Option valuation using the fast Fourier transform

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In this paper the authors show how the fast Fourier transform may be used to value options when the characteristic function of the return is known analytically.

1. INTRODUCTION

The Black–Scholes model and its extensions comprise one of the major developments in modern finance. Much of the recent literature on option valuation has successfully applied Fourier analysis to determine option prices (see e.g. Bakshi and Chen 1997, Scott 1997, Bates 1996, Heston 1993, Chen and Scott 1992). These authors numerically solve for the delta and for the risk-neutral probability of finishing in-the-money, which can be easily combined with the stock price and the strike price to generate the option value. Unfortunately, this approach is unable to harness the considerable computational power of the fast Fourier transform (FFT) (Walker 1996), which represents one of the most fundamental advances in scientific computing. Furthermore, though the decomposition of an option price into probability elements is theoretically attractive, as explained by Bakshi and Madan (1999), it is numerically undesirable owing to discontinuity of the payoffs.

The purpose of this paper is to describe a new approach for numerically determining option values, which is designed to use the FFT to value options efficiently. As is the case with all of the above approaches, our technique assumes that the characteristic function of the risk-neutral density is known analytically. Given any such characteristic function, we develop a simple analytic expression for the Fourier transform of the option value or its time value. We then use the FFT to numerically solve for the option price or its time value. Our use of the FFT in the inversion stage permits real-time pricing, marking, and hedging using realistic models, even for books with thousands of options.

To test the accuracy of our approach, we would like to use a model where the option price is known analytically. To illustrate the potential power of Fourier analysis, we would also like to use a model in which the density function is complicated, while the characteristic function of the log price is simple. Finally, we would like to use a model which is supported in a general equilibrium and which is capable of removing the biases of the standard Black–Scholes model. All of these requirements are met by the variance gamma (VG) model, which assumes that the log price obeys a one-dimensional pure jump Markov process with stationary independent increments. The mathematics of this process is detailed by Madan and Seneta (1990), while the economic motivation and empirical support for this model is described by Madan and Milne (1991) and by Madan, Carr, and Chang (1998) respectively.

The outline of this paper is as follows. In Section 2, we briefly review the current literature on the use of Fourier methods in option pricing. In Section 3, we present our approach for analytically determining the Fourier transform of the option value and of the time value in terms of the characteristic function of the risk-neutral density. Section 4 details the use of the FFT to numerically solve for the option price or time value. In Section 5, we illustrate our approach in the VG model. Section 6 concludes.

2. REVIEW OF FOURIER METHODS IN OPTION PRICING

Consider the problem of valuing a European call of maturity T, written on the terminal spot price S_T of some underlying asset. The characteristic function of $s_T = \ln(S_T)$ is defined by

$$\phi_T(u) = \mathsf{E}[\exp(\mathrm{i}us_T)]. \tag{1}$$

In many situations this characteristic function is known analytically. A wide class of examples arises when the the dynamics of the log price is given by an infinitely divisible process of independent increments. The characteristic function then arises naturally from the Lévy–Khintchine representation for such processes. Among this class of processes, we have the process of independent stable increments (McCulloch 1978), the VG process (Madan, Carr, and Chang 1998), the inverse Gaussian law (Barndorff-Nielsen 1997), and a wide range of other processes proposed by Geman, Yor, and Madan (1998). Characteristic functions have also been used in the pure diffusion context with stochastic volatility (Heston 1993) and with stochastic interest rates (Bakshi and Chen 1997). Finally, they have been used for jumps coupled with stochastic volatility (Bates 1996) and for jumps coupled with stochastic interst rates and volatility (Scott 1997). The solution methods can also be applied to average rate claims and to other exotic claims (Bakshi and Madan 1999). The methods are generally much faster than finite difference solutions to partial differential equations or integrodifferential equations, which led Heston (1993) to refer to them as closed-form solutions.

Assuming that the characteristic function is known analytically, many authors (e.g. Bakshi and Madan 1999, Scott 1997) have numerically determined the risk-neutral probability of finishing in-the-money as

$$\Pr(S_T > K) = \Pi_2 = \tfrac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{\mathrm{e}^{-\mathrm{i}u\ln(K)}\,\phi_T(u)}{\mathrm{i}u}\right)\mathrm{d}u.$$

Similarly, the delta of the option, denoted Π_1, is numerically obtained as

$$\Pi_1 = \tfrac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{\mathrm{e}^{-\mathrm{i}u\ln(K)}\phi_T(u-\mathrm{i})}{\mathrm{i}u\phi_T(-\mathrm{i})}\right)\mathrm{d}u.$$

Assuming no dividends and constant interest rates r, the initial option value is then determined as

$$C = S\Pi_1 - Ke^{-rT}\Pi_2.$$

Unfortunately, the FFT cannot be used to evaluate the integral, since the integrand is singular at the required evaluation point $u = 0$. Given the considerable speed advantages of the FFT, we examine two alternative approaches in the next section, both of which are amenable to evaluation by the FFT.

3. TWO NEW FOURIER METHODS

In this section, we develop analytic expressions for the Fourier transform of an option price and for the Fourier transform of the time value of an option. Both Fourier transforms are expressed in terms of the characteristic function of the log price.

3.1 *The Fourier Transform of an Option Price*

Let k denote the log of the strike price K, and let $C_T(k)$ be the desired value of a T-maturity call option with strike $\exp(k)$. Let the risk-neutral density of the log price s_T be $q_T(s)$. The characteristic function of this density is defined by

$$\phi_T(u) \equiv \int_{-\infty}^{\infty} e^{ius} q_T(s)\, ds. \tag{2}$$

The initial call value $C_T(k)$ is related to the risk-neutral density $q_T(s)$ by

$$C_T(k) \equiv \int_{k}^{\infty} e^{-rT}(e^s - e^k) q_T(s)\, ds.$$

Note that $C_T(k)$ tends to S_0 as k tends to $-\infty$, and hence the call pricing function is not square-integrable. To obtain a square-integrable function, we consider the modified call price $c_T(k)$ defined by

$$c_T(k) \equiv \exp(\alpha k)\, C_T(k) \tag{3}$$

for $\alpha > 0$. For a range of positive values of α, we expect that $c_T(k)$ is square-integrable in k over the entire real line. We comment later on the choice of α. Consider now the Fourier transform of $c_T(k)$ defined by

$$\psi_T(v) = \int_{-\infty}^{\infty} e^{ivk} c_T(k)\, dk. \tag{4}$$

We first develop an analytical expression for $\psi_T(v)$ in terms of ϕ_T and then obtain call prices numerically using the inverse transform

$$C_T(k) = \frac{\exp(-\alpha k)}{2\pi} \int_{-\infty}^{\infty} e^{-ivk} \psi_T(v)\, dv = \frac{\exp(-\alpha k)}{\pi} \int_0^{\infty} e^{-ivk} \psi(v)\, dv. \tag{5}$$

The second equality holds because $C_T(k)$ is real, which implies that the function $\psi_T(v)$ is odd in its imaginary part and even in its real part. The expression for $\psi_T(v)$ is determined as follows:

$$\begin{aligned}
\psi_T(v) &= \int_{-\infty}^{\infty} e^{ivk} \int_k^{\infty} e^{\alpha k} e^{-rT} (e^s - e^k) q_T(s)\, ds\, dk \\
&= \int_{-\infty}^{\infty} e^{-rT} q_T(s) \int_{-\infty}^{s} (e^{s+\alpha k} - e^{(1+\alpha)k}) e^{ivk}\, dk\, ds \\
&= \int_{-\infty}^{\infty} e^{-rT} q_T(s) \left(\frac{e^{(\alpha+1+iv)s}}{\alpha + iv} - \frac{e^{(\alpha+1+iv)s}}{\alpha + 1 + iv} \right) ds \\
&= \frac{e^{-rT} \phi_T\big(v - (\alpha+1)i\big)}{\alpha^2 + \alpha - v^2 + i(2\alpha+1)v}.
\end{aligned} \tag{6}$$

Call values are determined by substituting (6) into (5) and performing the required integration. We note that the integration (5) is a direct Fourier transform and lends itself to an application of the FFT. We also note that if $\alpha = 0$ then the denominator vanishes when $v = 0$, inducing a singularity in the integrand. Since the FFT evaluates the integrand at $v = 0$, the use of the factor $\exp(\alpha k)$ or something similar is required.

We now consider the issue of the appropriate choice of the coefficient α. Positive values of α assist the integrability of the modified call value over the negative log strike axis, but aggravate the same condition for the positive log strike axis. For the modified call value to be integrable in the positive log strike direction, and hence for it to be square-integrable as well, a sufficient condition is provided by $\psi(0)$ being finite. From (6), we observe that $\psi_T(0)$ is finite provided that $\phi_T\big(-(\alpha+1)i\big)$ is finite. From the definition of the characteristic function, this requires that

$$\mathsf{E}[S_T^{\alpha+1}] < \infty. \tag{7}$$

In practice, one may determine an upper bound on α from the analytical expression for the characteristic function and the condition (7). We find that one quarter of this upper bound serves as a good choice for α.

We now consider the issue of the infinite upper limit of integration in (5). Note that, since the modulus of ϕ_t is bounded by $\mathsf{E}[S_T^{\alpha+1}]$, which is independent of v, it follows that

$$\big|\psi_T(v)\big|^2 \leqslant \frac{\mathsf{E}[S_T^{\alpha+1}]}{(\alpha^2 + \alpha - v^2)^2 + (2\alpha+1)^2 v^2} \leqslant \frac{A}{v^4}$$

for some constant A, or that

$$\left|\psi(v)\right| < \frac{\sqrt{A}}{v^2}.$$

It follows that we may bound the integral of the upper tail by

$$\int_a^\infty \left|\psi(v)\right| \mathrm{d}v < \frac{\sqrt{A}}{a}. \tag{8}$$

This bound makes it possible to set up a truncation procedure. Specifically, the integral of the tail in computing the transform of (5) is bounded by $\sqrt{A}/a$, and hence the truncation error is bounded by

$$\frac{\exp(-\alpha k)}{\pi}\frac{\sqrt{A}}{a},$$

which can be made smaller than ε by choosing

$$a > \frac{\exp(-\alpha k)}{\pi}\frac{\sqrt{A}}{\varepsilon}.$$

3.2 *Fourier Transform of Out-of-the-Money Option Prices*

In the last section we multiplied call values by an exponential function to obtain a square-integrable function whose Fourier transform is an analytic function of the characteristic function of the log price. Unfortunately, for very short maturities, the call value approaches its nonanalytic intrinsic value causing the integrand in the Fourier inversion to be highly oscillatory, and therefore difficult to integrate numerically. The purpose of this section is to introduce an alternative approach that works with time values only. Again letting k denote the log of the strike and S_0 denote the initial spot price, we let $z_T(k)$ be the T-maturity put price when $k < \ln(S_0)$, and we let it be the T-maturity call price when $k > \ln(S_0)$. For any unimodal probability density function, the function $z_T(k)$ peaks at $k = \ln(S_0)$ and declines in both directions as k tends to positive or negative infinity. In this section, we develop an analytic expression for the Fourier transform of $z_T(k)$ in terms of the characteristic function of the log of the terminal stock price.

Let $\zeta_T(v)$ denote the Fourier transform of $z_T(k)$:

$$\zeta_T(v) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}vk} z_T(k)\,\mathrm{d}k. \tag{9}$$

The prices of out-of-the-money options are obtained by inverting this transform:

$$z_T(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}vk}\zeta_T(v)\,\mathrm{d}v. \tag{10}$$

For ease of notation, we will derive $\zeta_T(v)$ assuming that $S_0 = 1$ (one may always scale up to other values later). We may then define $z_T(k)$ by

$$z_T(k) = \mathrm{e}^{-rT} \int_{-\infty}^{\infty} [(\mathrm{e}^k - \mathrm{e}^s)\mathbf{1}_{s<k,\,k<0} + (\mathrm{e}^s - \mathrm{e}^k)\mathbf{1}_{s>k,\,k>0}] q_T(s)\,\mathrm{d}s. \tag{11}$$

The expression for $\zeta_T(v)$ follows on noting that

$$\zeta_T(v) = \int_{-\infty}^{0} \mathrm{e}^{\mathrm{i}vk}\mathrm{e}^{-rT} \int_{-\infty}^{k} (\mathrm{e}^k - \mathrm{e}^s) q_T(s)\,\mathrm{d}s\,\mathrm{d}k + \int_{0}^{\infty} \mathrm{e}^{\mathrm{i}vk}\mathrm{e}^{-rT} \int_{k}^{\infty} (\mathrm{e}^s - \mathrm{e}^k) q_T(s)\,\mathrm{d}s\,\mathrm{d}k. \tag{12}$$

Reversing the order of integration in (12) yields

$$\zeta_T(v) = \int_{-\infty}^{0} \mathrm{e}^{-rT} q_T(s) \int_{s}^{\infty} (\mathrm{e}^{(1+\mathrm{i}v)k} - \mathrm{e}^s\mathrm{e}^{\mathrm{i}vk})\,\mathrm{d}k\,\mathrm{d}s + \int_{0}^{\infty} \mathrm{e}^{-rT} q_T(s) \int_{0}^{s} (\mathrm{e}^s\mathrm{e}^{\mathrm{i}vk} - \mathrm{e}^{(1+\mathrm{i}v)k})\,\mathrm{d}k\,\mathrm{d}s. \tag{13}$$

Performing the inner integrations, simplifying, and writing the outer integration in terms of characteristic functions, we get

$$\zeta_T(v) = \mathrm{e}^{-rT}\left(\frac{1}{1+\mathrm{i}v} - \frac{\mathrm{e}^{rT}}{\mathrm{i}v} - \frac{\phi_T(v-\mathrm{i})}{v^2 - \mathrm{i}v}\right). \tag{14}$$

Although there is no issue regarding the behavior of $z_T(k)$ as k tends to positive or negative infinity, the time value at $k = 0$ can get quite steep as $T \to 0$, and this can cause difficulties in the inversion. The function $z_T(k)$ approximates the shape of a Dirac delta function near $k = 0$ when maturity is small (see Figure 1), and thus the transform is wide and oscillatory.

It is useful in this case to consider the transform of $\sinh(\alpha k) z_T(k)$ as this function vanishes at $k = 0$. Define

$$\begin{aligned}
\gamma_T(v) &= \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}vk} \sinh(\alpha k) z_T(k)\,\mathrm{d}k \\
&= \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}vk} \frac{\mathrm{e}^{\alpha k} - \mathrm{e}^{-\alpha k}}{2} z_T(k)\,\mathrm{d}k \\
&= \frac{\zeta_T(v - \mathrm{i}\alpha) - \zeta_T(v + \mathrm{i}\alpha)}{2}.
\end{aligned} \tag{15}$$

Thus, the time value is given by

$$z_T(k) = \frac{1}{\sinh(\alpha k)} \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}vk} \gamma_T(v)\,\mathrm{d}v.$$

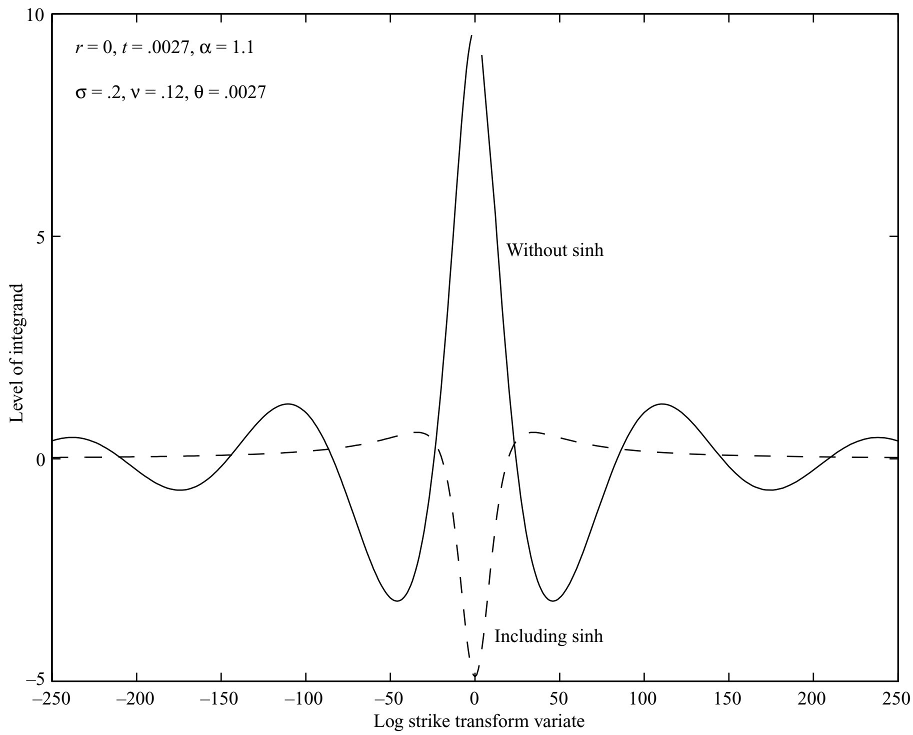

FIGURE 1. Fourier inversion integrands with and without the use of sinh.

The value of α can be chosen to control the steepness of the integrand near zero.

4. OPTION PRICING USING THE FFT

The FFT is an efficient algorithm for computing the sum

$$w(k) = \sum_{j=1}^{N} \mathrm{e}^{-\mathrm{i}\frac{2\pi}{N}(j-1)(k-1)} x(j) \quad \text{for } k = 1, \ldots, N, \tag{16}$$

where N is typically a power of 2. The algorithm reduces the number of multiplications in the required N summations from an order of N^2 to that of $N \ln_2(N)$, a very considerable reduction. We present in this section the details for writing the integration (5) as an application of the summation (16).

Using the trapezoid rule for the integral on the right-hand side of (5) and setting $v_j = \eta(j-1)$, an approximation for $C(k)$ is

$$C_T(k) \approx \frac{\exp(-\alpha k)}{\pi} \sum_{j=1}^{N} \mathrm{e}^{-\mathrm{i}v_j k} \psi_T(v_j)\eta. \tag{17}$$

The effective upper limit for the integration is now

$$a = N\eta. \tag{18}$$

We are mainly interested in at-the-money call values $C(k)$, which correspond to k near 0. The FFT returns N values of k and we employ a regular spacing of size λ, so that our values for k are

$$k_u = -b + \lambda(u-1) \quad \text{for } u = 1, \ldots, N. \tag{19}$$

This gives us log strike levels ranging from $-b$ to b, where

$$b = \tfrac{1}{2}N\lambda. \tag{20}$$

Substituting (19) into (17) yields

$$C_T(k_u) \approx \frac{\exp(-\alpha k_u)}{\pi} \sum_{j=1}^{N} \mathrm{e}^{-\mathrm{i}v_j[-b+\lambda(u-1)]} \psi_T(v_j)\eta \quad \text{for } u = 1, \ldots, N. \tag{21}$$

Noting that $v_j = (j-1)\eta$, we write

$$C_T(k_u) \approx \frac{\exp(-\alpha k_u)}{\pi} \sum_{j=1}^{N} \mathrm{e}^{-\mathrm{i}\lambda\eta(j-1)(u-1)} \mathrm{e}^{\mathrm{i}bv_j} \psi_T(v_j)\eta. \tag{22}$$

To apply the fast Fourier transform, we note from (16) that

$$\lambda\eta = \frac{2\pi}{N}. \tag{23}$$

Hence, if we choose η small in order to obtain a fine grid for the integration, then we observe call prices at strike spacings that are relatively large, with few strikes lying in the desired region near the stock price. We would therefore like to obtain an accurate integration with larger values of η and, for this purpose, we incorporate Simpson's rule weightings into our summation. With Simpson's rule weightings and the restriction (23), we may write our call price as

$$C(k_u) = \frac{\exp(-\alpha k_u)}{\pi} \sum_{j=1}^{N} \mathrm{e}^{-\mathrm{i}\frac{2\pi}{N}(j-1)(u-1)} \mathrm{e}^{\mathrm{i}bv_j} \psi(v_j) \frac{\eta}{3}[3 + (-1)^j - \delta_{j-1}], \tag{24}$$

where δ_n is the Kronecker delta function that is unity for $n = 0$ and zero otherwise. The summation in (24) is an exact application of the FFT. One needs to make the appropriate choices for η and α. The next section addresses these issues in the context of the VG option pricing model used to illustrate our approaches.

The use of the FFT for calculating out-of-the-money option prices is similar to (24). The only differences are that we replace the multiplication by $\exp(-\alpha k_u)$ with a division by $\sinh(\alpha k)$ and the function call to $\psi(v)$ is replaced by a function call to $\gamma(v)$ defined in (15).

5. THE FFT FOR VG OPTION PRICING

The VG option pricing model is described in detail in Madan, Carr, and Chang (1998), who document that this process effectively removes the smile observed when plotting Black–Scholes implied volatilities against strike prices. The VG process is obtained by evaluating arithmetic Brownian motion with drift θ and volatility σ at a random time given by a gamma process having a mean rate per unit time of 1 and a variance rate of ν. The resulting process $X_t(\sigma, \theta, \nu)$ is a pure jump process with two additional parameters θ and ν relative to the Black–Scholes model, providing control over skewness and kurtosis respectively. The resulting risk-neutral process for the stock price is

$$S_t = S_0 \exp[rt + X_t(\sigma, \theta, \nu) + \omega t], \quad t > 0, \tag{25}$$

where, by setting $\omega = (1/\nu)\ln(1 - \theta\nu - \frac{1}{2}\sigma^2\nu)$, the mean rate of return on the stock equals the interest rate r.

Madan, Carr, and Chang (1998) show that the characteristic function for the log of S_T is

$$\phi_T(u) = \exp[\ln(S_0) + (r + \omega)T](1 - \mathrm{i}\theta\nu u + \tfrac{1}{2}\sigma^2 u^2 \nu)^{-T/\nu}. \tag{26}$$

To obtain option prices, one can analytically invert this characteristic function to get the density function, and then integrate the density function against the option payoff. Madan, Carr, and Chang (1998) provide a closed-form formula for both the density function and the option price. Alternatively, the Fourier transform of the distribution functions can be numerically inverted as reviewed in Section 1. Finally, the Fourier transform of the modified call can be numerically inverted without using FFT. In this last case, one must set the damping coefficient α. To accomplish this, we evaluate the term $\phi_T\big(-(\alpha+1)\mathrm{i}\big)$ in (7) as

$$\phi_T\big(-(\alpha+1)\mathrm{i}\big) = \exp[\ln(S_0) + (r+\omega)T][1 - \theta\nu(\alpha+1) - \tfrac{1}{2}\sigma^2(\alpha+1)^2\nu]^{-T/\nu}.$$

For this expression to be finite, we must have

$$\alpha < \sqrt{\frac{\theta^2}{\sigma^4} + \frac{2}{\sigma^2\nu}} - \frac{\theta}{\sigma^2} - 1.$$

Generally, we anticipate in our estimates that the expectation of S_T^2 is finite and that this upper bound is above unity. A value of α above unity and well below the upper bound performs well.

For our FFT methods, we found that setting the spacing $\eta = 0.25$ delivers the speedup of the FFT without compromising the accuracy delivered by other methods. However, as a quality control, we recommend selective checking of the FFT output against other methods. We used $N = 4096$ points in our quadrature, implying a log strike spacing of $8\pi/4096 = .00613$, or a little over half a percentage point, which is adequate for practice. For the choice of the dampening coefficient in the transform of the modified call price, we used a value of $\alpha = 1.5$. For the modified time value, we used $\alpha = 1.10$.

We evaluated option prices using the FFT to invert the modified call price (termed VGFFTC) and using it to invert the modified time value (termed VGFFTTV). We used 160 strike levels at four combinations of parameter settings and compared the CPU times with those required by the following three other methods:

1. VGFIC—Fourier inversion of the modified call price without using FFT;
2. VGPS—computing delta and the risk-neutral probability of finishing in-the-money;
3. VGP—the analytic formula in Madan, Carr, and Chang (1998).

The results are presented in Table 1.

We see from Table 1 that both FFT methods are considerably faster than the other methods, computing 160 option prices in around 6.5 seconds and 11.5 seconds respectively. The analytical method of Madan, Carr, and Chang (1998) has a speed that is broadly comparable with that of direct Fourier inversion without invoking the fast Fourier transform. By far the slowest method is the practice of solving for the probability of finishing in-the-money and for the delta. Additionally we note that this method is not only slow but also inaccurate, with substantial errors in Case 4.

For a more detailed analysis of Case 4, we evaluate the option prices in this case for strikes ranging from 70 to 130 in steps of a dollar, with the spot set at \$100, the interest rate at .05, and the dividend yield at .03. At the strikes of 77, 78, and 79 the prices reported by VGPS were respectively −.2425, −.2299, and 1.5386. The correct price reported by all the other methods, VGP, VGFIC, and the time value (TV) approach, were in agreement to four decimal places and were respectively .6356, .6787, and .7244. For a more detailed evaluation of the pricing errors, we computed for the remaining strikes the mean errors and their standard deviations. The errors were measured as deviations from the analytical formula VGP. This mean and standard deviation for VGPS are .0005658 and .0057 respectively. The corresponding values for the modified call price are

TABLE 1. CPU times for VG pricing.

	Case 1	Case 2	Case 3	Case 4
σ	.12	.25	.12	.25
ν	.16	2.0	.16	2.0
θ	−.33	−.10	−.33	−.10
t	1	1	.25	.25
VGFFTC	6.09	6.48	6.72	6.52
VGFFTTV	11.53	11.48	11.57	11.56
VGFIC	29.90	23.74	23.18	22.63
VGPS	288.50	191.06	181.62	197.97
VGP	22.41	24.81	23.82	24.74

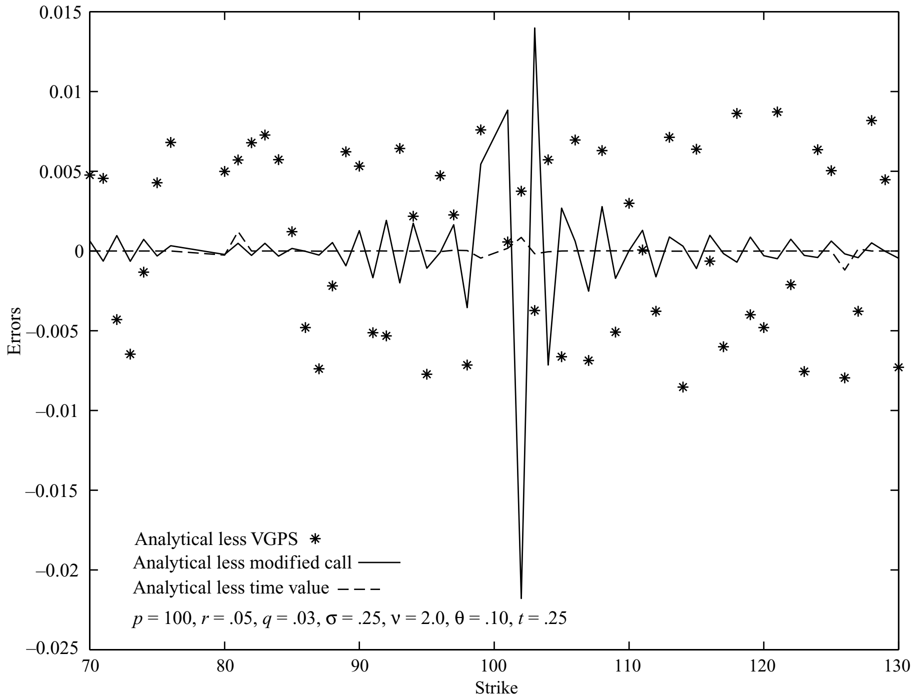

FIGURE 2. Pricing errors in Case 4 of Table 1.

.0001196 and .0041, while for the time value approach we have .000006059 and .0002662. Hence, we observe that the time value approach is an order of magnitude lower in its pricing errors compared with VGFIC, which is considerably better than VGPS. Figure 2 presents a graph of the pricing errors excluding the troublesome strikes for VGPS. The primary difficulty with VGPS comes from the behavior of the term iu in the denominator for values of u near zero.

6. SUMMARY AND CONCLUSIONS

We analytically developed two Fourier transforms in terms of the characteristic function of the log of the terminal stock price. The first is the Fourier transform of the modified call price written as a function of log strike, where the modification involves multiplying by an exponential. The second is the Fourier transform of the modified time value, where the modification involves multiplying by the hyperbolic sine function. Fourier inversion using the FFT yields the modified call price and the modified time value respectively. We illustrate our methods for the VG option pricing model and find that the use of the FFT is considerably faster than most available methods and, furthermore, that the traditional method described by Heston (1993), Bates (1996), Bakshi and Madan (1999), and Scott (1997) can be both slow and inaccurate. By focusing attention on delta claims, the traditional method sacrifices the advantages of the continuity of the call payoff and inherits in its place the problematic discontinuity of these claims. Thus, we recommend the use of the VGFFTC or VGFFTTV and in general the use of the FFT whenever the characteristic function of the underlying uncertainty is available in closed form.

We anticipate that the advantages of the FFT are generic to the widely known improvements in computation speed attained by this algorithm and is not connected to the particular characteristic function or process we chose to analyze. We have observed similar speed improvements when we work with generalizations of the VG model introduced by Geman, Madan, and Yor (1998), where a considerable variety of processes are developed with closed forms for the characteristic function of the log price.

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