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On solutions of backward stochastic differential equations with jumps and applications[1]

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Abstract

For backward stochastic differential equation (BSDE) with jumps and with non-Lipschitzian coefficient the existence and uniqueness of an adapted solution is obtained. By generalizing the existence result on partial differential and integral equations (PDIE) and Ito formula to the functions with only first and second Sobolev derivatives the probabilistic interpretation for solutions of PDIE (a new Feynman–Kac formula) by means of solutions of BSDE with jumps is got. With the help of this formula a new existence and uniqueness result for the solution of PDIE with non-Lipschitzian force is obtained. The convergence theorems of solutions to BSDE and PDIE are also derived.

Keywords: BSDE with jumps; Adapted solution; PDIE; Ito formula; Convergence theorem

1. Introduction

The adapted solution for a linear BSDE which appears as the adjoint process for a stochastic control problem was first investigated by Bismut (1973), then by Bensoussan (1982), and others, while the first result for the existence of an adapted solution to a continuous non-linear BSDE with Lipschitzian-coefficient was obtained by Pardoux and Peng (1990). Later Peng and Pardoux developed the theory and applications of continuous BSDEs in a series of papers (1991, 1992, 1993, 1994) under the assumption that the coefficients satisfy the Lipschitzian condition globally, or locally but with some additional condition. Tang and Li (1994) then applied the idea of Peng to get the first result on the existence of an adapted solution to a BSDE with Poisson jumps for a fixed terminal time and with Lipschitzian coefficients. Using this result, he generalized the general maximum principle of Peng (1990) for the continuous SDE

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system to the case of SDE with Poisson jumps, which also generalized the result got by Situ Rong (1991).

In this paper we derive an existence result for the adapted solution to BSDE with jumps, with bounded random stopping time as its terminal time and with non-Lipschitzian coefficient. Here the coefficient is assumed to be less than linear growth, jointly continuous and to satisfy some weak "monotone" condition locally. Since the same technique can be applied to the continuous SDE, our result implies Peng's (1993) for continuous SDE with local Lipschitzian coefficient. We also derive the convergence of solutons for such BSDE with jumps. Then by generalizing the existence result of PDIE from Marhno (1976) to some more general case (quasi-linear case) and the Ito formula to the functions having Sobolev derivatives only, we obtain a probabilistic interpretation for the solution of PDIE (a new Feyman–Kac formula) by means of the adapted solution of BSDE with jumps. With the help of this new F–K formula a new existence result for a PDIE with non-Lipschitzian and non-homogeneous term is obtained. Finally, we derive a Krylov type estimate in a weak form for such BSDE with jumps by means of its relation with PDIE.

2. BSDE with jumps and with non-Lipschitzian coefficients

Consider a BSDE in $\mathbb{R}^d$:

$$x_t = X + \int_{t\wedge\tau}^{\tau} b(s, x_s, q_s, p_s, \omega)\mathrm{d}s - \int_{t\wedge\tau}^{\tau} q_s \mathrm{d}w_s - \int_{t\wedge\tau}^{\tau}\int_Z p_s(z)(\tilde{N}_k(\mathrm{d}s, \mathrm{d}z), \quad t \geqslant 0, \tag{1}$$

where w_t is an r-dimensional standard Brownian motion process (BM), $k(\cdot)$ is a Poisson point process taking values in a measurable space $(Z, \mathscr{B}(Z))$, $N_k(\mathrm{d}s, \mathrm{d}z)$ is the Poisson counting measure defined by $k(\cdot)$ with compensator $\Pi(\mathrm{d}z)\mathrm{d}s$, $\tilde{N}_k(\mathrm{d}s, \mathrm{d}z)$ is the martingale measure such that

$$\tilde{N}_k(\mathrm{d}s, \mathrm{d}z) = N_k(\mathrm{d}s, \mathrm{d}z) - \Pi(\mathrm{d}z)\,\mathrm{d}s, \tag{2}$$

$\Pi(\cdot)$ is a σ-finite measure on $\mathscr{B}(Z)$, τ is a bounded $\mathscr{T}_t$-stopping time, and X is a $\mathscr{T}_\tau$-measurable and $\mathbb{R}^d$-valued random variable, where $\mathscr{T}_t$ is the σ-algebra generated (and completed) by all w_s, $s \leqslant t$, and $N_k((0, s], U)$, $s \leqslant t$, $U \in \mathscr{B}(Z)$.

We also use the following notation: $\mathscr{F}^2_{k,(\mathscr{T}_t)}([0, \tau]; \mathbb{R}^d)$ is the set of $\mathbb{R}^d$-valued $\mathscr{T}_t$-predictable processes $f(t, z, \omega)$ such that

$$E\int_0^{\tau}\int_Z |f(t, z, w)|^2 \Pi(\mathrm{d}z)\mathrm{d}t < \infty;$$

$L^2_{(\mathscr{T}_t)}([0, \tau]; \mathbb{R}^d)$ is the set of $f(t, \omega)$, which is $\mathscr{T}_t$-adapted, jointly measurable and $\mathbb{R}^d$-valued such that

$$E\int_0^{\tau} |f(t, \omega)|^2\,\mathrm{d}t < \infty;$$

and $L^2_{(\mathscr{T}_t)}$ $([0, \tau]; \mathbb{R}^{d\otimes r})$ is defined similarly.

Denote also $L^2_{\Pi(\cdot)}(\mathbb{R}^d)$ = The set of $\mathbb{R}^d$-valued functions $f(z)$, $z \in Z$, which is $\mathscr{B}(Z)$ measurable such that $|||f||| = (\int_Z |f(z)|^2 \Pi(\mathrm{d}z))^{1/2} < \infty$; $\langle a, b \rangle = a \cdot b$ = the inner product of a, $b \in \mathbb{R}^d$; $\| q \|$ = the norm of the matrix $q \in \mathbb{R}^{d \otimes r}$.
Introduce now

Definition 1. (x_t, q_t, p_t) is said to be a solution of (1), iff

$$1^\circ\ (x_t, q_t, p_t) \in L^2_{(\mathscr{F}_t)}([0, \tau]; \mathbb{R}^d) \times L^2_{(\mathscr{F}_t)}([0, \tau]; \mathbb{R}^{d \otimes r}) \times \mathscr{F}^2_{k,(\mathscr{F}_t)}([0, \tau]; \mathbb{R}^d),$$

2° (x_t, q_t, p_t) satisfies (1).

Definition 2. A real function $\rho(u)$, $u \geqslant 0$, is said to have the $\bar{A}$ property, iff $\rho(0) = 0$, $\rho(u) > 0$, as $u > 0$, and $\rho(u)$ is increasing, concave and continuous such that

$$\int_{0^+} \mathrm{d}u/\rho(u) = +\infty.$$

We have

Theorem 1. *Assume that $\tau \leqslant T$ and*

1° $b = b_1 + b_2$,

$$b_i = b_i(t, x, q, p, \omega)\colon [0, T] \times \mathbb{R}^d \times \mathbb{R}^{d \otimes r} \times L^2_{\Pi(\cdot)}(\mathbb{R}^d) x\Omega \to \mathbb{R}^d,$$

$i = 1, 2$ are $\mathscr{F}_t$-adapted and measurable processes such that P-a.s.

$$|b_1(t, x, q, p, \omega)| \leqslant c(t),$$

$$|b_2(t, x, q, p, \omega)| \leqslant c(t)(1 + |x| + \| q \| + |||p|||),$$

where $c(t) \geqslant 0$ is real and non-random such that

$$\int_0^T c(t)^2\, \mathrm{d}t < \infty;$$

2° *for all $t \in [0, T]$; $x, x_i \in \mathbb{R}^d$; $q, q_i \in \mathbb{R}^{d \otimes r}$; $p_i \in L^2_{\Pi(\cdot)}(\mathbb{R}^d)$, $i = 1, 2$,*

$$\langle x_1 - x_2, b_1(t, x_1, q_1, p_1, \omega) - b_1(t, x_2, q_2, p_2, \omega) \leqslant c(t)(\rho(|x_1 - x_2|^2) + |x_1 - x_2|(\| q_1 - q_2 \| + |||p_1 - p_2|||)),$$

$$|b_1(t, x, q, p_1, \omega) - b_1(t, x, q, p_2, \omega)| \leqslant c(t)\, |||p_1 - p_2|||,$$

$$|b_2(t, x_1, q_1, p_1, \omega) - b_2(t, x_2, q_2, p_2, \omega)| \leqslant c(t)(|x_1 - x_2|$$
$$+ \|q_1 - q_2\| + |||p_1 - p_2|||),$$

where $c(t)$ has the same property in 1°, *and $\rho(u)$ has the $\bar{A}$ property;*

3° *$b(t, x, q, p, \omega)$ is continuous in $(x, q, p) \in \mathbb{R}^d \times \mathbb{R}^{d \otimes r} \times L^2_{\Pi(\cdot)}(\mathbb{R}^d)$;*

4° X is *$\mathcal{T}_\tau$-measurable, and $E|X|^2 < \infty$.*

Then (1) *has a unique solution (x_t, q_t, p_t).*

Remark 1. "Uniqueness" here means that if (x_t^i, q_t^i, p_t^i), $i = 1, 2$ are two solutions of (1), then

$$E\int_0^\tau |x_t^1 - x_t^2|^2\,\mathrm{d}t = 0,\ E\int_0^\tau \|q_t^1 - q_t^2\|^2\,\mathrm{d}t = 0,$$

$$E\int_0^\tau |||p_t^1 - p_t^2|||^2\,\mathrm{d}t = 0.$$

Remark 2. In the one-dimensional space if b only depends on (t, x) and is decreasing in $x \in \mathbb{R}^l$, then 2° is satisfied by b. Hence 2° can be referred to as a weak "monotonicity condition".

By using the conditional expectation method of Peng (1993) and some localization technique one can relax condition 2° to a local condition provided that X satisfies

5° $|X|^2 \leqslant k_0$, P-a.s.

where $k_0 \geqslant 0$ is a constant, and X is $\mathcal{T}_\tau$-measurable.

Corollary 1. *Assume that all conditions except* 1° *and* 2° *in Theorem* 1 *hold, and assume that* 5° *also holds and*

1°′ $b = b_1 + b_2$,

$$|b_1(t, x, q, p, \omega)| \leqslant c(t)(1 + |x|),$$

$$|b_2(t, x, q, p, \omega)| \leqslant c(t)(1 + |x| + \|q\| + |||p|||),$$

where $c(t) \geqslant 0$, non-random and

$$\int_0^T c(t)^2\,\mathrm{d}t < \infty;$$

2°′ *For all $t \in [0, T]$; $q, q_i \in \mathbb{R}^{d \otimes r}$; $p, p_i \in L^2_{\Pi(\cdot)}(\mathbb{R}^d)$*

$$|b_2(t, x_1, q_1, p_1, \omega) - b_2(t, x_2, q_2, p_2, \omega)|$$
$$\leqslant c_N(t)(|x_1 - x_2| + \|q_1 - q_2\| + |||p_1 - p_2|||);$$

$$\langle x_1 - x_2, b_1(t, x_1, q_1, p_1, \omega) - b_1(t, x_2, q_2, p_2, \omega)\rangle$$
$$\leqslant c_N(t)(\rho_N(|x_1 - x_2|^2) + |x_1 - x_2|(\|q_1 - q_2\| + |||p_1 - p_2|||)),$$
$$|b_1(t, x, q, p_1, \omega) - b_1(t, x, q, p_2, \omega)| \leqslant c_N(t)|||p_1 - p_2|||,$$

as $|x| \leqslant N$, $|x_i| \leqslant N$, $i = 1, 2$, $N = 1, 2, \ldots$; *where* $c_N(t)$ *and* $\rho_N(u)$ *have the same property as* $c(t)$ *and* $\rho(u)$ *in* 2° *of Theorem* 1 *for each* N, *and no generality is lost to assume that*

$$c_N(t) \leqslant c_{N+1}(t),\ \rho_N(t) \leqslant \rho_{N+1}(t), \textit{for all } N.$$

Then (1) *has a unique solution* (x_t, q_t, p_t).

To show Corollary 1 we need the following interesting

Proposition 1. *Assume that* $\tau \leqslant T$, *and*

$$|b(t, x, q, p, \omega)| \leqslant c(t)(1 + |x| + \|q\| + |||p|||),$$

where $c(t)$ *satisfies conditions in* 1°, *and assume that* 5° *holds. If* (x_t, q_t, p_t) *is a solution of* (1), *then there exists a constant* N_0 *depending on* k_0, T *and* $c(\cdot)$ *only such that*

$$|x_t| \leqslant N_0, \quad \textit{for all } t \in [0, \tau],\ P\textit{-a.s.}$$

i.e. x_t *is uniformly bounded in* $t \in [0, \tau]$, $w \in \Omega$.

Proof. By the Ito formula

$$|x_{t\wedge\tau}|^2 = |X|^2 + \int_{t\wedge\tau}^{\tau} 2x_s b(s, x_s, q_s, p_s, \omega)\,\mathrm{d}s - \int_{t\wedge\tau}^{\tau} \|q_s\|^2\,\mathrm{d}s$$
$$- \int_{t\wedge\tau}^{\tau} |||p_s|||^2\,\mathrm{d}s + \int_{t\wedge\tau}^{\tau} \mathrm{d}M_s, \tag{3}$$

where M_t is a $\mathscr{T}_t$-martingale. From this as $0 \leqslant r \leqslant t \leqslant T$ ($E^{\mathscr{T}_t}(x) \triangleq E(x|\mathscr{T}_t)$)

$$E^{\mathscr{T}_{r\wedge\tau}}\left(|X_{t\wedge\tau}|^2 + \frac{1}{2}\int_{t\wedge\tau}^{\tau} \|q_s\|^2\,\mathrm{d}s + \frac{1}{2}\int_{t\wedge\tau}^{\tau} |||p_s|||^2\,\mathrm{d}s\right)$$
$$\leqslant E^{\mathscr{T}_{r\wedge\tau}}|X|^2 + T + 5E^{\mathscr{T}_{r\wedge\tau}}\int_{t\wedge\tau}^{\tau} c(s)^2|x_s|^2\,\mathrm{d}s + 2E^{\mathscr{T}_{r\wedge\tau}}\int_{t\wedge\tau}^{\tau} c(s)|x_s|^2\,\mathrm{d}s$$
$$\leqslant T + k_0 + \int_t^T (2c(s) + 5c(s)^2)E^{\mathscr{T}_{r\wedge\tau}}|x_{s\wedge\tau}|^2\,\mathrm{d}s.$$

Hence by the Cronwall inequality

$$E^{\mathscr{T}_{r\wedge\tau}}\left(|x_{t\wedge\tau}|^2 + \frac{1}{2}\int_{t\wedge\tau}^{\tau} \|q_s\|^2\,\mathrm{d}s + \frac{1}{2}\int_{t\wedge\tau}^{\tau} |||p_s|||^2\,\mathrm{d}s\right)$$
$$\leqslant (T + k_0)\exp\left(\int_t^T (2c(s) + 5c(s)^2)\,\mathrm{d}s\right) \leqslant N_0. \tag{4}$$

Let $r = t$. Then

$$|x_{t\wedge\tau}|^2 \leqslant N_0,$$

i.e. P-a.s. for all $t \in [0, \tau]$,

$$|x_t|^2 = |x_{t\wedge\tau}|^2 \leqslant N_0. \qquad \square$$

Now we are in a position to prove Corollary 1.

Proof. For each N let $W^N(x) \in C_0^\infty(\mathbb{R}^d)$ be such that

$$W^N(x) = \begin{cases} 1 & \text{for } |x| \leqslant N + 2, \\ 0 & \text{for } |x| \geqslant N + 3, \end{cases}$$

$$|W^N(x) - W^N(y)| \leqslant \bar{k}_0 |x - y| \quad \text{for all } x, y \in \mathbb{R}^d,$$

and $0 \leqslant W^N(x) \leqslant 1$, where $\bar{k}_0$ is a constant. Define

$$b^N(t, x, q, p, \omega) = b(t, x, q, p, \omega) \cdot W^N(x).$$

Then it is easily seen that if one lets

$$\bar{c}_N(t) = c(t)(1 + N + 3)\bar{k}_0 + c_{N+3}(t),$$

$$\bar{\rho}_N(u) = \rho_{N+3}(u) + u,$$

then for each N the global condition 2° in Theorem 1 holds for b^N with respect to such $\bar{c}_N(t)$ and $\bar{\rho}_N(u)$. Hence by Theorem 1 there exists a unique solution (x_t^N, q_t^N, p_t^N) of (1) with such b^N for each N.

Applying Proposition 1 one sees that there exists a natural number N_0

$$|x_t^N| \leqslant N_0 + 2 \quad \text{for all } N = 1, 2, \ldots .$$

Hence $(x_t^{N_0}, q_t^{N_0}, p_t^{N_0})$ is a solution of (1). The uniqueness will be derived by Lemma 2 below. $\square$

Let us give an example to show that under conditions of Theorem 1 or Corollary 1 coefficient b can be non-Lipschitzian in x.

Example 1. $b = b_1 + b_2$, where b_2 satisfies conditions of Theorem 1 and we assume that $0 \leqslant v_i(t, \omega)$, $i = 1, 2, 3$, are $\mathscr{F}_t$-progressive processes on $[0, T]$ and uniformly bounded.

Let

$$b_1 = \sum_{i=1}^{3} b_{1i},$$

where

$$b_{11}(t,x,q,p,\omega) = -|x|^{r_0-2}x\cdot v_1(t,\omega)I_{x\neq 0},$$

$$b_{12}(t,x,q,p,\omega) = -|x|^{r_1-2}x\cdot v_2(t,\omega)I_{x\neq 0}(\|q\|I_{\|q\|\leqslant k_0} + k_0 I_{\|q\|>k_0}),$$

$$b_{13}(t,x,q,p,\omega) = -|x|^{r_2-2}x\cdot v_3(t,\omega)I_{x\neq 0}(|||p|||I_{|||p|||\leqslant k_1} + k_1 I_{|||p|||>k_1}),$$

and $k_i \geqslant 0$, $i = 0, 1$; $r_j \in (1,2)$, $j = 0, 1, 2$; which are all constants. Then b_1 satisfies all conditions in 1°, 2° and 3°, but it is not Lipschitsian continuous.

Now we are going to prepare a series of lemmas and theorems, which are necessary for proving Theorem 1, and are of interest on their own.

Lemma 1. *Assume that $\tau \leqslant T$, $E|X|^2 < \infty$, and*

$$\langle x, b(t,x,q,p,\omega)\rangle \leqslant c(t)(1 + |x|^2 + |x|(\|q\| + |||p|||)),$$

where $c(t)$ has the property stated in 1° of Theorem 1.
If (x_t, q_t, p_t) is a solution of (1), then

$$E\left(\sup_{t\leqslant\tau}|x_t|^2 + \int_0^\tau \|q_t\|^2\,\mathrm{d}t + \int_0^\tau |||p_t|||^2\,\mathrm{d}t\right) \leqslant k_T < \infty,$$

where $k_T \geqslant 0$ is a constant depending on T, $\int_0^T c(t)^2\,\mathrm{d}t$ and $E|X|^2$ only.

Proof. Similarly as for (4), one gets that for all $t \in [0, T]$

$$E\left(|x_{t\wedge\tau}|^2 + \frac{1}{2}\int_{t\wedge\tau}^\tau \|q_s\|^2\,\mathrm{d}s + \frac{1}{2}\int_{t\wedge\tau}^\tau |||p_s|||^2\,\mathrm{d}s\right) \leqslant \tilde{k}_T,$$

where

$$\tilde{k}_T = (T + E|X|^2)\exp\left(\int_0^T (2c(t) + 5c(t)^2)\,\mathrm{d}t\right).$$

In particular,

$$E\left(|x_0|^2 + \frac{1}{2}\int_0^\tau \|q_s\|^2\,\mathrm{d}s + \frac{1}{2}\int_0^\tau |||p_s|||^2\,\mathrm{d}s\right) \leqslant \tilde{k}_T.$$

Note that

$$|x_{t\wedge\tau}|^2 \leqslant 4^2\left(|x_0|^2 + \left|\int_0^{t\wedge\tau} b(s,x_s,q_s,p_s,\omega)\,\mathrm{d}s\right|^2 + \left|\int_0^{t\wedge\tau} q_s\,\mathrm{d}w_s\right|^2 + \left|\int_0^{t\wedge\tau}\int_Z p_s(z)\tilde{N}_k(\mathrm{d}s,\mathrm{d}z)\right|^2\right) = 4\sum_{i=1}^4 I_i.$$

However, by the martingale inequality

$$E \sup_{t \leqslant T} \left| \int_0^{t \wedge \tau} Q_s \, \mathrm{d}w_s \right|^2 \leqslant 2E \int_0^{T \wedge \tau} \| q_s \|^2 \, \mathrm{d}s \leqslant 4\tilde{k}_T < \infty.$$

Similarly,

$$E \sup_{t \leqslant T} I_4 \leqslant c_0 \tilde{k}_T < \infty.$$

Moreover, by the Schwarz inequality

$$E\left(\int_0^{t \wedge \tau} c(s) \| q_s \| \, \mathrm{d}s \right)^2 \leqslant \int_0^T |c(s)|^2 \, \mathrm{d}s \, E \int_0^\tau \| q_s \|^2 \, \mathrm{d}s,$$

$$E\left(\int_0^{t \wedge \tau} c(s) |||p_s||| \, \mathrm{d}s \right)^2 \leqslant \int_0^T |c(s)|^2 \, \mathrm{d}s \, E \int_0^\tau |||p_s|||^2 \, \mathrm{d}s,$$

etc. Hence

$$E \sup_{t \leqslant T} |x_{t \wedge \tau}|^2 \leqslant 4^2 \sum_{i=1}^{4} \sup_{t \leqslant T} |I_i| \leqslant k_T < \infty. \qquad \square$$

Lemma 2. *Assume that all conditions in Lemma* 1 *hold and*

$$\bar{1}^0 \ \langle x_1 - x_2, b(t, x_1, q_1, p_1, \omega) - b(t, x_2, q_2, p_2, \omega) \rangle$$
$$\leqslant c(t)(\rho(|x_1 - x_2|^2) + |x_1 - x_2|(\| q_1 - q_2 \| + |||p_1 - p_2|||)),$$

for all $(x_i, q_i, p_i) \in \mathbb{R}^d \times \mathbb{R}^{d \otimes r} \times L^2_{\Pi(\cdot)}(\mathbb{R}^d)$, $i = 1, 2$,

where $c(t)$ *has the same property as that in* 1° *of Theorem* 1, *and* $\rho(u)$ *has the* $\bar{A}$ *property.*
Then the solution of (1) *is unique.*

Proof. Let

$$X_t = x_t^1 - x_t^2, \qquad Q_t = q_t^1 - q_t^2, \qquad P_t = p_t^1 - p_t^2.$$

Then by the Ito formula for X_t as in (3) one gets that

$$Z_t = E\left(|X_{t \wedge \tau}|^2 + \frac{1}{2} \int_{t \wedge \tau}^{\tau} \| Q_s \|^2 \, \mathrm{d}s + \frac{1}{2} \int_{t \wedge \tau}^{\tau} |||p_s|||^2 \, \mathrm{d}s \right)$$

$$\leqslant E \int_{t \wedge \tau}^{\tau} (2\rho(|X_s|^2) + 5|X_s|^2) c_1(s) \, \mathrm{d}s$$

$$\leqslant \int_t^T \rho_1(Z_s) c_1(s) \, \mathrm{d}s,$$

where

$$\rho_1(u) = 2\rho(u) + 5u, \qquad c_1(s) = c(s) + c(s)^2.$$

Hence (see Situ, 1985)

$$Z_t = 0, \text{ for all } t \in [0, T].$$

It implies that for all $0 \leqslant t \leqslant T$

$$E|X_{t\wedge\tau}|^2 = 0, \qquad E\int_0^\tau \|Q_s\|^2\,\mathrm{d}s = 0, \qquad E\int_0^\tau |||p_s|||^2\,\mathrm{d}s = 0. \qquad \square$$

Theorem 2. *If in Theorem* 1

$$b_1 = 0,$$

then (1) *has a unique solution* (x_t, q_t, p_t).

Theorem 2 will be proved in the next section. Let us prove Theorem 1 now.

Proof of Theorem 1. For simplicity we assume that $b_2 = 0$. (In case $b_2 \neq 0$ we can just smooth out b_1 and proceed as follows.) Let us smooth out b to get b^n, i.e. let

$$J_d(u) = \begin{cases} c_d \exp(-(1 - |u|^2)^{-1}) & \text{for } |u| < 1, \\ 0 & \text{otherwise}, \quad \text{for all } u \in \mathbb{R}^d, \end{cases}$$

such that constant c_d satisfies

$$\int_{\mathbb{R}^d} J_d(u)\,\mathrm{d}u = 1;$$

and for $u \in \mathbb{R}^{d\otimes r}$ (it is viewed as a $d\cdot r$-dimensional vector), $J_{dr}(u)$ is similarly defined. Set for $x \in \mathbb{R}^d$, $q \in \mathbb{R}^{d\otimes r}$

$$J(x, q) = J_d(x)\cdot J_{dr}(q).$$

Define

$$b^n(t, x, q, p, \omega) = \int_{\mathbb{R}^d \times \mathbb{R}^{dr}} b(t, x - n^{-1}\bar{x}, q - n^{-1}\bar{q}, p, \omega) J(\bar{x}, \bar{q})\,\mathrm{d}\bar{x}\,\mathrm{d}\bar{q}.$$

Since for each $n = 1, 2, \ldots$ it is easily seen that

$$\begin{aligned} &|b^n(t, x_1, q_1, p_1, \omega) - b^n(t, x_2, q_2, p_2, \omega)| \\ &\quad \leqslant k_n c(t)(|x_1 - x_2| + \|q_1 - q_2\| + |||p_1 - p_2|||), \end{aligned}$$

as $(x_i, q_i, p_i) \in \mathbb{R}^d \times \mathbb{R}^{d\otimes r} \times L^2_{\Pi(\cdot)}(\mathbb{R}^d)$, $i = 1, 2$. Hence by Theorem 2 there exists a unique solution (x_t^n, q_t^n, p_t^n) of the following BSDE (denote $t' = t \wedge \tau$ below)

$$x_{t'}^n = X + \int_{t'}^\tau b^n(s, x_s^n, q_s^n, p_s^n, \omega)\,\mathrm{d}s - \int_{t'}^\tau q_s^n\,\mathrm{d}w_s - \int_{t'}^\tau \int_Z p_s^n \tilde{N}_k(\mathrm{d}s, \mathrm{d}z). \tag{5}$$

By the Ito formula

$$|x^m_{t'} - x^n_{t'}|^2 = 2\int_{t'}^{\tau} (x^m_s - x^n_s)\cdot(b^m(s, x^m_s, q^m_s, p^m_s, \omega) - b^n(s, x^n_s, q^n_s, p^n_s, \omega))\,\mathrm{d}s$$

$$- 2\int_{t'}^{\tau} (x^m_s - x^n_s)\cdot(q^m_s - q^n_s)\,\mathrm{d}w_s - \int_{t'}^{\tau} \|q^m_s - q^n_s\|^2\,\mathrm{d}s$$

$$- 2\int_{t'}^{\tau}\int_Z (x^m_s - x^n_s)\cdot(p^m_s - p^n_s)\tilde{N}_k(\mathrm{d}s, \mathrm{d}z)$$

$$- \int_{t'}^{\tau}\int_Z |p^m_s - p^n_s|^2 N_k(\mathrm{d}s, \mathrm{d}z) = \sum_{i=1}^{5} I_i.$$

Note that

$$I_1 = 2\int_{t'}^{\tau} (x^m_s - x^n_s)\cdot\int(b(s, x^m_s - m^{-1}\bar{x}, q^m_s - m^{-1}\bar{q}, p^m_s, \omega)$$

$$- b(s, x^n_s - n^{-1}\bar{x}, q^n_s - n^{-1}\bar{q}, p^n_s, \omega))J(\bar{x}, \bar{q})\,\mathrm{d}\bar{x}\,\mathrm{d}\bar{q}\,\mathrm{d}s$$

$$\leqslant 2\int_{t'}^{\tau}\mathrm{d}s\int((\rho(|x^m_s - x^n_s - (m^{-1} - n^{-1})\bar{x}|^2) + |x^m_s - x^n_s - (m^{-1} - n^{-1})\bar{x}|$$

$$\times(\|q^m_s - q^n_s - (m^{-1} - n^{-1})\bar{q}\| + |||p^m_s - p^n_s|||)c(s)$$

$$+ |m^{-1} - n^{-1}||\bar{x}|\,2c(s))J(\bar{x}, \bar{q})\,\mathrm{d}\bar{x}\,\mathrm{d}\bar{q}.$$

Since by Lemma 1 for all n

$$E\left(\sup_{t\leqslant\tau}|x^n_t|^2 + \int_0^{\tau}\|q^n_t\|^2\,\mathrm{d}t + \int_0^{\tau}|||p^n_t|||^2\,\mathrm{d}t\right) \leqslant k_T < \infty.$$

Hence

$$E\left(|x^m_{t'} - x^n_{t'}|^2 + \int_{t'}^{\tau}\|q^m_s - q^n_s\|^2\,\mathrm{d}s + \int_{t'}^{\tau}|||p^m_s - p^n_s|||^2\,\mathrm{d}s\right)$$

$$\leqslant \bar{k}_T\int_t^T\int(\rho(E|x^m_{s'} - x^n_{s'} - (m^{-1} - n^{-1})\bar{x}|^2) + E|x^m_{s'} - x^n_{s'}|^2)$$

$$\times J(\bar{x}, \bar{q})\,\mathrm{d}\bar{x}\,\mathrm{d}\bar{q}\,c_1(s)^2\,\mathrm{d}s + \bar{k}_T(m^{-1} + n^{-1}),$$

where $c_1(t)^2 = c(t)^2 + c(t)$. Note that

$$\rho(2E(x^m_{s'} - x^n_{s'}|^2 + 2(m^{-1} - n^{-1})^2|\bar{x}|^2) \leqslant \rho(2k_T + 2|\bar{x}|^2).$$

But by assumption it yields that

$$\int\rho(2k_T + 2|\bar{x}|^2)J(\bar{x}, \bar{q})\,\mathrm{d}\bar{x}\,\mathrm{d}\bar{q} \leqslant \rho(2K_T + 2) < \infty.$$

Hence by Lemma 1 and by the Fatou lemma it is easily seen that

$$\overline{\lim_{m,n\to\infty}} E|x_{t'}^m - x_{t'}^n|^2 + \overline{\lim_{m,n\to\infty}} E\int_{t'}^{\tau} (|q_s^m - q_s^n|^2 + |||p_s^m - p_s^n|||^2)\,\mathrm{d}s$$

$$\leqslant \tilde{k}_T \int_t^T \rho_1\left(\overline{\lim_{m,n\to\infty}} 2E|x_{s'}^m - x_{s'}^n|^2\right) c_1(s)^2\,\mathrm{d}s,$$

where

$$\rho_1(u) = \rho(u) + u,$$

since $\rho_1(u)$ is increasing, therefore (cf. Situ, 1985)

$$\overline{\lim_{m,n\to\infty}} E|x_{t'}^m - x_{t'}^n|^2 = 0, \quad \text{for all } t\in[0,T],\ t' = t\wedge\tau$$

and

$$\overline{\lim_{m,n\to\infty}} E\int_0^{\tau} (|q_s^n - q_s^m|^2 + |||p_s^m - p_s^n|||^2)\,\mathrm{d}s = 0.$$

By Lemma 1

$$E\sup_{t\leqslant T} |x_{t\wedge\tau}^n|^2 \leqslant K_T < \infty.$$

Hence by the Lebesgue domination convergence theorem as $m, n\to\infty$

$$E\int_0^{\tau} |x_t^m - x_t^n|^2\,\mathrm{d}t \leqslant E\int_0^T |x_{t\wedge\tau}^m - x_{t\wedge\tau}^n|^2\,\mathrm{d}t$$

$$= \int_0^T E|x_{t\wedge\tau}^m - x_{t\wedge\tau}^n|^2\,\mathrm{d}t \to 0.$$

Let now for $(x.,q.,p.)\in L^2_{(\mathscr{F}_t)}([0,\tau];\mathbb{R}^d)\times L^2_{(\mathscr{F}_t)}[0,\tau];\mathbb{R}^{d\otimes r})$

$$\times\,\mathscr{F}^2_{k,(\mathscr{F}_t)}([0,\tau];\mathbb{R}^d) = \tilde{B} \tag{6}$$

$$\|(x.,p.,p.)\| = \|x.\|_1 + \|q.\|_2 + \|p.\|_3,$$

where

$$\|x.\|_1^2 = E\int_0^{\tau} |x_s|^2\,\mathrm{d}s = \iint_{[0,T]x\Omega} |x_s(\omega)|^2\,\mathrm{d}\mu(s,\omega),$$

$$\|q.\|_2^2 = E\int_0^{\tau} \|q_s\|^2\,\mathrm{d}s = \iint_{[0,T]x\Omega} \|q_s(\omega)\|^2\,\mathrm{d}\mu(s,\omega),$$

$$\|p.\|_3^2 = E\int_0^{\tau} |||p_s|||^2\,\mathrm{d}s = \iint_{[0,T]x\Omega} |||p_s(\omega)|||^2\,\mathrm{d}\mu(s,\omega),$$

where

$$\mathrm{d}\mu(t,\omega) = I_{(0<t\leqslant\tau(\omega))}(t,\omega)\,\mathrm{d}t\,P(\mathrm{d}\omega).$$

Then $(\tilde{B}, \|\cdot\|)$ is a Banach space. By the completeness of $\tilde{B}$ there exists a unique $(x_{\cdot}, q_{\cdot}, p_{\cdot}) \in \tilde{B}$ such that as $n \to \infty$

$$\|(x^n_{\cdot}, q^n_{\cdot}, p^n_{\cdot}) - (x_{\cdot}, q_{\cdot}, p_{\cdot})\| \to 0.$$

Therefore there exists a subsequence $\{n_k\}$ of $\{n\}$, denote it by $\{n\}$ again such that as $n \to \infty$

$$\begin{aligned} &x^n_t(\omega) \to x_t(\omega), \quad \text{in } \mathbb{R}^d, \\ &q^n_t(\omega) \to q_t(\omega), \quad \text{in } \mathbb{R}^{d\otimes r}, \\ &p^n_t(\omega) \to p_t(\omega), \quad \text{in } L^2_{\Pi(\cdot)}(\mathbb{R}^d),\ \mu\text{-a.e. } (t, \omega) \in [0, T] x \Omega. \end{aligned}$$

Hence as $n \to \infty$

$$\begin{aligned} &b^n(s, x^n_s(\omega), q^n_s(\omega), p^n_s(\omega), \omega) \to b(s, x_s(\omega), q_s(\omega), p_s(\omega), \omega), \\ &\mu\text{-a.e. } (s, \omega) \in [0, T] \times \Omega. \end{aligned}$$

In fact:

$$\begin{aligned} &|b^n(s, x^n_s(\omega), q^n_s(\omega), p^n_s(\omega), \omega) - b(s, x_s(\omega), q_s(\omega), p_s(\omega), \omega)| \\ &\quad \leqslant \int |b(s, x^n_s(\omega) - n^{-1}\bar{x}, q^n_s(\omega) - n^{-1}\bar{q}, p^n_s(\omega), \omega) \\ &\qquad - b(s, x_s(\omega), q_s(\omega), p_s(\omega), \omega)| J(\bar{x}, \bar{q})\, \mathrm{d}\bar{x}\, \mathrm{d}\bar{q} = I_n. \end{aligned}$$

But by assumption 1° and the continuity of $b = b_1$ in (x, q, p) as $n \to \infty$

$$I_n \to 0.$$

Since now we know that $|b^n(t, .)|$ and $|b(t, .)|$ are smaller than $c(t)$, by taking the limit in $L^1(\Omega, \mathscr{F}, P)$ one obtains that (x_t, q_t, p_t) is a solution of (1). The uniqueness of solution for (1) is derived by Lemma 2. □

3. Proof of Theorem 2 and convergence theorems of solutions

In this section we are going to prove Theorem 2. For this we first discuss a simpler BSDE as follows:

$$x_t = X + \int_{t\wedge\tau}^{\tau} f(s, \omega)\, \mathrm{d}s - \int_{t\wedge\tau}^{\tau} q_s\, \mathrm{d}w_s - \int_{t\wedge\tau}^{\tau} \int_Z p_s(z) \tilde{N}_k(\mathrm{d}s, \mathrm{d}z), \quad t \geqslant 0, \tag{7}$$

we have

Lemma 3. *If τ is a bounded stopping time, X is a $\mathscr{F}_\tau$-measurable and $\mathbb{R}^d$-valued random variable, $f(t, \omega)$ is a $\mathscr{F}_t$-adapted and $\mathbb{R}^d$-valued random process such that*

$$E|X|^2 < \infty, \quad E\int_0^{\tau} |f(s, \omega)|^2\, \mathrm{d}s < \infty,$$

then (7) have a unique solution.

Proof. Let

$$x_t = E\left(X + \int_{t\wedge\tau}^{\tau} f(s)\,\mathrm{d}s \,|\, \mathscr{F}_{t\wedge t}\right).$$

This makes sense, since by assumption

$$E\left|X + \int_{t\wedge\tau}^{\tau} f(s)\,\mathrm{d}s\right|^2 < \infty \quad \text{for all } t \geqslant 0.$$

Hence

$$x_0 = E\left(X + \int_0^{\tau} f(s)\,\mathrm{d}s \,\middle|\, \mathscr{F}_0\right).$$

Note that

$$M_t = E\left(X + \int_0^{\tau} f(s)\,\mathrm{d}s \,\middle|\, \mathscr{F}_t\right)$$

is a square integrable martingale. Hence by the martingale representation theorem (e.g. see [Tang and Li, Lemma 2.3]) there exists a unique (q_t, p_t) such that

$$M_t = M_0 + \int_0^t q_s \mathrm{d}w_s + \int_0^t \int_Z p_s(z)\tilde{N}_k(\mathrm{d}s, \mathrm{d}z), \quad t \geqslant 0, \tag{$*$}$$

and

$$E\int_0^T (\|q_s\|^2 + |||p_s|||^2)\mathrm{d}s < \infty \quad \text{as } t < \infty.$$

But

$$M_{t\wedge\tau} = E\left(X + \int_0^{\tau} f(s)\,\mathrm{d}s \,|\, \mathscr{F}_{t\wedge\tau}\right) = \int_0^{t\wedge\tau} f(s)\,\mathrm{d}s + x_t,$$

and by $(*)$

$$X + \int_0^{\tau} f(s)\,\mathrm{d}s = x_0 + \int_0^{\tau} q_s\,\mathrm{d}w_s + \int_0^{\tau}\int_Z p_s(z)\tilde{N}_k(\mathrm{d}s, \mathrm{d}z).$$

Hence

$$\begin{aligned}
&X + \int_{t\wedge\tau}^{\tau} f(s)\,\mathrm{d}s - \int_{t\wedge\tau}^{\tau} q_s\,\mathrm{d}w_s - \int_{t\wedge\tau}^{\tau}\int_Z p_s(z)\tilde{N}_k(\mathrm{d}s, \mathrm{d}z) \\
&\quad = x_0 + \int_0^{t\wedge\tau} q_s\,\mathrm{d}w_s + \int_0^{t\wedge\tau}\int_Z p_s(z)\tilde{N}_k(\mathrm{d}s, \mathrm{d}z) - \int_0^{t\wedge\tau} f(s)\,\mathrm{d}s \\
&\quad = M_{t\wedge\tau} - \int_0^{t\wedge\tau} f(s)\,\mathrm{d}s = x_t.
\end{aligned}$$

It is proved that (x_t, q_t, p_t) satisfies (7). Now by the Jensen inequality

$$E\int_0^\tau |x_t|^2\,\mathrm{d}t = E\int_0^\tau \left|E\left(X + \int_{t\wedge\tau}^\tau f(s)\,\mathrm{d}s \,\middle|\, \mathscr{T}_{t\wedge\tau}\right)\right|^2 \mathrm{d}t$$

$$\leqslant \int_0^T E\left|X + \int_{t\wedge\tau}^\tau f(s)\,\mathrm{d}s\right|^2 \mathrm{d}t < \infty,$$

where we assume that $0 \leqslant \tau \leqslant T$. Note also that

$$E\int_0^\tau (\|q_z\|^2 + |||p_s|||^2)\mathrm{d}s \leqslant E\int_0^T (\|q_s\|^2 + |||p_s|||^2))\,\mathrm{d}s < \infty.$$

Hence (x_t, q_t, p_t) is a solution of (7).

The uniqueness of solution to (7) is derived by the Ito formula. □

Now we are in a position to prove Theorem 2.

Proof of Theorem 2. Similarly to [Tang and Li, 1994] let us use the contractive mapping principle to show this result. Introduce a new norm as follows:

For $(x_\cdot, q_\cdot, p_\cdot)\in \tilde{B}(\tilde{B}$ is defined in (6)) let

$$\|(x_\cdot, q_\cdot, p_\cdot)\|_M^2 = \sup_{t\leqslant T} \mathrm{e}^{-bA(t)}E|x_{t\wedge\tau}|^2 + \sup_{t\leqslant T} \mathrm{e}^{-bA(t)}E\int_{t\wedge\tau}^\tau \|q_s\|^2\,\mathrm{d}s$$

$$+ \sup_{t\leqslant T} \mathrm{e}^{-bA(t)}E\int_{t\wedge\tau}^\tau |||p_s|||^2\,\mathrm{d}s,$$

where $b \geqslant 0$ is a constant, which will be determined later, and

$$A(t) = \int_t^T c(s)^2\,\mathrm{d}s.$$

Denote

$$H = \{(x_\cdot, q_\cdot, p_\cdot)\in \tilde{B}\colon \|(x_\cdot, q_\cdot, p_\cdot)\|_M < \infty\}.$$

Then $(H, \|\cdot\|_M)$ is a Banach space.

Now by Lemma 3 for any $(\bar{x}^i_\cdot, \bar{q}^i_\cdot, \bar{p}^i_\cdot)\in H$, $i = 1, 2$, there exist unique solutions (x^i_t, q^i_t, p^i_t), $i = 1, 2$, of the following BSDE

$$x^i_t = X + \int_{t\wedge\tau}^\tau b(s, \bar{x}^i_s, \bar{q}^i_s, \bar{p}^i_s, \omega)\,\mathrm{d}s - \int_{t\wedge\tau}^\tau q^i_s\,\mathrm{d}w_s - \int_{t\wedge\tau}^\tau\int_Z p^i_s(s)\tilde{N}_k(\mathrm{d}s, \mathrm{d}z),$$

$$\text{for all } t \geqslant 0,\ i = 1, 2;$$

moreover, by Lemma 1 $(x^i_\cdot, q^i_\cdot, p^i_\cdot)\in H$, $i = 1, 2$. By the Ito formula

$$E\left(|x^1_{t\wedge\tau} - x^2_{t\wedge\tau}|^2 + \int_{t\wedge\tau}^\tau \|q^1_s - q^2_s\|^2\,\mathrm{d}s + \int_{t\wedge\tau}^\tau |||p^1_s - p^2_s|||^2\,\mathrm{d}s\right)$$

$$\leqslant \hat{\gamma}^{-1}E\left(\int_{t\wedge\tau}^\tau |\bar{x}^1_s - \bar{x}^2_s|^2\,\mathrm{d}s + \int_{t\wedge\tau}^\tau \|\bar{q}^1_s - \bar{q}^2_s\|^2\,\mathrm{d}s\right.$$

$$+ \int_{t\wedge\tau}^{\tau} |||\bar{p}_s^1 - \bar{p}_s^2|||^2\,\mathrm{d}s) + \hat{\gamma} E \int_{t\wedge\tau}^{\tau} c(s)^2 |x_s^1 - x_s^2|^2\,\mathrm{d}s$$

$$= \hat{\gamma}^{-1} I_t^1 + \hat{\gamma} E \int_{t\wedge\tau}^{\tau} c(s)^2 |x_s^1 - x_s^2|^2\,\mathrm{d}s$$

$$\leqslant \hat{\gamma}^{-1} I_t^1 + \hat{\gamma} \int_t^T c(s)^2\, E(|x_{s\wedge\tau}^1 - x_{s\wedge\tau}^2|^2)\,\mathrm{d}s$$

$$\leqslant \hat{\gamma}^{-1} I_t^1 + \int_t^T \exp(\hat{\gamma}(A(t) - A(s)))\, c(s)^2\, I_s^1\,\mathrm{d}s,$$

where we have applied the following result.

Lemma 4. (Gronwall inequality). *If* $0 \leqslant y_t \leqslant \hat{\gamma}^{-1} v_t + \hat{\gamma}\int_t^T c(s)^2 y_s\,\mathrm{d}s,\ t \geqslant 0,$ *where* $\hat{\gamma} > 0$ *is a constant, then*

$$y_t \leqslant \hat{\gamma}^{-1} v_t + \int_t^T \exp\left(\hat{\gamma}\int_t^s c(r)^2\,\mathrm{d}r\right) c(s)^2 v_s\,\mathrm{d}s, \quad t \geqslant 0.$$

Denote

$$u_t = |\bar{x}_t^1 - \bar{x}_t^2|^2.$$

Note that $A(t)$ is decreasing, then

$$\mathrm{e}^{-b\cdot A(t)} E \int_{t\wedge\tau}^{\tau} u_s\,\mathrm{d}s \leqslant \mathrm{e}^{-A(t)b} \int_t^T E u_{s\wedge\tau}\,\mathrm{d}s$$

$$\leqslant T \sup_{s\leqslant T} \mathrm{e}^{-bA(s)} E u_{s\wedge\tau},$$

$$\mathrm{e}^{-b\cdot A(t)} \int_t^T \cdot\, \mathrm{e}^{\hat{\gamma}(A(t) - A(s))} c(s)^2\, E \int_{s\wedge\tau}^{\tau} u_r\,\mathrm{d}r\,\mathrm{d}s$$

$$\leqslant \int_t^T \mathrm{e}^{-(b-\hat{\gamma})(A(t) - A(s))} c(s)^2\,\mathrm{d}s\; T \sup_{r\leqslant T} \mathrm{e}^{-bA(r)} E u_{r\wedge\tau}$$

$$\leqslant T \sup_{r\leqslant T} \mathrm{e}^{-b\bar{A}(r)} E u_{r\wedge\tau} (b - \hat{\gamma})^{-1}, \quad \text{for } 0 < \hat{\gamma} < b.$$

Denote now $u_t = E\int_{t\wedge\tau}^{\tau} \|\bar{q}_s^1 - \bar{q}_s^2\|^2\,\mathrm{d}s$ or $E\int_{t\wedge\tau}^{\tau} |||\bar{p}_s^1 - \bar{p}_s^2|||^2\,\mathrm{d}s$. Then

$$\mathrm{e}^{-bA(t)} \int_t^T \mathrm{e}^{\hat{\gamma}(A(t) - A(s))} c(s)^2 u_s\,\mathrm{d}s$$

$$\leqslant \sup_{s\leqslant T} \mathrm{e}^{-bA(s)} u_s \int_t^T \mathrm{e}^{-(b-\hat{\gamma})(A(t) - A(s))} c(s)^2\,\mathrm{d}s$$

$$\leqslant \sup_{s\leqslant T} \mathrm{e}^{-bA(s)} u_s \cdot (b - \hat{\gamma})^{-1}.$$

Hence

$$\|(x^1_\cdot - x^2_\cdot, q^1_\cdot - q^2_\cdot, p^1_\cdot - p^2_\cdot)\|^2$$
$$\leqslant \mathrm{Max}(3\hat{\gamma}^{-1}(T+1), 3(b-\hat{\gamma})^{-1}(T+1))\|(\bar{x}^1_\cdot - \bar{x}^2_\cdot, \bar{q}^1_\cdot - \bar{q}^2_\cdot, \bar{p}^1_\cdot - \bar{p}^2_\cdot)\|^2.$$

After appropriately choosing $\hat{\gamma}$ and b, one can easily complete the proof. □

Now we have the following convergence theorem for solutions of BSDE (1).

Theorem 3. *Assume that τ is a $\mathscr{F}_t$-stopping time for $n = 0, 1, 2, \ldots$*

$1°$ $b^n = b^n(t, x, q, p, \omega)$: $[0, T] \times \mathbb{R}^d \times \mathbb{R}^{d\otimes r} \times L^2_{\Pi(\cdot)}(\mathbb{R}^d) \to \mathbb{R}^d$ *are $\mathscr{F}_t$-adapted such that* P-a.s.

$$\langle x, b^n(s, x, q, p, \omega)\rangle \leqslant c(s)(1 + |x|^2 + |x|(\|q\| + |||p|||)),$$

where $c(t)$ has the property as that of $1°$ *in Theorem* 1;

$2°$ $\langle x_1 - x_2, b^0(t, x_1, q_1, p_1, \omega) - b^0(t, x_2, q_2, p_2, \omega)\rangle$

$$\leqslant c(t)(\rho(|x_1 - x_2|^2) + |x_1 - x_2|(\|q_1 - q_2\| + |||p_1 - p_2|||)), \ P\text{-a.s.},$$

for all $(x_i, q_i, p_i) \in \mathbb{R}^d \times \mathbb{R}^{d\otimes r} \times L^2_{\Pi(\cdot)}(\mathbb{R}^d)$, $i = 1, 2$,

where $\rho(u)$ has the same property stated in $2°$ *of Theorem* 1,

$3°$ X^n *is $\mathscr{F}_{T\wedge t}$-adapted for all $n = 0, 1, 2, \ldots$, and*

$$E(X^n - X^0|^2 \to 0, \text{ as } n \to \infty, \ E|X^n|^2 < \infty;$$

$$4° \ \lim_{n\to\infty} \sup_{\substack{x\in\mathbb{R}^d, q\in\mathbb{R}^{d\otimes r}\\ p\in L^2_{\Pi(\cdot)}(\mathbb{R}^d), \omega\in\Omega}} \int_0^T |b^n(t, x, q, p, \omega) - b^0(t, x, q, p, \omega)|^2\, \mathrm{d}t = 0.$$

If (x^n_t, q^n_t, p^n_t) are solutions of the following BSDEs: as $0 \leqslant s \leqslant T$

$$x^n_{s\wedge\tau} = X^n + \int_{s\wedge\tau}^{T\wedge\tau} b^n(r, x^n_r, q^n_r, p^n_r, \omega)\,\mathrm{d}r - \int_{s\wedge\tau}^{T\wedge\tau} q^n_r\,\mathrm{d}w_r - \int_{s\wedge\tau}^{T\wedge\tau}\int_Z p^n_r \tilde{N}_k(\mathrm{d}r, \mathrm{d}z),$$
$$n = 0, 1, 2, \ldots,$$

then for all $0 \leqslant s \leqslant T$

$$\lim_{n\to\infty} E\left(|x^n_{s\wedge\tau} - x^0_{s\wedge\tau}|^2 + \int_{s\wedge\tau}^{T\wedge\tau} (\|q^n_r - q^0_r\|^2 + |||p^n_r - p^0_r|||^2)\,\mathrm{d}r\right) = 0.$$

Proof. By the Ito formula (3) one has

$$E\left(|x^n_{s\wedge\tau} - x^0_{s\wedge\tau}|^2 + \int_{s\wedge\tau}^{T\wedge\tau} (\|q^n_r - q^0_r\|^2 + |||p^n_r - p^0_r|||^2)\,\mathrm{d}r\right)$$
$$\leqslant k_0\left(\int_s^T c_1(r)^2 \rho_1(E|x^n_{r\wedge\tau} - x^0_{r\wedge\tau}|^2)\,\mathrm{d}r + E|X^n - X^0|^2\right)$$
$$+ k_0 E\int_s^T |b^n(r, x^n_r, q^n_r, p^n_r, \omega) - b^0(r, x^n_r, q^n_r, p^n_r, \omega)|^2\,\mathrm{d}r,$$

where

$$c_1(r)^2 = c(r)^2 + c(r) + 1, \qquad \rho_1(u) = \rho(u) + u.$$

Hence

$$\overline{\lim_{n\to\infty}}\, E|x^n_{s\wedge\tau} - x^0_{s\wedge\tau}|^2 \leqslant k_0 \int_s^T C_1(r)^2 \rho_1\Big(\overline{\lim_{n\to\infty}}\, E|x^n_{s\wedge\tau} - x^0_{s\wedge\tau}|^2\Big)\,\mathrm{d}r.$$

Therefore (cf. Situ, 1985)

$$\overline{\lim_{n\to\infty}}\, E|x^n_{s\wedge\tau} - x^0_{s\wedge\tau}|^2 = 0.$$

The other conclusion now is easily derived. □

We also have some other useful convergence theorem as follows:

Theorem 4. *Assume that for all* $(x, q, p), (y, \bar{q}, \bar{p}) \in \mathbb{R}^d \times \mathbb{R}^{d\otimes r} \times L^2_{\Pi(\cdot)}(\mathbb{R}^d)$

$1^{\circ\prime}$ $|b^n(t, x, q, p, \omega)| \leqslant k_0(1 + |x| + \|q\| + |||p|||)$, $n = 0, 1, \ldots$, *where* $k_0 \geqslant 0$ *is a constant*;

$2^{\circ\prime}$ $\langle x - y, b^n(t, x, q, p, \omega) - b^n(t, y, \bar{q}, \bar{p}, \omega)\rangle \leqslant c(t)(\rho(|x-y|^2)$

$$+ |x - y|(\|q - \bar{q}\| + |||p - \bar{p}|||)), \ P\text{-a.s. } n = 1, 2, \ldots$$

where $c(t)$ *and* $\rho(u)$ *have the same property as that in* 2° *of Theorem* 3;

$3^{\circ\prime}$ *The same as* 3° *in Theorem* 3;

$4^{\circ\prime}$ $\lim_{n\to\infty} b^n(t, x, q, p, \omega) = b^0(t, x, q, p, \omega)$, P-a.s.

Then the conclusion of Theorem 3 *still holds.*

The proof can be completed similarly as that of Theorem 3.

4. Application to PDIE

Applying the idea of Peng (1991) to explain the solution of partial differential equations (PDE) by the solution of BSDE one can have a probabilistic interpretation for the solution of partial differential and integral equations (PDIE) by means of the solution of BSDE with jumps.

For this we need some preparation on PDIE.

Let D be a bounded domain in $\mathbb{R}^d$, ∂D be the boundary of D, $D^c = \mathbb{R}^d - D$.

Consider the following first boundary problem (8):

$$\mathscr{L}_{b,\sigma,c}\, u(t, x) \triangleq \partial u(t, x)/\partial t + \sum_{i=1}^d b_i(t, x)\partial u(t, x)/\partial x_i$$

$$+ \frac{1}{2}\sum_{i,j=1}^d a_{ij}(t, x)\partial^2 u(t, x)/\partial x_i \partial x_j$$

$$+ \int_Z (u(t, x + c(t, x, z)) - u(t, x) - \sum_{i=1}^{d} c_i(t, x, z)\partial u(t, x)/\partial x_i)\, \Pi(\mathrm{d}z)$$

$$= f(t, x, u(t, x), u'_x(t, x)\cdot \sigma(t, x), u(t, x + c(t, x, \cdot)) - u(t, x)),$$

$$t \in [0, T), \quad x \in D; \tag{8$_1$}$$

$$u(T, x) = \varphi(x), \qquad u(t, x)|_{D^c} = \psi(t, x), \qquad \psi(T, x) = \varphi(x)|_{D^c}, \tag{8$_2$}$$

$$u \in W_p^{1,2}([0, T] \times \mathbb{R}^d) = W_p^{1,2}([0, T] \times \mathbb{R}^d; \mathbb{R}^m),$$

where for coefficients b, σ, c one makes the following assumption (A):

A.1. b: $[0, T] \times \mathbb{R}^d \to \mathbb{R}^d$,
is bounded measurable;

A.2. σ: $[0, T] \times \mathbb{R}^d \to \mathbb{R}^{d\otimes r}$

is bounded, measurable; continuous in x (uniformly with respect to t), and continuous in t such that there exist constants δ_0 and $k_0 > 0$ satisfying (here we denote $a = \sigma\sigma^* = (a_{ij})$)

$$\delta_0 |\lambda|^2 \leqslant \langle \lambda, a\lambda \rangle \leqslant k_0 |\lambda|^2 \quad \text{for all } \lambda \in \mathbb{R}^d;$$

A.3. for arbitrary $G \subset \mathbb{R}^d$-bounded domain there exists a measure $\bar{\Pi}(\cdot)$ such that as $(s, x) \in [0, T] \times G$

$$\Pi(s, x, A) \triangleq \int I_A(c(s, x, z))\Pi(dz) \leqslant \bar{\Pi}(A),$$

$$\int |y|^2 \bar{\Pi}(\mathrm{d}y) < \infty, \int |y| \bar{\Pi}(\mathrm{d}y) < \infty;$$

and

$$\lim_{x \to x'} \int_Z (|c(s, x, z)|^2 - |c(s, x', z)|^2 | \Pi(\mathrm{d}z) = 0;$$

for functions f, φ, ψ one makes the following assumption (B):

B.1. $f: [0, T] \times D \times \mathbb{R}^m \times \mathbb{R}^{m\otimes r} \times L^2_{\Pi(\cdot)}(\mathbb{R}^m) \to \mathbb{R}^m$

is measurable such that for all $u \in W_p^{1,2}([0, T] \times \mathbb{R}^d)$ satisfying $(8)_2$

$$g^u \in L_p([0, T] \times D),$$

where

$$g^u(t, x) = f(t, x, u(t, x), u'_x(t, x)\cdot \sigma(t, x), u(t, x + c(t, x, \cdot)) - u(t, x));$$

and for arbitrary $r_1, r_2 \in \mathbb{R}^m$; $q_1, q_2 \in \mathbb{R}^{m\otimes r}$; $p_1, p_2^- \in L^2_{\Pi(\cdot)}(\mathbb{R}^m)$

$$|f(t, x, r_1, q_1, p_1(.)) - f(t, x, r_2, q_2, p_2(.))|$$

$$\leqslant k_0(|r_1 - r_2| + \| q_1 - q_2 \| + |||p_1 - p_2|||),$$

B.2. $\varphi \in W_p^{2(1-1/p)}(\mathbb{R}^d)$; $(\varphi : \mathbb{R}^d \to \mathbb{R}^m)$;

B.3. $\psi \in W_p^{1,2}([0, \mathrm{T}] \times D^c)$; $(\psi\colon [0, T] \times D^c \to \mathbb{R}^m)$;

B.4. $\partial D \in O^2$ (for definition of 0^2 see Ladyzenskaja et al., 1968).
We have

Theorem 5. *Under the assumptions* (A) *and* (B) *if* $p > d + 2$, *then* (8) *has a unique solution* $u(t, x) \in W_p^{1,2}([0, T] \times \mathbb{R}^d)$ *such that*

$$\| u \|_{W_p^{1,2}([0,T]\times\mathbb{R}^d)} \leqslant c_0 (\| g^u \|_{L_p([0,T]\times D)} + \| \psi \|_{W_p^{1,2}([0,T]\times D^c)} + \| \varphi \|_{W_p^{2(1-1/p)}(\mathbb{R}^d)}),$$

where c_0 *is a constant.*

Remark 1. In case condition B.1 is strengthened to be

B.1′ $f\colon [0, T] \times D \to \mathbb{R}^m$ such that

$f \in L_p([0, T] \times D)$;

then Theorem 5 becomes Theorem 2 of Marhno (1976) in the case of $u(t, x)$-valued in m-dimensional space. Hence Theorem 5 is a generalization of it.

Proof of Theorem 5. For each $u_i \in W_p^{1,2}([0, T] \times \mathbb{R}^d)$ satisfying $(8)_2$ by Theorem 2 of Marhno (1976) there exists a unique $\tilde{u}_i \in W_p^{1,2}([T - h, T] \times \mathbb{R}^d)$ such that

$$\mathscr{L}_{b,\alpha,c} \tilde{u}_i = f(t, x, u_i(t, x), u_{ix}'(t, x) \cdot \sigma(t, x), u_i(t, x + c(t, x,.)) - u_i(t, x)),$$

$$\text{for all } t \in [T\text{-}h, T],\ x \in D,$$

$$\tilde{u}_i(T, x) = \varphi(x), \qquad \tilde{u}_i(t, x)|_{D^c} = \psi(t, x), \qquad \psi(T, x) = \varphi(x)|_{D^c}.$$

Moreover,

$$\begin{aligned} \| \tilde{u}_1 - \tilde{u}_2 \|^p_{W_p^{1,2}([T-h,T]\times D)} &\leqslant c_0 \| g^{u_1} - g^{u_2} \|^p_{L_p([T-h,T]\times D)} \\ &= c_0 \int_{T-h}^{T} \mathrm{d}t \int_D | g^{u_1} - g^{u_2} |^p(t, x)\,\mathrm{d}x \\ &\leqslant k_0' \int_{T-h}^{T} \mathrm{d}t \int_D | u_1 - u_2 \|^p(t, x)\,\mathrm{d}x \\ &\quad + \int_{T-h}^{T} \mathrm{d}t \int_D \| u_{1x}' - u_{2x}' \|^p(t, x)\,\mathrm{d}x \\ &\quad + \int_{T-h}^{T} \mathrm{d}t \int_D \left| \int_{\mathbb{R}^d} \int_0^1 \| u_{1x}' - u_{2x}' \|^2(t, x + \alpha y)\,\mathrm{d}\alpha\, |y|^2\, \Pi(\mathrm{d}y) \right|^{p/2} \mathrm{d}x) \\ &= \sum_{i=1}^{3} I_i, \end{aligned}$$

where $\| u_x' \|$ = the norm of $m \times r$-matrix.

By the Sobolev imbedding theorem (e.g. see Ladyzenskaja et al., 1968, Lemma 4.5, p. 305)

$$I_1^{1/p} \leqslant ch^1 \| u_1 - u_2 \|_{W_p^{1,2}([T-h,T]\times D)}.$$

Similarly,

$$I_2^{1/p} \leqslant ch^{1/2} \| u_1 - u_2 \|_{W_p^{1,2}([T-h,T]\times D)}.$$

Moreover,

$$I_3 \leqslant c \sup_{t\in(T-h,T),x\in D} |u'_{1x} - u'_{2x}|^p(t,x)\,\mathrm{meas}(D) | \int |y|^2 \bar{\Pi}(\mathrm{d}y)|^{p/2} h$$

$$\leqslant \tilde{k}_0 \| u_1 - u_2 \|^p_{W_p^{1,2}([T-h,T]\times D)} \cdot h^{1+\lambda p},$$

where $\tilde{k}_0$ is independent of λ and h, and

$$0 < 2\lambda = 1 - (d+2)/p < 1,$$

and we have applied Lemmas II.3.3 and IV.4.5 of [4]. (See Ladyzenskaja et al., 1968, p. 80 and p. 305).

It yields that as $0 < h < 1$

$$\| \tilde{u}_1 - \tilde{u}_2 \|^p_{W_p^{1,2}([T-h,T]\times D)} \leqslant \bar{k}'_0 \| u_1 - u_2 \|^p_{W_p^{1,2}([T-h,T]\times \mathbb{R}^d)} \cdot h.$$

On the other hand,

$$\| \tilde{u}_1 - \tilde{u}_2 \|^p_{W_p^{1,2}([T-h,T]\times D^c)} = 0.$$

Therefore

$$\| \tilde{u}_1 - \tilde{u}_2 \|_{W_p^{1,2}([T-h,T]\times \mathbb{R}^d)} \leqslant k_0 \| u_1 - u_2 \|_{W_p^{1,2}([T-h,T]\times \mathbb{R}^d)} \cdot h.$$

Now take $h > 0$ small enough, then there exists a unique

$$u(t,x) \in W_p^{1,2}([T-h,T]\times \mathbb{R}^d)$$

satisfying $(8)_1$ as $t \in [T-h, T]$, $x \in D$, and satisfying $(8)_2$. From this it is not difficult to derive the conclusion. □

Consider now the forward SDE (FSDE) with Poisson jumps in $\mathbb{R}^d$ for any given $(t,x) \in [0,T] \times D$

$$y_s = x + \int_t^s b(r, y_r)\,\mathrm{d}r + \int_t^s \sigma(r, y_r)\,\mathrm{d}w_r + \int_t^s \int_Z c(r, y_{r-}, z) \tilde{N}_k(\mathrm{d}r, \mathrm{d}z),$$

$$t \leqslant s \leqslant T, \tag{9}$$

where $\tilde{N}_k$ is as in (2). By Theorem 1 of [Marhno, 1976] under assumption (A) (but the following condition is not necessary:

$\sigma(s, x)$ is continuous in s and

$$\int |y| \tilde{\Pi}(\mathrm{d}y) < \infty,$$

see [Marhno, 1976]), (9) has a weak solution unique in law. Denote now

$$\mathscr{T}_{t,s} = \sigma(w(r) - w(t), N_k(t, r), U) \quad \text{for all } t \leqslant r \leqslant s,\ U \in \mathscr{B}(Z)).$$

Then (9) has a pathwise unique strong solution $y_s^{t,x}$ ("strong" means that it is $\mathscr{T}_{t,s}$-adapted) under additional assumption (A)′:

$$\text{(A)}' \ 2\langle y_1 - y_2, b(s, y_1) - b(s, y_2)\rangle + \|\sigma(s, y_1) - \sigma(s, y_2)\|^2$$
$$+ |||c(s, y_1) - c(s, y_2)|||^2 \leqslant k_N(s)\rho_N(|y_1 - y_2|^2) \quad \text{as} |y_1|, |y_2| \leqslant N,$$
$$\text{for each } N = 1, 2, \ldots,$$

where for each N $\rho_N(u)$ has the $\bar{A}$ property (see Definition 2) and $k_N(s) \geqslant 0$ such that $\int_0^T k_N(s)\,\mathrm{d}s < \infty$, for each $0 \leqslant T < \infty$ (see [Situ, 1985]).

For arbitrary $(t, x) \in [0, T] \times D$ let

$$\tau = \tau_x = \inf[s > t\colon y_s^{t,x} \notin D\}, \text{ and } \tau = \tau_x = T, \text{ for } \inf\{\phi\},$$

and make the additional assumption (C):

C.1. $|f(s, x, r, q, p)| \leqslant k_0(1 + |r| + \|q\| + |||p|||)$.

By Theorem 2 we have

Lemma 5. *Under the assumptions* (A), (A)′, (B) *and* (C) *there exists a unique*

$$(x_s, q_s, p_s) \in L^2_{(\mathscr{T}_{t,s})}([t, \tau]; \mathbb{R}^m) \times L^2_{(\mathscr{T}_{t,s})}([t, \tau]; \mathbb{R}^{m \otimes r}) \times \mathscr{F}^2_{k,(\mathscr{T}_{t,s})}([t, \tau]; \mathbb{R}^m)$$

satisfying (from now on we denote $y_s = y_s^{t,x}$ for simplicity)

$$x_s = I_{\tau < T}\psi(\tau, y_\tau) + I_{\tau = T}\varphi(y_T) - \int_{s\wedge\tau}^{\tau} f(r, y_r, x_r, q_r, p_r)\,\mathrm{d}r$$
$$- \int_{s\wedge\tau}^{\tau} q_r\,\mathrm{d}w_r - \int_{s\wedge\tau}^{\tau}\int_Z p_r(z)\tilde{N}_k(\mathrm{d}r, \mathrm{d}z) \quad \text{as } t \leqslant s \leqslant T. \tag{10}$$

Proof. Actually, it is seen that $X = I_{\tau<T}\ \psi(\tau, y_\tau) + I_{\tau=T}(y_T)$ is $\mathscr{T}_{t,\tau}$ measurable and $E|X|^2 < \infty$. Therefore Theorem 2 is applied. □

Now we have the following

Theorem 6 (Ito formula). *Under assumption* (A), *but the following condition is not necessary*:

$\sigma(s, x)$ is continuous in s,

if y_s satisfies (9), then for all $u(s,x)\in W_p^{1,2}([0,T]\times\mathbb{R}^d)$ as $p>2d+4$ one has that for $t\leqslant s\leqslant T$

$$u(s\wedge\tau_G, y_{s\wedge\tau_G}) - u(t,y_t) = \int_t^{s\wedge\tau_G} \mathscr{L}_{b,\sigma,c}u(r,y_r)\,\mathrm{d}r$$
$$+\int_t^{s\wedge\tau_G} u'_x(r,y_r)\cdot\sigma(r,y_r)\,\mathrm{d}w_r + \int_t^{s\wedge\tau_G}\int_Z (u(r,y_{r^-}+c(r,y_{r^-},z))$$
$$-u(r,y_{r^-}))\tilde{N}_k(\mathrm{d}r,\mathrm{d}z), \tag{11}$$

where $G\subset\mathbb{R}^d$ is a bounded domain, and

$$\tau_G=\inf(s>t\colon y_s\bar{\in}G),\quad \tau_G=T)\quad \textit{for } \inf\{\phi\}.$$

The Ito formula for functions with Sobolev derivatives in the continuous SDE case was obtained by Krylov (1980), in SDE with Poisson jumps but for formula with mathematical expectation by Marhno (1976). Here our Ito formula is without the mathematical expectation. Hence we need to assume that $p>2d+4$ (but not $p>d+2$ as in Marhno, 1976).

Proof. By the imbedding theorem and so on one can take $u_n\in C^{1,2}([0,T]\times\mathbb{R}^d)$ such that as $n\to\infty$

$$\sup_{t\in(0,T),x\in G}|u_n(t,x)-u(t,x)|\to 0,$$

and

$$\|u_n-u\|_{W_p^{1,2}([0,T]\times\mathbb{R}^d)}\to 0,$$

since (11) holds for u_n. Note that the following estimate holds: for $f\in L_p([0,T]\times G)$, $p>d+2$, $G\subset\mathbb{R}^d$ is a bounded domain, and $\int|y|\bar{\Pi}(\mathrm{d}y)<\infty$, one has

$$E_{t,x}\int_t^{\tau_G\wedge T}|f(s,y_s)|\,\mathrm{d}s\leqslant c_{p,T,d}\,\|f\|_{L_p([t,T]xG)}, \tag{12}$$

where we denote $E_{t,x}F(y.,\omega)=EF(y.,\omega)=EF(y_\cdot^{t,x},\omega)$. (See Marhno, 1976, Theorem 1). Hence it is not difficult to derive the formula for u by approximation, e.g. let us show that as $n\to\infty$

$$I_n=E\left|\int_t^{s\wedge\tau_G}\int_Z((u_n-u)(r,y_{r^-}+c(r,y_{r^-},z))-(u_n-u)(r,y_{r^-}))\tilde{N}_k(\mathrm{d}r,\mathrm{d}z)\right|$$
$$\to 0 \tag{13}$$

Indeed, since by (12)

$$E\int_t^{s\wedge\tau_G}\int_Z|((u_n-u)(r,y_r+c(r,y_r,z))-(u_n-u)(r,y_r))|^2\,\Pi(\mathrm{d}z)\mathrm{d}r$$
$$\leqslant E\int_t^{s\wedge\tau_G}\mathrm{d}r\left(\int|y|^2\bar{\Pi}(\mathrm{d}y)\int_0^1|(u_n-u)'_x(r,y_r+\alpha y)|^2\,\mathrm{d}\alpha\right)$$

$$\leqslant c_{p/2,T,d}\left(\int_t^T \int_G \left| \int |y|^2 \Pi(\mathrm{d}y)^{p/2} \int_0^1 |(u_n - u)'_x(r, x + \alpha y)|^p \,\mathrm{d}\alpha \,\mathrm{d}x \,\mathrm{d}r\right)^{2/p}\right.$$

$$\leqslant c_{p/2,T,d}\int |y|^2 \Pi(\mathrm{d}y) \, \|u_n - u\|^2_{W_p^{1,2}([t,T]\times\mathbb{R}^d)} \to 0 \quad \text{as } n \to +\infty.$$

Hence (13) holds. □

Now we are in a position to show

Theorem 7. *Under assumptions* (A), (B), (C) *and* $(A)'$ *if* $p > 2d + 4$ *then* x_t *solves* (8), *i.e.*

$$x_t = E_{t,x} x_t = u(t, x), \tag{14}$$

where $u(t, x)$ *satisfies* (8), *and* x_s *is the unique solution of BSDE* (10).

Proof. Applying the Ito formula (Theorem 6) to $u(r, y_r)$ on $r \in [s \wedge \tau, \tau]$, and using Lemma 5 the conclusion is derived. □

Remark. By the proof of above and the imbedding theorem it yields that in the solution (x_s, q_s, p_s) of (10) q_s has a cadlag version $u'_x(s, y_s)\sigma(s, y_s)$ in $L^2_{(\mathcal{F}_{t,\cdot})}(\mathbb{R}^{'m \times r})$ as $t \leqslant s \leqslant T$, provided that $\sigma(s, x)$ is jointly continuous.

Applying Theorems 3 and 6 one can get a convergence theorem on the solutions of PDIEs, which seems to be not easy to derive directly from the PDE theory under such a weak condition.

Assume that $u^n(t, x) \in W_p^{1,2}([0, T] \times \mathbb{R}^d)$, $n = 0, 1, 2, \dots,$ satisfy the following PDIEs

$$\mathcal{L}_{b,\sigma,c} u^n(t, x) = f^n(t, x, u^n(t, x), u_x^{n\prime}(t, x) \cdot \sigma(t, x), u^n(t, x + c(t, x, \cdot)) - u^n(t, x))$$

$$\text{as } t \in [0, T], \; x \in D,$$

$$u^n(T, x) = \varphi^n(x), u^n(t, x)|_{D^c} = \psi^n(t, x), \psi^n(T, x) = \varphi^n(x)|_{D^c},$$

where D is a bounded domain, $\mathcal{L}_{b,\sigma,c}$ is defined in $(8)_1$. Let

$$b^{n,y}(t, x, q, p) = -f^n(t, y, x, q, p).$$

We have

Theorem 8. *Assume that all conditions in Theorem* 6 *hold, and assumption* 1°, 2° *and* 4° *in Theorem* 3 *for* $b^{n,y}(s, x, q, p)$, $n = 0, 1, \dots,$ *are satisfied uniformly with respect to* y *(e.g.*

1° $\langle x, b^{n,y}(s, x, q, p)\rangle \leqslant c(s)(1 + |x|^2 + |x|(\|q\| + |||p|||)),$

uniformly for $y \in \mathbb{R}^d$, $n = 0, 1, 2, \dots$ *etc.*);

Moreover, assume that for all $(t, x) \in [0, T] \times \mathbb{R}^d$

$\bar{1}^\circ$ $I_{D^c}(x)|\psi^n(t, x)| + |\varphi^n(x)| \leqslant k_0, n = 0, 1, \ldots$

and

$\bar{2}^\circ$ $|\psi^n(t, x) - \psi^0(t, x)| I_{D^c}(x) + |\varphi^n(x) - \varphi^0(x)| \to 0$, *as* $n \to \infty$. *Then for all* $(t, x) \in [0, T] \times \mathbb{R}^d$

$$|u^n(t, x) - u^0(t, x)|^2 \to 0, \text{ as } n \to \infty.$$

Proof. For any given $(t, x) \in [0, T] \times D$ by Theorem 6 as $t \leqslant s \leqslant T$ (9) has a weak solution y_s, which is unique in law, and the Ito formula can be applied:

$$u^n(\tau, y_\tau) - u^n(s \wedge \tau, y_{s\wedge\tau})$$

$$= \int_{s\wedge\tau}^{\tau} f^n(r, y_r, u^n(r, y_r), u_x^{n\prime}(r, y_r) \cdot \sigma(r, y_r),$$

$$u^n(r, y_{r^-} + c(r, y_{r^-}, \cdot)) - u^n(r, y_{r^-}))\,\mathrm{d}r + \int_{s\wedge\tau}^{\tau} u_x^{n\prime}(r, y_r) \cdot \sigma(r, y_r)\,\mathrm{d}w_r$$

$$+ \int_{s\wedge\tau}^{\tau} \int_Z (u^n(r, y_{r-} + c(r, y_{r^-}, z)) - u^n(r, y_{r-}))\tilde{N}_k(\mathrm{d}r, \mathrm{d}z)$$

$$n = 0, 1, 2, \ldots,$$

where $\tau = \tau_x$ has been defined before. By virtue of Theorem 3 as $n \to \infty$

$$E_{t,x}|u^n(s \wedge \tau, y_{s\wedge\tau}) - u^0(s \wedge \tau, y_{s\wedge\tau})|^2 \to 0 \quad \text{for all } t \leqslant s \leqslant T.$$

Especially, let $s = t$, one has

$$|u^n(t, x) - u^0(t, x)|^2 \to 0, \text{ as } n \to \infty. \quad \square$$

Applying Theorems 4 and 6 we can also get another convergence theorem on solutions of PDIE as follows:

Theorem 9. *Assume that all conditions in Theorem* 6 *hold, and assumptions* $1^{\circ\prime}$, $2^{\circ\prime}$, *and* $4^{\circ\prime}$ *in Theorem* 4 *are satisfied, where for* $n = 0, 1, 2, \ldots$ *in* $1^{\circ\prime}$, *and for* $n = 1, 2, \ldots$ *in* $2^{\circ\prime}$, *both uniformly with respect to* $y \in \mathbb{R}^d$. *Moreover, assume that conditions* $\bar{1}^0$ *and* $\bar{2}^0$ *of Theorem* 8 *holds. Then the conclusion of Theorem* 8 *still holds.*

Since the proof is similar, we omit it here.

The expression of solution to PDIE (8) by (14) (we may call it a new Feynman–Kac formula) is a useful tool in the study of PDIE. We are now going to use it to derive a new existence theorem for the solution of (8) with non-Lipschitzian force f (the function in the right side of $(8)_1$), which seems to be difficult to derive it directly from the PDIE theory.

Theorem 10. *Under the assumption of Theorem* 5 *except* B.1 *but assuming that*
B.1′: *in the assumption of* B.1 *the Lipschitzian condition for f in (r, c, p) is relaxed to that f is continuous in (r, q, p) and*

$$-\langle r_1 - r_2, f(t, x, r_1, q_1, p_1) - f(t, x, r_2, q_2, p_2)\rangle$$
$$\leqslant k_0(\rho(|r_1 - r_2|^2) + |r_1 - r_2|(\|q_1 - q_2\| + |||p_1 - p_2|||)),$$

where $\rho(u)$ has the same property as that in 2° of Theorem 1, *and*

$$|f(\tau, x, r, q_1\ p_1) - f(t, x, r, q_2, p_2)| \leqslant k_0(\|q_1 - q_2\| + |||p_1 - p_2|||),$$

in addition, assume that

$$|f(t, x, r, q, p)| \leqslant k_0;$$

then the conclusion of Theorem 5 *still holds.*

Proof. Denote the smoothness of f with respect to the variable r by f^n. Using Theorem 5, one gets a unique solution $u^n(t, x) \in W_p^{1,2}(= W_p^{1,2}([0, T] \times \mathbb{R}^d])$ satisfying (8) with respect to the given (f^n, φ, ψ), and one can select a subsequence of $\{u^n\}$ (denote it by $\{u^n\}$ again) such that as $n \to \infty$

$$u^n \to u^0, \text{ weakly in } W_p^{1,2}.$$

Applying Lemma 5 and Theorem 7 for each $n = 0, 1\ 2, \ldots$, there exists a unique solution x_s^n for BSDE (10) with respect to the given (ψ, φ, f^n) such that $u^n(t, x) = x_t^n$, $n = 1, 2, \ldots$. Since by Theorem 4 it yields that $|x_t^n - x_t^0|^2 \to 0$, as $n \to \infty$. Hence it is not difficult to derive that $u^n \to u^0$, in $W_p^{1/2}$, as $n \to \infty$. Moreover, $u^0(t, x) = x_t^0$. Therefore, from the limit one obtains that (8) has a solution $u^0(t, x) \in W_p^{1,2}$ for the given (f, φ, ψ).

Uniqueness. Since by assumption on $f^0 = f$ for each $(t, x) \in [0, T] \times D$ the solution $(x_s^0, q_s^0, p_s^0) \in L^2_{(\mathscr{F}_{t,s})}([t, \tau]; \mathbb{R}^m) \times L^2_{(\mathscr{F}_{t,s})}([t, \tau]; \mathbb{R}^{m \otimes r}) \times \mathscr{F}^2_{(\mathscr{F}_{t,s})}([t, \tau]; \mathbb{R}^m)$, $t \leqslant s \leqslant \tau$, for (10) must be unique. Hence the solution of (8) $u(t, x) = x_t^0$ is also unique. $\square$

We can also get another existence theorem on (8) with non-Lipschitzian f but stronger assumption on a_{ij} and without Lipschitzian condition on f to q, which can be proved without the help of BSDE. But the new Feynman–Kac formula still holds.

Theorem 11. *Under the assumption of Theorem* 5 *except* B.1 *but assuming that* B.1″ *in the assumption* B.1 *the Lipschitzian condition for f in (r, q, p) is relaxed to that f is continuous in (r, q, p) and*

$$-\langle r_1 - r_2, f(t, x, r_1, q_1, p_1) - f(t, x, r_2, q_2, p_2)\rangle$$
$$\leqslant k_0(\rho(|r_1 - r_2|^2) + |r_1 - r_2|(\|q_1 - q_2\| + |||p_1 - p_2|||)),$$

where $\rho(u)$ has the same property as that in Theorem 10, *and*

$$|f(t, x, r, q, p_1) - f(t, x, r, q, p_2)\rangle \leqslant k_0|||p_1 - p_2|||; \tag{15}$$

in addition, assume that $\partial a_{ij}/\partial x_k$, $i, j, k = 1, 2, \ldots, d$ *exist and are bounded,*

$$|a'_x| \leqslant k_0;$$

moreover,

$$|f(t, x, r, q, p)| \leqslant k_0,$$

then the conclusion of Theorem 10 *still holds.*

Remark 1. Conditions in Theorem 11 except (15) are stronger than that in Theorem 10. But (15) is weaker than the condition that f is Lipschitzian with respect to q and p.

Since the proof of Theorem 11 is standard, we omit it here.

It is well known that the Krylov estimate (Krylov, 1980) is a powerful tool in the stochastic analysis and its applications. The crucial assumption in it is that the diffusion coefficient in the related SDE should be uniformly non-degenerate. Since now q_t in our BSDE can be degenerate, hence it seems to be impossible to get a Krylov type estimate related to the BSDE. Anyway, by means of the relation of PDIE and BSDE with jumps we do have a similar estimate, call it a weak form of Krylov estimate related to the BSDE, which may be useful for the stochastic analysis and its applications.

Theorem 12. (A weak form of Krylov estimate). *Under the assumption of Theorem* 7 *for the solution* (x_s, q_s, p_s) *of BSDE* (10) *one has*

$$E_{t,x}\int_t^\tau |f(s, y_s, x_s, q_s, p_s(\cdot))|\,\mathrm{d}s$$

$$\leqslant k_{T,p,d}\,\| f(r, y, u(r, y), u'_x(r, y)\cdot\sigma(r, y), u(r, y + c(r, y, \cdot)) - u(r, y))\|_{L_p},$$

where y_s *is the solution of* (9), $L_p = L_p([0, T] \times D)$,

$$\tau = \inf(s > t\colon y_s \bar{\in} D); \quad \tau = T, \textit{for } \inf\{\phi\}.$$

and $0 \leqslant k_{T,p,d}$ *is a universal constant,* $u(t, x)$ *is the solution of PDIE* (8) *related to* f, *and* D *is a bounded domain in* $\mathbb{R}^d$.

Proof. By Theorem 1 of Marhno (1976) and Theorem 7 here one has

$$E_{t,x}\int_t^\tau |f(s, y_s, x_s, q_s, p_s(\cdot))|\,\mathrm{d}s$$

$$= E_{t,v}\int_t^\tau |f(s, y_s, u(s, y_s), u'_x(s, y_s)\cdot\sigma(s, y_s), u(s, x_s + c(s, y_s, \cdot)) - u(s, y_s))|\,\mathrm{d}s,$$

$$\leqslant k_{T,p,d}\,\| f(r, y, u(r, y), u'_x(r, y)\cdot\sigma(r, y), u(r, y + c(r, y, \cdot)) - u(r, y))\|_{L_p}, \quad \square$$

Acknowledgements and additional notes

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After having completed this paper, the author visited the University of New South Wales in Australia and found that a new paper by Mao (1995) had also discussed the existence of adapted solutions for BSDE with non-Lipschitzian coefficients. But there are obvious differences between the two papers. Mao discussed the BSDE in the continuous case and with fixed deterministic terminal time, while in our paper we treated the problem for the BSDE with jumps and with bounded random stopping time as a terminal time.

Suppose that we keep the assumption on g in Mao's paper unchanged but replace the hypothesis on f in that paper with our weaker assumption, it is easily seen that the unique solution of BSDE with jumps can still be derived by means of the technique developed in our paper but cannot be solved by Mao's method. Moreover, since our technique can also be applied to the continuous BSDE and since our assumption of f is obviously weaker, it shows that our result implies that obtained by Mao, in case we make the same assumption on g. Finally, the author would like to thank professor Layton, the Dean of the Faculty of Commerce, UNSW for his kind invitation and hospitality, and the author would also like to thank the referee for his careful reading of the manuscript, pointing out the misprints, and making nice suggestions.

References

A. Bensoussan, Lectures on stochastic control, in: S.K. Mittler and A. Moro, eds., Nonlinear Filtering and Stochastic Control, Lecture Notes in Mathematics, Vol. 972. (Springer, Berlin 1982) pp. 1–62.

J.M. Bismut, Theorie probabiliste du controle des diffusions, Mem Amer Math. Soc. 176 (1973) 1–30.

N.V. Krylov, Controlled Diffusion Proceses (Springer, Berlin, 1980).

O.A. Ladyzenskaja, V.A. Solonnikov and N.N. Uralceva, Linear and Quasi-linear Equations of Parabolic Type, Translations of Monographs 23 (AMS, Providence, RI, 1968).

X. Mao, Adapted solutions of backward stochastic differential equations with non-Lipschitz coefficients, Stochastic Process. Appl. 58 (1995) 281–292.

S. Marhno, The first boundary problem for integral–differential equations, Theory of Stochastic Processes 4 (1976) Kiev, 66–72 (in Russian).

E. Pardoux and S. Peng, Adapted solution of a backward stochastic differential equation, Systems & Control Lett. 14 (1990) 55–61.

E. Pardoux and S. Peng, Backward doubly stochastic differential equations and systems of quasilinear stochastic partial differential equations, Probab. Theory Related Fields 98 (1994) 2

S. Peng, A general stochastic maximum principle for optimal control problems, SIAM J. Control Optim. 28 (1990) 966–979.

S. Peng, Probabilistic interpletation for systems of quasilinear parabolic partial differential equations, Stochastics and Stochastics Reports 32 (1991) 61–74.

S. Peng, A generalized dynamic programming principle and Hamilton–Jacobi–Bellman equation, Stochastics and Stochastics Reports 38 (1992) 119–134.

S. Peng, Backward stochastic differential equations and applications to optimal control, Appl. Math. Optim. 27 (1993) 125–144.

R. Situ, On strong solution, uniqueness, stability and comparison theorem for a stochastic system with Poisson jumps, Lecture Notes in Control & Inform. Sci; vol. 75, Distributed Parameter System (Springer, Berlin, 1985) 352–381.

R. Situ, Theory and application of stochastic differential equations in China, Contemporary Math. 118 (Amer. Math. Soci., Providence, RI, 1991) 263–280.

R. Situ, A maximum principle for optimal controls of stochastic systems with random jumps, Proc. National Conf. on Control Theory and its Applications, Qingdao (1991) 1–7.

R. Situ and W.L. Chen, Existence of solutions and optimal control for reflecting stochastic differential equations with applications to population control theory, Stochastic Anal. Appl. 10 (1992) 45–106.

S. Tang and X. Li, Necessary conditions for optimal control of stochastic systems with random jumps, SIAM J. Control Optim. 32 (1994) 1447–1475.