HV3, Thms. 2, 3, 4], Himmelberg and Van Vleck primarily assume Gr F is weakly Suslin; Theorems 2 and 3 are implicit function theorems and Theorem 4 is a lifting theorem.

Furukawa's [FU, Lem. 4.6] is a special case of Theorem 7.4 (iii) above with $X \subset R^n$ compact, $Y = R^m$, T a Borel subset of a Polish space, and $\mathcal{M} = B(T)$.

Dauer and Van Vleck [DV] apply Aumann's Theorem 5.8 above, assuming in part μ σ-finite, X Lusin, and g measurable, to obtain an a.e. measurable implicit function. This is generalized independently by Sainte-Beuve [SB3] in fashion similar to her generalization of Theorem 5.8.

Mägerl [MA, Kapitel III, Satz 3.1] has T σ-compact, T and X Hausdorff, μ Radon, Y separable metric, Gr F Suslin, and g a Borel function.

Götz [GZ] has given a schematic tabular summary of measurable implicit function results (and of general measurable selection results and bang–bang results).

8. Convex-valued functions. In this section we assume that X is a linear space and usually that F is convex-valued. Separate topics are discussed, not ordered by chronology or supersession.

We define X' to be the dual of X, $\langle \cdot, \cdot \rangle$ to be the pairing on $X' \times X$, and for $C \subset X$,

$$\varphi(x', C) = \sup \{\langle x', x\rangle : x \in C\} \quad \text{for } x' \in X';$$

thus $\varphi(\cdot, C)$ is the support function of C.

When F is compact-convex-valued, we say F is *scalarly measurable* if for $x' \in X$, $\varphi(x', F(\cdot))$ is a measurable function. A function $f: T \to X$ is *scalarly measurable* if for $x' \in X'$, $\langle x', f(\cdot)\rangle$ is measurable. Thus named by Valadier [VA1], the concept of scalar measurability was (see [VA3, pp. 270–271]) introduced by Kudō [KD] and subsequently used by Richter [RI] and then Kellerer [KE] and Olech [OL] to obtain measurability of the lexicographic maximum of F with $X = R^n$ (generalized by Leese [LE3, Thm. 16.15]). Debreu [DE, (5.10)] and Castaing [CA 4, 5, Chap. 6] gave early results relating measurability of F to scalar measurability of F. The following selection theorem was given by Valadier (for earlier versions see [VA1, 2]).

THEOREM 8.1 [VA3, Props. 7, 8]. *Suppose X is locally convex Hausdorff, F is compact-convex-valued and scalarly measurable, and either* (i) *X is separated by a countable subset of X' or* (ii) *$F(t) \subset g(t)Q$ for $t \in T$, for some convex compact metrizable $Q \subset X$ and measurable $g: T \to R$. Then F has a Castaing representation consisting of scalarly measurable selections.*

Castaing [CA15, 16, Thm. 2] obtained this conclusion assuming instead of (i) or (ii) that μ is a complete probability measure, X is a Lusin space and each $F(t)$ is weakly locally compact and line-free. Benemara [BN1, Lem. 2] also obtained this conclusion, collateral to characterizing extreme scalarly measurable selections of F. Castaing [CA17, 18] treats a scalarly measurable convex-compact-valued F, parameterized on $[0, 1]$ in an absolutely continuous manner, and he obtains parameterized well-behaved selections. Additional results on existence of scalarly measurable selections have been given by Ekelund and Valadier [EV] (see § 10) and Valadier [VA6] (see § 16). In [CA20] Castaing shows that the set of scalarly measurable selections (identified under a.e. equality) is nonempty and compact, when the support functions of F belong to a Köthe space and X is Suslin, among other assumptions.

Suppose $X = R^n$ and h is a selection of co F, where co $F(t)$ is the convex hull of $F(t)$ for $t \in T$. Then by Carathéodory's theorem, for $t \in T$, there exist $\lambda_0(t), \cdots, \lambda_n(t) \geqq 0$, and $g_0(t), \cdots, g_n(t) \in F(t)$ such that $\sum_{i=0}^n \lambda_i(t) = 1$ and $h(t) = \sum_{i=0}^n \lambda_i(t) g_i(t)$. If such λ_i's and g_i's can be chosen a.e. as measurable functions, we say h has a *measurable Carathéodory representation*. Existence of such a representation is a key to proving various versions of the LaSalle bang–bang principle of control theory. We have stated the desired choice of λ_i's and g_i's as a measurable selection problem. It is solved, of course, by applying more general selection theorems. Consider the following theorem given by Wagner. Under (ii) it is essentially [CA6, Thm. 3]; under (i) or (iii) one picks a natural $G: T \to \mathscr{P}(R^{(n+1)^2})$ somewhat as in [AU1, Thm. 3] and [CA5, Thm. 7.1], proves G is of Suslin type by remarks in § 6, and applies Theorem 6.1. Theorem 4.2(g) affords alternative hypotheses equivalent to (i). Still earlier versions were given by Sonneborn and Van Vleck [SV], who applied Aronszajn's generalization of Filippov's lemma, and in [CA3]. A related result for constant F is given as [HJ, Thm. III.16.14], credited to Hermes [HE1].

THEOREM 8.2 [WG1, Lem. 2.5(a)]. *Suppose μ is a σ-finite outer measure, $X = R^n$, $h \in \mathscr{S}(\text{co } F)$, and either* (i) *$F$ is measurable and closed-valued; or* (ii) *co F is measurable and compact-valued; or* (iii) *Gr $F \in \mathscr{M} \otimes \mathscr{B}(X)$. Then h has a measurable Carathéodory representation.*

In [CA22, Thms. 1, 2], Castaing obtains Carathéodory map selections of a suitably parametrized closed-convex-valued function into a separable Banach space or the weak dual of such.

In discussing Theorem 4.2, we have noted Rockafellar's [RC1–6] use of measurable convex-valued functions in the form of epigraph functions associated with convex normal integrands. In this work, explicit results on existence of measurable selections are mainly those referenced in Theorem 4.2 and its proof; however, in additional various ways he uses the equivalence (iii)$\Leftrightarrow$(ix) in Theorem 4.2(e) to obtain and apply Castaing representations. (See addendum (v).)

Let $\ddot{F}(t)$ be the set of extreme points of $F(t)$ (the profile of $F(t)$) for $t \in T$. Himmelberg and Van Vleck have treated measurability properties of $\ddot{F}$ in [HV5]. Their Theorem 4(a) is a finite-dimensional version of the first of the following two theorems of Leese, who notes that their methods may be used to prove it. The Suslin type conclusion of Theorem 8.3 affords a generalization of [HV5, Thm. 3], which includes implicit function results (note that $\ddot{F}$ need not be closed-valued).

THEOREM 8.3 [LE3, Thm. 16.10]. *Suppose X is a separable metrizable topological vector space and F is measurable and compact-convex-valued. Then, Gr $\ddot{F} \in \mathscr{M} \otimes \mathscr{B}(X)$. Hence if also $\mathscr{M}$ is a Suslin family and X is a Suslin space, then $\ddot{F}$ is of Suslin type.*

THEOREM 8.4 [LE3, Thms. 16.13, 16.16, 16.18]. *Suppose X is a Hausdorff locally convex real vector space, F is measurable and convex-valued, and one of the following holds:*

(i) *$\mathscr{M}$ is a Suslin family, X is Suslin, and F is compact-valued;*

(ii) *X is separated by a countable subset of X' and F is compact-valued; or*

(iii) *X is separable metric and F is weakly-compact-valued.*

Then there exist $f_1, f_2, \cdots \in \mathscr{S}(\ddot{F})$ such that for $t \in T$, $F(t)$ is the closed convex hull of $\{f_1(t), f_2(t), \cdots\}$. Hence under (i) *or* (iii), *F has a Castaing representation.*

Leese has given the following two theorems in [LE1, 6]. In [LE6] the σ-algebra $\mathscr{M}$ is replaced by more general structures (see § 13). Related results on conjugate Banach spaces and some unsolved problems are also given in [LE6, § 4]. Under the hypothesis of Theorem 8.5, each compact convex set has a unique element closest to the origin [LE3, p. 54], and under the hypothesis of Theorem 8.6 this is true for closed convex sets [LE3, Lem. 9.5].

THEOREM 8.5 [LE1, Thm. 1; LE6, Thm. 2.3]. *Suppose X has a strictly convex norm and F is compact-convex-valued and weakly measurable. Then $\mathcal{S}(F) \neq \varnothing$.*

THEOREM 8.6 [LE1, Thm. 2; LE6, Thm. 3.3]. *Suppose X is a Banach space and has a uniform norm $\|\cdot\|$ (i.e., $\|x_n\| \leqq 1$, $\|y_n\| \leqq 1$, and $\|x_n + y_n\| \to 2$ implies $\|x_n - y_n\| \to 0$), and F is closed-convex-valued and weakly measurable. Then $\mathcal{S}(F) \neq \varnothing$.*

Cole [CL1, 2] has shown that if X is a separable reflexive Banach space, and F is convex-closed-bounded-valued on $T = [0, 1]$ and obeys a condition like Cesari's Q (e.g., [CE]), then F has a strongly measurable selection (pointwise limit of simple functions), and the set of such selections is weakly compact in itself. An earlier result of Himmelberg, Jacobs, and Van Vleck [HJV, Thm. 4] has some hypotheses in common with [CL1]. In [CA7, Cor. 4], Castaing obtains a Lusin measurable selection of F (see § 14), without separability of X.

9. Pointwise optimal measurable selections. Here we consider the existence of a measurable selection of F such that a real-valued function on $\operatorname{Gr} F$ is maximized pointwise: We suppose $u: \operatorname{Gr} F \to R$ is $\mathcal{M}\otimes\mathcal{B}(X)$ measurable and $u(t, \cdot)$ is usc on $F(t)$ for $t \in T$, and we let $v(t) = \sup\{u(t, x): x \in F(t)\}$ for $t \in T$. Our concern is whether there exists $f \in \mathcal{S}(F)$ such that $u(\cdot, f(\cdot)) = v$, and to this end, whether v is measurable. These results are sometimes called Dubins–Savage type theorems, after [DS, Lem. 6] (1965). (See addendum (vii).)

The strongest result to date appears to be the following, which combines Leese's [LE3, Prop. 14.8], [HPV, Thm. 2] of Himmelberg, Parthasarathy, and Van Vleck, and Schäl [SC1, Thm. 2; SC2, Prop. 9.4 and Thm. 12.1].

THEOREM 9.1. *Suppose F is compact-valued, and either*

(i) *$\mathcal{M}$ is a Suslin family, X is Hausdorff and F is of Suslin type; or*

(ii) *T and X are Borel subsets of Polish spaces, $\mathcal{M} = \mathcal{B}(T)$, and F is measurable; or*

(iii) *X is separable metric, F is measurable, and u is the limit of a decreasing sequence of Carathéodory maps.*

Then v is measurable and there exists $f \in \mathcal{S}(F)$ such that $u(\cdot, f(\cdot)) = v$.

Under (i), this is proved in [LE3] by showing, without assuming that F is compact-valued or that each $u(t, \cdot)$ is usc, that $G|S$ is of Suslin type, where $G(t) = F(t) \cap \{x: u(t, x) = v(t)\}$ for $t \in T$ and $S = T \cap \{t: G(t) \neq \varnothing\}$. Under (ii), it is proved in [HPV] via the "Kunugui–Novikov" theorem. Under (iii) one puts together the cited statements of Schäl (brought to our attention by Robert Kertz).

Various facts related to the condition on u given in Theorem 9.1 (iii) are given in [SC2, § 11]. If this condition were implied by the hypothesis of Theorem 9.1 (ii) (which includes that u is $\mathcal{M}\otimes\mathcal{B}(X)$ measurable and each $u(t, \cdot)$ is usc), then 9.1 (ii) would follow from 9.1 (iii); this appears to be an open question.

Castaing [CA17, Lem.] gave a version of Theorem 9.1 (i) with X a Lusin space and μ complete. Furukawa [FU, Thm. 4.1] obtained Theorem 9.1 (ii) with the added assumptions that X is compact, $X \subset R^n$, and u is a bounded Carathéodory map. Darst [DR, Thm. 1] obtained a Borel selection as in (ii), assuming X is compact metric, T is Polish, and u (and not just each $u(t, \cdot)$) is usc. Dubins and Savage made the stronger assumption that F is usc, as did Maitra [MT], and Hinderer [HD1, 2] in separate generalizations of [DS]. Debreu [DE, (4.5)] (1965) obtained measurability of v and G mentioned above.

Brown and Purves have given a related result when F is σ-compact-valued.

THEOREM 9.2 [BP, Cor. 1]. *Suppose T is a Borel subset of a Polish space, $\mathcal{M} = \mathcal{B}(T)$, X is Polish, F is σ-compact-valued, $\operatorname{Gr} F$ is Borel, $I = \{t:$ for some $x \in F(t)$, $u(t, x) = v(t)\}$, and $\varepsilon > 0$. Then I is a Borel set and there exists $f \in \mathcal{S}(F)$ such that*

$$u(t, f(t)) = v(t) \quad \text{when } t \in I,$$

$$u(t, f(t)) \geqq v(t) - \varepsilon \quad \text{when } t \notin I \text{ and } v(t) \neq \infty,$$

$$u(t, f(t)) \geqq \frac{1}{\varepsilon} \quad \text{when } t \notin I \text{ and } v(t) = \infty.$$

The problem of finding $f \in \mathcal{S}(F)$ such that $u(\cdot, f(\cdot)) \geqq v(\cdot) - \varepsilon(\cdot)$ is treated by Schäl [SC1], Strauch [ST], and Furukawa [FU], for example.

10. Decomposition of Gr F into measurable selections. For the problem of decomposing $\operatorname{Gr} F$ into measurable selections, we cite principally a 1930 result of Lusin [LS] on countably-valued F, from the early beginnings of measurable selection theory, and a theorem from Wesley's thesis [WE1] which is probably the most profound result to date in measurable selections.

THEOREM 10.1 [LS, p. 244]. *Suppose $T = R^m$, $X = R$, $F(t)$ is countable for $t \in T$, and $\operatorname{Gr} F$ is Borel. Then there exists a Borel map $f_i: T \to X$ for $i = 1, 2, \cdots$, such that $\operatorname{Gr} F \subset \bigcup_{i=1}^{\infty} f_i$ and for $i, j = 1, 2, \cdots$, we have $f_i(t) < f_j(t)$ for $t \in T$ or $f_j(t) < f_i(t)$ for $t \in T$.*

COROLLARY 10.2. *Under the hypothesis of Theorem 10.1 with $\mathcal{M} = \mathcal{B}(T)$, there exist $g_1, g_2, \cdots \in \mathcal{S}(F)$ such that $\operatorname{Gr} F = \bigcup_{i=1}^{\infty} g_i$. If each $F(t)$ is infinite, the g_i's may be taken to be distinct.*

Wesley [WE1, Thm. 1] obtained a version of Corollary 10.2 (wherein the selections are Lebesgue measurable), having belatedly learned of Lusin's results as indicated by his footnote. We conjecture, but have not verified, that Corollary 10.2 holds for arbitrary $(T, \mathcal{M})$ and separable metrix X. Himmelberg has shown this when F is finite-valued [HM2, Thm. 5.4].

For certain F having uncountable values, Wesley has given a nice partitioning of $\operatorname{Gr} F$ into measurable selections, as stated next. In [WE2] he has applied his methods to mathematical economics, i.e., to showing existence of a well-behaved representation of continuous preference orders parameterized in Borel fashion over $2^{\aleph_0}$ traders; he avoids connectedness assumptions made by Aumann [AU3].

THEOREM 10.3 [WE1, Thm. 2]. *Suppose T and X are Lusin spaces, μ is the completion of a σ-finite measure on $\mathcal{B}(T)$, $\mu(T) > 0$, $\operatorname{Gr} F$ is Borel, and $F(t)$ is uncountable for $t \in T$. Let $\mathcal{L}$ be the σ-algebra of Lebesgue measurable subsets of $[0, 1]$. Then there exists $h: T \times [0, 1] \to \operatorname{Gr} F$ such that:*

(a) *for $t \in T$, $h(t, \cdot)$ is a one-to-one Borel function on $[0, 1]$ onto $F(t)$;*

(b) *for $y \in [0, 1]$, $h(\cdot, y) \in \mathcal{S}(F)$;*

(c) *h is an $\mathcal{M}\otimes\mathcal{L}$ measurable function.*

Wesley's statement of Theorem 10.3 has $T = X = [0, 1]$ and $\mathcal{M} = \mathcal{L}$. The generalization to Lusin spaces is straightforward, as pointed out to us by Aumann, by taking isomorphisms between the measurable spaces $([0, 1], \mathcal{L})$ and $(T, \mathcal{M})$ via, e.g., [AS, Lem. 6.2], and between $([0, 1], \mathcal{B}([0, 1]))$ and $(X, \mathcal{B}(X))$ (one also needs $\mathcal{B}(T \times X) = \mathcal{B}(T)\otimes\mathcal{B}(X)$, e.g., via [HJ, Props. I.6.A.4 and I.5.B.7]).

Wesley's proofs of [WE1, Thms. 1, 2] are based on the Cohen forcing methods of mathematical logic, which, without implying doubt, we do not understand. He recommends (personal communication) explanations in [WE2] for better understanding of [WE1]. He also states that by modifying his proof, conclusions (b) and (c) may be strengthened to assert universal measurability, i.e., measurability with respect to any σ-finite complete measure whose set of measurable sets includes the Borel sets. (See addendum (viii).)

It would be desirable to prove Theorem 10.3 without the use of metamathematics. This problem appears to be quite difficult. Wesley poses the problem of proving his result without the Zermelo–Frankel replacement axiom.

Ekeland and Valadier [EV] have given decomposition results in the vein of this section, in the form of representing a compact-convex-valued function which is a Carathéodory map (see § 2). The following is taken from their Corollary 5 and Theorem 2.

THEOREM 10.4. *Let X be a compact metrizable subset of a locally convex topological vector space, Z be a topological space, and $G: T\times Z\to \mathcal{P}(X)$ be compact-convex-valued and a Carathéodory map (with respect to the Hausdorff metric on the set of compact subsets of X). Then there exists a Carathéodory map $f: T\times(Z\times X)\to X$ such that*

$$G(t, z)=\{f(t, z, x): x\in X\} \quad \text{for } t\in T, \quad z\in Z,$$

and such that if $g: T\to Z$ is strongly measurable, and $h: T\to X$ is scalarly measurable and with $h(t)\in G(t, g(t))$ for $t\in T$, then there exists a measurable $u: T\to X$ for which

$$h(t)=f(t, g(t), u(t)) \quad \text{for } t\in T.$$

Of course, the decomposition of Gr G provided by f in this theorem need not be a partitioning of Gr G, i.e., we might have $f(\cdot, \cdot, x)$ and $f(\cdot, \cdot, x')$ overlapping and unequal. Included here is a measurable implicit function result. In [EV], these results are given for G more general than being a Carathéodory map.

Larman's result [LA1, 2] noted in §12 below provides an uncountable disjoint family of selections of F which are Borel sets—it is not asserted that these exhaust Gr F.

11. Selections of partitions. In this section we suppose that $\mathcal{Q}$ is a partition of T. A *selection* of $\mathcal{Q}$ is a set $S\subset T$ such that $S\cap E$ is singletonic whenever $E\in\mathcal{Q}$. Here we let $T=X$ and F be given by the requirement that $t\in F(t)\in\mathcal{Q}$ for $t\in T$. We see that a selection of $\mathcal{Q}$ is the range of a selection f of F; however, f must also be constant on each $F(t)$. Note that the members of $\mathcal{Q}$ are closed iff F is closed-valued, and that this situation is associated naturally with the inverse of a continuous map. Also, to any $G: T\to\mathcal{P}(X)$ (without $T=X$) corresponds a natural partition of Gr G, viz., $\{\{t\}\times G(t): t\in T\}$, so that the results of this section are also relevant to the next section on uniformization. We let $\mathcal{L}$ be a family of subsets of T.

Early results on Borel selections of partitions were obtained by Mackey [MC1] in 1952 (Theorem 11.6 below) and Dixmier [DI] in 1962 (Corollary 11.2(ii) below—see also remarks in § 3 and following 4.1 and 11.6).

We begin with 1970 results of Hoffman-Jørgensen [HJ]. Although the hypothesis of the following rather general theorem seems complicated, all selection results of [HJ] cited in this survey are derived from it. Conditions somewhat similar to those of Theorem 11.1 are given at the end of [LE5, § 3], in a selection statement (not referring to partitions per se).

THEOREM 11.1 [HJ, Thm. II.6.1; or CH, Thm. 4.1]. *Suppose $\mathcal{L}\supset\mathcal{M}$, $\mathcal{L}$ is closed under countable union and countable intersection, and there exists $A: \mathcal{V}^*\to\mathcal{M}$ (see § 2) such that:*

(i) $T=\bigcup_{n=1}^{\infty}A_{(n)}$;
(ii) $A_{(\sigma_1,\cdots,\sigma_k)}=\bigcup_{n=1}^{\infty}A_{(\sigma_1,\cdots,\sigma_k,n)}$ *for $\sigma\in\mathcal{V}$, $k=1, 2, \cdots$;*
(iii) *for $\sigma\in\mathcal{V}$ and $t\in T$, letting $D_k=F(t)\cap A_{\sigma|k}$ for $k=1, 2, \cdots$, we have $\bigcap_{k=1}^{\infty}D_k$ is singletonic or for some k, $D_k=\emptyset$;*
(iv) $F^-(A_{\sigma|k})\in\mathcal{L}$ *for $\sigma\in\mathcal{V}$, $k=1, 2, \cdots$.*

Then $\mathcal{Q}$ has a selection S such that $T\backslash S\subset\mathcal{L}$.

The following two corollaries are given in [HJ]; Corollary 11.2 (ii) was previously given by Dixmier [DI] (cited in [HJ]). Dixmier applied his results to show the Borel nature of equivalence classes of factorial representations of a separable involutive Banach algebra and to give a converse of a result of Mackey [MC2].

COROLLARY 11.2 [HJ, Thms. III. 8.3–8.6]. *If T is topologized, $\mathcal{M}=\mathcal{B}(t)$, and F is closed-valued, then $\mathcal{Q}$ has a selection $S\in\mathcal{M}$ providing one of the following holds:*

(i) *distinct points of T are separable by continuous functions into $[0, 1]$, T is Suslin, and F is weakly measurable and compact-valued;*
(ii) *T is Polish and F is weakly measurable;*
(iii) *F is measurable, T is a countable union of closed Polish subspaces, and $\bigcup_{S\in\mathcal{Q}}(S\times S)\in\mathcal{B}(T\times T)$; or*
(iv) *T is a Lusin space and $F^-(B)\in\mathcal{B}(T)$ for $B\in\mathcal{B}(T)$.*

COROLLARY 11.3 [HJ, Thm. III. 8.7]. *Suppose F is closed-valued, T is Suslin, $\mathcal{L}$ is closed under countable union and countable intersection, and $\mathcal{L}$ contains A and $F^-(A)$ whenever $A\subset T$ is Suslin. Then $\mathcal{Q}$ has a selection S such that $T\backslash S\in\mathcal{L}$.*

Christensen has further applied Theorem 11.1 to the Effros σ-algebra over the set of closed subsets of T, when T is metric Suslin [CH, Thm. 4.2]. As noted in § 4, Theorem 11.2 (ii) constitutes a special case of Theorem 4.1 above, with $\mathcal{M}=\mathcal{B}(T)$.

Turning next to work of Kuratowski, Maitra, and Rao, following [KMT] we say $\mathcal{Q}$ is an $\mathcal{L}^-$ partition {an $\mathcal{L}^+$ partition} of T if $F^-(A)\in\mathcal{L}$ for each open $A\subset T\{T\backslash F^-(A)\in\mathcal{L}$ for each closed $A\subset T\}$, with F as above. Kuratowski and Maitra have given the following.

THEOREM 11.4 [KMT, § 3]. *Suppose $\mathcal{L}$ is a Boolean algebra (i.e., field), T is Polish, the open sets of T belong to $\mathcal{L}_\sigma$ (see § 2), each member of $\mathcal{Q}$ is closed, and $\mathcal{Q}$ is an $\mathcal{L}^+$ or $\mathcal{L}^-$ partition. Then there is a selection S of $\mathcal{Q}$ such that $T\backslash S\in\mathcal{L}_\sigma$.*

One application of this in [KMT] is to find a Borel set selection of $\mathcal{Q}$ which intersects each member of an analytic set of compact sets of T.

Special cases of Theorem 11.4, when $\mathcal{L}$ is a σ-algebra, have been given in [KU4, Thm. B] and [KU5, Thm. 7.1].

Maitra and Rao [MR2] have taken a different approach, utilizing a linear order on T induced by a continuous open map from the irrationals with lexicographic ordering. Their main result follows.

THEOREM 11.5 [MR2, Thm. 4.1]. *Suppose T is Polish, $\mathscr{L}$ is a σ-lattice containing the closed subsets of T, each member of $\mathscr{Q}$ is closed, and $\mathscr{Q}$ is an $\mathscr{L}^-$ partition. Then there is a selection S of $\mathscr{Q}$ such that $T\setminus S \in \mathscr{L}$.*

Both [KMT] and [MR2] apply their results to the case where $\mathscr{L}$ is the σ-additive lattice of subsets of T of additive class α, with [MR2] having stronger results. Also given in [MR2] are several examples showing that the latter results cannot be improved in certain ways. They cite a 1927 antecedent by Mazurkiewicz [MK].

We conclude this section with results on topological groups.

THEOREM 11.6. *Suppose T is a locally compact topological group. H is a metrizable closed subgroup of T, and $\mathscr{Q} = \{Ht: t \in T\}$. Then $\mathscr{Q}$ has a Borel set selection.*

This result was obtained by Mackey [MC1, Lem. 1.1] in 1952 with metrizability of H strengthened to separability (the latter implies the former in this context), using in the proof [FM, Thm. 5.1] of Federer and Morse. It was obtained as given here by Feldman and Greenleaf [FG, Thm. 1]. Weaker versions were given as [HJ, Thm. III. 16.6] and earlier as [DI, Lem. 3].

The selection of $\mathscr{Q}$ obtained in Theorem 11.6 determines a selection f of p^{-1} where $p: T \to T/H$ is canonical; in [FG] it is added that $f^{-1}(C)$ is in the σ-algebra generated by the compact sets of T/M for compact $C \subset T$, and that if T has an open subgroup $U \supset H$ such that $p(U)$ is σ-compact, then f may be obtained to be measurable w.r.t. $\mathscr{B}(T/H)$.

Greenleaf has applied Theorem 11.6 in [GR] to prove that a closed subgroup of an amenable group is amenable (a locally compact group G is called amenable if there is a left invariant positive linear functional M on $L_\infty(G)$ such that $M(h) = 1$ if $h(g) = 1$ for $g \in G$).

In 1965, Baker [BA, Thm. 2] and Effros [EF, Thm. 2.9] independently showed that several conditions previously shown to be equivalent by Glimm [GL] were also equivalent to the existence of a Borel set selection of the partitioning of an "almost Hausdorff" space M (see Theorem 12.4 below) into orbits of a locally compact Hausdorff group G of transformations acting continuously on M, when G and M each have countable base; one of these conditions is merely that M/G is a T_0 space. This has recently been applied by Bondar [BR].

One cannot omit "H is closed" in Theorem 11.6 [HJ, p. 177]: Let T be the additive reals and H be the rationals. Then $\mathscr{Q}$ has no Lebesgue measurable selection (the selections of $\mathscr{Q}$ are the examples usually given of non-Lebesgue-measurable sets). This has been generalized by Kuratowski [KU6]. The following remarkable converse has been pointed out by Bondar (who brought Theorem 11.6 to our attention) as a consequence of [BA, Thm. 2] and [MC2, Thm. 7.2]: If T is Polish, and $\mathscr{Q}$ has a Borel set selection, then H is closed.

12. Uniformization. The term "uniformization" is a synonym for "selection." One usually refers to uniformizations of $\mathrm{Gr}\, F$ rather than of F, and with interest in properties of a selection as a subset of product space (such as being a Borel set) rather than properties of mappings (such as being a Borel function). It dates from the era of [LS] and [NO1], as noted in § 3, or perhaps earlier.

An early result is the following, proved independently by Lusin [LS2] and Sierpinski [SP1]. If $T = X = R$ and $\mathrm{Gr}\, F$ is Borel in R^2, then F has a selection (uniformization) f such that $(T \times X)\setminus f$ is Suslin, that is, f is a complementary Suslin (i.e., CA) subset of R^2. This was improved by Kondô [KN] in permitting $\mathrm{Gr}\, F$ to be complementary Suslin and in other ways (see Sampei [SM] or Suzuki [SZ] for a later proof). Kondô's results were further generalized by Rogers and Willmott [RW], [WI]. Related results are given by Kuratowski [KU6]. A variation is claimed by Hoffman-Jørgensen [HJ, Thm. III.9.5] under Suslin T, X, and $\mathrm{Gr}\, F$; Leese finds the supporting argument incomplete. Jankov [JN] has shown that a Suslin subset of R^2 has a uniformization which is in the σ-algebra generated by the Suslin sets of R^2.

Results on G_δ uniformizations of F have been given by Braun [BR, Thm. 1] (she also showed that a closed subset of R^2 need not have an F_σ uniformization), Engelking [EN], and Michael [MI].

Larman's main theorem of [LA1, 2] yields an uncountable disjoint family of Borel set uniformizations of F, requiring that each $F(t)$ be an uncountable σ-compact G_δ, among other conditions. Brown and Purves [BP] show that if X and T are Polish, $\mathrm{Gr}\, F$ is Borel, F is σ-compact-valued, and $\mathscr{M} = \mathscr{B}(T)$, then there exists $f \in \mathscr{S}(F)$ (this much follows from Sion [SN]) such that f is a Borel subset of $T \times X$; they thereby generalize a result of Stschegolkow, given in [AL]. A similar result with different conditions on the values of F has been given by Sarbadhikari [SR].

To relate measurable selection results to uniformization results, one wishes to know when certain properties of $f: T \to X$ as a subset of $T \times X$ imply that f is a measurable function, and conversely. Following are some facts of this kind. See also [LE5, Appendix to § 6].

THEOREM 12.1 (Hoffman-Jørgensen [HJ, pp. 8–9]). *Suppose $\mathscr{N}$ is a σ-algebra over X (X not necessarily topologized), some countably generated sub-σ-algebra of $\mathscr{N}$ separates X (equivalently, $\{(x, x): x \in X\} \in \mathscr{N} \otimes \mathscr{N}$), $f: T \to X$, and $f^{-1}(A) \in \mathscr{M}$ for $A \in \mathscr{N}$. Then $f \in \mathscr{M} \otimes \mathscr{N}$.*

This very general statement implies, in particular, [KU5, § 2, Thm. 8]. It is also given, essentially, as [SB3, Prop. 2].

THEOREM 12.2 (Leese—personal communication). *Suppose $\mathscr{M}$ is a Suslin family, X is weakly Suslin, $f: T \to X$, and $f \in \mathscr{M} \otimes \mathscr{B}(X)$. Then f is a measurable function.*

Proof. Note (iv) at the end of § 6. □

THEOREM 12.3 (Leese—personal communication). *Suppose T is topologized, $\mathscr{M}$ contains the Suslin family generated by the closed sets of T, X is analytic in the sense that there exist a Polish space P and a compact-valued u.s.c. $G: P \to \mathscr{P}(X)$ such that $X = G(P)$, $f: T \to X$, and f is in the Suslin family generated by the closed sets of $T \times X$. Then f is a measurable function.*

Proof. Apply [LE5, Thm. 8.2], originally due to Rogers and Willmott. □

THEOREM 12.4 (Baker [BA, Lem. 4]). *Suppose T is topologized and T and X each have a countable base and are almost Hausdorff, i.e., are locally compact T_0 spaces with every nonvoid locally compact subspace containing a nonvoid relatively open Hausdorff subspace. Suppose $B \in \mathscr{B}(T)$, $f: B \to X$, and $f \in \mathscr{B}(T \times X)$. Then f is a Borel function.*

Theorem 12.5 (Hoffman-Jørgensen [HJ, Thm. III. 4.1]). *Suppose T and X are Suslin and $f: T\to X$. Then f is a Borel function iff f is a Borel subset of $T\times X$ iff f is a Suslin subset of $T\times X$.*

Theorem 12.6 (Lehn [LN2]). *Suppose $(T, \mathcal{M}, \mu)$ is the completion of the measure space $(T, \mathcal{M}_0, \nu)$, $(T, \mathcal{M}_0)$ and $(X, \mathcal{N})$ are countably separated Blackwell spaces (see [HJ]), $f: T\to X$, and $f\in \mathcal{M}\otimes\mathcal{N}$. Then $f^{-1}(A)\in\mathcal{M}$ for $A\in\mathcal{N}$.*

Valadier [VA4, Cor.] relates scalar measurability of $f: T\to X$ (with X locally convex) to $f\in\mathcal{M}\otimes\mathcal{B}(X)$.

In [HJ, III. 16.3, 5] examples are given where (a) X is R^2, T is topologized but not Suslin, $g: X\to T$ is continuous and bijective, g^{-1} is a Borel subset of $T\times X$, and g^{-1} is not a Borel function; and (b) $f: R\to R$ is not a Lebesgue measurable function but f is a complementary Suslin subset of $R\times R$ and hence a Lebesgue measurable set ((b) assumes axiom of constructibility—see also [AU3]).

13. Measurability with other structures. In this section we replace the role of $\mathcal{M}$ with a family $\mathcal{L}$ of subsets of T not necessarily a σ-algebra and we define $\mathcal{Q}$ as a similar family of subsets of X. Our interest is in selections f of F which are $(\mathcal{L}, \mathcal{Q})$ measurable in the sense that $f^{-1}(A)\in\mathcal{L}$ for $A\in\mathcal{Q}$. Measurability of F is defined similarly. No role is played by μ in this section.

Let $\mathcal{K}$ and $\mathcal{G}$ be the respective families of closed and open subsets of X, and, when T is topologized, let $\mathcal{F}$ be the family of closed subsets of T.

The most important case where $\mathcal{L}$ is not a σ-algebra is when $\mathcal{L}$ and $\mathcal{Q}$ are both topologies. This is the subject of continuous selections, i.e., $(\mathcal{F}, \mathcal{K})$ measurable selections in the above terminology. This topic has extensive literature which is essentially topological, rather than measure-theoretical, in character, and which we do not review here. We merely cite three general references, [MI1], [FL], and the first half of [PR1], where numerous additional references may be found.

We have noted that Kuratowski and Ryll-Nardzewski [KRN] have shown that Theorem 4.1 holds with $\mathcal{M}=\mathcal{L}_\sigma$, where $\mathcal{L}$ is a Boolean algebra. This generality enables them to obtain selections which are continuous, continuous modulo first category sets, or of additive class α (i.e., $f^{-1}(U)$ is Borel of additive class α for open $U\subset X$). Leese [LE5, Thm. 3.2] has sharpened this slightly: Let $\mathcal{L}$ be closed under finite union and intersection,[4] $\varnothing\in\mathcal{L}$, and $\mathcal{D}=\{A\setminus B: A, B\in\mathcal{L}\}$; then if X is Polish and F is closed-valued and $(\mathcal{L}_\sigma, \mathcal{G})$ measurable, there exists a $(\mathcal{D}_\sigma, \mathcal{G})$ measurable selection of F. In fact several of Leese's results given above have been stated by him in this kind of generality, generalizing the σ-algebra $\mathcal{M}$ differently in the hypothesis and the conclusion, i.e., Theorems 4.10 [LE5, Thms. 4.1 and 4.2], 5.13 [LE5, Thm. 5.5], 5.14 [LE5, Thm. 6.2 or 6.3], 8.5 [LE6, Thm. 2.3], and 8.6 [LE6, Thm. 3.3]. Here is a companion result (when $\mathcal{L}$ is a σ-algebra and a Suslin family this is included in Theorem 5.13).

Theorem 13.1 [LE5, Thm. 5.2]. *Suppose $\varnothing\in\mathcal{L}$, X is Polish, $\mathcal{R}=\{S\times K: S\in\mathcal{L}, K\in\mathcal{K}\}$, $\mathrm{Gr}\,F$ is in the Suslin family generated by $\mathcal{R}$, $\mathcal{A}=\{A: A\subset T$ is in the Suslin family generated by $\mathcal{L}\}$, and $\mathcal{D}=\{A\setminus A': A, A'\in\mathcal{A}\}$. Then there exists a selection f of F such that $f^{-1}(U)\in\mathcal{D}_\sigma$ for $U\in\mathcal{G}$.*

[4] Leese has pointed out that [LE5, Thm. 3.2] should include the requirement that $\mathcal{L}$ be closed under finite intersection.

Engelking [EN, Thm. 1] obtains an $(\mathcal{F}_\sigma, \mathcal{G})$ measurable selection of F; here T is compact and perfectly normal, X is metrized, and F is usc and complete-separable-valued. If "usc" is replaced by "lsc," then "separability" may be omitted, as proved independently by Čoban. Čoban [CB1, 2] gives numerous theorems on $(\mathcal{F}_\sigma, \mathcal{G})$ measurable selections, and related selection results. In [CB3] he gives a variation on Theorem 4.1 with $F^{-}(U)$ a kind of complementary Suslin set for open $U\subset X$ and with the selection obtained a kind of Borel function. Rogers and Willmott [RGW, Thm. 20] find a selection f of F such that for open $U\subset X$, $f^{-1}(U)$ is the T projection of a complementary Suslin subset of $T\times X$; here $\mathrm{Gr}\,F$ is complementary Suslin among other conditions.

Maitra and Rao give the following result.

Theorem 13.2 [MR2, Thm. 2]. *Suppose $\varnothing\in\mathcal{L}$, $T\in\mathcal{L}$, and $\mathcal{L}$ is closed under countable union and finite intersection. Let $\mathcal{L}'=\{T\setminus D: D\in\mathcal{L}\}$. Then the following are equivalent:*

(a) *Whenever $A, B\in\mathcal{L}$ and $A\cap B=\varnothing$, there exists $D\in\mathcal{L}\cap\mathcal{L}'$ such that $A\subset D$ and $B\subset T\setminus D$ (i.e., $\mathcal{L}'$ satisfies the first principle of separation, equivalently the weak reduction principle).*

(b) *If X is compact metric, then any $(\mathcal{L}, \mathcal{G})$ measurable closed-valued $G: T\to \mathcal{P}(X)$ has an $(\mathcal{L}\cap\mathcal{L}')_\sigma$ measurable selection.*

Their Theorem 1 generalizes this statement to the use of higher ordinals and cardinals in the union closedness condition and in the weak reduction principle and to avoiding compactness in (b), thereby extending Theorem 4.1. From this Theorem 1, [MR2] further deduces, in addition to some known results, Theorem 4.8 above and a selection result (Theorem 6) which assumes that F is a countable union of weakly measurable closed-valued functions and that an "$\aleph_1$ weak reduction principle" holds for the "measurable" sets of T.

Kaniewski and Pol give the following result, which does not assume separability of X. They also present some related examples and pose some unsolved problems.

Theorem 13.3 [KP, Thm. 2]. *Suppose T is an absolutely analytic [HN] and F is compact-valued and $(\mathcal{L}, \mathcal{G})$ measurable, where $\mathcal{L}=\{S: S\subset T$ is a Borel of additive class $\alpha\}$ with $0<\alpha<\omega_1$. Then there exists an $(\mathcal{L}, \mathcal{G})$ measurable selection of F.*

Whitt [WH] gives conclusions in terms of $(\mathcal{F}_\sigma, \mathcal{G})$ measurable selections and of selections of third Baire class.

14. Lusin measurable set-valued functions and selections. Let us recall Lusin's theorem as given in [FE, § 2.3.4 and § 2.3.6].

Theorem 14.1. *Suppose μ is an outer measure. Suppose also μ is Borel regular and T is metric {μ is Radon and T is locally compact Hausdorff}, X is separable metric, $f: T\to X$ is measurable, $\mu(T)<\infty$, and $\varepsilon>0$. Then there is a closed {compact} $C\subset T$ such that $\mu(T\setminus C)<\varepsilon$ and $f|C$ is continuous. If also μ is σ-finite, f is a.e. equal to a Borel function.*

We note three related directions in which Lusin's theorem has been generalized.

First, there are formulations of Lusin's theorem for set-valued maps. Pliś [PL1] (1961) and Castaing [CA1, 2, 4, 5] have given such for compact-valued maps, in which case the Hausdorff metric is a natural tool. Extensions to closed-valued maps have been given by Jacobs [JC2], Himmelberg, Jacobs and Van Vleck [HJV], and Castaing [CA8]. In [HJV] and [CA8] the restricted maps obtained are semi-continuous; Castaing uses the term "approximately semi-continuous" maps.

Second, one may formulate Lusin type theorems for $g: T\times Y\to X$, where Y is a topological space and g is a Carathéodory map. These are called Scorza-Dragoni theorems, after [SD] (1948), the first result of this type. Van Vleck has pointed out to us that Krasnosel'skii's [KR] (first edition 1956) also gave such a result as Lemma 3.2. Subsequent generalizations have been given by Castaing [CA4, 5, 8], Goodman [GD], and Jacobs [JC1].

Third, we have Scorza-Dragoni type results for set-valued maps. Results of this kind have been given by Jacobs [JC2], Castaing [CA8, 13], Himmelberg, Jacobs, and Van Vleck [HJV], Brunovský [BV], Himmelberg [HM1], and Himmelberg and Van Vleck [HV4, 9], usually having the restricted set-valued map semi-continuous.

For the rest of this section, we assume T is topologized as a Hausdorff space and μ is an outer measure and is σ-finite and Radon. Castaing has defined F to be *Lusin measurable* if for some partition $\{S, C_1, C_2, \cdots\}$ of T, $\mu(S)=0$ and for $i=1,2,\cdots$, C_i is compact and $F|C_i$ is usc. If $f: T\to X$ and $F(t)=\{f(t)\}$ for $t\in T$, Lusin measurability of F coincides with "μ measurability" of f, here called *Lusin measurability* of f, as defined in [BO2, Chap. IV, § 5.1], since F is then usc iff f is continuous. If $f: T\to X$ is Lusin measurable, it is measurable as defined in § 2. If the hypothesis of Theorem 14.1 holds, then f is Lusin measurable.

As Théorème 8.4 of [CA4], Castaing has given the following selection result and a corollary (there not restricted to positive measure).

Theorem 14.2. *Suppose T is locally compact, X is separated by a sequence of continuous real-valued functions, and F is Lusin measurable and compact-valued. Then $\mathcal{S}(F)\neq\emptyset$.*

Castaing has given results on existence of Lusin measurable selections in [CA7, Cors. 1–4], with X a reflexive Banach space, not necessarily separable. Leese's Theorem 7.2 above uses a Lusin measurability hypothesis. In general, the main usefulness of Lusin measurability seems to be in dealing with nonseparable spaces.

15. Set-valued measures. Loosely speaking, one calls Φ a set-valued measure if X is (at least) an Abelian topological group and $\Phi: \mathcal{M}\to\mathcal{P}(X)$ is suitably countably additive. Central to the approaches that have been taken appear to be the definitions of convergence of an infinite sum in $\mathcal{P}(X)$. Our interest here is in the existence of a selection of such a Φ which is a measure on T.

Set-valued measures appear to have originated with Brooks' work [BK] on a finitely additive function Φ on $\mathcal{M}$ into the set of bounded convex sets of a real Banach space. From this point of departure, Godet-Thobie has developed the subject extensively during 1970–75 in a series of papers [GT1–4] and, with Pham The Lai, [GTP], culminating in her thesis [GT5]. She has X a Frechet space in [GT1], X a Banach space in [GT2], and Φ closed-bounded-convex-valued in both. In [GT3, 4], X is a locally convex Hausdorff real vector space; here a convex-compact-valued $\Phi: \mathcal{M}\to\mathcal{P}(X)$ is called a set-valued measure ("multimesure," in [PB2] "multi-mesure faible") if for each point in the dual of X, the associated support function of Φ is a (not necessarily positive) measure. Apparently, [GT4] supersedes [GT1]. Her results are further generalized and unified in [GT5], where, with substantially more abstraction and embedding, X is an Abelian topological group.

Artstein [AR1] (1972) deals with $X\subset R^n$, and his results seem more accessible. In [AR1], $\Phi: \mathcal{M}\to\mathcal{P}(X)$ is a *set-valued measure* if $\Phi(\bigcup_{j=1}^{\infty} S_j)=\sum_{j=1}^{\infty}\Phi(S_j)$ whenever $S_1, S_2, \cdots\in\mathcal{M}$ are mutually disjoint; the sum of a sequence of subsets of R^n is the set of absolutely convergent sums of selections of the sequence. His main selection result follows.

Theorem 15.1 [AR1, Theorem 8.1]. *Suppose $\mu(T)<\infty$, $X\subset R^n$, Φ is a set-valued measure with convex values, $\Phi\ll\mu$, i.e., $\mu(A)=0$ implies $\Phi(A)=\{0\}$, $S\in\mathcal{M}$, and $x\in\Phi(S)$. Then there exists a selection θ of Φ such that θ is a (vector-valued) measure on $\mathcal{M}$ and $\theta(S)=x$.*

Neither the convexity condition nor the condition $\Phi\ll\mu$ may be omitted, as shown in [AR1]. However, the conditions on μ and the convexity condition may be replaced by $\Phi(T)$ being bounded [AR1, Theorem 8.3]. The boundary condition $\theta(S)=x$ in this type of selection result originated in [AR1].

Pallu de la Barrière [PB1, Théorème 3] considers a compact-convex-valued Φ with X a reflexive vector space topologized compatibly with its dual; he uses the Hausdorff metric to define the above summation. With no further assumption he obtains the conclusion of Theorem 15.1.

Costé's [CS1, Thm. 1.2] is in the vein of [GT3, 4] with less assumption on X, but with locally compact line-free values of Φ. In [CS1, Thm. 2.1], [CS3, Thms. 1, 3], and [CS7, Thm.], X is a Banach space and results in the vein of Theorem 15.1 are given; Φ is closed-bounded-valued (and convex-valued in [CS7]) and the conclusion is of the form $x\in\mathrm{cl}\{\theta(T): \theta$ is a selection measure of $\Phi\}$. A similar conclusion is attained in [CS6, Thm. 2–1] with "σ-additive" $\mathcal{M}$ and Φ. In [CS6, Thm. 1–3], he generalizes [PB1, Thm. 3] to finitely additive Φ and selections of Φ, with $\mathcal{M}$ a Boolean algebra. He further obtains in [CS2, Prop. 1] a Radon selection of a compact-valued Φ with X a complete Hausdorff locally convex space.

Thiam [TH1] requires Φ to have positive values as determined by a fixed cone in X, a vector space. For a minimal extremal point x of $\Phi(T)$, by methods of [PB1], he finds a selection measure θ such that $\theta(T)=x$ and $\theta(A)$ is minimal extremal in $\Phi(A)$ for $A\in\mathcal{M}$. When X is locally convex Hausdorff and Φ is weakly-compact-valued such that sup $\Phi(A)$ exists for $A\in\mathcal{M}$, applying [CS 6], for $x\in\Phi(T)$ he finds a selection measure θ such that $\theta(T)=x$. In [TH2], he treats an additive function on a clan of subsets of T into a semi-group of subsets of X, assumed locally convex Hausdorff; additive selections are obtained.

In [GT-4, 5, 6], Godel-Thobie considers set-valued transition measures, i.e., set-valued measures measurably parameterized with respect to a second measure space. Selections are found in the form of transition measures analogous to those of Markov processes.

Selection results for set-valued measures are applied in [AR1, § 9], [GT2, 5], [CS1], and [CP1] to obtain Radon–Nikodym type results, extending earlier results of Debreu and Schmeidler [DES]. A counterexample to [AR1, Thm. 9.1] is asserted in [CP2].

16. Special topics. We note a few treatments of existence of measurable selections which do not come directly under our above topic headings.

Theorem 4.1, for example, may be used as follows to find a measurable extension of a measurable $f: S \to X$ where $S \in \mathcal{M}$ (e.g., see Maitra and Rao [MR2, Cor. 6]). For extension results without assuming $S \in \mathcal{M}$, see Himmelberg [HM2, § 8].

THEOREM 16.1. *Suppose $S \in \mathcal{M}$, $f: S \to X$ is measurable and X is a Lusin space. Then there exists a measurable $g: T \to X$ such that $g|S = f$.*

Proof. Let $F(t) = \{f(t)\}$ for $t \in S$ and $F(t) = X$ for $t \in T \backslash S$. Take a Polish space P, a continuous bijective $\varphi: P \to X$, and, by Theorem 4.1, $h \in \mathcal{S}(\varphi^{-1} \circ F)$. Let $g = \varphi \circ h$. □

Garnir and Garnir-Monjoie [GGM; GM] treat $T = R^m$, $X = R^n$, and F such that for some $S \in \mathcal{M}$, $\mu(S) = 0$ and $\mathrm{Gr}[F|(T\backslash S)]$ is Suslin. Measurable selections are readily found from known results.

Maritz' thesis [MZ] gives, first of all, an excellent history of the theory of set-valued functions, with an extensive bibliography. He develops a comprehensive treatment of the subject under F and μ having values in Banach spaces, including generalizations in this context of known selection results.

Nürnberger's thesis [NU] treats $T = X$ and F of the form

$$F(t) = \{x: d(t, A) = d(x, A)\} \quad \text{for } t \in T,$$

where d is a metric on X and $A \subset X$ is fixed. For such F, called a projection, he finds Borel function selections in Theorems 4, 5, 6, and 8.

In [VA6, Lem. 3 and Thm. 2], as a tool to generalizing Strassen's theorem, Valadier finds a "pseudo-selection" of F, i.e., a scalarly measurable (see § 8) $\sigma: T \to X'^*$ such that

$$\langle x', \sigma(t)\rangle \leqq \sup \{\langle x', z\rangle: z \in F(t)\} \quad \text{for a.e. } t \in T,$$

for $x' \in X'$. In fact, existence of σ for which equality holds is shown. Here X is a locally convex Hausdorff vector space, X' is its topological dual, X'^* is the algebraic dual of X', and F is convex-compact-valued with all of its support functions finitely integrable. Theorems 3 and 4 relate pseudo-selections to selections.

Blackwell and Dubins obtain the following result related to Theorem 5.7 (from [BRN]). When $\mathcal{M} = \mathcal{B}(X)$, the selection obtained is trivially the identity map of X, so the interest arises when $\mathcal{M}$ is a coarser σ-algebra than $\mathcal{B}(X)$.

THEOREM 16.2 [BD, Thm. 4]. *Suppose $T = X$, X is a Borel subset of a Polish space, $\mathcal{M} \subset \mathcal{B}(X)$, and there exists $g: X \times \mathcal{B}(X) \to R$ such that $g(x, \cdot)$ is a probability measure on $\mathcal{B}(X)$ for $x \in T$, and $g(\cdot, B)$ is a measurable function for $B \in \mathcal{B}(X)$. Then there exists a measurable $f: X \to X$ such that $f(x) \in S$ whenever $x \in S \in \mathcal{M}$.*

A structure more general than ours has been treated very recently by Delode [DL], using as foundation slightly earlier work (which generalizes on separable metrix X) by Delode, Arino, and Penot [DAP1, 2]. Suppose $p: E \to T$ is surjective, $p^{-1}(t)$ is topologized for $t \in T$ (E as a whole need not be topologized), $\mathcal{E}$ is a σ-algebra on E which induces $\mathcal{B}(p^{-1}(t))$ on $p^{-1}(t)$ for $t \in T$, and $p^{-1}(S) \in \mathcal{E}$ for $S \in \mathcal{M}$. Then $(E, \mathcal{E}, T, \mathcal{M}, p)$ is called a *measurable field of topological spaces*. It is *Suslin* if there exists another such object $(E', \mathcal{E}', T, \mathcal{M}, p')$ such that $p^{-1}(t)$ is a Suslin space for $t \in T$ and $f: E' \to E$ such that $p' = p \circ f$, $f|p'^{-1}(t)$ is continuous for $t \in T$ and $f^{-1}(B) \in \mathcal{E}'$ for $B \in \mathcal{E}$. This structure specializes to ours by letting $E = \mathrm{Gr}\, F$ and $p = \pi_T|\mathrm{Gr}\, F$. (Beyond this specialization [DL] and [DAP2] give examples in various spaces of interest in functional analysis.) In this specialization, a Suslin field (as a subfield of $T \times X$) is the graph of a set-valued function of Suslin type (§ 6). In [DAP1, 2], each $p^{-1}(t)$ is metric (usually separable) and existence of a subset of $\mathcal{S}(p^{-1})$ satisfying certain axioms is assumed. Relevance of [DAP2] and [DL] to Theorem 4.2(g) above is noted following Theorem 4.2.

17. Recommended introductory reading. We briefly outline a recommended sequence of reading for someone who is fairly new to the subject of measurable selections and who would like to acquire at least a moderately general knowledge.

The best starting point is Rockafellar's [RC2]. This has $X = R^n$ and takes one through several important fundamentals in an easily readable way. A comprehensive exposition of closed-valued $F: T \to \mathcal{P}(R^n)$ is given in his forthcoming [RC6, § 1], also easily readable.

We recommend next Himmelberg's [HM2]. This gives the principal fundamental results on measurable selections and related properties of measurable sive exposition of closed-valued $F: T \to \mathcal{P}(R^n)$ is given in his [RC6, § 1], also easily readable.

Recommended next are Kuratowski's and Ryll-Nardzewski's [KRN], whose main theorem and proof have not been greatly improved upon, and the main published portion of Castaing's widely referenced thesis [CA5]. The latter is the first comprehensive treatment of measurable set-valued functions and is still worthy of careful review. We emphasize that is is more easily read if preceded by [RC2] and [HM2]. (A comment in [RC2] to the effect that [CA5] primarily treats compact-valued functions is not applicable to the measurable selection portion of [CA5]). One expects that [CA5] will be superseded by the forthcoming Castaing-Valadier text [CV2]. (See addendum (iii).)

We consider that Leese's [LE2] on set-valued functions of Suslin type has considerable unifying effect and we recommend it next accordingly.

This much should give the reader a rather good general knowledge. A graduation piece for an ambitious reader is Wesley's [WE1] profound proof of his easily stated result, Theorem 10.3 above (see also [WE2]). (See addendum (viii).)

Needless to say, a very considerable amount of excellent work on measurable selections is not included in this short list. A general knowledge afforded by these recommended papers can be substantially illuminated in terms of historical development and of specialization in several directions, as may be surmised from the diversity of topics addressed in this survey. It is hoped that the survey itself will give guidance to such further reading, for which the survey is certainly no substitute.

Acknowledgment. Since this survey is by a user of, rather than a significant contributor to, measurable selection results, we have leaned to an unusual degree on help from others. It is a pleasure to acknowledge first of all our particularly heavy debt to Charles Himmelberg, Fred Van Vleck, Charles Castaing, Stephen Leese, Kasimir Kuratowski, Robert Aumann, R. Tyrrell Rockafellar, and James Bondar. Dr. Leese gave a most useful critique of a recent preliminary version of this survey, correcting several errors. Each of the above has provided especially helpful comments and source material. We further appreciate similar forms of help from (mentioned alphabetically) Zvi Artstein, David Blackwell, Herbert Federer, Robert Kertz, Dietrich Kölzow, John Oxtoby, Michel Valadier, and Eugene Wesley. Assistance from Roman Pol and Pawel Szeptycki is noted in § 4. Numerous others have provided helpful reprints. Among those acknowledged above are several leaders of schools of activity in measurable selection theory.

BIBLIOGRAPHY

The bibliography is composed of three categories, according to whether the bracketed coding has no prime, a single prime, or a double prime. References in the unprimed category contain one or more results on existence of measurable selections which contributed something new at the time presented. An effort at completeness has been made in this category. The single primed references are papers (not texts) which do not appear to belong in the preceding category but which contain properties of set-valued functions of a measurability nature. Moderate inclusiveness has been attempted here. The double primed category consists of (a) useful texts and (b) papers whose only connection with measurable selections is in applications—no attempt at completeness is made here. Several references in the single and double primed categories are not cited in the text.

[AL] W. J. ARSENIN AND A. A. LYAPUNOV, *Die Theorie der A-Mengen*, Arbeiten zur Deskriptiven Mengenlehre, H. Grell, ed., VEB Deutscher Verlag der Wissenschaften, Berlin, 1955, pp. 35–93.

[AR1] Z. ARTSTEIN, *Set-valued measures*, Trans. Amer. Math. Soc., 165 (1972), pp. 103–125.

[AR2] ———, *Weak convergence of set-valued functions and control*, this Journal, 13 (1975), pp. 865–878.

[AR3]' ———, *On the calculus of closed set-valued functions*, Indiana Univ. Math. J., 24 (1974), pp. 433–441.

[AR4]" ———, *On a variational problem*, J. Math. Anal. Appl., 45 (1974), pp. 404–415.

[AR5]" ———, *A lemma and some bang-bang applications*, Brown Univ., Providence, RI, 1975.

[AV]' Z. ARTSTEIN AND R. A. VITALE, *A strong law of large numbers for random compact sets*, Ann. Probability, 3 (1975), pp. 879–882.

[AU1] R. J. AUMANN, *Integrals of set-valued functions*, J. Math. Anal. Appl., 12 (1965), pp. 1–12.

[AU2]" ———, *Existence of competitive equilibria in markets with a continuum of traders*, Econometrica, 34 (1966), pp. 1–17.

[AU3] ———, *Measurable utility and the measurable choice theorem*, La Décision Actes Colloq. Internat. du Centre Nat. Recherche Sci., Aix-en-Provence, 1967, Paris, 1969, pp. 15–26.

[AP]" R. J. AUMANN AND M. PERLES, *A variational problem arising in economics*, J. Math. Anal. Appl., 11 (1965), pp. 488–503.

[AS]" R. J. AUMANN AND L. S. SHAPLEY, *Values of Non-Atomic Games*, Princeton University Press, Princeton, 1974.

[AG]" E. A. AZOFF AND F. GILFEATHER, *Measurable choice and the invariant subspace problem*, Bull. Amer. Math. Soc., 80 (1974), pp. 893–895.

[BA] K. A. BAKER, *Borel functions for transformation group orbits*, J. Math. Anal. Appl., 11 (1965), pp. 217–225.

[BJ]' H. T. BANKS AND M. Q. JACOBS, *A differential calculus for multifunctions*, Ibid., 29 (1970), pp. 246–272.

[BM1] M. BENAMARA, *Sections mesurables extrémales d'une multi-application*, C. R. Acad. Sci. Paris Sér. A, 278 (1974), pp. 1249–1252.

[BM2]' M. BENAMARA, *Points Extrémaux, Multi-applications et Functionelles Intégrales*, Thèse, Grenoble, 1975.

[BG]" C. BERGE, *Espaces Topologiques—Fonctions Multivoque*, Dunod, Paris, 1959.

[BI]' J. M. BISMUT, *Intégrales convexes et probabilités*, J. Math. Anal. Appl., 42 (1973), pp. 639–673.

[BL] D. BLACKWELL, *A Borel set not containing a graph*, Ann. Math. Statist., 39 (1968), pp. 1345–1347.

[BD] D. BLACKWELL AND L. E. DUBINS, *On existence and non-existence of proper, regular, conditional distributions*, Ann Probability, 3 (1975), pp. 741–752.

[BRN] D. BLACKWELL AND C. RYLL-NARDZEWSKI, *Non-existence of everywhere proper conditional distributions*, Ann. Math. Statist., 34 (1963), pp. 223–225.

[BR] J. V. BONDAR, *Borel cross-sections and maximal invariants*, Ann. Statist., 4 (1976), pp. 866–877.

[BO1]" N. BOURBAKI, *General Topology*, Part 2, Hermann, Paris, 1966.

[BO2]" ———, *Intégration*, Hermann, Paris, 1974.

[BN] S. BRAUN, *Sur l'uniformisation des ensembles fermés*, Fund. Math., 28 (1937), pp. 214–218.

[BK]' J. K. BROOKS, *An integration theory for set-valued measures I, II*, Bull. Soc. Roy. Sci. Liège, 37 (1968), pp. 312–319 and 375–380.

[BP] L. D. BROWN AND R. PURVES, *Measurable selections of extrema*, Ann. Statist., 1 (1973), pp. 902–912.

[BV]' P. BRUNOVSKÝ, *Scorza-Dragoni's theorem for unbounded set-valued functions and its applications to control problems*, Mat. Casopis Sloven, Akad. Vied., 20 (1970), pp. 205–213.

[BU]' J. J. BUCKLEY, *Graphs of measurable functions*, Proc. Amer. Math. Soc., 44 (1974), pp. 78–80.

[CA1] C. CASTAING, *Multi-applications mesurables, généralisation du principe de bang–bang*, Proc. Colloq. on Convexity (Copenhagen, 1975), W. Fenchel, ed., Copenhagen University, 1967.

[CA2] ———, *Quelques problèmes de mesurabilité liés à la théorie de la commande*, C. R. Acad. Sci. Paris Sér. A, 262 (1966), pp. 409–411.

[CA3] ———, *Sur une nouvelle extension du théorème de Ljapunov*, Ibid., 264 (1967), pp. 333–336.

[CA4] ———, *Sur les multi-applications mesurables*, Thèses, Caen, 1967.

[CA5] ———, *Sur les multi-applications mesurables*, Rev. Française Inf. Rech. Opéra., 1 (1967), pp. 91–126.

[CA6] ———, *Sur la mesurabilité du profil d'un ensemble convexe compact variant de façon mesurable*, Multilithed article, Université de Perpignan, 1968.

[CA7] ———, *Proximité et mesurabilité. Un théorème de compacité faible*, Colloque sur la Théorie Mathématique du Contrôle Optimal, Brussels, 1969, pp. 25–33.

[CA8] ———, *Sur le graphe d'une multi-application souslinienne (mesurable)*, Secrétariat des Math. de la Faculté des Sciences de Montpellier, Publication No. 55, 1969–1970.

[CA9] ———, *Le théorème de Dunford–Pettis généralisé*, Université de Montpellier, Secrétariat des Mathématiques, Publication No. 43, 1969.

[CA10] ———, *Le théorème de Dunford–Pettis généralisé*, C. R. Acad. Sci. Paris Sér. A, 268 (1969), pp. 327–329.

[CA11]' ———, *Quelques compléments sur le graphe d'une multi-application mesurable*, Travaux du Séminaire d'Analyse Unilatérale, Faculté des Sciences, Université de Montpellier, 1969.

[CA12] ———, *Quelques applications du théorème de Banach Dieudonné a l'intégration*, Université de Montpellier, Secrétariat des Mathématiques, Publication No. 67, 1969–1970.

[CA13] ———, *Une nouvelle extension du théorème de Scorza-Dragoni*, C. R. Acad. Sci. Paris Sér. A, 271 (1970), pp. 396–398.

[CA14]' ———, *Application d'un théorème de compacité à la désintégration des mesures*, Travaux de Séminaire d'Analyse Convexe, vol. 1, Exp. No. 12, Secrétariat des Mathematiques, U.E.R. de Math., Univ. Sci. Tech. Languedoc, Montpellier, 1971.

[CA15] ———, *Intégrales convexes duales*, C. R. Acad. Sci. Paris Sér. A, 275 (1972), pp. 1331–1334.

[CA16] ———, *Intégrales convexes duales*, Travaux du Séminaire d'Analyse Convexe, vol. 3, Exp. No. 6, Secrétariat des Mathématiques, Publication No. 125, U.E.R. de Math., Univ. Sci. Tech. Languedoc, Montpellier, 1973.

[CA17] ———, *Un théorème d'existence de sections séparément mesurables et séparément absolument continues*, Travaux du Séminaire d'Analyse Convexe, vol. 3, Exp. No. 3, Secrétariat des Mathématiques, Publication No. 125, U.E.R. de Math., Univ Sci. Tech. Languedoc, Montpellier, 1973.

[CA18] ———, *Un théorème d'existence de sections séparément mesurables et séparément absolument continues d'une multiapplication séparément mesurable et séparément absolument continue*, C. R. Acad. Sci. Paris Sér. A, 276 (1973), pp. 367–370.

[CA19]' ———, *Quelques proprietés du profil d'un convexe compact variable*, Travaux du Séminaire d'Analyse Convexe, vol. 15, Exp. No. 4, Secrétariat des Mathématiques, U.E.R. de Math., Univ. Sci. Tech. Languedoc, Montpellier, 1975.

[CA20] ———, *A compactness theorem of measurable selections of a measurable multifunction and its applications*, Actes du Colloque Intégration Vectorielle et Multivoque, Alain Costé, ed., Université de Caen, 1975.

[CA21]' ———, *Quelques propriétés du profil d'un convexe compact variable*, Travaux du Séminaire d'Analyse Convexe, vol. 5, Exp. No. 4, Secrétariat des Mathématiques, U.E.R. de Math., Univ. Sci. Tech. Languedoc, Montpellier, 1975.

[CA22] ———, *Sur l'existence des sections séparément mesurables et séparément continues d'une multi-application*, Travaux du Séminaire d'Analyse Convexe, vol. 5, Exp. No. 14, Secrétariat des Mathématiques, U.E.R., des Math., Univ. Sci. Tech., Languedoc, Montpellier, 1975.

[CV1]' C. CASTAING AND M. VALADIER, *Equations différentielles multivoques dans les espaces vectoriels localement convexes*, Rev. Française Automat. Inf. Rech. Opér., 3 (1969), pp. 3–16.

[CV2]" ———, *Measurable Multifunctions and Applications*, Springer-Verlag, to appear.

[CV3]" ———, *Convex Analysis and Measurable Multifunctions*, to appear.

[CH] J. P. R. CHRISTENSEN, *Topology and Borel Structure*, North-Holland, Amsterdam, 1974.

[CB1] M. M. ČOBAN, *Many-valued mappings and Borel sets I*, Trudy Moskov. Mat. Obšč., 22 (1970) = Trans. Moscow Math. Soc. 22 (1970) 258–280.

[CB2] ———, *Many-valued mappings and Borel sets II*, Trudy Moskov. Mat. Obšč., 23 (1970) = Trans. Moscow Math. Soc. 23 (1970), pp. 286–310.

[CB3] ———, *On B-measurable sections*, Dokl. Akad. Nauk. SSSR, 207 (1972), pp. 48–51 = Soviet Math. Dokl. 13 (1972), pp. 1473–1477.

[CL1] J. K. COLE, *A selector theorem in Banach spaces*, J. Optimization Theory Appl., 7 (1971), pp. 170–172.

[CL2]' ———, *A note on a selector theorem in Banach spaces*, Ibid., 9 (1972), pp. 214–215.

[CR]' R. R. CORNWALL, *Conditions for the graph and the integral of a correspondence to be open*, J. Math. Anal. Appl., 39 (1972), pp. 771–792.

[CS1] A. COSTÉ, *Set-valued measures*, Topology and Measure Theory, Zinnowitz (D.D.R), to appear.

[CS2] ———, *Sur l'intégration par rapport à une multimesure de Radon*, C. R. Acad. Sci. Paris Sér. A., 278 (1974), pp. 545–548.

[CS3] ———, *Sur les multimesures à valeurs fermées bornées d'un espace de Banach*, Ibid., 280 (1975), pp. 567–570.

[CS4]' ———, *La propriété de Radon–Nikodym en intégration multivoque*, Ibid., 280 (1975), pp. 1515–1518.

[CS5]" ———, *Applications de la théorie des probabilités cylindriques et des opérateurs radonifiants à l'étude des fonctions aléatoires sous-linéaires et des multimesures*, Ibid., 282 (1976), pp. 103–106.

[CS6] ———, *On multivalued additive set functions*, Ark. Mat., to appear.

[CS7] ———, *Densité des sélecteurs d'une multimesure à valeurs convexes fermées bornées d'un espace de Banach séparable*, C. R. Acad. Sci. Paris. Sér. A, 282 (1976), pp. 967–969.

[CP1]' A. COSTÉ AND R. PALLU DE LA BARRIÈRE, *Un théorème de Radon–Nikodym pour les multimesures à valeurs convexes fermées localement compactes sans droite*, Ibid., 280 (1975), pp. 225–258.

[CP2]' ———, *Radon–Nikodyms theorems for set-valued measures whose values are convex and closed*, to appear.

[CP3] ———, *Sur L'ensemble des sections d'une multimesure à valeurs convexes fermées*, C. R. Acad. Sci. Paris Sér. A-B, 282 (1976), no. 106, Ai, A 828–832.

[DR] R. B. DARST, *Remarks on measurable selections*, Ann. Statist., 2 (1974), pp. 845–847.

[DT]' R. DATKO, *Measurability properties of set-valued mappings in a Banach space*, this Journal, 8 (1970), pp. 226–238.

[DV] J. D. DAUER AND F. S. VAN VLECK, *Measurable selectors of multifunctions and applications*, Math. Systems Theory, 7 (1974), pp. 367–376.

[DE]' G. DEBREU, *Integration of Correspondences*, Proc. Fifth Berkeley Symp. Math. Statist. and Probability (1965/66), vol. II, University of California Press, Berkeley, CA, 1967, pp. 351–372.

[DES]' G. DEBREU AND D. SCHMEIDLER, *The Radon–Nikodym derivative of a correspondence*, Proc. Sixth Berkeley Symp. Math. Statist. and Probability, vol. II, Univ. of California Press, Berkeley, CA, 1971, pp. 41–56.

[DL] C. DELODE, *Champs mesurables d'espaces sousliniens*, Département de Mathématiques, Université de Pau, 1976.

[DAP1] C. DELODE, O. ARINO AND J. P. PENOT, *Champs mesurables d'espaces polonais*, C. R. Acad. Sci. Paris Sér. A., 281 (1975), pp. 617–620.

[DAP2] ———, *Champs mesurables et multisections*, Ann. Inst. H. Poincaré Sect. B, 12 (1976), pp. 11–42.

[DW]' M. DEWILDE, *A note on the bang–bang principle*, J. London Math. Soc. (2), 1 (1969), pp. 753–759.

[DI] J. DIXMIER, *Dual et quasi-dual d'une algebre de Banach involutive*, Trans. Amer. Math. Soc., 104 (1962), pp. 278–283.

[DS] L. E. DUBINS AND L. J. SAVAGE, *How to Gamble If You Must*, McGraw-Hill, New York, 1965.

[EV] I. EKELAND AND M. VALADIER, *Representation of set-valued mappings*, J. Math. Anal. Appl., 35 (1971), pp. 621–629.

[EN] R. ENGELKING, *Selectors of the first Baire class for semicontinuous set-valued functions*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys., 16 (1968), pp. 277–282.

[FE]' H. FEDERER, *Geometric Measure Theory*, Springer-Verlag, New York, 1969.

[FM]' H. FEDERER AND A. MORSE, *Some properties of measurable functions*, Bull. Amer. Math. Soc., 49 (1943), pp. 270–277.

[FG] J. FELDMAN AND F. P. GREENLEAF, *Existence of Borel transversals in groups*, Pacific J. Math., 25 (1968), pp. 455–461.

[FI] A. FILIPPOV, *On certain questions in the theory of optimal control*, Vestnik Moskov. Univ. Ser. Mat. Meh. Astronom. Fiz. Him. (1959), no. 2, pp. 25–32; English transl., this Journal, 1 (1962), pp. 76–84.

[FL] W. M. FLEISCHMAN, ed., *Set-valued Mappings, Selections, and Topological Properties of 2^X*, Springer-Verlag, New York, 1970.

[FU] N. FURUKAWA, *Markovian decision processes with compact action spaces*, Ann. Math. Statist., 43 (1972), pp. 1612–1622.

[GM] F. S. GARNIR-MONJOIE, *Sur certaines multifonctions μ-mesurables*, Bull. Soc. Roy. Sci. Liège, 42 (1973), pp. 183–194.

[GGM] H. G. GARNIR AND F. S. GARNIR-MONJOIE, *Multifonctions dans l'espace Euclidien*, Université de Liège, 1973.

[GL]" J. GLIMM, *Locally compact transformation groups*, Trans. Amer. Math. Soc., 101 (1961), pp. 124–138.

[GT1] C. GODET-THOBIE, *Sur les sélections de mesures généralisées*, C. R. Acad. Sci. Paris Sér. A, 271 (1970), pp. 153–156.

[GT2] ———, *Sur les intégrales de fonctions réelles par rapport à une mesure généralisée et ses sélections*, Ibid., 271 (1970), pp. 497–500.

[GT3] ———, *Sur les multimesures de transition*, Ibid., 278 (1974), pp. 1367–1369.

[GT4] ———, *Sélections de multimesures*, Ibid., 279 (1974), pp. 603–606.

[GT5] ———, *Multimesures et Multimesures de Transition*, Thèse, Montpellier, 1975.

[GT6] ———, *On transition multimeasures*, Actes de Colloque Intégration Vectorielle et Multioque, Alain Costé, ed., Université de Caen, 1975.

[GD]' G. S. GOODMAN, *On a theorem of Scorza-Dragoni and its application to optimal control*, Mathematical Theory of Control, A. V. Balakrishnan and L. W. Neustadt, eds., Academic Press, 1967, pp. 222–233.

[GHJ]' G. S. GOODMAN AND J. HOFFMANN-JØRGENSEN, *Support functions and the integration of convex sets in infinite-dimensional spaces*, Colloque sur la Théorie Mathématique du Côntrole Optimal, Brussels, 1969, pp. 83–97.

[GZ]' L. GÖTZ, *Bang–Bang Prinzipien*, Zulassungsarbeit zum Staatsexamen, Erlangen, 1972/73.

[GR]" F. P. GREENLEAF, *Invariant Means on Topological Groups*, Van Nostrand Mathematics Study Series No. 16, Van Nostrand–Reinhold, New York, 1969.

[HL]" P. R. HALMOS, *Measure Theory*, Van Nostrand, Princeton, 1954.

[HN]' R. W. HANSELL, *Borel measurable mappings for nonseparable metric spaces*, Trans. Amer. Math. Soc., 161 (1971), pp. 145–169.

[HF]" F. HAUSDORFF, *Set Theory*, Chelsea, New York, 1962 (second edition of translation from German).

[HE1] H. Hermes, *A note on the range of a vector measure, Application to the theory of optimal control*, J. Math. Anal. Appl., 8 (1964), pp. 78–83.

[HE2]' ———, *Calculus of set-valued functions and control*, J. Math. Mech., 18 (1968), pp. 47–60.

[HE3]' ———, *The generalized differential equations $\dot{x} \in R(t, x)$*, Advances in Math., 4 (1970), pp. 149–169.

[HI]" W. Hildebrand, *Core and Equilibrium of a Large Economy*, Princeton University Press, Princeton, 1974.

[HM1]' C. J. Himmelberg, *Precompact contraction of metric uniformities and the continuity of $F(t, x)$*, Rend. Sem. Mat. Univ. Padova, 50 (1973), pp. 185–188.

[HM2] ———, *Measurable relations*, Fund. Math., 87 (1975), pp. 53–72.

[HJV] C. J. Himmelberg, M. Q. Jacobs and F. S. Van Vleck, *Measurable multifunctions, selectors and Filippov's implicit functions lemma*, J. Math. Anal. Appl., 25 (1969), pp. 276–284.

[HPRV]" C. J. Himmelberg, T. Partasarathy, T. E. S. Raghavan and F. S. Van Vleck, *Existence of p-equilibrium and optimal stationary strategies in stochastic games*, Proc. Amer. Math. Soc., 60 (1976), pp. 245–251.

[HPV] C. J. Himmelberg, T. Parthasarathy and F. S. Van Vleck, *Optimal plans for dynamic programming problems*, Mathematics of Operations Research, to appear.

[HV1] C. J. Himmelberg and F. S. Van Vleck, *Some remarks on Filippov's lemma*, Proc. U.S.-Japan Seminar on Differential Functional Equations, W. A. Benjamin, New York, 1967, pp. 455–462.

[HV2] ———, *Some selection theorems for measurable functions*, Canad. J. Math., 21 (1969), pp. 394–399.

[HV3] ———, *Selections and implicit function theorems for multifunctions with Souslin graph*, Bull. Acad. Polon. Sci., 19 (1971), pp. 911–916.

[HV4]' ———, *Lipschitzian generalized differential equations*, Rend. Sem. Mat. Univ. Padova, 48 (1972), pp. 159–169.

[HV5] ———, *Extreme points of multifunctions*, Indiana Univ. Math. J., 22 (1973), pp. 719–729.

[HV6] ———, *Multifunctions on abstract measurable spaces and application to stochastic decision theory*, Ann. Mat. Pura. Appl., 101 (1974), pp. 229–236.

[HV7]' ———, *Multifunctions with values in a space of probability measures*, J. Math. Anal. Appl., 50 (1975), pp. 108–112.

[HV8]" ———, *Two existence theorems for generalized differential equations*, submitted for publication.

[HV9]' ———, *An extension of Brunovský's Scorza-Dragoni type theorem for unbounded set-valued functions*, Mat. Časopis Sloven. Akad. Vied., 26 (1976), pp. 179–188.

[HD1] K. Hinderer, *Zur Theorie Stochastischer Entscheidungsmodelle*, Habilitationsschrift, University of Stuttgart, 1967.

[HD2] ———, *Foundations of non-stationary dynamic programming with discrete time parameter*, Lecture Notes in Operations Research and Mathematical Systems, vol. 33, Springer-Verlag, New York, 1970.

[HJ] J. Hoffmann-Jørgensen, *The Theory of Analytic Spaces*, Various Publication Series, No. 10, Århus Universitet, Denmark, 1970.

[HU]' M. Hukahara, *Intégration des applications mesurables dont la valeur est un compact convexe*, Funkcial. Ekvac., 10 (1967), pp. 205–223.

[JC1] M. Q. Jacobs, *Remarks on some recent extensions of Filippov's implicit functions lemma*, this Journal, 5 (1967), pp. 622–627.

[JC2] ———, *Measurable multivalued mappings and Lusin's theorem*, Trans. Amer. Math. Soc., 134 (1968), pp. 471–481.

[JC3]' ———, *On the approximation of integrals of multivalued functions*, this Journal, 7 (1969), pp. 158–177.

[JN] V. Jankov, *Sur l'uniformisation des ensembles A*, C. R. Acad. Sci. URSS, 30 (1941), pp. 597–598.

[JO]' J. A. Johnson, *Extreme measurable selections*, Proc. Amer. Math. Soc., 44 (1974), pp. 107–112.

[KP] J. Kaniewski and R. Pol, *Borel-measurable selectors for compact-valued mappings in the non-separable case*, Bull. Acad. Polon. Sci. Sér. Math. Astronom. Phys., 23 (1975), pp. 1043–1050.

[KE] H. G. Kellerer, *Bemerkung zu einem Satz von H. Richter*, Arch. Math., 15 (1964), pp. 204–207.

[KO] M. Kondō, *Sur l'uniformisation des complémentaires analytiques et les ensembles projectifs de la seconde classe*, Japan. J. Math., 15 (1938), pp. 197–230.

[KR]' M. A. Krasnosel'skii, *Topological Methods in the Theory of Nonlinear Integral Equations*, Pergamon Press, New York, 1964.

[KD]' H. Kudō, *Dependent experiments and sufficient statistics*, Natur. Sci. Rep. Ochanomizu Univ., 4 (1954), pp. 151–163.

[KU1]" K. Kuratowski, *Topology*, vol. 1, Fifth ed., Academic Press, New York, 1966.

[KU2]" ———, *Topology*, vol. 2, Fifth ed., Academic Press, New York, 1968.

[KU3]' ———, *A general approach to the theory of set-valued mappings*, Proc. Third Prague Topological Symp., 1971, pp. 271–280.

[KU4] ———, *On the selector problems for the partitions of Polish spaces and for the compact-valued mappings*, Ann. Polon. Math., 29 (1974), pp. 421–427.

[KU5] ———, *The σ-algebra generated by Souslin sets and its applications to set-valued mappings and to selector problems*, Boll. Un. Mat. Ital., 11 (1975), pp. 285–298

[KU6] ———, *On partitions of complete spaces which do not admit analytic selectors and on some consequences of a theorem of Gödel*, Instituto Naxionale di Alta Matematica, Symposia Mathematica, 16 (1975), pp. 67–74.

[KMT] K. Kuratowski and A. Maitra, *Some theorems on selectors and their applications to semi-continuous decompositions*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys., 22 (1974), pp. 877–881.

[KMS]" K. Kuratowski and A. Mostowski, *Set Theory*, second ed., North-Holland, Amsterdam, 1976.

[KRN] K. Kuratowski and C. Ryll-Nardzewski, *A general theorem on selectors*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys., 13 (1965), pp. 397–403.

[LA1] D. G. Larman, *Projecting and uniformising Borel sets with $\mathcal{K}_\sigma$ sections I*, Mathematika, 19 (1972), pp. 231–244.

[LA2] ———, *Projecting and uniformising Borel sets with $\mathcal{K}_\sigma$ sections II*, Ibid., 20 (1973), pp. 233–246.

[LE1] S. J. Leese, *Measurable selections in normed spaces*, Proc. Edinburgh Math. Soc., 19 (1974), pp. 147–150.

[LE2] ———, *Multifunctions of Souslin type*, Bull. Austral. Math. Soc., 11 (1974), pp. 395–411.

[LE3] ———, *Set-valued functions and selectors*, Thesis, Keele, England, 1974.

[LE4] ———, *Multifunctions of Souslin type: Corrigendum*, Bull. Austral. Math. Soc., 13 (1975), pp. 159–160.

[LE5] ———, *Measurable selections and the uniformisation of Souslin sets*, Amer. J. Math., to appear.

[LE6] ———, *Continuous and Borel selectors in normed spaces*, Loughborough University Report No. 68, July 1975.

[LN1] J. Lehn, *Massfortsetzungen und Aumann's Selektionstheorem*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, 35 (1976), pp. 265–268.

[LN2]' ———, *Remark on measurable graph theorems*, Proc. Amer. Math. Soc., to appear.

[LM]" J. Lehn and G. Mägerl, *On the uniqueness of preimage measures*, submitted for publication.

[LB]" A. Lubin, *Extensions of measures and the von Neumann selection theorem*, Proc. Amer. Math. Soc., 43 (1974), pp. 118–122.

[LS] N. Lusin, *Leçons sur les Ensembles Analytiques et Leurs Applications*, Chelsea, New York (second edition), 1972. (First edition was 1930.)

[MC1] G. W. Mackey, *Induced representations of locally compact groups I*, Ann. of Math., 55 (1952), pp. 101–139.

[MC2]" ———, *Borel structure in groups and their duals*, Trans. Amer. Math. Soc., 85 (1957), pp. 134–165.

[MG] G. Mägerl, *Stetige und messbare Schnitte fur Korrespondenzen*, Diplomarbeit, Erlangen, 1973.

[MT] A. Maitra, *Discounted dynamic programming on compact metric spaces*, Sankhyā Ser. A, 27 (1968), pp. 241–248.

[MR1] A. Maitra and B. V. Rao, *Selection theorems and the reduction principle*, Trans. Amer. Math. Soc., 202 (1975), pp. 57–66.

[MR2] ———, *Selection theorems for partitions of Polish spaces*, Fund. Math., 93 (1976), pp. 47–56.

[MZ] P. MARITZ, *Integration of set-valued functions*, Thesis, Rijksuniversiteit te Leiden, 1975.

[MK] S. MAZURKIEWICZ, *Sur une propriété des ensembles $C(A)$*, Fund. Math., 10 (1927), pp. 172–174.

[MW] E. J. MCSHANE AND R. B. WARFIELD, JR., *On Filippov's implicit functions lemma*, Proc. Amer. Math. Soc., 18 (1967), pp. 41–47.

[MI1] E. A. MICHAEL, *Selected selection theorems*, Amer. Math. Monthly, 4 (1956), pp. 233–236.

[MI2] E. A. MICHAEL, *G_δ sections and compact-covering maps*, Duke. Math. J., 36 (1969), pp. 125–128.

[MU]' D. R. MUNOZ, *Integration of Correspondences*, Thesis, Univ. of California, San Diego, 1971.

[NE] J. VON NEUMANN, *On rings of operators, reduction theory*, Ann. of Math., 30 (1949), pp. 401–485.

[NO1] P. NOVIKOV, *Sur les fonctions implicites mesurables B*, Fund. Math., 17 (1931), pp. 8–25.

[NO2] ———, *Sur les projections de certain ensembles mesurables*, C. R. (Doklady) Acad. Sci. URSS, 23 (1939), pp. 864–865.

[NU] G. NÜRNBERGER, *Dualität von Schnitten für die metrische Projektion und von Fortsetzungen kompakter Operatoren*, Dissertation, Erlangen, 1975.

[OL] C. OLECH, *A note concerning set-valued measurable functions*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys., 13 (1965), pp. 317–321.

[PB1] R. PALLU DE LA BARRIÈRE, *Queques propriétés des multimesures*, Travaux du Séminaire d'Analyse Convexe, vol. 3, Exp. No. 11, Secrétariat des Mathématiques U.E.R. de Math., Univ. Sci. Tech. Languedoc, Montpellier, 1973.

[PB2] ———, *Multimesures à valeurs convexes fermées*, Actes du Colloque Intégration Vectorielle et Multivoque, Alain Costé, ed., Université de Caen, 1975.

[PR1]" T. PARTHASARATHY, *Selection Theorems and Their Applications*, Springer-Verlag, New York, 1972.

[PR2]" ———, *Discounted, positive, and noncooperative stochastic games*, Internat. J. Game Theory, 2 (1973), pp. 25–37.

[PL1]' A. PLIŚ, *Remark on measurable set-valued functions*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys., 9 (1961), pp. 857–859.

[PL2]' ———, *Measurable orientor fields*, Ibid., 13 (1965), pp. 565–569.

[RI] H. RICHTER, *Verallgemeinerung eines in der Statistik benötigten Satzes der Masstheorie*, Math. Ann., 150 (1963), pp. 85–90.

[RB] A. P. ROBERTSON, *On measurable selections*, Proc. Royal Soc. Edinburgh, 72 (1972/73), pp. 1–7.

[RC1]' R. T. ROCKAFELLAR, *Integrals which are convex functionals*, Pacific J. Math., 24 (1968), pp. 525–539.

[RC2] ———, *Measurable dependence of convex sets and functions on parameters*, J. Math. Anal. Appl., 28 (1969), pp. 4–25.

[RC3] ———, *Convex integral functionals and duality*, Contributions to Nonlinear Functional Analysis, Academic Press, New York, 1971.

[RC4]' ———, *Weak compactness of level sets of integral functionals*, Proc. Troisième Colloque d'Analyse Fonctionelle, H. G. Garnir, ed., Liège, 1971.

[RC5]' ———, *Integrals which are convex functionals, II*, Pacific J. Math., 39 (1971), pp. 439–469.

[RC6] ———, *Integral functionals, normal integrands, and measurable selections*, Proc. Colloq. Nonlinear Operators and the Calculus of Variations, Lecture Notes in Mathematics, 543, G. P. Gossez et al., ed., Springer-Verlag, New York, 1976, pp. 157–207.

[RCW]" R. T. ROCKAFELLAR AND J. B. WETS, *Continuous versus measurable recourse in N-stage stochastic programming*, J. Math. Anal. Appl., 48 (1974), pp. 836–859.

[RG]" C. A. ROGERS, *Hausdorff Measures*, Cambridge University Press, Cambridge, England, 1970.

[RGW] C. A. ROGERS AND R. C. WILLMOTT, *On the uniformization of sets in topological spaces*, Acta. Math., 120 (1968), pp. 1–52.

[RK1]" V. A. ROKHLIN, *On decomposition of a dynamical system into transitive components*, Mat. Sb., 25 (1949), pp. 235–249.

[RK2] ———, *Selected topics from the metric theory of dynamic systems*, Uspekhi Mat. Nauk. 4 (1949), pp. 57–128; English transl., Amer. Math. Soc. Transl., 49 (1966), pp. 171–240.

[RN] C. RYLL-NARDZEWSKI, *On Borel measurability of orbits*, Fund. Math., 56 (1964), pp. 129–130.

[SB1] M. F. SAINTE-BEUVE, *Sur la généralisation d'un théorème de section mesurable de von Neumann–Aumann*, Travaux du Séminaire d'Analyse Convexe, vol. 3, Exp. No. 7, Secrétariat des Mathématiques, U.E.R. de Math., Univ. Sci. Tech. Languedoc, Montpellier, 1973.

[SB2] ———, *Sur la généralisation d'un théorème de section mesurable de von Neumann–Aumann et applications à un théorème de fonctions implicites mesurables et à un théorème de représentation intégrale*, C. R. Acad. Sci. Paris Sér. A, 276 (1973), pp. 1297–1300.

[SB3] ———, *On the extension of von Neumann–Aumann's theorem*, J. Funct. Anal., 17 (1974), pp. 112–129.

[SB4]' ———, *Mesures vectorielles à variation bornée et représentation intégrale*, Actes du Colloque Intégration Vectorielle et Multivoque, Alain Costé, ed., Université de Caen, 1975.

[SK] S. SAKS, *Theory of the Integral*, Dover, New York, 1968.

[SM] Y. SAMPEI, *On the uniformization of the complement of an analytic set*, Comment. Math. Univ. St. Paul., 10 (1961), pp. 57–62.

[SR] H. SARBADAIKARI, *Some uniformization results*, Fund. Math., to appear.

[SC1] M. SCHÄL, *A selection theorem for optimization problems*, Arch. Math., 25 (1974), pp. 219–224.

[SC2] ———, *Conditions for optimality in dynamic programming and for the limit of n-stage optimal policies to be optimal*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, 32 (1975), pp. 179–196.

[SD]' G. SCORZA-DRAGONI, *Un teorema sulla funzioni continue rispetto ad una e misurabili rispetto ad un' altra variabile*, Rend. Sem. Math. Univ. Padova, 17 (1948), pp. 102–106.

[SP1] W. SIERPINSKI, *Sur l'uniformisation des ensembles mesurables (B)*, Fund. Math., 16 (1930), pp. 136–139.

[SP2] ———, *Sur deux complémentaires analytiques non separable B*, Ibid., 17 (1931), pp. 296–297.

[SN] M. SION, *On uniformization of sets in topological spaces*, Trans. Amer. Math. Soc., 96 (1960), pp. 237–245.

[SV] L. M. SONNEBORN AND F. S. VAN VLECK, *The bang–bang principle for linear control systems*, this Journal, 2 (1965), pp. 151–159.

[ST]" R. E. STRAUCH, *Negative dynamic programming*, Ann. Math. Statist., 37 (1966), pp. 871–890.

[SY]' C. SUNYACH, *Sur le théorème du graphe borélien*, C. R. Acad. Sci. Paris Sér. A, 269 (1969), pp. 233–234.

[SZ] T. SUZUKI, *On the uniformization principle*, Proc. Symposium on the Foundations of Mathematics, Katadaj, Japan, 1962, pp. 137–144.

[TH1] D. S. THIAM, *Multimesures positive*, C. R. Acad. Sci. Paris Sér. A, 280 (1975), pp. 993–995.

[TH2] ———, *Applications à l'intégration multivoque de l'intégrale de Daniell à valeurs dans un monoide ordonné*, Actes du Colloque Intégration Vectorielle et Multivoque, Alain Costé, ed., Université de Caen, 1975.

[VA1] M. VALADIER, *Sur l'intégration d'ensembles convexes compacts en dimension infinie*, C. R. Acad. Sci. Paris Sér. A, 266 (1968), pp. 14–16.

[VA2] ———, *Contributions à l'analyse convexe*, Thèse, Paris, 1970.

[VA3] ———, *Multi-applications mesurables à valeurs convexes compactes*, J. Math. Pures Appl., 50 (1971), pp. 265–297.

[VA4] ———, *Convex integrands on Souslin locally convex spaces*, Travaux du Séminaire d'Analyse Convexe, vol. 3, Exp. No. 2, Secrétariat des Math., Publication No. 125, U.E.R. de Math., Univ. Sci. Tech. Languedoc, Montpellier, 1973.

[VA5] ———, *Intégrandes sur des localement convexes sousliniens*, C. R. Acad. Sci. Paris Sér. A, 276 (1973), pp. 693–695.

[VA6] ———, *On the Strassen theorem*, Travaux du Séminaire d'Analyse Convexe, vol. 4, Exp. No. 4, Secrétariat des Math., U.E.R. de Math., Univ. Sci. Tech. Languedoc, Montpellier, 1974.

[VA7] ———, *Sur le théorème de Strassen*, C. R. Acad. Sci. Paris Sér, A. 278 (1974), pp. 1021–1024.

[VA8]" ———, *About the Strassen theorem for vector valued sublinear functions*, Actes du Colloque Intégration Vectorielle et Multivoque, Alain Costé, ed., Université de Caen, 1975.

[WG1] D. H. WAGNER, *Integral of a convex-hull-valued function*, J. Math. Anal. Appl., 50 (1975), pp. 548–559.

[WG2]" ———, *Integral of a set-valued function with semi-closed values*, Ibid., 55 (1976), pp. 616–633.

[WS] D. H. WAGNER AND L. D. STONE, *Necessity and existence results on constrained optimization of separable functionals by a multiplier rule*, this Journal, 12 (1974), pp. 356–372.

[WZ] T. WAZEWSKI, *Sur une condition d'existence des fonctions implicites mesurables*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys., 9 (1961), pp. 861–863.

[WE1] E. WESLEY, *Extensions of the measurable choice theorem by means of forcing*, Israel J. Math., 14 (1973), pp. 104–114.

[WE2]″ ———, *Borel preference orders in markets with a continuum of traders*, J. Math. Economics, 3 (1976), pp. 155–165.

[WH] W. WHITT, *Baire classification of measured selections of extrema*, School of Organization and Management, Yale Univ., 1976.

[WI] R. C. WILLMOTT, *On the uniformization of Souslin $\mathscr{F}$ sets*, Proc. Amer. Math. Soc., 22 (1969), pp. 148–155.

[YO]″ L. C. YOUNG, *Lectures on the Calculus of Variations and Optimal Control Theory*, W. B. Saunders, Philadelphia, 1969.

Addenda in proof. In the above, coverage of Russian contributions to measurable selection theory is inadequate. Items (i), (ii), and (iii) below were brought to our attention very recently by A. D. Ioffe via Rockafellar. We had just previously learned of item (i) from E. B. Dynkin via Aumann. It is hoped that Russian contributions will be further surveyed in subsequent publications by Ioffe and perhaps others.

Items (iv) through (xi), given in order of the sections to which they relate, note various additional matters of which we have recently learned. Of these, we consider the announcement by Cenzer and Mauldin in (viii) most important.

(i) Our comment in § 12 on Jankov's [JN] is particularly inadequate. (Henceforth, we transliterate "Yankov.") Statement (3) in the proof of his theorem is the main content of what has been widely called the "von Neumann selection theorem" (5.1 and 5.2 above). In our usages it says: if $T=X=R$ and $\mathrm{Gr}\,F$ is Suslin, then F has a selection which is a Lebesgue measurable function. (This does not follow from his theorem statement, which is given in § 12, by reason of the last sentence in § 12.) To understand the proof in [JN] (also given in [AL, Satz 32]) recourse must be made to usages of [LS], to which he refers, as follows (we are grateful to R. D. Mauldin, A. A. Yuškevič, and J. C. Oxtoby for clarifying these points): (a) all real domains are identified with the irrationals in $(0, 1)$, further identified with ω^ω as usual (e.g., see 5.1 above); (b) an "elementary G_δ" is (the graph of) a continuous map on ω^ω; and (c) "inferior point" means lexicographic minimum. He should probably have stated that $\sum_k \delta_{(n+1)k} \subset \sum_k \delta_{nk}$ (easily obtained), although that appears to be implicit in the definitions from [LS]. Reference in [JN] to the Baire property is redundant since the σ-algebra generated by the Suslin subsets of R^n is contained in the family of Baire property sets.

Yankov's [JN] was published in 1941 and was presented in 1940. Von Neumann's [NE] appeared in 1949, having been submitted in 1948; it states at the outset that the paper was written in 1937–38 and publication was delayed to make certain changes which are itemized and which do not pertain to the selection result, Lemma 5. Both authors obtained the same selection (lexicographic minimum), by different constructions. We have no doubt that these two works were independent of each other, having moreover consulted two former collaborators of von Neumann's, F. J. Murray and H. H. Goldstine. Murray observes

that Lemma 5 is of central importance to [NE] ("without it there is no paper"). He recalls a prewar conversation in which von Neumann spoke with pride over solving this selection problem (although it is not spotlighted in [NE] and was little known for several years). Of course, Yankov was the first to publish.

We conclude that a statement of the form of 5.1 or 5.2 above is appropriately called a "Yankov–von Neumann theorem." Subsequent improvements by Aumann, Sainte-Beuve, and Leese have resulted in 5.10 above.

In Russian literature (e.g., [AL, § 11], [NA, § 40.3 and App. IV], [IT1], [IT2]) statements such as 5.2 have been referred to as the "Lusin–Yankov theorem"; [RK1] credits Yankov. Having reviewed [LS2], which evidently inspired [JN], we do not conclude that Lusin should be credited with this result, despite his eminent pioneering contributions to the foundations of the subject (e.g., see § 3 above). It does appear that the construction on page 57 of [LS2] (which differs from those of [JN] and [NE]) if specialized in the most natural way, yields the Yankov–von Neumann selection. However, [LS2] does not prove that his selection is a Lebesgue *measurable function*, in fact, in contrast to [NE], neither he nor Yankov appeared to seek that kind of result. Again, Yankov did state and prove a measurable function result during his proof of his theorem.

(ii) A second important omission, pointed out by Ioffe, is Novikov's [NO2, Cor. 2] (1939), also quoted in [AL, § 14], which we render: if $T=X=R$, F is closed-valued, and $\mathrm{Gr}\,F$ is Borel, then F has what has now been termed a Castaing representation. Contrary to the end of § 3 above, this is the first result on existence of measurable selections without assuming countable or compact values.

(iii) Ioffe points out that Rokhlin's argument in [RK2] (also given in [RK1]), discussed in § 4, becomes a valid proof if the following changes are made (we concur): (a) replace 2^{-n} by 2^{-n+2} in (10_n), and (b) redefine A_i to be $B_i/\bigsqcup_{j=1}^{i-1} B_j$, where

$$B_i=\{x: r(Y_i, \Psi(x))<2^{-n} \text{ and } r(Y_i, \psi_{n-1}(x))<2^{-n+2}\}.$$

Moreover, this argument suffices for Theorem 4.1 as given above, without the Lebesgue space assumption made by Rokhlin.

Ioffe feels that the error in [RK2] was "insignificant and easily correctible." Were it only (a), we would agree. However, (b) is a substantive change, e.g., the new A_i involves the approximating function ψ_{n-1} and in [RK2] it did not. Therefore, we feel the argument in [RK2] should be regarded as incomplete. Thanks to Ioffe, we now know that it is completable within the main ideas of Rokhlin's reasoning. Thus, Rokhlin gave in 1949 a statement of the essence of Theorem 4.1 and the principal ideas of its proof.

From the facts on the origin of Theorem 4.1 given after its statement and from the observations just made, it appears that the credit for this result is somewhat diffuse among, chronologically, Rokhlin [RK2], Kuratowski and Ryll-Nardzewski [KRN], and Castaing [CA, 1, 2, 4]. Moreover, Novikov contributed a significant special case (see (ii)) in 1939, albeit with the strong assumption that $\mathrm{Gr}\,F$ is Borel. We propose that Theorem 4.1 be given the impersonal name "Fundamental Measurable Selection Theorem," which we believe is commensurate with its importance.

(iv) A new and fairly general exposition of measurable selections and continuous selections is given by Kuratowski and Mostowski [KMS, Chap. XIV]. A briefer discussion in similar vein (in Polish) is given in [KU7].

(v) In § 4 and § 8 we have noted Rockafellar's use of F such that $F(t)$ is the epigraph of a convex $f(t, \cdot)$ for $t \in T$, where $f: T \times X \to R \cup \{\infty, -\infty\}$ and μ is complete and σ-finite. He points out (personal communication) that for the most part, "convex" is weakened to "lsc" and "complete" is avoided in [RC6], which supersedes most of the finite-dimensional parts of [RC1–5]. In [RC6], the key condition for such an f to be normal, by definition, is that F be measurable, and the latter property is the focus of his manipulations, via Theorem 4.2(e) ((ii)$\Leftrightarrow$(ix)).

With this approach, Rockafellar obtains in a relatively easy way, within $X = R^n$, variants of several results reviewed above, e.g., his result in (vii) below and an implicit function result [RC6, Thm. 2J] in which the g constraint in § 7 is generalized to an infinite sequence of inequalities.

Evstigneev [ES] has applied Theorem 4.2(e) ((iii)$\Leftrightarrow$(ix)) to dynamic programming problems, generalizing certain results of Rockafellar and West [RCW] and of Dynkin.

(vi) Cenzer and Mauldin [CM1] give variants on Theorem 5.7 above from [BRN]. In one they replace $\mathscr{B}(X)$ by its completion. In another they assume Gr F is complementary Suslin and obtain $2^{\aleph_0}$ distinct Borel function selections of F (Larman [LA1, 2] obtained $\aleph_1$ with F σ-compact-valued—see § 12).

(vii) Schäl [SC3] has answered affirmatively the open question in § 9. Shreve and Bertsekas [SHB] have given a variant of a result of Brown and Purves [BP]. They assume Gr F and, for $a \in R$, $\{(t, x): u(t, x) > a\}$ are Suslin (u as in § 9). Rockafellar [RC6, Thm. 2K] gave the following variant on 9.1 with u and v as in § 9: If $X = R^n$, $-u$ is normal (see (v)), F is measurable and closed-valued, and $G(t) \equiv F(t) \cap \{x: u(t, x) = v(t)\}$ for $t \in T$, then v and G are measurable, and since also G is closed-valued, $\mathscr{S}(G) \neq \emptyset$.

(viii) In [WE3], Wesley proves his universal measurability assertion in § 10. Cenzer and Mauldin announce (personal communication) an extension [CM2] of this result and moreover a proof that uses only standard techniques of descriptive set theory, not requiring forcing or other metamathematical methods: In Theorem 10.3 they replace $\mathscr{L}$, $\mathscr{M}$, and $\mathscr{M} \otimes \mathscr{L}$ by the smaller σ-algebras $S([0, 1])$, $S(T)$, and $S(T \times [0, 1])$, where for a topological space Y, $S(Y)$ is the smallest σ-algebra which is a Suslin family and contains $\mathscr{B}(Y)$.

(ix) Kallman and Mauldin [KAM] have extended Corollary 11.2(ii) (due to Dixmier) as follows (under partition usages of § 11): If $X(=T)$ is a Borel subset of a Polish space, each $F(t)$ is an F_δ and a G_δ in X, $\mathscr{M} = \mathscr{B}(T)$, and F is weakly measurable, then $\mathscr{S}(F) \neq \emptyset$. Kaniewski [KA2] has obtained a Borel set selection of a partition into compact sets of a Borel subset of a metric Suslin space; he also generalizes Kunugui–Novikov [NO2].

(x) Kaniewski [KA1] has generalized Kondō's theorem (§ 12). Mauldin points out that ZFC + MA + not CH (without the constructibility axiom) denies (b) at the end of § 12 (this (b) is included in Aumann's discussion [AU3] of Gr F complementary Suslin.

(xi) Further contributions to selections of set-valued measures (§ 15) have been given by M. Rao [RA], Vincent-Smith [VS], and Talagrand [TA]. These relate to Choquet theory.

BIBLIOGRAPHY ADDED IN PROOF

[AN]″ V. I. ARKIN AND E. L. LEVIN, *The convexity of values of vector integrals, Theorems on measurable selections and variational problems*, Uspehi Mat. Nauk, 27 (1972), pp. 22–77, English transl. Russian Math. Surveys, London Math. Soc., 1972, pp. 21–85.

[BS] D. BERTSEKAS AND S. E. SHREVE, *Stochastic Optimal Control: The Discrete Time Case*, Academic Press, New York, to appear.

[DY]″ E. B. DYNKIN AND A. A. YUŠKEVIČ, *Markov Control Processes and their Applications*, Nauka, Moscow, 1975, English ed. Springer-Verlag, New York, to appear.

[CM1] D. CENZER AND R. D. MAULDIN, *Inductive definitions: Measure and category*, in preparation.

[CM2] ———, *Measurable parametrizations and selections*, submitted for publication.

[ES]′ I. V. EVSTIGNEEV, *Measurable selection and dynamic programming*, Math. Operations Res., 1 (1976), pp. 267–272.

[IT1]″ A. D. IOFFE AND V. M. TIKHOMIROV, *Duality of convex functions and extremal problems*, Uspehi Mat. Nauk, 22 (1968), pp. 51–116; English transl. Russian Math. Surveys, 23 (1968), pp. 53–125.

[IT2]″ ———, *On minimization of integral functionals*, Funkcional. Anal. i Priložen, 3 (1969), pp. 61–70; English transl. Plenum, New York.

[KLM] R. R. KALLMAN AND R. D. MAULDIN, *A cross section theorem and application to C^*-algebras*, Pure Appl. Math. Sci., to appear.

[KA1] J. KANIEWSKI, *A generalization of Kondo's uniformization theorem*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astron. Phys., 24 (1976), pp. 393–398.

[KA2] ———, *A selection theorem for partitions of Borel sets into compact subsets*, Dept. of Math. and Mech., Warsaw Univ., 1976–77.

[KU7]″ K. KURATOWSKI, *O selektorach w topologii i teorii miary* (On selectors in topology and measure theory), Wiadom. Mat., 19 (1975), pp. 3–9.

[LS2] N. LUSIN, *Sur le problème de M. Jacques Hadamard d'uniformisation des ensembles*, Mathematica, 4 (1930), pp. 54–66.

[NA]″ M. NAIMARK, *Normed Rings*, Nauka, Moscow, 1968.

[RA] M. RAO, *Measurable selection of representing measures*, Quart. J. Math. Ser. 2, 22 (1971), pp. 571–572.

[SC3] M. SCHÄL, *Addendum to* [SC1] *and* [SC2], Tech. Rpt., University of Bonn, 1977.

[TA] M. TALAGRAND, *Sélection measurable de mesures maximales*, Séminaire Choquet, 15th year, 1975/76, Communications No. 3, Secrétariat mathématique, Équipe d'Analyse, Université Paris.

[VS] G. F. VINCENT-SMITH, *Measurable selection of simplicial maximal measures*, J. London Math. Soc. Ser. 2, 7 (1973), pp. 427–428.

[WE3] E. WESLEY, *On the existence of absolutely measurable selection functions*, submitted for publication; see Abstract 76T-B208, Notices Amer. Math. Soc., 23 (1977), p. A-648.