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Stochastic Processes and their Applications 126 (2016) 1585–1621

stochastic processes and their applications

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Optimal inventory control with path-dependent cost criteria

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Received 27 March 2015; received in revised form 29 October 2015; accepted 29 November 2015
Available online 17 December 2015

Abstract

This paper deals with a stochastic control problem arising from inventory control, in which the cost structure depends on the current position as well as the running maximum of the state process. A control mechanism is introduced to control the growth of the running maximum which represents the required storage capacity. The infinite horizon discounted cost minimization problem is addressed and it is used to derive a complete solution to the long-run average cost minimization problem. An associated control cost minimization problem subject to a storage capacity constraint is also addressed. Finally, as an application of the above results, a related infinite-horizon discounted control problem with a regime-switching inventory model is also solved.

MSC: 93E20; 60H30

Keywords: Inventory control; Running maximum; Ergodic control; Constrained minimization; Regime-switching diffusions

1. Introduction

In this article, we address a stochastic control problem motivated by inventory control subject to capacity expansion. Consider an inventory model for a product where the *inventory netflow*

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http://dx.doi.org/10.1016/j.spa.2015.11.014
0304-4149/

fluctuates as a random process. Here the netflow process represents the difference between the supplies of the product to the storage facility and customer demands. A negative netflow represents the inventory backlog at the time. A storage capacity is used to store the product when the inventory level is positive. However, rapid expansion of the storage capacity is expensive and therefore a control is enforced to the netflow process in order to maintain a gradual capacity expansion. In practice, exercising the control amounts to providing customer sales discounts for the product. We introduce a cost structure associated with the controlled netflow process and it consists of two components: *a control cost* to represent the loss of profits due to the exercise of control and *a capacity expansion cost* that is proportional to the storage expansion.

Stochastic control problems motivated by inventory models and stochastic processing systems have a long history and is a common theme in the literature. Some early work can be found in [2,17,16,33] and the references therein. More recent work include [9,11,10,12,5,6,3,4,26], to name just a few. In [4], a discrete inventory system is studied, where the unmet demand is lost and the excess inventory is subject to shrinkage. In the recent work of [5,9,11,10,26,33], the inventory processes are modeled by drifted Brownian motions with or without Poisson demand or more general jump diffusions. The adjustments to inventory levels are represented by controlled impulse jumps (upward or downward) and each such impulse jump includes a fixed cost and a proportional cost. In addition, a state-dependent holding and/or backorder cost is also included. In these articles, they address the optimal inventory control problem under the long-run average or discounted cost minimization criteria. Often, the focus is to establish the optimality of an (s, S)-policy. The article of [12] considers a single inventory model subject to independent stochastic demand and item returns. Using an appropriate transformation, they derive the optimality of an (s, S)-policy under the average cost minimization criterion.

This work takes a different perspective toward inventory control problems. We begin with a netflow process modeled by a drifted Brownian motion process. Next, control policies are introduced in order to avoid costly rapid capacity expansion and they represent discount sales or discarding items in the context of a perishable inventory. But exercising such controls may result in loss of profits. Therefore, it is desirable to determine a control policy that minimizes a cost structure which constitutes control costs and capacity expansion costs. To this end, we first introduce an infinite horizon discounted cost functional J in Eq. (2.5), with a convex control cost and a linear capacity expansion cost.

It is natural to use the running maximum process of the netflow to model the storage expansion. Since the running maximum at any time $t \geq 0$ depends on the entire history of the underlying netflow process in the interval $[0, t]$, it is of path-dependent nature. With such a path-dependent term in the cost functional J of (2.5), the analysis of the problem is nontrivial. The usual methodology of stochastic control theory (such as [13,25,38]) is not applicable directly. Indeed, a heuristic application of the usual dynamical programming principle yields a Hamilton–Jacobi–Bellman (HJB) equation of an enlarged dimension; which does not lead to the value function and optimal control policies. The paper [18] deals with a stochastic control problem involving a running maximum process. Thanks to the special features of their cost structure, they were able to approach the problem by first solving a family of auxiliary regular stochastic control problems. This methodology is further developed in [19,20] to deal with optimal control and replacement problem with state-dependent failure rate. On the other hand, the literature on Russian options, e.g., [29,30] and the references therein, deal with exotic financial options that involve the running maximum process of the underlying stock process. In these works, the value function and optimal stopping policies were obtained through appropriate transformations.

Our approach is different from those in the aforementioned references. We first introduce a reflected process Y to formulate an equivalent stochastic control problem (2.13)–(2.15) in Section 2.2. This, in turn, leads to a solvable HJB equation (3.4). With the aid of the Legendre transformation Φ and its derivative Ψ, we show that the HJB equation (3.4) has a smooth bounded solution Q in Proposition 3.2. Then we establish in Theorem 3.4 that Q coincides with the value function for the problem (2.13)–(2.15). Moreover, we determine an explicit feed-back type optimal policy $u^*(\cdot)$ in (3.8). Finally, in Corollary 3.5 we obtain the value function as well as a non-Markovian optimal policy for the original problem given by (2.5), (2.8) and (2.9). In this approach, the pair (Φ, Ψ) is vital for our analysis. We note that [14,35] also use the pair (Φ, Ψ) in essential ways.

Next, we consider an associated constrained minimization problem. Let τ be an exponentially distributed random variable which is independent of the system dynamics. A controller would like to minimize the expected control cost during an observed time period $[0, \tau]$ subject to a mean capacity constraint. Such problems are quite common in real world applications. Here we employ a Lagrange multiplier type method together with our solution to the discounted problem in Section 3 to find an explicit feed-back type non-Markovian control strategy. In this process, we also derive many interesting qualitative properties of the value function $\tilde{V}$ of (2.15); see Lemma 4.3 and Propositions 4.4 and 4.5. The solution to the constrained minimization is spelled out in Proposition 4.2 and Theorem 4.7.

We also obtain an explicit solution to the long-run average cost minimization (ergodic control) problem (5.1)–(5.2) in Section 5 as an application of our results in Section 3. Using the vanishing discount factor technique (see, for example, [1,27]), we obtain a constant optimal control u^* in (5.3). The value of u^* is dependent on the control cost function and the rate of the storage expansion cost p in the cost structure. This dependence is made explicit in (5.3). In the derivation of these results, we need to treat carefully several different cases concerning the relationship between the drift rate θ in (2.1) and the value $\Psi(p)$.

The last section is motivated by an inventory control problem arising from a more realistic situation when the netflow process exhibits regime-switching behavior. We model the netflow by a Brownian motion whose drift and diffusion coefficients are subject to regime-switching. The rationale here is that the dynamics of the netflow process often display structural changes due to the effects of the economic cycles, monetary policy uncertainties, or changes of working shifts in a factory, etc. For example, as illustrated in [15,28], the production rates are different in different states of a failure-prone machine. Therefore, we introduce a regime-switching diffusion to model the netflow process. Then we formulate the associated infinite horizon discounted cost minimization problem in Section 6. Typically, the HJB equation satisfied by the value function of regime-switching control problems leads to a coupled system of second-order differential equations where explicit solutions are nearly impossible to obtain. However, here we provide a novel technique to obtain an *explicit* optimal feed-back type non-Markovian control strategy using the results in the discounted control problem in Section 3. The study of regime-switching diffusion processes has attracted growing attention in recent years. We refer to [24,37] and the references therein for recent progress in the investigation of such processes. Various stochastic control problems and their applications in mathematical finance and optimal harvesting problems involving regime-switching diffusion processes were considered in [40,31,39], among others. Our contribution further expands the application of regime-switching diffusion processes in the context of inventory control problems.

The rest of the paper is arranged as follows. The stochastic model, precise problem formulation, and the equivalent stochastic control problem involving the reflected process will be introduced in Section 2. Section 3 deals with the infinite horizon discounted problem. The constrained minimization problem is arranged in Section 4. Section 5 considers the long term average problem while Section 6 takes up the problem of infinite horizon discounted problem with regime switching. Several technical results are arranged in the Appendices A–D.

2. Stochastic model

2.1. Problem formulation

Let $Z = \{Z(t) : t \geq 0\}$ be a Brownian motion with drift θ, variance σ^2 and initial value $Z(0) = z$. It is defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ and is adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$. The process Z can be represented by

$$Z(t) = z + \theta t + \sigma W(t), \quad t \geq 0, \tag{2.1}$$

where $W = \{W(t) : t \geq 0\}$ is a standard $\{\mathcal{F}_t\}$-Brownian motion so that it is adapted to $\{\mathcal{F}_t\}$ and the increments $W(t + s) - W(t)$ are independent of $\mathcal{F}_t$ for each $s > 0$. We use the process Z to model the netflow process of an inventory of a certain product in the absence of any control mechanism. This netflow process captures the difference between regular supplies and customer demands of the product. At a given time $t \geq 0$, it occupies $Z(t)^+$ units of capacity in the storage and thus, $Z(0)$ is assumed to be nonnegative. If $Z(t)$ is negative, $|Z(t)|$ represents the inventory backlog at time t.

On one hand, a rapid capacity expansion to accommodate high inventory levels is costly and to remedy this situation, management may reduce the inventory levels by other means, such as discounted sales, etc. On the other hand, such inventory reduction strategies lower the profit margins. These opposing tendencies of the capacity expansion and the controlled inventory reduction naturally lead us to formulate a stochastic control problem. We introduce a control structure to the basic model described in (2.1) to reduce the inventory so that it would slow down the expansion of the capacity. Let $X = \{X(t) : t \geq 0\}$ be the controlled netflow process, where the inventory level at time $t \geq 0$ is $X(t)$ and $X(0) = x$. The dynamics of X is given by

$$X(t) = x + \theta t + \sigma W(t) - \int_0^t u(s)\,\mathrm{d}s, \quad t \geq 0, \tag{2.2}$$

where W is a standard $\{\mathcal{F}_t\}$-Brownian motion. The process $u = \{u(t) : t \geq 0\}$ represents the control exerted on the inventory and u is a nonnegative, non-anticipative process which is adapted to $\{\mathcal{F}_t\}$. Moreover, it satisfies the integrability condition

$$\int_0^t u(s)\,\mathrm{d}s < \infty, \quad \text{a.s. for each } t \geq 0. \tag{2.3}$$

The running maximum process $M = \{M(t) : t \geq 0\}$ is defined by

$$M(t) := \sup_{s\in[0,t]} X(s), \tag{2.4}$$

for each $t \geq 0$. The quantity $M(t)$ represents the total storage (or capacity) used during the time interval $[0, t]$. The integral $\int_0^t u(s)\,\mathrm{d}s$ in (2.2) represents the effort exercised during $[0, t]$ in reducing the inventory so that the expansion of the used capacity process M will grow slower. Intuitively, it is clear that $u(t)$ may take large values when $X(t)$ is close to $M(t)$.

There are two types of costs introduced in our cost structure: a control cost of $c(u(t))$ which represents the lost profits due to the reduction of the inventory (using discounted sales, etc.), and a linear cost proportional to the size of capacity expansion. We introduce an infinite-horizon discounted cost functional

$$J(x, u, X) = \mathbb{E}\left[\int_0^\infty e^{-\alpha t}[c(u(t))\,\mathrm{d}t + p\,\mathrm{d}M(t)]\right], \tag{2.5}$$

where $\alpha > 0$ and $p > 0$ are constants. The constant $p > 0$ represents the cost per unit capacity increase and $\alpha > 0$ is a discount factor.

We make the following assumptions on the running cost function c which will remain valid throughout the article.

Assumption 2.1. The running cost function $c : [0, \infty) \mapsto [0, \infty)$ is nonnegative, non-decreasing, strictly convex, and continuously differentiable. Moreover, it satisfies $c(0) = 0$ and

$$\lim_{x\to\infty} \frac{c(x)}{x} = \infty. \tag{2.6}$$

Remark 2.2. For a convex function c, the assumption (2.6) is equivalent to

$$\lim_{x\to\infty} c'(x) = \infty. \tag{2.7}$$

In fact, if (2.7) holds, then l'Hopital's rule implies (2.6). On the other hand, if c is convex, c' is increasing and $\lim_{x\to\infty} c'(x) = l_0$ exists. Then, using the mean value theorem, $\frac{c(x)}{x} \le l_0$ holds for all x. Thus, (2.6) implies that $l_0 = \infty$.

It can be shown that the above cost J in (2.5) is equivalent to the expected cost accrued during a random time interval $[0, \tau]$, where τ is an exponentially distributed random variable with parameter $\alpha > 0$ and τ is independent of system dynamics (see also [36]). To observe this, we use Fubini's theorem and obtain $\mathbb{E}[\int_0^\tau c(u(t))\,\mathrm{d}t] = \mathbb{E}[\int_0^\infty e^{-\alpha t} c(u(t))\,\mathrm{d}t]$ and $\mathbb{E}[M(\tau)] = \mathbb{E}[\int_0^\infty e^{-\alpha t}\,\mathrm{d}M(t)]$.

The computation of our cost functional $J(x, u, X)$ requires only the probability laws of the processes u and M. Hence, it suffices to formulate the admissible control systems (x, u, X) for our control problem using weak solutions to (2.2) and it also allows the use of a large class of control processes, such as discontinuous "bang–bang" type controls. This weak formulation described below is also in line with the formulation of stochastic control problems in chapters IV and V of [13].

In view of the above considerations, the processes u, W, and X satisfying (2.2) and (2.3) are allowed to be defined on their own filtered probability spaces $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\ge 0}, \mathbb{P})$. We let the initial value x to take any real number while in our motivating application it is nonnegative. For a given $x \in \mathbb{R}$, we call the quintuple $((\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\ge 0}, \mathbb{P}), x, u, W, X)$ an *admissible control system* if

(i) (x, u, W, X) is a weak solution to (2.2) with $X(0) = x$; and
(ii) the explosion time of X is infinite with probability one.

When there is no ambiguity, we simply use the triple (x, u, X) to represent an admissible control system. To define the value function, we introduce the set

$$\mathcal{A}(x) := \{(x, u, X) : (x, u, X) \text{ is admissible}\}. \tag{2.8}$$

This set is non-empty for each $x \in \mathbb{R}$, since (2.2) has a path-wise unique weak solution X with $X(0) = x$ when the control process u is identically zero. Moreover, the cost functional J is finite under the zero control. The value function of the stochastic control problem is thus defined by

$$V(x) = \inf_{\mathcal{A}(x)} J(x, u, X) \quad \text{for each } x \in \mathbb{R}. \tag{2.9}$$

Our objective is to find an explicit optimal control u^* and to characterize the value function V of (2.9). We can carefully observe that a control process u which is admissible for (x, u, X) is also admissible for $(0, u, X)$. Hence using the structure of $J(x, u, X)$ in (2.5) and the comparison method in dynamic programming, one can verify that the value function $V(x)$ of (2.9) is a constant. Our discussion in the next subsection makes this observation quite clear.

2.2. *Reflected processes*

We introduce a reflected process to formulate an equivalent control problem with the same value function. We let $Y(t) = M(t) - X(t)$, for $t \geq 0$, where X satisfies (2.2) and M is the maximum process defined in (2.4). Then Y satisfies $Y(t) = -x - \theta t - \sigma W(t) + \int_0^t u(s)\,\mathrm{d}s + M(t)$, for $t \geq 0$. Since $(-W(t))$ is also a standard Brownian motion adapted to the same Brownian filtration $\{\mathcal{F}_t\}_{t\geq 0}$, with a slight abuse of notation, we replace the term $-W(t)$ by $W(t)$ and write Y in the form

$$Y(t) = -\theta t + \sigma W(t) + \int_0^t u(s)\,\mathrm{d}s + M(t) - x, \quad t \geq 0. \tag{2.10}$$

Moreover, the process M increases only when $Y(t) = 0$, and it satisfies

$$M(t) = x + \int_0^t I_{\{Y(s)=0\}}\,\mathrm{d}M(s). \tag{2.11}$$

By (2.10) and (2.11), and observing that $Y(t) \geq 0$ for all $t \geq 0$, we can characterize the pair $(Y(\cdot), M(\cdot) - x)$ as the one-sided reflection map (or the Skorokhod map as in [8]) of the unrestricted process $(-\theta t + \sigma W(t) + \int_0^t u(s)\,\mathrm{d}s)$ with the reflection barrier at the origin. Given such an unrestricted process and the reflection barrier, such a decomposition always exists and is unique. For details, we refer to Section 2.2 in [16], or Lemma 6.14 in Chapter 3 of [23]. This decomposition is also known as "the Skorokhod decomposition" of the unrestricted process.

Our objective is to minimize the cost functional J in (2.5) when $Y(0) = 0$. Moreover, the sign of the drift coefficient θ is irrelevant in our analysis. Hence, we consider the general situation where Y satisfies an equation similar to (2.10) with drift μ and the initial value $Y(0) = y$ with y being a non-negative real number. Now we formulate a more general control problem.

With the standard notation for the local-time process, we consider

$$\begin{aligned} Y(t) &= y + \mu t + \sigma W(t) + \int_0^t u(s)\,\mathrm{d}s + L_Y(t), \\ L_Y(t) &= \int_0^t I_{\{Y(s)=0\}}\,\mathrm{d}L_Y(s), \end{aligned} \tag{2.12}$$

for $t \geq 0$. Here, the drift coefficient μ can be any real number. In this situation, the unique pair $(Y(\cdot), L_Y(\cdot))$ represents the one-sided reflection map of the unrestricted process $(y - \theta t + \sigma W(t) + \int_0^t u(s)\,\mathrm{d}s)$ with the reflection barrier at the origin, as described in Chapter 2 of [16].

Moreover, when $y = 0$ and $\mu = -\theta$, the local-time process is described by $L_Y(t) = M(t) - x$ for all $t \geq 0$, where M is given by (2.4).

Next, we introduce the cost functional

$$\tilde{J}(y, u, Y) = \mathbb{E}\left[\int_0^\infty e^{-\alpha t}[c(u(t))\,\mathrm{d}t + p\,\mathrm{d}L_Y(t)]\right]. \tag{2.13}$$

Notice that when $\mu = -\theta$ and $y = 0$, we have $\tilde{J}(0, u, Y) = J(x, u, X)$, where J is given in (2.5). Hence, we observe that J and the value function V in (2.9) are independent of x. Corresponding to an admissible (x, u, X) in $\mathcal{A}(x)$, using the same control u, we can define an admissible system (y, u, Y) if

(i) (y, u, Y) is a weak solution to (2.12) with $Y(0) = y$, and
(ii) the explosion time of Y is infinite with probability one.

Finally we let

$$\tilde{\mathcal{A}}(y) = \{(y, u, Y) : (y, u, Y) \text{ is admissible}\} \tag{2.14}$$

and the corresponding value function $\tilde{V}$ by

$$\tilde{V}(y) = \inf_{\tilde{\mathcal{A}}(y)} \tilde{J}(y, u, Y), \tag{2.15}$$

for each $y \in \mathbb{R}$. In the case $\mu = -\theta$, we have $V(x) = \tilde{V}(0)$ for all $x \in \mathbb{R}$ and hence the value function V in (2.9) is a constant. We intend to find an optimal control u^* for the value function $\tilde{V}(y)$, when $y \in \mathbb{R}$.

3. The discounted control problem

To address the discounted control problem defined in (2.13)–(2.15), first we introduce a pair of functions (Φ, Ψ) which will play an important role in our analysis. Let $\Phi : \mathbb{R} \mapsto [0, \infty)$ be the function defined by

$$\Phi(y) := \sup_{a \geq 0}\{ay - c(a)\}, \quad \text{for all } y \in \mathbb{R}. \tag{3.1}$$

Here $c(\cdot)$ is the running cost function in (2.5). The following lemma can be found in [14,35].

Lemma 3.1. *Let the cost function c satisfy Assumption 2.1. Then the function Φ is finite on $\mathbb{R}$, nonnegative and nondecreasing, with $\Phi(y) = 0$ for $y \leq 0$. Moreover, $\Phi(y)$ is differentiable with*

$$\Phi'(y) = \Psi(y) = \begin{cases} 0 & \text{if } y \leq c'(0), \\ (c')^{-1}(y) & \text{if } y > c'(0). \end{cases} \tag{3.2}$$

In addition, we have

$$\Phi(y) = y\Psi(y) - c(\Psi(y)). \tag{3.3}$$

A detailed discussion of the properties of the pair (Φ, Ψ) and their use in discrete-time control problem can be found in [14]. An application to a queuing control problem is in [35], and we follow a similar approach here.

The formal Hamilton–Jacobi–Bellman (HJB) equation associated with the stochastic control problem in (2.12)–(2.15) can be written as

$$\begin{cases} \inf_{u\geq 0}\left\{\frac{1}{2}\sigma^2 Q''(x)+\mu Q'(x)-\alpha Q(x)+uQ'(x)+c(u)\right\}=0, & \text{if } x\geq 0,\\ Q'(0)=-p. \end{cases} \tag{3.4}$$

Using the functions Φ and Ψ, (3.4) can be represented by

$$\begin{cases} \frac{\sigma^2}{2}Q''(x)+\mu Q'(x)-\Phi(-Q'(x))-\alpha Q(x)=0, & \text{if } x\geq 0,\\ Q'(0)=-p. \end{cases} \tag{3.5}$$

Furthermore, it follows from (3.3) that we can rewrite (3.5) as

$$\begin{cases} \frac{\sigma^2}{2}Q''(x)+\mu Q'(x)+\Psi(-Q'(x))Q'(x)-\alpha Q(x)+c(\Psi(-Q'(x)))=0,\\ Q'(0)=-p, \end{cases} \tag{3.6}$$

for all $x\geq 0$.

The following proposition guarantees a smooth solution to the HJB equation (3.5) and it will be proved in the Appendix A.

Proposition 3.2. *There is a function Q defined on $[0,\infty)$ which satisfies the following conditions:*

(a) *Q is a positive, twice differentiable on $(0,\infty)$ and it satisfies the HJB equation (3.4).*
(b) *The derivative Q' is negative, strictly increasing and bounded on the interval $[0,\infty)$. Consequently, Q is a bounded convex function which is strictly decreasing on $[0,\infty)$.*

Next we establish a verification lemma which verifies that the function Q obtained in Proposition 3.2 is a lower bound for the value function $\tilde{V}$ in (2.15).

Lemma 3.3. *Let $Q:[0,\infty)\mapsto(0,\infty)$ be the function obtained in Proposition 3.2. Then*

$$Q(x)\leq \tilde{V}(x) \tag{3.7}$$

holds for all $x\geq 0$.

Proof. The proof is quite similar to verification lemmas in [34] and [13, Theorem 8.4.1] and hence, only major steps are indicated here.

We consider any admissible process (y,u,Y) satisfying (2.12) and apply Itô's formula to $Q(Y(t))e^{-\alpha t}$. Notice that both Q and its derivative Q' are bounded on $[0,\infty)$. Hence, following the proof of [13, Theorem 8.4.1] and using $Q'(0)=-p$, we obtain

$$\mathbb{E}[e^{-\alpha T}Q(Y(T))]+\tilde{J}(y,u,Y)\geq Q(y),\quad \text{for any } T\geq 0.$$

Since Q is bounded, by letting $T\to\infty$ we have $\tilde{J}(y,u,Y)\geq Q(y)$ for each admissible process (y,u,Y). Now taking infimum over all admissible (y,u,Y), we obtain $\tilde{V}(y)\geq Q(y)$ for each $y\geq 0$, where $\tilde{V}$ is the value function given in (2.15). This completes the proof. □

In the next theorem, we show that the function Q described in Proposition 3.2 indeed coincides with the value function $\tilde{V}$ given in (2.15). Thus, we can consider the value function $\tilde{V}$

as the smooth bounded solution to the HJB equation (3.4) on $[0, \infty)$ with a bounded derivative. Moreover, we obtain a feed-back type optimal drift control policy $u^*(\cdot)$ given by

$$u^*(x) = \Psi(-Q'(x)), \quad \text{for all } x \geq 0, \tag{3.8}$$

where the nonnegative function Ψ is given in (3.2) and Q is as in Proposition 3.2. Since Q is convex, Q' is increasing and it satisfies $-p \leq Q'(x) < 0$ for all $x \geq 0$. Consequently, our candidate optimal feedback control is bounded and it satisfies

$$0 \leq u^*(x) \leq \Psi(p) \tag{3.9}$$

for all $x \geq 0$. Note that if $p = 0$, it is straightforward to verify that zero control is optimal since the growth of the maximum process is inconsequential to the cost functional. Indeed, when $p = 0$, the function Q also becomes the zero function and hence (3.8) and (3.9) yield $u^*(x) \equiv 0$ for all $x \geq 0$. The optimal control $u^*(\cdot)$ in (3.8) is described in terms of the value function. This is a common approach in the literature and we refer to Chapters 3 and 8 of [13]. Our next result is in agreement with this observation.

Using the description of u^* in (3.8), our candidate for the optimal state process can be considered as a weak solution to

$$Y^*(t) = y + \sigma W(t) + \mu t + \int_0^t u^*(Y^*(s))\,\mathrm{d}s + L_{Y^*}(t), \tag{3.10}$$

for all $t \geq 0$. Here $W = \{W(t) : t \geq 0\}$ is a standard Brownian motion adapted to a right-continuous Brownian filtration $\mathbb{F} := \{\mathcal{F}_t : t \geq 0\}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The following theorem is the main result of this section.

Theorem 3.4. *The following results hold:*

(i) *For each $y \geq 0$, the triple (y, u^*, Y^*) described above is an admissible control system with the associated cost functional*

$$\tilde{J}(y, u^*, Y^*) = Q(y). \tag{3.11}$$

(ii) *The triple (y, u^*, Y^*) represents an optimal control system for each $y \geq 0$, and u^* is an optimal feed-back control policy. Hence*

$$\tilde{V}(y) = Q(y) \tag{3.12}$$

for all $y \geq 0$ and the value function $\tilde{V}$ is the unique smooth solution to the HJB equation (3.4) that satisfies the assertions of Proposition 3.2.

Proof. The function $u^*(\cdot)$ described in (3.8) is a bounded, nonnegative continuous function. Therefore a weak solution to (3.10) exists and is unique in the sense of probability law. In fact, since the diffusion coefficient is a positive constant and the drift coefficient of (3.10) is a continuous function, it is straightforward to check that the corresponding sub-martingale problem is well-posed. Hence, the existence of such a weak solution Y^* which is unique in probability law is a simple consequence of the sub-martingale problem of [32].

Next, we apply Itô's formula to $|Y^*(t)|^2$ and mimic the proof of Theorem 5.2.9 in [23] to show that $\mathbb{E}[\sup_{t\in[0,T]} |Y^*(t)|^2] \leq C_1(1 + |y|^2)e^{C_2 T}$, for some positive constants C_1 and C_2. Hence the solution Y^* to (3.10) exists for all $t \geq 0$. Consequently (y, u^*, Y^*) is an admissible control system. Again, we apply Itô's formula to $e^{-\alpha t} Q(Y^*(t))$ and employ Proposition 3.2 together with (3.6) to obtain

$$\begin{aligned}\mathbb{E}[e^{-\alpha T} Q(Y^*(T))] &= Q(y) - \mathbb{E}\left[\int_0^T e^{-\alpha t}[c(u^*(Y^*(t)))\,\mathrm{d}t + p\,\mathrm{d}L_{Y^*}(t)]\right] \\ &= Q(y) - \tilde{J}(y, u^*, Y^*)\end{aligned}$$

for each $T > 0$. But Q is bounded and thus $\lim_{T\to\infty} \mathbb{E}[e^{-\alpha T} Q(Y^*(T))] = 0$. By letting $T \to \infty$, we have $Q(y) = \tilde{J}(y, u^*, Y^*)$ for each $y \geq 0$ and this yields (i).

For part (ii), since $Q(y) = \tilde{J}(y, u^*, Y^*)$ for each $y \geq 0$ and that (y, u^*, Y^*) is an admissible control system, it follows that $Q(y) \geq \tilde{V}(y)$ and hence thanks to Proposition 3.2, we have

$$\tilde{J}(y, u^*, Y^*) = Q(y) = \tilde{V}(y), \quad \text{for each } y \geq 0. \tag{3.13}$$

Consequently, (y, u^*, Y^*) is an admissible optimal control system with a feedback type optimal control function $u^*(\cdot)$. Hence, the value function $\tilde{V}(\cdot)$ can be characterized as the unique smooth bounded solution to the HJB equation (3.4) with a bounded derivative. This completes the proof. □

The optimal control system $(0, u^*, Y^*)$ yields an optimal strategy for the original control problem described in (2.9). To be precise, consider the stochastic control problem in (2.9) with an admissible state process X satisfying (2.2). Now introduce the constant $\mu = -\theta$ as explained prior to Eq. (2.12). Given the drift coefficient μ and diffusion coefficient $\sigma > 0$, there is an optimal process Y^* with $Y^*(0) = 0$ and which corresponds to $\tilde{V}(0) = \tilde{J}(0, u^*, Y^*)$. In this case, the feedback optimal control is given by $u^*(x) = \Psi(-Q'(x))$ for all $x \geq 0$. Notice that the function Q is dependent on the constants μ, σ, p, and α.

Using (3.10) with $y = 0$, and the uniqueness of the solution to the Skorokhod problem (see, e.g., Lemma 3.6.14 of [23]), we obtain that

$$L_{Y^*}(t) = \max\left\{0, \max_{s\in[0,t]}\left\{-\sigma W(s) - \mu s - \int_0^s u^*(Y^*(r))\,\mathrm{d}r\right\}\right\}.$$

Next introduce X^* by

$$X^*(t) = x + \theta t + \sigma W_1(t) - \int_0^t u^*(Y^*(s))\,\mathrm{d}s \tag{3.14}$$

where $\theta = -\mu$ and $W_1 \equiv -W$ is also a standard Brownian motion. We also define $M^*(t) = \sup_{s\in[0,t]} X^*(s)$ as in (2.4). Then $M^*(t) = x + L_{Y^*}(t)$ and $Y^*(t) = M^*(t) - X^*(t)$ for all $t \geq 0$. Hence X^* satisfies

$$X^*(t) = x + \theta t + \sigma W_1(t) - \int_0^t u^*(M^*(s) - X^*(s))\,\mathrm{d}s \tag{3.15}$$

and the cost functional $J(x, u^*, X^*) = \tilde{J}(0, u^*, Y^*)$. Thus, we have the following corollary:

Corollary 3.5. *The process X^* given in (3.14)–(3.15) is an optimal state process with the optimal control policy given by $u^*(M^*(t) - X^*(t)) = \Psi(-Q'(M^*(t) - X^*(t)))$ as in (3.8). Here Q satisfies (3.4)–(3.6) with $\mu = -\theta$. Moreover,*

$$V(x) = J(x, u^*, X^*) = \tilde{J}(0, u^*, Y^*) = \tilde{V}(0),$$

where the functions V and $\tilde{V}$ are defined in (2.9) and (2.15), respectively.

4. Cost minimization with a capacity constraint

In this section, we address a relevant cost minimization problem with a capacity constraint which is arising from the following situation. Let X be the inventory process which satisfies (2.2). Consider a random observation time period $[0, \tau]$ of the inventory process, where τ is an exponentially distributed random variable with parameter $\alpha > 0$ and is independent of the system dynamics. The system manager's objective is to minimize the expected control cost during $[0, \tau]$, subject to a capacity constraint, say $m > 0$, during this observation period. This is a constrained minimization problem and we can find an optimal strategy by using our solution to the discounted control problem.

Let M be the running maximum process of X as in (2.4) and then $M(t) - x$ represents the expanded capacity during $[0, t]$. Formally, this constrained minimization problem can be written in the following form:

$$\text{Minimize } \mathbb{E}\left[\int_0^{\tau} c(u(t))\,\mathrm{d}t\right] \tag{4.1}$$

$$\text{Subject to } \mathbb{E}\left[\int_0^{\tau} \mathrm{d}M(t)\right] \le m \tag{4.2}$$

where X is an admissible state process satisfying (2.2), $X(0) = x$, and is associated with the control $u(\cdot)$ satisfying (2.3). Here $m > 0$ is an exogenously given constant and the cost function c satisfies Assumption 2.1. Following the discussion in Section 2, here also we use the reflected process Y in (2.10) to formulate a more general constrained minimization problem

$$\text{Minimize } \mathbb{E}\left[\int_0^{\tau} c(u(t))\,\mathrm{d}t\right] \tag{4.3}$$

$$\text{Subject to } \mathbb{E}\left[L_Y(\tau)\right] \le m \tag{4.4}$$

over all admissible processes (y, u, Y) in (2.14), where L_Y represents the local-time process of Y at the origin as in (2.10). Similar to the discussion underneath (2.15), we observe that when $y = 0$ and the drift parameter $\mu = -\theta$ in (2.12), the problem in (4.1)–(4.2) is equivalent to the constrained minimization problem of (4.3)–(4.4). Since the observation time τ is exponentially distributed with parameter $\alpha > 0$ and is independent of the system dynamics, using Fubini's theorem we can write $\mathbb{E}[\int_0^{\tau} c(u(t))\,\mathrm{d}t] = \mathbb{E}[\int_0^{\infty} e^{-\alpha t} c(u(t))\,\mathrm{d}t]$ and $\mathbb{E}[L_Y(\tau)] = \mathbb{E}[\int_0^{\infty} e^{-\alpha t}\,\mathrm{d}L_Y(t)]$.

Consider the collection $\tilde{A}(y)$ of admissible control systems (y, u, Y) in (2.14). For each such (y, u, Y), the corresponding local-time process L_Y is given in (2.10). When the constant $m > 0$, we introduce

$$\tilde{A}(y; m) := \left\{(y, u, Y) \in \tilde{A}(y) : \mathbb{E}\left[\int_0^{\infty} e^{-\alpha t}\,\mathrm{d}L_Y(t)\right] \le m\right\}. \tag{4.5}$$

Consequently, we can restate the constrained minimization problem as follows:

$$\text{Minimize } \mathbb{E}\left[\int_0^{\infty} e^{-\alpha t} c(u(t))\,\mathrm{d}t\right] \tag{4.6}$$

over all admissible systems $(y, u, Y) \in \tilde{A}(y; m)$. We define the value function F_m related to (4.6) by

$$F_m(y) = \inf_{\tilde{A}(y;m)} \mathbb{E}\left[\int_0^\infty e^{-\alpha t} c(u(t))\,\mathrm{d}t\right]. \tag{4.7}$$

When $y = 0$ and the drift μ in (2.10) is equal to $-\theta$, the value $F_m(0)$ and optimal strategy for (4.6) yield the value and optimal strategy for the original problem (4.1)–(4.2).

When the constraint $m > 0$ is very large, it is evident that zero control will be a very good choice. To quantify this, we have the following lemma:

Lemma 4.1. *Let Y be the state process corresponding to zero control in (2.10) so that it satisfies*

$$Y(t) = y + \mu t + \sigma W(t) + L_Y(t), \tag{4.8}$$

where the process L_Y represents the local-time of Y at the origin. Then

$$\mathbb{E}_y\left[\int_0^\infty e^{-\alpha t}\,\mathrm{d}L_Y(t)\right] = l_0(y), \tag{4.9}$$

where

$$l_0(y) = \frac{1}{\lambda} e^{-\lambda y} \quad \text{and} \quad \lambda = \frac{1}{\sigma^2}\left[\mu + \sqrt{\mu^2 + 2\alpha\sigma^2}\right]. \tag{4.10}$$

Proof. It is easy to check that $\frac{\sigma^2}{2} l_0''(y) + \mu l_0'(y) - \alpha l_0(y) = 0$ for $y \in [0, \infty)$, $l_0'(0) = -1$ and that l_0 is a strictly decreasing positive function with $\lim_{y\to\infty} l_0(y) = 0$. Now a straightforward application of Itô's formula to $e^{-\alpha t} l_0(Y(t))$ yields the result. $\square$

The following proposition is a direct consequence of Lemma 4.1.

Proposition 4.2. *Let the initial value $y \geq 0$ be fixed and consider the minimization problem in (4.3)–(4.4), and let $l_0(y)$ be as in (4.10). If $m \geq l_0(y)$, then zero control is optimal and thus, the value function $F_m(y) = 0$.*

Proof. Since $m \geq l_0(y)$, using Lemma 4.1, we notice that the triple $(y, 0, Y)$ is in $\tilde{A}(y; m)$, where Y is given by (4.8). Then the corresponding optimal cost is zero. Hence the zero control (i.e., $u^*(t) \equiv 0$) is optimal and $F_m(y) = 0$. $\square$

When $0 < m < l_0(y)$, this constrained minimization problem is quite non-trivial and we rely on the optimal policies we have derived for the discounted control problem in Section 3. Let (y, u^*, Y^*) be the optimal control system we have derived in Theorem 3.4. Since the dependence on $p > 0$ is significant in our analysis, henceforth we denote this optimal feed-back control by $u_p^*(\cdot)$ and the state process by $Y_p^*(\cdot)$. Recall that $u_p^*(\cdot)$ is given in (3.8) and $Y_p^*(\cdot)$ satisfies (3.10). We also denote the value function $\tilde{V}(\cdot)$ in (3.13) by $\tilde{V}_p(\cdot)$. Hence for a fixed $p > 0$ and $y \geq 0$,

$$u_p^*(x) = \Psi(-W_p(x)), \tag{4.11}$$

where

$$W_p(x) = \tilde{V}_p'(x), \quad \text{for all } x \geq 0, \tag{4.12}$$

and the value function $\tilde{V}_p$ is the smooth solution to the HJB equation (3.4) as described in Proposition 3.2. Moreover, to describe the dependence on $p > 0$, we rewrite (3.10) in the form

$$Y_p^*(t) = y + \sigma W(t) + \mu t + \int_0^t u_p^*(Y_p^*(s))\,\mathrm{d}s + L_p^*(t), \tag{4.13}$$

for all $t \geq 0$. Then for each $p > 0$ and $y \geq 0$,

$$\tilde{V}_p(y) = \tilde{J}_p(y, u_p^*, Y_p^*) = \mathbb{E}\left[\int_0^\infty e^{-\alpha t} c(u_p^*(Y_p^*(t)))\,\mathrm{d}t + p\Gamma(p, y)\right], \tag{4.14}$$

where

$$\Gamma(p, y) = \mathbb{E}\left[\int_0^\infty e^{-\alpha t}\,\mathrm{d}L_p^*(t)\right]. \tag{4.15}$$

Our strategy to find an optimal control for the case $0 < m < l_0(y)$ can be summarized as follows. First we show that there is a unique $p_m > 0$ so that $\Gamma(p_m, y) = m$. In the second step, we establish the above described $(y, u_{p_m}^*, Y_{p_m}^*)$ is an optimal admissible policy. We begin with the following lemma which will establish several properties of the value function $\tilde{V}_p(y)$.

Lemma 4.3. *Let $y \geq 0$ be fixed. Then the value function $\tilde{V}_p(y)$ in (4.14) is a Lipschitz continuous, concave, and increasing function in the p variable.*

Proof. Since the collection $\tilde{A}(y)$ of admissible control systems defined in (2.14) is independent of p, by (2.13) and (2.15), it is evident that the value function $\tilde{V}_p(y)$ is an increasing function of p. Next let $p_1 < p_2$ and consider $\tilde{V}_{p_2}(y) - \tilde{V}_{p_1}(y)$. Then $\tilde{V}_{p_1}(y) = \tilde{J}(y, u_{p_1}^*, Y_{p_1}^*)$ as in Theorem 3.4. On the other hand, using $u_{p_1}^*$ and $Y_{p_1}^*$, we have

$$\tilde{V}_{p_2}(y) \leq \mathbb{E}\left[\int_0^\infty e^{-\alpha t} c(u_{p_1}^*(Y_{p_1}^*(t)))\,\mathrm{d}t\right] + p_2\Gamma(p_1, y).$$

Consequently, it follows that

$$0 \leq \tilde{V}_{p_2}(y) - \tilde{V}_{p_1}(y) \leq (p_2 - p_1)\mathbb{E}\left[\int_0^\infty e^{-\alpha t}\,\mathrm{d}L_{p_1}^*(t)\right]. \tag{4.16}$$

Notice that the pair $(Y_{p_1}^*, L_{p_1}^*)$ represents the Skorokhod decomposition of the process $y + \sigma W(t) + \mu t + \int_0^t u_{p_1}^*(Y_{p_1}^*(s))\,\mathrm{d}s, t \geq 0$. Similarly, (Y, L_Y) in (4.8) represents the Skorokhod decomposition of $y + \mu t + \sigma W(t), t \geq 0$. Since $u_{p_1}^*$ is a nonnegative function, we can use the comparison theorem for Skorokhod maps (Proposition 3.4 of [8]) to conclude that $L_{p_1}^*(t) \leq L_Y(t)$ for all $t \geq 0$. Moreover, using the Fubini theorem, we have $\mathbb{E}[\int_0^\infty e^{-\alpha t}\,\mathrm{d}L_{p_1}^*(t)] = \alpha\mathbb{E}[\int_0^\infty e^{-\alpha t} L_{p_1}^*(t)\,\mathrm{d}t]$. Thus, it follows that

$$\mathbb{E}\left[\int_0^\infty e^{-\alpha t}\,\mathrm{d}L_{p_1}^*(t)\right] \leq \mathbb{E}\left[\int_0^\infty e^{-\alpha t}\,\mathrm{d}L_Y(t)\right] = l_0(y) \leq \frac{1}{\lambda} = \frac{\sigma^2}{\mu + \sqrt{\mu^2 + 2\alpha\sigma^2}},$$

where $l_0(y)$ and λ are given in (4.10) of Lemma 4.1. This together with (4.16) yields that $0 \leq \tilde{V}_{p_2}(y) - \tilde{V}_{p_1}(y) \leq \frac{1}{\lambda}(p_2 - p_1)$. Hence, $\tilde{V}_p(y)$ is Lipschitz continuous in the p variable.

To obtain the concavity of $\tilde{V}_p(y)$ in the p-variable, keep $y \geq 0$ fixed and consider any admissible control system $(y, u, Y) \in \tilde{A}(y)$. Since $\tilde{A}(y)$ does not depend on p, the cost functional $\mathbb{E}[\int_0^\infty e^{-\alpha t}[c(u(t))\,\mathrm{d}t + p\,\mathrm{d}L_Y(t)]]$ is a linear function in p. Thus, $\tilde{V}_p(y)$ is the infimum of a collection of linear functions in p where the infimum is taken over $\tilde{A}(y)$. Therefore, $\tilde{V}_p(y)$ is concave in the p variable. This completes the proof. □

In the next two propositions, we collect several important properties of $\Gamma(p, y)$.

Proposition 4.4. *Consider $\Gamma(p, y)$ given in (4.15) for $p > 0$ and $y \ge 0$. Then*
(i) *for all $p > 0$ and $y \ge 0$, we have*

$$\Gamma(p, y) = \frac{-W_p(y)}{W_p'(0)} = \frac{\sigma^2}{2} \cdot \frac{(-\tilde{V}_p'(y))}{\alpha \tilde{V}_p(0) + \Phi(p) + \mu p}. \tag{4.17}$$

(ii) *$\Gamma(p, y)$ is jointly continuous in (p, y) and for any fixed $p > 0$, $\Gamma(p, y)$ is strictly decreasing in y. Moreover, $\lim_{y\to\infty} \Gamma(p, y) = 0$ for each $p > 0$.*
(iii) *For each $p > 0$ and $y \ge 0$, we have*

$$\frac{\partial}{\partial p} \tilde{V}_p(y) = \Gamma(p, y). \tag{4.18}$$

Proof. Consider the function $W_p(y) = \tilde{V}_p'(y)$ for $y \ge 0$. In view of Proposition 3.2 and (A.8), W_p satisfies the assertions of Theorem A.7. In particular, W_p satisfies (A.2). Using (4.11), (A.2) can be written as

$$\begin{cases} \dfrac{\sigma^2}{2} W_p''(x) + \mu W_p'(x) + u_p^*(x) W_p'(x) - \alpha W_p(x) = 0, & x \ge 0, \\ W_p(0) = -p, \qquad \lim\limits_{x\to\infty} W_p(x) = 0. \end{cases} \tag{4.19}$$

Moreover, $W_p(\cdot)$ is bounded and strictly increasing on $[0, \infty)$. Now a direct application of Itô's formula to $e^{-\alpha t} W_p(Y_p(t))$ yields the first equation of (4.17). Note that $W_p'(0) > 0$ as described in Theorem A.7 and Eq. (A.8) imply that $\alpha \tilde{V}_p(0) = \frac{\sigma^2}{2} W_p'(0) - \mu p - \Phi(p)$. Consequently, the second equation of (4.17) follows.

For part (ii), Lemma 4.3 says that the function $p \mapsto \tilde{V}_p(0)$ is continuous. In addition, by Proposition B.1, the function $(p, y) \mapsto W_p(y)$ is continuous. Hence, by the second equation of (4.17), it follows that $\Gamma(p, y)$ is jointly continuous in (p, y). On the other hand, by Theorem A.7, the function $y \mapsto W_p(y)$ is strictly increasing with $\lim_{y\to\infty} W_p(y) = 0$ for each fixed $p > 0$. In addition, we have $W_p'(0) > 0$. Thus, it follows that for every fixed $p > 0$, the function $y \mapsto \Gamma(p, y)$ is strictly decreasing with $\lim_{y\to\infty} \Gamma(p, y) = 0$ and this completes the proof of part (ii).

To prove part (iii), let $y \ge 0$ be fixed and consider $\tilde{V}_{p+h}(y) - \tilde{V}_p(y)$ for $h > 0$. Let $(y, u_p^*, Y_p^*) \in \tilde{A}(y)$ which satisfies (4.13)–(4.14). Hence, we have

$$\tilde{V}_p(y) = \tilde{J}_p(y, u_p^*, Y_p^*) = \mathbb{E}\left[\int_0^\infty e^{-\alpha t} c(u_p^*(Y_p^*(t)))\, dt + p\Gamma(p, y)\right].$$

For $\tilde{V}_{p+h}(y)$, the strategy $(y, u_p^*, Y_p^*) \in \tilde{A}(y)$ and hence

$$\tilde{V}_{p+h}(y) \le \mathbb{E}\left[\int_0^\infty e^{-\alpha t} c(u_p^*(Y_p^*(t)))\, dt + (p+h)\Gamma(p, y)\right].$$

Therefore we have

$$0 \le \tilde{V}_{p+h}(y) - \tilde{V}_p(y) \le h\Gamma(p, y) \quad \text{and thus,} \ \limsup_{h\to 0+} \frac{\tilde{V}_{p+h}(y) - \tilde{V}_p(y)}{h} \le \Gamma(p, y).$$

Similarly, by considering the optimal strategy $(y, u_{p+h}^*, Y_{p+h}^*)$ for $\tilde{V}_{p+h}(y)$, we can obtain $h\Gamma(p + h, y) \le \tilde{V}_{p+h}(y) - \tilde{V}_p(y)$. Then, using the continuity of $\Gamma(p, y)$ in p, we have $\liminf_{h\to 0+} \frac{\tilde{V}_{p+h}(y) - \tilde{V}_p(y)}{h} \ge \Gamma(p, y)$. Thus, we conclude $\lim_{h\to 0+} \frac{\tilde{V}_{p+h}(y) - \tilde{V}_p(y)}{h} = \Gamma(p, y)$.

In a similar fashion, we can show that $\Gamma(p, y) \leq \frac{\tilde{V}_p(y)-\tilde{V}_{p-h}(y)}{h} \leq \Gamma(p-h, y)$, for $h > 0$. This, together with the continuity of $\Gamma(p, y)$ in the p variable, implies that $\lim_{h\to 0+} \frac{\tilde{V}_p(y)-\tilde{V}_{p-h}(y)}{h} = \Gamma(p, y)$ and consequently, $\lim_{h\to 0} \frac{\tilde{V}_p(y)-\tilde{V}_{p-h}(y)}{h} = \Gamma(p, y)$. This proves that $\frac{\partial}{\partial p}\tilde{V}_p(y) = \Gamma(p, y)$ for each $y \geq 0$. $\square$

Proposition 4.5. *For each fixed $y \geq 0$, $\Gamma(p, y)$ is strictly decreasing as a function of the p variable and $\lim_{p\to\infty} \Gamma(p, y) = 0$. Moreover, for each $y \geq 0$, $\lim_{p\to 0+} \Gamma(p, y) = l_0(y)$, where $l_0(y)$ is given in (4.10).*

Proof. See Appendix C. $\square$

Remark 4.6. For a fixed $y \geq 0$, if we formally define $\Gamma(0, y) = l_0(y)$, then Propositions 4.4 and 4.5 imply that $\Gamma(\cdot, y) : [0, \infty) \to \mathbb{R}$ is a strictly decreasing and continuous function which takes all the values in the half-open interval $(0, l_0(y)]$.

Using Propositions 4.4 and 4.5, now we can find an optimal control policy for the constrained minimization problem described in (4.5)–(4.7). Recall from Proposition 4.2 that when $m \geq l_0(y)$, the zero control is optimal. Hence in the following theorem, we only address the remaining case $0 < m < l_0(y)$.

Theorem 4.7. *Let $m > 0$ be given in (4.4). If $y \geq 0$ and $l_0(y) > m$, then the following results hold:*

(i) *There exists a unique point $p_m > 0$ so that $\Gamma(p_m, y) = m$.*
(ii) *The optimal policy $(y, u^*_{p_m}(\cdot), Y^*_{p_m})$ in Theorem 3.4 is also optimal for the minimization problem (4.3) with constraint (4.4).*
(iii) *The value function $F_m(y)$ in (4.7) is given by $F_m(y) = \mathbb{E}_y[\int_0^\infty e^{-\alpha t} c(u^*_{p_m}(Y^*_{p_m}(t)))\,\mathrm{d}t]$ and it satisfies the relation*

$$F_m(y) + m = \tilde{V}_{p_m}(y), \tag{4.20}$$

where $\tilde{V}_{p_m}(\cdot)$ is given in (4.14).

Proof. Part (i) is a direct consequence of Propositions 4.4 and 4.5.

To prove part (ii), we consider $p_m > 0$ as in part (i) and the optimal strategy $(y, u^*_{p_m}, Y^*_{p_m})$ for the discounted control problem in Theorem 3.4 with $p = p_m$. Notice that $\Gamma(p_m, y) = \mathbb{E}_y[\int_0^\infty e^{-\alpha t}\,\mathrm{d}L^*_{p_m}(t)] = m$ and thus the strategy $(y, u^*_{p_m}, Y^*_{p_m})$ belongs to $\tilde{A}(y, m)$ described in (4.5). Note that p_m may depend on y as well. Now let $(y, u, Y) \in \tilde{A}(y, m)$ be any admissible strategy. Thus, $Y(t) = y + \sigma W(t) + \mu t + \int_0^t u(s)\,\mathrm{d}s + L_Y(t)$, and $\mathbb{E}\left[\int_0^\infty e^{-\alpha t}\,\mathrm{d}L_Y(t)\right] \leq m$. Certainly, (y, u, Y) is also an admissible strategy in $\tilde{A}(y)$ of (2.14) for the discounted control problem with $p = p_m$. Hence we have

$$\mathbb{E}\left[\int_0^\infty e^{-\alpha t}[c(u(t))\,\mathrm{d}t + p_m\,\mathrm{d}L_Y(t)]\right] \geq \mathbb{E}\left[\int_0^\infty e^{-\alpha t} c(u^*_{p_m}(Y^*_{p_m}(t)))\,\mathrm{d}t + p_m m\right],$$

since $\Gamma(p_m, y) = m$. Consequently, for any $(y, u, Y) \in \tilde{A}(y, m)$, it follows that

$$\mathbb{E}\left[\int_0^\infty e^{-\alpha t} c(u(t))\,\mathrm{d}t\right] \geq \mathbb{E}\left[\int_0^\infty e^{-\alpha t} c(u^*_{p_m}(Y^*_{p_m}(t)))\,\mathrm{d}t\right] + p_m\left(m - \mathbb{E}\left[\int_0^\infty e^{-\alpha t}\,\mathrm{d}L_Y(t)\right]\right)$$

$$\geq \mathbb{E}\left[\int_0^\infty e^{-\alpha t} c(u^*_{p_m}(Y^*_{p_m}(t)))\,\mathrm{d}t\right].$$

Thus, $(y, u^*_{p_m}, Y^*_{p_m})$ is a feed-back type optimal strategy for the constrained minimization problem in (4.3)–(4.4). This completes the proof for Part (ii).

From part (ii), the value function $F_m(y)$ satisfies $F_m(y) = \mathbb{E}_y\left[\int_0^\infty e^{-\alpha t} c(u^*_{p_m}(Y^*_{p_m}(t)))\,\mathrm{d}t\right]$ for each $y \geq 0$. Moreover, since $\Gamma(p_m, y) = m$, the above expression for F_m demonstrates the relationship between the value functions $\tilde{V}_{p_m}$ and F_m: $\tilde{V}_{p_m}(y) = F_m(y) + m$, for all $y \geq 0$. This completes the proof of part (iii). □

5. Ergodic control problem

5.1. Problem formulation

For our model described in (2.1)–(2.4), here we consider a long-term average cost (ergodic cost) minimization problem associated with the cost functional

$$I(x, u, X) = \liminf_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T c(u(t))\,\mathrm{d}t + p\,\mathrm{d}M(t)\right]. \tag{5.1}$$

We take the infimum of these cost functionals over all admissible control systems $\mathcal{A}(x)$ described in (2.8) to obtain the value function $U(x)$. Hence

$$U(x) = \inf_{\mathcal{A}(x)} I(x, u, X). \tag{5.2}$$

Since $c(0) = 0$, if the zero control is used, (5.1) becomes $\liminf_{T\to\infty} \frac{p}{T}\mathbb{E}[M(T)]$ and this quantity is finite. Hence $U(x)$ is also finite. Our aim here is to use the results in Section 3 to obtain an optimal policy for (5.2). Our main result related to (5.1) and (5.2) is given in the following theorem.

Theorem 5.1. *Consider the ergodic control problem described in (5.1)–(5.2). Then there is a constant optimal control described by*

$$u^* = \max\{0, \min\{\theta, \Psi(p)\}\}. \tag{5.3}$$

The corresponding value function in (5.2) is independent of x and is given by

$$U = \begin{cases} 0 & \text{if } \theta \leq 0, \\ c(\theta) & \text{if } 0 < \theta < \Psi(p), \\ \theta p - \Phi(p) & \text{if } \theta \geq \Psi(p). \end{cases} \tag{5.4}$$

Here the constant θ is as in the state equation (2.2) and the functions Φ and Ψ are given in (3.1) and (3.2), respectively.

Proof. This theorem is a direct consequence of the relations (5.7)–(5.8), Lemma 5.4, and Theorems 5.5, 5.9 and 5.12. □

Remark 5.2. It is interesting to observe that the constant optimal control u^* and the value U are continuous functions of θ.

Following our approach in Section 3, we consider the reflected process Y in (2.12) with the local-time process L_Y and introduce the associated ergodic cost functional $\tilde{I}(y, u, Y)$ described below:

$$\tilde{I}(y, u, Y) = \liminf_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T c(u(t))\,\mathrm{d}t + p\,\mathrm{d}L_Y(t)\right]. \tag{5.5}$$

The corresponding value function is defined by

$$\tilde{U}(y) = \inf_{\tilde{\mathcal{A}}(y)} \tilde{I}(y, u, Y), \tag{5.6}$$

where the collection $\tilde{\mathcal{A}}(y)$ of admissible system is given in (2.14). The value function $\tilde{U}(y)$ is also finite. For example, if the zero control (i.e. $u(t) \equiv 0$) is used, then $\lim_{T\to\infty} \frac{1}{T}\mathbb{E}[L_Y(T)]$ is finite. This limit is zero when $\mu \ge 0$ and is equal to $|\mu|$ when $\mu < 0$; see, for example, [23] and also the proofs of Theorem 5.5 and Lemma 5.6. Following the arguments in Section 3, we see that the cost functionals $I(x, u, X)$ and $\tilde{I}(0, u, Y)$ are the same when the drift μ in the reflected process is related to θ by $\mu = -\theta$. Hence

$$I(x, u, X) = \tilde{I}(0, u, Y) \tag{5.7}$$

holds. Using this identity and following the arguments similar to Section 3, we obtain

$$U(x) = \tilde{U}(0) \quad \text{for all } x, \tag{5.8}$$

when $\mu = -\theta$ and hence $U(\cdot)$ is a constant function.

Remark 5.3. In our definitions of the cost functionals I and $\tilde{I}$ in (5.1) and (5.5), respectively, we have used lim inf. But the same results will hold when lim sup is used and the proofs will remain the same.

The following lemma shows that $\tilde{U}(\cdot)$ in (5.6) is also a constant.

Lemma 5.4. $\tilde{U}(y) = \tilde{U}(0)$ *for each* $y \ge 0$.

Proof. Consider the admissible system (y, u, Y) satisfying (2.12). Using the same control process $u = \{u(s) : s \ge 0\}$, we construct $(0, u, \tilde{Y})$ satisfying (2.12) with $\tilde{Y}(0) = 0$ and the corresponding local-time process $\tilde{L}(\cdot)$. Thus $\tilde{Y}$ satisfies $\tilde{Y}(t) = \sigma W(t) + \mu t + \int_0^t u(s)\,\mathrm{d}s + \tilde{L}(t)$, for $t \ge 0$ and $\tilde{L}(\cdot)$ increases only when $\tilde{Y}(t) = 0$. Hence we have $Y(t) - \tilde{Y}(t) = y + L_Y(t) - \tilde{L}(t)$ and it is a process with continuous and bounded variation paths. We consider

$$\mathrm{d}(Y(t) - \tilde{Y}(t))^2 = 2(Y(t) - \tilde{Y}(t))\,\mathrm{d}L_Y(t) + 2(\tilde{Y}(t) - Y(t))\,\mathrm{d}\tilde{L}(t).$$

Notice that if $\mathrm{d}L_Y(t) > 0$ for some $t \ge 0$, then $Y(t) = 0$. But $\tilde{Y}(t) \ge 0$, then we have $2(Y(t) - \tilde{Y}(t))\,\mathrm{d}L_Y(t) \le 0$. Similarly $2(\tilde{Y}(t) - Y(t))\,\mathrm{d}\tilde{L}(t) \le 0$. Consequently, $(Y(t) - \tilde{Y}(t))^2$ has decreasing paths and we obtain $0 \le (Y(t) - \tilde{Y}(t))^2 \le y^2$ and therefore $(L_Y(t) - \tilde{L}(t))^2 \le 2y^2$ for all $t \ge 0$. This shows that $\lim_{T\to\infty} \frac{1}{T}\mathbb{E}[|L_Y(T) - \tilde{L}(T)|] = 0$. Since the same control process u is used for both processes, we see that $\tilde{I}(y, u, Y) = \tilde{I}(0, u, \tilde{Y})$. Consequently $\tilde{U}(y) \ge \tilde{U}(0)$. By repeating the same argument, we also obtain $\tilde{U}(0) \ge \tilde{U}(y)$. Hence $\tilde{U}(y) = \tilde{U}(0)$, and this completes the proof. □

Since $U(y)$ and $\tilde{U}(y)$ are constant functions, henceforth we simply write U and $\tilde{U}$ respectively to represent their values.

When the drift coefficient $\mu \geq 0$ in (2.12) (or correspondingly $\theta \leq 0$ in (2.1)), it is quite straightforward to show that the zero control is optimal. In this case, the value functions U and $\tilde{U}$ defined above are identically zero. These results are in the next theorem.

5.2. The case of non-negative drift

Theorem 5.5. *Let $\mu \geq 0$. Then for the ergodic control problem in (5.6), zero control is optimal. Moreover, the value function $\tilde{U}$ is zero.*

Proof. Consider the state process Y^* corresponding to zero control

$$Y^*(t) = y + \mu t + \sigma W(t) + L_{Y^*}(t), \quad t \geq 0,$$

where $L_{Y^*}(\cdot)$ is the local time process at the origin. Since zero control $u^*(t) \equiv 0$ is used here, the running cost $c(u^*(t)) = 0$, and $\tilde{I}(y, 0, Y^*)$ reduces to $\tilde{I}(y, 0, Y^*) = p \liminf_{T\to\infty} \frac{1}{T}\mathbb{E}[L_{Y^*}(T)]$. When $\mu \geq 0$, $\lim_{T\to\infty} \frac{1}{T}\mathbb{E}[L_{Y^*}(T)] = 0$ holds, since Y^* visits the origin quite infrequently.

In fact, when $\mu = 0$, from Lemma 3.6.14 of [23], $L_{Y^*}(\cdot)$ can be represented by $L_{Y^*}(t) = \max\{0, -\min_{0\leq s\leq t}(y + \sigma W(s))\}, t \geq 0$. Hence $0 \leq L_{Y^*}(t) \leq |y| + |\sigma| \max_{s\in[0,t]} |W(s)|$. Next, using Burkholder's inequality together with Jensen's inequality, we obtain $\mathbb{E}[\max_{s\in[0,t]} |W(s)|] \leq C\sqrt{t}$ and this yields $\lim_{T\to\infty} \frac{1}{T}\mathbb{E}[L_{Y^*}(T)] = 0$.

When $\mu > 0$, we let $\rho = \frac{\mu}{2\sigma} > 0$ and consider the bounded function $F(x) = \frac{1}{2\rho}(1 - e^{-2\rho x})$. Then we have $F''(x) + 2\rho F'(x) = 0$ for $x \geq 0$ and $F'(0) = 1$. We can apply Itô's formula to obtain $\mathbb{E}[F(Y^*(T))] = F(y) + \mathbb{E}[L_{Y^*}(T)]$. Since $0 < F(x) < \frac{1}{\rho}$, it is evident that $\lim_{T\to\infty} \frac{1}{T}\mathbb{E}[L_{Y^*}(T)] = 0$.

Consequently $\tilde{I}(y, 0, Y^*) = 0$. Therefore, zero control is optimal and $\tilde{U}$ is equal to zero. This completes the proof. □

5.3. Negative drift

Next we consider the case $\mu < 0$. Henceforth we write $\mu = -\theta$ where θ is a positive constant. This notation is also in agreement with the relationship of the processes X and Y described in (2.2) and (2.12), respectively. In this case, the value $\tilde{U} > 0$ and the derivation of our optimal control policies are non-trivial. We heavily rely on the results in Section 3 to obtain the value $\tilde{U}$, using the *vanishing discount factor technique* (see, for example, Section 2.7 of [1] or [27]). To represent the discounted factor $\alpha > 0$ in our computations, we write $Q_\alpha(\cdot)$ to denote the smooth bounded solution to the HJB equation (3.4) with the discounted factor $\alpha > 0$. We have obtained the existence of such a solution in Proposition 3.2. In Theorem 3.4, we showed that it coincides with the value function $\tilde{V}$ given in (2.15).

In the derivation of our solution to the ergodic control problem, we will show that

$$\lim_{\alpha\to 0+} \alpha Q_\alpha(0) = \tilde{U}, \tag{5.9}$$

where $\tilde{U}$ is given in (5.6). First, we obtain a preliminary result associated with a constant control u_0 where $0 \leq u_0 \leq \theta$. We let $m = \theta - u_0 \geq 0$. Then the corresponding state process can be written as

$$Y(t) = y + \sigma W(t) - mt + L_Y(t), \quad \text{for all } t \geq 0. \tag{5.10}$$

Thus, Y is a reflecting diffusion and $L_Y(\cdot)$ represents its local time process at the origin.

Lemma 5.6. *Let Y be the state process given in (5.10) which corresponds to constant control $u_0 \geq 0$. Then we have*

$$\lim_{T\to\infty} \frac{1}{T}\mathbb{E}[L_Y(T)] = \theta - u_0, \tag{5.11}$$

and the corresponding cost functional

$$\tilde{I}(y, u_0, Y) = c(u_0) + p(\theta - u_0). \tag{5.12}$$

Consequently,

$$\tilde{U} \leq \theta p - \sup_{0\leq a\leq\theta} \{pa - c(a)\} \tag{5.13}$$

holds.

Proof. The pair (Y, L_Y) represents the Skorokhod decomposition of the process $y+\sigma W(t)-mt$, for $t \geq 0$. Since $m \geq 0$, using the comparison theorem associated with the Skorokhod map (Propositions 3.4 and 3.5 of [8], and also Section 2.3 of [36]), it follows that $0 \leq Y(t) \leq Z(t)$, where $Z(t) = y + \sigma W(t) + L_Z(t)$ for all $t \geq 0$. Thus, Z is a reflected Brownian motion and following an argument similar to the proof of Theorem 5.5 in the case $\mu = 0$, we obtain $\lim_{T\to\infty} \frac{1}{T}\mathbb{E}[L_Z(T)] = 0$. Consequently, $\lim_{T\to\infty} \frac{1}{T}\mathbb{E}[Z(T)] = 0$ and this implies that $\lim_{T\to\infty} \frac{1}{T}\mathbb{E}[Y(T)] = 0$. Hence, using (5.10), we conclude $\lim_{T\to\infty} \frac{1}{T}\mathbb{E}[L_Y(T)] = m$ and thus (5.11) holds.

Since the constant control $u(t) \equiv u_0$ for all $t \geq 0$ is used in (5.10), it immediately follows that $\int_0^T c(u(t))\,\mathrm{d}t = c(u_0)T$ and hence (5.12) holds. Then we have $\tilde{U} \leq \tilde{I}(y, u_0, Y) = c(u_0) + p(\theta - u_0)$ for each $0 \leq u_0 \leq \theta$. Thus, we can take the infimum over $u_0 \in [0, \theta]$ in the right hand side of the above equation to obtain (5.13). □

Remark 5.7. From the definitions of Φ and Ψ in (3.1) and (3.2) respectively, we notice that, if $\Psi(p) \leq \theta$, then (5.13) yields $\tilde{U} \leq \theta p - \Phi(p)$. In fact, the function $a \mapsto pa - c(a)$ achieves its maximum value at $\Psi(p)$ (see Lemma 3.1) or equivalently $\Phi(p) = p\Psi(p) - c(\Psi(p))$. Thus, if $\Psi(p) \leq \theta$, then $\Phi(p) = \sup_{0\leq a\leq\theta}\{pa - c(a)\}$ holds.

In the next lemma, again we use the process Y in (5.10) corresponding to the zero control process $u(t) \equiv 0$ to show that the set $\{\alpha Q_\alpha(0) : \alpha > 0\}$ is bounded.

Lemma 5.8. *Let Q_α be the smooth bounded solution to the HJB equation (3.4) with the discount factor $\alpha > 0$. Then*

$$\limsup_{\alpha\to 0+} \alpha Q_\alpha(0) = \limsup_{\alpha\to 0+} \sup_{y\geq 0} \alpha Q_\alpha(y) \leq \theta p. \tag{5.14}$$

Proof. By Proposition 3.2, $Q_\alpha(\cdot)$ is strictly decreasing. Hence $\alpha Q_\alpha(0) = \sup_{y\geq 0} \alpha Q_\alpha(y)$ and it is enough to show that $\limsup_{\alpha\to 0+} \alpha Q_\alpha(0) \leq \theta p$ holds. We consider the process Y in (5.10) with $Y(0) = 0$ and $u_0 = 0$. Introduce the function $F(x) = e^{-\lambda x}$ for $x \geq 0$, where

$$\lambda = \frac{1}{\sigma^2}\left[\sqrt{\theta^2 + 2\alpha\sigma^2} - \theta\right] = \frac{2\alpha}{\sqrt{\theta^2 + 2\alpha\sigma^2} + \theta}. \tag{5.15}$$

Thus $F(0) = 1$, $F'(0) = -\lambda$ and F is a bounded solution to $\frac{\sigma^2}{2}F'' - \theta F' - \alpha F = 0$ on $[0, \infty)$. We apply Itô's formula to $e^{-\alpha t}F(Y(t))$ and obtain $\mathbb{E}[e^{-\alpha T}F(Y(T))] = 1 - \lambda\mathbb{E}[\int_0^T e^{-\alpha t}\,\mathrm{d}L_Y(t)]$.

Since F is bounded, by letting T tend to infinity, we have $\lambda\mathbb{E}\left[\int_0^\infty e^{-\alpha t}\,\mathrm{d}L_Y(t)\right] = 1$. Using (5.15), we obtain $\alpha\mathbb{E}\left[\int_0^\infty e^{-\alpha t}\,\mathrm{d}L_Y(t)\right] = \frac{1}{2}\left[\sqrt{\theta^2+2\alpha\sigma^2}+\theta\right]$. Since the zero control is used in (5.10) with $Y(0)=0$, the total discounted cost is given by $\frac{p}{2\alpha}[\sqrt{\theta^2+2\alpha\sigma^2}+\theta]$, and hence $Q_\alpha(0) \le \frac{p}{2\alpha}[\sqrt{\theta^2+2\alpha\sigma^2}+\theta]$ holds.

Consequently, $\limsup_{\alpha\to 0+}\alpha Q_\alpha(0) \le \theta p$ and thus (5.14) holds. This completes the proof. □

Theorem 5.9. *Let $\theta \ge \Psi(p)$ where Ψ is defined in (3.2). Then the constant control $u^*(t) \equiv \Psi(p)$ for all $t \ge 0$ is optimal. The corresponding optimal state process Y^* is given by*

$$Y^*(t) = y + \sigma W(t) - (\theta - \Psi(p))t + L^*(t), \quad t \ge 0. \tag{5.16}$$

Here L^* is the local-time process of Y^* at the origin. Moreover, the constant value function $\tilde{U}$ is given by

$$\tilde{U} = \theta p - \Phi(p), \tag{5.17}$$

where Φ is given in (3.1).

Proof. Under the condition $\theta \ge \Psi(p)$, thanks to Remark 5.7, we have

$$\tilde{U} \le \theta p - \Phi(p). \tag{5.18}$$

Next we use (5.14) in Lemma 5.8 and consider any limit point l_0 of the sequence $\{\alpha Q_\alpha(0) : \alpha > 0\}$. Since Q_α satisfies (3.5), by evaluating it at the origin, we obtain

$$\frac{\sigma^2}{2}Q_\alpha''(0+) = \alpha Q_\alpha(0+) + \Phi(-Q_\alpha'(0+)) + \theta Q_\alpha'(0+) = \alpha Q_\alpha(0+) + \Phi(p) - \theta p.$$

Since $Q_\alpha(\cdot)$ is convex, we have $Q_\alpha''(0+) \ge 0$. Thus using the limit point l_0 of $\{\alpha Q_\alpha(0) : \alpha > 0\}$, we obtain

$$l_0 \ge \theta p - \Phi(p). \tag{5.19}$$

Combining (5.18) and (5.19), we can write

$$l_0 \ge \theta p - \Phi(p) \ge \tilde{U}. \tag{5.20}$$

Next, we intend to show that $\tilde{U} \ge l_0$.

First notice that by Proposition 3.2, $-p \le Q_\alpha'(y) < 0$ holds for all $y \ge 0$. Hence $|Q_\alpha(y) - Q_\alpha(0)| \le py$ for each $y \ge 0$. Since l_0 is a limit point of $\{\alpha Q_\alpha(0) : \alpha > 0\}$, it follows that l_0 is also a limit point of $\{\alpha Q_\alpha(y) : \alpha > 0\}$. Moreover, we can find a single sequence $\{\alpha_n\}$ which is decreasing to 0 so that $\lim_{n\to\infty}\alpha_n Q_{\alpha_n}(y) = l_0$ for every $y \ge 0$.

Next we consider an arbitrary admissible state process Y starting from the origin: $Y(t) = \sigma W(t) - \theta t + \int_0^t u(s)\,\mathrm{d}s + L_Y(t)$, $t \ge 0$. Using Itô's formula for $Q_\alpha(Y(t))$, we obtain

$$\begin{aligned}&\mathbb{E}[Q_\alpha(Y(t))]\\ &\quad = Q_\alpha(0) + \mathbb{E}\left[\int_0^t\left(\frac{\sigma^2}{2}Q_\alpha''(Y(s)) - \theta Q_\alpha'(Y(s)) + u(s)Q_\alpha'(Y(s))\right)\mathrm{d}s - pL_Y(t)\right].\end{aligned}$$

But Q_α satisfies (3.4) with $\mu = -\theta$, and therefore

$$\mathbb{E}[Q_\alpha(Y(t)) - Q_\alpha(0)] \ge -\mathbb{E}\left[\int_0^t c(u(s))\,\mathrm{d}s + pL_Y(t)\right] + \mathbb{E}\left[\int_0^t \alpha Q_\alpha(Y(r))\,\mathrm{d}r\right].$$

By Proposition 3.2, Q_α is decreasing and hence $Q_\alpha(Y(t)) \le Q_\alpha(0)$. Thus

$$\mathbb{E}\left[\int_0^t c(u(s))\,\mathrm{d}s + pL_Y(t)\right] \ge \mathbb{E}\left[\int_0^t \alpha Q_\alpha(Y(r))\,\mathrm{d}r\right].$$

Since Q_α is bounded and $\lim_{n\to\infty} \alpha_n Q_{\alpha_n}(y) = l_0$ for every $y \ge 0$, by letting $n \to \infty$ and using the bounded convergence theorem, we obtain $\lim_{n\to\infty} \mathbb{E}[\int_0^t \alpha_n Q_{\alpha_n}(Y(r))\,\mathrm{d}r] = l_0 t$. Consequently, we have $\frac{1}{T}\mathbb{E}[\int_0^T c(u(s))\,\mathrm{d}s + pL_Y(T)] \ge l_0$. This leads to $\tilde{I}(0, u, Y) \ge l_0$, where $\tilde{I}$ is given in (5.5). Now taking the infimum over all admissible systems $(0, u, Y)$, we obtain

$$\tilde{U} \ge l_0 \tag{5.21}$$

as desired. Now a combination of (5.20) and (5.21) gives

$$\tilde{U} = l_0 = \theta p - \Phi(p), \tag{5.22}$$

which characterizes the value $\tilde{U}$ in terms of θ and p. Next, we can choose the constant control $u^*(t) \equiv \Psi(p)$. Then by (5.12) of Lemma 5.6, we have

$$\tilde{I}(y, \Psi(p), Y) = c(\Psi(p)) + p(\theta - \Psi(p)) = \theta p - \Phi(p), \tag{5.23}$$

where the second equality above follows from (3.3). In view of (5.22) and (5.23), it follows that the constant control $u^*(t) \equiv \Psi(p)$ is optimal and Y^* described in (5.16) yields the optimal state process. This completes the proof. □

Corollary 5.10. *When $\theta \ge \Psi(p)$,*

$$\lim_{\alpha\to 0+} \alpha Q_\alpha(y) = \theta p - \Phi(p) = \tilde{U}$$

holds. This convergence is uniform over compact sets.

Proof. The proof of Theorem 5.9 identifies that every limit point l_0 of the bounded set $\{\alpha Q_\alpha(0) : \alpha > 0\}$ is equal to $\tilde{U} = \theta p - \Phi(p)$. Moreover, we observed in the proof of Theorem 5.9 that $|Q_\alpha(y) - Q_\alpha(0)| \le py$ for all $y \ge 0$. Hence the desired limit holds and its convergence is uniform over compact sets. □

Next we consider the situation $0 < \theta < \Psi(p)$. Recall that Ψ defined in (3.2) is non-decreasing since $c(\cdot)$ is a strictly convex function. Hence, there is a unique value $r_\theta > 0$ so that

$$\Psi(r_\theta) = \theta \quad \text{and} \quad 0 < r_\theta < p. \tag{5.24}$$

In fact, the concave function $\theta x - \Phi(x)$ achieves its unique maximum at $x = r_\theta$. We begin with an existence result for a solution to an ordinary differential equation, whose proof is relegated to Appendix D.

Lemma 5.11. *Let $0 < \theta < \Psi(p)$ and r_θ be defined as in (5.24). Consider the first-order differential equation*

$$\begin{cases} \dfrac{\sigma^2}{2} W_\theta'(x) - \theta W_\theta(x) - \Phi(-W_\theta(x)) = c(\theta), \\ W_\theta(0) = -p. \end{cases} \tag{5.25}$$

Then there is a twice differentiable, strictly increasing solution W_θ defined on $[0, \infty)$. Moreover, $W_\theta(x) < 0$, and $\lim_{x\to\infty} W_\theta(x) = -r_\theta$.

We use Lemma 5.11 in the next theorem to obtain a constant optimal control for the case $0 < \theta < \Psi(p)$.

Theorem 5.12. *Let $0 < \theta < \Psi(p)$. Then the constant control policy $u^*(t) \equiv \theta$ is optimal, and the corresponding optimal state process Y^* is a reflected Brownian motion and is given by*

$$Y^*(t) = y + \sigma W(t) + L^*(t), \quad t \geq 0. \tag{5.26}$$

Here $L^*(\cdot)$ is the local-time process of $Y^*(\cdot)$. Moreover, the value $\tilde{U}$ of (5.6) is given by

$$\tilde{U} = c(\theta). \tag{5.27}$$

Proof. Introduce the function $F(x) = \int_0^x W_\theta(u)\,\mathrm{d}u$ for $x \geq 0$, where W_θ is given in Lemma 5.11. Then $F(x) \leq 0$ for all $x \geq 0$, $F(0) = 0$ and $F'(0) = -p$. Now consider any admissible process

$$Y(t) = y + \sigma W(t) - \theta t + \int_0^t u(s)\,\mathrm{d}s + L_Y(t), \quad t \geq 0. \tag{5.28}$$

The process L_Y represents the local-time of Y at the origin. We apply Itô's formula to $F(Y(t))$ and using $F'(x) = W_\theta(x)$, we obtain

$$\begin{aligned}&\mathbb{E}[F(Y(T))]\\ &\quad = F(y) + \mathbb{E}\left[\int_0^T \left(\frac{\sigma^2}{2} W_\theta'(Y(s)) - \theta W_\theta(Y(s)) + u(s) W_\theta(Y(s))\right)\mathrm{d}s - pL_Y(T)\right].\end{aligned} \tag{5.29}$$

Notice that

$$\begin{aligned}\frac{\sigma^2}{2} W_\theta'(y) - \theta W_\theta(y) + u W_\theta(y) &= \frac{\sigma^2}{2} W_\theta'(y) - \theta W_\theta(y) - [(-W_\theta(y))u - c(u)] - c(u)\\ &\geq \frac{\sigma^2}{2} W_\theta'(y) - \theta W_\theta(y) - \Phi(-W_\theta(y)) - c(u)\\ &= c(\theta) - c(u).\end{aligned} \tag{5.30}$$

In the above derivation, we have used (3.1) and (5.25). Next we use (5.30) in (5.29) to obtain $\mathbb{E}[F(Y(T))] \geq F(y) + c(\theta)T - \mathbb{E}[\int_0^T c(u(s))\,\mathrm{d}s + pL_Y(T)]$. Since $F(Y(T)) \leq 0$, this yields $\tilde{I}(y, u, Y) = \liminf_{T\to\infty} \frac{1}{T}\mathbb{E}[\int_0^T c(u(s))\,\mathrm{d}s + pL_Y(T)] \geq c(\theta)$. Hence $\tilde{U} \geq c(\theta)$ follows.

On the other hand, using the constant control $u^*(t) \equiv \theta$, we obtain the state process Y^* in (5.26). By (5.12) in Lemma 5.6, we have $\tilde{I}(y, u^*, Y^*) = c(\theta)$. Consequently, the constant control $u^*(t) \equiv \theta$ is optimal and $\tilde{U} = c(\theta)$ follows. This completes the proof. □

6. Regime-switching diffusion

6.1. Problem formulation

In this section, we address an infinite-horizon discounted cost minimization problem when the netflow of the inventory model exhibits the regime-switching behavior. To introduce the regime-switching state process, we let $\Lambda = \{\Lambda(t); t \geq 0\}$ be a continuous-time Markov chain with a discrete state space $\mathcal{M} = \{1, \ldots, m\}$ $(1 \leq m < \infty)$ and generator $Q = (q_{ij}) \in \mathbb{R}^{m\times m}$. That is,

$$\mathbb{P}\{\Lambda(t+\Delta t) = j|\Lambda(t) = i, \Lambda(s), s \leq t\} = \begin{cases} q_{ij}\Delta t + o(\Delta t), & \text{if } j \neq i,\\ 1 - \lambda_i \Delta t + o(\Delta t), & \text{if } j = i,\end{cases} \tag{6.1}$$

where for all $i, j = 1, \ldots, m$, $q_{ij} \geq 0$ with $j \neq i$ and $0 \leq \lambda_i := -q_{ii} = \sum_{j \neq i} q_{ij} < \infty$. Next we consider the state process X driven by the following controlled drifted Brownian motion subject to regime-switching, given by

$$X(t) = x - \int_0^t b(\Lambda(s))\,\mathrm{d}s - \int_0^t u(s)\,\mathrm{d}s - \int_0^t \sigma(\Lambda(s))\,\mathrm{d}W(s), \tag{6.2}$$

where $b(i)$ and $\sigma(i)$ are real constants for each $i \in \mathcal{M}$. The state process X represents the controlled netflow in the inventory model. Similar to the description in Section 2, the control process $u = \{u(t) : t \geq 0\}$ is nonnegative, non-anticipative which is adapted to a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$. Moreover, the integrability condition (2.3) is satisfied. As before, let $M(t) := \sup_{0 \leq s \leq t} X(s)$ denote the running maximum process. Next, as in Section 2, we introduce the infinite-horizon cost functional

$$J(x, i, u, X, \Lambda) := \mathbb{E}_{x,i}\left[\int_0^\infty e^{-\alpha t}[c(u(t))\,\mathrm{d}t + p\,\mathrm{d}M(t)]\right], \tag{6.3}$$

where $\alpha > 0$ denotes the discount factor and p is a positive constant. The control cost rate function c is assumed to satisfy Assumption 2.1. Here and below, $\mathbb{E}_{x,i}$ denotes the expectation with respect to the probability law $\mathbb{P}_{x,i}$ under which $(X(0), \Lambda(0)) = (x, i)$.

For a given $(x, i) \in \mathbb{R} \times \mathcal{M}$, we call the quintuple $((\Omega, \mathcal{F}, \{\mathcal{F}\}_{t \geq 0}, \mathbb{P}), x, i, u, W, X, \Lambda)$ an *admissible control system* if

(i) (x, i, u, W, X, Λ) is a weak solution to (6.2) with $X(0) = x$, $\Lambda(0) = i$; and
(ii) the explosion time of X is infinite with probability 1.

When there is no ambiguity, we simply use the quintuple (x, i, u, X, Λ) to represent an admissible control system. Next we introduce the set

$$\mathcal{A}(x, i) := \{(x, i, u, X, \Lambda) : (x, i, u, X, \Lambda) \text{ is admissible}\}. \tag{6.4}$$

If we let $u \equiv 0$ in (6.2), then using the Burkholder–Davis–Gundy inequality, we obtain

$$\begin{aligned}\mathbb{E}[M(t)] &\leq |x| + K_1 t + \mathbb{E}\left[\sup_{0 \leq s \leq t}\left|\int_0^s \sigma(\Lambda(r))\,\mathrm{d}W(r)\right|\right] \\ &\leq |x| + K_1 t + \mathbb{E}\left[\left(\int_0^t \sigma^2(\Lambda(r))\,\mathrm{d}r\right)^{1/2}\right] \\ &\leq |x| + K_1 t + K_2\sqrt{t},\end{aligned}$$

where $K_1 = \max\{|b(i)|, i \in \mathcal{M}\}$ and $K_2 := \max\{|\sigma(i)|, i \in \mathcal{M}\}$. In particular, this estimate shows that for each $x \in \mathbb{R}$ and $i \in \mathcal{M}$, under the zero control, (6.2) has a pathwise unique weak solution with $X(0) = x$, $\Lambda(0) = i$ and infinite explosion time almost surely. Moreover, the zero control has a finite cost and hence set $\mathcal{A}(x, i)$ of (6.4) is nonempty for every $(x, i) \in \mathbb{R} \times \mathcal{M}$. The value function is thus defined by

$$V(x, i) := \inf\{J(x, i, u, X, \Lambda) : u \in \mathcal{A}(x, i)\}, \tag{6.5}$$

and it is finite.

Next we observe that the value function V defined in (6.5) is constant with respect to the x variable. Indeed, let $\tilde{x} \neq x$ and consider an admissible control system $(x, i, u, X, \Lambda) \in \mathcal{A}(x, i)$.

Then we have $(\tilde{x}, i, u, \tilde{X}, \Lambda) \in \mathcal{A}(\tilde{x}, i)$, where $\tilde{X}(t) := X(t) + \tilde{x} - x$ for $t \geq 0$. Moreover, since

$$\tilde{M}(t) := \sup_{0 \leq s \leq t} \tilde{X}(s) = \sup_{0 \leq s \leq t} \{X(s) + \tilde{x} - x\} = M(t) + \tilde{x} - x,$$

we have $\mathrm{d}\tilde{M}(t) = \mathrm{d}M(t)$. Therefore, it follows that $J(x, i, u, X, \Lambda) = J(\tilde{x}, i, u, \tilde{X}, \Lambda)$ and consequently $V(x, i) = V(\tilde{x}, i)$. Hence, the value function $V(x, i)$ of (6.5) is independent of x. Henceforth we write $V(i)$ to represent $V(x, i)$.

Next we recall the dynamic programming principle [13,25]:

(i) For any $u \in \mathcal{A}(x, i)$ and any $\mathcal{F}_t$-stopping time ζ,

$$V(i) \leq \mathbb{E}_{x,i}\left[\int_0^{\zeta} e^{-\alpha t}[c(u(t))\,\mathrm{d}t + p\,\mathrm{d}M(t)] + e^{-\alpha\zeta} V(\Lambda(\zeta))\right], \tag{6.6}$$

where X is the controlled process under u and M is the running maximum process corresponding to X.

(ii) For any $\varepsilon > 0$, there exists a $u^{\varepsilon} \in \mathcal{A}(x, i)$ such that for all $\mathcal{F}_t$-stopping time ζ,

$$V(i) + \varepsilon \geq \mathbb{E}\left[\int_0^{\zeta} e^{-\alpha t}\left[c(u^{\varepsilon}(t))\,\mathrm{d}t + p\,\mathrm{d}M^{\varepsilon}(t)\right] + e^{-\alpha\zeta} V(\Lambda(\zeta))\right], \tag{6.7}$$

where X^{ε} is the controlled process under u^{ε} and M^{ε} is the running maximum process corresponding to X^{ε}.

Assume $(X(0), \Lambda(0)) = (x, i)$ and denote by $S_i := \inf\{t \geq 0 : \Lambda(t) \neq \Lambda(t-)\}$ the first transition time of the Markov chain Λ. Suppose first $\lambda_i > 0$. Then S_i is exponentially distributed with parameter $\lambda_i > 0$. In view of the dynamic programming principle (6.7), for each $\varepsilon > 0$, there exists a $u^{\varepsilon} \in \mathcal{A}(x, i)$ such that

$$V(i) + \varepsilon > \mathbb{E}_{x,i}\left[\int_0^{S_i} e^{-\alpha t}\left[c(u^{\varepsilon}(t))\,\mathrm{d}t + p\,\mathrm{d}M^{\varepsilon}(t)\right] + e^{-\alpha S_i} V(\Lambda(S_i))\right], \tag{6.8}$$

where X^{ε} is the controlled process under u^{ε} and M^{ε} is the associated running maximum process of X^{ε}. We analyze the two terms of (6.8) separately. Let τ be an independent exponential random variable with parameter $\alpha > 0$. Then $\tau \wedge S_i$ is exponentially distributed with parameter $\alpha + \lambda_i$. Consequently we have

$$\begin{aligned}
&\mathbb{E}_{x,i}\left[\int_0^{S_i} e^{-\alpha t}[c(u^{\varepsilon}(t))\,\mathrm{d}t + p\,\mathrm{d}M^{\varepsilon}(t)]\right] \\
&= \mathbb{E}_{x,i}\left[\int_0^{\infty} e^{-(\alpha+\lambda_i)t}[c(u^{\varepsilon}(t))\,\mathrm{d}t + p\,\mathrm{d}M^{\varepsilon}(t)]\right] \\
&= \mathbb{E}_{x,i}\left[\int_0^{\tau\wedge S_i} [c(u^{\varepsilon}(t))\,\mathrm{d}t + p\,\mathrm{d}M^{\varepsilon}(t)]\right].
\end{aligned}$$

Using Theorem 3.4 in Section 3, we obtain

$$\begin{aligned}
&\mathbb{E}_{x,i}\left[\int_0^{\tau\wedge S_i} [c(u^{\varepsilon}(t))\,\mathrm{d}t + p\,\mathrm{d}M^{\varepsilon}(t)]\right] \geq Q_{\alpha+\lambda_i}(0) \\
&= \frac{1}{\alpha + \lambda_i}\left[\frac{\sigma(i)^2}{2} r_i^* - b(i)p - \Phi(p)\right],
\end{aligned} \tag{6.9}$$

for any $x \in \mathbb{R}$, where $r_i^* > 0$ is a constant as in Theorem A.7, and $Q_{\alpha+\lambda_i}$ is the solution to (3.4) constructed in Proposition 3.2 corresponding to the parameters $\mu = b(i), \sigma = \sigma(i)$, and α is replaced by $\alpha + \lambda_i$.

On the other hand, using the strong Markov property for the Markov chain (see, for example, [7, Theorem 8.4.1]), we have

$$\mathbb{E}_{x,i}[e^{-\alpha S_i} V(\Lambda(S_i))] = \mathbb{E}_{x,i}[e^{-\alpha S_i}]\mathbb{E}_{x,i}[V(\Lambda(S_i))] = \frac{\lambda_i}{\alpha+\lambda_i}\sum_{j\neq i}\frac{q_{ij}}{\lambda_i}V(j)$$
$$= \frac{1}{\alpha+\lambda_i}\sum_{j\neq i} q_{ij}V(j). \tag{6.10}$$

Using (6.9) and (6.10) in (6.8) yields $V(i) + \varepsilon > Q_{\alpha+\lambda_i}(0) + \frac{1}{\alpha+\lambda_i}\sum_{j\neq i} q_{ij}V(j)$. But $\varepsilon > 0$ is arbitrary, and therefore, we obtain

$$V(i) \geq Q_{\alpha+\lambda_i}(0) + \frac{1}{\alpha+\lambda_i}\sum_{j\neq i} q_{ij}V(j). \tag{6.11}$$

On the other hand, we apply the dynamic programming principle (6.6) to obtain

$$\begin{aligned} V(i) &\leq \inf_{u\in\mathcal{A}(x,i)} \mathbb{E}_{x,i}\left[\int_0^{S_i} e^{-\alpha t}[c(u(t))\,\mathrm{d}t + p\,\mathrm{d}M(t)] + e^{-\alpha S_i}V(\Lambda(S_i))\right] \\ &= \inf_{u\in\mathcal{A}(x,i)} \mathbb{E}_{x,i}\left[\int_0^{S_i} e^{-\alpha t}[c(u(t))\,\mathrm{d}t + p\,\mathrm{d}M(t)]\right] + \frac{1}{\alpha+\lambda_i}\sum_{j\neq i} q_{ij}V(j), \end{aligned}$$

where we have used (6.10) in the last equality. But as we observed earlier,

$$\begin{aligned} &\inf_{u\in\mathcal{A}(x,i)} \mathbb{E}_{x,i}\left[\int_0^{S_i} e^{-\alpha t}[c(u(t))\,\mathrm{d}t + p\,\mathrm{d}M(t)]\right] \\ &= \inf_{u\in\mathcal{A}(x,i)} \mathbb{E}_{x,i}\left[\int_0^{\tau\wedge S_i} [c(u(t))\,\mathrm{d}t + p\,\mathrm{d}M(t)]\right] \\ &= Q_{\alpha+\lambda_i}(0). \end{aligned}$$

Hence,

$$V(i) \leq Q_{\alpha+\lambda_i}(0) + \frac{1}{\alpha+\lambda_i}\sum_{j\neq i} q_{ij}V(j). \tag{6.12}$$

Now a combination of (6.11) and (6.12) yields

$$(\alpha+\lambda_i)V(i) - \sum_{j\neq i} q_{ij}V(j) = (\alpha+\lambda_i)Q_{\alpha+\lambda_i}(0). \tag{6.13}$$

Obviously if $\lambda_i = 0$, then $q_{ij} = 0$ for $j \neq i$ and $S_i = \infty \mathbb{P}_{x,i}$-a.s. In such a case, (6.13) still holds by Theorem 3.4.

We introduce the matrix $R := \alpha I - \mathcal{Q} \in \mathbb{R}^{m\times m}$ and the column vector $\mathbf{b} := ((\alpha + \lambda_i)Q_{\alpha+\lambda_i}(0)) \in \mathbb{R}^m$, where I is the m-dimensional identity matrix. Then (6.13) can be written as

$$R\mathbf{V} = \mathbf{b},$$

where $\mathbf{V} = (V(j)) \in \mathbb{R}^m$ is the column vector which represents the value function. Note that the matrix R is strictly diagonally dominant and therefore non-singular. Thus the value function is given by

$$\mathbf{V} = R^{-1}\mathbf{b}. \tag{6.14}$$

In addition, the optimal policy can be described by

$$u^*(t) := \sum_{i\in\mathcal{M}} \Psi(-Q'_{\alpha+\lambda_i}(M^*(t) - X^*(t)))I_{\{\Lambda(t)=i\}}. \tag{6.15}$$

We summarize the above results into the following theorem.

Theorem 6.1. *For the infinite-horizon discounted control problem specified by (6.2), (6.3) and (6.5), the value function $V(x,i)$ of (6.5) is independent of x and is equal to the ith component of the vector $\mathbf{V}$ of (6.14). In addition, the policy given by (6.15) is an optimal non-Markovian control policy.*

Example 6.2. In this example, we focus on the case when $m = 2$ and for notational simplicity we will write the generator $\mathcal{Q}$ for $\Lambda(\cdot)$ as $\mathcal{Q} = \begin{bmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{bmatrix}$, in which the jump rates λ_1 and λ_2 are non-negative. In addition, we assume that $\lambda_1^2 + \lambda_2^2 > 0$ or at least one of λ_1 or λ_2 is positive. Then (6.13) which describes the value function reduces to the following system of equations:

$$\begin{cases} V(1) = Q_{\alpha+\lambda_1}(0) + \dfrac{\lambda_1}{\alpha+\lambda_1}V(2), \\ V(2) = Q_{\alpha+\lambda_2}(0) + \dfrac{\lambda_2}{\alpha+\lambda_2}V(1). \end{cases} \tag{6.16}$$

We solve (6.16) to obtain the value function

$$V(1) = \frac{Q_{\alpha+\lambda_1}(0)(\lambda_1+\alpha)(\lambda_2+\alpha) + Q_{\alpha+\lambda_2}(0)\lambda_1(\lambda_2+\alpha)}{(\lambda_1+\lambda_2)\alpha+\alpha^2}, \quad \text{and}$$
$$V(2) = \frac{Q_{\alpha+\lambda_1}(0)\lambda_2(\lambda_1+\alpha) + Q_{\alpha+\lambda_2}(0)(\lambda_1+\alpha)(\lambda_2+\alpha)}{(\lambda_1+\lambda_2)\alpha+\alpha^2}.$$

Acknowledgments

The research of A.W. is partially supported by the Simons Foundation grant 317381 and the research of C.Z. was partially supported by a grant from UW-Milwaukee Research Growth Initiative.

Appendix A. Proof of Proposition 3.2.

Consider the following family of differential equations parametrized by $r \geq 0$:

$$\begin{cases} \dfrac{\sigma^2}{2}Y_r'(x) + \mu Y_r(x) - \Phi(-Y_r(x)) = \dfrac{\sigma^2}{2}r - \mu p - \Phi(p) + \alpha\displaystyle\int_0^x Y_r(u)\,\mathrm{d}u, \\ Y_r(0) = -p, \quad Y_r'(0) = r. \end{cases} \tag{A.1}$$

We differentiate (A.1) to obtain

$$\begin{cases} \frac{\sigma^2}{2} Y_r''(x) + \mu Y_r'(x) + \Psi(-Y_r(x)) Y_r'(x) - \alpha Y_r(x) = 0, \\ Y_r(0) = -p, \qquad Y_r'(0) = r. \end{cases} \tag{A.2}$$

Eq. (A.1) can be written in the form

$$Y_r(x) = -p + \left(r - \frac{2\mu p}{\sigma^2} - \frac{2\Phi(p)}{\sigma^2} \right) x + \frac{2}{\sigma^2} \int_0^x [\Phi(-Y_r(u)) - \mu Y_r(u)] \, du + \frac{2\alpha}{\sigma^2} \int_0^x \int_0^y Y_r(u) \, du dy.$$

Since Φ is locally Lipschitz continuous, it follows that the above equation has a unique solution on an interval $[0, \omega(r))$, where $\omega(r)$ is the explosion time (see [21, Chapter 2]). Notice that (A.1) and (A.2) are equivalent, so that Y_r also satisfies (A.2). Moreover, $Y_r(x)$ is jointly continuous in (r, x).

We observe that if $Y_r'(\xi) = 0$ for some ξ, then $\frac{\sigma^2}{2} Y_r''(\xi) = \alpha Y_r(\xi)$. Hence

$$\begin{cases} \text{if } Y_r'(\xi) = 0 \quad \text{and} \quad Y_r(\xi) > 0, & \text{then } x = \xi \text{ is a strict local minimum point,} \\ \text{if } Y_r'(\xi) = 0 \quad \text{and} \quad Y_r(\xi) < 0, & \text{then } x = \xi \text{ is a strict local maximum point.} \end{cases} \tag{A.3}$$

If $r = 0$, since $Y_0'(0) = 0$ and $Y_0(0) = -p < 0$, it follows that $\frac{\sigma^2}{2} Y_0''(0) = -\alpha p < 0$. Since Y_0 satisfies (A.2) in a neighborhood of 0, $x = 0$ is an end-point maximum.

To establish Proposition 3.2, our aim here is to find a special value r^* for the parameter r so that the corresponding solution Y_{r^*} to (A.1) is defined on $[0, \infty)$ and bounded. Then we can find the function Q required in Proposition 3.2 so that $Q'(x) = Y_{r^*}(x)$ for all $x \geq 0$. To this end, we first show in Lemma A.5 that $Y_r(x)$ tends to $+\infty$ as x approaches $\omega(r)$ when r is large. Then in Lemma A.6, we establish that $Y_r(x)$ tends to $-\infty$ as x approaches $\omega(r)$ when $r > 0$ is small. Finally, in Theorem A.7, we show that these two solution profiles are separated by a special solution Y_{r^*} which is bounded and has infinite explosion time. This Y_{r^*} is the required solution to establish Proposition 3.2. These steps will be carried out in the next few results.

Lemma A.1. *There exists a $\delta_0 > 0$ such that for $0 < r < \delta_0$, Y_r has a negative local maximum.*

Proof. Since the origin is an end-point maximum for $Y_0(x)$, we can fix some $\varepsilon > 0$ and $x_0 > 0$ so that $Y_0(x_0) < -p - \varepsilon < -p$. Now by the continuity in (r, x), we can find some $\delta_0 > 0$ so that

$$|Y_r(x_0) - Y_0(x_0)| < \varepsilon/2, \quad \text{if } 0 < r < \delta_0.$$

Hence it follows that $Y_r(x_0) < -p - \varepsilon/2$. But $Y_r(0) = -p$ and $Y_r'(0) = r > 0$. So Y_r is increasing on $[0, \eta)$ for some $\eta > 0$ and $Y_r(x_0) < Y_r(0)$. This shows that Y_r has a local maximum when $0 < r < \delta_0$. In view of (A.3), the local maximum is necessarily negative. □

Lemma A.2. *Let $r_1 > r_2$, then $Y_{r_1}(x) > Y_{r_2}(x)$ for all $x \in [0, \hat{\omega})$, where $\hat{\omega} := \min\{\omega(r_1), \omega(r_2)\}$.*

Proof. Since $Y_{r_1}(0) = Y_{r_2}(0)$ and $Y_{r_1}'(0) = r_1 > r_2 = Y_{r_2}'(0)$, we have $Y_{r_1}(x) > Y_{r_2}(x)$ for all $x \in [0, \epsilon)$ for some $\epsilon > 0$. Suppose there exists some $x \in [0, \hat{\omega})$ so that $Y_{r_1}(x) = Y_{r_2}(x)$. Let

$$\xi_0 := \inf \left\{ x \in [0, \hat{\omega}) : Y_{r_1}(x) = Y_{r_2}(x) \right\}.$$

We have $\xi_0 > \epsilon$ and $Y_{r_1}(x) > Y_{r_2}(x)$ for $0 < x < \xi_0$. Moreover, using (A.1) on $[0, \xi_0]$, we obtain

$$\begin{aligned}&\frac{\sigma^2}{2}[Y'_{r_1}(x) - Y'_{r_2}(x)] + \mu[Y_{r_1}(x) - Y_{r_2}(x)] - [\Phi(-Y_{r_1}(x)) - \Phi(-Y_{r_2}(x))]\\&\quad = \frac{\sigma^2}{2}[r_1 - r_2] + \alpha \int_0^x [Y_{r_1}(u) - Y_{r_2}(u)]\,\mathrm{d}u.\end{aligned}$$

Evaluating the above equation at $x = \xi_0$ leads to $\frac{\sigma^2}{2}[Y'_{r_1}(\xi_0) - Y'_{r_2}(\xi_0)] = \frac{\sigma^2}{2}[r_1 - r_2] + \alpha \int_0^{\xi_0}[Y_{r_1}(u) - Y_{r_2}(u)]\,\mathrm{d}u$. Obviously, the right-hand side of the above equation is positive. But using the definition of ξ_0, we have $Y'_{r_1}(\xi_0) \le Y'_{r_2}(\xi_0)$. This is a contradiction and thus $Y_{r_1}(x) > Y_{r_2}(x)$ for all $x \in [0, \hat{\omega})$. Hence, the proof is complete. □

Lemma A.3. *There exists an $r_0 > 0$ so that for each $r > r_0$, we have $Y_r(x) > 0$ for some $x = x(r) > 0$.*

Proof. Using Lemma A.1, we can fix some $\hat{r} > 0$ so that $Y_{\hat{r}}(x)$ is increasing on $[0, \eta)$ and has a local maximum at some $\eta = \eta(\hat{r}) > 0$. Let $r > \hat{r}$. Suppose that the solution $Y_r(x) \le 0$ for all $0 \le x < \eta$. In addition, by Lemma A.2, we have $0 \ge Y_r(x) > Y_{\hat{r}}(x)$ for all $0 \le x < \eta$. Thus, $Y_r(x)$ is bounded: $-p \le Y_r(x) \le 0$ for all $0 \le x < \eta$ and $r > \hat{r}$. Then, by virtue of (A.1), $Y'_r(x)$ is also bounded. This shows that the explosion time $\omega(r) > \eta$. Otherwise, $Y_r(x)$ and therefore $Y'_r(x)$ are well-defined at $x = \omega(r)$ since Y_r can only have local maximum. Now the solution can be extended beyond $\omega(r)$ using $Y_r(\omega(r))$ and $Y'_r(\omega(r))$ as initial data, and we conclude $\omega(r) > \eta$.

For such $r > \hat{r}$, since $-p \le Y_r(x) \le 0$ for all $0 \le x \le \eta$, it follows that

$$\sup_{0 \le x \le \eta} \{|\mu Y_r(x)| + |\Phi(-Y_r(x))|\} \le M < \infty$$

for some positive M. This, together with (A.1), implies that $\frac{\sigma^2}{2}Y'_r(x) \ge \frac{\sigma^2}{2}r - \mu p - \Phi(p) - M - \alpha p \eta$. We can pick $r > \hat{r}$ sufficiently large so that $(\frac{\sigma^2}{2})\frac{r}{2} > \mu p + \Phi(p) + M + \alpha p \eta$. Consequently, for such r, we have $Y'_r(x) > \frac{r}{2}$ for $0 \le x \le \eta$. Then, from the mean value theorem, we have $0 \ge Y_r(\eta) = Y_r(0) + Y'_r(x)(\eta - 0) \ge -p + \frac{r}{2}\eta$ for some $x \in (0, \eta)$. This is a contradiction since the right-hand side tends to ∞ as $r \to \infty$. Thus there exists an $r_0 > 0$ so that $Y_{r_0}(x_0) > 0$ for some $0 < x_0 < \eta$.

For $x > x_0$, in view of (A.3), Y_{r_0} cannot have a positive local maximum. Therefore Y_{r_0} is increasing after it crosses the x-axis and hence $Y_{r_0}(x) \ge Y_{r_0}(x_0) > 0$. Finally, by Lemma A.2, for each $r > r_0$, we have $Y_r(x) > Y_{r_0}(x) > 0$. This completes the proof. □

Lemma A.4. *If $Y_r(x)$ is bounded, then $\omega(r) = \infty$.*

Proof. Let us fix an $r > 0$ and assume that $\omega(r)$ is finite. Suppose that for some $M < \infty$, we have $|Y_r(x)| \le M$, for all $x \in [0, \omega(r))$. In view of (A.3), Y_r can only have positive local minimum or negative local maximum. Therefore, the boundedness assumption implies that $\lim_{x \to \omega(r)} Y_r(x) = l_0$ exists and l_0 is finite. Using this together with (A.1), we see that $\lim_{x \to \omega(r)} Y'_r(x) = l_1$ also exists. Likewise, using (A.2), we observe that $\lim_{x \to \omega(r)} Y''_r(x) = l_2$ also exists. Now we can consider the differential equation (A.2) with new initial data $Y_r(\omega(r)) = l_0$ and $Y'_r(\omega(r)) = l_1$. By the existence and uniqueness theorem for ordinary differential equations, this new initial value problem has a unique solution in an interval $(\omega(r) - \varepsilon, \omega(r) + \varepsilon)$

for some $\varepsilon > 0$. Thus, we can extend the solution Y_r to the interval $[0, \omega(r) + \varepsilon)$. This is a contradiction since $\omega(r)$ is the explosion time. Therefore, we must have $\omega(r) = \infty$. $\square$

By Lemma A.3, there exists some $r_0 > 0$ so that for each $r > r_0$, $Y_r(x) > 0$ for some $x > 0$. Next we show the following result.

Lemma A.5. *For every $r > r_0$, we have $\lim_{x\to\omega(r)} Y_r(x) = \infty$.*

Proof. For $r > r_0$, using Lemma A.3, let x_r be such that $Y_r(x_r) > 0$. Recall that $Y_r(0) = -p < 0$. By the intermediate value theorem, there exists some $0 < p_r < x_r$ so that $Y_r(p_r) = 0$ and $Y_r(x) > 0$ for $x > p_r$. Thus we have $Y_r'(p_r) \geq 0$. Recall from (A.3) that Y_r cannot have positive local maxima. Therefore Y_r is increasing on $[x_r, \omega(r))$. By virtue of Lemma A.4, if $\omega(r)$ is finite, then we must have $\lim_{x\to\omega(r)} Y_r(x) = +\infty$.

Now let us consider the case when $\omega(r) = \infty$ and suppose $\lim_{x\to\infty} Y_r(x) = \lambda < \infty$. From the argument in the previous paragraph, we necessarily have $\lambda > 0$. In this case, using (A.1), we have

$$\begin{aligned}
&\lim_{x\to\infty} \frac{\sigma^2}{2}\frac{Y_r'(x)}{x} \\
&\quad = \lim_{x\to\infty}\left[-\frac{\mu Y_r(x)}{x} + \frac{\Phi(-Y_r(x))}{x} + \frac{\sigma^2}{2}\frac{r}{x} - \frac{\mu p}{x} - \frac{\Phi(p)}{x} + \frac{\alpha}{x}\int_0^x Y_r(u)\,\mathrm{d}u\right] = \alpha\lambda.
\end{aligned}$$

Consequently, for all large x, we have $Y_r'(x) > \frac{\alpha}{\sigma^2}\lambda$ and hence $\lim_{x\to\infty} Y_r(x) = \infty$. This is a contradiction and hence the proof is complete. $\square$

Lemma A.6. *There exists some $\delta_0 > 0$ so that $\lim_{x\to\omega(r)} Y_r(x) = -\infty$ for all $0 < r < \delta_0$.*

Proof. By Lemma A.1, there exists a $\delta_0 > 0$ so that Y_r has a negative local maximum at $x = \eta_r$ for $0 < r < \delta_0$. Then since $Y_r''(\eta_r) < 0$, Y_r is strictly decreasing on an interval $[\eta_r, \eta_r + \varepsilon)$ for some $\varepsilon > 0$. Moreover, Y_r cannot have any negative local minima. Hence $Y_r'(x) < 0$ and therefore Y_r is strictly decreasing on $[\eta_r, \omega(r))$. Note that (A.3) implies that $Y_r'(x) = 0$ is impossible on $[\eta_r, \omega(r))$. Therefore, either $\lim_{x\to\omega(r)} Y_r(x)$ is a finite negative number or $\lim_{x\to\omega(r)} Y_r(x) = -\infty$.

If $\lim_{x\to\omega(r)} Y_r(x) = l_1 < 0$ is finite, then $Y_r(x)$ is bounded on $[0, \omega(r))$ and hence by virtue of Lemma A.4, $\omega(r) = +\infty$. But then, using (A.1), similar to the argument in the proof of Lemma A.5, we obtain $\lim_{x\to\infty} \frac{Y_r'(x)}{x} = \frac{2\alpha}{\sigma^2} l_1 < 0$. This further implies that $Y_r'(x) < \frac{\alpha}{\sigma^2} l_1 < 0$ for all large x and hence $\lim_{x\to\infty} Y_r(x) = -\infty$. This is a contradiction and we must have $\lim_{x\to\omega(r)} Y_r(x) = -\infty$, as desired. $\square$

Theorem A.7. *There exist an $r^* > 0$ and a function $Y_{r^*} : [0,\infty) \mapsto [-p, 0]$ so that*

$$\frac{\sigma^2}{2} Y_{r^*}'(x) + \mu Y_{r^*}(x) - \Phi(-Y_{r^*}(x)) = \frac{\sigma^2}{2} r^* - \mu p - \Phi(p) + \alpha\int_0^x Y_{r^*}(u)\,\mathrm{d}u, \tag{A.4}$$

for all $0 < x < \infty$, with the initial data

$$Y_{r^*}(0) = -p, \quad Y_{r^*}'(0) = r^* > 0.$$

In addition, Y_{r^*} is monotonic increasing, with $\lim_{x\to\infty} Y_{r^*}(x) = 0$ and $\int_0^\infty Y_{r^*}(u)\,\mathrm{d}u$ being convergent. Moreover r^* satisfies the lower bound $\frac{\sigma^2}{2} r^* > \mu p + \Phi(p)$.

Proof. Let

$$r^* := \sup \{r > 0 : Y_r \text{ has a local maximum}\}.$$

Then by Lemma A.1, $r^* > 0$ and is finite. Note that

$$Y_{r^*}(x) \leq 0 \quad \text{for all } x \in [0, \omega(r^*)). \tag{A.5}$$

Otherwise, suppose $Y_{r^*}(x) > 0$ for some $x \in (0, \omega(r^*))$, then by the joint continuity in (x, r) of $Y_r(x)$, we can pick an $r < r^*$ so that $Y_r(x) > 0$ and Y_r has a local maximum. Recall from the proof of Lemma A.5 that Y_r is increasing on $[x, \omega(r))$ since $Y_r(x) > 0$. But thanks to (A.3) and Lemma A.6, the local maximum of Y_r is necessarily negative and $\lim_{x\to\omega(r)} Y_r(x) = -\infty$. This is a contradiction and thus (A.5) must hold.

Now let us suppose that Y_{r^*} has a local maximum at $x = \xi > 0$ with $Y_{r^*}(\xi) \leq 0$. Assume that $Y_{r^*}(\xi) < 0$. Then, again using the joint continuity of $Y_r(x)$ in (r, x) and Lemma A.2, we can find an $r > r^*$ so that $0 > Y_r(\xi) > Y_{r^*}(\xi)$ and Y_r has a local maximum. This contradicts with the definition of r^*. Hence the only possibility is $Y_{r^*}(\xi) = 0$ and $Y'_{r^*}(\xi) = 0$.

Next we define $c := \inf \{x > 0 : Y_{r^*}(x) = 0\}$. From the argument in the previous paragraph, we have $c \leq \xi$. Moreover, in view of (A.5), $x = c$ is also a local maximum. Thus we have $Y_{r^*}(c) = 0$ and $Y'_{r^*}(c) = 0$. Note also that $Y_{r^*}(x) < 0$ for $0 \leq x < c$.

Now let us consider the ODE (A.2) in a neighborhood $(c - \varepsilon, c + \varepsilon)$ with data $Y_r(c) = 0$ and $Y'_r(c) = 0$, where $\varepsilon > 0$ satisfies $c - \varepsilon > 0$. Clearly both $Y_{r^*}(x)$ and $Y \equiv 0$ satisfy the ODE (A.2) on the interval $(c - \varepsilon, c + \varepsilon)$. Because $Y_{r^*}(x) < 0$ for $c - \varepsilon < x < c$, these are two different solutions. This contradicts with the uniqueness of solutions. Hence such a local maximum at $x = \xi$ with $Y_{r^*}(\xi) = 0$ is also not possible.

Consequently, Y_{r^*} cannot have any local maximum. Recall that $Y'_{r^*}(0) = r^* > 0$. Thus Y_{r^*} is increasing and hence

$$-p \leq Y_{r^*}(x) \leq 0 \quad \text{for all } x \in [0, \omega(r^*)). \tag{A.6}$$

But then Lemma A.4 implies that $\omega(r^*) = \infty$. This shows that Y_{r^*} is a solution to (A.4) for all $0 \leq x < \infty$.

Let us assume that $\lim_{x\to\infty} Y_{r^*}(x) = \lambda \leq 0$. Since Y_{r^*} is bounded, λ is necessarily finite. As argued in the proof of Lemma A.6, if $\lambda < 0$, then we would have $\lim_{x\to\infty} Y_{r^*}(x) = -\infty$. This is impossible. Thus $\lambda = 0$.

The function $x \mapsto \alpha \int_0^x (-Y_{r^*}(u))\,\mathrm{d}u$ is increasing since Y_{r^*} is negative. In view of (A.1), we further have

$$\begin{aligned} 0 < \alpha \int_0^x (-Y_{r^*}(u))\,\mathrm{d}u &= \frac{\sigma^2}{2}[r^* - Y'_{r^*}(x)] - \mu[p + Y_{r^*}(x)] - \Phi(p) + \Phi(-Y_{r^*}(x)) \\ &\leq \frac{\sigma^2}{2} r^* - \mu[p + Y_{r^*}(x)] - \Phi(p) + \Phi(-Y_{r^*}(x)) \leq \frac{\sigma^2}{2} r^* \end{aligned} \tag{A.7}$$

where we have used (A.6), the monotonicity of Φ, and the fact that $Y'_{r^*}(x) \geq 0$ to obtain the inequalities above. This shows that $\int_0^\infty (-Y_{r^*}(u))\,\mathrm{d}u$ is convergent.

Passing to the limit as $x \to \infty$ in (A.7) and using $\lim_{x\to\infty} Y_{r^*}(x) = 0$ and the continuity of Φ, we further obtain $0 < \alpha \int_0^\infty (-Y_{r^*}(u))\,\mathrm{d}u \leq \frac{\sigma^2}{2} r^* - \mu p - \Phi(p)$. This implies that $\frac{\sigma^2}{2} r^* > \mu p + \Phi(p)$. $\square$

Proof of Proposition 3.2. Let $r^* > 0$ and Y_{r^*} as in Theorem A.7. We define

$$Q(x) = \frac{1}{\alpha}\left[\frac{\sigma^2}{2}r^* - \mu p - \Phi(p)\right] + \int_0^x Y_{r^*}(u)\,\mathrm{d}u. \tag{A.8}$$

Then we have $Q'(x) = Y_{r^*}(x) < 0$, $Q'(0) = Y_{r^*}(0) = -p$, and $Q''(x) = Y'_{r^*}(x) > 0$ for any $x \in [0, \infty)$. Hence, Q is a strictly decreasing and convex function. Moreover, from Theorem A.7, it follows that Q satisfies (3.5) or equivalently (3.4), with $\lim_{x\to\infty} Q(x)$ being finite.

Recall from Theorem A.7 that $\lim_{x\to\infty} Y_{r^*}(x) = 0$ and that Y_{r^*} is increasing. Thus we have $\lim_{x\to\infty} Y'_{r^*}(x) = 0$. Then passing to the limit as $x \to \infty$ in (A.1) with $r = r^*$, and noting that $Y_{r^*}(u) \le 0$ for all $u \in [0, \infty)$, we obtain $\alpha \int_0^\infty Y_{r^*}(u)\,\mathrm{d}u = \mu p + \Phi(p) - \frac{\sigma^2}{2}r^* < 0$. Using this together with (A.8) yields $Q(x) > 0$ for all $x \ge 0$ and that $\lim_{x\to\infty} Q(x) = 0$. Consequently, the function Q is bounded. This completes the proof of Proposition 3.2. □

Appendix B. Joint continuity of the function $W_p(y)$

To address the minimization problem with a constraint in Section 4, it is essential to understand the behavior of the value function $\tilde{V}(\cdot)$ in (2.15) with respect to the p variable. Hence we relabel it as $\tilde{V}_p(\cdot)$. Moreover, as in (4.12), its derivative $\frac{\mathrm{d}}{\mathrm{d}y}\tilde{V}_p(y)$ will be denoted by $W_p(\cdot)$. For each fixed $p > 0$, $W_p(y)$ is identical to $Y_{r^*}(y)$ in Theorem A.7. The following result will be used in the proof of Proposition 4.4.

Proposition B.1. *Let $W_p(\cdot)$ be the y-derivative of the value function $\tilde{V}_p(\cdot)$. Then the following results hold.*

(i) *If $0 < p_1 < p_2$, then $W_{p_1}(y) = W_{p_2}(\xi_{p_1} + y)$ for all $y \ge 0$, where ξ_{p_1} is the unique point which satisfies $W_{p_2}(\xi_{p_1}) = -p_1$.*
(ii) *If $0 < p_1 < p_2$, then $W_{p_2}(y) < W_{p_1}(y)$ for all $y \ge 0$.*
(iii) *$W_p(y)$ is jointly continuous in (p, y).*

Proof. Let $0 < p_1 < p_2$. Consider $W_{p_2}(\cdot)$ and introduce $\tilde{W}(\cdot)$ by $\tilde{W}(y) = W_{p_2}(\xi_{p_1} + y)$ for all $y \ge 0$, where $\xi_{p_1} > 0$ is the unique point which satisfies $W_{p_2}(\xi_{p_1}) = -p_1$. The unique existence of $\xi_{p_1} > 0$ is guaranteed by the facts that $W_{p_2}(0) = -p_2 < -p_1$, $\lim_{x\to\infty} W_{p_2}(x) = 0$, and $W_{p_2}(\cdot)$ is a continuous and strictly increasing function. Next since $W_{p_2}(\cdot)$ satisfies the assertions of Theorem A.7, it is straightforward to check that $\tilde{W}(\cdot)$ satisfies

$$\frac{\sigma^2}{2}\tilde{W}''(y) + \mu\tilde{W}'(y) + \Psi(-\tilde{W}(y))\tilde{W}'(y) - \alpha\tilde{W}(y) = 0, \quad \text{for } y \ge 0, \tag{B.1}$$

$\tilde{W}(0) = -p_1$, $\lim_{x\to\infty} \tilde{W}(x) = 0$, $\tilde{W}'(x) > 0$ for all $x \ge 0$, and $\int_0^\infty \tilde{W}(u)\,\mathrm{d}u$ is finite. Hence we can define $\tilde{Q}(x)$ by $\tilde{Q}(x) = \frac{1}{\alpha}[\frac{\sigma^2}{2}\tilde{W}'(0) - \mu p_1 - \Phi(p_1)] + \int_0^x \tilde{W}(u)\,\mathrm{d}u$, for $x \ge 0$. Using (B.1) and the above facts, it is easy to verify that $\tilde{Q}$ also satisfies the assertions of Proposition 3.2. Thus, by Theorem 3.4, we have $\tilde{Q}(y) = \tilde{V}_{p_1}(y)$ for all $y \ge 0$. Consequently, $\tilde{W}(y) = W_{p_1}(y)$ for all $y \ge 0$. Hence part (i) follows.

Since $0 < p_1 < p_2$, using the conclusion of part (i) and the strict monotonicity of $W_{p_2}(\cdot)$, we have $W_{p_2}(y) < W_{p_2}(\xi_{p_1} + y) = W_{p_1}(y)$, for all $y \ge 0$. This establishes part (ii).

For part (iii), we first show that $W_p(y)$ is continuous in the p variable. To this end, we consider a sequence $\{p_n\}$ with $0 < p_1 < p_2 < \cdots$ and $\lim_{n\to\infty} p_n = p_0 < \infty$. Hence for any $y \ge 0$, the sequence $\{W_{p_n}(y)\}$ is a decreasing sequence and is bounded below by $W_{p_0}(y) \geq -p_0$. Hence $\{W_{p_n}(\cdot)\}$ is convergent and denote the limit by $W_0(\cdot)$. By integrating $W_{p_n}(\cdot)$ in (A.4) with the aid of (A.8), we obtain

$$\frac{\sigma^2}{2}(W_{p_n}(x) + p_n) + \mu \int_0^x W_{p_n}(u)\,\mathrm{d}u - \int_0^x \Phi(-W_{p_n}(u))\,\mathrm{d}u$$
$$= \alpha \tilde{V}_{p_n}(0)x + \alpha \int_0^x \int_0^y W_{p_n}(u)\,\mathrm{d}u\,\mathrm{d}y.$$

Now letting $n \to \infty$ and by using Lemma 4.3 for the continuity of $\tilde{V}_p(0)$ in p variable, and the bounded convergence theorem, we obtain

$$\frac{\sigma^2}{2}(W_0(x) + p_0) + \mu \int_0^x W_0(u)\,\mathrm{d}u - \int_0^x \Phi(-W_0(u))\,\mathrm{d}u$$
$$= \alpha \tilde{V}_{p_0}(0)x + \alpha \int_0^x \int_0^y W_0(u)\,\mathrm{d}u\,\mathrm{d}y.$$

Thus, $W_0(0) = -p_0$ and $\frac{\sigma^2}{2} W_0'(0) = \alpha \tilde{V}_{p_0}(0) + \Phi(p_0) + \mu p_0$. Consequently, W_0 and W_{p_0} satisfy the same initial value problem and hence as argued in the first paragraph of the proof, we have $W_0(y) \equiv W_{p_0}(y)$ for all $y \geq 0$. Thus $W_0(y) = \lim_{p_n \to p_0-} W_{p_n}(y) = W_{p_0}(y)$ for all $y \geq 0$. A very similar argument shows that $\lim_{p_n \to p_0+} W_{p_n}(y) = W_{p_0}(y)$ for all $y \geq 0$. Thus, we conclude that $W_p(y)$ is continuous in the p variable.

To obtain the joint continuity, let $p_0 > 0$ and $y_0 \geq 0$. For any sequence (p_n, y_n) that converges to (p_0, y_0) as $n \to \infty$, we can pick two constants $K_0 > 0$ and $N > 0$ so that $0 < p_n < K_0$ and $0 \leq y_n < N$ for all n. Now we use Theorem A.7, (A.4) and (A.8) to obtain $\frac{\sigma^2}{2} W_{p_n}'(y) \leq \alpha \tilde{V}_{p_n}(0) + |\mu|\, p_n + \Phi(p_n) \leq \alpha \tilde{V}_{K_0}(0) + |\mu| K_0 + \Phi(K_0)$, for all $y \geq 0$, where the second inequality follows from the monotonicity of $p \mapsto \tilde{V}_p(0)$ as in Lemma 4.3 and the monotonicity of the function $\Phi(\cdot)$ in Lemma 3.1. Hence, we can find a constant $K_1 > 0$ which is independent of n such that $0 \leq W_{p_n}'(y) < K_1$, whenever $0 < p_n < K_0$ and $y \geq 0$. Thus, it follows that $\left|W_{p_n}(y_n) - W_{p_n}(y_0)\right| \leq K_1 |y_n - y_0|$ and consequently, we obtain

$$\begin{aligned}\left|W_{p_n}(y_n) - W_{p_0}(y_0)\right| &\leq \left|W_{p_n}(y_n) - W_{p_n}(y_0)\right| + \left|W_{p_n}(y_0) - W_{p_0}(y_0)\right| \\ &\leq K_1 |y_n - y_0| + \left|W_{p_n}(y_0) - W_{p_0}(y_0)\right| \to 0,\end{aligned}$$

as $n \to \infty$. Here we have used the fact that when y_0 is fixed, the function $p \mapsto W_p(y_0)$ is continuous, which was established earlier. This shows that $W_p(y)$ is jointly continuous and hence the proof is complete. □

Appendix C. Proof of Proposition 4.5

Proof of Proposition 4.5. When $y \geq 0$ is fixed, we can use the concavity of $\tilde{V}_p(y)$ in the p variable as in Lemma 4.3 together with (4.18) to conclude that $\Gamma(p, y)$ is monotone decreasing in p. However, to show that it is strictly decreasing in p, we need a quite careful argument and we divide the proof of this fact into two steps. In the first step, we establish the case $y > 0$ and then in step two below, we prove it for the case $y = 0$. Thereafter, in steps three and four, we establish the limits of $\Gamma(p, y)$ as p tends to infinity and to zero.

Step 1. Let $y > 0$ and define $\tau_p = \inf\left\{t \geq 0 : Y_p^*(t) = 0\right\}$, where Y_p^* is the process given by (4.13) with initial value $Y_p^*(0) = y$. Using Fubini's theorem, we have

$$\begin{aligned}\Gamma(p, y) &= \alpha\mathbb{E}\left[\int_0^\infty \int_t^\infty e^{-\alpha s}\,\mathrm{d}s\,\mathrm{d}L_p^*(t)\right] = \alpha\mathbb{E}\left[\int_0^\infty \int_0^s \mathrm{d}L_p^*(t)e^{-\alpha s}\,\mathrm{d}s\right]\\ &= \alpha\mathbb{E}\left[\int_0^\infty L_p^*(s)e^{-\alpha s}\,\mathrm{d}s\right] = \alpha\mathbb{E}\left[\int_0^{\tau_p} e^{-\alpha s}L_p^*(s)\,\mathrm{d}s + \int_{\tau_p}^\infty e^{-\alpha s}L_p^*(s)\,\mathrm{d}s\right].\end{aligned}$$

But, $L_p^*(s) = 0$ for all $0 \leq s \leq \tau_p$. Thus, using the strong Markov property and the fact that $Y_p^*(\tau_p) = 0$ when $\tau_p < \infty$, we have

$$\Gamma(p, y) = \alpha\mathbb{E}\left[e^{-\alpha\tau_p}\mathbb{E}_{Y_p^*(\tau_p)}\left[\int_0^\infty e^{-\alpha s}L_p^*(s)\,\mathrm{d}s\right]\right] = \alpha\mathbb{E}[e^{-\alpha\tau_p}]\Gamma(p, 0).$$

Let $0 < p_1 < p_2$ and keep $y > 0$ fixed. We intend to show that $\mathbb{E}_y[e^{-\alpha\tau_{p_1}}] > \mathbb{E}_y[e^{-\alpha\tau_{p_2}}]$, where $\tau_{p_i} := \inf\left\{t \geq 0 : Y_{p_i}^*(t) = 0\right\}$ for $i = 1, 2$, and $Y_{p_i}^*$ satisfies (4.13). By Proposition B.1, $u_{p_1}^*(y) = u_{p_2}^*(\xi_{p_1} + y)$ where $\xi_{p_1} > 0$ is the unique point which satisfies $W_{p_2}(\xi_{p_1}) = -p_1$, and $u_{p_2}^*$ is decreasing. Hence $u_{p_1}^*(y) \leq u_{p_2}^*(y)$ for all $y \geq 0$. Moreover $L_{p_1}^*(t) = 0$ for all $0 \leq t \leq \tau_{p_1}$. Hence when $y > 0$, we can write $Y_{p_1}^*$ of (4.13) in the form

$$\begin{aligned}Y_{p_1}^*(t) &= y + \sigma W(t) + \mu t + \int_0^t u_{p_1}^*(Y_{p_1}^*(s))\,\mathrm{d}s\\ &= y + \sigma W(t) + \mu t + \int_0^t u_{p_2}^*(\xi_{p_1} + Y_{p_1}^*(s))\,\mathrm{d}s,\end{aligned} \tag{C.1}$$

for $0 \leq t \leq \tau_{p_1}$. On the other hand, since $u_{p_2}^*(y)$ is locally Lipschitz continuous and bounded on $[0, \infty)$, we can smoothly extend it to $\mathbb{R}$ so that it is bounded and locally Lipschitz continuous on $\mathbb{R}$. Hence, we can construct a solution $\tilde{Y}_{p_2}$ with respect to the same Brownian motion W so that

$$\tilde{Y}_{p_2}(t) = y + \sigma W(t) + \mu t + \int_0^t u_{p_2}^*(\tilde{Y}_{p_2}(s))\,\mathrm{d}s, \quad t \geq 0. \tag{C.2}$$

Since $u_{p_2}^*(\cdot)$ is bounded, $\tilde{Y}_{p_2}$ does not explode in a finite time and $\tilde{Y}_{p_2}(t)$ is finite for all $t \geq 0$.

Using the comparison theorem for one-dimensional diffusion processes [23] or [22], we have $Y_{p_1}^*(t) \leq \tilde{Y}_{p_2}(t)$ for all $0 \leq t \leq \tau_{p_1}$ almost surely. Next we develop a pathwise argument to show that $Y_{p_1}^*(t) < \tilde{Y}_{p_2}(t)$ for all $0 < t < \tau_{p_1}$ almost surely. To this end, we define $Z(t) := Y_{p_1}^*(t) - \tilde{Y}_{p_2}(t)$ for $0 \leq t \leq \tau_{p_1}$. Then using (C.1) and (C.2), we have $Z(t) = \int_0^t [u_{p_2}^*(\xi_{p_1} + Y_{p_1}^*(s)) - u_{p_2}^*(\tilde{Y}_{p_2}(s))]\mathrm{d}s$, for $0 \leq t \leq \tau_{p_1}$. First notice that the function $u_{p_2}^*(\cdot)$ in (4.11) is strictly decreasing. Therefore, $\frac{\mathrm{d}}{\mathrm{d}t}Z(t) < 0$ if and only if $\xi_{p_1} + Y_{p_1}^*(t) > \tilde{Y}_{p_2}(t)$ or equivalently $Z(t) > -\xi_{p_1}$. Moreover, $\frac{\mathrm{d}}{\mathrm{d}t}Z(t) = 0$ if and only if $Z(t) = -\xi_{p_1}$. Note that $\xi_{p_1} > 0$. Since $Z(0) = 0$, we can find a $\delta > 0$ so that the derivative $Z'(t) < 0$ on the interval $[0, \delta)$. Consequently, $Z(t) < 0$ on $(0, \delta]$. Now suppose that $Z(t) = 0$ for some $0 < t \leq \tau_{p_1}$. We let $t_0 = \inf\{t > 0 : Z(t) = 0\}$ be the first zero of Z. Then $t_0 \geq \delta$ and $Z(t) < 0$ holds on the interval $(0, t_0)$. Therefore, $Z'(t_0) \geq 0$. But $Z(t_0) = 0 > -\xi_{p_1}$ and thus $Z'(t_0) < 0$. This is a contradiction and hence, $Z(t) < 0$ for all $0 < t \leq \tau_{p_1}$. Consequently, $Y_{p_1}^*(t) < \tilde{Y}_{p_2}(t)$ for all $0 < t \leq \tau_{p_1}$ almost surely.

Define $\tilde{\tau}_{p_2} := \inf\{t \geq 0 : \tilde{Y}_{p_2}(t) = 0\}$. Then we have $\tilde{\tau}_{p_2} > \tau_{p_1}$ almost surely. However, since $y > 0$, by (4.13) and (C.2), the processes $\{Y_{p_2}^*(t) : 0 \leq t \leq \tau_{p_2}\}$ and $\{\tilde{Y}_{p_2}(t) : 0 \leq t \leq \tilde{\tau}_{p_2}\}$

have the same probability law. Therefore we have $\mathbb{E}_y[e^{-\alpha\tau_{p_1}}] > \mathbb{E}_y[e^{-\alpha\tilde{\tau}_{p_2}}] = \mathbb{E}_y[e^{-\alpha\tau_{p_2}}]$ for all $y > 0$. Since $\Gamma(p, y)$ is non-increasing with respect to p for each $y \geq 0$, we have $\Gamma(p_1, 0) \geq \Gamma(p_2, 0)$. Hence, it follows that

$$\Gamma(p_1, y) = \Gamma(p_1, 0)\mathbb{E}_y[e^{-\alpha\tau_{p_1}}] > \Gamma(p_2, 0)\mathbb{E}_y[e^{-\alpha\tau_{p_2}}] = \Gamma(p_2, y), \quad \text{for each } y > 0.$$

Step 2. It remains to show that $\Gamma(p_1, 0) > \Gamma(p_2, 0)$. Let $Y^*_{p_1}$ and $Y^*_{p_2}$ satisfy (4.13) with initial condition $Y^*_{p_1}(0) = Y^*_{p_2}(0) = 0$, respectively. Hence we have

$$Y^*_{p_1}(t) - Y^*_{p_2}(t) = \int_0^t \Big[u^*_{p_2}(\xi_{p_1} + Y^*_{p_1}(s)) - u^*_{p_2}(Y^*_{p_2}(s))\Big]\,\mathrm{d}s + L^*_{p_1}(t) - L^*_{p_2}(t), \quad t \geq 0.$$

Let us also consider a sequence of increasing smooth functions $\{f_n\}$ so that $f_n(x) \equiv 0$ for $-\infty < x \leq \frac{1}{n}$, $f_n(x) > 0$ and $f_n'(x) > 0$ for $x > \frac{1}{n}$ and f_n' is bounded. Then for any $t \geq 0$, we have

$$\begin{aligned} 0 \leq\ & f_n(Y^*_{p_1}(t) - Y^*_{p_2}(t)) \\ =\ & \int_0^t I_{\{Y^*_{p_1}(s) - Y^*_{p_2}(s) \geq \frac{1}{n}\}} f_n'(Y^*_{p_1}(s) - Y^*_{p_2}(s)) \Big[u^*_{p_2}(\xi_{p_1} + Y^*_{p_1}(s)) - u^*_{p_2}(Y^*_{p_2}(s))\Big]\,\mathrm{d}s \\ & + \int_0^t I_{\{Y^*_{p_1}(s) - Y^*_{p_2}(s) \geq \frac{1}{n}\}} f_n'(Y^*_{p_1}(s) - Y^*_{p_2}(s)) \Big[\mathrm{d}L^*_{p_1}(t) - \mathrm{d}L^*_{p_2}(t)\Big]. \end{aligned} \tag{C.3}$$

On the set $\{Y^*_{p_1}(s) - Y^*_{p_2}(s) \geq \frac{1}{n}\}$, $Y^*_{p_1}(s) \geq Y^*_{p_2}(s) + \frac{1}{n} \geq \frac{1}{n}$ and thus, the term $\mathrm{d}L^*_{p_1}(t) = 0$. Since f_n' is non-negative, this in turn implies that the second integral of (C.3) reduces to $-\int_0^t I_{\{Y^*_{p_1}(s) - Y^*_{p_2}(s) \geq \frac{1}{n}\}} f_n'(Y^*_{p_1}(s) - Y^*_{p_2}(s))\,\mathrm{d}L^*_{p_2}(t) \leq 0$. Similarly, when $Y^*_{p_1}(s) - Y^*_{p_2}(s) \geq \frac{1}{n}$, we have $\xi_{p_1} + Y^*_{p_1}(s) > Y^*_{p_2}(s)$ and therefore $u^*_{p_2}(\xi_{p_1} + Y^*_{p_1}(s)) - u^*_{p_2}(Y^*_{p_2}(s)) < 0$. Thus, it follows that the first integral of (C.3) is also non-positive. Then, it must hold true that $f_n(Y^*_{p_1}(t) - Y^*_{p_2}(t)) = 0$ or equivalently $Y^*_{p_1}(t) - Y^*_{p_2}(t) \leq \frac{1}{n}$. But $n \in \mathbb{N}$ is arbitrary, we conclude that $Y^*_{p_1}(t) \leq Y^*_{p_2}(t)$ almost surely. Since both $Y^*_{p_1}$ and $Y^*_{p_2}$ have continuous sample paths, it follows that $Y^*_{p_1}(t) \leq Y^*_{p_2}(t)$ for all $0 \leq t < \infty$ almost surely. Recall that (see, for example, Section 3.6 of [23]), $L^*_{p_i}(t) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_0^t I_{\{0 \leq Y^*_{p_i}(s) \leq \varepsilon\}}(s)\,\mathrm{d}s$ for $i = 1, 2$. This, in particular, leads to $L^*_{p_1}(t) \geq L^*_{p_2}(t)$ for all $t \geq 0$ almost surely. Next, we fix $t > 0$ and consider

$$\begin{aligned} \Gamma(p_1, 0) &= \alpha\mathbb{E}\left[\int_0^\infty e^{-\alpha s} L^*_{p_1}(s)\,\mathrm{d}s\right] = \alpha\mathbb{E}\left[\int_0^t e^{-\alpha s} L^*_{p_1}(s)\,\mathrm{d}s + \int_t^\infty e^{-\alpha s} L^*_{p_1}(s)\,\mathrm{d}s\right] \\ &= \alpha\mathbb{E}\left[\int_0^t e^{-\alpha s} L^*_{p_1}(s)\,\mathrm{d}s + \int_t^\infty e^{-\alpha s}[L^*_{p_1}(s) - L^*_{p_1}(t)]\,\mathrm{d}s\right] + \mathbb{E}\Big[e^{-\alpha t} L^*_{p_1}(t)\Big]. \end{aligned}$$

Using the Markov property, we have

$$\begin{aligned} &\mathbb{E}\bigg[\int_t^\infty e^{-\alpha s}[L^*_{p_1}(s) - L^*_{p_1}(t)]\,\mathrm{d}s\bigg] = e^{-\alpha t}\mathbb{E}\bigg[\int_0^\infty e^{-\alpha u}[L^*_{p_1}(t+u) - L^*_{p_1}(t)]\,\mathrm{d}u\bigg] \\ &\quad = e^{-\alpha t}\mathbb{E}\bigg[\mathbb{E}\bigg[\int_0^\infty e^{-\alpha u}\tilde{L}^*_{p_1}(u)\,\mathrm{d}u \,\bigg|\, Y^*_{p_1}(t)\bigg]\bigg] = e^{-\alpha t}\mathbb{E}[\Gamma(p_1, Y^*_{p_1}(t))], \end{aligned}$$

where $\{\tilde{L}^*_{p_1}(u) : u \geq 0\}$ is the local time process for $\{Y^*_{p_1}(t+u) : u \geq 0\}$. Since $Y^*_{p_1}$ reflects instantaneously at the origin, it follows that $Y^*_{p_1}(t) > 0$ a.s. (see [32]). Hence, using the results in Step 1, we have $\Gamma(p_1, Y^*_{p_1}(t)) > \Gamma(p_2, Y^*_{p_1}(t))$ a.s. In addition, thanks to Proposition 4.4 and the fact that $Y^*_{p_1}(t) \leq Y^*_{p_2}(t)$ a.s., we have $\Gamma(p_1, Y^*_{p_1}(t)) > \Gamma(p_2, Y^*_{p_1}(t)) \geq \Gamma(p_2, Y^*_{p_2}(t))$ a.s.

Thus, it follows that

$$\begin{aligned}\Gamma(p_1,0) &= \mathbb{E}\left[\alpha\int_0^t e^{-\alpha s}L^*_{p_1}(s)\,\mathrm{d}s + e^{-\alpha t}\Gamma(p_1,Y^*_{p_1}(t)) + e^{-\alpha t}L^*_{p_1}(t)\right]\\ &> \mathbb{E}\left[\alpha\int_0^t e^{-\alpha s}L^*_{p_2}(s)\,\mathrm{d}s + e^{-\alpha t}\Gamma(p_2,Y^*_{p_2}(t)) + e^{-\alpha t}L^*_{p_2}(t)\right] = \Gamma(p_2,0).\end{aligned}$$

Hence, by combining Steps 1 and 2, we have shown that $\Gamma(p,y)$ is strictly decreasing for all $y \ge 0$.

Step 3. Now we show that $\lim_{p\to\infty}\Gamma(p,y) = 0$ for all $y \ge 0$. To this end, we notice that using the definition of Φ in (3.1), for any $a > 0$, we have $\Phi(p) \ge ap - c(a)$ and hence, $\liminf_{p\to\infty}\frac{\Phi(p)}{p} \ge \liminf_{p\to\infty}\left(a - \frac{c(a)}{p}\right) = a$. But, $a > 0$ is arbitrary, so we have $\lim_{p\to\infty}\frac{\Phi(p)}{p} = \infty$. On the other hand, by virtue of Theorem A.7, $W_p(y)$ is uniformly bounded. Then by (4.17), we obtain $\lim_{p\to\infty}\Gamma(p,y) \le \lim_{p\to\infty}\frac{\sigma^2}{2}\cdot\frac{-W_p(y)/p}{\Phi(p)/p} = 0$ for all $y \ge 0$, as desired.

Step 4. Finally, we show that $\lim_{p\to 0}\Gamma(p,y) = l_0(y)$ for all $y \ge 0$, where l_0 is defined in (4.10). Let $\varepsilon > 0$. Since the function Ψ is continuous with $\Psi(0) = 0$, we can find a $\delta > 0$ so that $\Psi(p) < \varepsilon$ if $0 < p < \delta$. For such a p, we let Y^*_p be the process satisfying (4.13). In addition, we introduce two processes Y_0 and Y_ε as follows:

$$Y_0(t) = y + \mu t + \sigma W(t) + L_0(t), \quad t \ge 0, \tag{C.4}$$

$$Y_\varepsilon(t) = y + (\mu+\varepsilon)t + \sigma W(t) + L_\varepsilon(t), \quad t \ge 0, \tag{C.5}$$

where L_0 and L_ε are the local time processes of Y_0 and Y_ε, respectively. Also, we have

$$\begin{aligned}Y_0(t) - Y^*_p(t) &= -\int_0^t u^*_p(Y^*_p(s))\,\mathrm{d}s + L_0(t) - L^*_p(t),\\ Y^*_p(t) - Y_\varepsilon(t) &= -\int_0^t [\varepsilon - u^*_p(Y^*_p(s))]\,\mathrm{d}s + L^*_p(t) - L_\varepsilon(t).\end{aligned}$$

Note that for all $y \ge 0$, we have $0 \le u^*(y) = \Psi(-W_p(y)) \le \Psi(p) < \varepsilon$. Now we can use almost the same computations in Step 2 to derive that $Y_0(t) \le Y^*_p(t) \le Y_\varepsilon(t)$ and $L_0(t) \ge L^*_p(t) \ge L_\varepsilon(t)$ for all $t \ge 0$ almost surely. Consequently,

$$\begin{aligned}l_0(y) &= \alpha\mathbb{E}\left[\int_0^\infty e^{-\alpha t}L_0(t)\,\mathrm{d}t\right] \ge \alpha\mathbb{E}\left[\int_0^\infty e^{-\alpha t}L^*_p(t)\,\mathrm{d}t\right]\\ &= \Gamma(p,y) \ge \alpha\mathbb{E}\left[\int_0^\infty e^{-\alpha t}L_\varepsilon(t)\,\mathrm{d}t\right].\end{aligned}$$

On the other hand, by (4.10),

$$\begin{aligned}l_\varepsilon(y) &= \mathbb{E}\left[\int_0^\infty e^{-\alpha t}\,\mathrm{d}L_\varepsilon(t)\right] = \alpha\mathbb{E}\left[\int_0^\infty e^{-\alpha t}L_\varepsilon(t)\,\mathrm{d}t\right]\\ &= \frac{\sigma^2}{(\mu+\varepsilon)+\sqrt{(\mu+\varepsilon)^2+2\alpha\sigma^2}}\exp\left\{-\frac{(\mu+\varepsilon)+\sqrt{(\mu+\varepsilon)^2+2\alpha\sigma^2}}{\sigma^2}y\right\}.\end{aligned}$$

It is obvious that $l_\varepsilon(y) \to l_0(y)$ as $\varepsilon \to 0$. Therefore, $\lim_{p\to 0}\Gamma(p,y) = l_0(y)$ as desired. This completes the proof. □

Appendix D. Proof of Lemma 5.11

Proof of Lemma 5.11. First, we introduce the function $h : [0, \infty) \to \mathbb{R}$ by $h(x) = \theta x - \Phi(x)$ for $x \geq 0$. Then h is a differentiable concave function and has its unique maximum, say, at $x = r_\theta$. Moreover, from (3.3) and (5.24), we have

$$h(r_\theta) = \theta r_\theta - \Phi(r_\theta) = \theta r_\theta - r_\theta \Psi(r_\theta) + c(\Psi(r_\theta)) = \theta r_\theta - r_\theta \theta + c(\theta) = c(\theta).$$

Using this fact and the definition of the function h, we can rewrite (5.25) in the form

$$\begin{cases} \dfrac{\sigma^2}{2} W_\theta'(x) + h(-W_\theta(x)) = h(r_\theta) \\ W_\theta(0) = -p. \end{cases} \tag{D.1}$$

Since the nonlinear function h is continuously differentiable, we can guarantee a solution in an interval $[0, \omega)$, where ω is the explosion time. We observe that $\frac{\sigma^2}{2} W_\theta'(0+) = h(r_\theta) - h(p) > 0$ and hence W_θ is strictly increasing on an interval $[0, \varepsilon)$ for some $\varepsilon > 0$. Let $\xi := \inf\{x > 0 : W_\theta'(x) = 0\}$ and suppose that ξ is finite. Then $W_\theta'(\xi) = 0$ and by (D.1), $W_\theta(\xi) = -r_\theta$ since h has a unique maximum as it is a concave function. Notice that $Y(x) \equiv -r_\theta$ is a constant solution to the differential equation in (D.1) in a neighborhood of ξ (with initial value $Y(\xi) = -r_\theta$). Since h is a continuously differentiable function, uniqueness of the solution for (D.1) holds and thus, $W_\theta(x) \equiv -r_\theta$ for all x. This is a contradiction since $W_\theta(0) = -p$. Consequently, ξ is infinite. Hence, $W_\theta'(x) > 0$ for all $0 < x < \omega$, and $W_\theta(\cdot)$ is strictly increasing.

Next we claim that $W_\theta(x) < -r_\theta$ for all x. Suppose not. Then there exists a ξ with $W_\theta(\xi) = -r_\theta$ and by (D.1), $W_\theta'(\xi) = 0$. Now again we have a contradiction as argued in the previous paragraph. Hence $-p \leq W_\theta(x) < -r_\theta$ holds for all $0 \leq x < \omega$. Since $W_\theta(x)$ is bounded, (D.1) yields that W_θ' is also bounded and consequently the explosion time ω is infinite. Finally, by integrating (D.1), we obtain $0 < \int_0^x [h(r_\theta) - h(-W_\theta(u))] \, du = \frac{\sigma^2}{2}(W_\theta(x) + p) \leq \frac{\sigma^2}{2}(p - r_\theta)$. Since $h(r_\theta) > h(-W_\theta(x))$, the integral $\int_0^\infty [h(r_\theta) - h(-W_\theta(u))] \, du$ is convergent. Consequently, $\lim_{x\to\infty} h(-W_\theta(x)) = h(r_\theta)$. Since $W_\theta(\cdot)$ is strictly increasing and bounded on $[0, \infty)$, we can conclude that $\lim_{x\to\infty} W_\theta(x) = -r_\theta$. This completes the proof. □

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