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EXIT TIME PROBLEMS IN OPTIMAL CONTROL AND VANISHING VISCOSITY METHOD*

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Abstract. The authors study the connections between deterministic exit time control problems and possibly discontinuous viscosity solutions of a first-order Hamilton-Jacobi (HJ) equation up to the boundary. This equation admits a maximum and a minimum solution that are the value functions associated to stopping time problems on the boundary. When these solutions are equal, they can be obtained through the vanishing viscosity method. Finally, when the HJ equation has a continuous solution, it is proved to be the value function for the first exit time of the domain. It is also the vanishing viscosity limit arising, in particular, in some large deviations problems.

Résumé. Les auteurs étudient les liens entre les problèmes de contrôle déterministe avec temps de sortie et les solutions de viscosité, généralement discontinues, d'une équation d'Hamilton-Jacobi du premier ordre posée jusqu'au bord. Cette équation admet une solution maximale et une minimale qui sont des fonctions valeur de problèmes de temps d'arrêt sur le bord. Quand ces solutions sont égales, elles peuvent être obtenues grâce à la méthode de viscosité évanescente. Enfin, quand l'équation de Hamilton-Jacobi a une solution continue, on découvre que celá est la fonction valeur pour le premier temps de sortie de l'ouvert. C'est aussi la limite par viscosité évanescente qui apparaît, en particulier, dans certains problèmes de grandes déviations.

Key words. deterministic optimal control, exit time, Hamilton-Jacobi equations, viscosity solutions, vanishing viscosity, large deviations

Mots-clé. contrôle optimal déterministe, temps de sortie, équations de Hamilton-Jacobi, solutions de viscosité, viscosité évanescente, grandes déviations

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Introduction. In this work, we are interested in a systematic understanding of the relations between Hamilton-Jacobi (HJ) equations and value functions that can be considered in deterministic exit time control problems, even (and mainly) when these functions are discontinuous. As a consequence of this study, we obtain that no uniqueness holds for discontinuous solutions to the HJ equation under consideration, since the control problem leads naturally to very different value functions solving it. Thus the question is, what happens in the vanishing viscosity method? Does it converge (as it can be thought intuitively) to the value function for the first exit time of the domain? We answer this question only for two cases: when the value functions are not too different, and when the HJ equation admits a continuous solution. Then we obtain a uniqueness result which allows us to conclude, even for Hamiltonians more general than the one appearing in optimal control theory (nonconvex Hamiltonians). As a byproduct of this result, we can simplify the proof (and weaken the assumptions) of convergence theorems of Wentzell-Freidlin type [29] following the new approach, initiated in Evans and Ishii [10], to the method of Fleming [11], [12].

This work is based on the notion of viscosity solutions introduced by Crandall and Lions [7] (see also Crandall, Evans, and Lions [8] or Lions [22]) and extended to different forms of boundary conditions by Lions [23], Perthame and Sanders [25], Soner [27], and Capuzzo-Dolcetta and Lions [6]. This notion has also been extended to discontinuous solutions by Ishii [16], [17] and Barles and Perthame [4].

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This paper is organized as follows. Section I is devoted to the study of the exit time problem and its relation to an HJ equation. In § II we prove a new comparison result between a continuous and a discontinuous solution and we apply it to some vanishing viscosity problem. Finally, the Appendix is devoted to a technical result on the two-sided obstacle problem. Sections I and II are nearly independent, although § I shows the pathology of discontinuous solutions.

I. Exit time problems in optimal control. In this section we consider a system, the state of which is given by the solution of

$$dy_x(t)+b(y_x(t), v(t))\,dt=0, \qquad y_x(0)=x\in\bar{\Omega}, \tag{1}$$

or, in the case of *relaxed controls*, by

$$d\hat{y}_x(t)+\left[\int_V b(\hat{y}_x(t), v(t))\,d\mu_t\right]dt=0, \qquad \hat{y}_x(0)=x\in\bar{\Omega}, \tag{1'}$$

and the basic cost functions are

$$J(x, v, \theta)=\int_0^\theta f(y_x(s), v(s))\,e^{-\lambda s}\,ds+\varphi(y_x(\theta))\,e^{-\lambda\theta}$$

and

$$\hat{J}(x, \mu, \theta)=\int_0^\theta\int_V f(\hat{y}_x(s), v)\,e^{-\lambda s}\,d\mu_s\,ds+\varphi(\hat{y}_x(\theta))\,e^{-\lambda\theta}.$$

Ω is a bounded domain in $\mathbb{R}^N$, f, b are given functions, $\lambda>0$, θ is a nonnegative number, and φ is a given function on the complementary set of Ω. $v(\cdot)\in L^\infty(\mathbb{R}^+, V)$, $\mu_s\in L^\infty(\mathbb{R}^+, P(V))$ are, respectively, the control and the relaxed control; V is a compact metric space and $P(V)$ is the set of probability measures on V. We assume that

$$\begin{aligned}&\text{For } \phi=f,\quad b_i\ (1\le i\le N),\quad \phi\in C(\mathbb{R}^N\times V),\quad \phi(\cdot, v)\in W^{1,\infty}(\mathbb{R}^N),\\ &\|\phi(\cdot, v)\|_{W^{1,\infty}}\leqq C\quad \text{(independent of } v)\end{aligned} \tag{2}$$

(so that (1) and (1′) have a unique solution).

$$\varphi\in \mathrm{BUC}\,(\Omega^C) \tag{3}$$

(BUC denotes the set of bounded uniformly continuous functions).

Since we are interested in the exit time problem of Ω, let us denote

$$\tau=\inf\{t\geqq 0, y_x(t)\notin\Omega\}, \tag{4}$$

$$\bar{\tau}=\inf\{t\geqq 0, y_x(t)\notin\bar{\Omega}\} \tag{5}$$

(and $\hat{\tau}$, $\hat{\bar{\tau}}$ are defined by (4) or (5) using $\hat{y}_x$). In particular we will use the following three value functions:

$$u^+(x)=\inf_{v(\cdot)\in L^\infty(\mathbb{R}^+;V)}[\operatorname{Sup}\{J(x, v, \theta);\ \tau\leqq\theta\leqq\bar{\tau},\ y_x(\theta)\in\partial\Omega\}], \qquad x\in\bar{\Omega}, \tag{6}$$

$$u_-(x)=\operatorname{Inf}\{\hat{J}(x, \mu, \theta);\ \hat{\tau}\leqq\theta\leqq\hat{\bar{\tau}},\ \hat{y}_x(\theta)\in\partial\Omega,\ \mu\in L^\infty(\mathbb{R}^+, P(V))\}, \qquad x\in\bar{\Omega}, \tag{7}$$

$$\begin{aligned}&u(x)=\operatorname{Inf}\{J(x, v, \tau);\ v(\cdot)\in L^\infty(\mathbb{R}^+, V)\}\quad \text{for } x\in\Omega,\\ &u(x)=\liminf_{y\to x,\ y\in\Omega} u(y)\quad \text{for } x\in\partial\Omega.\end{aligned} \tag{8}$$

The reason we introduce a different value for u on $\partial\Omega$ is that u_- is lower semicontinuous (l.s.c.) on $\bar{\Omega}$ and u^+ is upper semicontinuous (u.s.c.) on $\bar{\Omega}$, but u has no such property and this definition will simplify the notation. Let us mention that u_- has been studied by Quadrat [26] and that this type of value function has already been studied by Ishii [16] and the authors [4]. These works show the connections between the above control problems and the following HJ equation:

$$\begin{aligned} &H(x, u, Du)=0 \quad \text{in } \Omega,\\ &\operatorname{Min}\{H(x, u, Du);\ u-\varphi\} \leqq 0 \quad \text{on } \partial\Omega, \\ &\operatorname{Max}\{H(x, u, Du);\ u-\varphi\} \geqq 0 \quad \text{on } \partial\Omega,\end{aligned} \tag{9}$$

where

$$H(x, t, p)=\sup_{v \in V}\{b(x, v) \cdot p+\lambda t-f(x, v)\}. \tag{10}$$

Of course, these equations have to be understood in the viscosity sense of the following definition.

DEFINITION. A u.s.c. function u (on $\bar{\Omega}$) is called a viscosity subsolution (or simply a subsolution) of (9) if for any $\phi \in C^1(\bar{\Omega})$ and any $x_0 \in \bar{\Omega}$ such that $\operatorname{Max}_{\bar{\Omega}}(u-\phi)=(u-\phi)(x_0)$,

$$\begin{aligned} &H(x_0, u(x_0), D\phi(x_0)) \leqq 0 \quad \text{if } x_0 \in \Omega,\\ &\operatorname{Min}(H(x_0, u(x_0), D\phi(x_0)), u(x_0)-\varphi(x_0)) \leqq 0 \quad \text{if } x_0 \in \partial\Omega.\end{aligned}$$

We refer to the references in the Introduction for the motivation of this definition and for variants and properties of viscosity solutions. Let us note only that a supersolution may be defined in a similar fashion for l.s.c. functions. A function u is said to be a solution if u^* is a subsolution and u_* is a supersolution, where

$$\begin{aligned} u^*(x) &= \limsup_{y \in \bar{\Omega},\, y \to x} u(y),\\ u_*(x) &= \liminf_{y \in \bar{\Omega},\, y \to x} u(y).\end{aligned} \tag{11}$$

Our main results are the following. First, we prove that u_-, u_+, u are viscosity solutions of (9) and an example shows that they may be very different. Thus, no general uniqueness result holds for solutions that are discontinuous on the boundary (see however [16], [4]). Since u^+ (respectively, u_-) is the maximum (respectively, minimum) solution of (9), if we assume

$$(u^+)_* = u_- \quad \text{in } \Omega, \tag{12}$$

then the solution is, in some sense, unique and it can be recovered by vanishing viscosity. Let u_ε satisfy

$$-\varepsilon \Delta u_\varepsilon + H(x, u_\varepsilon, Du_\varepsilon)=0 \quad \text{in } \Omega, \qquad u_\varepsilon|_{\partial\Omega}=\varphi; \tag{13}$$

then

$$u_-(x)=\liminf_{y \to x,\, \varepsilon \to 0} u_\varepsilon(y) \quad \text{in } \Omega.$$

Equation (12) holds, in particular, if $u_-=\varphi$ on $\partial\Omega$. This is a compatibility condition close to that of Lions [22]. Finally, we notice that u has a remarkable regularity property:

$$[(u_*)^*]_* = u_* \quad \text{in } \bar{\Omega}. \tag{14}$$

Our conclusion is that such a property could be a criterion for uniqueness, as can be seen from the uniqueness theorem of § II, but that a general result is still lacking.

The section is divided into three smaller sections. In the first we study u_+ and u_-. The second is devoted to the properties of u, and the last gives examples which show that our results are nearly optimal and that u and u_- can be very different.

I.1. Properties of u_+ and u_-. In this section, we characterize u_+ and u_- as the maximal and minimal solutions of (9). We also prove the "regularity" property $(u^+)_* = u_*$ and that the vanishing viscosity method converges to u in the case when $u_* = u_-$. To do so, we need the following assumptions:

$$\Omega \text{ is a bounded open set satisfying } \mathring{\bar{\Omega}} = \Omega. \tag{15}$$

We assume that φ satisfies (3) and we define

$$\psi_1 = C \quad \text{in } \Omega, \qquad \psi_1 = \varphi \quad \text{in } \Omega^c,$$

$$\psi_2 = -C \quad \text{in } \Omega, \qquad \psi_2 = \varphi \quad \text{in } \Omega^c,$$

where C is large enough ($C \geqq \sup |f|/\lambda + \sup |\varphi|$).

We have the following theorem.

THEOREM I.1. *Under assumptions* (2), (3), (15), *let v be defined in $\bar{\Omega}$, and let us denote by $\hat{v}$ the extension of v by φ outside $\bar{\Omega}$. Then v is a viscosity subsolution* (*respectively, supersolution*) *of* (9) *if and only if $\hat{v}$ is a viscosity subsolution* (*respectively, supersolution*) *of*

$$\operatorname{Max}(u - \psi_1, \operatorname{Min}(u - \psi_2, H(x, u, Du))) = 0 \quad \text{in } \mathbb{R}^N. \tag{16}$$

The definition of solutions of (16) and its main properties are given in the Appendix. In particular, (16) admits a maximal subsolution $\bar{u}$ and a minimal supersolution $\underline{u}$ (explicit formulas are given for $\underline{u}$ and $\bar{u}$) and $\bar{u}$ and $\underline{u}$ are solutions of (16). Thus, we have the following corollary.

COROLLARY I.2. *$u^+ = \bar{u}|_{\bar{\Omega}}$ and $u_- = \underline{u}|_{\bar{\Omega}}$ are, respectively, the maximal subsolution and the minimal supersolution of* (9), *and u^+ and u_- are solutions of* (9). *u^+ is* u.s.c. *and u_- is* l.s.c. *in $\bar{\Omega}$.*

The following theorem shows that u^+ and u are not very different.

THEOREM I.3. *Under assumptions* (2), (3), (15), *we have*

$$(u^+)_* = u_* \quad \text{in } \bar{\Omega}.$$

COROLLARY I.4. *If $u_- = u_*$ in Ω, then there exists a unique* l.s.c. (*in $\bar{\Omega}$*) *viscosity solution of* (9) *satisfying* (14). *This is the case, for example, if $u_- = \varphi$ on $\partial\Omega$.*

COROLLARY I.5. *Assume that* (13) *has a solution $u^\varepsilon \in C^2(\Omega) \cap C(\bar{\Omega})$ for ε small enough. Then*

$$u_-(x) \leqq \liminf_{\varepsilon \to 0,\, y \to x} u^\varepsilon(y) \leqq \limsup_{\varepsilon \to 0,\, y \to x} u^\varepsilon(y) \leqq u^+(x),$$

and the function defined above by the lim inf (*respectively, the* lim sup) *is a supersolution* (*respectively, subsolution*) *of* (9) *and if $u_- = u_*$ in Ω*

$$u_-(x) = \liminf_{\varepsilon \to 0,\, y \to x} u^\varepsilon(y) \quad \text{in } \Omega.$$

Most of these results are mere adaptations of the method of [4] and Proposition A.1 in the Appendix and we will skip the proofs. Theorem I.1 is an easy consequence of the definition of viscosity solutions. Corollary I.2 can be proved without any difficulty using the Appendix. The only new point is Theorem I.3, which we prove below. Then, Corollary I.4 and I.5 are clear. Let us mention that Corollary I.5 is a consequence of the stability results of [16] and [4], which show that $\limsup_{\varepsilon \to 0, y \to x} u^\varepsilon(y)$ and

$\liminf_{\varepsilon\to 0, y\to x} u^\varepsilon(y)$ are, respectively, viscosity sub- and supersolutions of (9), together with Corollary I.2.

Proof of Theorem I.3. Since $u^+ \geqq u$ in Ω, it is clear that $(u^+)_* \geqq u_*$ and it remains to prove the other inequality. We will need the following lemma.

LEMMA I.6. *Let x be a point of Ω, $v(\cdot)$ a control, τ the exit time from Ω of the trajectory associated to x and $v(\cdot)$. There exists a sequence $x_n \in \Omega \to x$ such that*

$$\liminf_n u^+(x_n) \leqq J(x, v, \tau).$$

First, we prove Theorem I.3 by using Lemma I.6.

Let $x \in \Omega$. Then there exists a sequence $x_n \to x$ such that

$$\lim_n u(x_n) = u_*(x).$$

Now, we take $v_n(\cdot)$ such that

$$u(x_n) + \frac{1}{n} \geqq J(x_n, v_n, \tau_n),$$

τ_n being the exit time from Ω of the trajectory y_{x_n} associated to $v_n(\cdot)$. By Lemma I.6, there exists $x_n^p \to x_n$ such that

$$J(x_n, v_n, \tau_n) \geqq u^+(x_n^p) - \frac{1}{p}.$$

Therefore, by a diagonal procedure

$$u_*(x) = \lim_n u(x_n) \geqq \liminf_n \left(u^+(x_n^n) - \frac{2}{n} \right) \geqq (u^+)_*(x)$$

and the result is proved. □

Next, we prove Lemma I.6.

Proof of Lemma I.6. We treat only the case when $\tau < \infty$. If $\tau = +\infty$, then $u^+(x) \leqq J(x, v(\cdot), \tau)$ and we are done. We are going to build a sequence x_n converging to x such that the trajectory y_{x_n} associated to v and x_n satisfies

$$\tau - 1/n \leqq \tau_n \leqq \bar{\tau}_n \leqq \tau,$$

where τ_n is the first exit time of y_{x_n} from Ω and $\bar{\tau}_n$ from $\bar{\Omega}$. Since the map $Y: z \to y_z(\tau)$ is an homeomorphism from a neighborhood of x onto some neighborhood of $y_x(\tau)$, and since Ω satisfies (15) for ε small enough, the set $A = Y^{-1}(B_\varepsilon(y_x(\tau)) \cap \bar{\Omega}^C)$—where $B_\varepsilon(y_x(\tau))$ is the ball of center $y_x(\tau)$ and of radius ε—is not empty and is open, and $x \in \bar{A}$. Let us remark that for $z \in A$, the exit time of y_z from $\bar{\Omega}$ is less than τ. Now, since $\operatorname{Inf}\{d(y_x(s), \partial\Omega), 0 \leqq s \leqq \tau - 1/n\} > 0$ and since $z \to y_z(s)$ is uniformly Lipschitz in z for $0 \leqq s \leqq \tau$, there exists $\eta > 0$ such that for $|z - x| \leqq \eta$, we have

(i) $y_z(s) \in \Omega$ for $0 \leqq s \leqq \tau - 1/n$,

(ii) $|y_z(s) - y_x(s)| \leqq 1/n$.

Therefore, if we take $x_n \in B_\eta(x) \cap A$, we have

(iii) $\tau - 1/n \leqq \tau_n \leqq \bar{\tau} \leqq \tau$

and

$$u^+(x_n) \leqq \operatorname{Sup}\{J(x_n, v, \theta), \theta \in [\tau_n, \bar{\tau}_n], y_{x_n}(\theta) \in \partial\Omega\}.$$

But using the Lipschitz properties for b and f and the continuity of φ we easily deduce

$$u^+(x_n) \leqq J(x, v, \tau) + \alpha(n),$$

where $\alpha(n) \to 0$ when $n \to \infty$, and the lemma is proved. □

I.2. Some properties of u. In this section, we study some properties of the function u defined by (8). Let us begin by commenting on the value of u on $\partial\Omega$. If we define $\tilde{u}$ by

$$\tilde{u}(x) = u(x), \qquad x \in \Omega,$$
$$\tilde{u}(x) = \varphi(x), \qquad x \in \partial\Omega,$$

then it is easy to check that $\tilde{u}^*$ (respectively, $\tilde{u}_*$) is a viscosity subsolution (respectively, supersolution) of (9) if and only if u^* (respectively, u_*) is. We are going to prove that u and u_* are solutions of (9) and some regularity property of u. Finally, let us note that Ishii [16] has already proved that $\tilde{u}$ is a viscosity solution of (9).

THEOREM I.7. *Assume* (2), (3), (15); *then u and u_* are viscosity solutions of* (9). *Moreover, u_* satisfies*

$$[(u_*)^*]_* = u_* \quad \text{in } \bar{\Omega}. \tag{14}$$

Remark I.8. It is known that the functions satisfying (14) are continuous almost everywhere in the Baire sense; this justifies the term regularity. Let us mention that u^+ satisfies (14) since it is u.s.c. The new point here is that u_* is a viscosity solution with our definition of u on $\partial\Omega$. In general, this would be wrong for u_- as we will see in the next section.

Proof of Theorem I.7. Let us first show the regularity property. Since $u \leqq u^+$ in Ω and u^+ is u.s.c., we have

$$u_* \leqq (u_*)^* \leqq u^+ \quad \text{in } \bar{\Omega};$$

therefore, since u_* is l.s.c.

$$u_* \leqq [(u_*)^*]_* \leqq (u^+)_* \quad \text{in } \bar{\Omega}.$$

But, using Theorem I.3, we have

$$(u^+)_* = u_* \quad \text{in } \bar{\Omega}.$$

Thus (14) is proved. To prove that u is a solution of (9), we give a more direct proof than that of Ishii [16]. We prove only that u is a supersolution on $\partial\Omega$. The other properties may be obtained by the same method. Let x be a point of $\partial\Omega$ and x_n a sequence of points of Ω such that $x_n \to x$ and $u(x_n) \to u_*(x)$. Using the Dynamic Programming Principle (cf. Fleming and Rishel [13], Lions [22], Ishii [16]), we know that

$$u(x_n) = \operatorname*{Inf}_{v(\cdot)} \left(\int_0^{T \wedge \tau} f(y_{x_n}(t), v(t))\, e^{-\lambda t}\, dt + \varphi(y_{x_n}(\tau))\, e^{-\lambda \tau} \cdot 1_{\{\tau \leqq T\}} \right.$$
$$\left. + u(y_{x_n}(T))\, e^{-\lambda T} \cdot 1_{\{T < \tau\}} \right).$$

If $u_*(x) \geqq \varphi(x)$, we have nothing to prove. In the other case, we choose T such that

$$T \|f\|_\infty + \rho_\varphi(\|b\|_\infty T) \leqq [\varphi(x) - u_*(x)]/2, \tag{17}$$

where ρ_φ is the modulus of continuity of φ. Now, we choose $v_n(\cdot)$ such that

$$u(x_n) + 1/n \geqq \int_0^{T \wedge \tau_n} f(y_{x_n}(t), v_n(t))\, e^{-\lambda t}\, dt + \varphi(y_{x_n}(\tau_n))\, e^{-\lambda \tau_n} 1_{\{\tau_n \leqq T\}}$$
$$+ u(y_{x_n}(T))\, e^{-\lambda T} 1_{\{T < \tau_n\}}.$$

It is easy to check that for n large enough, $\tau_n > T$, and using the compactness of relaxed controls (cf. [5], [28]) we get

$$u_*(x) \geqq \int_0^T \int_V f(\hat{y}_x(t), v)\, e^{-\lambda t}\, d\mu_t\, dt + u_*(\hat{y}_x(T))\, e^{-\lambda T},$$

where $\hat{y}_x$ is defined by (1′). We have used the fact that

$$y_{x_n}(T) \to \hat{y}_x(T)$$

and $\liminf_n u(y_{x_n}(T)) \geqq u_*(\hat{y}_x(T))$ by definition. So, we finally obtain that, for T small enough,

$$u_*(x) \geqq \operatorname*{Inf}_{\mu} \left(\int_0^T \int_V f(\hat{y}_x(t), v)\, e^{-\lambda t}\, d\mu_t\, dt + u_*(\hat{y}_x(T))\, e^{-\lambda T} \right).$$

This is a superoptimality principle of dynamic programming (cf. Lions and Souganidis [24]) and when it is used, classical arguments lead to the result. Therefore, we skip the end of the proof. □

We can use the same kind of ideas to prove that $(u_*)^*$ is a subsolution of (9). Again we consider only the case of a point $x \in \partial\Omega$ and we choose a sequence $x_n \in \Omega$ such that $x_n \to x$ and $u_*(x_n) \to (u_*)^*(x)$. For any control $v(\cdot)$ and $y \in \Omega$ we have, for some modulus of continuity ρ_n,

$$u_*(x_n) - \rho_n(|x_n - z|) \leqq u(z) \leqq \int_0^{T\wedge\tau} f(y_z(s), v(s))\, e^{-\lambda s}\, ds$$
$$+ \varphi(y_z(\tau))\, e^{-\lambda\tau} \cdot 1_{\{\tau \leqq T\}} + u(y_z(T))\, e^{-\lambda T} \cdot 1_{\{T<\tau\}}.$$

As before, we must consider the case when $(u_*)^*(x) > \varphi(x)$ and we choose T such that

$$T\|f\|_\infty + \rho_\varphi(\|b\|_\infty T) \leqq \frac{(u_*)^*(x) - \varphi(x)}{2}. \tag{17′}$$

Then, for n large enough and $|z - x_n|$ small enough (depending on n), we easily obtain $T < \tau(y)$, and thus

$$u_*(x_n) - \rho_n(|x_n - z|) \leqq \int_0^T f(y_z(s), v(s))\, e^{-\lambda s}\, ds + u(y_z(T))\, e^{-\lambda T}.$$

Since the range of a small neighbourhood of x_n by the application $z \to y_z(T)$ is a neighbourhood of $y_{x_n}(T)$, we may choose a sequence z_p such that $z_p \to x_n$ as $p \to \infty$ and $u(y_{z_p}(T)) \to u_*(y_{x_n}(T))$. This gives

$$u_*(x_n) \leqq \int_0^T f(y_{x_n}(s), v(s))\, e^{-\lambda s}\, ds + u_*(y_{x_n}(T))\, e^{-\lambda T},$$

and, letting n go to infinity, we obtain

$$(u_*)^*(x) \leqq \int_0^T f(y_x(s), v(s))\, e^{-\lambda s}\, ds + (u_*)^*(y_x(T))\, e^{-\lambda T}.$$

And since this holds for any control $v(\cdot)$ we have proved a suboptimality of dynamic programming. This is enough to conclude that $(u_*)^*$ is a subsolution by classical arguments. □

I.3. An example. The aim of this section is to describe a situation where u_- is very different from u. For (9), it is the only real nonuniqueness feature, since $(u^+)_* = u_*$

in $\bar{\Omega}$. We shall make some comments concerning the connections between the boundary values, the boundary conditions in (9), and the nonuniqueness feature for (9). Let us describe our example. With our notation, we take the following:

$\Omega=\{(x, y)\in[-1, 1]^2/x<0 \text{ or } y<0\}$,

$f\equiv 0$, $\lambda=0$ (but is not relevant for our example),

$b(x, v)=-v$ where $v\in V=\{(v_1, v_2)\in\mathbb{R}^2/v_1^2+v_2^2\leqq 1,\ v_1\geqq 0,\ v_2\leqq 0\}$.

Finally, we take $\varphi\equiv 0$ on $\partial\Omega$, except on $[0, 1]\times\{0\}$, on which we define φ by

$$\varphi(x, y)=-4x(1-x).$$

It is now easy to compute u^+, u, and u_-, and we obtain

$$u^+\equiv u\equiv 0 \quad \text{in } \bar{\Omega}$$

and

$$u_-(x, y)=\begin{cases}0 & \text{if } y<0,\\ -1 & \text{if } x\leqq\frac{1}{2},\quad y\geqq 0,\\ \varphi & \text{if } x>\frac{1}{2},\quad y=0.\end{cases}$$

Of course, u and u_- are viscosity solutions of (3) in $\bar{\Omega}$. However, we can remark that the values of u_- on $[0, 1]\times\{0\}$ are very different from the behavior of u_- on $[0, 1]\times\,]-\varepsilon, 0[$. This remark leads to two comments.

(i) It does not seem possible to use the idea of Soner [27] (and used for this type of problem by Ishii [16]) to prove uniqueness results for (9) with functions such as u_- since the behavior at the boundary is completely different from the behavior of the interior points; this will motivate the uniqueness result of § II.

(ii) If we define the l.s.c. and u.s.c. envelopes of u_- in the same way as for u in (8), we must consider the function ω defined by

$$\omega(x, y)=\begin{cases}-1 & \text{if } x\leqq 0,\quad y\geqq 0,\\ 0 & \text{otherwise.}\end{cases}$$

$\omega(x)=\liminf_{y\in\Omega,\, y\to x} u_-(y)$, and it is easy to check that ω *is not a viscosity supersolution* at the point $(0, 0)$. So, for u_-, the boundary values play an essential role for the viscosity solution property. This is a striking difference from u. It seems that u_- is a singular solution and that the "good" solutions are those which have the regularity property (14).

Remark I.9. In this example, u_- has the regularity property (14) in Ω. But this is not the case in general. (In the above example replace b by $\tilde{b}$ defined by $\tilde{b}(x, v)\equiv(1, 0)$.)

Remark I.10. In this example, u_* is the maximum continuous viscosity subsolution of (9). It is easy to construct examples in which the supremum of the continuous subsolution is u_-, and u_* is different from u_- at some points of Ω. Finally, let us point out that Theorem II.1 below may be applied to this example and provides a uniqueness theorem for u.

II. Uniqueness and vanishing viscosity method for HJ equations. In this section, we prove an extension of the uniqueness result of Ishii [16] and we apply it to simplify the proof and weaken the assumptions for asymptotic theorems of large deviations type.

II.1. A uniqueness result. We prove the uniqueness result for a general HJ equation for which we need the following assumptions:

(18) $H(x, t, p)$ is uniformly continuous in p for $x\in\bar{\Omega}$, $t\in\mathbb{R}$;

(19) There exists a continuous nondecreasing function $m\colon (0, \infty) \to [0, \infty)$ such that $m(0) = 0$ and $|H(x, t, p) - H(y, t, p)| \leqq m(|x-y|(1+|p|))$;

(20) $\exists \gamma \geqq 0, \forall x \in \bar{\Omega}, p \in \mathbb{R}^N, s \leqq t, H(x, t, p) - H(x, s, p) \geqq \gamma(t-s)$.

Moreover, we need an assumption for the bounded set Ω, which is a classical assumption introduced by Soner [27]:

(21) There exists a continuous function η defined on a neighborhood of $\partial\Omega$ with values in $\mathbb{R}^N$ and a constant $b > 0$ such that $B(x + t\eta(x), bt) \subset \Omega$ for $x \in \bar{\Omega}$, $0 < t < b$.

THEOREM II.1. *Assume* (18)-(21), *and either* $\gamma > 0$ *in* (20) *and* $a = 0$, *or* $\gamma = 0$ *and* $a > 0$. *Let* v, *an l.s.c.* (*in* $\bar{\Omega}$) *function, be a supersolution of* (9). *Let* $w \in C(\bar{\Omega})$ *satisfy* (*in the viscosity sense*)

$$
(22) \qquad \begin{aligned} &H(x, w, Dw) \leqq -a \quad \text{in } \Omega, \\ &\operatorname{Min}(w - \varphi, H(x, w, Dw) + a) \leqq 0 \quad \text{on } \partial\Omega. \end{aligned}
$$

Finally, assume that

$$
(23) \qquad \forall x \in \partial\Omega \quad \text{if } w(x) > \varphi(x) \quad \text{then } \liminf_{y \to x,\, y \in \Omega} v(y) = v(x).
$$

Then $w \leqq v$ *in* $\bar{\Omega}$. □

Remark II.2. Of course, the same result holds for a continuous supersolution and a u.s.c. subsolution. It is also possible to weaken slightly the assumption of continuity of the subsolution by assuming that it is continuous only at the points of $\partial\Omega$. But when we deal with solutions this is of no use, since a solution that is continuous at the points of $\partial\Omega$ is continuous on $\bar{\Omega}$. Indeed, let w be a solution that is continuous at the points of $\partial\Omega$, choose two continuous functions w_1, w_2 such that

$$
\begin{aligned} &w_1 \leqq w \leqq w_2 \quad \text{in } \bar{\Omega}, \\ &w_1 = w_2 = w \quad \text{on } \partial\Omega, \end{aligned}
$$

and extend w, w_1, w_2 in $\mathbb{R}^N$ such that

$$
w, w_1, w_2 \in \mathrm{BUC}(\mathbb{R}^N), \qquad w = w_1 = w_2 \quad \text{in } \Omega^C
$$

(BUC denotes the space of uniformly bounded continuous functions). It is easy to check that w is a solution of

$$
(24) \qquad u + \inf\{-w_1; \sup(H(x, u, Du) - u, -w_2)\} = 0 \quad \text{in } \mathbb{R}^N,
$$

and the unique solution of (24) belongs to BUC $(\mathbb{R}^N)$ (cf. [9], [2]). Therefore, w is continuous in $\bar{\Omega}$.

Thus, the main difference between Theorem II.1 and Theorem 2.1 of [16] is that w is not assumed to satisfy $w \leqq \varphi$ on $\partial\Omega$.

Proof of Theorem II.1. The proof is based on the idea of Soner [27] and is almost the same as that of Ishii [16]. So, we just indicate the adaptation required for this proof and we treat the case $\gamma > 0$, $a = 0$. We are interested in $M = \operatorname{Max}_{x \in \bar{\Omega}}(w(x) - v(x))$, which is achieved, say, at $\bar{x}$. The only new argument is used when $\bar{x} \in \partial\Omega$, $w(\bar{x}) > \varphi(\bar{x})$, and $v(\bar{x}) \geqq \varphi(\bar{x})$. It is contained in the proof of the following lemma.

LEMMA II.3. *Let $\bar{x}_n \in \Omega$ satisfy (as in (23)), $\bar{x}_n \to \bar{x}$, $v(x_n) \to v(\bar{x})$ as $n \to \infty$ and let ε_n and z_n be defined by*

$$\varepsilon_n = \operatorname{Inf}\{t \geqq 0, \bar{x}_n - t\eta(\bar{x}) \notin \Omega\},$$
$$z_n = \bar{x}_n - \varepsilon_n \eta(\bar{x}).$$

Then $\varepsilon_n \to 0$ and if ψ_n is defined by

$$\psi_n(x, y) = w(x) - v(y) - \left|\frac{y-x}{\varepsilon_n} - \eta(\bar{x})\right|^2 - |x - \bar{x}|^2;$$

then, denoting (x_n, y_n) a maximum point of ψ_n, we have

$$|x_n - \bar{x}| \to 0, \quad \left|\frac{y_n - x_n}{\varepsilon_n} - \eta(\bar{x})\right| \to 0, \quad v(y_n) \to v(\bar{x}). \qquad \square$$

Proof of Lemma II.3. Using (21), it is not difficult to check that $\varepsilon_n \to 0$. Therefore, for n large enough, $\varepsilon_n \leqq b$. First, it is easy to check that

$$|y_n - x_n| \leqq C \cdot \varepsilon_n.$$

And so

$$\psi_n(x_n, y_n) \leqq (w(x_n) - w(y_n)) + (w(y_n) - v(y_n))$$

which gives

$$\psi_n(x_n, y_n) \leqq \rho_w(C \cdot \varepsilon_n) + M, \tag{25}$$

where ρ_w is the modulus of continuity of w. Moreover,

$$\psi_n(x_n, y_n) \geqq \psi(z_n, \bar{x}_n)$$

since (x_n, y_n) is a maximum point of ψ_n. We obtain

$$\psi_n(x_n, y_n) \geqq w(z_n) - v(\bar{x}_n) - |z_n - \bar{x}|^2.$$

But $z_n \to \bar{x}$ and $v(\bar{x}_n) \to v(\bar{x})$, so there exists $\alpha(n) \to 0$ such that

$$\psi_n(x_n, y_n) \geqq M - \alpha(n). \tag{26}$$

From (25) and (26), we deduce that $\psi_n(x_n, y_n) \to M$. Moreover, since Ω is bounded, we can consider subsequences—still denoted by x_n, y_n—such that $x_n, y_n \to z \in \bar{\Omega}$. Hence

$$M \leqq \limsup_n \psi_n(x_n, y_n) \leqq w(z) - \liminf_n v(y_n) - \liminf_n \left|\frac{x_n - y_n}{\varepsilon_n} - \eta(\bar{x})\right|^2 - |z - \bar{x}|^2.$$

The right-hand side is estimated by

$$w(z) - v(z) \leqq M.$$

From the above we deduce the following:

(i) $z = \bar{x}$,

(ii) $\liminf_n v(y_n) = v(\bar{x})$,

(iii) $\liminf_n \left|\frac{x_n - y_n}{\varepsilon_n} - \eta(\bar{x})\right|^2 = 0.$

Since (i)-(iii) are true for every converging subsequence, the proof is complete.

Using this lemma, we easily conclude the proof, as in Ishii [16]. $\square$

II.2. A general case of uniqueness. In this section we consider the case when H satisfies the assumption

(27) $\forall R>0$, $H(x,t,p)\to+\infty$ as $|p|\to+\infty$, uniformly for $x\in\bar\Omega$, $|t|\leqq R$.

Our purpose is to show that Theorem II.1 applies to getting a general uniqueness theorem. Indeed, we will prove that any subsolution is continuous in Ω. We refer to Ishii [16] for other continuity criteria.

THEOREM II.4. *Under assumptions* (18)-(21) *and* (27), *any bounded u.s.c.* (*in* $\bar\Omega$) *subsolution* v *of* (9) *is uniformly Lipschitz continuous in* Ω *and satisfies* $v\leqq\varphi$ *on* $\partial\Omega$.

Proof of Theorem II.4. The proof of this theorem uses classical arguments and we only sketch it. First, we prove that $v\leqq\varphi$ on $\partial\Omega$. Take a sequence of smooth functions φ_n on $\mathbb{R}^N$, $\varphi+2/n\geqq\varphi_n\geqq\varphi+1/n$ on $\partial\Omega$, and define for $\lambda>0$,

$$w(x)=\lambda d(x,\partial\Omega)+\varphi_n(x).$$

On $E_\alpha=\{x/d(x,\partial\Omega)\geqq\alpha\}$, $w(x)$ is continuous and satisfies, for a proper choice of α, λ,

$$\begin{aligned}&H(x,u,Du)\geqq 1/n \quad \text{in } E_\alpha,\\ &u\geqq\varphi+1/n \quad \text{on } \partial\Omega,\\ &u\geqq v+1/n \quad \text{on } \partial E_\alpha\backslash\partial\Omega.\end{aligned}\tag{28}$$

Indeed, for any $\phi\in C^1(\bar\Omega)$ and any minimum point $x_0\in\Omega$ of $w-\phi$, we have, for $y_0\in\partial\Omega$, $d(x_0,\partial\Omega)=|x_0-u_0|$,

$$\begin{aligned}\phi(x_0)-\phi(x)&\geqq w(x_0)-w(x)\\ &\geqq-|x_0-x|\cdot\|D\varphi_n\|_\infty+\lambda(d(x_0,\partial\Omega)-d(x,\partial\Omega))\\ &\geqq-|x_0-x|\cdot\|D\varphi_n\|_\infty+\lambda(|x_0-y_0|-|x-y_0|)\\ &\geqq-|x_0-x|\cdot\|D\varphi_n\|_\infty+\lambda|x-x_0|,\end{aligned}$$

choosing $x\in[x_0,y_0]$, and thus $D\phi(x_0)\geqq\lambda-\|D\varphi_n\|_\infty$. Now, we choose α (depending upon λ) such that $\alpha\lambda-\|\varphi_n\|_\infty\geqq\max v+1$ so that the third inequality of (28) holds and w remains bounded. Then we choose λ such that (27), the bound on w and the estimate on $D\phi(x_0)$, implies the first equation of (28). Finally, the condition $w\geqq\varphi+1/n$ on $\partial\Omega$ is clear enough. Thus, we may apply Theorem II.1 to compare v and w (notice that (21) is not needed on $\partial E_\alpha/\partial\Omega$, since on this part of the boundary we have a pure Dirichlet condition). Therefore, we obtain $v(x)\leqq\varphi_n\leqq 2/n+\varphi(x)$ on $\partial\Omega$ and we have proved that $v(x)\leqq\varphi(x)$ on $\partial\Omega$.

Now, we prove that v is uniformly Lipschitz continuous in Ω. We introduce v_ε defined by

$$v_\varepsilon(x)=\operatorname*{Max}_{y\in\bar\Omega}\left\{v(y)-\frac{|x-y|^2}{\varepsilon}\right\}.$$

(This operation is called sup-convolution and is studied in Lasry and Lions [20].) v_ε is Lipschitzian, is nondecreasing to v, and satisfies, for any strict open subset of Ω and ε small enough,

$$H(x,v_\varepsilon,Dv_\varepsilon)\leqq C \quad (\text{independent of } \varepsilon).$$

Since v_ε remains uniformly bounded, (27) gives a uniform bound on Dv_ε and Theorem II.4 is proved.

Remark II.5. With the notation of Theorem II.4, setting $v^0(x)=\limsup_{y\in\Omega,y\to x}v(y)$, we define a Lipschitzian function on $\bar\Omega$ and v^0 is still a subsolution.

II.3. Application to large deviations problems. Assumption (27) is naturally satisfied when we look at the asymptotic behavior for exponentially small probabilities and expectations associated with processes with small diffusion term. We refer the reader to Evans and Ishii [10], Fleming and Souganidis [14], or Bardi [1] for the motivations and the main results obtained recently from the method initiated by Fleming [11], [12].

In order to illustrate how the theory above may simplify the approach to these problems, we have, for ε small enough, a solution $u_\varepsilon \in C^1(\Omega) \cap C(\bar{\Omega})$ of

$$-\varepsilon a_{ij}(x) \frac{\partial^2 u_\varepsilon}{\partial x_i \, \partial x_j} + H(x, u_\varepsilon, Du_\varepsilon) = 0, \qquad u_\varepsilon|_{\partial\Omega} = \varphi, \tag{29}$$

where $a_{ij}(x)$ is a positive symmetric continuous matrix.

(Here we assume that u_ε exists. Generally, it is explicitly given by the logarithmic transformation of a "cost function." Above, it is possible to assume that $u_\varepsilon \in C(\bar{\Omega})$ is a viscosity solution of (29) in the sense of Lions [21]).

In the context of large deviations the following Hamiltonian is relevant:

$$H(x, p) = p' \cdot a(x) \cdot p + b(x) \cdot p. \tag{30}$$

Generally, the question of studying the behavior of u_ε is performed as follows (see [10]):

(1) Uniform bounds in L^∞ are proved on u_ε;

(2) Uniform bounds on the first derivatives of u_ε are proved;

(3) Then, it is possible to pass to the limit in (29);

(4) Identify the limit of u_ε by a uniqueness theorem for (9).

We remark that step (2) may be hard to establish (and either wrong if, as below, a boundary layer appears in (29) and a part of the boundary condition is lost). The following proposition shows that this step is not necessary to obtain uniform convergence of u_ε inside Ω. Therefore, it simplifies the program described above, weakens the assumptions on H, and makes the program applicable to more general situations.

PROPOSITION II.6. *Let u_ε be a viscosity solution of* (29) *in $C(\bar{\Omega})$ satisfying $\|u_\varepsilon\|_\infty \leqq C$ (independent of ε) and assume that H satisfies* (18)–(21).

(i) *Then the functions*

$$\bar{u}(x) = \limsup_{y \to x, \varepsilon \to 0} u_\varepsilon(y), \qquad \underline{u}(x) = \liminf_{y \to x, \varepsilon \to 0} u_\varepsilon(y),$$

are, respectively, viscosity sub- and supersolutions of (9).

(ii) *If* (27) *holds, then $\tilde{u}(x) = \limsup_{y \in \Omega, y \to x} \bar{u}(y)$ is a Lipschitzian subsolution of* (9) *and $\tilde{u} \leqq \varphi$ on $\partial\Omega$.*

(iii) *Assume* (27). *Then, assume $\gamma > 0$ in* (20), *or that $H(x, t, p)$ is convex in p and that* (9) *admits a strict subsolution in the sense of* (31) *below; then $\underline{u} = \bar{u}$ in Ω and u_ε converges uniformly on every compact subset of Ω to $\bar{u}$ (or $\underline{u}$).* □

(By a strict subsolution we mean a function $w \in C^1(\bar{\Omega})$, such that, for all $t \in \mathbb{R}$,

$$H(x, t, Dw) \leqq -a < 0 \quad \text{in } \bar{\Omega}). \tag{31}$$

Proof of Proposition II.6. Much of this proposition is an adaptation of the above results and classical ones. Item (i) is proved in [16] and is an adaptation of the stability result of [4]. Item (ii) is nothing but Theorem II.4 and Remark II.5. Item (iii) is an adaptation to discontinuous solutions of a result of Kruskov [19], Crandall and Lions [7], Lions [22], Ishii [15], Lasry and Lions [20], and Barles [3]. Let us just recall the proof of [15] (we consider the case $\gamma = 0$ only). Let $\theta \in (0, 1)$ and set $u_\theta = \theta\tilde{u} + (1 - \theta)w$.

Thanks to the convexity of H in p and to (31), we get

$$H(x, u_\theta, Du_\theta) \leqq -(1-\theta)a < 0 \quad \text{in } \Omega.$$

Since this still holds if we replace w by $w - M$ we get also

$$\text{Min}\,(u_\theta - \varphi, H(x, u_\theta, Du_\theta)) \leqq -(1-\theta)a \quad \text{on } \partial\Omega.$$

Thus, we may apply Theorem II.1 and obtain

$$u_\theta \leqq \underline{u},$$

and we conclude the uniqueness statement by letting θ go to 1. Finally, we have obtained

$$\limsup_{\varepsilon \to 0, y \to x} u_\varepsilon(y) = \liminf_{\varepsilon \to 0, y \to x} u_\varepsilon(y) \quad \forall x \in \Omega,$$

and this implies the uniform convergence of u_ε on every compact subset of Ω. □

Remark II.7. For the particular Hamiltonian given by (30), the existence of a strict subsolution is known to hold if

$$\forall x(s) \in \bar{\Omega}, x(\cdot) \in H^1_{\text{loc}}([0, \infty); \mathbb{R}^N) \quad \int_0^\infty |\dot{x}(s) - b(x(s))|^2\, dx = \infty.$$

Proposition II.6 may be applied to this example as soon as b and a are continuous (and a is uniformly positive). Indeed, it is well known that (18)-(20) must hold only for bounded p since (27) gives the Lipschitz continuity of the subsolutions.

Appendix. On the two-sided obstacle problem. In this Appendix, we consider the problem

(16) $$\text{Max}\,(u - \psi_1, \text{Min}\,(u - \psi_2, H(x, u, Du))) = 0 \quad \text{in } \mathbb{R}^N$$

for discontinuous obstacles ψ_1, ψ_2. We prove that it has a maximum subsolution and a minimum supersolution which are solutions. These results have been used in § I, since the exit time problem is a particular case of (16).

Let us recall what we mean by a solution of (16).

DEFINITION. A u.s.c. u on $\mathbb{R}^N$ is called a viscosity subsolution of (16) if, for any $\phi \in C^1(\mathbb{R}^N)$ and any point x_0 such that $\text{Max}\,(u - \phi) = (u - \phi)(x_0)$, then

$$H^*(x_0, u(x_0), D\phi(x_0)) \leqq 0.$$

The definition of supersolutions or solutions is obtained as usual (see § I).

In the following, H is given by (10) with b and f satisfying (2).

To state our result, we need the following assumption:

(32) ψ_1 and ψ_2 are bounded and satisfy $(z^*)_* = z_*$, $(z_*)^* = z^*$ (where $z = \psi_1$ or ψ_2).

PROPOSITION A.1. *Under assumptions* (2), (32) *the functions* $\bar{u}$ *and* $\underline{u}$ *defined by*

$$\bar{u}(x) = \inf_{v(\cdot), \theta_1} \sup_{\theta_2} \left[\int_0^{\theta_1 \wedge \theta_2} f(y_x(t), v(t))\, e^{-\lambda t}\, dt \right.$$
$$\left. + \psi_1^*(y_x(\theta_1))\, e^{-\lambda\theta_1} 1_{\{\theta_1 < \theta_2\}} + \psi_2^*(y_x(\theta_2))\, e^{-\lambda\theta_2} 1_{\{\theta_2 \leqq \theta_1\}} \right],$$

$$\underline{u}(x) = \sup_{\theta_2} \inf_{(\mu, \theta_1)} \left[\int_0^{\theta_1 \wedge \theta_2} f(\hat{y}_x(t), v(t))\, e^{-\lambda t}\, dt \right.$$
$$\left. + \psi_{1*}(\hat{y}_x(\theta_1))\, e^{-\lambda\theta_1} 1_{\{\theta_1 < \theta_2\}} + \psi_{2*}(\hat{y}_x(\theta_2))\, e^{-\lambda\theta_2} 1_{\{\theta_2 \leqq \theta_1\}} \right]$$

(where y_x and $\hat{y}_x$ are, respectively, defined by (1) *and* (1′) *in* $\mathbb{R}^N$*) are, respectively, the maximum viscosity subsolution and solution of* (16) *and the minimum viscosity supersolution and solution of* (16).

Proof of Proposition A.1. Since the arguments are routine adaptations of the arguments of [4], we only sketch the proof for $\bar{u}$. Let $(\psi_1^n)_n$ and $(\psi_2^n)_n$ be two nonincreasing sequences of functions of $\mathrm{BUC}(\mathbb{R}^N)$ such that

$$\psi_1^* = \operatorname*{Inf}_n \psi_1^n, \qquad \psi_2^* = \operatorname*{Inf}_n \psi_2^n.$$

Let v be a bounded u.s.c. viscosity subsolution of (16). By classical results (see [2], [7]-[9], [22]), we know that there exists a unique bounded and continuous solution u^n of

$$\text{(33)} \qquad \operatorname{Max}\,[(u-\psi_1^n), \operatorname{Min}\,((u-\psi_2^n), H(x,u,Du))]=0 \quad \text{in } \mathbb{R}^N.$$

Moreover, since (33) satisfies Isaac's condition [24], u^n is given by

$$u^n(x) = \operatorname*{Inf}_{(v(\cdot),\theta_1)} \operatorname*{Sup}_{\theta_2} \left[\int_0^{\theta_1 \wedge \theta_2} f(y_x(t), v(t))\, e^{-\lambda t} \right.$$

$$\left. + \psi_1^n(y_x(\theta_1))\, e^{-\lambda\theta_1} 1_{\{\theta_1<\theta_2\}} + \psi_2^n(y_n(\theta_2))\, e^{-\lambda\theta_2} 1_{\{\theta_2 \leqq \theta_1\}} \right].$$

Let us only note that, in this particular case, u belongs to $\mathrm{BUC}(\mathbb{R}^N)$ and that we could invert the Sup and Inf in the above formula. Moreover, the comparison results for viscosity sub- and supersolution give

$$\text{(34)} \qquad v \leqq u^{n+1} \leqq u^n \quad \text{in } \mathbb{R}^N,$$

since v is still a subsolution of (33). So, it is enough to identify the function

$$w = \operatorname*{Inf}_n u^n.$$

It is clear enough that

$$w \geqq \bar{u}.$$

To prove the other inequality we denote by

$$J^n(v,\theta_1,\theta_2) = \int_0^{\theta_1 \wedge \theta_2} f(y_x(t), v(t))\, e^{-\lambda t} + \psi_1^n(y_x(\theta_1))\, e^{-\lambda\theta_1} 1_{\{\theta_1<\theta_2\}}$$

$$+ \psi_2^n(y_x(\theta_2))\, e^{-\lambda\theta_2} 1_{\{\theta_2 \leqq \theta_1\}}$$

and by $J(v,\theta_1,\theta_2)$ the same expression, where ψ_1^n and ψ_2^n are replaced by ψ_1^* and ψ_2^*. We have

$$w(x) = \operatorname*{Inf}_{(v,\theta_1)} \operatorname*{Inf}_n \operatorname*{Sup}_{\theta_2} J^n(v,\theta_1,\theta_2).$$

So, it is enough to compute $\lim_n \sup_{\theta_2} J^n(v,\theta_1,\theta_2)$, v, θ_1 being fixed with $\theta_1<+\infty$. Let θ_2^n be a maximum point of $J^n(v,\theta_1,\theta_2)$; we may assume that $\theta_2^n \leqq \theta_1+1$ and therefore we can take a subsequence, still denoted by θ_2^n, which converges to θ_2:

$$\operatorname*{Inf}_n \operatorname*{Sup}_{\theta_2} J^n(v,\theta_1,\theta_2) = \lim_{n\to+\infty} J^n(v,\theta_1,\theta_2^n) \leqq \limsup_{\substack{n\to\infty \\ \theta\to\theta_2}} J^n(v,\theta_1,\theta)$$

and it is easy to see that the right-hand side of the inequality is exactly $J(v, \theta_1, \theta_2)$. Finally

$$w(x) \leqq \operatorname*{Inf}_{(v,\theta_1)} \operatorname*{Sup}_{\theta_2} J(x, \theta_1, \theta_2) = \bar{u}(x)$$

which is the result we want. Let us just note that this proof shows that $\bar{u}$ is u.s.c. and it is easy to check that

$$\bar{u}(x) = \limsup_{n\to\infty, y\to x} u^n(y),$$

$$(\bar{u})_*(x) = \liminf_{n\to\infty, y\to x} u^n(y).$$

So, by the stability result of [4], we conclude that $\bar{u}$ is the viscosity solution of (16). Moreover, by (34), we see that $\bar{u}$ is the maximum viscosity subsolution of (6). Assumption (32) is used to identify the limits of the Hamiltonians and to ensure $\bar{u}$ and $\underline{u}$ are solutions of the same Hamiltonian. The proof is complete. □

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