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Option pricing and Esscher transform under regime switching*

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Summary. We consider the option pricing problem when the risky underlying assets are driven by Markov-modulated Geometric Brownian Motion (GBM). That is, the market parameters, for instance, the market interest rate, the appreciation rate and the volatility of the underlying risky asset, depend on unobservable states of the economy which are modelled by a continuous-time Hidden Markov process. The market described by the Markov-modulated GBM model is incomplete in general and, hence, the martingale measure is not unique. We adopt a regime switching random Esscher transform to determine an equivalent martingale pricing measure. As in Miyahara [33], we can justify our pricing result by the minimal entropy martingale measure (MEMM).

Keywords and Phrases: Option pricing, Regime switching, Hidden Markov chain model, Esscher transform, MEMM.

JEL Classification Numbers: G13.

1 Introduction

Over the past three decades, academic researchers and market practitioners have developed and adopted different models and techniques for option valuation. The path-breaking work on option pricing was done by Black and Scholes [2] and Merton [32]. The seminal work by Black and Scholes [2] and Merton [32] assumed that the asset price dynamics are governed by a Geometric Brownian Motion (GBM). Under the additional assumptions of a perfect market and the absence

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of arbitrage opportunities, they are able to hedge an option perfectly by constructing a replicating portfolio consisting of primary assets. Harrison and Kreps [26], Harrison and Pliska [27, 28] showed that the absence of arbitrage opportunities is equivalent to the existence of an equivalent martingale measure. Delbaen and Schachermayer [9] mentioned that this equivalence is "essentially" valid. They introduced some technical concepts, like σ-martingale, which is a generalization of local martingale, and free lunches when a general probability space is considered. When the market is incomplete, there are infinitely many equivalent martingale measures and, hence, a range of no-arbitrage prices for a contingent claim.

Föllmer and Sondermann [19], Föllmer and Schweizer [20] and Schweizer [36] determined a martingale pricing measure by minimizing a quadratic function of the losses from an imperfectly hedged position. Davis [10] considered a traditional economic argument for valuation, namely the marginal rate of substitution, and determined a pricing measure for option valuation by solving a utility maximization problem. The pioneering work by Gerber and Shiu [23] adopted a time-honored tool in actuarial science, namely the Esscher transform introduced by Esscher [18], to choose a pricing measure for option valuation (see Yang [41] for an overview). This pricing result can be justified by maximizing the expected power utility of an economic agent. Their work highlights the interplay between financial and insurance pricing problems in an incomplete market. This is an important issue in actuarial science and finance as pointed out by Bühlmann et al. [6], and Embrechts [17]. Some other works on the use of the Esscher transform and its variants for option valuation and risk management include Siu et al. [38] for risk measurement, Bühlmann et al. [7] for discrete finance models, Pafumi [34], Shiryaev [37] and McLeish and Reesor [31] for some theoretical issues, Yao [42] for pricing interest-rate derivatives and Siu et al. [39] for GARCH option pricing models.

In recent years, there is a considerable interest in the applications of regime switching models driven by a hidden Markov Chain process to various financial problems. For an overview of hidden Markov Chain processes and their financial applications, see Elliott et al. [11] and Elliott and Kopp [13]. Some works on the use of hidden Markov Chain models in finance include Elliott and van der Hoek [12] for asset allocation, Pliska [35] and Elliott et al. [14] for short rate models, Elliott and Hinz [15] for portfolio analysis and chart analysis, Guo [25] for option pricing under market incompleteness, Buffington and Elliott [4, 5] for pricing European and American options, Elliott et al. [16] for volatility estimation and the working paper by Elliott and Chan in 2004 for a dynamic portfolio selection problem. Much of the work in the literature focus on the use of the Esscher transform for option valuation under incomplete markets induced by Lévy-type processes. There is a relatively little amount of work on the use of the Esscher transform for option valuation under incomplete markets generated by other asset price dynamics, such as Markov regime switching processes.

In this paper, we investigate the option pricing problem when the risky underlying assets are driven by Markov-modulated Geometric Brownian Motion (GBM). In particular, the market interest rate, the appreciation rate and the volatility of the risky asset depend on the unobservable states of the economy which are modelled by a continuous-time Hidden Markov process. The market described by the Markov-modulated GBM model is incomplete in general, and, hence, the martingale measure is not unique. Instead of using the argument by Guo [25] that the market is completed by Arrow-Debreu securities, we adopt the regime switching Esscher transform which is the modification of the random Esscher transform introduced by Siu et al. [38]. We assume that the process of the parameters for the regime switching random Esscher transform is driven by the hidden Markov Chain model. By using the result in Miyahara [33], we can justify our pricing result by the minimal entropy martingale measure, (MEMM). We organize our paper as follows:

Section two presents the main idea of our paper. Section three justifies our pricing result by the MEMM. The final section concludes the paper and proposes some topics for further investigation.

2 The model

Suppose $(\Omega, \mathcal{F}, \mathcal{P})$ is a complete probability space, where $\mathcal{P}$ is a real-world probability measure. Let $\mathcal{T}$ denote the time index set $[0, T]$ of the model. Let $\{W_t\}_{t\in\mathcal{T}}$ denote a standard Brownian Motion on $(\Omega, \mathcal{F}, \mathcal{P})$. We assume that the states of the economy are modelled by a continuous-time hidden Markov Chain process $\{X_t\}_{t\in\mathcal{T}}$ on $(\Omega, \mathcal{F}, \mathcal{P})$ with a finite state space $\mathcal{X} := (x_1, x_2, \ldots, x_N)$. Without loss of generality, we can identify the state space of $\{X_t\}_{t\in\mathcal{T}}$ to be a finite set of unit vectors $\{e_1, e_2, \ldots, e_N\}$, where $e_i = (0, \ldots, 1, \ldots, 0) \in \mathcal{R}^N$. We suppose that $\{X_t\}_{t\in\mathcal{T}}$ and $\{W_t\}_{t\in\mathcal{T}}$ are independent.

Write $A(t)$ for the generator $[a_{ij}(t)]_{i,j=1,2,\ldots,N}$. Then, from Elliott et al. [11], we have the following semi-martingale representation theorem for $\{X_t\}_{t\in\mathcal{T}}$:

$$X_t = X_0 + \int_0^t A(s) X_s ds + M_t \ , \tag{2.1}$$

where $\{M_t\}_{t\in\mathcal{T}}$ is an $\mathcal{R}^N$-valued martingale increment process with respect to the filtration generated by $\{X_t\}_{t\in\mathcal{T}}$.

We consider a financial model consisting of two risky underlying assets, namely a bank account and a stock, that are tradable continuously. The instantaneous market interest rate $\{r(t, X_t)\}_{t\in\mathcal{T}}$ of the bank account is given by:

$$r_t := r(t, X_t) = < r, X_t > \ , \tag{2.2}$$

where $r := (r_1, r_2, \ldots, r_N)$ with $r_i > 0$ for each $i = 1, 2, \ldots, N$ and $< \cdot, \cdot >$ denotes the inner product in $\mathcal{R}^N$.

In this case, the dynamics of the price process $\{B_t\}_{t\in\mathcal{T}}$ for the bank account are described by:

$$dB_t = r_t B_t dt \ , \quad B_0 = 1 \ . \tag{2.3}$$

We suppose that the stock appreciation rate $\{\mu_t\}_{t\in\mathcal{T}}$ and the volatility $\{\sigma_t\}_{t\in\mathcal{T}}$ of S also depend on $\{X_t\}_{t\in\mathcal{T}}$ and are described by:

$$\mu_t := \mu(t, X_t) = < \mu, X_t > \ , \quad \sigma_t := \sigma(t, X_t) = < \sigma, X_t > \ , \tag{2.4}$$

where $\mu := (\mu_1, \mu_2, \ldots, \mu_N)$ and $\sigma := (\sigma_1, \sigma_2, \ldots, \sigma_N)$ with $\sigma_i > 0$ for each $i = 1, 2, \ldots, N$.

The dynamics of the stock price process $\{S_t\}_{t \in \mathcal{T}}$ are then given by the following Markov-modulated Geometric Brownian Motion:

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t \ , \quad S_0 = s \ . \tag{2.5}$$

Let Z_t denote the logarithmic return $\ln(S_t/S_0)$ from S over the interval $[0,t]$. Then, the stock price dynamics can be written as:

$$S_t = S_u \exp(Z_t - Z_u) \ , \tag{2.6}$$

where

$$Z_t = \int_0^t \left(\mu_s - \frac{1}{2}\sigma_s^2 \right) ds + \int_0^t \sigma_s dW_s \ . \tag{2.7}$$

The additional uncertainty described by regime switching makes the market incomplete. Guo [25] used Arrow-Debreu securities related to the cost of switching to complete the market by hedging away the additional uncertainty induced by regime switching.

Write $\{\mathcal{F}_t^X\}_{t \in \mathcal{T}}$ and $\{\mathcal{F}_t^Z\}_{t \in \mathcal{T}}$ for the $\mathcal{P}$-augmentation of the natural filtrations generated by $\{X_t\}_{t \in \mathcal{T}}$ and $\{Z_t\}_{t \in \mathcal{T}}$, respectively. For each $t \in \mathcal{T}$, we define $\mathcal{G}_t$ as the σ-algebra $\mathcal{F}_t^X \vee \mathcal{F}_t^Z$. Let $\theta_t := \theta(t, X_t)$ denote the regime switching Esscher parameter, which can be written as follows:

$$\theta(t, X_t) = <\theta, X_t> \ , \tag{2.8}$$

where $\theta := (\theta_1, \theta_2, \ldots, \theta_N) \in \mathcal{R}^N$.

Then, the regime switching Esscher transform $\mathcal{Q}_\theta \sim \mathcal{P}$ on $\mathcal{G}_t$ with respect to a family of parameters $\{\theta_s\}_{s \in [0,t]}$ is given by:

$$\left.\frac{d\mathcal{Q}_\theta}{d\mathcal{P}}\right|_{\mathcal{G}_t} = \frac{\exp\left(\int_0^t \theta_s dZ_s \right)}{E_{\mathcal{P}}\left[\exp\left(\int_0^t \theta_s dZ_s \right) \middle| \mathcal{F}_t^X \right]} \ , \quad t \in \mathcal{T} \ . \tag{2.9}$$

Since $\int_0^t \theta_s dZ_s | \mathcal{F}_t^X \sim N(\int_0^t \theta_s(\mu_s - \frac{1}{2}\sigma_s^2)ds, \int_0^t \theta_s^2 \sigma_s^2 ds)$, which is a normal distribution with mean $\int_0^t \theta_s(\mu_s - \frac{1}{2}\sigma_s^2)ds$ and variance $\int_0^t \theta_s^2 \sigma_s^2 ds$, under $\mathcal{P}$,

$$\begin{aligned} &E_{\mathcal{P}}\left[\exp\left(\int_0^t \theta_s dZ_s \right) \middle| \mathcal{F}_t^X \right] \\ &= \exp\left[\int_0^t \theta_s \left(\mu_s - \frac{1}{2}\sigma_s^2 \right) ds + \frac{1}{2} \int_0^t \theta_s^2 \sigma_s^2 ds \right] . \end{aligned} \tag{2.10}$$

The Radon-Nikodym derivative of the regime switching Esscher transform is given by:

$$\left.\frac{d\mathcal{Q}_\theta}{d\mathcal{P}}\right|_{\mathcal{G}_t} = \exp\left(\int_0^t \theta_s \sigma_s dW_s - \frac{1}{2} \int_0^t \theta_s^2 \sigma_s^2 ds \right) . \tag{2.11}$$

Let $\{\tilde{\theta}_t\}_{t\in\mathcal{T}}$ denote a family of risk-neutral regime switching Esscher parameters. By the fundamental theorem of asset pricing (see Harrison and Kreps [26], Harrison and Pliska [27, 28] and Delbaen and Schachermayer [9]), the absence of arbitrage opportunities is "essentially" equivalent to the existence of an equivalent martingale measure under which the discounted stock price process is a martingale. In our setting, due to the presence of the uncertainty generated by the hidden Markov Chain process, the martingale condition is given by considering an enlarged filtration as follows:

$$S_0 = E_{\mathcal{Q}_{\tilde{\theta}}}\left[\exp\left(-\int_0^t r_s ds\right) S_t \middle| \mathcal{F}_t^X\right], \quad \text{for any } t \in \mathcal{T} . \tag{2.12}$$

The martingale condition on $\mathcal{F}_t^X$ can be interpreted as the martingale condition when the information for the hidden Markov Chain process is accessible to the market's agent in advance. By the tower law, if one can find a probability measure $\mathcal{Q}$ satisfying the martingale condition on $\mathcal{F}_t^X$, $\mathcal{Q}$ also satisfies the martingale condition without knowing $\mathcal{F}_t^X$.

Let $b_s := \mu_s - \frac{1}{2}\sigma_s^2$. By Bayes' rule and (2.11), the martingale condition can be written as:

$$E_{\mathcal{P}}\left\{\exp\left[\int_0^t \left(b_s - r_s - \frac{1}{2}\tilde{\theta}_s^2\sigma_s^2\right)ds + \int_0^t (\tilde{\theta}_s + 1)\sigma_s dW_s\right] \middle| \mathcal{F}_t^X\right\} = 1, \quad \text{for any } t \in \mathcal{T} . \tag{2.13}$$

Let $L_t := \int_0^t (b_s - r_s - \frac{1}{2}\tilde{\theta}_s^2\sigma_s^2)ds + \int_0^t (\tilde{\theta}_s + 1)\sigma_s dW_s$. Then, $L_t|\mathcal{F}_t^X \sim N(\int_0^t (b_s - r_s - \frac{1}{2}\tilde{\theta}_s^2\sigma_s^2)ds, \int_0^t (\tilde{\theta}_s + 1)^2\sigma_s^2 ds)$ under $\mathcal{P}$. Therefore,

$$\begin{aligned} &E_{\mathcal{P}}\left\{\exp\left[\int_0^t \left(b_s - r_s - \frac{1}{2}\tilde{\theta}_s^2\sigma_s^2\right)ds + \int_0^t (\tilde{\theta}_s + 1)\sigma_s dW_s\right] \middle| \mathcal{F}_t^X\right\} \\ &= \exp\left\{\int_0^t \left[b_s - r_s + \left(\frac{2\tilde{\theta}_s + 1}{2}\right)\sigma_s^2\right] ds\right\} . \end{aligned} \tag{2.14}$$

Hence, the martingale condition can be written:

$$\int_0^t \left(b_s - r_s + \frac{1}{2}\sigma_s^2 + \tilde{\theta}_s\sigma_s^2\right)ds = 0 , \quad \text{for any } t \in \mathcal{T} . \tag{2.15}$$

Therefore,

$$b_t - r_t + \frac{1}{2}\sigma_t^2 + \tilde{\theta}_t\sigma_t^2 = 0 , \tag{2.16}$$

and $\tilde{\theta}_t$ can be determined uniquely as follows:

$$\tilde{\theta}_t = \frac{r_t - \mu_t}{\sigma_t^2} = -\frac{\lambda_t}{\sigma_t}, \quad t \in \mathcal{T} , \tag{2.17}$$

where λ_t is the market price of risk at time t.

Then, $\tilde{\theta}_t = <\tilde{\theta}, X_t>$, where $\tilde{\theta} = (\frac{r_1 - \mu_1}{\sigma_1^2}, \frac{r_2 - \mu_2}{\sigma_2^2}, \ldots, \frac{r_N - \mu_N}{\sigma_N^2})$.

Write $\tilde{\mathcal{G}}_{t,s}$ for the double indexed σ-field $\mathcal{F}_t^X \vee \mathcal{F}_s^Z$, for any $s, t \in \mathcal{T}$ with $s \le t$. Then, it can be shown that the discounted stock price process $\{e^{-\int_0^t r_s ds} S_t\}_{t\in\mathcal{T}}$ is a martingale with respect to $\{\tilde{\mathcal{G}}_{t,s}\}_{t\in\mathcal{T}, s\in[0,t]}$ under $\mathcal{Q}_{\tilde{\theta}}$. Hence, at any time $t \in \mathcal{T}$, the price of a European-style option written on S with payoff $V(S_T)$ at maturity T is given by:

$$V(t,T,S_t,P_{t,T},U_{t,T}) = E_{\mathcal{Q}_{\tilde{\theta}}}\left[\exp\left(-\int_t^T r_s ds\right) V(S_T)\middle|\tilde{\mathcal{G}}_{T,t}\right], \quad (2.18)$$

where $P_{t,T} = \int_t^T < r, X_s > ds$ and $U_{t,T} = \int_t^T < \sigma, X_s >^2 ds$.

For pricing the option, we first need to know the conditional probability distribution of Z_T given $\tilde{\mathcal{G}}_{T,t}$ under $\mathcal{Q}_{\tilde{\theta}}$. Since $P_{t,T}$ and $U_{t,T}$ are unknown in practice, the price $V(t,T,S_t,P_{t,T},U_{t,T})$ is also unknown. As in Buffington and Elliott [4, 5], we can take a second expectation of $V(t,T,S_t,P_{t,T},U_{t,T})$ with respect to the probability distributions of $P_{t,T}$ and $U_{t,T}$, which can be interpreted as a statistical estimation of the unobservable price $V(t,T,S_t,P_{t,T},U_{t,T})$ given observable market information.

First, we determine the conditional probability distribution of Z_T given $\tilde{\mathcal{G}}_{T,t}$ under $\mathcal{Q}_{\tilde{\theta}}$. By the martingale condition, the Radon-Nikodym derivative of $\mathcal{Q}_{\tilde{\theta}}$ is given by:

$$\left.\frac{d\mathcal{Q}_{\tilde{\theta}}}{d\mathcal{P}}\right|_{\mathcal{G}_t} = \exp\left[\int_0^t \left(\frac{r_s - \mu_s}{\sigma_s}\right) dW_s - \frac{1}{2}\int_0^t \left(\frac{r_s - \mu_s}{\sigma_s}\right)^2 ds\right]. \quad (2.19)$$

Using Girsanov's theorem, $\tilde{W}_t = W_t + \int_0^t (\frac{r_s - \mu_s}{\sigma_s}) ds$ is a standard Brownian motion with respect to $\{\mathcal{G}_t\}_{t\in\mathcal{T}}$ under $\mathcal{Q}_{\tilde{\theta}}$. Hence, we can write the stock price dynamics under $\mathcal{Q}_{\tilde{\theta}}$ as follows:

$$dS_t = r_t S_t dt + \sigma_t S_t dW_t\ . \quad (2.20)$$

Hence, $Z_T|\tilde{\mathcal{G}}_{T,t} \sim N(\int_t^T \left(r_s - \frac{1}{2}\sigma_s^2\right) ds, \int_t^T \sigma_s^2 ds)$. This is in complete agreement with the result in Guo [25] and Buffington and Elliott [4, 5].

Let $J_i(t,T)$ denote the occupation time of $\{X_t\}_{t\in\mathcal{T}}$ in state i over the time duration $[t,T]$, where $t \in \mathcal{T}$. Then,

$$P_{t,T} = \int_t^T < r, X_s > ds = \sum_{i=1}^N r_i J_i(t,T)\ ,$$
$$U_{t,T} = \int_t^T < \sigma, X_s >^2 ds = \sum_{i=1}^N \sigma_i^2 J_i(t,T)\ . \quad (2.21)$$

Hence, in order to find the distribution of $P_{t,T}$ and $U_{t,T}$, we need to determine the joint distribution of the occupation times $(J_1(t,T), J_2(t,T), \ldots, J_N(t,T))$.

Write $J(t,T) := (J_1(t,T), J_2(t,T), \ldots, J_N(t,T))$ for the vector of occupation times. Let D denote a diagonal matrix consisting of the elements in the vector

$\zeta := (\zeta_1, \zeta_2, \ldots, \zeta_N)$ as its diagonal. Then, for any ζ, the characteristic function of $J(t,T)$ is given by:

$$E[\exp(i < \zeta, J(t,T) >)|\mathcal{F}_t^Z] = < \exp[(A+iD)(T-t)]X_t, \mathbf{I} > , \tag{2.22}$$

where $i = \sqrt{-1}$ and $\mathbf{I} := (1,1,\ldots,1) \in \mathcal{R}^N$.

Let $\phi(J_1, J_2, \ldots, J_N)$ denote the joint probability distribution for the occupation times $(J_1(t,T), J_2(t,T), \ldots, J_N(t,T))$. Note that $\phi(J_1, J_2, \ldots, J_N)$ can be completely determined by the characteristic function $E[\exp(i < \zeta, J(t,T) >)|\mathcal{F}_t^Z]$. Write $N(\cdot)$ for the probability distribution function of a standard normal distribution. From Buffington and Elliott [4, 5], the price of a European call option at time t with strike price K and maturity at time T is given by:

$$C(t,T,S_t) := \int_t^T \int_t^T \cdots \int_t^T C(t,T,S_t,P_{t,T},U_{t,T})\phi(J_1,J_2,\ldots,J_N) \, dJ_1 dJ_2 \ldots dJ_N , \tag{2.23}$$

where

$$C(t,T,S_t,P_{t,T},U_{t,T}) = S_t N(d_{1,t,T}) - K\exp(-P_{t,T})N(d_{2,t,T}) , \tag{2.24}$$

and

$$d_{1,t,T} = (U_{t,T})^{-1/2}\left(\ln\frac{S_t}{K} + P_{t,T} + \frac{1}{2}U_{t,T}\right) , \quad d_{2,t,T} = d_{1,t,T} - (U_{t,T})^{1/2} . \tag{2.25}$$

The same procedure can be applied to evaluate the price of a European put option. When the number of regimes N is two and the market interest rate of the bank account is a constant, the pricing formula for a European put option obtained from our approach is in complete agreement with the formula in Hardy [29]. It has been mentioned in Hardy [29] that the market described by a regime switching model is incomplete and the risk-neutral measure is not uniquely determined. We adopt the regime switching Esscher transform to choose the risk-neutral pricing measure in the incomplete setting and justify the choice by the MEMM in the next section.

Since there can be substantial changes in the states or conditions of an economy over a long time period, it is of practical importance and relevance to incorporate the switching behavior of the states of the economy for the valuation of a relatively long-dated option. The option pricing formula with regime switching can be applied in this case. The option pricing result based on the regime switching Esscher transform can be extended easily to the case that there is more than one risky underlying asset in the model.

3 Regime switching Esscher transform and relative entropy

The minimial entropy martingale measure (MEMM) is one of the major tools for option valuation in an incomplete market. Buchen and Kelly [3] investigated the use of the principle of maximum entropy to the estimation of the distribution for the underlying assets from a set of market option data. Stutzer [40] studied the problem of derivative valuation using a simple non-parametric approach based on the MEMM. Avellaneda [1] presented an algorithm for calibrating asset pricing models to benchmark prices by minimizing relative entropy with respect to a prior distribution. Delbaen et al. [8] studied the hedging problem of a contingent claim by maxmizing the expected exponential utility on the terminal wealth and established the duality between the MEMM and the maximization of the expected exponential utility. Kitamura and Stutzer [30] established the relationship between the relative entropy approach and the linear projection approach to the estimation and diagnostic checking for the stochastic discount factor models in asset pricing.

Here, we justify the choice of the equivalent martingale measure $\mathcal{Q}_{\tilde{\theta}}$ using the regime switching Esscher transform by minimizing the relative entropy with respect to $\mathcal{P}$. Given $\mathcal{P}$, the relative entropy $I_{\mathcal{P}}(\mathcal{Q})$ of any absolutely continuous probability measure $\mathcal{Q}$ with respect to $\mathcal{P}$ is defined by:

$$I_{\mathcal{P}}(\mathcal{Q}) := E_{\mathcal{P}}\left[\ln\left(\frac{d\mathcal{Q}}{d\mathcal{P}}\right)\middle|\mathcal{F}_t^X\right] . \tag{3.1}$$

A probability measure $\mathcal{Q}^*$ equivalent to $\mathcal{P}$ on $\mathcal{G}_t$ is the MEMM if it satisfies:

$$I_{\mathcal{P}}(\mathcal{Q}^*) \leq I_{\mathcal{P}}(\mathcal{Q}) , \quad \text{for all } \mathcal{Q} \in \mathcal{S}(\mathcal{P}) , \tag{3.2}$$

where $\mathcal{S}(\mathcal{P})$ is the space of all martingale measures equivalent to $\mathcal{P}$.

Miyahara [33] provided the sufficient condition for the MEMM under a geometric Lévy process (see Theorem 1 therein). We adopt Theorem 1 in Miyahara [33] in the case of Markov-modulated GBM. In this case, the sufficient condition for MEMM with respect to $\mathcal{F}_t^X$ is stated in the following proposition.

Proposition 3.1. *Suppose there exists a β_t such that the following equation is satisfied:*

$$\beta_t = \frac{r_t - \mu_t}{\sigma_t^2} . \tag{3.3}$$

Let $\mathcal{Q}^*$ be a probability measure equivalent to the measure $\mathcal{P}$ on $\mathcal{G}_t$ defined by the following Radon-Nikodym derivative:

$$\left.\frac{d\mathcal{Q}^*}{d\mathcal{P}}\right|_{\mathcal{G}_t} := \exp\left(\int_0^t \beta_s \sigma_s dW_s - \frac{1}{2}\int_0^t \beta_s^2 \sigma_s^2 ds\right) . \tag{3.4}$$

Then,

1. *$\mathcal{Q}^*$ is well defined and uniquely determined by the above Radon-Nikodym derivative*
2. *$\mathcal{Q}^*$ is the MEMM for the Markov-modulated GBM*

The proof of Proposition 3.1 is similar to the proof of Theorem 1 in Miyahara [33]. So, we do not repeat it here.

Let $\beta_t = \tilde{\theta}_t$, for each $t \in \mathcal{T}$. Then, $\mathcal{Q}_{\tilde{\theta}}$ is exactly the same as $\mathcal{Q}^*$. By proposition 3.1, $\mathcal{Q}_{\tilde{\theta}}$ is the MEMM. This justifies our choice of pricing measure based on the regime-switching Esscher transform. It was mentioned in Miyahara [33] that the MEMM is related to the expected exponential utility maximization and to the theory of large deviations. This further illustrates the advantages of using the Esscher transform.

4 Conclusion and further investigation

We have developed a method to price options when the risky underlying assets are driven by Markov-modulated Geometric Brownian Motion (GBM) based on a modification of the random Esscher transform by Siu et al. [38], namely the regime switching random Esscher transform. The choice of this martingale pricing measure is justified by the minimization of the relative entropy.

We may explore the applications of our models to other types of exotic options or hybrid financial products, such as barrier options, lookback options, Asian options, game options, passport options and option-embedded insurance products, etc. We may extend our framework to deal with interest rate products and credit derivatives. Finally, we may consider our model in the framework of quantile hedging by Föllmer and Leukert [21].

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