GRAPHON MEAN FIELD GAMES AND THE GMFG EQUATIONS

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ABSTRACT. The emergence of the graphon theory of large networks and their infinite limits has enabled the formulation of a theory of the centralized control of dynamical systems distributed on asymptotically infinite networks [16, 19]. Furthermore, the study of the decentralized control of such systems was initiated in [6, 7], where Graphon Mean Field Games (GMFG) and the GMFG equations were formulated for the analysis of non-cooperative dynamic games on unbounded networks. In that work, existence and uniqueness results were introduced for the GMFG equations, together with an ϵ-Nash theory for GMFG systems which relates infinite population equilibria on infinite networks to finite population equilibria on finite networks. Those results are rigorously established in this paper.

1. INTRODUCTION

One response to the problems arising in the analysis of systems of great complexity is to pass to an appropriately formulated infinite limit. This approach has a distinguished history since it is the conceptual principle underlying the celebrated Boltzmann Equation of statistical mechanics and that of the fundamental Navier-Stokes equation of fluid mechanics (see e.g. [37, 22, 14, 15]). Similarly the Fokker-Plank-Kolmogorov (FPK) equation for the macroscopic flow of probabilities [12, 27] is used to describe a vast range of phenomena which at a micro or mezzo level are modelled via the random interactions of discrete entities.

The work in this paper is formulated within two recent theories which were developed with an analogous motive to that above, namely Mean Field Game (MFG) theory for the analysis of equilibria in very large populations of non-cooperative agents (see [25, 23, 30, 31, 9, 10, 8]), and the graphon theory of the infinite limits of graphs and networks (see [33, 2, 3, 4, 32]).

A mathematically rigorous study of MFG systems with state values in finite graphs is provided in [21], and MFG systems where the agent subsystems are defined at the nodes (vertices) of finite random Erdös-Rényi graphs are treated in [11]. The system behaviour in [21] is subject to a fixed underlying network. The random graphs in [11] have unbounded growth but do not create spatial distinction of the agents due to symmetry properties of the interactions. However, graphon theory gives a rigorous formulation of the notion of limits for infinite sequences of networks of increasing size, and the first application of graphon theory in dynamics appears to be in the work of Medvedev [34, 35], and Kaliuzhnyi-Verbovetskyi and Medvedev [26]. The law of large numbers for graphon mean field systems is proven in [1] as a generalization of results for standard interacting particle systems. Furthermore, the work [38] derives the McKean-Vlasov limit for a network of

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agents described by delay stochastic differential equations that are coupled by randomly generated connections.

The first applications of graphon theory in systems and control theory are those in [17, 18, 16, 19, 20] which treat the centralized and distributed control of arbitrarily large networks of linear dynamical control systems for which a direct solution would be intractable. Approximate control is achieved by solving control problems on the infinite limit graphon and then applying control laws derived from those solutions on the finite network of interest. The analogy with the strategies for finding feedback laws resulting in ϵ-Nash equilibria in the MFG framework is obvious. In this connection we note that work on static game theoretic equilibria for infinite populations on graphons was reported in [36].

A natural framework for the formulation of game theoretic problems involving large populations of agents distributed over large networks is given by Mean Field Game theory defined on graphons. The resulting basic idea and the associated fundamental equations for what we term graphon Mean Field Game (GMFG) systems and the GMFG equations are the subject of the current paper and its predecessors [6, 7]. The GMFG equations are of significant generality since they permit the study, in the limit, of both dense and sparse, infinite networks of non-cooperative dynamical agents. Moreover the classical MFG equations are retrieved as a special case. We observe that an early analysis of linear quadratic models in mean field games on networks with non-uniform edge weightings can be found in [24]. However, in that work there was no application of graphon theory, and in the uniform system parameter case there is one agent per node and a single mean field, whereas in the present work there is a subpopulation with its own mean field at each node.

The basic ϵ-Nash equilibrium result in MFG theory and its corresponding form in GMFG theory are vital for the application of MFG derived control laws. This is the case since the solution of the MFG and GMFG equations is necessarily simpler than the effectively intractable task of finding the solution to the game problems for the large finite population systems. Indeed, this was one of the original motives for the creation of MFG theory to tackle complexity. Furthermore it is a basic feature of graphon systems control theory [17].

The paper is organized as follows. Section 2 provides preliminary materials on graphons. Section 3 introduces the GMFG equation system and proves the existence and uniqueness of a solution. For the decentralized strategies determined by the GMFG equations, an ϵ-Nash equilibrium theorem is proved in Section 4. The GMFG equations are illustrated by an LQ example in Section 5.

Table 1: Notation

G_k	the k-th graph in a sequence of graphs
g_k	weights of G_k as a step function
M_k	the number of nodes in G_k
$\mathcal{C}_i$	the cluster of agents residing at node i of G_k
$\mathcal{C}(i)$	the cluster that agent i belongs to
I_i^*, $I^*(i)$	the midpoint of an interval of length $1/M_k$
g	the graphon function
$\mu_\alpha(t)$	the graphon local mean field generated by agents at vertex $\alpha \in [0,1]$
$\mu_G(t)$	an ensemble of local mean fields $(\mu_\alpha(t))_{0\le\alpha\le1}$
$\mathcal{M}_{[0,T]}$	a class of $\mu_G(\cdot)$ satisfying a Hölder continuity condition
C_T	the space of continuous functions on $[0,T]$

$\mathcal{F}_T$	σ-algebra induced by cylindrical sets in C_T
$(C_T, \mathcal{F}_T, m_\alpha)$	probability measure space for the path space at vertex α
$\mathbf{M}_T$	the set of probability measures on $(C_T, \mathcal{F}_T)$
D_T	Wasserstein metric on $\mathbf{M}_T$
$\mathbf{M}_T^G$	the product space $\prod_{\alpha\in[0,1]} \mathbf{M}_T$
$\mathbf{M}_T^{G0}, \mathbf{M}_T^{G1}$	subsets of $\mathbf{M}_T^G$
m_G	an ensemble of measures $(m_\alpha)_{0\le\alpha\le1} \in \mathbf{M}_T^G$
$\mathrm{Proj}_\alpha(m_G)$	the component m_α at vertex α
$\mathrm{Marg}_t(m_\alpha)$	the time t-marginal of m_α
x_α	the state of a generic agent at vertex $\alpha \in [0,1]$
w^α	a generic standard Brownian motion at vertex α
$\varphi(t, x_\alpha \vert \mu_G(\cdot); g_\alpha)$	the best response at vertex α with $\mu_G(\cdot)$ given by the GMFG system; abbreviated as $\varphi(t, x_\alpha, g_\alpha)$ or φ_α
$\phi(t, x_\alpha \vert \mu_G(\cdot); g_\alpha)$	the best response at vertex α with respect to a general $\mu_G(\cdot)$; abbreviated as $\phi_\alpha(t, x_\alpha \vert \mu_G(\cdot))$ or ϕ_α

2. The Concept of a Graphon

The basic idea of the theory of graphons is that the edge structure of each finite cardinality network is represented by a step function density on the unit square in $\mathbb{R}^2$ on which the so-called cut norm and cut metrics are defined. The set of finite graphs endowed with the cut metric then gives rise to a metric space, and the completion of this space is the space of graphons. The space $\mathbf{G^{sp}}$ of graphons are represented by bounded symmetric Lebesgue measurable functions $W : [0,1]^2 \to [0,1]$ which can be interpreted as weighted graphs on the vertex set $[0,1]$. We note that functions W taking values in finite sets satisfy this definition and so, in particular, graphons are defined on finite graphs.

The cut norm of a graphon then has the expression:

$$\|W\|_\Box = \sup_{M,T\subset[0,1]} \Big| \int_{M\times T} W(x,y)dxdy \Big|$$

with the supremum taking over all measurable subsets M and T of $[0,1]$. Denote the set of measure preserving bijections $[0,1] \to [0,1]$ by $S_{[0,1]}$. The *cut metric* between two graphons V and W is then given by $\delta_\Box(W,V) = \inf_{\phi\in S_{[0,1]}} \|W^\phi - V\|_\Box$, where $W^\phi(x,y) := W(\phi(x),\phi(y))$ and any pair of graphons at zero distance are identified with each other.

The space $(\mathbf{G^{sp}}, \delta_\Box)$ is compact in the topology given by the cut metric [32]. Furthermore, sets in $(\mathbf{G^{sp}}, \delta_\Box)$ which are compact with respect to the L^2 metric are compact with respect to the cut metric. Since $\mathbf{G^{sp}}$ is compact in the cut metric all sequences of graphons have subsequential limits and throughout this paper we assume that all graphon sequences under consideration possess unique limits in this topology.

In this paper, we start with the modeling of the game of a finite population based on a finite graph. Specifically, the population resides on a weighted finite graph G_k with a set of nodes (or vertices) $\mathcal{V}_k = \{1,\ldots,M_k\}$ and weights $g_{ij}^k \in [0,1]$ for $(i,j) \in \mathcal{V}_k \times \mathcal{V}_k$, where a value g_{ii}^k is assigned in the case $i=j$. We call $g_i^k := (g_{i1}^k, \cdots, g_{iM_k}^k)$ a section of g^k at i. Each node l is occupied by a set of agents which is called a cluster of the population and hence the number of clusters is M_k. We list the clusters as $\mathcal{C}_1,\ldots,\mathcal{C}_{M_k}$. Without loss of generality, we assume the lth cluster occupies node l. Let $\mathcal{C}(i)$ denote the cluster that agent i belongs to. So $i \in \mathcal{C}(i)$. Our further analysis in the paper is based on the convergence of

g^k to a graphon limit g. To indicate its arguments, we may write $g(\alpha, \beta)$ or alternatively $g_{\alpha,\beta}$. We define the section of g at α by $g_\alpha : \beta \mapsto g_{\alpha,\beta}$, $\beta \in [0, 1]$.

Since clusters $\mathcal{C}_{i_1}$ and $\mathcal{C}_{i_2}$ reside on nodes i_1 and i_2 of G_k, respectively, we define $g^k_{\mathcal{C}_{i_1}\mathcal{C}_{i_2}} = g^k_{i_1 i_2}$. Similarly, we define the section $g^k_{\mathcal{C}_i} = g^k_i$.

We partition $[0, 1]$ into M_k subintervals of equal length. Here $I^k_l = [(l-1)/M_k, l/M_k]$ for $1 \le l \le M_k$. When it is clear from the context, we omit the superscript k and write I_l. To relate the clusters of agents to the vertex set $[0, 1]$, we let the cluster $\mathcal{C}_l$ correspond to I_l.

Throughout this paper, $C, C_0, C_1, \cdots$ denote generic constants, which do not depend on the network size k and population size N and may vary from place to place.

3. Graphon MFG Systems and the MFG Equations

3.1. **The Standard MFG Model and Its Graphon Generalization.** In the diffusion based models of large population games the state evolution of a collection of N agents $\mathcal{A}_i, 1 \le i \le N < \infty$, is specified by a set of N controlled stochastic differential equations (SDEs). A simplified form of the general case is given by the following set of controlled SDEs which for each agent $\mathcal{A}_i$ includes state coupling with *all* other agents:

$$dx_i(t) = \frac{1}{N}\sum_{j=1}^{N} f(x_i(t), u_i(t), x_j(t))dt + \sigma dw_i(t), \tag{3.1}$$

where $x_i \in \mathbb{R}^n$ is the state, $u_i \in \mathbb{R}^{n_u}$ the control input, and $w_i \in \mathbb{R}^{n_w}$ a standard Brownian motion, and where $\{w_i, 1 \le i \le N\}$ are independent processes. For simplicity, all collections of system initial conditions are taken to be independent and have finite second moment. The cost of agent $\mathcal{A}_i$ is given by

$$J^N_i(u_i, u_{-i}) = E\int_0^T \frac{1}{N}\sum_{j=1}^{N} l(x_i(t), u_i(t), x_j(t))dt, \tag{3.2}$$

where $l(\cdot)$ is the pairwise running cost, and u_{-i} denotes the controls of all other agents.

The dynamics of a generic agent $\mathcal{A}_i$ in the infinite population limit of this system is then described by the controlled McKean-Vlasov (MV) equation

$$dx_i = f[x_i, u_i, \mu_t]dt + \sigma dw_i, \quad 0 \le t \le T, \tag{3.3}$$

where μ_t is the distribution of $x_i(t)$, $f[x, u, \mu_t] := \int_{\mathbb{R}^n} f(x, u, y)\mu_t(dy)$ and where the initial distribution μ^x_0 of $x_i(0)$ is specified. Setting $l[x, u, \mu_t] = \int_{\mathbb{R}^n} l(x, u, y)\mu_t(dy)$, the corresponding infinite population cost for $\mathcal{A}_i$ takes the form

$$J_i(u_i; \mu(\cdot)) := E\int_0^T l[x_i(t), u_i(t), \mu_t]dt. \tag{3.4}$$

The key feature of the generalized graphon MFG construction beyond the standard MFG scheme is that at any agent in a network the averaged dynamics (3.1) and cost function (3.2) decompose into averages of neighbouring subpopulations distributed on the network edges incident upon that agent's node plus an average term for the local cluster. In the limit, the summed subpopulation averages are given by an integral over the local mean field measures of the neighbouring agents. For notational simplicity, we present the graphon MFG framework with scalar individual states and controls, i.e., $n = n_u = n_w = 1$. Its extension to the vector case is evident.

Now we consider a finite population distributed over the finite graph G_k. Let $\boldsymbol{x}_{G_k} = \bigoplus_{l=1}^{M_k}\{x_i | i \in \mathcal{C}_l\}$ denote the states of all agents in the total set of clusters of the population. This gives a total of $N = \sum_{l=1}^{M_k} |\mathcal{C}_l|$ individual states.

For $\mathcal{A}_i$ in the cluster $\mathcal{C}(i)$, two coupling terms in the dynamics take the form

$$f_0(x_i, u_i, \mathcal{C}(i)) = \frac{1}{|\mathcal{C}(i)|} \sum_{j \in \mathcal{C}(i)} f_0(x_i, u_i, x_j),$$

$$f_{G_k}(x_i, u_i, g^k_{\mathcal{C}(i)}) = \frac{1}{M_k} \sum_{l=1}^{M_k} g^k_{\mathcal{C}(i)\mathcal{C}_l} \frac{1}{|\mathcal{C}_l|} \sum_{j \in C_l} f(x_i, u_i, x_j). \tag{3.5}$$

They model intra- and inter-cluster couplings, respectively. The specification of f_{G_k} relies on the sectional information $g^k_{\mathcal{C}(i)\bullet}$. Concerning the coupling structure in (3.5) we observe that with respect to $\mathcal{A}_i$, all individuals residing in cluster $\mathcal{C}_l$ are symmetric and their state average generates the overall impact of that cluster on $\mathcal{A}_i$ mediated by the graphon weighting $g^k_{\mathcal{C}(i)\bullet}$. The two coupling terms are combined additively resulting in the local dynamics

$$\tilde{f}_{G_k}(x_i, u_i, g^k_{\mathcal{C}(i)}) = f_0(x_i, u_i, \mathcal{C}(i)) + f_{G_k}(x_i, u_i, g^k_{\mathcal{C}(i)}).$$

Note that $\mathcal{A}_i$ interacts with the overall population through a function of the complete system state $\boldsymbol{x}_{G_k}$ and the cluster sizes. These details shall be suppressed in this paper and we only indicate the graph G_k and the section $g^k_{\mathcal{C}(i)}$. The state process of $\mathcal{A}_i$ is then given by the stochastic differential equation

$$dx_i(t) = \tilde{f}_{G_k}(x_i, u_i, g^k_{\mathcal{C}(i)})dt + \sigma dw_i, \quad 1 \le i \le N,$$

where $\sigma > 0$ and the initial states $\{x_i(0), 1 \le i \le N\}$ are i.i.d. with distribution $\mu_0^x \in \mathcal{P}_1(\mathbb{R})$, the set of probability measures on $\mathbb{R}$ with finite mean.

The limit of the two dynamic coupling terms of an agent at a node α, as the number of nodes of the graph G_k and the subpopulation at each node tends to infinity, is described by the following expressions:

$$f_0[x_\alpha, u_\alpha, \mu_\alpha] := \int_{\mathbb{R}^n} f_0(x_\alpha, u_\alpha, z)\mu_\alpha(dz), \tag{3.6}$$

$$f[x_\alpha, u_\alpha, \mu_G; g_\alpha] := \int_0^1 \int_{\mathbb{R}^n} f(x_\alpha, u_\alpha, z) g(\alpha, \beta)\mu_\beta(dz)d\beta, \tag{3.7}$$

which give the complete local graphon dynamics via

$$\widetilde{f}[x_\alpha, u_\alpha, \mu_G; g_\alpha] := f_0[x_\alpha, u_\alpha, \mu_\alpha] + f[x_\alpha, u_\alpha, \mu_G; g_\alpha]. \tag{3.8}$$

We call μ_β the local mean field at node β, which is interpreted as the limit of the empirical distributions of agents at node β. And $\mu_G = \{\mu_\beta, 0 \le \beta \le 1\}$ is the set of local mean fields. Due to the integration with respect to β, the dependence of $\widetilde{f}$ on the graphon limit g is through the section g_α. Since μ_G contains μ_α, we do not list μ_α as an argument of $\widetilde{f}$.

Parallel to the standard MFG case, in the graphon case the stochastic differential equation

$$\begin{aligned} &[\text{MV-SDE}](\alpha) \quad dx_\alpha(t) = \widetilde{f}[x_\alpha(t), u_\alpha(t), \mu_G(t); g_\alpha]dt + \sigma dw_t^\alpha, \\ &\quad 0 \le t \le T, \quad \alpha \in [0, 1], \end{aligned} \tag{3.9}$$

generalizes the standard controlled MV equation (3.3). We note that in a parallel development of graphon based stochastic dynamical populations [1] the system disturbance intensity σ is also a function of graphon weighted state functions at other clusters. For simplicity, we consider a constant σ and our analysis may be generalized to the case of a state and mean field dependent diffusion term. Similarly, for simplicity our dynamics and cost doe not include a separate parametrization by α.

Analogously, in the GMFG case, we define the cost coupling terms for $\mathcal{A}_i$ to be

$$l_0(x_i, u_i, \mathcal{C}(i)) = \frac{1}{|\mathcal{C}(i)|} \sum_{j \in \mathcal{C}(i)} l_0(x_i, u_i, x_j),$$
$$l_{G_k}(x_i, u_i, g^k_{\mathcal{C}(i)}) = \frac{1}{M_k} \sum_{l=1}^{M_k} g^k_{\mathcal{C}(i)\mathcal{C}_l} \frac{1}{|\mathcal{C}_l|} \sum_{j \in \mathcal{C}_l} l(x_i, u_i, x_j).$$

Define $\tilde{l}_{G_k}(x_i, u_i, g^k_{\mathcal{C}(i)}) = l_0(x_i, u_i, \mathcal{C}(i)) + l_{G_k}(x_i, u_i, g^k_{\mathcal{C}(i)})$. The cost of $\mathcal{A}_i$ in a finite population on a finite graph G_k is given in the form

$$J_i = E \int_0^T \tilde{l}_{G_k}(x_i, u_i, g^k_{\mathcal{C}(i)}) dt. \tag{3.10}$$

Denote

$$l_0[x_\alpha, u_\alpha, \mu_\alpha] = \int_{\mathbb{R}^n} l_0(x_\alpha, u_\alpha, z) \mu_\alpha(dz),$$
$$l[x_\alpha, u_\alpha, \mu_G; g_\alpha] = \int_0^1 \int_{\mathbb{R}^n} l(x_\alpha, u_\alpha, z) g(\alpha, \beta) \mu_\beta(dz) d\beta,$$
$$\widetilde{l}[x_\alpha, u_\alpha, \mu_G; g_\alpha] = l_0[x_\alpha, u_\alpha, \mu_\alpha] + l[x_\alpha, u_\alpha, \mu_G; g_\alpha].$$

Then in the infinite population graphon case, the individual agent α has the cost function given by

$$J_\alpha(u_\alpha; \mu_G(\cdot)) = E \int_0^T \widetilde{l}[x_\alpha(t), u_\alpha(t), \mu_G(t); g_\alpha] dt. \tag{3.11}$$

3.2. **The Graphon MFG Model and Its Equations.** In this section the standard MFG equations (see e.g. [5, 8]) will be generalized so that they subsume the standard (implicitly uniform totally connected) dense network case and cover the fully general graphon limit network case. Specifically, agent $\mathcal{A}_i$ in a population of N agents will be located at the lth node in an M_k node network (identified with its graphon) and in the infinite population graphon limit that node will be taken to map to $\alpha \in [0, 1]$. It is important to note here that *although the limit network is assumed dense it is not assumed to be uniformly totally connected*; indeed, the connection structure of the infinite network is represented precisely by its graphon $g = \{g(\alpha, \beta), 0 \leq \alpha, \beta \leq 1\}$.

The generalized Graphon MFG scheme below on $[0, T]$ is given by the linked equations for (i) the value function V^α for a generic agent's stochastic control problem when all other agents' control laws are fixed and generating the given local mean field μ_α and the graphon local mean field μ_β, (ii) the FPK equation for the MV-SDE for the local mean field of the generic agent, and (iii) the specification of the best response (BR) feedback law.

Suppressing the time index on the measures for simplicity of notation, we have the *Graphon Mean Field Game (GMFG) equations*:

$$\begin{aligned}
&[\mathrm{HJB}](\alpha) \quad -\frac{\partial V^\alpha(t,x)}{\partial t} = \inf_{u\in U}\Big\{\widetilde{f}[x,u,\mu_G;g_\alpha]\frac{\partial V^\alpha(t,x)}{\partial x} \\
&\qquad\qquad + \widetilde{l}[x,u,\mu_G;g_\alpha]\Big\} + \frac{\sigma^2}{2}\frac{\partial^2 V^\alpha(t,x)}{\partial x^2}, \\
&V^\alpha(T,x)=0, \quad (t,x)\in[0,T]\times\mathbb{R}, \quad \alpha\in[0,1],
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
&[\mathrm{FPK}](\alpha) \quad \frac{\partial p_\alpha(t,x)}{\partial t} = -\frac{\partial\{\widetilde{f}[x,u^0,\mu_G;g_\alpha]p_\alpha(t,x)\}}{\partial x} \\
&\qquad\qquad + \frac{\sigma^2}{2}\frac{\partial^2 p_\alpha(t,x)}{\partial x^2},
\end{aligned} \tag{3.13}$$

$$[\mathrm{BR}](\alpha) \quad u^0 := \varphi(t,x|\mu_G;g_\alpha).$$

Here $p_\alpha(t,x)$ with initial condition $p_\alpha(0)$ is used to formally denote the density of the measure $\mu_\alpha(t)$, which is assumed to exist. The FPK equation may be replaced by the following closed-loop MV-SDE:

$$[\mathrm{MV}](\alpha) \quad dx_\alpha(t) = \widetilde{f}[x_\alpha(t),\varphi(t,x_\alpha(t)|\mu_G;g_\alpha),\mu_G(t);g_\alpha]dt + \sigma dw_t^\alpha, \tag{3.14}$$

where $x_\alpha(0)$ has distribution μ_0^x. Our subsequent analysis will directly treat the pair $(V^\alpha(t,x),\mu_\alpha(t))$, where $\mu_\alpha(t)$ is specified as the law of $x_\alpha(t)$ in (3.14).

When a solution exists for the GMFG equations, the resulting BR feedback controls depend upon the set of graphon mean fields μ_G and an agent's individual state. This is a natural generalization of the standard case. The standard MFG case is simply obtained by setting $g(\alpha,\beta)\equiv 0, 0\le\alpha,\beta\le 1$, which totally disconnects the network and results in $\widetilde{f}[x,u,\mu_G;g_\alpha] = f_0[x,u,\mu]$, and $\widetilde{l}[x,u,\mu_G;g_\alpha] = l_0[x,u,\mu]$ [5, 8].

A collection of measures on some measurable space which are indexed by the vertex set $[0,1]$ is called a measure ensemble. Thus, for each fixed t, $\mu_G(t)$ is a measure ensemble.

On $\mathcal{P}_1(\mathbb{R})$ we endow the Wasserstein metric W_1: for any $\mu,\nu\in\mathcal{P}_1(\mathbb{R})$, $W_1(\mu,\nu) = \inf_{\widehat{\gamma}}\int|x-y|\widehat{\gamma}(dx,dy)$, where $\widehat{\gamma}$ is a probability measure on $\mathbb{R}^2$ with marginals μ,ν.

Let $C([0,1],\mathcal{P}_1(\mathbb{R}))$ be the set of measure ensembles $\nu_G = (\nu_\beta)_{\beta\in[0,1]}$ satisfying $\nu_\beta\in\mathcal{P}_1(\mathbb{R})$, and $\lim_{\beta'\to\beta}W_1(\nu_{\beta'},\nu_\beta)=0$ for any $\beta\in[0,1]$.

In order to analyze the solvability of the GMFG equations, we need to restrict $\mu_G(\cdot)$ to a certain class. We say $\{\mu_G(t), 0\le t\le T\}$ is from the admissible set $\mathcal{M}_{[0,T]}$ if:

(C1) For each fixed t, $\mu_G(t)$ is in $C([0,1],\mathcal{P}_1(\mathbb{R}))$.

(C2) There exists $\eta\in(0,1]$ such that for any bounded and Lipschitz continuous function ϕ on $\mathbb{R}$,

$$\sup_{\beta\in[0,1]}\Big|\int_{\mathbb{R}}\phi(y)\mu_\beta(t_1,dy) - \int_{\mathbb{R}}\phi(y)\mu_\beta(t_2,dy)\Big| \le C_h|t_1-t_2|^\eta,$$

where C_h may be selected to depend only on the Lipschitz constant $\mathrm{Lip}(\phi)$ for ϕ.

Condition (C1) ensures that integration with respect to $d\beta$ in (3.7) is well defined. Condition (C2) ensures that the drift term in the HJB equation (3.12) has a certain time continuity, which facilitates the subsequent existence analysis of the best response.

3.3. **Existence Analysis.** We introduce the following assumptions:

(H1) U is a compact set.

(H2) $f_0(x,u,y)$, $f(x,u,y)$, $l_0(x,u,y)$ and $l(x,u,y)$ are continuous and bounded functions on $\mathbb{R}\times U\times\mathbb{R}$ and are Lipschitz continuous in (x,y) uniformly with respect to u.

(H3) $f_0(x,u,y)$ and $f(x,u,y)$ are Lipschitz continuous in u, uniformly with respect to (x,y).

(H4) For any $q\in\mathbb{R}$, $\alpha\in[0,1]$ and probability measure ensemble $\nu_G\in C([0,1],\mathcal{P}_1(\mathbb{R}))$, the set

$$S_\alpha^{\nu_G}(x,q)=\arg\min_{u\in U}\{q(\widetilde{f}[x,u,\nu_G;g_\alpha])+\widetilde{l}[x,u,\nu_G;g_\alpha]\} \tag{3.15}$$

is a singleton, and for any given compact interval $\mathcal{I}=[\underline{q},\bar{q}]$, the resulting u as a function of $(x,q)\in\mathbb{R}\times\mathcal{I}$ is Lipschitz continuous in (x,q), uniformly with respect to ν_G and g_α, $0\le\alpha\le1$.

The next two assumptions will be used to ensure that the best responses have continuous dependence on α. In particular, (H5) is a continuity assumption on the graphon function $g(\alpha,\beta)$. Under (H5), $\widetilde{f}$ and $\widetilde{l}$ have continuity in α.

(H5) For any bounded and measurable function $h(\beta)$, the function $\int_0^1 g(\alpha,\beta)h(\beta)d\beta$ is continuous in $\alpha\in[0,1]$.

(H6) For given $\nu_G\in C([0,1],\mathcal{P}_1(\mathbb{R}))$, $S_\alpha^{\nu_G}(x,q)$ is continuous in (α,x,q).

Although the GMFG equation system only involves $\{\mu_G(t),0\le t\le T\}$, which may be viewed as a collection of marginals at different vertices, it is necessary to develop the existence analysis in the underlying probability spaces (see related discussions in [25, p.240]).

We begin by introducing some analytic preliminaries. For the space $C_T=C([0,T],\mathbb{R})$, we specify a σ-algebra $\mathcal{F}_T$ induced by all cylindrical sets of the form $\{x(\cdot)\in C_T: x(t_i)\in B_i, 1\le i\le j \text{ for some } j\}$, where B_i is a Borel set. Let $\mathbf{M}_T$ denote the space of all probability measures on $(C_T,\mathcal{F}_T)$. The canonical process X is defined by $X_t(\omega)=\omega_t$ for $\omega\in C_T$. On C_T, we introduce the metric $\rho(x,y)=\sup_t|x(t)-y(t)|\wedge 1$. Then (C_T,ρ) is a complete metric space. Based on ρ, we introduce the Wasserstein metric on $\mathbf{M}_T$. For $m_1,m_2\in\mathbf{M}_T$, denote

$$D_T(m_1,m_2)=\inf_{\widehat{m}}\int_{C_T\times C_T}(\sup_{s\le T}|X_s(\omega_1)-X_s(\omega_2)|\wedge 1)d\widehat{m}(\omega_1,\omega_2),$$

where $\widehat{m}$ is called a coupling as a probability measure on $(C_T,\mathcal{F}_T)\times(C_T,\mathcal{F}_T)$ with the pair of marginals m_1 and m_2, respectively. Then $(\mathbf{M}_T,D_T)$ is a complete metric space [40].

We introduce the product of probability measure spaces $\prod_{\alpha\in[0,1]}(C_T,\mathcal{F}_T,m_\alpha)$, where each individual space is interpreted as the path space of the agent at vertex α with a corresponding probability measure m_α. Denote the product of spaces of probability measures $\mathbf{M}_T^G=\prod_{\alpha\in[0,1]}\mathbf{M}_T$. An element in $\mathbf{M}_T^G$ is a measure ensemble. Given $m_G\in\mathbf{M}_T^G$, the projection operator Proj_α picks out its component m_α associated with $\alpha\in[0,1]$. Let $\mathbf{M}_T^{G0}$ consist of all $(m_\alpha)_{\alpha\in[0,1]}\in\mathbf{M}_T^G$ such that for any $\alpha\in[0,1]$, $D_T(m_{\alpha'},m_\alpha)\to0$ as $\alpha'\to\alpha$.

For two measure ensembles $m_G:=(m_\alpha)_{\alpha\in[0,1]}$ and $\bar{m}_G:=(\bar{m}_\alpha)_{\alpha\in[0,1]}$ in $\mathbf{M}_T^G$, define $d(m_G,\bar{m}_G)=\sup_{\alpha\in[0,1]}D_T(m_\alpha,\bar{m}_\alpha)$.

Lemma 3.1. $(\mathbf{M}_T^G,d)$ *is a complete metric space.*

Proof. If $\{m_G^k, k \geq 1\}$ is a Cauchy sequence in $\mathbf{M}_T^G$, then for each given α, the sequence $\{\mathrm{Proj}_\alpha(m_G^k), k \geq 1\}$ (of probability measures) is a Cauchy sequence in the complete metric space $\mathbf{M}_T$ and so it contains a limit. This in turn determines a limit in $\mathbf{M}_T^G$. $\square$

Given the probability measure $m_\alpha \in \mathbf{M}_T$, we determine the t-marginal $\mu_\alpha(t)$ by $\mu_\alpha(t, B) = m_\alpha(\{x(\cdot) \in C_T : x(t) \in B\})$ for any Borel set $B \subset \mathbb{R}$, and denote the mapping from $\mathbf{M}_T$ to $\mathcal{P}(\mathbb{R})$ (the set of probability measures on $\mathbb{R}$):

$$\mu_\alpha(t) = \mathrm{Marg}_t(m_\alpha). \tag{3.16}$$

Consider the measure ensemble $m_G = (m_\alpha)_{\alpha\in[0,1]} \in \mathbf{M}_T^G$ with $\mu_\alpha(t)$ given by (3.16). Define the time t marginals by the following mapping

$$\mathrm{Marg}_t(m_G) = (\mu_\alpha(t))_{\alpha\in[0,1]}, \tag{3.17}$$

where the right hand side is simply written as $\mu_G(t)$. For a given t, $\mu_G(t)$ may be interpreted as a measure valued function defined on the vertex set $[0, 1]$. Further denote the mapping $\mathrm{Marg}(m_G) = (\mu_G(t))_{t\in[0,T]} = \mu_G(\cdot)$.

Take a fixed

$$\mu_G(\cdot) \in \mathcal{M}_{[0,T]} \tag{3.18}$$

with its associated Hölder parameter η in (C2), and denote

$$\widetilde{f}_\alpha^*(t, x, u) = \widetilde{f}[x, u, \mu_G(t); g_\alpha], \quad \widetilde{l}_\alpha^*(t, x, u) = \widetilde{l}[x, u, \mu_G(t); g_\alpha].$$

Lemma 3.2. *Assume* (H1)–(H2). *For $h_\alpha = \widetilde{f}_\alpha^*(t, x, u)$ or $\widetilde{l}_\alpha^*(t, x, u)$, there exist constants C and C_{μ_G} which depends on $\mu_G(\cdot)$ such that*

$$\sup_{t,u,\alpha} |h_\alpha(t, x, u) - h_\alpha(t, y, u)| \leq C|x - y|,$$
$$\sup_{x,u,\alpha} |h_\alpha(t, x, u) - h_\alpha(s, x, u)| \leq C_{\mu_G}|t - s|^\eta,$$

where the supremum is taken over $t \in [0, T]$, $x \in \mathbb{R}$, $u \in U$ and $\alpha \in [0, 1]$.

Proof. The Lipschitz continuity of $\widetilde{f}_\alpha^*$ with respect to x follows from (H2) and (3.6)–(3.7). For $t_1, t_2 \in [0, T]$, we estimate $|\widetilde{f}[x, u, \mu_G(t_1); g_\alpha] - \widetilde{f}[x, u, \mu_G(t_2); g_\alpha]|$ by using the Lipschitz condition of f_0, f and condition (C2) for $\mathcal{M}_{[0,T]}$. This establishes the Hölder continuity of $\widetilde{f}_\alpha^*$ in t. The other cases can be similarly checked. $\square$

In order to analyze the best response of the α-agent, we introduce the HJB equation

$$-V_t^\alpha(t, x) = \inf_{u\in U}\{\widetilde{f}_\alpha^*(t, x, u)V_x^\alpha(t, x) + \widetilde{l}_\alpha^*(t, x, u)\} + \frac{\sigma^2}{2}V_{xx}^\alpha(t, x), \tag{3.19}$$

where $V^\alpha(T, 0) = 0$. It differs from (3.12) by allowing a general $\mu_G(\cdot) \in \mathcal{M}_{[0,T]}$.

For studying (3.19), we introduce some standard definitions. Denote $Q_T = (0, T) \times \mathbb{R}$, and $\overline{Q}_T = [0, T] \times \mathbb{R}$. Let $C^{1,2}(\overline{Q}_T)$ (resp., $C^{1,2}(Q_T)$) denote the set of functions with continuous derivatives v_t, v_x, v_{xx} on $\overline{Q}_T$ (resp., Q_T). Let $C_b^{1,2}(\overline{Q}_T)$ be the set of bounded functions in $C^{1,2}(\overline{Q}_T)$, and let the open (or closed) set Q_b be a bounded subset of Q_T. $W_\lambda^{1,2}(Q_b)$, $1 \leq \lambda < \infty$, shall denote the Sobolev space consisting of functions v such that each v and its generalize derivatives v_t, v_x, v_{xx} are in $L^\lambda(Q_b)$; denote the norm

$$\|v\|_{\lambda,Q_b}^{(2)} = \|v\|_{\lambda,Q_b} + \|v_t\|_{\lambda,Q_b} + \|v_x\|_{\lambda,Q_b} + \|v_{xx}\|_{\lambda,Q_b}, \tag{3.20}$$

where $\|v\|_{\lambda,Q_b} = (\int_{Q_b} |v(t,x)|^\lambda dtdx)^{1/\lambda}$. Denote $|v|_{Q_b} = \sup_{(t,x)\in Q_b} |v(t,x)|$. Now for $Q_b = (T_1,T_2)\times\mathcal{I}$, where $\mathcal{I}$ is a bounded open subset of $\mathbb{R}$, and $\beta\in(0,1)$, denote the Hölder norms

$$\begin{aligned}
|v|^\beta_{Q_b} &= |v|_{Q_b} + \sup_{t\in(T_1,T_2),x,y\in\mathcal{I}} |v(t,x)-v(t,y)|\cdot|x-y|^{-\beta}\\
&\qquad + \sup_{s,t\in(T_1,T_2),x\in\mathcal{I}} |v(s,x)-v(t,x)|\cdot|s-t|^{-\beta/2},\\
|v|^{1+\beta}_{Q_b} &= |v|^\beta_{Q_b} + |v_x|^\beta_{Q_b},\\
|v|^{2+\beta}_{Q_b} &= |v|^{1+\beta}_{Q_b} + |v_t|^\beta_{Q_b} + |v_{xx}|^\beta_{Q_b}.
\end{aligned}$$

Lemma 3.3. *Under* (H1)–(H4)*, the following holds:*

(i) *Equation* (3.19) *has a unique solution V^α in $C_b^{1,2}(\overline{Q}_T)$ and* $\sup_{\overline{Q}_T} |V^\alpha_{xx}| \le C$.

(ii) *The best response*

$$u_\alpha = \phi_\alpha(t,x|\mu_G(\cdot)), \quad \alpha\in[0,1] \tag{3.21}$$

as the optimal control law solved from (3.19) *is bounded and Borel measurable on* $[0,T]\times\mathbb{R}$*, and Lipschitz continuous in x, uniformly with respect to α for the given $\mu_G(\cdot)$.*

Proof. (i) Denote

$$\boldsymbol{H}_\alpha(t,x,q) = \min_{u\in U}\{q\widetilde{f}^*_\alpha(t,x,u) + \widetilde{l}^*_\alpha(t,x,u)\}.$$

Then (3.19) may be rewritten as

$$-V^\alpha_t(t,x) = \boldsymbol{H}_\alpha(t,x,V^\alpha_x) + \frac{\sigma^2}{2}V^\alpha_{xx}, \qquad V^\alpha(T,x)=0. \tag{3.22}$$

As in the proof of [25, Theorem 5], we use Hölder and Lipschitz continuity (with respect to t and x, respectively) of $\widetilde{f}^*_\alpha$ and $\widetilde{l}^*_\alpha$ in Lemma 3.2, and follow the method in the proof of Theorem IV.6.2 of [13, p. 210] to show that (3.19) has a unique solution $V^\alpha\in C_b^{1,2}(\overline{Q}_T)$, where uniqueness follows from a verification theorem using the closed-loop state process.

We continue with the proof that V^α_{xx} is bounded on $\overline{Q}_T$. Take any $x_0\in\mathbb{R}$. Denote $B_r(x_0) = (x_0-r, x_0+r)$ for $r>0$, and $Q_T^{x_0,r} = (0,T)\times B_r(x_0)$. We use two steps involving local estimates. Each step gets refined information about V^α in a region based on available bound information in a larger region. It suffices to obtain a bound of V^α_{xx} on $Q_T^{x_0,1}$ as long as this bound does not change with x_0.

Step 1. First, there exists a constant C_1 such that

$$\sup_{t,x,\alpha}|V^\alpha|\le C_1, \quad \sup_{t,x,\alpha}|V^\alpha_x|\le C_1. \tag{3.23}$$

The first inequality is obtained using (H1)–(H2) and the fact that V^α is the value function of the associated optimal control problem. The second inequality is proved by the difference estimate of $|V^\alpha(t,x)-V^\alpha(t,y)|$ as in [13, p. 209].

By (H1), (H2) and (3.23), we have

$$\sup_\alpha \sup_{(t,x)\in\overline{Q}_T} |\boldsymbol{H}_\alpha(t,x,V^\alpha_x(t,x))| \le C_2.$$

We use a typical method for analyzing semilinear parabolic equations. Once V^α is known to be a solution of (3.22), we view V^α as the solution of a linear equation with the free term $\boldsymbol{H}_\alpha(t,x,V^\alpha_x)$. For further estimates, we need $\lambda>n+2$ when using the norm (3.20). Fix $\lambda=n+3=4$. We find a bound

$$\|V^\alpha\|^{(2)}_{\lambda,Q_T^{x_0,2}} \le C_3,$$

where C_3 depends on (C_2, T, σ), and the bound of (f, f_0, l, l_0) but not on x_0, α; see [13, p. 207] and also [29, p. 342] for local estimates of the Sobolev norm of solutions defined on unbounded domain using a cut-off function. Take $\beta = 1 - \frac{n+2}{\lambda} = \frac{1}{4}$. Subsequently, since $\lambda > n+2$, we have the Hölder estimate

$$|V^\alpha|^{1+\beta}_{Q_T^{x_0,2}} \le C_4 \|V^\alpha\|^{(2)}_{\lambda, Q_T^{x_0,2}} \le C_3 C_4, \tag{3.24}$$

where C_4 is determined by $\lambda = 4$ without depending on x_0, α; see [13, p. 207], [29, p. 343].

Step 2. On $[0,T] \times \mathbb{R} \times [-C_1, C_1]$, we can show $\boldsymbol{H}_\alpha(t,x,q)$ is Hölder continuous in t and Lipschitz continuous in (x,q). Denote $\beta_1 = \min\{\eta, \beta\}$. Next we view $\boldsymbol{H}_\alpha(t,x,V_x^\alpha(t,x))$ as a function of (t,x). Then by use of (3.24) we further obtain

$$\sup_\alpha \sup_{x_0} |\boldsymbol{H}_\alpha(t,x,V_x^\alpha)|^{\beta_1}_{Q_T^{x_0,2}} \le C_5. \tag{3.25}$$

Subsequently, by the method in [13, p. 207-208] with its cut-off function technique and [29, p. 351-352], we use (3.25) and local Hölder estimates of (3.22) to obtain

$$|V^\alpha|^{2+\beta_1}_{Q_T^{x_0,1}} \le C_6, \tag{3.26}$$

where C_6 depends on C_5 but not on x_0, α. Since x_0 is arbitrary, it follows that

$$\sup_\alpha \sup_{\overline{Q}_T} |V_{xx}^\alpha| \le C_6. \tag{3.27}$$

(ii) By (H4), the optimal control law (3.21) as a function of (t,x) is well defined and is bounded on $[0,T] \times \mathbb{R}$ by compactness of U. It is Borel measurable on $\overline{Q}_T$; see [13, p.168]. Since $S_\alpha^{\nu_G}(x,q)$ is Lipschitz continuous in $(x,q) \in \mathbb{R} \times [-C_1, C_1]$ and $V_x^\alpha(t,x)$ is Lipschitz continuous in $x \in \mathbb{R}$ by (3.27), uniformly with respect to α in each case, ϕ_α is uniformly Lipschitz continuous in x. □

Denote

$$\Psi^\alpha(t,x) = (V^\alpha(t,x), V_t^\alpha(t,x), V_x^\alpha(t,x), V_{xx}^\alpha(t,x)), \quad (t,x) \in \overline{Q}_T.$$

We prove the following continuity lemma for the solution of (3.19). For $\overline{Q}_T$, define the compact subsets $B_j = \{(t,x) | 0 \le t \le T, |x| \le j\}$, $j \in \mathbb{N}$.

Lemma 3.4. *Assume* (H1)–(H5) *hold and let* $\mu_G(\cdot)$ *in* (3.18) *be fixed. Then the following holds:*

(i) *For each compact set* B_j, $\lim_{\alpha' \to \alpha} |\Psi^{\alpha'} - \Psi^\alpha|_{B_j} = 0$.

(ii) $\lim_{\alpha' \to \alpha} V_x^{\alpha'}(t,x) = V_x^\alpha(t,x)$, $\forall (t,x)$.

Proof. It suffices to show (i) as (ii) follows immediately from (i).

Step 1. By (3.26) and the fact that the constant C_6 can be selected without depending on α, there exists a constant C such that $\sup_\alpha |V^\alpha|^{2+\beta_1}_{B_j} \le C$, which implies that $\{\Psi^\alpha, \alpha \in [0,1]\}$ is uniformly bounded and equicontinuous on B_j. For any sequence $\{\alpha_k, k \ge 0\}$ converging to α, by Ascoli–Arzela's lemma, for $j=1$, there exists a subsequence denoted by $\{\bar\alpha_k, k \ge 1\}$ such that $\Psi^{\bar\alpha_k}$ converges uniformly on B_1. By a diagonal argument, we may further extract a subsequence of $\{\bar\alpha_k, k \ge 0\}$, denoted by $\{\hat\alpha_k, k \ge 1\}$, such that $\Psi^{\hat\alpha_k}$ converges uniformly on each set B_j, $j \ge 1$. Hence there exists a function V^* with continuous derivatives V_t^*, V_x^*, V_{xx}^* on $\overline{Q}_T$ such that

$$\lim_{k\to\infty} \Psi^{\hat\alpha_k}(t,x) = \Psi^*(t,x), \qquad \forall (t,x) \in \overline{Q}_T, \tag{3.28}$$

where $\Psi^* = (V^*, V_t^*, V_x^*, V_{xx}^*)$. Since

$$-V_t^{\hat{\alpha}_k}(t,x) = \boldsymbol{H}_{\alpha_k}(t,x,V_x^{\hat{\alpha}_k}) + \frac{\sigma^2}{2} V_{xx}^{\hat{\alpha}_k}, \quad V^{\alpha_k}(T,x) = 0,$$

it follows from (3.28) that

$$-V_t^*(t,x) = \boldsymbol{H}_{\alpha}(t,x,V_x^*) + \frac{\sigma^2}{2} V_{xx}^*, \qquad V^*(T,x) = 0.$$

We have used the fact that $\boldsymbol{H}_\alpha(t,x,q)$ is continuous in α due to (H5) and condition (C1) of $\mathcal{M}_{[0,T]}$. It is clear that $V^* = V^\alpha$ by uniqueness of the solution of (3.22). So $\Psi^* = \Psi^\alpha$. Now it follows that

$$\lim_{k\to\infty} |\Psi^{\hat{\alpha}_k} - \Psi^\alpha|_{B_j} = 0, \quad \forall j. \tag{3.29}$$

Step 2. Suppose (i) does not hold so that for some $\hat{j}$ we have $|\Psi^{\alpha'} - \Psi^\alpha|_{B_{\hat{j}}}$ does not converge to 0 as $\alpha' \to \alpha$, which implies that there exist some $\epsilon_0 > 0$ and a sequence $\{\alpha_k^0\}$ converging to α such that for each k,

$$|\Psi^{\alpha_k^0} - \Psi^\alpha|_{B_{\hat{j}}} \ge \epsilon_0. \tag{3.30}$$

Step 3. Recall that $\{\alpha_k\}$ in Step 1 is arbitrary as long as it converges to α. Now we just take $\{\alpha_k\}$ in Step 1 as $\{\alpha_k^0\}$. By Step 1, there exists a subsequence of $\{\alpha_k^0\}$, denoted by $\{\hat{\alpha}_k^0\}$, such that $\lim_{k\to\infty} |\Psi^{\hat{\alpha}_k^0} - \Psi^\alpha|_{B_{\hat{j}}} = 0$, which contradicts (3.30). Hence (i) holds. $\square$

Lemma 3.5. *Assume* (H1)–(H6). *For given $\mu_G(t) \in \mathcal{M}_{[0,T}$, the best response $\phi_\alpha(t,x|\mu_G(\cdot))$ in* (3.21) *continuously depends on α. Specifically, for any $\alpha \in [0,1]$,*

$$\lim_{\alpha'\to\alpha} \phi_{\alpha'}(t,x|\mu_G(\cdot)) = \phi_\alpha(t,x|\mu_G(\cdot)), \quad \forall t,x. \tag{3.31}$$

Proof. The best response can be written as

$$\begin{aligned}
\phi_\alpha(t,x|\mu_G(\cdot)) &= S_\alpha^{\mu_G(t)}(x, V_x^\alpha(t,x)),\\
\phi_{\alpha'}(t,x|\mu_G(\cdot)) &= S_{\alpha'}^{\mu_G(t)}(x, V_x^{\alpha'}(t,x)).
\end{aligned}$$

It follows that

$$\begin{aligned}
&|S_\alpha^{\mu_G(t)}(x, V_x^\alpha(t,x)) - S_{\alpha'}^{\mu_G(t)}(x, V_x^{\alpha'}(t,x))|\\
\le &|S_\alpha^{\mu_G(t)}(x, V_x^\alpha(t,x)) - S_{\alpha}^{\mu_G(t)}(x, V_x^{\alpha'}(t,x))|\\
&+ |S_\alpha^{\mu_G(t)}(x, V_x^{\alpha'}(t,x)) - S_{\alpha'}^{\mu_G(t)}(x, V_x^{\alpha'}(t,x))|.
\end{aligned}$$

Given $\mu_G(\cdot)$ we have prior upper bound $\sup_{\alpha,t,x} |V_x^\alpha(t,x)| \le C$. It suffices to show that (3.31) holds for any given $C_0 > 0$ and $t \in [0,T]$, $|x| \le C_0$. By (H6), for the given $\mu_G(t)$, $S_\alpha^{\mu_G(t)}(x,q)$ is uniformly continuous in $\alpha \in [0,1]$, $|x| \le C_0$, $q \in [-C, C]$. For any $\epsilon > 0$, there exists $\delta > 0$ such that $|\alpha - \alpha'| < \delta$ implies $\sup_{|x|\le C_0, |q|\le C} |S_\alpha^{\mu_G(t)}(x,q) - S_{\alpha'}^{\mu_G(t)}(x,q)| \le \epsilon/2$, and moreover,

$$\sup_{|x|\le C_0} |S_\alpha^{\mu_G(t)}(x, V_x^\alpha(t,x)) - S_\alpha^{\mu_G(t)}(x, V_x^{\alpha'}(t,x))| \le \frac{\epsilon}{2}$$

in view of Lemma 3.4 (i). Therefore (3.31) holds. $\square$

We proceed to show the existence of a solution to the GMFG equations (3.12) and (3.14) in terms of $\{(V^\alpha, \mu_\alpha(\cdot))|\alpha \in [0,1]\}$. For $\mu_G \in \mathcal{M}_{[0,T]}$, denote the mapping

$$(\phi_\alpha)_{\alpha\in[0,1]} := \Gamma(\mu_G(\cdot)),$$

where the left hand side is given by (3.21) as the set of best responses with respect to $\mu_G(\cdot)$. Next, we combine $(\phi_\alpha)_{\alpha\in[0,1]}$ with $\mu_G(\cdot)$ to determine the distribution m_α of the closed-loop state process

$$dx_\alpha(t) = \widetilde{f}[x_\alpha(t), \phi_\alpha(t, x_\alpha(t)|\mu_G(\cdot)), \mu_G(t); g_\alpha]dt + \sigma dw_t^\alpha,$$

where $x_\alpha(0)$ has distribution μ_0^x. The choice of the Brownian motion for x_α is immaterial. For m_α above, denote the mapping from $\mathcal{M}_{[0,T]}$ to $\mathbf{M}_T^G$:

$$(m_\alpha)_{\alpha\in[0,1]} = \widehat{\Gamma}(\mu_G(\cdot)).$$

Define the set

$$\mathbf{M}_T^{G1} := \widehat{\Gamma}(\mathcal{M}_{[0,T]}) \subset \mathbf{M}_T^G.$$

Now the existence analysis may be formulated as a fixed point problem

$$m_G = \widehat{\Gamma} \circ \mathrm{Marg}(m_G), \tag{3.32}$$

where $m_G \in \mathbf{M}_T^{G1}$. Note that $\mathrm{Marg}(m_G) = \{(\mathrm{Marg}_t(m_\alpha))_{\alpha\in[0,1]}, 0 \le t \le T\}$.

Remark 3.6. The fixed point problem requires m_G to be from the subset $\mathbf{M}_T^{G1}$ of $\mathbf{M}_T^G$. If it is simply to look for $m_G \in \mathbf{M}_T^G$, the resulting $\mu_G(\cdot) = \mathrm{Marg}(m_G)$ lacks necessary properties such as Hölder continuity in (C2), and will cause difficulty establishing Lemma 3.3 for the HJB equation.

Lemma 3.7. *Under* (H1)–(H6)*, the following assertions hold:*

(i) $\mathbf{M}_T^{G1} \subset \mathbf{M}_T^{G0}$.

(ii) *For any* $m_G \in \mathbf{M}_T^{G1}$*,* $\mu_G(\cdot) = \mathrm{Marg}(m_G) \in \mathcal{M}_{[0,T]}$.

(iii) *The best response* $\phi_\alpha(t, x|\mu_G(\cdot))$ *with* $\mu_G(\cdot)$ *given in* (ii) *is Lipschitz in* x*, uniformly with respect to* $\alpha \in [0,1]$*,* $m_G \in \mathbf{M}_T^{G1}$.

Proof. (i) and (ii) For $m_G \in \mathbf{M}_T^{G1}$, there exists $\mu'_G \in \mathcal{M}_{[0,T]}$ such that $m_G = \widehat{\Gamma}(\mu'_G(\cdot))$. To estimate $D_T(m_\alpha, m_{\bar\alpha})$ and $W_1(\mu_\alpha(t), \mu_{\bar\alpha}(t))$, let x_α and $x_{\bar\alpha}$ be state processes generated by (3.9) with μ'_G, the same initial state and Brownian motion under the control laws $\phi_\alpha(t, x|\mu'_G(\cdot))$ and $\phi_{\bar\alpha}(t, x|\mu'_G(\cdot))$, respectively. Then $D_T(m_\alpha, m_{\bar\alpha}) \le E\sup_{t\le T}|x_\alpha(t) - x_{\bar\alpha}(t)|$ and $W_1(\mu_\alpha(t), \mu_{\bar\alpha}(t)) \le E|x_\alpha(t) - x_{\bar\alpha}(t)|$. Fixing $\bar\alpha$, we have

$$\begin{aligned}|x_\alpha(t) - x_{\bar\alpha}(t)| \le \int_0^t |&\widetilde{f}[x_\alpha(s), \phi_\alpha(s, x_\alpha(s)|\mu'_G(\cdot)), \mu'_G(s); g_\alpha] \\ &- \widetilde{f}[x_{\bar\alpha}(s), \phi_{\bar\alpha}(s, x_{\bar\alpha}(s)|\mu'_G(\cdot)), \mu'_G(t); g_{\bar\alpha}]|ds.\end{aligned} \tag{3.33}$$

Denote

$$\delta_1 = |f_0[x_{\bar\alpha}(s), \phi_{\bar\alpha}(s, x_{\bar\alpha}(s)|\mu'_G(\cdot)), \mu'_\alpha(s)] - f_0[x_{\bar\alpha}(s), \phi_{\bar\alpha}(s, x_{\bar\alpha}(s)|\mu'_G(\cdot)), \mu'_{\bar\alpha}(s)]|,$$

$$\delta_2 = |f[x_{\bar\alpha}(s), \phi_{\bar\alpha}(s, x_{\bar\alpha}(s)|\mu'_G(\cdot)), \mu'_G(s); g_\alpha] - f[x_{\bar\alpha}(s), \phi_{\bar\alpha}(s, x_{\bar\alpha}(s)|\mu'_G(\cdot)), \mu'_G(s); g_{\bar\alpha}]|.$$

Then by (3.33) and the Lipschitz continuity in x of ϕ_α in Lemma 3.3 (ii), we obtain

$$\begin{aligned}|x_\alpha(t) - x_{\bar\alpha}(t)| &\le C_1\int_0^t |x_\alpha(s) - x_{\bar\alpha}(s)|ds \\ &+ C_2\int_0^t \{|\phi_\alpha(s, x_{\bar\alpha}(s)|\mu'_G(\cdot)) - \phi_{\bar\alpha}(s, x_{\bar\alpha}(s)|\mu'_G(\cdot))| + \delta_1(s) + \delta_2(s)\}ds,\end{aligned} \tag{3.34}$$

where C_2 only depends on the Lipschitz constants of f_0, f; and C_1 does not change with α for the fixed μ'_G. Since $W_1(\mu'_\alpha(s), \mu'_{\bar\alpha}(s)) \to 0$ as $\alpha \to \bar\alpha$, by (H2) $E\delta_1(s) \to 0$ as $\alpha \to \bar\alpha$. By (H5), we have $E\delta_2(s) \to 0$ as $\alpha \to \bar\alpha$. Then using Lemma 3.5 and boundedness of the integrand below, we obtain

$$\lim_{\alpha\to\bar\alpha} E\int_0^T \{|\phi_\alpha(s, x_{\bar\alpha}(s)|\mu'_G(\cdot)) - \phi_{\bar\alpha}(s, x_{\bar\alpha}(s)|\mu'_G(\cdot))| + \delta_1(s) + \delta_2(s)\}ds = 0.$$

By Gronwall's lemma and (3.34), it follows that

$$\lim_{\alpha\to\bar\alpha} E \sup_{0\le t\le T} |x_\alpha(t) - x_{\bar\alpha}(t)| = 0. \tag{3.35}$$

Subsequently, as $\alpha \to \bar\alpha$, we obtain $D_T(m_\alpha, m_{\bar\alpha}) \to 0$, which implies (i); in addition, $W_1(\mu_\alpha(t), \mu_{\bar\alpha}(t)) \to 0$, which verifies condition (C1) of $\mathcal{M}_{[0,T]}$ for μ_G. Since each m_α is the distribution of x_α, for $\mu_G(\cdot)$ we take the Hölder parameter $\eta = 1/2$ and a constant C_h independent of μ'_G for (C2). So (ii) holds.

(iii) Due to the choice of η and C_h for $\mu_G(\cdot)$ in (ii), we may select a fixed constant C_5 in (3.25), which does not change with $(\alpha, \mu_G(\cdot))$. Subsequently the upper bound C_6 for V^α_{xx} does not change with $\alpha \in [0,1], \mu_G(\cdot) \in \mathrm{Marg}(\widehat{\varGamma}(\mathcal{M}_{[0,T]}))$. This ensures a uniform bound for the Lipschitz constant for x in ϕ_α. □

We introduce the sensitivity condition.

(H7) For $m_G, \bar m_G \in \mathbf{M}_T^{G1} = \widehat{\varGamma}(\mathcal{M}_{[0,T]})$, there exists a constant c_1 such that

$$\sup_{t,x,\alpha} |\phi_\alpha(t, x|\mu_G(\cdot)) - \bar\phi_\alpha(t, x|\bar\mu_G(\cdot))| \le c_1 d(m_G, \bar m_G), \tag{3.36}$$

where the set of control laws $\{\phi_\alpha(t, x|\mu_G(\cdot)), \alpha \in [0,1]\}$ (resp., $\{\bar\phi_\alpha(t, x|\bar\mu_G(\cdot)), \alpha \in [0,1]\}$) is determined by use of $\mu_G = \mathrm{Marg}(m_G)$ (resp., $\bar\mu_G = \mathrm{Marg}(m_G)$) in the optimal control problem specified by (3.9) and (3.11) with the graphon section g_α.

Assumption (H7) is a generalization from the finite class model in [25] where an illustration via a linear model is presented. Related sensitivity conditions are studied in [28].

Let $(\phi_\alpha)_{\alpha\in[0,1]}$ in (3.21) be applied by all agents, where $\mu_G(\cdot) \in \mathcal{M}_{[0,T]}$. We consider the following generalized McKean-Vlasov equation

$$dx_\alpha(t) = \widetilde{f}[x_\alpha(t), \phi_\alpha(t, x_\alpha(t)|\mu_G), \nu_G(t); g_\alpha]dt + \sigma dw^\alpha_t, \tag{3.37}$$

where $x_\alpha(0)$ is given with distribution μ^x_0. For this equation, ν_G is part of the solution. If ν_G is determined, we have a unique solution x_α on $[0,T]$ which further determines its law as the measure m_α on $(C_T, \mathcal{F}_T)$. Note that m_α does not depend on the choice of the standard Brownian motion w^α. We look for $\nu_G \in \mathcal{M}_{[0,T]}$ to satisfy the condition:

$$\mathrm{Marg}_t(m_\alpha) = \nu_\alpha(t), \quad \forall\alpha \in [0,1],\ t \in [0,T], \tag{3.38}$$

i.e., $\nu_\alpha(t)$ is the law of $x_\alpha(t)$ for all α, t (and we say $(x_\alpha)_{0\le\alpha\le1}$ is consistent with ν_G).

Lemma 3.8. *Assume* (H1)–(H6). *For the best response control law $\phi_\alpha(t, x_\alpha|\mu_G(\cdot))$ in* (3.21)*, where $\mu_G(\cdot) \in \mathcal{M}_{[0,T]}$, there exists a unique $\nu_G(\cdot)$ for* (3.37) *satisfying* (3.38)*.*

Proof. In order to solve (x_α, ν_G) in (3.37), we specify the law of the process x_α instead of just its marginal $\nu_\alpha(t)$. This extends the fixed point idea for treating standard McKean-Vlasov equations [40].

For $(m_\alpha)_{\alpha\in[0,1]} \in \mathbf{M}_T^{G0}$, we determine ν^1_G according to $\nu^1_\alpha(t) = \mathrm{Marg}_t(m_\alpha)$, which is used in (3.37) by taking $\nu_G = \nu^1_G$ to solve x_α on $[0,T]$. Let m^{new}_α denote the law of x_α. It in general does not satisfy $\mathrm{Marg}_t(m^{\mathrm{new}}_\alpha) = \nu_\alpha(t)$ for all t. Denote the mapping

$$(m^{\mathrm{new}}_\alpha)_{\alpha\in[0,1]} = \varPhi_{\mathbf{M}_T^{G0}}((m_\alpha)_{\alpha\in[0,1]}).$$

By (H5) and Lemma 3.5, $\Phi_{\mathbf{M}_T^{G0}}$ is a mapping from $\mathbf{M}_T^{G0}$ to itself. Similarly, from $(\bar{m}_\alpha)_{\alpha\in[0,1]} \in \mathbf{M}_T^{G0}$ we determine $\bar{\nu}_G^1$ for (3.37) and solve $\bar{x}_\alpha$ with its law $\bar{m}_\alpha^{\text{new}}$. Denote

$$(\bar{m}_\alpha^{\text{new}})_{\alpha\in[0,1]} = \Phi_{\mathbf{M}_T^{G0}}((\bar{m}_\alpha)_{\alpha\in[0,1]}).$$

If $h(x,y)$ a bounded Lipschitz continuous function with $|h(x,y)-h(\bar{x},\bar{y})| \le C_1|x-\bar{x}| + C_2(|y-\bar{y}|\wedge 1)$, we have

$$\begin{aligned}
&|\int h(x,y)g(\alpha,\beta)\nu_\beta^1(t,dy)d\beta - \int h(\bar{x},\bar{y})g(\alpha,\beta)\nu_\beta^2(t,d\bar{y})d\beta| \\
\le& C_1|x-\bar{x}| + \sup_\beta |\int h(\bar{x},y)\nu_\beta^1(t,dy) - \int h(\bar{x},\bar{y})\nu_\beta^2(t,d\bar{y})| \\
=& C_1|x-\bar{x}| + \sup_\beta |\int_{C_T} h(\bar{x},X_t(\omega))dm_\beta(\omega) - \int_{C_T} h(\bar{x},X_t(\bar{\omega}))d\bar{m}_\beta(\bar{\omega})| \\
\le& C_1|x-\bar{x}| + C_2\sup_\beta \int_{C_T\times C_T} (|X_t(\omega)-X_t(\bar{\omega})|\wedge 1)d\widehat{m}_\beta(\omega,\bar{\omega}),
\end{aligned}$$

where X is the canonical process, $\omega,\bar{\omega}\in C_T$, and $\widehat{m}_\beta$ is any coupling of m_β and $\bar{m}_\beta$. Hence

$$\begin{aligned}
&|\int h(x,y)g(\alpha,\beta)\nu_\beta^1(t,dy)d\beta - \int h(\bar{x},\bar{y})g(\alpha,\beta)\nu_\beta^2(t,d\bar{y})d\beta| \\
&\le C_1|x-\bar{x}| + C_2\sup_\beta D_t(m_\beta,\bar{m}_\beta).
\end{aligned}\tag{3.39}$$

By (H2), (H3), the uniform Lipschitz continuity of ϕ_α in x by Lemma 3.3 (ii), and (3.39), we obtain

$$\begin{aligned}
&|\widetilde{f}[x_\alpha,\phi_\alpha(t,x_\alpha|\mu_G),\nu_G^1(t);g_\alpha] - \widetilde{f}[\bar{x}_\alpha,\phi_\alpha(t,\bar{x}_\alpha|\mu_G),\nu_G^2(t);g_\alpha]| \\
\le& C_1(|x_\alpha-\bar{x}_\alpha|\wedge 1) + C_2\sup_\beta D_t(m_\beta,\bar{m}_\beta).
\end{aligned}$$

Hence by (3.37),

$$\begin{aligned}
\sup_{s\le t}|x_\alpha(s)-\bar{x}_\alpha(s)| \le\ & C_1\int_0^t |x_\alpha(s)-\bar{x}_\alpha(s)|\wedge 1 ds \\
&+ C_3\int_0^t \sup_\beta |D_s(m_\beta,\bar{m}_\beta)|ds.
\end{aligned}$$

Therefore, by Gronwall's lemma,

$$\sup_{s\le t}|x_\alpha(s)-\bar{x}_\alpha(s)|\wedge 1 \le C_4\int_0^t \sup_\beta |D_s(m_\beta,\bar{m}_\beta)|ds,$$

which combined with the definition of the Wasserstein metric $D_t(\cdot,\cdot)$ implies that

$$\sup_\beta |D_t(m_\beta^{\text{new}},\bar{m}_\beta^{\text{new}})| \le C_4\int_0^t \sup_\beta |D_s(m_\beta,\bar{m}_\beta)|ds.\tag{3.40}$$

By iterating (3.40) as in [40, p. 174], we can show that for a sufficiently large k_0, $\Phi_{\mathbf{M}_T^{G0}}^{k_0}$ is a contraction. We can further show that $\{\Phi_{\mathbf{M}_T^{G0}}^k(m_G), k\ge 1\}$ is a Cauchy sequence, and we obtain a unique fixed point m_G^* for $\Phi_{\mathbf{M}_T^{G0}}$. Then we obtain a solution of (3.37) by taking $\nu_\alpha(t) = \mathrm{Marg}_t(m_\alpha^*)$. If there are two different solutions with $\nu_G \ne \nu_G'$, we can derive a contradiction by using uniqueness of the fixed point of $\Phi_{\mathbf{M}_T^{G0}}$. $\square$

Now we consider two sets of best response control laws $(\phi_\alpha(t, x_\alpha|\mu_G))_{\alpha\in[0,1]}$ and $(\bar\phi_\alpha(t, x_\alpha|\bar\mu_G))_{\alpha\in[0,1]}$, where $\mu_G = \mathrm{Marg}(m_G), \bar\mu_G = \mathrm{Marg}(\bar m_G)$ for $m_G, \bar m_G \in \mathbf{M}_T^{G1}$ (then clearly $\mu_G, \bar\mu_G \in \mathcal{M}_{[0,T]}$), and use Lemma 3.8 to solve (x_α, ν_G) and $(x'_\alpha, \bar\nu_G)$ from the generalized MV SDEs

$$dx_\alpha = \widetilde f[x_\alpha, \phi_\alpha(t, x_\alpha|\mu_G), \nu_G(t); g_\alpha]dt + \sigma dw_t^\alpha, \tag{3.41}$$

$$dx'_\alpha = \widetilde f[x'_\alpha, \bar\phi_\alpha(t, x'_\alpha|\bar\mu_G), \bar\nu_G(t); g_\alpha]dt + \sigma dw_t^\alpha, \tag{3.42}$$

where $x'_\alpha(0) = x_\alpha(0)$ is given. Let m_α^{mv} (resp., $\bar m_\alpha^{\mathrm{mv}}$) denote the law of x_α (resp., x'_α). The following lemma is a generalization of [25, Lemma 9] to the graphon network case.

Lemma 3.9. *For* (3.41) *and* (3.42) *there exists a constant* c_2 *independent of* $(\alpha, m_G, \bar m_G)$ *such that*

$$D_T(m_\alpha^{\mathrm{mv}}, \bar m_\alpha^{\mathrm{mv}}) \le c_2 \sup_{t,x} |\phi_\alpha(t, x|\mu_G(\cdot)) - \bar\phi_\alpha(t, x|\bar\mu_G(\cdot))|.$$

Proof. For (3.41)–(3.42), denote

$$\varDelta_s = \widetilde f[x_\alpha(s), \phi_\alpha(s, x_\alpha(s)|\mu_G), \nu_G(s); g_\alpha] - \widetilde f[x'_\alpha(s), \bar\phi_\alpha(s, x'_\alpha(s)|\bar\mu_G), \bar\nu_G(s); g_\alpha].$$

We have

$$x_\alpha(t) - x'_\alpha(t) = \int_0^t \varDelta_s ds. \tag{3.43}$$

Noting $\nu_\alpha(t) = \mathrm{Marg}_t(m_\alpha^{\mathrm{mv}})$ and $\bar\nu_\alpha(t) = \mathrm{Marg}_t(\bar m_\alpha^{\mathrm{mv}})$, we have

$$\begin{aligned} |\varDelta_s| \le& |\widetilde f[x_\alpha(s), \phi_\alpha(s, x_\alpha(s)|\mu_G), \nu_G(s); g_\alpha] - \widetilde f[x'_\alpha(s), \phi_\alpha(s, x'_\alpha(s)|\mu_G), \bar\nu_G(s); g_\alpha]| \\ &+ |\widetilde f[x'_\alpha(s), \phi_\alpha(s, x'_\alpha(s)|\mu_G), \bar\nu_G(s); g_\alpha] - \widetilde f[x'_\alpha(s), \bar\phi_\alpha(s, x'_\alpha(s)|\bar\mu_G), \bar\nu_G(s); g_\alpha]| \\ \le& C_1|x_\alpha(s) - x'_\alpha(s)| + C_2 D_s(m_\alpha^{\mathrm{mv}}, \bar m_\alpha^{\mathrm{mv}}) \\ &+ C_3 \sup_{t,x} |\phi_\alpha(t, x|\mu_G(\cdot)) - \bar\phi_\alpha(t, x|\bar\mu_G(\cdot))|. \end{aligned} \tag{3.44}$$

The difference term on the first line is estimated by the method in (3.39). We have used the fact that ϕ_α is uniformly Lipschitz in x by Lemma 3.7 (iii). Therefore, by (3.43)–(3.44),

$$\begin{aligned} |x_\alpha(t) - x'_\alpha(t)| \le& \int_0^t [C_1|x_\alpha(s) - x'_\alpha(s)| + C_2 D_s(m_\alpha^{\mathrm{mv}}, \bar m_\alpha^{\mathrm{mv}})]ds \\ &+ C_3 t \sup_{t,x} |\phi_\alpha(t, x|\mu_G(\cdot)) - \bar\phi_\alpha(t, x|\bar\mu_G(\cdot))|. \end{aligned}$$

By Gronwall's lemma, we obtain

$$\begin{aligned} \sup_{0\le s\le t} |x_\alpha(s) - x'_\alpha(s)| \wedge 1 \le& e^{C_1 t} C_2 \int_0^t D_s(m_\alpha^{\mathrm{mv}}, \bar m_\alpha^{\mathrm{mv}})ds \\ &+ e^{C_1 t} C_3 t \sup_{t,x} |\phi_\alpha(t, x|\mu_G(\cdot)) - \bar\phi_\alpha(t, x|\bar\mu_G(\cdot))|, \end{aligned}$$

which again by the definition of the metric $D_t(\cdot, \cdot)$ leads to

$$\begin{aligned} D_t(m_\alpha^{\mathrm{mv}}, \bar m_\alpha^{\mathrm{mv}}) \le& e^{C_1 t} C_2 \int_0^t D_s(m_\alpha^{\mathrm{mv}}, \bar m_\alpha^{\mathrm{mv}})ds \\ &+ e^{C_1 t} C_3 t \sup_{t,x} |\phi_\alpha(t, x|\mu_G(\cdot)) - \bar\phi_\alpha(t, x|\bar\mu_G(\cdot))|. \end{aligned} \tag{3.45}$$

The lemma follows from applying Gronwall's lemma to (3.45). $\square$

3.4. **Existence Theorem.** We state the main result on the existence and uniqueness of solutions to the GMFG equation system. We introduce a contraction condition:

(H8) $c_1 c_2 < 1$, where c_1 is the constant in the sensitivity condition (H7) and c_2 is specified in Lemma 3.9.

Remark 3.10. By SDE estimates, one can obtain refined bound information on c_2. When the coupling effect is weak or T is small, a small value for c_2 can be obtained.

Remark 3.11. For linear models, a verification of the contraction condition can be done under reasonable model parameters, as in [25].

Theorem 3.12. *Under* (H1)–(H8)*, there exists a unique solution* $(V^\alpha, \mu_\alpha(\cdot))_{\alpha\in[0,1]}$ *to the GMFG equations* (3.12) *and* (3.14)*, which* (i) *gives the feedback control best response* (BR) *strategy* $\varphi(t, x_\alpha|\mu_G(\cdot); g_\alpha)$ *depending only upon the agent's state and the graphon local mean fields (i.e.* (x_α, μ_G)*), and* (ii) *generates a Nash equilibrium.*

Proof. Step 1 – We return to the fixed point equation (3.32), which is redisplayed below:

$$m_G = \widehat{\Gamma} \circ \mathrm{Marg}(m_G), \tag{3.46}$$

where $m_G = (m_\alpha)_{\alpha\in[0,1]} \in \mathbf{M}_T^{G1}$. For $m_G \in \mathbf{M}_T^{G1}$, the Hölder continuity in t of the regenerated $\mu_G(\cdot) = \mathrm{Marg}(m_G)$ can be checked by elementary SDE estimates by adapting the proof of [25, Lemma 7].

Step 2 – Take a general $m_G \in \mathbf{M}_T^{G1}$ to determine $\mu_G = \mathrm{Marg}(m_G)$ and $\phi_\alpha(t, x_\alpha|\mu_G(\cdot))$. When $\bar{m}_G \in \mathbf{M}_T^{G1}$ is used, we determine $\bar{\mu}_G$ and $\bar{\phi}_\alpha(t, x_\alpha|\bar{\mu}_G(\cdot))$. Once the set of strategies $(\phi_\alpha)_{\alpha\in[0,1]}$ is applied to the generalized MV equation (3.37), by Lemma 3.8, we may solve for $(x_\alpha, \nu_G(\cdot))$ such that x_α has the law $m_\alpha^{\mathrm{new}} = \nu_\alpha(t)$. This is done in parallel for $\bar{m}_G$ to generate $\bar{m}_\alpha^{\mathrm{new}}$. We accordingly determine m_G^{new} and $\bar{m}_G^{\mathrm{new}}$.

Step 3 – By (3.36) and Lemma 3.9, we obtain

$$D_T(m_\alpha^{\mathrm{new}}, \bar{m}_\alpha^{\mathrm{new}}) \le c_1 c_2 d(m_G, \bar{m}_G).$$

Since α is arbitrary, it follows that

$$d(m_G^{\mathrm{new}}, \bar{m}_G^{\mathrm{new}}) \le c_1 c_2 d(m_G, \bar{m}_G).$$

Based on the above contraction property, we construct a Cauchy sequence in the complete metric space $\mathbf{M}_T^G$ by iterating with m_G and establish existence of a solution to the GMFG equation system. To show uniqueness, suppose m_G and $\tilde{m}_G$ are two fixed points to (3.46). We obtain $d(m_G, \tilde{m}_G) \le c_1 c_2 d(m_G, \tilde{m}_G)$, which implies $m_G = \tilde{m}_G$.

The Nash equilibrium property follows from the best response property of ϕ_α for a given vertex α. □

3.5. **An Example on Lipschitz feedback.** The main analysis in Section 3 relies on (H4) to ensure Lipschitz feedback. We provide a concrete model to check this assumption.

Example 3.13. The dynamics and cost have

$$f_0(x,u,y) = f_0(x,y)u, \quad f(x,u,y) = f(x,y)u,$$
$$l_0(x,u,y) = l_1(x,y) + l_2(x,y)u^2, \quad l(x,u,y) = l_3(x,y) + l_4(x,y)u^2,$$

where $x, y \in \mathbb{R}$ and $u \in U = [a,b]$. The functions f_0, f, l_1, l_2, l_3, l_4 satisfy (H1)–(H3), and there exists $c_0 > 0$ such that $l_2, l_4 \ge c_0$ for all x, y.

Given $\nu_G \in C([0,1], \mathcal{P}_1(\mathbb{R}))$, we check the minimizer of

$$S_\alpha^{\nu_G}(x,q) = \arg\min_{u\in U}\{q(f_0[x,\nu_\alpha] + f[x,\nu_G;g_\alpha])u + (l_2[x,\nu_\alpha] + l_4[x,\nu_G;g_\alpha])u^2\},$$

where $x, q \in \mathbb{R}$.

Proposition 3.14. *Given any compact interval $\mathcal{I}$, $S_\alpha^{\nu_G}(x,q)$ in Example 3.13 is a singleton and Lipschitz in (x,q), where $x \in \mathbb{R}$ and $q \in \mathcal{I}$, uniformly with respect to (ν_G, α).*

Proof. Consider the function $\Phi(u) = u^2 - 2su$, where $u \in U$ and s is a parameter. Its minimum is attained at the unique point

$$\hat{u} = \Theta(s) := \begin{cases} a & \text{if} \quad s \le a, \\ s & \text{if} \quad a < s < b, \\ b & \text{if} \quad s \ge b. \end{cases}$$

Denote the function

$$h_{\alpha,\nu_G}(x) = -\frac{f_0[x,\mu_\alpha] + f[x,\nu_G;g_\alpha]}{2(l_2[x,\mu_\alpha] + l_4[x,\nu_G;g_\alpha])}.$$

By elementary estimates we can show

$$|h_{\alpha,\nu_G}(x) - h_{\alpha,\nu_G}(y)| \le C_0|x-y|,$$

where C_0 does not depend on (ν_G, α). We have

$$\begin{aligned} S_\alpha^{\nu_G}(x,q) &= \arg\min_u (u^2 - 2qh_{\alpha,\nu_G}(x)u) \\ &= \Theta(qh_{\alpha,\nu_G}(x)). \end{aligned}$$

It is clear that $S_\alpha^{\nu_G}(x,q)$ is a continuous function of (x,q). For $(x,q) \in \mathbb{R}\times\mathcal{I}$,

$$\begin{aligned} &|S_\alpha^{\nu_G}(q_1,x_1) - S_\alpha^{\nu_G}(q_2,x_2)| \\ &\le \mathrm{Lip}(\Theta)(q_1 h_{\alpha,\nu_G}(x_1) - q_2 h_{\alpha,\nu_G}(x_2)) \\ &\le \mathrm{Lip}(\Theta)(|q_1 - q_2|\sup_x |h_{\alpha,\nu_G}(x)| + C_0|x_1 - x_2||q_2|). \end{aligned}$$

Note that there exists a fixed constant C such that $|h_{\alpha,\nu_G}(x)| \le C$ for all α, ν_G. This proves the proposition. □

If (H1)–(H3) and (H5) hold for Example 3.13, they further imply (H4) and (H6) so that the best response is Lipschitz continuous in x by Lemma 3.3 and Proposition 3.14.

4. Performance Analysis

In the MFG case it is shown [25, 8] that the joint strategy $\{u_i^\circ(t) = \varphi_i(t, x_i(t)|\mu_\cdot), 1 \le i \le N\}$ yields an ϵ-Nash equilibrium, i.e. for all $\epsilon > 0$, there exists $N(\epsilon)$ such that for all $N \ge N(\epsilon)$

$$J_i^N(u_i^\circ, u_{-i}^\circ) - \epsilon \le \inf_{u_i\in\mathcal{U}_i} J_i^N(u_i, u_{-i}^\circ) \le J_i^N(u_i^\circ, u_{-i}^\circ). \tag{4.1}$$

This form of approximate Nash equilibrium is a principal result of the MFG analyses in the sequence [25, 8, 39] and in many other studies. The importance of (4.1) is that it states that the cost function of any agent in a finite population can be reduced by at most ϵ *if it changes unilaterally from the infinite population MFG feedback law* while all other agents remain with the infinite population based control strategies. The main result of this section is that the same property holds for GMFG systems.

Throughout this section, let $\mu_G(\cdot)$ be solved from the GMFG equations (3.12) and (3.14).

4.1. **The ϵ-Nash Equilibrium.** The analysis of GMFG systems as limits of finite objects necessarily involves the consideration of graph limits and double limits in population and graph order. A corresponding set of assumptions is given below.

(H9) $M_k \to \infty$ and $\min_{1\le l\le M_k} |\mathcal{C}_l| \to \infty$ as $k \to \infty$.

(H10) All agents have i.i.d. initial states with distribution μ_0^x and $E|x_i(0)| \le C_0$.

Remark 4.1. (H10) is a simplifying assumption to keep further notation light. It may be generalized to α dependent initial distributions.

(H11) The sequence $\{G_k; 1 \le k < \infty\}$ and the graphon limit satisfy

$$\lim_{k\to\infty} \max_i \sum_{j=1}^{M_k} \Big| \frac{1}{M_k} g^k_{\mathcal{C}_i,\mathcal{C}_j} - \int_{\beta\in I_j} g_{I_i^*,\beta} d\beta \Big| = 0,$$

where I_i^* is the midpoint of the subinterval $I_i \in \{I_1 \cdots I_{M_k}\}$ of length $1/M_k$.

Remark 4.2. Assumption (H11) specifies the nature of the approximation error between g^k for the finite graph and the graphon function g.

For the ϵ-Nash equilibrium analysis, we consider a sequence of games each defined on a finite graph G_k. Recall that there is a total of $N = \sum_{l=1}^{M_k} |\mathcal{C}_l|$ agents.

Suppose the cluster $\mathcal{C}(i)$ of agent $\mathcal{A}_i$ corresponds to the subinterval $I(i) \in \{I_1, \cdots, I_{M_k}\}$. The agent $\mathcal{A}_i$ takes the midpoint $I^*(i)$ of the subinterval $I(i)$ and use the GMFG equations to determine its control law

$$\hat{u}_i = \varphi(t, x_i | \mu_G(\cdot); g_{I^*(i)}), \quad 1 \le i \le N, \tag{4.2}$$

which we simply write as $\varphi(t, x_i, g_{I^*(i)})$. Denote the resulting state process by $\hat{x}_i$, $1 \le i \le N$. Recall that

$$f_0(x_i^N, u_i^N, \mathcal{C}(i)) = \frac{1}{|\mathcal{C}(i)|} \sum_{j\in\mathcal{C}(i)} f(x_i^N, u_i^N, x_j^N),$$

$$f_{G_k}(x_i^N, u_i^N, g^k_{\mathcal{C}(i)}) = \frac{1}{M_k} \sum_{l=1}^{M_k} g^k_{\mathcal{C}(i)\mathcal{C}_l} \frac{1}{|\mathcal{C}_l|} \sum_{j\in\mathcal{C}_l} f(x_i^N, u_i^N, x_j^N),$$

where the superscript N is added to indicate the population size. The closed-loop system of N agents on the finite graph G_k under the set of strategies (4.2) is given by

$$\begin{aligned} \textit{System A:} \quad d\hat{x}_i^N =& f_0(\hat{x}_i^N, \varphi(t, \hat{x}_i^N, g_{I^*(i)}), \mathcal{C}(i))dt \\ &+ f_{G_k}(\hat{x}_i^N, \varphi(t, \hat{x}_i^N, g_{I^*(i)}), g^k_{\mathcal{C}(i)})dt + \sigma dw_i, \end{aligned} \tag{4.3}$$

where $1 \le i \le N$ and $\hat{x}_i^N(0) = x_i^N(0)$. Note that $g^k_{\mathcal{C}(i)}$ appears in f_{G_k} as determined by the finite population system dynamics. We state the following main result.

Theorem 4.3. (**ϵ-Nash equilibrium**) *Assume* (H1)–(H11). *When the strategies* (4.2) *determined by the GMFG equations* (3.12) *and* (3.14) *are applied to a sequence of finite graph systems* $\{G_k; 1 \le k < \infty\}$ *with limit* G *in cut metric, the ϵ-Nash equilibrium property holds where* $\epsilon \to 0$ *as* $k \to \infty$*, and where the unilateral agent* $\mathcal{A}_i$ *uses a centralized Lipschitz feedback strategy* $\psi(t, x_i, x_{-i})$*, with* x_{-i} *being the states of all other agents.*

We first explain the basic idea for demonstrating the ϵ-Nash equilibrium property. Suppose all other players, except agent $\mathcal{A}_i$, employ the control strategies based on the GMFG equation system. When $\mathcal{A}_i$ employs a different strategy, the resulting change in its performance can be measured using a limiting stochastic control problem where both the system dynamics and the cost are subject to small perturbation due to the mean field approximation of the effects of all other agents. The proof is technical and preceded by some lemmas.

4.2. **Proof of Theorem 4.3.** Suppose x_i^N is determined from a general feedback control law u_i^N instead of the GMFG best response. With the exception of agent $\mathcal{A}_i$ with its unilateral strategy, all other agents $\mathcal{A}_j$, $j \neq i$, still have strategies determined by (4.2). We introduce the system:

$$\text{(4.4)} \quad \textit{System B:} \quad \begin{cases} dx_i^N = f_0(x_i^N, u_i^N, \mathcal{C}(i))dt + f_{G_k}(x_i^N, u_i^N, g^k_{\mathcal{C}(i)})dt + \sigma dw_i, \\ dx_j^N = f_0(x_j^N, \varphi(t, x_j^N, g_{I^*(j)}), \mathcal{C}(j))dt \\ \qquad\qquad + f_{G_k}(x_j^N, \varphi(t, x_j^N, g_{I^*(j)}), g^k_{\mathcal{C}(j)})dt + \sigma dw_j, \\ \qquad\qquad j \neq i, \quad 1 \le j \le N. \end{cases}$$

We note that x_j^N is affected by x_i^N due to the coupling in f_0 and f_{G_k}. For this reason, x_j^N differs from $\hat{x}_j^N$ in (4.3) although the control law of $\mathcal{A}_j$, $j \neq i$, remains the same. The central task is to estimate by how much $\mathcal{A}_i$ can reduce its cost.

To facilitate the performance estimate in System B, we introduce two auxiliary systems below. Consider

$$\begin{aligned} \textit{System C:} \quad dy_i^N &= \int_{\mathbb{R}} f_0(y_i^N, \varphi(t, y_i^N, g_{I^*(i)}), z) m_{y_i^N}(dz)dt \\ &\quad + \frac{1}{M_k}\sum_{l=1}^{M_k} g^k_{\mathcal{C}(i)\mathcal{C}_l} \frac{1}{|C_l|}\sum_{j\in\mathcal{C}_l}\int_{\mathbb{R}} f(y_i^N, \varphi(t, y_i^N, g_{I^*(i)}), z) m_{y_j^N}(dz)dt \\ &\quad + \sigma dw_i \\ &= \int_{\mathbb{R}} f_0(y_i^N, \varphi(t, y_i^N, g_{I^*(i)}), z) m_{y_i^N}(dz)dt \\ &\quad + \frac{1}{M_k}\sum_{l=1}^{M_k} g^k_{\mathcal{C}(i)\mathcal{C}_l} \int f(y_i^N, \varphi(t, y_i^N, g_{I^*(i)}), z) m_l^N(t, dz)dt \\ \text{(4.5)} \qquad &\quad + \sigma dw_i, \end{aligned}$$

where $1 \le i \le N$ and $y_i^N(0) = x_i^N(0)$, and $m_{y_j^N(t)}$ denotes the law of $y_j^N(t)$. Each Brownian motion w_i is the same as in (4.3). The second equality holds since all processes in cluster $\mathcal{C}_l$ have the same distribution denoted by $m_l^N(t, dz)$ at time t. It is clear that the processes $y_1^N, \cdots, y_N^N$ are independent, and $\{y_j^N, j \in \mathcal{C}_l\}$ are i.i.d. for any given l.

Next we introduce

$$\text{(4.6)} \quad \textit{System D:} \quad dy_i^\infty(t) = \widetilde{f}[y_i^\infty(t), \varphi(t, y_i^\infty(t), g_{I^*(i)}), \mu_G(t); g_{I^*(i)}\,]dt + \sigma dw_i(t),$$

where $1 \le i \le N$ and $y_i^\infty(0) = x_i^N(0)$. Here w_i is the same as in (4.3). The process y_i^∞ is generated by the closed-loop dynamics for an agent at the node $I^*(i)$ associated with the cluster $\mathcal{C}(i)$ using the GMFG based control law (4.2) while situated in an infinite population represented by the mean fields $\mu_G(\cdot)$. We view (4.6) as an instance of the generic equation (3.9) under the control law (4.2). By Theorem 3.12, $y_i^\infty(t)$ has the law $\mu_{I^*(i)}(t)$. Note that if $j \in \mathcal{C}(i)$, y_i^∞ and y_j^∞ are two processes of the same distribution.

We shall denote the A to C system deviation by $\epsilon_{1,N}$, the C to D deviation by $\epsilon_{2,N}$ and the (non-unilateral agent) B to D deviation by $\epsilon_{3,N}$. Specifically, denote

$$\epsilon_{1,N} = \sup_{i \le N, t} E|\hat{x}_i^N(t) - y_i^N(t)|, \qquad \epsilon_{2,N} = \sup_{i \le N, t} E|y_i^N(t) - y_i^\infty(t)|,$$
$$\epsilon_{3,N} = \sup_{u_i^N, t, i \neq j \le N} E|x_j^N(t) - y_j^\infty(t)|,$$

where x_j^N is given by (4.4).

Lemma 4.4. *The SDE system* (4.5) *has a unique solution* $(y_1^N, \cdots, y_N^N)$.

Proof. The proof is similar to [25, Theorem 6]. □

Lemma 4.5. $\epsilon_{1,N} \to 0$ *as* $N \to \infty$ *(due to* $k \to \infty$*).*

Proof. We write

$$\hat{x}_i^N(t) - y_i^N(t) = \int_0^t \frac{1}{|\mathcal{C}(i)|} \sum_{j \in \mathcal{C}(i)} \xi_{ij}^0(s) ds + \int_0^t \frac{1}{M_k} \sum_{l=1}^{M_k} g^k_{\mathcal{C}(i)\mathcal{C}_l} \frac{1}{|\mathcal{C}_l|} \sum_{j \in \mathcal{C}_l} \xi_{ij}(s) ds, \tag{4.7}$$

where

$$\xi_{ij}^0(s) = f_0(\hat{x}_i^N, \varphi(s, \hat{x}_i^N, g_{I^*(i)}), \hat{x}_j^N) - \int_{\mathbb{R}} f_0(y_i^N, \varphi(s, y_i^N, g_{I^*(i)}), z) m_{y_j^N(s)}(dz),$$
$$\xi_{ij}(s) = f(\hat{x}_i^N, \varphi(s, \hat{x}_i^N, g_{I^*(i)}), \hat{x}_j^N) - \int_{\mathbb{R}} f(y_i^N, \varphi(s, y_i^N, g_{I^*(i)}), z) m_{y_j^N(s)}(dz).$$

We check the second line of (4.7) first. Write

$$\begin{aligned}\xi_{ij}(s) =& f(\hat{x}_i^N, \varphi(s, \hat{x}_i^N, g_{I^*(i)}), \hat{x}_j^N) - f(y_i^N, \varphi(s, y_i^N, g_{I^*(i)}), y_j^N) \\ &+ f(y_i^N, \varphi(s, y_i^N, g_{I^*(i)}), y_j^N) - \int_{\mathbb{R}} f(y_i^N, \varphi(s, y_i^N, g_{I^*(i)}), z) m_{y_j^N(s)}(dz).\end{aligned}$$

Denote

$$\zeta_{ij} = f(y_i^N, \varphi(s, y_i^N, g_{I^*(i)}), y_j^N) - \int_{\mathbb{R}} f(y_i^N, \varphi(s, y_i^N, g_{I^*(i)}), z) m_{y_j^N(s)}(dz).$$

By the Lipschitz conditions (H2), (H3) and the best response's uniform Lipschitz continuity in x by Lemma 3.7, we obtain

$$\begin{aligned}&|\frac{1}{M_k} \sum_{l=1}^{M_k} g^k_{\mathcal{C}(i)\mathcal{C}_l} \frac{1}{|\mathcal{C}_l|} \sum_{j \in C_l} \xi_{ij}(s)| \\ \le& C|\hat{x}_i^N - y_i^N| + \frac{C}{M_k} \sum_{l=1}^{M_k} g^k_{\mathcal{C}(i)\mathcal{C}_l} \frac{1}{|\mathcal{C}_l|} \sum_{j \in \mathcal{C}_l} |\hat{x}_j^N - y_j^N| \\ &+ |\frac{1}{M_k} \sum_{l=1}^{M_k} g^k_{\mathcal{C}(i)\mathcal{C}_l} \frac{1}{|\mathcal{C}_l|} \sum_{j \in \mathcal{C}_l} \zeta_{ij}|.\end{aligned}$$

Then by independence of y_i^N, $1 \le i \le N$,

$$E|\frac{1}{M_k}\sum_{l=1}^{M_k} g^k_{\mathcal{C}(i)\mathcal{C}_l}\frac{1}{|\mathcal{C}_l|}\sum_{j\in\mathcal{C}_l}\zeta_{ij}|^2 \le C\sum_{l=1}^{M_k}\sum_{j\in\mathcal{C}_l}\frac{|g^k_{\mathcal{C}(i)\mathcal{C}_l}|^2}{M_k^2|\mathcal{C}_l|^2}$$
$$\le \frac{C}{M_k \min_l |\mathcal{C}_l|}.$$

The estimate for $\frac{1}{|\mathcal{C}(i)|}\sum_{j\in\mathcal{C}(i)}\xi^0_{ij}(s)$ can be done similarly. Now it follows from (4.7) that

$$\begin{aligned}
&E|\hat{x}_i^N(t)-y_i^N(t)| \le C\int_0^t E|\hat{x}_i^N(s)-y_i^N(s)|ds\\
&+\frac{C}{M_k}\sum_{l=1}^{M_k}\frac{g^k_{\mathcal{C}(i)\mathcal{C}_l}}{|\mathcal{C}_l|}\sum_{j\in\mathcal{C}_l}\int_0^t E|\hat{x}_j^N(s)-y_j^N(s)|ds\\
&+\frac{C}{|\mathcal{C}(i)|}\sum_{j\in\mathcal{C}(i)}\int_0^t E|\hat{x}_j^N(s)-y_j^N(s)|ds+\frac{C_1}{\sqrt{M_k\min_l|\mathcal{C}_l|}}+\frac{C}{\sqrt{|\mathcal{C}(i)|}}\\
&\le C_2\int_0^t \Delta^N(s)ds+\frac{C_3}{\sqrt{\min_l|\mathcal{C}_l|}},
\end{aligned}$$

where $\Delta^N(t) = \max_{1\le i\le N} E|\hat{x}_i^N(t)-y_i^N(t)|$. The above further implies

$$\Delta^N(t) \le C_2\int_0^t \Delta^N(s)ds+\frac{C_3}{\sqrt{\min_l|\mathcal{C}_l|}}.$$

The lemma follows from (H9) and Gronwall's lemma. □

Lemma 4.6. *We have $\epsilon_{2,N}\to 0$ as $N\to\infty$.*

Proof. For System D and $1\le i\le N$, we write

$$\begin{aligned}
dy_i^\infty =& \int_{\mathbb{R}} f_0(y_i^\infty,\varphi(t,y_i^\infty,g_{I^*(i)}),z)\mu_{I^*(i)}(t,dz)dt+\sigma dw_i\\
&+\int_0^1\int_{\mathbb{R}} f(y_i^\infty,\varphi(t,y_i^\infty,g_{I^*(i)}),z)g(I^*(i),\beta)\mu_\beta(t,dz)d\beta\ dt.
\end{aligned}\tag{4.8}$$

Denote

$$\begin{aligned}
&\int_0^1\int_{\mathbb{R}} f(y_i^\infty,\varphi(t,y_i^\infty,g_{I^*(i)}),z)g(I^*(i),\beta)\mu_\beta(t,dz)d\beta\\
=&\sum_{l=1}^{M_k}\int_{\beta\in I_l}\int_{\mathbb{R}} f(y_i^\infty,\varphi(t,y_i^\infty,g_{I^*(i)}),z)g(I^*(i),\beta)\mu_\beta(t,dz)d\beta\\
=&\xi_k+\zeta_k,
\end{aligned}$$

where

$$\xi_k = \sum_{l=1}^{M_k} \int_{\beta \in I_l} g(I^*(i),\beta)d\beta \int_{\mathbb{R}} f(y_i^\infty, \varphi(t, y_i^\infty, g_{I^*(i)}), z)\mu_{I_l^*}(t,dz),$$

$$\zeta_k = \sum_{l=1}^{M_k} \zeta_{kl},$$

$$\zeta_{kl} \coloneqq \int_{\beta\in I_l}\int_{\mathbb{R}} f(y_i^\infty, \varphi(t, y_i^\infty, g_{I^*(i)}), z) g(I^*(i),\beta)[\mu_\beta(t,dz) - \mu_{I_l^*}(t,dz)]d\beta. \tag{4.9}$$

We rewrite

$$\begin{aligned}\xi_k =& \sum_{l=1}^{M_k} \frac{g^k_{\mathcal{C}(i)\mathcal{C}_l}}{M_k} \int_{\mathbb{R}} f(y_i^\infty, \varphi(t, y_i^\infty, g_{I^*(i)}), z)\mu_{I_l^*}(t,dz)\\ &+ \sum_{l=1}^{M_k}\left[\int_{\beta\in I_l} g(I^*(i),\beta)d\beta - \frac{g^k_{\mathcal{C}(i)\mathcal{C}_l}}{M_k}\right]\int_{\mathbb{R}} f(y_i^\infty, \varphi(t, y_i^\infty, g_{I^*(i)}), z)\mu_{I_l^*}(t,dz)\\ =&: \xi_{k,1} + \xi_{k,2}.\end{aligned}$$

By (H11) and boundedness of f, we have $\lim_{k\to\infty}\sup_{t,\omega}|\xi_{k,2}| = 0$ so that

$$\lim_{k\to\infty}\int_0^T E|\xi_{k,2}(t)|dt = 0. \tag{4.10}$$

Now (4.8) may be rewritten in the form

$$\begin{aligned}dy_i^\infty =& \int_{\mathbb{R}} f_0(y_i^\infty, \varphi(t, y_i^\infty, g_{I^*(i)}), z)\mu_{I^*(i)}(t,dz)dt + \sigma dw_i\\ &+ (\xi_{k,1}+\xi_{k,2}+\zeta_k)dt.\end{aligned}$$

In view of (4.5), we have

$$\begin{aligned}& y_i^\infty(t) - y_i^N(t)\\ =& \int_0^t\int_{\mathbb{R}} [f_0(y_i^\infty, \varphi(s, y_i^\infty, g_{I^*(i)}), z)\mu_{I^*(i)}(s,dz) - f_0(y_i^N, \varphi(s, y_i^N, g_{I^*(i)}), z)m_{y_i^N(s)}(dz)]ds\\ &+ \frac{1}{M_k}\sum_{l=1}^{M_k} g^k_{\mathcal{C}(i)\mathcal{C}_l}\int_0^t\int_{\mathbb{R}} f(y_i^\infty, \varphi(s, y_i^\infty, g_{I^*(i)}), z)\mu_{I_l^*}(s,dz)ds\\ &- \frac{1}{M_k}\sum_{l=1}^{M_k} g^k_{\mathcal{C}(i)\mathcal{C}_l}\int_0^t\int_{\mathbb{R}} f(y_i^N, \varphi(s, y_i^N, g_{I^*(i)}), z)m_l^N(s,dz)ds\\ &+ \int_0^t(\xi_{k,2}+\zeta_k)ds.\end{aligned}$$

Denote

$$\begin{aligned}\Delta_{il}(s) =& \Big|\int_{\mathbb{R}} f(y_i^\infty, \varphi(s, y_i^\infty, g_{I^*(i)}), z)\mu_{I_l^*}(s,dz)\\ &- \int_{\mathbb{R}} f(y_i^N, \varphi(s, y_i^N, g_{I^*(i)}), z)m_l^N(s,dz)\Big|.\end{aligned}$$

It follows that

$$\begin{aligned}\mathit{\Delta}_{il}(s) \le & \Big|\int_{\mathbb{R}} f(y_i^\infty, \varphi(s, y_i^\infty, g_{I^*(i)}), z)\mu_{I_l^*}(s, dz) \\ & - \int f(y_i^N, \varphi(s, y_i^N, g_{I^*(i)}), z)\mu_{I_l^*}(s, dz)\Big| \\ & + \Big|\int_{\mathbb{R}} f(y_i^N, \varphi(s, y_i^N, g_{I^*(i)}), z)\mu_{I_l^*}(s, dz) \\ & - \int_{\mathbb{R}} f(y_i^N, \varphi(s, y_i^N, g_{I^*(i)}), z)m_l^N(s, dz)\Big| \\ =: & \mathit{\Delta}_{il1}(s) + \mathit{\Delta}_{il2}(s).\end{aligned}$$

By the Lipschitz condition (H2), for any fixed $y \in \mathbb{R}$, we have

$$\begin{aligned}&\Big|\int_{\mathbb{R}} f(y, \varphi(s, y, g_{I^*(i)}), z)\mu_{I_l^*}(s, dz) - \int_{\mathbb{R}} f(y, \varphi(s, y, g_{I^*(i)}), z)m_l^N(s, dz)\Big| \\ =&|Ef(y, \varphi(s, y, g_{I^*(i)}), y_j^\infty) - Ef(y, \varphi(s, y, g_{I^*(i)}), y_j^N)| \\ \le& CE|y_j^\infty(s) - y_j^N(s)|,\end{aligned}$$

where $j \in \mathcal{C}_l$ and we have used the fact that $y_i^\infty(t)$ in (4.8) has the law $\mu_{I^*(i)}(t)$. Consequently, we have for $j \in \mathcal{C}_l$, with probability one,

$$\mathit{\Delta}_{il2}(s) \le CE|y_j^\infty(s) - y_j^N(s)|. \tag{4.11}$$

We estimate $\mathit{\Delta}_{kl1}$ using the Lipschitz property of f and $\varphi_{I^*(i)}$. Now it follows that

$$E\mathit{\Delta}_{il}(s) \le CE|y_i^\infty(s) - y_i^N(s)| + CE|y_j^\infty(s) - y_j^N(s)|, \quad j \in \mathcal{C}_l.$$

We similarly estimate the difference term involving f_0. Therefore,

$$\begin{aligned}E|y_i^\infty(t) - y_i^N(t)| \le & C\int_0^t E|y_i^\infty - y_i^N|ds + \int_0^t E(|\xi_{k,2}| + |\zeta_k|)ds \\ & + \frac{1}{M_k}\sum_{l=1}^{M_k} g^k_{\mathcal{C}(i)\mathcal{C}_l}\int_0^t E\mathit{\Delta}_{il}ds \\ \le & C_1\int_0^t \max_i E|y_i^\infty - y_i^N|ds + \int_0^t E(|\xi_{k,2}| + |\zeta_k|)ds \\ & + \frac{C}{M_k}\sum_{l=1}^{M_k} g^k_{\mathcal{C}(i)\mathcal{C}_l}\int_0^t \max_j E|y_j^\infty - y_j^N|ds \\ \le & 2C_2\int_0^t \max_i E|y_i^\infty - y_i^N|ds + \int_0^t E(|\xi_{k,2}| + |\zeta_k|)ds.\end{aligned}$$

Consequently,

$$\max_i E|y_i^\infty(t) - y_i^N(t)| \le 2C_2\int_0^t \max_i E|y_i^\infty - y_i^N|ds + \int_0^t E(|\xi_{k,2}| + |\zeta_k|)ds.$$

By Gronwall's lemma,

$$\sup_{0\le t\le T}\max_i E|y_i^\infty(t) - y_i^N(t)| \le C\int_0^T E(|\xi_{k,2}| + |\zeta_k|)ds. \tag{4.12}$$

To estimate (4.9), by (H2) we derive

$$
\begin{aligned}
\zeta_{kl,\beta} :=& \Big| \int_{\mathbb{R}} f(y_i^\infty, \varphi(t, y_i^\infty, g_{I^*(i)}), z)[\mu_\beta(t,dz) - \mu_{I_l^*}(t,dz)] \Big| \\
=& \Big| \int_{\mathbb{R}^2} [f(y_i^\infty, \varphi(t, y_i^\infty, g_{I^*(i)}), z_1) - f(y_i^\infty, \varphi(t, y_i^\infty, g_{I^*(i)}), z_2)] \widehat{\gamma}(dz_1, dz_2) \Big| \\
\leq& C \int_{\mathbb{R}^2} |z_1 - z_2| \widehat{\gamma}(dz_1, dz_2),
\end{aligned}
$$

where the probability measure $\widehat{\gamma}$ is any coupling of $\mu_\beta(t)$ and $\mu_{I_l^*}(t)$ and C is the Lipschitz constant of f. Since the coupling $\widehat{\gamma}$ is arbitrary, it follows that $\zeta_{kl,\beta} \leq CW_1(\mu_\beta(t), \mu_{I^*(i)}(t))$. Denote $\delta_k^\mu = \sup_{l \leq M_k} \sup_{\beta \in I_l, t \leq T} W_1(\mu_\beta(t), \mu_{I_l^*}(t))$. Then with probability one,

$$
|\zeta_{kl}(t)| \leq C\delta_k^\mu / M_k,
$$

and therefore $|\zeta_k(t)| \leq C\delta_k^\mu$. Note that $\delta_k^\mu \to 0$ as $k \to \infty$ by Lemma A.1. Recalling (4.10), the right hand side of (4.12) tends to 0 as $k \to \infty$. This completes the proof. □

Lemma 4.7. $\lim_{N\to\infty} \sup_{t,j} E|\hat{x}_j^N - y_j^\infty| = 0$.

Proof. The lemma follows from Lemmas 4.5 and 4.6. □

Lemma 4.8. $\lim_{N\to\infty} \epsilon_{3,N} = 0$.

Proof. For $(\hat{x}_1^N, \cdots, \hat{x}_N^N)$ in System A and $(x_1^N, \cdots, x_N^N)$ in System B, we compare the SDEs of $\hat{x}_j^N$ and x_j^N and apply Gronwall's lemma to obtain

$$
\sup_{u_i^N, t, j \neq i} |x_j^N - \hat{x}_j^N| \leq \frac{C}{\sqrt{\min_l |\mathcal{C}_l|}}.
$$

Next by Lemma 4.7, we obtain the desired estimate. □

Consider the limiting optimal control problem with dynamics and cost

$$
dx_i^\infty = \widetilde{f}[x_i^\infty, u_i, \mu_G; g_{I^*(i)}]dt + \sigma dw_i, \tag{4.13}
$$

$$
J_i^* = E\int_0^T \widetilde{l}[x_i^\infty, u_i, \mu_G; g_{I^*(i)}]dt, \tag{4.14}
$$

where $x_i^\infty(0) = x_i^N(0)$ and $\mu_G(\cdot)$ is given by the GMFG equation system.

To establish the ϵ-Nash equilibrium property, the cost of agent $\mathcal{A}_i$ within the N agents can be written using the mean field limit dynamics and cost, both involving $\mu_G(\cdot)$, up to a small error term that can be bounded uniformly with respect to u_i^N, while $\mathcal{A}_i$ chooses its control u_i^N. It can further have little improvement due to the best response property of $\varphi(t, x_i|\mu_G(\cdot), g_{I^*(i)})$. For (4.4) of System B, we rewrite

$$
dx_i^N = \widetilde{f}[x_i^N, u_i^N, \mu_G; g_{I^*(i)}]dt + (\delta_{f_0}^k(t) + \delta_f^k(t))dt + \sigma dw_i, \tag{4.15}
$$

where $\delta_{f_0}^k = f_0(x_i^N, u_i^N, \mathcal{C}(i)) - f_0[x_i^N, u_i^N, \mu_{I^*(i)}]$ and $\delta_f^k = f_{G_k}(x_i^N, u_i^N, g_{\mathcal{C}(i)}^k) - f[x_i^N, u_i^N, \mu_G; g_{I^*(i)}]$. Similarly the cost of $\mathcal{A}_i$ in System B is written as

$$
J_i^N(u_i^N) = E\int_0^T (\widetilde{l}[x_i^N, u_i^N, \mu_G; g_{I^*(i)}] + \delta_{l_0}^k(t) + \delta_l^k(t))dt,
$$

where we have $\delta_{l_0}^k = l_0(x_i^N, u_i^N, \mathcal{C}(i)) - l_0[x_i^N, u_i^N, \mu_{I^*(i)}]$ and $\delta_l^k = l_{G_k}(x_i^N, u_i^N, g_{\mathcal{C}(i)}^k) - l[x_i^N, u_i^N, \mu_G; g_{I^*(i)}]$. Note that all other agents have applied the control laws $\varphi(t, x_j^N, g_{I^*(j)})$, $j \neq i$. So we only indicate u_i^N within J_i^N. It is clear that $\delta_{f_0}^k$, δ_f^k, $\delta_{l_0}^k$, and δ_l^k are all affected by the control law u_i^N. Let $\boldsymbol{y}_t^\infty = (y_1^\infty(t), \cdots, y_N^\infty(t))$ for System D. Our next step is to derive a uniform upper bounded for $E|\delta_f^k|$ and $E|\delta_l^k|$ with respect to u_i^N.

Define the two random variables

$$\Delta_f^k(z, u, \boldsymbol{y}_t^\infty) = \frac{1}{M_k} \sum_{l=1}^{M_k} g_{\mathcal{C}(i)\mathcal{C}_l}^k \frac{1}{|\mathcal{C}_l|} \sum_{j \in \mathcal{C}_l} f(z, u, y_j^\infty(t)) - f[z, u, \mu_G(t); g_{I^*(i)}],$$

$$\Delta_l^k(z, u, \boldsymbol{y}_t^\infty) = \frac{1}{M_k} \sum_{l=1}^{M_k} g_{\mathcal{C}(i)\mathcal{C}_l}^k \frac{1}{|\mathcal{C}_l|} \sum_{j \in \mathcal{C}_l} l(z, u, y_j^\infty(t)) - l[z, u, \mu_G(t); g_{I^*(i)}],$$

where $z \in \mathbb{R}$ and $u \in U$ are deterministic and fixed.

Lemma 4.9. *We have*

$$\lim_{k\to\infty} \sup_{z,u,t} E(|\Delta_f^k(z, u, \boldsymbol{y}_t^\infty)|^2 + |\Delta_l^k(z, u, \boldsymbol{y}_t^\infty)|^2) = 0. \tag{4.16}$$

Proof. As in the proof of Lemma 4.6, we approximate μ_β, $\beta \in [0, 1]$, by using a finite number of points of β, and next expand the two quadratic terms in (4.16). The estimate is carried out using (H11) and Lemma A.1. $\square$

Lemma 4.10. *For any given constant $C_z > 0$ and any $\epsilon \in (0, 1)$,*

$$\lim_{k\to\infty} \sup_t P(\cap_{(z,u)\in[-C_z,C_z]\times U}\{|\Delta_f^k(z, u, \boldsymbol{y}_t^\infty)| \le \epsilon\}) = 1,$$
$$\lim_{k\to\infty} \sup_t P(\cap_{(z,u)\in[-C_z,C_z]\times U}\{|\Delta_l^k(z, u, \boldsymbol{y}_t^\infty)| \le \epsilon\}) = 1.$$

Proof. We show the first limit, and may deal with the second one in the same way. Note that the event

$$\mathcal{E}_{fC_z}^k := \cap_{(z,u)\in[-C_z,C_z]\times U}\{|\Delta_f^k(z, u, \boldsymbol{y}_t^\infty)| \le \epsilon\} \tag{4.17}$$

is well defined since Δ_f^k is continuous in (z, u) and the intersection may be equivalently expressed using only a countable number of values of (z, u) in $[-C_z, C_z] \times U$.

Take any $\epsilon \in (0, 1)$. By (H2) and (H3), we can find $\delta_\epsilon > 0$ such that $|\Delta_f^k(z, u, \boldsymbol{y}_t^\infty) - \Delta_f^k(z', u', \boldsymbol{y}_t^\infty)| \le \epsilon/2$ whenever $|z - z'| + |u - u'| \le \delta_\epsilon$. For the selected δ_ϵ, we can find a fixed p_0 and $(z^j, u^j) \in [-C_z, C_z] \times U$, $j = 1, \cdots, p_0$ such that for any $(z, u) \in [-C_z, C_z] \times U$, there exists some j_0 ensuring $|z - z^{j_0}| + |u - u^{j_0}| \le \delta_\epsilon$.

By Lemma 4.9 and Markov's inequality, for any $\delta > 0$, there exists K_{δ,p_0} such that for all $k \ge K_{\delta,p_0}$,

$$P(\{|\Delta_f^k(z^j, u^j, \boldsymbol{y}_t^\infty)| \le \epsilon/2\}) \ge 1 - \delta/p_0, \quad \forall j, t. \tag{4.18}$$

Denote the event $\mathcal{E}_j^k = \{|\Delta_f^k(z^j, u^j, \boldsymbol{y}_t^\infty)| \le \epsilon/2\}$. By (4.18), $P(\cap_{i=1}^{p_0}\mathcal{E}_i^k) \ge 1 - \delta$ for $k \ge K_{\delta,p_0}$. Now if $\omega \in \mathcal{E}^k := \cap_{i=1}^{p_0}\mathcal{E}_i^k$, $k \ge K_{\delta,p_0}$, then for any $(z, u) \in [-C_z, C_z] \times U$, we have $|\Delta_f^k(z, u, \boldsymbol{y}_t^\infty)| \le \epsilon$. Hence $\mathcal{E}^k \subset \mathcal{E}_{fC_z}^k$. It follows that for all $k \ge K_{\delta,p_0}$, $P(\mathcal{E}_{fC_z}^k) \ge 1 - \delta$. Since $\delta \in (0, 1)$ is arbitrary and K_{δ,p_0} does not depend on t, the first limit follows. $\square$

Lemma 4.11. *We have*

$$\lim_{k\to\infty} \sup_{t,u_i^N} E(|\Delta_f^k(x_i^N(t), u_i^N(t), \boldsymbol{y}_t^\infty)| + |\Delta_l^k(x_i^N(t), u_i^N(t), \boldsymbol{y}_t^\infty)|) = 0.$$

Proof. Fix any $\epsilon \in (0,1)$, by (H1) and (H2) we can find a sufficiently large C_z, independent of (k,N), such that for all $u_i^N(\cdot)$,

$$P(\sup_{0\le t\le T} |x_i^N(t)| \le C_z) \ge 1-\epsilon.$$

Denote $\mathcal{E}_x = \{\sup_{0\le t\le T} |x_i^N(t)| \le C_z\}$. By Lemma 4.10, for the above ϵ and $\mathcal{E}^k_{fC_z}$ given by (4.17), there exists K_0 independent of t such that for all $k \ge K_0$,

$$P(\mathcal{E}^k_{fC_z}) \ge 1-\epsilon.$$

Now if $\omega \in \mathcal{E}_x \cap \mathcal{E}^k_{fC_z}$, then $|\Delta^k_f(x_i^N(t), u_i^N(t), \boldsymbol{y}_t^\infty)| \le \epsilon$. We have $P(\mathcal{E}_x \cap \mathcal{E}^k_{fC_z}) \ge 1-2\epsilon$, and so

$$P(|\Delta^k_f(x_i^N(t), u_i^N(t), \boldsymbol{y}_t^\infty)| \le \epsilon) \ge P(\mathcal{E}_x \cap \mathcal{E}^k_{fC_z}) \ge 1-2\epsilon.$$

It follows that for all $k \ge K_0$,

$$E|\Delta^k_f(x_i^N(t), u_i^N(t), \boldsymbol{y}_t^\infty)| \le \epsilon + 2\epsilon C,$$

where C does not depend on $(u_i^N(\cdot), t)$. The bound for Δ^k_l is similarly obtained. □

Lemma 4.12. *We have*

$$\lim_{k\to\infty} \sup_{t, u_i^N(\cdot)} E(|\delta^k_f| + |\delta^k_l|) = 0.$$

Proof. By Lipschitz continuity of (f,l), we estimate $E|\delta^k_f - \Delta^k_f(x_i^N, u_i^N, \boldsymbol{y}_t^\infty)|$ and $E|\delta^k_l - \Delta^k_l(x_i^N, u_i^N, \boldsymbol{y}_t^\infty)|$, and next apply Lemma 4.8 to show that they converge to zero as $k \to \infty$. Recalling Lemma 4.11, we complete the proof. □

Lemma 4.13. *We have*

$$\lim_{k\to\infty} \sup_{t, u_i^N(\cdot)} E(|\delta^k_{f_0}| + |\delta^k_{l_0}|) = 0.$$

Proof. The proof is similar to that of Lemma 4.12 and the details are omitted. □

Denote

$$\epsilon^k_{fl} = \sup_{t, u_i^N(\cdot)} E(|\delta^k_{f_0}| + |\delta^k_{l_0}| + |\delta^k_f| + |\delta^k_l|).$$

Lemma 4.14. *For any admissible control u_i^N in System B and J_i^* in* (4.14)*,*

$$J_i^N(u_i^N) \ge \inf_{u_i} J_i^*(u_i) - C\epsilon^k_{fl},$$

where the constant C does not depend on u_i^N.

Proof. Take any full state based Lipschitz feedback control u_i^N. It together with the other agents's control laws generates the closed-loop state processes $x_1^N(t), \cdots, x_N^N(t)$. Let $u_i^N(t,\omega)$ denote the realization as a non-anticipative process. Now we take $\check{u}_i = u_i^N(t,\omega)$ in (4.13) and let $\check{x}_i^\infty$ be the resulting state process. It is clear from (4.14) that

$$J_i^*(\check{u}_i) \ge \inf_{u_i} J_i^*(u_i). \tag{4.19}$$

Recalling (4.15) and applying Gronwall's lemma to estimate the difference $\check{x}_i^\infty - x_i^N$, we can show there exists C independent of u_i^N such that $|J_i^N(u_i^N) - J_i^*(\check{u}_i)| \le C\epsilon^k_{fl}$, which combined with (4.19) completes the proof. □

Lemma 4.15. *Let $\varphi_{I^*(i)} = \varphi(t, x, g_{I^*(i)})$ be the GMFG based control law* (4.2)*. We have*

$$J_i^N(\varphi_{I^*(i)}) \le \inf_{u_i} J_i^*(u_i) + C\epsilon^k_{fl}.$$

Proof. Let $\varphi_{I^*(i)}$ be applied to the two systems (4.13) and (4.15). We further use Gronwall's lemma to estimate $E|x_i^\infty - x_i^N|$. We obtain $|J_i^N(\varphi_{I^*(i)}) - J_i^*(\varphi_{I^*(i)})| \le C\epsilon_{fl}^k$. Note that $J_i^*(\varphi_{I^*(i)}) = \inf_{u_i} J_i^*(u_i)$. This completes the proof. □

Proof of Theorem 4.3. It follows from Lemmas 4.12, 4.13, 4.14 and 4.15. □

5. The LQ Case

This section considers a special class of linear-quadratic-Gaussian (LQG) GMFG models. Consider the graph G_k with vertices $\mathcal{V}_k = \{1, \cdots, M_k\}$ and graph adjacency matrix $g^k = [g_{jl}^k]$. For agent $\mathcal{A}_i$ in subpopulation cluster C_q situated in node q, let the graph averaged mean value of the system state at node q be denoted by z_i, where

$$z_i = \frac{1}{|M_k|} \sum_{l \in \mathcal{V}_k} g_{ql}^k \frac{1}{|\mathcal{C}_l|} \sum_{j \in C_l} x_j, \quad x_i,\ z_i \in \mathbb{R}^n.$$

The dynamics of $\mathcal{A}_i$ are given by the linear system

$$dx_i = (Ax_i + Dz_i + Bu_i)dt + \Sigma dw_i, \quad 1 \le i \le N,$$

where $u_i \in \mathbb{R}^{n_u}$ is the control input, $w_i \in \mathbb{R}^{n_w}$ is a standard Brownian motion, and A, B, D, Σ are conformally dimensioned matrices. We assume $Ex_i(0) = x_0$ for all i.

The individual agent's cost function takes the form

$$\begin{aligned} J_i(u_i; \nu_i) =& E \int_0^T \big[(x_i - \nu_i)^T Q (x_i - \nu_i) + u_i^T R u_i\big] dt \\ &+ E\big[(x_i(T) - \nu_i(T))^T Q_T (x_i(T) - \nu_i(T))\big], \quad 1 \le i \le N, \end{aligned}$$

where $Q, Q_T \ge 0$, $R > 0$, and $\nu_i = \gamma(z_i + \eta)$ is the process tracked by $\mathcal{A}_i$. Here $\eta \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}$.

In the infinite population and graphon limit case, the mean field coupling at α-agent (i.e., an agent situated at the α-vertex in $[0, 1]$) is given by

$$z_\alpha = \int_{[0,1]} g(\alpha, \beta) \int_{\mathbb{R}^n} x \mu_\beta(dx) d\beta, \quad \alpha, \beta \in [0, 1].$$

The individual agent's state equation is given by

$$dx_\alpha = (Ax_\alpha + Dz_\alpha + Bu_\alpha)dt + \Sigma dw_\alpha, \quad \alpha \in [0, 1].$$

The individual agent's cost function is

$$\begin{aligned} J_\alpha(u_\alpha, \nu_\alpha) = E \int_0^T & \big[(x_\alpha - \nu_\alpha)^T Q (x_\alpha - \nu_\alpha) + u_\alpha^T R u_\alpha\big] dt \\ & + E\big[(x_\alpha(T) - \nu_\alpha(T))^T Q_T (x_\alpha(T) - \nu_\alpha(T))\big], \end{aligned}$$

where $\nu_\alpha = \gamma(z_\alpha + \eta)$.

Consider the Riccati equation

$$-\dot{\Pi}_t = A^T \Pi_t + \Pi_t A - \Pi_t B R^{-1} B^T \Pi_t + Q,$$

where $\Pi_T = Q_T$, and

$$-\dot{s}_\alpha(t) = (A - BR^{-1}B^T \Pi_t)^T s_\alpha(t) + \Pi_t D z_\alpha(t) - Q\nu_\alpha(t),$$

where $s_\alpha(T) = -Q_T \nu_\alpha(T)$. The best response for an α-agent is given by

$$u_\alpha(t) = -R^{-1} B^T [\Pi_t x_\alpha(t) + s_\alpha(t)].$$

Here the graphon local mean field and tracked process from cost coupling are

$$z_\alpha = \int_{[0,1]} g(\alpha,\beta)\bar{x}_\beta d\beta, \quad \nu_\alpha = \gamma(z_\alpha+\eta), \quad \alpha\in[0,1],$$

where $\bar{x}_\beta = \int_{\mathbb{R}^n} x\mu_\beta(dx)$. The mean state process of x_α is

$$\dot{\bar{x}}_\alpha = (A-BR^{-1}B^T\Pi_t)\bar{x}_\alpha + Dz_\alpha - BR^{-1}B^T s_\alpha, \quad \alpha\in[0,1].$$

The existence analysis reduces to checking the equation system

$$\dot{\bar{x}}_\alpha = (A-BR^{-1}B^T\Pi_t)\bar{x}_\alpha - BR^{-1}B^T s_\alpha + D\int_0^1 g(\alpha,\beta)\bar{x}_\beta d\beta \tag{5.1}$$

$$\dot{s}_\alpha = -(A-BR^{-1}B^T\Pi_t)^T s_\alpha + (\gamma Q - \Pi_t D)\int_0^1 g(\alpha,\beta)\bar{x}_\beta d\beta + \gamma Q\eta, \tag{5.2}$$

where $\bar{x}_\alpha(0)=x_0$ and $s_\alpha(T) = -\gamma Q_T(\int_0^1 g(\alpha,\beta)\bar{x}_\beta(T)d\beta+\eta)$.

To analyze (5.1)–(5.2), let $\Phi(t,s)$ and $\Psi(t,s)$ be the fundamental solution matrix of

$$\dot{x} = (A-BR^{-1}B^T\Pi_t)x, \qquad \dot{y} = -(A-BR^{-1}B^T\Pi_t)^T y$$

for $x(t), y(t)\in\mathbb{R}^n$. Then we have $\Psi(t,s)=\Phi^T(s,t)$. We convert the existence analysis into a fixed point problem. For (5.2), we view $\bar{x}_\beta(t)$ as a function of (t,β). First we solve s_α in terms of $\bar{x}_\alpha$. We get

$$\begin{aligned} s_\alpha(t) = &-\int_0^T \Psi(t,\tau)\Big[(\gamma Q-\Pi_\tau D)\int_0^1 g(\alpha,\beta)\bar{x}_\beta(\tau)d\beta + \gamma Q\eta\Big]d\tau \\ &- \gamma\Psi(t,T)Q_T(\int_0^1 g(\alpha,\beta)\bar{x}_\beta(T)d\beta+\eta). \end{aligned} \tag{5.3}$$

Substituting (5.3) into (5.1), we further calculate

$$\begin{aligned} \bar{x}_\alpha(t) =& \Phi(t,0)x_0 + \int_0^t \Phi(t,r)BR^{-1}B^T \\ & \Big\{\int_r^T \Psi(r,\tau)[(\gamma Q-\Pi_\tau D)\int_0^1 g(\alpha,\beta)\bar{x}_\beta(\tau)d\beta+\gamma Q\eta]d\tau \\ & + \gamma\Psi(r,T)Q_T(\int_0^1 g(\alpha,\beta)\bar{x}_\beta(T)d\beta+\eta)\Big\}dr \\ & + \int_0^t \Phi(t,r)D\int_0^1 g(\alpha,\beta)\bar{x}_\beta(r)d\beta dr. \end{aligned}$$

Denote the function space D_Λ consisting of continuous $\mathbb{R}^n$-valued functions on $[0,1]\times[0,T]$ with norm $\|\check{x}\| = \sup_{\alpha,t}|\check{x}(\alpha,t)|$. We use $|\cdot|$ to denote the Frobenius norm of a vector or matrix. Define the operator Λ as follows: For $\check{x}\in D_\Lambda$,

$$\begin{aligned} (\Lambda\check{x})(\alpha,t) = & \int_0^t \Phi(t,r)BR^{-1}B^T \\ & \Big\{\int_r^T \Psi(r,\tau)(\gamma Q-\Pi_\tau D)\int_0^1 g(\alpha,\beta)\check{x}(\beta,\tau)d\beta d\tau \\ & + \gamma\Psi(r,T)Q_T\int_0^1 g(\alpha,\beta)\check{x}(\beta,T)d\beta\Big\}dr \\ & + \int_0^t \Phi(t,r)D\int_0^1 g(\alpha,\beta)\check{x}(\beta,r)d\beta dr. \end{aligned}$$

If (H5) holds, $\varLambda$ is from $D_\varLambda$ to itself.

The solution of the LQG GMFG reduces to finding a fixed point to the equation

$$\check{x}(\alpha,t)=(\varLambda\check{x})(\alpha,t)+\varPhi(t,0)x_0+\gamma\int_0^t\varPhi(t,r)BR^{-1}B^T\Big\{\int_r^T\varPsi(r,\tau)Qd\tau+\varPsi(r,T)Q_T\Big\}\eta dr.$$

Denote $c_g=\max_\alpha\int_0^1 g(\alpha,\beta)d\beta$. We have the bound for the operator norm:

$$\begin{aligned}\|\varLambda\| \le & c_g\Big[\int_0^T\int_r^T|\varPhi(t,r)BR^{-1}B^T\varPsi(r,\tau)(\gamma Q-\varPi_\tau D)|d\tau dr\\ &+\int_0^T(\gamma|\varPhi(t,r)BR^{-1}B^T\varPsi(r,T)Q_T|+|\varPhi(t,r)D|)dr\Big]\\ =:&\ c_gc_\varLambda.\end{aligned}$$

If $c_gc_\varLambda<1$, $\varLambda$ is a contraction and has a unique solution.

As an example for illustration, we assume the graphon local mean field at α-agent arises from an underlying *uniform attachment graphon*, and consequently

$$z_\alpha=\int_{[0,1]}\big[(1-\max(\alpha,\beta))\int_{\mathbb{R}^n}x\mu_\beta(dx)\big]d\beta,$$

where $\alpha,\beta\in[0,1]$, where it is readily verified that the uniform attachment graphon satisfies (H5).

Appendix

Lemma A.1. *Assume* (H1)–(H8). *Let φ_α be the GMFG based best response* (4.2) *and $\mu_\alpha(t)$ the distribution of the closed-loop process $x_\alpha(t)$, $\alpha\in[0,1]$, in* (3.14) *with initial distribution μ_0^x. Then we have*

$$\lim_{r\to0}\sup_{|t-t^*|+|\beta-\beta^*|<r}W_1(\mu_\beta(t),\mu_{\beta^*}(t^*))=0,$$

where $t,t^*\in[0,T]$ and $\beta,\beta^*\in[0,1]$.

Proof. Step 1. Take any $\beta,\beta^*\in[0,1]$. For $\mu_G(\cdot)$ determined from the GMFG equations (3.12) and (3.14), define two processes

$$\begin{aligned}dy_{\beta^*}&=\widetilde{f}[y_{\beta^*},\varphi(t,y_{\beta^*},g_{\beta^*}),\mu_G;g_{\beta^*}]dt+\sigma dw_{\beta^*},\\ dy_\beta&=\widetilde{f}[y_\beta,\varphi(t,y_\beta,g_\beta),\mu_G;g_\beta]dt+\sigma dw_{\beta^*},\end{aligned}$$

where $y_{\beta^*}(0)=y_\beta(0)=x_i^N(0)$ and the same Brownian motion is used. Then the distributions of $y_{\beta^*}(t)$ and $y_\beta(t)$ are $\mu_\beta(t)$ and $\mu_{\beta^*}(t)$, respectively. We obtain

$$\begin{aligned}&y_\beta(t)-y_{\beta^*}(t)\\ &=\int_0^t\varDelta^0_{\beta,\beta^*}(s)ds+\int_0^t\int_{[0,1]}\int_{\mathbb{R}}\varDelta_{\beta,\beta^*}(s,z,\lambda)\mu_\lambda(s,dz)d\lambda ds,\end{aligned}$$

where

$$\begin{aligned}\varDelta^0_{\beta,\beta^*}(s)&=\int_{\mathbb{R}}f_0(y_\beta,\varphi(s,y_\beta,g_\beta),z)\mu_\beta(s,dz)-\int_{\mathbb{R}}f_0(y_{\beta^*},\varphi(s,y_{\beta^*},g_{\beta^*}),z)\mu_{\beta^*}(s,dz),\\ \varDelta_{\beta,\beta^*}(s,z,\lambda)&=f(y_\beta,\varphi(s,y_\beta,g_\beta),z)g(\beta,\lambda)\\ &\quad-f(y_{\beta^*},\varphi(s,y_{\beta^*},g_{\beta^*}),z)g(\beta^*,\lambda).\end{aligned}$$

We will simply write $\mu_\lambda(s,dz)$ as $\mu_\lambda(dz)$ if the time argument is clear, where λ is the vertex index. Denote $\kappa_{\beta,\beta^*}(s) = |\varphi(s, y_{\beta^*}, g_\beta) - \varphi(s, y_{\beta^*}, g_{\beta^*})|$, where the time argument s in y_β, y_{β^*} has been suppressed. It follows that

$$
\begin{aligned}
&|\Delta^0_{\beta,\beta^*}(s)| \le \\
&|\int_{\mathbb{R}} f_0(y_\beta, \varphi(s, y_\beta, g_\beta), z)\mu_\beta(s,dz) - \int_{\mathbb{R}} f_0(y_\beta, \varphi(s, y_\beta, g_\beta), z)\mu_{\beta^*}(s,dz)| \\
&+ |\int_{\mathbb{R}} f_0(y_\beta, \varphi(s, y_\beta, g_\beta), z)\mu_{\beta^*}(s,dz) - \int_{\mathbb{R}} f_0(y_{\beta^*}, \varphi(s, y_{\beta^*}, g_{\beta^*}), z)\mu_{\beta^*}(s,dz)| \\
&\le CE|y_\beta - y_{\beta^*}| + C|y_\beta - y_{\beta^*}| + C|\varphi(s, y_\beta, g_\beta) - \varphi(s, y_{\beta^*}, g_{\beta^*})| \\
&\le CE|y_\beta - y_{\beta^*}| + C_1|y_\beta - y_{\beta^*}| + C\kappa_{\beta,\beta^*}(s),
\end{aligned}
$$

where the second inequality is obtained using (H2), (H3), and the method in (4.11). The last inequality has used the uniform Lipschitz continuity of φ_β in the space variable (see Lemma 3.7). It follows that

$$
E|\Delta^0_{\beta,\beta^*}(s)| \le C_2 E|y_\beta(s) - y_{\beta^*}(s)| + CE\kappa_{\beta,\beta^*}(s). \tag{A.1}
$$

Next, we have

$$
\begin{aligned}
&\Big|\int_0^1 \int_{\mathbb{R}} \Delta(s,z,\lambda)\mu_\lambda(dz)d\lambda\Big| \\
&\le \Big|\int_0^1 \int_{\mathbb{R}} [f(y_\beta, \varphi(s, y_\beta, g_\beta), z) - f(y_{\beta^*}, \varphi(s, y_{\beta^*}, g_{\beta^*}), z)] g(\beta,\lambda)\mu_\lambda(dz)d\lambda\Big| \\
&\quad + \Big|\int_0^1 \int_{\mathbb{R}} f(y_{\beta^*}, \varphi(s, y_{\beta^*}, g_{\beta^*}), z)[g(\beta,\lambda) - g(\beta^*,\lambda)]\mu_\lambda(dz)d\lambda\Big| \\
&=: I_f(s) + I_g(s).
\end{aligned} \tag{A.2}
$$

We have

$$
\begin{aligned}
I_f(s) &\le \int_0^1 \int_{\mathbb{R}} C(|y_\beta - y_{\beta^*}| + \kappa_{\beta,\beta^*}) g(\beta,\lambda)\mu_\lambda(dz)d\lambda \\
&\le C(|y_\beta - y_{\beta^*}| + \kappa_{\beta,\beta^*})(s),
\end{aligned}
$$

where we have used the Lipschitz property of f and φ_β. Therefore,

$$
EI_f(s) \le C(E|y_\beta(s) - y_{\beta^*}(s)| + E\kappa_{\beta,\beta^*}(s)). \tag{A.3}
$$

For any fixed value $y_{\beta^*}(s,\omega)$, denote

$$
\xi_{\beta^*,s,\omega}(\lambda) = \int_{\mathbb{R}} f(y_{\beta^*}, \varphi(s, y_{\beta^*}, g_{\beta^*}), z)\mu_\lambda(dz).
$$

We have

$$
I_g(s) = |\int_0^1 \xi_{\beta^*,s,\omega}(\lambda) g(\beta,\lambda)d\lambda - \int_0^1 \xi_{\beta^*,s,\omega}(\lambda) g(\beta^*,\lambda)d\lambda|.
$$

Hence, by (H5), $I_g(s) \to 0$ (ω, s)-a.e. as $\beta \to \beta^*$. It is clear $I_g(s)$ is bounded by a fixed constant since f is a bounded function. For the fixed β^*, by Lemma 3.5, the random variable $\kappa_{\beta,\beta^*}(s)$ is bounded and converges to zero with probability one. Denote $\delta_g = \int_0^T EI_g(s)ds$ and $\delta_\kappa = \int_0^T E\kappa_{\beta,\beta^*}(s)ds$. By dominated convergence, we have

$$
\lim_{\beta\to\beta^*} (\delta_g + \delta_\kappa) = 0.
$$

By (A.1)–(A.3), it follows that

$$E|y_\beta(t) - y_{\beta^*}(t)| \le C \int_0^t E|y_\beta(s) - y_{\beta^*}(s)|ds + C(\delta_\kappa + \delta_g).$$

By Gronwall's lemma, we have

$$\sup_{0\le t\le T} E|y_\beta(t) - y_{\beta^*}(t)| \le Ce^{CT}(\delta_\kappa + \delta_g).$$

Since $W_1(\mu_\beta(t), \mu_{\beta^*}(t)) \le E|y_\beta(t) - y_{\beta^*}(t)|$, then

$$\sup_t W_1(\mu_\beta(t), \mu_{\beta^*}(t)) \le C_1(\delta_\kappa + \delta_g), \tag{A.4}$$

where δ_κ and δ_g depend on β^*.

Step 2. Now we consider given $(\beta^*, t^*) \in [0, 1] \times [0, T]$. By use of the SDE of y_β and elementary estimates, we obtain

$$\lim_{|t-t^*|\to 0} \sup_\beta W_1(\mu_\beta(t^*), \mu_\beta(t)) = 0. \tag{A.5}$$

We have

$$W_1(\mu_\beta(t), \mu_{\beta^*}(t^*)) \le W_1(\mu_\beta(t), \mu_\beta(t^*)) + W_1(\mu_\beta(t^*), \mu_{\beta^*}(t^*)).$$

Given any $\epsilon > 0$, by (A.4) and (A.5) there exists $\delta_{\epsilon,\beta^*} > 0$ such that whenever $|t - t^*| + |\beta - \beta^*| \le \delta_{\epsilon,\beta^*}$, we have

$$W_1(\mu_\beta(t), \mu_\beta(t^*)) \le \frac{\epsilon}{2}, \qquad W_1(\mu_\beta(t^*), \mu_{\beta^*}(t^*)) \le \frac{\epsilon}{2}.$$

Therefore, $W_1(\mu_\beta(t), \mu_{\beta^*}(t^*)) \le \epsilon$. We conclude that $\mu_\beta(t)$ as a mapping from the compact space $[0, 1] \times [0, T]$ to $\mathcal{P}_1(\mathbb{R})$ with the metric $W_1(\cdot, \cdot)$ is continuous and hence must be uniformly continuous. The lemma follows. □

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