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# Backward Stochastic Differential Equation, Nonlinear Expectation and Their Applications

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**Abstract**

We give a survey of the developments in the theory of Backward Stochastic Differential Equations during the last 20 years, including the solutions' existence and uniqueness, comparison theorem, nonlinear Feynman-Kac formula, $g$-expectation and many other important results in BSDE theory and their applications to dynamic pricing and hedging in an incomplete financial market.

We also present our new framework of nonlinear expectation and its applications to financial risk measures under uncertainty of probability distributions. The generalized form of law of large numbers and central limit theorem under sublinear expectation shows that the limit distribution is a sublinear $G$-normal distribution. A new type of Brownian motion, $G$-Brownian motion, is constructed which is a continuous stochastic process with independent and stationary increments under a sublinear expectation (or a nonlinear expectation). The corresponding robust version of Itô's calculus turns out to be a basic tool for problems of risk measures in finance and, more general, for decision theory under uncertainty. We also discuss a type of "fully nonlinear" BSDE under nonlinear expectation.

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The theory of backward stochastic differential equations (BSDEs in short) and nonlinear expectation has gone through rapid development in so many different areas of research and applications, such as probability and statistics, partial differential equations (PDE), functional analysis, numerical analysis and stochastic computations, engineering, economics and mathematical finance, that it is impossible in this paper to give a complete review of all the important progresses of recent 20 years. I only limit myself to talk about my familiar subjects. The book edited by El Karoui and Mazliark (1997) provided excellent introductory lecture, as well as a collection of many important research results before 1996, see also [35] with applications in finance. Chapter 7 of the book of Yong and Zhou (1999) is also a very good reference.

Recently, using the notion of sublinear expectations, we have developed systematically a new mathematical tool to treat the problem of risk and randomness under the uncertainty of probability measures. This framework is particularly important for the situation where the involved uncertain probabilities are singular with respect to each other thus we cannot treat the problem within the framework of a given "reference" probability space. The well-known volatility model uncertainty in finance is a typical example. We present a new type of law of large numbers and central limit theorem as well as $G$-Brownian motion and the corresponding stochastic calculus of Itô's type under such new sublinear expectation space. A more systematical presentation with detailed proofs and references can be found in Peng (2010a).

This paper is organized as follows. In Section 1 we present BSDE theory and the corresponding $g$-expectations with some applications in super-hedging and risk measuring in finance; In Section 2 we give a general notion of nonlinear expectations and a new law of large numbers combined with a central limit theorem under a sublinear expectation space. $G$-Brownian motion under a sublinear expectation–$G$-expectation, which is a nontrivial generalization of the notion of $g$ expectation, and the related stochastic calculus will be given in Section 3. We also discuss a type of fully "nonlinear BSDE" under $G$-expectation. For a systematic presentation with detailed proofs of the results on $G$-expectation, $G$-Brownian motion and the related calculus, see Peng (2010a).

# 1. BSDE and $g$-expectation

**1.1. Recall: SDE and related Itô's stochastic calculus.** We consider a typical probability space $(\Omega, \mathcal{F}, P)$ where $\Omega = C([0,\infty), \mathbb{R}^d)$, each element $\omega$ of $\Omega$ is a $d$-dimensional continuous path on $[0,\infty)$ and $\mathcal{F} = \mathcal{B}(\Omega)$, the Borel $\sigma$-algebra of $\Omega$ under the distance defined by

$$\rho(\omega,\omega') = \sup_{i\geq 1}\max_{0\leq t\leq i}|\omega_t - \omega'_t| \wedge 1, \;\; \omega,\omega' \in \Omega.$$

We also denote $\{(\omega_{s\wedge t})_{s\geq 0} : \omega \in \Omega\}$ by $\Omega_t$ and $\mathcal{B}(\Omega_t)$ by $\mathcal{F}_t$. Thus an $\mathcal{F}_t$-measurable random variable is a Borel measurable function of continuous paths defined on $[0,t]$. For an easy access by a wide audience I will not bother readers with too special vocabulary such as $P$-null sets, augmentation, etc. We say $\xi \in L_P^p(\mathcal{F}_t, \mathbb{R}^n)$ if $\xi$ is an $\mathbb{R}^n$-valued $\mathcal{F}_t$-measurable random variable such that $E_P[|\xi|^p] < \infty$. We also say $\eta \in M_P^p(0,T,\mathbb{R}^n)$ if $\eta$ is an $\mathbb{R}^n$-valued stochastic process on $[0,T]$ such that $\eta_t$ is $\mathcal{F}_t$-measurable for each $t \in [0,T]$ and $E_P[\int_0^T |\eta_t|^p dt] < \infty$. Sometimes we omit the space $\mathbb{R}^n$, if no confusion will be caused.

We assume that under the probability $P$ the canonical process $B_t(\omega) = \omega_t$, $t \geq 0$, $\omega \in \Omega$ is a d-dimensional standard Brownian motion, namely, for each $t$, $s \geq 0$,

(i) $B_0 = 0$, $B_{t+s} - B_s$ is independent of $B_{t_1}, \cdots, B_{t_n}$, for $t_1, \cdots, t_n \in [0,s]$, $n \geq 1$;

(ii) $B_{t+s} - B_s \overset{d}{=} N(0, I_d t)$, $s, t \geq 0$, where $I_d$ is the $d \times d$ identical matrix.

$P$ is called a Wiener measure on $(\Omega, \mathcal{F})$.

In 1942, Japanese mathematician Kiyosi Itô had laid the foundation of stochastic calculus, known as Itô's calculus, to solve the following *stochastic differential equation* (SDE):

$$dX_s = \sigma(X_s)dB_s + b(X_s)ds \tag{1.1}$$

with initial condition $X_s|_{s=0} = x \in \mathbb{R}^n$. Its integral form is:

$$X_t(\omega) = x + \int_0^t \sigma(X_s(\omega))dB_s(\omega) + \int_0^t b(X_s(\omega))ds, \tag{1.2}$$

where $\sigma : \mathbb{R}^n \mapsto \mathbb{R}^{n \times d}$, $b : \mathbb{R}^n \mapsto \mathbb{R}^n$ are given Lipschitz functions. The key part of this formulation is the stochastic integral $\int_0^t \sigma(X_s(\omega))dB_s(\omega)$. In fact, Wiener proved that the typical path of Brownian motion has no bounded variation and thus this integral is meaningless in the Lebesgue-Stieljes sense. Itô's deep insight is that, at each fixed time $t$, the random variable $\sigma(X_t(\omega))$ is a function of path depending only on $\omega_s$, $0 \leq s \leq t$, or in other words, it is an $\mathcal{F}_t$-measurable random variable. More precisely, the process $\sigma(X_\cdot(\omega))$ can be in the space $M_P^2(0,T)$. The definition of Itô integral is perfectly applied to a stochastic process $\eta$ in this space. The integral is defined as a limit of Riemann sums in a "non-anticipating" way: $\int_0^t \eta_s(\omega)dB_s(\omega) \approx \sum \eta_{t_i}(B_{t_{i+1}} - B_{t_i})$. It has zero expectation and satisfies the following Itô's isometry:

$$E\left[\left|\int_0^t \eta_s dB_s\right|^2\right] = E\left[\int_0^t |\eta_s|^2 ds\right]. \tag{1.3}$$

These two key properties allow Kiyosi Itô to obtain the existence and uniqueness of the solution of SDE (1.2) in a rigorous way. He has also introduced the well-known Itô formula: if $\eta$, $\beta \in M_P^2(0,T)$, then the following continuous process

$$X_t = x + \int_0^t \eta_s dB_s + \int_0^t \beta_s ds \tag{1.4}$$

is also in $M_P^2(0,T)$ and satisfies the following *Itô formula*: for a smooth function $f$ on $\mathbb{R}^n \times [0,\infty)$,

$$df(X_t,t) = \partial_t f(X_t,t)dt + \nabla_x f(X_t,t)dX_t + \frac{1}{2}\sum_{i,j=1}^n (\eta\eta^*)_{ij} D_{x_i x_j} f(X_t,t)dt. \quad (1.5)$$

Based on this formula, Kiyosi Itô proved that the solution $X$ of SDE (1.1) is a diffusion process with the infinitesimal generator

$$\mathcal{L} = \sum_{i=1}^n b_i(x) D_{x_i} + \frac{1}{2}\sum_{i,j=1}^n (\sigma(x)\sigma^*(x))_{ij} D_{x_i x_j}. \quad (1.6)$$

**1.2. BSDE: existence, uniqueness and comparison theorem.** In Itô's SDE (1.1) the initial condition can be also defined at any initial time $t_0 \geq 0$, with a given $\mathcal{F}_{t_0}$-measurable random variable $X_t|_{t=t_0} = \xi \in L_P^2(\mathcal{F}_{t_0})$. The solution $X_T^{t_0,\xi}$ at time $T > t_0$ is $\mathcal{F}_T$-measurable. This equation (1.1) in fact leads to a family of mappings $\phi_{T,t}(\xi) = X_T^{t,\xi} : L_P^2(\mathcal{F}_t, \mathbb{R}^n) \mapsto L_P^2(\mathcal{F}_T, \mathbb{R}^n)$, $0 \leq t \leq T < \infty$, determined uniquely by the coefficients $\sigma$ and $b$. This family forms what we called *stochastic flow* in the way that the following semigroup property holds: $\phi_{T,t}(\xi) = \phi_{T,s}(\phi_{s,t}(\xi))$, $\phi_{t,t}(\xi) = \xi$, for $t \leq s \leq T < \infty$.

But in many situations we can also meet an inverse type of problem to find a family of mappings $\mathcal{E}_{t,T} : L_P^2(\mathcal{F}_T, \mathbb{R}^m) \mapsto L_P^2(\mathcal{F}_t, \mathbb{R}^m)$ satisfying the following *backward semigroup property:* (see Peng (1997a)) for each $s \leq t \leq T < \infty$ and $\xi \in L_P^2(\mathcal{F}_T, \mathbb{R}^m)$,

$$\mathcal{E}_{s,t}[\mathcal{E}_{t,T}[\xi]] = \mathcal{E}_{s,T}[\xi], \text{ and } \mathcal{E}_{T,T}[\xi] = \xi.$$

$\mathcal{E}_{t,T}$ maps an $\mathcal{F}_T$-measurable random vector $\xi$, which can only be observed at time $T$, backwardly to an $\mathcal{F}_t$-measurable random vector $\mathcal{E}_{t,T}[\xi]$ at $t < T$. A typical example is the calculation of the value, at the current time $t$, of the risk capital reserve for a risky position with maturity time $T > t$. In fact this type of problem appears in many decision making problems.

But, in general, Itô's stochastic differential equation (1.1) cannot be applied to solve this type of problem. Indeed, if we try to use (1.1) to solve $X_t$ at time $t < T$ for a given terminal value $X_T = \xi \in L_P^2(\mathcal{F}_T)$, then

$$X_t = X_T - \int_t^T b(X_s)ds - \int_t^T \sigma(X_s)dB_s.$$

In this case the "solution" $X_t$ is still, in general, $\mathcal{F}_T$-measurable and thus $b(X)$ and $\sigma(X)$ become anticipating processes. It turns out that not only this formulation cannot ensure $X_t \in L_P^2(\mathcal{F}_t)$, the stochastic integrand $\sigma(X)$ also becomes illegal within the framework of Itô's calculus.

After the exploration over a long period of time, we eventually understand that what we need is the following new type of backward stochastic differential equation

$$Y_t = Y_T + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s, \tag{1.7}$$

or in its differential form

$$dY_s = -g(s, Y_s, Z_s)ds + Z_s dB_s, \quad s \in [0, T].$$

In this equation $(Y, Z)$ is a pair of unknown non-anticipating processes and the equation has to be solved for a given terminal condition $Y_T \in L_P^2(\mathcal{F}_T)$ (but $Z_T$ is not given). In contrast to SDE (1.1) in which two coefficients $\sigma$ and $b$ are given functions of one variable $x$, here we have only one coefficient $g$, called the generator of the BSDE, which is a function of two variables $(y, z)$. Bismut (1973) was the first to introduce a BSDE for the case where $g$ is a linear or (for $m = 1$) a convex function of $(y, z)$ in his pioneering work on maximum principle of stochastic optimal control systems with an application in financial markets (see Bismut (1975)). See also a systematic study by Bensoussan (1982) on this subject. The following existence and uniqueness theorem is a fundamental result:

**Theorem 1.1.** *(Pardoux and Peng (1990)) Let $g : \Omega \times [0, \infty) \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$ be a given function such that $g(\cdot, y, z) \in M_P^2(0, T, \mathbb{R}^m)$ for each $T$ and for each fixed $y \in \mathbb{R}^m$ and $z \in \mathbb{R}^{m\times d}$, and let $g$ be a Lipschitz function of $(y, z)$, i.e., there exists a constant $\mu$ such that*

$$|g(\omega, t, y, z) - g(\omega, t, y', z')| \leq \mu(|y - y'| + |z - z'|), \quad y, y' \in \mathbb{R}^m, \quad z, z' \in \mathbb{R}^{m\times d}.$$

*Then, for each given $Y_T = \xi \in L_P^2(\mathcal{F}_T, \mathbb{R}^m)$, there exists a unique pair of processes $(Y, Z) \in M_P^2(0, T, \mathbb{R}^m \times \mathbb{R}^{m\times d})$ satisfying BSDE (1.7). Moreover, $Y$ has continuous path, a.s. (almost surely).*

We denote $\mathcal{E}_{t,T}^g[\xi] = Y_t$, $t \in [0, T]$. From the above theorem, we have obtained a family of mappings

$$\mathcal{E}_{s,t}^g : L_P^2(\mathcal{F}_t) \mapsto L_P^2(\mathcal{F}_s), \quad 0 \leq s \leq t < \infty, \tag{1.8}$$

with "backward semigroup property" (see Peng (2007a)):

$$\mathcal{E}_{s,t}^g[\mathcal{E}_{t,T}^g[\xi]] = \mathcal{E}_{s,T}^g[\xi], \ \mathcal{E}_{T,T}^g[\xi] = \xi, \ \text{for } s \leq t \leq T < \infty, \ \forall \xi \in L^2(\mathcal{F}_T).$$

In 1-dimensional case, i.e., $m = 1$, the above property is called "recursive" in utility theory in economics. In fact, independent of the above result, Duffie and Epstein (1992) introduced the following class of recursive utilities:

$$-dY_t = \left[ f(c_t, Y_t) - \frac{1}{2} A(Y_t) Z_t^T Z_t \right] dt - Z_t dB_t, \quad Y_T = \xi, \tag{1.9}$$

where the function $f$ is called a generator, and $A$ a "variance multiplier".

In 1-dimensional case, we have the *comparison theorem* of BSDE, introduced by Peng (1992b) and improved by El Karoui, Peng and Quenez (1997).

**Theorem 1.2.** *We assume the same condition as in the above theorem for two generators $g_1$ and $g_2$. We also assume that $m = 1$. If $\xi_1 \geq \xi_2$ and $g_1(t, y, z) \geq g_2(t, y, z)$ for each $(t, y, z)$, a.s., then we have $\mathcal{E}^{g_1}_{t,T}[\xi_1] \geq \mathcal{E}^{g_2}_{t,T}[\xi_2]$, a.s.*

This theorem is a powerful tool in the study of 1-dimensional BSDE theory as well as in many applications. In fact it plays the role of "maximum principle" in the PDE theory. There are two typical theoretical situations where this comparison theorem plays an essential role. The first one is the existence theorem of BSDE, obtained by Lepeltier and San Martin (1997), for the case when $g$ is only a continuous and linear growth function in $(y, z)$ (the uniqueness under the condition of uniform continuity in $z$ was obtained by Jia (2008)).

The second one is also the existence and uniqueness theorem, in which $g$ satisfies quadratic growth condition in $z$ and some local Lipschitz conditions, obtained by Kobylanski (2000) for the case where the terminal value $\xi$ is bounded. The existence for unbounded $\xi$ was solved only very recently by Briand and Hu (2006).

A specially important model of symmetric matrix valued BSDEs with a quadratic growth in $(y, z)$ is the so-called stochastic Riccati equation. This equation is applied to solve the optimal feedback for linear-quadratic stochastic control system with random coefficients. Bismut (1976) solved this problem for a situation where there is no control variable in the diffusion term, and then raised the problem for the general situation. The problem was also listed as one of several open problems in BSDEs in Peng (1999a). It was finally completely solved by Tang (2003), whereas other problems in the list are still open. Only few results have been obtained for multi-dimensional BSDEs of which the generator $g$ is only assumed to be (bounded or with linear growth) continuous function of $(y, z)$, see Hamadène, Lepeltier and Peng (1997) for a proof in a Markovian case. Recently Buckdahn, Engelbert and Rascanu (2004) introduced a notion of weak solutions for BSDEs and obtained the existence for the case where $g$ does not depend on $z$.

The above mentioned stochastic Riccati equation is used to solve a type of backward stochastic partial differential equations (BSPDEs), called stochastic Hamilton-Jacobi-Bellman equation (SHJB equations) in order to solve the value function of an optimal controls for non-Markovian systems, see Peng (1992). Englezos and Karatzas (2009) characterized the value function of a utility maximization problem with habit formation as a solution of the corresponding stochastic HJB equation. A linear BSPDE was introduced by Bensoussan

(1992). It serves as the adjoint equation for optimal control systems with partial information, see Nagai (2005), Oksendal, Proske and Zhang (2005), or for optimal control system governed by a stochastic PDE, see Zhou (1992). For the existence, uniqueness and regularity of the adapted solution of a BSPDE, we refer to the above mentioned papers as well as Hu and Peng (1991), Ma and Yong (1997,1999), Tang (2005) among many others. The existence and uniqueness of a fully nonlinear backward HJB equation formulated in Peng (1992) was then listed in Peng (1999a) as one of open problems in BSDE theory. The problem is still open.

The problem of multi-dimensional BSDEs with quadratic growth in $z$ was partially motivated from the heat equation of harmonic mappings, see Elworthy (1993). Dynamic equilibrium pricing models and non-zero sum stochastic differential games also lead to such type of BSDE. There have been some very interesting progresses of existence and uniqueness in this direction, see Darling (1995), Blache (2005). But the main problem remains still largely open. One possible direction is to find a tool of "comparison theorem" in the multi-dimensional situation. An encouraging progress is the so called backward viability properties established by Buckdahn, Quincampoix and Rascanu (2000).

**1.3. BSDE, PDE and stochastic PDE.** It was an important discovery to find the relation between BSDEs and (systems of) quasilinear PDEs of parabolic and elliptic types. Assume that $X_s^{t,x}$, $s \in [t,T]$, is the solution of SDE (1.1) with initial condition $X_s^{t,x}|_{s=t} = x \in \mathbb{R}^n$, and consider a BSDE defined on $[t,T]$ of the following type

$$dY_s^{t,x} = -g(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})ds + Z_s^{t,x}dB_s, \tag{1.10}$$

with terminal condition $Y_T^{t,x} = \varphi(X_T^{t,x})$. Then we can use this BSDE to solve a quasilinear PDE. We consider a typical case $m = 1$:

**Theorem 1.3.** *Assume that $b$, $\sigma$, $\varphi$ are given Lipschitz functions on $\mathbb{R}^n$ with values in $\mathbb{R}^n$, $\mathbb{R}^{n\times d}$ and $\mathbb{R}$ respectively, and that $g$ is a real valued Lipschitz function on $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d$. Then we have the following relation $Y_s^{t,x} = \mathcal{E}_{s,T}^g[\varphi(X_T^{t,x})] = u(s, X_s^{t,x})$. In particular, $u(t,x) = Y_t^{t,x}$, where $u = u(t,x)$ is the unique viscosity solution of the following parabolic PDE defined on $(t,x) \in [0,T] \times \mathbb{R}^n$:*

$$\partial_t u + \mathcal{L}u + g(x, u, \sigma^* Du) = 0, \tag{1.11}$$

*with terminal condition $u|_{t=T} = \varphi$. Here $Du = (D_{x_1}u, \cdots, D_{x_n}u)$*

The relation $u(t,x) = Y_t^{t,x}$ is called a nonlinear Feynman-Kac formula. Peng (1991a) used a combination of BSDE and PDE method and established this relation for non-degenerate situations under which (1.11) has a classical solution. In this case (1.11) can also be a system of PDE, i.e., $m > 1$, and we also have $Z_s^{t,x} = \sigma^* Du(s, X_s^{t,x})$. Later Peng (1991b), (1992a) used a stochastic control argument and the notion of viscosity solution to prove a more general version of above theorem for $m = 1$. Using a simpler argument, Pardoux and Peng (1992) provided a proof for a particular case, which is the above theorem. They have introduced a new probablistic method to prove the regularity of $u$, under the condition that all coefficients are regular enough, but the PDE is possibly degenerate. They then proved that the function $u$ is also a classical regular solution of (1.11). This proof is also applied to the situation $m > 1$.

The above nonlinear Feynman-Kac formula is not only valid for a system of parabolic equation (1.11) with Cauchy condition but also for the corresponding elliptic PDE $\mathcal{L}u + g(x, u, \sigma^* Du) = 0$ defined on an open subset $\mathcal{O} \subset \mathbb{R}^n$ with boundary condition $u|_{x\in\mathcal{O}} = \varphi$. In fact, $u = u(x)$, $x \in \mathcal{O}$ can be solved by defining $u(x) = \mathcal{E}^g_{0,\tau_x}[\varphi(X^{0,x}_{\tau_x})]$, where $\tau_x = \inf\{s \geq 0 : X^{0,x}_s \notin \mathcal{O}\}$. In this case some type of non-degeneracy condition of the diffusion process $X$ and a monotonicity condition of $g$ with respect to $y$ are required, see Peng (1991a). The above results imply that we can solve PDEs by using BSDEs and, conversely, solve some BSDEs by PDEs.

In principle, once we have obtained a BSDE driven by a Markov process $X$ in which the final condition $\xi$ at time $T$ depends only on $X_T$, and the generator $g$ also depends on the state $X_t$ at each time $t$, then the corresponding solution is also state dependent, namely $Y_t = u(t, X_t)$, where $u$ is the solution of the corresponding quasilinear evolution equation. Once $\xi$ and $g$ are path functions of $X$, then the solution $Y_t = \mathcal{E}^g_{t,T}[\xi]$ of the BSDE becomes also path dependent. In this sense, we can say that PDE (1.11) is in fact a "state dependent BSDE", and BSDE gives us a new generalization of PDE—"path-dependent PDE" of parabolic and elliptic types.

The following backward doubly stochastic differential equation (BDSDE) smartly combines two essentially different SDEs, namely, an SDE and a BSDE into one equation:

$$dY_t = -\bar{g}_t(Y_t, Z_t)dt - \bar{h}_t(Y_t, Z_t) \downarrow dW_t + Z_t dB_t, \quad Y_T = \xi, \tag{1.12}$$

where $W$ and $B$ are two mutually independent Brownian motions. In (1.12) all processes at time $t$ are required to be measurable functions on $\Omega_t \times \Omega^W_t$ where $\Omega^W_t$ is the space of the paths of $(W_T - W_s)_{t\leq s\leq T}$ and $\downarrow dW_t$ denotes the "backward Itô's integral" ($\approx \sum_i h_{t_i}(W_{t_i} - W_{t_{i-1}})$). We also assume that $\bar{g}$ and $\bar{h}$ are Lipschitz functions of $(y, z)$ and, in addition, the Lipschitz constant of $\bar{h}$ with respect to $z$ is assumed to be strictly less than 1. Pardoux and Peng (1994) obtained the existence and uniqueness of (1.12) and proved that, under a further assumption:

$$\bar{g}_t(\omega, y, z) = g(X_t(\omega), y, z), \quad \bar{h}_t(y, z) = h(X_t(\omega), y, z), \quad \xi(\omega) = \varphi(X_T(\omega)), \tag{1.13}$$

where $X$ is the solution of (1.1) and where $g$, $h$, $b$, $\sigma$, $\varphi$ are sufficiently regular with $|\partial_z \bar{g}| < \mu$, $\mu < 1$, then $Y_t = u(t, X_t)$, $Z_t = \sigma^* Du(t, X_t)$. Here $u$ is a smooth solution of the following stochastic PDE:

$$du_t(x,\omega) = -(\mathcal{L}u + g(x,u,\sigma^* Du))dt + h(x,u,\sigma^* Du) \downarrow dW_t \tag{1.14}$$

with terminal condition $u|_{t=T} = \varphi(X_T)$. Here we see again a path-interpretation of a nonlinear stochastic PDE.

Another approach to give a probabilistic interpretation of some infinite dimensional Hamilton-Jacobi-Bellman equations is to consider a generator of a BSDE of the form $g(X_t, y, z)$ where $X$ is a solution of the following type of infinite dimensional SDE

$$dX_s = [\mathcal{A}X_s + b(X_s)]ds + \sigma(X_s)dB_s, \tag{1.15}$$

where $\mathcal{A}$ is some given infinitesimal generator of a semigroup and $B$ is, in general, an infinite dimensional Brownian motion. We refer to Fuhrman and Tessitore (2002) for the related references.

Up to now we have only discussed BSDEs driven by a Brownian motion. In principle a BSDE can be driven by a more general martingale. See Kabanov (1978), Tang and Li (1994) for optimal control system with jumps, where the adjoint equation is a linear BSDE with jumps. For results of the existence, uniqueness and regularity of solutions, see Situ (1996), El Karoui and Huang (1997), Barles, Buckdahn and Pardoux (1997), Nualart and Schoutens (2001) and many other results on this subject.

**1.4. Forward-backward SDE.** Nonlinear Feynman-Kac formula can be used to solve a nonlinear PDE of form (1.11) by a BSDE (1.10) coupled with an SDE (1.1). In this situation BSDE (1.10) and forward SDE (1.1) are only partially coupled. A fully coupled system of SDE and BSDE is called a forward-backward stochastic differential equation (FBSDE). It has the following form:

$$\begin{aligned} dX_t &= b(t, X_t, Y_t, Z_t)dt + \sigma(t, X_t, Y_t, Z_t)dB_t, \quad X_0 = x \in \mathbb{R}^n, \\ -dY_t &= f(t, X_t, Y_t, Z_t)dt - Z_t dB_t, \quad Y_T = \varphi(X_T). \end{aligned}$$

Note that it is not realistic to only assume, as in a BSDE framework, that the coefficients $b$, $\sigma$, $f$ and $\varphi$ are just Lipschitz functions in $(x, y, z)$. A counterexample can be easily constructed. Therefore additional conditions are needed for the well-posedness of the problem. Antonelli (1993) provided a counterexample and solved a special type of FBSDE. Then Ma, Protter and Yong (1994) have proposed a four-step scheme method of FBSDE. This method uses some classical result of PDE for which $\sigma$ is assumed to be independent of $z$ and strictly non-degenerate. The coefficients $f$, $b$, $\sigma$ and $\varphi$ are also assumed to be deterministic functions. For the case $\dim(x) = \dim(y) = n$, Hu and Peng (1995) proposed a new type of monotonicity condition: the function $A = (-f, b, \sigma)$ is said to be a monotone function in $\gamma = (x, y, z)$ if there exists a positive constant $\mu$ such that

$$(A(\gamma) - A(\gamma'), \gamma - \gamma') \leq -\mu|\gamma - \gamma'|^2, \quad \gamma, \gamma' \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n \times d}.$$

With this condition and $(\varphi(x) - \varphi(x'), x - x') \geq 0$, for each $x$, $x' \in \mathbb{R}^n$, the above FBSDE has a unique solution. The proof of the uniqueness is immediate and the existence was established by using a type of continuation method (see Peng (1991a), and Yong (1997)). This method does not need to assume coefficients to be deterministic. Peng and Wu (1999) have weakened the monotonicity condition and the constraint $\dim(x) = \dim(y)$, Wu (1999) provided a new type of comparison theorem. Another type of existence and uniqueness theorem under different conditions was obtained by Pardoux and Tang (1999). We also refer to the book of Ma and Yong (1999) for a systematic presentation on this subject. For time-symmetric forward-backward stochastic differential equations and its relation with stochastic optimality, see Peng and Shi (2003), Han, Peng and Wu (2010).

**1.5. Reflected BSDE and other types of constrained BSDE.** If $(Y, Z)$ solves

$$-dY_s = g(s, Y_s, Z_s)ds - Z_s dB_s + dK_s, \quad Y_T = \xi, \tag{1.16}$$

where $K$ is a càdlàg (continu à droite avec limite à gauche, or in English, right continuous with left limit) and increasing process with $K_0 = 0$ and $K_t \in L^2_P(\mathcal{F}_t)$, then $Y$ or $(Y, Z, K)$ is called a *supersolution* of the BSDE, or $g$*-supersolution*. This notion is often used for constrained BSDEs. A typical one is, for a given terminal condition $\xi$ and a continuous adapted process $(L_t)_{t\in[0,T]}$ to find a smallest $g$-supersolution $(Y, Z, K)$ such that $Y \geq L$, and $Y_T = \xi$. This probelm was initialed in El Karoui, Kapoudjian, Pardoux, Peng and Quenez (1997). They have proved that this problem is equivalent to finding a triple $(Y, Z, K)$ satisfying (1.16) and the following reflecting condition of Skorohod type:

$$Y_s \geq L_s, \quad \int_0^T (Y_s - L_s) dK_s = 0. \tag{1.17}$$

The existence, uniqueness and continuous dependence theorems were obtained. Moreover, a new type of nonlinear Feynman-Kac formula was introduced: if all coefficients are given as in Theorem 1.3 and $L_s = \Phi(X_s)$ where $\Phi$ satisfies the same condition as $\varphi$, then we have $Y_s = u(s, X_s)$, where $u = u(t, x)$ is the solution of the following variational inequality:

$$\min\{\partial_t u + \mathcal{L}u + g(x, u, \sigma^* Du), u - \Phi\} = 0, \quad (t, x) \in [0, T] \times \mathbb{R}^n, \tag{1.18}$$

with terminal condition $u|_{t=T} = \varphi$. They also proved that this reflected BSDE is a powerful tool to deal with contingent claims of American types in a financial market with constraints.

BSDE reflected within two barriers, for a lower one $L$ and an upper one $U$ was first investigated by Cvitanic and Karatzas (1996) where a type of nonlinear Dynkin games was formulated for a two-player model with zero-sum utility, each player chooses his own optimal exit time. See also Rascano (2009).

There are many other generalizations on the above problem of RBSDEs, e.g. $L$ and $U$ can be càdlàg or even $L^2$-processes and $g$ admits a quadratic growth condition, see e.g. Hamadene (2002), Lepeltier and Xu (2005), Peng and Xu (2005) and many other related results. For BSDEs applied to optimal switching, see Hamadene and Jeanblanc (2007). For the related multi-dimensional BSDEs with oblique reflection, see Ramasubramanian (2002), Carmona and Ludkovski (2008), Hu and Tang (2010) and Hamadene and Zhang (2010).

A more general case of constrained BSDE is to find the smallest $g$-supersolution $(Y, Z, K)$ with constraint $(Y_t, Z_t) \in \Gamma_t$ where, for each $t \in [0, T]$, $\Gamma_t$ is a given closed subset in $\mathbb{R} \times \mathbb{R}^d$. This problem was studied in El Karoui and Quenez (1995) and in Cvitanic and Karatzas (1993), El Karoui et al (1997) for the problem of superhedging in a market with constrained portfolios, in Cvitanic, Karatzas and Soner (1998) for BSDE with a convex constraint and in Peng (1999) with an arbitrary closed constraint.

**1.6. *g*-expectation and *g*-martingales.** Let $\mathcal{E}^g_{t,T}[\xi]$ be the solution of a real valued BSDE (1.7), namely $m = 1$, for a given generator $g$ satisfying an additional assumption $g|_{z=0} \equiv 0$. Peng (1997b) studied this problem by introducing a notion of $g$-expectation:

$$\mathcal{E}^g[\xi] := \mathcal{E}^g_{0,T}[\xi] : \xi \in \bigcup_{T \geq 0} L^2_P(\mathcal{F}_T) \mapsto \mathbb{R}. \tag{1.19}$$

$\mathcal{E}^g$ is then a monotone functional preserving constants: $\mathcal{E}^g[c] = c$. A significant character of this nonlinear expectation is that, thanks to the backward semigroup properties of $\mathcal{E}^g_{s,t}$, it keeps all dynamic properties of classical linear expectations: the corresponding conditional expectation, given $\mathcal{F}_t$, is uniquely defined by $\mathcal{E}^g[\xi|\mathcal{F}_t] = \mathcal{E}^g_{t,T}[\xi]$. It satisfies:

$$\mathcal{E}^g[\mathcal{E}^g[\xi|\mathcal{F}_s]|\mathcal{F}_t] = \mathcal{E}^g[\xi|\mathcal{F}_{t\wedge s}], \quad \mathcal{E}^g[\mathbf{1}_A\xi|\mathcal{F}_t] = \mathbf{1}_A\mathcal{E}^g[\xi|\mathcal{F}_t], \quad \forall A \in \mathcal{F}_t. \tag{1.20}$$

This notion allows us to establish a nonlinear $g$-martingale theory, which plays the same important role as the martingale theory in the classical probability theory. An important theorem is the so-called $g$-supermartingale decomposition theorem obtained in Peng (1999). This theorem does not need to assume that $g|_{z=0} = 0$. It claims that, if $Y$ is a càdlàg $g$-supermartingale, namely,

$$\mathcal{E}^g_{t,T}[Y_T] \leq Y_t, \text{ a.s. } 0 \leq t \leq T,$$

then it has the following unique decomposition: there exists a unique predictable, increasing and càdlàg process $A$ such that $Y$ solves

$$-dY_t = g(t, Y_t, Z_t)dt + dA_t - Z_t dB_t.$$

In other words, $Y$ is a $g$-supersolution of type (1.16).

A theoretically very interesting and practically important question is: given a family of expectations $\mathcal{E}_{s,t}[\cdot] : L^2_P(\mathcal{F}_t) \mapsto L^2_P(\mathcal{F}_s)$, $0 \leq s \leq t < \infty$, satisfying the same backward dynamically consistent properties of a $g$-expectation (1.20), can we find a function $g$ such that $\mathcal{E}_{s,t} \equiv \mathcal{E}^g_{s,t}$? The first result was obtained in Coquet, Hu, Memin and Peng (2001) (see also lecture notes of a CIME course of Peng (2004a)): under an additional condition such that $\mathcal{E}$ is dominated by a $g_\mu$-expectation with $g_\mu(z) = \mu|z|$ for a large enough constant $\mu > 0$, namely

$$\mathcal{E}_{s,t}[\xi] - \mathcal{E}_{s,t}[\xi'] \leq \mathcal{E}^{g_\mu}_{s,t}[\xi - \xi'], \tag{1.21}$$

then there exists a unique function $g = g(t, \omega, z)$ satisfying

$$g(\cdot, z) \in M^2_P(0, T), \quad g(t, z) - g(t, z') \leq \mu|z - z'|, \quad z, z' \in \mathbb{R}^d,$$

such that $\mathcal{E}_{s,t}[\xi] \equiv \mathcal{E}^g_{s,t}[\xi]$, for all $\xi \in L^2_P(\mathcal{F}_t)$, $s \leq t$. For a concave dynamic expectation with an assumption much weaker than the above domination condition, we can still find a function $g = g(t, z)$ with possibly singular values, see Delbaen, Peng and Rosazza Gianin (2009). Peng (2005a) proved the case without the assumption of constant preservation, the domination condition of $\mathcal{E}^{g_\mu}$ was also weakened by $g_\mu = \mu(|y| + |z|)$. The result is: there is a unique function $g = g(t, \omega, y, z)$ such that $\mathcal{E}_{s,t} \equiv \mathcal{E}^g_{s,t}$, where $g$ is a Lipschitz function:

$$g(t, y, z) - g(t, y', z') \leq \mu(|y - y'| + |z - z'|), \quad y, y' \in \mathbb{R}, \quad z, z' \in \mathbb{R}^d.$$

In practice, the above criterion is very useful to test whether a dynamic pricing mechanism of contingent contracts can be represented by a concrete function $g$. Indeed, it is an important test in order to establish and maintain a system of dynamically consistent risk measure in finance as well as in other industrial domains. We have collected some data in financial markets and realized a large scale computation. The results of the test strongly support the criterion (1.21) (see Peng (2006b) with numerical calculations and data tests realized by Chen and Sun).

Chen, Chen and Davison (2005) proved that there is an essential difference between $g$-expectation and the well-known Choquet-expectation, which is obtained via the Choquet integral. Since $g$-expectation is essentially equivalent to a dynamical expectation under a Wiener probability space, their result seems to tell us that, in general, a nontrivially nonlinear Choquet expectation cannot be a dynamical one. This point of view is still to be clarified.

**1.7. BSDE applied in finance.** The above problem of constrained BSDE was motivated from hedging problem with constrained portfolios in a financial market. El Karoui et al (1997) initiated this BSDE approach in finance and stimulated many very interesting results. We briefly present a typical model of continuous asset pricing in a financial market: the basic securities consist of $1 + d$ assets, a riskless one, called bond, and $d$ risky securities, called stocks. Their prices are governed by

$$dP^0_t = P^0_t r dt, \quad \text{for the bond,}$$

and

$$dP_t^i = P_t^i \left[ b^i dt + \sum_{j=1}^{d} \sigma^{ij} dB_t^j \right], \quad \text{for the } i\text{th stock, } i = 1, \cdots, d.$$

Here we only consider the situation where the matrix $\sigma = (\sigma^{ij})_{i,j=1}^d$ is invertible. The degenerate case can be treated by constrained BSDE. We consider a small investor whose investment behavior cannot affect market prices and who invests at time $t \in [0, T]$ the amount $\pi_t^i$ of his wealth $Y_t$ in the $i$th security, for $i = 0, 1, \cdots, d$, thus $Y_t = \pi_t^0 + \cdots + \pi_t^d$. If his investment strategy is self-financing, then we have $dY_t = \sum_{i=0}^d \pi_t^i dP_t^i / P_t^i$, thus

$$dY_t = rY_t dt + \pi_t^* \sigma\theta dt + \pi_t^* \sigma dB_t, \quad \theta^i = \sigma^{-1}(b^i - r), \quad i = 1, \cdots, d.$$

Here we always assume that all involved processes are in $M_P^2(0, T)$. A strategy $(Y_t, \{\pi_t^i\}_{i=1}^d)_{t\in[0,T]}$ is said to be feasible if $Y_t \geq 0$, $t \in [0, T]$, a.s. A European contingent claim settled at time $T$ is a non-negative random variable $\xi \in L_P^2(\mathcal{F}_T)$. A feasible strategy $(Y, \pi)$ is called a hedging strategy against a contingent claim $\xi$ at the maturity $T$ if it satisfies:

$$dY_t = rY_t dt + \pi_t^* \sigma\theta dt + \pi_t^* \sigma dB_t, \quad Y_T = \xi.$$

Observe that $(Y, \pi^* \sigma)$ can be regarded as a solution of BSDE and the solution is automatically feasible by the comparison theorem (Theorem 1.2). It is called a super-hedging strategy if there exists an increasing process $K_t$, often called an accumulated consumption process, such that

$$dY_t = rY_t dt + \pi_t^* \sigma\theta dt + \pi_t^* \sigma dB_t - dK_t, \quad Y_T = \xi.$$

This type of strategy are often applied in a constrained market in which certain constraint $(Y_t, \pi_t) \in \Gamma$ are imposed. Observe that a real market has many frictions and constraints. An example is the common case where interest rate $R$ for borrowing money is higher than the bond rate $r$. The above equation for hedging strategy becomes

$$dY_t = rY_t dt + \pi_t^* \sigma\theta dt + \pi_t^* \sigma dB_t - (R - r) \left[ \sum_{i=1}^{d} \pi_t^i - Y_t \right]^+ dt, \quad Y_T = \xi,$$

where $[\cdot]^+ = \max\{[\cdot], 0\}$. A short selling constraint $\pi_t^i \geq 0$ is also very typical. The method of constrained BSDE can be applied to this type of problems. BSDE theory provides powerful tools to the robust pricing and risk measures for contingent claims. For more details see El Karoui et al. (1997). For the dynamic risk measure under Brownian filtration see Rosazza Gianin (2006), Delbaen et al (2009). Barrieu and El Karoui (2004) revealed the relation between the inf-convolution of dynamic convex risk measures and the corresponding one for the generators of the BSDE, Rouge and El Karoui (2000) solved a utility maximization problem by using a type of quadratic BSDEs. Hu, Imkeller and Müller (2005) further considered the problem under a non-convex portfolio constraint where BMO martingales play a key role. For investigations of BMO martingales in BSDE and dynamic nonlinear expectations see also Barrieu, Cazanave, and El Karoui (2008), Hu, Ma, Peng and Yao (2008) and Delbaen and Tang (2010).

There are still so many important issues on BSDE theory and its applications. The well-known paper of Chen and Epstein (2002) introduced a continuous time utility under probability model uncertainty using $g$-expectation. The Malliavin derivative of a solution of BSDE (see Pardoux and Peng (1992), El Karoui et al (1997)) leads to a very interesting relation $Z_t = D_t Y_t$. There are actually very active researches on numerical analysis and calculations of BSDE, see Douglas, Ma and Protter (1996), Ma and Zhang (2002), Zhanng (2004), Bouchard and Touzi (2004), Peng and Xu (2003), Gobet, Lemor and Warin (2005), Zhao et al (2006), Delarue and Menozzi (2006). We also refer to stochastic differential maximization and games with recursive or other utilities (see Peng (1997a), Pham (2004), Buckdahn and Li (2008)), Mean-field BSDE (see Buckdahn et al (2009)).

## 2. Nonlinear Expectations and Nonlinear Distributions

The notion of $g$ expectations introduced via BSDE can be used as an idea tool to treat the randomness and risk in the case of the uncertainty of probability models, see Chen and Epstein (2002), but with the following limitation: all the involved uncertain probability measures are absolutely continuous with respect to the "reference probability" $P$. But for the well-known problem of volatility model uncertainty in finance, there is an uncountable number of unknown probabilities which are singular from each other.

The notion of sublinear expectations is a powelful tool to solve this problem. We give a survey on the recent development of $G$-expectation theory. More details with proofs and historical remarks can be found in a book of Peng (2010a). For references of decision theory under uncertainty in economics, we refer to the collection of papers edited by Gilboa (2004).

**2.1. Sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$.** We define from a very basic level of a nonlinear expectation.

Let $\Omega$ be a given set. A vector lattice $\mathcal{H}$ is a linear space of real functions defined on $\Omega$ such that all constants are belonging to $\mathcal{H}$ and if $X \in \mathcal{H}$ then $|X| \in \mathcal{H}$. This lattice is often denoted by $(\Omega, \mathcal{H})$. An element $X \in \mathcal{H}$ is called a random variable.

We denote by $C_{Lat}(\mathbb{R}^n)$ the smallest lattice of real functions defined on $\mathbb{R}^n$ containing the following $n+1$ functions (i) $\varphi_0(x) \equiv c$, (ii) $\varphi_i(x) = x_i$, for $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, $i = 1, \cdots, n$.

We also use $C_{Lip}(\mathbb{R}^n)$ (resp. $C_{l.Lip}(\mathbb{R}^n)$) for the space of all Lipschitz (resp. locally Lipschitz) real functions on $\mathbb{R}^n$. It is clear that $C_{Lat}(\mathbb{R}^n) \subset C_{Lip}(\mathbb{R}^n) \subset C_{l.Lip}(\mathbb{R}^n)$. Any elements of $C_{l.Lip}(\mathbb{R}^n)$ can be locally uniformly approximated by a sequence in $C_{Lat}(\mathbb{R}^n)$.

It is clear that if $X_1, \cdots, X_n \in \mathcal{H}$, then $\varphi(X_1, \cdots, X_n) \in \mathcal{H}$, for each $\varphi \in C_{Lat}(\mathbb{R}^n)$.

**Definition 2.1.** *A* ***nonlinear expectation*** $\hat{\mathbb{E}}$ *defined on* $\mathcal{H}$ *is a functional* $\hat{\mathbb{E}} : \mathcal{H} \mapsto \mathbb{R}$ *satisfying the following properties for all* $X, Y \in \mathcal{H}$*:*

- Monotonicity: *If* $X \geq Y$ *then* $\hat{\mathbb{E}}[X] \geq \hat{\mathbb{E}}[Y]$.
- Constant preserving: $\hat{\mathbb{E}}[c] = c$.

$\hat{\mathbb{E}}$ *is called a* ***sublinear expectation*** *if it furthermore satisfies*

$$\hat{\mathbb{E}}[X + \lambda Y] \leq \hat{\mathbb{E}}[X] + \lambda \hat{\mathbb{E}}[Y], \ \forall X, Y \in \mathcal{H}, \ \lambda \geq 0.$$

*If it further satisfies* $\hat{\mathbb{E}}[-X] = -\hat{\mathbb{E}}[X]$ *for* $X \in \mathcal{H}$*, then* $\hat{\mathbb{E}}$ *is called a* ***linear expectation****. The triple* $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ *is called a* ***nonlinear (resp. sublinear, linear) expectation space.***

We are particularly interested in sublinear expectations. In statistics and economics, this type of functionals was studied by, among many others, Huber (1981) and then explored by Walley (1991).

Recently a new notion of coherent risk measures in finance caused much attention to the study of such type of sublinear expectations and applications to risk controls, see the seminal paper of Artzner, Delbaen, Eber and Heath (1999) as well as Föllmer and Schied (2004).

The following result is well-known as representation theorem. It is a direct consequence of Hahn-Banach theorem (see Delbaen (2002), Föllmer and Schied (2004), or Peng (2010a)).

**Theorem 2.2.** *Let* $\hat{\mathbb{E}}$ *be a sublinear expectation defined on* $(\Omega, \mathcal{H})$*. Then there exists a family of linear expectations* $\{E_\theta : \theta \in \Theta\}$ *on* $(\Omega, \mathcal{H})$ *such that*

$$\hat{\mathbb{E}}[X] = \max_{\theta \in \Theta} E_\theta[X].$$

A sublinear expectation $\hat{\mathbb{E}}$ on $(\Omega, \mathcal{H})$ is said to be regular if for each sequence $\{X_n\}_{n=1}^{\infty} \subset \mathcal{H}$ such that $X_n(\omega) \downarrow 0$, for $\omega$, we have $\hat{\mathbb{E}}[X_n] \downarrow 0$. If $\hat{\mathbb{E}}$ is regular then from the above representation we have $E_\theta[X_n] \downarrow 0$ for each $\theta \in \Theta$. It follows from Daniell-Stone theorem that there exists a unique ($\sigma$-additive) probability measure $P_\theta$ defined on $(\Omega, \sigma(\mathcal{H}))$ such that

$$E_\theta[X] = \int_\Omega X(\omega) dP_\theta(\omega), \quad X \in \mathcal{H}.$$

The above representation theorem of sublinear expectation tells us that to use a sublinear expectation for a risky loss $X$ is equivalent to take the upper expectation of $\{E_\theta : \theta \in \Theta\}$. The corresponding model uncertainty of probabilities, or ambiguity, is the subset $\{P_\theta : \theta \in \Theta\}$. The corresponding uncertainty of distributions for an $n$-dimensional random variable $X$ in $\mathcal{H}$ is

$$\{F_X(\theta, A) := P_\theta(X \in A) : A \in \mathcal{B}(\mathbb{R}^n)\}.$$

**2.2. Distributions and independence.** We now consider the notion of the distributions of random variables under sublinear expectations. Let $X = (X_1, \cdots, X_n)$ be a given $n$-dimensional random vector on a nonlinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$. We define a functional on $C_{Lat}(\mathbb{R}^n)$ by

$$\hat{\mathbb{F}}_X[\varphi] := \hat{\mathbb{E}}[\varphi(X)] : \varphi \in C_{Lat}(\mathbb{R}^n) \mapsto \mathbb{R}.$$

The triple $(\mathbb{R}^n, C_{Lat}(\mathbb{R}^n), \hat{\mathbb{F}}_X[\cdot])$ forms a nonlinear expectation space. $\hat{\mathbb{F}}_X$ is called the distribution of $X$. If $\hat{\mathbb{E}}$ is sublinear, then $\hat{\mathbb{F}}_X$ is also sublinear. Moreover, $\hat{\mathbb{F}}_X$ has the following representation: there exists a family of probability measures $\{F_X(\theta, \cdot)\}_{\theta \in \Theta}$ on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ such that

$$\hat{\mathbb{F}}_X[\varphi] = \sup_{\theta \in \Theta} \int_{\mathbb{R}^n} \varphi(x) F_X(\theta, dx), \text{ for each bounded continuous function } \varphi.$$

Thus $\hat{\mathbb{F}}_X$ indeed characterizes the distribution uncertainty of $X$.

Let $X_1$ and $X_2$ be two $n$–dimensional random vectors defined on nonlinear expectation spaces $(\Omega_1, \mathcal{H}_1, \hat{\mathbb{E}}_1)$ and $(\Omega_2, \mathcal{H}_2, \hat{\mathbb{E}}_2)$, respectively. They are called identically distributed, denoted by $X_1 \overset{d}{=} X_2$, if

$$\hat{\mathbb{E}}_1[\varphi(X_1)] = \hat{\mathbb{E}}_2[\varphi(X_2)], \quad \forall \varphi \in C_{Lat}(\mathbb{R}^n).$$

In this case $X_1$ is also said to be a copy of $X_2$. It is clear that $X_1 \overset{d}{=} X_2$ if and only if they have the same distribution uncertainty. We say that the distribution of $X_1$ is stronger than that of $X_2$ if $\hat{\mathbb{E}}_1[\varphi(X_1)] \geq \hat{\mathbb{E}}_2[\varphi(X_2)]$, for $\varphi \in C_{Lat}(\mathbb{R}^n)$. The meaning is that the distribution uncertainty of $X_1$ is stronger than that of $X_2$.

The distribution of $X \in \mathcal{H}$ has the following two typical parameters: the upper mean $\bar{\mu} := \hat{\mathbb{E}}[X]$ and the lower mean $\underline{\mu} := -\hat{\mathbb{E}}[-X]$. If $\bar{\mu} = \underline{\mu}$ then we say that $X$ has no mean uncertainty.

In a nonlinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ a random vector $Y = (Y_1, \cdots, Y_n)$, $Y_i \in \mathcal{H}$ is said to be independent from another random vector $X =$

$(X_1, \cdots, X_m)$, $X_i \in \mathcal{H}$ under $\hat{\mathbb{E}}[\cdot]$ if for each test function $\varphi \in C_{Lat}(\mathbb{R}^m \times \mathbb{R}^n)$ we have

$$\hat{\mathbb{E}}[\varphi(X, Y)] = \hat{\mathbb{E}}[\hat{\mathbb{E}}[\varphi(x, Y)]_{x=X}].$$

Under a sublinear expectation $\hat{\mathbb{E}}$, the independence of $Y$ from $X$ means that the uncertainty of distributions of $Y$ does not change with each realization of $X = x$, $x \in \mathbb{R}^n$. It is important to note that under nonlinear expectations the condition "$Y$ is independent from $X$" does not imply automatically that "$X$ is independent from $Y$".

A sequence of $d$-dimensional random vectors $\{\eta_i\}_{i=1}^{\infty}$ in a nonlinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is said to converge in distribution (or in law) under $\hat{\mathbb{E}}$ if for each $\varphi \in C_b(\mathbb{R}^n)$ the sequence $\{\hat{\mathbb{E}}[\varphi(\eta_i)]\}_{i=1}^{\infty}$ converges, where $C_b(\mathbb{R}^n)$ denotes the space of all bounded and continuous functions on $\mathbb{R}^n$. In this case it is easy to check that the functional defined by

$$\hat{\mathbb{F}}[\varphi] := \lim_{i \to \infty} \hat{\mathbb{E}}[\varphi(\eta_i)], \quad \varphi \in C_b(\mathbb{R}^n)$$

forms a nonlinear expectation on $(\mathbb{R}^n, C_b(\mathbb{R}^n))$. If $\hat{\mathbb{E}}$ is a sublinear (resp. linear) expectation, then $\hat{\mathbb{F}}$ is also sublinear (resp. linear).

### 2.3. Normal distributions under a sublinear expectation.

We begin by defining a special type of distribution, which plays the same role as the well-known normal distribution in classical probability theory and statistics. Recall the well-known classical characterization: $X$ is a zero mean normal distribution, i.e., $X \overset{d}{=} N(0, \Sigma)$ if and only if

$$aX + bX' \overset{d}{=} \sqrt{a^2 + b^2} X, \quad \text{for } a, b \geq 0,$$

where $X'$ is an independent copy of $X$. The covariance matrix $\Sigma$ is defined by $\Sigma = E[XX^*]$.

We now consider the so-called $G$-normal distribution under a sublinear expectation space. A $d$-dimensional random vector $X = (X_1, \cdots, X_d)$ in a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called $G$**-normally distributed** with zero mean if for each $a$, $b \geq 0$ we have

$$aX + b\bar{X} \overset{d}{=} \sqrt{a^2 + b^2} X, \quad \text{for } a, b \geq 0, \tag{2.1}$$

where $\bar{X}$ is an independent copy of $X$.

It is easy to check that, if $X$ satisfies (2.1), then any linear combination of $X$ also satisfies (2.1). From $\hat{\mathbb{E}}[X_i + \bar{X}_i] = 2\hat{\mathbb{E}}[X_i]$ and $\hat{\mathbb{E}}[X_i + \bar{X}_i] = \hat{\mathbb{E}}[\sqrt{2}X_i] = \sqrt{2}\hat{\mathbb{E}}[X_i]$ we have $\hat{\mathbb{E}}[X_i] = 0$, and similarly, $\hat{\mathbb{E}}[-X_i] = 0$ for $i = 1, \cdots, d$.

We denote by $\mathbb{S}(d)$ the linear space of all $d \times d$ symmetric matrices and by $\mathbb{S}_+(d)$ all non-negative elements in $\mathbb{S}(d)$. We will see that the distribution of $X$ is characterized by a sublinear function $G : \mathbb{S}(d) \mapsto \mathbb{R}$ defined by

$$G(A) = G_X(A) := \frac{1}{2}\hat{\mathbb{E}}[\langle AX, X \rangle], \quad A \in \mathbb{S}(d). \tag{2.2}$$

It is easy to check that $G$ is a sublinear and monotone function on $\mathbb{S}(d)$. Thus there exists a bounded and closed subset $\Theta$ in $\mathbb{S}_+(d)$ such that (see e.g. Peng (2010a))

$$\frac{1}{2}\hat{\mathbb{E}}[\langle AX, X\rangle] = \hat{G}(A) = \frac{1}{2}\max_{Q\in\Theta} tr[AQ], \quad A \in \mathbb{S}(d). \tag{2.3}$$

If $\Theta$ is a singleton: $\Theta = \{Q\}$, then $X$ is normally distributed in the classical sense, with mean zero and covariance $Q$. In general $\Theta$ characterizes the covariance uncertainty of $X$. We denote $X \overset{d}{=} N(\{0\} \times \Theta)$.

A $d$-dimensional random vector $Y = (Y_1, \cdots, Y_d)$ in a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called **maximally distributed** if we have

$$a^2 Y + b^2 \bar{Y} \overset{d}{=} (a^2 + b^2)Y, \quad \forall a, b \in \mathbb{R}, \tag{2.4}$$

where $\bar{Y}$ is an independent copy of $Y$. A maximally distributed $Y$ is characterized by a sublinear function $g = g_Y(p) : \mathbb{R}^d \mapsto \mathbb{R}$ defined by

$$g_Y(p) := \hat{\mathbb{E}}[\langle p, Y\rangle], \quad p \in \mathbb{R}^d. \tag{2.5}$$

It is easy to check that $g$ is a sublinear function on $\mathbb{R}^d$. Thus, as for (2.3), there exists a bounded closed and convex subset $\bar{\Theta} \in \mathbb{R}^d$ such that

$$g(p) = \sup_{q\in\bar{\Theta}} \langle p, q\rangle, \quad p \in \mathbb{R}^d. \tag{2.6}$$

It can be proved that the maximal distribution of $Y$ is given by

$$\hat{\mathbb{F}}_Y[\varphi] = \hat{\mathbb{E}}[\varphi(Y)] = \max_{v\in\bar{\Theta}} \varphi(v), \quad \varphi \in C_{Lat}(\mathbb{R}^d).$$

We denote $Y \overset{d}{=} N(\bar{\Theta} \times \{0\})$.

The above two types of distributions can be nontrivially combined together to form a new distribution. We consider a pair of random vectors $(X, Y) \in \mathcal{H}^{2d}$ where $X$ is $G$-normally distributed and $Y$ is maximally distributed.

In general, a pair of $d$-dimensional random vectors $(X, Y)$ in a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called $G$-distributed if for each $a$ , $b \geq 0$ we have

$$(aX + b\bar{X}, a^2 Y + b^2 \bar{Y}) \overset{d}{=} (\sqrt{a^2 + b^2} X, (a^2 + b^2)Y\ ), \quad \forall a, b \geq 0, \tag{2.7}$$

where $(\bar{X}, \bar{Y})$ is an independent copy of $(X, Y)$.

The distribution of $(X, Y)$ can be characterized by the following function:

$$G(p, A) := \hat{\mathbb{E}}\left[\frac{1}{2}\langle AX, X\rangle + \langle p, Y\rangle\right], \quad (p, A) \in \mathbb{R}^d \times \mathbb{S}(d). \tag{2.8}$$

It is easy to check that $G : \mathbb{R}^d \times \mathbb{S}(d) \mapsto \mathbb{R}$ is a sublinear function which is monotone in $A \in \mathbb{S}(d)$. Clearly $G$ is also a continuous function. Therefore there exists a bounded and closed subset $\Gamma \subset \mathbb{R}^d \times \mathbb{S}_+(d)$ such that

$$G(p, A) = \sup_{(q,Q)\in\Gamma} \left[\frac{1}{2}\mathrm{tr}[AQ] + \langle p, q\rangle\right], \quad \forall (p, A) \in \mathbb{R}^d \times \mathbb{S}(d). \tag{2.9}$$

The following result tells us that for each such type of function $G$, there exists a unique $G$-normal distribution.

**Proposition 2.3.** *(Peng (2008b, Proposition 4.2)) Let $G : \mathbb{R}^d \times \mathbb{S}(d) \mapsto \mathbb{R}$ be a given sublinear function which is monotone in $A \in \mathbb{S}(d)$, i.e., $G$ has the form of (2.9). Then there exists a pair of $d$-dimensional random vectors $(X, Y)$ in some sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ satisfying (2.7) and (2.8). The distribution of $(X, Y)$ is uniquely determined by the function $G$. Moreover the function $u$ defined by*

$$u(t, x, y) := \hat{\mathbb{E}}[\varphi(x + \sqrt{t}X, y + tY)], \ (t, x, y) \in [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d, \tag{2.10}$$

*for each given $\varphi \in C_{Lat}(\mathbb{R}^{2d})$, is the unique (viscosity) solution of the parabolic PDE*

$$\partial_t u - G(D_y u, D_x^2 u) = 0, \quad u|_{t=0} = \varphi, \tag{2.11}$$

*where $D_y = (\partial_{y_i})_{i=1}^d$, $D_x^2 = (\partial^2_{x_i, x_j})_{i,j=1}^d$.*

In general, to describe a possibly degenerate PDE of type (2.11), one needs the notion of viscosity solutions. But readers also can only consider non-degenerate situations (under strong elliptic condition). Under such condition, equation (2.11) has a unique smooth solution $u \in C^{1+\frac{\alpha}{2}, 2+\alpha}((0, \infty) \times \mathbb{R}^d)$ (see Krylov (1987) and Wang (1992)). The notion of viscosity solution was introduced by Crandall and Lions. For the existence and uniqueness of solutions and related very rich references we refer to a systematic guide of Crandall, Ishii and Lions (1992) (see also the Appendix of Peng (2007b, 2010a) for more specific parabolic cases). In the case where $d = 1$ and $G$ contains only the second order derivative $D_x^2 u$, the $G$-heat equation is the well-known Baronblatt equation (see Avellanaeda, Levy and Paras (1995)).

If both $(X, Y)$ and $(\bar{X}, \bar{Y})$ are $G$-normal distributed with the same $G$, i.e.,

$$G(p, A) := \hat{\mathbb{E}}\left[\frac{1}{2}\langle AX, X\rangle + \langle p, Y\rangle\right] = \hat{\mathbb{E}}\left[\frac{1}{2}\left\langle A\bar{X}, \bar{X}\right\rangle + \left\langle p, \bar{Y}\right\rangle\right],$$
$$\forall (p, A) \in \mathbb{S}(d) \times \mathbb{R}^d,$$

then $(X, Y) \overset{d}{=} (\bar{X}, \bar{Y})$. In particular, $X \overset{d}{=} -X$.

Let $(X, Y)$ be $G$-normally distributed. For each $\psi \in C_{Lat}(\mathbb{R}^d)$ we define a function

$$v(t, x) := \hat{\mathbb{E}}[\psi(x + \sqrt{t}X + tY)], \ (t, x) \in [0, \infty) \times \mathbb{R}^d.$$

Then $v$ is the unique solution of the following parabolic PDE

$$\partial_t v - G(D_x v, D_x^2 v) = 0, \qquad v|_{t=0} = \psi. \tag{2.12}$$

Moreover we have $v(t, x + y) \equiv u(t, x, y)$, where $u$ is the solution of the PDE (2.11) with initial condition $u(t, x, y)|_{t=0} = \psi(x + y)$.

**2.4. Central limit theorem and law of large numbers.** We have a generalized central limit theorem together with the law of large numbers:

**Theorem 2.4.** *(**Central Limit Theorem,** Peng (2007a, 2010a)) Let $\{(X_i, Y_i)\}_{i=1}^{\infty}$ be a sequence of $\mathbb{R}^d \times \mathbb{R}^d$-valued random variables in $(\mathcal{H}, \hat{\mathbb{E}})$. We assume that $(X_{i+1}, Y_{i+1}) \overset{d}{=} (X_i, Y_i)$ and $(X_{i+1}, Y_{i+1})$ is independent from $\{(X_1, Y_1), \cdots, (X_i, Y_i)\}$ for each $i = 1, 2, \cdots$. We further assume that $\hat{\mathbb{E}}[X_1] = \hat{\mathbb{E}}[-X_1] = 0$ and $\hat{\mathbb{E}}[|X_1|^{2+\delta}] + \hat{\mathbb{E}}[|Y_1|^{1+\delta}] < \infty$ for some fixed $\delta > 0$. Then the sequence $\{\bar{S}_n\}_{n=1}^{\infty}$ defined by $\bar{S}_n := \sum_{i=1}^n (\frac{X_i}{\sqrt{n}} + \frac{Y_i}{n})$ converges in law to $\xi + \zeta$:*

$$\lim_{n\to\infty} \hat{\mathbb{E}}[\varphi(\bar{S}_n)] = \hat{\mathbb{E}}[\varphi(\xi + \zeta)], \tag{2.13}$$

*for all functions $\varphi \in C(\mathbb{R}^d)$ satisfying a linear growth condition, where $(\xi, \zeta)$ is a pair of $G$-normal distributed random vectors and where the sublinear function $G : \mathbb{S}(d) \times \mathbb{R}^d \mapsto \mathbb{R}$ is defined by*

$$G(p, A) := \hat{\mathbb{E}}\left[\langle p, Y_1\rangle + \frac{1}{2}\langle AX_1, X_1\rangle\right], \quad A \in \mathbb{S}(d), \quad p \in \mathbb{R}^d.$$

The proof of this theorem given in Peng (2010) is very different from the classical one. It based on a deep $C^{1,2}$-estimate of solutions of fully nonlinear parabolic PDEs initially given by Krylov (1987) (see also Wang (1992)). Peng (2010b) then introduced another proof, involving a nonlinear version of weak compactness based on a nonlinear version of tightness.

**Corollary 2.5.** *The sum $\sum_{i=1}^n \frac{X_i}{\sqrt{n}}$ converges in law to $N(\{0\} \times \hat{\Theta})$, where the subset $\hat{\Theta} \subset \mathbb{S}_+(d)$ is defined in (2.3) for $\hat{G}(A) = G(0, A)$, $A \in \mathbb{S}(d)$. The sum $\sum_{i=1}^n \frac{Y_i}{n}$ converges in law to $N(\bar{\Theta} \times \{0\})$, where the subset $\bar{\Theta} \subset \mathbb{R}^d$ is defined in (2.6) for $\bar{G}(p) = G(p, 0)$, $p \in \mathbb{R}^d$. If we take, in particular, $\varphi(y) = d_{\bar{\Theta}}(y) = \inf\{|x-y| : x \in \bar{\Theta}\}$, then we have the following generalized law of large numbers:*

$$\lim_{n\to\infty} \hat{\mathbb{E}}\left[d_{\bar{\Theta}}\left(\sum_{i=1}^n \frac{Y_i}{n}\right)\right] = \sup_{\theta\in\bar{\Theta}} d_{\bar{\Theta}}(\theta) = 0. \tag{2.14}$$

If $Y_i$ has no mean-uncertainty, or in other words, $\bar{\Theta}$ is a singleton: $\bar{\Theta} = \{\bar{\theta}\}$ then (2.14) becomes $\lim_{n\to\infty} \hat{\mathbb{E}}[|\sum_{i=1}^n \frac{Y_i}{n} - \bar{\theta}|] = 0$. To our knowledge, the law of large numbers with non-additive probability measures have been investigated under a framework and approach quite different from ours, where no convergence in law is obtained (see Marinacci (1999) and Maccheroni & Marinacci (2005)). For a strong version of LLN under our new framework of independence, see Chen (2010).

**2.5. Sample based sublinear expectations.** One may feel that the notion of the distribution of a $d$-dimensional random variable $X$ introduced through $\hat{\mathbb{E}}[\varphi(X)]$ is somewhat abstract and complicated. But in practice this maybe the simplest way for applications: in many cases what we want to get from the distribution of $X$ is basically the expectation of $\varphi(X)$. Here $\varphi$ can be a financial contract, e.g., a call option $\varphi(x) = \max\{0, x - k\}$, a consumer's utility function, a cost function in optimal control problems, etc. In a classical probability space $(\Omega, \mathcal{F}, P)$, we can use the classical LLN to calculate $E[\varphi(X)]$, by using

$$E[\varphi(X)] = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \varphi(x_i),$$

where $x_i$ , $i = 1, 2, \cdots$ is an i.i.d. sample from the random variable $X$. This means that in practice we can use the mean operator

$$\mathbb{M}[\varphi(X)] := \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \varphi(x_i) : C_{Lat}(\mathbb{R}^d) \mapsto \mathbb{R}$$

to obtain the distribution of $X$. This defines what we call "sample distribution of $X$". In fact the well-known Monté-Carlo approach is based on this convergence.

We are interested in the corresponding situation in a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$. Let $x_i$, $i = 1, 2, \cdots$ be an i.i.d. sample from $X$, meaning that $x_i \overset{d}{=} X$ and $x_{i+1}$ is independent from $x_1, \cdots, x_i$ under $\hat{\mathbb{E}}$. Under this much weaker assumption we have that $\frac{1}{n}\sum_{i=1}^{n}\varphi(x_i)$ converges in law to a maximal distribution $N([\underline{\mu}, \overline{\mu}] \times \{0\})$, with $\overline{\mu} = \hat{\mathbb{E}}[\varphi(X)]$ and $\underline{\mu} = -\hat{\mathbb{E}}[-\varphi(X)]$. A direct meaning of this result is that, when $n \to \infty$, the number $\frac{1}{n}\sum_{i=1}^{n}\varphi(x_i)$ can take any value inside $[\underline{\mu}, \overline{\mu}]$. Then we can calculate $\hat{\mathbb{E}}[\varphi(X)]$ by introducing the following upper limit mean operator of $\{\varphi(x_i)\}_{i=1}^{\infty}$:

$$\hat{\mathbb{M}}_{\{x_i\}}[\varphi] := \limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \varphi(x_i), \quad \varphi \in C_{b.Lat}(\mathbb{R}^d).$$

On the other hand, it is easy to check that for any arbitrarily given sequence of data $\{x_i\}_{i=1}^{\infty}$, the above defined $\hat{\mathbb{M}}_{\{x_i\}}[\varphi]$ still forms a sublinear expectation on $(\mathbb{R}^d, C_{b.Lat}(\mathbb{R}))$. We call $\hat{\mathbb{M}}_{\{x_i\}}$ the sublinear distribution of the data $\{x_i\}_{i=1}^{\infty}$. $\hat{\mathbb{M}}_{\{x_i\}}$ gives us the statistics and statistical uncertainty of the random data $\{x_i\}_{i=1}^{\infty}$. This also provides a new "nonlinear Monté-Carlo" approach (see Peng (2009)).

In the case where $\hat{\mathbb{M}}_{\{x_i\}}[\varphi] < \infty$ for $\varphi(x) \equiv |x|$, we can prove that $\hat{\mathbb{M}}_{\{x_i\}}[\varphi]$ is also well-defined for $\varphi \in \mathbb{L}^{\infty}(\mathbb{R}^d)$. This allows us to calculate the capacity $\hat{c}(B) := \hat{\mathbb{M}}_{\{x_i\}}[\mathbf{1}_B]$, $B \in \mathcal{B}(\mathbb{R}^d)$, of $\{x_i\}_{i=1}^{\infty}$ which is the "upper relative frequency" of $\{x_i\}_{i=1}^{\infty}$ in $B$.

For a sample with relatively finite size $\{x_i\}_{i=1}^{N}$, we can also introduce the following form of sublinear expectation:

$$\hat{\mathbb{F}}[\varphi] := \sup_{\theta\in\Theta} \sum_{i=1}^{N} p_i(\theta)\varphi(x_i), \text{ with } p_i(\theta) \geq 0, \ \sum_{i=1}^{N} p_i(\theta) = 1.$$

Here $\{(p_i(\theta))_{i=1}^N : \theta \in \Theta\}$ is regarded as the subset of distribution uncertainty. Conversely, from the representation theorem of sublinear expectation, each sublinear expectation based on a sample $\{x_i\}_{i=1}^N$ also has the above representation.

In many cases we are concerned with some $\mathbb{R}^d$-valued continuous time data $(x_t)_{t\geq 0}$. It's upper mean expectation can be defined by

$$\hat{\mathbb{M}}_{(x_t)}[\varphi] = \limsup_{T\to\infty} \frac{1}{T}\int_0^T \varphi(x_t)dt, \quad \varphi \in C_{Lat}(\mathbb{R}^d),$$

or, in some circumstances,

$$\hat{\mathbb{M}}_{(x_t)}[\varphi] = \limsup_{T\to\infty} \int_0^T \varphi(x_t)\mu_T(dt),$$

where, for each $T > 0$, $\mu_T(\cdot)$ is a given non-negative measure on $([0,T], \mathcal{B}([0,T]))$ with $\mu_T([0,T]) = 1$. $\hat{\mathbb{M}}_{(x_t)}$ also forms a sublinear expectations on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. This notion also links many other research domains such as dynamical systems, particle systems.

## 3. *G*-Brownian Motion and its Stochastic Calculus

**3.1. Brownian motion under a sublinear expectation.** In this section we discuss $G$-Brownian motion under a nonlinear expectation, called $G$-expectation which is a natural generalization of $g$-expectation to a fully nonlinear case, i.e., the martingale under $G$-expectation is in fact a path-dependence solution of fully nonlinear PDE, whereas $g$-martingale corresponds to a quasilinear one. $G$-martingale is very useful to measure the risk of path-dependent financial products.

We introduce the notion of Brownian motion related to the $G$-normal distribution in a space of a sublinear expectation. We first give the definition of the $G$-Brownian motion introduced in Peng (2006a). For simplification we only consider 1-dimensional $G$-Brownian motion. Multidimensional case can be found in Peng (2008a, 2010a).

**Definition 3.1.** *A process $\{B_t(\omega)\}_{t\geq 0}$ in a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a* ***Brownian motion*** *under $\hat{\mathbb{E}}$ if for each $n \in \mathbb{N}$ and $0 \leq t_1, \cdots, t_n < \infty$, $B_{t_1}, \cdots, B_{t_n} \in \mathcal{H}$ and the following properties are satisfied:*

*(i)* $B_0(\omega) = 0$,

*(ii) For each $t, s \geq 0$, the increments satisfy $B_{t+s} - B_t \overset{d}{=} B_s$ and $B_{t+s} - B_t$ is independent from $(B_{t_1}, B_{t_2}, \cdots, B_{t_n})$, for each $0 \leq t_1 \leq \cdots \leq t_n \leq t$.*

*(iii)* $|B_t|^3 \in \mathcal{H}$ *and* $\hat{\mathbb{E}}[|B_t|^3]/t \to 0$ *as* $t \downarrow 0$.

*$B$ is called a symmetric Brownian motion if $\hat{\mathbb{E}}[B_t] = -\hat{\mathbb{E}}[-B_t] = 0$. If moreover, there exists a nonlinear expectation $\widetilde{\mathbb{E}}$ defined on $(\Omega, \mathcal{H})$ dominated by $\hat{\mathbb{E}}$, namely,*

$$\widetilde{\mathbb{E}}[X] - \widetilde{\mathbb{E}}[Y] \leq \hat{\mathbb{E}}[X - Y], \quad X, Y \in \mathcal{H}$$

*and such that the above condition (ii) also holds for $\widetilde{\mathbb{E}}$, then $B$ is also called a Brownian motion under $\widetilde{\mathbb{E}}$.*

Condition (iii) is to ensure that $B$ has continuous trajectories. Without this condition, $B$ may become a $G$-Lévy process (see Hu and Peng (2009b)).

**Theorem 3.2.** *Let $(B_t)_{t\geq 0}$ be a symmetric $G$-Brownian motion defined on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$. Then $B_t/\sqrt{t} \overset{d}{=} N(0, [\underline{\sigma}^2, \overline{\sigma}^2])$ with $\overline{\sigma}^2 = \hat{\mathbb{E}}[\tilde{B}_1^2]$ and $\underline{\sigma}^2 = -\hat{\mathbb{E}}[-\tilde{B}_1^2]$. Moreover, if $\underline{\sigma}^2 = \overline{\sigma}^2 > 0$, then the finite dimensional distribution of $(B_t/\overline{\sigma})_{t\geq 0}$ coincides with that of classical one dimensional standard Brownian motion.*

A Brownian motion under a sublinear expectation space is often called a *$G$-Brownian motion*. Here the letter $G$ indicates that the $B_t$ is $G$-normal distributed with

$$G(\alpha) := \frac{1}{2}\hat{\mathbb{E}}[\alpha B_1^2], \qquad \alpha \in \mathbb{R}.$$

We can prove that, for each $\lambda > 0$ and $t_0 > 0$, both $(\lambda^{-\frac{1}{2}} B_{\lambda t})_{t\geq 0}$ and $(B_{t+t_0} - B_{t_0})_{t\geq 0}$ are symmetric $G$-Brownian motions with the same generating function $G$. That is, a $G$-Brownian motion enjoys the same type of scaling as in the classical situation.

## 3.2. Construction of a $G$-Brownian motion.

Since each increment of a $G$-Brownian motion $B$ is $G$-normal distributed, a natural way to construct this process is to follow Kolmogorov's method: first, establish the finite dimensional (sublinear) distribution of $B$ and then take a completion. The completion will be in the next subsection.

We briefly explain how to construct a symmetric $G$-Brownian. More details were given in Peng (2006a, 2010a). Just as at the beginning of this paper, we denote by $\Omega = C([0, \infty))$ the space of all real–valued continuous paths $(\omega_t)_{t\in\mathbb{R}^+}$ with $\omega_0 = 0$, by $L^0(\Omega)$ the space of all $\mathcal{B}(\Omega)$-measurable functions and by $C_b(\Omega)$ all bounded and continuous functions on $\Omega$. For each fixed $T \geq 0$, we consider the following space of random variables:

$$\mathcal{H}_T = C_{Lat}(\Omega_T) := \{X(\omega) = \varphi(\omega_{t_1\wedge T}, \cdots, \omega_{t_m\wedge T}),\ \forall m \geq 1, \varphi \in C_{l.Lat}(\mathbb{R}^m)\},$$

where $C_{l.Lat}(\mathbb{R}^m)$ is the smallest lattice on $\mathbb{R}^m$ containing $C_{Lat}(\mathbb{R}^m)$ and all polynomials of $x \in \mathbb{R}^m$. It is clear that $C_{Lat}(\Omega_t) \subseteq C_{Lat}(\Omega_T)$, for $t \leq T$. We also denote

$$\mathcal{H} = C_{Lat}(\Omega) := \bigcup_{t\geq 0}^{\infty} C_{Lat}(\Omega_t).$$

We will consider the canonical space and set $B_t(\omega)=\omega_t$, $t\in[0,\infty)$, for $\omega\in\Omega$. Then it remains to introduce a sublinear expectation $\hat{\mathbb{E}}$ on $(\Omega,\mathcal{H})$ such that $B$ is a $G$-Brownian motion, for a given sublinear function $G(a)=\frac{1}{2}(\underline{\sigma}^2a^+-\overline{\sigma}^2a^-)$, $a\in\mathbb{R}$. Let $\{\xi_i\}_{i=1}^\infty$ be a sequence of $G$-normal distributed random variables in some sublinear expectation space $(\bar{\Omega},\bar{\mathcal{H}},\bar{\mathbb{E}})$: such that $\xi_i\overset{d}{=}N(\{0\}\times[\underline{\sigma}^2,\overline{\sigma}^2])$ and such that $\xi_{i+1}$ is independent from $(\xi_1,\cdots,\xi_i)$ for each $i=1,2,\cdots$. For each $X\in\mathcal{H}$ of the form

$$X=\varphi(B_{t_1}-B_{t_0},B_{t_2}-B_{t_1},\cdots,B_{t_m}-B_{t_{m-1}})$$

for some $\varphi\in C_{l.Lat}(\mathbb{R}^m)$ and $0=t_0<t_1<\cdots<t_m<\infty$, we set

$$\hat{\mathbb{E}}[X]=\bar{\mathbb{E}}[\varphi(\sqrt{t_1-t_0}\xi_1,\cdots,\sqrt{t_m-t_{m-1}}\xi_m)],$$

and

$$\hat{\mathbb{E}}_{t_k}[X]=\Phi(B_{t_1},\cdots,B_{t_k}-B_{t_{k-1}}),\quad\text{where}$$
$$\Phi(x_1,\cdots,x_k)=\bar{\mathbb{E}}[\varphi(x_1,\cdots,x_k,\sqrt{t_{k+1}-t_k}\xi_{k+1},\cdots,\sqrt{t_m-t_{m-1}}\xi_m)].$$

It is easy to check that $\hat{\mathbb{E}}:\mathcal{H}\mapsto\mathbb{R}$ consistently defines a sublinear expectation on $(\Omega,\mathcal{H})$ and $(B_t)_{t\geq 0}$ is a (symmetric) $G$-Brownian motion in $(\Omega,\mathcal{H},\hat{\mathbb{E}})$. In this way we have also defined the conditional expectations $\hat{\mathbb{E}}_t:\mathcal{H}\mapsto\mathcal{H}_t$, $t\geq 0$, satisfying

**(a')** If $X\geq Y$, then $\hat{\mathbb{E}}_t[X]\geq\hat{\mathbb{E}}_t[Y]$.

**(b')** $\hat{\mathbb{E}}_t[\eta]=\eta$, for each $t\in[0,\infty)$ and $\eta\in C_{Lat}(\Omega_t)$.

**(c')** $\hat{\mathbb{E}}_t[X]+\hat{\mathbb{E}}_t[Y]\leq\hat{\mathbb{E}}_t[X+Y]$.

**(d')** $\hat{\mathbb{E}}_t[\eta X]=\eta^+\hat{\mathbb{E}}_t[X]+\eta^-\hat{\mathbb{E}}_t[-X]$, for each $\eta\in C_{Lat}(\Omega_t)$.

Moreover, we have

$$\hat{\mathbb{E}}_t[\hat{\mathbb{E}}_s[X]]=\hat{\mathbb{E}}_{t\wedge s}[X],\ \text{ in particular }\quad\hat{\mathbb{E}}[\hat{\mathbb{E}}_t[X]]=\hat{\mathbb{E}}[X].$$

**3.3. *G*-Brownian motion in a complete sublinear expectation space.** Our construction of a $G$-Brownian motion is very simple. But to obtain the corresponding Itô's calculus we need a completion of the space $\mathcal{H}$ under a natural Banach norm. Indeed, for each $p\geq 1$, $\|X\|_p:=\hat{\mathbb{E}}[|X|^p]^{\frac{1}{p}}$, $X\in C_{Lat}(\Omega_T)$ (respectively, $C_{Lat}(\Omega)$) forms a norm under which $C_{Lat}(\Omega_T)$ (resp. $C_{Lat}(\Omega)$) can be continuously extended to a Banach space, denoted by

$$\mathcal{H}_T=L_G^p(\Omega_T)\quad\text{ (resp. }\mathcal{H}=L_G^p(\Omega)).$$

For each $0\leq t\leq T<\infty$ we have $L_G^p(\Omega_t)\subseteq L_G^p(\Omega_T)\subset L_G^p(\Omega)$. It is easy to check that, in $L_G^p(\Omega_T)$ (respectively, $L_G^p(\Omega)$), the extension of $\hat{\mathbb{E}}[\cdot]$ and its conditional expectations $\hat{\mathbb{E}}_t[\cdot]$ are still sublinear expectation and conditional expectations on $(\Omega, L_G^p(\Omega))$. For each $t \geq 0$, $\hat{\mathbb{E}}_t[\cdot]$ can also be extended as a continuous mapping $\hat{\mathbb{E}}_t[\cdot] : L_G^1(\Omega) \mapsto L_G^1(\Omega_t)$. It enjoys the same type of properties as $\hat{\mathbb{E}}_t[\cdot]$ defined on $\mathcal{H}_t$.

There are mainly two approaches to introduce $L_G^p(\Omega)$, one is the above method of finite dimensional nonlinear distributions, introduced in Peng (2005b: for more general nonlinear Markovian case, 2006a: for $G$-Brownian motion). The second one is to take a super-expectation with respect to the related family of probability measures, see Denis and Martini (2006) (a similar approach was introduced in Peng (2004) to treat more nonlinear Markovian processes). They introduced $\hat{c}$-quasi surely analysis, which is a very powerful tool. These two approaches were unified in Denis, Hu and Peng (2008), see also Hu and Peng (2009a).

**3.4. $L_G^p(\Omega)$ is a subspace of measurable functions on $\Omega$.** The following result was established in Denis, Hu and Peng (2008), a simpler and more direct argument was then obtained in Hu and Peng (2009a).

**Theorem 3.3.** *We have*

*(i) There exists a family of ($\sigma$-additive) probability measures $\mathcal{P}_G$ defined on $(\Omega, \mathcal{B}(\Omega))$, which is weakly relatively compact, $P$ and $Q$ are mutually singular from each other for each different $P, Q \in \mathcal{P}_G$ and such that*

$$\hat{\mathbb{E}}[X] = \sup_{P \in \mathcal{P}_G} E_P[X] = \sup_{P \in \mathcal{P}_G} \int_\Omega X(\omega) dP, \quad \textit{for each } X \in C_{Lat}(\Omega).$$

*Let $\hat{c}$ be the Choquet capacity induced by*

$$\hat{c}(A) = \hat{\mathbb{E}}[\mathbf{1}_A] = \sup_{P \in \mathcal{P}_G} E_P[\mathbf{1}_A], \ \textit{for } A \in \mathcal{B}(\Omega).$$

*(ii) Let $C_b(\Omega)$ be the space of all bounded and continuous functions on $\Omega$; $L^0(\Omega)$ be the space of all $\mathcal{B}(\Omega)$-measurable functions and let*

$$\mathbb{L}^p(\Omega) := \left\{ X \in L^0(\Omega) : \sup_{P \in \mathcal{P}_G} E_P[|X|^p] < \infty \right\}, \ p \geq 1.$$

*Then every element $X \in L_G^p(\Omega)$ has a $\hat{c}$-quasi continuous version, namely, there exists a $Y \in L_G^p(\Omega)$, with $X = Y$, quasi-surely such that, for each $\varepsilon > 0$, there is an open set $O \subset \Omega$ with $\hat{c}(O) < \varepsilon$ such that $Y|_{O^c}$ is continuous. We also have $\mathbb{L}^p(\Omega) \supset L_G^p(\Omega) \supset C_b(\Omega)$. Moreover,*

$$L_G^p(\Omega) = \{X \in \mathbb{L}^p(\Omega) : X \textit{ has a } \hat{c}\textit{-quasi-continuous version and } \lim_{n\to\infty} \hat{\mathbb{E}}[|X|^p \mathbf{1}_{\{|X|>n\}}] = 0\}.$$

**3.5. Itô integral of $G$–Brownian motion.** Itô integral with respect to a $G$-Brownian motion is defined in an analogous way as the classical one, but in a language of "$\hat{c}$-quasi-surely", or in other words, under $L_G^2$-norm. The following definition of Itô integral is from Peng (2006a). Denis and Martini (2006) independently defined this integral in the same space. For each $T > 0$, a partition $\Delta$ of $[0, T]$ is a finite ordered subset $\Delta = \{t_1, \cdots, t_N\}$ such that $0 = t_0 < t_1 < \cdots < t_N = T$. Let $p \geq 1$ be fixed. We consider the following type of simple processes: For a given partition $\{t_0, \cdots, t_N\} = \Delta$ of $[0, T]$, we set

$$\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega)\mathbf{I}_{[t_j, t_{j+1})}(t),$$

where $\xi_i \in L_G^p(\Omega_{t_i})$, $i = 0, 1, 2, \cdots, N-1$, are given. The collection of processes of this form is denoted by $M_G^{p,0}(0, T)$.

**Definition 3.4.** *For each $p \geq 1$, we denote by $M_G^p(0, T)$ the completion of ${M_G}^{p,0}(0, T)$ under the norm*

$$\|\eta\|_{M_G^p(0,T)} := \left\{ \hat{\mathbb{E}}\left[ \int_0^T |\eta_t|^p dt \right] \right\}^{1/p}.$$

Following Itô, for each $\eta \in M_G^{2,0}(0, T)$ with the above form, we define its Itô integral by

$$I(\eta) = \int_0^T \eta(s) dB_s := \sum_{j=0}^{N-1} \xi_j (B_{t_{j+1}} - B_{t_j}).$$

It is easy to check that $I : M_G^{2,0}(0, T) \longmapsto L_G^2(\Omega_T)$ is a linear continuous mapping and thus can be continuously extended to $I : M_G^2(0, T) \longmapsto L_G^2(\Omega_T)$. Moreover, this extension of $I$ satisfies

$$\hat{\mathbb{E}}[I] = 0 \text{ and } \hat{\mathbb{E}}[I^2] \leq \overline{\sigma}^2 \hat{\mathbb{E}}[\int_0^T (\eta(t))^2 dt], \ \eta \in M_G^2(0, T).$$

Therefore we can define, for a fixed $\eta \in M_G^2(0, T)$, the stochastic integral

$$\int_0^T \eta(s) dB_s := I(\eta).$$

We list some main properties of the Itô integral of $G$–Brownian motion. We denote for some $0 \leq s \leq t \leq T$,

$$\int_s^t \eta_u dB_u := \int_0^T \mathbf{I}_{[s,t]}(u) \eta_u dB_u.$$

We have

**Proposition 3.5.** *Let $\eta, \theta \in M_G^2(0,T)$ and $0 \le s \le r \le t \le T$. Then we have*

*(i)* $\int_s^t \eta_u dB_u = \int_s^r \eta_u dB_u + \int_r^t \eta_u dB_u$,

*(ii)* $\int_s^t (\alpha\eta_u + \theta_u) dB_u = \alpha \int_s^t \eta_u dB_u + \int_s^t \theta_u dB_u$, *if* $\alpha$ *is bounded and in* $L_G^1(\Omega_s)$,

*(iii)* $\hat{\mathbb{E}}_t[X + \int_t^T \eta_u dB_u] = \hat{\mathbb{E}}_t[X]$, $\forall X \in L_G^1(\Omega)$.

**3.6. Quadratic variation process.** The quadratic variation process of a $G$–Brownian motion is a particularly important process, which is not yet fully understood. But its definition is quite classical: Let $\pi_t^N$, $N = 1, 2, \cdots$, be a sequence of partitions of $[0,t]$ such that $|\pi_t^N| \to 0$. We can easily prove that, in the space $L_G^2(\Omega)$,

$$\langle B \rangle_t = \lim_{|\pi_t^N| \to 0} \sum_{j=0}^{N-1} (B_{t_{j+1}^N} - B_{t_j^N})^2 = B_t^2 - 2\int_0^t B_s dB_s.$$

From the above construction, $\{\langle B \rangle_t\}_{t \ge 0}$ is an increasing process with $\langle B \rangle_0 = 0$. We call it the *quadratic variation process* of the $G$–Brownian motion $B$. It characterizes the part of statistical uncertainty of $G$–Brownian motion. It is important to keep in mind that $\langle B \rangle_t$ is not a deterministic process unless $\underline{\sigma}^2 = \overline{\sigma}^2$, i.e., when $B$ is a classical Brownian motion.

A very interesting point of the quadratic variation process $\langle B \rangle$ is, just like the $G$–Brownian motion $B$ itself, the increment $\langle B \rangle_{t+s} - \langle B \rangle_s$ is independent of $\langle B \rangle_{t_1}, \cdots, \langle B \rangle_{t_n}$ for all $t_1, \cdots, t_n \in [0,s]$ and identically distributed: $\langle B \rangle_{t+s} - \langle B \rangle_s \overset{d}{=} \langle B \rangle_t$. Moreover $\hat{\mathbb{E}}[|\langle B \rangle_t|^3] \le Ct^3$. Hence the quadratic variation process $\langle B \rangle$ of the $G$-Brownian motion is in fact a $G$-Brownian motion, but for a different generating function $G$.

We have the following isometry:

$$\hat{\mathbb{E}}\left[\left(\int_0^T \eta(s) dB_s\right)^2\right] = \hat{\mathbb{E}}\left[\int_0^T \eta^2(s) d\langle B \rangle_s\right], \quad \eta \in M_G^2(0,T).$$

Furthermore, the distribution of $\langle B \rangle_t$ is given by $\hat{\mathbb{E}}[\varphi(\langle B \rangle_t)] = \max_{v \in [\underline{\sigma}^2, \overline{\sigma}^2]} \varphi(vt)$ and we can also prove that $\hat{c}$-quasi-surely, $\underline{\sigma}^2 t \le \langle B \rangle_{t+s} - \langle B \rangle_s \le \overline{\sigma}^2 t$. It follows that

$$\hat{\mathbb{E}}[|\langle B \rangle_{s+t} - \langle B \rangle_s|^2] = \sup_{P \in \mathcal{P}_G} E_P[|\langle B \rangle_{s+t} - \langle B \rangle_s|^2] = \max_{v \in [\underline{\sigma}^2, \overline{\sigma}^2]} |vt|^2 = \overline{\sigma}^4 t^2.$$

We then can apply Kolmogorov's criteria to prove that $\langle B \rangle_s(\omega)$ $\hat{c}$-q.s. has continuous paths.

**3.7. Itô's formula for $G$–Brownian motion.** We have the corresponding Itô formula of $\Phi(X_t)$ for a "$G$-Itô process" $X$. The following form of Itô's formula was obtained by Peng (2006a) and improved by Gao (2009). The following result of Li and Peng (2009) significantly improved the previous ones. We now consider an Itô process

$$X_t^\nu = X_0^\nu + \int_0^t \alpha_s^\nu ds + \int_0^t \eta_s^\nu d\langle B\rangle_s + \int_0^t \beta_s^\nu dB_s.$$

**Proposition 3.6.** *Let $\alpha^\nu$, $\eta^\nu \in M_G^1(0,T)$ and $\beta^\nu \in M_G^2(0,T)$, $\nu = 1,\cdots,n$. Then for each $t \geq 0$ and each function $\Phi$ in $C^{1,2}([0,t]\times\mathbb{R}^n)$ we have*

$$\begin{aligned}\Phi(t,X_t)-\Phi(s,X_s) = &\sum_{\nu=1}^n \int_s^t \partial_{x^\nu}\Phi(u,X_u)\beta_u^\nu dB_u + \int_s^t [\partial_u \Phi(u,X_u) \\ &+ \partial_{x_\nu}\Phi(u,X_u)\alpha_u^\nu]du \\ &+ \int_s^t \left[\sum_{\nu=1}^n \partial_{x^\nu}\Phi(u,X_u)\eta_u^\nu \right. \\ &\left. + \frac{1}{2}\sum_{\nu,\mu=1}^n \partial^2_{x^\mu x^\nu}\Phi(u,X_u)\beta_u^\mu\beta_u^\nu\right] d\langle B\rangle_u .\end{aligned}$$

In fact Li and Peng (2009) allows all the involved processes $\alpha^\nu$, $\eta^\nu$ to belong to a larger space $M_\omega^1(0,T)$ and $\beta^\nu$ to $M_\omega^2(0,T)$.

**3.8. Stochastic differential equations.** We have the existence and uniqueness result for the following SDE:

$$X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t h(X_s)d\langle B\rangle_s + \int_0^t \sigma(X_s)dB_s,\ t\in[0,T],$$

where the initial condition $X_0 \in \mathbb{R}^n$ is given and $b,h,\sigma : \mathbb{R}^n \mapsto \mathbb{R}^n$ are given Lipschitz functions, i.e., $|\varphi(x)-\varphi(x')| \leq K|x-x'|$, for each $x$, $x' \in \mathbb{R}^n$, $\varphi = b$, $h$ and $\sigma$, respectively. Here the interval $[0,T]$ can be arbitrarily large. The solution of the SDE is a continuous process $X \in M_G^2(0,T;\mathbb{R}^n)$.

**3.9. Brownian motions, martingales under nonlinear expectation.** We can also define a non-symmetric $G$-Brownian under a sublinear or nonlinear expectation space. Let $G(p,A) : \mathbb{R}^d \times \mathbb{S}(d) \mapsto \mathbb{R}$ be a given sublinear function monotone in $A$, i.e., in the form (2.9). It is proved in Peng (2010, Sec.3.7, 3.8) that there exists an $\mathbb{R}^{2d}$–valued Brownian motion $(B_t,b_t)_{t\geq 0}$ such that $(B_1,b_1)$ is $G$-distributed. In this case $\Omega = C([0,\infty),\mathbb{R}^{2d})$, $(B_t(\omega),b_t(\omega))$ is the canonical process, and the completion of the random variable space is $(\Omega, L_G^1(\Omega))$. $B$ is a symmetric Brownian motion and $b$ is non-symmetric. Under the sublinear expectation $\hat{\mathbb{E}}$, $B_t$ is normal distributed and $b_t$ is maximal distributed. Moreover for each fixed nonlinear function $\tilde{G}(p,A):\mathbb{R}^d\times\mathbb{S}(d)\mapsto\mathbb{R}$ which is dominated by $G$ in the following sense:

$$\tilde{G}(p,A)-\tilde{G}(p',A')\le G(p-p',A-A'),\ \ p,p'\in\mathbb{R},\ \ A,A'\in\mathbb{S}(d),$$

we can construct a nonlinear expectation $\widetilde{\mathbb{E}}$ on $(\Omega,L_G^1(\Omega))$ such that

$$\widetilde{\mathbb{E}}[X]-\widetilde{\mathbb{E}}[Y]\le\hat{\mathbb{E}}[X-Y],\quad X,Y\in L_G^1(\Omega)$$

and that the pair $(B_t,b_t)_{t\ge 0}$ is an $\mathbb{R}^{2d}$-valued Brownian motion under $\widetilde{\mathbb{E}}$. We have

$$\tilde{G}(p,A)=\widetilde{\mathbb{E}}[\langle b_1,p\rangle+\frac{1}{2}\langle AB_1,B_1\rangle],\ \ p\in\mathbb{R}^d,\ \ A\in\mathbb{S}(d).$$

This formula gives us a characterization of the change of expectations (a generalization of the notion of change of measures in probability theory) from one Brownian motion to another one, using different generator $G$.

Moreover, $\widetilde{\mathbb{E}}$ allows conditional expectations $\widetilde{\mathbb{E}}_t:L_G^p(\Omega)\mapsto L_G^p(\Omega_t)$ which is still dominated by $\hat{\mathbb{E}}_t$: $\widetilde{\mathbb{E}}_t[X]-\widetilde{\mathbb{E}}_t[Y]\le\hat{\mathbb{E}}_t[X-Y]$, for each $t\ge 0$, satisfying:

1. $\widetilde{\mathbb{E}}_t[X]\ge\widetilde{\mathbb{E}}_t[Y]$, if $X\ge Y$,
2. $\widetilde{\mathbb{E}}_t[X+\eta]=\widetilde{\mathbb{E}}_t[X]+\eta$, for $\eta\in L_G^p(\Omega_t)$,
3. $\widetilde{\mathbb{E}}_t[X]-\widetilde{\mathbb{E}}_t[Y]\le\hat{\mathbb{E}}_t[X-Y]$,
4. $\widetilde{\mathbb{E}}_t[\widetilde{\mathbb{E}}_s[X]]=\widetilde{\mathbb{E}}_{s\wedge t}[X]$, in particular, $\widetilde{\mathbb{E}}[\widetilde{\mathbb{E}}_s[X]]=\widetilde{\mathbb{E}}[X]$.

In particular, the conditional expectation of $\hat{\mathbb{E}}_t:L_G^p(\Omega)\mapsto L_G^p(\Omega_t)$ is still sublinear in the following sense:

5. $\hat{\mathbb{E}}_t[X]-\hat{\mathbb{E}}_t[Y]\le\hat{\mathbb{E}}_t[X-Y]$,
6. $\hat{\mathbb{E}}_t[\eta X]=\eta^+\hat{\mathbb{E}}_t[X]+\eta^-\hat{\mathbb{E}}_t[-X]$, $\eta$ is a bounded element in $L_G^1(\Omega_t)$.

A process $(Y_t)_{t\ge 0}$ is called a $\tilde{G}$-martingale (respectively, $\tilde{G}$-supermartingale; $\tilde{G}$-submartingale) if for each $t\in[0,\infty)$, $M_t\in L_G^1(\Omega_t)$ and for each $s\in[0,t]$, we have

$$\widetilde{\mathbb{E}}_s[M_t]=M_s,\quad(\text{respectively,}\quad\le M_s;\quad\ge M_s).$$

It is clear that for each $X\in L_G^1(\Omega_T)$, $M_t:=\widetilde{\mathbb{E}}_t[X]$ is a $\tilde{G}$-martingale. In particular, if $X=\varphi(b_T+B_T)$, for a bounded and continuous real function $\varphi$ on $\mathbb{R}^d$, then

$$M_t=\widetilde{\mathbb{E}}_t[X]=u(t,b_t+B_t)$$

where $u$ is the unique viscosity solution of the PDE

$$\partial_t u+\tilde{G}(D_x u,D_{xx}^2 u)=0,\quad t\in(0,T),\quad x\in\mathbb{R}^d,$$

with the terminal condition $u|_{t=T} = \varphi$. We have discussed the relation between BSDEs and PDEs in the last section. Here again we can claim that in general $\tilde{G}$-martingale can be regarded as a path-dependent solution of the above fully nonlinear PDE. Also a solution of this PDE is a state-dependent $\tilde{G}$-martingale.

We observe that, even with the language of PDE, the above construction of Brownian motion and the related nonlinear expectation provide a new norm which is useful in the point view of PDE. Indeed, $\|\varphi\|_{L_G^p} := \hat{\mathbb{E}}[|\varphi(B_T)|^p]^{1/p}$ forms an norm for real functions $\varphi$ on $\mathbb{R}^d$. This type of norm was proposed in Peng (2005b). In general, a sublinear monotone semigroup (or, nonlinear Markovian semigroup of Nisio's type) $Q_t(\cdot)$ defined on $C_b(\mathbb{R}^n)$ forms a norm $\|\varphi\|_Q = (Q_t(|\varphi|^p))^{1/p}$. A viscosity solution of the form

$$\partial_t u - G(Du, D^2u) = 0,$$

forms a typical example of such a semigroup if $G = G(p, A)$ is a sublinear function which is monotone in $A$. In this case $\|\varphi\|_Q^p = u(t, 0)$, where $u$ is the solution of the above PDE with initial condition given by $u|_{t=0} = |\varphi|^p$.

Let us give an explanation, for a given $X \in L_G^p(\Omega_T)$, how a $\tilde{G}$-martingale $(\widetilde{\mathbb{E}}_t[X])_{t\in[0,T]}$, rigorously obtained in Peng from (2005a,b) to (2010a), can be regarded as the solution of a new type of "fully nonlinear" BSDE which is also related to a very interesting martingale representation problem. By using a technique given in Peng (2007b,2010a), it is easy to prove that, for given $Z \in M_G^2(0, T)$ and $p, q \in M_G^1(0, T)$, the process $Y$ defined by

$$Y_t = Y_0 + \int_0^t Z_s dB_s + \int_0^t p_s db_s + \int_0^t q_s d\langle B\rangle_s - \int_0^t \tilde{G}(p_s, 2q_s)ds, \quad t \in [0, T], \tag{3.1}$$

is a $\tilde{G}$-martingale. The inverse problem is the so-called nonlinear martingale representation problem: to find a suitable subspace $\mathcal{M}$ in $L_G^1(\Omega_T)$ such that $Y_t := \widetilde{\mathbb{E}}_t[X]$ has expression (3.1) for each fixed $X \in \mathcal{M}$. This also implies that the quadruple of the processes $(Y, Z, p, q) \in M_G^2(0, T)$ satisfies a new structure of the following BSDE:

$$-dY_t = \tilde{G}(p_t, 2q_t)dt - Z_t dB_t - p_t db_t - q_t d\langle B\rangle_t, \quad Y_T = X. \tag{3.2}$$

For a particular case where $\tilde{G} = G = G(A)$ (thus $b_t \equiv 0$) and $G$ is sublinear, this martingale representation problem was raised in Peng (2007, 2008 and 2010a). In this case the above formulation becomes:

$$-dY_t = 2G(q_t)dt - q_t d\langle B\rangle_t - Z_t dB_t, \quad Y_T = X.$$

Actually, this representation can be only proved under a strong condition where $X \in \mathcal{H}_T$, see Peng (2010a), Hu, Y. and Peng (2010). For a more general $X \in L_G^2(\Omega_T)$ with $\mathbb{E}[X] = -\mathbb{E}[-X]$, Xu and Zhang (2009) proved the following representation: there exists a unique process $Z \in M_G^2(0, T)$ such that $\mathbb{E}_t[X] =$

$\mathbb{E}[X]+\int_0^t Z_s dB_s$, $t \in [0,T]$. In more general case, we observe that the process $K_t = \int_0^t G(2q_s)ds - \int_0^t q_s d\langle B\rangle_s$ is an increasing process with $K_0 = 0$ such that $-K$ is a $G$-martingale. Under the assumption $\hat{\mathbb{E}}[\sup_{t\in[0,T]} \mathbb{E}_t[|X|^2]] < \infty$, Soner, Touzi and Zhang (2009) first proved the following result: there exists a unique decomposition $(Z, K)$ such that

$$\mathbb{E}_t[X] = \mathbb{E}[X] + \int_0^t Z_s dB_s - K_t, \quad t \in [0,T].$$

The above assumption was weakened by them to $\mathbb{E}[|X|^2] < \infty$ in their 2010 version and also, independently, by Song (2010) with an even weaker assumption $\mathbb{E}[|X|^\beta] < \infty$, for a given $\beta > 1$, by using a quite different method. Our problem of representation is then reduced to prove $K_t = \int_0^t G(2q_s)ds - \int_0^t q_s d\langle B\rangle_s$. Hu and Peng (2010) introduced an a prior estimate for the unknown process $q$ to get a uniqueness result for $q$. Soner, Touzi and Zhang (2010) proved the well-posenes of the following type of BSDE, called 2BSDE, or 2nd order BSDE,

$$-dY_t = F(t, Y_t, Z_t)dt - Z_t dB_t - dK_t, \quad Y_T = X.$$

This 2BSDE is in fact quite different from the first paper by Cheridito, Soner, Touzi and Victoir (2007) which was within the framework of classical probability space.

We prefer to call (3.2) a BSDE under nonlinear expectation, (see Peng (2005b)), or a fully nonlinear BSDE, instead of 2BSDE. Indeed, in a typical situation where $\tilde{G} = g(p)$ (thus $B_t \equiv 0$, $Z_t \equiv 0$), the solution $Y_t = \widetilde{\mathbb{E}}_t[X]$ is in fact related to a first order fully nonlinear PDE of the form $\partial_t u - g(Du) = 0$. Generally speaking, with different generators $\tilde{G}$, $Y_t = \widetilde{\mathbb{E}}_t[X]$ gives us 'path-dependent' solutions of a very large type of quasi-linear or fully nonlinear parabolic PDEs of the first and second order.

Note that for a given $X \in L_G^1(\Omega_T)$, the $\tilde{G}$-martingale $Y_t := \widetilde{\mathbb{E}}_t[X]$ has solved the part $Y$ of the fully nonlinear BSDE (3.2). Furthermore, we can follow the domination approach introduced in Peng (2005b, Theorem 6.1) to consider the following type of multi-dimensional fully nonlinear BSDE:

$$Y_t^i = \widetilde{\mathbb{E}}_t^i\left[X^i + \int_t^T f^i(s, Y_s)ds\right], \quad i = 1, \cdots, m, \quad Y = (Y^1, \cdots, Y^m), \tag{3.3}$$

where, as for a $\tilde{G}$-expectation, for each $i = 1, \cdots, m$, $\widetilde{\mathbb{E}}^i$ is a $\tilde{G}_i$-expectation and $\tilde{G}_i$ is a real function on $\mathbb{R}^d \times \mathbb{S}(d)$ dominated by $G$. Then it can be proved that if $f^i(\cdot, y) \in M_G^1(0,T)$, $y \in \mathbb{R}^d$, and is Lipschitz in $y$, for each $i$, then for each given terminal condition $X = (X^1, \cdots, X^m) \in L_G^1(\Omega_T, \mathbb{R}^m)$, there exists a unique solution $Y \in M_G^1(0, T, \mathbb{R}^m)$ of BSDE (3.3).

Another problem is for stopping times. It is known that stopping times play a fundamental role in classical stochastic analysis. But up to now it is difficult to apply stopping time techniques in $G$-expectation space since the stopped process may not belong to the class of processes which are meaningful in the $G$-framework. Song (2010b) considered the properties of hitting times for $G$-martingale and the stopped processes. He proved that the stopped processes for G-martingales are still G-martingales and that the hitting times for symmetric G-martingales with strictly increasing quadratic variation processes are quasi-continuous.

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