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ASYMPTOTIC PERRON'S METHOD AND SIMPLE MARKOV STRATEGIES IN STOCHASTIC GAMES AND CONTROL*

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Abstract. We introduce a modification of Perron's method, where semisolutions are considered in a carefully defined asymptotic sense. With this definition, we can show, in a rather elementary way, that in a zero-sum game or a control problem (with or without model uncertainty), the value function over all strategies coincides with the value function over Markov strategies discretized in time. Therefore, there are always discretized Markov ε-optimal strategies (uniform with respect to the bounded initial condition). With a minor modification, the method produces a value and approximate saddle points for an asymmetric game of feedback strategies versus counterstrategies.

Key words. asymptotic Perron's method, stochastic games, Markov strategies, viscosity solutions

AMS subject classifications. 91A05, 91A15, 49L20, 49L25

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1. Introduction.

The aim of the paper is to introduce the asymptotic Perron's method, i.e., constructing a solution of the Hamilton–Jacobi–Belman–Isaacs (HJBI) equation as the supremum/infimum of carefully defined *asymptotic semisolutions*. Using this method we show, in a rather elementary way, that the value functions of zero-sum games/control problems can be (uniformly) approximated by some simple Markov strategies for the weaker player (the player in the exterior of the sup/inf or inf/sup). From this point of view, we can think of the method as an alternative to the shaken coefficients method of Krylov [Kry00] (in the case of only one player, under slightly different technical assumptions) or to the related method of regularization of solutions of HJBIs by Święch in [Świ96a] and [Świ96b] (for control problems or games in Elliott–Kalton formulation). The method of shaken coefficients has been recently used to study games in Elliott–Kalton formulation in [BN] under a convexity assumption (not needed here).

While the result on zero-sum games (under our standing assumptions) is rather new, but certainly expected, the goal of the paper is to present the method. To the best of our knowledge, this modification of Perron's method does not appear in the literature. In addition, we believe that it applies to more general situations than we consider here and using either a stochastic formulation (as in the present work) or an analytic one (see Remark 3.1). Compared to the method of shaken coefficients of Krylov, or to the regularization of solutions by Święch, the analytic approximation of the value function/solution of HJB by smooth approximate solutions is replaced by the Perron construction. The careful definition of asymptotic semisolutions allows us to prove that such semisolutions work well with Markov strategies. The idea of restricting actions to a deterministic time grid is certainly not new; we just provide a method that works well for such strategies/controls. The arguments display once again the

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robustness of the Perron construction, combined with viscosity comparison. There is basically a large amount of freedom in choosing the definition of semisolutions, as long as they compare (as a simple consequence of their definition) to the value functions. Here, we consider such *asymptotic* semisolutions.

Perron's method and its possible modifications seem to be rather useful in dynamic programming analysis. One could use Ishii's arguments (see [Ish87]) to construct a viscosity solution and later upgrade its regularity (if possible) for the purpose of verification (as in [JS12]) or employ a modification of Perron's method over stochastic semisolutions if one does not expect smooth solutions (see [BS12], [BS14], and [BS13]). Furthermore, with an appropriate definition of semisolutions in the stochastic sense and a suitable model, one can even treat games and problems with model uncertainty, where dynamic programming is particularly cumbersome (see [SÎ14c], [SÎ14a]). The present work represents an additional step in this program of using Perron's method and some possible offshoots to approach dynamic programming *without having to first prove the dynamic programming principle* (DPP). This step, unlike the previous (nonsmooth) ones actually provides a stronger conclusion than the usual dynamic programming approach. More precisely, it proves the existence of approximate simple Markov strategies (like [Kry00], or [Świ96a], [Świ96b]) something that is unclear from the DPP alone.

The formulation of the games/control problems we consider is similar to the formulation of deterministic games/control problems in the seminal work [KS88b]. What we do here is to propose a novel method to study such models in a stochastic framework.

Beyond the particular results obtained here, the (analytic version of the) method also seems to be useful in proving convergence of numerical schemes (but probably not the rate of convergence) in the spirit of Barles and Souganidis [BS91], or in the study of discretized games with mixed strategies as in [BLQ14]. We do not pursue this direction here.

Compared to the so-called stochastic Perron method employed in [SÎ14c] or [BS13], the method we introduce here is quite different. The idea in [SÎ14c] and [BS13] was to use *exact* semisolutions in the stochastic sense, then do Perron. In order to do so, the flexibility of stopping rules or stopping times was needed, leading to the possibility to complete the analysis *only* over general feedback strategies (elementary for the purpose of well posedness of the state equation) or general predictable controls. Here, we propose instead to use *asymptotic* semisolutions in the Perron construction. The flexibility on this end allows us to work with a deterministic time grid, resulting in approximation over Markov strategies. Obviously, the analytic part of the proof (the "bump-up/down" argument) is similar to [SÎ14c] or [BS13], but those parts of the proof were already similar to the (purely analytic) arguments of Ishii for viscosity Perron [Ish87].

2. Setup and main results.

We use the state equation and the standing assumptions from [SÎ14c].

The stochastic state system follows the dynamics:

$$\begin{cases} dX_t = b(t, X_t, u_t, v_t)dt + \sigma(t, X_t, u_t, v_t)dW_t, \\ X_s = x \in \mathbb{R}^d. \end{cases} \tag{2.1}$$

We assume that u and v belong to some compact metric spaces (U, d_u) and (V, d_V), respectively. For each s, the problem comes with a fixed probability space $(\Omega, \mathcal{P}, \mathcal{F})$, a fixed filtration $\mathbb{F} = (\mathcal{F}_t)_{s\leq t\leq T}$ satisfying the usual conditions, and a fixed Brownian motion $(W_t)_{s\le t\le T}$, with respect to the filtration $\mathbb{F}$. We suppress the dependence on s throughout the paper. We emphasize that the filtration may be strictly larger than the saturated filtration generated by the Brownian motion. The coefficients $b : [0,T]\times\mathbb{R}^d\times U\times V\to\mathbb{R}^d$ and $\sigma : [0,T]\times\mathbb{R}^d\times U\times V\to\mathcal{M}^{d\times d'}$ satisfy the following standing assumptions:

1. (C) b,σ are jointly continuous on $[0,T]\times\mathbb{R}^d\times U\times V$,
2. (L) b,σ satisfy a uniform local Lipschitz condition in x, i.e.,

$$|b(t,x,u,v)-b(t,y,u,v)|+|\sigma(t,x,u,v)-\sigma(t,y,u,v)|\le L(K)|x-y|$$

   $\forall\ |x|,|y|\le K, t\in[0,T],\ u\in U, v\in V$ for some $L(K)<\infty$, and
3. (GL) b,σ satisfy a global linear growth condition in x

$$|b(t,x,u,v)|+|\sigma(t,x,u,v)|\le C(1+|x|)$$

   $\forall\ |x|,|y|\in\mathbb{R}^d, t\in[0,T],\ u\in U, v\in V$ for some $C<\infty$.

Perron's method is a local method (for differential operators), so we only need local assumptions, except for the global growth which ensures nonexplosion of the state equation and comparison for the Isaacs equation (see [SÎ4c]).

Consider a bounded and continuous reward (for the player u) function $g:\mathbb{R}^d\to\mathbb{R}$. Fix an initial time $s\in[0,T]$. We are interested in the optimization problem

$$\sup_u \inf_v \mathbb{E}[g(X_T^{s,x;u,v}]$$

which is formally associated to the lower Isaacs equation

$$\begin{cases} -v_t - H^-(t,x,v_x,v_{xx}) = 0, \\ v(T,\cdot)=g(\cdot). \end{cases} \tag{2.2}$$

Above, we use the notation

$$H^-(t,x,p,M)\triangleq \sup_{u\in U}\inf_{v\in V} L(t,x,u,v,p,M),$$
$$L(t,x,u,v,p,M)\triangleq b(t,x,u,v)\cdot p+\frac{1}{2}Tr\big(\sigma(t,x,u,v)\sigma^T(t,x,u,v)M\big).$$

It is well known that making sense of what u, v should be above is a nontrivial issue, aside from relating the optimization problem to the Isaacs equation. We expect that three possible models are actually represented as solutions of the lower Isaacs equation. They are

1. the lower value of a symmetric game over feedback strategies (as in [PZ14] or [SÎ4c]),
2. the value function of a robust control problem where u is an intelligent maximizer and v is a possible worst case scenario modeling Knightian uncertainty (see [SÎ4a]), or
3. the genuine value of a sup-inf/inf-sup non symmetric game over feedback strategies versus feedback counterstrategies (as in [KS88b, section 10] or [FHH11], [FHH12]).

Although the main goal of the paper is to present the novel modification of Perron's method, we also provide a unified treatment for the three models above. Therefore, some modeling considerations need to be taken into account (on top of the ones we simply borrow from [SÎ4c] or [SÎ4a]). More precisely, depending on the information structure, one has two types of zero-sum games. The first, and fully symmetric one is where both players observe the past of the state process X (only) and make decisions based on this. In this model, both players use feedback strategies, i.e., nonanticipative functionals of the path X. (this is the case in [PZ14] or [SÎ4c]). On the other hand, one can envision a game where player u can see only the state, but player v observes the state and, in addition, the control u (in real time). Intuition corresponding to the nature of the problem tells us that the advantage that player v can gain from observing the whole past of the control u actually comes *only* from being able to adjust instantaneously to the observation u_t. In other words, in such a model we use a (counter) strategy for player v that depends on

1. the whole past of the state process X up to the present time t,
2. (only) the current value of the adverse control u_t.

This modeling avenue for deterministic games is used in section 10 of the seminal monograph [KS88b] and is followed up in the important work on stochastic games [FHH11], [FHH12].

Definition 2.1 (feedback strategies and counterstrategies). *Fix a time s.*

1. *A feedback strategy for player u is a mapping*

$$\alpha : [s,T] \times C[s,T] \to U,$$

which is predictable with respect to the (raw) filtration on the path space

$$\mathcal{B}_t \triangleq \sigma(y(q), s \le q \le t) \quad \forall t \in [s,T].$$

A similar definition holds for the player v.

2. *A feedback counterstrategy for the player v is a mapping*

$$\gamma : [s,T] \times C[s,T] \times U \to V,$$

which is measurable with respect to $\mathcal{P}^s \otimes \mathcal{U}^b$. The second player uses, at time t, the action

$$v_t = \gamma(t, X_\cdot, u_t),$$

where u_t is the action of the player u at time t. Above, $\mathcal{P}^s$ is the predictable sigma-field on $[s,T] \times C[s,T]$ with respect to the raw filtration on the path space and $\mathcal{U}^b$ is the Borel sigma-field on U with respect to the metric d_U.

The definition of counterstrategies goes back to [KS88b] (see, for example, the definition on p. 431 for a Markovian version) and the name is also borrowed from there. Feedback strategies have been used in control/games for a long time, and it is hard to trace their exact origin. It is clear that, for a fixed pair of feedback strategies (α, β) (or a feedback strategy versus a feedback counterstrategy (α, γ)), the state equation may fail to have a solution. Therefore, we need some restrictions on strategies (and counterstrategies). We use here (for strategies, there is) the restriction to elementary strategies in [SÎ4c].

Definition 2.2 (elementary feedback strategies and counterstrategies). *Fix s.*

1. *(See [SÎ4c].) An elementary feedback strategy (for the u-player) is a predictable functional of the path $\alpha : [s,T] \times C[s,T] \to U$, which is only "rebalanced" at some stopping rules $\tau_1 \le \tau_2 \le \cdots \le \tau_n$. More precisely, there exists n and some*

$$\tau_k : C[s,T] \to [s,T], \quad k = 1, 2, \ldots, n, \quad \textit{such that } \forall k, \quad \{\tau_k \le t\} \in \mathcal{B}_t, \; s \le t \le T,$$

and there also exist some

$$\xi_k : C[s,T] \to U \quad \textit{with} \quad \xi_k \in \mathcal{B}_{\tau_{k-1}}$$

so that

$$\alpha(t,y) = \sum_{k=1}^{n} 1_{\{\tau_{k-1}(y) < t \leq \tau_k(y)\}} \xi_k(y) \ \forall \ s \leq t \leq T, \ y \in C[s,T].$$

A similar definition holds for player v. We denote by $\mathcal{A}(s)$ and $\mathcal{B}(s)$ the collections of elementary (pure) feedback strategies for the u-player and the v-player, respectively.

2. *An elementary feedback counterstrategy for the v-player is a mapping*

$$\gamma : [s,T] \times C[s,T] \times U \to V,$$

for which there exist stopping rules $\tau_1 \leq \tau_2 \leq \cdots \leq \tau_n$ as above and some

$$\eta_k : C[s,T] \times U \to V \quad \textit{with} \quad \eta_k \in \mathcal{B}_{\tau_{k-1}} \otimes \mathcal{U}^b$$

so that

$$\gamma(t,y,u) = \sum_{k=1}^{n} 1_{\{\tau_{k-1}(y) < t \leq \tau_k(y)\}} \eta_k(y,u) \ \forall \ s \leq t \leq T, \ y \in C[s,T], \ u \in U.$$

We denote by $\mathcal{C}(s)$ the set of elementary counterstrategies of the v-player.

We also denote by $\mathcal{U}(s)$ and $\mathcal{V}(s)$ the set of open-loop controls for the u-player and the v-player, respectively. Precisely,

$$\mathcal{V}(s) \triangleq \{v : [s,T] \times \Omega \to V | \text{ predictable with respect to} \mathbb{F} = (\mathcal{F}_t)_{s \leq t \leq T}\},$$

and a similar definition is made for $\mathcal{U}(s)$. Since the number of symbols may become overwhelming, we will use for notation

1. α, β for the feedback strategies of players u and v,
2. u, v for the open-loop controls,
3. γ for the feedback counterstrategy of the second player v.

The *only* reason to restrict feedback strategies or counterstrategies to be elementary is to have well posedness of the state equation (actually in the strong sense).

LEMMA 2.3. *Fix s, x. Assume that the first player uses an open-loop control $u \in \mathcal{U}(s)$ or an elementary feedback strategy $\alpha \in \mathcal{A}(s)$. Assume also that the second player uses either an open-loop control $v \in \mathcal{V}(s)$ or a feedback strategy $\beta \in \mathcal{B}(s)$ or a counterstrategy $\gamma \in \mathcal{C}(s)$. Then the state equation has a unique strong solution.*

The result has been briefly proved in [SÎ4c] and [SÎ4a] with the possible exception of both players using open-loop controls (where the result is both obvious and not really used here) and the case when the second player uses a counterstrategy γ. We treat here the case when the second player uses an elementary counterstrategy γ.

1. (u, γ) If an open-loop control u and a counterstrategy γ are fixed, one has to solve the state equation, iteratively, between stopping rules τ_{k-1}, τ_k. This is possible since, between these stopping rules, the state equation has no "feedback," but it really looks like an open-loop versus open-loop equation. In other words, assume that the state equation is well posed up to the stopping

rule τ_{k-1} and the value of the process at that time (i.e., the random variable $X_{\tau_{k-1}(X.)}$) is L^2). Once the state process "has arrived" to the stopping rule τ_{k-1} one can start solving the SDE with an initial random time (rather than stopping rule) $\tau_{k-1}(X.)$ and initial condition $X_{\tau_{k-1}(X.)}$ and using the predictable open-loop controls

$$(u_t)_{\tau_{k-1}(X.)\leq t\leq T}, \quad (\eta_k(X., u_t))_{\tau_{k-1}(X.)\leq t\leq T},$$

on the stochastic interval $\tau_{k-1}(X.) \leq t \leq T$. This can be solved due to the locally Lipschitz and linear growth assumptions and results in a square integrable process X defined from $\tau_{k-1}(X.)$ forward. We let this run up to the next stopping rule τ_k and then continue as above. The result is a unique strong and square integrable solution X.

2. (α, γ) If an elementary feedback strategy $\alpha \in \mathcal{A}(s)$ and an elementary counterstrategy $\gamma \in \mathcal{C}(s)$ are chosen, one has to simply see that the superposition of the counterstrategy over the strategy is an elementary strategy. In other words,

$$\gamma[\alpha](t, y) \triangleq \gamma(t, y, \alpha(t, y)) \ \ \forall s \leq t \leq T,\ y \in \mathcal{C}[s, T]$$

defines a $\gamma[\alpha] \in \mathcal{B}(s)$. For $\alpha, \gamma[\alpha]$ the state equation is well posed (see [SÎ14c]), and therefore it is well posed over α, γ.

The state process X will be used with some explicit (and obvious) superscript notation $X^{s,x;\cdot,\cdot}$. Following [SÎ14c] we first define the lower value function (only) for the symmetric game in [SÎ14c] between two feedback players:

$$V^-(s, x) \triangleq \sup_{\alpha\in\mathcal{A}(s)} \left(\inf_{\beta\in\mathcal{B}(s)} \mathbb{E}[g(X_T^{s,x;\alpha,\beta})] \right). \tag{2.3}$$

Next, following [SÎ14a] we consider a robust control problem where the intelligent player u uses feedback strategies and the open-loop controls v parameterize worst case scenarios/Knightian uncertainty:

$$v^-(s, x) \triangleq \sup_{\alpha\in\mathcal{A}(s)} \left(\inf_{v\in\mathcal{V}(s)} \mathbb{E}[g(X_T^{s,x;\alpha,v})] \right). \tag{2.4}$$

We recall that the filtration may be larger than the one generated by the Brownian motion. Finally, following [KS88b] or [FHH11] we consider a genuine game (with a lower and an upper value) between two intelligent players who can both observe the state, but the second player has the advantage of also observing the first player's actions in real time, i.e.,

$$\begin{aligned} W^-(s, x) &\triangleq \sup_{\alpha\in\mathcal{A}(s)} \left(\inf_{\gamma\in\mathcal{C}(s)} \mathbb{E}[g(X_T^{s,x;\alpha,\gamma})] \right) \\ &\leq \inf_{\gamma\in\mathcal{C}(s)} \left(\sup_{\alpha\in\mathcal{A}(s)} \mathbb{E}[g(X_T^{s,x;\alpha,\gamma})] \right) \triangleq W^+(s, x). \end{aligned} \tag{2.5}$$

In addition to this, for mathematical reasons, we define yet another value function

$$v^+(s, x) \triangleq \inf_{\gamma\in\mathcal{C}(s)} \left(\sup_{u\in\mathcal{U}(s)} \mathbb{E}[g(X_T^{s,x;u,\gamma})] \right) \geq W^+(s, x). \tag{2.6}$$

We could attach to v^+ the meaning of some robust optimization problem, but this is not natural, since the intelligent optimizer v can see in real time the "worst case scenario". By simple observation we have $v^- \leq W^- \leq W^+ \leq v^+$. In addition, since $\mathcal{B}(s) \subset \mathcal{C}(s)$, and according to the second part of the proof of Lemma 2.3, for any fixed $\alpha \in \mathcal{A}(s), \gamma \in \mathcal{C}(s)$ we have $\gamma[\alpha] \in \mathcal{B}(s)$, we actually see that

$$\begin{aligned}\sup_{\alpha\in\mathcal{A}(s)}\left(\inf_{\gamma\in\mathcal{C}(s)}\mathbb{E}[g(X_T^{s,x;\alpha,\gamma})]\right) &\leq \sup_{\alpha\in\mathcal{A}(s)}\left(\inf_{\beta\in\mathcal{B}(s)}\mathbb{E}[g(X_T^{s,x;\alpha,\beta})]\right)\\ &\leq \sup_{\alpha\in\mathcal{A}(s)}\left(\inf_{\gamma\in\mathcal{C}(s)}\mathbb{E}[g(X_T^{s,x;\alpha,\gamma[\alpha]})]\right),\end{aligned}$$

i.e., $W^- = V^-$. Altogether, we know that

$$v^- \leq W^- = V^- \leq W^+ \leq v^+.$$

Remark 2.1. Since the state equation is well posed, the counterstrategies $\gamma \in \mathcal{C}(s)$ (and, therefore, the feedback strategies $\beta \in \mathcal{B}(s)$) are all strategies in the sense of Elliott and Kalton [EK72] (or [FS89] for the stochastic case). Therefore, there is a natural question: since, in the nonsymmetric game (2.5) the player v observes u, why not formulate the game as a sup/inf and inf/sup over Elliott–Kalton strategies versus open-loop controls? In other words, we could set up the problem (with little rigor in the formulation) as

$$\sup_u \inf_e \mathbb{E}[g(X_T^{s,x;u,e})] \leq \inf_e \sup_u \mathbb{E}[g(X_T^{s,x;u,e})],$$

where e is an Elliott–Kalton strategy. The lower value of the game above is, heuristically (quite obvious in the deterministic case), equal to the lower value of the symmetric game over open-loop controls and the upper value above is expected (according to [FS89]) to be the solution of the lower Isaacs equation (or, the unified Isaacs equation if the Isaacs condition holds). Therefore, (a possible modification of) the well-known example of Buckdahn (Example 8.1 in [PZ14]) shows that such a game may fail to have a value. On the other hand, the nonsymmetric game over feedback strategies versus counterstrategies will have a value (see Theorem 2.5 below) showing that the information structure considered in (2.5) is better suited to analyze a nonsymmetric game of this type. Elliott–Kalton strategies are designed to be considered only in the exterior of the inf-sup/sup-inf, i.e., only in the upper value above.

We now define a very special class of elementary strategies, namely, simple Markov strategies. Actually, what we call simple Markov strategies below are called "positional strategies" in [KS88b] (see the definition on p. 12 and relation (3) on p. 6, for example) and are used extensively in deterministic games. Similar strategies/counterstrategies are used under different names in the more recent interesting contributions [FHH11], [FHH12] on asymmetric zero-sum games.

DEFINITION 2.4 (time grids, simple Markov strategies, and counterstrategies). *Fix* $0 \leq s \leq T$.

1. *A time grid for* $[s,T]$ *is a finite sequence* π *of* $s = t_0 < t_1 < \cdots < t_n = T$.
2. *Fix a time grid* π *as above. A strategy* $\alpha \in \mathcal{A}(s)$ *is called a simple Markov strategy over* π *if there exist some functions* $\xi_k : \mathbb{R}^d \to U, k = 1, \ldots, n$, *measurable, such that*

$$\alpha(t, y(\cdot)) = \sum_{k=1}^n 1_{\{t_{k-1}<t\leq t_k\}}\xi_k(y(t_{k-1})).$$

The set of all simple Markov strategies over π is denoted by $\mathcal{A}^M(s,\pi)$. Define the set of all simple Markov strategies over all possible time grids as

$$\mathcal{A}^M(s) \triangleq \bigcup_{\pi} \mathcal{A}^M(s,\pi).$$

3. *Fix a time grid π as above. A counterstrategy $\gamma \in \mathcal{C}(s)$ is called a simple Markov counterstrategy over π if there exist some functions $\eta_k : \mathbb{R}^d \times U \to V, k = 1, \dots, n$, measurable, such that*

$$\gamma(t, y(\cdot), u) = \sum_{k=1}^{n} 1_{\{t_{k-1} < t \leq t_k\}} \eta_k(y(t_{k-1}), u).$$

The set of all simple Markov counterstrategies over π is denoted by $\mathcal{C}^M(s,\pi)$. Define the set of all simple Markov counterstrategies over all possible time grids as

$$\mathcal{C}^M(s) \triangleq \bigcup_{\pi} \mathcal{C}^M(s,\pi).$$

In words, a "simple Markov" strategy means that the player only changes actions over the time grid, and any time he/she does so, the new control depends on the current position only. However, for "simple Markov counterstrategies," the situation stands in stark contrast: actions are changed continuously in time based on the instantaneous observation of u, but the information from observing the state is updated only discretely over π.

Remark 2.2. If we attempt to fully discretize counterstrategies as well, allowing for the v-player to only update his/her actions over the time grid π, one cannot expect the game (2.5) to have a value, since the opponent u can change actions many times in between t_{k-1} and t_k. One would need to restrict both players to the same time grid π, then to pass to the limit over π (in the spirit of [BLQ14]) to obtain a value over fully discretized counterstrategies.

Consider now the same optimization problems as above, but where the weaker player is restricted to using *only* simple Markov strategies/counterstrategies, restricted to a fixed time grid, or not. Denote

$$v_\pi^-(s,x) \triangleq \sup_{\alpha \in \mathcal{A}^M(s,\pi)} \left(\inf_{v \in \mathcal{V}(s)} \mathbb{E}[g(X_T^{s,x;\alpha,v})] \right) \leq v^-(s,x) \leq W^-(s,x) = V^-(s,x)$$

and

$$\begin{aligned} v_M^-(s,x) &\triangleq \sup_{\alpha \in \mathcal{A}^M(s)} \left(\inf_{v \in \mathcal{V}(s)} \mathbb{E}[g(X_T^{s,x;\alpha,v})] \right) \\ &= \sup_{\pi} v_\pi^-(s,x) \leq v^-(s,x) \leq W^-(s,x) = V^-(s,x) \end{aligned}$$

as well as

$$v_\pi^+(s,x) \triangleq \inf_{\gamma \in \mathcal{C}^M(s,\pi)} \left(\sup_{u \in \mathcal{U}(s)} \mathbb{E}[g(X_T^{s,x;u,\gamma})] \right) \geq v^+(s,x) \geq W^+(s,x)$$

and

$$v_M^+(s,x) \triangleq \inf_{\gamma \in \mathcal{C}^M(s)} \left(\sup_{u \in \mathcal{U}(s)} \mathbb{E}[g(X_T^{s,x;u,\gamma})] \right) = \inf_{\pi} v_\pi^+(s,x) \geq v^+(s,x) \geq W^+(s,x).$$

The main result is that the u-player (in either setting) cannot do better with general feedback strategies than with simple Markov strategies, and the v-player can do as well over simple Markov counterstrategies as over general ones. The nonsymmetric game (2.5) has a value and admits approximate saddle points over simple Markov strategies/simple Markov counterstrategies. This is certainly expected. Under slightly stronger technical assumptions (natural filtration, global Lipschitz conditions) this is proved in [FHH11] with different methods and is also pursued in section 10 of [KS88b] for deterministic games. The one player (i.e., control) case is well studied (again with different methods) in the seminal monograph [Kry09] (see subsection 4.1 for more comments).

The whole idea of approximating the problem over controls/strategies that are fixed over time grids is by no means novel. It goes much further than [KS88b] for games or [Kry09] in control. The main contribution of the paper is to obtain such results with a different (and more elementary) method (and under slightly different assumptions).

THEOREM 2.5. *Under the standing assumptions, we have that*

$$v_M^- = v^- = W^- = V^- = W^+ = v^+ = v_M^+$$

and the common value is the unique bounded continuous viscosity solution of the lower Isaacs equation. In particular, the nonsymmetric game (2.5) *has a value which is equal to the lower value of the symmetric game* (2.3)*. For each N, and each $\varepsilon > 0$, there exists $\delta(N, \varepsilon) > 0$ such that*

$$\forall s \in [0,T], \forall\ |\pi| \leq \delta,\ \exists\ \hat{\alpha} \in \mathcal{A}^M(s,\pi), \hat{\gamma} \in \mathcal{C}^M(s,\pi)$$

such that $\forall\ |x| \leq N$, we have

$$0 \leq W^-(s,x) - \underbrace{\inf_{v \in \mathcal{V}(s)} \mathbb{E}[g(X_T^{s,x;\hat{\alpha},v})]}_{\leq v_\pi^-(s,x)} \leq \varepsilon \text{ and } 0 \leq \underbrace{\sup_{u \in \mathcal{U}(s)} \mathbb{E}[g(X_T^{s,x;u,\hat{\gamma}})]}_{\geq v_\pi^+(s,x)} - W^+(s,x) \leq \varepsilon.$$

Therefore, $v_\pi^-, v_\pi^+ \to V^- = W^- = W^+$ as $|\pi| \to 0$ uniformly on compacts in $[0,T] \times \mathbb{R}^d$.

The result above can be rewritten as

$$\mathbb{E}[g(X_T^{s,x;u,\hat{\gamma}})] - \varepsilon \leq W^+(s,x) = V^-(s,x) = W^-(s,x) \leq \mathbb{E}[g(X_T^{s,x;\hat{\alpha},v})] + \varepsilon$$

for $(\forall)\ \ (u,v) \in \mathcal{U}(s) \times \mathcal{V}(s), |x| \leq N$ (this is the way the result is phrased in the very interesting paper [FHH11]). Plugging in above the open-loop controls

$$\hat{u}_t = \hat{\alpha}(t, X_\cdot^{s,x;\hat{\alpha},\hat{\gamma}}), \hat{v}_t = \hat{\gamma}(t, X_\cdot^{s,x;\hat{\alpha},\hat{\gamma}}, \hat{u}_t)$$

we obtain that $|\mathbb{E}[g(X_T^{s,x;\hat{\alpha},\hat{\gamma}})] - V^-(s,x)| \leq \varepsilon$. This yields

$$\mathbb{E}[g(X_T^{s,x;u,\hat{\gamma}})] - 2\varepsilon \leq \mathbb{E}[g(X_T^{s,x;\hat{\alpha},\hat{\gamma}})] \leq \mathbb{E}[g(X_T^{s,x;\hat{\alpha},v})] + 2\varepsilon$$

$(\forall)\ \ (u,v) \in \mathcal{U}(s) \times \mathcal{V}(s), |x| \leq N$, which is stronger than

$$\mathbb{E}[g(X_T^{s,x;\alpha,\hat{\gamma}})] - 2\varepsilon \leq \mathbb{E}[g(X_T^{s,x;\hat{\alpha},\hat{\gamma}})] \leq \mathbb{E}[g(X_T^{s,x;\hat{\alpha},\gamma})] + 2\varepsilon$$

$(\forall)\ \ (\alpha,\gamma) \in \mathcal{A}(s) \times \mathcal{C}(s), |x| \leq N$. This means that $(\hat{\alpha}, \hat{\gamma})$ is actually a 2ε-saddle point for the (genuine) nonsymmetric game (2.5).

3. Proof: The asymptotic Perron's method. We introduce here the new version of Perron's method, where semisolutions of the (lower) Isaacs equations are replaced by asymptotic semisolutions. In our particular framework, we use asymptotic stochastic semisolutions (so the Perron method here is asymptotic in the stochastic sense of [SÎ4c] or [BS13]). However, we claim that a similar asymptotic Perron's method can be designed in the analytic framework (see Remark 3.1).

The definition of asymptotic semisolutions is different from the definition of stochastic semisolutions in [SÎ4a] or [SÎ4c] and, consequently, so are the proofs. The analytic part of the proof still resembles Ishii [Ish87] and the probabilistic part uses Itô along the smooth test function, but this is where similarities stop. As mentioned, the method we introduce here, since it is closely related to Markov strategies, can be viewed as an alternative to the powerful method of shaken coefficients of Krylov [Kry00] or to the work of Święch [Świ96a] and [Świ96b]. Anyway, in the case of games, the method of Święch actually works for a slightly different game, defined on a space accommodating an independent Brownian motion, and using Elliott–Kalton strategies. Since we are ultimately studying a nonsymmetric game (2.5), the analysis has to be done separately for the two value functions.

3.1. Asymptotic Perron over strategies. We perform here a sup Perron construction lying below the value function v_M^-. If one only cares about the lower value of the game V^- in (2.3) or the robust control problem (2.4), this is the only construction we need. Together with the results in [SÎ4c] this provides a full approximation of the two problems by elementary Markov strategies of the player u. We regard this as the most important result.

DEFINITION 3.1 (asymptotic stochastic subsolutions). *A function $w : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ is called an asymptotic (stochastic) subsolution of the (lower) Isaacs equation if it is bounded and continuous and satisfies $w(T,\cdot) \le g(\cdot)$. In addition, there exists a gauge function $\varphi = \varphi_w : (0,\infty) \to (0,\infty)$, depending on w such that*

1. $\lim_{\varepsilon \searrow 0} \varphi(\varepsilon) = 0$,
2. *for each s (and the optimization problem coming with it), for each time $s \le r \le T$, there exists a measurable function $\xi : \mathbb{R}^d \to U$ such that, for each x, each $\alpha \in \mathcal{A}(s)$, and $v \in \mathcal{V}(s)$, if we make the notation $\alpha[r,\xi] \in \mathcal{A}(s)$, defined by*

$$\alpha[r,\xi](t, y(\cdot)) = 1_{\{s<t\le r\}}\alpha(t, y(\cdot)) + 1_{\{r<t\le T\}}\xi(y(r)),$$

then for each $r \le t \le T$ we have

$$\begin{aligned} w(r, X_r^{s,x;\alpha,v}) &= w(r, X_r^{s,x;\alpha[r,\xi],v}) \\ &\le \mathbb{E}[w(t, X_t^{s,x;\alpha[r,\xi],v})|\mathcal{F}_r] + (t-r)\varphi(t-r) \ a.s. \end{aligned} \tag{3.1}$$

Denote by $\mathcal{L}$ the set of asymptotic subsolutions.

Remark 3.1. The definition of asymptotic solutions tells us that for each r there exists a Markov control at that time such that if the control is held constant until later, the state equation plugged inside w will almost have the submartingale property between r and any later time (reasonably close) for any choice of open-loop controls v. Since v can change wildly after r, in this framework it is very convenient to consider asymptotic subsolutions in the stochastic sense, resembling [SÎ4c] or [BS13]. However, if the adverse controls are restricted to not change very soon after r, one could consider asymptotic subsolutions in analytic formulation. Without pursuing this direction here, in the definition of such a subsolution, the inequality (3.1) could be replaced by

$$w(r,x_1) \le \int_{\mathbb{R}^d} w(t,z)p(r,t,x_1,x_2;\xi(x_1),v)dx_2 + (t-r)\varphi(t-r)\ \forall x_1 \in \mathbb{R}^d, v \in V.$$

Here, $p(r,t,x_1,x_2,u,v)dx_2$ is the transition law from time r to t of the state process X where u, v are held constant (which is obviously a Markov process). Such a definition, as mentioned above, would work well if controls v do not change between r and t, and would amount to an "analytic asymptotic Perron's method," as opposed to the stochastic setup we follow below.

Compared to [SÎ4c], the next proposition is not entirely trivial, but it is not hard either.

PROPOSITION 3.2. *Any $w \in \mathcal{L}$ satisfies $w \le v_M^- \le v^- \le W^- = V^-$.*

Proof. Fix ϵ and let δ such that $\varphi(\delta) \le \varepsilon$. Choose a time partition such that $t_k - t_{k-1} \le \delta$. For this particular partition, we construct, recursively, going from time t_{k-1} to time t_k, some measurable $\xi_k : \mathbb{R}^d \to U$ satisfying Definition 3.1. Now, we have, with α formally defined as in Definition 3.1 of simple Markov strategies, that for the simple Markov strategy α we have constructed,

$$w(t_{k-1}, X_{t_{k-1}}^{s,x;\alpha,v}) \le \mathbb{E}[w(t_k, X_{t_k}^{s,x;\alpha,v})|\mathcal{F}_{t_{k-1}}] + (t_k - t_{k-1})\underbrace{\varphi(t_k - t_{k-1})}_{\le\varepsilon} \text{ a.s.} \quad \forall k.$$

This happens for any x and any open-loop control v. Taking expectations and summing up, we conclude that

$$w(s,x) \le \mathbb{E}[w(T, X_T^{s,x;\alpha,v})] + \varepsilon \times (T-s)\ \forall v \in \mathcal{V}(s).$$

Taking the infimum over v, since $w(T,\cdot) \le g(\cdot)$, we conclude that if $|\pi| \le \delta$ there exists $\alpha \in \mathcal{A}^M(s,\pi)$ such that

$$w(s,x) \le \inf_{v\in\mathcal{V}} \mathbb{E}[g(X_T^{s,x;\alpha,v})] + \varepsilon \times (T-s) \le v_\pi^-(s,x) + \varepsilon \times (T-s) \ \ \forall x \in \mathbb{R}^d.$$

Letting $\varepsilon \searrow 0$ we obtain the conclusion. ∎

Remark 3.2. We could let the gauge function φ in Definition 3.1 depend on the control ξ (or even the time r) as well, to make the method even more flexible, but we just don't need that here. However, dependence on ξ makes a difference if one wants to treat unbounded controls, rather than the compact case we consider here.

The next lemma is rather obvious.

LEMMA 3.3. *The set of asymptotic subsolutions is directed upward, i.e., $w_1, w_2 \in \mathcal{L}$ implies $w_1 \vee w_2 \in \mathcal{L}$.*

Proof. The only important thing in the proof is to notice that one can choose the gauge function $\varphi = \varphi_1 \vee \varphi_2$ for $w = w_1 \vee w_2$. The choice of ξ is obvious. ∎

Asymptotic Perron's method for strategies. We define

$$w^- \triangleq \sup_{w\in\mathcal{L}} w \le v_M^- \le v^- \le W^-.$$

PROPOSITION 3.4 (asymptotic Perron). *The function w^- is an LSC viscosity supersolution of the (lower) Isaacs equation.*

Proof. From Proposition 4.1 in [BS12], there exist $\tilde{w}_n \in \mathcal{L}$ such that $w^- = \sup_n \tilde{w}_n$. We define the increasing sequence $w_n = \tilde{w}_1 \vee \cdots \vee \tilde{w}_n \in \mathcal{L} \nearrow w^-$.

1. Interior supersolution property. Let ψ touch w^- strictly below at some $(t_0,x_0) \in [0,T) \times \mathbb{R}^d$. Let us assume, by contradiction, that the viscosity supersolution property fails at (t_0,x_0). This means that

$$\psi_t(t_0,x_0) + \sup_u \inf_v L(t_0,x_0,u,v;\psi_x(t_0,x_0),\psi_{xx}(t_0,x_0)) > 0.$$

Since L is continuous and V is compact, we can choose a small neighborhood $B(t_0,x_0;\varepsilon) \subset [0,T) \times \mathbb{R}^d$ and some $\hat{u} \in U$ such that over this neighborhood we have

$$\psi_t(t,x) + \inf_v L(t,x,\hat{u},v;\psi_x(t,x)\psi_{xx}(t,x)) > \varepsilon.$$

This ε will be kept fixed, including in Lemma 3.5 below. From here, we follow the usual Perron construction. The very different part will be to show that after we "bump up" (an approximation of) w^-, it still stays an asymptotic subsolution. More precisely, we know that since ψ touches w^- below in a strict sense, there exists room of size $\delta > 0$ in between w^- and ψ over the compact (rectangular) torus

$$\mathbb{T} \triangleq \overline{B(t_0,x_0,\varepsilon)} - B(t_0,x_0,\varepsilon/2),$$

i.e., $w^- \geq \psi + \delta$ on $\mathbb{T}$. A Dini type argument (see, for example, [BS14]) shows that one of the terms $w \triangleq w_n$ actually satisfies $w \geq \psi + \delta/2$ on $\mathbb{T}$. Define now for $0 < \rho << \delta/2$ the function

$$\hat{v} = \begin{cases} w \vee (\psi+\rho), & \text{on } B(t_0,x_0;\varepsilon), \\ w \text{ outside } B(t_0,x_0;\varepsilon). \end{cases}$$

Note that $\hat{v} = w$ on the overlapping $\mathbb{T}$ (so, it is continuous) and $\hat{v}(t_0,x_0) = w^-(t_0,x_0) + \rho > w^-(t_0,x_0)$. The proof would be finished if we can show that $\hat{v}$ is an asymptotic subsolution.

The idea of the proof is quite simple:

1. If, at time r, we have $w \geq \psi + \rho$ (at that particular position $y(r)$), then we follow from r forward the nearly optimal strategy corresponding to the asymptotic subsolution w (which depends only on $y(r)$).
2. If, at time r, we have instead $w < \psi + \rho$ (again, at that particular position $y(r)$) we follow from r forward the strategy $\hat{u}$. Between r and any later time t, the process $\psi + \gamma$ superposed to the state equation is not a true submartingale, but is an asymptotic one. The reason is that it is a submartingale until the first time it exits $B(t_0,x_0;\varepsilon)$. However, since $w < \psi + \rho$ holds only inside $B(t_0,x_0;\varepsilon/2)$, the chance that such an exit actually occurs before t can be estimated in terms of the size of the interval $t-r$ and bounded above by a gauge function.

We develop rigorously below the arguments described above. Fix $s \leq r \leq T$. Since w is an asymptotic subsolution, there exists a Markov strategy ξ at time r corresponding to Definition 3.1 for the subsolution w (for the initial time s). Now, we define

$$\hat{\xi}(x) = 1_{\{(r,x)\notin B(t_0,x_0;\varepsilon/2) \vee w(r,x) \geq \psi(r,x)+\rho\}}\xi(x) + 1_{\{(r,x)\in B(t_0,x_0;\varepsilon/2) \wedge w(r,x) < \psi(r,x)+\rho\}}\hat{u}.$$

We want to show that $\hat{\xi}$ satisfies the desired property in the Definition 3.1 for the (expected) subsolution $\hat{v}$ at r, with an appropriate choice of the gauge function φ independent of r, s, or $\hat{\xi}$. Let φ_w be the gauge function of the subsolution w. Consider any $\alpha \in \mathcal{A}(s)$ and any $v \in \mathcal{V}(s)$. By the definition of the subsolution w, and taking into account that $w \leq \hat{v}$, we have that on the event

$$A \triangleq \{(r, X_r^{s,x;\alpha,v}) \notin B(t_0, x_0; \varepsilon/2)\} \cup \{w(r, X_r^{s,x;\alpha,v}) \geq \psi(r, X_r^{s,x;\alpha,v}) + \rho\} \in \mathcal{F}_r$$

we have that $X_t^{s,x;\alpha[r,\xi],v} = X_t^{s,x;\alpha[r,\hat{\xi}],v}$ a.s. for $r \leq t \leq T$, and therefore

$$\begin{aligned} 1_A \hat{v}(r, X_r^{s,x;\alpha,v}) &= 1_A w(r, X_r^{s,x;\alpha,v}) \leq \mathbb{E}[1_A w(t, X_t^{s,x;\alpha[r,\xi],v}) | \mathcal{F}_r] \\ &\quad + 1_A \times (t-r)\varphi_w(t-r) \leq \mathbb{E}[1_A \hat{v}(t, X_t^{s,x;\alpha[r,\hat{\xi}],v}) | \mathcal{F}_r] + 1_A \times (t-r)\varphi_w(t-r). \end{aligned} \tag{3.2}$$

On the complement of A, the process $\psi(t, X_t^{s,x;\alpha[r,\hat{\xi}],v})$ is a submartingale (by Itô) up to the first time τ where the process gets out of $B(t_0, x_0; \varepsilon)$, i.e., up to

$$\tau \triangleq \inf\{t \geq r | (t, X_t^{s,x;\alpha[r,\hat{\xi}]}) \notin B(t_0, x_0; \varepsilon)\}.$$

The submartingale property says that

$$\begin{aligned} 1_{A^c}(\psi + \rho)(r, X_r^{s,x;\alpha,v}) &\leq \mathbb{E}[1_{A^c}(\psi + \rho)(\tau \wedge t, X_{\tau\wedge t}^{s,x;\alpha[r,\hat{\xi}],v}) | \mathcal{F}_r] \\ &\leq \mathbb{E}[1_{A^c} \hat{v}(\tau \wedge t, X_{\tau\wedge t}^{s,x;\alpha[r,\hat{\xi}],v}) | \mathcal{F}_r]. \end{aligned}$$

Fix t such that $r \leq t \leq r + \varepsilon/2$. Denote now the event

$$B \triangleq \{|X_{t'}^{s,x;\alpha[r,\hat{\xi}],v} - x_0| < \varepsilon \ \forall \ r \leq t' \leq t\}.$$

We use here the norm $|(t,x)) \triangleq \max\{|t|, |x|\}$ so that $A^c \cap B = A^c \cap \{\tau > t\}$ and $A^c \cap B^c = A^c \cap \{\tau \leq t\}$. Consequently, we have

$$\begin{aligned} \mathbb{E}[1_{A^c} \hat{v}(\tau \wedge t, X_{\tau\wedge t}^{s,x;\alpha[r,\hat{\xi}],v}) | \mathcal{F}_r] &= \mathbb{E}[1_{A^c} 1_B \hat{v}(t, X_t^{s,x;\alpha[r,\hat{\xi}],v}) | \mathcal{F}_r] \\ &\quad + \mathbb{E}[1_{A^c} 1_{B^c} \hat{v}(\tau, X_\tau^{s,x;\alpha[r,\hat{\xi}],v}) | \mathcal{F}_r]. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbb{E}[1_{A^c} \hat{v}(\tau \wedge t, X_{\tau\wedge t}^{s,x;\alpha[r,\hat{\xi}],v}) | \mathcal{F}_r] &= \mathbb{E}[1_{A^c} \hat{v}(t, X_t^{s,x;\alpha[r,\hat{\xi}],v}) | \mathcal{F}_r] \\ &\quad + \mathbb{E}[1_{A^c} 1_{B^c} \big(\hat{v}(\tau, X_\tau^{s,x;\alpha[r,\hat{\xi}],v}) - \hat{v}(t, X_t^{s,x;\alpha[r,\hat{\xi}],v})\big) | \mathcal{F}_r]. \end{aligned}$$

Since $\hat{v}$ is bounded by some constant $\|\hat{v}\|_\infty$, we conclude that for $r \leq t \leq r + \varepsilon/2$ we have

$$\begin{aligned} 1_{A^c} \hat{v}(r, X_r^{s,x;\alpha,v}) &= 1_{A^c}(\psi + \gamma)(r, X_r^{s,x;\alpha,v}) \\ &\leq \mathbb{E}[1_{A^c} \hat{v}(t, X_t^{s,x;\alpha[r,\hat{\xi}],v}) | \mathcal{F}_r] + 2\|\hat{v}\|_\infty \mathbb{P}[A^c \cap B^c | \mathcal{F}_r] \text{ a.s.} \end{aligned} \tag{3.3}$$

We can now put together (3.2) and (3.3). If we can find a gauge function $\tilde{\varphi}$ such that

$$2\|\hat{v}\|_\infty \mathbb{P}[A^c \cap B^c | \mathcal{F}_r] \leq 1_{A^c} \tilde{\varphi}(t-r) \times (t-r) \text{ a.s.}$$

we are done, as one can choose the gauge function for $\hat{v}$ as

$$\varphi_{\hat{v}} \triangleq \varphi_w \vee \tilde{\varphi}. \qquad \square$$

We do that in Lemma 3.5 below and finish the proof of the interior subsolution property.

LEMMA 3.5. *There exist constants C, C' (dependent only on (t_0, x_0), the fixed ε, and the coefficients b, σ of the state equation) such that, for any s and any $r \geq s$, if $r \leq t \leq r + \frac{\varepsilon}{2} \wedge \frac{\varepsilon}{4C}$ we have*

$$\mathbb{P}[A^c \cap B^c | \mathcal{F}_r] = 1_{A^c} \mathbb{P}[A^c \cap B^c | \mathcal{F}_r] \leq 1_{A^c} C' \mathbb{P}\left(N(0,1) \geq \frac{1}{C\sqrt{t-r}} \frac{\varepsilon}{4}\right) \quad a.s.$$

for all strategies $\alpha \in \mathcal{A}(s)$ and controls $v \in \mathcal{V}$. The function

$$\tilde{\varphi}(t) \triangleq \frac{2\|\hat{v}\|_\infty C'}{t} \mathbb{P}\left(N(0,1) \geq \frac{1}{C\sqrt{t}} \frac{\varepsilon}{4}\right)$$

satisfies $\lim_{t \searrow 0} \tilde{\varphi}(t) = 0$ and therefore is a gauge function.

Proof. To begin with, we emphasize that we do not need such a precise bound on conditional probabilities to finish the proof of Theorem 2.5 (both the interior supersolution part or the terminal condition). The simple idea of the proof is to see that, conditioned on A^c, the event we care about amounts to a continuous semimartingale with bounded volatility and bounded drift to exit from a fixed box in the interval of time $[r, t]$. If the size of $t - r$ is small enough, that amounts to just the martingale part exiting from a smaller fixed box between t and r. This can be rephrased, through a time change, in terms of a Brownian motion, and estimated very precisely to be of the order $\mathbb{P}(N(0,1) \geq \frac{1}{C\sqrt{t-r}} \frac{\varepsilon}{4}) = o(t-r)$, where $N(0, a^2)$ is a normal with mean zero and standard deviation a. This is basically the whole proof, in words. The precise mathematics below follows exactly these lines. We first notice that

$$A^c \cap B^c \subset \Big\{(r, X_r^{s,x;\alpha,v}) \in B(t_0, x_0; \varepsilon/2)\Big\} \cap \Big\{|X_{t'}^{s,x;\alpha[r,\hat{\xi}],v} - x_0| \geq \varepsilon \text{ for some } r \leq t' \leq t\Big\}.$$

If $t - r < \varepsilon/2$ we therefore have (from previous considerations) that $A^c \cap B^c \subset \{\tau \leq t\}$ and (even more important for our proof) that

$$A^c \cap B^c \subset \Big\{|X_{t'}^{s,x;\alpha[r,\hat{\xi}],v} - X_r^{s,x;\alpha[r,\xi],v}| \geq \varepsilon/2 \text{ for some } r \leq t' \leq t \wedge \tau\Big\}.$$

Over $B(t_0, x_0; \varepsilon)$ both the drift and the volatility of the state system are uniformly bounded by some constant C. Therefore, if we choose $t - r \leq \frac{\varepsilon}{4C}$ the integral of the drift part cannot exceed $\varepsilon/4$ in size. We therefore conclude that if $t - r \leq \frac{\varepsilon}{2} \wedge \frac{\varepsilon}{4C}$, then $A^c \cap B^c$ is a subset of the event

$$\left\{\left|\int_r^{t'} \sigma(q, X_q^{s,x;\alpha[r,\hat{\xi}],v}, \alpha[r,\hat{\xi}](q, X_\cdot^{s,x;\alpha[r,\hat{\xi}],v}), v_q) dW_q\right| \geq \varepsilon/4 \text{ for some } r \leq t' \leq t \wedge \tau\right\}.$$

Denote by

$$M_{t'} \triangleq \int_s^{t'} \sigma(q, X_q^{s,x;\alpha[r,\hat{\xi}],v}, \alpha[r,\hat{\xi}](q, X_\cdot^{s,x;\alpha[r,\hat{\xi}],v}), v_q) dW_q \;\forall\;\; s \leq t' \leq T.$$

We study separately the coordinates of M. More precisely we consider on $\mathbb{R}^d$ the max norm as well, and, for $M_{t'} = (M_{t'}^1, \ldots, M_{t'}^d)$, we have

$$A^c \cap B^c \subset \bigcup_{l=1}^d \Big\{|M_{t'}^l - M_r^l| \geq \varepsilon/4 \text{ for some } r \leq t' \leq t \wedge \tau\Big\}.$$

We estimate the probabilities above (conditioned on $\mathcal{F}_r$) individually. We want to use the Dambis–Dubins–Schwarz result to translate the computation into the probability of exiting from a fixed box of a Brownian motion. In order to do so rigorously, and without enlarging the probability space to accommodate an additional Brownian motion, we first need to make the volatility explode at T. Choose any function

$$f \to [s,T) \to (0,\infty), \quad \int_s^t f^2(q)dq < \infty \ \forall q < T, \ \int_s^T f^2(q)dq = \infty.$$

Fix some $\varepsilon' > 0$ such that $t_0 + \varepsilon < T - \varepsilon'$, so $\tau \le T - \varepsilon'$. Define

$$\begin{aligned} M_{t'}^{\varepsilon'} \triangleq \int_s^{t'} \Big(& 1_{\{s \le q \le T-\varepsilon'\}} \sigma(q, X_q^{s,x;\alpha[r,\hat{\xi}],v}, \alpha[r,\hat{\xi}](q, X_\cdot^{s,x;\alpha[r,\hat{\xi}],v}), v_q) \\ & + 1_{\{T-\varepsilon' < q \le T\}} f(q) \Big) dW_q \ \forall s \le t' < T \end{aligned}$$

such that $M_{t'} = M_{t'}^{\varepsilon'}$ for $r \le t' \le T - \varepsilon'$. According to the Dambis–Dubins–Schwarz theorem (see [KS88a, p. 174]) applied with the obvious shift of time origin from r to zero, the process $(Z_{s'})_{0 \le s' < \infty}$ defined as

$$Z_{s'} = M_{A_{s'}}^{\varepsilon',l} - M_r^{\varepsilon',l}, 0 \le s' < \infty, \quad \text{for } A_{s'} = \inf\{t' \ge r | \langle M^{\varepsilon',l}\rangle_{t'} - \langle M^{\varepsilon',l}\rangle_r > s'\},$$

is a standard (i.e., $Z_0 = 0$) one-dimensional Brownian motion with respect to the filtration $\mathcal{G}_{s'} \triangleq \mathcal{F}_{A_{s'}}$, $0 \le s' < \infty$. Therefore, since $\mathcal{F}_r \subset \mathcal{G}_0$, Z is independent of $\mathcal{F}_r$. In addition, still from Dambis-Dubins-Schwarz, we have

$$M_{t'}^{\varepsilon',l} - M_r^{\varepsilon',l} = Z_{\langle M^{\varepsilon',l}\rangle_{t'} - \langle M^{\varepsilon',l}\rangle_r}, \ r \le t' < T.$$

On the other hand, since σ is bounded by C (independently on the strategy and the control) before τ (the exit time from $B(t_0, x_0; \varepsilon)$) we have

$$\langle M^{\varepsilon',l}\rangle_{t'} - \langle M^{\varepsilon',l}\rangle_r \le C^2(t'-r), \quad r \le t' \le \tau \le T - \varepsilon'.$$

Since $M_{t'} = M_{t'}^{\varepsilon'}$ for $r \le t' \le \tau \le T - \varepsilon'$ we conclude that

$$\Big\{ |M_{t'}^l - M_r^l| \ge \varepsilon/4 \ \text{ for } \ \text{some } r \le t' \le t \wedge \tau \Big\} \subset \left\{ \max_{0 \le s' \le C^2(t-r)} |Z_{s'}| \ge \varepsilon/4 \right\}.$$

Let M_t^+ and M_t^- be the distributions of the running max and the (negative) running min of a standard Brownian motion starting at time zero. Since the Brownian motion Z is independent of $\mathcal{F}_r$, we can obtain that

$$\begin{aligned} \mathbb{P}\Big(\Big\{ |M_{t'}^l - M_r^l| \ge \varepsilon/4 \ \text{ for } \ \text{some } r \le t' \le t \wedge \tau \Big\} | \mathcal{F}_r \Big) & \\ \le \mathbb{P}(M_{C^2(t-r)}^+ \ge \varepsilon/4) + \mathbb{P}(M_{C^2(t-r)}^- \ge \varepsilon/4) &= 2\mathbb{P}(M_{C^2(t-r)}^+ \ge \varepsilon/4). \end{aligned}$$

It is well known, from the reflection principle (see, for example, Karatzas and Shreve [KS88a, p. 80]) that this is the same as

$$\begin{aligned} \mathbb{P}\Big(\Big\{ |M_{t'}^l - M_r^l| \ge \varepsilon/4 \ \text{ for } \ \text{some } r \le t' \le t \wedge \tau \Big\} | \mathcal{F}_r \Big) & \\ \le 4\mathbb{P}(N(0, C^2(t-r)) \ge \varepsilon/4) &= 4\mathbb{P}\left(N(0,1) \ge \frac{1}{C\sqrt{t-r}} \frac{\varepsilon}{4} \right). \end{aligned}$$

We emphasize that all estimates are independent of the initial time s, the later time r, and the even later time t as long as $t-r \leq \frac{\varepsilon}{2} \wedge \frac{\varepsilon}{4C}$ (and independent of $\alpha \in \mathcal{A}(s)$ and of $v \in \mathcal{V}(s)$). Summing all the terms, we obtain

$$\mathbb{P}[A^c \cap B^c | \mathcal{F}_r] \leq 4d\mathbb{P}\left(N(0,1) \geq \frac{1}{C\sqrt{t-r}}\frac{\varepsilon}{4}\right) \text{ if } t-r \leq \frac{\varepsilon}{2} \wedge \frac{\varepsilon}{4C}.$$

We can multiply now with 1_{A^c} which is measurable with respect to $\mathcal{F}_r$ to obtain the conclusion. It is well known that

$$\frac{\mathbb{P}\left(N(0,1) \geq \frac{1}{C\sqrt{t}}\frac{\varepsilon}{4}\right)}{t} \to 0 \text{ as } t \searrow 0,$$

and this finishes the proof of the lemma. Instead of appealing to the Dambis–Dubins–Schwarz theorem, we could also use the (smooth solution) characterizing the exit probability of a Brownian motion from a box and superpose it to the state process, resulting in a supermartingale (just using the Itô formula). This would still bound the probability we are interested in by the exit probability of a standard Brownian motion from a box. $\square$

The proof of Proposition 3.4 continues.

2. The terminal condition $w^-(T,\cdot) \geq g(\cdot)$. The proof of this is done again by contradiction. The "bump-up" analytic construction we use is similar to [BS13], and the rest is based on arguments similar to the interior supersolution property and a very similar estimate to Lemma 3.5 above. The only minor difference (if the Dambis–Dubins–Schwarz route is followed) is that, with all notation as above, if $r \leq t < T$ and $t \leq r + \frac{\varepsilon}{4C}$, we have to choose an $\varepsilon' > 0$ dependent on t (unlike in the proof of the interior supersolution property) such that $r \leq t < T - \varepsilon'$. For this ε' we again have

$$\left\{|M^l_{t'} - M^l_r| \geq \varepsilon/4 \text{ for some } r \leq t' \leq \tau \wedge t\right\}$$
$$= \left\{|M^{\varepsilon',l}_{t'} - M^{\varepsilon',l}_r| \geq \varepsilon/4 \text{ for some } r \leq t' \leq \tau \wedge t\right\}.$$

Therefore, with a gauge function similar to the one we just chose in the proof of the Lemma 3.5 above), we have

$$\hat{v}(r, X^{s,x;\alpha,v}_r) \leq \mathbb{E}[\hat{v}(t, X^{s,x;\alpha[r,\hat{\xi}],v}_t)|\mathcal{F}_r] + (t-r)\varphi(t-r) \text{ a.s.}$$

$\forall\, r \leq t < T$ such that $t - r \leq \frac{\varepsilon}{4C}$. Obviously, letting $t \nearrow T$ we obtain the same for $t = T$ as long as $T - r \leq \frac{\varepsilon}{4C}$. The proof is now complete. $\square$

3.2. Asymptotic Perron over counterstrategies. One only has to go through this construction if the genuine (nonsymmetric) game in (2.5) is studied. The notion of counterstrategies (even with the Markov discretization) is still not so easily implementable, since actions do change continuously in time (if u does so, in a situation of counterstrategies versus open-loop controls as in the definition (2.6) of v^+, which describes some strange model of the worst case scenario, analyzed here for mathematical reasons only). On the other hand, there is no way one can genuinely discretize counterstrategies to obtain a value in (2.5) (see Remark 2.2). Therefore, we go over this analysis for mathematical completeness, emphasizing that the basic method and the more important result are contained in subsection 3.1. We view this as a simple additional application of the asymptotic Perron's method. The definitions and proofs follow in lockstep with subsection 3.1, with minor appropriate modifications to account for counterstrategies.

DEFINITION 3.6 (asymptotic stochastic supersolutions). *A function $w : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ is called an asymptotic (stochastic) supersolution of the (lower) Isaacs equation if it is bounded and continuous and satisfies $w(T,\cdot) \geq g(\cdot)$. In addition, there exists a gauge function $\varphi = \varphi_w : (0,\infty) \to (0,\infty)$, depending on w such that*

1. $\lim_{\varepsilon \searrow 0} \varphi(\varepsilon) = 0$,
2. *for each s (and the optimization problem coming with it), for each time $s \leq r \leq T$, there exists a measurable function $\eta : \mathbb{R}^d \times U \to V$ such that, for each x, each $\gamma \in \mathcal{C}(s)$, and $u \in \mathcal{U}(s)$, if we make the notation $\gamma[r,\eta] \in \mathcal{C}(s)$, defined by*

$$\gamma[r,\eta](t,y(\cdot),u) = 1_{\{s<t\leq r\}}\gamma(t,y(\cdot),u) + 1_{\{r<t\leq T\}}\eta(y(r),u),$$

then for each $r \leq t \leq T$ we have

$$\begin{aligned} w(r, X_r^{s,x;u,\gamma}) &= w(r, X_r^{s,x;u,\gamma[r,\eta]}) \\ &\geq \mathbb{E}[w(t, X_t^{s,x;u,\gamma[r,\eta],})|\mathcal{F}_r] - (t-r)\varphi(t-r) \text{ a.s.} \end{aligned} \tag{3.4}$$

Denote by $\mathcal{U}$ the set of asymptotic supersolutions.

We again have the following.

PROPOSITION 3.7. *Any $w \in \mathcal{U}$ satisfies $w \geq v_M^+$.*

Proof. Fix ϵ and let δ such that $\varphi(\delta) \leq \varepsilon$. Choose π such that $\|\pi\| \leq \delta$. For this partition π, we construct, recursively, going from time t_{k-1} to time t_k, some measurable $\eta_k : \mathbb{R}^d \times U \to V$ satisfying Definition 3.6. We put the η_k's together to obtain a Markov counterstrategy γ for which

$$w(t_{k-1}, X_{t_{k-1}}^{s,x;u,\gamma}) \geq \mathbb{E}[w(t_k, X_{t_k}^{s,x;u,\gamma})|\mathcal{F}_{t_{k-1}}] - (t_k - t_{k-1})\underbrace{\varphi(t_k - t_{k-1})}_{\leq \varepsilon} \text{ a.s.} \quad \forall k.$$

This happens for any x and any open-loop control u. Taking expectations and summing up, we conclude that

$$w(s,x) \geq \mathbb{E}[w(T, X_T^{s,x;u,\gamma})] - \varepsilon \times (T-s) \; \forall u \in \mathcal{U}(s).$$

Taking the supremum over u, since $w(T,\cdot) \geq g(\cdot)$, we conclude that if $|\pi| \leq \delta$ there exists $\gamma \in \mathcal{C}^M(s,\pi)$ such that

$$w(s,x) \geq \sup_{u \in \mathcal{U}(s)} \mathbb{E}[g(X_T^{s,x;u,\gamma})] - \varepsilon \times (T-s) \geq v_\pi^+(s,x) - \varepsilon \times (T-s) \;\; \forall x \in \mathbb{R}^d.$$

Letting $\varepsilon \searrow 0$ we obtain the conclusion. ∎

The next lemma is, once again, obvious.

LEMMA 3.8. *The set of asymptotic supersolutions is directed downward, i.e., $w_1, w_2 \in \mathcal{U}$ implies $w_1 \wedge w_2 \in \mathcal{U}$.*

Proof. The only thing needed is to notice that one can choose the gauge function $\varphi = \varphi_1 \vee \varphi_2$ for $w = w_1 \vee w_2$. The choice of η is, again, obvious. ∎

Asymptotic Perron's method for counterstrategies. We define

$$w^+ \triangleq \inf_{w \in \mathcal{U}} w \geq v_+^M \geq v^+ \geq W^+.$$

PROPOSITION 3.9 (asymptotic Perron). *The function w^+ is a USC viscosity subsolution of the (lower) Isaacs equation.*

Proof. 1. Interior subsolution property. Let ψ touch w^+ strictly above at some $(t_0, x_0) \in [0,T) \times \mathbb{R}^d$. Assume, by contradiction, that

$$\psi_t(t_0,x_0) + \sup_u \inf_v L(t_0,x_0,u,v;\psi_x(t_0,x_0),\psi_{xx}(t_0,x_0)) < 0.$$

This means that there exists a small $\varepsilon > 0$ and a (measurable) function $h : U \to V$ such that

$$\psi_t(t_0,x_0) + L(t_0,x_0,u,h(u);\psi_x(t_0,x_0),\psi_{xx}(t_0,x_0)) < -\varepsilon.$$

Since L is continuous and U, V are compact (so L is uniformly continuous over (t,x,u,v,p,M) as long as (t,x) is close to (t_0,x_0) and $(p.M)$ is close to $(\psi_x(t_0,x_0), \psi_{xx}(t_0,x_0))$, we can choose an even smaller ε such that

$$\psi_t(t,x) + L(t,x,u,h(u);\psi_x(t,x),\psi_{xx}(t,x)) < -\varepsilon$$

over the (smaller) neighborhood $B(t_0,x_0;\varepsilon) \subset [0,T) \times \mathbb{R}^d$. From here on, we follow the usual Perron construction (with ε fixed as above). We need to show that after we "bump down" (an approximation of) w^+, it still stays an asymptotic supersolution. Since ψ touches w^+ above in a strict sense, there exists room of size $\delta > 0$ between w^+ and ψ over the compact (rectangular) torus

$$\mathbb{T} \triangleq \overline{B(t_0,x_0,\varepsilon)} - B(t_0,x_0,\varepsilon/2),$$

i.e., $w^+ \le \psi - \delta$ on $\mathbb{T}$. Again a Dini type argument (see, for example, [BS14]) shows that one of the terms of the sequence $w_n \searrow w^+$, which we simply denote by w, actually satisfies $w \le \psi - \delta/2$ on $\mathbb{T}$. Define now for $0 < \rho << \delta/2$ the function

$$\hat{v} = \begin{cases} w \wedge (\psi - \rho) & \text{on } B(t_0,x_0;\varepsilon), \\ w \text{ outside } B(t_0,x_0;\varepsilon). \end{cases}$$

We have that $\hat{v} = w$ on the overlapping $\mathbb{T}$ (so, $\hat{v}$ is continuous) and $\hat{v}(t_0,x_0) = w^+(t_0,x_0) - \rho < w^+(t_0,x_0)$. We only need to show that $\hat{v}$ is an asymptotic supersolution to have a full proof. Fix $s \le r \le T$. Since w is an asymptotic supersolution in the sense of Definition 3.6, there exists an $\eta : \mathbb{R}^d \times U \to V$ at time r corresponding to the Definition 3.6 for the supersolution w (for the initial time s). Define

$$\begin{aligned} \hat{\eta}(x,u) = {} & 1_{\{(r,x)\notin B(t_0,x_0;\varepsilon/2) \vee w(r,x) \le \psi(r,x)-\rho\}}\eta(x,r) \\ & + 1_{\{(r,x)\in B(t_0,x_0;\varepsilon/2) \wedge w(r,x) > \psi(r,x)-\rho\}} h(u). \end{aligned} \tag{3.5}$$

Remark 3.3. The choice of $h(u)$ together with the Itô formula tells as that as long as the player v always adjusts his/her control (observing continuously the other player's actions) to be $h(u_t)$ and (t, X_t) is inside $B(t_0,x_0;\varepsilon)$, then $(\psi - \rho)(t, X_t)$ is a supermartingale.

Now, some arguments very similar to the considerations in subsection 3.1 based on the remark above and a (next to) identical result to Lemma 3.5 finish the proof of $\hat{v} \in \mathcal{U}$, resulting in a contradiction.

2. *The terminal condition* $w^+(T,\cdot) \le g(\cdot)$ One has to use arguments identical to the proof of $w^-(T,\cdot) \ge g(\cdot)$ in subsection 3.1 (which was, in turn, apparent), with the only difference of constructing a counterstrategy similar to (3.5) in order to reach a contradiction to the assumption that $w^+(T,x_0) > g(x_0)$ for some x_0.

3.3. Proof of Theorem 2.5. Recall that $w^- \leq v_M^- \leq v^- \leq W^- = V^- \leq W^+ \leq v^+ \leq v_M^+ \leq w^+$. Since w^- is an LSC viscosity supersolution and w^+ is a USC subsolution of the lower Isaacs equation, the comparison result in [SÎ4c] ensures that

$$v_M^- = v^- = W^- = V^- = W^+ = v^+ = v_M^+.$$

As mentioned before, if one does not really care about the genuine nonsymmetric game (2.5) and its value/saddle points, then only the lower Perron construction

$$w^- \leq v_M^- \leq v^- \leq V^-$$

is needed. The viscosity supersolution property of w^- together with the viscosity property of V^- from [SÎ4c] (which is actually reproved above) yields the more important half of the Theorem 2.5, which is

$$v_M^- = v^- = V^-.$$

Now, in order to prove the second part of Theorem 2.5, we note that we have constructed $\mathcal{L} \ni w^n \nearrow W^-$. By continuity and the Dini's criterion, the above convergence is uniform on compacts. This means that for each ε there exists $w \in \mathcal{L}$ (one of the terms of the increasing sequence of asymptotic subsolutions) such that

$$W^- - \varepsilon \leq w \text{ on } C = [0,T] \times \ \{|x| \leq N\}.$$

Let φ be the gauge function of this particular w, and let δ such that $\varphi(\delta) \leq \varepsilon$. According to the proof of Proposition 3.2, if $|\pi| \leq \delta$, there exists $\hat{\alpha} \in \mathcal{A}^M(s,\pi)$ such that

$$w(s,x) \leq \inf_{v \in \mathcal{V}} \mathbb{E}[g(X_T^{s,x;\hat{\alpha},v})] + \varepsilon \times (T-s) \ \forall x.$$

This implies that

$$W^-(s,x) - \varepsilon \times (1 + (T-s)) \leq \inf_{v \in \mathcal{V}(s)} \mathbb{E}[g(X_T^{s,x;\hat{\alpha},v})] \leq v_\pi^-(s,x) \quad \forall |x| \leq N.$$

A very similar argument based on Dini, together with the proof of Proposition 3.7, shows that for $\|\pi\| \leq \delta(\varepsilon)$ (here $\delta(\varepsilon)$ may have to be modified) there exists a counterstrategy $\hat{\gamma} \in \mathcal{C}(\pi, s)$ such that

$$W^+(s,x) + \varepsilon \times (1 + (T-s)) \geq \sup_{u \in \mathcal{U}(s)} \mathbb{E}[g(X_T^{s,x;u,\hat{\gamma}})] \geq v_\pi^+(s,x) \quad \forall |x| \leq N.$$

Not only are the approximations uniform on C, but, for fixed time s, the uniform approximations can be realized over the same simple Markov strategy $\hat{\alpha} \in \mathcal{A}^M(s,\pi)$ or the same Markov counterstrategy $\hat{\gamma} \in \mathcal{C}^M(s,\pi)$ for $|\pi| \leq \delta$. ∎

4. Final considerations.

4.1. One player/control problems. In case the state system only depends on u and not on v (i.e., we have a control problem rather than a game), then, with the obvious observation that

$$v_M^-(s,x) \leq V^-(s,x) \leq \sup_{u \in \mathcal{U}(s)} \mathbb{E}[g(X_T^{s,x;u})],$$

one can use our result about games to conclude that in a control problem (one-player) like in [BS13] (but under our stronger standing assumptions here), the value functions over open-loop controls, elementary feedback strategies, and simple Markov strategies coincide. In addition, the approximation with simple Markov strategies is uniform over the mesh of the grid, uniform on compacts. We remind the reader that, in [BS13], the value function studied was defined over open-loop controls, i.e.,

$$V_{ol}(s,x) \triangleq \sup_{u \in \mathcal{U}(s)} E[g(X_T^{s,x;u})].$$

In this case, as pointed out in Remark 3.1, we can actually use the analytic formulation of asymptotic solutions. Up to some considerations related to the Markov property of SDEs and some other small technical considerations (filtration, local Lipschitz condition), this result is the same as Theorem 2 on p. 148 in the seminal monograph [Kry09]. Again, we just present a novel method to prove such a result.

4.2. Values for symmetric feedback games. In the case of symmetric feedback games, if the Isaacs condition is satisfied, we know from [SÎ4c] that the game has a value. Applying the asymptotic Perron's method over strategies in subsection 3.1 to both players (on both sides) we obtain that, for each ε, there actually exist ε-saddle points within the class of simple Markov strategies, uniformly in bounded x, which means $(\alpha(\varepsilon), \beta(\varepsilon)) \in \mathcal{A}^M(s) \times \mathcal{B}^M(s)$ such that

$$\mathbb{E}[g(X_T^{s,x;u,\beta(\varepsilon)})] - \varepsilon \le \mathbb{E}[g(X_T^{s,x;\alpha(\varepsilon),\beta(\varepsilon)})] \le \mathbb{E}[g(X_T^{s,x;\alpha(\varepsilon),v})] + \varepsilon$$

for $(\forall) \quad (u,v) \in \mathcal{U}(s) \times \mathcal{V}(s), |x| \le N$.

If the Isaacs condition fails, we can still model the game, in a martingale formulation, as in [SÎ4b], and a value over feedback mixed/relaxed strategies does exist. Using again the asymptotic Perron's method, for both players, we can obtain the existence of ε-saddle point within the class of mixed/relaxed strategies of simple Markov type, uniformly in bounded x. A mixed strategy μ of simple Markov type (for the player u) is defined by a time grid π and some functions $\xi_k : \mathbb{R}^d \to \mathcal{P}(U), k = 1, \ldots, n$, measurable, such that

$$\mu(t, y(\cdot)) = \sum_{k=1}^{n} 1_{\{t_{k-1} < t \le t_k\}} \xi_k(y(t_{k-1}) \in \mathcal{P}(U).$$

In other words, player u decides at time t_{k-1} based only on the position at that time what distribution he/she will be sampling continuously from until t_k. Obviously, one can define similarly mixed strategies of Markov type for the v-player. In order to do the analysis and obtain the approximate mixed Markov saddle strategies, one would have to go inside the short proofs in [SÎ4b] and apply the asymptotic Perron's method for the auxiliary (and strongly defined) games in the proofs there. In other words, the above paragraph for games over pure strategies satisfying the Isaacs condition applies to the auxiliary game in [SÎ4b], leading to ε-saddle points in the class of mixed strategies of Markov type for the original game.

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