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# Quantile hedging for a jump-diffusion financial market model

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**Abstract.** The paper is devoted to the problem of hedging contingent claims in the framework of a jump-diffusion model. Based on the results of H. Föllmer and P. Leukert [1]-[2] in a general semimartingale setting, we study the question how an investor can maximize the probability of a successful hedge under the constraint that he invests not more than a fixed amount of capital which is strictly less than the price of the option. We derive explicit formulas for this so-called quantile hedging strategy.

## 1. Introduction

One of the basic problems in *Contingent Claim Analysis* is the problem of hedging options. A number of papers, including the famous paper by Black and Scholes, are devoted to the analysis of hedges which succeed with probability one (see, for example Shiryaev[3]). The solution of such a problem in the case of a complete market yields to the so-called *fair price* as the minimal capital that is required to replicate the contingent claim. But in general the initial capital of an investor can be less this price. The natural question arises: What kind of hedging strategy should an investor pursue who is short the option in this situation? One answer to this question is the following: The investor establishes a self-financing hedging strategy that successfully replicates the option *with maximal probability* over all self-financing strategies that do not require more capital than he has at disposal. General results concerning this type of hedging (quantile hedging, hedging with a given probability) were given by H. Föllmer and P. Leukert [1]-[2] when the price process of the underlying asset is a semimartingale. We consider here the special case of a jump-diffusion market model firstly introduced by Aase[4] and derive the corresponding stochastic differential equations for hedging strategy, its value and the price of call-option.

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## 2. Description of the Model and Auxiliary results

Let $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\geqslant 0}, \mathrm{P})$ be a standard stochastic basis. We assume there are two risky assets (stocks) $S^i, i = 1,2$, whose price-processes are described by the following stochastic differential equations

$$dS_t^i = S_{t-}^i(\mu^i dt + \sigma^i dW_t - \nu^i d\Pi_t), i = 1,2, \tag{1}$$

where $W$ is a standard Wiener process, $\Pi$ is a Poisson process with positive intensity $\lambda$. Suppose also that $W$ and $\Pi$ are independent and the filtration $\mathbf{F}$ is generated by $W$ and $\Pi$, $\mu^i \in \mathbb{R}, \sigma^i > 0, \nu^i < 1$.

There is a non-risky asset $B$ (bond or bank account) which satisfies the equation

$$dB_t = rB_t dt, B_0 = 1, r \in \mathbb{R} \tag{2}$$

Every predictable process $\pi = (\pi_t)_{t\geqslant 0} = ((\beta_t, \gamma_t^1, \gamma_t^2))_{t\geqslant 0}$ can be regarded as *trading strategy* or portfolio. The value of such a portfolio is given by

$$X_t^\pi = \beta_t B_t + \gamma_t^1 S_t^1 + \gamma_t^2 S_t^2. \tag{3}$$

A strategy with non-negative value is called *admissible*. If the discounted value $\frac{X^\pi}{B}$ of such a strategy $\pi$ can be represented in the form

$$\frac{X_t^\pi}{B_t} = \frac{X_0^\pi}{B_0} + \int_0^t \sum_{i=1}^2 \gamma_u^i d\left(\frac{S_u^i}{B_u}\right) (\mathrm{P}-\text{a.s.}), \tag{4}$$

then $\pi$ is called by *self-financing* ($\pi \in \mathbb{SF}$).

The market (1)-(2) with the class $\mathbb{SF}$ is *complete* if the following conditions are fulfiled

$$\sigma^2\nu^1 - \sigma^1\nu^2 \neq 0, \frac{(\mu^1 - r)\sigma^2 - (\mu^2 - r)\sigma^1}{\sigma^2\nu^1 - \sigma^1\nu^2} > 0. \tag{5}$$

Under condition (5) there exists a unique equivalent martingale measure $\mathrm{P}^*$ with local density

$$Z_t = \frac{d\mathrm{P}_t^*}{d\mathrm{P}_t} = \frac{d\mathrm{P}^*}{d\mathrm{P}}\bigg|\mathcal{F}_t = exp(\alpha^* W_t - \frac{\alpha^{*2}}{2}t + (\lambda - \lambda^*)t + (ln\lambda^* - ln\lambda)\Pi_t), \tag{6}$$

where the pair $(\alpha^*, \lambda^*)$ is given by the unique solution of the equation (see, for instance Melnikov and Shiryaev [8], Volkov and Kramkov [8])

$$\begin{cases} \mu^1 - r = & -\sigma^1\alpha^* + \nu^1\lambda^* \\ \mu^2 - r = & -\sigma^2\alpha^* + \nu^2\lambda^* \end{cases}, \lambda^* > 0. \tag{7}$$

Under the measure $\mathrm{P}^*$, $W_t^* = W_t - \alpha^* t$ is a Wiener process, $\Pi$ is a Poisson process with intensity $\lambda^* > 0$, and $W^*$ is independent of $\Pi$.

A non-negative $\mathcal{F}_T$-measurable function $f_T$ is called contingent claim. For a *perfect* hedge, we have to find a self-financing strategy $\pi$ that eliminates the risk completely (i.e. $\mathrm{P}\{X_T^\pi \geqslant f_T\} = 1$) and requires minimal initial capital $X_0^\pi = \mathbb{C}(T, S_0^1)$.



We consider classical options of the form $f_T = f(S_T^1)$. According to the general theory of perfect hedging (see [3]-[7]) the fair price of the option is given by

$$\mathbb{C}(T, S_0^1) = \mathbf{E}^* f_T e^{-rT}, \tag{8}$$

where $\mathbf{E}^*$ denotes expectation w.r. to $\mathrm{P}^*$.

Let us note by the Ito formula that

$$\begin{aligned} S_t^i &= S_0^i exp(\sigma^i W_t + (\mu^i - \tfrac{1}{2}(\sigma^i)^2)t)(1-\nu^i)^{\Pi_t} \\ &= S_0^i exp(\sigma^i W_t^* + (\mu^i + \sigma^i\alpha^* - \tfrac{1}{2}(\sigma^i)^2)t)(1-\nu^i)^{\Pi_t} \\ &= S_0^i exp(\sigma^i W_t^* + (r + \nu^i\lambda^* - \tfrac{1}{2}(\sigma^i)^2)t)(1-\nu^i)^{\Pi_t}, \end{aligned} \tag{9}$$

$$Y_t^i = \frac{S_t^i}{B_t} = Y_0^i exp(\sigma W_t^* + (\nu^i\lambda^* - \frac{1}{2}(\sigma^i)^2)t)(1-\nu^i)^{\Pi_t}$$

Using (8)-(9) and the independence of $W^*$ and $\Pi$ yields

$$\mathbb{C}(T, S_0^1) = \sum_{n=0}^\infty \mathbf{E}^*\left[f\left(S_0^1 e^{\nu^1\lambda^* T} e^{(\sigma^1 W_T^* + (r - \frac{1}{2}(\sigma^1)^2)T)}(1-\nu^1)^n\right)e^{-rT}\right] e^{-\lambda^* T}\frac{(\lambda^* T)^n}{n!} \tag{10}$$

In the case of a call option we obtain from (10) formula (see [4],[7]):

$$\mathbb{C}(T, S_0^1) = e^{-\lambda^* T}\sum_{n=0}^\infty \mathbb{C}^{BS}(S_0^1(1-\nu^1)^n e^{\nu^1\lambda^* T}, K, T, r, \sigma^1)\frac{(\lambda^* T)^n}{n!} \tag{11}$$

where $\mathbb{C}^{BS}$ is the *Black-Scholes price*

$$\mathbb{C}^{BS}(S_0, K, T, r, \sigma) = S_0\Phi(y_+(T)) - Ke^{-rT}\Phi(y_-(T))$$

$$y_\pm(T-t) = \frac{ln\frac{S_t}{K} + (T-t)(r \pm \frac{\sigma^2}{2})}{\sigma\sqrt{T-t}}$$

and $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{-\frac{y^2}{2}}dy$.

The value of the corresponding hedging strategy $\pi$ at time $t$ is given by

$$X_t^\pi = \mathbb{C}(T-t, S_t^1) =$$
$$e^{-\lambda^*(T-t)}\sum_{n=0}^\infty \mathbb{C}^{BS}(S_t^1(1-\nu^1)^n e^{\nu^1\lambda^*(T-t)}, K, T-t, r, \sigma^1)\frac{(\lambda^*(T-t))^n}{n!}.$$

The components $\gamma_t^1, \gamma_t^2$ of this hedging strategy are solutions of the equations

$$\begin{cases} \gamma_t^1\sigma^1 S_{t-}^1 + \gamma_t^2\sigma^2 S_{t-}^2 = S_{t-}^1\sigma^1\frac{\partial}{\partial x}\mathbb{C}(S_{t-}^1, t) \\ \gamma_t^1\nu^1 S_{t-}^1 + \gamma_t^2\nu^2 S_{t-}^2 = \mathbb{C}(S_{t-}^1, t) - \mathbb{C}(S_{t-}^1(1-\nu^1), t). \end{cases} \tag{17}$$

We note that the equation (17) has a unique solution in view (5).

The first component of the hedge can be recognized from the *balance equation*

(18) $$\beta_t = \frac{\mathbb{C}(S^1_{t-}, t) - \gamma^1_t S^1_{t-} - \gamma^2_t S^2_{t-}}{B_t}$$

The value $X^\pi_t = \mathbb{C}(S^1_t, t)$ of such a strategy satisfies to the following equation

(19)
$$\left[\mathbb{C}(S^1_{t-}(1-\nu^1), t) - \mathbb{C}(S^1_{t-}, t)\right]\lambda^* + \frac{\partial}{\partial t}\mathbb{C}(S^1_{t-}, t) + \frac{1}{2}(\sigma^1 S^1_{t-})^2 \frac{\partial^2}{\partial x^2}\mathbb{C}(S^1_{t-}, t)$$
$$+ rS^1_{t-}\frac{\partial}{\partial x}\mathbb{C}(S^1_{t-}, t) - r\mathbb{C}(S^1_{t-}, t) + \frac{\partial}{\partial x}\mathbb{C}(S^1_{t-}, t)\nu^1\lambda^* S^1_{t-} = 0.$$

We now introduce a constraint which is a crucial motivation for *quantile hedging*: the initial capital $X_0$ of the investor is less than $\mathbb{C}(T, S^1_0)$. Faced with the impossibility to hedge the option with certainty the investor chooses a strategy $\pi \in \mathbb{SF}$ that maximizes the success-probability $\mathrm{P}\{X^\pi_T \geqslant f_T\}$ over all self-financing strategies with initial value $X^\pi_0 \leqslant X_0 < \mathbb{C}(T, S^1_0)$. We paraphrase this as the following problem

(20) $$1 - \varepsilon = \sup_\pi P(X^\pi_T \geqslant f_T), X^\pi_0 \leqslant X_0 < \mathbb{C}(T, S^1_0)$$

where the "optimal" $\varepsilon \in [0,1]$ and $\pi$ should be found.

The problem (20) was considered by H. Föllmer and P. Leukert [1]-[2] in a "semimartingale" $(1, S)$ - setting. They derived the following general methodology to solve the problem (20):

*If $\bar{\pi}$ is the minimal hedging strategy for the contingent claim $\tilde{\phi_T} f_T$, where*

$$\tilde{\phi_T} = I_{\{\frac{dP}{dP^*} > \tilde{a} f_T\}} + \gamma I_{\{\frac{dP}{dP^*} = \tilde{a} f_T\}},$$

$$\gamma = \frac{X_0 - \mathbf{E}^*\left[e^{-rT} f_T I_{\{\frac{dP}{dP^*} > \tilde{a} f_T\}}\right]}{\mathbf{E}^*\left[e^{-rT} f_T I_{\{\frac{dP}{dP^*} = \tilde{a} f_T\}}\right]},$$

$$\tilde{a} = \inf\{a : \mathbf{E}^*\left[e^{-rT} f_T I_{\{\frac{dP}{dP^*} > a f_T\}}\right] \leqslant X_0\},$$

*then $\bar{\pi}$ is the solution to the quantile hedging problem (20) and*

$$1 - \varepsilon = \mathbf{E}(\tilde{\phi_T}).$$

**Remark 1.** *The critical value $\tilde{a}$ can be determined by condition*

(21) $$\mathbf{E}^*\left[e^{-rT}\tilde{\phi_T} f_T\right] = X_0.$$

*If $P\left[\frac{dP}{dP^*} = const \cdot f_T\right] = 0$ holds then we obtain $\tilde{\phi_T} = I_{\{\frac{dP}{dP^*} > \tilde{a} f_T\}}$.*



## 3. Main results

We consider the quantile hedging problem (20) in the case of the model (1)-(2), applying the methodology described above. The specification of the jump-diffusion model by (1)-(2) allows us to give the explicit solution to problem (20).

**Theorem 1.** *Assume that $f_T = f(S^1_T)$. Then the value process $\mathbb{C}(S^1_t, S^2_t, t) := X^\pi_t$ and the components $\beta, \gamma^1, \gamma^2$ of the optimal hedge for the problem (20) satisfy the equations*

(22)
$$\left[\mathbb{C}(S^1_{t-}(1-\nu^1), S^2_{t-}(1-\nu^2), t) - \mathbb{C}(S^1_{t-}, S^2_{t-}, t)\right]\lambda^* + rS^1_{t-}\frac{\partial}{\partial x}\mathbb{C}(S^1_{t-}, S^2_{t-}, t)$$
$$+ rS^2_{t-}\frac{\partial}{\partial y}\mathbb{C}(S^1_{t-}, S^2_{t-}, t) + \frac{\partial}{\partial t}\mathbb{C}(S^1_{t-}, S^2_{t-}, t) + \frac{1}{2}(\sigma^1 S^1_{t-})^2\frac{\partial^2}{\partial x^2}\mathbb{C}(S^1_{t-}, S^2_{t-}, t)$$
$$+ \frac{1}{2}(\sigma^2 S^2_{t-})^2\frac{\partial^2}{\partial y^2}\mathbb{C}(S^1_{t-}, S^2_{t-}, t) + (\sigma^1\sigma^2 S^1_{t-}S^2_{t-})\frac{\partial^2}{\partial x \partial y}\mathbb{C}(S^1_{t-}, S^2_{t-}, t)$$
$$- r\mathbb{C}(S^1_{t-}, S^2_{t-}, t) + \frac{\partial}{\partial x}\mathbb{C}(S^1_{t-}, S^2_{t-}, t)\nu^1\lambda^* S^1_{t-} + \frac{\partial}{\partial y}\mathbb{C}(S^1_{t-}, S^2_{t-}, t)\nu^2\lambda^* S^2_{t-} = 0,$$

(23) $$\begin{cases} \gamma^1_t\sigma^1 S^1_{t-} + \gamma^2_t\sigma^2 S^2_{t-} = S^1_{t-}\sigma^1\frac{\partial}{\partial x}\mathbb{C}(S^1_{t-}, S^2_{t-}, t) + S^2_{t-}\sigma^2\frac{\partial}{\partial y}\mathbb{C}(S^1_{t-}, S^2_{t-}, t) \\ \gamma^1_t\nu^1 S^1_{t-} + \gamma^2_t\nu^2 S^2_{t-} = \mathbb{C}(S^1_{t-}, S^2_{t-}, t) - \mathbb{C}(S^1_{t-}(1-\nu^1), S^2_{t-}(1-\nu^2), t), \end{cases}$$

(24) $$\beta_t = \frac{\mathbb{C}(S^1_{t-}, S^2_{t-}, t) - \gamma^1_t S^1_{t-} - \gamma^2_t S^2_{t-}}{B_t}.$$

Note that the value $\mathbb{C}$ of the quantile hedging strategy depends on $S^1_t$ and $S^2_t$ whereas the value of the perfect hedging strategy depends on $S^1_t$ only. Nevertheless we use here the same letter for the value.

Consider $f = (S^1_T - K)^+$. We have to distinguish two cases:
a) $-\frac{\alpha^*}{\sigma^1} \leqslant 1$ and b) $-\frac{\alpha^*}{\sigma^1} > 1$

The following theorem gives the full answer to problem (20).

**Theorem 2.**

*Case a.*

*1) The maximal probability for "success hedging" equals*

(25) $$1 - \varepsilon = e^{-\lambda T}\sum_{n=0}^{\infty}\Phi\left(\frac{c'_n(\tilde{a}) + \alpha^* T}{\sqrt{T}}\right)\frac{(\lambda T)^n}{n!}$$

*where $\tilde{a}$ is the solution of the equation*

(26)
$$X_0 = e^{-\lambda^* T}\sum_{n=0}^{\infty}\Big[\mathbb{C}^{BS}(S^1_0(1-\nu^1)^n e^{\nu^1\lambda^* T}, K, T, r, \sigma^1)$$
$$-\mathbb{C}^{BS}(S^1_0(1-\nu^1)^n e^{\nu^1\lambda^* T}, c_n(a), T, r, \sigma^1) - (c_n(a) - K)e^{-rT}\Phi\left(\frac{-c'_n(a)}{\sqrt{T}}\right)\Big]\frac{(\lambda^* T)^n}{n!},$$

*$c_n(a)$ is the unique root of* $x^{-\frac{\alpha^*}{\sigma^1}} = g \cdot b^n a e^{-rT}(x-K)$,

$$c_n'(a) = \frac{1}{\sigma^1}\left(\ln\left(\frac{c_n(a)}{(1-\nu^1)^n S_0^1}\right) - (r+\nu^1\lambda^* - \frac{1}{2}(\sigma^1)^2)T\right), \tag{27}$$

*and*

$$g = \frac{1}{S_0^{1\frac{\alpha^*}{\sigma^1}}} \exp\left(-\frac{\alpha^*\mu^1}{\sigma^1}T + \frac{\sigma^1\alpha^*}{2}T - \frac{\alpha^{*2}}{2}T + (\lambda-\lambda^*)T\right), \tag{28}$$

$$b = \frac{\lambda^*}{\lambda(1-\nu^1)^{\frac{\alpha^*}{\sigma^1}}}. \tag{29}$$

*2) The value of the optimal strategy is given by*

$$\begin{aligned} X_t^\pi &= e^{-\lambda^*(T-t)} \sum_{n=0}^\infty \Big[\mathbb{C}^{BS}(S_t^1(1-\nu^1)^n e^{\nu^1\lambda^*(T-t)}, K, (T-t), r, \sigma^1) \\ &- \mathbb{C}^{BS}(S_t^1(1-\nu^1)^n e^{\nu^1\lambda^*(T-t)}, \tilde{c}_n, (T-t), r, \sigma^1) \\ &- (\tilde{c}_n - K)e^{-r(T-t)} \cdot \Phi(-\frac{\tilde{c}_n'}{\sqrt{T-t}})\Big] \frac{(\lambda^*(T-t))^n}{n!}, \end{aligned} \tag{30}$$

*where $\tilde{c}_n$ is the root of the equation:*

$$x^{-\frac{\alpha^*}{\sigma^1}} = g \cdot b^n b^{\Pi(S_t^1, S_t^2)} \tilde{a} e^{-rT}(x-K)^+, \tag{31}$$

$$\tilde{c}_n' = \frac{1}{\sigma^1}\left(\ln\left(\frac{\tilde{c}_n}{(1-\nu^1)^n S_t^1}\right) - (r+\nu^1\lambda^* - \frac{1}{2}(\sigma^1)^2)(T-t)\right), \tag{32}$$

(33)

$$\Pi(S_t^1, S_t^2) = \frac{\frac{1}{\sigma^1}\ln\frac{S_t^1}{S_0^1} - (r+\nu^1\lambda^* - \frac{(\sigma^1)^2}{2})\frac{t}{\sigma^1} - \frac{1}{\sigma^2}\ln\frac{S_t^2}{S_0^2} + (r+\nu^2\lambda^* - \frac{(\sigma^2)^2}{2})\frac{t}{\sigma^2}}{\frac{1}{\sigma^1}\ln(1-\nu^1) - \frac{1}{\sigma^2}\ln(1-\nu^2)}.$$

*Case b.*

*1) The maximal probability for "success hedging" equals*

$$1-\varepsilon = e^{-\lambda T}\sum_{n=0}^\infty \left[\Phi\left(\frac{c^{1'}_n(\tilde{a}) + \alpha^* T}{\sqrt{T}}\right) + \Phi\left(-\frac{c^{2'}_n(\tilde{a}) + \alpha^* T}{\sqrt{T}}\right)\right] \frac{(\lambda T)^n}{n!}, \tag{34}$$

*where $\tilde{a}$ is the solution of*

$$\begin{aligned} X_0 &= e^{-\lambda^* T}\sum_{n=0}^\infty \Big[\mathbb{C}^{BS}(S_0^1(1-\nu^1)^n e^{\nu^1\lambda^* T}, K, T, r, \sigma^1) \\ &- \mathbb{C}^{BS}(S_0^1(1-\nu^1)^n e^{\nu^1\lambda^* T}, c_n^1(a), T, r, \sigma^1) \\ &+ \mathbb{C}^{BS}(S_0^1(1-\nu^1)^n e^{\nu^1\lambda^* T}, c_n^2(a), T, r, \sigma^1) \\ &- (c_n^1(a) - K)e^{-rT} \cdot \Phi(\frac{-c^{1'}_n(a)}{\sqrt{T}}) + (c_n^2(a) - K)e^{-rT} \cdot \Phi(\frac{-c^{2'}_n(a)}{\sqrt{T}})\Big] \frac{(\lambda^* T)^n}{n!}, \end{aligned} \tag{35}$$



*$c_n^1(a), c_n^2(a)$ are the roots of the equation* $x^{-\frac{\alpha^*}{\sigma^1}} = g \cdot b^n a e^{-rT}(x-K)$,

$$c^{i'}_n(a) = \frac{1}{\sigma^1}\left(\ln\left(\frac{c_n^i(a)}{(1-\nu^1)^n S_0^1}\right) - (r+\nu^1\lambda^* - \frac{1}{2}(\sigma^1)^2)T\right), i = 1, 2 \tag{37}$$

*$g$ , $b$ were defined by (28)-(29).*

*2) The value of the optimal strategy is given by*

(38)

$$\begin{aligned} X_t^\pi &= e^{-\lambda^*(T-t)} \sum_{n=0}^\infty \Big[\mathbb{C}^{BS}(S_t^1(1-\nu^1)^n e^{\nu^1\lambda^*(T-t)}, K, (T-t), r, \sigma^1) \\ &- \mathbb{C}^{BS}(S_t^1(1-\nu^1)^n e^{\nu^1\lambda^*(T-t)}, \tilde{c}_n^1, (T-t), r, \sigma^1) \\ &+ \mathbb{C}^{BS}(S_t^1(1-\nu^1)^n e^{\nu^1\lambda^*(T-t)}, \tilde{c}_n^2, (T-t), r, \sigma^1) \\ &- (\tilde{c}_n^1 - K)e^{-r(T-t)} \cdot \Phi(-\frac{\tilde{c}^{1'}_n}{\sqrt{T-t}}) + (\tilde{c}_n^2 - K)e^{-r(T-t)} \cdot \Phi(-\frac{\tilde{c}^{2'}_n}{\sqrt{T-t}})\Big] \frac{(\lambda^*(T-t))^n}{n!}, \end{aligned}$$

*where $\tilde{c}_n^1, \tilde{c}_n^2$ are roots of the equation:*

$$x^{-\frac{\alpha^*}{\sigma^1}} = g \cdot b^n b^{\Pi(S_t^1, S_t^2)} \tilde{a} e^{-rT}(x-K)^+, \tag{39}$$

$$\tilde{c}^{i'}_n = \frac{1}{\sigma^1}\left(\ln\left(\frac{\tilde{c}_n^i}{(1-\nu^1)^n S_t^1}\right) - (r+\nu^1\lambda^* - \frac{1}{2}(\sigma^1)^2)(T-t)\right), i = 1, 2, \tag{40}$$

$\Pi(S_t^1, S_t^2)$, *$g$, $b$ were defined by (33), (28), (29), $\tilde{a}$ was defined by (35).*

Let us note that the *boundary condition* to equation (22) in case of a call option $f_T = (S_T^1 - K)^+$ is given by

$$\mathbb{C}(S_T^1, S_T^2, T) = (S_T^1 - K)^+ I_{\left\{S_T^{1\,-\frac{\alpha^*}{\sigma^1}} > g \cdot b^{\Pi(S_T^1, S_T^2)} \tilde{a} e^{-rT}(S_T^1 - K)^+\right\}}, \tag{41}$$

where $g$, $b$, $\Pi(S_T^1, S_T^2)$, $\tilde{a}$ were defined by (28), (29), (33) and (26) (resp. (35)).

The proofs of these theorems are given in the Appendix.

**Appendix.**

We prove Theorem 2 first.

Taking into account that in the jump-diffusion model (1)-(2) the condition $P\left[\frac{dP}{dP^*} = const \cdot f\right] = 0$ is satisfied we obtain from Remark1:

$$\tilde{\phi_T} = I_{\{\frac{dP}{dP^*} > \tilde{a} f\}}. \tag{A.1}$$

Let us paraphrase $Z_T$ in terms $S_T^1$:

$$\begin{aligned}\frac{dP^*}{dP} &= \exp(\alpha^* W_T - \tfrac{\alpha^{*2}}{2}T + (\lambda - \lambda^*)T + (ln\lambda^* - ln\lambda)\Pi_T)\\ &= \left(S_0^1 \exp\left\{\sigma^1 W_T + (\mu^1 - \tfrac{\sigma^{12}}{2})T\right\}(1-\nu^1)^{\Pi_T}\right)^{\frac{\alpha^*}{\sigma^1}} \times\\ &\times \frac{1}{S_0^{1\frac{\alpha^*}{\sigma^1}}} \exp\left(-\tfrac{\alpha^*\mu^1}{\sigma^1}T + \tfrac{\sigma^1\alpha^*}{2}T - \tfrac{\alpha^{*2}}{2}T + (\lambda-\lambda^*)T\right) \times\\ &\times \left(\frac{\lambda^*}{\lambda(1-\nu^1)^{\frac{\alpha^*}{\sigma^1}}}\right)^{\Pi_T}\\ &= g \cdot (S_T^1)^{\frac{\alpha^*}{\sigma^1}} \cdot b^{\Pi_T},\end{aligned} \tag{A.2}$$

where $g = \frac{1}{S_0^{1\frac{\alpha^*}{\sigma^1}}} \exp\left(-\frac{\alpha^*\mu^1}{\sigma^1}T + \frac{\sigma^1\alpha^*}{2}T - \frac{\alpha^{*2}}{2}T + (\lambda-\lambda^*)T\right)$, $b = \frac{\lambda^*}{\lambda(1-\nu^1)^{\frac{\alpha^*}{\sigma^1}}}$.

Using (A.1)-(A.2) we can represent $\tilde{\phi}$ in the form

$$\tilde{\phi} = I_{\{\frac{dP}{dP^*} > \tilde{a} f\}} = I_{\left\{S_T^{1-\frac{\alpha^*}{\sigma^1}} > g \cdot b^{\Pi_T} \tilde{a} e^{-rT}(S_T - K)^+\right\}}. \tag{A.3}$$

We show how to compute $\tilde{a}$ by means of the condition $\mathbf{E}^*\left[e^{-rT}\phi f\right] = X_0$. It follows from (A.3) that

(A.4)

$$\begin{aligned}\mathbf{E}^*\left[e^{-rT}\phi f\right] &= \mathbf{E}^*\left[e^{-rT}(S_T^1 - K)^+ I_{\left\{S_T^{1-\frac{\alpha^*}{\sigma^1}} > g \cdot b^{\Pi_T}\tilde{a}e^{-rT}(S_T-K)^+\right\}}\right]\\ &= \sum_{n=0}^\infty \mathbf{E}^*\left[e^{-rT}(S_T^1 - K)^+ I_{\left\{S_T^{1-\frac{\alpha^*}{\sigma^1}} > g \cdot b^n \tilde{a} e^{-rT}(S_T-K)^+\right\}}\right] \cdot \frac{(\lambda^* T)^n}{n!} e^{-\lambda^* T}.\end{aligned}$$

Case a, $-\frac{\alpha^*}{\sigma^1} \leqslant 1$:

In this case, the equation

$$x^{-\frac{\alpha^*}{\sigma^1}} = g \cdot b^n a e^{-rT}(x - K)^+$$

has a unique root $c_n(a)$. Thus the inequality

$$x^{-\frac{\alpha^*}{\sigma^1}} > g \cdot b^n a e^{-rT}(x - K)^+.$$

is equivalent to $x < c_n(a)$. This implies

$$I_{\left\{S_T^{1-\frac{\alpha^*}{\sigma^1}} > g \cdot b^n a e^{-rT}(S_T^1 - K)^+\right\}} = I_{\{S_T^1 < c_n(a)\}}. \tag{A.5}$$



Using (A.5) we can transform (A.4) as follows:

(A.6)

$$\begin{aligned}\mathbf{E}^*\left[e^{-rT}\phi f\right] &= \sum_{n=0}^\infty \mathbf{E}^*\left[e^{-rT}(S_T^1 - K)^+ I_{\{S_T^1 < c_n(a)\}}\right] \cdot \frac{(\lambda^* T)^n}{n!} e^{-\lambda^* T}\\ &= \sum_{n=0}^\infty \mathbf{E}^*\left[e^{-rT}\left((S_T^1 - K)^+ - (S_T^1 - c_n(a))^+ - (c_n(a) - K) I_{\{S_T^1 \geqslant c_n(a)\}}\right)\right] \times\\ &\times \frac{(\lambda^* T)^n}{n!} e^{-\lambda^* T}\\ &= e^{-\lambda^* T} \sum_{n=0}^\infty \Big[\mathbb{C}^{BS}(S_0^1(1-\nu^1)^n e^{\nu^1\lambda^* T}, K, T, r, \sigma^1)\\ &- \mathbb{C}^{BS}(S_0^1(1-\nu^1)^n e^{\nu^1\lambda^* T}, c_n(a), T, r, \sigma^1) - (c_n(a) - K)e^{-rT} \times\\ &\times P^*\left(S_0^1 exp(\sigma^1 W_T^* + (r + \nu^1\lambda^* - \tfrac{1}{2}(\sigma^1)^2)T)(1-\nu^1)^n \geqslant c_n(a)\right)\Big] \frac{(\lambda^* T)^n}{n!}\\ &= e^{-\lambda^* T} \sum_{n=0}^\infty \Big[\mathbb{C}^{BS}(S_0^1(1-\nu^1)^n e^{\nu^1\lambda^* T}, K, T, r, \sigma^1)\\ &- \mathbb{C}^{BS}(S_0^1(1-\nu^1)^n e^{\nu^1\lambda^* T}, c_n(a), T, r, \sigma^1) - (c_n(a) - K)e^{-rT}\Phi\left(\tfrac{-c_n'(a)}{\sqrt{T}}\right)\Big] \frac{(\lambda^* T)^n}{n!}\end{aligned}$$

where

$$c_n'(a) = \frac{1}{\sigma^1}\left(\ln\left(\frac{c_n(a)}{(1-\nu^1)^n S_0^1}\right) - (r + \nu^1\lambda^* - \frac{1}{2}(\sigma^1)^2)T\right).$$

Let $\tilde{a}$ be determined by the condition $\mathbf{E}^*\left[e^{-rT}\phi f\right] = X_0$ by means of (A.6). The maximal probability for a successful hedge is given by (cf. (A.5))

$$\begin{aligned}1 - \varepsilon &= P\left\{S_T^{1-\frac{\alpha^*}{\sigma^1}} > g \cdot b^{\Pi_T}\tilde{a}e^{-rT}(S_T^1 - K)^+\right\}\\ &= e^{-\lambda T}\sum_{n=0}^\infty P\{S_T^1 < c_n(\tilde{a})\}\frac{(\lambda T)^n}{n!}\\ &= e^{-\lambda T}\sum_{n=0}^\infty \Phi\left(\frac{c_n'(\tilde{a}) + \alpha^* T}{\sqrt{T}}\right)\frac{(\lambda T)^n}{n!}.\end{aligned} \tag{A.7}$$

The quantile hedging strategy is given by the perfect hedge of the modified claim $\tilde{\phi_T} f_T$. We calculate the value of this strategy:

$$\begin{aligned}X_t^\pi &= \mathbf{E}^*\left[\tilde{\phi_T} f_T e^{-r(T-t)} | \mathcal{F}_t\right]\\ &= \mathbf{E}^*\left[e^{-r(T-t)}(S_T^1 - K)^+ I_{\left\{S_T^{1-\frac{\alpha^*}{\sigma^1}} > g \cdot b^{\Pi_T}\tilde{a}e^{-rT}(S_T^1 - K)^+\right\}} | \mathcal{F}_t\right].\end{aligned} \tag{A.8}$$

We want to describe $\Pi_t$ in terms of the observable variables $S_t^i, i = 1, 2$. Since

$$S_t^i = S_0^i exp(\sigma^i W_t^* + (r + \nu^i\lambda^* - \frac{1}{2}(\sigma^i)^2)t)(1-\nu^i)^{\Pi_t},$$

$$W_t^* = \frac{1}{\sigma^i}\left(\ln\frac{\frac{S_t^i}{S_0^i}}{(1-\nu^i)^{\Pi_t}} - (r + \nu^i\lambda^* - \frac{(\sigma^i)^2}{2})t\right)$$

we obtain

$$\frac{1}{\sigma^1}\left(\ln\frac{S_t^1}{S_0^1}-\Pi_t\ln(1-\nu^1)-(r+\nu^1\lambda^*-\frac{(\sigma^1)^2}{2})t\right)$$
$$=\frac{1}{\sigma^2}\left(\ln\frac{S_t^2}{S_0^2}-\Pi_t\ln(1-\nu^2)-(r+\nu^2\lambda^*-\frac{(\sigma^2)^2}{2})t\right),$$

which implies

$$(A.9)\quad \Pi_t=\frac{\frac{1}{\sigma^1}\ln\frac{S_t^1}{S_0^1}-(r+\nu^1\lambda^*-\frac{(\sigma^1)^2}{2})\frac{t}{\sigma^1}-\frac{1}{\sigma^2}\ln\frac{S_t^2}{S_0^2}+(r+\nu^2\lambda^*-\frac{(\sigma^2)^2}{2})\frac{t}{\sigma^2}}{\frac{1}{\sigma^1}\ln(1-\nu^1)-\frac{1}{\sigma^2}\ln(1-\nu^2)}.$$

Hence we arrive at $\Pi_t=\Pi(S_t^1,S_t^2)$. Taking into account (A.8) we obtain
(A.10)

$$X_t^\pi=\mathbf{E}^*\left[e^{-r(T-t)}(S_T^1-K)^+I_{\left\{S_T^{1}{}^{-\frac{\alpha^*}{\sigma^1}}>g\cdot b^{\Pi_{T-t}}b^{\Pi(S_t^1,S_t^2)}\tilde a e^{-rT}(S_T^1-K)^+\right\}}|\mathcal{F}_t\right]$$
$$=\sum_{n=0}^\infty\mathbf{E}^*\left[e^{-r(T-t)}(S_T^1-K)^+I_{\left\{S_T^{1}{}^{-\frac{\alpha^*}{\sigma^1}}>g\cdot b^n b^{\Pi(S_t^1,S_t^2)}\tilde a e^{-rT}(S_T^1-K)^+\right\}}|\mathcal{F}_t\right]\times$$
$$\times\frac{(\lambda^*(T-t))^n}{n!}e^{-\lambda^*(T-t)}.$$

As in case (A.6) the equality (A.10) can be represented in the form
(A.11)

$$X_t^\pi=\sum_{n=0}^\infty\mathbf{E}^*\left[e^{-r(T-t)}\left((S_T^1-K)^+-(S_T^1-\tilde c_n)^+-(\tilde c_n-K)I_{\{S_T^1\geqslant\tilde c_n\}}\right)|\mathcal{F}_t\right]\times$$
$$\times\frac{(\lambda^*(T-t))^n}{n!}e^{-\lambda^*(T-t)},$$

where $\tilde c_n$ is the unique root of

$$x^{-\frac{\alpha^*}{\sigma^1}}=g\cdot b^n b^{\Pi(S_t^1,S_t^2)}\tilde a e^{-rT}(x-K)^+.$$

Equation (A.11) implies the final formula
(A.12)

$$X_t^\pi=e^{-\lambda^*(T-t)}\sum_{n=0}^\infty\Big[\mathbb{C}^{BS}(S_t^1(1-\nu^1)^n e^{\nu^1\lambda^*(T-t)},K,(T-t),r,\sigma^1)$$
$$-\mathbb{C}^{BS}(S_t^1(1-\nu^1)^n e^{\nu^1\lambda^*(T-t)},\tilde c_n,(T-t),r,\sigma^1)-(\tilde c_n-K)e^{-r(T-t)}\times$$
$$\times P^*(S_t^1\exp(\sigma^1W^*_{(T-t)}+(r+\nu^1\lambda^*-\tfrac{1}{2}(\sigma^1)^2)(T-t))(1-\nu^1)^n\geqslant\tilde c_n|\mathcal{F}_t)\Big]\times$$
$$\times\frac{(\lambda^*(T-t))^n}{n!}$$
$$=e^{-\lambda^*(T-t)}\sum_{n=0}^\infty\Big[\mathbb{C}^{BS}(S_t^1(1-\nu^1)^n e^{\nu^1\lambda^*(T-t)},K,(T-t),r,\sigma^1)$$
$$-\mathbb{C}^{BS}(S_t^1(1-\nu^1)^n e^{\nu^1\lambda^*(T-t)},\tilde c_n,(T-t),r,\sigma^1)$$
$$-(\tilde c_n-K)e^{-r(T-t)}\cdot\Phi(-\frac{\tilde c_n'}{\sqrt{T-t}})\Big]\frac{(\lambda^*(T-t))^n}{n!},$$



where

$$\tilde c_n'=\frac{1}{\sigma^1}(\ln(\frac{\tilde c_n}{(1-\nu^1)^nS_t^1})-(r+\nu^1\lambda^*-\frac{1}{2}(\sigma^1)^2)(T-t)).$$

Case b, $-\frac{\alpha^*}{\sigma^1}>1$.
Now the equation

$$x^{-\frac{\alpha^*}{\sigma^1}}=g\cdot b^n a e^{-rT}(x-K)^+$$

has two roots. Thus the inequality

$$x^{-\frac{\alpha^*}{\sigma^1}}>g\cdot b^n a e^{-rT}(x-K)^+$$

is equivalent to $x<c_n^1(a)$ or $x>c_n^2(a)$, where $K<c_n^1(a)\leqslant c_n^2(a)$ are the solutions of the equation $x^{-\frac{\alpha^*}{\sigma^1}}=g\cdot b^n a e^{-rT}(x-K)$. Hence

$$(A.13)\qquad I_{\left\{S_T^{1}{}^{-\frac{\alpha^*}{\sigma^1}}>g\cdot b^n a e^{-rT}(S_T^1-K)^+\right\}}=I_{\{S_T^1<c_n^1(a)\}}+I_{\{S_T^1>c_n^2(a)\}}.$$

Equations (A.6), (A.7), (A.13) imply
(A.14)

$$\mathbf{E}^*\left[e^{-rT}\phi f\right]=$$
$$\sum_{n=0}^\infty\mathbf{E}^*\left[e^{-rT}(S_T^1-K)^+(I_{\{S_T^1<c_n^1(a)\}}+I_{\{S_T^1>c_n^2(a)\}}\right]\frac{(\lambda^*T)^n}{n!}e^{-\lambda^*T}$$
$$=\sum_{n=0}^\infty\mathbf{E}^*\Big[e^{-rT}\big((S_T^1-K)^+-(S_T^1-c_n^1(a))^++(S_T^1-c_n^2(a))^+$$
$$-(c_n^1(a)-K)I_{\{S_T^1\geqslant c_n^1(a)\}}+(c_n^2(a)-K)I_{\{S_T^1>c_n^2(a)\}}\big)\Big]\cdot\frac{(\lambda^*T)^n}{n!}e^{-\lambda^*T}$$
$$=e^{-\lambda^*T}\sum_{n=0}^\infty\Big[\mathbb{C}^{BS}(S_0^1(1-\nu^1)^n e^{\nu^1\lambda^*T},K,T,r,\sigma^1)$$
$$-\mathbb{C}^{BS}(S_0^1(1-\nu^1)^n e^{\nu^1\lambda^*T},c_n^1(a),T,r,\sigma^1)$$
$$+\mathbb{C}^{BS}(S_0^1(1-\nu^1)^n e^{\nu^1\lambda^*T},c_n^2(a),T,r,\sigma^1)$$
$$-(c_n^1(a)-K)e^{-rT}\cdot\Phi(\frac{-c^{1'}_n(a)}{\sqrt T})+(c_n^2(a)-K)e^{-rT}\cdot\Phi(\frac{-c^{2'}_n(a)}{\sqrt T})\Big]\frac{(\lambda^*T)^n}{n!},$$

where

$$c^{i'}_n(a)=\frac{1}{\sigma^1}(\ln(\frac{c_n^i(a)}{(1-\nu^1)^nS_0^1})-(r+\nu^1\lambda^*-\frac{1}{2}(\sigma^1)^2)T).$$

Now we can determine $\tilde\phi$ because $\tilde a$ can be calculated by the condition $\mathbf{E}^*\left[e^{-rT}\phi f\right]=X_0$ and (A.14).

The maximal probability for a successful hedge is given by

$$(A.15)\qquad 1-\varepsilon=e^{-\lambda T}\sum_{n=0}^\infty\left[\Phi(\frac{c^{1'}_n(\tilde a)+\alpha^*T}{\sqrt T})+\Phi(-\frac{c^{2'}_n(\tilde a)+\alpha^*T}{\sqrt T})\right]\frac{(\lambda T)^n}{n!}.$$

As in (A.10)-(A.12) the value of the quantile hedging strategy is given by

$$\sum_{n=0}^{\infty} \mathbf{E}^* \Big[e^{-r(T-t)} \Big((S_T^1 - K)^+ - (S_T^1 - \tilde{c}_n^1)^+ + (S_T^1 - \tilde{c}_n^2)^+ -(\tilde{c}_n^1 - K)I_{\{S_T^1 \geqslant \tilde{c}_n^1\}} + (\tilde{c}_n^2 - K)I_{\{S_T^1 > \tilde{c}_n^2\}}\Big) |\mathcal{F}_t\Big] \cdot \frac{(\lambda^*(T-t))^n}{n!} e^{-\lambda^*(T-t)},$$

where $\tilde{c}_n^i$ are the roots of the equation

$$x^{-\frac{\alpha^*}{\sigma^1}} = g \cdot b^n b^{\Pi(S_t^1, S_t^2)} \tilde{a} e^{-rT} (x-K)^+.$$

This yields

(A.16)

$$X_t^\pi = e^{-\lambda^*(T-t)} \sum_{n=0}^{\infty} \Big[\mathbb{C}^{BS}(S_t^1(1-\nu^1)^n e^{\nu^1\lambda^*(T-t)}, K, (T-t), r, \sigma^1)$$
$$- \mathbb{C}^{BS}(S_t^1(1-\nu^1)^n e^{\nu^1\lambda^*(T-t)}, \tilde{c}_n^1, (T-t), r, \sigma^1)$$
$$+ \mathbb{C}^{BS}(S_t^1(1-\nu^1)^n e^{\nu^1\lambda^*(T-t)}, \tilde{c}_n^2, (T-t), r, \sigma^1)$$
$$- (\tilde{c}_n^1 - K)e^{-r(T-t)} \cdot \Phi(-\frac{\tilde{c}^{1'}_n}{\sqrt{T-t}}) + (\tilde{c}_n^2 - K)e^{-r(T-t)} \cdot \Phi(-\frac{\tilde{c}^{2'}_n}{\sqrt{T-t}})\Big] \frac{(\lambda^*(T-t))^n}{n!},$$

where

$$\tilde{c}^{i'}_n = \frac{1}{\sigma^1}(\ln(\frac{\tilde{c}_n^i}{(1-\nu^1)^n S_t^1}) - (r + \nu^1\lambda^* - \frac{1}{2}(\sigma^1)^2)(T-t)).$$

The value given by (A.12) (resp. (A.16)) depends on $S_t^1$ and $S_t^2$. We can apply Theorem 1 to determine the components $\beta, \gamma^1, \gamma^2$ of the hedging strategy (see (23)-(24)).

We now prove Theorem 1.

By equations ((A.10)-(A.12)) the value of the quantile hedging strategy for $f_T = f(S_T^1)$ is given by

(A.17)

$$X_t^\pi = \mathbf{E}^* \left[e^{-r(T-t)} f(S_T^1) I_{\left\{S_T^{1\,-\frac{\alpha^*}{\sigma^1}} > g \cdot b^{\Pi_{T-t}} b^{\Pi(S_t^1, S_t^2)} \tilde{a} e^{-rT} f(S_T^1)\right\}} |\mathcal{F}_t\right] =$$

$$\sum_{n=0}^{\infty} \mathbf{E}^* \left[e^{-r(T-t)} f(S_T^1) I_{\left\{S_T^{1\,-\frac{\alpha^*}{\sigma^1}} > g \cdot b^n b^{\Pi(S_t^1, S_t^2)} \tilde{a} e^{-rT} f(S_T^1)\right\}} |\mathcal{F}_t\right] \frac{(\lambda^*(T-t))^n e^{-\lambda^*(T-t)}}{n!}$$

$$= \sum_{n=0}^{\infty} e^{-r(T-t)} \frac{(\lambda^*(T-t))^n}{n!} e^{-\lambda^*(T-t)} \times$$
$$\times \left[\int_{-\infty}^{\infty} \left(\phi_{T-t}(x) f(S_t^1 \exp(\sigma^1 x + (r + \nu^1\lambda^* - \tfrac{1}{2}(\sigma^1)^2)(T-t))(1-\nu^1)^n) \times \right.\right.$$
$$\times I\left\{\left[S_t^1 \exp(\sigma^1 x + (r + \nu^1\lambda^* - \tfrac{1}{2}(\sigma^1)^2)(T-t))(1-\nu^1)^n\right]^{-\frac{\alpha^*}{\sigma^1}} > \right.$$
$$\left.\left.\left. > g b^n b^{\Pi(S_t^1, S_t^2)} \tilde{a} e^{-rT} f(S_t^1 e^{(\sigma^1 x + (r+\nu^1\lambda^* - \frac{1}{2}(\sigma^1)^2)(T-t))}(1-\nu^1)^n)\right\}\right) dx\right]$$



where $\tilde{a}$, $b$, $g$, $\Pi(S_t^1, S_t^2)$ were defined in the proof of Theorem 2 and

$$\phi_{T-t}(x) = \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{x^2}{2(T-t)}}.$$

Thus the value of the optimal strategy is a function of $S_t^1$, $S_t^2$ and $t$.

We are going to derive the components of the hedge and the partial differential equation (22) from the representation

$$\frac{X_t^\pi}{B_t} = \frac{X_0^\pi}{B_0} + \int_0^t \sum_{i=1}^{2} \gamma_u^i d(\frac{S_u^i}{B_u}). \tag{A.18}$$

Let us denote

$$\mathbb{C}(S_t^1, S_t^2, t) := X_t^\pi$$

and rewrite (A.18) in the form

$$\frac{\mathbb{C}(S_t^1, S_t^2, t)}{B_t} = \frac{\mathbb{C}(S_0^1, S_0^2, 0)}{B_0} + \int_0^t \gamma_u^1 d\left(\frac{S_u^1}{B_u}\right) + \int_0^t \gamma_u^2 d\left(\frac{S_u^2}{B_u}\right). \tag{A.19}$$

The discounted price processes $Y_t^i = \frac{S_t^i}{B_t}, i = 1, 2$ satisfy the equations

$$dY_t^i = Y_{t-}^i(\sigma^i dW_t^* - \nu^i d(\Pi_t - \lambda^* t)), i = 1, 2. \tag{A.20}$$

Appling (A.19) and (A.20) we get

(A.21)

$$\frac{\mathbb{C}(S_t^1, S_t^2, t)}{B_t} = $$
$$= \frac{\mathbb{C}(S_0^1, 0)}{B_0} + \int_0^t \frac{\gamma^1\sigma^1 S_{u-}^1 + \gamma^2\sigma^2 S_{u-}^2}{B_u} dW_u^* - \int_0^t \frac{\gamma^1\nu^1 S_{u-}^1 + \gamma^2\nu^2 S_{u-}^2}{B_u} d(\Pi_u - \lambda^* u).$$

The Ito formula yields

$$d\frac{\mathbb{C}(S_t^1, S_t^2, t)}{B_t} = e^{-rt} d\mathbb{C}(S_t^1, S_t^2, t) - re^{-rt}\mathbb{C}(S_{t-}^1, S_{t-}^2, t)dt, \tag{A.22}$$

$$d\frac{S_t^i}{B_t} = e^{-rt} dS_t^i - re^{-rt} S_{t-}^i dt. \tag{A.23}$$

Applying the Ito formula to the value process yields

(A.24)

$$\mathbb{C}(S_t^1, S_t^2, t) = \mathbb{C}(S_0^1, S_0^2, 0) + \int_0^t \frac{\partial}{\partial x}\mathbb{C}(S_{u-}^1, S_{u-}^2, u) dS_u^1 + \int_0^t \frac{\partial}{\partial y}\mathbb{C}(S_{u-}^1, S_{u-}^2, u) dS_u^2$$
$$+ \int_0^t \frac{\partial}{\partial t}\mathbb{C}(S_{u-}^1, S_{u-}^2, u) du + \frac{1}{2}\int_0^t \frac{\partial^2}{\partial x^2}\mathbb{C}(S_{u-}^1, S_{u-}^2, u) d\langle S^{1^c}, S^{1^c}\rangle_u$$
$$+ \frac{1}{2}\int_0^t \frac{\partial^2}{\partial y^2}\mathbb{C}(S_{u-}^1, S_{u-}^2, u) d\langle S^{2^c}, S^{2^c}\rangle_u + \int_0^t \frac{\partial^2}{\partial x \partial y}\mathbb{C}(S_{u-}^1, S_{u-}^2, u) d\langle S^{1^c}, S^{2^c}\rangle_u$$
$$+ \sum_{0<u\leqslant t}\Big[\mathbb{C}(S_u^1, S_u^2, u) - \mathbb{C}(S_{u-}^1, S_{u-}^2, u)$$
$$- \frac{\partial}{\partial x}\mathbb{C}(S_{u-}^1, S_{u-}^2, u)\Delta S_u^1 - \frac{\partial}{\partial y}\mathbb{C}(S_{u-}^1, S_{u-}^2, u)\Delta S_u^2\Big].$$

We know from (9) that

$$\Delta S_u^i = -\nu^i S_{u-}^i \Delta\Pi_u \tag{A.25}$$

and

$$\mathbb{C}(S_u^1, S_u^2, u) - \mathbb{C}(S_{u-}^1, S_{u-}^2, u) = \left[\mathbb{C}(S_{u-}^1(1-\nu^1), S_{u-}^2(1-\nu^2), u) - \mathbb{C}(S_{u-}^1, S_{u-}^2, u)\right] \cdot \Delta\Pi_u. \tag{A.26}$$

Furthermore, the properties of $W^*$ and $\Pi$ imply

$$d\left\langle S^{i^c}, S^{j^c}\right\rangle_u = (\sigma^i\sigma^j)(S_{u-}^i S_{u-}^j)du, i, j = 1, 2 \tag{A.27}$$

Using (A.22)-(A.27) we obtain

$$d\mathbb{C}(S_t^1, S_t^2, t) = \tfrac{\partial}{\partial x}\mathbb{C}(S_{t-}^1, S_{t-}^2, t)dS_t^1 + \tfrac{\partial}{\partial y}\mathbb{C}(S_{t-}^1, S_{t-}^2, t)dS_t^2 + \tfrac{\partial}{\partial t}\mathbb{C}(S_{t-}^1, t)dt$$

$$+ \tfrac{1}{2}(\sigma^1 S_{t-}^1)^2 \tfrac{\partial^2}{\partial x^2}\mathbb{C}(S_{t-}^1, S_{t-}^2, t)dt + \tfrac{1}{2}(\sigma^2 S_{t-}^2)^2 \tfrac{\partial^2}{\partial y^2}\mathbb{C}(S_{t-}^1, S_{t-}^2, t)dt$$

$$+ (\sigma^1\sigma^2 S_{t-}^1 S_{t-}^2)\tfrac{\partial^2}{\partial x\partial y}\mathbb{C}(S_{t-}^1, S_{t-}^2, t)dt$$

$$+ \left[\mathbb{C}(S_{t-}^1(1-\nu^1), S_{t-}^2(1-\nu^2), t) - \mathbb{C}(S_{t-}^1, S_{t-}^2, t)\right] d\Pi_t$$

$$- \tfrac{\partial}{\partial x}\mathbb{C}(S_{t-}^1, S_{t-}^2, t)(-\nu^1 S_{t-}^1)d\Pi_t - \tfrac{\partial}{\partial y}\mathbb{C}(S_{t-}^1, S_{t-}^2, t)(-\nu^2 S_{t-}^2)d\Pi_t.$$

Finally we arrive at
(A.28)

$$d\frac{\mathbb{C}(S_t^1, S_t^2, t)}{B_t} = \left(\tfrac{\partial}{\partial x}\mathbb{C}(S_{t-}^1, S_{t-}^2, t)\tfrac{S_{t-}^1}{B_t}\sigma^1 + \tfrac{\partial}{\partial y}\mathbb{C}(S_{t-}^1, S_{t-}^2, t)\tfrac{S_{t-}^2}{B_t}\sigma^2\right)dW_t^*$$

$$+ \left[\mathbb{C}(S_{t-}^1(1-\nu^1), S_{t-}^2(1-\nu^2), t) - \mathbb{C}(S_{t-}^1, S_{t-}^2, t)\right] e^{-rt} d(\Pi_t - \lambda^* t)$$

$$+ \Big( \left[\mathbb{C}(S_{t-}^1(1-\nu^1), S_{t-}^2(1-\nu^2), t) - \mathbb{C}(S_{t-}^1, S_{t-}^2, t)\right] \lambda^* + rS_{t-}^1 \tfrac{\partial}{\partial x}\mathbb{C}(S_{t-}^1, S_{t-}^2, t)$$

$$+ rS_{t-}^2 \tfrac{\partial}{\partial y}\mathbb{C}(S_{t-}^1, S_{t-}^2, t) + \tfrac{\partial}{\partial t}\mathbb{C}(S_{t-}^1, S_{t-}^2, t) + \tfrac{1}{2}(\sigma^1 S_{t-}^1)^2 \tfrac{\partial^2}{\partial x^2}\mathbb{C}(S_{t-}^1, S_{t-}^2, t)$$

$$+ \tfrac{1}{2}(\sigma^2 S_{t-}^2)^2 \tfrac{\partial^2}{\partial y^2}\mathbb{C}(S_{t-}^1, S_{t-}^2, t) + (\sigma^1\sigma^2 S_{t-}^1 S_{t-}^2)\tfrac{\partial^2}{\partial x\partial y}\mathbb{C}(S_{t-}^1, S_{t-}^2, t) -$$

$$r\mathbb{C}(S_{t-}^1, S_{t-}^2, t) + \tfrac{\partial}{\partial x}\mathbb{C}(S_{t-}^1, S_{t-}^2, t)\nu^1\lambda^* S_{t-}^1 + \tfrac{\partial}{\partial y}\mathbb{C}(S_{t-}^1, S_{t-}^2, t)\nu^2\lambda^* S_{t-}^2 \Big) e^{-rt}dt$$

The proof of Theorem 1 now follows from the comparison (A.28) with the representation (A.21).

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