VALUATION EQUATIONS FOR STOCHASTIC VOLATILITY MODELS

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ABSTRACT. We study the valuation partial differential equation for European contingent claims in a general framework of stochastic volatility models. The standard Feynman-Kac theorem cannot be directly applied because the diffusion coefficients may degenerate on the boundaries of the state space and grow faster than linearly. We allow for various types of model behavior; for example, the volatility process in our model can potentially reach zero and either stay there or instantaneously reflect, and asset-price processes may be strict local martingales under a given risk-neutral measure. Our main result is an extension of the standard Feynman-Kac theorem in the context of stochastic volatility models. Sharp results on the existence and uniqueness of classical solutions to the valuation equation are obtained using a combination of probabilistic and analytical techniques. The role of boundary conditions is also discussed.

0. INTRODUCTION

In financial practice, the widely-documented volatility smile (see e.g. [34]) is undeniable evidence that the Black-Scholes model is too simplistic to realistically capture the empirical features of option prices in the market. The need for more elaborate modeling is, therefore, unavoidable. Stochastic volatility models have become a very popular and attractive alternative to the Black-Scholes model, mostly due to their analytic tractability and their capability to capture stylized features of market-observed option prices; see e.g. [17] and references therein.

Unlike the Black-Scholes model, stochastic volatility models are incomplete. For the purpose of valuing contingent claims written on the underlying asset, one typically postulates a diffusion model for the asset price and its volatility, formulated under a risk-neutral measure that makes the asset-price process a (possibly local) martingale. Models are then calibrated to market data in a way so that model-specified values of liquid derivatives, such as European call option prices, match the observed trading prices.

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Due to the Markovian structure of stochastic volatility models, valuing a European contingent claim boils down to determining a value function, which is plainly the expectation (under the chosen risk-neutral measure) of the terminal payoff evaluated at the market's current configuration, including the current asset price, the level of the factor that drives the volatility, as well as the time-to-maturity. A way to determine this value function is by solving a partial differential equation (PDE), heuristically derived by formally applying Itô's formula and utilizing a martingale argument. We call this PDE the *valuation equation*. Given that the value function is the unique solution to the valuation equation in a certain class of functions, one typically approximates this value function by numerically solving the valuation equation.

Economic intuition suggests that the aforementioned strategy should work nicely. However, as was pointed out in [20], it is surprisingly tricky to rigorously prove the heuristic connection. To begin with, it is not easy to show that the value function is sufficiently smooth in order for Itô's formula to be applicable and ensure that the value function indeed solves the valuation equation. On the other hand, valuation equations in stochastic volatility models are typically degenerate on the boundaries of state space. Therefore, the assumptions that are present in standard versions of the Feynman-Kac formula (see e.g. Chapter 6 of [19]) are not satisfied for many stochastic volatility models used in practice.

A further source of difficulty is that the asset-price process in many stochastic volatility models is a strict local martingale; see [1], [36], [31], [23], and [30]. (The loss of the martingale property relates to the notion of stock price *bubbles*; see [22], [6], [27] and [28]. Similar situations have also been studied in markets without local martingale measures; see [15], [14], and [35].) An important consequence of losing the martingale property, mentioned in [22], is that the valuation equation may have multiple solutions. An issue of considerable theoretical, as well as practical, importance is to understand how the value function can be chosen amongst the many solutions of the valuation equation.

In this paper, and in the framework of general stochastic volatility models, we focus on the following questions:

(Q1) How should one formulate the concept of a solution of the valuation equation (regarding smoothness and boundary conditions) in order to ensure that the value function is one such solution?

(Q2) Given that (Q1) has been answered, what is a natural condition under which the value function is the unique solution in a certain class of candidate functions?

(Q3) Given that existence in (Q1) holds, but uniqueness in (Q2) fails, how could one identify the value function among all possible solutions?

The above questions have been tackled for the case of local volatility models in [26], [10], and [4]; for the case of interest rate models in [12]. For stochastic volatility models, these questions have been discussed in [20], [11], and [13]. However, it is our belief that a more scrutinized study is needed for general stochastic volatility models. Necessary and sufficient conditions for the existence of a unique classical solution still needs to be determined, which we set out to do in this paper. Valuation equations such as the ones discussed here are still nonstandard in the PDE literature, even though equations with degenerating coefficients have been studied extensively. To the best of our knowledge, only [7] and [8] studied equations which can be considered as special cases of valuation equations in Heston models.

The contributions of the present paper as compared to previous literature are multiple:

- The stochastic volatility models we study come with minimal assumptions on the coefficients, which degenerate on boundaries of the state space and allow the volatility to grow faster than linearly.
- The volatility process can potentially reach zero. This extends results in [20], where it is assumed that the underlying processes never hit boundaries of the state space. We classify the local behavior of the volatility process near zero and introduce notions of classical solutions in each scenario to answer (Q1).
- The asset-price process can be a strict local martingale. We give an analytic condition which is necessary and sufficient for the martingale property of the asset price. This condition generalizes results in [31] and it is a stronger version of the condition in [36]. Meanwhile, it is exactly the loss of martingale property that leads us to a sharp answer to (Q2): uniqueness holds in the class of at most linear growth functions *if and only if* the asset-price process is a martingale.
- Prescribing boundary conditions to the valuation equations may be unnecessary. When the volatility process reaches zero and instantaneously reflects back, a boundary condition at the vanishing volatility which is widely used in numerical computations (see [9]), is, surprisingly, not needed to ensure the uniqueness of a solution. In this paper, we clarify both the theoretical and practical role of this boundary condition.
- The payoff function can be unbounded, generalizing results in [11], where it is assumed that the payoff function is bounded. As the payoff function corresponding to a call option, the most famous example of a contingent claim, is indeed unbounded, this point is important both from a practical and a theoretical point of view.

Our main result is presented in Theorem 3.2, which can be considered as a Feynman-Kac-type result. It extends the standard Feynman-Kac formula in the context of stochastic volatility models. The probabilistic representation of the value function is extensively used to study the valuation equation. As a result, the probabilistic results feed back to the analysis of the valuation equation to provide weaker conditions on the existence and uniqueness of classical solutions.

The remainder of the paper is organized as follows. Section 1 introduces the stochastic volatility model under the risk-neutral probability. In Section 2, we explore necessary and sufficient conditions on the martingale (as opposed to strict local martingale) property of the asset-price process. In Section 3 we state our main results. Section 4 answers (Q3) by discussing issues related with numerical computations and provides some examples. Our main findings are proved progressively in Sections 5, 6 and 7. In particular, a notion of a stochastic solution is introduced in Section 6 to bridge the analytic and the probabilistic properties of solutions to the valuation equation.

1. The Stochastic Volatility Model

1.1. **The set-up.** All stochastic processes in the sequel are defined on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, \mathbb{P})$, satisfying the usual conditions. All relationships between random variables are understood in the $\mathbb{P}$-a.s. sense. We denote $\mathbb{R}_+ = [0,\infty)$ and $\mathbb{R}_{++} = (0,\infty)$. The following stochastic volatility model will be considered, written for the time being formally in differential form:

$$dS_t = S_t\, b(Y_t)\, dW_t, \quad S_0 = x \in \mathbb{R}_+, \tag{STOCK}$$

$$dY_t = \mu(Y_t)\, dt + \sigma(Y_t)\, dB_t, \quad Y_0 = y \in \mathbb{R}_+. \tag{VOL}$$

Above, W and B are two standard Wiener processes with constant instantaneous correlation $\rho \in (-1,1)$. In this model, the asset price is modeled by the dynamics of S, whose volatility is driven by an auxiliary factor Y. To simplify notation, we assume the instantaneous short rate to be zero; we note, however, that all our results carry for the case of nonzero constant short rate, with obvious modifications. The dynamics in (STOCK) imply that $\mathbb{P}$ is a local martingale measure for the asset-price process $(S_t)_{t\in\mathbb{R}_+}$. As mentioned in the Introduction, we allow for the possibility that the latter process is a strict local martingale.

Standing Assumption 1.1. It will be tacitly assumed throughout the paper that the coefficients of (STOCK) and (VOL) satisfy the following:

(i) The function $\mu : \mathbb{R}_+ \to \mathbb{R}$ is locally Lipschitz on $\mathbb{R}_+$ and $\mu(0) \geq 0$. The function $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ is locally (1/2)-Hölder continuous on $\mathbb{R}_+$, strictly positive on $\mathbb{R}_{++}$, and satisfies $\sigma(0) = 0$. Also, μ and σ have at most linear growth, i.e., there exists a positive constant C such that

$$|\mu(y)| + \sigma(y) \leq C(1+y) \quad \text{for } y \in \mathbb{R}_+. \tag{1.1}$$

(ii) $b : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies $b(y) > 0$ for $y \in \mathbb{R}_{++}$ and $b(0) = 0$. Furthermore, b is locally Hölder continuous on $\mathbb{R}_+$ with some exponent α, and σb is locally Lipschitz on $\mathbb{R}_+$. Finally, b has at most polynomial growth, i.e., there exist positive constants C and m such that

$$b(y) \leq C\,(1+y^m) \quad \text{for } y \in \mathbb{R}_+. \tag{1.2}$$

Remark 1.2. Our standing assumptions are satisfied by most diffusion stochastic volatility models that are used in practice. For example:

- in the *Hull-White* model [25], $\mu(y) = ay$ with $a < 0$, $\sigma(y) = \sigma y$ with $\sigma > 0$;
- in the *Heston* model [21], $\mu(y) = \mu_0 - ay$ with $\mu_0 > 0$ and $a > 0$, $\sigma(y) = \sigma\sqrt{y}$ with $\sigma > 0$;
- in the *GARCH(1,1)* model, $\mu(y) = \mu_0 - ay$ with $\mu_0 > 0$ and $a > 0$, $\sigma(y) = \sigma y$ with $\sigma > 0$.

In all of the above models, $b(y) = \sqrt{y}$ for $y \in \mathbb{R}_+$. When $b(y) = y$ for $y \in \mathbb{R}_+$, we have the model suggested in [38].

We now proceed in showing that the informal dynamics for the asset price and the volatility indeed produce a well-defined model.

Proposition 1.3. *For any $y \in \mathbb{R}_+$, the stochastic differential equation* (VOL) *has a unique nonexplosive and nonnegative strong solution Y^y. Moreover, for any $m \geq 1$ and $T \in \mathbb{R}_+$, there exists $C_{m,T} \in \mathbb{R}_{++}$ such that*

$$\mathbb{E}\left[\sup_{t\in[0,T]} |Y_t^y|^m\right] \leq C_{m,T}(1+y^m). \tag{1.3}$$

Proof. In the course of the proof we fix $y \in \mathbb{R}_+$ and drop superscripts involving y.

For any $n > 0$, define

$$\mu^n(y) = \begin{cases} \mu(y), & 0 \leq y < n \\ \mu(n), & y \geq n \end{cases}, \quad \sigma^n(y) = \begin{cases} \sigma(y), & 0 \leq y < n \\ \sigma(n), & y \geq n \end{cases},$$

and consider the stochastic differential equation (SDE):

$$dY_t^n = \mu^n(Y_t^n)\, dt + \sigma^n(Y_t^n)\, dB_t, \quad Y_0^n = y \in \mathbb{R}_+. \tag{1.4}$$

From the assumptions on μ and σ in item (i) of our standing assumptions, it follows that μ^n is Lipschitz and σ^n is (1/2)-Hölder continuous on $\mathbb{R}_+$. Thanks to the Yamada-Watanabe's theorem (see e.g. Theorem 40.1 in Chapter 5 of [33]), pathwise uniqueness holds for (1.4). As μ^n and σ^n are continuous and bounded on $\mathbb{R}_+$, it follows from Skorokhod's theorem (see e.g. Theorem 23.5 in Chapter 5 of [33]) that (1.4) has a weak solution. Combined with pathwise uniqueness, we obtain that (1.4) has a unique strong solution, which we denote by Y^n. Define $\eta_n := \inf\{t \geq 0 : Y_t^n = n\}$. Thanks to the pathwise uniqueness, Y^n and Y^{n+1} agree until η_n. Hence $\eta_n = \inf\left\{t \geq 0 : Y_t^{n+1} = n\right\}$ from which it follows that $\{\eta_n\}_{n\in\mathbb{N}}$ is an increasing sequence of stopping times. Let us define $Y_t := Y_t^n$ for $0 \leq t \leq \eta_n$; Y solves (VOL) until its explosion time $\eta := \lim_{n\to\infty} \eta_n$. Uniqueness follows from the Yamada-Watanabe's theorem, since any other solution must agree with Y on $[0, \eta_n]$.

The fact that $\mathbb{P}[\eta = \infty] = 1$ and the moment estimate (1.3) follows from classical results for SDEs with at most linear growth coefficients (see e.g. Problem 5.3.15 in [29]). □

For given $(x, y) \in \mathbb{R}_+^2$, and with Y^y established in Proposition 1.3 above, the solution of (STOCK) is given by the process $S^{x,y} := xH^y$, where

$$H^y := \exp\left\{\int_0^\cdot b(Y_t^y)\, dW_t - \frac{1}{2}\int_0^\cdot b^2(Y_t^y)\, dt\right\}. \tag{1.5}$$

As b is locally bounded on $\mathbb{R}_+$ (thanks to item (ii) of our standing assumptions) and Y^y is nonexplosive, $\int_0^t b^2(Y_u)\, du < \infty$, hence $S_t > 0$, holds for any $t \in \mathbb{R}_+$.

1.2. **Zero volatility.** For $y \in \mathbb{R}_+$, we define $\tau_0^y := \inf\{t \in \mathbb{R}_{++} \mid Y_t^y = 0\}$. It is possible that $\mathbb{P}[\tau_0^y < \infty] > 0$, i.e., the volatility factor Y^y can potentially reach zero in finite time. In this case:

- when $\mu(0) = 0$, $Y_t^y = 0$ for $\tau_0^y \leq t < \infty$, thus the point 0 is *absorbing*;
- when $\mu(0) > 0$, Y^y is lead back into $\mathbb{R}_{++}$ after τ_0^y.

More precisely, in the latter case the point 0 is *instantaneously reflecting* (see Definition 3.11 in Chapter VII of [32]). We show this in the following result, which will be useful later.

Lemma 1.4. *Fix $y \in \mathbb{R}_+$. If $\mu(0) > 0$, then $\int_{\mathbb{R}_+} \mathbb{I}_{\{Y_t^y = 0\}} dt = 0$.*

Remark 1.5. As the proof below suggests, for the validity of Lemma 1.4 we only use that σ is locally (1/2)-Hölder continuous on $\mathbb{R}_+$, it is strictly positive on $\mathbb{R}_{++}$, and it satisfies $\sigma(0) = 0$.

Proof. We fix $y \in \mathbb{R}_+$ and drop superscripts y from Y^y for the ease of notation.

By $L_t(a)$ we denote the local time of Y at the level $a \in \mathbb{R}_+$ accumulated by time $t \in \mathbb{R}_+$. We take the version of the local time that is $\mathbb{P}$-a.s. jointly continuous in t and càdlàg in a — see Theorem 3.7.1 in [29]. Since $\langle Y, Y \rangle = \int_0^\cdot \sigma^2(Y_t)\, dt$, it follows from the occupation time formula (see e.g. Theorem 3.7.1 (iii) in [29]) that

$$t \geq \int_0^t \mathbb{I}_{(0,\infty)}(Y_u)\, du = \int_0^t \mathbb{I}_{(0,\infty)}(Y_u)\, \sigma^{-2}(Y_u)\, d\langle Y, Y \rangle_u = 2 \int_{(0,\infty)} \sigma^{-2}(a)\, L_t(a)\, da, \tag{1.6}$$

in which the first equality follows since $\sigma(y) > 0$ for $y > 0$. Since $\sigma(0) = 0$ and σ is (1/2)-Hölder continuous in a neighborhood of 0, we have that $\sigma(a) \leq C a^{1/2}$ for $a \in [0, a_0]$, where C and a_0 are $\mathbb{R}_{++}$-valued constants. Hence, σ^{-2} is not integrable in this neighborhood of 0. Combining the last fact with the càdlàg property of L in the spatial variable, it can be seen that if $L_t(0)$ were not zero, the right-hand-side of (1.6) would be equal to infinity. This, however, contradicts with the bound on the leftmost-side of (1.6). It then follows from Problem 3.7.6 in [29] and $L_t(0) = 0 = L_t(0-)$ that

$$0 = L_t(0) - L_t(0-) = \mu(0) \int_0^t \mathbb{I}_{\{Y_u = 0\}}\, du.$$

Since $\mu(0) > 0$, we obtain the result. □

2. Characterizing the Martingale Property of the Asset-Price Process

In this section, we shall present a necessary and sufficient condition for the martingale property of the asset price process, which is essentially H^y (up to normalization with respect to the initial asset price) for the corresponding initial factor $y \in \mathbb{R}_+$. As will be later revealed, the latter martingale property is important in our discussion.

Let us first consider an auxiliary diffusion $\widetilde{Y}$ governed by the following formal dynamics:

$$d\widetilde{Y}_t = \widetilde{\mu}(\widetilde{Y}_t)\, dt + \sigma(\widetilde{Y}_t)\, dB_t, \quad \widetilde{Y}_0 = y, \tag{2.1}$$

where $\widetilde{\mu} := \mu + \rho b \sigma$. By our standing assumptions, $\widetilde{\mu}$ is locally Lipschitz and σ is (1/2)-Hölder continuous. An argument similar to the one used in the proof of Proposition 1.3 shows that (2.1) has a unique nonnegative strong solution $\widetilde{Y}^y$ for all $y \in \mathbb{R}_+$; however, due to the fact that $\widetilde{\mu}$ is only locally Lipschitz, the solution $\widetilde{Y}^y$ is defined up to an explosion time ζ^y, and it might be the case that $\mathbb{P}[\zeta^y < \infty] > 0$. This has important consequences on the stochastic behavior of the asset-price process, as the following result demonstrates.

Proposition 2.1. *We have*

$$\mathbb{E}\left[S_T^{x,y}\right] = x\,\mathbb{E}\left[H_T^y\right] = x\,\mathbb{P}\left[\zeta^y > T\right], \qquad \text{for all } (x,y,T) \in \mathbb{R}_+^3. \tag{2.2}$$

Moreover, $\mathbb{P}\left[\zeta^{y_1} \leq \zeta^{y_2}\right] = 1$*, holds whenever* $y_1 \in \mathbb{R}_+$ *and* $y_2 \in [0, y_1]$.

Proof. It follows from Proposition 1.3 that Y in nonexploding. As a result, (2.2) follows from an argument similar to the one used in the proof of Lemma 4.2 in [36]. Also, see Lemma 2.3 and its proof in [1]. The fact that $\mathbb{P}\left[\zeta^{y_1} \leq \zeta^{y_2}\right] = 1$ holds whenever $y_1 \in \mathbb{R}_+$ and $y_2 \in [0, y_1]$ follows from standard comparison theorems for SDEs — see, e.g., Proposition 5.2.18 of [29]. □

Remark 2.2. When $S^{x,y}$ is a strict local martingale, Proposition 2.1 shows that $\mathbb{R}_+ \ni y \mapsto \mathbb{E}\left[H_T^y\right]$ is decreasing in y for fixed $T \in \mathbb{R}_+$. Therefore, when the asset price is a strict local martingale, the value of an European call option with zero strike is not increasing with respect to volatility. A similar phenomenon has been observed in [10] for local volatility models.

Whether an explosion of $\widetilde{Y}$ happens or not is fully characterized by Feller's test, which we now revisit. With a fixed $c \in \mathbb{R}_{++}$, the scale function $\mathfrak{s}$ for the diffusion described in (2.1) is defined as

$$\mathfrak{s}(y) := \int_c^y \exp\left\{-2\int_c^\xi \frac{\widetilde{\mu}(z)}{\sigma^2(z)}\,dz\right\} d\xi, \qquad \text{for } y \in \mathbb{R}_{++}.$$

We set

$$\mathfrak{v}(y) := \int_c^y \frac{\mathfrak{s}(y) - \mathfrak{s}(\xi)}{\mathfrak{s}'(\xi)\sigma^2(\xi)}\,d\xi \qquad \text{for } y \in \mathbb{R}_{++}.$$

Note that $\mathfrak{v}$ is increasing on (c, ∞). Therefore, $\mathfrak{v}(\infty) := \lim_{y\uparrow\infty} \mathfrak{v}(y)$ is well defined. Feller's test (see Theorem 5.5.29 in [29]) states that $\mathbb{P}\left[\zeta^y < \infty\right] > 0$ for $y \in \mathbb{R}_{++}$ if and only if

$$\mathfrak{v}(\infty) < \infty. \tag{2.3}$$

As was pointed out by [5] in Section 4.1, it is sometimes easier to check the following equivalent condition:

$$\mathfrak{s}(\infty) < \infty \quad \text{and} \quad \frac{\mathfrak{s}(\infty) - \mathfrak{s}}{\mathfrak{s}'\sigma^2} \in L^1_{loc}(\infty-), \tag{2.4}$$

where $L^1_{loc}(\infty-)$ denotes the class of functions $f : \mathbb{R}_+ \to \mathbb{R}$ that are Lebesgue integrable on (y, ∞) for some $y > 0$.

Combining (2.2) and the above discussion, one obtains the following corollary of Proposition 2.1, which is due to [36]: H^y *is a martingale for all* $y \in \mathbb{R}_{++}$ *if and only if* (2.3) *fails to hold (or, equivalently, if and only if* (2.4) *fails to hold).* The previous statement implies that H^y is a strict local martingale for some, and then all, $y \in \mathbb{R}_{++}$ if and only if (2.3) (or (2.4)) is satisfied. However, given that H^y is a strict local martingale, it is not clear whether $H^y_{\cdot\wedge T}$ is still a strict local martingale for any $T > 0$. The next result, which is a stronger statement than the one previously made, is one of our main findings. Its proof requires some later results of this paper; therefore, we defer it to Section 5.

Proposition 2.3. *The following statements are equivalent:*

(1) $H^y_{\cdot\wedge T}$ *is a strict local martingale for some, and then all,* $(y,T) \in \mathbb{R}^2_{++}$.
(2) (2.3) *(or, equivalently,* (2.4)*) is satisfied.*

Note when H^y is a martingale for all $y \in \mathbb{R}_{++}$, H^0 is a martingale as well because of the monotonicity of $\mathbb{R}_+ \ni y \mapsto \mathbb{P}[\zeta^y > T]$ in y for fixed $T \in \mathbb{R}_+$ — see Proposition 2.1. In view of Proposition 2.3, when we are referring to the martingale property of the asset-price process, we mean that H^y is a martingale for all $y \in \mathbb{R}_+$.

Remark 2.4. Notice that the dynamics of S and Y are time-homogeneous. In this case, Proposition 2.3 implies that if H^y is going to lose its martingale property eventually, it must lose its martingale property immediately. This result generalizes Theorem 2.4 in [31], where a sufficient condition and a different necessary condition are given such that $H^y_{\cdot\wedge T}$ is a strict local martingale for any fixed $T \in \mathbb{R}_{++}$. Proposition 2.3 closes the gap between these two conditions in [31].

In contrast, when the dynamics in the stochastic volatility model are not time-homogeneous, the asset price may lose its martingale property only at a later time, as can be seen from an example in Section 2.2.1 in [6].

3. Main Results on the Valuation Equation for European Options

3.1. **The value function.** In all that follows, $g : \mathbb{R}^2_+ \to \mathbb{R}_+$ will be a continuous function with at most linear growth in its first variable and at most polynomial growth in the second variable; in other words, we assume that there exist positive constants C and m such that $g(x,y) \le C(1+x+y^m)$ for all $(x,y) \in \mathbb{R}^2_+$.

The value function $u : \mathbb{R}^3_+ \to \mathbb{R}_+$ of a European option with payoff function g is defined via

$$u(x,y,T) := \mathbb{E}\left[g\left(S^{x,y}_T, Y^y_T\right)\right], \quad \text{for } (x,y,T) \in \mathbb{R}^3_+.$$

Thanks to the growth assumption on g, the supermartingale property of H^y, and the moment estimate (1.3), there exists $C_{m,T} \in \mathbb{R}_{++}$ such that

$$u(x,y,t) \le C_{m,T}\left(1+x+y^m\right), \quad \text{for any } (x,y) \in \mathbb{R}^2_+ \text{ and } t \in [0,T]. \tag{3.1}$$

3.2. **The valuation equation.** For $(x,y,T) \in \mathbb{R}^3_+$, define a process $U^{x,y,T} = (U^{x,y,T}_t)_{t\in[0,T]}$ via $U^{x,y,T}_t = u(S^{x,y}_t, Y^y_t, T-t)$ for $t \in [0,T]$. The Markov property of $(S^{x,y}, Y^y)$ gives

$$U^{x,y,T}_t = \mathbb{E}\left[g\left(S^{x,y}_T, Y^y_T\right) \mid \mathcal{F}_t\right], \quad \mathbb{P}\text{-a.s. for } t \in [0,T]. \tag{3.2}$$

As $\mathbb{E}\left[g\left(S^{x,y}_T, Y^y_T\right)\right] < \infty$, we conclude that $U^{x,y,T}$ is a martingale on $[0,T]$. If u is sufficiently smooth (at the moment, we are being intensionally vague on this point; we shall have more to say in Theorem 3.2), a formal application of Itô's formula implies that the value function u is a solution to the following PDE (in v):

$$\begin{aligned} &\partial_T v(x,y,T) = \mathcal{L}v(x,y,T), \quad (x,y,T) \in \mathbb{R}^3_{++}, \\ &v(x,y,0) = g(x,y), \quad (x,y) \in \mathbb{R}^2_+, \end{aligned} \tag{BS-PDE}$$

in which

$$\mathcal{L} := \mu(y)\partial_y + \frac{1}{2}b^2(y)x^2\partial^2_{xx} + \frac{1}{2}\sigma^2(y)\partial^2_{yy} + \rho b(y)\sigma(y)x\partial^2_{xy}$$

is the infinitesimal generator of (S, Y).

The above valuation equation (BS-PDE) is usually supplied by further conditions that guarantee that u is a solution, hopefully unique in a certain class of functions. Then, one can use some numerical method on (BS-PDE) in order to calculate u. If a solution v of (BS-PDE) is to be identified with the value function u, it is clearly necessary that the process $V^{x,y,T} = (V^{x,y,T}_t)_{t\in[0,T]}$ for $(x, y, T) \in \mathbb{R}^3_{++}$ defined via $V^{x,y,T}_t = v(S^{x,y}_t, Y^y_t, T-t)$ for $t \in [0, T]$ is at least a local martingale on $[0, T]$. Given $v \in C^{2,2,1}(\mathbb{R}^3_{++})$, Itô's lemma implies that $V^{x,y,T}$ is a local martingale up to $\tau^y_0 \wedge T$. When $\mathbb{P}[\tau^y_0 < T] > 0$, it is reasonable to expect that some boundary condition is needed to ensure that $V^{x,y,T}$ is still a local martingale after τ^y_0 and up to T. When $\mu(0) = 0$, the point 0 is absorbing for Y^y. Since $b(0) = 0$, we have $(S^{x,y}_t, Y^y_t) = (S^{x,y}_{\tau^y_0}, 0)$ for $\tau^y_0 \le t < \infty$. Therefore, we enforce the following Dirichlet boundary condition,

$$v(x, 0, T) = g(x, 0), \quad (x, T) \in \mathbb{R}^2_{++}. \tag{3.3}$$

When $\mu(0) > 0$, a *rule of thumb*, which is widely used in numerical computations (see [9]), is to formally send y to zero in (BS-PDE), i.e.,

$$\partial_T v(x, 0, T) = \mu(0)\partial_y v(x, 0, T), \quad (x, T) \in \mathbb{R}^2_{++}. \tag{3.4}$$

where $\partial_T v$ and $\partial_y v$ at $y = 0$ are defined as

$$\partial_T v(x, 0, T) := \lim_{y\downarrow 0} \partial_T v(x, y, T) \quad \text{and } \partial_y v(x, 0, T) := \lim_{y\downarrow 0} \partial_y v(x, y, T), \quad \text{for } (x, T) \in \mathbb{R}^2_{++},$$

if both limits exist. Actually, imposing (3.4) as the boundary condition at $y = 0$ implicitly assumes that both limits in (3.2) exist, and that the partial derivatives $\partial_T v$ and $\partial_y v$ are continuous up to the boundary $y = 0$.

In fact, we shall show that (3.4) is not necessary to guarantee that $V^{x,y,T}$ to be a local martingale beyond τ_0. This is mainly due to Lemma 1.4, which shows that Y spends Lebesgue measure zero time at 0 when $\mu(0) > 0$; for this reason, boundary conditions at $y = 0$ do not seem to affect the local martingale property of $V^{x,y,T}$.

3.3. **The main result.** In this section, the existence and uniqueness of classical solutions for (BS-PDE) (with a possible boundary condition (3.3)) are presented. The notion of classical solutions for (BS-PDE) depends on whether Y^y reaches 0 in finite time. Recall that the latter characterized by applying Feller's test to Y^y at 0 — see Theorem 5.5.29 in [29]. When Y^y cannot reach 0 in finite time, no boundary condition at $y = 0$ is needed; also see [20]. Otherwise, when Y^y reaches 0 in finite time and $\mu(0) = 0$, the boundary condition (3.3) is needed.

Definition 3.1. A function $v \in C(\mathbb{R}^3_+) \cap C^{2,2,1}(\mathbb{R}^3_{++})$, that solves (BS-PDE), is called a classical solution in all the following cases (below, y is arbitrary in $\mathbb{R}_{++}$):

(A) When $\mathbb{P}[\tau^y_0 = \infty] = 1$.

(B) When $\mathbb{P}[\tau_0^y < \infty] > 0$, $\mu(0) = 0$, and v further satisfies the boundary condition (3.3).

(C) When $\mathbb{P}[\tau_0^y < \infty] > 0$, $\mu(0) > 0$, and v belongs to

$$\mathfrak{C} := \left\{ f \in C(\mathbb{R}_+^3) \cap C^{2,2,1}(\mathbb{R}_{++}^3) \,|\, \text{all } \partial_T f,\ \partial_y f,\ y^{2\alpha}\partial_{xx}^2 f \text{ are locally bounded on } \mathbb{R}_{++} \times \mathbb{R}_+ \times \mathbb{R}_{++} \right\},$$

where α is the Hölder exponent for b. For any $f \in C(\mathbb{R}_+^3) \cap C^{2,2,1}(\mathbb{R}_{++}^3)$, by saying that $\partial_T f$, $\partial_y f$, and $y^{2\alpha}\partial_{xx}^2 f$ are all locally bounded on the boundary $y = 0$, we mean that

$$\limsup_{y \downarrow 0} \sup_{(x,T)\in[n^{-1},n]^2} \left[|\partial_T f(x,y,T)| + |\partial_y f(x,y,T)| + y^{2\alpha} |\partial_{xx}^2 f(x,y,T)| \right] < \infty$$

holds for all $n \in \mathbb{N}$.

Let us also recall that a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is of *strictly sublinear growth* if $\lim_{x\to\infty} f(x)/x = 0$. Now we are ready to present our main result, whose proof is given in Section 7.

Theorem 3.2. *The value function $u \in C(\mathbb{R}_+^3) \cap C^{2,2,1}(\mathbb{R}_{++}^3)$. Furthermore, u is the smallest nonnegative classical solution to* (BS-PDE) *in each of the following cases (where $y \in \mathbb{R}_{++}$ below is arbitrary):*

(A) *When $\mathbb{P}[\tau_0^y = \infty] = 1$.*

(B) *When $\mathbb{P}[\tau_0^y < \infty] > 0$ and $\mu(0) = 0$.*

(C) *When $\mathbb{P}[\tau_0^y < \infty] > 0$, $\mu(0) > 0$, and $u \in \mathfrak{C}$.*

In all of the above cases, the following two statements hold:

(i) *If g is of strictly sublinear growth in x and polynomial in y, then u is the unique classical solution within the same class of functions.*

(ii) *If g is of linear growth in x and polynomial in y, then u is the unique classical solution within the same class of functions if and only if the asset-price process is a martingale.*

Our main contribution in the above theorem is the uniqueness part. This theorem generalizes Theorems 6.1 and 6.2 in [11]. In [11], the boundary condition (3.4) is needed to establish the uniqueness via the maximum principle. In contrast, our proof relies on probabilistic arguments; thanks to Lemma 1.4, (3.4) is not needed for our uniqueness result.

Theorem 3.2 also extends the results obtained in [4] for local volatility models to the case of stochastic volatility models.

4. Remarks on Numerical Computation and Examples

4.1. **Numerical computation of the value function.** When g is of linear growth in x and the asset-price process is a strict local martingale, uniqueness of the solution of BS-PDE among functions with linear growth fails. On the positive side, it was observed in [10] that u can be identified by the following limit:

$$u(x,y,T) = \lim_{n\to\infty} u_n(x,y,T), \quad \text{where } u_n(x,y,T) := \mathbb{E}\left[g(S_T^{x,y}, Y_T^y) \wedge n \right]. \tag{4.1}$$

Indeed this statement follows from the dominated convergence theorem since $\mathbb{E}[g(S_T^{x,y}, Y_T^y)] < \infty$. Now thanks to item (i) in Theorem 3.2 (given that the first part of the theorem is satisfied), each u_n can be identified as the unique classical solution of (BS-PDE) with a bounded initial condition given by $g \wedge n$.

However, if we choose a different approximating sequence, its limit may not converge to the value of the European option.

Example 4.1. Consider a standard call option payoff $g(x, y) = (x - K)_+$ for $K \in \mathbb{R}_{++}$. We choose a localizing sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ for $S^{x,y}$, for example $\sigma_n = \inf\{t \geq 0 : S_t^{x,y} \geq n\}$. Let us define

$$\widetilde{u}(x, y, T) := \lim_{n\to\infty} \mathbb{E}[g(S_{\sigma_n\wedge T}^{x,y})].$$

It was observed in [3] that $\widetilde{u}$ is a classical solution to (BS-PDE) in cases (A) and (B). But when $S^{x,y}$ is a strict local martingale, $\widetilde{u}$ is the American call value, and it is strictly larger than the European call value u.

Even though the boundary condition (3.4) is not needed in Theorem 3.2, its validity (if true) certainly helps to solve (BS-PDE) numerically using finite difference methods — see [9]. For the rest of this section, we shall discuss how to approximate u by a sequence of bounded functions $\{u^\epsilon\}_{\epsilon>0}$ in the spirit of (4.1), and give a sufficient condition so that each u^ϵ is an element of $\mathfrak{C}$ in Theorem 3.2 case (C). This condition is the main result of [11].

Only for the remainder of this part on numerical computations, we shall need additional assumptions on the coefficients of (STOCK) and (VOL) and on the payoff function g.

Assumptions 4.2. (i) μ, σ^2, b^2, and $b\sigma$ are all continuously differentiable on $\mathbb{R}_+$ with α-Hölder continuous derivatives for some $\alpha \in (0, 1]$. $(b^2)'$ has at most polynomial growth on $\mathbb{R}_+$.
(ii) The payoff function g only depends on x. The derivatives g' and g'' exist and are bounded on $[0, \delta)$ for some $\delta > 0$.

Given that g satisfies item (ii) in Assumption 4.2, there exists a sequence of functions $\{g^\epsilon\}_{\epsilon>0}$, such that, for each ϵ:

(1) g^ϵ is bounded;
(2) $g^\epsilon \in C^\infty(\mathbb{R}_{++})$;
(3) $(g^\epsilon)'$ and $(g^\epsilon)''$ have compact support in $\mathbb{R}_+$, and both derivatives are finite at $x = 0$;
(4) $g^\epsilon(x) \leq g(x) + 1$ for $x \in \mathbb{R}_+$, and
(5) $\lim_{\epsilon\downarrow 0} g^\epsilon(x) = g(x)$ for $x \in \mathbb{R}_+$.

Indeed, for sufficient small ϵ, one can consider $g^\epsilon := \eta^\epsilon * \widetilde{g}^\epsilon$, where $\widetilde{g}^\epsilon(x) := g(x \wedge \frac{1}{\epsilon})$ for $x \in \mathbb{R}_+$, η^ϵ is the standard mollifier, and $*$ denotes the convolution operator.

Define $u^\epsilon(x, y, T) := \mathbb{E}[g^\epsilon(S_T^{x,y})]$ for $(x, y, T) \in \mathbb{R}_+^3$. It then follows from the dominated convergence theorem that

$$u(x, y, T) = \lim_{\epsilon\downarrow 0} u^\epsilon(x, y, T), \qquad \text{for } (x, y, T) \in \mathbb{R}_+^3.$$

Proposition 4.3. *Under Assumptions 1.1 and 4.2, $u^\epsilon \in \mathfrak{C}$ and* (3.4) *is satisfied for u^ϵ.*

Proof. It follows from the dominated convergence theorem that

$$x^2\, \partial^2_{xx} u^\epsilon(x,y,T) = \mathbb{E}\left[(S^{x,y}_T)^2\, (g^\epsilon)''(S^{x,y}_T)\right].$$

Noting that $(g^\epsilon)''$ has compact support and it is finite at $x=0$, we obtain that $|x^2\,\partial^2_{xx}u^\epsilon|$ is bounded on $\mathbb{R}^3_+$. As a result, $\limsup_{y\downarrow 0} y^{2\alpha}|\partial^2_{xx}u^\epsilon(x,y,T)| = 0$ for $(x,T)\in\mathbb{R}^2_{++}$.

It then follows from the main result in [11] that $\partial_T u^\epsilon, \partial_y u^\epsilon \in C(\mathbb{R}_{++}\times\mathbb{R}_+\times\mathbb{R}_{++})$ and u^ϵ satisfies (3.4). As a result, $\partial_T u^\epsilon$ and $\partial_y u^\epsilon$ are clearly locally bounded on $\mathbb{R}_{++}\times\mathbb{R}_+\times\mathbb{R}_{++}$. The assumptions on the payoff function in [11] are satisfied by our g^ϵ. In [11], $b(y)$ is chosen as $\sqrt{y}$. However, the same proof still goes through if $b^2 \in C^1(\mathbb{R}_+)$, $(b^2)'$ is α-Hölder continuous, and has at most polynomial growth. In particular, (22) in [11] is replaced by $1 \le b^2(\frac{y}{m}) x_0^2 \frac{k^2}{m} \le 2$. For a sequence $\{m_n\}_{n\in\mathbb{N}} \uparrow \infty$, a sequence $\{k_n\}_{n\in\mathbb{N}}$ can still be chosen appropriately so that above inequalities are satisfied. Moreover, Proposition 4.1 of [11] still holds. Indeed, for any $(x,y)\in\mathbb{R}^2_+$ and a sequence $\{(x_n,y_n)\}_{n\in\mathbb{N}}$ in a bounded neighborhood of (x,y), there exists a constant $C_{T,\epsilon}$ such that

$$\left|\exp\left(\int_0^\nu \mu'(Y^{y_n}_\sigma)\,d\sigma\right)\left(b^2(Y^{y_n}_\nu)\right)'(S^{x_n,y_n}_\nu)^2\,\partial^2_{xx}u^\epsilon\left(S^{x_n,y_n}_\nu, Y^{y_n}_\nu, T-\nu\right)\right| \le C_{T,\epsilon}\left|\left(b^2(Y^{y_n}_\nu)\right)'\right|,$$

for any $n\in\mathbb{N}$ and $\nu\in[0,T]$. Thanks to the growth assumption on $(b^2)'$ and the moment estimate (1.3), $\left\{\left|\left(b^2(Y^{y_n}_\nu)\right)'\right|\right\}_{n\in\mathbb{N}}$ is a uniformly integrable family. Therefore, the function v^ϵ defined as

$$v^\epsilon(x,y,T) := \mathbb{E}\left[\int_0^T \exp\left(\int_0^\nu \mu'(Y^y_\sigma)\,d\sigma\right)\left(b^2(Y^y_\nu)\right)'(S^{x,y}_\nu)^2\,\partial^2_{xx}u^\epsilon\left(S^{x,y}_\nu, Y^y_\nu, T-\nu\right)\,d\nu\right],$$

is still a continuous function on $\mathbb{R}^3_+$. □

4.2. **Examples.** In this section, we will give several examples where (3.4), understood as asymptotic behavior of u near $y=0$, is satisfied. The reader will notice that the discussed payoff functions are unbounded.

Example 4.4 (Heston model). Let us choose $b(y)=\sqrt{y}$, $\mu(y)=\mu(0)+\gamma y$ with $\mu(0)>0$, and $\sigma(y)=\sigma\sqrt{y}$. It is well known that Heston model is an affine model. Therefore there exist $\mathbb{C}$-valued functions ϕ, ψ_1, and ψ_2 such that

$$\mathbb{E}\left[\exp\left(v\log S^{x,y}_T + wY^y_T\right)\right] = \exp\left(\phi(v,w,T)+\psi_1(v,w,T)\,\log x + \psi_2(v,w,T)\,y\right),$$

for any $(v,w)\in\mathbb{C}^2$ and $T\ge 0$ such that either side of the previous identity is finite. Here ϕ, ψ_1, and ψ_2 solve a system of Riccati equations, in particular,

$$\partial_T\phi(v,w,T) = \mu(0)\,\psi_2(v,w,T). \tag{4.2}$$

For a European call with the payoff $g(x)=(x-K)_+$, it follows from (6.4) in [16] that u has the following form

$$u(x,y,T) = x + \int_{\mathbb{R}} \exp\left(\phi(p+iz,0,T)+\psi_2(p+iz,0,T)y + (p+iz)\log x\right)\widetilde{g}(z)\,dz, \tag{4.3}$$

in which $p \in (0,1)$ and $\widetilde{g}(z) = \frac{K^{1-p-iz}}{2\pi(p+iz)(p+iz-1)}$. It follows from explicit formulas in (6.2) of [16] that $|\phi(p+iz,0,T)| \sim -\mu(0)|z|$ when $|z| \to \infty$ and $|\phi(p+iz,0,T)| \sim 1$ when $|z| \to 0$, moreover, $|\psi_2(p+iz,0,T)| \sim -|z|$ when $|z| \to \infty$ and $|\psi_2(p+iz,0,T)| \sim 1$ when $|z| \to 0$. Using these estimates and (4.3), one can check that $\partial_T u(x,0,T)$ and $\partial_y u(x,0,T)$ exist. Moreover, (3.4) is satisfied, since it is equivalent to (4.2).

The following example shows that if we follow the rule of thumb to derive the boundary behavior of u, it may be wrong when the uniqueness of (VOL) fails beyond τ_0.

Example 4.5. Choosing $\gamma = -1$ and $\sigma = 1$ in above example, we have seen that both $\partial_T u$ and $\partial_y u$ exist on $y = 0$, and (3.4) is satisfied.

Now, let us consider $Z_t := Y_t^2$ and $b(z) = z^{\frac{1}{4}}$. The new factor satisfies

$$dZ_t = \left((2\mu(0)+1)\sqrt{Z_t} - 2Z_t\right) dt + 2Z_t^{\frac{3}{4}} dB_t. \tag{4.4}$$

The value function for a European call is given by $\widetilde{u}(x,z,T) = u(x,y,T)$. If we followed the rule of thumb, we would obtain $\partial_T \widetilde{u}(x,0,T) = 0$ as the boundary behavior. However, this contradicts with (3.4) in the Heston model, because $\partial_T \widetilde{u}(x,0,T) = \partial_T u(x,0,T) = \mu(0)\partial_y u(x,0,T) \neq 0$. Moreover, note that $\partial_z \widetilde{u} = \frac{1}{2} z^{-\frac{1}{2}} \partial_y u$. Although $\lim_{y\downarrow 0} \partial_y u(x,y,T)$ exists, $\lim_{z\downarrow 0} \partial_z \widetilde{u}(x,z,T)$ does not.

It seems that following the rule of thumb is problematic in this case. However, the uniqueness in law no longer holds for (4.4) beyond τ_0 (item (i) of Assumption 1.1 is not satisfied for (4.4) since its drift is not Lipschitz in a neighborhood of 0). Due to Feller's test, $\mathbb{P}(\tau_0 < \infty) > 0$ when $\mu(0) < \frac{1}{2}$. Therefore, on $\{\tau_0 < \infty\}$, we have $Z_t = Y_t^2$ as a solution of (4.4) for $t \geq \tau_0$ and $Z_t \equiv 0$ as another solution.

In the following two examples, the payoff function is not continuously differentiable at the boundary $y = 0$.

Example 4.6. Let $\mu(y) = \frac{1}{4}y$ and $\sigma(y) = y$. The process $Y_\cdot^y$ for $y \in \mathbb{R}_{++}$ does not hit 0 in finite time. Let $g(x,y) = \sqrt{y}$ for $(x,y) \in \mathbb{R}_+^2$. It follows from Itô's formula that $g(S_t^{x,y}, Y_t^y) = g(x,y) + \exp(-\frac{1}{8}t + \frac{1}{2}B_t)$. Hence $u(x,y,T) \equiv \sqrt{y}$ for any $(x,y,T) \in \mathbb{R}_{++}^3$. Clearly $\partial_y u$ does not exist at $y = 0$. But $\lim_{y\downarrow 0} \mu(y)\partial_y u(x,y,T) = 0$. Therefore the boundary asymptote in this case is $\partial_T u(x,0,T) = 0$.

Example 4.7. Let $\mu(y) = 1$ and $\sigma(y) = \sqrt{y}$. Then $Y_\cdot^y$ is the squared 1-dimensional Bessel process. Let $g(x,y) = \sqrt{y}$ for $(x,y) \in \mathbb{R}_+^2$. Then $g(Y_\cdot^y)$ is a standard Brownian motion reflected at 0 — see Section 1 in Chapter XI of [32]. As a result,

$$u(x,y,T) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |\sqrt{y} + \sqrt{T} z| \, e^{-\frac{z^2}{2}} \, dz, \quad \text{for } (x,y,T) \in \mathbb{R}_+^3.$$

We then obtain

$$\partial_y u(x,y,T) = \frac{1}{2\sqrt{y}} \left[\Phi\left(\sqrt{y/T}\right) - \Phi\left(-\sqrt{y/T}\right)\right], \quad \text{for } (x,y,T) \in \mathbb{R}_+ \times \mathbb{R}_{++}^2,$$

where $\Phi(\cdot) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\cdot} e^{-z^2/2}dz$. It follows from applying d'Hospital rule that $\partial_y u(x,0,T) = \frac{1}{\sqrt{2\pi T}}$. On the other hand, we also obtain $\partial_T u(x,0,T) = \frac{1}{\sqrt{2\pi T}}$. Therefore, both $\partial_T u(x,0,T)$ and $\partial_y u(x,0,T)$ are continuous on $\mathbb{R}_{++}$ and (3.4) is satisfied.

5. Smoothness of the Value Function

In this section we shall prove the part of Theorem 3.2 concerning smoothness of the value function, as well as Proposition 2.3, an important corollary of this result.

5.1. Stability of (STOCK) and (VOL).

We shall start with a technical result on the stability of solutions of (STOCK) and (VOL) with respect to their initial values.

Lemma 5.1. *Pick any $(x,y,T) \in \mathbb{R}^3_+$, and any sequence $\{(x_n,y_n,T_n)\}_{n\in\mathbb{N}}$ which converges to (x,y,T). Then,*

$$\mathbb{P}\text{-}\lim_{n\to\infty} Y^{y_n}_{T_n} = Y^y_T \quad \text{and} \quad \mathbb{P}\text{-}\lim_{n\to\infty} S^{x_n,y_n}_{T_n} = S^{x,y}_T, \tag{5.1}$$

where "$\mathbb{P}$-lim" denotes limit in $\mathbb{P}$-measure.

Proof. The stability properties of solutions for (VOL) have been well-studied under the linear growth assumption (1.1) (see e.g. [2]). In fact, the stability of Y in (5.1) follows from Theorem 2.4 in [2], which shows that

$$\lim_{n\to\infty} \mathbb{E}\left[\sup_{0\le u\le t+\delta} |Y^{y_n}_u - Y^y_u|^2\right] = 0, \qquad \text{for any } \delta>0, \tag{5.2}$$

and the fact that $\mathbb{E}\left[\left|Y^y_{t_n} - Y^y_t\right|^2\right] \le C(1+y^2)|t-t_n|$ for some $C>0$ — see Problem 5.3.15 in [29].

For the stability of S, it suffices to show that $\mathbb{P}\text{-}\lim_{n\to\infty} \log H^{y_n}_{t_n} = \log H^y_t$. In the next paragraph, we will prove that

$$\lim_{n\to\infty} \mathbb{E}\left[\left|\int_0^t b\left(Y^y_u\right)\, dW_u - \int_0^{t_n} b\left(Y^{y_n}_u\right)\, dW_u\right|^2\right] = 0. \tag{5.3}$$

The fact that $\lim_{n\to\infty}\mathbb{E}\left[\left|\int_0^t b^2\left(Y^y_u\right)\, du - \int_0^{t_n} b^2\left(Y^{y_n}_u\right)\, du\right|\right] = 0$ can be shown in a similar fashion. Then, $\mathbb{P}\text{-}\lim_{n\to\infty} \log H^{y_n}_{t_n} = \log H^y_t$ follows from these two identities.

To estimate the left-hand-side of (5.3), use Itô's isometry and the fact that $(a+b)^2 \le 2\left(a^2+b^2\right)$ for any $a,b\in\mathbb{R}$ to get

$$\mathbb{E}\left[\left|\int_0^t b\left(Y^y_u\right)\, dW_u - \int_0^{t_n} b\left(Y^{y_n}_u\right)\, dW_u\right|^2\right] \le 2\,\mathbb{E}\left[\int_0^{t_n}\left(b(Y^y_u)-b(Y^{y_n}_u)\right)^2\, du\right] + 2\,\mathbb{E}\left[\left|\int_{t_n}^t b^2(Y^y_u)\, du\right|\right].$$

Let n be large enough (greater than or equal to, say, some $N(\delta)$) so that $t_n \le t+\delta$ and $y_n \le y+\delta$ for some $\delta>0$. Since b is of at most polynomial growth (see (1.2)), it then follows from (1.3) that $\mathbb{E}\left[\sup_{u\le t+\delta} b^2(Y^y_u)\right] \le C(\delta,y)$, for some constant $C(\delta,y)$. As a result, $\lim_{n\to\infty}\mathbb{E}\left[\left|\int_{t_n}^t b^2(Y^y_u)\, du\right|\right] \le \lim_{n\to\infty} C(\delta,y)|t-t_n| = 0$. On the other hand, since b is locally α-Hölder continuous on $\mathbb{R}_+$, for any $M > 0$, there exist a constant C_M such that $|b(x) - b(y)|^2 \leq C_M |x-y|^{2\alpha}$ for any $x, y \leq M$. As a result, for any $u \leq t_n$

$$\begin{aligned} &\mathbb{E}\left[(b(Y_u^y) - b(Y_u^{y_n}))^2\right] \\ &= \mathbb{E}\left[(b(Y_u^y) - b(Y_u^{y_n}))^2 \, \mathbb{I}_{\left\{Y_u^y \leq M, \, Y_u^{y_n} \leq M\right\}}\right] + \mathbb{E}\left[(b(Y_u^y) - b(Y_u^{y_n}))^2 \, \mathbb{I}_{\left\{Y_u^y > M \text{ or } Y_u^{y_n} > M\right\}}\right] \\ &\leq C_M \mathbb{E}\left[|Y_u^y - Y_u^{y_n}|^{2\alpha}\right] + C\, \mathbb{E}\left[\left(1 + (Y_u^y)^{2m} + (Y_u^{y_n})^{2m}\right) \mathbb{I}_{\left\{Y_u^y > M \text{ or } Y_u^{y_n} > M\right\}}\right]. \end{aligned} \tag{5.4}$$

Since $\alpha \leq 1$, $\mathbb{E}\left[|Y_u^y - Y_u^{y_n}|^{2\alpha}\right] \leq \mathbb{E}\left[|Y_u^y - Y_u^{y_n}|^2\right]^\alpha$ holds by Jensen's inequality; sending $n \to \infty$, it follows from (5.2) that the first term on the right-hand-side of (5.4) converges to zero. For the second term, observe that $\sup_{n \in \mathbb{N}} \mathbb{E}\left[(Y_u^{y_n})^{4m}\right] < \infty$ implies that $\left\{(Y_u^{y_n})^{2m}\right\}_{n \in \mathbb{N}}$ is a uniformly integrable family; therefore,

$$\limsup_{n \to \infty} \mathbb{E}\left[\left(1 + (Y_u^y)^{2m} + (Y_u^{y_n})^{2m}\right) \mathbb{I}_{\left\{Y_u^y > M \text{ or } Y_u^{y_n} > M\right\}}\right] \leq \mathbb{E}\left[\left(1 + 2\,(Y_u^y)^{2m}\right) \mathbb{I}_{\left\{Y_u^y \geq M\right\}}\right]$$

and the last expression is further dominated by

$$\mathbb{E}\left[\left(1 + 2 \sup_{u \in [0, t+\delta]} (Y_u^y)^{2m}\right) \mathbb{I}_{\left\{\sup_{u \in [0,t+\delta]} Y_u^y \geq M\right\}}\right]$$

Then, it follows that

$$\limsup_{n \to \infty} \int_0^{t_n} \mathbb{E}\left[(b(Y_u^y) - b(Y_u^{y_n}))^2\right] du \leq C(t+\delta)\, \mathbb{E}\left[\left(1 + 2 \sup_{u \in [0,t+\delta]} (Y_u^y)^{2m}\right) \mathbb{I}_{\left\{\sup_{u \in [0,t+\delta]} Y_u^y \geq M\right\}}\right]$$

for some constant C. Sending $M \to \infty$, we have that the right-hand-side of last inequality converges 0 thanks to (1.3) and the dominated convergence theorem. This concludes the proof of (5.3). $\square$

5.2. **Smoothness of the value function.** Now comes the first important step towards proving Theorem 3.2.

Lemma 5.2. *$u \in C(\mathbb{R}^3_+) \cap C^{2,2,1}(\mathbb{R}^3_{++})$ and it satisfies* (BS-PDE)*.*

Before we begin with the proof of Lemma 5.2, let us remark on some of the issues involved. The differential operator $\mathcal{L}$ in (BS-PDE) degenerates on the boundaries of the state space, in particular on $y = 0$; therefore, the continuity of u does not follow directly from the standard results for parabolic differential equations (which may be applicable when $\mathcal{L}$ is nondegenerate). On the other hand, for payoffs of linear growth, Lemma 5.1 cannot be used to give a direct proof. Instead, we shall apply regularity results for nondegenerate parabolic differential equations to show that u is continuous in the interior of the state space, and then use probabilistic arguments to show that u continuously extends to the boundaries. As a byproduct, we obtain that u is differentiable up to the second order in the interior of the state space, and that it satisfies (BS-PDE).

Proof of Lemma 5.2. We decompose the proof into three steps. First, we show that u is continuous in the interior of $\mathbb{R}^3_+$. Then assuming that $g(x,y) \equiv x$ we prove that u extends continuously to the boundaries of $\mathbb{R}^3_+$. Finally, we generalize the result to general payoff functions.

Step 1. Let us consider a sequence of payoff functions $g^m := g \wedge m$, for $m \in \mathbb{N}$, and define $u^m(x,y,T) := \mathbb{E}[g^m(S_T^{x,y}, Y_T^y)]$ for $(x,y,T) \in \mathbb{R}_+^3$. The monotone convergence theorem implies that $\lim_{m\to\infty} u^m(x,y,T) = u(x,y,T)$ for every $(x,y,T) \in \mathbb{R}_+^3$. For each u^m, since g^m is bounded and continuous, it follows from the bounded convergence theorem that the continuity of u^m is equivalent to the stability properties in (5.1). Therefore, we obtain $u^m(x,y,T) \in C(\mathbb{R}_+^3)$.

Now let us consider a cylindrical domain $\mathcal{D} = A \times (t_1, t_2)$ such that its closure $\overline{\mathcal{D}}$ is a bounded subset of $\mathbb{R}_{++}^3$. Since $\overline{\mathcal{D}}$ avoids the boundaries $x = 0$ and $y = 0$, it follows from a verification argument (see e.g. Theorem 2.7 in [26]) that u^m satisfies a uniformly parabolic differential equation $u_T^m = \mathcal{L}u^m$ in $\mathcal{D}$. Note that the coefficients of these equations are the same for all m and that u^m are uniformly bounded above by u. Moreover, since $\overline{\mathcal{D}}$ is bounded, (3.1) implies that u is bounded on $\overline{\mathcal{D}}$. It then follows from the *interior Schauder estimate* (see e.g. Theorem 15 in page 80 of [18]) that for any subsequence $\{u^{m'}\}$ of $\{u^m\}$, there exists a further subsequence $\{u^{m''}\}$ such that $\{u^{m''}\}$ uniformly converges to u in any compact subdomain in $\mathcal{D}$. Thanks to the continuity of $u_{m''}$ and the uniform convergence, we obtain $u \in C(\mathcal{D})$. Therefore, $u \in C(\mathbb{R}_{++}^3)$ since $\mathcal{D}$ is arbitrarily chosen. On the other hand, Theorem 15 in page 80 of [18] also implies that u satisfies (BS-PDE) and $u \in C^{2,2,1}(\mathbb{R}_{++}^3)$.

Step 2. Consider the special case of g satisfying $g(x,y) = x$ for $(x,y) \in \mathbb{R}_+^2$; in this case, u satisfies $u(x,y,T) = x\,\mathbb{E}\left[H_T^y\right]$ for $(x,y,T) \in \mathbb{R}_+^3$. We are going to show that u extends continuously to the boundaries $x = 0$, $y = 0$, and $T = 0$. (If H^y is a martingale for $y \in \mathbb{R}_+$, this step is entirely trivial. Indeed $x\mathbb{E}\left[H_T^y\right] = x$ clearly indicates that u is continuous on $\mathbb{R}_+^3$.)

Take an $\mathbb{R}_+$-valued sequence $(x_k)_{k\in\mathbb{N}}$ such that $\downarrow \lim_{k\to\infty} x_k = 0$. It follows from the supermartingale property of H^y that $|u(x_k,y,T) - u(0,y,T)| = x_k\,\mathbb{E}\left[H_T^y\right] \le x_k$ for all $(y,T) \in \mathbb{R}_+^2$. Therefore, $u(x_k,y,T)$ converges uniformly in (y,T) to $u(0,y,T)$. This ensures that u extends continuously to the boundary $x = 0$.

Let us prove the continuity at $T = 0$. Given any sequence $\mathbb{R}_+^3 \ni (x_k, y_k, T_k) \to (x, y, 0)$, it follows from Fatou's lemma and (5.1) that $\liminf_{k\to\infty} u(x_k,y_k,T_k) \ge x\,\mathbb{E}[\liminf_{k\to\infty} H_{T_k}^{y_k}] = x$. On the other hand, note that since $\mathbb{E}[H_{T_k}^{y_k}] \le 1$ holds for all k, $\limsup_{k\to\infty} u(x_k,y_k,T_k) \le x$. We then conclude that u extends continuously to $T = 0$.

Since $\lim_{k\to\infty} u(x_k,y,T) = u(x,y,T)$ uniformly in (y,T), in order to show that u extends continuously to $y = 0$, it suffices to show that for any $\mathbb{R}_+$-valued sequence $\{y_\ell\} \downarrow 0$, $\mathbb{E}\left[H_T^{y_\ell}\right]$ converges to $\mathbb{E}\left[H_T^0\right]$ uniformly, and that $\mathbb{R}_+ \ni T \mapsto \mathbb{E}\left[H_T^0\right]$ is continuous.

Let us prove the continuity of $\mathbb{R}_+ \ni T \mapsto \mathbb{E}\left[H_T^0\right]$ first. Recall $\mathbb{E}\left[H_T^0\right] = \mathbb{P}\left[\zeta^0 > T\right]$ from (2.2). It is clear that $\mathbb{R}_+ \ni T \mapsto \mathbb{P}\left[\zeta^0 > T\right]$ is right continuous. In order to show the left continuity of this map, it suffices to show $\mathbb{P}\left[\zeta^0 = T\right] = 0$ for any $T \in \mathbb{R}_+$. To this end, set $\tau = \inf\left\{t \ge 0 \,|\, Y_t^0 = 1\right\}$. It follows from the strong Markov property that

$$\mathbb{P}\left[\zeta^0 = T\right] = \int_0^T \mathbb{P}\left[\zeta^1 = T - s\right] \mathbb{P}\left[\tau \in ds\right]. \tag{5.5}$$

We have shown that $T \mapsto \mathbb{E}[H^1_T]$ is continuous at $T = 0$, moreover we also conclude from Step 1 that the last map is continuous at $T > 0$. Therefore, $\mathbb{R}_+ \ni T \mapsto \mathbb{E}\left[H^1_T\right]$ is continuous, which implies that $\mathbb{P}\left[\zeta^1 = t\right] = 0$ for any $t \in \mathbb{R}_+$. Combining the last fact with (5.5), we obtain that $\mathbb{P}\left[\zeta^0 = T\right] = 0$, which confirms the left continuity of $\mathbb{R}_+ \ni T \mapsto \mathbb{E}\left[H^0_T\right]$.

Now we prove $\lim_{\ell\to\infty} \mathbb{E}[H^{y_\ell}_T] = \mathbb{E}[H^0_T]$ for fixed T. On one hand, it follows from Fatou's lemma that $\mathbb{E}\left[H^0_T\right] \le \liminf_{\ell\to\infty} \mathbb{E}\left[H^{y_\ell}_T\right]$. On the other hand, it follows from Proposition 2.1 that $\left\{\mathbb{E}\left[H^{y_\ell}_T\right]\right\}_{\ell\in\mathbb{N}}$ is a nondecreasing sequence. This implies that $\limsup_{\ell\to\infty} \mathbb{E}[H^{y_\ell}_T] \le \mathbb{E}[H^0_T]$. Therefore we have shown $\uparrow \lim_{\ell\to\infty} \mathbb{E}[H^{y_\ell}_T] = \mathbb{E}[H^0_T]$.

To show that the convergence $\uparrow \lim_{\ell\to\infty} \mathbb{E}[H^{y_\ell}_T] = \mathbb{E}[H^0_T]$ is uniform, recall that $\mathbb{R}_+ \ni T \mapsto \mathbb{E}\left[H^0_T\right]$ is continuous. On the other hand, $\mathbb{R}_+ \ni T \mapsto \mathbb{E}[H^y_T]$ is continuous for $y > 0$. It then follows from Dini's theorem that the convergence of $\left\{\mathbb{E}[H^{y_\ell}_T]\right\}_{\ell\in\mathbb{N}}$ is uniform in T.

Step 3. The results of the previous two steps imply that $\mathbb{R}^3_+ \ni (x, y, T) \mapsto \mathbb{E}[S^{x,y}_T]$ is continuous on $\mathbb{R}^3_+$. Hence, for any sequence $\{(x_n, y_n, T_n)\}_{n\in\mathbb{N}}$ converging to (x, y, T) with (x_n, y_n, T_n) inside a bounded neighborhood of (x, y, T) for $n \in \mathbb{N}$, $\left\{S^{x_n,y_n}_{T_n}\right\}_{n\in\mathbb{N}}$ is a uniformly integrable family. Due to (1.3), $\sup_n \mathbb{E}[(Y^{y_n}_{T_n})^{2m}] \le C(1 + \sup_n y_n^{2m})$, which implies that $\{(Y^{y_n}_{T_n})^m\}_{n\in\mathbb{N}}$ is also a uniformly integrable family. Therefore, for a nonnegative payoff g which is at most linear growth in x and polynomial growth in y, $\left\{g(S^{x_n,y_n}_{T_n}, Y^{y_n}_{T_n})\right\}_{n\in\mathbb{N}}$ is bounded from above by a uniform integrable family $\left\{C\left(1 + S^{x_n,y_n}_{T_n} + (Y^{y_n}_{T_n})^m\right)\right\}_{n\in\mathbb{N}}$, which along with (5.1) implies that $u \in C(\mathbb{R}^3_+)$. $\square$

5.3. **Proof of Proposition 2.3.** Let $y \in \mathbb{R}_{++}$. When (2.3) is violated, it follows from Feller's test that $\mathbb{P}[\zeta^y = \infty] = 1$. Then, (2.2) implies that $H^y_{\cdot\wedge T}$ is a martingale for any $T \ge 0$. This confirms the implication (1) $\implies$ (2).

The proof of the implication (2) $\implies$ (1) is motivated by the proof of Proposition 3 in [14]. Let us define $I(y, T) := \mathbb{E}[H^y_T] = \mathbb{P}[\zeta^y > T]$ for $(y, T) \in \mathbb{R}^2_+$. Since $\mathbb{E}[S^{x,y}_T] = x\, I(y, T)$, it follows from Lemma 5.2 (choosing g such that $g(x, y) = x$ for $(x, y) \in \mathbb{R}^2_+$) that $I \in C(\mathbb{R}^2_+) \cap C^{2,1}(\mathbb{R}^2_{++})$ and that I satisfies

$$\begin{aligned} &\partial_T I - \frac{1}{2}\sigma^2(y)\, \partial^2_{yy} I - (\mu(y) + \rho b(y)\sigma(y))\, \partial_y I = 0, \quad (y, T) \in \mathbb{R}^2_{++}, \\ &I(y, 0) = 1, \quad y \in \mathbb{R}_+. \end{aligned} \tag{5.6}$$

When (2.3) is satisfied, it follows from Feller's test for explosions that $\lim_{T\to\infty} I(y, T) < 1$ for all $y \in \mathbb{R}_{++}$. Pick sufficiently large T^* such that $I(1, T^*) < 1$. We claim that

$$I(y, T^*) < 1 \quad \text{for all } y \in \mathbb{R}_{++}. \tag{5.7}$$

We shall prove this by contradiction. Suppose that there exists $y^* \in \mathbb{R}_{++}$ such that $I(y^*, T^*) = 1$. For any $y > 0$, consider an open domain A which contains both 1 and y^* and whose closure $\overline{A}$ is a compact subset of $\mathbb{R}_{++}$. Then I attains its maximum at (y^*, T^*) over the cylindrical domain $A \times [0, T^* + 1]$. Note that I satisfies the uniformly parabolic equation (5.6) in $A \times (0, T^* + 1)$. Then the maximum principle for parabolic partial differential equations (see e.g. Chapter 2 in [18]) implies that $I(y,T)=1$ for any $0\le T\le T^*$ and $y\in A$. Therefore $I(1,T^*)=1$, which clearly contradicts with the choice of T^*.

Now define $\mathcal{S}(T)=\{y\in\mathbb{R}_{++}: I(y,T)=1\}$ and

$$T_* := \sup\left\{T\ge 0 : \mathcal{S}(T)\neq\emptyset\right\}, \tag{5.8}$$

with the convention that $T_*=\infty$ when the above set is empty. In fact, (5.7) implies $T_*<\infty$. We shall show $T_*=0$ in what follows.

Suppose $T_*>0$. Then for any $\delta\in(0,T_*/2)$, there exists a $y\in\mathbb{R}_{++}$ such that $I(y,T_*-\delta)=1$. Using the maximum principle as we did above, we obtain that

$$I(y,T)=1,\qquad \text{for any } 0\le T\le T_*-\delta \text{ and } y\in\mathbb{R}_+. \tag{5.9}$$

(Note that $I(0,T)=1$ follows because $I(\cdot,T)$ is nonincreasing for fixed $T\in\mathbb{R}_+$ — see Proposition 2.1.) Now, from the definition of I and the Markov property, we have $\mathbb{E}\left[H^y_T\,|\,\mathcal{F}_t\right]=I(Y^y_t,T-t)$ for all $(y,T)\in\mathbb{R}^2_{++}$. When $0\le t\le T_*-\delta$ and $0\le T-t\le T_*-\delta$, applying (5.9) to the previous identities, we obtain $I(y,T)=1$ for every $T\in[0,2(T_*-\delta)]$ and $y\in\mathbb{R}_{++}$. Note that $2(T_*-\delta)>T_*$, this contradicts with the definition of T_*. Therefore, $T_*=0$, which implies that $I(y,T)<1$ for any $(y,T)\in\mathbb{R}^2_{++}$. □

6. The Notion of a Stochastic Solution

Here, we shall introduce the notion of a stochastic solution of (BS-PDE). Its definition is motivated by Definition 3.1 in [24], Definition 2.2 in [26], and [37]. In [24], a similar notion of stochastic solutions was introduced to study a Neumann problem associated with a reflected Brownian motion residing in a compact domain.

Definition 6.1. Consider a continuous function $v:\mathbb{R}^3_+\to\mathbb{R}_+$. For $(x,y,T)\in\mathbb{R}^3_{++}$, define $V^{x,y,T}=(V^{x,y,T}_t)_{t\in[0,T]}$ via $V^{x,y,T}_t=v(S^{x,y}_t,Y^y_t,T-t)$ for $t\in[0,T]$. Then, v is a *stochastic solution* of (BS-PDE), if for each $(x,y,T)\in\mathbb{R}^3_{++}$:

(i) $V^{x,y,T}$ is a local martingale on $[0,T]$,
(ii) $v\left(x,y,0\right)=g\left(x,y\right)$.

Proposition 6.2. *The value function u, defined in* (3.1)*, is a stochastic solution. In fact, u is the smallest stochastic solution.*

Proof. We have already shown in Lemma 5.2 that $u\in C(\mathbb{R}^3_+)$. Define $U^{x,y,T}=(U^{x,y,T}_t)_{t\in[0,T]}$ via $U^{x,y,T}_t=u(S^{x,y}_t,Y^y_t,T-t)$ for $t\in[0,T]$. In (3.2) we established that $U^{x,y,T}$ is a martingale on $[0,T]$. Therefore, u is a stochastic solution.

To show the second statement, we take another stochastic solution v and let $V^{x,y,T}$ be as in Definition 6.1. Since $V^{x,y,T}$ is a nonnegative local martingale, hence a supermartingale, we have

$$v(x,y,T)=V^{x,y,T}_0\ge\mathbb{E}\left[V^{x,y,T}_T\right]=\mathbb{E}\left[v\left(S^{x,y}_T,Y^y_T,0\right)\right]=\mathbb{E}\left[g\left(S^{x,y}_T,Y^y_T\right)\right]=u(x,y,T).$$

Therefore $v \geq u$ holds on $\mathbb{R}^3_{++}$. Thanks to the continuity of v and u on $\mathbb{R}^3_+$, the last inequality then holds on $\mathbb{R}^3_+$, which completes the proof. $\square$

The uniqueness of stochastic solutions for (BS-PDE) ties naturally to the martingale property of the asset-price process.

Proposition 6.3. *There exists a unique stochastic solution in the class of functions which are of at most linear growth in x and polynomial in y if and only if the asset-price process is a martingale. In that case, u is this unique solution.*

Proof. Let us define a function $\delta : \mathbb{R}^3_+ \mapsto \mathbb{R}$ via $\delta(x,y,T) := x - \mathbb{E}\left[S^{x,y}_T\right] = x - x\,\mathbb{E}\left[H^y_T\right]$ for $(x,y,T) \in \mathbb{R}^3_+$. Since H^y is a nonnegative local martingale for $y \in \mathbb{R}_+$, δ is nonnegative. Also,

$$\delta\left(S^{x,y}_t, Y^y_t, T-t\right) = S^{x,y}_t - \mathbb{E}\left[S^{x,y}_T \,|\, \mathcal{F}_t\right]$$

holds for all $(x,y,T) \in \mathbb{R}^3_+$ and $t \in [0,T]$, in view of the Markov property. It follows that $\left(\delta\left(S^{x,y}_t, Y^y_t, T-t\right)\right)_{t\in[0,T]}$ is a local martingale for all $(x,y,T) \in \mathbb{R}^3_+$. Since u is a stochastic solution of at most linear growth in x and polynomial in y (see (3.1)), $u+\delta$ is clearly another stochastic solution with the same growth property. As a result, if the stochastic solution is unique, the asset-price process must be a martingale. Otherwise, it follows from Proposition 2.3 that $S^{x,y}_{\cdot\wedge T}$ is a strict local martingale for any $(x,y,T) \in \mathbb{R}^3_{++}$; therefore, $\delta(x,y,T) > 0$ for $(x,y,T) \in \mathbb{R}^3_{++}$ and u and $u+\delta$ are two different stochastic solutions.

Now assume that the asset-price process is a martingale and take a stochastic solution v which is of at most linear growth in x and polynomial in y. The uniqueness follows once we show $v \equiv u$. We shall establish below that $v(x,y,T) = u(x,y,T)$ for all $(x,y,T) \in \mathbb{R}^3_{++}$. Then, the last identity can be extended to $\mathbb{R}^3_+$ thanks to the continuity of v and u.

Fix $(x,y,T) \in \mathbb{R}^3_{++}$, and take a localizing sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ of the local martingale $V^{x,y,T}$. Then,

$$v(x,y,T) = V^{x,y,T}_0 = \mathbb{E}\left[V^{x,y,T}_{\sigma_n\wedge T}\right] = \mathbb{E}\left[v\left(S^{x,y}_{\sigma_n\wedge T}, Y^y_{\sigma_n\wedge T}, T-\sigma_n\wedge T\right)\right]$$

holds for all $n \in \mathbb{N}$. The growth assumption on v implies that there exist constants C and m such that

$$v\left(S^{x,y}_{\sigma_n\wedge T}, Y^y_{\sigma_n\wedge T}, T-\sigma_n\wedge T\right) \leq C\left(1 + xH^y_{\sigma_n\wedge T} + \left(Y^y_{\sigma_n\wedge T}\right)^m\right).$$

Since H^y is a martingale, $\left\{H^y_{\sigma_n\wedge T}\right\}_{n\in\mathbb{N}}$ is a uniformly integrable family. On the other hand, it follows from (1.3) that $\sup_n \mathbb{E}_y\left[\left(Y_{\sigma_n\wedge T}\right)^{2m}\right] < \infty$, hence $\left\{Y^m_{\sigma_n\wedge T}\right\}_{n\in\mathbb{N}}$ is also a uniformly integrable family. As a result, $\{v\left(S_{\sigma_n\wedge T}, Y_{\sigma_n\wedge T}, T-\sigma_n\wedge T\right)\}_{n\in\mathbb{N}}$ is a uniformly integrable family, which along with the continuity of v implies that

$$\begin{aligned}
v(x,y,T) &= \lim_{n\to\infty} \mathbb{E}\left[v\left(S^{x,y}_{\sigma_n\wedge T}, Y^y_{\sigma_n\wedge T}, T-\sigma_n\wedge T\right)\right] \\
&= \mathbb{E}\left[\lim_{n\to\infty} v\left(S^{x,y}_{\sigma_n\wedge T}, Y^y_{\sigma_n\wedge T}, T-\sigma_n\wedge T\right)\right] \\
&= \mathbb{E}\left[v\left(S^{x,y}_T, Y^y_T, 0\right)\right] = \mathbb{E}\left[g\left(S^{x,y}_T, Y^y_T\right)\right] = u(x,y,T).
\end{aligned}$$

This completes the proof. $\square$

When the payoff g is of strictly sublinear growth in x and polynomial in y, the uniqueness for stochastic solutions always holds, no matter whether the asset-price process is a martingale or not.

Proposition 6.4. *Let g be of strictly sublinear growth in x and polynomial in y, i.e., suppose that there exist positive constants C, m, and a strictly sublinear growth function f such that $g(x, y) \leq C(1 + f(x) + y^m)$ for all $(x, y) \in \mathbb{R}_+^2$. Then u is the unique stochastic solution.*

Proof. Fix $T \in \mathbb{R}_+$. Since $\lim_{x\to\infty} f(x)/x = 0$, there exists a nondecreasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{\infty\}$ with $\lim_{x\to\infty} \phi(x)/x = \infty$, such that $\phi(f(x)) \leq x$ holds for all $x \in \mathbb{R}_+$. Therefore, for any localizing sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ of the local martingale $V^{x,y,T}$, we have

$$\mathbb{E}\left[\phi\left(f\left(S^{x,y}_{\sigma_n\wedge T}\right)\right)\right] \leq \mathbb{E}\left[xH^y_{\sigma_n\wedge T}\right] \leq x, \qquad \text{for all } n \in \mathbb{N}.$$

From de la Vallée-Poussin criterion, $\left\{f\left(S^{x,y}_{\sigma_n\wedge T}\right)\right\}_{n\in\mathbb{N}}$ is a uniformly integrable family. The rest follows from arguments similar to the ones used in the proof of Proposition 6.3. □

7. Proof of Theorem 3.2

In order to prove Theorem 3.2, we shall show in this section that classical solutions according to Definition 3.1 are in fact stochastic solutions, as introduced in the last section. Then, the existence part of Theorem 3.2 follows from Lemma 5.2 and Proposition 6.2. The boundary condition (3.3) is satisfied by u in case (B) since $(S^{x,0}_T, Y^0_T) = (x, 0)$ for any $T \in \mathbb{R}_+$ in this case. The uniqueness part follows from Propositions 6.3 and 6.4.

Let us start with the following useful result.

Lemma 7.1. *Let σ be a stopping time and Z be a nonnegative continuous-path process with $Z = Z_{\sigma\wedge\cdot}$. If there exists a nondecreasing sequence of stopping times $\{\sigma_n\}_{n\in\mathbb{N}}$ with $\mathbb{P}[\lim_{n\to\infty} \sigma_n = \sigma] = 1$ such that $Z_{\sigma_n\wedge\cdot}$ is a martingale for all $n \in \mathbb{N}$, then Z is a local martingale.*

Proof. As $Z_{\sigma_n\wedge\cdot}$ is a nonnegative martingale, we have $\mathbb{P}\left[\sup_{t\in[0,\sigma_n]} Z_t > \ell\right] \leq 1/\ell$ for all $n \in \mathbb{N}$ and $\ell \in \mathbb{R}_+$. Since $Z = Z_{\sigma\wedge\cdot}$ and $\mathbb{P}[\lim_{n\to\infty} \sigma_n = \sigma] = 1$, we get $\mathbb{P}\left[\sup_{t\in\mathbb{R}_+} Z_t < \infty\right] = 1$. Thefore, defining $\widetilde{\sigma}_k := \inf\{t \in \mathbb{R}_+ \,|\, Z_t \geq k\}$ for $k \in \mathbb{N}$, we have $\mathbb{P}[\lim_{k\to\infty} \widetilde{\sigma}_k = \infty] = 1$. Furthermore, $\left\{Z_{\sigma_n\wedge\widetilde{\sigma}_k}\right\}_{n\in\mathbb{N}}$ is a uniformly integrable family for each k; indeed, this follows because $\mathbb{P}\left[\sup_{t\in[0,\widetilde{\sigma}_k]} Z_t \leq k\right] = 1$. We infer that $Z_{\widetilde{\sigma}_k\wedge\cdot}$ is a martingale for each $k \in \mathbb{N}$, which concludes the proof. □

For fixed $(x, y, T) \in \mathbb{R}^3_{++}$, recall that $V^{x,y,T}_t = v(S^{x,y}_t, Y^y_t, T - t)$ for $t \in [0, T]$. We define $V^{x,y,T}_\cdot$ on $\mathbb{R}_+$ via $V^{x,y,T}_\cdot = V^{x,y,T}_{\cdot\wedge T}$. Thanks to the previous lemma, to show that $V^{x,y,T}_\cdot$ is a local martingale on $[0, T]$, it suffices to find a sequence of stopping times $\{\sigma_n\}_{n\in\mathbb{N}}$ such that $\mathbb{P}[\lim_{n\to\infty} \sigma_n = T] = 1$ and $V^{x,y,T}_{\cdot\wedge\sigma_n}$ is a martingale for each n. We shall use this observation to study the relationship between classical solutions and stochastic solutions in the following three cases.

7.1. **Case (A).** We show here that a classical solution in the sense of Definition 3.1 case (A) is a stochastic solution. Fix $(x, y, T) \in \mathbb{R}^3_{++}$, consider the sequence of stopping times $(\sigma_n)_{n\in\mathbb{N}}$ defined via $\sigma_n := \inf\left\{t \in \mathbb{R}_+ \,|\, (S^{x,y}_t, Y^y_t) \notin [n^{-1}, n]^2\right\} \wedge (T - T/n)$ for each $n \in \mathbb{N}$. Given a classical solution v, it follows from Itô's formula that $V^{x,y,T}_{\cdot\wedge\sigma_n}$ is a martingale. As $\mathbb{P}[\tau_0 = \infty] = 1$, $\mathbb{P}[\lim_{n\to\infty} \sigma_n = T] = 1$; therefore, $V^{x,y,T}_{\cdot}$ is a local martingale on $[0, T]$ thanks to Lemma 7.1.

7.2. **Case (B).** We shall show that a classical solution in the sense of Definition 3.1 case (B) is a stochastic solution. Given such a classical solution v, the same argument as in case (A) implies that $V^{x,y,T}_{\cdot\wedge\sigma_n}$ is a martingale on $[0, T]$ for any $n \in \mathbb{N}$. Since $\mathbb{P}[\tau^y_0 < \infty] > 0$ in this case, we have $\mathbb{P}[\lim_{n\to\infty} \sigma_n = \tau^y_0]$. However, the boundary condition (3.3) implies that $V^{x,y,T} = V^{x,y,T}_{\tau^y_0\wedge\cdot}$. Invoking Lemma 7.1, we conclude that $V^{x,y,T}$ is a local martingale, which shows that v is a stochastic solution.

7.3. **Case (C).** We shall show that a classical solution in the sense of Definition 3.1 case (C) is a stochastic solution.

In the sequel, we fix $(x, y, T) \in \mathbb{R}_{++}$ and drop all superscripts involving x, y and T in order to ease notation. For $\epsilon \in [0, 1]$, let $L_t(\epsilon)$ denote the local time process for Y at the level ϵ accumulated up to time $t \in \mathbb{R}_+$. Recall that we choose L to be $\mathbb{P}$-a.s. jointly continuous in the time variable and càdlàg in the spatial variable; see Theorem 3.7.1 of [29]. For $n \in \mathbb{N}$, set

$$\sigma_n := \inf\left\{t \in \mathbb{R}_+ \;\middle|\; Y_t = n, \text{ or } S_t = n^{-1}, \text{ or } S_t = n\right\} \wedge (T - T/n).$$

It is clear that $\{\sigma_n\}_{n\in\mathbb{N}}$ is a nondecreasing sequence of stopping times with $\mathbb{P}[\lim_{n\to\infty} \sigma_n = T] = 1$.

Lemma 7.2. *With the above notation,* $\sup_{\epsilon\in(0,1]} \mathbb{E}\left[(L_{\sigma_n}(\epsilon))^2\right] < \infty$.

Proof. Let $C := \sup_{y\in[0,n]} \left(|\mu(y)| + \sigma^2(y)\right) < \infty$. From the Itô-Tanaka-Meyer formula, we obtain

$$\begin{aligned} L_{\sigma_n}(\epsilon) &\le \max\{Y_{\sigma_n} - \epsilon, 0\} - \int_0^{\sigma_n} \mathbb{I}_{(\epsilon,\infty)}(Y_t)\mu(Y_t)dt - \int_0^{\sigma_n} \mathbb{I}_{(\epsilon,\infty)}(Y_t)\sigma(Y_t)dB_t \\ &\le n + CT - \int_0^{\sigma_n} \mathbb{I}_{(\epsilon,\infty)}(Y_t)\sigma(Y_t)dB_t. \end{aligned}$$

Furthermore, we have from Itô isometry that

$$\mathbb{E}\left[\left|\int_0^{\sigma_n} \mathbb{I}_{(\epsilon,\infty)}(Y_t)\sigma(Y_t)dB_t\right|^2\right] \le \mathbb{E}\left[\int_0^T \mathbb{I}_{(\epsilon,\infty)}(Y_t)\sigma^2(Y_t)dt\right] \le CT.$$

Combining the last two bounds, we conclude that $\sup_{\epsilon\in(0,1]} \mathbb{E}\left[(L_{\sigma_n}(\epsilon))^2\right] < \infty$. $\square$

Let v be a classical solution in the sense of Definition 3.1 case (C). Fixing $n \in \mathbb{N}$, we shall show starting from the next paragraph that $V_{\sigma_n\wedge\cdot}$ is a martingale on $[0, T]$. (Recall that $V_t = v(S_t, Y_t, T - t)$ for $t \in [0, T]$.) As $\mathbb{P}[\lim_{n\to\infty} \sigma_n = T] = 1$, it will follow from Lemma 7.1 that V is a local martingale, which enables us to conclude that v is a stochastic solution.

A classical solution v is only expected to be smooth in the interior of the domain $\mathbb{R}^3_+$. Since Y hits zero with positive probability in this case, one cannot directly apply Itô's Lemma to V_t for $t > \tau_0$. Instead, we apply Itô's formula to a process that approximates V.

For $\epsilon \in (0, 1]$, we define ${}^{\epsilon}Y := \max\{Y, \epsilon\}$. It follows from the Itô-Tanaka-Meyer formula that

$$d\,{}^{\epsilon}Y_t = \mathbb{I}_{(\epsilon,\infty)}(Y_t)\left(\mu(Y_t)dt + \sigma(Y_t)dB_t\right) + dL_t(\epsilon).$$

Let ${}^{\epsilon}V$ be defined via ${}^{\epsilon}V_t := v(S_t, {}^{\epsilon}Y_t, T-t)$ for $t \in [0, T]$. Since $v \in C^{2,2,1}(\mathbb{R}^3_{++})$ and $(S, {}^{\epsilon}Y)$ takes values in $[n^{-1}, n] \times [\epsilon, n]$ for $t \in [0, \sigma_n]$, we can apply Itô's formula on $t \in [0, \sigma_n]$ and obtain

(7.1)

$$\begin{aligned}
{}^{\epsilon}V_{\cdot\wedge\sigma_n} &= v(x, y, T) - \int_0^{\cdot\wedge\sigma_n} \partial_T v(S_u, {}^{\epsilon}Y_u, T-u)du + \int_0^{\cdot\wedge\sigma_n} \partial_x v(S_u, {}^{\epsilon}Y_u, T-u) S_u b(Y_u)\, dW_u \\
&+ \int_0^{\cdot\wedge\sigma_n} \mathbb{I}_{(\epsilon,\infty)}(Y_u)\partial_y v(S_u, {}^{\epsilon}Y_u, T-u)\mu(Y_u)\, du + \int_0^{\cdot\wedge\sigma_n} \mathbb{I}_{(\epsilon,\infty)}(Y_u)\partial_y v(S_u, {}^{\epsilon}Y_u, T-u)\sigma(Y_u)\, dB_u \\
&+ \int_0^{\cdot\wedge\sigma_n} \partial_y v(S_u, {}^{\epsilon}Y_u, T-u)\, dL_u(\epsilon) + \frac{1}{2}\int_0^{\cdot\wedge\sigma_n} \mathbb{I}_{(\epsilon,\infty)}(Y_u)\partial^2_{yy} v(S_u, {}^{\epsilon}Y_u, T-u)\sigma^2(Y_u)\, du \\
&+ \int_0^{\cdot\wedge\sigma_n} \mathbb{I}_{(\epsilon,\infty)}(Y_u)\partial^2_{xy} v(S_u, {}^{\epsilon}Y_u, T-u)\rho\sigma(Y_u)b(Y_u)S_u\, du \\
&+ \frac{1}{2}\int_0^{\cdot\wedge\sigma_n} \partial^2_{xx} v(S_u, {}^{\epsilon}Y_u, T-u)S_u^2 b^2(Y_u)\, du.
\end{aligned}$$

Since $\{Y > \epsilon\} \subseteq \{{}^{\epsilon}Y = Y\}$, it follows from (BS-PDE) that

$$\int_0^{\cdot\wedge\sigma_n} \mathbb{I}_{(\epsilon,\infty)}(Y_u)\left[(\partial_T - \mathcal{L})v(S_u, Y_u, T-u)\right] du = 0. \tag{7.2}$$

On the other hand, $\int_0^{\cdot\wedge\sigma_n} \partial_y v(S_u, {}^{\epsilon}Y_u, T-u)\, dL_u(\epsilon) = \int_0^{\cdot\wedge\sigma_n} \partial_y v(S_u, \epsilon, T-u)\, dL_u(\epsilon)$, following from the fact that $\int_0^{\cdot\wedge\sigma_n} \mathbb{I}_{\{Y_u \neq \epsilon\}} dL_u(\epsilon) = 0$. Moreover, the two stochastic integrals in (7.1) are martingales thanks to the choice of σ_n. As a result, combining (7.1) and (7.2), and setting

$$\begin{aligned}
{}^{\epsilon}M := {}^{\epsilon}V &+ \int_0^{\cdot} \mathbb{I}_{[0,\epsilon]}(Y_u)\partial_T v(S_u, \epsilon, T-u)du - \int_0^{\cdot} \partial_y v(S_u, \epsilon, T-u)\, dL_u(\epsilon) \\
&- \frac{1}{2}\int_0^{\cdot} \mathbb{I}_{[0,\epsilon]}(Y_u)\partial^2_{xx} v(S_u, \epsilon, T-u)S_u^2 b^2(Y_u)\, du,
\end{aligned}$$

we have that ${}^{\epsilon}M_{\cdot\wedge\sigma_n}$ is a martingale for each $\epsilon \in (0, 1]$.

Next we shall study the limit of ${}^{\epsilon}M$ as $\epsilon \downarrow 0$ and establish

$$\mathbb{P}\text{-}\lim_{\epsilon\downarrow 0} \sup_{t\in[0,T]} \left|{}^{\epsilon}M_{t\wedge\sigma_n} - V_{t\wedge\sigma_n}\right| = 0. \tag{7.3}$$

First observe that

$$\begin{aligned}
&\sup_{t\in[0,T]} |\mathcal{M}_{t\wedge\sigma_n} - V_{t\wedge\sigma_n}| \\
&\quad\le \sup_{t\in[0,T]} |\mathcal{V}_{t\wedge\sigma_n} - V_{t\wedge\sigma_n}| + \sup_{t\in[0,T]} \int_0^{t\wedge\sigma_n} \mathbb{I}_{[0,\epsilon]}(Y_u)\,|\partial_T v(S_u,\epsilon,T-u)|\,du \\
&\qquad + \sup_{t\in[0,T]} \int_0^{t\wedge\sigma_n} |\partial_y v(S_u,\epsilon,T-u)|\;dL_u(\epsilon) \\
&\qquad + \frac{1}{2}\sup_{t\in[0,T]} \int_0^{t\wedge\sigma_n} \mathbb{I}_{[0,\epsilon]}(Y_u)\left|\partial^2_{xx} v(S_u,\epsilon,T-u)\right| S_u^2 b^2(Y_u)\,du,
\end{aligned}$$

We will show that each term on the right-hand-side of the previous inequality converges to zero in probability as $\epsilon \downarrow 0$. Let us denote $\mathcal{D}_n = [n^{-1}, n] \times [0, n] \times [T/n, T]$. First, the convergence of the first term follows from the continuity of v. Second, since $v \in \mathfrak{C}$, we have from the bounded convergence theorem and Lemma 1.4 that

$$\mathbb{P}\text{-}\lim_{\epsilon\downarrow 0} \sup_{t\in[0,T]} \int_0^{t\wedge\sigma_n} \mathbb{I}_{[0,\epsilon]}(Y_u)\,|\partial_T v(S_u,\epsilon,T-u)|\,du \le \sup_{(x,y,s)\in\mathcal{D}_n} |\partial_T v(x,y,s)| \int_0^{T\wedge\sigma_n} \mathbb{I}_{\{Y_u=0\}}\,du = 0.$$

On the other hand, recall that b is α-Hölder continuous with $b(0) = 0$, there exists $C \in \mathbb{R}_{++}$ such that $b^2(y) \le Cy^{2\alpha}$ for $y \in [0, \epsilon]$. As a result, an argument similar to the previous estimate shows that the fourth term in (7.3) also converges to zero. Finally, using Lemma 1.4 again, we have the following estimate for the third term,

$$\mathbb{P}\text{-}\lim_{\epsilon\downarrow 0} \sup_{t\in[0,T]} \int_0^{t\wedge\sigma_n} |\partial_y v(S_u,\epsilon,T-u)|\;dL_u(\epsilon) \le \sup_{(x,y,s)\in\mathcal{D}_n} |\partial_y v(x,y,s)| \cdot \mathbb{P}\text{-}\lim_{\epsilon\downarrow 0} L_{T\wedge\sigma_n}(\epsilon) = 0,$$

where the last identity follows from the right-continuity of $\epsilon \mapsto L_\cdot(\epsilon)$ and $L(0) = \mu(0)\int_0^\cdot \mathbb{I}_{\{Y_u=0\}}du$ — see Theorem 3.7.1 (iv) and Problem 3.7.6 in [29]. As a result, (7.3) follows combining all the previous estimates.

To finish the proof, we shall show that $V_{\cdot\wedge\sigma_n}$ is a martingale. Using again the facts that $v \in \mathfrak{C}$ and $(S, \mathcal{Y})$ takes values in $[n^{-1}, n] \times [\epsilon, n]$, we obtain the existence of $C \in \mathbb{R}_{++}$ (depending on v as well as n but independent of ϵ), such that

$$\sup_{t\in[0,T]} |\mathcal{M}_{t\wedge\sigma_n} - V_{t\wedge\sigma_n}| \le C(1 + L_{\sigma_n}(\epsilon)), \qquad \text{for } \epsilon \in (0, 1].$$

An application of Lemma 7.2 ensures the uniform integrability of $\left\{\sup_{t\in[0,T]} |\mathcal{M}_{t\wedge\sigma_n} - V_{t\wedge\sigma_n}|\right\}_{\epsilon\in(0,1]}$. As a result, we obtain

$$\lim_{\epsilon\downarrow 0} \mathbb{E}\left[\sup_{t\in[0,T]} |\mathcal{M}_{t\wedge\sigma_n} - V_{t\wedge\sigma_n}|\right] = 0.$$

Combining it with the martingale property of $\mathcal{M}$ for each $\epsilon \in (0, 1]$, we conclude that $V_{\cdot\wedge\sigma_n}$ is a martingale.

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