Regularity for Fully Nonlinear Elliptic Equations and Motion by Mean Curvature

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Introduction

These are the notes for a series of six lectures I gave at Montecatini Terme, Italy, during June 12–17, 1995, as part of a CIME program on "Viscosity Solutions and Applications".

Each speaker presented a minicourse of six lectures on a variety of topics. As I was asked to speak on two essentially distinct subjects (fully nonlinear elliptic PDE and mean curvature motion), I in fact presented two microcourses, or more precisely nanocourses, of three lectures each.

Part I. Regularity for fully nonlinear elliptic PDE of second order

A. Introduction, viscosity solutions, linear estimates

B. $C^{1,\alpha}$ estimates

C. $C^{2,\alpha}$ estimates

Part II. Motion by mean curvature

D. Introduction, distance function

E. Level set method

F. Geometric properties

For these notes I have changed the oral presentation somewhat, mostly to be consistent with the other speakers' lectures.

I am pleased to thank I. Capuzzo-Dolcetta and P.-L. Lions for inviting me to speak.

Part I
Regularity Theory for Fully Nonlinear Elliptic PDE of Second Order

A. Introduction, viscosity solutions, linear estimates

1. Introduction. This and the next two lectures concern regularity for ***fully nonlinear second-order elliptic PDE*** and in particular for the model problem:

$$F(D^2u) = f \qquad \text{in } U\ . \tag{1}$$

Here U denotes an open subset of $\mathbb{R}^n$, $f : U \to \mathbb{R}$ is given, $u : \overline{U} \to \mathbb{R}$ is the unknown, and

$$D^2u = \begin{pmatrix} u_{x_1x_1} & \cdots & u_{x_1x_n} \\ & \ddots & \\ u_{x_nx_1} & & u_{x_nx_n} \end{pmatrix}$$

is the Hessian matrix of u. We are given the nonlinearity

$$F : \mathbb{S}^n \to \mathbb{R}\ ,$$

$\mathbb{S}^n$ denoting the space of real, $n \times n$ symmetric matrices.

The overall problem is to study solutions u of the PDE (1), subject say to the boundary condition

$$u = 0 \quad \text{on} \quad \partial U\ . \tag{2}$$

As the left-hand side of (1) is in general a nonlinear combination of the various second derivatives of u, this PDE is called ***fully nonlinear***.

We will require of course various assumptions concerning F, the most important of which is a kind of (reverse) monotonicity condition with respect to matrix ordering.

Definition. (i) We say that F is ***elliptic*** provided

$$A \geq B \quad \text{in} \quad \mathbb{S}^n \tag{3}$$

implies

$$F(B) \geq F(A) \qquad (A, B \in \mathbb{S}^n)\ . \tag{4}$$

(ii) We say F is ***uniformly elliptic*** if there exist constants $0 < \theta \leq \Theta < \infty$ such that (3) implies

$$\theta\|A - B\| \leq F(B) - F(A) \leq \Theta\|A - B\| \ . \tag{5}$$

Notation. (a) Condition (3) means $C = A - B$ is a nonnegative definite symmetric $n \times n$ matrix.

(b) In (5), we take the norm

$$\|A - B\| = \|C\| = \sup_{|\xi|=1} |\xi \cdot C\xi|$$

(c) If $\xi \in \mathbb{R}^n$, $\xi = (\xi_1, \ldots, \xi_n)$, then

$$\xi \otimes \xi = \begin{pmatrix} \xi_1\xi_1 & \cdots & \xi_1\xi_n \\ & \ddots & \\ \xi_n\xi_1 & \cdots & \xi_n\xi_n \end{pmatrix} \in \mathbb{S}^n$$

□

Examples of nonlinear elliptic PDE arise in stochastic control theory (cf. Soner's lectures) and in geometry (cf. Lecture D below).

We will assume hereafter that F, f are smooth. We can consequently reinterpret (4),(5) in differential form as follows. Fix a matrix $R \in \mathbb{S}^n$ and a vector $\xi \in \mathbb{R}^n$. Taking

$$B = R \ , \qquad A = R + \lambda(\xi \otimes \xi) \qquad (\lambda > 0) \ ,$$

we deduce from (4) that

$$\frac{1}{\lambda} \left[F(R + \lambda(\xi \otimes \xi)) - F(R)\right] \leq 0$$

Upon letting $\lambda \to 0^+$ we conclude:

$$-\frac{\partial F}{\partial r_{ij}} (R)\xi_i\xi_j \geq 0 \qquad (R \in \mathbb{S}^n, \ \ \xi \in \mathbb{R}^n) \ . \tag{6}$$

Similarly (5) implies

$$\theta|\xi|^2 \leq -\frac{\partial F}{\partial r_{ij}} (R)\xi_i\xi_j \leq \Theta|\xi|^2 \qquad (R \in \mathbb{S}^n, \ \ \xi \in \mathbb{R}^n) \ . \tag{7}$$

Hence the strict monotonicity condition of uniform ellipticity (5) means that the "linearization" of F about any fixed matrix $R \in \mathbb{S}^n$ is a uniformly elliptic operator. Our study of regularity for solutions of the fully nonlinear PDE (1) will therefore profit from good estimates for linear elliptic PDE.

2. Viscosity solutions. A main concern of Lectures A–C will be the possibility of estimating a smooth solution of (1),(2). It will be helpful however to first introduce an appropriate notion of a weak solution.

Motivation. We intend to find a way to interpret a merely continuous function $u : \overline{U} \to \mathbb{R}$ as solving (1). Since u will not necessarily have second derivatives, we must somehow "move" the second derivatives onto a smooth test function ϕ. The idea is to assume temporarily that u is a smooth solution of (1) and then to use the maximum principle.

So let ϕ be a smooth function and suppose that at some point $x_0 \in U$ we have

$$u(x_0) = \phi(x_0)$$

with

$$u(x) \leq \phi(x) \qquad \text{for all } x \text{ near } x_0 \ .$$

Thus $u-\phi$ has a local maximum at x_0. Geometrically the graph of u is touched from above at x_0 by the graph of ϕ. As we are temporarily supposing u to be smooth, we deduce

$$D^2u(x_0) \leq D^2\phi(x_0) \ .$$

Hence the ellipticity condition (4) implies

$$F(D^2u(x_0)) \geq F(D^2\phi(x_0)) \ .$$

But then the PDE (1) gives

$$F(D^2\phi(x_0)) \leq f(x_0) \ .$$

The opposite inequality would occur if the graph of ϕ were to touch the graph of u from below at x_0.

These considerations motivate the following:

Definitions. (i) We say $u \in C(\overline{U})$ is a ***subsolution of (1) in the viscosity sense*** if

$$F(D^2\phi(x_0)) \leq f(x_0)$$

whenever ϕ is smooth and $u - \phi$ has a local maximum at a point $x_0 \in U$.

(ii) We say $u \in C(\overline{U})$ is a ***supersolution*** of (1) ***in the viscosity sense*** if

$$F(D^2\phi(x_0)) \geq f(x_0)$$

whenever ϕ is smooth and $u - \phi$ has a local minimum at a point $x_0 \in U$.

(iii) We say u is a ***viscosity solution*** of (1) if u is both a viscosity subsolution and supersolution.

The great advantage of this definition is that it allows us to interpret a nonsmooth function u as solving (1). The point is that by weakening what we mean by a solution of (1),(2) we are more likely to be able to construct a solution.

The accompanying lectures of M. G. Crandall explain in more detail the stability, uniqueness and existence properties of viscosity solutions.

I will in these notes mostly provide various formal calculations leading to estimates of $|u|$, $|Du|$, $|D^2u|$, etc. under the assumption that u is a smooth solution of (1),(2). Work, sometimes lots of work, is needed to obtain the same estimate for viscosity solutions, and for this I refer readers to the very clear book [C-C] of Cabre and Caffarelli. I think however it is extremely important for students to understand also that unjustified, formal calculations can serve as guides for rigorous proofs.

Example: Jensen's regularizations. To illustrate this point, let me show you the following heuristic calculation. Suppose u is a smooth solution of

$$F(D^2u) = 0 \qquad \text{in } \mathbb{R}^n , \tag{8}$$

F being smooth and elliptic. Consider the Hamilton-Jacobi equation

$$\begin{cases} v_t + H(Dv) = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ v = u & \text{on } \mathbb{R}^n \times \{t = 0\} , \end{cases} \tag{9}$$

where $H : \mathbb{R}^n \to \mathbb{R}$ is smooth, convex. I will show that the effect of the flow (9) is to convert u into a supersolution. Indeed, set

$$w = F(D^2v) .$$

Then

$$\begin{aligned} w_t &= F_{r_{ij}}(D^2v) v_{x_i x_j t} \\ w_{x_k} &= F_{r_{ij}}(D^2v) v_{x_i x_j x_k} , \qquad (1 \le k \le n) \end{aligned}$$

Now (9) implies

$$\begin{aligned} &v_{x_i t} + H_{p_k}(Dv) v_{x_k x_i} = 0 \qquad (1 \le i \le n) \\ &v_{x_i x_j t} + H_{p_k}(Dv) v_{x_k x_i x_j} + H_{p_k p_l}(Dv) v_{x_k x_i} v_{x_l x_j} = 0 \quad (1 \le i, j \le n) \end{aligned}$$

Thus

$$
\begin{aligned}
w_t + H_{p_k} w_{x_k} &= F_{r_{ij}}(-H_{p_k} v_{x_k x_i x_j} - H_{p_k p_l} v_{x_k x_i} v_{x_l x_j}) \\
&\quad + H_{p_k}(F_{r_{ij}} v_{x_i x_j x_k}) \\
&= -H_{p_k p_l} F_{r_{ij}} v_{x_k x_i} v_{x_l x_j} \geq 0 \ ,
\end{aligned} \tag{10}
$$

as F is elliptic and H is convex. Since $w = F(D^2u) = 0$ at $t = 0$, we deduce from (10) that

$$w = F(D^2 v) \geq 0 \quad \text{in} \ \ \mathbb{R}^n \times (0, \infty) \ .$$

In particular, for each $t > 0$, $v(\cdot, t)$ is a supersolution.

As a special case, let $H(p) = \frac{|p|^2}{2}$. Then the (unique viscosity) solution of (9) turns out to be

$$v(x,t) = \min_{y \in \mathbb{R}^n} \left(u(y) + \frac{|x-y|^2}{2t} \right)$$

Consequently we can expect upon letting $t = \varepsilon$ that

$$u_\varepsilon(x) = \min_{y \in \mathbb{R}^n} \left(u(y) + \frac{|x-y|^2}{2\varepsilon} \right)$$

will satisfy

$$F(D^2 u_\varepsilon) \geq 0 \quad \text{in} \ \ \mathbb{R}^n \ .$$

This is indeed true, provided we interpret everything in the viscosity sense — see Jensen-Lions-Souganidis [J-L-S]. My point here is that the formal calculations lead us to guess the correct result.

Taking $H(p) = (1 + |p|^2)^{\frac{1}{2}}$, we deduce

$$\begin{aligned} \tilde{u}_\varepsilon = {} & \text{function whose graph in } \mathbb{R}^{n+1} \text{ is at a} \\ & \text{distance } \varepsilon \text{ below the graph of } u \end{aligned}$$

is also a supersolution. This was Jensen's original observation [J]. □

3. Review of linear estimates. As noted above, our estimates for the fully nonlinear PDE will depend upon deep estimates for linear elliptic PDE. We therefore review here the basic estimates for ***nondivergence structure*** equations of the form

$$Lu = -a_{ij}(x) u_{x_i x_j} = f \quad \text{in} \ \ U \ ,$$

where the coefficients $a_{ij} \in L^\infty(U)$ $(1 \le i,j \le n)$ satisfy the uniform ellipticity condition

$$\theta|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \Theta|\xi|^2 \qquad (x \in U, \ \ \xi \in \mathbb{R}^n) \tag{12}$$

for constants $0 < \theta \le \Theta < \infty$. We assume the u, f, ∂U, etc., are smooth. It will be extremely important for our applications that the following estimates do not depend in any way on the smoothness or even continuity of the a_{ij}.

Estimate 1: ***(Sup-norm bound)***

$$\|u\|_{L^\infty(U)} \le \|u\|_{L^\infty(\partial U)} + C\|f\|_{L^\infty(U)} \ . \tag{13}$$

Estimate 2: ***(Estimate of gradient on ∂U).*** If $u = 0$ on ∂U, then

$$\|Du\|_{L^\infty(\partial U)} \le C(\|u\|_{L^\infty(U)} + \|f\|_{L^\infty(U)}) \tag{14}$$

Estimate 3: ***(Alexandroff-Bakelman-Pucci).*** We have

$$\sup_U u \le \sup_{\partial U} u + C\|f\|_{L^n(U)} \tag{15}$$

Estimate 4: ***(Krylov-Safonov Hölder estimates).*** (i) For each $V \subset\subset U$ there exists $\alpha > 0$ such that

$$\|u\|_{C^{0,\alpha}(V)} \le C(\|u\|_{L^\infty(U)} + \|f\|_{L^n(U)}) \tag{16}$$

(ii) If $u|_{\partial U}$ is Hölder continuous with exponent β, there exists $0 < \alpha \le \beta$ such that

$$\|u\|_{C^{0,\alpha}(U)} \le C(\|u\|_{C^{0,\beta}(\partial U)} + \|f\|_{L^n(U)}) \tag{17}$$

Estimate 5: ***(Krylov-Safonov Harnack inequality).*** If u solves (1) and $u \ge 0$, then for each $V \subset\subset U$ there exists a constant C such that

$$\sup_V u \le C(\inf_V u + \|f\|_{L^n(U)}) \ . \tag{18}$$

These inequalities are proved in [G-T] for sufficiently smooth solutions u of $Lu = f$. The various constants C depend on n, θ, Θ, the smoothness of ∂U, etc., but not on u and f.

See [C-C] for viscosity solution interpretations and proofs of the foregoing.

B. $C^{1,\alpha}$ estimates

In this lecture we obtain *a priori* $C^{1,\alpha}$ estimates for a smooth solution of our fully nonlinear PDE

1. A boundary Hölder estimate. We will require first a rather delicate linear estimate at the boundary. We reproduce the following recent proof of Caffarelli, taken from [G-T 2].

Notation. $B^+(0,r) = \{x \in \mathbb{R}^n \mid |x| \le r, x_n \ge 0\}$.

Theorem (Krylov). *Let u solve*

$$(1) \qquad \begin{cases} -a_{ij}u_{x_ix_j} = f & \text{in } B^+(0,1) \\ u = 0 & \text{on } \partial B^+(0,1) \cap \{x_n = 0\} . \end{cases}$$

Then for $0 < r \le 1$,

$$(2) \qquad \operatorname*{osc}_{B^+(0,r)} \left(\frac{u}{x_n}\right) \le C\, r^\alpha \left(\operatorname*{osc}_{B^+(0,1)} \left(\frac{u}{x_n}\right) + \|f\|_{L^\infty} \right) ,$$

the constants $0 < \alpha < 1$, $C > 0$ *depending only on* n, θ, Θ, *etc.*

Outline of proof. **1.** An argument using barriers (cf. Estimate 1 above) shows

$$v = \frac{u}{x_n}$$

is locally bounded in $B^+(0,1)$. Assume for the moment $u \ge 0$.

2. Write $x = (x', x_n)$, $(x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1})$ and define

$$C_{r,\delta} = \{x \mid |x'| \le r,\ 0 \le x_n \le \delta r\} .$$

We ***claim*** that there exists $\delta > 0$ such that

$$(3) \qquad \inf_{\substack{|x'| \le r \\ x_n = \delta r}} v \le C \left(\inf_{C_{r/2,\delta}} v + r\|f\|_{L^\infty} \right)$$

To see this assume without loss $r = 1$,

$$\inf_{\substack{|x'| \le 1 \\ x_n = \delta}} v = 1 .$$

Define

$$w(x) = \left[(1 - |x'|^2) + (1 + \|f\|_{L^\infty}) \frac{(x_n - \delta)}{\delta^{\frac{1}{2}}} \right] x_n$$

and check $Lw \le f$ if $\delta > 0$ is small enough and $0 \le x_n \le \delta$. Also $w \le u$ on $\partial C_{1,\delta}$. Thus the maximum principle implies

$$w \le u \quad \text{on} \quad C_{1,\delta} \; ;$$

and so on $C_{\frac{1}{2},\delta}$,

$$\begin{aligned} v = \frac{u}{x_n} &\ge (1 - |x'|^2) + (1 + \|f\|_{L^\infty}) \frac{x_n - \delta}{\delta^{\frac{1}{2}}} \\ &\ge \tfrac{1}{2} - C\|f\|_{L^\infty} \quad \text{for small } \; \delta > 0 \; . \end{aligned}$$

This gives (3) (for $r = 1$, $\inf_{\substack{|x'| \le 1 \\ x_n = \delta}} v = 1$).

3. Now set

$$D_{r,\delta} = \left\{ x \mid |x'| \le r, \; \frac{\delta r}{2} \le x_n \le \frac{3\delta r}{2} \right\} \; .$$

Note $D_{r,\delta} \subset C_{2r,\delta}$. Observe also

$$\frac{2u}{3\delta r} \le v = \frac{u}{x_n} \le \frac{2u}{\delta r}$$

on $D_{r,\delta}$. Thus the Harnack inequality (Estimate 5) gives

$$\sup_{D_{r,\delta}} u \le C(\inf_{D_{r,\delta}} u + r^2 \|f\|_{L^\infty}) \; .$$

Hence

$$\begin{aligned} \sup_{D_{r,\delta}} v &\le C(\inf_{D_{r,\delta}} v + r\|f\|_{L^\infty}) \\ &\le C(\inf_{\substack{|x'| \le r \\ x_n = \delta r}} v + r\|f\|_{L^\infty}) \\ &\le C(\inf_{C_{r/2,\delta}} v + r\|f\|_{L^\infty}) \; , \qquad \text{by (3)} \end{aligned} \tag{4}$$

4. Now drop the assumption $u \ge 0$. Set

$$M_{2r} = \sup_{C_{2r,\delta}} v \; , \qquad m_{2r} = \inf_{C_{2r,\delta}} v \; , \qquad \operatorname*{osc}_{C_{2r,\delta}} v = M_{2r} - m_{2r} \; .$$

Apply (4) to $M_{2r} - v \ge 0$, $v - m_{2r} \ge 0$ in place of v:

$$\begin{aligned} \sup_{D_{r,\delta}} (M_{2r} - v) &\le C[\inf_{C_{r/2,\delta}} (M_{2r} - v) + r\|f\|_{L^\infty}] \; , \\ \sup_{D_{r,\delta}} (v - m_{2r}) &\le C[\inf_{C_{r/2,\delta}} (v - m_{2r}) + r\|f\|_{L^\infty}] \; . \end{aligned}$$

Adding these inequalities we deduce:

$$\begin{aligned} M_{2r} - m_{2r} &\le \sup_{D_{r,\delta}}(M_{2r} - v) + \sup_{D_{r,\delta}}(v - m_{2r}) \\ &\le C[(M_{2r} - M_{r/2}) + (m_{r/2} - m_{2r}) + r\|f\|_{L^\infty}] \end{aligned}$$

Rearrange this inequality to discover

$$\operatorname*{osc}_{C_{r/2,\delta}} v \le \eta \operatorname*{osc}_{C_{2r,\delta}} v + Cr\|f\|_{L^\infty}$$

for $\eta = (1 - \frac{1}{C}) < 1$. This inequality and a standard iteration yield (2). □

As a corollary we have:

Estimate 6: Let u solve $Lu = f$ in U, $u = 0$ on ∂U. Then

$$\|Du\|_{C^{0,\alpha}(\partial U)} \le C\|f\|_{L^\infty(U)} , \tag{5}$$

for constants $C, \alpha > 0$ depending only on n, θ, Θ, ∂U, etc.

The proof of (5) follows by locally straightening out ∂U to convert to the geometric setting of Theorem 1 and letting $x_n \to 0$.

2. $C^{1,\alpha}$ estimates for fully nonlinear elliptic equations. Let us return to the fully nonlinear PDE

$$\begin{cases} F(D^2u) = f & \text{in } U \\ u = 0 & \text{on } \partial U . \end{cases} \tag{7}$$

We will next show how the linear theory discussed above gives us *a priori* estimates on the solution u. The idea is to "linearize" in various ways.

a. First linearization. We may as well suppose $F(0) = 0$. Then (7) implies

$$\begin{aligned} f = F(D^2u) - F(0) &= \int_0^1 \frac{d}{dt} F(tD^2u)dt \\ &= \int_0^1 F_{r_{ij}}(tD^2u)dt\ u_{x_ix_j} \end{aligned}$$

Setting

$$a_{ij}(x) = -\int_0^1 F_{r_{ij}}(tD^2u(x))dt ,$$

we see

$$\begin{cases} -a_{ij}u_{x_ix_j} = f & \text{in } U \\ u = 0 & \text{on } \partial U . \end{cases} \tag{8}$$

Then according to Estimates 1, 2 and 6 we have:

$$\|u\|_{C^{0,\alpha}(U)} \leq C\|f\|_{L^\infty(U)} , \tag{9}$$

$$\|Du\|_{C^{0,\alpha}(\partial U)} \leq C\|f\|_{L^\infty(U)} . \tag{10}$$

b. Second linearization. We use a different trick to invoke once more the estimates above for linear equations. Let us fix $k \in \{1, \dots, n\}$ and differentiate the PDE (7) with respect to x_k. We find

$$F_{r_{ij}}(D^2u)u_{x_k x_i x_j} = f_{x_k} \quad \text{in } U \quad (1 \leq k \leq n) .$$

Thus

$$-\tilde{a}_{ij}\tilde{u}_{x_i x_j} = \tilde{f} \quad \text{in } U , \tag{11}$$

for

$$\tilde{u} = u_{x_k} , \qquad \tilde{f} = f_{x_k}$$
$$\tilde{a}_{ij} = -F_{r_{ij}}(D^2u) .$$

In view of (10) and Estimates 1–4, we deduce

$$\|\tilde{u}\|_{C^{0,\alpha}(U)} \leq C\|\tilde{f}\|_{L^\infty(U)}$$

for some $\alpha > 0$. As $\tilde{u} = u_{x_k}$ $(k = 1, \dots, n)$, we conclude that

$$\|Du\|_{C^{0,\alpha}(U)} \leq C\|f\|_{C^{0,1}(U)} . \tag{12}$$

Let us summarize:

Theorem. *If u is a smooth solution of the uniformly elliptic, fully nonlinear PDE (7), we have the estimate*

$$\|u\|_{C^{1,\alpha}(U)} \leq C\|f\|_{C^{0,1}(U)} \tag{13}$$

for constants $C > 0$, $0 < \alpha < 1$, depending only on n, θ, Θ, ∂U, etc.

Remark. See [C-C, Chapters 5,9] for a proof of estimate (13) (and improvements) if u is only a viscosity solution of (7). □

C. $C^{2,\alpha}$ estimates

It is unknown in dimensions $n \geq 3$ whether the (unique) viscosity solution of

$$\text{(1)} \qquad \begin{cases} F(D^2u) = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

belongs to $C^{2,\alpha}_{\text{loc}}$ or $C^{1,1}_{\text{loc}}$. This is a major unsolved problem.

To continue further we will therefore need to assume, in addition to uniform ellipticity, that

$$\text{(2)} \qquad F \quad \text{is convex.}$$

Many, if not most, of the important examples of fully nonlinear elliptic PDE satisfy this condition. We will see that the "one-sided control" on D^2F (due to convexity) will give us "two-sided control" on D^2u.

1. Sup-norm bounds on D^2u.

Theorem. *There exists a constant C such that*

$$\text{(3)} \qquad \|D^2u\|_{L^\infty(U)} \leq C\|f\|_{C^{0,1}(U)}$$

OUTLINE OF PROOF. 1. Assume for simplicity

$$\begin{cases} F(D^2u) = f & \text{in } B^+(0,1) \\ u = 0 & \text{on } \partial B^+(0,1) \cap \{x_n = 0\} \end{cases}$$

Let $k \in \{1, \dots, n-1\}$ and differentiate with respect to x_k:

$$F_{r_{ij}}(D^2u)u_{x_k x_i x_j} = f_{x_k} \quad \text{in} \quad B^+(0,1)$$

Let

$$\text{(4)} \qquad \begin{cases} \tilde{u} = u_{x_k}\,, \quad \tilde{f} = f_{x_k}\,, \\ \tilde{a}_{ij} = -F_{r_{ij}}(D^2u)\,. \end{cases}$$

Then

$$\text{(5)} \qquad \begin{cases} -\tilde{a}_{ij}\tilde{u}_{x_i x_j} = \tilde{f} & \text{in } B^+(0,1) \\ \tilde{u} = 0 & \text{on } \partial B^+(0,1) \cap \{x_n = 0\} \end{cases}$$

Then a local version of Estimates 1,2 provide us with the bound

$$\|u_{x_k x_n}\|_{L^\infty(\partial B^+(0,\frac{1}{2}) \cap \{x_n = 0\})} \leq C(\|Du\|_{L^\infty} + \|f\|_{C^{0,1}})$$

for $k = 1, \dots n-1$. We next employ the PDE $F(D^2u) = f$ to obtain a bound on $u_{x_n x_n}|_{x_n=0}$. Indeed the uniform ellipticity implies

$$\Theta \geq -\frac{\partial F}{\partial r_{nn}}(D^2u) \geq \theta$$

and so we can solve for $u_{x_n x_n}$ in terms of f, $u_{x_k x_l}$ $(k = 1, \dots, n-1,\ l = 1, \dots, n)$.

Locally straightening ∂U to reduce to the case above gives us

$$\begin{aligned} \|D^2u\|_{L^\infty(\partial U)} &\leq C(\|u\|_{C^{0,1}(U)} + \|f\|_{C^{0,1}}) \\ &\leq C\|f\|_{C^{0,1}} \end{aligned} \tag{6}$$

So far we have not used the convexity of F.

2. We use the maximum principle trick to extend (6) to an estimate of D^2u inside U. Let ξ denote some unit vector in $\mathbb{R}^n$ and differentiate the PDE (1) twice in the direction ξ. We obtain

$$F_{r_{ij}}(D^2u)u_{\xi\xi x_i x_j} + F_{ij,kl}(D^2u)u_{x_i x_j \xi}u_{x_k x_l \xi} = f_{\xi\xi}$$

for $u_{\xi\xi} = u_{x_i x_j}\xi_i\xi_j$. Since F is convex, the matrix D^2F is nonnegative. Hence

$$-\hat{a}_{ij}\hat{u}_{x_i x_j} \leq \hat{f} \tag{7}$$

where

$$\begin{cases} \hat{u} = u_{\xi\xi}\ , \quad \hat{f} = f_{\xi\xi} \\ \hat{a}_{ij} = -F_{r_{ij}}(D^2u)\ . \end{cases}$$

Thus $\hat{u}$ is a ***sub***solution of a linear elliptic PDE, and consequently (cf. Estimate 1):

$$\sup_U \hat{u} \leq \max_{\partial U} \hat{u} + C\|\hat{f}\|_{L^\infty}\ .$$

Consequently (6),(8) imply

$$\sup_U u_{\xi\xi} \leq C\|f\|_{C^{1,1}(U)}\ .$$

This is a one-sided upper estimate for $u_{\xi\xi}$ in any direction ξ.

We next employ as follows the PDE $F(D^2u) = f$ to obtain a two-sided estimate. Since F is convex and $F(0) = 0$,

$$F(R) \geq DF(0) : R\ .$$

Thus

$$F_{r_{ij}}(0)u_{x_i x_j} \leq f \quad \text{in} \quad U\ .$$

Rotating and stretching coordinates we deduce

$$-\Delta u \le C\|f\|_{L^\infty} ,$$

and so

$$\begin{aligned} u_{x_1x_1} &\ge -\sum_{i=2}^{n} u_{x_ix_i} - C\|f\|_{L^\infty} \\ &\ge -C\|f\|_{C^{1,1}} . \end{aligned}$$

This inequality is valid with any $u_{\xi\xi}$ replacing $u_{x_1x_1}$:

$$\inf_U u_{\xi\xi} \ge -C\|f\|_{C^{1,1}}$$

□

2. $C^{2,\alpha}$ estimates. We next modify the foregoing calculations to obtain a far more subtle bound on the Hölder modulus of continuity of D^2u.

Theorem. *There exist constants* $C > 0,\ 0 < \alpha < 1$ *such that*

(9) $$\|u\|_{C^{2,\alpha}(\bar U)} \le C\|f\|_{C^{1,1}(U)}$$

OUTLINE OF PROOF. 1. Again for simplicity assume

$$\begin{cases} F(D^2u) = f & \text{in } \ B^+(0,1) \\ u = 0 & \text{on } \ \partial B^+(0,1) \cap \{x_n = 0\} \end{cases}$$

Take $k \in \{1, \dots, n-1\}$ and differentiate with respect to x_k to obtain (4),(5).

By Estimate 6 (from Lecture 2) we deduce

$$\|u_{x_kx_n}\|_{C^{0,\alpha}(\partial B^+(0,\frac{1}{2})\cap\{x_n=0\})} \le C\|f\|_{C^{0,1}(U)} ,$$

for $k = 1, \dots, n-1$. We use the PDE $F(D^2u) = f$ to obtain control on $u_{x_nx_n}|_{x_n=0}$.

Locally straightening ∂U to reduce to the case above gives us

(8) $$\|D^2u\|_{C^{0,\alpha}(\partial U)} \le C\|f\|_{C^{0,1}(U)} .$$

This bond does not require the convexity of F.

2. We must extend (8) to give a bound on the Hölder modulus of D^2u within U. I will sketch next a derivation of interior Hölder estimates on D^2u. These can be combined with the boundary estimate (8) to give global bounds on $\|D^2u\|_{C^{0,\alpha}}$.

The idea is this: Using (7) we deduce from variants of the Krylov-Safonov estimates "one-sided" control on the oscillation of $\hat{u} = u_{\xi\xi}$ on balls. Then the PDE $F(D^2u) = f$ itself, which gives a functional relation among the various second derivatives, allows us to obtain "two-sided" control on the oscillation.

I will present from [C-C] a recent proof due to Caffarelli. We first record a technical fact from the linear theory:

$$\text{(9)} \qquad \begin{cases} \text{If } \; -\hat{a}_{ij}v_{x_ix_j} \geq 0 \; \text{ in } \; B(0,1) \text{ and } \; v \geq 0, \; \text{ then} \\ \inf_{B(0,\frac{1}{2})} v \geq C|\{v \geq 1\} \cap B(0,\frac{1}{2})|^{\delta} \end{cases}$$

for constants C, δ.

3. From now on we assume $B(0,1) \subset U$ and for simplicity assume as well $f \equiv 0$ on $B(0,1)$. Let us suppose

$$1 \leq \text{diam } D^2u(B(0,1)) \leq 2$$

and that $D^2u(B(0,1)) \subset \mathbb{S}^n$ is covered by N balls $\tilde{B}_1, \tilde{B}_2, \tilde{B}_N$ of radius $\varepsilon > 0$. (Notation: $\tilde{B}$ = ball in $\mathbb{S}^n$.)

We ***claim*** that if $\varepsilon > 0$ is small enough, then

$$\text{(10)} \qquad \begin{cases} D^2u(B(0,\frac{1}{2})) \quad \text{can be covered by} \\ N-1 \quad \text{of the balls} \quad \tilde{B}_1, \ldots, \tilde{B}_N \; . \end{cases}$$

To substantiate this claim, for each $i \in \{1, \ldots, N\}$ we first select $x_i \in B(0,1)$ so that

$$\tilde{B}_i \subset \tilde{B}(R_i, 2\varepsilon)$$

where $R_i = D^2u(x_i)$. Now let $\lambda > 0$ be a constant (to be selected later). Take ε so small that $2\varepsilon \leq \frac{\lambda}{2}$. Then

$$\text{(11)} \qquad \text{the balls} \;\; \{\tilde{B}(R_i, \frac{\lambda}{2})\}_{i=1}^N \;\; \text{cover} \;\; D^2u(B(0,1)) \subset \mathbb{S}^n \; ,$$

and so

$$\text{(12)} \qquad \text{the sets} \;\; \{(D^2u)^{-1}(\tilde{B}(R_i, \frac{\lambda}{2}))\}_{i=1}^N \;\; \text{cover} \;\; B(0,1) \subset \mathbb{R}^n \; .$$

We may assume that N is bounded above by a fixed constant (depending only on λ). Thus at least one of the sets in (12) intersects $B(0,\frac{1}{2})$ on a set of measure bounded away from zero. So without loss we may assume

$$\text{(13)} \qquad |(D^2u)^{-1}(\tilde{B}(R_1, \frac{\lambda}{2}) \cap B(0,\frac{1}{2})| \geq \eta$$

for some universal constant $\eta > 0$.

4. Since diam $D^2u(B(0,1)) \geq 1$, we may assume that

$$\|R_1 - R_2\| \geq \tfrac{1}{2}\ .$$

Now $F(R_1) = F(D^2u(x_1)) = 0$, $F(R_2) = F(D^2u(x_2)) = 0$. This implies — owing to the uniform ellipticity of F — that there exists a unit vector ξ such that

$$u_{\xi\xi}(x_2) \geq u_{\xi\xi}(x_1) + \lambda$$

for some positive constant λ depending only on θ, Θ (see [C-C]). We take this to be the constant λ in step 3. Hence

$$M = \sup_{B(0,1)} u_{\xi\xi} \geq u_{\xi\xi}(x_1) + \lambda\ . \tag{14}$$

Now if $x \in (D^2u)^{-1}(\tilde{B}(R_1, \frac{\lambda}{2}))$, then (14) implies

$$u_{\xi\xi}(x) \leq u_{\xi\xi}(x_1) + \frac{\lambda}{2} \leq M - \lambda + \frac{\lambda}{2} = M - \frac{\lambda}{2}\ . \tag{15}$$

Set

$$v = M - u_{\xi\xi}\ . \tag{16}$$

Then (13),(15) imply

$$|\{v \geq \frac{\lambda}{2}\} \cap B(0, \tfrac{1}{2})| \geq \eta\ . \tag{17}$$

Next recall from (7) that $-\hat{a}_{ij}\hat{u}_{x_ix_j} \leq 0$ in $B(0,1)$ for $\hat{u} = u_{\xi\xi}$, $\hat{f} \equiv 0$. Hence $-\hat{a}_{ij}v_{x_ix_j} \geq 0$, $v \geq 0$ in $B(0,1)$. Therefore (9) and (17) imply

$$\inf_{B(0,\frac{1}{2})} v \geq \sigma\ ,$$

σ an absolute constant. Hence

$$\max_{B(0,\frac{1}{2})} u_{\xi\xi} \leq M - \sigma\ . \tag{18}$$

5. Since the balls $\{\tilde{B}(R_i, 2\varepsilon)\}$ cover $D^2u(B(0,1))$, there exists $j \in \{1, \ldots, N\}$ such that

$$M - u_{\xi\xi}(x_j) \leq 3\varepsilon\ . \tag{19}$$

Take ε so small that $6\varepsilon < \sigma$. Then (18),(19) imply

(20) $$D^2u(B(0,\tfrac{1}{2})) \cap \tilde{B}(R_j, 2\varepsilon) = \emptyset \ .$$

Indeed if $x \in B(0,\frac{1}{2})$ and $D^2u(x) \in \tilde{B}(R_j, 2\varepsilon)$, then

$$|u_{\xi\xi}(x) - u_{\xi\xi}(x_j)| \leq 2\varepsilon \ .$$

But (19) implies

$$u_{\xi\xi}(x) \geq M - 5\varepsilon > M - \sigma \ ,$$

in contradiction to (18).

This verifies (20) and thus proves claim (10).

6. Next we show that there exists $\delta > 0$ such that

(20) $$\begin{cases} \text{diam } D^2u(B(0,1)) = 2 \\ \text{implies} \\ \text{diam } D^2u(B(0,\delta)) \leq 1 \end{cases}$$

To see this let us cover $D^2u(B(0,1))$ with N balls of radius ε, ε as above. Then

$$D^2u(B(0,\tfrac{1}{2}))$$

is covered by $N-1$ balls. Thus

$$D^2u(B(0,\tfrac{1}{4}))$$

is covered with $N-2$ balls and so

$$D^2u(B(0,\frac{1}{2^k}))$$

is covered with $N-k$ balls. Hence for some k

$$D^2u(B(0,\frac{1}{2^k})) \leq 1 \ .$$

Interior Hölder continuity for D^2u follows upon our iterating estimate (20). □

This proof is technically simpler than the original proofs of $C^{2,\alpha}$ estimates (done independently by Krylov and myself). I strongly suggest again that interested readers consult the book of Cabre-Caffarelli [C-C].

Part II
Mean Curvature Motion

Lecture D: Introduction, distance function

The next three lectures concern a completely different topic, the flow of hypersurfaces in $\mathbb{R}^n$ by mean curvature. We will first discuss the local existence of a classical solution of such an evolution, following the method of [E-S II]. This requires analysis of a fully nonlinear parabolic PDE (satisfied by the signed distance function) and so provides a kind of bridge with the mathematics discussed in Part I.

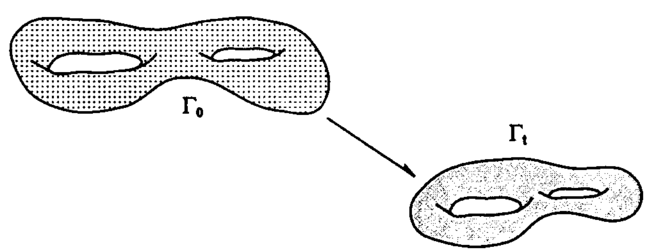
These are meant to be elementary, expository lectures, which serve as an introduction to Souganidis' more advanced talks.

1. Mean curvature motion

Let Γ_0 denote a smooth, $(n-1)$-dimensional hypersurface lying in $\mathbb{R}^n$. We propose to evolve Γ_0 in time, generating thereby a family

$$\{\Gamma_t\}_{t\geq 0}$$

of hypersurfaces, by moving the surface Γ_t at time t so that its normal velocity is the mean curvature vector. This is in fact the "geometric heat equation".



Geometric Notation.

$$\begin{aligned} \mathrm{v} &= \text{ normal velocity} \\ \boldsymbol{\nu} &= \text{ unit normal vector field to } \Gamma_t \\ \mathbf{H} &= \text{ mean curvature vector } = H\boldsymbol{\nu} \\ H &= \text{ mean curvature, computed with respect to } \boldsymbol{\nu} \end{aligned}$$

From differential geometry we recall the formula

$$H = -\operatorname{div}(\boldsymbol{\nu}) \,, \tag{1}$$

where $\boldsymbol{\nu}$ is extended arbitrarily off Γ_t and "div" is the usual divergence in $\mathbb{R}^n$.

Hence our geometric law of motion is

$$\mathrm{v} = H\boldsymbol{\nu} = -\operatorname{div}(\boldsymbol{\nu})\boldsymbol{\nu} \tag{2}$$

2. The distance function

We now reinterpret (2) in terms of the signed distance function to Γ_t. Let us therefore for the moment assume that the geometric motion problem described in §1 has a smooth solution, existing for times $0 \le t < t_\star$. We will assume that

$$\Gamma_0 = \partial U_0 \,,$$

U_0 denoting a bounded open set, and further that

$$\Gamma_t = \partial U_t \qquad 0 \le t < t_\star$$

U_t denoting a bounded open set at time $t > 0$.

Definition. For each time $t \ge 0$, define

$$d(x,t) = \begin{cases} \operatorname{dist}(x,\Gamma_t) & x \in \mathbb{R}^n - \bar{U}_t \\ -\operatorname{dist}(x,\Gamma_t) & x \in U_t \end{cases} \tag{3}$$

Then d is the ***signed distance*** function to $\{\Gamma_t\}_{0 \le t < t_\star}$.

Note that even though we are assuming the surfaces $\{\Gamma_t\}_{0 \le t < t_\star}$ are smooth, the mapping $x \mapsto d(x,t)$ need not be smooth. However, d is smooth in some sufficiently small region around the $\{\Gamma_t\}$, to which we hereafter restrict our attention.

Since d is determined by Γ_t, d "contains geometric information about Γ_t," which we now extract by differentiating d.

If x is sufficiently close to Γ_t and $d(x,t) > 0$, there exists a unique point $y \in \Gamma_t$ for which

$$d(x,t) = |x - y| \,.$$

Then

$$\boldsymbol{\nu} = Dd(x,t) \tag{4}$$

is the outer unit normal to Γ_t at y.

Information about the curvature of Γ_t is contained in D^2d.

Notation. (a) The eigenvalues of D^2d will be written as $\lambda_1, \ldots, \lambda_n$.
(b) The ***principal curvatures*** of Γ_t at y will be denoted $\kappa_1, \ldots, \kappa_{n-1}$.

In particular, $H = \kappa_1 + \cdots + \kappa_{n-1}$. As in [G-T, §14.6], we have after a possible reordering

$$\lambda_i = \frac{-\kappa_i}{1-\kappa_i d} \qquad (1 \le i \le n-1)\ , \quad \lambda_n = 0\ . \tag{5}$$

Inverting this relation we obtain

$$\kappa_i = \frac{\lambda_i}{\lambda_i d - 1} \qquad (1 \le i \le n-1)\ .$$

3. A PDE satisfied by the distance function

We propose next to convert the geometric law of motion (2) into a PDE satisfied by d. Now for a fixed point x such that $d(x,t) > 0$ we have

$$\begin{aligned} d_t &= -\mathbf{v}\cdot\boldsymbol{\nu} = -H \\ &= -\sum_{i=1}^{n-1} \kappa_i \\ &= \sum_{i=1}^{n-1} \frac{\lambda_i}{1-\lambda_i d} \end{aligned}$$

The same formula is valid if $d(x,t) \le 0$. Consequently,

$$d_t = F(D^2 d, d) \qquad \text{near } \Gamma\ , \tag{7}$$

where

$$\begin{aligned} F(R,z) &= f(\boldsymbol{\lambda}, z) \\ &= \sum_{i=1}^{n} \frac{\lambda_i}{1-\lambda_i z} \end{aligned} \tag{8}$$

and $\boldsymbol{\lambda} = \boldsymbol{\lambda}(R) = (\lambda_1, \ldots, \lambda_n)$ are the eigenvalues of R, ordered so that $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$. We will always stay so close to Γ_t that $\lambda_i d < 1$ $(i = 1, \ldots, n)$.

Remark. In particular, consider the region $\{d > 0\}$. Then if $\lambda_i \geq 0$, we have $\frac{\lambda_i}{1-\lambda_i d} \geq \lambda_i$. Furthermore if $\lambda_i \leq 0$, then $\frac{\lambda_i}{1-\lambda_i d} \geq \lambda_i$ as well. Thus

$$0 = d_t - \sum_{i=1}^{n} \frac{\lambda_i}{1-\lambda_i d} \leq d_t - \sum_{i=1}^{n} \lambda_i = d_t - \Delta d \ .$$

Hence

$$d_t - \Delta d \geq 0 \quad \text{in} \quad \{d \geq 0\} \ ; \tag{9}$$

and likewise,

$$d_t - \Delta d \leq 0 \quad \text{in} \quad \{d \leq 0\} \ . \tag{10}$$

We deduce in particular

$$d_t = \Delta d \quad \text{on} \quad \Gamma_t \ .$$

These observations, generalized to hold in the viscosity sense on all of $\mathbb{R}^n \times (0,\infty)$, are important in the study of moving fronts for reaction-diffusion equations. (See the lectures of Souganidis.) □

Let us next examine the nonlinearity (8). Note

$$\frac{\partial f}{\partial \lambda_i}(\boldsymbol{\lambda}, z) = \frac{1}{(1-\lambda_i z)^2} > 0 \qquad (1 \leq i \leq n) \ .$$

Thus if $z \in \mathbb{R}$ is fixed and $A, B \in \mathbb{S}^n$ with $A \geq B$, we have

$$F(A,z) = f(\boldsymbol{\lambda}(A), z) \geq f(\boldsymbol{\lambda}(B), z) = F(B,z) \ .$$

Here we used the fact that $A \geq B$ implies

$$\lambda_i(A) \geq \lambda_i(B) \qquad \text{for} \ \ i = 1, \ldots, n \ .$$

This follows from Courant's min-max representation formula for eigenvalues.

Consequently the fully nonlinear operator $-F(D^2u, u)$ is elliptic. In particular then the ***PDE*** (7) ***is a fully nonlinear parabolic PDE for*** d.

4. Local existence of smooth solutions for mean-curvature flow

We next show that we can reverse the reasoning above and solve the PDE (7) in order to build a smooth solution $\{\Gamma_t\}_{0\le t\le t_0}$ of the mean curvature flow, provided $t_0 > 0$ is sufficiently small.

Notation.

(a) $g(x) = \begin{cases} \operatorname{dist}(x,\Gamma_0) & \text{if } x \in \mathbb{R}^n - \bar{U}_0 \\ -\operatorname{dist}(x,\Gamma_0) & \text{if } x \in U_0 \end{cases}$

(b) $V = \{x \in \mathbb{R}^n \mid -\delta_0 < g(x) < \delta_0\}$

(c) $W = V \times (0,t_0)$, $\Sigma = \partial V \times [0,t_0]$.

We choose δ_0 so small that g is smooth within $\overline{V}$. We then consider the initial/boundary value problem

$$\begin{cases} v_t = F(D^2 v, v) & \text{in } W \\ |Dv|^2 = 1 & \text{on } \Sigma \\ v = g & \text{on } V \times \{t=0\} \end{cases} \tag{12}$$

The point is that *if* $\{\Gamma_t\}_{0<t\le t_0}$ is a classical mean curvature motion, then $v = d$ solves (12). On the other hand, [E-S II] shows that there exists a solution v of (12) if t_0 is sufficiently small, with

$$v \in C^{2+\alpha,\frac{2+\alpha}{2}}(\overline{W}) \cap C^\infty(W) .$$

The proof proceeds by looking for v in the form

$$v = g + th + w , \qquad \text{where } \; h = F(D^2 g, g) .$$

Then w satisfies a nonlinear PDE (with nonlinear boundary conditions), which is "close to linear" for small w. A contraction mapping argument yields the existence of w: see [E-S II].

It remains to show that $v(\cdot\,,t)$ is actually the signed distance from a surface Γ_t, which in turn is evolving by mean curvature flow. We require this

Lemma. *The function*

$$w = |Dv|^2 - 1$$

is identically equal to zero in $\overline{W}$.

PROOF. Clearly $w = 0$ on the parabolic boundary of W, owing to the boundary condition in (12). Let us also compute

$$v_{tx_k} = F_{r_{ij}}(D^2 v, v) v_{x_k x_i x_j} + F_z(D^2 v, v) v_{x_k} .$$

Hence

$$
\begin{aligned}
w_t &= 2v_{x_k} v_{x_k t} \\
&= 2F_{r_{ij}} v_{x_k} v_{x_k x_i x_j} + 2F_z |Dv|^2 \\
&= F_{r_{ij}} w_{x_i x_j} - 2F_{r_{ij}} v_{x_k x_i} v_{x_k x_j} + 2F_z |Dv|^2 \; .
\end{aligned} \tag{13}
$$

Now $F(R, z) = f(\boldsymbol{\lambda}, z)$, $\boldsymbol{\lambda} = \boldsymbol{\lambda}(R)$, and so

$$
F_z(D^2 v, v) = f_z(\boldsymbol{\lambda}(D^2 v), v) = \sum_{i=1}^{n} \frac{\lambda_i^2}{(1 - \lambda_i v)^2} \; .
$$

In addition,

$$
\begin{aligned}
F_{r_{ij}}(D^2 v, v) v_{x_k x_i} v_{x_k x_j} &= \sum_{i=1}^{n} \frac{\partial f}{\partial \lambda_i} (\boldsymbol{\lambda}, v) \lambda_i^2 \\
&= \sum_{i=1}^{n} \frac{\lambda_i^2}{(1 - \lambda_i v)^2} = F_z(D^2 v, v) \; .
\end{aligned}
$$

(We prove the second equality by rotating coordinates to diagonalizable $D^2 v$.) Then (13) implies

$$
w_t = F_{r_{ij}} w_{x_i x_j} + 2F_z w \quad \text{in} \quad W \; . \tag{14}
$$

As $w = 0$ on the parabolic boundary of W, (14) and uniqueness imply $w \equiv 0$ in W. □

Next ***define***

$$
\Gamma_t = \{x \in V \mid v(x, t) = 0\} \qquad (0 < t \leq t_0)
$$

Then the Lemma implies $v(\cdot, t)$ is the signed distance function to Γ_t, and the PDE (12) implies the surfaces $\{\Gamma_t\}_{0 < t \leq t_0}$ move by curvature flow.

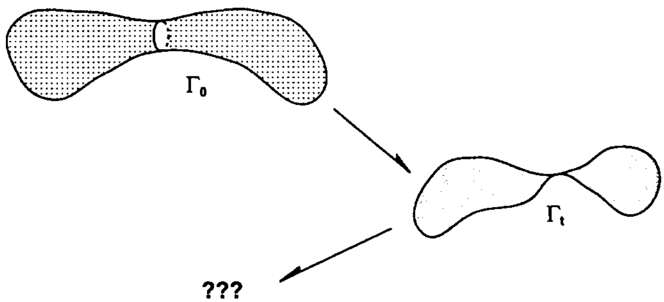
E. The level set method

The foregoing lecture provides us with a smooth solution of the mean curvature flow problem, at least for small times $0 \leq t \leq t_0$. Such a smooth flow definitely will cease to exist after a finite time however. If, for instance $\Gamma_0 = \partial B(0, R)$, a sphere of radius R, then

$$
\Gamma_t = B(0, R(t)) \quad \text{if} \quad 0 \leq t < t^* \; ,
$$

where $R(t) = (R^2 - 2(n-1)t)^{\frac{1}{2}}$, $t^* = R^2/2(n-1)$. The sphere consequently shrinks to the point 0 at time t^*, and so $\Gamma_{t^*} = \{0\}$ is not a smooth surface.

Even worse, a smooth flow $\{\Gamma_t\}_{0\leq t<t_*}$ existing on some interval $[0,t_*)$ can cease to be smooth in the limit $t\to t_*$, even if the Γ_t are not collapsing to a point. See for example the illustration



A fascinating question thus arises:

> How is it possible to define a "generalized" flow $\{\Gamma_t\}_{t>0}$ which extends the classical flow $\{\Gamma_t\}_{0<t<t_0}$ past the first time t_* of singularity?

The level set method, introduced by Osher-Sethian for numerical purposes, provides an answer.

1. Motivation for viscosity solution approach. The theory of viscosity solutions is designed to resolve a similar problem for certain PDE (e.g. Hamilton-Jacobi equations), for which a classical solution exists only during a short initial time interval, after which singularities arise. We can expect that viscosity solution methods apply to those PDE for which there is — formally at least — a maximum principle or, more precisely, a comparison principle for solutions.

To see that there is some hope of applying such ideas in the geometric setting, consider the following heuristics. Let $\{\Gamma_t\}_{0\leq t\leq t_*}$, $\{\Delta_t\}_{0\leq t\leq t_*}$ be two smooth mean curvature flows, with

$$\begin{cases} \Delta_0 \text{ lying within the bounded region } U_0 \\ \Gamma_0 = \partial U_0 \end{cases}$$

Then Δ_t lies within U_t, where $\Gamma_t = \partial U_t$ for $0\leq t\leq t_0$. In other words, ***the flows*** $\{\Gamma_t\}_{0\leq t\leq t_0}$, $\{\Delta_t\}_{0\leq t\leq t_0}$ ***preserve the initial "ordering".*** This is a kind of analogue of the comparison principle for PDE. To see informally why Δ_t lies within U_t, suppose instead that there exists a first time $s\in[0,t]$ for which Δ_s intersects Γ_s at a point y. We then expect that the mean curvature of Δ_t at y is greater than the mean curvature of Γ_t at y. Consequently the flow will tend to push Δ_s back inside U_s. A careful proof uses the strong parabolic maximum principle.

2. Level set method. To gain further insight, recall from Lecture D that the (smooth) evolving surfaces $\{\Gamma_t\}_{0\leq t\leq t_0}$ are the zero level sets of the signed distance function:

$$\Gamma_t = \{x \in \mathbb{R}^n \mid d(x,t) = 0\} \qquad (0 \leq t \leq t_0)$$

The key to the so-called ***level set method*** for studying generalized mean curvature flow is to consider instead of d a new function u with the property that — formally at least — ***each level set of*** u ***evolves via mean curvature flow.*** In other words, we seek a function u such that for each $\gamma \in \mathbb{R}$ the level sets

$$\Gamma_t^\gamma = \{x \in \mathbb{R}^n \mid u(x,t) = \gamma\} \tag{1}$$

flow according to their mean curvature.

Following the general principle that we should always seek an "infinitesimal" version of any concept, let us now ask what PDE such a function u will satisfy. Let us assume for the moment u is smooth and $|Du| \neq 0$ in some region. Then for each $\gamma \in \mathbb{R}$, Γ_t^γ is a smooth hypersurface, with normal

$$\boldsymbol{\nu} = \frac{Du}{|Du|}\,. \tag{2}$$

The mean curvature of Γ_t^γ is then

$$H = -\mathrm{div}(\boldsymbol{\nu}) = -\mathrm{div}\left(\frac{Du}{|Du|}\right) \tag{3}$$

Now the normal velocity of the level surface Γ_t^γ in the direction $\boldsymbol{\nu}$ is

$$\mathbf{v} = -\frac{u_t}{|Du|}\,\boldsymbol{\nu}\,.$$

Thus our geometric law of motion

$$\mathbf{v} = H\boldsymbol{\nu}$$

becomes

$$-\frac{u_t}{|Du|} = H = -\mathrm{div}\left(\frac{Du}{|Du|}\right)\,,$$

and so

$$u_t = |Du|\mathrm{div}\left(\frac{Du}{|Du|}\right) = \left(I - \frac{Du \otimes Du}{|Du|^2}\right) : D^2u \tag{4}$$

along Γ_t^γ. Since the level surfaces fill up the region where $|Du| \neq 0$, we conclude the PDE (4) holds everywhere in this region.

In summary, if u is smooth, then

$$(5) \qquad u_t = \left(\delta_{ij} - \frac{u_{x_i} u_{x_j}}{|Du|^2}\right) u_{x_i x_j}$$

provided $|Du| \neq 0$.

The idea is now to study the PDE (5) and to use it to ***define*** a generalized mean curvature motion existing for all time $t \geq 0$. Here is the procedure:

Construction of generalized mean curvature flow

Step 1: Given a compact set $\Gamma_0 \subset \mathbb{R}^n$, find a continuous bounded function $g : \mathbb{R}^n \to \mathbb{R}$ such that

$$\Gamma_0 = \{x \mid g(x) = 0\}$$

Step 2: Solve the initial value problem

$$(6) \qquad \begin{cases} u_t = \left(\delta_{ij} - \frac{u_{x_i} u_{x_j}}{|Du|^2}\right) u_{x_i x_j} & \text{in } \mathbb{R}^n \times (0, \lambda) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases}$$

Step 3: Then define

$$(7) \qquad \Gamma_t = \{x \mid u(x,t) = 0\} \qquad (t \geq 0)$$

Assuming this all works we call

$$\{\Gamma_t\}_{t \geq 0}$$

the generalized mean curvature flow starting from Γ_0. This flow will exist for all times $t \geq 0$, although it may develop singularities (even if Γ_0 is a smooth hypersurface) and will certainly become the empty set after a finite time.

3. Justification of level set method

Considerable work is needed to justify the foregoing plan. The main issue is that we must show the PDE (6) has some kind of unique weak solution u, the uniqueness being necessary to justify the article "the" in "the generalized mean curvature flow". The overall problems are that the PDE (6) is degenerate parabolic and, worse, is undefined where $|Du| = 0$. And yet, since we definitely want to allow for the possibility that the level sets $\{\Gamma_t\}$ change topological type, the regions where $|Du| = 0$ are important.

The solution is to invoke viscosity solution methods to interpret the PDE (6), to solve this initial value problem, and to establish uniqueness ([E-S-I],[C-G-G]).

Following [E-S-I] we introduce the following

Definition. A bounded, continuous function u is a ***viscosity solution*** of (6) provided $u = g$ on $\mathbb{R}^n \times \{t = 0\}$, and for each $\phi \in C^\infty(\mathbb{R}^n \times (0,\infty))$ if $u - \phi$ has a local maximum (minimum) at a point $(x_0, t_0) \in \mathbb{R}^n \times (0,\infty)$, then

$$\phi_t \underset{(\geq)}{\leq} \left(\delta_{ij} - \frac{\phi_{x_i}\phi_{x_j}}{|D\phi|^2}\right)\phi_{x_i,x_j} \quad \text{at } (x_0,t_0) \quad \text{if} \quad D\phi(x_0,t_0) \neq 0 \; , \tag{8}$$

and

$$\phi_t \underset{(\geq)}{\leq} (\delta_{ij} - \eta_i\eta_j)\phi_{x_ix_j} \quad \text{at } (x_0,t_0) \quad \text{for some} \quad |\eta| \leq 1, \text{ if } \; D\phi(x_0,t_0) = 0 \; . \tag{9}$$

This definition is motivated by the proof of the following

Theorem 1. *There exists a viscosity solution of (6).*

PROOF. 1. We consider for $\varepsilon > 0$ the approximation

$$\begin{cases} u_t^\varepsilon = \left(\delta_{ij} - \frac{u^\varepsilon_{x_i}u^\varepsilon_{x_j}}{|Du^\varepsilon|^2+\varepsilon^2}\right)u^\varepsilon_{x_ix_j} & \text{in } \mathbb{R}^n \times (0,\infty) \\ u^\varepsilon = g & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases} \tag{10}$$

Problem (10) has a unique, smooth, bounded solution u^ε. We can differentiate the PDE (10) with respect to x_k ($k = 1,\dots,n$) and t, and apply the maximum principle to derive the bounds

$$\sup_{0<\varepsilon\leq 1} \|u^\varepsilon, Du^\varepsilon, u_t^\varepsilon\|_{L^\infty} \leq C \; ,$$

the constant C depending only on $\|g\|_{C^{1,1}}$. Consequently we have for some sequence $\varepsilon_j \to 0$:

$$u^{\varepsilon_j} \to u \quad \text{locally uniformly on } \; \mathbb{R}^n \times [0,\infty) \; ,$$

where u is bounded, Lipschitz.

2. We will now demonstrate that u is a viscosity solution of (6). So let $\phi \in C^\infty(\mathbb{R}^n \times (0,\infty))$ and suppose $u - \phi$ has a (strict) local maximum at (x_0, t_0). Then

$$u^{\varepsilon_j} - \phi \quad \text{has a local maximum at} \;\; (x_j, t_j) \tag{11}$$

with

$$(x_j, t_j) \to (x_0, t_0) \tag{12}$$

Owing to (10),(11),

$$\phi_t \leq \left(\delta_{ij} - \frac{\phi_{x_i}\phi_{x_j}}{|D\psi|^2+\varepsilon^2}\right)\phi_{x_ix_j} \quad \text{at } (x_j,t_j) \; ; \tag{13}$$

since (11) implies

$$D\phi = Du^\varepsilon \ , \quad \phi_t = u_t^{\varepsilon_j} \ , \quad D^2u^{\varepsilon_j} \leq D^2\phi \quad \text{at } (x_j,t_j) \ .$$

Now if $D\phi(x_0,t_0) \neq 0$, then our sending $x_j \to x_0$, $t_j \to t_0$ in (13) gives (8). If $D\phi(x_0,t_0) = 0$, we likewise derive (9) for

$$\eta = \lim_{j\to\infty} \frac{D\phi(x_j,t_j)}{(|D\phi(x_j,t_j)|^2+\varepsilon_j^2)^{\frac{1}{2}}} \ ,$$

where we pass to a subsequence for which this limit exists. □

The main assertion is then that

Theorem 2. *A viscosity solution of (6) is unique.*

The proof (cf [E-S I]) uses the technology described in Crandall's lectures, suitably modified to handle the case (9). The basic technical trick is, roughly speaking, to take two viscosity solutions, call them u and v, and maximize

$$u(x,t) - v(y,s) - \frac{1}{\varepsilon}[|x-y|^4 + (t-s)^2] \ .$$

The point is that the term "$|x-y|^4$", rather than "$|x-y|^2$" allows us to ignore the right-hand side of (9) should this case arise. See [E-S I] for details.

Theorems 1 and 2 assure us that the PDE (6) has a unique solution, but our definition (7) of Γ_t is still worrisome. The problem is this. Suppose we select a different function $\hat{g}:\mathbb{R}^n \to \mathbb{R}$ such that

$$\Gamma_0 = \{x \mid \hat{g}(x) = 0\} \ ,$$

and we solve

$$\begin{cases} \hat{u}_t = \left(\delta_{ij} - \frac{\hat{u}_{x_i}\hat{u}_{x_j}}{|D\hat{u}|^2}\right)\hat{u}_{x_ix_j} & \text{in } \mathbb{R}^n \times (0,\infty) \\ \hat{g} = \hat{y} & \text{on } \mathbb{R}^n \times (t=0) \ . \end{cases}$$

How do we know

$$\Gamma_t = \{x \mid u(x,t) = 0\} = \{x \mid \hat{u}(x,t) = 0\} \ \ ? \tag{14}$$

If this is false, then Γ_t is not well-defined by Steps 1–3 above.

It turns out that (14) is in fact true. The proof ([E-S I; Theorem 5.1]) depends upon

Theorem 3. *Let u be the unique viscosity solution of (6). Assume $\Phi : \mathbb{R} \to \mathbb{R}$ is continuous. Then*

$$\tilde{u} = \Phi(u) \tag{15}$$

is the unique viscosity solution of

$$\begin{cases} \tilde{u}_t = \left(\delta_{ij} - \frac{\tilde{u}_{x_i}\tilde{u}_{x_j}}{|D\tilde{u}|^2}\right)\tilde{u}_{x_i x_j} & \text{in } \mathbb{R}^n \times (0,\infty) \\ \tilde{u} = \tilde{g} & \text{on } \mathbb{R}^n \times \{t=0\}\ , \end{cases}$$

where $\tilde{g} = \Phi(g)$.

In other words an arbitrary nonlinear function of a solution to our mean curvature PDE is still a solution. This is because (15) amounts merely to a relabeling of the level sets, each of which moves via the mean curvature flow. A formal analytic proof is this:

$$\begin{aligned} \left(\delta_{ij} - \frac{\tilde{u}_{x_i}\tilde{u}_{x_j}}{|D\tilde{u}|^2}\right)\tilde{u}_{x_i x_j} &= \left(\delta_{ij} - \frac{u_{x_i}u_{x_j}}{|Du|^2}\right)[u_{x_i x_j}\Phi' + u_{x_i}u_{x_j}\Phi''] \\ &= \Phi' u_t \\ &= \tilde{u}_t\ . \end{aligned}$$

See [E-S I] for a proof in the viscosity sense, and for a proof of (14).

4. Consistency

Finally we must verify that the level set motion agrees with the classical evolution.

Theorem 4. *Assume $\Gamma_0 \subset \mathbb{R}^n$ is a smooth, embedded $(n-1)$-dimensional surface. Then the generalized flow $\{\Gamma_t\}_{t\geq 0}$ agrees with the unique classical flow starting from Γ_0, so long as the latter exists.*

Idea of proof. Let $\{\Sigma_t\}_{0\leq t<t_0}$ denote the smooth flow starting with Γ_0. Let $d(x,t)$ be the (signed) distance function to Σ_t. Then, as in Lecture D,

$$d_t = \sum_{i=1}^{n} \frac{\lambda_i}{1-\lambda_i d} \qquad (\lambda = \lambda(D^2 d) = \text{ eigenvalues of } D^2 d)\ .$$

Thus

$$\begin{aligned} d_t - \Delta d &= \sum_{i=1}^{n} \frac{\lambda_i}{1-\lambda_i d} - \lambda_i \\ &= \left(\sum_{i=1}^{n} \frac{\lambda_i^2}{1-\lambda_i d}\right) d \qquad \text{near } \Sigma_t\ . \end{aligned}$$

Consequently,

$$\underline{d} = \alpha\, e^{-\lambda t} d$$

satisfies

$$\underline{d}_t - \Delta \underline{d} \le 0 \quad \text{for appropriate } \alpha, \lambda$$

Since $|D\underline{d}|^2 = \alpha^2 e^{-2\lambda t}$, we have as well

$$\underline{d}_t - \left(\delta_{ij} - \frac{\underline{d}_{x_i}\underline{d}_{x_j}}{|D\underline{d}|^2}\right)\underline{d}_{x_i x_j} \le 0\ .$$

We now employ the comparison principle: see [E-S I]/ □

5. Questions

Having rigorously justified Steps 1–3 above, and so unambiguously defined the generalized level set flow, the real question is to study $\{\Gamma_t\}_{t\ge 0}$. We ask:

> Given various geometric properties of Γ_0, what can be said about the geometric properties of the flow $\Gamma_0 \mapsto \Gamma_t \ \ (t \ge 0)$?

F. Geometric properties

We discuss next some partial answers to the question posed at the end of the preceding lecture. We are especially interested in recovering from the level set method various classical differential geometric assertions.

1. Decrease of surface area

Hereafter let $\mathcal{H}^{n-1}$ denote $(n-1)$-dimensional Hausdorff measure. Let us temporarily assume $\{\Gamma_t\}_{t\ge 0}$ is a smooth flow of hypersurfaces moving by mean curvature flow. Then differential geometry tells us that

$$(1) \qquad \begin{aligned} \frac{d}{dt}\mathcal{H}^{n-1}(\Gamma_t) &= -\int_{\Gamma_t} \mathbf{v}\cdot\mathbf{H}\; d\mathcal{H}^{n-1} \\ &= -\int_{\Gamma_t} H^2 d\mathcal{H}^{n-1} \end{aligned}$$

In particular therefore, $t \mapsto \mathcal{H}^{n-1}(\Gamma_t)$ is nondecreasing.

How can we mimic this calculation in our level set approach? The point is that all the information must somehow be contained in the PDE

$$(2) \qquad u_t = |Du|\mathrm{div}\left(\frac{Du}{|Du|}\right) = \left(\delta_{ij} - \frac{u_{x_i}u_{x_j}}{|Du|^2}\right)u_{x_i x_j}\ .$$

What we need is a way to recover from the evolution equation (2) information about the evolution of the level sets of u. The technical tools that allow for this are:

(a) Theorem 3 from Lecture E, which asserts that $\tilde{u} = \Phi(u)$ solves (2) for each continuous Φ,

and

(b) the ***coarea formula***, which asserts

$$\int_{\mathbb{R}^n} f|Du|dx = \int_{-\infty}^{\infty} \int_{\{u=s\}} f \, d\mathcal{H}^{n-1} ds \tag{3}$$

for each Lipschitz $u : \mathbb{R}^n \to \mathbb{R}$ and continuous $f : \mathbb{R}^n \to \mathbb{R}$.

Suppose for the moment u is a smooth solution of (2). We now compute in analogy with (1) that

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^n} |Du|dx &= \int_{\mathbb{R}^n} \frac{Du}{|Du|} \cdot Du_t \, dx \\ &= - \int_{\mathbb{R}^n} \operatorname{div}\left(\frac{Du}{|Du|}\right) u_t \, dx^* \\ &= - \int_{\mathbb{R}^n} H^2 |Du| dx \end{aligned} \tag{4}$$

where

$$H = -\operatorname{div}\left(\frac{Du}{|Du|}\right) \qquad \text{(cf. (1) in Lecture D)} . \tag{5}$$

Integrating we find

$$\int_{\mathbb{R}^n} |Du(x,t_2)|dx \leq \int_{\mathbb{R}^n} |Du(x,t_1)|dx \tag{6}$$

if $0 \leq t_1 \leq t_2$. Now apply (6) with $\tilde{u} = \Phi(u)$ replacing u:

$$\int_{\mathbb{R}^n} |\Phi'|(u(x,t_2))|Du(x,t_2)|dx \leq \int_{\mathbb{R}^n} |\Phi'|(u(x,t_1))|Du(x,t_1)|dx . \tag{7}$$

By an approximation we see that (7) obtains for

$$\Phi(z) = \begin{cases} 0 & z \leq \gamma \\ \text{linear} & \gamma \leq z \leq \gamma + h \\ 1 & z \geq \gamma + h \end{cases}$$

where $\gamma \in \mathbb{R}$, $h > 0$. Substituting into (3), and using the coarea formula we find

$$\frac{1}{h} \int_{\gamma}^{\gamma+h} \mathcal{H}^{n-1}(\Gamma_{t_2}^s) ds \leq \frac{1}{h} \int_{\gamma}^{\gamma+h} \mathcal{H}^{n-1}(\Gamma_{t_1}^s) ds . \tag{8}$$

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[0]I believe (4) contains the only occurrence of integration-by-parts in this entire CIME course.

Then if we let $h \to 0^+$ we expect that

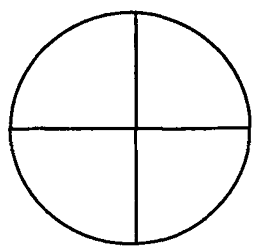
$$\mathcal{H}^{n-1}(\Gamma_{t_2}^{\gamma}) \leq \mathcal{H}^{n-1}(\Gamma_{t_1}^{\gamma}) \tag{9}$$

for each level γ at times $0 \leq t_1 \leq t_2$.

This computation suggests that each level set of u has decreasing surface area. This is of course consistent with the classical calculation (1) (applied to each $\{\Gamma_t^{\gamma}\}$). However we must be careful, as our weak solution of (2) is not smooth and the passage to limits from (8) to (9) need not be true for all levels γ and all times $0 \leq t_1 < t_2$.

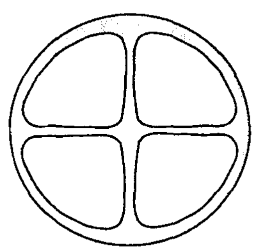
Example. Consider in fact the case that $n = 2$ and Γ_0 looks like this:

Γ_0



It turns out that ***under the generated mean curvature flow the*** $\{\Gamma_t\}$ ***develop interiors.***

Γ_t



Thus $\mathcal{H}^1(\Gamma_t) = +\infty$ for $t > 0$, whereas $\mathcal{H}^1(\Gamma_0) < \infty$. Consequently (9) is certainly false here. □

To recover some rigorous deduction from the heuristics above we turn to the approximations u^ε from Lecture E. Recall these solve

$$\begin{cases} u_t^\varepsilon = \left(\delta_{ij} - \frac{u_{x_i}^\varepsilon u_{x_j}^\varepsilon}{|Du^\varepsilon|^2}\right) u_{x_i x_j}^\varepsilon & \text{in } \mathbb{R}^n \times (0,\infty) \\ u^\varepsilon = g & \text{on } \mathbb{R}^n \times \{t=0\}\,. \end{cases} \tag{10}$$

Let

$$\phi(x) = \phi_\delta(x) = e^{-\delta(1+|x|^2)^{\frac{1}{2}}}$$

and compute

$$\frac{d}{dt}\int_{\mathbb{R}^n} \phi^2(|Du^\varepsilon|^2+\varepsilon^2)^{\frac{1}{2}}\ dx = \int_{\mathbb{R}^n} \phi^2 \frac{Du^\varepsilon}{(|Du^\varepsilon|^2+\varepsilon^2)^{\frac{1}{2}}} \cdot Du_t^\varepsilon\ dx$$

$$= -\int_{\mathbb{R}^n} \phi^2 \mathrm{div}\left(\frac{Du^\varepsilon}{(|Du^\varepsilon|^2+\varepsilon^2)^{\frac{1}{2}}}\right) u_t^\varepsilon\ dx$$

$$-2\int \phi \frac{D\phi\cdot Du^\varepsilon}{(|Du^\varepsilon|^2+\varepsilon^2)^{\frac{1}{2}}}\ u_t^\varepsilon\ dx$$

$$= -\int_{\mathbb{R}^n} \phi^2 (H^\varepsilon)^2(|Du^\varepsilon|^2+\varepsilon^2)^{\frac{1}{2}} dx$$

$$+2\int_{\mathbb{R}^n} \phi\ D\phi\cdot Du^\varepsilon H^\varepsilon\ dx\ ,$$

where

$$H^\varepsilon = -\mathrm{div}\left(\frac{Du^\varepsilon}{(|Du^\varepsilon|^2+\varepsilon^2)^{\frac{1}{2}}}\right)\ .$$

Consequently

$$\frac{d}{dt}\int_{\mathbb{R}^n} \phi^2(|Du^\varepsilon|^2+\varepsilon^2)^{\frac{1}{2}} dx \le C\int_{\mathbb{R}^n} |D\phi|^2(|Du^\varepsilon|^2+\varepsilon^2)^{\frac{1}{2}} dx\ .$$

Since $|D\phi| \le \delta\phi$, we may apply Gronwall's inequality to obtain

$$\int_{\mathbb{R}^n} \phi^2(|Du^\varepsilon(x,t)|^2+\varepsilon^2)^{\frac{1}{2}} dx \le e^{C\delta^2 t}\int_{\mathbb{R}^n} \phi^2(|Dg|^2+\varepsilon^2)^{\frac{1}{2}} dx\ .$$

Let $\varepsilon \to 0$ and then $\delta \to 0$ to deduce

$$\int_{\mathbb{R}^n} |Du(x,t)| dx \le \int_{\mathbb{R}^n} |Dg| dx$$

Then, as before,

$$\frac{1}{h}\int_\gamma^{\gamma+h} \mathcal{H}^{n-1}(\Gamma_t^s) ds \le \frac{1}{h}\int_\gamma^{\gamma+h} \mathcal{H}^{n-1}(\Gamma_0^s) ds$$

Now for a fixed t and a.e. $\gamma \in \mathbb{R}$, the left-hand side converges to $\mathcal{H}^{n-1}(\Gamma_t^\gamma)$ as $h \to 0$. Thus

$$\mathcal{H}^{n-1}(\Gamma_t^\gamma) \le \mathcal{H}^{n-1}(\Gamma_0^\gamma) \quad \text{for a.e. } \gamma \in \mathbb{R}\ .$$

A more subtle estimate shows for the particular level set $\Gamma_t = \Gamma_t^0$ that

$$\sup_{t>0} \mathcal{H}^{n-1}(\partial\Gamma_t) \le C\mathcal{H}^{n-1}(\Gamma_0)\ ,$$

provided Γ_0 is nice enough. See [E-S III] for more details.

2. Estimates of extinction time

We turn now to the question of estimating the time t^* of extinction for a flow $\{\Gamma_t\}_{0\le t<\infty}$.

If $\Gamma_0 \subset B(0,R)$ for some $R>0$, then $\Gamma_t \subset B(0,R(t))$ where $R(t) = (R^2 - 2(n-1)t)^{\frac{1}{2}}$. Since $R(t)=0$ for $t = R^2/2(n-1)$ we see that $t^* \le R^2/2(n-1)$. In particular we have the estimate

$$t^* \le C\mathrm{diam}(\Gamma_0)^2 \; . \tag{11}$$

This is quite a crude geometric estimate.

To do better let us proceed again by supposing $\{\Gamma_t\}_{0\le t<t^*}$ is a smooth flow. Then (1) says

$$\frac{d}{dt}\mathcal{H}^{n-1}(\Gamma_t) = -\int_{\Gamma_t} H^2 \, d\mathcal{H}^{n-1} \; . \tag{12}$$

Now a Sobolev-type inequality for manifolds states

$$\left(\int_{\Gamma_t} |f|^{\frac{n-1}{n-2}} d\mathcal{H}^{n-1}\right)^{\frac{n-2}{n-1}} \le C\int_{\Gamma_t} |Df| + |f||H| d\mathcal{H}^{n-1} \tag{13}$$

for any smooth function f with compact support. As Γ_t is bounded we can take $f\equiv 1$:

$$\begin{aligned}(\mathcal{H}^{n-1}(\Gamma_t))^{\frac{n-2}{n-1}} &\le C\int_{\Gamma_t} |H| d\mathcal{H}^{n-1} \\ &\le C\left(\int_{\Gamma_t} H^2 \, d\mathcal{H}^{n-1}\right)^{\frac{1}{2}} \mathcal{H}^{n-1}(\Gamma_t)^{\frac{1}{2}} \; .\end{aligned}$$

Thus

$$(\mathcal{H}^{n-1}(\Gamma_t))^{\frac{n-3}{n-1}} \le C\int_{\Gamma_t} H^2 \, d\mathcal{H}^{n-1} \; ,$$

and so (12) implies

$$\frac{d}{dt}\mathcal{H}^{n-1}(\Gamma_t) \le C\mathcal{H}^{n-1}(\Gamma_t)^{\frac{n-3}{n-1}} \; .$$

We integrate this differential inequality to deduce

$$(\mathcal{H}^{n-1}(\Gamma_t))^{\frac{2}{n-1}} - (\mathcal{H}(\Gamma_0))^{\frac{2}{n-1}} \le -Ct \; .$$

Letting $t \to t^*$ we conclude

$$t^* \le C\mathcal{H}^{n-1}(\Gamma_0)^{\frac{2}{n-1}} \; . \tag{14}$$

This estimates t^* in terms of the surface area of Γ_0.

To make a rigorous proof for the generalized flow we quote without proof a "clearing out" result proved in [E-S III]:

Theorem. *There exist constants $\alpha, \beta, \eta > 0$ such that if*

(15) $$\mathcal{H}^{n-1}(\Gamma_{t_0} \cap B(x_0, r)) \leq \eta r^{n-1}$$

for some time $t_0 \geq 0$ and some ball $B(x_0, r) \subset \mathbb{R}^n$, then

(16) $$\Gamma_t \cap B(x_0, \frac{r}{2}) = \emptyset \quad \textit{for} \quad \alpha r^2 \leq t - t_0 \leq \beta r^2 .$$

This result, which is based upon some fundamental observations of Brakke, states that if Γ_t has very little surface area within some ball at some time, then Γ_t does not intersect at all the ball with half the radius at certain later times. The proof of the theorem is a kind of "localized" version of the proof of (14), applied to the level surfaces of the approximations u^ε.

As an application, take $t_0 = 0$ and set

$$r = \left[\frac{\mathcal{H}^{n-1}(\Gamma_0)}{\eta}\right]^{\frac{1}{n-1}} .$$

Then

$$\mathcal{H}^{n-1}(\Gamma_0 \cap B(x_0, r)) \leq \mathcal{H}^{n-1}(\Gamma_0) = \eta r^{n-1}$$

for each point x_0. Owing to (16) we have

$$\Gamma_t \cap B(x_0, \frac{r}{2}) = \emptyset \quad \text{for} \quad t = \alpha r^2 .$$

As this holds for all x_0, we deduce

$$t^\star \leq \alpha r^2 = C\mathcal{H}^{n-1}(\Gamma_0)^{\frac{2}{n-1}} .$$

This is (14).

Remark. Giga and Yama-uchi [G-Y] have derived the extremely interesting lower bound

$$t^\star \geq 2|U_0|^2\mathcal{H}^{n-1}(\Gamma_0)^{-2}$$

where $\Gamma_0 = \partial U_0$. □

3. Geometric structure for a.e. level set

The computations in §1,2 above strongly suggest that

$$\nu = \frac{Du}{|Du|}$$

should be the unit normal vector field to Γ_t^γ and

$$H = \begin{cases} u_t/|Du| & \text{if } |Du| \neq 0 \\ 0 & \text{if } |Du| = 0 \end{cases}$$

should act like the mean curvature. Formally this is so, but as u is not smooth it is difficult to make rigorous computations. The paper [E-S IV] is devoted to showing that such interpretations are valid, at least for a "generic" level set Γ_t^γ.

There are many open questions concerning the regularity of u and its level sets, other geometric properties of the flow $\Gamma_0 \mapsto \Gamma_t$ $(t \geq 0)$, etc. It is also extremely important to understand how this notion of generalized mean curvature motion relates to other theories, especially those of Brakke and of De Giorgi.

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