Mean Field Game ε-Nash Equilibria for Partially Observed Optimal Execution Problems in Finance

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Abstract— **Partially observed Mean Field Game (PO MFG) theory was introduced and developed in (Caines and Kizilkale, 2013, 2014, Şen and Caines 2014, 2015), where it is assumed the major agent's state is partially observed by each minor agent, and the major agent completely observes its own state. Accordingly, each minor agent can recursively estimate the major agent's state, compute the system's mean field and thence generate the feedback control which yields the ε-Nash property. This PO MM LQG MFG theory was further extended in recent work (Firoozi and Caines, 2015) to major-minor LQG systems in which both the major agent and the minor agents partially observe the major agent's state. The existence of ε-Nash equilibria, together with the individual agents' control laws yielding the equilibria, were established wherein each minor agent recursively generates (i) an estimate of the major agent's state, and (ii) an estimate of the major agent's estimate of its own state (in order to estimate the major agent's control feedback), and hence generates a version of the system's mean field. In the current work, PO MM LQG MFG theory is applied to the optimal execution problem in the financial sector where an institutional investor, interpreted as a major agent, has partial observations of its own inventories, and high frequency traders (HFTs), interpreted as minor agents, have partial observations of the major agent's inventories. The objective for each agent is to maximize its own wealth and to avoid the occurrence of large execution prices, large rates of trading and large trading accelerations which are appropriately weighted in the agent's performance function. PO LQG MFG theory is utilized to establish the existence of ε-Nash equilibria and a simulation example is provided.**

I. INTRODUCTION

Partially observed Mean Field Game (PO MFG) theory was introduced and developed in [4], [5], [12], [13] where it is assumed the major agent's state is partially observed by each minor agent, and the major agent completely observes its own state. Accordingly, each minor agent can recursively estimate the major agent's state, compute the system's mean field and thence generate the feedback control which yields the ε-Nash property. This PO MM LQG MFG theory was further extended in recent work [8] to major-minor LQG systems in which both the major agent and the minor agents partially observe the major agent's state. The existence of ε-Nash equilibria, together with the individual agents' control laws yielding the equilibria, were established wherein each minor agent recursively generates (i) an estimate of the major agent's state, and (ii) an estimate of the major agent's estimate of its own state (in order to estimate the major agent's control feedback), and hence generates a version of the system's mean field.

It is to be noted that the work [4], [5], [12], [13] and the work here does not cover the case where each agent has only partial observations on its own state; the extension of MFG to that case was addressed in the LQG case in [11] and in the nonlinear case is analyzed in (Sen and Caines, 2016 a, b)

In the current work, PO MM LQG MFG theory is applied to the optimal execution problem in the financial sector where an institutional investor, interpreted as a major agent, has partial observations of its own inventories, and high frequency traders (HFTs), interpreted as minor agents, have partial observations of the major agent's inventories (see [1], [2], [3]). The objective for each agent is to maximize its own wealth and to avoid the occurrence of large execution prices, large rates of trading and large trading accelerations which are appropriately weighted in the agent's performance function. PO LQG MFG theory is utilized to establish the existence of ε-Nash equilibria and a simulation example is provided.

The terms major agent (respectively minor agent), major trader (respectively, minor trader), and institutional trader (respectively, HFT) are used interchangeably in this paper.

II. TRADING DYNAMICS OF AGENTS IN THE MARKET

As stated in the Introduction, the institutional investor is considered as a major agent in the mean field model of the market and the HFTs are considered as minor agents, where the state dynamics of the trading process of the major agent and any generic minor agent are described by the time evolution of the inventories, the prices and cash levels of each agent.

A. Inventory Dynamics

It is assumed that the institutional investor liquidates its inventory of shares, $Q_0(t)$, by trading at a rate $\nu_0(t)$ during the trading period $[0,T]$. Hence the major agent's inventory dynamics is given by

$$dQ_0(t) = \nu_0(t)dt + \sigma_0^Q dw_0^Q, \quad 0 \le t \le T, \tag{1}$$

where w_0^Q is a Wiener process, σ_0^Q is a positive scalar and we assume that $Q_0(0) \gg 1$. The same dynamical model is adopted for the trading dynamics of a generic HFT

$$dQ_i(t) = \nu_i(t)dt + \sigma_i^Q dw_i^Q, \quad 1 \le i \le N, \quad 0 \le t \le T, \tag{2}$$

where w_i^Q is a Wiener Process, σ_i^Q is a positive scalar, $Q_i(t)$ is the minor agent's remaining shares at time t, and $\nu_i(t)$ is its

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rate of trading which can be positive or negative depending on whether the agent is buying or selling its shares. However, the initial share stock of the HFTs, $\{Q_i(0), 1 \le i \le N\}$, are not considered to be large, furthermore they are not motivated to retain share stocks and are assumed to trade them quickly.
We assume that the trading rate of the major agent is controlled via $u_0(t)$ as

$$d\nu_0(t) = u_0(t)dt + \sigma_0^\nu dw_0^\nu, \quad 0 \le t \le T, \tag{3}$$

where w_0^ν, is a Wiener process, σ_0^ν is a positive scalar and the trading strategy $u_0(t)$ can be seen to be the trading acceleration of the major trader. Correspondingly, $u_i(t)$ controls the trading rate of minor agent, $\mathscr{A}_i$, by

$$d\nu_i(t) = u_i(t)dt + \sigma_i^\nu dw_i^\nu, \quad 1 \le i \le N \quad 0 \le t \le T, \tag{4}$$

where w_i^ν is a Wiener process and σ_i^ν is a positive scalar.

B. Price Dynamics

The trading rate of the major agent and the average trading rates of the minor agents give rise to the fundamental asset price which models the permanent effect of agents' trading rates on the market price. Further, each agent has a temporary effect on the asset price which only persists during the action of the trade and which determines the execution price, that is to say the price at which each agent can trade.

1) Fundamental Asset Price: We model the dynamics of the fundamental asset price, as seen from the major agent's viewpoint, by

$$dF_0(t) = \big(\lambda_0 \nu_0(t) + \frac{\lambda}{N}\sum_{i=1}^{N} \nu_i(t)\big)dt + \sigma dw_0^F(t),\ 0 \le t \le T, \tag{5}$$

where the Wiener process $w_0^F(t)$ models the aggregate effect of all traders in the market which - unlike the major and minor agents $\mathscr{A}_0$, $\mathscr{A}_i$, - have no partial observations on any of the state variables appearing in the dynamical market model (these are termed uninformed traders). Further, σ denotes the intensity of the market volatility and $\lambda_0, \lambda \ge 0$ denote the strength of the linear permanent impact of the major and minor agents' tradings on the fundamental asset price, respectively. Similarly, we model the fundamental asset price dynamics, as seen by a minor agent $\mathscr{A}_i$, by

$$dF_i(t) = \big(\lambda_0 \nu_0(t) + \frac{\lambda}{N}\sum_{i=1}^{N} \nu_i(t)\big)dt + \sigma dw_i^F(t),\ 0 \le t \le T, \tag{6}$$

where $1 \le i \le N$, and the Wiener process, $w_i^F(t)$, represents the mass effect of all uninformed traders.

2) Execution Price: The major agent's execution price, $S_0(t)$, is assumed to be given by

$$S_0(t) = F_0(t) + a_0 \nu_0(t), \quad 0 \le t \le T, \tag{7}$$

where $a_0 \ge 0$ controls the temporary impact strength of the major agent on fundamental asset price. Likewise, a minor agent's execution price, $S_i(t)$, is assumed to be determined by

$$S_i(t) = F_i(t) + a\nu_i(t), \quad 1 \le i \le N, \quad 0 \le t \le T, \tag{8}$$

where a models the temporary impact of a minor agent's trading on its execution price.

C. Cash Process

The cash process for the major agent and a generic minor agent, $Z_0(t)$, $Z_i(t)$, are given by

$$dZ_0(t) = -S_0(t)\nu_0(t)dt, \quad 0 \le t \le T, \tag{9}$$
$$dZ_i(t) = -S_i(t)\nu_i(t)dt, \quad 1 \le i \le N, \quad 0 \le t \le T, \tag{10}$$

where each agent has no cash at the beginning of trading interval, i.e. $Z_0(0) = 0$, and $Z_i(0) = 0,\ 1 \le i \le N$, and we note that the value of the trading velocity $\nu_0(t)$ in a stock sale is negative and hence for positive $S_0(t)$, $Z_0(t)$ increases.

D. Cost Function

The objective for the major trader is to maximize the cash it holds at the end of the trading horizon, i.e. maximize $Z_0(T)$, and if the remaining inventory at the final time T is $Q_0(T)$, the resulting loss is modeled by the penalty $Q_0(T)(F_0(T) + \alpha Q_0(T)$. Further, the major trader's utility in minimizing the inventory over the period $[0,T]$ is modeled by including the penalty $\phi \int_0^T Q_0^2(s)ds$ in its objective function, and the utility of avoiding very high execution prices, large trading intensities and large trading accelerations by including the terms $\varepsilon S_0^2(T)$, $\int_0^T \delta S_0^2(s)ds$, $\beta \nu_0^2(T)$, $\int_0^T \theta \nu_0^2(s)ds$ and $\int_0^T R_0 u_0^2(s)ds$ in the objective function. Therefore, its cost function to be minimized is given by

$$\begin{aligned} J_0(u_0,u_{-0}) = \mathbb{E}\Big[&-Z_0(T) + Q_0(T)\big(F_0(T) + \alpha Q_0(T)\big) \\ &+ \varepsilon S_0^2(T) + \beta \nu_0^2(s) \\ &+ \int_0^T \big(\phi Q_0^2(s) + \delta S_0^2(s) + \theta \nu_0^2(s) + R_0 u_0^2(s)\big)ds\Big], \end{aligned} \tag{11}$$

where α, ε, β, ϕ, δ, θ, and R_0 are positive scalars, and $u_{-0} := (u_1, ..., u_N)$ are trading strategies of the minor traders. Note that for large values of ϕ the trader attempts to liquidate its inventory quickly whereas with $\phi = 0$ it is indifferent to the level of its share stock.
In a similar way, the objective function to be minimized for an HFT is given by

$$\begin{aligned} J_i(u_i,u_{-i}) = \mathbb{E}\Big[&-Z_i(T) + Q_i(T)\big(F_i(T) + \psi Q_i(T)\big) \\ &+ \xi S_i^2(T) + \mu \nu_i^2(T) \\ &+ \int_0^T \big(\gamma S_i^2(s) + \rho \nu_i^2(s) + R u_i^2(s)\big)ds\Big],\ 1 \le i \le N, \end{aligned} \tag{12}$$

where ψ, ξ, μ, γ, ρ and R are positive scalars, $u_{-i} := (u_0, u_1, ..., u_{i-1}, u_{i+1}, ..., u_N)$, $Z_i(T)$ is the minor agent's total cash at the end of the trading horizon T, $Q_i(T)\big(F_i(T) + \psi Q_i(T)\big)$ is the penalty to be paid by an HFT if it keeps $Q_i(T)$ shares at the terminal time T, and its tendency to avoid high execution prices, large trading intensities and

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large trading accelerations is modeled by including $\xi S_i^2(T) + \mu \nu_i^2(T) + \int_0^T \big(\gamma S_i^2(s) + \rho \nu_i^2(s) + R u_i^2(s)\big) ds$ in the objective function. Note that there is no inventory cost for the HFT if it trades rapidly and does not hold shares.

III. MFG Formulation of the Optimal Execution Problem

In this section we put the optimal execution problem into the MM LQG MFG framework.

A. Finite population

1) Major Agent: The stochastic optimal control problem for major trader is modeled

$$d\nu_0(t) = u_0(t)dt + \sigma_0^\nu dw_0^\nu, \tag{13}$$
$$dQ_0(t) = \nu_0(t)dt + \sigma_0^Q dw_0^Q, \tag{14}$$
$$dF_0(t) = \big(\lambda_0 \nu_0(t) + \frac{\lambda}{N}\sum_{i=1}^N \nu_i(t)\big)dt + \sigma dw_0^F(t), \tag{15}$$

with the cost function

$$\begin{aligned} J_0(u_0, u_{-0}) = \mathbb{E}\Big[& Q_0(T)\big(F_0(T) + \alpha Q_0(T)\big) \\ & + \varepsilon\big(F_0(T) + a_0\nu_0(T)\big)^2 + \beta \nu_0^2(T) \\ & + \int_0^T \Big(\phi Q_0^2(s) + \nu_0(s)\big(F_0(s) + a_0\nu_0(s)\big) \\ & + \delta\big(F_0(s) + a_0\nu_0(s)\big)^2 + \theta\nu_0^2(s) + R_0 u_0^2(s)\Big) ds\Big], \end{aligned} \tag{16}$$

wherein the final cash process in (11) was replaced by $Z_0(T) = -\int_0^T S_0(s)\nu_0(s)ds = -\int_0^T \big(F_0(t) + a_0\nu_0(s)\big)\nu_0(s)ds$, and the execution prices $S_0(T)$, $S_0(t)$ were replaced using (7).

As can be seen, the major agent is coupled with the minor agents by the average term $\frac{\lambda}{N}\sum_{i=1}^N \nu_i$ in the fundamental asset price dynamics (15).

Now let the major agent's state be denoted by

$$X_0 = \begin{bmatrix} \nu_0 \\ Q_0 \\ F_0 \end{bmatrix}, \tag{17}$$

and subsequently, the major agent's cost function will be written in the standard quadratic form

$$J_0(u_0) = \mathbb{E}\Big[\|X_0(T)\|_{M_0}^2 + \int_0^T \big(\|X_0(s)\|_{N_0}^2 + \|u_0(s)\|_{R_0}^2\big)ds\Big], \tag{18}$$

with

$$M_0 = \begin{bmatrix} \varepsilon a_0^2 + \beta & 0 & a_0\varepsilon \\ 0 & \alpha & \frac{1}{2} \\ a_0\varepsilon & \frac{1}{2} & \varepsilon \end{bmatrix},$$

and

$$N_0 = \begin{bmatrix} a_0 + \delta a_0^2 + \theta & 0 & a_0\delta + \frac{1}{2} \\ 0 & \phi & 0 \\ \frac{1}{2} + a_0\delta & 0 & \delta \end{bmatrix}, \quad R_0 > 0.$$

The equations (13)-(15) together with the cost function (18) form the standard stochastic LQG problem for the major agent. It should be remarked that for M_0, N_0 to be positive semi definite matrices, the conditions $\varepsilon \geq \frac{1}{4\alpha}$, $\beta(\alpha\varepsilon - \frac{1}{4}) \geq \frac{1}{4}\varepsilon a_0^2$ and $\delta \geq \frac{1}{4\theta}$ must hold, respectively, and this will be assumed throughout this paper.

2) Minor Agent: Similarly, the stochastic optimal control problem for a minor trader $\mathscr{A}_i$, $1 \leq i \leq N$, is given by the set of dynamical equations

$$d\nu_i(t) = u_i(t)dt + \sigma_i^\nu dw_i^\nu, \tag{19}$$
$$dQ_i(t) = \nu_i(t)dt + \sigma_i^Q dw_i^Q, \tag{20}$$
$$dF_i(t) = \big(\lambda_0\nu_0(t) + \frac{\lambda}{N}\sum_{i=1}^N \nu_i(t)\big)dt + \sigma dw_i^F, \tag{21}$$

with the cost function

$$\begin{aligned} J_i(u_i, u_{-i}) = \mathbb{E}\Big[& Q_i(T)\big(F_i(T) + \psi Q_i(T)\big) \\ & + \xi\big(F_i(t) + a\nu_i(T)\big)^2 + \mu\nu_i^2(T) \\ & + \int_0^T \nu_i(s)\big(F_i(s) + a\nu_i(s)\big) \\ & + \gamma\big(F_i(s) + a\nu_i(s)\big)^2 + \rho\nu_i^2(s) + R u_i^2(s)\big)ds\Big], \end{aligned} \tag{22}$$

where the final cash process in (12) has been replaced using (10) by $Z_i(T) = -\int_0^T S_i(s)\nu_i(s)ds = -\int_0^T \big(F_i(t) + a\nu_i(s)\big)\nu_i(s)ds$, and the execution prices $S_i(T)$, $S_i(t)$ were replaced using (8).

The equations above show that a minor agent is coupled with the major agent and other minor agents through the fundamental asset price dynamics (21).

Similar to the major trader, we define a generic minor trader's state vector as

$$X_i = \begin{bmatrix} \nu_i \\ Q_i \\ F_i \end{bmatrix} \tag{23}$$

and its quadratic cost function is given by

$$J(u_i, u_{-i}) = \mathbb{E}\Big[\|X_i(T)\|_M^2 + \int_0^T \big(\|X_i(s)\|_N^2 + \|u_i(s)\|_R^2\big)ds\Big], \tag{24}$$

where

$$M = \begin{bmatrix} \xi a^2 + \mu & 0 & a\xi \\ 0 & \psi & \frac{1}{2} \\ a\xi & \frac{1}{2} & \xi \end{bmatrix},$$

and

$$N = \begin{bmatrix} a + \gamma a^2 + \rho & 0 & \frac{1}{2} + a\gamma \\ 0 & 0 & 0 \\ \frac{1}{2} + a\gamma & 0 & \gamma \end{bmatrix}, \quad R > 0.$$

The set of equations (19)-(21) and the cost function (24) constitute the standard stochastic LQG problem for a minor trader. Again, for the matrices M, N to be positive semi definite, $\xi > \frac{1}{4\psi}$, $\mu(\psi\xi - \frac{1}{4} \geq \frac{1}{4}\xi a^2$ and $\gamma > \frac{1}{4\rho}$ must be, respectively, satisfied and this is adopted as an assumption.

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B. Mean Field Evolution

Following the LQG MFG methodology [9], the mean field, $\bar{X}$, is defined as the L^2 limit, when it exists, of the average of minor agents' states when population size goes to infinity

$$\bar{X}(t) = \lim_{N\to\infty} X^N(t) = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} X_i(t),\ a.s.$$

Now, if the control strategy for each minor agent is considered to have the general feedback form

$$u_i = L_1 X_i + L_2 X_0 + \sum_{j\neq i, j=1}^{N} L_4 X_j + L_3, \quad 1 \le i \le N, \tag{25}$$

then mean field dynamics can be obtained by substituting (25) in the minor agents' dynamics (19)-(21) and taking the average and then its limit as $N \to \infty$. However, the only element of the mean field directly active in the dynamics in our setup is

$$\bar{\nu} = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} \nu_i. \tag{26}$$

We now derive the evolution equation of $\bar{\nu}$. Substituting (25) in the trading dynamics (19), we get

$$d\nu_i = \big[L_1 X_i + L_2 X_0 + L_3\big]dt + \sum_{j\neq i, j=1}^{N} L_4 X_j dt + \sigma_i^{\nu} dw_i^{\nu}, \quad 1 \le i \le N.$$

Summing up over $\{i : 1 \le i \le N\}$ yields to

$$\begin{aligned} N d\nu^N = N\big[&L_{1,1}\nu^N + L_{1,2}Q^N + L_{1,3}F^N \\ &+ L_{2,1}\nu_0 + L_{2,2}Q_0 + L_{2,3}F_0 + L_3\big]dt \\ + \Big[\sum_{i=1}^{N}\sum_{j\neq i, j=1}^{N} L_{4,1}\nu_j &+ \sum_{i=1}^{N}\sum_{j\neq i, j=1}^{N} L_{4,2}Q_j dt + \sum_{i=1}^{N}\sum_{j\neq i, j=1}^{N} L_{4,3}F_j\Big]dt \\ &+ \sigma_i^{\nu}\sum_{i=1}^{N} dw_i^{\nu}. \end{aligned}$$

Then with $N \to \infty$ (as in[9], [4]), the mean field equation for $\bar{\nu}$ is given by

$$\begin{aligned} d\bar{\nu} &= [\bar{L}_{1,1}\bar{\nu} + \bar{L}_{1,2}\bar{Q} + \bar{L}_{1,3}\bar{F}]dt \\ &\quad + [\bar{L}_{2,1}\nu_0 + \bar{L}_{2,2}Q_0 + \bar{L}_{2,3}F_0]dt + \bar{L}_3 dt, \quad a.s. \end{aligned} \tag{27}$$

where $\bar{L}_{1,1}$, $\bar{L}_{1,2}$, $\bar{L}_{1,3}$, $\bar{L}_{2,1}$, $\bar{L}_{2,2}$ $\bar{L}_{2,3}$ and $\bar{L}_3$ are the constants which can be calculated from consistency condition, and by (26), and the strong law of large numbers,

$$\begin{aligned} d\bar{Q} &= \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N} dQ_i = \lim_{N\to\infty}\Big[\frac{1}{N}\sum_{i=1}^{N}\nu_i dt + \frac{\sigma_i^Q}{N}\sum_{i=1}^{N} dw_i^Q\Big] \\ &= \bar{\nu}dt, \quad a.s. \\ d\bar{F} &= \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N} F_i = \lim_{N\to\infty}\Big[\big(\lambda_0\nu_0 + \frac{\lambda}{N}\sum_{i=1}^{N}\nu_i\big)dt + \frac{\sigma}{N}\sum_{i=1}^{N} dw_i\Big] \\ &= \big(\lambda_0\nu_0 + \lambda\bar{\nu}\big)dt, \quad a.s. \end{aligned}$$

where we have used a differential increments notation for the sake of brevity. Equivalently, the set of mean field equations can be written as

$$\begin{aligned} \begin{bmatrix} d\bar{\nu} \\ d\bar{Q} \\ d\bar{F} \end{bmatrix} &= \begin{bmatrix} \bar{L}_{1,1} & \bar{L}_{1,2} & \bar{L}_{1,3} \\ 1 & 0 & 0 \\ \lambda & 0 & 0 \end{bmatrix}\begin{bmatrix} \bar{\nu} \\ \bar{Q} \\ \bar{F} \end{bmatrix} dt \\ &+ \begin{bmatrix} \bar{L}_{2,1} & \bar{L}_{2,2} & \bar{L}_{2,3} \\ 0 & 0 & 0 \\ \lambda_0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \nu_0 \\ Q_0 \\ F_0 \end{bmatrix} dt + \begin{bmatrix} \bar{L}_3 \\ 0 \\ 0 \end{bmatrix} dt. \end{aligned} \tag{28}$$

The matrices in the mean field dynamics shall be denoted as follows

$$\bar{A} = \begin{bmatrix} \bar{L}_{1,1} & \bar{L}_{1,2} & \bar{L}_{1,3} \\ 1 & 0 & 0 \\ \lambda & 0 & 0 \end{bmatrix}, \quad \bar{m} = \begin{bmatrix} \bar{L}_3 \\ 0 \\ 0 \end{bmatrix},$$

$$\bar{G} = \begin{bmatrix} \bar{L}_{2,1} & \bar{L}_{2,2} & \bar{L}_{2,3} \\ 0 & 0 & 0 \\ \lambda_0 & 0 & 0 \end{bmatrix}.$$

C. Infinite Population

1) Major Agent:

$$\begin{aligned} \begin{bmatrix} d\nu_0 \\ dQ_0 \\ dF_0 \end{bmatrix} &= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ \lambda_0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \nu_0 \\ Q_0 \\ F_0 \end{bmatrix} dt + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u_0(t)dt \\ + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \lambda & 0 & 0 \end{bmatrix}&\begin{bmatrix} \bar{\nu} \\ \bar{Q} \\ \bar{F} \end{bmatrix} dt + \begin{bmatrix} \sigma_0^{\nu} & 0 & 0 \\ 0 & \sigma_0^{Q} & 0 \\ 0 & 0 & \sigma \end{bmatrix}\begin{bmatrix} dw_0^{\nu} \\ dw_0^{Q} \\ dw_0^{F} \end{bmatrix} \end{aligned} \tag{29}$$

The matrices shall be denoted by

$$A_0 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ \lambda_0 & 0 & 0 \end{bmatrix},\ B_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ W_0 = \begin{bmatrix} w_0^{\nu} \\ w_0^{Q} \\ w_0^{F} \end{bmatrix}$$

$$E_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \lambda & 0 & 0 \end{bmatrix},\ D_0 = \begin{bmatrix} \sigma_0^{\nu} & 0 & 0 \\ 0 & \sigma_0^{Q} & 0 \\ 0 & 0 & \sigma \end{bmatrix}.$$

2) Minor Agent:

$$\begin{aligned} \begin{bmatrix} d\nu_i \\ dQ_i \\ dF_i \end{bmatrix} &= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \nu_i \\ Q_i \\ F_i \end{bmatrix} dt \\ &+ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \lambda & 0 & 0 \end{bmatrix}\begin{bmatrix} \bar{\nu} \\ \bar{Q} \\ \bar{F} \end{bmatrix} dt + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u_i(t)dt \\ + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \lambda_0 & 0 & 0 \end{bmatrix}&\begin{bmatrix} \nu_0 \\ Q_0 \\ F_0 \end{bmatrix} dt + \begin{bmatrix} \sigma_i^{\nu} & 0 & 0 \\ 0 & \sigma_i^{Q} & 0 \\ 0 & 0 & \sigma \end{bmatrix}\begin{bmatrix} dw_i^{\nu} \\ dw_i^{Q} \\ dw_i^{F} \end{bmatrix}. \end{aligned} \tag{30}$$

and the matrices in the minor agent's dynamics are denoted as follows

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \lambda & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$G = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \lambda_0 & 0 & 0 \end{bmatrix},\ D = \begin{bmatrix} \sigma_i^{\nu} & 0 & 0 \\ 0 & \sigma_i^{Q} & 0 \\ 0 & 0 & \sigma \end{bmatrix},\ W_i = \begin{bmatrix} w_i^{\nu} \\ w_i^{Q} \\ w_i^{F} \end{bmatrix}.$$

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IV. Full Observation Optimal Execution Problem

Following the mean field game methodology with the major agent [9], [10], the optimal execution problem first is solved in the infinite population case yielding the best accelerations of trading for each agent. For this purpose, major agent's state is extended with the mean field, and minor agent's state is extended with the mean field and major agent's state.

A. Major Agent

The Major agent's extended dynamic in the infinite population is given by

$$\begin{bmatrix} X_0 \\ d\bar{X} \end{bmatrix} = \begin{bmatrix} A_0 & E_0 \\ \bar{G} & \bar{A} \end{bmatrix} \begin{bmatrix} X_0 \\ \bar{X} \end{bmatrix} dt + \begin{bmatrix} 0 \\ \bar{m} \end{bmatrix} dt + \begin{bmatrix} B_0 \\ 0 \end{bmatrix} u_0(t)dt + \begin{bmatrix} D_0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} dW_0 \\ 0 \end{bmatrix} \quad (31)$$

Accordingly, the following matrices are defined

$$\mathbb{A}_0 = \begin{bmatrix} A_0 & E_0 \\ \bar{G} & \bar{A} \end{bmatrix}, \quad \mathbb{M}_0 = \begin{bmatrix} 0 \\ \bar{m} \end{bmatrix},$$
$$\mathbb{B}_0 = \begin{bmatrix} B_0 \\ 0_{3\times 1} \end{bmatrix}, \quad \mathbb{D}_0 = \begin{bmatrix} D_0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Following [9], [10], the infinite population best response control is given by

$$u_0^*(t) = -R_0^{-1}\mathbb{B}_0^T \Pi_0 \big(X_0^T, \bar{X}\big)^T. \quad (32)$$

Let us define $\bar{N}_0 = [I_{3\times 3}, 0_{3\times 3}]^T N_0 [I_{3\times 3}, 0_{3\times 3}]$, then Π_0 is calculated by the following Riccati equation as

$$-\frac{d\Pi_0}{dt} = \Pi_0\mathbb{A}_0 + \mathbb{A}_0^T\Pi_0 - \Pi_0\mathbb{B}_0 R_0^{-1}\mathbb{B}_0^T\Pi_0 + \bar{N}_0,\ \Pi_0(T) = M_0.$$

B. Minor Agent

The minor agent's extended dynamics are

$$\begin{bmatrix} dX_i \\ dX_0 \\ d\bar{X} \end{bmatrix} = \begin{bmatrix} A & \begin{bmatrix} G & E \end{bmatrix} \\ 0_{6\times 3} & \mathbb{A}_0 - \mathbb{B}_0 R_0^{-1}\mathbb{B}_0^T\Pi_0 \end{bmatrix} \times \begin{bmatrix} X_i \\ X_0 \\ \bar{X} \end{bmatrix} dt + \begin{bmatrix} 0_{3\times 1} \\ \mathbb{M}_0 \end{bmatrix} dt + \begin{bmatrix} B \\ 0_{6\times 1} \end{bmatrix} u_i(t)dt + \begin{bmatrix} D & 0_{3\times 6} \\ 0_{6\times 3} & \mathbb{D}_0 \end{bmatrix} \begin{bmatrix} dW_i \\ dW_0 \\ 0 \end{bmatrix}. \quad (33)$$

Substituting the major agent's control action (32) into (33), we define

$$\mathbb{A} = \begin{bmatrix} A & \begin{bmatrix} G & E \end{bmatrix} \\ 0_{6\times 3} & \mathbb{A}_0 - \mathbb{B}_0 R_0^{-1}\mathbb{B}_0^T\Pi_0 \end{bmatrix}, \quad \mathbb{M} = \begin{bmatrix} 0_{3\times 1}, \\ \mathbb{M}_0 \end{bmatrix}, \quad (34)$$
$$\mathbb{B} = \begin{bmatrix} B \\ 0_{6\times 1} \end{bmatrix}, \quad \mathbb{D} = \begin{bmatrix} D & 0_{3\times 1} \\ 0_{6\times 1} & \mathbb{D}_0 \end{bmatrix}.$$

Then the best response control for a generic minor agent is

$$u_i = -R^{-1}\mathbb{B}^T\Pi\big(X_i^T, X_0^T, \bar{X}\big)^T, \quad (35)$$

where

$$-\frac{d\Pi}{dt} = \Pi\mathbb{A} + \mathbb{A}^T\Pi - \Pi\mathbb{B}R^{-1}\mathbb{B}^T\Pi + \bar{N},\ \Pi(T) = M(T).$$

with $\bar{N} = [I_{3\times 3}, 0_{3\times 6}]^T N [I_{3\times 3}, 0_{3\times 6}]$.

C. Consistency Condition

The closed loop trading dynamics of a generic minor agent applying (35) is consequently

$$d\nu_i = -R^{-1}\mathbb{B}^T\Pi\big(X_i^T, X_0^T, \bar{X}\big)^T dt + \sigma_i^\nu dw_i^\sigma, \quad 1 \le i \le N,$$

and so we obtain the mean field $\bar{\nu}$ process as follows.

$$\sum_{i=1}^N d\nu_i = -\sum_{i=1}^N R^{-1}\mathbb{B}^T\Pi\big(X_i^T, X_0^T, \bar{X}\big)^T dt + \sigma_i^\nu \sum_{i=1}^N dw_i^\sigma,$$

which yields

$$\lim_{N\to\infty} d\nu^N = -R^{-1}\mathbb{B}^T\Pi \lim_{N\to\infty}\big((X^N)^T, X_0^T, \bar{X}\big)^T dt$$

and hence

$$\begin{aligned} d\bar{\nu} &= -R^{-1}\mathbb{B}^T\Pi(\bar{X}^T, \bar{X}_0^T, \bar{X})^T dt \\ &= -R^{-1}\big(\Pi_{1,1}\bar{\nu} + \Pi_{1,2}\bar{Q} + \Pi_{1,3}\bar{F} + \Pi_{1,4}\nu_0 \\ &\quad + \Pi_{1,5}Q_0 + \Pi_{1,6}F_0 + \Pi_{1,7}\bar{\nu} + \Pi_{1,8}\bar{Q} + \Pi_{1,9}\bar{F}\big)dt, \quad a.s. \end{aligned}$$

So the consistency equations become

$$\begin{aligned} \bar{L}_{1,1} &= -R^{-1}(\Pi_{1,1} + \Pi_{1,7}), \\ \bar{L}_{1,2} &= -R^{-1}(\Pi_{1,2} + \Pi_{1,8}), \\ \bar{L}_{1,3} &= -R^{-1}(\Pi_{1,3} + \Pi_{1,9}), \\ \bar{L}_{2,1} &= -R^{-1}\Pi_{1,4}, \\ \bar{L}_{2,2} &= -R^{-1}\Pi_{1,5}, \\ \bar{L}_{2,3} &= -R^{-1}\Pi_{1,6}, \\ \bar{L}_3 &= 0 \quad \equiv \quad \mathbb{M}_0 = 0, \quad \mathbb{M} = 0, \end{aligned} \quad (36)$$

where the $L_\cdot$ scalars were defined in (27).

V. Partially Observed MM LQG MFG Problem

We now follow the general development in [8] for PO MM LQG MFG systems where the major agent has only partial observations on its own states.

A. Major Agent

let the major agent's observation process be

$$dy_0 = \mathbb{H}_0[X_0^T, \bar{X}]^T dt + dv_0 \quad (37)$$

where $\mathbb{H}_0$ is a constant matrix with appropriate dimension. Then the corresponding Kalman filter equation to generate the estimates of the major agent's states are given by

$$d\hat{\mathscr{X}}_{0|\mathscr{F}_0^y} = \mathbb{A}_0\hat{\mathscr{X}}_{0|\mathscr{F}_0^y}dt + \mathbb{B}_0\hat{u}_0 + K_0(t)[dy_0 - \mathbb{H}_0\hat{\mathscr{X}}_{0|\mathscr{F}_0^y}dt] \quad (38)$$

where the filter gain is given by

$$K_0(t) = -V_0(t)\mathbb{H}_0^T R_{v_0}^{-1}, \quad (39)$$

and the associated Riccati equation is

$$\dot{V}_0(t) = \mathbb{A}_0 V_0(t) + V_0(t)\mathbb{A}_0^T - K_0(t)R_{v_0}K_0(t)^T + Q_{w_0}. \quad (40)$$

Assumption: $[\mathbb{A}_0, Q_{w_0}]$ is controllable and $[H_0, \mathbb{A}_0]$ is observable.

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1) Cost Function: The system states can be decomposed into

$$\begin{bmatrix} v_0 \\ Q_0 \\ F_0 \end{bmatrix} = \begin{bmatrix} \hat{v}_{0|\mathscr{F}_0^y} \\ \hat{Q}_{0|\mathscr{F}_0^y} \\ \hat{F}_{0|\mathscr{F}_0^y} \end{bmatrix} + \begin{bmatrix} v_0 - \hat{v}_{0|\mathscr{F}_0^y} \\ Q_0 - \hat{Q}_{0|\mathscr{F}_0^y} \\ F_0 - \hat{F}_{0|\mathscr{F}_0^y} \end{bmatrix} \tag{41}$$

So by the Separation Principle

$$J_0(u_0) = \mathbb{E}\Big[\|\hat{X}_{0|\mathscr{F}_0^y}(T)\|^2_{M_0} + \int_0^T \big(\|\hat{X}_{0|\mathscr{F}_0^y}(s)\|^2_{N_0} + \|u_0(s)\|^2_{R_0}\big)ds + \|X_0(T) - \hat{X}_{0|\mathscr{F}_i^y}(T)\|^2_{M_0} + \int_0^T \big(\|X_0(s) - \hat{X}_{0|\mathscr{F}_0^y}(s)\|^2_{N_0}\big)ds\Big] \tag{42}$$

and the corresponding infinite population best response control action is given by

$$\hat{u}_0^\circ = -R_0^{-1}\mathbb{B}_0^T\Pi_0\big(\hat{X}^T_{0|\mathscr{F}_0^y}, \hat{\bar{X}}_{|\mathscr{F}_0^y}\big)^T \tag{43}$$

B. Minor Agent

The extended state shall be denoted by

$$\mathscr{X}_i = [X_i^T, X_0^T, \bar{X}, \hat{X}^T_{0|\mathscr{F}_0^y}, \hat{\bar{X}}_{|\mathscr{F}_0^y}]^T. \tag{44}$$

Let the minor agent's observation process be given by

$$dy_i(t) = \mathbb{H}[X_i^T, X_0^T, \bar{X}, \hat{X}^T_{0|\mathscr{F}_0^y}, \hat{\bar{X}}_{0|\mathscr{F}_0^y}]^T dt + dv_i \tag{45}$$

with $\mathbb{H}$ constant matrix. Then the extended dynamics of minor agent are given by

$$\begin{bmatrix} dX_i \\ dX_0 \\ d\bar{X} \\ d\hat{X}_{0|\mathscr{F}_0^y} \\ d\hat{\bar{X}}_{|\mathscr{F}_0^y} \end{bmatrix} = \begin{bmatrix} A & [G,E] & 0_{3\times 6} \\ 0_{6\times 3} & \mathbb{A}_0 & -\mathbb{B}_0 R_0^{-1}\mathbb{B}_0^T\Pi_0 \\ 0_{6\times 3} & K_0H_0 & \mathbb{A}_0 - \mathbb{B}_0R_0^{-1}\mathbb{B}_0^T\Pi_0 - K_0H_0 \end{bmatrix} \times \begin{bmatrix} X_i \\ X_0 \\ \bar{X} \\ \hat{X}_{0|\mathscr{F}_0^y} \\ \hat{\bar{X}}_{|\mathscr{F}_0^y} \end{bmatrix} dt + \begin{bmatrix} \mathbb{B} \\ 0_{6\times 1} \end{bmatrix} u_i(t)dt + \begin{bmatrix} \mathbb{D} & 0 \\ 0 & K_0 \end{bmatrix} \begin{bmatrix} dw_i \\ dw_0 \\ 0 \\ dv_0 \end{bmatrix}, \tag{46}$$

or equivalently

$$d\mathscr{X}_i = \mathbb{A}\mathscr{X}_i dt + \mathbb{B}u_i dt + \Sigma \begin{bmatrix} dw_i \\ dw_0 \\ 0 \\ dv_0 \end{bmatrix}. \tag{47}$$

The Kalman filter which generates the estimates of the minor agent's states is

$$d\hat{\mathscr{X}}_{i|\mathscr{F}_i^y} = \mathbb{A}\hat{\mathscr{X}}_{i|\mathscr{F}_i^y}dt + \mathbb{B}\hat{u}_i dt + K(t)\big[dy_i - \mathbb{H}\hat{\mathscr{X}}_{i|\mathscr{F}_i^y}dt\big] \tag{48}$$

where the filter gain is given as

$$K(t) = V(t)\mathbb{H}^T R_{v_i}^{-1}. \tag{49}$$

Assumption: $[\mathbb{A}, Q_w]$ is controllable (respectively stabilizable) and $[\mathbb{H}, \mathbb{A}]$ is observable (respectively detectable).
The corresponding Riccati equation is

$$\dot{V}(t) = \mathbb{A}V(t) + V(t)\mathbb{A} - K(t)R_v K^T(t) + Q_w. \tag{50}$$

1) Cost Function: We can decompose a generic minor agent's state as

$$\begin{bmatrix} v_i \\ Q_i \\ F_i \end{bmatrix} = \begin{bmatrix} \hat{v}_{i|\mathscr{F}_i^y} \\ \hat{Q}_{i|\mathscr{F}_i^y} \\ \hat{F}_{i|\mathscr{F}_i^y} \end{bmatrix} + \begin{bmatrix} v_i - \hat{v}_{i|\mathscr{F}_i^y} \\ Q_i - \hat{Q}_{i|\mathscr{F}_i^y} \\ F_i - \hat{F}_{i|\mathscr{F}_i^y} \end{bmatrix} \tag{51}$$

By Separation Theorem we have

$$J_i(u_i, u_{-i}) = \mathbb{E}\Big[\|\hat{X}_{i|\mathscr{F}_i^y}(T)\|^2_M + \int_0^T \big(\|\hat{X}_{i|\mathscr{F}_i^y}(s)\|^2_N + \|u_i(s)\|^2_R\big)ds \; \|X_i(T) - \hat{X}_{i|\mathscr{F}_i^y(T)}\|^2_M + \int_0^T \|X_i(s) - \hat{X}_{i|\mathscr{F}_i^y}(s)\|^2_N ds\Big]. \tag{52}$$

So the corresponding infinite population best response control is given by

$$\hat{u}_i^\circ(t) = -R^{-1}\Pi\big(\hat{X}^T_{i|\mathscr{F}_i^y}, \hat{X}_{0|\mathscr{F}_i^y}, \hat{\bar{X}}_{|\mathscr{F}_i^y}\big). \tag{53}$$

The infinite population best response control laws applied to a finite population system yield to ε-Nash equillibrium.

Theorem: *ε-Nash Equilibria for PO MM-MF Systems*: The KF-MF state estimation scheme (38),(40) and (48),(50) together with the MM-MFG equation scheme (36) generate the set of control laws $\hat{\mathscr{U}}_{MF}^N \triangleq \{\hat{u}_i^\circ; 0 \le i \le N\}$, $1 \le N < \infty$, given by

$$\hat{u}_0^\circ = -R_0^{-1}\mathbb{B}_0^T\Pi_0(\hat{X}^T_{0|\mathscr{F}_0^y}, \hat{\bar{X}}^T_{|\mathscr{F}_0^y}),$$
$$\hat{u}_i^\circ = -R^{-1}\mathbb{B}^T\Pi(\hat{X}^T_{i|\mathscr{F}_i^y}, \hat{X}^T_{0|\mathscr{F}_i^y}, \hat{\bar{X}}^T_{|\mathscr{F}_i^y})^T, \; 1 \le i \le N$$

such that

(i) All agent systems $0 \le i \le N$, are $e^{-\frac{\rho}{2}t}$discounted second order stable in the sense that $\sup_{t\ge 0, 0\le i\le N} e^{-\frac{\rho}{2}t}\mathbb{E}\Big(\|\hat{X}_{i|\mathscr{F}_i^y}(t)\|^2 + \|\hat{\bar{X}}_{|\mathscr{F}_i^y}(t)\|^2\Big) < C$, with C independent of N;
(ii) $\{\hat{\mathscr{U}}_{MF}^N; 1 \le N < \infty\}$ yields an ε-Nash equilibrium for all ε, i.e. for all $\varepsilon > 0$, there exists $N(\varepsilon)$ such that for all $N \ge N(\varepsilon)$;

$$J_i^{s,N}(\hat{u}_i^\circ, \hat{u}_{-i}^\circ) - \varepsilon \le \inf_{u_i \in \mathscr{U}_{i,y}^N} J_i^{s,N}(u_i, \hat{u}_{-i}^\circ) \le J_i^{s,N}(\hat{u}_i^\circ, \hat{u}_{-i}^\circ),$$

VI. Simulation Results for MFG Optimal Execution Problem

In numerical experiments, it is assumed that the trading takes place $T = 500$ seconds. The parameters which are used are taken from [3]. We assume that the temporary impact parameters for the major agent and a generic minor agent are $a_0 = a = 0.1434$, the permanent impact parameters of the major agent's and the HFTs' trading rates on the fundamental asset price are $\lambda_0 = \lambda = 7.2 \times 10^{-4}$. We take the penalty coefficient for the remaining inventory stocks $\alpha = 10a$ for the major trader and $\psi = 10a_0$ for a minor trader. The inventory penalty for the major agent is assumed to be $\phi = 10^{-1}a$. Furthermore, the volatility is $\sigma = 0.6565$, the initial asset price is taken to be $F_0(0) = F_i(0) = \$35$, the initial inventory stock of the major trader and an HFT are assumed to be $Q_0(0) = 5000$ and $Q_i(0) = 500$, respectively. The resulting ε-Nash equilibrium trajectories for the major trader and a generic minor agent are displayed in Fig. 1-2 and Fig. 3-4 for the full observation case and partially observed case, respectively.

VII. Conclusion

An initial application of PO MM LQG MFG Theory to the financial execution problem was presented in this paper. Current and future work will expand the dynamical models of the financial execution systems, explore the robustness of the model with respect to its modeling hypothesis and take into account the nonlinear dynamical effects, where the latter constitutes a very challenging problem [6], [7].

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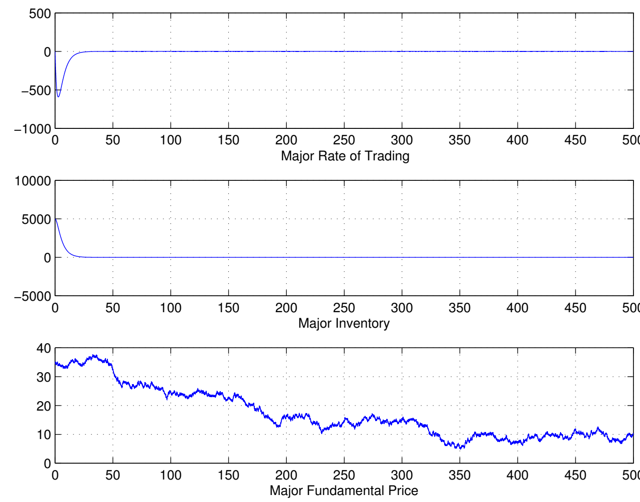

Fig. 1. Major Agent's States in Full Observation Case

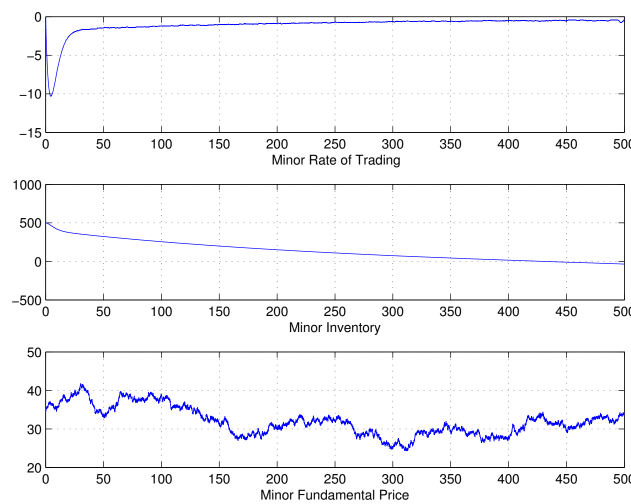

Fig. 2. A Generic Minor Agent's States in Full Observation Case

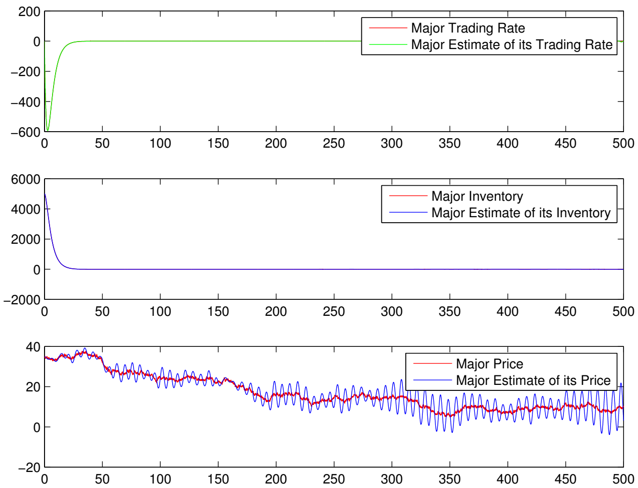

Fig. 3. Major Agent's States and Major Agent's Estimates of its Own States

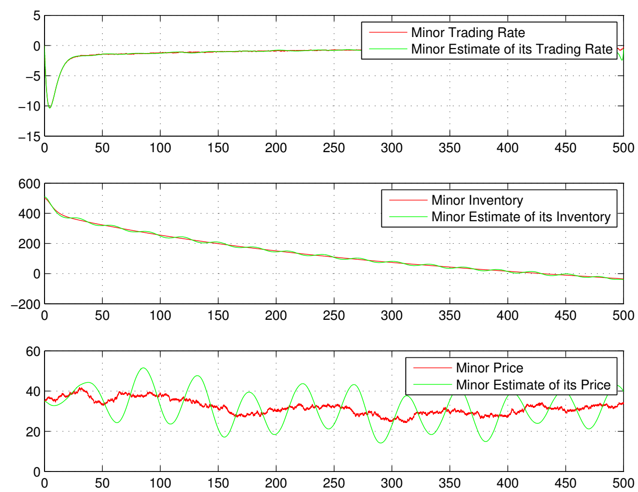

Fig. 4. A Generic Minor Agent's States and Minor Agent's Estimates of its Own States

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