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Probabilistic approach to the Dirichlet problem of perturbed stable processes

Renming Song

Department of Mathematics, University of Florida, Gainesville, FL 32611, USA

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Summary. Let X_t be a symmetric stable process of index α, $0 < \alpha < 2$, in R^d $(d \geqq 2)$. In this paper we deal with the perturbation of X_t by a multiplicative functional of the following form:

$$M_t = \exp\left\{\sum_{s \leqq t} F(X_{s-}, X_s)\right\}$$

with F being a function on $R^d \times R^d$ satisfying certain conditions. First we prove the following gauge theorem: If D is a bounded open domain of R^d, then the function $g(x) = E^x\{M(\tau_D)\}$ is either identically infinite on D or bounded on D, where τ_D is the first exit time from D. Then we formulate the Dirichlet problem associated with the perturbed symmetric stable process by using Dirichlet form theory. Finally we apply the gauge theorem to prove the existence and uniqueness of solutions to the Dirichlet problem mentioned above.

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It is well known that if L is a nice elliptic differential operator on R^d and $X = (X_t, P^x)$ is the corresponding diffusion process, then for any bounded regular open subset D of R^d and any continuous function f on the boundary ∂D of D, the unique solution to the Dirichlet problem of $Lu(x) = 0$ in D with boundary function f is given by

$$u(x) = E^x f(X_{\tau_D}),$$

where $\tau_D = \inf\{t > 0 : X_t \notin D\}$. Recently this probabilistic approach has been extended by many authors (see, for instance, [2, 7, 14] and the references therein), to treat the Dirichlet problem for equations of the form $(L + \mu)u(x) = 0$ by considering diffusion processes "killed" by a multiplicative functional associated with μ. For example, the solution of the Dirichlet problem of $(\Delta + \mu)u(x) = 0$ in D with boundary function f is given by

$$u(x) = E^x\{\exp(A(\tau_D)) f(X(\tau_D))\},$$

where X_t is the standard d-dimensional Brownian motion and A_t is the additive functional of X_t having μ as its Revuz measure. One of the most important questions in this extension was a criterion about the boundedness of the following gauge function

$$g(x) = E^x\{\exp(A(\tau_D))\} .$$

This was first studied by Chung and Rao in [5] and later generalized by many others (see [2, 6, 18] and their references).

While the continuous case has been studied extensively, there has been little systematic discussion about the Dirichlet problem for discontinuous Markov processes. There is a brief discussion in [12] for solving the Dirichlet problem for the symmetric stable process by using a balayage method. In [15] the Dirichlet problem for a class of infinitely divisible processes is discussed, but the formulation there is purely probabilistic and is not interpreted in analytic language. Another reference is [13] in which the Dirichlet problem for general discontinuous Markov processes is formulated and solved by using characteristic operators; and when the underlying process is a symmetric stable process an equivalent analytical formulation is also given. References [8] and [9] also contain some related discussions.

The purpose of this paper is to give a probabilistic treatment of the Dirichlet problem for the following equation

$$-(-\Delta)^{\frac{\alpha}{2}}u(x) + \int_{R^d} \frac{G(x, y)u(y)}{|x-y|^{d+\alpha}}\,dy = 0 \tag{0.1}$$

in $D \subset R^d$, where $G(x, y)$ is a bounded function on $R^d \times R^d$ satisfying some additional conditions. More precisely, we are going to show that if X_t is a symmetric stable process of index α, $0 < \alpha < 2$, in R^d with the characteristic function $e^{-t|z|^\alpha}$ and if D is a regular bounded open subset of R^d such that the function

$$g(x) = E^x\left\{\exp\left(\sum_{s \leqq \tau_D} \ln(G+1)(X_{s-}, X_s)\right)\right\}$$

is not identically infinite on D, then for suitable bounded continuous functions f on D^c,

$$u(x) = E^x\left\{\exp\left(\sum_{s \leqq \tau_D} \ln(G+1)(X_{s-}, X_s)\right) f(X(\tau_D))\right\}$$

is the unique solution to (0.1) such that (a) $u|_{D^c} = f|_{D^c}$ and (b) $\lim_{D \ni x \to z} u(x) = f(z)$ for every $z \in \partial D$. (See Theorem 2.11 below.)

When compared with the case of a second order differential operator L, the Dirichlet problem for (0.1) in an open set D has some new features. First, instead of a boundary function as in the diffusion case, we have to use a function which is defined on all of the complement of D (which we call an exterior function) and we have to require that the solution to the Dirichlet problem coincide with the exterior function on the complement of D (because when X_t leaves D it can jump to any point in the complement of D). Secondly, we have to be careful about the phrase "u is a solution of (0.1) in D", especially when we encounter it for the first time. In the historical case where L is a differential operator, the phrase "u is a solution to the equation $Lu = 0$ in D" means that u is defined in D and satisfies the equation

$Lu(x) = 0$ for every $x \in D$. But in our case, the function u has to be defined everywhere on R^d because the operator in (0.1) is not a differential operator but rather an integral operator and the integration extends over the whole space, not just over D.

The new feature of this paper is that, for the first time, we have treated the perturbation of a symmetric stable process by a discontinuous multiplicative functional and formulated the associated Dirichlet problem very naturally using Dirichlet forms for symmetric stable processes. In all the previous studies of the Dirichlet problem, only continuous multiplicative functionals are considered, or equivalently, only local perturbations of the generator of the Markov process are considered.

The organization of this paper is as follows. In Sect. 1 we check that under our concrete assumptions all the conditions in the general Gauge Theorem (Theorem 3.4) of [18] are satisfied, so g is either identically infinite or bounded on D; some useful consequences of the boundedness of g are also given in this section. In Sect. 2 we give a new formulation of the Dirichlet problem by using Dirichlet form terminology and then the existence and uniqueness of the solution to the Dirichlet problem of (0.1) is proved.

1 The Gauge Theorem

In this paper we shall always assume that $X = (\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \theta_t, P^x)$ is a symmetric stable process of index α, $0 < \alpha < 2$, on R^d ($d \geqq 2$), with the following characteristic function

$$e^{-t|z|^\alpha}.$$

Then the pseudo-differential operator corresponding to X is $-(-\Delta)^{\frac{\alpha}{2}}$. (See, for instance, [4] or [19].)

For convenience, we shall assume that Ω is the space of all the right continuous maps ω from $[0, \infty)$ to R^d. Set $X_t(\omega) = \omega(t)$, and let $\mathscr{F}_t$ and $\mathscr{F}$ be the appropriate completions of $\mathscr{F}_t^0 = \sigma\{X_s : s \leqq t\}$ and $\mathscr{F}^0 = \sigma\{X_s : s \geqq 0\}$. For each $t \geqq 0$, $\theta_t : \Omega \to \Omega$ is the shift operator characterized by $X_s \circ \theta_t = X_{s+t}$.

As usual, we use (P_t) to denote the transition semigroup of X_t and U^λ, $\lambda \geqq 0$, to denote the λ-potential of (P_t),

$$U^\lambda f(x) = \int_0^\infty e^{-\lambda t} P_t f(x) dt; \quad 0 \leqq f \in L^\infty .$$

It is well known that X has a transition density $p(t, x, y)$ such that $p(t, \cdot, \cdot)$ is a bounded continuous function with $\lim_{|y| \to \infty} p(t, 0, y) = 0$.

(1.1) Definition. A bounded Borel function $F(\cdot, \cdot)$ on $R^d \times R^d$ is said to be *admissible with respect to X* if

(1) F vanishes on the diagonal;

(2) for any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|F(x, y)| < \varepsilon$$

whenever the distance between (x, y) and the diagonal of $R^d \times R^d$ is less than δ;

(3) the function

$$z \mapsto \int \frac{|F(z, y)|}{|z-y|^{d+\alpha}} dy$$

is bounded on R^d.

We are going to use $\mathscr{A}(X)$ to denote the collection of all the admissible functions with respect to X. When there is no confusion, we shall simply write $\mathscr{A}$ for $\mathscr{A}(X)$.

It is easy to see from the definition that if $F_1, F_2 \in \mathscr{A}$, then $cF_1, F_1 + F_2, F_1 F_2$ all belong to $\mathscr{A}$. Furthermore, we have the following

(1.2) Lemma. *If $F \in \mathscr{A}$, then $e^F - 1 \in \mathscr{A}$.*

Proof. It is easy to see that the bounded function $G = e^F - 1$ satisfies the first two conditions in the above definition, so we need only to check the third one. It follows from the elementary fact

$$\lim_{a \to 0} \frac{e^a - 1}{a} = 1$$

that there exists an $\varepsilon > 0$ such that

$$|e^a - 1| < 2|a|$$

whenever $|a| < \varepsilon$. Now by the second condition in the above definition we know that there exists a $\delta > 0$ such that whenever the distance between (x, y) and the diagonal of $R^d \times R^d$ is less than δ we have $|F(x, y)| < \varepsilon$. Consequently we have

$$|e^{F(x, y)} - 1| < 2|F(x, y)| .$$

Therefore for any $z \in R^d$,

$$\int_{R^d} \frac{|G(z, y)|}{|z-y|^{d+\alpha}} dy \leqq \|G\|_\infty \cdot \int_{\{|y-z| \geqq \delta\}} \frac{1}{|z-y|^{d+\alpha}} dy + 2 \int_{\{|y-z| < \delta\}} \frac{|F(z, y)|}{|z-y|^{d+\alpha}} dy ,$$

hence the third condition in the definition above is also satisfied. □

In the remainder of this section we are going to fix an $F \in \mathscr{A}$. Let F^+ and F^- be the positive and negative parts of F respectively. Put

$$A_t = \sum_{0 < s \leqq t} F(X_{s-}, X_s) ,$$

$$A_t^+ = \sum_{0 < s \leqq t} F^+(X_{s-}, X_s) ,$$

$$A_t^- = \sum_{0 < s \leqq t} F^-(X_{s-}, X_s) ,$$

$$M_t = e^{A_t} ,$$

$$L_t = e^{A_t^+} ,$$

$$K_t = e^{-A_t^-} .$$

(Notice that the finiteness of A_t, A_t^+ and A_t^- is implied by Definition 1.1) Then $\{M_t, t \geqq 0\}$, $\{L_t, t \geqq 0\}$ and $\{K_t, t \geqq 0\}$ are multiplicative functionals of X and furthermore for all $t \geqq 0$, $M_t = K_t \cdot L_t$, $K_t \leqq 1$, $L_t \geqq 1$. The result below is essential in proving the gauge theorem and its consequences.

(1.3) Theorem. *There exists a $b > 0$ such that*

$$\sup_{x \in R^d} E^x\{M_t\} \leqq e^{bt} .$$

Proof. We need only to prove the theorem for a positive $F \in \mathscr{A}$. Put $G = e^F - 1$ and

$$B_t = \sum_{0 < s \leqq t} G(X_{s-}, X_s) .$$

Then

$$B_t^p := \int_0^t \int_{R^d} \frac{G(X_s, y)}{|X_s - y|^{d+\alpha}} dyds$$

is the dual predictable projection of B_t. Therefore $B_t - B_t^p$ is a P^x-martingale for every $x \in R^d$. Now it follows from the exponential formula (see, for instance, [7]) that

$$\begin{aligned} N_t &:= e^{B_t - B_t^p} \prod_{s \leqq t} (1 + G(X_{s-}, X_s)) e^{-G(X_{s-}, X_s)} \\ &= e^{-B_t^p} \prod_{s \leqq t} (1 + G(X_{s-}, X_s)) \\ &= e^{A_t - B_t^p} \end{aligned}$$

is a local martingale under P^x for any $x \in R^d$, where the convergence of the infinite product in the first line above is also guaranteed by the exponential formula. $\{N_t\}$ is clearly a multiplicative functional of X, so it is a supermartingale multiplicative functional of X. Therefore by Theorem 62.19 of [17] we know that there exists a unique Markov kernel Q^x from $(R^d, \mathscr{B})$ to $(\Omega, \mathscr{F})$ rendering the coordinate maps X Markov with the semigroup

$$Q_t f(x) = E^x[N_t f(X_t)]$$

and $Q^x(X_0 = x) = 1$.

And now we have

$$\begin{aligned} E^x(M_t) &= E^x(e^{A_t - B_t^p} e^{B_t^p}) \\ &= E_Q^x(e^{B_t^p}) , \end{aligned}$$

where E_Q^x denotes the expectation with respect to Q^x. The fact that $F \in \mathscr{A}$ implies that $G \in \mathscr{A}$, so the function

$$z \mapsto \int \frac{G(z, y)}{|z - y|^{d+\alpha}}$$

is bounded by a constant b, and consequently $B_t^p \leqq bt$. $\square$

(1.4) Corollary. *Let D be a bounded open subset of R^d. There exists a $\delta > 0$ such that for any Borel subset B of D whose Lebesgue measure is less than δ,*

$$\sup_{x \in R^d} E^x\{M(\tau_B)\} < \infty .$$

Proof. The proof is similar to that of Lemma 4 of [6], we give the complete proof here for the readers' convenience.

If $P^x\{\tau_B = 0\} = 1$, then $E^x\{M(\tau_B)\} = 1$. Otherwise $P^x\{\tau_B = 0\} = 0$ and we have

$$\begin{aligned} E^x\{M(\tau_B)\} &\leqq E^x\{L(\tau_B)\} \\ &= \sum_{n=0}^{\infty} E^x\{n < \tau_B; L_n E^{X_n}\{0 < \tau_B \leqq 1; L(\tau_B)\}\} \\ &= \sum_{n=0}^{\infty} E^x\{n < \tau_B; L_n E^{X_n}\{L_1\}\} \\ &\leqq C \sum_{n=0}^{\infty} E^x\{n < \tau_B; \exp(A_n^+)\} \end{aligned}$$

where

$$C = \sup_{x \in R^d} E^x\{L_1\} < \infty$$

by Theorem 1.3. The sum above is bounded by

$$\sum_{n=0}^{\infty} P^x\{n < \tau_B\}^{\frac{1}{2}} E^x\{\exp(2A_n^+)\}^{\frac{1}{2}} .$$

Applying Theorem 1.3 to $\{2A_t^+\}$ we get

$$\sup_{x \in R^d} E^x\{\exp(2A_n^+)\} \leqq e^{nb} ,$$

for some $b > 0$. Hence the sum in question is bounded by

$$\sum_{n=0}^{\infty} P^x\{n < \tau_B\}^{\frac{1}{2}} e^{\frac{nb}{2}} . \tag{1.5}$$

It is easy to see that there exists a $\delta > 0$ such that for any Borel subset B of D whose Lebesgue measure is less than δ we have

$$\sup_{x \in R^d} P^x\{1 < \tau_B\} \leqq e^{-2b} .$$

It follows by the Markov property that

$$P^x\{n < \tau_B\} \leqq e^{-2nb} .$$

Using this in (1.5) we see that the series converges. □

Now fix a bounded open domain D in R^d and define

$$g(x) = E^x(M(\tau_D)) .$$

The function g is called the gauge function of (A, D) with respect to the process X, or simply the gauge function of (A, D) if there is no confusion.

(1.6) Proposition. $\inf_{x \in R^d} g(x) > 0$.

Proof. Since $F \in \mathscr{A}$, the function

$$z \mapsto \int_{R^d} \frac{|F(z, y)|}{|z - y|^{d+\alpha}} dy$$

is bounded, by a constant C_1. It follows from the transience of the process X that

$$\sup_{x \in R^d} E^x \tau_D = C_2 < \infty .$$

Therefore

$$\sup_{x \in R^d} E^x \int_0^{\tau_D} \int_{R^d} \frac{|F(X_t, y)|}{|X_t - y|^{d+\alpha}} dy dt \leqq C_1 C_2 .$$

Hence by Jensen's inequality

$$\begin{aligned} g(x) &= E^x\{M(\tau_D)\} \\ &\geqq E^x\left\{\exp\left(- \sum_{0 < t \leqq \tau_D} |F|(X_{t-}, X_t)\right)\right\} \\ &\geqq \exp\left\{ - E^x \sum_{0 < t \leqq \tau_D} |F|(X_{t-}, X_t)\right\} \\ &= \exp\left\{ - E^x \int_0^{\tau_D} \int_{R^d} \frac{|F(X_t, y)|}{|X_t - y|^{d+\alpha}} dy dt\right\} \\ &\geqq e^{-C_1 C_2} > 0 . \end{aligned}$$

The proof is now complete. □

(1.7) Lemma. *There exists an* $\beta_0 > 0$ *such that*

$$\sup_{x \in R^d} E^x \int_0^{\tau_D} e^{-\beta t} dL_t < \infty ,$$

whenever $\beta > \beta_0$.

Proof. Since

$$\begin{aligned} \int_0^{\tau_D} e^{-\beta t} dL_t &= \sum_{t \leqq \tau_D} e^{-\beta t}\{L_t - L_{t-}\} \\ &= \sum_{0 < t \leqq \tau_D} e^{-\beta t} L_{t-}\{\exp(F^+(X_{t-}, X_t)) - 1\} \end{aligned}$$

we know that for any x,

$$E^x \int_0^{\tau_D} e^{-\beta t} dL_t = E^x \int_0^{\tau_D} e^{-\beta t} L_{t-} \int_{R^d} \frac{\exp(F^+(X_t, y)) - 1}{|X_t - y|^{d+\alpha}} dy dt .$$

It follows from $F^+ \in \mathscr{A}$ that $\exp(F^+) - 1$ is also in $\mathscr{A}$, thus the function

$$z \mapsto \int_{R^d} \frac{\exp(F^+(z, y)) - 1}{|z - y|^{d+\alpha}} dy$$

is bounded by a constant C. Applying Theorem 1.3 to F^+ we know that there exists a $b > 0$ such that

$$E^x L_t \leqq e^{bt} .$$

Take $\beta_0 = b$. Then for any $\beta > \beta_0$,

$$\begin{aligned} E^x \int_0^{\tau_D} e^{-\beta t} dL_t &\leqq C \cdot E^x \int_0^{\tau_D} e^{-\beta t} L_{t-} dt \\ &\leqq C \cdot \int_0^{\infty} e^{-\beta t} E^x L_t dt \\ &\leqq C \cdot \int_0^{\infty} e^{-(\beta - \beta_0)t} dt \\ &= C \cdot \frac{1}{\beta - \beta_0} . \end{aligned}$$

The proof is now complete. □

Let $X^D = (X_t^D, P^x)$ be X_t killed when it exits D:

$$X_t^D(\omega) = \begin{cases} X_t(\omega), & \text{if } t < \tau_D ; \\ \delta, & \text{if } \tau_D \leqq t , \end{cases}$$

and let $D_\delta = D \cup \{\delta\}$ be the one-point compactification of D.

We are going to use (P_t^D) to denote the transition semigroup of X_t^D and $U^{D,\lambda}$, $\lambda \geqq 0$ to denote the λ-potential of (P_t^D). It is known that X^D has a transition density function $p^D(t, x, y)$ such that $p^D(t, \cdot, \cdot)$ is bounded for every $t > 0$.

The proof of the following result is due to Murali Rao. I am very grateful to him for allowing me to use it here.

(1.8) Lemma. *For any $t > 0$, P_t^D is a compact operator from $L^\infty(D)$ to $L^\infty(D)$.*

Proof. Let $\{f_n\}$ be a bounded sequence in $L^\infty(D)$, i.e., there exists a $C_1 > 0$ such that for any $n > 0$, $\| f_n \|_\infty \leqq C_1$. Then $\{f_n\}$ is also bounded in $L^k(D)$ for any $k > 1$. Since for any $k > 1$, the unit ball in $L^k(D)$ is weakly compact, by applying the diagonal argument we see that there is a subsequence of $\{f_n\}$, say, $\{f_n\}$ itself, which is weakly convergent to an $f \in L^\infty(D)$. Hence for any $x \in D$ and any $t > 0$,

$$\lim_{n \to \infty} P_t^D f_n(x) = P_t^D f(x) .$$

Consequently for any $t > 0$, $\{P_t^D f_n\}$ converges in measure with respect to the Lebesgue measure m on D. Therefore for any $\varepsilon > 0$, there exists an integer N such that

$$m(\{y \in D : |P_{\frac{t}{2}}^D f_n(y) - P_{\frac{t}{2}}^D f(y)| > \varepsilon\}) < \varepsilon ,$$

whenever $n \geqq N$. Thus for any $x \in D$ and any $t > 0$,

$$
\begin{aligned}
|P_t^D f_n(x) - P_t^D f(x)| &= \left| \int_D p^D\left(\frac{t}{2}, x, y\right)(P^D_{\frac{t}{2}} f_n(y) - P^D_{\frac{t}{2}} f(y))\,dy \right| \\
&\leqq \int_D p^D\left(\frac{t}{2}, x, y\right) |P^D_{\frac{t}{2}} f_n(y) - P^D_{\frac{t}{2}} f(y)|\,dy \\
&= \int_{\{|P^D_{\frac{t}{2}} f_n - P^D_{\frac{t}{2}} f| > \varepsilon\}} p^D\left(\frac{t}{2}, x, y\right) |P^D_{\frac{t}{2}} f_n(y) - P^D_{\frac{t}{2}} f(y)|\,dy \\
&\quad + \int_{\{|P^D_{\frac{t}{2}} f_n - P^D_{\frac{t}{2}} f| \leqq \varepsilon\}} p^D\left(\frac{t}{2}, x, y\right) |P^D_{\frac{t}{2}} f_n(y) - P^D_{\frac{t}{2}} f(y)|\,dy \\
&\leqq C_2 \varepsilon + \varepsilon\,,
\end{aligned}
$$

whenever $n \geqq N$, where

$$C_2 = 2 \max\left(C_1, \|f\|_\infty, \left\| p^D\left(\frac{t}{2}, \cdot, \cdot\right)\right\|_\infty\right).$$

The proof is now complete. □

(1.9) The Gauge Theorem. *The gauge function g is either identically infinite on D or bounded on D.*

Proof. Lemma 1.7 implies that M_t is compatible in the sense of [18]:

$$\sup_{x \in D} E^x \int_0^{\tau_D} e^{-\beta t} \cdot K_t dL_t < \infty$$

for some $\beta \in \mathrm{R}$. It follows from Lemma 1.8 above that $U^{D,\lambda}$ is compact as a map from $L^\infty(D)$ to $L^\infty(D)$ for any $\lambda \geqq 0$. The operator U_λ^K defined by

$$f \mapsto E^{\cdot} \int_0^{\tau_D} e^{-\lambda t} \cdot K_t f(X_t) dt$$

is dominated by $U^{D,\lambda}$ for any $\lambda \geqq 0$, consequently U_λ^K is compact from $L^\infty(D)$ to $L^\infty(D)$ for any $\lambda \geqq 0$ by Remark 3.5b of [18]. It again follows from Remark 3.5 of [18] that for any $\lambda \geqq 0$, U_λ^K is non-degenerate and irreducible in the following sense: $U_\lambda^K 1$ is not identically zero and for any nonnegative f, $U_\lambda^K f$ is either identically zero or strictly positive on D. Theorem 1.6 above guarantees that g is strictly positive. Thus all the conditions of the general Gauge Theorem (Theorem 3.4) of [18] are satisfied, and so the gauge function is either identically infinite or bounded on D. □

Now we are going to derive a consequence of the boundedness of the gauge function which will be very useful in the next section. But first we state a lemma which we will use in proving that result.

(1.10) Lemma. *If the gauge function g is bounded, then for any $\delta > 0$, there exists a constant $C(\delta)$ such that for any $x \in D$,*

$$c^{-1} g(x) \leqq \sum_{n=0}^{\infty} E^x\{n\delta < \tau_D; M(n\delta)\} \leqq C(\delta)\,.$$

Proof. The proof is similar to that of Lemma 9 of [6], we give the complete proof here for the readers' convenience.

Since the proof is the same for any $\delta > 0$ we take $\delta = 1$. For each $x \in D$ the quantity $E^x\{n < \tau_D; M(\tau_D)\}$ decreases to zero. Hence for any $\varepsilon > 0$, we can choose an integer N and a set F such that the Lebesgue measure of $D \backslash F$ is less than ε and

$$\sup_{x \in F} E^x\{N < \tau_D; M(\tau_D)\} < \varepsilon .$$

By choosing a compact subset of F if necessary we may assume that F itself is compact. Here ε is such that

$$\sup_{x \in R^d} E^x\{M^2(\tau_{D\backslash F})\} := \lambda < \infty .$$

This is possible by Corollary 1.4. Now

$$\begin{aligned}
&E^x\{2N < \tau_D; M(\tau_D)\} \\
&\quad \leqq E^x\{N < \tau_{D\backslash F}; M(\tau_D)\} + E^x\{N \geqq \tau_{D\backslash F}, 2N < \tau_D; M(\tau_D)\} \\
&\quad \leqq \|g\|_\infty E^x\{N < \tau_{D\backslash F}; M(\tau_{D\backslash F})\} \\
&\qquad + E^x\{\tau_{D\backslash F} < \tau_D; M(\tau_{D\backslash F}) E^{X(\tau_{D\backslash F})}\{N < \tau_D; M(\tau_D)\}\} .
\end{aligned}$$

The first term in the third line above can be estimated by

$$E^x\{M^2(\tau_{D\backslash F})\}^{\frac{1}{2}} P^x\{\tau_{D\backslash F} > N\}^{\frac{1}{2}}$$

which is uniformly small if N is large, by the transience of X. On the set $\{\tau_{D\backslash F} < \tau_D\}$, $X_{\tau_{D\backslash F}} \in F$, so the second term in the third line above is bounded by $\sqrt{\lambda\varepsilon}$. Thus for large N,

$$\sup_{x \in R^d} E^x\{N < \tau_D; M(\tau_D)\}$$

is small. Now from Proposition 1.6 g is bounded from below, say $g \geqq m > 0$. Then

$$E^x\{N < \tau_D; M(N)\} \leqq \frac{1}{m} E^x\{N < \tau_D; M(N) g(X_N)\} \leqq \frac{1}{m} E^x\{N < \tau_D; M(\tau_D)\} .$$

Therefore we can see that for N large enough

$$\sup_{x \in R^d} E^x\{N < \tau_D; M(N)\} \leqq \beta < 1 .$$

By the Markov property we have for any k,

$$\sup_{x \in R^d} E^x\{kN < \tau_D; M(kN)\} \leqq \beta^k .$$

Further if $j < N$,

$$\begin{aligned}
E^x\{kN + j < \tau_D; M(kN + j)\} &= E^x\{j < \tau_D; M(j) E^{x_j}\{kN < \tau_D; M(kN)\}\} \\
&\leqq \beta^k \sup_{j \leqq N} \sup_{x \in R^d} E^x\{M(j)\} := \beta^k \gamma .
\end{aligned}$$

All these inequalities give

$$\sum_{n=0}^{\infty} E^x\{n < \tau_D; M(n)\} = \sum_{k=0}^{\infty} \sum_{j=1}^{\infty} E^x\{kN + j < \tau_D; M(kN + j)\}$$

$$\leqq N\gamma \sum_{k=1}^{\infty} \beta^k = \frac{N\gamma}{1-\beta} := C(1) .$$

Finally we have

$$g(x) = \sum_{n=0}^{\infty} E^x\{n < \tau_D \leqq n + 1; M(\tau_D)\}$$

$$\leqq \sum_{n=0}^{\infty} E^x\{n < \tau_D; M(n) g(X_1)\}$$

$$\leqq \|g\|_\infty \sum_{n=0}^{\infty} E^x\{n < \tau_D; M(n)\}$$

which proves the result. □

(1.11) Theorem. *If the gauge function is bounded on D, then for any nonnegative $H \in \mathscr{A}$ we have*

$$\sup_{x \in D} E^x \sum_{0 < t \leqq \tau_D} e^{A_t}(1 - e^{-H(X_{t-}, X_t)}) < \infty .$$

Proof. Put

$$\tilde{A}_t = \sum_{0 < s \leqq t} (|F| + H)(X_{s-}, X_s) .$$

Then

$$e^{\tilde{A}_t} - 1 = \sum_{s \leqq t} (e^{\tilde{A}(s)} - e^{\tilde{A}(s-)}) = \sum_{0 < s \leqq t} e^{\tilde{A}(s)} (1 - e^{-(|F| + H)(X_{s-}, X_s)}) .$$

Since $A_t \leqq \tilde{A}_t$ and $\exp(-(|F| + H)(X_{s-}, X_s)) \leqq \exp(-H(X_{s-}, X_s))$, it follows that

$$E^x \sum_{0 < s \leqq t} e^{A(s)}(1 - e^{-H(X_{s-}, X_s)}) \leqq E^x\{e^{\tilde{A}(t)}(1 - e^{-(|F| + H)(X_{s-}, X_s)})\}$$

$$= E^x(e^{\tilde{A}(t)}) - 1 \leqq C(t)$$

where $C(t)$ is a constant independent of x, by Theorem 1.3 applied to $\tilde{A}$. Now we have

$$E^x \sum_{0 < t \leqq \tau_D} e^{A_t}(1 - e^{-H(X_{t-}, X_t)})$$

$$= E^x \left\{ \sum_{n=0}^{\infty} \sum_{\tau_D \wedge n < t \leqq \tau_D \wedge (n+1)} e^{A(t)}(1 - e^{-H(X_{t-}, X_t)}) \right\}$$

$$= \sum_{n=0}^{\infty} E^x \left\{ n < \tau_D; \sum_{n < t \leqq \tau_D \wedge (n+1)} e^{A(t)}(1 - e^{-H(X_{t-}, X_t)}) \right\}$$

$$= \sum_{n=0}^{\infty} E^x \left\{ n < \tau_D; e^{A(n)} E^{X_n} \left[\sum_{t \leqq \tau_D \wedge 1} e^{A(t)} (1 - e^{-H(X_{t-}, X_t)}) \right] \right\}$$

$$\leqq \sum_{n=0}^{\infty} E^x \left\{ n < \tau_D; M_n E^{X_n} \left[\sum_{t \leqq \tau_D \wedge 1} e^{\tilde{A}(t)} (1 - e^{-(|F| + H)(X_{t-}, X_t)}) \right] \right\}$$

$$\leqq C(1) \sum_{n=0}^{\infty} E^x \{ n < \tau_D; M_n \}$$

$$< \infty$$

where the last step follows from Lemma 1.10. □

Now here is the consequence of the boundedness of the gauge function we are going to use in the next section. It looks much stronger than the assumption that the gauge function is bounded.

(1.12) Theorem. *If the gauge function g is bounded on D*

$$\sup_{x \in D} E^x \left\{ \sup_{0 < t \leqq \tau_D} M_t \right\} < \infty .$$

Proof. We have

$$e^{A(t)} = 1 + \sum_{0 < s \leqq t} e^{A(s)} (1 - e^{-F(X_{s-}, X_s)})$$
$$\leqq 1 + \sum_{0 < s \leqq t} e^{A(s)} (1 - e^{-|F|(X_{s-}, X_s)}) ,$$

hence

$$\sup_{0 < t \leqq \tau_D} e^{A(t)} \leqq 1 + \sum_{0 < t \leqq \tau_D} e^{A(t)} (1 - e^{-|F|(X_{t-}, X_t)}) .$$

And now the conclusion follows from Theorem 1.11. □

(1.13) Remark. The results of this section are still true when A_t is replaced by the additive functional

$$\sum_{0 < s \leqq t} F(X_{s-}, X_s) + \int_0^t q(X_s) ds$$

for some Borel function q on R^d satisfying the following condition

$$\lim_{r \downarrow 0} \sup_{x \in R^d} \int_{|y - x| \leqq r} \frac{|q|(y) dy}{|y - x|^{d - \alpha}} = 0 .$$

2 The Dirichlet problem

As an application of the gauge theorem obtained in Sect. 1, we are going to formulate and solve the Dirichlet problem for the following generalized Schrödinger equation

$$(2.1) \qquad -(-\Delta)^{\frac{\alpha}{2}} u(x) + \int_{R^d} \frac{G(x, y) u(y)}{|x - y|^{d + \alpha}} dy = 0$$

on a bounded regular open subset D in R^d, where $G = e^F - 1$ for some $F \in \mathscr{A}$. But first we are going to introduce the Dirichlet form associated with our symmetric stable process X.

Put

$$\mathscr{E}(u, v) = \frac{1}{2} \int_{R^d} \int_{R^d} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{d+\alpha}} dxdy ,$$

$$\mathscr{C} = \left\{ u \in L^2(R^d): \int_{R^d} \int_{R^d} \frac{(u(x) - u(y))^2}{|x - y|^{d+\alpha}} dxdy < \infty \right\} .$$

Then $(\mathscr{E}, \mathscr{C})$ is the Dirichlet form of the symmetric stable process X of index α on R^d. Denote by $(\mathscr{E}, \mathscr{C}_e)$ the extended Dirichlet form of X. Then one of the characterizations of $\mathscr{C}_e$ is as follows (see [9]).

$$\mathscr{C}_e = \left\{ u \in L^1_{\text{loc}}(R^d): u \text{ is a tempered distribution and } \int |\hat{u}(x)|^2 |x|^\alpha dx < \infty \right\}$$

where

$$\hat{u}(x) = (2\pi)^{-\frac{d}{2}} \int_{R^d} e^{i(x,y)} u(y) dy ,$$

Another characterization of $\mathscr{C}_e$ is (see [9] again)

$$\mathscr{C}_e = \{ u = R^{\frac{\alpha}{2}} f: f \in L^2(R^d) \}$$

where $R^{(\alpha)}$ denotes the Riesz kernel of index α:

$$R^{(\alpha)}(x) = \frac{\Gamma\left(\frac{d - \alpha}{2}\right)}{2^\alpha \pi^{\frac{d}{2}} \Gamma\left(\frac{\alpha}{2}\right)} |x|^{\alpha - d} .$$

For a bounded open set D in R^d, we define

$$\mathscr{C}_D = \{ u \in \mathscr{C} : \tilde{u} = 0 \text{ q.e. on } D^c \}$$

where $\tilde{u}$ denotes a quasicontinuous version of u and "q.e." stands for quasi-everywhere. Then $(\mathscr{E}, \mathscr{C}_D)$ is the Dirichlet form of X^D.

Before we discuss the Dirichlet problem of (2.1), we are going to recall some useful facts for the reader's convenience.

(2.2) Definition. A bounded open set D in R^d is said to be *regular* if every point z on ∂D is regular for D^c, i.e., $P^z(\tau_D = 0) = 1$.

(2.3) Definition. A function $u \in \mathscr{C}_e$ is said to be *harmonic with respect to X* on D if $\mathscr{E}(u, v) = 0$ for any $v \in C_0(D) \cap \mathscr{C}_e = C_0(D) \cap \mathscr{C}_D$.

In the remainder of this paper a function which is harmonic with respect to X and D will simply said to be harmonic on D. Here we need to keep in mind that a function that is harmonic on D is required to be defined on all of R^d and even the part of the function on the complement of D is involved in the above definition.

(2.4) Lemma. *If $f \in \mathscr{C}_e$ is bounded and D is a bounded open set in R^d, then the function*

$$u(x) := E^x \tilde{f}(X(\tau_D))$$

is harmonic and continuous on D. Furthermore, if f is continuous on D^c and D is regular, then u is continuous everywhere and coincides with f on D^c.

Proof. That u is harmonic follows from the transient version of Theorem 4.4.1 of [9]. The rest are proven in [15]. □

(2.5) Lemma. *Let D be a bounded open set in R^d. If a bounded function $u \in \mathscr{C}_e$ is harmonic on D, then*

$$u_1(x) := E^x \tilde{u}(X(\tau_D))$$

is a version of u which is continuous in D.

Proof. Follows immediately from the transient version of Theorem 4.4.1 of [9] and Lemma 2.4 above. □

(2.6) Lemma. *Let D be a bounded open set in R^d. If f is a bounded function on R^d, then*

$$u(x) := E^x \int_0^{\tau_D} f(X_t)dt$$

is a bounded function which belongs to $\mathscr{C}$, continuous in D and

$$\mathscr{E}(u, v) = \int_{Rd} v(x)f(x)dx$$

for any $v \in C_0(D) \cap \mathscr{C}_e = C_0(D) \cap \mathscr{C}_D$. Furthermore if D is regular, then u is continuous on R^d and vanishes on D^c.

Proof. The first assertion is proven in [10] and the rest are proven in [15]. □

As in Sect. 1, we will fix an $F \in \mathscr{A}$, not necessarily nonnegative, and put

$$A_t = \sum_{0 < s \leqq t} F(X_{s-}, X_s) ;$$

$$M_t = e^{A_t} ;$$

$$G = e^F - 1 .$$

(2.7) Definition. A bounded function $u \in \mathscr{C}_e$ is called a *solution to the Eq.* (2.1) in D if

$$\mathscr{E}(u, v) - \int_{R^d} \int_{R^d} \frac{v(x)G(x, y)u(y)}{|x - y|^{d+\alpha}} dxdy = 0 \tag{2.8}$$

for any $v \in C_0(D) \cap \mathscr{C}_D$.

Again we need to bear in mind that a solution to the Eq. (2.1) in D is required to be defined on all of R^d and even the part of u on the complement of D plays a role in the definition above.

The definition above is a generalization of that of a solution to the Laplace equation. In fact, when the underlying process X is the standard Brownian motion, then for any F vanishing on the diagonal of $R^d \times R^d$, the additive functional

A_t above is identically zero, or equivalently, F can be replaced by the function which is zero everywhere, thus G can be replaced by the function which is zero everywhere and so the second term on the right hand side of (2.8) disappears and (2.8) becomes

$$\int_D \nabla u \cdot \nabla v dx = 0$$

for any $C_0(D) \cap H^1(D)$ and is easily seen to be equivalent to the definition of a solution to the Laplace equation by using the Green formula.

(2.9) Theorem. *Any solution u of* (2.1) *in D has a version which is continuous in D.*

Proof. Put

$$u_1(x) = E^x \int_0^{\tau_D} \left(\int_{R^d} \frac{G(X_t, y)u(y)}{|X_t - y|^{d+\alpha}} dy \right) dt .$$

Then it follows from Lemma 2.6 above the $u_1 \in \mathscr{C}$ is a bounded function which is continuous in D and

$$\mathscr{E}(u_1, v) = \int_{R^d} \int_{R^d} \frac{v(x)\, G(x, y)u(y)}{|x_t - y|^{d+\alpha}} dxdy$$

for any $v \in C_0(D) \cap \mathscr{C}_D$. Consequently the function $u_2 := u - u_1 \in \mathscr{C}_e$ satisfies the following condition

$$\mathscr{E}(u_2, v) = 0 ,$$

for any $v \in C_0(D) \cap \mathscr{C}_D$, i.e., u_2 is harmonic on D. Therefore, by Lemma 2.5 above, we know that

$$u_3(x) := E^x \tilde{u}_2(X_{\tau_D})$$

is a version of u_2 which is continuous in D. Thus the function $u_3 + u_1$ is a version of u that is continuous in D. □

(2.10) Definition. Suppose f is a bounded function in $\mathscr{C}_e$. We say that a bounded function u in $\mathscr{C}_e$ is a *solution to the Dirichlet problem of* (2.1) *with the exterior function* f if the following three conditions are satisfied:

(1) u is a solution of (2.1) in D;
(2) $u|_{D^c} = f|_{D^c}$;
(3) for every $z \in \partial D$,

$$\lim_{D \ni x \to z} u(x) = f(z) .$$

(2.11) Theorem. *Suppose that D is regular and that the gauge function g for* (D, A) *is bounded. Then for any $f \in \mathscr{C}_e$ which is bounded continuous on D^c,*

$$u(x) := E^x(M(\tau_D)f(X(\tau_D)))$$

is the unique continuous solution to the Dirichlet problem of (2.1) *in D with the exterior function f.*

Proof. It follows from the regularity of D, the boundedness of f and the boundedness of the gauge function that the function u defined above is bounded on R^d. It follows easily from the definition of u that

$$R_t := M(t \wedge \tau_D)u(X_{t \wedge \tau_D})$$

is a P^x-martingale for each $x \in D$. Using the integration by parts formula we get

$$\begin{aligned}
u(X_{t \wedge \tau_D}) &= e^{-\sum_{s \leqq t \wedge \tau_D} F(X_{s-}, X_s)} R_t \\
&= u(X_0) + \int_0^t e^{-\sum_{r < s \wedge \tau_D} F(X_{r-}, X_r)} dR_s \\
&\quad + \sum_{s \leqq t \wedge \tau_D} R_s \left(e^{-\sum_{r \leqq s \wedge \tau_D} F(X_{r-}, X_r)} - e^{-\sum_{r < s \wedge \tau_D} F(X_{r-}, X_r)}\right) \\
&= u(X_0) + \int_0^{t \wedge \tau_D} e^{-\sum_{r < s} F(X_{r-}, X_r)} dR_s \\
&\quad + \sum_{s \leqq t \wedge \tau_D} u(X_s)\left(1 - e^{F(X_{s-}, X_s)}\right).
\end{aligned}$$

Thus

$$\begin{aligned}
S_t &:= u(X_{t \wedge \tau_D}) - \sum_{s \leqq t \wedge \tau_D} u(X_s)\left(1 - e^{F(X_{s-}, X_s)}\right) \\
&= u(X_{t \wedge \tau_D}) + \sum_{s \leqq t \wedge \tau_D} u(X_s) G(X_{s-}, X_s)
\end{aligned}$$

is a local martingale with respect to P^x for each $x \in D$. Now, for any $t > 0$,

$$S_t \leqq \|u\|_\infty \left(1 + \sum_{s \leqq \tau_D} |G|(X_{s-}, X_s)\right),$$

and for each $x \in D$,

$$E^x \sum_{s \leqq \tau_D} |G|(X_{s-}, X_s) = E^x \int_0^{\tau_D} \int_{R^d} \frac{|G|(X_s, y)}{|X_s - y|^{d+\alpha}} dyds < \infty,$$

thus S_t is uniformly P^x-integrable for any $x \in D$, consequently N_t is a uniformly integrable P^x-martingale for any $x \in D$. Hence

$$\begin{aligned}
u(x) &= E^x u(X_{\tau_D}) + E^x \sum_{s \leqq \tau_D} G(X_{s-}, X_s) u(X_s) \\
&= E^x f(X_{\tau_D}) + E^x \int_0^{\tau_D} \int_{R^d} \frac{G(X_s, y) u(y)}{|X_s - y|^{d+\alpha}} dyds \\
&:= u_1(x) + u_2(x).
\end{aligned}$$

Now it follows from Lemma 2.6 that

(1) $u_2(x) \in \mathscr{C}_D$ is bounded continuous function which vanishes outside D;

(2) for any $v \in C_0(D) \cap \mathscr{C}_D$,

$$\mathscr{E}(u_2, v) = \int_{R^d} \int_{R^d} \frac{v(x) G(x, y) u(y)}{|x - y|^{d+\alpha}} dxdy.$$

And by Lemma 2.4 we know that

(1) $u_1 \in \mathscr{C}_e$ is a bounded function which is continuous in D;
(2) $u_1|_{D^c} = f|_{D^c}$;
(3) for any $z \in \partial D$,

$$\lim_{x \to z} u_1(x) = f(z)\,;$$

(4) for any $v \in C_0(D) \cap \mathscr{C}_D$,

$$\mathscr{E}(u_1, v) = 0\,.$$

Therefore $u = u_1 + u_2$ is a continuous solution to the Dirichlet problem of (2.1) with exterior function f.

Now let us assume that $\bar{u}$ is a continuous solution to the Dirichlet problem of (2.1) with the exterior function f. Then by Lemma 2.6 we know that the function

$$\bar{u}_2(x) := E^x \int_0^{\tau_D} \int_{R^d} \frac{G(X_s, y)\bar{u}(y)}{|X_s - y|^{d+\alpha}}\, dyds$$

satisfies the following

(1) $\bar{u}_2 \in \mathscr{C}$ is a bounded continuous function;
(2) for any $v \in C_0(D) \cap \mathscr{C}_D$,

$$\mathscr{E}(\bar{u}_2, v) = \int_{R^d} \int_{R^d} \frac{v(x)G(x, y)\bar{u}(y)}{|x - y|^{d+\alpha}}\, dxdy\,.$$

Therefore $\bar{u} - \bar{u}_2$ is a bounded continuous function in $\mathscr{C}_e$ such that

$$\mathscr{E}(\bar{u} - \bar{u}_2, v) = 0$$

for any $v \in C_0(D) \cap \mathscr{C}_D$, i.e., $\bar{u} - \bar{u}_2$ is harmonic on D. Hence by Lemma 2.5 we know that

$$(\bar{u}(x) - \bar{u}_2)(x) = E^x((\bar{u} - \bar{u}_2)(X_{\tau_D})) = E^x f(X_{\tau_D})\,,$$

consequently

$$\bar{u}(x) = E^x f(X_{\tau_D}) + E^x \int_0^{\tau_D} \int_{R^d} \frac{G(X_s, y)\bar{u}(y)}{|X_s - y|^{d+\alpha}}\, dyds\,.$$

From this we can easily get that

$$Z_t := \bar{u}(X_{t \wedge \tau_D}) + \sum_{s \leqq t \wedge \tau_D} \bar{u}(X_s)\, G(X_{s-}, X_s)$$

is a P^x-martingale for each $x \in D$. Using the integration by parts formula we get

$$M(t \wedge \tau_D)\bar{u}(X_{t \wedge \tau_D}) = \bar{u}(X_0) + \int_0^t M(s \wedge \tau_D -)d\bar{u}(X_{s \wedge \tau_D})$$
$$+ \sum_{s \leqq t \wedge \tau_D} \bar{u}(X_{s \wedge \tau_D})(M(s \wedge \tau_D) - M(s \wedge \tau_D -))$$

$$= \bar{u}(X_0) + \int_0^t M(s \wedge \tau_D -) d\bar{u}(X_{s \wedge \tau_D})$$

$$+ \sum_{s \leqq t \wedge \tau_D} M(s \wedge \tau_D -) \bar{u}(X_s) G(X_{s-}, X_s)$$

$$= \bar{u}(X_0) + \int_0^t M(s \wedge \tau_D -) dZ_s .$$

Therefore

$$Y_t := M(t \wedge \tau_D) \bar{u}(X_{t \wedge \tau_D})$$

is a local martingale under P^x for each $x \in D$. It follows from Theorem 1.11 $\{Y_t\}$ is uniformly P^x-integrable for each $x \in D$, thus $\{Y_t\}$ is a uniformly integrable P^x-martingale for each $x \in D$, consequently

$$\bar{u}(x) = E^x(M(\tau_D)\bar{u}(X_{\tau_D}))$$

$$= E^x(M(\tau_D) f(X_{\tau_D})) = u(x)$$

for each $x \in D$. The proof is now complete. □

(2.12) Remark. When (2.1) is replaced by the equation

$$-(-\Delta)^{\frac{\alpha}{2}} u(x) + q(x)u(x) + \int_{R^d} \frac{G(x, y)u(y)}{|x - y|^{d+\alpha}} dy = 0$$

and A_t is replaced by the additive functional

$$\sum_{0 < s \leqq t} F(X_{s-}, X_s) + \int_0^t q(X_s) ds$$

for some Borel function q on R^d satisfying the following condition

$$\lim_{r \downarrow 0} \sup_{x \in R^d} \int_{|y - x| \leqq r} \frac{|q|(y) dy}{|y - x|^{d-\alpha}} = 0 ,$$

the results of this section are still true.

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