Options and Bubbles

Steven L. Heston
University of Maryland

Mark Loewenstein
University of Maryland

Gregory A. Willard
University of Maryland

The Black-Scholes-Merton option valuation method involves deriving and solving a partial differential equation (PDE). But this method can generate multiple values for an option. We provide new solutions for the Cox-Ingersoll-Ross (CIR) term structure model, the constant elasticity of variance (CEV) model, and the Heston stochastic volatility model. Multiple solutions reflect asset pricing bubbles, dominated investments, and (possibly infeasible) arbitrages. We provide conditions to rule out bubbles on underlying prices. If they are not satisfied, put-call parity might not hold, American calls have no optimal exercise policy, and lookback calls have infinite value. We clarify a longstanding conjecture of Cox, Ingersoll, and Ross. (*JEL* G12 and G13)

The Black and Scholes (1973) and Merton (1973) methodology has become the dominant paradigm for valuing options and other derivative assets. This method uses a delta-hedging argument to value options based on the absence of arbitrage strategies that profit instantaneously. It establishes that option values must satisfy a particular partial differential equation (PDE).

But this method does not uniquely determine option values. There may be multiple solutions to the usual valuation PDE. In Section 1, we illustrate this with new closed-form solutions for the Cox, Ingersoll, and Ross (1985) (CIR) term structure model, the constant elasticity of variance (CEV) model, and the Heston (1993) stochastic volatility model. Multiple PDE solutions imply distinct strategies that exactly replicate identical option payoffs at different costs. For any two distinct replication strategies, the return of one solution will be dominated by the other at the option's maturity.

We thank Yacine Aït-Sahalia (the editor) and an anonymous referee, as well as seminar participants at the University of Maryland, Princeton University, Washington University in Saint Louis, Carnegie Mellon, and George Washington University, including Kerry Back, Gurdip Bakshi, Hong Liu, and Dilip Madan for their comments. We are responsible for any errors. Address correspondence to Steve Heston, Robert H. Smith School of Business, Van Munching Hall, University of Maryland, College Park, MD 20742, or e-mail: sheston@rhsmith.umd.edu.

doi:10.1093/rfs/hhl005 Advance Access publication May 15, 2006

An asset with dominated returns has an asset pricing bubble because its payouts can be replicated by a cheaper investment strategy. Spot and option prices can have finite-lived bubbles, a possibility noted by Loewenstein and Willard (2000b). Shorting bubbles generates arbitrage profits by short selling dominated assets and purchasing cheaper replicating strategies. As we explain shortly, the feasibility of shorting bubbles crucially depends on assumptions about trading opportunities, particularly those used to make doubling strategies infeasible.

Section 2 identifies three necessary and sufficient conditions that prevent the underlying assets—the "bond" and the "stock"—from being dominated in diffusion models.[1] The first is the familiar Black-Scholes-Merton condition ensuring no instantaneous profit without risk. The second ensures that buying and holding a discount bond is undominated. Together, the first and second conditions are necessary and sufficient for an equivalent local martingale measure, as defined by Delbaen and Schachermayer (1994). We use the second condition to clarify a longstanding arbitrage conjecture of Cox, Ingersoll, and Ross and show that it allows parameter values previously dismissed for admitting arbitrage opportunities. In their independent and ongoing work, Cheridito, Filipović, and Kimmel (2003) derived a similar condition as being necessary and sufficient for an equivalent local martingale measure for the CIR model.[2] Our third and final condition ensures that buying and holding the stock is undominated. The CEV model violates the third condition (Section 1.2). All three conditions are necessary and sufficient for an equivalent martingale measure for the underlying assets, as defined by Harrison and Kreps (1979). These conditions also provide further parameter restrictions to prevent bubbles in models estimated by Broadie et al. (2000), Chernov and Ghysels (2000), Chernov et al. (2003), Eraker, Johannes, and Polson (2003), and Jones (2003).

Section 3 examines relative spot and option prices given just an equivalent local martingale measure (i.e., when Conditions 1 and 2 hold). Option values in the presence of bubbles have unusual properties. With stock bubbles, put-call parity and risk-neutral option pricing are mutually exclusive, American options do not have optimal exercise times, and lookback call options have infinite value. When option values independently have bubbles, covered option strategies imply (possibly infeasible) arbitrage

[1] The mathematical conditions we use appear in the stochastic process literature [see, e.g., Stroock and Varadhan (1979), Karatzas and Shreve (1988), and Lipster and Shiryaev (2000)]; however, they have been only sporadically applied to finance applications, notable exceptions being Sin (1998), Lewis (2000), and Cheridito, Filipović, and Kimmel (2003). None of these studies connect these conditions to specific dominated investment strategies and limits of arbitrage.

[2] Cheridito, Filipović, and Kimmel (2003) further demonstrated the empirical importance of allowing the broader set of parameters. Our analysis also resolves related questions by Chernov and Ghysels (2000) for a stochastic volatility model.

opportunities. Understanding these properties helps connect appropriate models of stock prices and solutions for option values for practical applications. Our theoretical predictions of option values in the presence of bubbles resemble the empirical violations of put-call parity and other no-arbitrage relations documented for dot-com stocks by Mitchell, Pulvino, and Stafford (2002), Lamont and Thaler (2003), and Ofek, Richardson, and Whitelaw (2003). Our results bridge the theory of bubbles and the empirical properties of derivative prices.

Section 4 shows that the theoretical relevance of bubbles depends on what types of trades are feasible given assumptions used for making doubling strategies infeasible. Delbaen and Schachermayer (1994) assumed that negative wealth is bounded for this purpose. Under their assumption, attempts to arbitrage stock and option bubbles by shorting them share the main characteristic of a doubling strategy by risking unbounded temporary losses before producing an arbitrage profit. Our Proposition 4.1 shows that bounds on negative wealth make these attempts infeasible and permit models of bubbles, including the CEV model we analyze in Section 1.2. By contrast, Harrison and Kreps (1979) did not limit negative wealth but required strategies to be locally buy-and-hold ("simple") to make doubling strategies infeasible. Although this gives an equivalent martingale measure and therefore rules out the CEV model, it also makes continuous replicating strategies infeasible, typically including the strategies having returns that dominate the returns of the assets with bubbles.

1. Three Examples

We begin with three examples to motivate our analysis. Each example yields multiple solutions of the valuation PDE but has unique economic interpretations. In general, multiple solutions imply different replicating costs for assets. Later, we define a bubble as the difference between a given solution and the cheapest solution. Our first example uses the CIR term structure model with a linear risk premium to provide an explicit bubble on the CIR discount bond price for particular parameter values. Our second example uses the CEV stock-price model to provide explicit bubbles on stock and option values in the absence of bond bubbles. Our third example uses the stochastic volatility models of Heston (1993) and Grünbichler and Longstaff (1996) to provide option bubbles in the absence of stock and bond bubbles. Later sections provide a general analysis of each type of bubble.

1.1 Example: The CIR model

Our first example examines multiple solutions of the valuation PDE in the CIR term structure model with a linear risk premium—that is the case in which the excess expected rate of return on a bond is linear in the locally riskless rate [Cox, Ingersoll, and Ross (1985, Section 5)]. It illustrates the possibility of bubbles on bond prices. We assume that the locally riskless interest rate r is described under the empirical, or true, measure P by

$$dr = \alpha(\beta - r)dt + \sigma\sqrt{r}dZ, \tag{1}$$

where α and β are positive constants and Z is a P-Brownian motion. So that r cannot hit zero in finite time under P, we also assume that

$$2\alpha\beta \geq \sigma^2. \tag{2}$$

We focus on a linear risk premium $\psi_0 + \psi_1 r$, where ψ_0 and ψ_1 are constants.

Inequality (2) rules out the arbitrages identified by CIR caused by the interest rate hitting zero when $\psi_0 \neq 0$.[3] Cox, Ingersoll, and Ross further conjectured that all parameter values lead to arbitrage given the linear risk premium even if inequality (2) holds and $\psi_0 \neq 0$. Our later analysis in Section 2 (Condition 2) shows that this conjecture is not generally correct. Nonetheless, some parameter values produce arbitrage opportunities corresponding to multiple PDE solutions and bond bubbles, as we now show.

Example 1.1. *A unit discount bond has a payout equal to one at maturity T. CIR show the bond's value G(r,t) satisfies the valuation PDE*

$$\frac{\sigma^2}{2} rG_{rr} + \alpha(\beta - r)G_r + G_t - rG = \left(\psi_0 + \psi_1 r\right)G_r \text{ subject to } G(r,T) = 1. \tag{3}$$

Define $\hat{\alpha} = \alpha + \psi_1$ *and* $\hat{\beta} = (\alpha\beta - \psi_0)/(\alpha + \psi_1)$.

One solution of (3) *is the standard CIR formula* $G^1(r,t) = A(t)e^{-B(t)r}$, *where*

$$A(t) = \left[\frac{2\gamma e^{(\hat{\alpha}-\gamma)(T-t)/2}}{2\gamma + (\hat{\alpha} - \gamma)[1 - e^{-\gamma(T-t)}]}\right]^{\frac{2\hat{\alpha}\hat{\beta}}{\sigma^2}},$$

[3] Inequality (2) prevents the local price of risk from becoming infinite. We address this in Section 2 (Condition 1).

$$B(t) = \frac{2(1 - e^{-\gamma(T-t)})}{2\gamma + (\hat{\alpha} - \gamma)[1 - e^{-\gamma(T-t)}]},$$

and $\gamma = \sqrt{\hat{\alpha}^2 + 2\sigma^2}$.

For some values of ψ_0, *the original CIR solution is not the cheapest nonnegative solution. If inequality* (2) *holds and*

$$2\hat{\alpha}\hat{\beta} < \sigma^2, \tag{4}$$

then a cheaper solution is

$$G^2(r,t) = A(t)e^{-B(t)r}\left[1 - \frac{\Gamma(\nu, r\zeta(t))}{\Gamma(\nu, 0)}\right], \tag{5}$$

where $\Gamma(\nu, m) = \int_m^\infty e^{-l} l^{\nu-1} dl$ *is the incomplete gamma function,* $\nu = 1 - 2\hat{\alpha}\hat{\beta}/\sigma^2$, *and*

$$\zeta(t) = \frac{2}{\sigma^2} \frac{e^{-\hat{\alpha}(T-t)} A(t)^{-\frac{1}{\hat{\alpha}\hat{\beta}}}}{\int_t^T e^{-\hat{\alpha}(T-s)} A(s)^{-\frac{1}{\hat{\alpha}\hat{\beta}}} ds}.$$

Inequality (4) *ensures* ν *is positive in* $\Gamma(\nu, \cdot)$. *Note that* G^2 *is nonnegative, agrees with* G^1 *at maturity for positive values of r, and is strictly less than* G^1 *prior to maturity.*

Section 2 later reveals that the CIR bond price is indeed the lowest cost nonnegative solution if $2\hat{\alpha}\hat{\beta} \geq \sigma^2$. *This ensures that there is an equivalent martingale measure Q so that r has zero probability of hitting zero under both P and Q. Thus, Cox, Ingersoll, and Ross's conjecture that linear risk premiums always allow arbitrage is not generally true. In their independent work, Cheridito, Filipović, and Kimmel (2003) showed that this condition leads to an equivalent local martingale measure and showed that the value of* ψ_0 *is empirically important.*

We note some important features of this example when $2\hat{\alpha}\hat{\beta} < \sigma^2$.

- **There is no equivalent (local) martingale measure**. Note that the valuation PDE can be rewritten as

$$\frac{\sigma^2}{2} r G_{rr} + \hat{\alpha}(\hat{\beta} - r)G_r + G_t - rG = 0.$$

  If there were an equivalent (local) martingale measure Q, r would have drift $\hat{\alpha}(\hat{\beta} - r)$ under Q. Inequality (4) indicates r can hit zero under Q even though it cannot under P (2). The measures P and Q are not equivalent because their zero probability events differ. Intuitively, we can interpret the solution G^2 as the value of a "knockout" bond that pays zero if the interest rate hits zero prior to maturity and 1 otherwise. Thus, $G^1 - G^2$ would represent the price of a claim that pays a dollar if the interest rate hits zero prior to maturity. Although under P the origin is inaccessible, the measure Q assigns this a positive probability and hence a positive price.

- **There is an arbitrage with bounded temporary losses**. The arbitrage involves short selling G^1 and purchasing G^2 (perhaps using replicating strategies), thereby selling payoffs in the event r hits zero under Q.[4] But when r is sufficiently close to zero, $G^2 - G^1$ is negative, implying that the arbitrage risks temporary losses prior to closure. These temporary losses are bounded because $G^2 - G^1 \geq -1$. Whether the arbitrage is feasible at a large scale depends on the set of feasible trades: Indeed, it would not even be considered to be a feasible arbitrage according to Dybvig and Huang (1988, Definition 1 and Footnote 7), who assume fixed bounds on negative wealth; in Delbaen and Schachermayer's study (1994), it would be unlimited because an investor may choose a wealth bound simultaneously with a strategy.
- **The original CIR bond price has a bounded asset pricing bubble**. The replicating cost of G^1 exceeds the replicating cost of G^2, yet the payouts at maturity are the same. Section 2 explains that this feature corresponds to the classical definition of a bubble. The bubble is the nonnegative difference $G^1 - G^2 = A(T-t)e^{-B(T-t)r}\frac{\Gamma(\nu, r\zeta(t))}{\Gamma(\nu,0)}$ and is bounded above by one.

We present another example with different economic features.

1.2 Example: The CEV model

Our second example shows multiple solutions in CEV model of asset prices.[5] It shows the possibility of bubbles on stock and option values even when bond prices do not have bubbles and an equivalent local martingale measure exists. The stock-price process is described by

$$dS = rSdt + \sigma S^{\alpha} dZ^{Q}, \tag{6}$$

[4] Because locally riskless lending and borrowing are possible, replicating strategies exist if the discount bond price is described by either solution when $2\alpha\beta \geq \sigma^2$. Thus, there are two portfolios with the same payoffs but different costs.

[5] This model has been studied empirically by Beckers (1980), MacBeth and Merville (1980), Emmanuel and MacBeth (1982), Gibbons and Jacklin (1988), and Jones (2003).

where r is the instantaneous riskless rate. We assume that r and σ are positive constants. The Brownian motion Z^Q makes the local stock return equal to r under a given equivalent change of measure Q. The coefficient α is the elasticity of variance. Our analysis focuses on $\alpha > 1$, which we assume for this example.[6]

The stochastic differential equation (6) defines a unique and "well-behaved" strictly positive price process that does not explode to infinity. Following Emmanuel and MacBeth (1982), the probability density of S_T conditional on $S_t = S$ at time $t < T$ is

$$f\left(S_T,T|S,t\right) = 2(\alpha-1)k^{\frac{1}{2(1-\alpha)}}\left(xz^{1-\alpha}\right)^{\frac{1}{4-\alpha}}e^{-x-z}I_{\frac{1}{2(\alpha-1)}}\left(2\sqrt{xz}\right), \qquad (7)$$

where

$$\begin{aligned} k &= \frac{2r}{\sigma^2 2(1-\alpha)[e^{2r(1-\alpha)(T-t)}-1]}, \\ x &= kS^{2(1-\alpha)}e^{2r(1-\alpha)(T-t)}, \\ z &= kS_T^{2(1-\alpha)}, \end{aligned}$$

and I_q is the modified Bessel function of the first kind of order q. The moments $E^Q[S_T^p]$ exist for $p < 2\alpha - 1$. In particular, the variance of S_T is finite for $\alpha > 3/2$. However, the expected instantaneous variance $\sigma^2 E^Q[S_T^{2\alpha}]$ is infinite. Thus, the probability density has extremely fat tails.

Unlike the preceding CIR example, here there is an equivalent change of probability equating the instantaneous stock return and the riskless rate [an equivalent local martingale measure, as defined by Delbaen and Schachermayer (1994)]. There are no bond bubbles. Nonetheless, we show the option valuation PDE has multiple solutions with properties economically different from the CIR example.[7]

Example 1.2. *A European call option pays* $\max(S_T - K,0)$ *at maturity* T. *Its value must satisfy the valuation PDE*

$$\frac{1}{2}\sigma^2 S^{2\alpha}G_{SS} + rSG_S + G_t - rG = 0 \qquad (8)$$

[6] The special case of $\alpha = 1$ corresponds to the original Black and Scholes (1973) model. Cox (1975) and MacBeth and Merville (1980) focused on the case $\alpha < 1$, whereas Emmanuel and MacBeth's study (1982) and subsequent literature explicitly addressed the case $\alpha > 1$.

[7] Lewis (2000) also recognized the possibility of multiple solutions for the CEV model but does not connect them to dominated strategies, bubbles, or arbitrage.

subject to the boundary conditions $G(S,T) = \max(S - K,0)$, $G(0,t) = 0$, *and* $G(\infty,t) = \infty$. *For* $\alpha > 1$, *Emmanuel and MacBeth (1982), Schroder (1989, Footnote 2), and Hull (2003, Chapter 20) present a solution of the form*

$$G^1(S,t) = Sp_1(S,t) - e^{-r(T-t)}Kp_2(S,t), \tag{9}$$

where

$$p_1(S,t) = Q\left[2x, \frac{1}{\alpha - 1}, 2kK^{2(1-\alpha)}\right]$$
$$p_2(S,t) = 1 - Q\left[2kK^{2(1-\alpha)}, 2 + \frac{1}{\alpha - 1}, 2x\right],$$

where x and k are defined above, $Q[a,\nu,\lambda]$ *is the complementary chi-square cumulative distribution function with degrees of freedom* ν *and noncentrality parameter* λ,

$$Q[a,\nu,\lambda] = \int_a^\infty p[b,\nu,\lambda]db$$

and

$$p[b,\nu,\lambda] = \frac{1}{2}e^{\frac{-(b+\lambda)}{2}}\left(\frac{b}{\lambda}\right)^{\frac{(\nu-2)}{4}} I_{(\nu-2)/2}\left(\sqrt{\lambda b}\right).$$

Note that p_1 *satisfies the PDE*

$$\frac{1}{2}\sigma^2 S^{2\alpha}p_{1SS} + \left(rS + \sigma^2 S^{2\alpha-1}\right)p_{1S} + p_{1t} - rp_1 = 0 \tag{10}$$

subject to the condition $p_1(S,T) = 1_{\{S>K\}}$. *Thus,* p_1 *can be interpreted as a solution to the backward equation for the probability the option expires in the money when the spot price process follows the dynamics*

$$dS = \left(rS + \sigma^2 S^{2\alpha-1}\right)dt + \sigma S^\alpha dZ, \tag{11}$$

where Z is a Brownian motion. This raises an immediate issue. Although the CEV process (6) *remains finite almost surely, the process* (11) *can explode to infinity in finite time (we show this formally in Example 2.3). Depending on the boundary conditions at infinity (absorbing, reflecting, etc.), there are* *different solutions for* p_1 *that satisfy* (10) *and consequently imply different solutions for the call option.*

In particular, Emmanuel and MacBeth (1982) and Hull (2003, Chapter 20) incorrectly implied that their solution is the risk-neutral expected discounted value of the payoff to the call option. It is not. Instead, direct calculation using the probability density (7) *produces a new formula for the CEV model*

$$G^2(S,t) = S\left(p_1(S,t) - \frac{\Gamma(v,u)}{\Gamma(v,0)}\right) - e^{-r(T-t)}Kp_2(S,t),$$

where $\Gamma(v,m) = \int_m^\infty e^{-l} l^{v-1} dl$ *is the incomplete gamma function,* $v = \frac{1}{2(\alpha-1)}$, *and*

$$u = \frac{2r}{\sigma^2(1-\alpha)} \frac{e^{2r(1-\alpha)(T-t)}}{\left(e^{2r(1-\alpha)(T-t)} - 1\right)} S^{2(1-\alpha)}.$$

This solution satisfies the same boundary conditions as the published solution (9). *It is the risk-neutral expected discounted payoff of the call option and is also the cheapest nonnegative solution subject to the boundary conditions.*

We note several features of this example.

- **There is an arbitrage even though an equivalent local martingale measure exists**. The solutions' difference $G^1 - G^2 = S\frac{\Gamma(v,u)}{\Gamma(v)}$ satisfies the PDE subject to a terminal condition of zero at time T. Thus, it represents the value of a dynamic trading strategy that turns a positive amount of money into nothing almost surely by time T, termed a "suicide strategy" by Harrison and Pliska (1981). A reverse position in this strategy shares the main characteristic of a doubling strategy; namely, it risks unbounded temporary losses prior to closure. This arbitrage would be infeasible for an investor prohibited from risking unbounded marked-to-market dollar losses, perhaps to make doubling strategies infeasible (Proposition 4.1 shows this for general models). The arbitrage is also infeasible for an investor with no ability to short sell the stock.
- **There are asset pricing bubbles on option values, as well as on the stock price**. Emmanuel and MacBeth's solution G^1 exceeds our solution G^2, the minimum quantity needed to replicate the option payoff. Thus, $G^1 - G^2$ is a bubble. The same bubble exists on the stock price [we can set $K = 0$ in (9) so that $G^1 = S$]. There is a portfolio with returns that almost surely dominates the stock return at time T but risks underperforming at any time $t < T$. The bubble is unbounded, unlike that in the preceding CIR example.

- **One can choose either put-call parity or risk-neutral option pricing, but not both**. Emmanuel and MacBeth's solution G^1 does not satisfy risk-neutral pricing, yet as we show in Section 3, put-call parity will hold for G^1 if the put is risk-neutral priced [Emmanuel and MacBeth and Hull (2003, Problem 20.1) use the risk-neutral put price]. The solution G^2 is less than G^1, so put-call parity cannot hold if the put and call are risk-neutral priced. We also show in Section 3 that one must choose among other "no-arbitrage" relations. For example, the limit of the call price G^1 as the strike price approaches infinity is not zero, but the limit of the call price G^2 *is* zero.

Example 1.2 shows that the criteria for selecting a "correct" solution might not be obvious. In particular, both solutions are bounded by linear functions of the stock price. Because the PDE method by itself does not identify a unique solution, we will ultimately need to impose additional economic considerations.

Our third example has additional economic differences.

1.3. Example: Stochastic volatility

This section presents a particularly simple closed-form solution for a bubble in the Heston (1993) model. In this example, the stock price is described by

$$dS_t = rS_t dt + \sqrt{V_t} S_t dZ_t^Q,$$

with stochastic variance V_t described by

$$dV_t = \sigma^2 dt + \sigma\sqrt{V_t} dZ_t^Q,$$

where r and σ are positive constants. The variance process V cannot hit zero. By construction, an equivalent risk-neutral measure Q exists, so there are no bond bubbles. One can also show that the discounted stock price is a martingale under Q, so there are no stock price bubbles either. Nonetheless, option values can have bubbles associated with multiple PDE solutions.

Example 1.3. *Heston (1993) provided a formula for European equity option values, and Grünbichler and Longstaff (1996) provided a formula for European volatility derivatives (defined, e.g., by the payoff* $\max\{V_T - K, 0\}$ *at maturity). To handle both studies' formulas simultaneously, we denote the time T payout of a European derivative by* $F(S_T, V_T)$. *The formulas solve the valuation PDE*

$$\frac{1}{2}S^2VG_{SS}+\frac{1}{2}\sigma^2VG_{VV}+\sigma SVG_{SV}+rSG_S+\sigma^2G_V+G_t-rG=0 \quad (12)$$

subject to the appropriate terminal condition $G(S,V,T)=F(S,V)$.

We provide solutions more expensive than those of Heston (1993) and Grünbichler and Longstaff (1996). Our bubble in this example is the formula

$$\Pi(V,t)=\frac{1}{V}e^{-r(T-t)-\frac{2V}{\sigma^2(T-t)}}.$$

It satisfies the PDE

$$\frac{1}{2}\sigma^2V\Pi_{VV}+\sigma^2\Pi_V+\Pi_t-r\Pi=0$$

subject to the condition $\Pi(V,T)=0$. *As for the CEV example, the bubble corresponds to a suicide strategy that starts with wealth* $\Pi(V_0,0)>0$ *and ends with wealth* $\Pi(V_T,T)=0$. *An alternate solution for the valuation PDE* (12) *is* $G^2(S,V,t)=G^1(S,V,t)+\Pi(V,t)$, *where* G^1 *appropriately corresponds to the Heston (1993) equity option pricing formula or the Grünbichler and Longstaff (1996) volatility derivative pricing formula.*[8]

There is an equivalent local martingale measure for the stock, the option, and the money market account. There are no stock or bond bubbles. Yet Π is an unbounded option bubble, and there is an arbitrage involving a short position in the replicating strategy for G^2 and a long position in that for G^1. The arbitrage shares the main characteristic of a doubling strategy in that it risks unbounded marked-to-market losses when V approaches zero. Thus, it may be infeasible given bounds on negative wealth that rule out doubling strategies (Proposition 4.1).

Here are this example's special features.

- **Stock bubbles are not (mathematically) necessary for option bubbles**. Option bubbles are not ruled out by an equivalent martingale measure for the stock. Even seemingly "well-behaved" models can admit option bubbles. Whether bubbles are relevant for practice or empirical estimation requires additional economic considerations we present in Section 3.
- **If G^2 describes the equity option values, a covered call strategy risks unbounded temporary losses**. The solution G^2 must exceed the stock price with positive probability. In principle, a covered call strategy

[8] The bubble solution in this example does not satisfy a Lipshitz growth condition. The bubble solutions in our previous examples do. Section 4 analyzes economic restrictions on solutions.

offers an arbitrage opportunity whenever its value is negative, but Proposition 4.1 can be used to show the strategy is also infeasible given typical bounds on negative wealth that rule out doubling strategies. That institutional arrangements often permit covered call trades without margin might suggest appropriate solutions for option values.

The preceding examples illustrate that multiple PDE solutions are possible and are connected to the existence of bubbles, dominated asset prices, and (possibly infeasible) arbitrage strategies. Bubbles can appear on any combination of bond prices, stock prices, or option values. We now turn to our general analysis.

2. Conditions for the Absence of Stock and Money Market Bubbles

We now study bubbles, arbitrage, and multiple PDE solutions for standard diffusion models that have no frictions other than lower bounds on negative wealth. The model should be understood to include the by-now standard conditions on portfolios meant to ensure the existence of stochastic integrals.[9] In this section, we focus on the primitive assets—a risky stock and a money market account. We provide three step-by-step necessary and sufficient conditions to rule out bubbles on these assets. The first ensures that the local price of risk is finite. The second additionally ensures an equivalent local martingale measure, which further restricts the local price of risk as a function of state variables. It ensures that run money market and bond returns are undominated. The third additionally checks the martingale properties of the stock price to ensure that its returns are not dominated and restricts its volatility as a function of the state variables.[10] It rules out stock bubbles. These conditions are nested and progressively rule out specific types of bubbles by enlarging the set of feasible arbitrage strategies. In particular, the conditions classify bubbles and arbitrage by their types.

We now provide a formal definition of a bubble consistent with our later assumption that assets pay no intermediate dividends.

Definition 2.1 (Asset Pricing Bubble). *An asset with a nonnegative price has a "bubble" if there is a self-financing portfolio with pathwise nonnegative wealth that costs less than the asset and replicates the asset's price at a fixed*

[9] See, for example, Dybvig and Huang (1988, pp. 380–384) for a complete description of these conditions.

[10] The mathematical conditions we use are similar to those developed for the "martingale problem" [Stroock and Varadhan (1979, Chapter 10)]. Sin (1998) and Cheridito, Filipović, and Kimmel (2003) used similar techniques for specific financial models. We remark that the often-assumed Novikov condition is only sufficient and is typically difficult to verify Karatzas and Shreve (1988, Corollary 3.5.13).

future date.[11] *The bubble's value is the difference between the asset's price and the lowest cost replicating strategy.*

The economic importance of a bubble relates to the Law of One Price, which says that two portfolios with identical payouts have identical prices. A bubble would seem to imply an arbitrage, but the corresponding strategy might be made infeasible by bounds on negative wealth [Delbaen and Schachermayer (1994), Loewenstein and Willard (2000a, b)], short-sales constraints [Basak and Croitoru (2000)], or other limits of trade. Our definition is used in general equilibrium theory [Santos and Woodford (1997), Loewenstein and Willard (2000b)] and is related to "limits of arbitrage" as studied by De Long et al. (1990), Shleifer and Vishny (1997), Lamont and Thaler (2003), and others.

2.1 Framework for model specification.

We first specify a general class of models for asset prices. We focus on one-factor models, those in which the investment opportunity set is driven by a single state variable Y.[12] We assume the state variable Y is the unique strong solution of the stochastic differential equation

$$dY_t = a(Y_t)dt + b(Y_t)dB_t, \tag{13}$$

where Y_0 is strictly positive. The process B is a Brownian motion under the empirical probability measure P. We assume that $P\{Y_t = 0\} = 0$ for all t. Moreover, we assume that $a(\cdot)$ and $b(\cdot)$ are continuous functions on $(0,\infty)$ with $b(\cdot)^2 > 0$.

The stock price is denoted S_t, and the value of a rolled over investment in a money market account is denoted R_t. We assume that

$$dS_t = \mu(Y_t)S_t dt + \sigma(Y_t)S_t dZ_t$$

where S_0 is a positive initial stock price and μ and σ are continuous functions on $(0,\infty)$ with $\sigma^2(\cdot) > 0$, and

$$dR_t = r(Y_t)R_t dt$$

where $R_0 = 1$. We assume that the locally riskless interest rate $r(\cdot)$ and the instantaneous correlation $\rho(\cdot)$ between Z and B are continuous functions of Y_t. We can alternatively write the asset values as

[11] See Dybvig and Huang [1988, Relation (8), p. 383] for the self-financing constraint. The definition of a bubble given intermediate dividends would require the self-financing replicating strategy to replicate those dividends.

[12] Much of our analysis could also include multiple state variables.

$$S_t = S_0 \exp\left[\int_0^t \left(\mu(Y_s) - \frac{1}{2}\sigma(Y_s)^2\right) ds + \int_0^t \sigma(Y_s) dZ_s\right]$$

and

$$R_t = \exp\left[\int_0^t r(Y_s) ds\right].$$

Under this specification, markets would be incomplete if $|\rho| \not\equiv 1$. To simplify matters, we assume that either $|\rho| \equiv 1$ or there is a third traded asset that earns the locally riskless rate and is perfectly correlated with the Brownian motion risk unspanned by the stock. This assumption is typical for studies of option pricing and can be relaxed if the local price of unspanned risk is a function of Y.

We first make sure the solution of (13) in fact satisfies our assumptions. This introduces explosion methods useful for confirming the presence of bubbles as we explain shortly. The procedure begins by choosing an arbitrary $c \in (0,\infty)$ and constructing the "scale measure" for Y defined by

$$p_c(x) = \int_c^x \exp\left[-2\int_c^\eta \frac{a(\phi)}{b(\phi)^2} d\phi\right] d\eta$$

and the "speed measure" for Y defined by

$$v_c(x) = \int_c^x p_c^{'}(y) \int_c^y \frac{2}{p_c^{'}(z) b(z)^2} dzdy.$$

The scale measure and speed measure give useful information about the solutions to (13). In particular, Y can take on any value in $(0,\infty)$. Moreover, Karatzas and Shreve (1988, Proposition 5.5.29) (Feller's test for explosions) states that if

$$\lim_{x\downarrow 0} v_c(x) = \lim_{x\uparrow\infty} v_c(x) = \infty,$$

then the process Y cannot hit the origin or explode to ∞ in finite time, consistent with our assumptions. However, if this condition fails, then the origin or ∞ can be reached in finite time, a case we do not formally examine.[13]

2.2 Absence of instantaneously profitable arbitrage

Our first condition ensures that the price of risk is finite to rule out arbitrages that profit instantaneously without the risk of loss. Instantaneously profitable arbitrages imply the existence of portfolios with positive risk premium and zero volatility. These arbitrages would never lose money and, consequently, would never violate a nonnegative wealth constraint starting from a nonnegative initial investment. The PDE method of valuing options relies on the absence of these arbitrages, as discussed by Black and Scholes (1973), Merton (1973), Cox, Ingersoll, and Ross (1985), and others.

Denote the instantaneous local price of risk (or Sharpe ratio or return per unit risk) by $\kappa(Y_t) \equiv \frac{\mu(Y_t)-r(Y_t)}{\sigma(Y_t)}$. Here is our first condition.

Condition 1: $\boldsymbol{\kappa}(Y_t)$ is finite valued.

Ruling out instantaneously profitable arbitrages is economically sensible when one could take a long position in the higher drift portfolio and a short position in the lower drift portfolio and earn a locally riskless profit. Condition 1 is automatically satisfied under our assumption that Y cannot hit zero or infinity and our assumptions about the model's coefficients. However, Condition 1 might not be automatically satisfied in other models. In those models, it would represent an important restriction on parameters and would be useful for classifying those restrictions by their implied assumption about feasible portfolios. The following well-known example illustrates an instantaneously profitable arbitrage.

Example 2.1. *Consider the CIR term structure example in Section 1.1. When $\alpha\beta < \sigma^2$, the interest rate r hits zero under P with positive probability. For the linear risk premium*

$$\kappa = \frac{\psi_0 + \psi_1 r}{\sigma\sqrt{r}},$$

CIR show an instantaneously profitable arbitrage would exist if $\psi_0 \neq 0$. Note in this case $|\kappa| = \infty$ whenever $r = 0$ and $\psi_0 \neq 0$, so Condition 1 is not satisfied.

We now present our second condition.

[13] Because of the time homogeneity of our coefficients of Y, if the probability of explosion in finite time is positive, the probability of explosion at any fixed time is also positive.

2.3 Absence of money market bubbles

Even if the price of risk is finite, money market bubbles are a concern, as shown by the CIR example of Section 1.1. A money market bubble corresponds to an arbitrage strategy that risks temporary losses before it generates an ultimate profit. The arbitrage strategy would be infeasible given a nonnegative wealth constraint; it would be infeasible at some scale given a fixed lower bound on negative wealth. Neither restriction on the feasible portfolio set allows an investor to arbitrage a money market bubble at sufficiently large scales (they still allow investors to fully exploit instantaneously profitable arbitrages). An investor could generate unlimited profits from the arbitrage strategy only if the feasible portfolio set allows the temporary losses and the arbitrage strategy to be scaled together [as is the case in Delbaen and Schachermayer (1994); see Section 4].

To rule out money market bubbles, we provide Condition 2 to additionally ensure the existence of a change of probability measure Q equivalent to P, under which the stock price is given by

$$dS_t = r(Y_t)S_t dt + \sigma(Y_t)S_t dZ_t^Q,$$

where Z_t^Q is a Q-Brownian motion. Delbaen and Schachermayer (1994) call Q an equivalent *local* martingale measure because the discounted stock price generally remains a Q-local martingale.

Condition 2: The exponential local martingale

$$M_t \equiv \exp\left[-\frac{1}{2}\int_0^t |\kappa(Y_s)|^2 ds - \int_0^t \kappa(Y_s) dZ_s\right]$$

is a strictly positive martingale.

Note that Condition 2 nests Condition 1 by requiring M to be strictly positive [Revuz and Yor (1994, Exercise IV.3.25)]. Thus, Condition 2 rules out instantaneously profitable arbitrages and the arbitrages associated with money market bubbles. Given Condition 2, we can use M to define an equivalent change of measure Q under which the state variable has the form

$$dY_t = \{a(Y_t) - \rho(Y_t)\kappa(Y_t)b(Y_t)\}dt + b(Y_t)dB_t^Q, \tag{14}$$

where $B_t^Q \equiv B_t + \int_0^t \rho(Y_s)\kappa(Y_s)ds$ is a Q-Brownian motion [Girsanov's theorem; Karatzas and Shreve (1988, Theorem 3.5.1)]. We show rigorously in the Appendix that Condition 2 is equivalent to the condition that solutions of (14) cannot explode to infinity or hit zero. If these solutions explode, Q obviously could not be equivalent to P because this behavior is not possible under P. This gives us Condition $2'$, which can be checked by applying the test for explosions in a Section 2.1 to (14).

Condition 2′. The solution of (14) does not explode or hit 0 under Q.

To connect Condition 2 to money market bubbles, define a stochastic discount factor by $\xi_T = M_T/R_T$. This gives the well-known asset pricing relations

$$E[\xi_T R_T] \leq R_0 = 1 \quad \text{and} \quad E[\xi_T S_T] \leq S_0$$

[see, e.g., Loewenstein and Willard (2000a)]. Condition 1 ensures that $\xi_T > 0$ almost surely [Revuz and Yor (1994, Exercise IV.3.25)]. Condition 2 additionally ensures that $E\left[\xi_T R_T\right] = R_0$, which is equivalent to $dQ/dP = M_T$ being a probability density (because the martingale property is equivalent to $E[M_T] = M_0 = 1$). It is still possible, however, that $E^Q\left[S_T/R_T\right] < S_0$, meaning the discounted stock price is only a local martingale, not a martingale.

A failure of Condition 2 implies there is a trading strategy that replicates the payout R_T at a cost lower than $R_0 = 1$; that is, there is a money market bubble. Section 1.1 provides an example, as we now explain.

Example 2.2. *In the CIR example of Section 1.1, there is a bubble on the CIR discount bond price when $2\alpha\beta \geq \sigma^2$ and $2\hat{\alpha}\hat{\beta} < \sigma^2$. The state variable Y is the locally riskless rate r. Condition 2 does not hold. The bond bubble is $G^1(r,t) - G^2(r,t)$ and is bounded by one. There is also a money market bubble. Consider the strategy that shorts the replicating strategy for G^1, goes long the replicating strategy for G^2, and invests 1 in the money market. The nonnegative initial cost is $1 - G^1(r_0,0) + G^2(r_0,0) < 1$, and the time t value is $e^{\int_0^t r_s ds} + G^2(r_t,t) - G^1(r_t,t) \geq 0$. The payout at maturity is $R_T = e^{\int_0^T r_t dt}$, the same as investing 1 directly in the money market.*

The CIR bond price does not have a bubble when $2\alpha\beta \geq \sigma^2$ and $2\hat{\alpha}\hat{\beta} \geq \sigma^2$. In that case, there is an equivalent martingale measure Q for the CIR bond price, making it the lowest cost solution. There are no money market bubbles either. The mathematical condition ensures that r cannot hit zero under both P and Q. Moreover, r cannot hit infinity. Cox, Ingersoll, and Ross conjectured that there is an arbitrage even if the origin is inaccessible. Condition 2 shows that this conjecture is not general.[14] The absence of money market bubbles would imply the absence

[14] After obtaining this result, we learned it is also independently proven by Cheridito, Filipović, and Kimmel (2003).

of bubbles on discount bond prices if r is nonnegative. This follows from the fact that G^1/R is a bounded Q-local martingale and, hence, a Q-martingale.

2.4 Absence of stock bubbles

Our third and final condition additionally ensures that the discounted stock price is a martingale under an equivalent local martingale measure Q. That is, it ensures that there are no stock bubbles. The arbitrage strategies associated with stock bubbles risk unbounded losses, as illustrated in the CEV example in Section 1.2. Thus, our third condition effectively enlarges the feasible portfolio set to include strategies that can short the stock at an unlimited scale.

Together, the three conditions imply there is a stochastic discount factor ξ_T such that

$$E[\xi_T R_T] = R_0 = 1 \quad \text{and} \quad E[\xi_T S_T] = S_0,$$

and the discounted stock price and the money market account are martingales under the probability measure Q defined by $dQ/dP = \xi_T R_T$.

Here is Condition 3.

Condition 3: There exists an equivalent local martingale measure Q, and the Q-exponential local martingale

$$\frac{S_t}{R_t} = \exp\left[-\frac{1}{2}\int_0^t \sigma(Y_s)^2 ds + \int_0^t \sigma(Y_s) dZ_s^Q\right] \tag{15}$$

is a Q-martingale.

Condition 3 nests Condition 2 by requiring an equivalent local martingale measure. Given Condition 2, a failure of Condition 3 would imply that there is a dynamic portfolio that dominates the stock return on $[0,T]$. In other words, there would be a bubble on the stock price. Section 1.2 provides an explicit stock bubble for the CEV model. An investor in principle could exploit the bubble by shorting the stock and investing the proceeds in a cheaper strategy, which would guarantee a profit by time T. The combination always risks temporary losses before ultimately profiting. These temporary losses cannot be bounded by fixed dollar amounts, unlike those associated with bond bubbles.

Because $\sigma(\cdot)$ is continuous on $(0,\infty)$ and Y cannot explode or hit 0, S_t/R_t is strictly positive [Revuz and Yor (1994, Exercise IV.3.25)]. Using logic similar to that of Condition 2, if the exponential local martingale is a martingale, then it defines a probability measure H, equivalent to Q and P, such that

$$dY_t = \{a^Q(Y_t) + \rho(Y_t)\sigma(Y_t)b(Y_t)\}dt + b(Y_t)dB_t^H, \tag{16}$$

where $a^Q(Y_t) \equiv a(Y_t) - \rho(Y_t)\kappa(Y_t)b(Y_t)$ and $B_t^H \equiv B_t^Q - \int_0^t \rho(Y_s)\sigma(Y_s)ds$ is a H-Brownian motion by Girsanov's theorem (in fact, H is related to the probability measure using the stock price as the *numeraire*).

We can again use the test for explosions for solutions of (16). If the solution of (16) cannot explode or hit 0 in finite time then the discounted stock price is a Q-martingale. We show rigorously in the Appendix that Condition 3 is equivalent to the following condition.

Condition 3′ The solution of (16) does not explode or hit 0 under Q.

In the following example, we connect the explosion criterion in Condition 3 to the CEV model in Section 1.

Example 2.3. *For the CEV model in Section 1, let $r(Y_t) = r$, $Y_t = S_t$, and*

$$dS_t = rS_t dt + \sigma S_t^\alpha dZ_t^Q.$$

Note that $\sigma(Y) = \sigma S^{\alpha-1}$. We focus on $\alpha > 1$, for which Conditions 1 and 2 hold. To confirm the stock bubble calculated directly in Section 1.2, we verify Condition $3'$ is violated, meaning the solution of

$$dS_t = (rS_t + \sigma^2 S_t^{\alpha-1} S_t^\alpha)dt + \sigma S_t^\alpha dZ_t$$

explodes with positive probability. We compute the speed measure

$$\begin{aligned} v_c(x) &= \int_c^x p_c'(y) \int_c^y \frac{2}{p_c'(z)\sigma^2 z^{2\alpha}} dz dy \\ &= 2\int_c^x \exp\left(-2\int_c^y \frac{r\phi + \sigma^2\phi^{2\alpha-1}}{\sigma^2\phi^{2\alpha}} d\phi\right) \int_c^y \frac{\exp\left(2\int_c^z \frac{r\phi+\sigma^2\phi^{2\alpha-1}}{\sigma^2\phi^{2\alpha}} d\phi\right)}{\sigma^2 z^{2\alpha}} dz dy. \end{aligned}$$

If $\lim_{x\downarrow 0} v_c(x) = \lim_{x\uparrow\infty} v_c(x) = \infty$ (no explosion), then the discounted stock price will be a Q-martingale. However, if this fails (solutions explode), then the discounted stock price will be a local martingale but not a martingale.

Assume $r = 0$.[15] *We have*

$$v_c(x) = \frac{2}{\sigma^2(3-2\alpha)}\left(\frac{x^{2-2\alpha}}{2-2\alpha} + \frac{c^{3-2\alpha}}{x} - \frac{c^{2-2\alpha}}{2-2\alpha} - c^{2-2\alpha}\right)$$

When $\alpha > 1$, $\lim_{x\uparrow\infty} v_c(x)$ *is finite so the process explodes, and the discounted stock price is a strict local martingale. Thus, the stock price has a bubble.*

Condition 3 has other important applications. For example, it proves Sin's (1998) stochastic volatility results that show the Hull and White (1987) model would have a bubble if one allowed positive correlation between stock returns and volatility.

2.5 Economic consequences of the conditions

The following proposition summarizes the economic consequences of our conditions.

Proposition 2.1. *Condition 2 is satisfied if and only if the money market account has no bubbles. Condition 3 is satisfied if and only if Condition 2 is satisfied and the stock has no bubbles.*

Proof. See Appendix B. □

We now turn to option bubbles.

3. Options and Bubbles

Option textbooks typically present "universal" properties of option values based on seemingly weak assumptions about underlying spot prices. These properties include the ideas that option values should not exceed the stock price, put-call parity should hold, and other relations between option values and intrinsic values should hold [Merton (1973)]. Such properties were originally motivated by the absence of arbitrage or by dominated asset arguments, but the possibility of multiple solutions of the valuation PDE might mean that some solutions satisfy these properties, whereas others do not. In some cases like the CEV model in Section 1.2, there may be no PDE solutions that simultaneously satisfy all these properties.

This section examines this issue and the general relations between option and stock bubbles. Observable option properties can suggest

[15] If a solution explodes when $r = 0$, the corresponding solution with $r > 0$ must also explode. This may be verified using comparison theorems for solutions of SDEs [Karatzas and Shreve (1988, Proposition 5.2.18)].

whether the models of stock prices with bubbles are appropriate for a given practical application.

3.1 Multiple PDE solutions as bubbles

We consider a European-style derivative security that pays $F(S_T)$ at time T, where $F(\cdot) \geq 0$. We postulate that the derivative's value is a nonnegative function of time, the underlying stock price, and the state variable.[16] We write this function as $G(S,Y,t)$, and assume that G is a $C^{2,2,1}((0,\infty) \times (0,\infty) \times [0,T))$-function of its arguments. We study nonnegative solutions of the valuation PDE

$$\frac{1}{2}\sigma^2 S^2 G_{SS} + \frac{1}{2}b^2 G_{YY} + \rho b \sigma S G_{YS} + rSG_S + a^Q G_Y + G_t - rG = 0$$
$$\text{subject to } G(S,Y,T) = F(S) \quad (17)$$

where $a^Q = a - \rho\kappa b$.

Assumption 3.1. *In this section, we assume that Conditions 1 and 2 are satisfied and that a nonnegative solution to the valuation PDE* (17) *exists.*[17] *This means that there are neither instantaneously profitable arbitrages nor bubbles on the money market account. Stock and option values may have bubbles.*

Assumption 3.1 implies there is an equivalent local martingale measure Q as defined by Delbaen and Schachermayer (1994). We do not assume Condition 3. If Condition 3 does not hold, the stock has a bubble: the discounted stock price is strictly a Q-local martingale, consistent with Delbaen and Schachermayer's analysis.

The CEV example in Section 1.2 and the stochastic volatility example in Section 1.3 show the valuation PDE can have multiple solutions even when Assumption 3.1 holds. We can rewrite a solution G of the PDE (17) as

$$G(S,Y,t) = E^Q_{t,S,Y}\left[\frac{R_t}{R_T}F(S_T)\right] + \Pi(S,Y,t) \quad \text{with } \Pi(S,Y,T) \equiv 0. \quad (18)$$

We now show that $\Pi(S,Y,t)$ is the bubble for the solution G.

[16] We consider only nonnegative prices for derivatives with nonnegative payoffs even though negative solutions to the PDE are mathematically possible. Although economically implausible, the mathematical possibility helps illustrate differences in the standard methods of ruling out doubling strategies as we discuss in Section 4.

[17] The solution must be finite to exist. The proof of Proposition 3.1 shows that the existence of a nonnegative solution implies that the expectations of quantities we use in this section exist.

Proposition 3.1. *The risk-neutral value of a derivative, defined by*

$$E_t^Q\left[\frac{R_t}{R_T}F(S_T)\right],$$

is the lowest cost of a replicating strategy with nonnegative value.

Proof. See Appendix B. □

The solution G solves the valuation PDE, and standard arguments produce a trading strategy that both replicates the derivative payout and has value $G(S_t, Y_t, t)$ at all times t. Thus, $G(S_t, Y_t, t)$ cannot be less than the risk-neutral value, and Π is nonnegative. Moreover, Π has local return equal to r under Q because both G and the risk-neutral value do. Its discounted value is a Q-local martingale that fails to be a martingale when it is nonzero (because $\Pi(S_T, Y_T, T) \equiv 0$). This means Q is an equivalent local martingale measure for the bond, stock, and the derivative.

3.2 Put-Call Parity

We now study the properties of bubbles related to put-call parity, which connects option values to stock prices [Stoll (1969)]. We consider European calls and puts with strike price K. Equation (18) says we can write

$$\begin{aligned} S_t &= E_t^Q\left[\frac{R_t}{R_T}S_T\right] + \Pi_t^S, \\ C_t &= E_t^Q\left[\frac{R_t}{R_T}(S_T - K)^+\right] + \Pi_t^C, \\ \text{and} \qquad P_t &= E_t^Q\left[\frac{R_t}{R_T}(K - S_T)^+\right] + \Pi_t^P, \end{aligned} \tag{19}$$

where Π^S, Π^C, and Π^P represent bubbles on the stock, the call, and the put, respectively. Put-call parity says

$$C_t + KE_t^Q\left[\frac{R_t}{R_T}\right] = P_t + S_t \tag{20}$$

and implies

$$\Pi_t^S + \Pi_t^P = \Pi_t^C. \tag{21}$$

Equality (21) connects the properties of stock and option bubbles.

Put-call parity might or might not hold if prices include bubbles. For the CEV example in Section 1.2, the solution of Emmanuel and MacBeth (1982) and Hull (2003) satisfies put-call parity; our new formula corresponding to the risk-neutral value does not. Our next result connects put-call parity and the general failure of risk-neutral pricing when a stock price has a bubble.

Proposition 3.2. *Suppose put-call parity holds for a given strike and maturity, and Condition 3 is violated. Because the stock price has a bubble, the call price must also have one. The call price cannot equal its risk-neutral value.*

Proof. The proof follows directly from (21) and the nonnegativity of bubbles. □

For the CEV stock-price model of Section 1.2, we argue that one must choose between risk-neutral pricing and put-call parity: choosing both is impossible. This follows directly from Proposition 3.2 and applies to any model in which the stock price has a bubble.

We now show that nonzero value for a call option with infinite strike price is a pure bubble when put-call parity holds and Condition 3 fails.

Proposition 3.3. *Suppose the stock price has a bubble. If put-call parity holds at all strike levels, the prices of European call options do not converge to zero as the strike price tends to infinity. That is,* $\liminf_{K\to\infty} C_t^K \neq 0$, *where* C^K *emphasizes the European call's dependence on its strike price* K.

Proof. The proof follows immediately from (19). □

Proposition 3.3 explains the findings of Emmanuel and MacBeth (1982) for options on a CEV process. They observe that call values from their formula do not tend to zero as the strike price tended to infinity. This is entirely caused by the fact that the discounted CEV stock price is not a martingale under Q. More generally, we can use this restriction on option values to diagnose the presence of bubbles in spot prices. Well-behaved option prices rule out the possibility of bubbles in spot prices and imply a "correct" underlying stock-price model ought to satisfy Condition 3.

Corollary 3.1. *Suppose put option prices are bounded by the strike price* K.[18] *If put-call parity holds at all strike prices, and the limit of the call price is zero as the strike price goes to infinity, then the stock and European call prices do not have bubbles.*

[18] Bubbles on put values are mathematically possible. A put bubble would cause the price of the put to exceed the strike price with positive probability. Our assumption rules out bubbles on puts.

We now connect other option properties and stock-price bubbles.

3.3 American options

A well-known result in option theory is that it is not optimal to exercise early an American call option on a stock that pays no dividends. This result remains valid whether or not the stock price includes a bubble, but we show that what is different is there is no optimal exercise policy at all if Condition 3 is violated. Unlike the no-bubble case, instead of waiting until maturity one would prefer to sell the American option before the stock bubble "bursts." But there is no optimal time prior to the bursting to exercise the option, as our next result shows.

Proposition 3.4. *Suppose $r = 0$ and $K > 0$.*[19] *Also suppose the stock has a bubble and pays no dividends. Then an American call option written on the stock has no optimal exercise time.*

Proof. See Appendix B. □

Most numerical methods determine the optimal exercise policy jointly with the option value. The numerical method used by Detemple and Tian (2002) to determine optimal exercise for the CEV model with $\alpha > 1$ needs further examination because that model has a bubble (Section 1.2).

3.4 Exotic derivatives

The previous subsections show that bubbles in spot prices manifest themselves in the values of "plain vanilla" options. Exotic options can be even more sensitive to bubbles. For example, a lookback call option is the right to sell a share at a prespecified time at the maximum stock price attained over the maturity. Its value is very sensitive to the explosive nature of a stock bubble.

Proposition 3.5. *Suppose for simplicity that $r = 0$. A European lookback call option with a payout function $\max_{0 \le t \le T} S_t - K$ has infinite value if the stock price has a bubble.*

Proof. By Protter (1992, Theorem I.47), the fact that S has drift of zero (the riskless rate) under Q but is not a martingale implies that $E^Q[\max_{0 \le t \le T} S_t] = \infty$, which immediately implies the result. □

We emphasize that Proposition 3.5 does not suggest that either S_t or $\max_{0 \le t \le T} S_t$ somehow becomes infinite. Indeed, Doob's maximal inequality for nonnegative local martingales (supermartingales) shows that the probability of the maximum exceeding a level λ approaches zero; in other words,

[19] Our proof easily extends to the case where r is nonnegative.

$$Q\left(\max_{0\le t\le T} S_t \ge \lambda\right) \le \frac{S_0}{\lambda}$$

for all real numbers λ [Revuz and Yor (1994, Theorem II.1.7)]. Instead, the bubble causes the stock price to become large enough (it remains finite) on a small probability set to make the expectation infinite. The payout function of the lookback makes its value highly sensitive to this property.

4. Limits to Arbitrage

Our previous analysis shows that bubbles and multiple PDE solutions are associated with dominated trading strategies and arbitrage strategies. The key issue, however, is whether these arbitrages are feasible at all scales, allowing an investor to obtain arbitrarily large payouts at no cost. If so, the bubble is inconsistent with partial equilibrium when investors have appropriately monotone preferences. It is well-known, however, that continuous-trade models require some trading restriction to make doubling strategies infeasible as unlimited arbitrage opportunities. Our preceding discussion indicates that these restrictions can also prevent bubble arbitrage.

In particular, our analysis reveals tension between two methods often used to make doubling strategies infeasible—namely, those of Harrison and Kreps (1979) and Delbaen and Schachermayer (1994). Harrison and Kreps require local buy-and-hold (simple) strategies, such as those that exploit violations of put-call parity. Nontrivial dynamic strategies are not locally buy-and-hold but are typically needed for replicating option payouts. They may also be needed for replicating the payouts of a dominated asset in forming an arbitrage strategy. Under the additional assumption that preferences are L^2-continuous, Harrison and Kreps avoid this difficulty by ruling out approximate arbitrages in the L^2-norm. The absence of these "L^2-approximate arbitrages" gives an equivalent martingale measure for all assets. All three of our conditions in Section 2 are necessary for an equivalent martingale measure.

By contrast, Delbaen and Schachermayer allow continuous trade but bound negative wealth to make doubling strategies infeasible. This approach is appealing because it allows continuous replicating strategies and standard preferences. However, when stock and option values can have bubbles, bounds on negative wealth can prevent the shorting of dominated assets in arbitrage strategies. Thus, some L^2-approximate arbitrages cannot be ruled out in the Delbaen and Schachermayer framework. This gives only an equivalent local martingale measure, and both multiple PDE solutions and bubbles can persist. Only Conditions 1 and 2 are necessary in this case. Our next result formalizes our point.

Proposition 4.1. *Assume an equivalent local martingale measure Q exists (i.e., Conditions 1 and 2 of Section 2 hold). Suppose G^1 and G^2 are distinct solutions of the valuation PDE (17) with $G^1(S_0, Y_0, 0) > G^2(S_0, Y_0, 0)$. A short position in the replicating strategy for G^1 and a long position in the replicating strategy for G^2 is infeasible given a bound on negative wealth defined by $G^2(S_t, Y_t, t) - G^1(S_t, Y_t, t) \geq L_t R_t$ for some process L that satisfies $E^Q[\inf_{t \in [0,T]} L_t] > -\infty$.*[20]

Proof. See Appendix B. □

Harrison and Kreps' assumption does not allow the CEV process for $\alpha > 1$ because there is no equivalent martingale measure (Section 1.2); in fact, an L^2-approximate arbitrage exists. Delbaen and Schachermeyer allow the CEV model because bounds on negative wealth prevent the short sales of the stock needed for an actual arbitrage—each locally buy-and-hold strategy in the L^2-approximate arbitrage risks unbounded and permanent losses and is not an arbitrage. Literally interpreted, bounds on negative wealth allow an excessively wide range of option prices, including even negative option prices because buying negatively priced options risks unbounded negative wealth when future option prices become even more negative. Appropriate solutions depend on the actual feasibility of certain trading strategies.[21,22]

5. Conclusion

We study the connections between stock and option bubbles, multiple solutions of the valuation PDE, dominated investments, and arbitrage. We provide several examples to illustrate these connections. One might conclude that the models of stock and option bubbles are economically unpalatable, but in light of our theoretical results, it is interesting to look at empirical results about options where bubbles are suspected. Mitchell, Pulvino, and Stafford (2002) documented mispricing in many dot-com

[20] Delbaen and Schachermayer studied that the constraint that time t wealth exceeds γR_t for some $\gamma \leq 0$. They showed that this constraint would allow an investor to short any scale of the money market account (by choosing progressively more negative values of γ) and an equivalent local martingale measure Q would exist. Taking this as given (i.e., assuming Conditions 1 and 2 are already satisfied), our constraint still rules out doubling strategies and is more general than that studied by Dybvig and Huang (1988), Delbaen and Schachermayer (1995), and Loewenstein and Willard (2000a).

[21] Dammon, Dunn, and Spatt (1993, Footnote 33) documented a case where a warrant sold for slightly more than the common stock. Because warrants are not standardized options contracts, it may not have been feasible to buy the stock and hold a short warrant position.

[22] Some studies impose $L^2(Q)$ integrability constraints on strategies, where Q is an equivalent martingale measure. Dybvig and Huang (1988) discussed the economic shortcomings of this constraint. There are several further problems in our context. First, we do not know a priori that an equivalent martingale measure exists. Second, imposing an analagous constraint using an equivalent local martingale measure would make both purchases and short sales of assets with bubbles infeasible. Thus, it would not change our conclusions about the feasibility of arbitrages that exploit bubbles.

stocks during the "NASDAQ bubble" period. Over a similar time period, Ofek, Richardson, and Whitelaw (2003) contemporaneously documented persistent and widespread violations of put-call parity among certain stocks. These ideas are linked by Lamont and Thaler (2003), who provided evidence that options on Palm and other stocks violated put-call parity at the same time the stocks clearly had bubbles.

In these cases, the techniques we develop would be helpful in determining appropriate models for the purposes of option valuation. A consistent theoretical description of the dot-com stocks is they violated Condition 3; that is, they had stock bubbles. At the same time, there were no significant impediments to shorting listed call options, so the call options did not have bubbles. Therefore, put-call parity was violated, as Section 3 indicates that it is necessary. Our theory of bubbles provides important connections between stock and option markets in the presence of bubbles.

It is also important to consider our theoretical results for numerical applications. There has been interest in solving for American options [Detemple and Tian (2002)] and lookback options [Boyle and Tian (2001)] in models that potentially allow bubbles. Some methods compute risk-neutral values, some impose put-call parity, whereas others give approximations. Section 3 indicates that these methods generally correspond to different solutions for option values. Indeed, Proposition 3.2 indicates some cases where these methods must give different solutions.

Appendix A: Specification Tests for Martingales

In this section, we state a theorem that provides the specification tests in Section 2. Results with substantially similar conclusions but varying assumptions appear in Stroock and Varadhan (1979), Karatzas and Shreve (1988), Sin (1998), Lewis (2000), Lipster and Shiryaev (2000), and Cheridito, Filipović, and Kimmel (2003). None of these results readily encompasses our theorem as a special case. We provide a complete proof of our theorem that exactly matches the assumptions we make.

In the following theorem, we assume that the domain of the state variable Y is $(-\infty,\infty)$. This is consistent with our earlier analysis if we transform the model so that the state variable is $\log(Y)$ and appropriately reinterpret the coefficients.

Theorem A.1. *Suppose Y_t is the unique nonexploding strong (or weak) solution of the stochastic differential equation*

$$dY_t = a(Y_t)dt + b(Y_t)dB_t \tag{A1}$$

where $a(\cdot)$ and $b(\cdot)$ are continuous and $b(\cdot)^2 > 0$. Then for a continuous function $f(\cdot)$, the exponential local martingale M defined by

$$M_t \equiv \exp\left[\int_0^t -\frac{f(Y_s)^2}{2}ds + \int_0^t f(Y_s)dZ_s\right] \tag{A2}$$

is a martingale if and only if there is a nonexploding weak solution of

$$dY_t = \{a(Y_t) + \rho(Y)f(Y_t)b(Y_t)\}dt + b(Y_t)d\tilde{B}_t, \tag{A3}$$

where $\text{corr}(dB,dZ) = \rho(Y)$.

Proof of Theorem A.1. Suppose (A2) is a martingale for a given continuous function $f(\cdot)$. Define the probability measure Q by

$$\frac{dQ}{dP} \equiv \exp\left[-\int_0^T \frac{f(Y_s)^2}{2} ds + \int_0^T f(Y_s)dZ_s\right].$$

Then Girsanov's theorem [Karatzas and Shreve (1988, Theorem 3.5.1)] implies that the process

$$\tilde{B}_t \equiv B_t - \int_0^t \rho(Y_u)f(Y_u)du$$

is a Q-Brownian motion. It immediately follows that the process Y_t defined by (A1) also satisfies (A3). Notice Y cannot explode on $[0,T]$ because Q is equivalent to P.

Now let us assume that solutions to (A3) do not explode for a given continuous function $f(\cdot)$. Define $a^n(Y) = a(Y)$ for $Y \in [-n,n]$ and $a^n(Y) = a(-n)$ for $Y < -n$ and $a^n(Y) = a(n)$ for $Y > n$. Define $b^n(Y)$ and $f^n(Y)$ likewise. Now consider the sequence of exponential martingales

$$\frac{dQ^n}{dP} = \exp\left[-\int_0^T \frac{(f^n(Y_s^n))^2}{2} ds + \int_0^T f^n(Y_s^n)dZ_s\right]$$

and the corresponding sequence of Q^n-Brownian motions defined by

$$\tilde{B}_t^n = B_t - \int_0^t \rho(Y_u^n)f^n(Y_u^n)du.$$

The martingale (as opposed to local martingale) and Brownian motion properties follow from the fact that, for a given n each process has bounded coefficients [Karatzas and Shreve (1988, Corollary 3.5.13)]. For a given n, these bounded coefficients also imply that there is a unique strong solution of the process

$$dY_t^n = \{a^n(Y_t^n) + b^n(Y_t^n)f^n(Y_t^n)\}dt + b^n(Y_t^n)d\tilde{B}_t^n. \tag{A4}$$

Define the stopping time $\tau_n = \inf\{t \,|\, |Y_t| \geq n\} \wedge n$, where $\wedge$ denotes minimum. By construction, $Y = Y^n = Y^{n+1}$ on $\mathcal{F}_{\tau_n}$ and Q^{n+1} agrees with Q^n on $\mathcal{F}_{\tau_n}$. The assumed nonexplosiveness of Y implies that

$$\lim_{n\to\infty} Q^n\{\tau_n \leq t\} = \lim_{n\to\infty} Q^n\left\{\sup_{0\leq s\leq t} |Y_s| \geq n\right\} = 0.$$

Now we find that the exponential local martingale (A2) is really a martingale because

$$\begin{aligned}0 \le E[|M_t - M_{t\wedge\tau_n}|] &= E\big[|(M_t - M_{t\wedge\tau_n})1_{\{\tau_n\le t\}}|\big]\\ &\le E\big[M_t 1_{\{\tau_n\le t\}}\big] + E\big[M_{\tau_n} 1_{\{\tau_n\le t\}}\big]\\ &= E[M_t] - E\big[M_t 1_{\{\tau_n>t\}}\big] + Q^n\{\tau_n \le t\}\\ &\le 1 - E\big[M_t 1_{\{\tau_n>t\}}\big] + Q^n\{\tau_n \le t\}\\ &\to 0\end{aligned}$$

because $E[M_t] \le 1$ and $E[M_t 1_{\{\tau_n>t\}}] = E[M_{\tau_n} 1_{\{\tau_n>t\}}] = Q^n\{\tau_n > t\} \to 1$. This implies $M_{t\wedge\tau_n} \to M_t$ in L^1. □

Appendix B: Proofs

Proof of Proposition 2.1. Let W_t be a nonnegative wealth process corresponding to a self-financing portfolio of the stock, the bond, and possibly the third asset completing the market (as described in Section 2.1). Standard arguments, much the same as those of Dybvig and Huang (1988), imply $\xi_t W_t$ is a P-local martingale, where $\xi_t = M_t/R_t$.[23] Because ξ_t is strictly positive under our assumptions, Fatou's lemma says $E[\xi_T W_T] \le W_0$. We use this inequality to prove our result.

We first prove equivalence between Condition 2 and an undominated money market account. Given Condition 2, suppose that a strategy generates terminal wealth $W_T = R_T$. Then we have $W_0 \ge E[\xi_T R_T] = R_0 = 1$, so the money market is undominated. To prove the converse, we argue by contradiction. Suppose that Condition 2 fails. Then M_t is a local martingale, which implies $1 = R_0 > E[\xi_T R_T]$. Karatzas et al. (1991) and Loewenstein and Willard (2000a) showed that the minimum investment needed to replicate R_T with pathwise nonnegative wealth is $W_0 = E[\xi_T R_T]$. This implies that the money market would be dominated, which would contradict our assumption.

Using $W_T = S_T$, the proof of the second statement is virtually identical. □

Proof of Proposition 3.1. Define the risk-neutral value process $\tilde{g}$ by

$$\tilde{g}_t = E_t^Q\left[\frac{R_t}{R_T}F(S_T)\right].$$

Then $\tilde{g}_t/R_t$ is a Q-martingale. Standard PDE methods give a self-financing trading strategy that replicates the derivative payout, requires initial wealth of $\tilde{g}_0$, and has value $\tilde{g}_t$ at every time t. We remark that this trading strategy might involve a third asset used to complete the market (as described in Section 2.1).

For any nonnegative solution $G(S,Y,t)$ of the valuation PDE, the process $g_t = G(S_t,Y_t,t)$ satisfies the boundary condition $g_T = F(S_T)$ and has the property that g_t/R_t has zero drift under Q. This implies that g_t/R_t is a supermartingale under Q [Karatzas and Shreve (1988, Problem 1.5.19)]. Consequently,

$$g_t \ge E_t^Q[g_T] = E_t^Q\left[\frac{R_t}{R_T}F(S_T)\right] = \tilde{g}_t. \tag{B1}$$

Standard PDE methods also produce a self-financing trading strategy that replicates the derivative payout, requires initial wealth of g_0, and has value g_t at every time t. Inequality

[23] Strictly speaking, Dybvig and Huang assumed that an equivalent martingale measure exists. Loewenstein and Willard (2000a, Proposition 1) showed that substantially the same arguments apply even when neither an equivalent martingale measure nor an equivalent local martingale measure exists.

(B1) says that the risk-neutral value is the lowest possible cost of a nonnegative PDE solution. □

Proof of Proposition 3.4. The American call value C_t^{AM} satisfies the inequality $C_t^{\mathrm{AM}} \geq \mathrm{essup}_{\tau \in \mathcal{T}_t} E_t^Q[(S_\tau - K)^+]$, where $\mathcal{T}_t$ is the collection of exercise (stopping) times on the interval $[t,T]$.[24] We must have $C_t^{AM} \geq (S_t - K)^+ \geq S_t - K$, which reflects the known result that the price of the American call option value should always be at least as great as its intrinsic value [Merton (1973)].

We first show that it is not optimal to exercise the option before maturity. Consider a candidate exercise time $\tau \in \mathcal{T}_t$, and suppose that $\{\tau < T\}$ has positive probability. Because the stock price has no drift under Q, there exists a sequence of stopping times $\{\tau_n\}$ such that $\tau_n \uparrow T$ almost surely and $S_\tau = E_\tau^Q[S_{\tau_n \wedge \tau}]$. Then for sufficiently large n, the event $\{\tau < \tau_n\}$ must have positive probability because $\tau_n \uparrow T$ almost surely as $n \to \infty$. We have $C_\tau^{AM} \geq E_\tau^Q[(S_{\tau_n} - K)^+] > E_\tau^Q[(S_{\tau_n} - K)] = (S_\tau - K)$, which implies waiting to exercise improves on immediately exercising. We remark that we are using the assumption that $K > 0$ and that the fact that the probability the option is out of the money at time τ_n is strictly positive but not equal to one.

Given that it is not optimal to exercise prior to maturity, the only candidate for an optimal exercise time would be $\tau = T$. However, if the optimal stopping time were $\tau = T$, we would have $E_t^Q[(S_T - K)^+] \geq S_t - K$. This would say that the stock price is dominated by a "very well-behaved" martingale, which would be inconsistent with a bubble on the stock price.[25] It is not optimal to wait until maturity to exercise an American call when the stock has a bubble. □

Proof of Proposition 4.1. We argue by contradiction: Suppose that the arbitrage strategy is feasible, meaning there exists a bound $L \in \mathcal{L}$ such that

$$G^2(S_t, Y_t, t) - G^1(S_t, Y_t, t) \geq L_t R_t \tag{B2}$$

along Q-almost all paths. Because G^1 and G^2 solve the Black-Scholes PDE, their local returns under Q equal r, and the process

$$M_t \equiv \big(G^1(S_t, Y_t, t) - G^2(S_t, Y_t, t)\big)/R_t,$$

has zero drift under Q (but is not necessarily nonnegative). Recalling $G^1(S_0, Y_0, 0) > G^2(S_0, Y_0, 0)$ and $G^1(S_T, Y_T, T) = G^2(S_T, Y_T, T)$, define the stopping time $\tau = \inf\{t \in [0,T] : G^1(S_t, Y_t, t) = G^2(S_t, Y_t, t)\}$. The stopped process $M_{t\wedge\tau}$ also has zero drift under Q, is nonnegative, and satisfies $E^Q[\sup_{t\in[0,T]} M_{t\wedge\tau}] < \infty$ by inequality (B2). Therefore, the stopped process $M_{t\wedge\tau}$ must be a Q-martingale [Protter (1992, Theorem I.6.47)]. Because $M_{T\wedge\tau} \equiv 0$, $M_{t\wedge\tau}$ can be a martingale only if $M_{0\wedge\tau} = M_0 = 0$, which is a contradiction of $G^1(S_0, Y_0, 0) > G^2(S_0, Y_0, 0)$. Therefore, the arbitrage strategy described in Proposition 4.1 is infeasible given the wealth constraint. □

[24] We are allowing the possibility that the American call value has bubble. We remark that the essential supremum is finite because applying Doob's optional sampling theorem [Karatzas and Shreve (1988, Theorem 1.3.22)] to the nonnegative local martingale (supermartingale) S_t says that it satisfies $\mathrm{essup}_{\tau\in\mathcal{T}} E_t^Q[(S_\tau - K)^+] \leq \mathrm{essup}_{\tau\in\mathcal{T}} E_t^Q[S_\tau] \leq S_t < \infty$.

[25] Formally, this would imply that the stock price is dominated by class D Q-martingale, which in turn would imply that the stock price S_t is also a class D Q-martingale, which would contradict our original assumption that the stock price has a bubble. See Revuz and Yor (1994, Proposition IV.1.7).

References

Basak, S., and B. Croitoru, 2000, "Equilibrium Mispricing in a Capital Market with Portfolio Constraints," *Review of Financial Studies*, 13(3), 715–748.

Beckers, S., 1980, "The Constant Elasticity of Variance Model and Its Implications for Option Pricing," *Journal of Finance*, 35(3), 661–673.

Black, F., and M. Scholes, 1973, "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, 81, 637–654.

Boyle, P., and Y. S. Tian, 2001, "Pricing Lookback and Barrier Options under the CEV Process," *Journal of Financial and Quantitative Analysis*, 34(2), 241–264.

Broadie, M., et al., 2000, "American Options with Stochastic Dividends and Volatility: A Nonparametric Investigation," *Journal of Econometrics*, 94, 53–92.

Cheridito, P., D. Filipović, and R. L. Kimmel, 2003, "Market Price of Risk Specifications for Affine Models: Theory and Evidence," working paper, Princeton University.

Chernov, M., et al., 2003, "Alternative Models for Stock Price Dynamics," *Journal of Econometrics*, 116, 225–257.

Chernov, M., and E. Ghysels, 2000, "A Study Towards a Unified Approach to the Joint Estimation of Objective and Risk Neutral Measures for the Purpose of Option Valuation," *Journal of Financial Economics*, 56, 407–458.

Cox, J. C., 1975, "Notes on Option Pricing I: Constant Elasticity of Variance Diffusions," mimeo, Stanford University.

Cox, J. C., J. Ingersoll, and S. A. Ross, 1985, "A Theory of the Term Structure of Interest Rates," *Econometrica*, 53, 385–407.

Dammon, R. M., K. B. Dunn, and C. S. Spatt, 1993, "The Relative Pricing of High-Yield Debt: The Case of RJR Nabisco Holdings Capital Corporation," *American Economic Review*, 83, 1090–1111.

Delbaen, F., and W. Schachermayer, 1994, "A General Version of the Fundamental Theorem of Asset Pricing," *Mathematiche Annalen*, 300, 463–520.

Delbaen, F., and W. Schachermayer, 1995, "The Existence of Absolutely Continuous Local Martingale Measures," *The Annals of Applied Probability*, 5(4), 926–945.

De Long, J. B., et al., 1990, "Noise Trader Risk in Financial Markets," *Journal of Political Economy*, 98(4), 703–738.

Detemple, J., and W. Tian, 2002, "The Valuation of American Options for a Class of Diffusion Processes," *Management Science*, 48(7), 917–937.

Dybvig, P. H., and C. Huang, 1988, "Nonnegative Wealth, Absence of Arbitrage, and Feasible Consumption Plans," *The Review of Financial Studies*, 1(4), 377–401.

Emmanuel, D. C., and J. D. MacBeth, 1982, "Further Results on the Constant Elasticity of Variance Call Option Pricing Model," *Journal of Financial and Quantitative Analysis*, 17, 533–554.

Eraker, B., M. Johannes, and N. Polson, 2003, "The Impact of Jumps in Volatility and Returns," *Journal of Finance*, 58, 1269–1300.

Gibbons, M. R., and C. Jacklin, 1988, "CEV Diffusion Estimation: The Untold Story," mimeo, Stanford University.

Grünbichler, A., and F. A. Longstaff, 1996, "Valuing Futures and Options on Volatility," *Journal of Banking and Finance*, 20, 985–1001.

Harrison, J. M., and D. M. Kreps, 1979, "Martingales and Arbitrage in Multiperiod Securities Markets," *Journal of Economic Theory*, 20, 381–408.

Harrison, J. M., and S. R. Pliska, 1981, "Martingales and Stochastic Integrals in the Theory of Continuous Trading," *Stochastic Processes and Their Applications*, 11, 215–260.

Heston, S., 1993, "A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options," *Review of Financial Studies*, 6, 327–344.

Hull, J. C., 2003, *Options, Futures, and Other Derivatives* (5th ed.), Prentice Hall, New Jersey.

Hull, J., and A. White, 1987, "The Pricing of Options on Assets with Stochastic Volatilities," *Journal of Finance*, 42, 281–300.

Jones, C. S., 2003, "The Dynamics of Stochastic Volatility: Evidence from Underlying and Options Markets," *Journal of Econometrics*, 116, 181–224.

Karatzas, I., et al., 1991, "Martingale and Duality Methods for Utility Maximization in an Incomplete Market," *SIAM Journal on Control and Optimization*, 29(3), 702–730.

Karatzas, I., and S. E. Shreve, 1988, *Brownian Motion and Stochastic Calculus* (2d ed.), Springer-Verlag, New York.

Lamont, O. A., and R. H. Thaler, 2003, "Can the Market Add and Subtract? Mispricing in Tech Stock Carve-Outs," *Journal of Political Economy*, 111, 227–268.

Lewis, A. L., 2000, *Option Valuation Under Stochastic Volatility with Mathematica Code*, Finance Press, Newport Beach, California.

Lipster, R. S., and A. N. Shiryaev, 2000, *Statistics of Random Processes*, Springer-Verlag, Berlin.

Loewenstein, M., and G. A. Willard, 2000a, "Local Martingales, Arbitrage, and Viability: Free Snacks and Cheap Thrills," *Economic Theory*, 16, 135–161.

Loewenstein, M., and G. A. Willard, 2000b, "Rational Equilibrium Asset-Pricing Bubbles in Continuous Trading Models," *Journal of Economic Theory*, 91(1), 17–58.

MacBeth, J. D., and L. J. Merville, 1980, "Tests of the Black-Scholes and Cox Call Option Valuation Models," *Journal of Finance*, 35, pp. 285–303.

Merton, R. C., 1973, "Theory of Rational Option Pricing," *Bell Journal of Economics and Management Science*, 4, 141–183.

Mitchell, M., T. Pulvino, and E. Stafford, 2002, "Limited Arbitrage in Equity Markets," *Journal of Finance*, 57(2), 551–584.

Ofek, E., M. Richardson, and R. F. Whitelaw, 2003, "Limited Arbitrage and Short Sales Restrictions," working paper. New York University Stern School of Business.

Protter, P. E., 1992, *Stochastic Integration and Differential Equations*, Springer-Verlag, New York.

Revuz, D., and M. Yor, 1994, *Continuous Martingales and Brownian Motion* (2d ed.), Springer-Verlag, New York.

Santos, M. S., and M. Woodford, 1997, "Rational Asset Pricing Bubbles," *Econometrica*, 65(1), 19–57.

Schroder, M., 1989, "Computing the Constant Elasticity of Variance Option Pricing Formula," *Journal of Finance*, 44, 211–219.

Shleifer, A., and R. W. Vishny, 1997, "The Limits of Arbitrage," *The Journal of Finance*, 52(1), 35–55.

Sin, C., 1998, "Complications with Stochastic Volatility Models," *Advances in Applied Probability*, 30, 256–268.

Stoll, H. R., 1969, "The Relationship Between Put and Call Prices," *Journal of Finance*, 24, 801–824.

Stroock, D. W., and S. R. S. Varadhan, 1979, *Multidimensional Diffusion Processes*, Springer-Verlag, Berlin.