Solvability of Dirichlet problem with Nonlinear Integro-differential Operator

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Abstract

This paper studies the solvability of a class of Dirichlet problem associated with a non-linear integro-differential operator. The main ingredient is the use of Perron's method together with the probabilistic construction of continuous supersolution via the identification of the continuity set of the exit time operators in the path space under Skorohod topology.

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1 Introduction and problem setup

1.1 Problem setup

Consider an equation of the form

$$F(u,x) + u(x) - \ell(x) = 0, \ x \in O \tag{1}$$

with the boundary value

$$u(x) = g(x), \ x \in O^c. \tag{2}$$

In the above, the operator

$$F(u,x) = -\inf_{a\in[\underline{a},\overline{a}]} H(u,x,a) - \mathcal{I}(u,x)$$

is defined via operators given by

$$\mathcal{I}(u,x) = \int_{\mathbb{R}^d} (u(x+y) - u(x) - Du(x)\cdot y I_{B_1}(y))\nu(dy) \ ;$$
$$H(u,x,a) = \frac{1}{2} tr(A(a)D^2u(x)) + b(a)\cdot Du(x) \text{ with } A(a) = \sigma'(a)\sigma(a).$$

In the above, $\underline{a} \leq \overline{a}$ are given two real numbers, $\nu(\cdot)$ is a Levy measure on $\mathbb{R}^d$, $B_r(x)$ is a ball of radius r with center x, and $B_r = B_r(0)$ for simplicity. Recall that, we say ν is a Levy measure, if $\int_{\mathbb{R}^d}(1 \wedge |y|^2)\nu(dy) < \infty$ holds. To simplify our presentation, we will use the following additional assumptions throughout the paper.

Assumption 1 *1. O is a connected open bounded set in $\mathbb{R}^d$.*

2. $\sigma, b \in C^{0,1}(\mathbb{R})$; $\ell, g \in C_0(\mathbb{R}^d)$.

3. $\nu(dy) = \hat{\nu}(y)dy$ is a Levy measure satisfying $\hat{\nu} \in C_0(\mathbb{R}^d \setminus \{0\})$.

For some $\alpha \in (0,2)$, if ν is given by

$$\nu(dy) = \frac{dy}{|y|^{d+\alpha}},$$

then ν satisfies Assumption 1, and the integral operator is denoted by $\mathcal{I}(u,x) = -(-\Delta)^{\alpha/2}u(x)$ as convention. For convenience, we write $-(-\Delta)^0 u = 0$.

1.2 Literature review and an example

A function u is said to be a solution of Dirichlet problem (1)-(2), if $u \in C(\bar{O})$ satisfies (1) in the viscosity sense and $u = g$ on ∂O. It is worth to note that, as far as Dirichlet problem (1)-(2) concerned, one can generalize the boundary condition (2) by

$$\max\{F(u,x) + u(x) - \ell(x), u - g\} \geq 0 \geq \min\{F(u,x) + u(x) - \ell(x), u - g\} \text{ on } O^c \tag{3}$$

without loss of uniqueness in the viscosity sense.

In contrast to the (classical) Dirichlet problem (1)-(2), Dirichlet problem (1)-(3) is referred to a generalized Dirichlet problem. For the generalized Dirichlet problem without nonlocal operator, there were many excellent discussions on the solvability with the comparison principle and Perron's method, see for instance, [5], [6], [3], and Section 7 of [12]. Recently, the solvability result has been extended to nonlinear equations associated to Integro-differential operators, see [7], [4], [1], [18], and the references therein.

Compared to the generalized Dirichlet problem, there are relatively less discussions available on the classical Dirichlet problem associated with the Integral operators in the aforementioned references. For the illustration purpose, we will use the following example, which will be used throughout the paper.

Example 1 *Justify the the uniqueness and existence of the viscosity solution for Dirichlet problem given by, with* $\alpha \in [0, 2]$

$$|\partial_{x_1} u| + (-\Delta)^{\alpha/2} u + u - 1 = 0, \ \forall x \in O = (-1, 1) \times (-1, 1) \tag{4}$$

with

$$u(x) = 0, \ \forall x \in O^c.$$

A partial answer of Example 1 from the existing literature is given as this below:

- If $\alpha = 0$, there is no solution. In fact, one can directly check that $u(x) = 1 - e^{-1+|x_1|}$ is the unique solution of the generalized Dirichlet problem, but not a solution of classical Dirichlet problem due to its loss of boundary at $\{(x_1, x_2) : |x_2| = 1, |x_1| < 1\}$.
- If $\alpha \in [1, 2]$, there is a unique solution by [4].
- If $\alpha \in (0, 1)$, although there is unique solution of generalized Dirichlet problem by [18], it is remained unanswered whether there is a solution of classical Dirichlet problem.

1.3 Work outline

This work focuses on the sufficient condition of the existence and uniqueness of the viscosity solution for Dirichlet problem of (1)-(2). Formally, the solution of (1)-(2), if it exists, is expected to be equal to the value function of a stochastic exit control problem, see for instance [14]. However, a rigorous proof on the equivalence between the solution of (1)-(2) and the value function associated to exit control problem is not an easy task due to the lack of dynamic programming principle, see more discussions in [8] for Hamilton-Jacobi-Bellman equation without non-local operator.

Alternatively, our approach here is to construct the subsolution and supersolution, and then the unique solvability follows by the comparison principle and Perron's method. The comparison principle and Perron's method are already available in [7]. In this connection, Section 3 establishes the main result in Theorem 17 by constructing subsolution and supersolution using a particular controlled process and boundary data g. Since the subsolution and supersolution of Section 3 are given in the sense of Definition 2, while the comparison principle and Perron's method are proved under Definition 3, we show their equivalence of two definitions in Section 2.

Finally, we emphasize that the proof of Theorem 17 relies on the continuity of the value function of the exit problem, whose proof is separated in Section 4. In general, due to the non-local property, continuity of the value function up to a stopping time is much more delicate than the counterpart of the purely differential form. Indeed, by investigating the continuity of the exit mappings on path space under Skorohod metric, we conclude that the regularity of the boundary guarantees the continuity of the value function. This part is crucial for the main result, and as far as the solvability of Dirichlet problem concerned, the methodology is original to the best of our knowledge.

The contribution of this work is therefore the sufficient condition on the existence and uniqueness of the solution for (1)-(2) provided by Theorem 17. In particular, the sufficient condition in Theorem

17 requires the regularity of the boundary with respect to some controlled process. When Dirichlet problem is given by purely differential elliptic operators with C^2-smooth boundary, our result is consistent to Example 4.6 of [12] and [3]. Nevertheless, it is also useful for Dirichlet problem when the regularity of the boundary is known for a nonsmooth domain. Back to Example 1, one can easily show that existence and uniqueness holds for any constant $\alpha \in (0, 2]$ by Theorem 17, see more explanations in Example 18. It is also noted that existence and uniqueness still holds for $\alpha \in (0, 2]$ as long as the boundary satisfies exterior cone condition, and this is also a new result.

2 Two equivalent definitions for the viscosity property

In this section, we first give two different definitions of viscosity subsolution and supersolution properties, Definition 2 and Definition 3 respectively. Definition 2 involves only with C^2 smooth test functions, which will be used later in Section 3 to verify the supersolution property of a certain value function associated to some exit control problem. Compared to Definition 2, Definition 3 is given with more test functions including non-smooth functions, and it's much harder to be used directly in Section 3 to verify viscosity solution property. However, Definition 3 of this paper is exactly Definition 2 of [7], where it was used to provide the proof of comparison principle and Perron's method.

In this connection, we shall prove the equivalence of Definition 3 and Definition 2. Note that viscosity property of Definition 3 automatically implies the viscosity property of Definition 2, since the test functions for the viscosity subsolution (resp supersolution) property of Definition 3 includes the test function space $J^+(u, x)$ (resp. $J^-(u, x)$) adopted for Definition 2.

On the other hand, to show that the Definition 2 implies Definition 3, we first define the closure $\bar{J}^+(u, x)$ (resp. $\bar{J}^-(u, x)$) of test function space $J^+(u, x)$ (resp. $J^-(u, x)$), see Definition 4. Note that it is exactly the closure of semijet defined in [12] in the case with only differential operator, and it is designed in that way to make (5) (resp. (6)) true for all $\phi \in \bar{J}^+(u, x)$ (resp. $\phi \in \bar{J}^-(u, x)$) whenever u is a subsolution (resp. supersolution) under Definition 2. In other words, it is enough to show that any test function adopted for subsolution (resp. supersolution) of Definition 3 belongs to $\bar{J}^+(u, x)$ (resp. $\bar{J}^-(u, x)$).

2.1 Two different definitions of viscosity properties

Definition 2 below is consistent to the Definition 1 of [7], which will be used to establish the existence of the solution in Section 3. To proceed, for a function $u : \bar{O} \mapsto \mathbb{R}$, we define its extension by

$$u^g = (uI_{\bar{O}} + gI_{\bar{O}^c})^*, \quad u_g = (uI_{\bar{O}} + gI_{\bar{O}^c})_*,$$

where f^* and f_* stand for USC (upper semicontinuous) and LSC (Lower semicontinuous) envelopes of the function f, respectively. We also define the supertest function space, for $u \in USC$ and $x \in \mathbb{R}^d$

$$J^+(u, x) = \{\phi \in C_b^\infty(\mathbb{R}^d), \text{ such that } \phi \geq u^g \text{ and } \phi(x) = u(x)\}.$$

Analogously, the subtest function space is given by, for $u \in LSC$ and $x \in \mathbb{R}^d$

$$J^-(u, x) = \{\phi \in C_b^\infty(\mathbb{R}^d), \text{ such that } \phi \leq u_g \text{ and } \phi(x) = u(x)\}.$$

Definition 2 *1. We say a function $u \in USC(\bar{O})$ satisfies the viscosity subsolution property at $x \in O$, if the following inequality holds,*

$$F(\phi, x) + u(x) - \ell(x) \leq 0, \ \forall \phi \in J^+(u, x). \tag{5}$$

2. *We say a function $u \in LSC(\bar{O})$ satisfies the viscosity supersolution property at $x \in O$, if the following inequality holds for all $\phi \in J^-(u, x)$,*

$$F(\phi, x) + u(x) - \ell(x) \geq 0, \ \forall \phi \in J^-(u, x). \tag{6}$$

Next, we observe that $F(\phi, x) + u(x) - \ell(x)$ of (5) and (6) could be well defined for a function only C^∞-smooth at some neighborhood of x. Indeed, for an arbitrary $x \in \mathbb{R}^d$, if we define a function space C_x by

$$C_x = \{\phi : \exists \hat{r} > 0, \ \phi_1 \in C^\infty, \ \phi_2 \in L^1, \text{ s.t. } \phi = \phi_1 I_{\bar{B}_{\hat{r}}(x)} + \phi_2(1 - I_{\bar{B}_{\hat{r}}(x)})\}, \tag{7}$$

one can directly verify that $\phi \mapsto \mathcal{I}(\phi, x)$ is well defined for $\phi \in C_x$, with a property

$$\mathcal{I}(\phi, x) = b_r \cdot D\phi(x) + \mathcal{I}_{r,1}(\phi, x) + \mathcal{I}_{r,2}(\phi, x), \ \forall r > 0 \tag{8}$$

where

1. $b_r = \int_{B_1 \setminus B_r} y \nu(dy)$.
2. $\mathcal{I}_{r,1}(\phi, x) = \int_{B_r} (\phi(x+y) - \phi(x) - D\phi(x) \cdot y) \nu(dy)$
3. $\mathcal{I}_{r,2}(\phi, x) = \int_{\mathbb{R} \setminus B_r} (\phi(x+y) - \phi(x)) \nu(dy)$.

In the above, $\int_{B_1 \setminus B_r}$ for $r > 1$ is understood as $-\int_{B_r \setminus B_1}$. Note that, r in (8) could be larger than $\hat{r}$ of (7), and this observation allows us to use more test functions from C^∞ to C_x compared to Definition 2. In this below, Definition 3 is consistent to Definition 2 of [7], see also other relevant definitions in [11] and [18].

Definition 3 *1. We say a function $u \in USC(\bar{O})$ satisfies the viscosity subsolution property at $x \in O$, if for all $\phi \in C_x$ with (1) $\phi(x) = u(x)$; (2) $\phi - u \geq 0$ on $\bar{O}$, satisfies*

$$-b_r \cdot D\phi(x) - \mathcal{I}_{r,1}(\phi, x) - \mathcal{I}_{r,2}(u^g, x) - \inf_{a \in [\underline{a}, \overline{a}]} H(\phi, x, a) + u(x) - \ell(x) \leq 0, \ \forall r > 0, \tag{9}$$

2. *We say a function $u \in LSC(\bar{O})$ satisfies the viscosity supersolution property at $x \in O$, if for all $\phi \in C_x$ with (1) $\phi(x) = u(x)$; (2) $\phi - u \leq 0$ on $\bar{O}$*

$$-b_r \cdot D\phi(x) - \mathcal{I}_{r,1}(\phi, x) - \mathcal{I}_{r,2}(u_g, x) - \inf_{a \in [\underline{a}, \overline{a}]} H(\phi, x, a) + u(x) - \ell(x) \geq 0, \ \forall r > 0. \tag{10}$$

2.2 Closure of the test function space

Next, we shall define the closure of test function space $J^\pm(u, x)$ in the sense of non-local version of closure of semijets of [12], and provide the sufficient condition for a function ϕ to be in the closure $\bar{J}^\pm(u, x)$.

Definition 4 *A set $\bar{J}^+(u, x)$* (respectively $\bar{J}^-(u, x)$) *is given by all functions $\phi \in C_x$ satisfying the following conditions:*

There exists $x_\epsilon \to x$ and $\phi_\epsilon \in J^+(u, x_\epsilon)$ (respectively $\phi_\epsilon \in J^-(u, x_\epsilon)$) *satisfying*

$$(x_\epsilon, \phi_\epsilon(x_\epsilon), D\phi_\epsilon(x_\epsilon), D^2\phi_\epsilon(x_\epsilon), \mathcal{I}(\phi_\epsilon, x_\epsilon)) \to (x, \phi(x), D\phi(x), D^2\phi(x), \mathcal{I}(\phi, x)).$$

For notational simplicity, we define a shifted Levy measure ν_x by $\nu_x(dy) = \hat{\nu}(y-x)dy$ for any $x \in \mathbb{R}^d$. Accordingly, we say $\phi \in L^1(\nu_x, B)$ for some Lebesgue measurable set B of $\mathbb{R}^d$, if $\int_B |\phi(y)|\nu_x(dy) < \infty$ is well defined.

Lemma 5 *For a given $x \in \mathbb{R}^d$ and $\phi \in C_x$, if there exists $\{(\phi_\epsilon, x_\epsilon) : \epsilon > 0\}$ and $r > 0$ such that*

1. $\lim_\epsilon x_\epsilon = x$;
2. $\phi_\epsilon \in C^\infty(B_{2r}(x))$ *such that* $\|\phi_\epsilon - \phi\|_{W^{2,\infty}(B_r(x))} \to 0$ *as* $\epsilon \to 0$;
3. $\exists \hat{\phi} \in L^1(\nu_x, B_r^c(x))$ *such that* $|\phi_\epsilon| \leq \hat{\phi}$ *and* $\lim_{\epsilon\to 0} \|\phi_\epsilon - \phi\|_{L^1(\nu_x, B_r^c(x))} = 0$;

Then, we have,

$$\mathcal{I}_{r,1}(\phi_\epsilon, x_\epsilon) \to \mathcal{I}_{r,1}(\phi, x), \text{ and } \mathcal{I}_{r,2}(\phi_\epsilon, x_\epsilon) \to \mathcal{I}_{r,2}(\phi, x), \quad \text{as } \epsilon \to 0^+.$$

Proof: Without loss of generality, we assume r is small enough such that $\phi \in C^\infty(B_{2r}(x))$. For an arbitrary ϵ satisfying $|x_\epsilon - x| < r/3$, using f_ϵ defined by

$$f_\epsilon(y) = \phi_\epsilon(x_\epsilon + y) - \phi(x + y),$$

we can write the following inequalities:

$$\Big|\mathcal{I}_{r,1}(\phi_\epsilon, x_\epsilon) - \mathcal{I}_{r,1}(\phi, x)\Big| = \Big| \int_{B_r} (f_\epsilon(y) - f_\epsilon(0) - Df_\epsilon(0)\cdot y)\nu(dy)\Big| \leq \frac{1}{2}\|D^2 f_\epsilon\|_{L^\infty(\bar{B}_r)} \int_{B_r} |y|^2\nu(dy).$$

Note that $x_\epsilon + y \in \bar{B}_r(x)$ whenever $y \in B_r$.

- Since $D^2\phi_\epsilon \to D^2\phi$ holds uniformly in $B_r(x)$, we have
$$\sup_{y\in B_r} |D^2\phi_\epsilon(x_\epsilon + y) - D^2\phi(x_\epsilon + y)| \to 0^+;$$
- $\phi \in C^\infty(B_{2r})$ implies that $D^2\phi$ is uniformly continuous in B_r and
$$\sup_{y\in B_r} |D^2\phi(x_\epsilon + y) - D^2\phi(x_\epsilon + y)| \to 0^+;$$

we conclude that $\frac{1}{2}\|D^2 f_\epsilon\|_{L^\infty(\bar{B}_r)} \to 0$ and $\mathcal{I}_{r,1}(\phi_\epsilon, x_\epsilon) \to \mathcal{I}_{r,1}(\phi, x)$ as $\epsilon \to 0^+$.

Next, we write

$$|\mathcal{I}_{r,2}(\phi_\epsilon, x_\epsilon) - \mathcal{I}_{r,2}(\phi, x)| \leq TERM1 + TERM2 + TERM3,$$

where three terms are followed by

1. Due to the property of Levy measure, it yields $\nu(B_r^c) < \infty$, and uniform convergence of ϕ_ϵ on $B_{2r}(x)$ leads to
$$TERM1 = \Big| \int_{B_r^c} (\phi_\epsilon(x_\epsilon) - \phi(x))\nu(dy)\Big| = |\phi_\epsilon(x_\epsilon) - \phi(x)|\nu(B_r^c) \to 0, \text{ as } \epsilon \to 0^+;$$
2. Since $\hat{\nu} \in C_b(B_r^c)$, we have
$$TERM2 = \Big| \int_{B_r^c} (\phi_\epsilon - \phi)(x + y)\nu(dy)\Big| \leq \|\phi_\epsilon - \phi\|_{L^1(\nu_x, B_r^c(x))} \to 0, \text{ as } \epsilon \to 0^+;$$

3. One can write

$$\begin{aligned} TERM3 &= \Big| \int_{B_r^c} (\phi_\epsilon(x_\epsilon + y) - \phi_\epsilon(x+y))\nu(dy) \Big| \\ &= \Big| \int_{B_r^c(x_\epsilon)} \phi_\epsilon(z)\hat\nu(z - x_\epsilon) dz - \int_{B_r^c(x)} \phi_\epsilon(z)\hat\nu(z-x)dz \Big| \\ &\le TERM31 + TERM32 + TERM33 \end{aligned}$$

where $TERM3$ is again divided by three terms as such:

- Since $|\phi_\epsilon| \le \hat\phi \in L^1(\nu_x, B_r^c(x))$, $\hat\nu \in C_b(B_r^c)$ and $|z - x_\epsilon| \wedge |z - x| \ge r$, one can use Dominated Convergence Theorem to conclude that

$$TERM31 = \int_{B_r^c(x_\epsilon)\cap B_r^c(x)} |\phi_\epsilon(z)(\hat\nu(z - x_\epsilon) - \hat\nu(z-x))| dz \to 0$$

as $\epsilon \to 0$;

- Note that $x_\epsilon + y \in B_r(x)$ whenever $y \in B_r^c \cap B_r(x - x_\epsilon)$. Together with $\|\phi_\epsilon\|_{L^\infty(B_r(x))} \to \|\phi\|_{L^\infty(B_r(x))}$ as $\epsilon \to 0$ due to the uniform convergence on $B_{2r}(x)$, it yields

$$\begin{aligned} TERM32 &= \int_{B_r^c(x_\epsilon)\cap B_r(x)} |\phi_\epsilon(z)|\hat\nu(z - x_\epsilon)dz \\ &= \int_{B_r^c \cap B_r(x - x_\epsilon)} |\phi_\epsilon(x_\epsilon + y)|\hat\nu(y)dy \\ &\le \|\phi_\epsilon\|_{L^\infty(B_r(x))}\nu(B_r^c \cap B_r(x - x_\epsilon)) \to 0, \text{ as } \epsilon \to 0^+; \end{aligned}$$

- Similarly, we have $x + y \in B_r(x_\epsilon) \subset B_{4r/3}(x)$ whenever $y \in B_r^c \cap B_r(x_\epsilon - x)$. Thus, we have

$$\|\phi_\epsilon\|_{L^\infty(B_r(x_\epsilon))} \le \|\phi_\epsilon\|_{L^\infty(B_{2r}(x))} \to \|\phi\|_{L^\infty(B_{2r}(x))} \quad \text{as } \epsilon \to 0$$

due to the uniform convergence on $B_{2r}(x)$, and it yields

$$\begin{aligned} TERM33 &= \int_{B_r(x_\epsilon)\cap B_r^c(x)} |\phi_\epsilon(z)|\hat\nu(z-x)dz \\ &= \int_{B_r(x_\epsilon - x)\cap B_r^c} |\phi_\epsilon(x+y)|\hat\nu(y)dy \\ &\le \|\phi_\epsilon\|_{L^\infty(B_r(x_\epsilon))}\nu(B_r(x_\epsilon - x)\cap B_r^c) \\ &\le \|\phi_\epsilon\|_{L^\infty(B_{2r}(x))}\nu(B_r(x_\epsilon - x)\cap B_r^c) \to 0 \text{ as } \epsilon \to 0^+; \end{aligned}$$

Therefore, $TERM3$ is also converging to zero as ϵ goes to zero.

This completes the proof of $|\mathcal{I}_{r,2}(\phi_\epsilon, x_\epsilon) - \mathcal{I}_{r,2}(\phi, x)| \to 0$.

□

Now we can simplify the statement of Lemma 5 for the convenience of the later use.

Proposition 6 *For a given $x \in \mathbb{R}^d$ and $\phi \in C_x$, if there exists $\{(\phi_\epsilon, x_\epsilon) : \epsilon > 0\}$ and $r > 0$ such that*

1. $\lim_\epsilon x_\epsilon = x$;
2. $\phi_\epsilon \in C^\infty(B_r(x))$ *such that* $\|\phi_\epsilon - \phi\|_{W^{2,\infty}(B_r(x))} \to 0$ *as* $\epsilon \to 0$;
3. $\exists \hat\phi \in L^1(\nu_x, B_r^c(x))$ *such that* $|\phi_\epsilon| \le \hat\phi$ *and* $\lim_{\epsilon\to 0} \|\phi_\epsilon - \phi\|_{L^1(\nu_x, B_r^c(x))} = 0$;

Then, we have, for any $\hat{r} > 0$

$$\mathcal{I}_{\hat{r},1}(\phi_\epsilon, x_\epsilon) \to \mathcal{I}_{\hat{r},1}(\phi, x), \text{ and } \mathcal{I}_{\hat{r},2}(\phi_\epsilon, x_\epsilon) \to \mathcal{I}_{\hat{r},2}(\phi, x), \quad \text{as } \epsilon \to 0^+. \tag{11}$$

Proof: Let $\hat{r} = r/2$, then $(\phi_\epsilon, x_\epsilon)$ satisfies all conditions of Lemma 5 by switching r by $\hat{r}$ and $\hat{\phi}$ by $\hat{\phi} I_{B^c_r(x)} + (|\phi| + 1) I_{\overline{B}_r(x)}$. Therefore, the conclusion (11) holds for $\hat{r} = r/2$. Together with (8), we have

$$\mathcal{I}(\phi_\epsilon, x_\epsilon) := \mathcal{I}(\phi_\epsilon, x_\epsilon; \nu) \to \mathcal{I}(\phi, x) := \mathcal{I}(\phi, x; \nu).$$

This convergence is always valid for all ν, and we apply this convergence to $I_{B_{\hat{r}}}(y)\nu(dy)$, which yields

$$\forall \hat{r} > 0, \ \mathcal{I}_{\hat{r},2}(\phi_\epsilon, x_\epsilon) \to \mathcal{I}_{\hat{r},2}(\phi, x), \quad \text{as } \epsilon \to 0^+.$$

This in turn implies, due to (8)

$$\forall \hat{r} > 0, \ \mathcal{I}_{\hat{r},1}(\phi_\epsilon, x_\epsilon) \to \mathcal{I}_{\hat{r},1}(\phi, x), \quad \text{as } \epsilon \to 0^+.$$

□

We will give a sufficient condition for $\phi \in \bar{J}^{\pm} u(x)$ in this below.

Proposition 7 1. *For a given $x \in \mathbb{R}^d, \phi \in C_x$ and $u \in USC(\mathbb{R}^d)$, if there exists*

$$\{(\phi_\epsilon, x_\epsilon) : \phi_\epsilon \in J^+(u, x_\epsilon), \ \epsilon > 0\}$$

satisfying all conditions in Proposition 6, then we have $\phi \in \bar{J}^+(u, x)$.

2. *For a given $x \in \mathbb{R}^d, \phi \in C_x$ and $u \in LSC(\mathbb{R}^d)$, if there exists*

$$\{(\phi_\epsilon, x_\epsilon) : \phi_\epsilon \in J^-(u, x_\epsilon), \ \epsilon > 0\}$$

satisfying all conditions in Proposition 6, then we have $\phi \in \bar{J}^-(u, x)$.

Proof: L^1-convergence implies, with a subsequence, $\phi_\epsilon \to \phi$ pointwisely, and so $\phi \geq u$. Uniform convergence in $B_r(x)$ also implies that

$$(x_\epsilon, \phi_\epsilon(x_\epsilon), D\phi_\epsilon(x_\epsilon), D^2\phi_\epsilon(x_\epsilon)) \to (x, \phi(x), D\phi(x), D^2\phi(x)).$$

Moreover, $\phi(x) = u(x)$ holds by the facts of $\phi_\epsilon \in J^+(u, x_\epsilon)$ and upper semicontinuity of u, i.e.

$$\phi(x) = \lim_\epsilon \phi_\epsilon(x_\epsilon) = \limsup_\epsilon \phi_\epsilon(x_\epsilon) = \limsup_\epsilon u(x_\epsilon) = u(x).$$

In view of the relation of (8) and Proposition 6, we also have $\mathcal{I}(\phi_\epsilon, x_\epsilon) \to \mathcal{I}(\phi, x)$ and $\phi \in \bar{J}^+(u, x)$. Similarly, we can show $\phi \in \bar{J}^-(u, x)$. □

Finally, we present the continuity of $(\phi, \cdot)$, which will be later used several times.

Proposition 8 *For a given $x \in \mathbb{R}^d$ and $\phi \in C_x$, the mapping $\mathcal{I}(\phi, \cdot)$ is continuous at x.*

Proof:

If $x_\epsilon \to x$, then we can take $\phi_\epsilon = \phi$ and apply Proposition 6 and the relation of (8) to conclude the result.

□

2.3 Proof of equivalence between two definitions

Proposition 9 *Definition 2 is equivalent to Definition 3.*

Proof: If u is a subsolution of Definition 3, then it automatically satisfies subsolution properties of Definition 2. In the reverse direction, in view of Assumption 1 (2), we shall show that, arbitrary $\phi \in J^+(u,x)$ and $r > 0$ implies that

$$w := \phi I_{\bar{B}_r(x)} + u^g I_{\bar{B}_r^c(x)} \in \bar{J}^+(u,x).$$

In the rest of the proof, we fix $x \in O$ and $r = \frac{1}{2} dist(x, \partial O)$. According to Proposition 7, we shall construct $\{\phi_\epsilon \in J^+(u, x_\epsilon) : \epsilon > 0\}$ satisfying all conditions of Proposition 6. We establish this in the following steps: with restriction on $\epsilon \in (0, 1 \wedge \frac{r^4}{4})$,

1. Set $\hat{\phi}(y) = \phi(y) + \sqrt{\epsilon}|y - x|^2$. Note that

$$\|\hat{\phi} - w\|_{W^{2,\infty}(B_r(x))} \leq \sqrt{\epsilon}(r^2 + 2rd + 2d). \tag{12}$$

2. Let

$$w_1(y) = \hat{\phi}(y) I_{\bar{B}_r(x)}(y) + (\epsilon + u^g(y)) I_{\bar{B}_r^c(x)}(y),$$

then $w_1 \in USC$ due to $\hat{\phi} > u^g$ on $\partial B_r(x)$. Also, we have

$$w_1 = \hat{\phi} \ \text{ on } B_r; \quad \|w_1 - w\|_{L^1(\nu_x, B_r^c(x))} \leq \epsilon \ \nu(B_r^c). \tag{13}$$

3. Next, w_2 is chosen from the continuous functions dominating w_1 from its above, and sufficiently close to w_1 in the following sense. Let $\mathcal{C}_2$ be

$$\mathcal{C}_2 = \{\bar{w} : \bar{w} - \epsilon \in C_0(\mathbb{R}^d); \ \bar{w} \geq w_1 \text{ on } \mathbb{R}^d; \ \bar{w} = w_1 \text{ on } \bar{B}_r(x)\}.$$

Since $w_1 \in USC(\mathbb{R}^d)$, $w_1(y) = g(y) + \epsilon$ for $y \notin O$, and $g \in C_0$, the set $\mathcal{C}_2$ is not empty. If we let $\bar{w}$ run over all such functions, then $\inf_{\bar{w} \in \mathcal{C}_2} (\bar{w} - w)(x) = 0$ for all $x \in B_r^c(x)$. Then, we can apply the monotone convergence theorem to have

$$\inf_{\bar{w} \in \mathcal{C}_2} \|\bar{w} - w_1\|_{L^1(\nu_x, B_r^c(x))} = 0.$$

Therefore, we can take $w_2 \in \mathcal{C}_2$

$$w_2 = w_1 \text{ on } B_r(x), \quad \|w_2 - w_1\|_{L^1(\nu_x, B_r^c(x))} \leq \epsilon. \tag{14}$$

4. $w_3 = \eta_{\epsilon'} * w_2$ is the convolution with a mollifier (see Appendix C.4 of [13]) of radius $\epsilon' = \epsilon'(\epsilon)$, satisfying

$$w_3 \in C_b^\infty(\mathbb{R}^d); \ \|w_3 - w_2\|_\infty \leq \frac{1}{4}\epsilon; \ \text{ and } \|w_3 - w_2\|_{W^{2,\infty}(B_{r/2}(x))} \leq \sqrt{\epsilon}. \tag{15}$$

Indeed, $w_2 - \epsilon \in C_0(\mathbb{R}^d)$ ensures that

$$\text{As } \epsilon' \to 0, \ w_3 = \eta_{\epsilon'} * w_2 = \eta_{\epsilon'} * (w_2 - \epsilon) + \epsilon \to w_2 \ \text{ uniformly on } \mathbb{R}^d.$$

Moreover, due to $w_2 \in C^\infty(B_r(x))$, for any $\epsilon' < r/2$ and $y \in B_{r/2}(x)$, we have $\partial_{x_i} w_3 = \eta_{\epsilon'} * \partial_{x_i} w_2$ and $\partial_{x_i x_j} w_3 = \eta_{\epsilon'} * \partial_{x_i x_j} w_2$. This implies that

$$\text{As } \epsilon' \to 0, \ (Dw_3, D^2 w_3) \to (Dw_2, D^2 w_2), \ \text{ uniformly on } B_{r/2}(x).$$

This explains the existence of ϵ' satisfying (15). In addition, it also implies that

$$\|w_3 - w_2\|_{L^1(\nu_x, B_r^c(x))} \le \frac{1}{4}\epsilon\ \nu(B_r^c(x)). \tag{16}$$

Moreover, we have, for any $y \in \mathbb{R}^d$

$$\begin{aligned} w_3(y) &\ge w_2(y) - \tfrac{1}{4}\epsilon \ge w_1(y) - \tfrac{1}{4}\epsilon \\ &\ge (\phi(y) + \sqrt{\epsilon}|y-x|^2 - \frac{1}{4}\epsilon) I_{\bar{B}_r(x)}(y) + (\frac{3}{4}\epsilon + u^g) I_{\bar{B}_r^c(x)}(y). \end{aligned} \tag{17}$$

5. Since u^g is USC, there exists x_ϵ at which $u^g - w_3$ attains maximum over $\bar{B}_r(x)$. We denote

$$x_\epsilon \in \arg\max_{\bar{B}_r(x)} (u^g - w_3), \text{ and } \phi_\epsilon = w_3 + (u^g - w_3)(x_\epsilon).$$

We observe the following two useful estimations:

$$(u_g - w_3)(x_\epsilon) \ge (u_g - w_3)(x) \ge (u_g - w_2)(x) - \frac{1}{4}\epsilon = (u_g - \hat{\phi})(x) - \frac{1}{4}\epsilon = -\frac{1}{4}\epsilon, \tag{18}$$

and

$$(u_g - w_3)(x_\epsilon) \le (u_g - w_2)(x_\epsilon) + \frac{1}{4}\epsilon \le (u_g - \hat{\phi})(x_\epsilon) + \frac{1}{4}\epsilon \le -\sqrt{\epsilon}|x_\epsilon - x|^2 + \frac{1}{4}\epsilon. \tag{19}$$

Next, we shall verify that ϕ_ϵ belongs to $J^+(u, x)$ and also satisfies all conditions of Proposition 6 as well.

1. ϕ_ϵ is a constant shift of the smooth mollification w_3, and hence $\phi_\epsilon \in C^\infty(\mathbb{R}^d)$ holds. Moreover, $\phi_\epsilon(x_\epsilon) = u_g(x_\epsilon)$ is valid by its definition. In addition, we conclude $\phi_\epsilon \in J^+(u, x_\epsilon)$, since
   - if $y \in \bar{B}_r(x)$, then $(\phi_\epsilon - u_g)(y) = (u^g - w_3)(x_\epsilon) - (u^g - w_3)(y) \ge 0$ since x_ϵ is maximum point of $u^g - w_3$ on $B_r(x)$.
   - if $y \in B_r^c(x)$, then we have, by (18) and (17)

   $$(\phi_\epsilon - u_g)(y) = (u^g - w_3)(x_\epsilon) + (-u^g + w_3)(y) \ge (u^g - w_3)(x_\epsilon) + \frac{3}{4}\epsilon \ge \frac{1}{2}\epsilon > 0.$$

2. From (18) and (19), we immediately write $-\sqrt{\epsilon}|x_\epsilon - x|^2 + \frac{1}{4}\epsilon \le -\frac{1}{4}\epsilon$ or equivalently $|x_\epsilon - x|^2 \le \frac{1}{2}\sqrt{\epsilon}$. This implies $\lim_{\epsilon \to 0} x_\epsilon = x$.

3. If $y \in B_r(x)$, then (18) and (19) again implies that ϕ_ϵ is a constant shift from w_3 with

   $$|\phi_\epsilon(y) - w_3(y)| < \frac{1}{4}\epsilon.$$

   Together with (12), (13), (14), and (15), we obtain

   $$\|\phi_\epsilon - w\|_{W^{2,\infty}(B_{r/2}(x))} \le \sqrt{\epsilon}(r^2 + 2rd + 2d + 1) + \frac{1}{4}\epsilon \to 0, \text{ as } \epsilon \to 0.$$

4. Finally, we shall check $\|\phi_\epsilon - w\|_{L^1(\nu_x, B_r^c(x))} \to 0$. First, we write from definition of ϕ_ϵ that

   $$\|\phi_\epsilon - w\|_{L^1(\nu_x, B_r^c(x))} \le \|w_3 - w\|_{L^1(\nu_x, B_r^c(x))} + |(u^g - w_3)(x_\epsilon)| \cdot \nu(B_r^c)$$

   The first term $\|w_3 - w\|_{L^1(\nu_x, B_r^c(x))} \to 0$ holds due to (13), (14), and (16). The second term $|(u^g - w_3)(x_\epsilon)| \cdot \nu(B_r^c) \to 0$ holds due to (18) and (19).

We finish the proof by applying Proposition 7. □

3 Existence of the unique solution for Dirichlet problem

Recall that a subsolution (resp. supersolution) property at a point $x \in O$ has been defined by Definition 2, and it's proved to be equivalent to Definition 3 by Proposition 9.

Definition 10 *A function $u \in USC(\bar{O})$ (resp. $u \in LSC(\bar{O})$) is said to be a viscosity subsolution* (resp. supersolution) *of* (1) - (2)*, if u satisfies the subsolution* (resp. supersolution) *property at each $x \in O$ and $u = g$ at ∂O. A function $u \in C(\bar{O})$ is said to be a solution of* (1) - (2)*, if it is a sub and supersolution of* (1) - (2) *at the same time.*

We first show Comparison principle and Perron's method in Section 3.1, then define an exit control problem in Section 3.2. The existence will be concluded in Section 3.3.

3.1 Comparison principle and Perron's method

Comparison principle and Perron's method under Definition 3 are well-studied, see [7] for instance. For the completeness, we will show their proofs within our problem setting.

In the proof of comparison principle, a globalized version of Ishii's lemma is crucial. The following lemma is a generalization of the original version of Ishii's lemma in [12].

Lemma 11 *For $u, -v \in USC(\bar{O})$, let w be*

$$w(x, y) = u(x) - v(y) - \frac{\gamma}{2}|x - y|^2,$$

and $(\hat{x}, \hat{y})$ be a maximum point of w on $\bar{O} \times \bar{O}$. Then for any $\epsilon > 0$, there exist $\phi_1 \in \bar{J}^+(u, \hat{x})$ and $\phi_2 \in \bar{J}^-(v, \hat{y})$ such that

1. *$D\phi_1(\hat{x}) = D\phi_2(\hat{y}) = \gamma(\hat{x} - \hat{y})$;*
2. *$D^2\phi_1(\hat{x}) \leq D^2\phi_2(\hat{y})$;*
3. *$\mathcal{I}(\phi_1, \hat{x}) - \mathcal{I}(\phi_2, \hat{y}) \leq \epsilon$.*

Proof: By the original version of Ishii's lemma (see Theorem 3.2 of [12]), there exists $\hat{\phi}_1 \in \bar{J}^+(u, \hat{x})$ and $\hat{\phi}_2 \in \bar{J}^-(v, \hat{y})$ such that

1. $D\hat{\phi}_1(\hat{x}) = D\hat{\phi}_2(\hat{y}) = \gamma(\hat{x} - \hat{y})$;
2. $D^2\hat{\phi}_1(\hat{x}) \leq D^2\hat{\phi}_2(\hat{y})$.

Therefore, we write

$$\mathcal{I}(\phi_1^r, \hat{x}) - \mathcal{I}(\phi_2^r, \hat{y}) = \mathcal{I}_{r,1}(\hat{\phi}_1, \hat{x}) + \mathcal{I}_{r,2}(u^g, \hat{x}) - \mathcal{I}_{r,1}(\hat{\phi}_2, \hat{y}) - \mathcal{I}_{r,2}(v_g, \hat{y}).$$

where

$$\phi_1^r := \phi_1 I_{\bar{B}_r(\hat{x})} + u^g I_{\bar{B}_r^c(\hat{x})}, \ \phi_2^r := \phi_1 I_{\bar{B}_r(\hat{y})} + v_g I_{\bar{B}_r^c(\hat{y})}.$$

In view of Proposition 9, $\phi_1^r \in \bar{J}^+(u, \hat{x}) \bigcap C^\infty(B_r(\hat{x}))$ and $\phi_2^r \in \bar{J}^-(v, \hat{y}) \bigcap C^\infty(B_r(\hat{y}))$ hold for all sufficiently small choices of $r > 0$.

If we denote $f(z) = u^g(\hat{x}+z) - v_g(\hat{y}+z)$, then $f(z) \leq f(0)$ by the definition of $(\hat{x}, \hat{y})$, and we have

$$\mathcal{I}_{r,2}(u^g, \hat{x}) - \mathcal{I}_{r,2}(v_g, \hat{y}) = \int_{\bar{B}_r^c} (f(z) - f(0))\nu(dz) \leq 0. \tag{20}$$

On the other hand, if we denote $f(z) = \phi_1(\hat{x}+z) - \phi_2(\hat{y}+z)$, then we can take $r > 0$, such that

$$\max_{\bar{B}_r} D^2 f \int_{B_r} |z|^2 \nu(dz) \leq \epsilon I.$$

The existence of such r is due to the fact of $D^2 f(0) \leq 0$, the continuity of $D^2 f$ on B_r, and the property of Levy measure ν. Hence, we can write

$$\mathcal{I}_{r,1}(\phi_1, \hat{x}) - \mathcal{I}_{r,1}(\phi_2, \hat{y}) = \int_{B_r} (f(z) - f(0) - z \cdot Df(0))\nu(dz) = \frac{1}{2}\int_{B_r} \langle D^2 f(\xi_z) z, z\rangle \nu(dz) \leq \frac{1}{2}\epsilon \tag{21}$$

for some $\xi(z) \in B_r$. Combining the inequalities (21) and (20), with the above choice of small enough r, we have the desired $\phi_1^r \in \bar{J}^+(u, \hat{x})$ and $\phi_2^r \in \bar{J}^-(v, \hat{y})$ satisfying all three conditions of the lemma.
□

Remark 12 *The choice of ϕ_1 and ϕ_2 may be dependent on ϵ in the above Lemma 11. In the proof of Lemma 11, we actually constructed $r(\epsilon) \to 0^+$ such that $\phi_1^r \in \bar{J}^+(u, \hat{x})$ and $\phi_2^r \in \bar{J}^-(v, \hat{y})$ satisfying all three conditions of Lemma 11 with given $\epsilon > 0$. However, one may not have the existence of $\phi_1^0 \in \bar{J}^+(u, \hat{x})$ and $\phi_2^0 \in \bar{J}^-(v, \hat{y})$ satisfying all three conditions of Lemma 11 with $\epsilon = 0$.*

Proposition 13 *If u and v are subsolution and supersolution of (1) - (2), then $u \leq v$.*

Proof: Let u and v are sub and super solutions, respectively. We define $\delta := \sup_{\bar{O}}(u - v)$, and

$$w(x, y) = u(x) - v(y) - \frac{\gamma}{2}|x - y|^2,$$

and let (x_γ, y_γ) be a maximum point of w on $\bar{O} \times \bar{O}$. By Lemma 3.1 of [12] and Lemma 11, we have the following properties on the maximum points: There exist $\gamma > 0$, $\phi_1 \in \bar{J}^+(u, x_\gamma)$ and $\phi_2 \in \bar{J}^-(v, y_\gamma)$, such that

1. $|\ell(x_\gamma) - \ell(y_\gamma)| \leq \delta/4$;
2. $D\phi_1(x_\gamma) = D\phi_2(y_\gamma) = \gamma(x_\gamma - y_\gamma)$;
3. $D^2\phi_1(x_\gamma) \leq D^2\phi_2(y_\gamma)$;
4. $\mathcal{I}(\phi_1, x_\gamma) - \mathcal{I}(\phi_2, y_\gamma) \leq \delta/4$.

If $\delta > 0$, then we shall find a contradiction in this below.

$$\begin{aligned}
0 \quad & < \delta \leq u(x_\gamma) - v(y_\gamma) \\
& \leq F(\phi_2, y_\gamma) - F(\phi_1, x_\gamma) + \ell(x_\gamma) - \ell(y_\gamma) \\
& = \inf_a\left(\frac{1}{2}A(a)D^2\phi_1(x_\gamma) + b(a) \cdot D\phi_1(x_\gamma)\right) - \inf_a\left(\frac{1}{2}A(a)D^2\phi_2(y_\gamma) + b(a) \cdot D\phi_2(y_\gamma)\right) \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \mathcal{I}(\phi_1, x_\gamma) - \mathcal{I}(\phi_2, y_\gamma) + \ell(x_\gamma) - \ell(y_\gamma) \\
& \leq \tfrac{1}{2}\sup_a (A(a)(D^2\phi_1(x_\gamma) - D^2\phi_2(x_\gamma))) + b(a) \cdot (D\phi_1(x_\gamma) - D\phi_2(x_\gamma)) \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \mathcal{I}(\phi_1, x_\gamma) - \mathcal{I}(\phi_2, y_\gamma) + \ell(x_\gamma) - \ell(y_\gamma) \\
& \leq \delta/2.
\end{aligned}$$

□

Proposition 14 *If u and v are both subsolutions of* (1) - (2)*, then* $\max\{u, v\}$ *is also a subsolution of* (1) - (2)*.*

The proof of Proposition 14 is referred to Theorem 2 of [7]. Next, Proposition 13 and 14 enables us to follow the same *bump construction* to as of Lemma 4.4 of [12], which eventually leads to Perron's method via Lemma 15 in this below.

Lemma 15 *Let u be a subsolution of* (1) - (2)*, and u_* fail to be a supersolution at some $\hat{x} \in O$. Then, for any small enough $\kappa > 0$, there exists a subsolution u_κ such that*

$$u_\kappa \geq u(x); \ \sup_O (u_\kappa - u) > 0; \quad \textit{and } u_\kappa = u \textit{ on } B_\kappa(\hat{x}).$$

Proof: For simplicity $\hat{x} = 0$ and there exists $\phi \in J^-(u_*, 0)$ such that

$$\hat{F}(\phi, 0) := F(\phi, 0) + \phi(0) - \ell(0) = -\epsilon < 0.$$

Since $\hat{F}(\phi, \cdot)$ is continuous, there exists $\kappa_0 > 0$ such that

$$\sup_{x \in B_{\kappa_0}} \hat{F}(\phi, x) < -\frac{\epsilon}{2}.$$

We fix arbitrary $\kappa < \kappa_0$. Let u_γ be a function of

$$u_\gamma(x) = \phi(x) + \gamma(\kappa^2 - |x|^2) I_{B_{2\kappa}}(x) := \phi(x) + \psi_\kappa(x).$$

If $x \in B_\kappa$, then we have

1. $$H(u_\gamma, x, a) = H(u_\gamma, x, a) - \gamma(tr(A(a)) + b(a) \cdot x) \geq H(u_\gamma, x, a) - \gamma c_{\kappa,1},$$
   where $c_{\kappa,1}$ is a number defined by $c_{\kappa,1} := \sup_{x \in B_\kappa, a \in [\underline{a}, \overline{a}]} |tr(A(a)) + b(a) \cdot x| < \infty$. This means
   $$-\inf_{a \in [\underline{a}, \overline{a}]} H(u_\gamma, x, a) \leq -\inf_{a \in [\underline{a}, \overline{a}]} H(\phi, x, a) + \gamma c_{\kappa,1}.$$
2. On the other hand, we also have
   $$-\mathcal{I}(u_\gamma, x) = -\mathcal{I}(\phi, x) + \gamma \mathcal{I}(\psi_\kappa, x) \leq -\mathcal{I}(\phi, x) + \gamma c_{\kappa,2},$$
   where $c_{\kappa,2} := \sup_{x \in B_\kappa} |\mathcal{I}(\psi_k, x)| < \infty$ holds due to the continuity of $\mathcal{I}(\psi_\kappa, \cdot)$, see Proposition 8.

Therefore, we conclude that, with $c_\kappa := c_{\kappa,1} + c_{\kappa,2}$

$$\hat{F}(u_\gamma, x) \leq F(\phi, x) + \gamma c_\kappa + \phi(x) - \ell(x) = \hat{F}(\phi, x) + \gamma c_\kappa.$$

Now we take $\gamma = \frac{\epsilon}{2c_\kappa}$ and we have u_γ be a subsolution on B_κ. Then, we have

1. if $x \in B_\kappa$, then
   $$u_\gamma(x) = \phi(x) + \gamma(\kappa^2 - |x|^2) I_{B_{2\kappa}}(x) \leq \phi(x) \leq u_*(x) \leq u(x),$$
2. and $u_\gamma(0) = \phi(0) + \gamma\kappa^2 > \phi(0) = u_*(0)$ implies that there exists $x_n \to 0$ such that $u_\gamma(x_n) > u(x_n)$.

Finally, we take $u_\kappa = \max\{u_\gamma, u\}$ to finish the proof by Proposition 14. $\square$

A direct consequence of Lemma 15 is Perron's method below.

Proposition 16 *If $\underline{u}$ and $\overline{u}$ are subsolution and supersolution of* (1) - (2)*, then*

$$w(x) = \inf\{u \in LSC(\bar{O}) : u \text{ is subsolution}\}$$

is the unique solution in $C(\bar{O})$.

According to Proposition 16, given Comparison principle and Perron's Method above, the remaining task for the existence of the solution is to construct subsolution $\underline{u}$ and supersolution $\overline{u}$. In general, as far as the classical Dirichlet boundary concerned, one shall not expect the existence of subsolution and supersolution for free due to Example 7.8 of [12]. In this regard, some sufficient conditions of the existence of subsolution and supersolution of Dirichlet problem is provided by Example 4.6 of [12], and the general case is remained open.

In this below, we will establish the existence and some sufficient condition in Theorem 17 and provide some examples for the illustration purpose.

3.2 Stochastic exit control problem on Markovian policy

To proceed, we consider an exit control problem with Markovian policy. We consider a fixed filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t, t > 0\})$, on which W is a standard Brownian motion and L is a Levy process with generating triplet $(0, \nu, 0)$, see notions of Levy process in [15] or [9]. We consider a stochastic differential equation controlled by a Lipschitz continuous function $m : \mathbb{R}^d \mapsto [\underline{a}, \overline{a}]$,

$$X_t = x + \int_0^t b(m(X_s))dt + \int_0^t \sigma(m(X_s))dW_s + L_t, \tag{22}$$

By [2], (22) admits a unique solution which has a Càdlàg version, and we assume X to be a Càdlàg process. Next, we define the first exit time

$$\tau = \inf\{t > 0, X_t \notin O\} \text{ and } \hat{\tau} = \inf\{t > 0, X_t \notin \bar{O}\}.$$

Let $\mathbb{D}^d_\infty$ be the space of Càdlàg functions on $[0, \infty)$ with Skorohod metric given by d^o_∞, see detailed definition in Section A.1. For a given $(x, m) \in \mathbb{R}^d \times \mathcal{M}$, we use $\mathbb{P}^{m,x}$ to denote the probability measure on $\mathbb{D}^d_\infty$ induced by X_t, i.e. $\mathbb{P}^{m,x}(B) = \mathbb{P}(X \in B)$ for all Borel set B of $(\mathbb{D}^d_\infty, d^o_\infty)$. We also use $\mathbb{E}^{m,x}$ to denote the expectation operator with respect to $\mathbb{P}^{m,x}$.

We are interested in the following subset of Markovian policy space $\mathcal{M}$ defined by

$$\mathcal{M} = \{m \in C^{0,1}(\mathbb{R}^d, [\underline{a}, \overline{a}]) : \mathbb{P}^{m,x}(\hat{\tau} = 0) = 1, \ \forall x \in \partial O\}. \tag{23}$$

The value function V_m is defined as

$$V_m(x) = \mathbb{E}^{m,x}\big[\int_0^\tau e^{-s}\ell(X_s)ds + g(X_\tau)\big].$$

3.3 The existence of the continuous solution

In the proof of Theorem 17, we will use the fact of

$$V_m \in C(\bar{O}) \text{ with } V_m = g \text{ on } \partial O \text{ for any } m \in \mathcal{M}$$

according to Proposition 22, which will be proved later in Section 4.

Theorem 17 *If (1) $\mathcal{M} \neq \emptyset$; and (2) $u = g$ is a subsolution of (1) - (2), then there exists a continuous solution of (1) - (2).*

PROOF:

By Perron's method, to establish the existence, we shall find out sub and supersolution. Note that g is a subsolution and we will show that V_m is a supersolution for any $m \in \mathcal{M}$ in this below.

We fix a policy $m \in \mathcal{M}$. By Proposition 22, we have $V_m \in C(\bar{O})$ with $V_m(x) = g(x)$ for all $x \in \partial O$. So, it's enough to show that V_m satisfies the supersolution property in O, i.e.

$$F_m(\phi, x) + \phi(x) - \ell(x) \geq 0, \ \forall x \in O, \phi \in J^-(V_m, x).$$

where $F_m(\phi, x) = -H(\phi, x, m(x)) - \mathcal{I}(\phi, x)$. To the contrary, let's assume

$$F_m(\phi, x) + \phi(x) - \ell(x) = -\epsilon < 0$$

for some $x \in O$ and $\phi \in J^-(V_m, x)$. By Proposition 8 and the continuity of m, the function $F_m(\phi, \cdot)$ is continuous at x, and there exists $h > 0$ that

$$\sup_{|y-x|<h} F_m(\phi, y) + \phi(y) - \ell(y) < -\epsilon/2. \tag{24}$$

Since X of (22) is a Càdlàg process, the first exit time satisfies $\mathbb{P}^{m,x}\{\tau > 0\} = 1$. By the strong Markov property of the process X, we rewrite the value function V_m as, for any stopping time $\theta \in (0, \tau]$

$$V_m(x) = \mathbb{E}^{m,x}\Big[e^{-\theta} V(X_\theta) + \int_0^\theta e^{-s} \ell(X_s) ds\Big],$$

which in turn implies that, with the fact of $\phi \in J^-(V_m, x)$

$$\phi(x) \geq \mathbb{E}^{m,x}\Big[e^{-\theta} \phi(X_\theta) + \int_0^\theta e^{-s} \ell(X_s) ds\Big],$$

On the other hand, one can use Dynkin's formula on ϕ to write

$$\mathbb{E}^{m,x}[e^{-\theta}\phi(X_\theta)] = \phi(x) - \mathbb{E}^{m,x}\Big[\int_0^\theta e^{-s}(F_m(\phi, X_s) + \phi(X_s)) ds\Big].$$

By adding up the above two formulas together, it yields that

$$\mathbb{E}^{m,x}\Big[\int_0^\theta e^{-s}(F_m(\phi, X_s) + \phi(X_s) - \ell(X_s)) ds\Big] \geq 0.$$

Finally we take $\theta = \inf\{t > 0 : X(t) \notin B_h(x)\} \wedge \tau$ in the above and note that $\theta > 0$ almost surely in $\mathbb{P}^{m,x}$. This leads to a contradiction to (24).

□

The sufficient condition of Theorem 17 consists of (1) $\mathcal{M} \neq \emptyset$; and (2) subsolution property g to ensure the uniqueness and existence of the solution. In this below, we shall give two examples to illustrate the two conditions above.

Example 18 (A remark on $\mathcal{M} \neq \emptyset$) Fix a constant $\alpha \in (0,2)$ and $O := (-1,1) \times (-1,1)$ as of Example 1, and we prove the uniqueness and existence of by Theorem 17 in this below. This proof would not be affected when the domain O is replaced by any open connected set satisfying exterior cone condition. We first rewrite the equation (4) by

$$- \inf_{a \in [-1,1]} \{a \, \partial_{x_1} u\} + (-\Delta)^{\alpha/2} u + u - 1 = 0 \quad \text{on } O.$$

For $m \in \mathcal{M}$, we set

$$X_t = x + \int_0^t m(X_s) e_1 ds + L_t^\alpha$$

where $e_1 = (1,0)'$ is a unit vector and L^α is a symmetric α-stable process with the generating triplet $(0, \nu(dy) = \frac{dy}{|y|^{d+\alpha}}, 0)$. The corresponding value function is

$$V_m(x) = \mathbb{E}^{m,x}\Big[\int_0^\tau e^{-s} ds\Big] = \mathbb{E}^{m,x}[1 - e^{-\tau}]$$

with the first exit time $\tau = \inf\{t > 0, X_t \notin O\}$. One can directly check both conditions required by Theorem 17

- If $\alpha > 0$, then we take $m(x) = 0$ and corresponding X is given by
$$X_t = x + L_t^\alpha;$$
In this case, $\mathbb{P}^{m,x}\{\hat{\tau} = 0\} = 1$ for all $x \in \partial O$ and $\mathcal{M} \neq \emptyset$.
- $u = 0$ is subsolution.

This implies unique solvability.

Example 19 (A remark on subsolution property of g) *One can check $u(x) = 1 - e^{-1+|x|}$ is the unique solution of*

$$|u'| + u - 1 = 0, \ \forall x \in (-1,1) \text{ with } u(\pm 1) = 0.$$

However, there is no solution for

$$|u'| + u + 1 = 0, \ \forall x \in (-1,1) \text{ with } u(\pm 1) = 0.$$

Indeed, if there were a solution u, the boundary condition $u(1) = 0$ implies that $|u'| + u + 1 > 1/2$ in some neighborhood of 1 due to the continuity of u, which leads to a contradiction. One can see that this equation does not satisfy the second condition, i.e. $u = 0$ is not subsolution.

4 Continuous mappings in Skorohod space

We denote by $(\mathbb{D}_t^d, d_t^o)$ the complete space of Càdlàg functions on $[0,t)$ taking values in $\mathbb{R}^d$ with Skorohod metric d_t^o, and by $(\mathbb{D}_\infty^d, d_\infty^o)$ the space of Càdlàg functions on $[0,\infty)$. See detailed definition in Section A.1 taken from [10].

We also define the entrance time operator $T_A : \mathbb{D}_\infty^d \mapsto \mathbb{R}$ by, for a set $A \in \mathbb{R}^d$ and $a \in (0,\infty)$

$$T_A(\omega) = \inf\{t \geq 0 : \omega(t) \in A\}, \ T_A^a(\omega) = \inf\{t \geq 0 : \omega(t) \in A\} \wedge a, \tag{25}$$

By convention, $T_A(\omega) = \infty$ if $\omega(t) \notin A$ for all $t \geq 0$. Given a set O, we will call $T_{O^c}(\omega)$ as the exit time of ω from the set O.

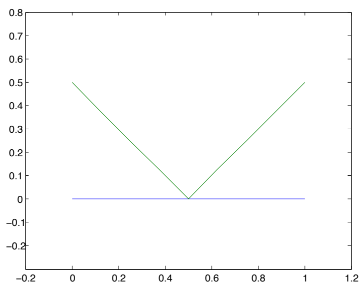

Figure 1: Shift up

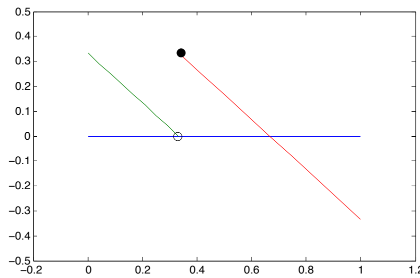

Figure 2: Shift down

Figure 3: Illustration for Example 29

Recall [10] that a projector $\Pi : \mathbb{D}^d_\infty \times [0, \infty) \mapsto \mathbb{R}^d$ defined by $\Pi(\omega, t) = \omega(t)$. Similarly, we can define projection operator on the exit time by

$$\Pi_O(\omega) = \omega(T_{O^c}(\omega)). \tag{26}$$

Our goal is to investigate the sufficient condition such that the mappings T_{O^c} and Π_O are continuous for a given set O, and this will serve as an important tool for the existence of the solution. In this below, we first show the main results in Theorem 20 and Theorem 21, and provide an important application in Proposition 22, which will be used in the proof of Theorem 17.

4.1 Main results and their applications

4.1.1 Main results

Example 29 and Figure 3 show that T_{O^c} is neither upper semicontinuous nor lower semicontinuous in general. Moreover, Example 30 and Figure 4 indicates that the situation for the continuity of Π_O is even worse than the mapping T_{O^c}.

Now we present the sufficient condition of the continuity of Π_O and T_{O^c}. To proceed, let's denote the càglàd modification of process ω by ω^-, i.e.

$$\omega^-(t) = \lim_{s\to t-} \omega(s), \ \forall \omega \in \mathbb{D}^d_\infty.$$

The following theorems are the main results of this section on the continuity of two mappings T_{O^c} and Π_O, and their proofs will be relegated to Section 4.4 and 4.5. Roughly speaking, both T_{O^c} and Π_O are continuous at some ω if, at the first exit time

1. either ω exits from O to $\bar{O}$ continuously by crossing ∂O;
2. or ω jumps from a point of O to another point of $\bar{O}^c$.

Theorem 20 *T_{O^c} is continuous w.r.t. Skorohod metric at any $\omega \in \Gamma_O$ where*

$$\Gamma_O := \{\omega \in \mathbb{D}^d_\infty : T_{O^c}(\omega^-) = T_{O^c}(\omega) = T_{\bar{O}^c}(\omega)\}. \tag{27}$$

Theorem 21 *Π_O is continuous w.r.t. Skorohod metric on*

$$\hat{\Gamma}_O := \{\omega \in \Gamma_O : \ \textit{if } \Pi_O(\omega^-) \in \partial O, \ \textit{then } \Pi_O(\omega^-) = \Pi_O(\omega)\} \tag{28}$$

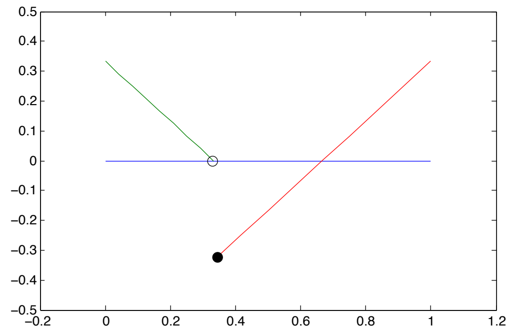

Figure 4: A small down shift makes a big change in the state at the first exit time

4.1.2 Applications

For a fixed $m \in \mathcal{M}$, we consider X satisfying (22), i.e.

$$X_t = x + \int_0^t b_m(X_s)ds + \int_0^t \sigma_m(X_s)dW_s + L_t,$$

where $b_m = b \circ m$ and $\sigma_m = \sigma \circ m$. Recall that, the objective function V_m is defined as

$$V_m(x) = \mathbb{E}^{m,x}\Big[\int_0^\tau e^{-s}\ell(X_s)ds + g(X_\tau)\Big].$$

Proposition 22 *If $m \in \mathcal{M}$ of (23), then $V_m \in C(\bar{O})$ with $V_m(x) = g(x)$ for all $x \in \partial O$.*

Proof: If $x \in \partial O$, then $\tau = 0$ $\mathbb{P}^{m,x}$-almost surely by definition and $V_m(x) = g(x)$. In the rest of the proof, let $x_n \to x \in \bar{O}$, and we will show the continuity of V_m at x.

1. *In this step, we will show $\mathbb{P}^{m,x}(\hat{\Gamma}_O) = 1$ for all $x \in \bar{O}$.* Since both b_m and σ_m are Lipschitz continuous, there exists unique strong solution X, which is Càdlàg process with strong Markovian property, see Example 6.4.7 of [2]. Therefore, $m \in \mathcal{M}$ implies

$$\mathbb{P}^{m,x}\{\tau = \hat{\tau}\} = 1, \ \forall x \in \bar{O}. \tag{29}$$

Hence, for all $x \in \partial O$, we have $\Gamma_O = \hat{\Gamma}_O$ and $\mathbb{P}^{m,x}(\hat{\Gamma}_O) = 1$. Now, it remains to show $\mathbb{P}^{m,x}(\hat{\Gamma}_O) = 1, \ \forall x \in O$. Let $x \in O$ and $\bar{\tau} = T_{O^c}(X^-)$. We define

$$\bar{\tau}_A = \begin{cases} \bar{\tau} & \text{if } \omega \in A; \\ \infty & \text{otherwise} \end{cases} \quad \text{and } \bar{\tau}_B = \begin{cases} \bar{\tau} & \text{if } \omega \in B; \\ \infty & \text{otherwise.} \end{cases}$$

where

$$A = \{X^-(\bar{\tau}) \in \partial O\} \text{ and } B = \{X^-(\bar{\tau}) \neq X(\bar{\tau})\}.$$

The left continuity of X^- implies $A \in \mathcal{F}_{\bar{\tau}-}$ and the hitting time $\bar{\tau}_A$ is a predictable stopping time, while $\bar{\tau}_B$ is totally inaccessible stopping time due to the jump by Meyer's theorem, see Theorem III.4 of [17]. Therefore, we conclude $\mathbb{P}^{m,x}(\bar{\tau}_A = \bar{\tau}_B) = 0$ by Theorem III.3 of [17], and further we have $\mathbb{P}^{m,x}(A \cap B) = 0$. Therefore, X is continuous at $\bar{\tau}$ almost surely in $\mathbb{P}^{m,x}$. Together with (29), we conclude $\mathbb{P}^{m,x}(\hat{\Gamma}_O) = 1$.

2. *Next, we will show that $f_1 + f_2$ is continuous at all $\omega \in \hat{\Gamma}_O$*, where

$$f_1(\omega) = \int_0^{T_{O^c}(\omega)} e^{-s}\ell(\omega_s)ds, \quad \text{and} \quad f_2(\omega) = e^{-T_{O^c}(\omega)}g(\Pi_O(\omega)), \ \forall \omega \in \mathbb{D}^d_\infty.$$

The continuity of f_2 is the direct consequence of Theorem 20 and Theorem 21. So, it remains to show the continuity of f_1.

Suppose $\omega^n \to \omega \in \hat{\Gamma}_O$ in Skorohod metric, and we denote $T_n = T_{O^c}(\omega^n)$ and $T = T_{O^c}(\omega)$, we conclude $f_1(\omega^n) \to f_1(\omega)$ as $n \to \infty$, since

(a) $T_n \to T$ due to Theorem 20;

(b) $\omega^n \to \omega$ in $\mathbb{D}^d_\infty$ means that $\omega^n(s) \to \omega(s)$ for all $s \in D^c_\omega$, where D^c_ω is the complement of the countable set
$$D_\omega := \{s \in (0,\infty) : \omega \text{ is discontinuous at } s\}.$$
Together with the continuity of ℓ, we have $\ell(\omega^n(s)) \to \ell(\omega(s))$ almost everywhere on $(0,t)$ w.r.t. Lebesgue measure.

(c) For any $\epsilon > 0$, we have
$$|f_1(\omega^n) - f_1(\omega)| \leq \int_0^{T_n} e^{-qs}|\ell(\omega^n(s)) - \ell(\omega(s))|ds + K|T_n - T| \to 0, \quad \text{as } n \to \infty.$$

3. *Finally we can show $V_m(x_n) \to V_m(x)$ if $x_n \to x \in \bar{O}$.* We first conclude $\mathbb{P}^{m,x_n}$ is weakly convergent to $\mathbb{P}^{m,x}$, since

   By Theorem 3.2 of [16], X satisfies
   $$\mathbb{E}\Big[\sup_{0\leq s\leq t}|X^{m,x_n}_s - X^{m,x}_s|^2\Big] \leq K_t|x_n - x|^2 \to 0, \text{ as } n \to \infty.$$
   This means $\{X^{m,x_n}_s : 0 \leq s \leq t\}$ is convergent to $\{X^{m,x}_s : 0 \leq s \leq t\}$ $\mathbb{P}$-almost surely with respect to L^∞, and hence convergent in distribution with respect to Skorohod metric. Weak convergence on any finite time interval implies the weak convergence on the entire time interval by Theorem 16.7 of [10].

Moreover, in the above two steps, we established $f_1 + f_2$ is continuous $\mathbb{P}^{m,x}$-almost surely. Finally, we apply the continuous mapping theorem and Bounded Convergence Theorem to obtain
$$V_m(x_n) = \mathbb{E}^{m,x_n}[(f_1 + f_2)(X)] \to \mathbb{E}^{m,x}[(f_1 + f_2)(X)] = V_m(x).$$

□

4.2 Working on a series of simpler topologies

Let Λ_∞ be the set of continuous and strictly increasing maps of $[0,\infty)$ to itself. Let
$$\|\omega\|_m = \sup_{0\leq t\leq m}|\omega(t)|, \ \|\omega\| = \sup_{0\leq t<\infty}|\omega(t)|.$$

The topology induced by the above supnorm is finer than Skorohod topology. Therefore, the continuity of Π_O at ω with respect to Skorohod topology automatically implies the continuity with respect to uniform topology. In this below, we will prove that the converse is also true: The continuity with respect to uniform topology implies the continuity of Π_O with respect to Skorohod topology. This enables us to simplify our subsequent analysis by working on a series of simpler metrics.

Proposition 23 *$T_{O^c}(\omega \circ \lambda) = \lambda^{-1} \circ T_{O^c}(\omega)$ for all $\omega \in \mathbb{D}^d_\infty$ and $\lambda \in \Lambda_\infty$.*

PROOF: $T_{O^c}(\omega \circ \lambda) = \inf\{t > 0 : \omega \circ \lambda(t) \notin O\} = \lambda^{-1}(\inf\{\lambda(t) > 0 : \omega(\lambda(t)) \notin O\}) = \lambda^{-1} \circ T_{O^c}(\omega)$. □

Lemma 24 *1. If $T^m_{O^c}$ is lower semicontinuous w.r.t $\|\cdot\|_m$ for all integer m, then T_{O^c} is lower semicontinuous w.r.t d^o_∞.*

2. If $T^m_{O^c}$ is upper semicontinuous w.r.t $\|\cdot\|_m$ for all integer m, then T_{O^c} is upper semicontinuous w.r.t d^o_∞.

PROOF: We assume $T_{O^c}(\omega) \in (0, \infty)$, otherwise it's obvious. Let $\lim_n d^o_\infty(\omega_n, \omega) = 0$. By Theorem 16.1 of [10], there exists $\lambda_n \in \Lambda_\infty$ such that

$$\lim_n \|\lambda_n - 1\| = 0$$

and

$$\lim_n \|\omega_n \circ \lambda_n - \omega\|_m = 0, \ \forall m \in \mathbb{N}.$$

1. We suppose $T^m_{O^c}$ is lower semicontinuous w.r.t. $\|\cdot\|_m$ for every integer m. Then, we have

$$\liminf_n T^m_{O^c}(\omega_n \circ \lambda_n) \geq T^m_{O^c}(\omega).$$

Also, we have by Proposition 23

$$|T^m_{O^c}(\omega_n) - T^m_{O^c}(\omega_n \circ \lambda_n)| = |T_{O^c}(\omega_n) \wedge m - \lambda_n^{-1} \circ T_{O^c}(\omega_n) \wedge m| \leq \|1 - \lambda_n^{-1}\| \to 0.$$

Thus, we have

$$\liminf_n T^m_{O^c}(\omega_n) \geq T^m_{O^c}(\omega).$$

Therefore, for a big enough m such that $m > T_{O^c}(\omega)$ holds, we have

$$\liminf_n T_{O^c}(\omega_n) \geq \liminf_n T^m_{O^c}(\omega_n) \geq T^m_{O^c}(\omega) = T_{O^c}(\omega).$$

This implies T_{O^c} is also lower semicontinuous w.r.t. d^o_∞.

2. We suppose $T^m_{O^c}$ is upper semicontinuous w.r.t. $\|\cdot\|_m$ for every integer m. Then, we have

$$\limsup_n T^m_{O^c}(\omega_n \circ \lambda_n) \leq T^m_{O^c}(\omega).$$

Also, we have similarly $|T^m_{O^c}(\omega_n) - T^m_{O^c}(\omega_n \circ \lambda_n)| \to 0$ as $n \to \infty$ by Proposition 23, and conclude

$$\limsup_n T^m_{O^c}(\omega_n) \leq T^m_{O^c}(\omega) \leq T_{O^c}(\omega) \quad \text{for all integer } m.$$

Now we fix an integer $m > T_{O^c}(\omega) + 1$. This means, $\forall \epsilon \in (0, 1)$, there exists N_ϵ such that

$$T_{O^c}(\omega_n) \wedge m \leq T_{O^c}(\omega) + \epsilon, \quad \forall n \geq N_\epsilon.$$

Since $m > T_{O^c}(\omega) + \epsilon$, the left hand side $T_{O^c}(\omega_n) \wedge m$ must be equal to $T_{O^c}(\omega_n)$, i.e.

$$T_{O^c}(\omega_n) \leq T_{O^c}(\omega) + \epsilon, \quad \forall n \geq N_\epsilon.$$

This implies T_{O^c} is also upper semicontinuous w.r.t. d^o_∞.

□

4.3 Working with dimension one

In this below, we will identify the continuity set in one dimensional Càdlàg space for the mapping T_{O^c} with respect to uniform topology induced by supnorm.

Lemma 25 *The mapping $\omega \mapsto T^m_{(-\infty,0)}(\omega)$ is upper semicontinuous in $\mathbb{D}^1_\infty$ w.r.t. $\|\cdot\|_m$ for every $m \in \mathbb{N}$.*

PROOF: For convenience, we denote $\hat{T}_m(\omega) = T_{(-\infty,0)}(\omega) \wedge m$. It's enough to show that

$$\text{If } \|\omega_n - \omega\|_m \to 0, \text{ then } \limsup_n \hat{T}_m(\omega_n) \leq \hat{T}_m(\omega).$$

We prove it in two cases separately:

1. Assume $\inf_{0\leq t\leq m} \omega(t) > 0$. This implies $\hat{T}_m(\omega) = m$. Given $\|\omega_n - \omega\|_m \to 0$, there exists N, such that
$$\forall n > N,\ \|\omega_n - \omega\|_m < \frac{1}{2} \inf_{0\leq t\leq m} \omega(t).$$
This yields
$$\forall n > N,\ \forall s \in [0, m],\ \omega_n(s) - \omega(s) > -\frac{1}{2} \inf_{0\leq t\leq m} \omega(t).$$
Therefore,
$$\forall n > N,\ \forall s \in [0, m],\ \omega_n(s) > 0,$$
or equivalently, $\hat{T}_m(\omega_n) = m$ for all $n > N$. This proves the conclusion of the first case.

2. Assume $\inf_{0\leq t\leq m} \omega(t) \leq 0$. Fix arbitrary $\epsilon > 0$, then
$$\exists t_\epsilon \in [\hat{T}_m(\omega), \hat{T}_m(\omega) + \epsilon) \text{ such that } \omega(t_\epsilon) < 0.$$
Given $\|\omega_n - \omega\|_m \to 0$,
$$\exists N \ \text{ such that } \ \|\omega_n - \omega\|_m < \frac{1}{2}|\omega(t_\epsilon)|,\ \forall n \geq N.$$
In particular, one can write $\omega_n(t_\epsilon) - \omega(t_\epsilon) < -\frac{1}{2}\omega(t_\epsilon)$, or equivalently
$$\exists N \text{ such that } \ \omega_n(t_\epsilon) < 0,\ \forall n \geq N.$$
Therefore, $\hat{T}_m(\omega_n) \leq t_\epsilon \leq \hat{T}_m(\omega) + \epsilon$ for all $n \geq N$. By taking $\limsup_n$ both sides, we have
$$\limsup_n \hat{T}_m(\omega_n) \leq \hat{T}_m(\omega) + \epsilon$$
and the conclusion follows due to the arbitrary selection of ϵ.

□

Lemma 26 *$\omega \mapsto T^m_{(-\infty,0]}(\omega_*)$ is lower semicontinuous in $\mathbb{D}^1_\infty$ w.r.t. $\|\cdot\|_m$ for every $m \in \mathbb{N}$, where*

$$\omega_*(t) = \liminf_{s\to t} \omega(s),\ \forall t > 0$$

is the lower envelope of ω.

PROOF: For simplicity, we denote $\tilde{T}_m(\omega) = T_{(-\infty,0]}(\omega_*) \wedge m$ and $M[\omega](t) = \inf_{0\leq s\leq t} \omega(s)$. Note that $M[\omega] = M[\omega_*]$ is a non-increasing process. It's enough to show that

$$\text{If } \|\omega_n - \omega\|_m \to 0, \text{ then } \liminf_n \tilde{T}_m(\omega_n) \geq \tilde{T}_m(\omega).$$

1. Assume $\tilde{T}_m(\omega) = m$. This implies $M[\omega](m) = M[\omega_*](m) > 0$, otherwise $\tilde{T}_m(\omega) < m$. Given $\|\omega_n - \omega\|_m \to 0$,
$$\exists N, \text{ such that } \|\omega_n - \omega\|_m < \frac{1}{2} M[\omega](m), \ \forall n \geq N,$$
which implies
$$\exists N, \text{ such that } \omega_n(t) > \omega(t) - \frac{1}{2} M[\omega](m) \geq \frac{1}{2} M[\omega](m) > 0, \ \forall t \in (0, m), \ \forall n \geq N.$$
Hence, $\tilde{T}_m(\omega_n) = m$ for all $n \geq N$, and this proves the continuity at ω for this case.

2. Assume $\tilde{T}_m(\omega) < m$. Since ω_* is lower semicontinuous , we have
$$M[\omega](\tilde{T}_m(\omega)) \leq 0, \text{ and } M[\omega](t) > 0, \forall t < \tilde{T}_m(\omega).$$
Fix arbitrary $\epsilon > 0$. Then, we have $M[\omega](\tilde{T}_m(\omega) - \epsilon) > 0$, and
$$\exists N, \text{ such that } \|\omega_n - \omega\|_m < \frac{1}{2} M[\omega](\tilde{T}_m(\omega) - \epsilon), \ \forall n \geq N.$$
This leads to
$$\forall n \geq N, \ \forall t < \tilde{T}_m(\omega) - \epsilon, \ \omega_n(t) > \omega(t) - \frac{1}{2} M[\omega](\tilde{T}_m(\omega) - \epsilon) \geq \frac{1}{2} M[\omega](\tilde{T}_m(\omega) - \epsilon) > 0.$$
In other words, we have $\tilde{T}_m(\omega_n) \geq \tilde{T}_m(\omega) - \epsilon$ for all $n \geq N$. So we conclude $\liminf_n \tilde{T}_m(\omega_n) \geq \tilde{T}_m(\omega)$ for the this case.

□

Proposition 27 *1.* $T^m_{(-\infty,0]}$ *is upper semicontinuous on* $\{\omega \in \mathbb{D}^1_\infty : T^m_{(-\infty,0]}(\omega) = T^m_{(-\infty,0)}(\omega)\}$ *w.r.t.* $\|\cdot\|_m$*;*

2. $T^m_{(-\infty,0]}$ *is lower semicontinuous on* $\{\omega \in \mathbb{D}^1_\infty : T^m_{(-\infty,0]}(\omega) = T^m_{(-\infty,0]}(\omega_*)\}$ *w.r.t.* $\|\cdot\|_m$.

PROOF: If (a) $\omega_n \to \omega$ w.r.t. $\|\cdot\|_m$; and (b) $T^m_{(-\infty,0]}(\omega) = T^m_{(-\infty,0)}(\omega)$, then Lemma 25 implies

$$\limsup_n T^m_{(-\infty,0]}(\omega_n) \leq \limsup_n T^m_{(-\infty,0)}(\omega_n) \leq T^m_{(-\infty,0)}(\omega) = T^m_{(-\infty,0]}(\omega),$$

which asserts the upper semicontinuity.

Similarly, if (a) $\omega_n \to \omega$ w.r.t. $\|\cdot\|_m$; and (b) $T^m_{(-\infty,0]}(\omega) = T^m_{(-\infty,0]}(\omega_*)$, then Lemma 26 implies

$$\liminf_n T^m_{(-\infty,0]}(\omega_n) \geq \liminf_n T^m_{(-\infty,0]}(\omega_{n,*}) \geq T^m_{(-\infty,0]}(\omega_*) = T^m_{(-\infty,0]}(\omega),$$

which asserts the lower semicontinuity.

□

4.4 Continuity of the exit time operator: The proof of of Theorem 20

Finally, we can identify the continuity set in multidimensional Càdlàg space for the mapping T_{O^c} with respect to Skorohod metric. We also define the signed distance function $\rho(x)$ of (30)

$$\rho(x) = \begin{cases} dist(x, \partial O) & \text{if } x \in O; \\ -dist(x, \partial O) & \text{otherwise} \end{cases} \tag{30}$$

Note that, if O is open, then

$$T_{O^c}(\omega) = \inf\{t \geq 0 : \omega(t) \notin O\} = \inf\{t \geq 0 : \rho \circ \omega(t) \leq 0\} = T_{(-\infty,0]}(\rho \circ \omega),$$

and

$$T_{\bar{O}^c}(\omega) = \inf\{t \geq 0 : \omega(t) \notin \bar{O}\} = \inf\{t \geq 0 : \rho \circ \omega(t) < 0\} = T_{(-\infty,0)}(\rho \circ \omega).$$

In other words, we have

$$T_{O^c} = T_{(-\infty,0]} \circ \rho, \ T_{\bar{O}^c} = T_{(-\infty,0)} \circ \rho, \ \forall \omega \in \mathbb{D}^d_\infty \text{ for all open set } O. \tag{31}$$

This simple fact enables us to generalize 1-d result of Proposition 27 to the multidimensional case.

Proof:[of Theorem 20] First assume $d = 1$ and $O = (0, \infty)$. Proposition 27 and Lemma 24 implies $T_{(-\infty,0]}$ is continuous on

$$B = \{\omega \in \mathbb{D}^1_\infty : T_{(-\infty,0]}(\omega_*) = T_{(-\infty,0]}(\omega) = T_{(-\infty,0)}(\omega)\}.$$

Recall that, we want to show $T_{(-\infty,0]}$ is continuous on

$$\Gamma_{(0,\infty)} = \{\omega \in \mathbb{D}^1_\infty : T_{(-\infty,0]}(\omega^-) = T_{(-\infty,0]}(\omega) = T_{(-\infty,0)}(\omega)\}.$$

Hence, it's enough to show $B = \Gamma_{(0,\infty)}$.

1. By an inequality of $T_{(-\infty,0]}(\omega_{-*}) \leq T_{(-\infty,0]}(\omega^-) \leq T_{(-\infty,0]}(\omega)$, we have $B \subset \Gamma_{(0,\infty)}$.
2. If there exists $\omega \in \Gamma_{(0,\infty)} \setminus B$, then $T_{(-\infty,0]}(\omega_*) < T_{(-\infty,0]}(\omega^-)$. This yields that

   $$\omega_*(T_{(-\infty,0]}(\omega_*)) \leq 0 < \omega^-(T_{(-\infty,0]}(\omega_*)),$$

   which again implies, with the notion of $\Delta\omega(t) = \omega(t) - \omega(t-)$

   $$\Delta\omega(T_{(-\infty,0]}(\omega_*)) < 0, \quad \omega(T_{(-\infty,0]}(\omega_*)) = \omega_*(T_{(-\infty,0]}(\omega_*)) \leq 0.$$

   Hence, we have $T_{(-\infty,0]}(\omega) = T_{(-\infty,0]}(\omega_*)$, which is a contradiction to $\omega \notin B$.

In conclusion, we obtain $B = \Gamma_{(0,\infty)}$ and $T_{(-\infty,0]}$ is continuous at any $\omega \in \Gamma_{(0,\infty)}$.

Now we turn to the general case of $d \geq 1$. If $\omega_n \to \omega \in \Gamma_O$, then $\rho \circ \omega_n \to \rho \circ \omega \in \Gamma_{(0,\infty)}$ by the continuity of ρ. Thanks to (31) and the continuity of $T_{(-\infty,0]}$ on $\Gamma_{(0,\infty)}$, we conclude,

$$T_{O^c}(\omega_n) = T_{(-\infty,0]}(\rho(\omega_n)) \to T_{(-\infty,0]}(\rho(\omega)) = T_{O^c}(\omega).$$

□

4.5 Continuity of the projection operator: The proof of Theorem 21

We will explore the continuity of the mapping Π_O, which was defined as the projector on the exit time in (26).

Recall that Γ_O is the subset of $\mathbb{D}^d_\infty$ defined in (27) as $\Gamma_O = \{T_{O^c}(\omega^-) = T_{O^c}(\omega) = T_{\bar{O}^c}(\omega)\}$. Although the exit time T_{O^c} is continuous at all $\omega \in \Gamma_O$, the projection Π_O may not be continuous at some $\omega \in \Gamma_O$.

PROOF: [of Theorem 21] Let $\omega^n \to \omega \in \hat{\Gamma}_O$ in Skorohod topology, and denote for simplicity that

$$T = T_{O^c}(\omega), T_n = T_{O^c}(\omega^n).$$

Then, we can write $\omega(T) = \Pi_O(\omega)$ and $\omega(T_n) = \Pi_O(\omega^n)$. We want to show that $\omega(T_n) \to \omega(T)$ as $n \to \infty$.

1. If $\Pi_O(\omega^-) = \Pi_O(\omega)$, then $\Pi_O(\omega) \in \partial O$. Since ω is continuous at T, $\omega^n \to \omega$ in Skorohod metric implies that $\omega^n \to \omega$ uniformly on some interval $(T - \epsilon, T + \epsilon)$ for $\epsilon > 0$, i.e.

$$\sup_{|s-T|<\epsilon} |\omega^n(s) - \omega(s)| \to 0, \text{ as } n \to \infty.$$

Sine $T_n \to T$ by Theorem 20, there exists N such that $T_n \in (T - \epsilon, T + \epsilon)$ for all $n \geq N$. Together with the continuity of ω at T, we conclude that

$$\begin{aligned} |\omega^n(T_n) - \omega(T)| &\leq |\omega^n(T_n) - \omega(T_n)| + |\omega(T_n) - \omega(T)| \\ &\leq \sup_{|s-T|<\epsilon} |\omega^n(s) - \omega(s)| + |\omega(T_n) - \omega(T)| \to 0, \text{ as } n \to 0 \end{aligned}$$

2. If $\Pi_O(\omega^-) \neq \Pi_O(\omega)$, then $\omega \in \hat{\Gamma}_O$ means that $\omega^-(T) \in O$ and $\omega(T) \in O^c$.

   (a) If $\|\omega^n - \omega\|_m \to 0$ for some $m > T + 1$, then there exists N_1 such that $T_n < m$ for all $n \geq N_1$. Since $T_{O^c}(\omega^-) = T_{O^c}(\omega)$, we can also define

$$\epsilon := \sup_{0 \leq s \leq T} \rho(\omega^-(s)) > 0,$$

   where ρ is the signed distance to the boundary as of (30). Note that, there exists $N_2 > N_1$ such that

$$\|\omega^n - \omega\|_m < \frac{1}{2}\epsilon, \ \forall n > N_2.$$

   Therefore, $\sup_{0 \leq s \leq T} \rho(\omega^n(s)) > 0$ and $T_n \geq T$. Hence, $T_n \downarrow T$ as $n \to \infty$, and the right continuity of ω leads to

$$\begin{aligned} |\omega^n(T_n) - \omega(T)| &\leq |\omega^n(T_n) - \omega(T_n)| + |\omega(T_n) - \omega(T)| \\ &\leq \sup_{|s-T|<\epsilon} |\omega^n(s) - \omega(s)| + |\omega(T_n) - \omega(T)| \to 0, \text{ as } n \to 0 \end{aligned}$$

   (b) If $d^o_\infty(\omega^n, \omega) \to 0$, then there exists $\lambda_n \in \Lambda_\infty$ such that

$$\lim_n \|\lambda_n - 1\| = 0$$

   and

$$\lim_n \|\omega^n \circ \lambda_n - \omega\|_m = 0, \ \forall m \in \mathbb{N}.$$

   Applying Proposition 23, we have

$$\omega^n(T_{O^c}(\omega^n)) = \omega^n(\lambda_n \circ T_{O^c}(\omega^n \lambda_n)) = \hat{\omega}^n(T_{O^c}(\hat{\omega}^n)),$$

   where $\hat{\omega}^n = \omega^n \circ \lambda_n$. Since $\lim_n \|\hat{\omega}^n - \omega\|_m = 0$ for all $m \in \mathbb{N}$, we can repeat the same proof of Step 2a, and obtain $\hat{\omega}^n(T_{O^c}(\hat{\omega}^n)) \to \omega(T)$, which in turn implies that $\omega^n(T_n) \to \omega(T)$.

□

A Appendices

A.1 Skorohod metric in Càdlàg space

We denote by $\mathbb{D}_t^d$ the collection of Càdlàg functions on $[0, t)$ taking values in $\mathbb{R}^d$. In particular, $\mathbb{D}_\infty^d$ is the collection of Càdlàg functions on $[0, \infty)$. According to [10], one can impose Skorohod metric d_t^o in the space $\mathbb{D}_t^d$ as of below to make the space complete.

1. For $t \in [0, \infty)$, we define the sup norm
$$\|x\| = \sup_{0 \le s < t} |x(t)|. \tag{32}$$

2. For $t \in [0, \infty)$, we denote by Λ_t by the class of strictly increasing continuous mappings of $[0, t]$ onto itself. In particular, $\lambda(0) = 0$ and $\lambda(t) = t$ for all $\lambda \in \Lambda$. The identity I on $[0, t]$ also belongs to Λ_t. We can define a functional in Λ_t by
$$\|\lambda\|^o = \sup_{0 \le s < r \le t} \left| \log \frac{\lambda \circ r - \lambda \circ s}{r - s} \right|, \ \forall \lambda \in \Lambda_t.$$
Note that $\|\lambda\|^o$ may not be necessarily finite in Λ_t.

3. For $t \in [0, \infty)$, define the distance function $d_t^o(x, y)$ in $\mathbb{D}_t^d$ by
$$d_t^o(x, y) = \inf_{\lambda \in \Lambda_t} \{\|\lambda\|^o \vee \|x - y \circ \lambda\|\}, \ \forall x, y \in \mathbb{D}_t^d.$$

4. We define the distance function $d_\infty^o(x, y)$ in $\mathbb{D}_\infty^d$ by
$$d_\infty^o(x, y) = \sum_{m=1}^\infty 2^{-m}(1 \wedge d_m^o(x^m, y^m)) \ \forall x, y \in \mathbb{D}_\infty^d,$$
where $x^m(t) = g_m(t)x(t)$ for all $t \ge 0$ with a continuous function g_m given by
$$g_m(t) = \begin{cases} 1, & \text{if } t \le m - 1, \\ m - t, & \text{if } m - 1 \le t \le m, \\ 0, & \text{otherwise.} \end{cases}$$

Define a projector $\Pi : \mathbb{D}_\infty^d \times [0, \infty) \mapsto \mathbb{R}^d$ by
$$\Pi(\omega, t) = \omega(t). \tag{33}$$

Proposition 28 *$\omega \mapsto \Pi(\omega, t)$ is continuous at ω_0 if $t \mapsto \omega_0(t)$ is continuous at t.*

Proof: It's a consequence of Theorem 12.5 of [10]. □

Finally, we give two useful examples.

Example 29 *For simplicity, consider $O = (0, 1) \subset \mathbb{R}$.*

- *T_{O^c} is not upper semicontinuous at ω given by*
$$\omega(t) = |t - 1/2|$$
since $\lim_n T_{O^c}(\omega_n) = 3/2 > 1/2 = T_{O^c}(\omega)$ *where $\omega_n = \omega + 1/n$.*

- *T_{O^c} is not lower semicontinuous at ω given by*

$$\omega(t) = (-t + 1/3)I(t < 1/3) + (-t + 2/3)I(t \geq 1/3).$$

In fact, setting $\omega_n = \omega - 1/n$, we have $\lim_n T_{O^c}(\omega_n) = 1/3 < 2/3 = T_{O^c}(\omega)$.

Example 30 *Let $O = (0, 1)$ and*

$$\omega(t) = 1 - t - I(t \geq 1).$$

Since $\omega \in \Gamma_O$, we have the continuity of T_{O^c} at ω by Theorem 20. If we take $\omega_n = \omega - 1/n$ for all $n \in \mathbb{N}$, we have $\omega_n \to \omega$ in uniform topology, hence in Skorohod topology. Therefore, $T_{O^c}(\omega_n) = 1 - 1/n \to 1 = T_{O^c}(\omega)$, which supports Theorem 20. However, we have

$$\Pi_O(\omega_n) = 0 \not\to -1 = \Pi_O(\omega).$$

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