

LINEAR-QUADRATIC OPTIMAL CONTROL PROBLEMS FOR MEAN-FIELD STOCHASTIC DIFFERENTIAL EQUATIONS — TIME-CONSISTENT SOLUTIONS

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ABSTRACT. Linear-quadratic optimal control problems are considered for mean-field stochastic differential equations with deterministic coefficients. Time-inconsistency feature of the problems is carefully investigated. Both open-loop and closed-loop equilibrium solutions are presented for such kinds of problems. Open-loop solutions are presented by means of the variational method with decoupling of forward-backward stochastic differential equations, which leads to a Riccati equation system lack of symmetry. Closed-loop solutions are presented by means of multi-person differential games, the limit of which leads to a Riccati equation system with a symmetric structure.

1. INTRODUCTION

Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ be a complete filtered probability space, on which a one-dimensional standard Brownian motion $W(\cdot)$ is defined with $\mathbb{F} \equiv \{\mathcal{F}_t\}_{t \geq 0}$ being its natural filtration augmented by all the \mathbb{P} -null sets. Consider the following controlled linear stochastic differential equation (SDE, for short):

$$(1.1) \quad \begin{cases} dX(s) = [A(s)X(s) + B(s)u(s)]ds \\ \quad + [C(s)X(s) + D(s)u(s)]dW(s), \quad s \in [t, T], \\ X(t) = x, \end{cases}$$

with a quadratic cost functional

$$(1.2) \quad J(t, x; u(\cdot)) = \mathbb{E}_t \left\{ \int_t^T [\langle Q(s)X(s), X(s) \rangle + \langle R(s)u(s), u(s) \rangle] ds + \langle GX(T), X(T) \rangle \right\}.$$

Here, $A(\cdot), B(\cdot), C(\cdot), D(\cdot), Q(\cdot), R(\cdot)$ are suitable matrix-valued (deterministic) functions, G is a matrix, and $\mathbb{E}_t = \mathbb{E}[\cdot | \mathcal{F}_t]$ stands for the conditional expectation operator. In the above, $X(\cdot)$, valued in \mathbb{R}^n , is called the *state process*; $u(\cdot)$, valued in \mathbb{R}^m , is called the *control process*; and $(t, x) \in \mathcal{D}$ is called the *initial pair* where

$$\mathcal{D} = \{(t, x) \mid t \in [0, T], x \text{ is } \mathcal{F}_t\text{-measurable, } \mathbb{E}|x|^2 < \infty\}.$$

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Denote

$$\begin{aligned}\mathcal{U}[t, T] &\equiv L^2_{\mathbb{F}}(t, T; \mathbb{R}^m) \\ &= \{u : [t, T] \times \Omega \rightarrow \mathbb{R}^m \mid u(\cdot) \text{ is } \mathbb{F}\text{-progressively measurable,} \\ &\quad \mathbb{E} \int_t^T |u(s)|^2 ds < \infty\},\end{aligned}$$

which is the set of all control processes. Under some mild conditions on the coefficients, for any initial pair $(t, x) \in \mathcal{D}$ and a control $u(\cdot) \in \mathcal{U}[t, T]$, the state equation (1.1) admits a unique solution $X(\cdot) = X(\cdot; t, x, u(\cdot))$, and the cost functional $J(t, x; u(\cdot))$ is well-defined. A standard linear-quadratic (LQ, for short) optimal control problem can be stated as follows.

Problem (LQ). For any given initial pair $(t, x) \in \mathcal{D}$, find a $u^*(\cdot) \in \mathcal{U}[t, T]$ such that

$$(1.3) \quad J(t, x; u^*(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}[t, T]} J(t, x; u(\cdot)) \triangleq V(t, x).$$

Any $u^*(\cdot) \in \mathcal{U}[t, T]$ satisfying (1.3) is called an *optimal control* for the given initial pair (t, x) , the corresponding state process $X^*(\cdot)$ is called an *optimal state process* for (t, x) , $(X^*(\cdot), u^*(\cdot))$ is called an *optimal pair* for (t, x) , and $V(\cdot, \cdot)$ is called the *value function* of Problem (LQ).

It is well-known that [29] under proper conditions, the above Problem (LQ) admits a unique optimal pair $(X^*(\cdot), u^*(\cdot))$. Moreover, the following *Riccati equation*

$$(1.4) \quad \begin{cases} \dot{P}(s) + P(s)A(s) + A(s)^T P(s) + C(s)^T P(s)C(s) + Q(s) \\ \quad - [P(s)B(s) + C(s)^T P(s)D(s)][R(s) + D(s)^T P(s)D(s)]^{-1} \\ \quad \cdot [B(s)^T P(s) + D(s)^T P(s)C(s)] = 0, & s \in [0, T], \\ P(T) = G, \\ \det [R(s) + D(s)^T P(s)D(s)] > 0, & s \in [0, T], \end{cases}$$

has a unique solution $P(\cdot)$ which is positive definite, and the optimal control $u^*(\cdot)$ admits the following state feedback representation:

$$(1.5) \quad u^*(s) = -\Theta(s)X^*(s), \quad s \in [t, T],$$

with

$$(1.6) \quad \begin{aligned} \Theta(s) &= [R(s) + D(s)^T P(s)D(s)]^{-1} [B(s)^T P(s) + D(s)^T P(s)C(s)], \\ &\quad s \in [0, T], \end{aligned}$$

and $X^*(\cdot)$ being the solution to the following closed-loop system:

$$(1.7) \quad \begin{cases} dX^*(s) = [A(s) - B(s)\Theta(s)]X^*(s)ds \\ \quad + [C(s) - D(s)\Theta(s)]X^*(s)dW(s), & s \in [t, T] \\ X^*(t) = x. \end{cases}$$

Further, the value function $V(\cdot, \cdot)$ is explicitly given by

$$(1.8) \quad V(t, x) = \langle P(t)x, x \rangle, \quad \forall (t, x) \in \mathcal{D}.$$

Next, if $\Phi(\cdot; \cdot)$ is the *fundamental matrix* of (1.7), i.e.,

$$(1.9) \quad \begin{cases} d\Phi(s; t) = [A(s) - B(s)\Theta(s)]\Phi(s; t)ds \\ \quad + [C(s) - D(s)\Theta(s)]\Phi(s; t)dW(s), \quad s \in [t, T], \\ \Phi(t; t) = I, \end{cases}$$

then

$$(1.10) \quad X^*(s) = \Phi(s; t)x, \quad \forall s \in [t, T], \quad (t, x) \in \mathcal{D},$$

and

$$(1.11) \quad u^*(s) \equiv u^*(s; t, x) = -\Theta(s)\Phi(s; t)x, \quad s \in [t, T], \quad (t, x) \in \mathcal{D}.$$

Note that $\Theta(\cdot)$ is independent of the initial pair (t, x) . According to the above representation of the optimal pair, if we start from the initial pair $(\tau, X^*(\tau)) \in \mathcal{D}$ with $\tau \in (t, T]$ being any \mathbb{F} -stopping time, then we have

$$\begin{aligned} u^*(s; \tau, X^*(\tau)) &= -\Theta(s)\Phi(s; \tau)X^*(\tau) = -\Theta(s)\Phi(s; \tau)\Phi(\tau; t)x \\ &= -\Theta(s)\Phi(s; t)x = u^*(s; t, x), \quad s \in [\tau, T]. \end{aligned}$$

This means that the restriction $u^*(\cdot; t, x)|_{[\tau, T]}$ of the optimal control $u^*(\cdot; t, x)$ for the initial pair $(t, x) \in \mathcal{D}$ on a later time interval $[\tau, T]$ coincides with the optimal control $u^*(\cdot; \tau, X^*(\tau))$ for the initial pair $(\tau, X^*(\tau)) \in \mathcal{D}$. Such a phenomenon is called the *time-consistency* of Problem (LQ).

We now look at the situation for the following cost functional:

$$(1.12) \quad J^\lambda(t, x; u(\cdot)) = \mathbb{E}_t \left\{ \int_t^T e^{-\lambda(s-t)} [\langle Q(s)X(s), X(s) \rangle + \langle R(s)u(s), u(s) \rangle] ds + e^{-\lambda(T-t)} \langle GX(T), X(T) \rangle \right\},$$

for some constant $\lambda > 0$ which is referred to as a *discount rate*. We also call the function $s \mapsto e^{-\lambda s}$ an *exponential discounting*. If we denote

$$\tilde{Q}(s) = e^{-\lambda s}Q(s), \quad \tilde{R}(s) = e^{-\lambda s}R(s), \quad \tilde{G} = e^{-\lambda T}G,$$

then

$$(1.13) \quad \tilde{J}(t, x; u(\cdot)) \triangleq \mathbb{E}_t \left[\int_t^T [\langle \tilde{Q}(s)X(s), X(s) \rangle + \langle \tilde{R}(s)u(s), u(s) \rangle] ds + \langle \tilde{G}X(T), X(T) \rangle \right] = e^{-\lambda t} J^\lambda(t, x; u(\cdot)).$$

Therefore, the optimal controls for the problems with respective cost functionals (1.12) and (1.13) coincide. Since the problem with cost functional (1.13) is time-consistent, so is the problem with cost functional (1.12).

Note that the significant difference between the cost functionals $J^\lambda(t, x; u(\cdot))$ defined by (1.12) and $J(t, x; u(\cdot))$ defined by (1.2) is that the running cost rate and the terminal cost of $J^\lambda(t, x; u(\cdot))$ explicitly depend on the initial time moment t . However, from the above, we see that such a special dependence on t does not change the time-consistency of the problem. The reason is that exponential discounting exhibits a time-consistent memory effect.

It then will be interesting to know what will happen if we replace the exponential discounting by some general discounting function. To get some feeling, let us look at the following simple example.

Example 1.1. Consider a one-dimensional controlled linear SDE:

$$(1.14) \quad \begin{cases} dX(s) = u(s)ds + X(s)dW(s), & s \in [t, T], \\ X(t) = x, \end{cases}$$

with cost functional

$$(1.15) \quad J(t, x; u(\cdot)) = \mathbb{E}_t \left[\int_t^T \rho(s, t) |u(s)|^2 ds + g(t) |X(T)|^2 \right],$$

where $\rho(s, t)$ and $g(t)$ are deterministic non-constant, continuous and positive functions. Let $P(\cdot, t)$ be the unique solution to the following Riccati differential equation:

$$(1.16) \quad \begin{cases} P_s(s, t) + P(s, t) - \frac{P(s, t)^2}{\rho(s, t)} = 0, & s \in [t, T], \\ P(T, t) = g(t). \end{cases}$$

Observe the following:

$$\begin{aligned} & \mathbb{E}_t [g(t) |X(T)|^2] - P(t, t)x^2 \\ &= \mathbb{E}_t \int_t^T (P_s(s, t)X(s)^2 + 2P(s, t)X(s)u(s) + P(s, t)X(s)^2) ds. \end{aligned}$$

Hence,

$$\begin{aligned} J(t, x; u(\cdot)) &= \mathbb{E}_t \left[\int_t^T \rho(s, t) |u(s)|^2 ds + g(t) |X(T)|^2 \right] \\ &= P(t, t)x^2 + \mathbb{E}_t \int_t^T (\rho(s, t) |u(s)|^2 + 2P(s, t)X(s)u(s) \\ &\quad + [P_s(s, t) + P(s, t)]X(s)^2) ds \\ &= P(t, t)x^2 + \mathbb{E}_t \int_t^T \rho(s, t) \left| u(s) + \frac{P(s, t)}{\rho(s, t)} X(s) \right|^2 ds. \end{aligned}$$

Let

$$(1.17) \quad u^*(s; t, x) = -\frac{P(s, t)}{\rho(s, t)} X^*(s; t, x), \quad s \in [t, T],$$

with $X^*(\cdot; t, x)$ being the solution of the following closed-loop system:

$$(1.18) \quad \begin{cases} dX^*(s; t, x) = -\frac{P(s, t)}{\rho(s, t)} X^*(s; t, x) ds + X^*(s; t, x) dW(s), & s \in [t, T], \\ X^*(t; t, x) = x. \end{cases}$$

Then

$$\inf_{u(\cdot) \in \mathcal{U}[t, T]} J(t, x; u(\cdot)) = J(t, x; u^*(\cdot; t, x)) = P(t, t)x^2.$$

Hence, for any $\tau \in (t, T)$, if we let $P(\cdot, \tau)$ be the solution of (1.16) with t replaced by τ , then

$$\begin{aligned} J(\tau, X^*(\tau; t, x); u(\cdot)) &= \mathbb{E}_\tau \left[\int_\tau^T \rho(s, \tau) |u(s)|^2 ds + g(\tau) |\tilde{X}(T)|^2 \right] \\ &= P(\tau, \tau) X^*(\tau; t, x)^2 + \mathbb{E}_\tau \int_\tau^T \rho(s, \tau) \left| u(s) + \frac{P(s, \tau)}{\rho(s, \tau)} \tilde{X}(s) \right|^2 ds, \end{aligned}$$

where

$$\tilde{X}(\cdot) \equiv X(\cdot; \tau, X^*(\tau; t, x), u(\cdot)), \quad u(\cdot) \in \mathcal{U}[\tau, T].$$

Consequently, by denoting

$$\tilde{X}^*(\cdot) = X(\cdot; \tau, X^*(\tau; t, x), u^*(\cdot)|_{[\tau, T]}),$$

we have

$$\begin{aligned} & J(\tau, X^*(\tau; t, x); u^*(\cdot)|_{[\tau, T]}) \\ (1.19) \quad &= P(\tau, \tau)X^*(\tau; t, x)^2 + \mathbb{E}_\tau \int_\tau^T \rho(s, \tau) |u^*(s; t, x) + \frac{P(s, \tau)}{\rho(s, \tau)} \tilde{X}^*(s)|^2 ds \\ &\geq P(\tau, \tau)X^*(\tau; t, x)^2 = \inf_{u(\cdot) \in \mathcal{U}[\tau, T]} J(\tau, X^*(\tau; t, x), u(\cdot)) \\ &= J(\tau, X^*(\tau; t, x), u^*(\cdot; \tau, X^*(\tau; t, x))), \end{aligned}$$

and the inequality actually becomes an equality if and only if

$$(1.20) \quad \begin{cases} u^*(s; t, x) = u^*(s; \tau, X^*(\tau; t, x)), \\ X^*(s; t, x) = X^*(s; \tau, X^*(\tau; t, x)), \end{cases} \quad s \in [\tau, T].$$

Similar to (1.17), we should also have

$$(1.21) \quad u^*(s; \tau, X^*(\tau; t, x)) = -\frac{P(s, \tau)}{\rho(s, \tau)} X^*(s; \tau, X^*(\tau; t, x)), \quad s \in [\tau, T].$$

Combining (1.17) and (1.20)–(1.21), we obtain

$$(1.22) \quad \frac{P(s, \tau)}{\rho(s, \tau)} = \frac{P(s, t)}{\rho(s, t)}, \quad \forall t \leq \tau \leq s \leq T.$$

In particular, evaluating at $s = T$, one has

$$\frac{g(\tau)}{\rho(T, \tau)} = \frac{g(t)}{\rho(T, t)}.$$

This means that if $t \mapsto \frac{g(t)}{\rho(T, t)}$ is not a constant function, then a strict inequality in (1.19) will hold for some $\tau \in (t, T]$, which means that the restriction $u^*(\cdot)|_{[\tau, T]}$ of the optimal control $u^*(\cdot)$ (for the initial pair (t, x)) is not optimal on $[\tau, T]$ with initial pair $(\tau, X^*(\tau; t, x)) \in \mathcal{D}$. Such a feature is called *time-inconsistency* of the problem.

The next example exhibits a different situation.

Example 1.2. Consider state equation (1.14) with cost functional

$$(1.23) \quad J(t, x; u(\cdot)) = \mathbb{E}_t \left[\int_t^T |u(s)|^2 ds + |\mathbb{E}_t[X(T)]|^2 \right].$$

Note that in the above, $\mathbb{E}_t[X(T)]$ appears non-linearly, which is non-classical. To solve the corresponding LQ problem, let us introduce the following Riccati equation:

$$\begin{cases} \dot{\hat{P}}(s) - \hat{P}(s)^2 = 0, & s \in [0, T], \\ \hat{P}(T) = 1, \end{cases}$$

whose solution is given by

$$\hat{P}(s) = \frac{1}{T - s + 1}, \quad s \in [0, T].$$

For any given $(t, x) \in \mathcal{D}$ and $u(\cdot) \in \mathcal{U}[t, T]$, let $X(\cdot) = X(\cdot; t, x, u(\cdot))$ be the corresponding state process. Then

$$\begin{cases} d\mathbb{E}_t[X(s)] = \mathbb{E}_t[u(s)]ds, & s \in [t, T], \\ \mathbb{E}_t[X(t)] = x. \end{cases}$$

Observe the following:

$$\begin{aligned} J(t, x; u(\cdot)) &= \mathbb{E}_t \left[\int_t^T |u(s)|^2 ds + \widehat{P}(T) |\mathbb{E}_t[X(T)]|^2 \right] \\ &= \widehat{P}(0) |x|^2 + \mathbb{E}_t \int_t^T (|u(s)|^2 + \widehat{P}(s)^2 |\mathbb{E}_t[X(s)]|^2 \\ &\quad + 2P(s) \mathbb{E}_t[X(s)] u(s)) ds \\ &= \widehat{P}(0) |x|^2 + \mathbb{E}_t \int_t^T |u(s) + \widehat{P}(s) \mathbb{E}_t[X(s)]|^2 ds \geq \widehat{P}(0) |x|^2, \end{aligned}$$

with the equality holding when

$$u^*(s) = -\widehat{P}(s) \mathbb{E}_t[X^*(s)], \quad s \in [t, T],$$

where $X^*(\cdot)$ is the solution to the following closed-loop system:

$$\begin{cases} dX^*(s) = -\widehat{P}(s) \mathbb{E}_t[X^*(s)] ds + X^*(s) dW(s), & s \in [t, T], \\ X^*(t) = x. \end{cases}$$

Such an equation is called a mean-field SDE. The above yields

$$\begin{cases} d(\mathbb{E}_t[X^*(s)]) = -\widehat{P}(s) \mathbb{E}_t[X^*(s)] ds, & s \in [t, T], \\ \mathbb{E}_t[X^*(t)] = x. \end{cases}$$

Consequently,

$$\mathbb{E}_t[X^*(s)] = e^{-\int_t^s \widehat{P}(r) dr} x = e^{-\int_t^s \frac{dr}{T-r+1}} x = \frac{T-s+1}{T-t+1} x, \quad s \in [t, T].$$

Then

$$u^*(s) \equiv u^*(s; t, x) = -\widehat{P}(s) \mathbb{E}_t[X^*(s)] = -\frac{x}{T-t+1}, \quad s \in [t, T].$$

Note that $u^*(\cdot)$ stays as a constant on $[t, T]$, with (t, x) as a parameter. With such a control, the closed-loop system reads

$$\begin{cases} dX^*(s) = -\frac{x}{T-t+1} ds + X^*(s) dW(s), & s \in [t, T], \\ X^*(t) = x, \end{cases}$$

whose solution is given explicitly by the following:

$$\begin{aligned} X^*(s) &= e^{-\frac{1}{2}(s-t)+W(s)-W(t)} x \\ &\quad + \frac{x}{T-t+1} \int_t^s e^{-\frac{1}{2}(s-r)+W(s)-W(r)} dr, \quad s \in [t, T]. \end{aligned}$$

Consequently, for any $\tau \in (t, T]$, the optimal control starting from $(\tau, X^*(\tau))$ should be

$$u^*(s; \tau, X^*(\tau)) = -\frac{X^*(\tau)}{T-\tau+1}, \quad s \in [\tau, T],$$

which is \mathcal{F}_τ -measurable and is not \mathcal{F}_t -measurable for any $t < \tau$, whereas $u^*(s; t, x)$ is merely \mathcal{F}_t -measurable, with $t < \tau$. Hence,

$$u^*(s; \tau, X^*(\tau)) = u^*(s; t, x), \quad s \in [\tau, T], \text{ a.s.,}$$

cannot be true. This implies that the problem associated with (1.14) and (1.23) is time-inconsistent.

The above two examples present two different situations for which the associated problems are time-inconsistent. In the first example, the time-inconsistency is due to the fact that the discounting is non-exponential, and in the second example, the time-inconsistency is due to the appearance of the conditional expectation of the state in a non-linear way. The non-exponential discounting situation has been discussed in [8–10, 19, 25–27]. The main motivation for that is to try to catch people's subjective preferences on the discounting (see [6, 11–13, 15, 17, 20–24]). The case with non-linear appearance of conditional expectation in the terminal cost of the cost functional has been studied in [2, 3, 14]. A main motivation of that is a kind of conditional/dynamic mean-variance problem (see [1, 4]). See also [5, 7, 16] for relevant results. On the other hand, in [28], an LQ problem for mean-field SDEs in a fixed time-duration was investigated (without mentioning the time-inconsistency), for which a motivation is from the optimal control theory point of view: People might like to have the optimal control, as well as the optimal state, to be not too “random”. To achieve that, one could include conditional variation $\text{var}_t[X(s)]$ of the state process and/or conditional variation $\text{var}_t[u(s)]$ of the control process in the cost functional, where

$$\begin{aligned} \text{var}_t[X(s)] &= \mathbb{E}_t |X(s) - \mathbb{E}_t[X(s)]|^2 = \mathbb{E}_t |X(s)|^2 - |\mathbb{E}_t[X(s)]|^2, \\ \text{var}_t[u(s)] &= \mathbb{E}_t |u(s) - \mathbb{E}_t[u(s)]|^2 = \mathbb{E}_t |u(s)|^2 - |\mathbb{E}_t[u(s)]|^2. \end{aligned}$$

Then $\mathbb{E}_t[X(s)]$ and $\mathbb{E}_t[u(s)]$ naturally appear quadratically.

The purpose of this paper is to study a general linear quadratic optimal control problem for mean-field stochastic differential equations (MF-SDEs, for short), which include the cases of non-exponential discounting and conditional mean-variance problems. We point out that the problem can be stated for MF-SDEs with random coefficients. However, at the moment, we only have a relatively complete theory for the deterministic coefficient case. Therefore, for the definiteness of the presentation, we restrict ourselves to the deterministic coefficients. A theory for the general random coefficient case will be reported in a future publication.

The rest of the paper is organized as follows. Section 2 will collect some basic preliminary results. In Section 3, we formulate our linear-quadratic optimal control problem for mean-field SDEs (MF-LQ problem for short), and the so-called pre-commitment solution to such a problem (which is a result from [28]). In Section 4, we introduce open-loop and closed-loop time-consistent solutions to our MF-LQ problem. We derive the Riccati equation system for open-loop equilibrium solutions by means of the variational method and decoupling a forward-backward stochastic differential equation (FBSDE, for short). We show that as long as the Riccati equation system has a solution, the open-loop equilibrium control can be constructed. It turns out that the Riccati equation system for the open-loop equilibrium solution does not have a desired symmetric structure. Therefore, it seems to be difficult to establish the well-posedness for such a system in general. Further, in this section,

we state the main result of the paper: the existence of a closed-loop equilibrium solution via a Riccati equation system which has a symmetric structure. Some interesting special cases are indicated. In Section 5, a multi-person differential game is introduced from which we find a kind of approximate time-consistent strategy. Convergence of the approximate time-consistent strategy is established in Section 6, which leads to the Riccati equation system and its well-posedness.

2. PRELIMINARIES

In this section, we present some preliminary results which will be useful in the following sections. Let us first introduce some notation. For any $0 \leq t < T$, let

$$\begin{aligned}\mathcal{X}_t &= \{\xi : \Omega \rightarrow \mathbb{R}^n \mid \xi \text{ is } \mathcal{F}_t\text{-measurable, } \mathbb{E}_t|\xi|^2 < \infty\}, \\ \mathcal{X}[t, T] &= \left\{ X : [t, T] \times \Omega \rightarrow \mathbb{R}^n \mid X(\cdot) \text{ is } \mathbb{F}\text{-adapted,} \right. \\ &\quad \left. t \mapsto X(t) \text{ is continuous, } \mathbb{E}_t \left[\sup_{s \in [t, T]} |X(s)|^2 \right] < \infty \right\}.\end{aligned}$$

For any Euclidean space $H = \mathbb{R}^n, \mathbb{R}^{n \times m}, \mathbb{S}^n$ (where \mathbb{S}^n is the set of all $(n \times n)$ symmetric matrices), we let

$$\begin{aligned}C([t, T]; H) &= \{\varphi : [t, T] \rightarrow H \mid \varphi(\cdot) \text{ is continuous}\}, \\ C^1([t, T]; H) &= \{\varphi : [t, T] \rightarrow H \mid \varphi(\cdot), \dot{\varphi}(\cdot) \text{ are continuous}\}, \\ L^\infty(t, T; H) &= \{\varphi : [t, T] \rightarrow H \mid \|\varphi(\cdot)\| \equiv \operatorname{esssup}_{s \in [t, T]} |\varphi(s)| < \infty\}.\end{aligned}$$

Now, we consider the following linear mean-field SDE (MF-SDE, for short):

$$(2.1) \quad \begin{cases} dX(s) = [\mathcal{A}(s)X(s) + \bar{\mathcal{A}}(s)\mathbb{E}_t[X(s)] + b(s)]ds \\ \quad + [\mathcal{C}(s)X(s) + \bar{\mathcal{C}}(s)\mathbb{E}_t[X(s)] + \sigma(s)]dW(s), \quad s \in [t, T], \\ X(t) = x, \end{cases}$$

with suitable coefficients $\mathcal{A}(\cdot), \bar{\mathcal{A}}(\cdot), \mathcal{C}(\cdot), \bar{\mathcal{C}}(\cdot)$, and non-homogeneous terms $b(\cdot), \sigma(\cdot)$, as well as the initial pair $(t, x) \in \mathcal{D}$. We refer to $\{\mathcal{A}(\cdot), \bar{\mathcal{A}}(\cdot), \mathcal{C}(\cdot), \bar{\mathcal{C}}(\cdot)\}$ as the *generator* of the linear MF-SDE (2.1).

The following result is concerned with the well-posedness of (2.1).

Proposition 2.1. *Let $\mathcal{A}(\cdot), \bar{\mathcal{A}}(\cdot), \mathcal{C}(\cdot), \bar{\mathcal{C}}(\cdot)$ be in $L^\infty(0, T; \mathbb{R}^{n \times n})$. Then for any $(t, x) \in \mathcal{D}$ and $b(\cdot), \sigma(\cdot) \in L^2_{\mathbb{F}}(0, T; \mathbb{R}^n)$, MF-SDE (2.1) admits a unique solution $X(\cdot) \in \mathcal{X}[t, T]$. Moreover, the following variation of constants formula holds:*

$$(2.2) \quad \begin{aligned}X(s) &= \Psi(s, t) \left\{ \left[I + \int_t^s \Psi(r, t)^{-1} (\bar{\mathcal{A}}(r) - \mathcal{C}(r)\bar{\mathcal{C}}(r)) \bar{\Phi}(r, t) dr \right. \right. \\ &\quad \left. \left. + \int_t^s \Psi(r, t)^{-1} \bar{\mathcal{C}}(r) \bar{\Phi}(r, t) dW(r) \right] x \right. \\ &\quad + \int_t^s \Psi(r, t)^{-1} \left[b(r) + (\bar{\mathcal{A}}(r) - \mathcal{C}(r)\bar{\mathcal{C}}(r)) \int_t^r \bar{\Phi}(r, t) \mathbb{E}_t[b(\tau)] d\tau - \mathcal{C}(r)\sigma(r) \right] dr \\ &\quad \left. + \int_t^s \Psi(r, t)^{-1} \left[\bar{\mathcal{C}}(r) \int_t^r \bar{\Phi}(r, \tau) \mathbb{E}_t[b(\tau)] d\tau + \sigma(r) \right] dW(r) \right\}, \quad s \in [t, T],\end{aligned}$$

where $\bar{\Phi}(\cdot, \cdot)$ is the fundamental matrix of $\mathcal{A}(\cdot) + \bar{\mathcal{A}}(\cdot)$, and $\Psi(\cdot, \cdot)$ is the fundamental matrix of $\{\mathcal{A}(\cdot), \mathcal{C}(\cdot)\}$, i.e.,

$$(2.3) \quad \begin{cases} \dot{\bar{\Phi}}(s, t) = [\mathcal{A}(s) + \bar{\mathcal{A}}(s)]\bar{\Phi}(s, t), & s \in [t, T], \\ \bar{\Phi}(t, t) = I, \end{cases}$$

and

$$(2.4) \quad \begin{cases} d\Psi(s, t) = \mathcal{A}(s)\Psi(s, t)ds + \mathcal{C}(s)\Psi(s, t)dW(s), & s \in [t, T], \\ \Psi(t, t) = I. \end{cases}$$

Proof. First, we introduce the following linear ordinary differential equation (ODE, for short):

$$(2.5) \quad \begin{cases} d\bar{X}(s) = \{[\mathcal{A}(s) + \bar{\mathcal{A}}(s)]\bar{X}(s) + \mathbb{E}_t[b(s)]\}ds, & s \in [t, T], \\ \bar{X}(t) = x, \end{cases}$$

whose solution is given by

$$\bar{X}(s) = \bar{\Phi}(s, t)x + \int_t^s \bar{\Phi}(s, r)\mathbb{E}_t[b(r)]dr, \quad s \in [t, T],$$

where $\bar{\Phi}(\cdot, \cdot)$ is the fundamental matrix of $\mathcal{A}(\cdot) + \bar{\mathcal{A}}(\cdot)$ (which is deterministic). Then we consider the following linear non-homogeneous SDE:

$$\begin{cases} dX(s) = [\mathcal{A}(s)X(s) + \bar{\mathcal{A}}(s)\bar{X}(s) + b(s)]ds \\ \quad + [\mathcal{C}(s)X(s) + \bar{\mathcal{C}}(s)\bar{X}(s) + \sigma(s)]dW(s), & s \in [t, T], \\ X(t) = x. \end{cases}$$

It is standard that the above admits a unique solution $X(\cdot) \equiv X(\cdot; t, x, b(\cdot), \sigma(\cdot)) \in \mathcal{X}[t, T]$. Clearly, for such a solution, one has

$$\mathbb{E}_t[X(s)] = \bar{X}(s), \quad s \in [t, T].$$

Hence, $X(\cdot)$ is the unique solution to (2.1). The rest of the conclusion follows easily. \square

It is known that if $X(\cdot; t, x)$ is the solution to the following SDE:

$$(2.6) \quad \begin{cases} dX(s) = [\mathcal{A}(s)X(s) + b(s)]ds + [\mathcal{C}(s)X(s) + \sigma(s)]dW(s), & s \in [t, T], \\ X(t) = x, \end{cases}$$

for any given $(t, x) \in \mathcal{D}$, then

$$(2.7) \quad X(s; \tau, X(\tau; t, x)) = X(s; t, x), \quad t \leq \tau \leq s \leq T.$$

In other words, the map $\tau \mapsto (\tau, X(\tau; t, x))$ has the so-called semigroup property. We point out that this is not necessarily the case for MF-SDEs, in general. Here is a simple example.

Example 2.2. Consider

$$\begin{cases} dX(s) = \mathbb{E}_t[X(s)]ds + \mathbb{E}_t[X(s)]dW(s), & s \in [t, T], \\ X(t) = x. \end{cases}$$

Then

$$d\mathbb{E}_t[X(s)] = \mathbb{E}_t[X(s)]ds, \quad s \in [t, T],$$

which leads to

$$\mathbb{E}_t[X(s)] = e^{s-t}x, \quad s \in [t, T].$$

Hence, the original equation becomes

$$\begin{cases} dX(s) = e^{s-t}xds + e^{s-t}x dW(s), & s \in [t, T], \\ X(t) = x. \end{cases}$$

Then one has

$$\begin{aligned} X(s; t, x) &= \left[1 + \int_t^s e^{r-t}dr + \int_t^s e^{r-t}dW(r) \right] x \\ &= \left[e^{s-t} + \int_t^s e^{r-t}dW(r) \right] x, \quad s \in [t, T]. \end{aligned}$$

Now, for any $t < \tau < s$, we have

$$\begin{aligned} X(s; \tau, X(\tau; t, x)) &= \left[e^{s-\tau} + \int_\tau^s e^{r-\tau}dW(r) \right] X(\tau; t, x) \\ &= \left[e^{s-\tau} + \int_\tau^s e^{r-\tau}dW(r) \right] \left[e^{\tau-t} + \int_t^\tau e^{r-t}dW(r) \right] x \\ &= \left[e^{s-t} + \int_\tau^s e^{r-t}dW(r) + \int_t^\tau e^{s-\tau+r-t}dW(r) \right. \\ &\quad \left. + \left(\int_\tau^s e^{r-\tau}dW(r) \right) \left(\int_t^\tau e^{r-t}dW(r) \right) \right] x. \end{aligned}$$

Hence,

$$X(s; \tau, X(\tau; t, x)) - X(s; t, x) = \left[e^{s-\tau} + \int_\tau^s e^{r-\tau}dW(r) \right] \left(\int_t^\tau e^{r-t}dW(r) \right) x.$$

Consequently, as long as $x \neq 0$,

$$\begin{aligned} &\mathbb{E}_t |X(s; \tau, X(\tau; t, x)) - X(s; t, x)|^2 \\ &= \mathbb{E}_t \left\{ \mathbb{E}_\tau \left[e^{s-\tau} + \int_\tau^s e^{r-\tau}dW(r) \right]^2 \left(\int_t^\tau e^{r-t}dW(r) \right)^2 \right\} x^2 \\ &= \left[e^{2(s-\tau)} + \int_\tau^s e^{2(r-\tau)}dr \right] \left(\int_t^\tau e^{2(r-t)}dr \right) x^2 > 0. \end{aligned}$$

This shows that (2.7) fails in general.

The following result will be useful later.

Proposition 2.3.

$$(2.8) \quad \begin{cases} \mathcal{A}(\cdot), \mathcal{C}(\cdot) \in L^\infty(t, T; \mathbb{R}^n), & \mathcal{B}(\cdot), \mathcal{D}(\cdot), \mathcal{S}(\cdot)^T \in L^\infty(t, T; \mathbb{R}^{n \times m}), \\ \mathcal{Q}(\cdot) \in L^\infty(t, T; \mathbb{S}^n), & \mathcal{R}(\cdot) \in L^\infty(t, T; \mathbb{S}^m), \quad \mathcal{G} \in \mathbb{S}^n. \end{cases}$$

Suppose that for some $\delta > 0$,

$$(2.9) \quad \begin{cases} \mathcal{R}(s) \geq \delta I, & \mathcal{Q}(s) - \mathcal{S}(s)^T \mathcal{R}(s)^{-1} \mathcal{S}(s) \geq 0, & s \in [t, T], \\ \mathcal{G} \geq 0. \end{cases}$$

Then the following Riccati equation admits a unique solution $P(\cdot) \in C^1([t, T]; \mathbb{S}^n)$:

$$(2.10) \quad \begin{cases} \dot{P}(s) + P(s)\mathcal{A}(s) + \mathcal{A}(s)^T P(s) + \mathcal{C}(s)^T P(s)\mathcal{C}(s) + \mathcal{Q}(s) \\ - [P(s)\mathcal{B}(s) + \mathcal{S}(s)^T + \mathcal{C}(s)^T P(s)\mathcal{D}(s)] [\mathcal{R}(s) + \mathcal{D}(s)^T P(s)\mathcal{D}(s)]^{-1} \\ \cdot [\mathcal{B}(s)^T P(s) + \mathcal{S}(s) + \mathcal{D}(s)^T P(s)\mathcal{C}(s)] = 0, \quad s \in [t, T], \\ P(T) = \mathcal{G}, \\ \mathcal{R}(s) + \mathcal{D}(s)^T P(s)\mathcal{D}(s) > 0. \end{cases}$$

Further,

$$(2.11) \quad 0 \leq P(s) \leq \Pi(s) \leq K_0 I, \quad s \in [t, T],$$

where $\Pi(\cdot) \in C^1([0, T]; \mathbb{S}^n)$ is the unique solution to the following Lyapunov equation:

$$(2.12) \quad \begin{cases} \dot{\Pi}(s) + \Pi(s)\mathcal{A}(s) + \mathcal{A}(s)^T \Pi(s) + \mathcal{C}(s)^T \Pi(s)\mathcal{C}(s) + \mathcal{Q}(s) = 0, \quad s \in [t, T], \\ \Pi(T) = \mathcal{G}, \end{cases}$$

and

$$K_0 = (|\mathcal{G}| + T\|\mathcal{Q}(\cdot)\|) e^{T\|\mathcal{A}(\cdot) + \mathcal{A}(\cdot)^T + \mathcal{C}(\cdot)^T \mathcal{C}(\cdot)\|}.$$

Proof. For any $(r, x) \in [t, T] \times \mathbb{R}^n$, we introduce the following controlled SDE:

$$(2.13) \quad \begin{cases} dX(s) = [\mathcal{A}(s)X(s) + \mathcal{B}(s)u(s)]ds \\ \quad + [\mathcal{C}(s)X(s) + \mathcal{D}(s)u(s)]dW(s), \quad s \in [r, T], \\ X(r) = x, \end{cases}$$

with cost functional

$$J(r, x; u(\cdot)) = \mathbb{E} \left[\int_r^T (\langle \mathcal{Q}(s)X(s), X(s) \rangle + 2\langle \mathcal{S}(s)X(s), u(s) \rangle + \langle \mathcal{R}(s)u(s), u(s) \rangle) ds + \langle \mathcal{G}X(T), X(T) \rangle \right].$$

From [29], we know that under our conditions (2.9), the associated LQ problem admits a unique optimal control $u^*(\cdot)$ which has the following state feedback representation:

$$u^*(s) = -\Theta(s)X^*(s), \quad s \in [r, T],$$

with $\Theta(\cdot)$ given by

$$(2.14) \quad \begin{aligned} \Theta(s) &= [\mathcal{R}(s) + \mathcal{D}(s)^T P(s)\mathcal{D}(s)]^{-1} \\ &\cdot [\mathcal{B}(s)^T P(s) + \mathcal{S}(s) + \mathcal{D}(s)^T P(s)\mathcal{C}(s)], \quad s \in [t, T], \end{aligned}$$

and $X^*(\cdot)$ is the solution to the following closed-loop system:

$$(2.15) \quad \begin{cases} dX^*(s) = [\mathcal{A}(s) - \mathcal{B}(s)\Theta(s)]X^*(s)ds \\ \quad + [\mathcal{C}(s) - \mathcal{D}(s)\Theta(s)]X^*(s)dW(s), \quad s \in [r, T], \\ X^*(r) = x. \end{cases}$$

Here, $P(\cdot)$ is the unique solution to the Riccati equation (2.10). Moreover,

$$(2.16) \quad \begin{aligned} 0 \leq \langle P(r)x, x \rangle &= \inf_{u(\cdot) \in \mathcal{U}[r, T]} J(r, x; u(\cdot)) = J(r, x; u^*(\cdot)) \\ &\leq J(r, x; 0) = \mathbb{E} \left[\int_r^T \langle \mathcal{Q}(s)X^0(s), X^0(s) \rangle ds + \langle \mathcal{G}X^0(T), X^0(T) \rangle \right], \end{aligned}$$

where $X^0(\cdot)$ is the solution to (2.13) with $u(\cdot) = 0$, i.e.,

$$(2.17) \quad \begin{cases} dX^0(s) = \mathcal{A}(s)X^0(s)ds + \mathcal{C}(s)X^0(s)dW(s), & s \in [r, T], \\ X^0(r) = x. \end{cases}$$

On the other hand, since $\Pi(\cdot)$ is the solution to (2.12), applying Itô's formula to $\langle \Pi(\cdot)X^0(\cdot), X^0(\cdot) \rangle$, we have

$$(2.18) \quad \begin{aligned} \langle \Pi(r)x, x \rangle &= \mathbb{E} [\langle \Pi(T)X^0(T), X^0(T) \rangle \\ &\quad - \int_r^T \langle [\dot{\Pi}(s) + \Pi(s)\mathcal{A}(s) + \mathcal{A}(s)^T\Pi(s) + \mathcal{C}(s)^T\Pi(s)\mathcal{C}(s)]X^0(s), X^0(s) \rangle ds] \\ &= \mathbb{E} \left[\langle \mathcal{G}X^0(T), X^0(T) \rangle + \int_r^T \langle \mathcal{Q}(s)X^0(s), X^0(s) \rangle ds \right] \\ &= J(r, x; 0) \geq \langle P(r)x, x \rangle. \end{aligned}$$

Next, applying Itô's formula to $|X^0(\cdot)|^2$, one has

$$\begin{aligned} \mathbb{E}|X^0(s)|^2 &= |x|^2 + \mathbb{E} \int_r^s \langle [\mathcal{A}(\tau) + \mathcal{A}(\tau)^T + \mathcal{C}(\tau)^T\mathcal{C}(\tau)]X^0(\tau), X^0(\tau) \rangle d\tau \\ &\leq |x|^2 + \|\mathcal{A}(\cdot) + \mathcal{A}(\cdot)^T + \mathcal{C}(\cdot)^T\mathcal{C}(\cdot)\| \int_r^s \mathbb{E}|X^0(r)|^2 dr. \end{aligned}$$

Then, by Gronwall's inequality, together with (2.16) and (2.18), we obtain

$$0 \leq P(r) \leq \Pi(r) \leq K_0 I, \quad r \in [t, T].$$

This completes the proof. \square

Note that with the function $\Theta(\cdot)$ defined by (2.14), Riccati equation (2.10) can be written as

$$(2.19) \quad \begin{cases} \dot{P}(s) + P(s)[\mathcal{A}(s) - \mathcal{B}(s)\Theta(s)] + [\mathcal{A}(s) - \mathcal{B}(s)\Theta(s)]^T P(s) + \mathcal{Q}(s) \\ \quad + [\mathcal{C}(s) - \mathcal{D}(s)\Theta(s)]^T P(s)[\mathcal{C}(s) - \mathcal{D}(s)\Theta(s)] = 0, & s \in [t, T], \\ P(T) = \mathcal{G}. \end{cases}$$

If $\Theta(\cdot)$ is given, the above can be regarded as a Lyapunov equation for $P(\cdot)$. The following lemma will play an interesting role in our later investigations.

Lemma 2.4. *Let*

$$(2.20) \quad \begin{cases} \mathcal{A}(\cdot), \mathcal{C}(\cdot) \in L^\infty(t, T; \mathbb{R}^n), & \mathcal{B}(\cdot), \mathcal{D}(\cdot) \in L^\infty(t, T; \mathbb{R}^{n \times m}), \\ \mathcal{Q}(\cdot), \tilde{\mathcal{Q}}(\cdot), \bar{\mathcal{Q}}(\cdot) \in L^\infty(t, T; \mathbb{S}^n), & \mathcal{G}, \bar{\mathcal{G}} \in \mathbb{S}^n. \end{cases}$$

Then there are $\tilde{\Gamma}(\cdot)$, $\Gamma(\cdot)$ and $\bar{\Gamma}(\cdot)$ that uniquely solve the following three Lyapunov equations:

$$(2.21) \quad \begin{cases} \dot{\tilde{\Gamma}}(s) + \tilde{\Gamma}(s)\mathcal{A}(s) + \mathcal{A}(s)^T\tilde{\Gamma}(s) + \mathcal{C}(s)^T\tilde{\Gamma}(s)\mathcal{C}(s) + \mathcal{Q}(s) = 0, & s \in [t, T], \\ \tilde{\Gamma}(T) = \mathcal{G}, \end{cases}$$

$$(2.22) \quad \begin{cases} \dot{\Gamma}(s) + \Gamma(s)[\mathcal{A}(s) + \bar{\mathcal{A}}(s)] + [\mathcal{A}(s) + \bar{\mathcal{A}}(s)]^T\Gamma(s) \\ \quad + [\mathcal{C}(s) + \bar{\mathcal{C}}(s)]^T\tilde{\Gamma}(s)[\mathcal{C}(s) + \bar{\mathcal{C}}(s)] + \mathcal{Q}(s) + \tilde{\mathcal{Q}}(s) = 0, & s \in [t, T], \\ \Gamma(T) = \mathcal{G}, \end{cases}$$

and

$$(2.23) \quad \begin{cases} \dot{\bar{\Gamma}}(s) + \bar{\Gamma}(s)[\mathcal{A}(s) + \bar{\mathcal{A}}(s)] + [\mathcal{A}(s) + \bar{\mathcal{A}}(s)]^T\bar{\Gamma}(s) + \bar{\mathcal{Q}}(s) = 0, & s \in [t, T], \\ \bar{\Gamma}(T) = \bar{\mathcal{G}}. \end{cases}$$

Let $(t, x) \in \mathcal{D}$ and $X(\cdot) \equiv X(\cdot; t, x)$ be the solution to (2.1) with $b(\cdot) = \sigma(\cdot) = 0$. Then for any $\tau \in [0, t]$,

$$(2.24) \quad \begin{aligned} & \mathbb{E}_\tau \left[\int_t^T \langle \mathcal{Q}(s)X(s), X(s) \rangle + \langle \tilde{\mathcal{Q}}(s)\mathbb{E}_t[X(s)], \mathbb{E}_t[X(s)] \rangle \right. \\ & \quad + \langle \bar{\mathcal{Q}}(s)\mathbb{E}_\tau[X(s)], \mathbb{E}_\tau[X(s)] \rangle ds \\ & \quad \left. + \langle \mathcal{G}X(T), X(T) \rangle + \langle \bar{\mathcal{G}}\mathbb{E}_\tau[X(T)], \mathbb{E}_\tau[X(T)] \rangle \right] \\ & = \mathbb{E}_\tau [\langle \Gamma(t)x, x \rangle] + \langle \bar{\Gamma}(t)\mathbb{E}_\tau[x], \mathbb{E}_\tau[x] \rangle. \end{aligned}$$

Further, if

$$(2.25) \quad \begin{cases} \mathcal{Q}(s), \mathcal{Q}(s) + \tilde{\mathcal{Q}}(s), \mathcal{Q}(s) + \tilde{\mathcal{Q}}(s) + \bar{\mathcal{Q}}(s) \geq 0, & s \in [0, T], \\ \mathcal{G}, \mathcal{G} + \bar{\mathcal{G}} \geq 0, \end{cases}$$

then

$$(2.26) \quad \tilde{\Gamma}(s), \Gamma(s), \Gamma(s) + \bar{\Gamma}(s) \geq 0, \quad s \in [0, T].$$

Note that in the left hand side of (2.24), the conditional expectation terms $\mathbb{E}_t[X(s)]$, $\mathbb{E}_\tau[X(s)]$ and $\mathbb{E}_\tau[X(T)]$ (with $0 \leq \tau \leq t$) appear quadratically (not linearly). It turns out that one can represent them in terms of x and $\mathbb{E}_\tau[x]$ by using $\Gamma(\cdot)$ and $\bar{\Gamma}(\cdot)$ (which also involves $\tilde{\Gamma}(\cdot)$ indirectly).

Proof. First of all, since Lyapunov equations are linear, under our conditions, $\tilde{\Gamma}(\cdot)$, $\Gamma(\cdot)$, and $\bar{\Gamma}(\cdot)$ uniquely exist. Note that for $0 \leq \tau \leq t < T$ and $x \in \mathcal{X}_t$, we let $X(\cdot) = X(\cdot; t, x)$ be the solution (2.1) with $b(\cdot) = \sigma(\cdot) = 0$. We have

$$\begin{aligned} & \begin{cases} d(\mathbb{E}_\tau[X(s)]) = [\mathcal{A}(s) + \bar{\mathcal{A}}(s)]\mathbb{E}_\tau[X(s)]ds, & s \in [t, T], \\ \mathbb{E}_\tau[X(t)] = \mathbb{E}_\tau[x], \end{cases} \\ & \begin{cases} d(\mathbb{E}_t[X(s)] - \mathbb{E}_\tau[X(s)]) = [\mathcal{A}(s) + \bar{\mathcal{A}}(s)](\mathbb{E}_t[X(s)] - \mathbb{E}_\tau[X(s)])ds, & s \in [t, T], \\ \mathbb{E}_t[X(t)] - \mathbb{E}_\tau[X(t)] = x - \mathbb{E}_\tau[x], \end{cases} \end{aligned}$$

and

$$\begin{cases} d(X(s) - \mathbb{E}_t[X(s)]) = \mathcal{A}(s)(X(s) - \mathbb{E}_t[X(s)])ds \\ \quad + \{\mathcal{C}(s)X(s) + \bar{\mathcal{C}}(s)\mathbb{E}_t[X(s)]\}dW(s), \quad s \in [t, T], \\ X(t) - \mathbb{E}_t[X(t)] = 0. \end{cases}$$

Denote $\hat{\Gamma}(\cdot) = \Gamma(\cdot) + \bar{\Gamma}(\cdot)$. Then $\hat{\Gamma}(\cdot)$ satisfies the following:

$$\begin{cases} \dot{\hat{\Gamma}}(s) + \hat{\Gamma}(s)[\mathcal{A}(s) + \bar{\mathcal{A}}(s)] + [\mathcal{A}(s) + \bar{\mathcal{A}}(s)]^T \hat{\Gamma}(s) \\ \quad + [\mathcal{C}(s) + \bar{\mathcal{C}}(s)]^T \tilde{\Gamma}(s)[\mathcal{C}(s) + \bar{\mathcal{C}}(s)] + \mathcal{Q}(s) + \tilde{\mathcal{Q}}(s) + \bar{\mathcal{Q}}(s) = 0, \quad s \in [t, T], \\ \hat{\Gamma}(T) = \mathcal{G} + \bar{\mathcal{G}}. \end{cases}$$

By Itô's formula, we have (with the time variable s being suppressed)

$$\begin{aligned} & \mathbb{E}_\tau \{ \langle \mathcal{G}X(T), X(T) \rangle + \langle \bar{\mathcal{G}}\mathbb{E}_\tau[X(T)], \mathbb{E}_\tau[X(T)] \rangle \} \\ &= \mathbb{E}_\tau \{ \langle \mathcal{G}(X(T) - \mathbb{E}_t[X(T)]), X(T) - \mathbb{E}_t[X(T)] \rangle \\ & \quad + \langle \mathcal{G}(\mathbb{E}_t[X(T)] - \mathbb{E}_\tau[X(T)]), \mathbb{E}_t[X(T)] - \mathbb{E}_\tau[X(T)] \rangle \\ & \quad + \langle (\mathcal{G} + \bar{\mathcal{G}})\mathbb{E}_\tau[X(T)], \mathbb{E}_\tau[X(T)] \rangle \} \\ &= \mathbb{E}_\tau \{ \langle \tilde{\Gamma}(T)(X(T) - \mathbb{E}_t[X(T)]), X(T) - \mathbb{E}_t[X(T)] \rangle \\ & \quad + \langle \Gamma(T)(\mathbb{E}_t[X(T)] - \mathbb{E}_\tau[X(T)]), \mathbb{E}_t[X(T)] - \mathbb{E}_\tau[X(T)] \rangle \\ & \quad + \langle \hat{\Gamma}(T)\mathbb{E}_\tau[X(T)], \mathbb{E}_\tau[X(T)] \rangle \} \\ &= \mathbb{E}_\tau \left[\langle \Gamma(t)(x - \mathbb{E}_\tau[x]), x - \mathbb{E}_\tau[x] \rangle + \langle \hat{\Gamma}(t)\mathbb{E}_\tau[x], \mathbb{E}_\tau[x] \rangle \right] \\ & \quad + \mathbb{E}_\tau \int_t^T \left[\langle (\dot{\hat{\Gamma}} + \tilde{\Gamma}\mathcal{A} + \mathcal{A}^T\tilde{\Gamma})(X - \mathbb{E}_t[X]), X - \mathbb{E}_t[X] \rangle \right. \\ & \quad + \langle \tilde{\Gamma}(\mathcal{C}X + \bar{\mathcal{C}}\mathbb{E}_t[X]), \mathcal{C}X + \bar{\mathcal{C}}\mathbb{E}_t[X] \rangle \\ & \quad + \langle [\dot{\Gamma} + \Gamma(\mathcal{A} + \bar{\mathcal{A}}) + (\mathcal{A} + \bar{\mathcal{A}})^T\Gamma](\mathbb{E}_t[X] - \mathbb{E}_\tau[X]), \mathbb{E}_t[X] - \mathbb{E}_\tau[X] \rangle \\ & \quad \left. + \langle \{\dot{\hat{\Gamma}} + \hat{\Gamma}(\mathcal{A} + \bar{\mathcal{A}}) + (\mathcal{A} + \bar{\mathcal{A}})^T\hat{\Gamma}\}\mathbb{E}_\tau[X], \mathbb{E}_\tau[X] \rangle \right] ds \\ &= \mathbb{E}_\tau \left[\langle \Gamma(t)x, x \rangle + \langle \bar{\Gamma}(t)\mathbb{E}_\tau[x], \mathbb{E}_\tau[x] \rangle \right] \\ & \quad + \mathbb{E}_\tau \int_t^T \left[\langle (\dot{\hat{\Gamma}} + \tilde{\Gamma}\mathcal{A} + \mathcal{A}^T\tilde{\Gamma} + \mathcal{C}^T\tilde{\Gamma}\mathcal{C})(X - \mathbb{E}_t[X]), X - \mathbb{E}_t[X] \rangle \right. \\ & \quad + \langle [\dot{\Gamma} + \Gamma(\mathcal{A} + \bar{\mathcal{A}}) + (\mathcal{A} + \bar{\mathcal{A}})^T\Gamma + (\mathcal{C} + \bar{\mathcal{C}})^T\tilde{\Gamma}(\mathcal{C} + \bar{\mathcal{C}})](\mathbb{E}_t[X] - \mathbb{E}_\tau[X]), \\ & \quad \quad \quad \mathbb{E}_t[X] - \mathbb{E}_\tau[X] \rangle \\ & \quad \left. + \langle \{\dot{\hat{\Gamma}} + \hat{\Gamma}(\mathcal{A} + \bar{\mathcal{A}}) + (\mathcal{A} + \bar{\mathcal{A}})^T\hat{\Gamma} + (\mathcal{C} + \bar{\mathcal{C}})^T\tilde{\Gamma}(\mathcal{C} + \bar{\mathcal{C}})\}\mathbb{E}_\tau[X], \mathbb{E}_\tau[X] \rangle \right] ds. \end{aligned}$$

On the other hand,

$$\begin{aligned} & \mathbb{E}_\tau \int_t^T \left[\langle \mathcal{Q}X, X \rangle + \langle \tilde{\mathcal{Q}}\mathbb{E}_t[X], \mathbb{E}_t[X] \rangle + \langle \bar{\mathcal{Q}}\mathbb{E}_\tau[X], \mathbb{E}_\tau[X] \rangle \right] ds \\ &= \mathbb{E}_\tau \int_t^T \left[\langle \mathcal{Q}(X - \mathbb{E}_t[X]), X - \mathbb{E}_t[X] \rangle \right. \\ & \quad + \langle (\mathcal{Q} + \tilde{\mathcal{Q}})(\mathbb{E}_t[X] - \mathbb{E}_\tau[X]), \mathbb{E}_t[X] - \mathbb{E}_\tau[X] \rangle \\ & \quad \left. + \langle (\mathcal{Q} + \tilde{\mathcal{Q}} + \bar{\mathcal{Q}})\mathbb{E}_\tau[X], \mathbb{E}_\tau[X] \rangle \right] ds. \end{aligned}$$

Hence,

$$\begin{aligned} & \mathbb{E}_\tau \left[\int_t^T (\langle \mathcal{Q}X, X \rangle + \langle \tilde{\mathcal{Q}}\mathbb{E}_t[X], \mathbb{E}_t[X] \rangle + \langle \bar{\mathcal{Q}}\mathbb{E}_\tau[X], \mathbb{E}_\tau[X] \rangle) ds \right. \\ & \quad \left. + \langle \mathcal{G}X(T), X(T) \rangle + \langle \bar{\mathcal{G}}\mathbb{E}_\tau[X(T)], \mathbb{E}_\tau[X(T)] \rangle \right] \\ &= \mathbb{E}_\tau [\langle \Gamma(t)x, x \rangle] + \langle \bar{\Gamma}(t)\mathbb{E}_\tau[x], \mathbb{E}_\tau[x] \rangle \\ & \quad + \mathbb{E}_\tau \int_t^T \left[\langle (\tilde{\Gamma} + \tilde{\Gamma}\mathcal{A} + \mathcal{A}^T\tilde{\Gamma} + \mathcal{C}^T\tilde{\Gamma}\mathcal{C} + \mathcal{Q})(X - \mathbb{E}_\tau[X]), X - \mathbb{E}_\tau[X] \rangle \right. \\ & \quad + \langle [\tilde{\Gamma} + \Gamma(\mathcal{A} + \bar{\mathcal{A}}) + (\mathcal{A} + \bar{\mathcal{A}})^T\tilde{\Gamma} + (\mathcal{C} + \bar{\mathcal{C}})^T\tilde{\Gamma}(\mathcal{C} + \bar{\mathcal{C}}) + \mathcal{Q} + \tilde{\mathcal{Q}}] (\mathbb{E}_t[X] - \mathbb{E}_\tau[X]), \\ & \quad \quad \quad \mathbb{E}_t[X] - \mathbb{E}_\tau[X] \rangle \\ & \quad \left. + \langle \{\hat{\Gamma} + \hat{\Gamma}(\mathcal{A} + \bar{\mathcal{A}}) + (\mathcal{A} + \bar{\mathcal{A}})^T\hat{\Gamma} + (\mathcal{C} + \bar{\mathcal{C}})^T\hat{\Gamma}(\mathcal{C} + \bar{\mathcal{C}}) + \mathcal{Q} + \tilde{\mathcal{Q}} + \bar{\mathcal{Q}}\} \mathbb{E}_\tau[X], \mathbb{E}_\tau[X] \rangle \right] ds \\ &= \mathbb{E}_\tau [\langle \Gamma(t)x, x \rangle] + \langle \bar{\Gamma}(t)\mathbb{E}_\tau[x], \mathbb{E}_\tau[x] \rangle. \end{aligned}$$

This proves (2.24). Finally, if (2.25) holds, by Proposition 2.3, we obtain (2.18). \square

3. LINEAR-QUADRATIC OPTIMAL CONTROL PROBLEM FOR MF-SDES

We now consider the following controlled linear MF-SDE:

$$(3.1) \quad \begin{cases} dX(s) = \{A(s)X(s) + \bar{A}(s)\mathbb{E}_t[X(s)] + B(s)u(s) + \bar{B}(s)\mathbb{E}_t[u(s)]\}ds \\ \quad + \{C(s)X(s) + \bar{C}(s)\mathbb{E}_t[X(s)] + D(s)u(s) + \bar{D}(s)\mathbb{E}_t[u(s)]\}dW(s), \\ \quad \quad \quad s \in [t, T], \\ X(t) = x \in \mathcal{X}_t. \end{cases}$$

Note that for any $(t, x) \in \mathcal{D}$ and $u(\cdot) \in \mathcal{U}[t, T]$, the corresponding state process $X(\cdot) = X(\cdot; t, x, u(\cdot))$ depends on $(t, x, u(\cdot))$. The cost functional is as follows:

$$(3.2) \quad \begin{aligned} J(t, x; u(\cdot)) = & \mathbb{E}_t \left\{ \int_t^T [\langle Q(s, t)X(s), X(s) \rangle + \langle \bar{Q}(s, t)\mathbb{E}_t[X(s)], \mathbb{E}_t[X(s)] \rangle \right. \\ & + \langle R(s, t)u(s), u(s) \rangle + \langle \bar{R}(s, t)\mathbb{E}_t[u(s)], \mathbb{E}_t[u(s)] \rangle] ds \\ & \left. + \langle G(t)X(T), X(T) \rangle + \langle \bar{G}(t)\mathbb{E}_t[X(T)], \mathbb{E}_t[X(T)] \rangle \right\}. \end{aligned}$$

Let us introduce the following hypotheses:

(H1) The following hold:

$$(3.3) \quad \begin{cases} A(\cdot), \bar{A}(\cdot), C(\cdot), \bar{C}(\cdot) \in C([0, T]; \mathbb{R}^{n \times n}), \\ B(\cdot), \bar{B}(\cdot), D(\cdot), \bar{D}(\cdot) \in C([0, T]; \mathbb{R}^{n \times m}). \end{cases}$$

(H2) The following hold:

$$(3.4) \quad \begin{cases} Q(\cdot, \cdot), \bar{Q}(\cdot, \cdot) \in C([0, T]^2; \mathbb{S}^n), & R(\cdot, \cdot), \bar{R}(\cdot, \cdot) \in C([0, T]^2; \mathbb{S}^m), \\ G(\cdot), \bar{G}(\cdot) \in C([0, T]; \mathbb{S}^n), \end{cases}$$

and for some $\delta > 0$,

$$(3.5) \quad \begin{cases} Q(s, t), Q(s, t) + \bar{Q}(s, t) \geq 0, & 0 \leq t \leq s \leq T, \\ R(s, t), R(s, t) + \bar{R}(s, t) \geq \delta I, \\ G(t), G(t) + \bar{G}(t) \geq 0, & 0 \leq t \leq T. \end{cases}$$

(H3) The following monotonicity conditions are satisfied:

$$(3.6) \quad \begin{cases} Q(s, t) \leq Q(s, \tau), & Q(s, t) + \bar{Q}(s, t) \leq Q(s, \tau) + \bar{Q}(s, \tau), \\ R(s, t) \leq R(s, \tau), & R(s, t) + \bar{R}(s, t) \leq R(s, \tau) + \bar{R}(s, \tau), \\ G(t) \leq G(\tau), & G(t) + \bar{G}(t) \leq G(\tau) + \bar{G}(\tau), \\ & 0 \leq t \leq \tau \leq s \leq T. \end{cases}$$

It is clear that under (H1)–(H2), for any $(t, x) \in \mathcal{D}$ and $u(\cdot) \in \mathcal{U}[t, T]$, state equation (3.1) admits a unique solution $X(\cdot) \equiv X(\cdot; t, x, u(\cdot))$, and the cost functional $J(t, x; u(\cdot))$ is well-defined. Then we can state the following problem.

Problem (MF-LQ). For any $(t, x) \in \mathcal{D}$, find a $u^*(\cdot) \in \mathcal{U}[t, T]$ such that

$$(3.7) \quad J(t, x; u^*(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}[t, T]} J(t, x; u(\cdot)) \equiv V(t, x).$$

For given $(t, x) \in \mathcal{D}$, any $u^*(\cdot) \in \mathcal{U}[t, T]$ satisfying the above is called a *pre-commitment optimal control* for Problem (MF-LQ) at (t, x) . The corresponding $X^*(\cdot)$ and $(X^*(\cdot), u^*(\cdot))$ are called *pre-commitment optimal state process* and *pre-commitment optimal pair* of Problem (MF-LQ), respectively, and $V(\cdot, \cdot)$ is called the *pre-commitment value function*.

In the case that

$$(3.8) \quad \bar{A}(s) = \bar{C}(s) = 0, \quad \bar{B}(s) = \bar{D}(s) = 0, \quad s \in [0, T],$$

and

$$(3.9) \quad \bar{Q}(s, t) = 0, \quad \bar{R}(s, t) = 0, \quad \bar{G}(t) = 0, \quad 0 \leq t \leq s \leq T,$$

Problem (MF-LQ) is reduced to the problem studied in [25], which is an LQ problem with purely general (possibly non-exponential) discounting. In addition, if

$$(3.10) \quad Q(s, t) = Q(s), \quad R(s, t) = R(s), \quad G(t) = G, \quad 0 \leq t \leq s \leq T,$$

then Problem (MF-LQ) is further reduced to a classical LQ problem for SDEs with deterministic coefficients (see [29] for details). In the case that (3.8) and (3.10) hold, and

$$(3.11) \quad \bar{Q}(s, t) = 0, \quad \bar{R}(s, t) = 0, \quad \bar{G}(t) = \bar{G}, \quad 0 \leq t \leq T,$$

Problem (MF-LQ) is a case of those studied in [14], where the problem was investigated by means of the variational method together with forward-backward SDEs, and the time-consistent solution studied is of open-loop type (see the next section for further details).

We also point out that if

$$\begin{cases} Q(s, t) = Q(s)e^{-\lambda(s-t)}, & \bar{Q}(s, t) = e^{-\lambda(s-t)}, \\ R(s, t) = R(s)e^{-\lambda(s-t)}, & \bar{R}(s, t) = e^{-\lambda(s-t)}, \\ G(t) = Ge^{-\lambda(T-t)}, & \bar{G}(t) = \bar{G}e^{-\lambda(T-t)}, \end{cases} \quad 0 \leq t \leq s \leq T,$$

with $\lambda > 0$ and with

$$\begin{cases} Q(s), Q(s) + \bar{Q}(s) \geq 0, & R(s), R(s) + \bar{R}(s) \geq 0, & s \in [0, T], \\ G, G + \bar{G} \geq 0, \end{cases}$$

then monotonicity condition (H3) holds. Hypothesis (H3) will be used later.

In what follows, we will denote

$$(3.12) \quad \begin{cases} \hat{A}(s) = A(s) + \bar{A}(s), & \hat{B}(s) = B(s) + \bar{B}(s), \\ \hat{C}(s) = C(s) + \bar{C}(s), & \hat{D}(s) = D(s) + \bar{D}(s), \\ \hat{Q}(s, t) = Q(s, t) + \bar{Q}(s, t), & \hat{R}(s, t) = R(s, t) + \bar{R}(s, t), \\ \hat{G}(t) = G(t) + \bar{G}(t), \end{cases} \quad 0 \leq t \leq s \leq T.$$

We now recall a result from [28] for the pre-commitment solution of Problem (MF-LQ).

Proposition 3.1. *Let (H1)–(H2) hold. Then for any fixed $t \in [0, T]$, the following Riccati equation system admits a unique solution $(P(\cdot), \hat{P}(\cdot)) \in C^1([t, T]; \mathbb{S}^n)^2$ (suppressing s):*

$$(3.13) \quad \begin{cases} \dot{P} + PA + A^T P + C^T PC + Q(t) \\ \quad - (PB + C^T PD)[R(t) + D^T PD]^{-1}(B^T P + D^T PC) = 0, \\ \dot{\hat{P}} + \hat{P}\hat{A} + \hat{A}^T \hat{P} + \hat{C}^T \hat{P}\hat{C} + \hat{Q}(t) \\ \quad - (\hat{P}\hat{B} + \hat{C}^T \hat{P}\hat{D})[\hat{R}(t) + \hat{D}^T \hat{P}\hat{D}]^{-1}(\hat{B}^T \hat{P} + \hat{D}^T \hat{P}\hat{C}) = 0, \\ P(T) = G(t), \quad \hat{P}(T) = \hat{G}(t). \end{cases} \quad s \in [t, T],$$

Further, let $x \in \mathcal{X}_t$ and $X^*(\cdot) \equiv X^*(\cdot; t, x)$ be the solution to the following closed-loop system:

$$(3.14) \quad \begin{cases} dX^*(s) = \left\{ [A(s) - B(s)\Theta(s)]X^*(s) \right. \\ \quad + [\bar{A}(s) + B(s)[\Theta(s) - \hat{\Theta}(s)] - \bar{B}(s)\hat{\Theta}(s)] \mathbb{E}_t[X^*(s)] \Big\} ds \\ \quad + \left\{ [C(s) - D(s)\Theta(s)]X^*(s) \right. \\ \quad + [\bar{C}(s) + D(s)[\Theta(s) - \hat{\Theta}(s)] - \bar{D}(s)\hat{\Theta}(s)] \mathbb{E}_t[X^*(s)] \Big\} dW(s), \\ X^*(t) = x, \end{cases} \quad s \in [t, T],$$

with

$$(3.15) \quad \begin{cases} \Theta(s) = [R(s, t) + D(s)^T P(s) D(s)]^{-1} [B(s)^T P(s) + D(s)^T P(s) C(s)], \\ \hat{\Theta}(s) = [\hat{R}(s, t) + \hat{D}(s)^T P(s) \hat{D}(s)]^{-1} [\hat{B}(s)^T \hat{P}(s) + \hat{D}(s)^T P(s) \hat{C}(s)], \\ s \in [t, T], \end{cases}$$

and define $u^*(\cdot)$ as follows:

$$(3.16) \quad u^*(s) = -\Theta(s)X^*(s) + [\Theta(s) - \hat{\Theta}(s)]\mathbb{E}_t[X^*(s)], \quad s \in [t, T].$$

Then $(X^*(\cdot), u^*(\cdot))$ is the pre-commitment optimal pair of Problem (MF-LQ) at (t, x) , and

$$(3.17) \quad V(t, x) = \inf_{u(\cdot) \in \mathcal{U}[t, T]} J(t, x; u(\cdot)) = J(t, x; u^*(\cdot)) = \langle \hat{P}(t)x, x \rangle, \quad \forall x \in \mathcal{X}_t.$$

We point out that system (3.13) is decoupled. With $\Theta(\cdot)$ and $\hat{\Theta}(\cdot)$ given by (3.15), we can write (3.13) as follows:

$$(3.18) \quad \begin{cases} \dot{P}(s) + P(s)[A(s) - B(s)\Theta(s)] + [A(s) - B(s)\Theta(s)]^T P(s) + Q(s, t) \\ \quad + [C(s) - D(s)\Theta(s)]^T P(s)[C(s) - D(s)\Theta(s)] + \Theta(s)^T R(s, t)\Theta(s) = 0, \\ \dot{\hat{P}}(s) + \hat{P}(s)[\hat{A}(s) - \hat{B}(s)\hat{\Theta}(s)] + [\hat{A}(s) - \hat{B}(s)\hat{\Theta}(s)]^T \hat{P}(s) + \hat{Q}(s) \\ \quad + [\hat{C}(s) - \hat{D}(s)\hat{\Theta}(s)]^T P(s)[\hat{C}(s) - \hat{D}(s)\hat{\Theta}(s)] + \hat{\Theta}(s)^T \hat{R}(s, t)\hat{\Theta}(s) = 0, \\ s \in [t, T], \\ P(T) = G(t), \quad \hat{P}(T) = \hat{G}(t). \end{cases}$$

Next, we note that in the case

$$\bar{A}(\cdot) = \bar{C}(\cdot) = 0, \quad \bar{B}(\cdot) = \bar{D}(\cdot) = 0,$$

the conditional expectation terms are all absent in the state equation, and

$$(3.19) \quad \begin{cases} \Theta(s) = [R(s, t) + D(s)^T P(s) D(s)]^{-1} [B(s)^T P(s) + D(s)^T P(s) C(s)], \\ \hat{\Theta}(s) = [\hat{R}(s, t) + D(s)^T P(s) D(s)]^{-1} [B(s)^T \hat{P}(s) + D(s)^T P(s) C(s)], \\ s \in [t, T]. \end{cases}$$

In this case, the equation for $\hat{P}(\cdot)$ becomes

$$(3.20) \quad \begin{cases} \dot{\hat{P}}(s) + \hat{P}(s)A(s) + A(s)^T \hat{P}(s) + C(s)^T P(s)C(s) + \hat{Q}(t, s) \\ \quad - [\hat{P}(s)B(s) + C(s)^T P(s)D(s)][\hat{R}(s, t) + D(s)^T P(s)D(s)]^{-1} \\ \quad \cdot [B(s)^T \hat{P}(s) + D(s)^T P(s)C(s)] = 0, \quad s \in [t, T], \\ \hat{P}(T) = \hat{G}(t). \end{cases}$$

We see that as long as

$$\bar{Q}(\cdot, \cdot) = 0, \quad \bar{R}(\cdot, \cdot) = 0, \quad \bar{G}(\cdot) = 0$$

are not true,

$$\Theta(s) = \hat{\Theta}(s), \quad s \in [t, T],$$

is not true in general. Hence, the term $\mathbb{E}_t[X^*(\cdot)]$ will appear in the state feedback representation of $u^*(\cdot)$ (see (3.16)), and the closed-loop system reads

$$\begin{cases} dX^*(s) = \left\{ [A(s) - B(s)\Theta(s)]X^*(s) + B(s)[\Theta(s) - \widehat{\Theta}(s)]\mathbb{E}_t[X^*(s)] \right\} ds \\ \quad + \left\{ [C(s) - D(s)\Theta(s)]X^*(s) + D(s)[\Theta(s) - \widehat{\Theta}(s)]\mathbb{E}_t[X^*(s)] \right\} dW(s), \\ \quad \quad \quad s \in [t, T], \\ X^*(t) = x, \end{cases}$$

which is a linear MF-SDE. The above shows that even if the original state equation does not contain the conditional expectation terms, as long as some conditional expectation terms appear in the cost functional, the closed-loop system for the pre-committed solution of the problem will still be an MF-SDE. This is why we start with a controlled MF-SDE.

We now present the following result, which is relevant to the Riccati equation system (3.13).

Proposition 3.2. *Let (H1)–(H2) hold. For a fixed $t \in [0, T)$, let $(P(\cdot), \widehat{P}(\cdot))$ be the solution to the Riccati equation system (3.13), and let $\Pi(\cdot)$ and $\widehat{\Pi}(\cdot)$ be the solutions to the following Lyapunov equations:*

$$(3.21) \quad \begin{cases} \dot{\Pi}(s) + \Pi(s)A(s) + A(s)^T\Pi(s) + C(s)^T\Pi(s)C(s) + Q(s, t) = 0, & s \in [t, T], \\ \Pi(T) = G(t), \end{cases}$$

and

$$(3.22) \quad \begin{cases} \dot{\widehat{\Pi}}(s) + \widehat{\Pi}(s)\widehat{A}(s) + \widehat{A}(s)^T\widehat{\Pi}(s) + \widehat{C}(s)^T\Pi(s)\widehat{C}(s) + \widehat{Q}(s, t) = 0, & s \in [t, T], \\ \widehat{\Pi}(T) = \widehat{G}(t). \end{cases}$$

Then

$$(3.23) \quad 0 \leq P(s) \leq \Pi(s), \quad s \in [t, T],$$

and

$$(3.24) \quad 0 \leq \widehat{P}(s) \leq \widehat{\Pi}(s), \quad s \in [t, T].$$

Proof. Fix $t \in [0, T)$. Applying Proposition 2.3 with

$$\begin{cases} \mathcal{A}(s) = A(s), & \mathcal{B}(s) = B(s), & \mathcal{C}(s) = C(s), & \mathcal{D}(s) = D(s), & t \leq s \leq T, \\ \mathcal{Q}(s) = Q(s, t), & \mathcal{S}(s) = 0, & \mathcal{R}(s) = R(s, t), & \mathcal{G} = G(t), & t \leq s \leq T, \end{cases}$$

we obtain (3.23). Next, we let

$$\begin{cases} \mathcal{A}(s) = \widehat{A}(s), & \mathcal{B}(s) = \widehat{B}(s), & \mathcal{C}(s) = \widehat{C}(s), & \mathcal{D}(s) = \widehat{D}(s), \\ \mathcal{Q}(s) = \widehat{Q}(s, t) + \mathcal{C}(s)^T P(s) \mathcal{C}(s), & \mathcal{S}(s) = \mathcal{D}(s)^T P(s) \mathcal{C}(s), \\ \mathcal{R}_0(s) = \widehat{R}(s, t), & \mathcal{R}(s) = \mathcal{R}_0(s) + \mathcal{D}(s)^T P(s) \mathcal{D}(s), & \mathcal{G} = \widehat{G}(t). \end{cases}$$

Then the equation for $\widehat{P}(\cdot)$ can be written as

$$(3.25) \quad \begin{cases} \dot{\widehat{P}}(s) + \widehat{P}(s)\mathcal{A}(s) + \mathcal{A}(s)^T\widehat{P}(s) + \mathcal{Q}(s) \\ \quad - [\widehat{P}(s)\mathcal{B}(s) + \mathcal{S}(s)^T]\mathcal{R}(s)^{-1}[\mathcal{B}(s)\widehat{P}(s) + \mathcal{S}(s)] = 0, & s \in [t, T], \\ \widehat{P}(T) = \mathcal{G}, \end{cases}$$

which is the Riccati equation for the deterministic LQ problem with the state equation

$$\begin{cases} \dot{X}(s) = \mathcal{A}(s)X(s) + \mathcal{B}(s)u(s), & s \in [r, T], \\ X(r) = x, \end{cases}$$

and the cost functional

$$J(r, x; u(\cdot)) = \int_r^T [\langle \mathcal{Q}(s)X(s), X(s) \rangle + 2\langle \mathcal{S}(s)X(s), u(s) \rangle + \langle \mathcal{R}(s)u(s), u(s) \rangle] ds \\ + \langle \mathcal{G}X(T), X(T) \rangle.$$

We now look at the following (suppressing s):

$$\begin{aligned} \mathcal{Q} - \mathcal{S}^T \mathcal{R}^{-1} \mathcal{S} &\geq \mathcal{C}^T P \mathcal{C} - \mathcal{C}^T P \mathcal{D} (\mathcal{R}_0 + \mathcal{D}^T P \mathcal{D})^{-1} \mathcal{D}^T P \mathcal{C} \\ &= \mathcal{C}^T P^{\frac{1}{2}} \left[I - P^{\frac{1}{2}} \mathcal{D} (\mathcal{R}_0 + \mathcal{D}^T P \mathcal{D})^{-1} \mathcal{D}^T P^{\frac{1}{2}} \right] P^{\frac{1}{2}} \mathcal{C} \\ &= \mathcal{C}^T P^{\frac{1}{2}} \left[I - P^{\frac{1}{2}} \mathcal{D} \mathcal{R}_0^{-\frac{1}{2}} \left(I + \mathcal{R}_0^{-\frac{1}{2}} \mathcal{D}^T P^{\frac{1}{2}} P^{\frac{1}{2}} \mathcal{D} \mathcal{R}_0^{-\frac{1}{2}} \right)^{-1} \mathcal{R}_0^{-\frac{1}{2}} \mathcal{D}^T P^{\frac{1}{2}} \right] P^{\frac{1}{2}} \mathcal{C} \\ &\equiv \mathcal{C}^T P^{\frac{1}{2}} \left[I - \Lambda (I + \Lambda^T \Lambda)^{-1} \Lambda^T \right] P^{\frac{1}{2}} \mathcal{C} = \mathcal{C}^T P^{\frac{1}{2}} (I + \Lambda \Lambda^T)^{-1} P^{\frac{1}{2}} \mathcal{C} \geq 0. \end{aligned}$$

In the above, we have denoted

$$\Lambda = P^{\frac{1}{2}} \mathcal{D} \mathcal{R}_0^{-\frac{1}{2}}$$

and used the fact that

$$I - \Lambda (I + \Lambda^T \Lambda)^{-1} \Lambda^T = (I + \Lambda \Lambda^T)^{-1}.$$

Then by Proposition 2.3 with $\mathcal{C}(\cdot) = 0$ and $\mathcal{D}(\cdot) = 0$, we have

$$\widehat{P}(s) \leq \bar{P}^0(s), \quad s \in [t, T],$$

where $\bar{P}^0(\cdot)$ is the solution to the following Lyapunov equation:

$$(3.26) \quad \begin{cases} \dot{\bar{P}}^0(s) + \bar{P}^0(s) \mathcal{A}(s) + \mathcal{A}(s)^T \bar{P}^0(s) + \mathcal{Q}(s) = 0, & s \in [t, T], \\ \bar{P}^0(T) = \mathcal{G}. \end{cases}$$

Next, by the proved (3.23), we see that

$$\mathcal{Q}(s) \equiv \widehat{Q}(s, t) + \mathcal{C}(s)^T P(s) \mathcal{C}(s) \leq \widehat{Q}(s, t) + \mathcal{C}(s)^T \Pi(s) \mathcal{C}(s).$$

Thus, by Proposition 2.3 again, we obtain

$$\bar{P}^0(s) \leq P^0(s),$$

with $P^0(\cdot)$ being the solution to the following Lyapunov equation (note $\mathcal{A}(\cdot) = \widehat{A}(\cdot)$, $\mathcal{C}(\cdot) = \widehat{C}(\cdot)$, and $\mathcal{G} = \widehat{G}(t)$):

$$\begin{cases} \dot{P}^0(s) + P^0(s) \widehat{A}(s) + \widehat{A}(s)^T P^0(s) + \widehat{C}(s)^T \Pi(s) \widehat{C}(s) + \widehat{Q}(s, t) = 0, & s \in [t, T], \\ P^0(T) = \widehat{G}(t). \end{cases}$$

Comparing with (3.22), by uniqueness, we obtain

$$P^0(\cdot) = \widehat{\Pi}(\cdot).$$

This completes the proof. □

4. TIME-CONSISTENT EQUILIBRIUM SOLUTIONS

From Examples 1.1 and 1.2, we see that in general, Problem (MF-LQ) is time-inconsistent. Our goal is to find time-consistent (equilibrium) solutions to Problem (MF-LQ).

4.1. Open-loop equilibrium control. In this subsection, we consider the case that

$$(4.1) \quad \bar{A}(\cdot) = \bar{C}(\cdot) = 0, \quad \bar{B}(\cdot) = \bar{D}(\cdot) = 0.$$

Thus, the state equation reads as (1.1). However, we still consider cost functional (3.2), in which all the conditional expectation terms appear and all the weighting matrices depend on (s, t) . Note that in [14], the state equation is similar to ours (with additional non-homogeneous terms which do not bring additional essential difficulties), and the cost functional has the only conditional expectation appearing at the terminal cost, and all the weighting matrices are independent of t (the initial time). Therefore, in some sense, the case of this subsection can be regarded as an extension of relevant cases that appeared in [14]. Similar to [14], we introduce the following notion.

Definition 4.1. For given $x \in \mathbb{R}^n$, a state-control pair $(X^*(\cdot), u^*(\cdot)) \in \mathcal{X}[0, T] \times \mathcal{U}[0, T]$ is called an *open-loop equilibrium pair* of Problem (MF-LQ) for the initial state x if

$$X^*(0) = x,$$

and for almost all $t \in [0, T)$ and any $u(\cdot) \in \mathcal{U}[t, T]$,

$$(4.2) \quad \lim_{\varepsilon \downarrow 0} \frac{J(t, X^*(t); u^\varepsilon(\cdot)) - J(t, X^*(t); u^*(\cdot))}{\varepsilon} \geq 0,$$

where

$$(4.3) \quad u^\varepsilon(\cdot) = u(\cdot)I_{[t, t+\varepsilon)}(\cdot) + u^*(\cdot)I_{[t+\varepsilon, T]}(\cdot).$$

In this case, $X^*(\cdot)$ and $u^*(\cdot)$ are called an *open-loop equilibrium state process* and an *open-loop equilibrium control*, respectively.

We refer to (4.2) as a *local optimality condition* at $t \in [0, T)$. One sees that if $(X^*(\cdot), u^*(\cdot))$ is an open-loop equilibrium pair of Problem (MF-LQ) for the initial state x , then along the open-loop equilibrium state $X^*(\cdot)$, the open-loop equilibrium control $u^*(\cdot)$ stays locally optimal. On the other hand, since $(X^*(\cdot), u^*(\cdot))$ is a fixed state-control pair of (1.1) on $[0, T]$, in which the conditional expectation terms are absent, the above defined open-loop equilibrium pair is time-consistent. Note that if we consider the general state equation (3.1), in which some conditional expectation terms appear, we do not know if one can directly define time-consistent state-control pairs. This is why for open-loop equilibrium solutions of Problem (MF-LQ), we only consider (1.1). The following result, in some sense, is an extension of a relevant result found in [14].

Proposition 4.2. Let (H1)–(H2) and (4.1) hold. Suppose $(X^*(\cdot), u^*(\cdot)) \in \mathcal{X}[0, T] \times \mathcal{U}[0, T]$ is a state-control pair starting from initial state x . For each $t \in [0, T)$,

let $(Y(\cdot, t), Z(\cdot, t))$ be the adapted solution of the following BSDE:

$$(4.4) \quad \begin{cases} dY(s, t) = -\{A(s)^T Y(s, t) + C(s)^T Z(s, t) + Q(s, t)X^* \\ \quad + \bar{Q}(s, t)\mathbb{E}_t[X^*(s)]\}ds + Z(s, t)dW(s), & s \in [t, T], \\ Y(T, t) = G(t)X^*(T) + \bar{G}(t)\mathbb{E}_t[X^*(T)]. \end{cases}$$

Suppose $(s, t) \mapsto (Y(s, t), Z(s, t))$ is continuous on $0 \leq t \leq s \leq T$ and suppose

$$(4.5) \quad u^*(t) = -\hat{R}(t, t)^{-1}\{B(t)^T Y(t, t) + D(t)^T Z(t, t)\}, \quad t \in [0, T].$$

Then $(X^*(\cdot), u^*(\cdot))$ is an open-loop equilibrium pair of Problem (MF-LQ) for initial state x .

Proof. For any fixed $t \in [0, T]$ and $u(\cdot) \in \mathcal{U}[t, T]$, define $u^\varepsilon(\cdot)$ by (4.3) and let $X^\varepsilon(\cdot) \equiv X(\cdot; t, X^*(\cdot), u^\varepsilon(\cdot))$ be the solution to the following:

$$(4.6) \quad \begin{cases} dX^\varepsilon(s) = [A(s)X^\varepsilon(s) + B(s)u^\varepsilon(s)]ds \\ \quad + [C(s)X^\varepsilon(s) + D(s)u^\varepsilon(s)]dW(s), & s \in [t, T], \\ X^\varepsilon(t) = X^*(t). \end{cases}$$

Then (suppressing s)

$$\begin{aligned} & J(t, X^*(t); u^\varepsilon(\cdot)) - J(t, X^*(t), u^*(\cdot)) \\ &= \mathbb{E}_t \left\{ \int_t^T [\langle Q(t)(X^\varepsilon + X^*), X^\varepsilon - X^* \rangle + \langle \bar{Q}(t)\mathbb{E}_t[X^\varepsilon + X^*], \mathbb{E}_t[X^\varepsilon - X^*] \rangle \right. \\ & \quad + \langle R(t)(u^\varepsilon + u^*), u^\varepsilon - u^* \rangle + \langle \bar{R}(t)\mathbb{E}_t[u^\varepsilon + u^*], \mathbb{E}_t[u^\varepsilon - u^*] \rangle] ds \\ & \quad + \langle G(t)[X^\varepsilon(T) + X^*(s)], X^\varepsilon(T) - X^*(T) \rangle \\ & \quad \left. + \langle \bar{G}(t)\mathbb{E}_t[X^\varepsilon(T) + X^*(T)], \mathbb{E}_t[X^\varepsilon(T) - X^*(T)] \rangle \right\} \\ &= \mathbb{E}_t \left\{ \int_t^T \langle Q(t)(X^\varepsilon + X^*) + \bar{Q}(t)\mathbb{E}_t[X^\varepsilon + X^*], X^\varepsilon - X^* \rangle ds \right. \\ & \quad + \int_t^{t+\varepsilon} \langle R(t)(u + u^*) + \bar{R}(t)\mathbb{E}_t[u + u^*], u - u^* \rangle ds \\ & \quad \left. + \langle G(t)[X^\varepsilon(T) + X^*(s)] + \bar{G}(t)\mathbb{E}_t[X^\varepsilon(T) + X^*(T)], X^\varepsilon(T) - X^*(T) \rangle \right\}. \end{aligned}$$

Let $(Y(\cdot, t), Z(\cdot, t))$ be the adapted solution to BSDE (4.4). Then one has (suppressing (s, t))

$$\begin{aligned} d[\langle X^\varepsilon - X^*, Y \rangle] &= [\langle A(X^\varepsilon - X^*) + B(u^\varepsilon - u^*), Y \rangle \\ & \quad - \langle X^\varepsilon - X^*, \{A^T Y + C^T Z + QX^* + \bar{Q}\mathbb{E}_t[X^*]\} \rangle \\ & \quad + \langle C(X^\varepsilon - X^*) + D(u^\varepsilon - u^*), Z \rangle] ds \\ & \quad + [\langle C(X^\varepsilon - X^*) + D(u^\varepsilon - u^*), Y \rangle + \langle X^\varepsilon - X^*, Z \rangle] dW \\ &= [\langle u^\varepsilon - u^*, B^T Y + D^T Z \rangle - \langle X^\varepsilon - X^*, QX^* + \bar{Q}\mathbb{E}_t[X^*] \rangle] ds \\ & \quad + [\langle C(X^\varepsilon - X^*) + D(u^\varepsilon - u^*), Y \rangle + \langle X^\varepsilon - X^*, Z \rangle] dW. \end{aligned}$$

Thus,

$$\begin{aligned} & \mathbb{E}_t [\langle X^\varepsilon(T) - X^*(T), GX^*(T) + \bar{G}\mathbb{E}_t[X^*(T)] \rangle] \\ &= \mathbb{E}_t \int_t^T [\langle u^\varepsilon - u^*, B^T Y + D^T Z \rangle - \langle X^\varepsilon - X^*, QX^* + \bar{Q}\mathbb{E}_t[X^*] \rangle] ds. \end{aligned}$$

Consequently,

$$\begin{aligned} & J(t, X^*(t); u^\varepsilon(\cdot)) - J(t, X^*(t), u^*(\cdot)) \\ &= \mathbb{E}_t \left\{ \int_t^T \left\{ \langle Q(X^\varepsilon + X^*) + \bar{Q}\mathbb{E}_t[X^\varepsilon + X^*] - 2QX^* - 2\bar{Q}\mathbb{E}_t[X^*], X^\varepsilon - X^* \rangle \right. \right. \\ & \quad + \langle R(u^\varepsilon + u^*) + \bar{R}\mathbb{E}_t(u^\varepsilon + u^*) + 2B^T Y + 2D^T Z, u^\varepsilon - u^* \rangle \Big\} ds \\ & \quad \left. + \langle G[X^\varepsilon(T) + X^*(T)] + \bar{G}\mathbb{E}_t[X^\varepsilon(T) + X^*(T)] - 2GX^*(T) - \bar{G}\mathbb{E}_t[X^*(T)], \right. \\ & \quad \left. X^\varepsilon(T) - X^*(T) \rangle \right\} \\ &= \mathbb{E}_t \left\{ \int_t^T \left\{ \langle Q(X^\varepsilon - X^*) + \bar{Q}\mathbb{E}_t[X^\varepsilon - X^*], X^\varepsilon - X^* \rangle \right. \right. \\ & \quad + \langle R(u^\varepsilon + u^*) + \bar{R}\mathbb{E}_t(u^\varepsilon + u^*) + 2B^T Y + 2D^T Z, u^\varepsilon - u^* \rangle \Big\} ds \\ & \quad \left. + \langle G[X^\varepsilon(T) - X^*(T)] + \bar{G}\mathbb{E}_t[X^\varepsilon(T) - X^*(T)], X^\varepsilon(T) - X^*(T) \rangle \right\} \\ &= \mathbb{E}_t \left\{ \int_t^{t+\varepsilon} \langle R(t)[u + u^*] + \bar{R}(t)\mathbb{E}_t[u + u^*] + 2B^T Y + 2D^T Z, u - u^* \rangle ds \right. \\ & \quad + \int_t^T |Q(t)^{\frac{1}{2}} \{X^\varepsilon - X^* - \mathbb{E}_t[X^\varepsilon - X^*]\}|^2 + |\hat{Q}(t)^{\frac{1}{2}} \mathbb{E}_t[X^\varepsilon - X^*]|^2 ds \\ & \quad + |G(t)^{\frac{1}{2}} \{X^\varepsilon(T) - X^*(T) - \mathbb{E}_t[X^\varepsilon(T) - X^*(T)]\}|^2 \\ & \quad \left. + |\hat{G}(t)^{\frac{1}{2}} \mathbb{E}_t[X^\varepsilon(T) - X^*(T)]|^2 \right\}. \end{aligned}$$

Let us denote

$$\begin{cases} \hat{u}(s) = u(s) - u^*(s), \\ \theta(s, t) = R(s, t)u^*(s) + \bar{R}(s, t)\mathbb{E}_t[u^*(s)] + B(s)^T Y(s, t) + D(s)^T Z(s, t). \end{cases}$$

Then

$$\begin{aligned} & \mathbb{E}_t \left\{ \langle R(t)(u + u^*) + \bar{R}(t)\mathbb{E}_t[u + u^*] + 2B^T Y + 2\bar{B}^T \mathbb{E}_t[Y] + 2D^T Z + 2\bar{D}^T \mathbb{E}_t[Z], \right. \\ & \quad \left. u - u^* \rangle \right\} \\ &= \mathbb{E}_t \left\{ \langle R(t)\hat{u} + \bar{R}(t)\mathbb{E}_t[\hat{u}] + 2\theta, \hat{u} \rangle \right\} \\ &= \mathbb{E}_t \left[\langle R(t)\{\hat{u} - \mathbb{E}_t[\hat{u}]\} + \hat{R}(t)\mathbb{E}_t[\hat{u}] + 2\theta, \hat{u} - \mathbb{E}_t[\hat{u}] + \mathbb{E}_t[\hat{u}] \rangle \right] \\ &= \mathbb{E}_t \left\{ |R(t)^{\frac{1}{2}} \{\hat{u} - \mathbb{E}_t[\hat{u}]\}|^2 + 2 \langle \theta, \hat{u} - \mathbb{E}_t[\hat{u}] \rangle + |\hat{R}(t)^{\frac{1}{2}} \mathbb{E}_t[\hat{u}]|^2 + 2 \langle \theta, \mathbb{E}_t[\hat{u}] \rangle \right\} \\ &= \mathbb{E}_t \left\{ |R(t)^{\frac{1}{2}} \{\hat{u} - \mathbb{E}_t[\hat{u}]\}|^2 + |R(t)^{-\frac{1}{2}} \theta|^2 + |\hat{R}(t)^{\frac{1}{2}} \mathbb{E}_t[\hat{u}] + \hat{R}(t)^{-\frac{1}{2}} \mathbb{E}_t[\theta]|^2 \right. \\ & \quad \left. - \langle R(t)^{-1} \theta, \theta \rangle - \langle \hat{R}(t)^{-1} \mathbb{E}_t[\theta], \mathbb{E}_t[\theta] \rangle \right\}. \end{aligned}$$

Hence,

$$\begin{aligned}
& J(t, X^*(t); u^\varepsilon(\cdot)) - J(t, X^*(t); u^*(\cdot)) \\
&= \mathbb{E}_t \left\{ \int_t^{t+\varepsilon} \left[|\widehat{R}(t)^{\frac{1}{2}} \mathbb{E}_t[u - u^*] + \widehat{R}(t)^{-\frac{1}{2}} \mathbb{E}_t[\theta]|^2 + |R(t)^{\frac{1}{2}} \{\widehat{u} - \mathbb{E}_t[\widehat{u}]\} + R(t)^{-\frac{1}{2}} \theta|^2 \right] ds \right. \\
&\quad + \int_t^T \left[|Q(t)^{\frac{1}{2}} \{X^\varepsilon - X^* - \mathbb{E}_t[X^\varepsilon - X^*]\}|^2 + |\widehat{Q}(t)^{\frac{1}{2}} \mathbb{E}_t[X^\varepsilon - X^*]|^2 \right] ds \\
&\quad + |G(t)^{\frac{1}{2}} \{X^\varepsilon(T) - X^*(T) - \mathbb{E}_t[X^\varepsilon(T) - X^*(T)]\}|^2 \\
&\quad + |\widehat{G}(t)^{\frac{1}{2}} \mathbb{E}_t[X^\varepsilon(T) - X^*(T)]|^2 \\
&\quad \left. - \int_t^{t+\varepsilon} \left(\langle R(t)^{-1} \theta, \theta \rangle + \langle \widehat{R}(t)^{-1} \mathbb{E}_t[\theta], \mathbb{E}_t[\theta] \rangle \right) ds \right\} \\
&\geq -\mathbb{E}_t \int_t^{t+\varepsilon} \left(\langle R(s, t)^{-1} \theta(s, t), \theta(s, t) \rangle + \langle \widehat{R}(s, t)^{-1} \mathbb{E}_t[\theta(s, t)], \mathbb{E}_t[\theta(s, t)] \rangle \right) ds.
\end{aligned}$$

Note that under our continuity condition (see (4.5)),

$$(4.7) \quad \lim_{s \downarrow t} \mathbb{E}_t |\theta(s, t)|^2 = |\theta(t, t)|^2 = 0, \quad t \in [0, T].$$

Hence, (4.2) holds and $(X^*(\cdot), u^*(\cdot))$ is an open-loop equilibrium pair of Problem (MF-LQ) for the initial state x . \square

The above result leads to the following FBSDE family (parameterized by $t \in [0, T]$):

$$(4.8) \quad \begin{cases} dX^*(s) = [A(s)X^*(s) + B(s)u^*(s)]ds \\ \quad + [C(s)X^*(s) + D(s)u^*(s)]dW(s), \quad s \in [0, T], \\ dY(s, t) = -[A(s)^T Y(s, t) + C(s)^T Z(s, t) + Q(s, t)X^*(s) \\ \quad + \bar{Q}(s, t)\mathbb{E}_t[X^*(s)]]ds + Z(s, t)dW(s), \quad 0 \leq t \leq s \leq T, \\ X^*(0) = x, \quad Y(T, t) = G(t)X^*(T) + \bar{G}(t)\mathbb{E}_t[X^*(T)], \\ \widehat{R}(t, t)u^*(t) + B(t)^T Y(t, t) + D(t)^T Z(t, t) = 0, \quad t \in [0, T]. \end{cases}$$

Inspired by [18], we suppose

$$Y(s, t) = P(s, t)X^*(s) + \bar{P}(s, t)\mathbb{E}_t[X^*(s)], \quad s \in [t, T],$$

for some deterministic functions $P(\cdot, \cdot)$ and $\bar{P}(\cdot, \cdot)$. Then

$$\begin{aligned}
& -\{A(s)^T Y(s, t) + C(s)^T Z(s, t) + Q(s, t)X^*(s) + \bar{Q}(s, t)\mathbb{E}_t[X^*(s)]\}ds \\
& \quad + Z(s, t)dW(s) = dY(s, t) \\
&= \left\{ P_s(s, t)X^*(s) + P(s, t)[A(s)X^*(s) + B(s)u^*(s)] + \bar{P}_s(s, t)\mathbb{E}_t[X^*(s)] \right. \\
& \quad \left. + \bar{P}(s, t)(A(s)\mathbb{E}_t[X^*(s)] + B(s)\mathbb{E}_t[u^*(s)]) \right\} ds \\
& \quad + P(s, t)(C(s)X^*(s) + D(s)u^*(s))dW(s).
\end{aligned}$$

Hence, we need

$$Z(s, t) = P(s, t)(C(s)X^*(s) + D(s)u^*(s)).$$

Consequently,

$$\begin{aligned}\widehat{R}(t, t)u^*(t) + B(t)^T Y(t, t) &= -D(t)^T Z(t, t) \\ &= -D(t)^T P(t, t)[C(t)X^*(t) + D(t)u^*(t)].\end{aligned}$$

Thus, (with $\widehat{P}(s, t) = P(s, t) + \bar{P}(s, t)$)

$$(4.9) \quad \begin{aligned}u^*(t) &= -[\widehat{R}(t, t) + D(t)^T P(t, t)D(t)]^{-1} \\ &\quad \cdot [B(t)^T \widehat{P}(t, t) + D(t)^T P(t, t)C(t)]X^*(t),\end{aligned}$$

and

$$(4.10) \quad \begin{aligned}Z(s, t) &= P(s, t)[C(s)X^*(s) + D(s)u^*(s)] \\ &= P(s, t)C(s)X^*(s) - P(s, t)D(s)[\widehat{R}(s, s) + D(s)^T P(s, s)D(s)]^{-1} \\ &\quad \cdot [B(s)^T \widehat{P}(s, s) + D(s)^T P(s, s)C(s)]X^*(s).\end{aligned}$$

Therefore,

$$\begin{aligned}0 &= P_s(s, t)X^*(s) + P(s, t)[A(s)X^*(s) + B(s)u^*(s)] + \bar{P}_s(s, t)\mathbb{E}_t[X^*(s)] \\ &\quad + \bar{P}(s, t)(A(s)\mathbb{E}_t[X^*(s)] + B(s)\mathbb{E}_t[u^*(s)]) \\ &\quad + A(s)^T Y(s, t) + C(s)^T Z(s, t) + Q(s, t)X^*(s) + \bar{Q}(s, t)\mathbb{E}_t[X^*(s)] \\ &= P_s(s, t)X^*(s) + P(s, t)A(s)X^*(s) \\ &\quad - P(s, t)B(s)[\widehat{R}(s, s) + D(s)^T P(s, s)D(s)]^{-1} \\ &\quad \cdot [B(s)^T \widehat{P}(s, s) + D(s)^T P(s, s)C(s)]X^*(s) \\ &\quad + \bar{P}_s(s, t)\mathbb{E}_t[X^*(s)] + \bar{P}(s, t)A(s)\mathbb{E}_t[X^*(s)] \\ &\quad - \bar{P}(s, t)B(s)[\widehat{R}(s, s) + D(s)^T P(s, s)D(s)]^{-1} \\ &\quad \cdot [B(s)^T \widehat{P}(s, s) + D(s)^T P(s, s)C(s)]\mathbb{E}_t[X^*(s)] \\ &\quad + A(s)^T P(s, t)X^*(s) + A(s)^T \bar{P}(s, t)\mathbb{E}_t[X^*(s)] + C(s)^T P(s, t)C(s)X^*(s) \\ &\quad - C(s)^T P(s, t)D(s)[\widehat{R}(s, s) + D(s)^T P(s, s)D(s)]^{-1} \\ &\quad \cdot [B(s)^T \widehat{P}(s, s) + D(s)^T P(s, s)C(s)]X^*(s) + Q(s, t)X^*(s) + \bar{Q}(s, t)\mathbb{E}_t[X^*(s)] \\ &= \left\{ P_s(s, t) + P(s, t)A(s) + A(s)^T P(s, t) + C(s)^T P(s, t)C(s) + Q(s, t) \right. \\ &\quad - [P(s, t)B(s) + C(s)^T P(s, t)D(s)][\widehat{R}(s, s) + D(s)^T P(s, s)D(s)]^{-1} \\ &\quad \cdot [B(s)^T \widehat{P}(s, s) + D(s)^T P(s, s)C(s)] \left. \right\} X^*(s) \\ &\quad + \left\{ \bar{P}_s(s, t) + \bar{P}(s, t)A(s) + A(s)^T \bar{P}(s, t) + \bar{Q}(s, t) \right. \\ &\quad - \bar{P}(s, t)B(s)[\widehat{R}(s, s) + D(s)^T P(s, s)D(s)]^{-1} \\ &\quad \cdot [B(s)^T \widehat{P}(s, s) + D(s)^T P(s, s)C(s)] \left. \right\} \mathbb{E}_t[X^*(s)].\end{aligned}$$

The above will be true if we let $P(s, t)$ and $\hat{P}(s, t) \equiv P(s, t) + \bar{P}(s, t)$ be the solutions to the following coupled Riccati equations:

$$(4.11) \quad \begin{cases} P_s(s, t) + P(s, t)A(s) + A(s)^T P(s, t) + C(s)^T P(s, t)C(s) + Q(s, t) \\ \quad - [P(s, t)B(s) + C(s)^T P(s, t)D(s)][\hat{R}(s, s) + D(s)^T P(s, s)D(s)]^{-1} \\ \quad \cdot [B(s)^T \hat{P}(s, s) + D(s)^T P(s, s)C(s)] = 0, \\ \hat{P}_s(s, t) + \hat{P}(s, t)A(s) + A(s)^T \hat{P}(s, t) + C(s)^T P(s, t)C(s) + \hat{Q}(s, t) \\ \quad - [\hat{P}(s, t)B(s) + C(s)^T P(s, t)D(s)][\hat{R}(s, s) + D(s)^T P(s, s)D(s)]^{-1} \\ \quad \cdot [B(s)^T \hat{P}(s, s) + D(s)^T P(s, s)C(s)] = 0, \quad s \in [t, T], \\ P(T, t) = G(t), \quad \hat{P}(T, t) = \hat{G}(t). \end{cases}$$

To summarize the above, we state the following result.

Theorem 4.3. *Let (H1)–(H2) hold. Suppose Riccati equation system (4.11) admits a unique solution $(P(\cdot, \cdot), \hat{P}(\cdot, \cdot))$ which is continuous in both variables. Then $u^*(\cdot) \in \mathcal{U}[0, T]$ defined by (4.9) is an open-loop equilibrium control and the corresponding $X^*(\cdot)$ is an open-loop equilibrium state process.*

On top of the above result, one may study the solvability of Riccati equation system (4.11). In principle, the above gives a time-consistent solution to Problem (MF-LQ) of open-loop type. Let us look at a couple of special cases:

Case 1. The case of general discounting only. Let

$$(4.12) \quad \begin{cases} \bar{A}(\cdot) = \bar{C}(\cdot) = 0, \quad \bar{B}(\cdot) = \bar{D}(\cdot) = 0, \\ \bar{Q}(\cdot, \cdot) = 0, \quad \bar{R}(\cdot, \cdot) = 0, \quad \bar{G}(\cdot) = 0. \end{cases}$$

In this case, we have general discounting without conditional expectations. Clearly,

$$\begin{cases} \hat{A}(\cdot) = A(\cdot), \quad \hat{B}(\cdot) = B(\cdot), \quad \hat{C}(\cdot) = C(\cdot), \quad \hat{D}(\cdot) = D(\cdot), \\ \hat{Q}(\cdot, \cdot) = Q(\cdot, \cdot), \quad \hat{R}(\cdot, \cdot) = R(\cdot, \cdot), \quad \hat{G}(\cdot) = G(\cdot), \end{cases}$$

and $P(\cdot, \cdot) = \hat{P}(\cdot, \cdot)$ satisfying

$$(4.13) \quad \begin{cases} P_s(s, t) + P(s, t)A(s) + A(s)^T P(s, t) + C(s)^T P(s, t)C(s) + Q(s, t) \\ \quad - [P(s, t)B(s) + C(s)^T P(s, t)D(s)][R(s, s) + D(s)^T P(s, s)D(s)]^{-1} \\ \quad \cdot [B(s)^T P(s, s) + D(s)^T P(s, s)C(s)] = 0, \quad s \in [t, T], \\ P(T, t) = G(t). \end{cases}$$

The open-loop equilibrium control $u^*(\cdot)$ is given by

$$(4.14) \quad u^*(t) = -[R(t, t) + D(t)^T P(t, t)D(t)]^{-1} \cdot [B(t)^T P(t, t) + D(t)^T P(t, t)C(t)]X^*(t), \quad t \in [0, T].$$

In addition, if $C(\cdot) = 0$ and $D(\cdot) = 0$, the problem becomes a deterministic LQ problem with general discounting, and we have

$$(4.15) \quad \begin{cases} P_s(s, t) + P(s, t)A(s) + A(s)^T P(s, t) + Q(s, t) \\ \quad - P(s, t)B(s)R(s, s)^{-1}B(s)^T P(s, s) = 0, \quad s \in [t, T], \\ P(T, t) = G(t), \end{cases}$$

with the open-loop equilibrium control $u^*(\cdot)$ given by

$$u^*(t) = -R(t, t)^{-1}B(t)^T P(t, t)X^*(t), \quad t \in [0, T].$$

This result is comparable with those in [25], where a closed-loop equilibrium solution was obtained for deterministic LQ problems with general discounting (without conditional expectation terms).

Case 2. The case of conditional expectation at terminal only. Let

$$\begin{cases} Q(s, t) = Q(s), & \bar{Q}(s, t) = 0, & R(s, t) = R(s), & \bar{R}(s, t) = 0, \\ G(t) = G, & \bar{G}(t) = \bar{G}. \end{cases}$$

In this case, the conditional expectation appears at the terminal cost only without general discounting. This is a case studied in [14]. For such a case, both $P(s, t)$ and $\hat{P}(s, t)$ are independent of t , and (4.11) becomes

$$(4.16) \quad \begin{cases} \dot{P}(s) + P(s)A(s) + A(s)^T P(s) + C(s)^T P(s)C(s) + Q(s) \\ \quad - [P(s)B(s) + C(s)^T P(s)D(s)] [R(s) + D(s)^T P(s)D(s)]^{-1} \\ \quad \cdot [B(s)^T \hat{P}(s) + D(s)^T P(s)C(s)] = 0, \\ \dot{\hat{P}}(s) + \hat{P}(s)A(s) + A(s)^T \hat{P}(s) + C(s)^T P(s)C(s) + Q(s) \\ \quad - [\hat{P}(s)B(s) + C(s)^T P(s)D(s)] [R(s) + D(s)^T P(s)D(s)]^{-1} \\ \quad \cdot [B(s)^T \hat{P}(s) + D(s)^T P(s)C(s)] = 0, \quad s \in [t, T], \\ P(T) = G, \quad \hat{P}(T) = \hat{G}, \end{cases}$$

and the open-loop equilibrium control is given by

$$u^*(t) = -[R(t) + D(t)^T P(t)D(t)]^{-1} \cdot [B(t)^T \hat{P}(t) + D(t)^T P(t)C(t)]X^*(t), \quad t \in [0, T].$$

This essentially covers a relevant result in [14]. Note that as long as $\bar{G} \neq 0$, $P(\cdot) \neq \hat{P}(\cdot)$. Then the equation for $P(\cdot)$ contains $\hat{P}(\cdot)$ and it does not have the desired symmetry. Therefore, $P(\cdot)$ is not expected to be symmetric. Consequently, $\hat{P}(\cdot)$ will not be symmetric either. Finally, if, in addition, $\bar{G} = 0$, then $P(\cdot) = \hat{P}(\cdot)$ and it satisfies

$$(4.17) \quad \begin{cases} \dot{P}(s) + P(s)A(s) + A(s)^T P(s) + C(s)^T P(s)C(s) + Q(s) \\ \quad - [P(s)B(s) + C(s)^T P(s)D(s)] [R(s) + D(s)^T P(s)D(s)]^{-1} \\ \quad \cdot [B(s)^T P(s) + D(s)^T P(s)C(s)] = 0, \quad s \in [t, T], \\ P(T) = G. \end{cases}$$

In this case, the problem is reduced to a classical stochastic LQ problem and it is time-consistent. For this case, the open-loop equilibrium control $u^*(\cdot)$ is the optimal control and is given by

$$u^*(t) = -[R(t) + D(t)^T P(t) D(t)]^{-1} \cdot [B(t)^T P(t) + D(t)^T P(t) C(t)] X^*(t), \quad t \in [0, T],$$

which recovers the result for the classical stochastic LQ problem.

We now make a couple of comments on this.

The advantages: The approach is direct and the derivation of the equilibrium pair is not very complicated. Moreover, the open-loop equilibrium control $u^*(\cdot)$ admits a closed-loop representation (4.9).

The disadvantages: (i) The Riccati equations in (4.11) do not have symmetry structure. Therefore the solutions $P(\cdot, \cdot)$ and $\hat{P}(\cdot, \cdot)$ of the system are not necessarily symmetric. This leads to some difficulties in establishing the well-posedness of the system. (ii) If the state equation contains conditional expectation, even the definition of the open-loop equilibrium pair is not clear to us.

4.2. Close-loop equilibrium strategy. In this subsection, we introduce closed-loop equilibrium strategies. To this end, we first introduce the following: For any $t \in [0, T]$,

$$(4.18) \quad \begin{aligned} & \tilde{J}(t; X(\cdot), u(\cdot)) \\ &= \mathbb{E}_t \left\{ \int_t^T [\langle Q(s, t) X(s), X(s) \rangle + \langle \bar{Q}(s, t) \mathbb{E}_t[X(s)], \mathbb{E}_t[X(s)] \rangle \right. \\ & \quad + \langle R(s, t) u(s), u(s) \rangle + \langle \bar{R}(s, t) \mathbb{E}_t[u(s)], \mathbb{E}_t[u(s)] \rangle] ds \\ & \quad \left. + \langle G(t) X(T), X(T) \rangle + \langle \bar{G}(t) \mathbb{E}_t[X(T)], \mathbb{E}_t[X(T)] \rangle \right\}, \end{aligned}$$

for any $(X(\cdot), u(\cdot)) \in \mathcal{X}[t, T] \times \mathcal{U}[t, T]$. We point out that in the above $(X(\cdot), u(\cdot))$ does not have to be a state-control pair of the original control system. Thus, $\tilde{J}(t; X(\cdot), u(\cdot))$ is an extension of the cost functional $J(t, x; u(\cdot))$, and

$$\tilde{J}(t; X(\cdot; t, x, u(\cdot)), u(\cdot)) = J(t, x; u(\cdot)), \quad \forall (t, x) \in \mathcal{D}, \quad u(\cdot) \in \mathcal{U}[t, T].$$

Next, and hereafter, we denote any partition of $[0, T]$ by Δ :

$$\Delta = \{t_k \mid 0 \leq k \leq N\} \equiv \{0 = t_0 < t_1 < t_2 < \cdots < t_{N-1} < t_N = T\},$$

with N being some natural number, and define its *mesh size* by the following:

$$\|\Delta\| = \max_{0 \leq k \leq N-1} (t_{k+1} - t_k).$$

For the above Δ , we define

$$(4.19) \quad J_k^\Delta(X(\cdot), u(\cdot)) = \tilde{J}(t_k, X(\cdot), u(\cdot)),$$

for any $(X(\cdot), u(\cdot)) \in \mathcal{X}[t_k, T] \times \mathcal{U}[t_k, T]$, $k = 0, 1, 2, \dots, N-1$.

Now, we introduce some notions.

Definition 4.4. Let $\Delta = \{0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T\}$ be a partition of $[0, T]$, and let $\Theta^\Delta, \hat{\Theta}^\Delta : [0, T] \rightarrow \mathbb{R}^{m \times n}$ be two given maps, possibly depending on Δ .

(i) For any $x \in \mathbb{R}^n$ fixed, let $X^\Delta(\cdot) \equiv X^\Delta(\cdot; x)$ be the solution to the following linear MF-SDE:

$$\left\{ \begin{array}{l} dX^\Delta(s) = \left\{ [A(s) - B(s)\Theta^\Delta(s)]X^\Delta(s) \right. \\ \quad + [\bar{A}(s) + B(s)[\Theta^\Delta(s) - \hat{\Theta}^\Delta(s)] - \bar{B}(s)\hat{\Theta}^\Delta(s)] \mathbb{E}_{\rho^\Delta(s)}[X^\Delta(s)] \Big\} ds \\ \quad + \left\{ [C(s) - D(s)\Theta^\Delta(s)]X^\Delta(s) \right. \\ \quad + [\bar{C}(s) + D(s)[\Theta^\Delta(s) - \hat{\Theta}^\Delta(s)] - \bar{D}(s)\hat{\Theta}^\Delta(s)] \mathbb{E}_{\rho^\Delta(s)}[X^\Delta(s)] \Big\} dW(s), \\ \quad \quad \quad s \in [0, T], \\ X^\Delta(0) = x, \end{array} \right.$$

where

$$\rho^\Delta(s) = \sum_{k=0}^{N-1} t_k I_{[t_k, t_{k+1})}(s), \quad s \in [0, T],$$

and let $u^\Delta(\cdot) \equiv u^\Delta(\cdot; x)$ be defined by

$$(4.20) \quad u^\Delta(s) = -\Theta^\Delta(s)X^\Delta(s) + [\Theta^\Delta(s) - \hat{\Theta}^\Delta(\cdot)] \mathbb{E}_{\rho^\Delta(s)}[X^\Delta(s)], \quad s \in [0, T].$$

The pair $(X^\Delta(\cdot), u^\Delta(\cdot))$ is called the *closed-loop pair* associated with Δ and $(\Theta^\Delta(\cdot), \hat{\Theta}^\Delta(\cdot))$, starting from x .

(ii) For each $t_k \in \Delta$ and any $u_k(\cdot) \in \mathcal{U}[t_k, t_{k+1}]$, let $X_k(\cdot)$ be the solution to the following system:

$$\left\{ \begin{array}{l} dX_k(s) = \left\{ A(s)X_k(s) + \bar{A}(s)\mathbb{E}_{t_k}[X_k(s)] + B(s)u_k(s) + \bar{B}(s)\mathbb{E}_{t_k}[u_k(s)] \right\} ds \\ \quad + \left\{ C(s)X_k(s) + \bar{C}(s)\mathbb{E}_{t_k}[X_k(s)] + D(s)u_k(s) + \bar{D}(s)\mathbb{E}_{t_k}[u_k(s)] \right\} dW(s), \\ \quad \quad \quad s \in [t_k, t_{k+1}], \\ X_k(t_k) = X^\Delta(t_k), \end{array} \right.$$

and let $X_{k+1}^\Delta(\cdot)$ be the solution to the following:

$$\left\{ \begin{array}{l} dX_{k+1}^\Delta(s) = \left\{ [A(s) - B(s)\Theta^\Delta(s)]X_{k+1}^\Delta(s) \right. \\ \quad + [\bar{A}(s) + B(s)[\Theta^\Delta(s) - \hat{\Theta}^\Delta(s)] - \bar{B}(s)\hat{\Theta}^\Delta(s)] \mathbb{E}_{\rho^\Delta(s)}[X_{k+1}^\Delta(s)] \Big\} ds \\ \quad + \left\{ [C(s) - D(s)\Theta^\Delta(s)]X_{k+1}^\Delta(s) \right. \\ \quad + [\bar{C}(s) + D(s)[\Theta^\Delta(s) - \hat{\Theta}^\Delta(s)] - \bar{D}(s)\hat{\Theta}^\Delta(s)] \mathbb{E}_{\rho^\Delta(s)}[X_{k+1}^\Delta(s)] \Big\} dW(s), \\ \quad \quad \quad s \in [t_{k+1}, T], \\ X_{k+1}^\Delta(t_{k+1}) = X_k(t_{k+1}). \end{array} \right.$$

Denote

$$\left\{ \begin{array}{l} X_k(\cdot) \oplus X^\Delta(\cdot) \equiv X_k(\cdot)I_{[t_k, t_{k+1})}(\cdot) + X_{k+1}^\Delta(\cdot)I_{[t_{k+1}, T]}(\cdot), \\ u_k(\cdot) \oplus u^\Delta(\cdot) = u_k(\cdot)I_{[t_k, t_{k+1})}(\cdot) \\ \quad - \{\Theta^\Delta(\cdot)X_{k+1}^\Delta(\cdot) + [\Theta^\Delta(\cdot) - \hat{\Theta}^\Delta(\cdot)] \mathbb{E}_{\rho^\Delta(\cdot)}[X_{k+1}^\Delta(\cdot)]\} I_{[t_{k+1}, T]}(\cdot). \end{array} \right.$$

We call $(X_k(\cdot) \oplus X^\Delta(\cdot), u_k(\cdot) \oplus u^\Delta(\cdot))$ a *local variation* of $(X^\Delta(\cdot), u^\Delta(\cdot))$ on $[t_k, t_{k+1}]$. Suppose the following *local optimality condition* holds:

$$J_k^\Delta(X_k^\Delta(\cdot), u_k^\Delta(\cdot)) \leq J_k^\Delta(X_k(\cdot) \oplus X^\Delta(\cdot), u_k(\cdot) \oplus u^\Delta(\cdot)),$$

$$\forall u_k(\cdot) \in \mathcal{U}[t_k, t_{k+1}].$$

Then we call $(\Theta^\Delta(\cdot), \hat{\Theta}^\Delta(\cdot))$ a *closed-loop Δ -equilibrium strategy* of Problem (MF-LQ), and we call $(X^\Delta(\cdot; x), u^\Delta(\cdot; x))$ a *closed-loop Δ -equilibrium pair* of Problem (MF-LQ) for the initial state x .

(iii) If

$$(4.21) \quad \lim_{\|\Delta\| \rightarrow 0} [\|\Theta^\Delta(\cdot) - \Theta(\cdot)\| + \|\hat{\Theta}^\Delta(\cdot) - \hat{\Theta}(\cdot)\|] = 0$$

holds for some $\Theta, \hat{\Theta} \in C([0, T]; \mathbb{R}^{m \times n})$, then $(\Theta(\cdot), \hat{\Theta}(\cdot))$ is called a *closed-loop equilibrium strategy* of Problem (MF-LQ). For any $(t, x) \in \mathcal{D}$, let $\hat{X}^*(\cdot) \equiv \hat{X}^*(\cdot; t, x)$ be the solution to the following system:

$$(4.22) \quad \begin{cases} d\hat{X}^*(s) = [\hat{A}(s) - \hat{B}(s)\hat{\Theta}(s)]\hat{X}^*(s)ds \\ \quad + [\hat{C}(s) - \hat{D}(s)\hat{\Theta}(s)]\hat{X}^*(s)dW(s), \quad s \in [t, T], \\ \hat{X}^*(t) = x, \end{cases}$$

and define $\hat{u}^*(\cdot) \equiv \hat{u}^*(\cdot; t, x)$ as follows:

$$(4.23) \quad \hat{u}^*(s) = -\hat{\Theta}(s)\hat{X}^*(s), \quad s \in [t, T].$$

Then $(t, x) \mapsto (\hat{X}^*(\cdot; t, x), \hat{u}^*(\cdot; t, x))$ is called a *closed-loop equilibrium pair flow* of Problem (MF-LQ). Further,

$$(4.24) \quad \hat{V}(t, x) = \tilde{J}(t, x; \hat{X}^*(\cdot; t, x), \hat{u}^*(\cdot; t, x)), \quad (t, x) \in \mathcal{D},$$

is called a *closed-loop equilibrium value function* of Problem (MF-LQ).

We point out that $(\Theta^\Delta(\cdot), \hat{\Theta}^\Delta(\cdot))$ and $(\Theta(\cdot), \hat{\Theta}(\cdot))$ are independent of the initial state $x \in \mathbb{R}^n$. To state the main result of this paper, we need one more assumption.

(H4) There exists a $\tilde{C}(\cdot) \in C([0, T]; \mathbb{R}^{n \times n})$ such that

$$(4.25) \quad C(s) + \bar{C}(s) = [D(s) + \bar{D}(s)]\tilde{C}(s), \quad s \in [0, T].$$

The above (H4) is technically necessary below. When $m = n$ and $[D(s) + \bar{D}(s)]^{-1}$ exists for all $s \in [0, T]$, the above condition holds. It seems to us that our result should hold without such a technical condition. However, at the moment, we need this result. Hopefully, in our future study, such a condition can be removed. Now, we state our main result.

Theorem 4.5. *Let (H1)–(H4) hold. Then there exists a unique pair $(\Gamma(\cdot, \cdot), \hat{\Gamma}(\cdot, \cdot))$ of \mathbb{S}^n -valued functions solving the following system of equations:*

$$(4.26) \quad \begin{cases} \Gamma_s(s, t) + \Gamma(s, t) [\hat{A}(s) - \hat{B}(s) \hat{\Theta}(s)] + [\hat{A}(s) - \hat{B}(s) \hat{\Theta}(s)]^T \Gamma(s, t) + Q(s, t) \\ \quad + [\hat{C}(s) - \hat{D}(s) \hat{\Theta}(s)]^T \Gamma(s, t) [\hat{C}(s) - \hat{D}(s) \hat{\Theta}(s)] + \hat{\Theta}(s)^T R(s, t) \hat{\Theta}(s) = 0, \\ \hat{\Gamma}_s(s, t) + \hat{\Gamma}(s, t) [\hat{A}(s) - \hat{B}(s) \hat{\Theta}(s)] + [\hat{A}(s) - \hat{B}(s) \hat{\Theta}(s)]^T \hat{\Gamma}(s, t) + \hat{Q}(s, t) \\ \quad + [\hat{C}(s) - \hat{D}(s) \hat{\Theta}(s)]^T \hat{\Gamma}(s, t) [\hat{C}(s) - \hat{D}(s) \hat{\Theta}(s)] + \hat{\Theta}(s)^T \hat{R}(s, t) \hat{\Theta}(s) = 0, \\ \hspace{15em} 0 \leq t \leq s \leq T, \\ \Gamma(T, t) = G(t), \quad \hat{\Gamma}(T, t) = \hat{G}(t), \quad 0 \leq t \leq T, \end{cases}$$

where $\hat{\Theta}(\cdot)$ is given by the following:

$$(4.27) \quad \begin{aligned} \hat{\Theta}(s) &= [\hat{R}(s, s) + \hat{D}(s)^T \Gamma(s, s) \hat{D}(s)]^{-1} \\ &\quad \cdot [\hat{B}(s)^T \hat{\Gamma}(s, s) + \hat{D}(s)^T \Gamma(s, s) \hat{C}(s)], \quad s \in [0, T]. \end{aligned}$$

The closed-loop equilibrium state process $X^*(\cdot)$ is the solution to the following system:

$$(4.28) \quad \begin{cases} dX^*(s) = [\hat{A}(s) - \hat{B}(s) \hat{\Theta}(s)] X^*(s) ds \\ \quad + [\hat{C}(s) - \hat{D}(s) \hat{\Theta}(s)] X^*(s) dW(s), \quad s \in [0, T], \\ X^*(0) = x; \end{cases}$$

the closed-loop equilibrium control admits the following representation:

$$(4.29) \quad u^*(s) = -\hat{\Theta}(s) X^*(s), \quad s \in [0, T];$$

and the closed-loop equilibrium value function is given by the following:

$$(4.30) \quad \hat{V}(t, x) = \langle \hat{\Gamma}(t, t) x, x \rangle, \quad \forall (t, x) \in \mathcal{D}.$$

A proof of the above theorem will be carried out in the following sections. Let us make some remarks.

- Although $\Theta^\Delta(\cdot)$ is an important part in the closed-loop Δ -equilibrium pair (see (4.20)), at the limit as $\|\Delta\| \rightarrow 0$, (4.20) becomes (4.29), in which $\Theta(\cdot)$ becomes irrelevant. From the proof below, we will see that $\Theta(\cdot)$ will not appear in (4.26) either! Hence, the definition of $\Theta(\cdot)$ is not necessary in the statement of the above theorem.

- In (4.26), the equations for $\Gamma(\cdot, \cdot)$ and $\hat{\Gamma}(\cdot, \cdot)$ are different: $(Q(\cdot, \cdot), R(\cdot, \cdot), G(\cdot))$ appears in the former and $(\hat{Q}(\cdot, \cdot), \hat{R}(\cdot, \cdot), \hat{G}(\cdot))$ appears in the latter. Also, we see that the system is fully coupled.

Let us now look at two special cases.

Case 0. The case of general discounting only. As in the previous subsection, let

$$\bar{A}(\cdot) = \bar{C}(\cdot) = 0, \quad \bar{B}(\cdot) = \bar{D}(\cdot) = 0, \quad \bar{Q}(\cdot, \cdot) = 0, \quad \bar{R}(\cdot, \cdot) = 0, \quad \bar{G}(\cdot) = 0.$$

In this case, we have

$$\begin{cases} \hat{A}(\cdot) = A(\cdot), \quad \hat{B}(\cdot) = B(\cdot), \quad \hat{C}(\cdot) = C(\cdot), \quad \hat{D}(\cdot) = D(\cdot), \\ \hat{Q}(\cdot, \cdot) = Q(\cdot, \cdot), \quad \hat{R}(\cdot, \cdot) = R(\cdot, \cdot), \quad \hat{G}(\cdot) = G(\cdot). \end{cases}$$

Thus,

$$(4.31) \quad \hat{\Theta}(s) = [R(s, s) + D(s)^T \Gamma(s, s) D(s)]^{-1} \cdot [B(s)^T \hat{\Gamma}(s, s) + D(s)^T \Gamma(s, s) C(s)], \quad s \in [0, T].$$

Hence, (4.26) becomes

$$\begin{cases} \Gamma_s(s, t) + \Gamma(s, t) [A(s) - B(s) \hat{\Theta}(s)] + [A(s) - B(s) \hat{\Theta}(s)]^T \Gamma(s, t) + Q(s, t) \\ \quad + [C(s) - D(s) \hat{\Theta}(s)]^T \Gamma(s, t) [C(s) - D(s) \hat{\Theta}(s)] + \hat{\Theta}(s)^T R(s, t) \hat{\Theta}(s) = 0, \\ \hat{\Gamma}_s(s, t) + \hat{\Gamma}(s, t) [A(s) - B(s) \hat{\Theta}(s)] + [A(s) - B(s) \hat{\Theta}(s)]^T \hat{\Gamma}(s, t) + Q(s, t) \\ \quad + [C(s) - D(s) \hat{\Theta}(s)]^T \hat{\Gamma}(s, t) [C(s) - D(s) \hat{\Theta}(s)] + \hat{\Theta}(s)^T R(s, t) \hat{\Theta}(s) = 0, \\ \Gamma(T, t) = \hat{\Gamma}(T, t) = G(t). \end{cases} \quad s \in [t, T],$$

Thus, we see that

$$\hat{\Gamma}(s, t) = \Gamma(s, t), \quad 0 \leq t \leq s \leq T.$$

Consequently, the equation for $\Gamma(\cdot, \cdot)$ becomes

$$\begin{cases} \Gamma_s(s, t) + \Gamma(s, t) [A(s) - B(s) \Theta(s)] + [A(s) - B(s) \Theta(s)]^T \Gamma(s, t) + Q(s, t) \\ \quad + [C(s) - D(s) \Theta(s)]^T \Gamma(s, t) [C(s) - D(s) \Theta(s)] + \Theta(s)^T R(s, t) \Theta(s) = 0, \\ \Gamma(T, t) = G(t), \end{cases} \quad s \in [t, T],$$

with

$$\Theta(s) = [R(s, s) + D(s)^T \Gamma(s, s) D(s)]^{-1} [B(s)^T \Gamma(s, s) + D(s)^T \Gamma(s, s) C(s)].$$

This coincides with a special case of the general result found in [27].

Case 1. The case of conditional expectation only. Let

$$\begin{cases} \bar{A}(\cdot) = \bar{C}(\cdot) = 0, \quad \bar{B}(\cdot) = \bar{D}(\cdot) = 0, \\ Q(s, t) = \bar{Q}(s, t), \quad \bar{Q}(s, t) = \bar{Q}(s), \quad R(s, t) = \bar{R}(s, t), \quad \bar{R}(s, t) = \bar{R}(s), \\ G(t) = \bar{G}, \quad \bar{G}(t) = \bar{G}. \end{cases}$$

Note that unlike Case 2 in the previous subsection, we allow conditional expectation terms to appear in the running costs. In this case, one has

$$\begin{cases} \hat{A}(\cdot) = A(\cdot), \quad \hat{B}(\cdot) = B(\cdot), \quad \hat{C}(\cdot) = C(\cdot), \quad \hat{D}(\cdot) = D(\cdot), \\ \hat{Q}(s, t) = \hat{Q}(s), \quad \hat{R}(s, t) = \hat{R}(s), \quad \hat{G}(t) = \hat{G}. \end{cases}$$

Thus, both $\Gamma(s, t)$ and $\hat{\Gamma}(s, t)$ are independent of t and system (4.26) becomes

$$(4.32) \quad \begin{cases} \dot{\Gamma}(s) + \Gamma(s)[A(s) - B(s)\hat{\Theta}(s)] + [A(s) - B(s)\hat{\Theta}(s)]^T \Gamma(s) + Q(s) \\ \quad + [C(s) - D(s)\hat{\Theta}(s)]^T \Gamma(s)[C(s) - D(s)\hat{\Theta}(s)] + \hat{\Theta}(s)^T R(s)\hat{\Theta}(s) = 0, \\ \dot{\hat{\Gamma}}(s) + \hat{\Gamma}(s)[A(s) - B(s)\hat{\Theta}(s)] + [A(s) - B(s)\hat{\Theta}(s)]^T \hat{\Gamma}(s) + \hat{Q}(s) \\ \quad + [C(s) - D(s)\hat{\Theta}(s)]^T \hat{\Gamma}(s)[C(s) - D(s)\hat{\Theta}(s)] + \hat{\Theta}(s)^T \hat{R}(s)\hat{\Theta}(s) = 0, \\ \quad \quad \quad 0 \leq s \leq T, \\ \Gamma(T) = G, \quad \hat{\Gamma}(T) = \hat{G}, \end{cases}$$

where $\hat{\Theta}(\cdot)$ is given by the following:

$$\hat{\Theta}(s) = [\hat{R}(s) + D(s)^T \Gamma(s) D(s)]^{-1} [B(s)^T \hat{\Gamma}(s) + D(s)^T \Gamma(s) C(s)], \quad s \in [0, T].$$

Further, if

$$\bar{Q}(\cdot) = 0, \quad \bar{R}(\cdot) = 0,$$

then we are in the case of conditional expectation at terminal cost only, and (4.32) becomes

$$\begin{cases} \dot{\Gamma}(s) + \Gamma(s)[A(s) - B(s)\hat{\Theta}(s)] + [A(s) - B(s)\hat{\Theta}(s)]^T \Gamma(s) + Q(s) \\ \quad + [C(s) - D(s)\hat{\Theta}(s)]^T \Gamma(s)[C(s) - D(s)\hat{\Theta}(s)] + \hat{\Theta}(s)^T R(s)\hat{\Theta}(s) = 0, \\ \dot{\hat{\Gamma}}(s) + \hat{\Gamma}(s)[A(s) - B(s)\hat{\Theta}(s)] + [A(s) - B(s)\hat{\Theta}(s)]^T \hat{\Gamma}(s) + \hat{Q}(s) \\ \quad + [C(s) - D(s)\hat{\Theta}(s)]^T \hat{\Gamma}(s)[C(s) - D(s)\hat{\Theta}(s)] + \hat{\Theta}(s)^T \hat{R}(s)\hat{\Theta}(s) = 0, \\ \quad \quad \quad 0 \leq s \leq T, \\ \Gamma(T) = G, \quad \hat{\Gamma}(T) = \hat{G}. \end{cases}$$

The closed-loop equilibrium state process $X^*(\cdot)$ satisfies

$$\begin{cases} dX^*(s) = [A(s) - B(s)\hat{\Theta}(s)]X^*(s)ds \\ \quad + [C(s) - D(s)\hat{\Theta}(s)]X^*(s)dW(s), \quad s \in [0, T], \\ X^*(0) = x; \end{cases}$$

the closed-loop equilibrium control admits the following representation:

$$u^*(s) = -\hat{\Theta}(s)X^*(s), \quad s \in [0, T];$$

and the closed-loop equilibrium value function is given by the following:

$$\hat{V}(t, x) = \langle \hat{\Gamma}(t)x, x \rangle, \quad \forall(t, x) \in \mathcal{D}.$$

Clearly, in the case that $\bar{G} = 0$, the problem is reduced to a classical LQ problem. In this case, $\Gamma(\cdot) = \hat{\Gamma}(\cdot)$, and our result recovers the classical one.

Directly comparing the results of this subsection with those in the previous subsection, we see that the open-loop and closed-loop equilibrium solutions are different for Problem (MF-LQ), even for the above two special cases. The results coincide when the problem is reduced to classical LQ problems.

5. MULTI-PERSON DIFFERENTIAL GAMES

In this section, we fix a partition $\Delta : 0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$ of $[0, T]$, and construct a closed-loop Δ -equilibrium strategy for Problem (MF-LQ). To this end, we introduce an associated N -person differential game. For notational convenience, we label the players by $0, 1, 2, \dots, N-1$. The k -th player controls the system on $[t_k, t_{k+1})$, with the cost functional being “sophisticatedly” constructed (see below). The main rules are the following:

(i) Each player will play optimally based on the assumption that the later players will play optimally.

(ii) The k -th player will affect the $(k+1)$ -th player’s action through her terminal state (which is the initial state of the $(k+1)$ -th player).

(iii) Although the k -th player will not be able to control the system from t_{k+1} on, she will still “discount” the cost functional in her own way on the interval $[t_{k+1}, T]$.

Let us now look at the N -person differential games in detail. In what follows, we denote

$$\begin{cases} Q_k(s) = Q(s, t_k), & R_k(s) = R(s, t_k), & G_k = G(t_k), \\ \bar{Q}_k(s) = \bar{Q}(s, t_k), & \bar{R}_k(s) = R(s, t_k), & \bar{G}_k = \bar{G}(t_k), \\ \hat{Q}_k(s) = Q_k(s) + \bar{Q}_k(s), & \hat{R}_k(s) = R_k(s) + \bar{R}_k(s), & \hat{G}_k = G_k + \bar{G}_k, \\ & & k = 0, 1, \dots, N-1. \end{cases}$$

We will carefully look at Players $(N-1)$, $(N-2)$, and $(N-3)$, who will have different features. Once we have done that, the situations for the rest of the players will be clear and can be treated inductively. We now begin with Player $(N-1)$. The state equation for this player is the following:

$$(5.1) \quad \begin{cases} dX_{N-1}(s) = \left\{ A(s)X_{N-1}(s) + \bar{A}(s)\mathbb{E}_{t_{N-1}}[X_{N-1}(s)] \right. \\ \quad \left. + B(s)u_{N-1}(s) + \bar{B}(s)\mathbb{E}_{t_{N-1}}[u_{N-1}(s)] \right\} ds \\ \quad + \left\{ C(s)X_{N-1}(s) + \bar{C}(s)\mathbb{E}_{t_{N-1}}[X_{N-1}(s)] \right. \\ \quad \left. + D(s)u_{N-1}(s) + \bar{D}(s)\mathbb{E}_{t_{N-1}}[u_{N-1}(s)] \right\} dW(s), \\ \quad \quad \quad s \in [t_{N-1}, t_N], \\ X_{N-1}(t_{N-1}) = x_{N-1} \in \mathcal{X}_{t_{N-1}}, \end{cases}$$

and the cost functional is given by

$$(5.2) \quad \begin{aligned} & J_{N-1}^\Delta(x_{N-1}; u_{N-1}(\cdot)) \\ &= \mathbb{E}_{t_{N-1}} \left\{ \int_{t_{N-1}}^{t_N} [\langle Q_{N-1}(s)X_{N-1}(s), X_{N-1}(s) \rangle \right. \\ &\quad + \langle \bar{Q}_{N-1}(s)\mathbb{E}_{t_{N-1}}[X_{N-1}(s)], \mathbb{E}_{t_{N-1}}[X_{N-1}(s)] \rangle \\ &\quad + \langle R_{N-1}(s)u_{N-1}(s), u_{N-1}(s) \rangle \\ &\quad + \langle \bar{R}_{N-1}(s)\mathbb{E}_{t_{N-1}}[u_{N-1}(s)], \mathbb{E}_{t_{N-1}}[u_{N-1}(s)] \rangle] ds \\ &\quad + \langle G_{N-1}X_{N-1}(t_N), X_{N-1}(t_N) \rangle \\ &\quad \left. + \langle \bar{G}_{N-1}\mathbb{E}_{t_{N-1}}[X_{N-1}(t_N)], \mathbb{E}_{t_{N-1}}[X_{N-1}(t_N)] \rangle \right\}. \end{aligned}$$

Player $(N-1)$ wants to solve the following problem.

Problem (MF-LQ)_{N-1}. For any $x_{N-1} \in \mathcal{X}_{t_{N-1}}$, find a $u_{N-1}^*(\cdot) \in \mathcal{U}[t_{N-1}, t_N]$ such that

$$(5.3) \quad J_{N-1}^\Delta(x_{N-1}; u_{N-1}^*(\cdot)) = \inf_{u_{N-1}(\cdot) \in \mathcal{U}[t_{N-1}, t_N]} J_{N-1}^\Delta(x_{N-1}; u_{N-1}(\cdot)).$$

This is a standard LQ problem for MF-SDEs. According to Proposition 3.1, under (H1)–(H2), one has a pair of \mathbb{S}^n -valued functions $(P_{N-1}(\cdot), \hat{P}_{N-1}(\cdot))$ uniquely solving the following Riccati equations (s is suppressed):

$$(5.4) \quad \begin{cases} \dot{P}_{N-1} + P_{N-1}A + A^T P_{N-1} + C^T P_{N-1}C + Q_{N-1} \\ \quad - (P_{N-1}B + C^T P_{N-1}D)(R_{N-1} + D^T P_{N-1}D)^{-1} \\ \quad \cdot (B^T P_{N-1} + D^T P_{N-1}C) = 0, \\ \dot{\hat{P}}_{N-1} + \hat{P}_{N-1}\hat{A} + \hat{A}^T \hat{P}_{N-1} + \hat{C}^T P_{N-1}\hat{C} + \hat{Q}_{N-1} \\ \quad - (\hat{P}_{N-1}\hat{B} + \hat{C}^T P_{N-1}\hat{D})(\hat{R}_{N-1} + \hat{D}^T P_{N-1}\hat{D})^{-1} \\ \quad \cdot (\hat{B}^T \hat{P}_{N-1} + \hat{D}^T P_{N-1}\hat{C}) = 0, \quad s \in [t_{N-1}, t_N], \\ P_{N-1}(t_N) = G_{N-1}, \quad \hat{P}_{N-1}(t_N) = \hat{G}_{N-1}. \end{cases}$$

Define

$$(5.5) \quad \begin{cases} \Theta_{N-1} = (R_{N-1} + D^T P_{N-1}D)^{-1}(B^T P_{N-1} + D^T P_{N-1}C), \\ \hat{\Theta}_{N-1} = (\hat{R}_{N-1} + \hat{D}^T P_{N-1}\hat{D})^{-1}(\hat{B}^T \hat{P}_{N-1} + \hat{D}^T P_{N-1}\hat{C}), \end{cases} \quad s \in [t_{N-1}, t_N].$$

Then the optimal state process $X_{N-1}^*(\cdot) \equiv X_{N-1}^*(\cdot; t_{N-1}, x_{N-1})$ solves the following closed-loop system:

$$(5.6) \quad \begin{cases} dX_{N-1}^* = \left\{ [A - B\Theta_{N-1}]X_{N-1}^* \right. \\ \quad \left. + [\bar{A} + B(\Theta_{N-1} - \hat{\Theta}_{N-1}) - \bar{B}\hat{\Theta}_{N-1}] \mathbb{E}_{t_{N-1}}[X_{N-1}^*] \right\} ds \\ \quad + \left\{ [C - D\Theta_{N-1}]X_{N-1}^* \right. \\ \quad \left. + [\bar{C} + D(\Theta_{N-1} - \hat{\Theta}_{N-1}) - \bar{D}\hat{\Theta}_{N-1}] \mathbb{E}_{t_{N-1}}[X_{N-1}^*] \right\} dW(s), \\ \quad \quad \quad s \in [t_{N-1}, t_N], \\ X_{N-1}^*(t_{N-1}) = x_{N-1}, \end{cases}$$

and the optimal control $u_{N-1}^*(\cdot) \equiv u_{N-1}^*(\cdot; t_{N-1}, x_{N-1})$ admits the following state feedback representation:

$$(5.7) \quad u_{N-1}^*(s) = -\Theta_{N-1}(s)X_{N-1}^*(s) + [\Theta_{N-1}(s) - \hat{\Theta}_{N-1}(s)] \mathbb{E}_{t_{N-1}}[X_{N-1}^*(s)], \\ s \in [t_{N-1}, t_N].$$

Finally,

$$(5.8) \quad \begin{aligned} \inf_{u_{N-1}(\cdot) \in \mathcal{U}[t_{N-1}, t_N]} J_{N-1}^\Delta(x_{N-1}; u_{N-1}(\cdot)) &= J_{N-1}^\Delta(x_{N-1}; u_{N-1}^*(\cdot)) \\ &= \langle \hat{P}_{N-1}(t_{N-1})x_{N-1}, x_{N-1} \rangle, \quad \forall x_{N-1} \in \mathcal{X}_{t_{N-1}}. \end{aligned}$$

This way, Player $(N-1)$ has solved Problem (MF-LQ)_{N-1}.

Next, we consider Player $(N-1)$, whose state equation is the following:

$$(5.9) \quad \begin{cases} dX_{N-2}(s) = \left\{ A(s)X_{N-2}(s) + \bar{A}(s)\mathbb{E}_{t_{N-2}}[X_{N-2}(s)] \right. \\ \quad \left. + B(s)u_{N-2}(s) + \bar{B}(s)\mathbb{E}_{t_{N-2}}[u_{N-2}(s)] \right\} ds \\ \quad + \left\{ C(s)X_{N-2}(s) + \bar{C}(s)\mathbb{E}_{t_{N-2}}[X_{N-2}(s)] \right. \\ \quad \left. + D(s)u_{N-2}(s) + \bar{D}(s)\mathbb{E}_{t_{N-2}}[u_{N-2}(s)] \right\} dW(s), \\ \quad \quad \quad s \in [t_{N-2}, t_{N-1}], \\ X(t_{N-2}) = x_{N-2} \in \mathcal{X}_{t_{N-2}}. \end{cases}$$

Let $X_{N-2}(\cdot) = X(\cdot; t_{N-2}, x_{N-2}, u_{N-2}(\cdot))$ be the corresponding solution. At $s = t_{N-1}$, Player $(N-1)$ takes over the system and will use her optimal control $u_{N-1}^*(\cdot)$ of state feedback from (5.7) on $[t_{N-1}, t_N]$, where $X_{N-1}^*(\cdot)$ satisfies closed-loop system (5.6) with initial state $x_{N-1} = X_{N-2}(t_{N-1})$. Because of this, Player $(N-2)$ considers the following (sophisticated) cost functional (suppressing s):

$$\begin{aligned} J_{N-2}^\Delta(x_{N-2}; u_{N-2}(\cdot)) \\ = \mathbb{E}_{t_{N-2}} \left\{ \int_{t_{N-2}}^{t_{N-1}} [\langle Q_{N-2}X_{N-2}, X_{N-2} \rangle + \langle \bar{Q}_{N-2}\mathbb{E}_{t_{N-2}}[X_{N-2}], \mathbb{E}_{t_{N-2}}[X_{N-2}] \rangle \right. \\ \quad + \langle R_{N-2}u_{N-2}, u_{N-2} \rangle + \langle \bar{R}_{N-2}\mathbb{E}_{t_{N-2}}[u_{N-2}], \mathbb{E}_{t_{N-2}}[u_{N-2}] \rangle] ds \\ \quad + \int_{t_{N-1}}^{t_N} [\langle Q_{N-2}X_{N-1}^*, X_{N-1}^* \rangle + \langle \bar{Q}_{N-2}\mathbb{E}_{t_{N-2}}[X_{N-1}^*], \mathbb{E}_{t_{N-2}}[X_{N-1}^*] \rangle \\ \quad + \langle R_{N-2}u_{N-1}^*, u_{N-1}^* \rangle + \langle \bar{R}_{N-2}\mathbb{E}_{t_{N-2}}[u_{N-1}^*], \mathbb{E}_{t_{N-2}}[u_{N-1}^*] \rangle] ds \\ \quad + \langle G_{N-2}X_{N-1}^*(t_N), X_{N-1}^*(t_N) \rangle \\ \quad \left. + \langle \bar{G}_{N-2}\mathbb{E}_{t_{N-2}}[X_{N-1}^*(t_N)], \mathbb{E}_{t_{N-2}}[X_{N-1}^*(t_N)] \rangle \right\}. \end{aligned}$$

Note that the appearance of $Q_{N-2}, \bar{Q}_{N-2}, R_{N-2}, \bar{R}_{N-2}, G_{N-2}, \bar{G}_{N-2}$ and $\mathbb{E}_{t_{N-2}}$ in the running cost over $[t_{N-1}, t_N]$ and in the terminal cost exactly explains the meaning of “discounting in her own way” for Player $(N-2)$ mentioned earlier (see rule (iii) at the beginning of the section). We now want to rewrite the above cost functional so that Player $(N-2)$ will face a standard LQ problem for an MF-SDE on $[t_{N-2}, t_{N-1}]$. To this end, we observe the following (noting (5.7)):

$$(5.10) \quad \mathbb{E}_{t_{N-2}}[u_{N-1}^*(s)] = -\hat{\Theta}_{N-1}(s)\mathbb{E}_{t_{N-2}}[X_{N-1}^*(s)], \quad s \in [t_{N-1}, t_N],$$

$$(5.11) \quad \begin{cases} d(\mathbb{E}_{t_{N-1}}[X_{N-1}^*]) = (\hat{A} - \hat{B}\hat{\Theta}_{N-1})\mathbb{E}_{t_{N-1}}[X_{N-1}^*]ds, & s \in [t_{N-1}, t_N], \\ \mathbb{E}_{t_{N-1}}[X_{N-1}^*(t_{N-1})] = x_{N-1}, \end{cases}$$

and

$$(5.12) \quad \begin{cases} d(\mathbb{E}_{t_{N-2}}[X_{N-1}^*]) = (\hat{A} - \hat{B}\hat{\Theta}_{N-1})\mathbb{E}_{t_{N-2}}[X_{N-1}^*]ds, & s \in [t_{N-1}, t_N], \\ \mathbb{E}_{t_{N-2}}[X_{N-1}^*(t_{N-1})] = \mathbb{E}_{t_{N-2}}[x_{N-1}]. \end{cases}$$

Thus

$$\begin{aligned}
\mathbb{I}_{N-2} &\equiv \mathbb{E}_{t_{N-2}} \left\{ \int_{t_{N-1}}^{t_N} [\langle Q_{N-2} X_{N-1}^*, X_{N-1}^* \rangle + \langle \bar{Q}_{N-2} \mathbb{E}_{t_{N-2}}[X_{N-1}^*], \mathbb{E}_{t_{N-2}}[X_{N-1}^*] \rangle \right. \\
&\quad + \langle R_{N-2} u_{N-1}^*, u_{N-1}^* \rangle + \langle \bar{R}_{N-2} \mathbb{E}_{t_{N-2}}[u_{N-1}^*], \mathbb{E}_{t_{N-2}}[u_{N-1}^*] \rangle] ds \\
&\quad + \langle G_{N-2} X_{N-1}^*(t_N), X_{N-1}^*(t_N) \rangle \\
&\quad \left. + \langle \bar{G}_{N-2} \mathbb{E}_{t_{N-2}}[X_{N-1}^*(t_N)], \mathbb{E}_{t_{N-2}}[X_{N-1}^*(t_N)] \rangle \right\} \\
&= \mathbb{E}_{t_{N-2}} \left\{ \int_{t_{N-1}}^{t_N} [\langle Q_{N-2} X_{N-1}^*, X_{N-1}^* \rangle + \langle \bar{Q}_{N-2} \mathbb{E}_{t_{N-2}}[X_{N-1}^*], \mathbb{E}_{t_{N-2}}[X_{N-1}^*] \rangle \right. \\
&\quad + \langle R_{N-2} \{ \Theta_{N-1} X_{N-1}^* + (\hat{\Theta}_{N-1} - \Theta_{N-1}) \mathbb{E}_{t_{N-1}}[X_{N-1}^*] \}, \\
&\quad \quad \quad \Theta_{N-1} X_{N-1}^* + (\hat{\Theta}_{N-1} - \Theta_{N-1}) \mathbb{E}_{t_{N-1}}[X_{N-1}^*] \rangle \\
&\quad + \langle \bar{R}_{N-2} \hat{\Theta}_{N-1} \mathbb{E}_{t_{N-2}}[X_{N-1}^*], \hat{\Theta}_{N-1} \mathbb{E}_{t_{N-2}}[X_{N-1}^*] \rangle] ds \\
&\quad + \langle G_{N-2} X_{N-1}^*(t_N), X_{N-1}^*(t_N) \rangle \\
&\quad \left. + \langle \bar{G}_{N-2} \mathbb{E}_{t_{N-2}}[X_{N-1}^*(t_N)], \mathbb{E}_{t_{N-2}}[X_{N-1}^*(t_N)] \rangle \right\} \\
&= \mathbb{E}_{t_{N-2}} \left\{ \int_{t_{N-1}}^{t_N} [\langle (Q_{N-2} + \Theta_{N-1}^T R_{N-2} \Theta_{N-1}) X_{N-1}^*, X_{N-1}^* \rangle \right. \\
&\quad + \langle (\hat{\Theta}_{N-1}^T R_{N-2} \hat{\Theta}_{N-1} - \Theta_{N-1}^T R_{N-2} \Theta_{N-1}) \mathbb{E}_{t_{N-1}}[X_{N-1}^*], \mathbb{E}_{t_{N-1}}[X_{N-1}^*] \rangle \\
&\quad + \langle (\bar{Q}_{N-2} + \hat{\Theta}_{N-1}^T \bar{R}_{N-2} \hat{\Theta}_{N-1}) \mathbb{E}_{t_{N-2}}[X_{N-1}^*], \mathbb{E}_{t_{N-2}}[X_{N-1}^*] \rangle] ds \\
&\quad + \langle G_{N-2} X_{N-1}^*(t_N), X_{N-1}^*(t_N) \rangle \\
&\quad \left. + \langle \bar{G}_{N-2} \mathbb{E}_{t_{N-2}}[X_{N-1}^*(t_N)], \mathbb{E}_{t_{N-2}}[X_{N-1}^*(t_N)] \rangle \right\}.
\end{aligned}$$

Note that on the right hand side of the above, both $\mathbb{E}_{t_{N-1}}$ and $\mathbb{E}_{t_{N-2}}$ appear quadratically. Suggested by Lemma 2.4, we introduce the following three Lyapunov equations on $[t_{N-1}, t_N]$:

$$(5.13) \quad \begin{cases} \dot{\tilde{\Gamma}}_{N-2} + \tilde{\Gamma}_{N-2}(A - B\Theta_{N-1}) + (A - B\Theta_{N-1})^T \tilde{\Gamma}_{N-2} + Q_{N-2} \\ \quad + (C - D\Theta_{N-1})^T \tilde{\Gamma}_{N-2}(C - D\Theta_{N-1}) + \Theta_{N-1}^T R_{N-2} \Theta_{N-1} = 0, \\ \quad \quad \quad s \in [t_{N-1}, t_N), \\ \tilde{\Gamma}_{N-2}(t_N) = G_{N-2}, \end{cases}$$

$$(5.14) \quad \begin{cases} \dot{\Gamma}_{N-2} + \Gamma_{N-2}(\hat{A} - \hat{B}\hat{\Theta}_{N-1}) + (\hat{A} - \hat{B}\hat{\Theta}_{N-1})^T \Gamma_{N-2} + Q_{N-2} \\ \quad + (\hat{C} - \hat{D}\hat{\Theta}_{N-1})^T \tilde{\Gamma}_{N-2}(\hat{C} - \hat{D}\hat{\Theta}_{N-1}) + \hat{\Theta}_{N-1}^T R_{N-2} \hat{\Theta}_{N-1} = 0, \\ \quad \quad \quad s \in [t_{N-1}, t_N), \\ \Gamma_{N-2}(t_N) = G_{N-2}, \end{cases}$$

and

$$(5.15) \quad \begin{cases} \dot{\bar{\Gamma}}_{N-2} + \bar{\Gamma}_{N-2}(\hat{A} - \hat{B}\hat{\Theta}_{N-1}) + (\hat{A} - \hat{B}\hat{\Theta}_{N-1})^T \bar{\Gamma}_{N-2} \\ \quad + \bar{Q}_{N-2} + \hat{\Theta}_{N-1}^T \bar{R}_{N-2} \hat{\Theta}_{N-1} = 0, \\ \quad \quad \quad s \in [t_{N-1}, t_N), \\ \bar{\Gamma}_{N-2}(t_N) = \bar{G}_{N-2}. \end{cases}$$

Then, using Lemma 2.4, the sophisticated cost functional of Player $(N-2)$ can be written as

$$\begin{aligned} & J_{N-2}^{\Delta}(x_{N-2}; u_{N-2}(\cdot)) \\ &= \mathbb{E}_{t_{N-2}} \left\{ \int_{t_{N-2}}^{t_{N-1}} [\langle Q_{N-2} X_{N-2}, X_{N-2} \rangle + \langle \bar{Q}_{N-2} \mathbb{E}_{t_{N-2}}[X_{N-2}], \mathbb{E}_{t_{N-2}}[X_{N-2}] \rangle \right. \\ &\quad + \langle R_{N-2} u_{N-2}, u_{N-2} \rangle + \langle \bar{R}_{N-2} \mathbb{E}_{t_{N-2}}[u_{N-2}], \mathbb{E}_{t_{N-2}}[u_{N-2}] \rangle] ds \\ &\quad + \langle \Gamma_{N-2}(t_{N-1}) X_{N-2}(t_{N-1}), X_{N-2}(t_{N-1}) \rangle \\ &\quad \left. + \langle \bar{\Gamma}_{N-2}(t_{N-1}) \mathbb{E}_{t_{N-2}}[X_{N-2}(t_{N-1})], \mathbb{E}_{t_{N-2}}[X_{N-2}(t_{N-1})] \rangle \right\}. \end{aligned}$$

This is a standard cost functional of an LQ problem for MF-SDEs on the interval $[t_{N-2}, t_{N-1}]$ now. We see that Lemma 2.4 plays an interesting role here.

Let us make two observations. First, we note that, in general,

$$\begin{aligned} & \langle \hat{P}_{N-1}(t_{N-1}) X_{N-2}(t_{N-1}), X_{N-2}(t_{N-1}) \rangle = J_{N-1}^{\Delta}(X_{N-2}(t_{N-1}); u_{N-1}^*(\cdot)) \\ &= \mathbb{E}_{t_{N-1}} \left\{ \int_{t_{N-1}}^{t_N} [\langle Q_{N-1} X_{N-1}^*, X_{N-1}^* \rangle + \langle \bar{Q}_{N-1} \mathbb{E}_{t_{N-1}}[X_{N-1}^*], \mathbb{E}_{t_{N-1}}[X_{N-1}^*] \rangle \right. \\ &\quad + \langle R_{N-1} u_{N-1}^*, u_{N-1}^* \rangle + \langle \bar{R}_{N-1} \mathbb{E}_{t_{N-1}}[u_{N-1}^*], \mathbb{E}_{t_{N-1}}[u_{N-1}^*] \rangle] ds \\ &\quad + \langle G_{N-1} X_{N-1}^*(t_N), X_{N-1}^*(t_N) \rangle \\ &\quad \left. + \langle \bar{G}_{N-1} \mathbb{E}_{t_{N-1}}[X_{N-1}^*(t_N)], \mathbb{E}_{t_{N-1}}[X_{N-1}^*(t_N)] \rangle \right\} \neq \mathbb{T}_{N-2} \\ &= \mathbb{E}_{t_{N-2}} \left\{ \int_{t_{N-1}}^{t_N} [\langle Q_{N-2} X_{N-1}^*, X_{N-1}^* \rangle + \langle \bar{Q}_{N-2} \mathbb{E}_{t_{N-2}}[X_{N-1}^*], \mathbb{E}_{t_{N-2}}[X_{N-1}^*] \rangle \right. \\ &\quad + \langle R_{N-2} u_{N-1}^*, u_{N-1}^* \rangle + \langle \bar{R}_{N-2} \mathbb{E}_{t_{N-2}}[u_{N-1}^*], \mathbb{E}_{t_{N-2}}[u_{N-1}^*] \rangle] ds \\ &\quad + \langle G_{N-2} X_{N-1}^*(t_N), X_{N-1}^*(t_N) \rangle \\ &\quad \left. + \langle \bar{G}_{N-2} \mathbb{E}_{t_{N-2}}[X_{N-1}^*(t_N)], \mathbb{E}_{t_{N-2}}[X_{N-1}^*(t_N)] \rangle \right\} \\ &= \langle \Gamma_{N-2}(t_{N-1}) X_{N-2}(t_{N-1}), X_{N-2}(t_{N-1}) \rangle \\ &\quad + \langle \bar{\Gamma}_{N-2}(t_{N-1}) \mathbb{E}_{t_{N-2}}[X_{N-2}(t_{N-1})], \mathbb{E}_{t_{N-2}}[X_{N-2}(t_{N-1})] \rangle, \end{aligned}$$

since the weighting matrices are possibly different, and conditional expectation terms are different.

Second, if we let

$$\begin{cases} \mathcal{A}(s) = A(s) - B(s)\Theta_{N-1}(s), & \mathcal{C}(s) = C(s) - D(s)\Theta_{N-1}(s), \\ \bar{\mathcal{A}}(s) = \bar{A}(s) + B(s)[\Theta_{N-1}(s) - \hat{\Theta}_{N-1}(s)] - \bar{B}(s)\hat{\Theta}_{N-1}(s), \\ \bar{\mathcal{C}}(s) = \bar{C}(s) + D(s)[\Theta_{N-1}(s) - \hat{\Theta}_{N-1}(s)] - \bar{D}(s)\hat{\Theta}_{N-1}(s), \\ \mathcal{Q}(s) = Q_{N-2}(s) + \Theta_{N-1}(s)^T R_{N-2}(s) \Theta_{N-1}(s), \\ \tilde{\mathcal{Q}}(s) = \hat{\Theta}_{N-1}^T R_{N-2}(s) \hat{\Theta}_{N-1}(s) - \Theta_{N-1}(s)^T R_{N-2}(s) \Theta_{N-1}(s), \\ \bar{\mathcal{Q}}(s) = \bar{Q}_{N-2}(s) + \hat{\Theta}_{N-1}(s)^T \bar{R}_{N-2}(s) \hat{\Theta}_{N-1}(s), \end{cases}$$

then, under (H1)–(H2), one has

$$\begin{cases} \mathcal{Q}(s) \geq 0, \\ \mathcal{Q}(s) + \tilde{\mathcal{Q}}(s) = Q_{N-2}(s) + \hat{\Theta}_{N-1}(s)^T R_{N-2}(s) \hat{\Theta}_{N-1}(s) \geq 0, \\ \mathcal{Q}(s) + \tilde{\mathcal{Q}}(s) + \bar{\mathcal{Q}}(s) = \hat{Q}_{N-2}(s) + \hat{\Theta}_{N-1}(s)^T \hat{R}_{N-2}(s) \hat{\Theta}_{N-1}(s) \geq 0. \end{cases}$$

Thus, by Lemma 2.4, we have

$$(5.16) \quad \Gamma_{N-2}(t_{N-1}), \Gamma_{N-2}(t_{N-1}) + \bar{\Gamma}_{N-2}(t_{N-1}) \geq 0.$$

Having the above, we now pose the following problem for Player $(N-2)$.

Problem (MF-LQ) $_{N-2}$. For $x_{N-2} \in \mathcal{X}_{t_{N-2}}$, find a $u_{N-2}^*(\cdot) \in \mathcal{U}[t_{N-2}, t_{N-1}]$ such that

$$(5.17) \quad J_{N-2}^\Delta(x_{N-2}; u_{N-2}^*(\cdot)) = \inf_{u_{N-2}(\cdot) \in \mathcal{U}[t_{N-2}, t_{N-1}]} J_{N-2}^\Delta(x_{N-2}; u_{N-2}(\cdot)).$$

Similar to the case of Player $(N-1)$ above, under (H1)–(H2), there exists a pair $(P_{N-2}(\cdot), \hat{P}_{N-2}(\cdot))$ uniquely solving the following Riccati equation system:

$$\begin{cases} \dot{P}_{N-2} + P_{N-2}A + A^T P_{N-2} + C^T P_{N-2}C + Q_{N-2} \\ \quad - (P_{N-2}B + C^T P_{N-2}D)(R_{N-2} + D^T P_{N-2}D)^{-1}(B^T P_{N-2} + D^T P_{N-2}C) = 0, \\ \dot{\hat{P}}_{N-2} + \hat{P}_{N-2}\hat{A} + \hat{A}^T \hat{P}_{N-2} + \hat{C}^T P_{N-2}\hat{C} + \hat{Q}_{N-2} \\ \quad - (\hat{P}_{N-2}\hat{B} + \hat{C}^T P_{N-2}\hat{D})(\hat{R}_{N-2} + \hat{D}^T P_{N-2}\hat{D})^{-1}(\hat{B}^T \hat{P}_{N-2} + \hat{D}^T P_{N-2}\hat{C}) = 0, \\ \quad \quad \quad s \in [t_{N-2}, t_{N-1}], \\ P_{N-2}(t_{N-1}) = \Gamma_{N-2}(t_{N-1}), \quad \hat{P}_{N-2}(t_{N-1}) = \Gamma_{N-2}(t_{N-1}) + \bar{\Gamma}_{N-2}(t_{N-1}). \end{cases}$$

Define

$$\begin{cases} \Theta_{N-2} = (R_{N-2} + D^T P_{N-2}D)^{-1}(B^T P_{N-2} + D^T P_{N-2}C), \\ \hat{\Theta}_{N-2} = (\hat{R}_{N-2} + \hat{D}^T P_{N-2}\hat{D})^{-1}(\hat{B}^T \hat{P}_{N-2} + \hat{D}^T P_{N-2}\hat{C}), \end{cases} \quad s \in [t_{N-2}, t_{N-1}].$$

Then the optimal state process $X_{N-2}^*(\cdot)$ solves the following closed-loop system:

$$\begin{cases} dX_{N-2}^* = \left\{ (A - B\Theta_{N-2})X_{N-2}^* \right. \\ \quad + [\bar{A} + B(\Theta_{N-2} - \hat{\Theta}_{N-2}) - \bar{B}\hat{\Theta}_{N-2}]\mathbb{E}_{t_{N-2}}[X_{N-2}^*] \Big\} ds \\ \quad + \left\{ (C - D\Theta_{N-2})X_{N-2}^* \right. \\ \quad \quad \quad + [\bar{C} + D(\Theta_{N-2} - \hat{\Theta}_{N-2}) - \bar{D}\hat{\Theta}_{N-2}]\mathbb{E}_{t_{N-2}}[X_{N-2}^*] \Big\} dW(s), \\ \quad \quad \quad s \in [t_{N-2}, t_{N-1}], \\ X_{N-2}^*(t_{N-2}) = x_{N-2}, \end{cases}$$

and the optimal control $u_{N-2}^*(\cdot) \equiv u_{N-2}^*(\cdot; t_{N-2}, x_{N-2})$ admits the following state feedback representation:

$$(5.18) \quad u_{N-2}^*(s) = -\Theta_{N-2}(s)X_{N-2}^*(s) + [\Theta_{N-2}(s) - \hat{\Theta}_{N-2}(s)]\mathbb{E}_{t_{N-2}}[X_{N-2}^*(s)], \quad s \in [t_{N-2}, t_{N-1}].$$

Finally, the following holds:

$$(5.19) \quad \inf_{u_{N-2}(\cdot) \in \mathcal{U}[t_{N-2}, t_{N-1}]} J_{N-2}^\Delta(x_{N-2}; u_{N-2}(\cdot)) = J_{N-2}^\Delta(x_{N-2}; u_{N-2}^*(\cdot)) = \langle \hat{P}_{N-2}(t_{N-2})x_{N-2}, x_{N-2} \rangle, \quad \forall x_{N-2} \in \mathcal{X}_{t_{N-2}}.$$

Thus, Player $(N-2)$ has solved Problem (FM-LQ) $_{N-2}$. Different from Player $(N-1)$, we need to solve three Lyapunov equations in order to transform the complicated-looking sophisticated cost functional into a standard cost functional for a controlled linear MF-SDE. Thus, the situation of Player $(N-2)$ is interestingly different from that of Player $(N-1)$.

Next, we consider Player $(N - 3)$ on $[t_{N-3}, t_{N-2}]$. For any $x_{N-3} \in \mathcal{X}_{t_{N-3}}$, consider

$$(5.20) \quad \left\{ \begin{array}{l} dX_{N-3}(s) = \left\{ A(s)X_{N-3}(s) + \bar{A}(s)\mathbb{E}_{t_{N-3}}[X_{N-3}(s)] \right. \\ \quad \left. + B(s)u_{N-3}(s) + \bar{B}(s)\mathbb{E}_{t_{N-3}}[u_{N-3}(s)] \right\} ds \\ \quad + \left\{ C(s)X_{N-3}(s) + \bar{C}(s)\mathbb{E}_{t_{N-3}}[X_{N-3}(s)] \right. \\ \quad \left. + D(s)u_{N-3}(s) + \bar{D}(s)\mathbb{E}_{t_{N-3}}[u_{N-3}(s)] \right\} dW(s), \\ \quad \quad \quad s \in [t_{N-3}, t_{N-2}], \\ X_{N-3}(t_{N-3}) = x_{N-3} \in \mathcal{X}_{t_{N-3}}. \end{array} \right.$$

Due to the fact that Players $(N - 2)$ and $(N - 1)$ will play optimally on $[t_{N-2}, t_{N-1})$ and $[t_{N-1}, t_N]$, respectively, the resulting state process, denoted by $X^\Delta(\cdot)$, on $[t_{N-2}, t_N]$ will satisfy the following:

$$\left\{ \begin{array}{l} dX^\Delta = \{ (A - B\Theta^\Delta)X^\Delta + [\bar{A} + B(\Theta^\Delta - \hat{\Theta}^\Delta) - \bar{B}\hat{\Theta}^\Delta]\mathbb{E}_{\rho^\Delta(s)}[X^\Delta] \} ds \\ \quad + \{ (C - D\Theta^\Delta)X^\Delta + [\bar{C} + D(\Theta^\Delta - \hat{\Theta}^\Delta) - \bar{D}\hat{\Theta}^\Delta]\mathbb{E}_{\rho^\Delta(s)}[X^\Delta] \} dW(s), \\ \quad \quad \quad s \in [t_{N-2}, t_N], \\ X^\Delta(t_{N-2}) = X_{N-3}(t_{N-2}), \end{array} \right.$$

where

$$\left\{ \begin{array}{l} \Theta^\Delta(s) = \Theta_{N-2}(s)I_{[t_{N-2}, t_{N-1})}(s) + \Theta_{N-1}(s)I_{[t_{N-1}, t_N]}(s), \\ \hat{\Theta}^\Delta(s) = \hat{\Theta}_{N-2}(s)I_{[t_{N-2}, t_{N-1})}(s) + \hat{\Theta}_{N-1}(s)I_{[t_{N-1}, t_N]}(s), \end{array} \quad s \in (t_{N-2}, t_N], \right.$$

and

$$\rho^\Delta(s) = t_{N-2}I_{[t_{N-2}, t_{N-1})}(s) + t_{N-1}I_{[t_{N-1}, t_N]}(s), \quad s \in [t_{N-2}, t_N].$$

The corresponding control on $[t_{N-2}, t_N]$ takes the following form:

$$\begin{aligned} u^\Delta(s) &= -\Theta^\Delta(s)X^\Delta(s) + [\Theta^\Delta(s) - \hat{\Theta}^\Delta(s)]\mathbb{E}_{\rho^\Delta(s)}[X^\Delta(s)] \\ &= \left\{ \begin{array}{l} -\Theta_{N-1}(s)X^\Delta(s) + [\Theta_{N-1}(s) - \hat{\Theta}_{N-1}(s)]\mathbb{E}_{t_{N-1}}[X^\Delta(s)], \quad s \in [t_{N-1}, t_N], \\ -\Theta_{N-2}(s)X^\Delta(s) + [\Theta_{N-2}(s) - \hat{\Theta}_{N-2}(s)]\mathbb{E}_{t_{N-2}}[X^\Delta(s)], \quad s \in [t_{N-2}, t_{N-1}). \end{array} \right. \end{aligned}$$

Similar to Player $(N - 2)$, Player $(N - 3)$ considers the following (sophisticated) cost functional:

$$\begin{aligned} &J_{N-3}^\Delta(x_{N-3}; u_{N-3}(\cdot)) \\ &= \mathbb{E}_{t_{N-3}} \left\{ \int_{t_{N-3}}^{t_{N-2}} [\langle Q_{N-3}X_{N-3}, X_{N-3} \rangle + \langle \bar{Q}_{N-3}\mathbb{E}_{t_{N-3}}[X_{N-3}], \mathbb{E}_{t_{N-3}}[X_{N-3}] \rangle \right. \\ &\quad \left. + \langle R_{N-3}u_{N-3}, u_{N-3} \rangle + \langle \bar{R}_{N-3}\mathbb{E}_{t_{N-3}}[u_{N-3}], \mathbb{E}_{t_{N-3}}[u_{N-3}] \rangle] ds \right. \\ &\quad \left. + \int_{t_{N-2}}^{t_N} [\langle Q_{N-3}X^\Delta, X^\Delta \rangle + \langle \bar{Q}_{N-3}\mathbb{E}_{t_{N-3}}[X^\Delta], \mathbb{E}_{t_{N-3}}[X^\Delta] \rangle \right. \\ &\quad \left. + \langle R_{N-3}u^\Delta, u^\Delta \rangle + \langle \bar{R}_{N-3}\mathbb{E}_{t_{N-3}}[u^\Delta], \mathbb{E}_{t_{N-3}}[u^\Delta] \rangle] ds \right. \\ &\quad \left. + \langle G_{N-3}X^\Delta(t_N), X^\Delta(t_N) \rangle + \langle \bar{G}_{N-3}\mathbb{E}_{t_{N-3}}[X^\Delta(t_N)], \mathbb{E}_{t_{N-3}}[X^\Delta(t_N)] \rangle \right\} \end{aligned}$$

$$\begin{aligned} &\equiv \mathbb{E}_{t_{N-3}} \left\{ \int_{t_{N-3}}^{t_{N-2}} [\langle Q_{N-3} X_{N-3}, X_{N-3} \rangle + \langle \bar{Q}_{N-3} \mathbb{E}_{t_{N-3}}[X_{N-3}], \mathbb{E}_{t_{N-3}}[X_{N-3}] \rangle \right. \\ &\quad \left. + \langle R_{N-3} u_{N-3}, u_{N-3} \rangle + \langle \bar{R}_{N-3} \mathbb{E}_{t_{N-3}}[u_{N-3}], \mathbb{E}_{t_{N-3}}[u_{N-3}] \rangle] ds + \mathbb{I}_{N-3} \right\}, \end{aligned}$$

where

$$\begin{aligned} \mathbb{I}_{N-3} &\equiv \mathbb{E}_{t_{N-3}} \left\{ \int_{t_{N-2}}^{t_N} \left[\langle Q_{N-3} X^\Delta, X^\Delta \rangle + \langle \bar{Q}_{N-3} \mathbb{E}_{t_{N-3}}[X^\Delta], \mathbb{E}_{t_{N-3}}[X^\Delta] \rangle \right. \right. \\ &\quad \left. + \langle R_{N-3} u^\Delta, u^\Delta \rangle + \langle \bar{R}_{N-3} \mathbb{E}_{t_{N-3}}[u^\Delta], \mathbb{E}_{t_{N-3}}[u^\Delta] \rangle \right] ds \\ &\quad \left. + \langle G_{N-3} X^\Delta(t_N), X^\Delta(t_N) \rangle + \langle \bar{G}_{N-3} \mathbb{E}_{t_{N-3}}[X^\Delta(t_N)], \mathbb{E}_{t_{N-3}}[X^\Delta(t_N)] \rangle \right\} \\ &= \mathbb{E}_{t_{N-3}} \left\{ \int_{t_{N-2}}^{t_N} \left[\langle Q_{N-3} X^\Delta, X^\Delta \rangle + \langle \bar{Q}_{N-3} \mathbb{E}_{t_{N-3}}[X^\Delta], \mathbb{E}_{t_{N-3}}[X^\Delta] \rangle \right. \right. \\ &\quad + \langle R_{N-3} \{ \Theta^\Delta X^\Delta + (\hat{\Theta}^\Delta - \Theta^\Delta) \mathbb{E}_{\rho^\Delta(s)}[X^\Delta] \}, \Theta^\Delta X^\Delta + (\hat{\Theta}^\Delta - \Theta^\Delta) \mathbb{E}_{\rho^\Delta(s)}[X^\Delta] \rangle \\ &\quad + \langle \bar{R}_{N-3} \hat{\Theta}^\Delta \mathbb{E}_{t_{N-3}}[X^\Delta], \hat{\Theta}^\Delta \mathbb{E}_{t_{N-3}}[X^\Delta] \rangle \left. \right] ds \\ &\quad + \langle G_{N-3} X^\Delta(t_N), X^\Delta(t_N) \rangle + \langle \bar{G}_{N-3} \mathbb{E}_{t_{N-3}}[X^\Delta(t_N)], \mathbb{E}_{t_{N-3}}[X^\Delta(t_N)] \rangle \left. \right\} \\ &= \mathbb{E}_{t_{N-3}} \left\{ \int_{t_{N-2}}^{t_{N-1}} \left[\langle [Q_{N-3} + (\Theta^\Delta)^T R_{N-3} \Theta^\Delta] X^\Delta, X^\Delta \rangle \right. \right. \\ &\quad + \langle [(\hat{\Theta}^\Delta)^T R_{N-3} \hat{\Theta}^\Delta - (\Theta^\Delta)^T R_{N-3} \Theta^\Delta] \mathbb{E}_{t_{N-2}}[X^\Delta], \mathbb{E}_{t_{N-2}}[X^\Delta] \rangle \\ &\quad + \langle [\bar{Q}_{N-3} + (\hat{\Theta}^\Delta)^T \bar{R}_{N-3} \hat{\Theta}^\Delta] \mathbb{E}_{t_{N-3}}[X^\Delta], \mathbb{E}_{t_{N-3}}[X^\Delta] \rangle \left. \right] ds \left. \right\} \\ &\quad + \mathbb{E}_{t_{N-3}} \left\{ \int_{t_{N-1}}^{t_N} \left[\langle [Q_{N-3} + (\Theta^\Delta)^T R_{N-3} \Theta^\Delta] X^\Delta, X^\Delta \rangle \right. \right. \\ &\quad + \langle [(\hat{\Theta}^\Delta)^T R_{N-3} \hat{\Theta}^\Delta - (\Theta^\Delta)^T R_{N-3} \Theta^\Delta] \mathbb{E}_{t_{N-1}}[X^\Delta], \mathbb{E}_{t_{N-1}}[X^\Delta] \rangle \\ &\quad + \langle [\bar{Q}_{N-3} + (\hat{\Theta}^\Delta)^T \bar{R}_{N-3} \hat{\Theta}^\Delta] \mathbb{E}_{t_{N-3}}[X^\Delta], \mathbb{E}_{t_{N-3}}[X^\Delta] \rangle \left. \right] ds \\ &\quad + \langle G_{N-3} X^\Delta(t_N), X^\Delta(t_N) \rangle + \langle \bar{G}_{N-3} \mathbb{E}_{t_{N-3}}[X^\Delta(t_N)], \mathbb{E}_{t_{N-3}}[X^\Delta(t_N)] \rangle \left. \right\}. \end{aligned}$$

To take care of the above, we apply Lemma 2.4 by introducing the following three Lyapunov equations on $[t_{N-2}, t_N]$:

$$(5.21) \quad \begin{cases} \dot{\tilde{\Gamma}}_{N-3} + \tilde{\Gamma}_{N-3}(A - B\Theta^\Delta) + (A - B\Theta^\Delta)^T \tilde{\Gamma}_{N-3} + Q_{N-3} \\ \quad + (C - D\Theta^\Delta)^T \tilde{\Gamma}_{N-3}(C - D\Theta^\Delta) + (\Theta^\Delta)^T R_{N-3} \Theta^\Delta = 0, \\ \quad \quad \quad s \in (t_{N-2}, t_{N-1}) \cup (t_{N-1}, t_N), \\ \tilde{\Gamma}_{N-3}(t_N) = G_{N-3}, \quad \tilde{\Gamma}_{N-3}(t_{N-1}) = \Gamma_{N-3}(t_{N-1}), \end{cases}$$

$$(5.22) \quad \begin{cases} \dot{\Gamma}_{N-3} + \Gamma_{N-3}(\hat{A} - \hat{B}\hat{\Theta}^\Delta) + (\hat{A} - \hat{B}\hat{\Theta}^\Delta)^T \Gamma_{N-3} + Q_{N-3} \\ \quad + (\hat{C} - \hat{D}\hat{\Theta}^\Delta)^T \tilde{\Gamma}_{N-3}(\hat{C} - \hat{D}\hat{\Theta}^\Delta) + (\hat{\Theta}^\Delta)^T R_{N-3} \hat{\Theta}^\Delta = 0, \\ \quad \quad \quad s \in [t_{N-2}, t_N], \\ \Gamma_{N-3}(t_N) = G_{N-3}, \end{cases}$$

and

$$(5.23) \quad \begin{cases} \dot{\bar{\Gamma}}_{N-3} + \bar{\Gamma}_{N-3}(\hat{A} - \hat{B}\hat{\Theta}^\Delta) + (\hat{A} - \hat{B})(\hat{\Theta}^\Delta)^T \bar{\Gamma}_{N-3} \\ \quad + \bar{Q}_{N-3} + (\hat{\Theta}^\Delta)^T \hat{R}_{N-3} \hat{\Theta}^\Delta = 0, & s \in [t_{N-2}, t_N], \\ \bar{\Gamma}_{N-3}(t_N) = \bar{G}_{N-3}. \end{cases}$$

Note that different from the situation of Player $(N-2)$, the Lyapunov equation for $\bar{\Gamma}_{N-3}(\cdot)$ is formulated on two intervals and there might be a jump at t_{N-1} for $\bar{\Gamma}_{N-3}(\cdot)$. Then

$$\begin{aligned} & \mathbb{E}_{t_{N-3}} \left\{ \int_{t_{N-1}}^{t_N} [\langle [Q_{N-3} + (\Theta^\Delta)^T R_{N-3} \Theta^\Delta] X^\Delta, X^\Delta \rangle \right. \\ & \quad + \langle [(\hat{\Theta}^\Delta)^T R_{N-3} \hat{\Theta}^\Delta - (\Theta^\Delta)^T R_{N-3} \Theta^\Delta] \mathbb{E}_{t_{N-1}}[X^\Delta], \mathbb{E}_{t_{N-1}}[X^\Delta] \rangle \\ & \quad + \langle [\bar{Q}_{N-3} + (\hat{\Theta}^\Delta)^T \bar{R}_{N-3} \hat{\Theta}^\Delta] \mathbb{E}_{t_{N-3}}[X^\Delta], \mathbb{E}_{t_{N-3}}[X^\Delta] \rangle ds \\ & \quad \left. + \langle G_{N-3} X^\Delta(t_N), X^\Delta(t_N) \rangle + \langle \bar{G}_{N-3} \mathbb{E}_{t_{N-3}}[X^\Delta(t_N)], \mathbb{E}_{t_{N-3}}[X^\Delta(t_N)] \rangle \right\} \\ & = \mathbb{E}_{t_{N-3}} \left\{ \langle \Gamma_{N-3}(t_{N-1}) X^\Delta(t_{N-1}), X^\Delta(t_{N-1}) \rangle \right. \\ & \quad \left. + \langle \bar{\Gamma}_{N-3}(t_{N-1}) \mathbb{E}_{t_{N-3}}[X^\Delta(t_{N-1})], \mathbb{E}_{t_{N-3}}[X^\Delta(t_{N-1})] \rangle \right\}. \end{aligned}$$

In the same manner, we have

$$\begin{aligned} \mathbb{I}_{N-3} & = \mathbb{E}_{t_{N-3}} \left\{ \int_{t_{N-2}}^{t_{N-1}} [\langle [Q_{N-3} + (\Theta^\Delta)^T R_{N-3} \Theta^\Delta] X^\Delta, X^\Delta \rangle \right. \\ & \quad + \langle [(\hat{\Theta}^\Delta)^T R_{N-3} \hat{\Theta}^\Delta - (\Theta^\Delta)^T R_{N-3} \Theta^\Delta] \mathbb{E}_{t_{N-2}}[X^\Delta], \mathbb{E}_{t_{N-2}}[X^\Delta] \rangle \\ & \quad + \langle [\bar{Q}_{N-3} + (\hat{\Theta}^\Delta)^T \bar{R}_{N-3} \hat{\Theta}^\Delta] \mathbb{E}_{t_{N-3}}[X^\Delta], \mathbb{E}_{t_{N-3}}[X^\Delta] \rangle ds \\ & \quad + \langle \Gamma_{N-3}(t_{N-1}) X^\Delta(t_{N-1}), X^\Delta(t_{N-1}) \rangle \\ & \quad \left. + \langle \bar{\Gamma}_{N-3}(t_{N-1}) \mathbb{E}_{t_{N-3}}[X^\Delta(t_{N-1})], \mathbb{E}_{t_{N-3}}[X^\Delta(t_{N-1})] \rangle \right\} \\ & = \mathbb{E}_{t_{N-3}} \left\{ \langle \Gamma_{N-3}(t_{N-2}) X^\Delta(t_{N-2}), X^\Delta(t_{N-2}) \rangle \right. \\ & \quad \left. + \langle \bar{\Gamma}_{N-3}(t_{N-2}) \mathbb{E}_{t_{N-3}}[X^\Delta(t_{N-2})], \mathbb{E}_{t_{N-3}}[X^\Delta(t_{N-2})] \rangle \right\}. \end{aligned}$$

Hence, the sophisticated cost functional for Player $(N-3)$ can be written as

$$\begin{aligned} & J_{N-3}^\Delta(x_{N-3}; u_{N-3}(\cdot)) \\ & = \mathbb{E}_{t_{N-3}} \left\{ \int_{t_{N-3}}^{t_{N-2}} [\langle Q_{N-3} X_{N-3}, X_{N-3} \rangle + \langle \bar{Q}_{N-3} \mathbb{E}_{t_{N-3}}[X_{N-3}], \mathbb{E}_{t_{N-3}}[X_{N-3}] \rangle \right. \\ & \quad + \langle R_{N-3} u_{N-3}, u_{N-3} \rangle + \langle \bar{R}_{N-3} \mathbb{E}_{t_{N-3}}[u_{N-3}], \mathbb{E}_{t_{N-3}}[u_{N-3}] \rangle] ds \\ & \quad + \langle \Gamma_{N-3}(t_{N-2}) X_{N-3}(t_{N-2}), X_{N-3}(t_{N-2}) \rangle \\ & \quad \left. + \langle \bar{\Gamma}_{N-3}(t_{N-2}) \mathbb{E}_{t_{N-3}}[X_{N-3}(t_{N-2})], \mathbb{E}_{t_{N-3}}[X_{N-3}(t_{N-2})] \rangle \right\}. \end{aligned}$$

Note that in solving (5.21)–(5.23), we first solve (5.21) on $[t_{N-1}, t_N]$, then solve (5.22) on $[t_{N-1}, t_N]$. After that, we solve (5.21) on $[t_{N-2}, t_{N-1}]$, and (5.22) on $[t_{N-2}, t_{N-1}]$. Solving (5.23) is more direct.

Similar to the situation for Player $(N-2)$ (see the derivation of (5.16)), we first have

$$\Gamma_{N-3}(t_{N-1}), \Gamma_{N-3}(t_{N-1}) + \bar{\Gamma}_{N-3}(t_{N-1}) \geq 0.$$

Then applying the same argument, we have

$$\Gamma_{N-3}(t_{N-2}), \Gamma_{N-3}(t_{N-2}) + \bar{\Gamma}_{N-3}(t_{N-2}) \geq 0.$$

Therefore, we can pose the following problem for Player $(N-3)$.

Problem (MF-LQ) $_{N-3}$. For $x_{N-3} \in \mathcal{X}_{t_{N-3}}$, find a $u_{N-3}^*(\cdot) \in \mathcal{U}[t_{N-3}, t_{N-2}]$ such that

$$(5.24) \quad J_{N-3}^\Delta(x_{N-3}; u_{N-3}^*(\cdot)) = \inf_{u_{N-3}(\cdot) \in \mathcal{U}[t_{N-3}, t_{N-2}]} J_{N-3}^\Delta(x_{N-3}; u_{N-3}(\cdot)).$$

This is a standard LQ problem for MF-SDEs now. Under (H1)–(H2), there exists a pair $(P_{N-3}(\cdot), \hat{P}_{N-3}(\cdot))$ uniquely solving the following Riccati equation system:

$$\begin{cases} \dot{P}_{N-3} + P_{N-3}A + A^T P_{N-3} + C^T P_{N-3}C + Q_{N-3} \\ \quad - (P_{N-3}B + C^T P_{N-3}D)(R_{N-3} + D^T P_{N-3}D)^{-1}(B^T P_{N-3} + D^T P_{N-3}C) = 0, \\ \dot{\hat{P}}_{N-3} + \hat{P}_{N-3}\hat{A} + \hat{A}^T \hat{P}_{N-3} + \hat{C}^T P_{N-3}\hat{C} + \hat{Q}_{N-3} \\ \quad - (\hat{P}_{N-3}\hat{B} + \hat{C}^T P_{N-3}\hat{D})(\hat{R}_{N-3} + \hat{D}^T P_{N-3}\hat{D})^{-1}(\hat{B}^T \hat{P}_{N-3} + \hat{D}^T P_{N-3}\hat{C}) = 0, \\ \quad \quad \quad s \in [t_{N-3}, t_{N-2}], \\ P_{N-3}(t_{N-2}) = \Gamma_{N-3}(t_{N-2} + 0), \quad \hat{P}_{N-3}(t_{N-2}) = \Gamma_{N-3}(t_{N-2}) + \bar{\Gamma}_{N-3}(t_{N-2}). \end{cases}$$

Define

$$\begin{cases} \Theta_{N-3} = (R_{N-3} + D^T P_{N-3}D)^{-1}(B^T P_{N-3} + D^T P_{N-3}C), \\ \hat{\Theta}_{N-3} = (\hat{R}_{N-3} + \hat{D}^T P_{N-3}\hat{D})^{-1}(\hat{B}^T \hat{P}_{N-3} + \hat{D}^T P_{N-3}\hat{C}), \end{cases} \quad s \in [t_{N-3}, t_{N-2}].$$

Then the optimal state process $X_{N-3}^*(\cdot)$ solves the following closed-loop system:

$$\begin{cases} dX_{N-3}^* = \{ (A - B\Theta_{N-3})X_{N-3}^* \\ \quad + [\bar{A} + B(\Theta_{N-3} - \hat{\Theta}_{N-3}) - \bar{B}\hat{\Theta}_{N-3}] \mathbb{E}_{t_{N-3}}[X_{N-3}^*] \} ds \\ \quad + \{ (C - D\Theta_{N-3})X_{N-3}^* \\ \quad + [\bar{C} + D(\Theta_{N-3} - \hat{\Theta}_{N-3}) - \bar{D}\hat{\Theta}_{N-3}] \mathbb{E}_{t_{N-3}}[X_{N-3}^*] \} dW(s), \\ \quad \quad \quad s \in [t_{N-3}, t_{N-2}], \\ X_{N-3}^*(t_{N-3}) = x_{N-3}, \end{cases}$$

and the optimal control $u_{N-3}^*(\cdot) \equiv u_{N-3}^*(\cdot; t_{N-3}, x_{N-3})$ admits the following representation:

$$\begin{aligned} u_{N-3}^*(s) &= -\Theta_{N-3}(s)X_{N-3}^*(s) \\ &\quad + [\Theta_{N-3}(s) - \hat{\Theta}_{N-3}(s)] \mathbb{E}_{t_{N-3}}[X_{N-3}^*(s)], \quad s \in [t_{N-3}, t_{N-2}]. \end{aligned}$$

Finally, the following holds:

$$\begin{aligned} \inf_{u_{N-3}(\cdot) \in \mathcal{U}[t_{N-3}, t_{N-2}]} J_{N-3}^\Delta(x_{N-3}; u_{N-3}(\cdot)) &= J_{N-3}^\Delta(x_{N-3}; u_{N-3}^*(\cdot)) \\ &= \langle \hat{P}_{N-3}(t_{N-3})x_{N-3}, x_{N-3} \rangle, \quad \forall x_{N-3} \in \mathcal{X}_{t_{N-3}}. \end{aligned}$$

As a result, Player $(N-3)$ solves Problem (MF-LQ) $_{N-3}$. The major difference between the situations of Player $(N-3)$ and Player $(N-2)$ is that the Lyapunov

equation for $\tilde{\Gamma}_{N-3}(\cdot)$ has a possible *impulse* at t_{N-1} . We actually have used Lemma 2.4 on $[t_{N-1}, t_N]$ and $[t_{N-2}, t_{N-1}]$ separately.

By induction, we can construct the finite sequences

$$\begin{cases} (P_k(\cdot), \hat{P}_k(\cdot)), & 0 \leq k \leq N-1, \\ (\tilde{\Gamma}_\ell(\cdot), \Gamma_\ell(\cdot), \bar{\Gamma}_\ell(\cdot)), & 0 \leq \ell \leq N-2, \end{cases}$$

where $(P_k(\cdot), \hat{P}_k(\cdot))$ defined on $(t_k, t_{k+1}]$ satisfies the following Riccati equation system:

$$(5.25) \quad \begin{cases} \dot{P}_k + P_k A + A^T P_k + C^T P_k C + Q_k \\ \quad - (P_k B + C^T P_k D)(R_k + D^T P_k D)^{-1}(B^T P_k + D^T P_k C) = 0, \\ \dot{\hat{P}}_k + \hat{P}_k \hat{A} + \hat{A}^T \hat{P}_k + \hat{C}^T P_k \hat{C} + \hat{Q}_k \\ \quad - (\hat{P}_k \hat{B} + \hat{C}^T P_k \hat{D})(\hat{R}_k + \hat{D}^T P_k \hat{D})^{-1}(\hat{B}^T \hat{P}_k + \hat{D}^T P_k \hat{C}) = 0, \\ \quad \quad \quad s \in (t_k, t_{k+1}), \\ P_k(t_{k+1}) = \Gamma_k(t_{k+1}), \quad \hat{P}_k(t_{k+1}) = \Gamma_k(t_{k+1}) + \bar{\Gamma}_k(t_{k+1}), \end{cases}$$

with the convention that

$$\Gamma_{N-1}(t_N) = G_{N-1}, \quad \Gamma_{N-1}(t_N) + \bar{\Gamma}_{N-1}(t_N) = \hat{G}_{N-1},$$

and $(\tilde{\Gamma}_\ell(\cdot), \Gamma_\ell(\cdot), \bar{\Gamma}_\ell(\cdot))$ defined on $(t_{\ell+1}, t_N]$ satisfies the following Lyapunov equation systems:

$$(5.26) \quad \begin{cases} \dot{\tilde{\Gamma}}_\ell + \tilde{\Gamma}_\ell(A - B\Theta^\Delta) + (A - B\Theta^\Delta)^T \tilde{\Gamma}_\ell + Q_\ell \\ \quad + (C - D\Theta^\Delta)^T \tilde{\Gamma}_\ell(C - D\Theta^\Delta) + (\Theta^\Delta)^T R_\ell \Theta^\Delta = 0, \quad s \in \bigcup_{i=\ell+1}^{N-1} (t_i, t_{i+1}), \\ \tilde{\Gamma}_\ell(t_N) = G_\ell, \quad \tilde{\Gamma}_\ell(t_{N-1}) = \Gamma_\ell(t_{N-1}), \quad \dots, \quad \tilde{\Gamma}_\ell(t_{\ell+2}) = \Gamma_\ell(t_{\ell+2}), \end{cases}$$

$$(5.27) \quad \begin{cases} \dot{\Gamma}_\ell + \Gamma_\ell(\hat{A} - \hat{B}\hat{\Theta}^\Delta) + (\hat{A} - \hat{B}\hat{\Theta}^\Delta)^T \Gamma_\ell + Q_\ell \\ \quad + (\hat{C} - \hat{D}\hat{\Theta}^\Delta)^T \tilde{\Gamma}_\ell(\hat{C} - \hat{D}\hat{\Theta}^\Delta) + (\hat{\Theta}^\Delta)^T R_\ell \hat{\Theta}^\Delta = 0, \quad s \in [t_{\ell+1}, t_N], \\ \Gamma_\ell(t_N) = G_\ell, \end{cases}$$

and

$$(5.28) \quad \begin{cases} \dot{\bar{\Gamma}}_\ell + \bar{\Gamma}_\ell(\hat{A} - \hat{B}\hat{\Theta}^\Delta) + (\hat{A} - \hat{B}\hat{\Theta}^\Delta)^T \bar{\Gamma}_\ell + \bar{Q}_\ell + (\hat{\Theta}^\Delta)^T \bar{R}_\ell \hat{\Theta}^\Delta = 0, \\ \quad \quad \quad s \in [t_{\ell+1}, t_N], \\ \bar{\Gamma}_\ell(t_N) = \bar{G}_\ell, \end{cases}$$

where, for $0 \leq \ell \leq N-1$,

$$\begin{cases} \Theta^\Delta(s) = [R_\ell(s) + D(s)^T P_\ell(s) D(s)]^{-1} [B(s)^T P_\ell(s) + D(s)^T P_\ell(s) C(s)], \\ \hat{\Theta}^\Delta(s) = [\hat{R}_\ell(s) + \hat{D}(s)^T P_\ell(s) \hat{D}(s)]^{-1} [\hat{B}(s)^T \hat{P}_\ell(s) + \hat{D}(s)^T P_\ell(s) \hat{C}(s)], \\ \quad \quad \quad s \in [t_\ell, t_{\ell+1}]. \end{cases}$$

Next, we define

$$(5.29) \quad u^\Delta(s) = -\Theta^\Delta(s) X^\Delta(s) + [\Theta^\Delta(s) - \hat{\Theta}^\Delta(s)] \mathbb{E}_{\rho^\Delta(s)}[X^\Delta(s)], \quad s \in [0, t_N],$$

where

$$(5.30) \quad \rho^\Delta(s) = \sum_{k=0}^{N-2} t_k I_{[t_k, t_{k+1})}(s) + t_{N-1} I_{[t_{N-1}, t_N]}(s), \quad s \in [0, T],$$

and define $X^\Delta(\cdot)$ by the solution of the following closed-loop system:

$$(5.31) \quad \begin{cases} dX^\Delta = \left\{ (A - B\Theta^\Delta)X^\Delta + [\bar{A} + B(\Theta^\Delta - \hat{\Theta}^\Delta) - \bar{B}\hat{\Theta}^\Delta] \mathbb{E}_{\rho^\Delta(s)}[X^\Delta] \right\} ds \\ \quad + \left\{ (C - D\Theta^\Delta)X^\Delta + [\bar{C} + D(\Theta^\Delta - \hat{\Theta}^\Delta) - \bar{D}\hat{\Theta}^\Delta] \mathbb{E}_{\rho^\Delta(s)}[X^\Delta] \right\} dW(s), \\ \quad s \in [0, t_N], \\ X^\Delta(0) = x. \end{cases}$$

Then one has that

$$(5.32) \quad \inf_{u_k(\cdot) \in \mathcal{U}[t_k, t_{k+1}]} J_k^\Delta(X^\Delta(t_k); u_k(\cdot)) = J_k^\Delta(X^\Delta(t_k); u^\Delta(\cdot)|_{[t_k, t_N]}) \\ = \langle \hat{P}^\Delta(t_k) X^\Delta(t_k), X^\Delta(t_k) \rangle, \quad 0 \leq k \leq N-1.$$

From the above, we see that $(\Theta^\Delta(\cdot), \hat{\Theta}^\Delta(\cdot))$ is a closed-loop Δ -equilibrium strategy of Problem (MF-LQ), and $(X^\Delta(\cdot), u^\Delta(\cdot))$ is the associated closed-loop Δ -equilibrium pair.

To conclude this section, we note that $\hat{\Gamma}_\ell(\cdot) \equiv \Gamma_\ell(\cdot) + \bar{\Gamma}_\ell(\cdot)$ satisfies the following Lyapunov equation:

$$(5.33) \quad \begin{cases} \dot{\hat{\Gamma}}_\ell + \hat{\Gamma}_\ell(\hat{A} - \hat{B}\hat{\Theta}^\Delta) + (\hat{A} - \hat{B}\hat{\Theta}^\Delta)^T \hat{\Gamma}_\ell + \hat{Q}_\ell \\ \quad + (\hat{C} - \hat{D}\hat{\Theta}^\Delta)^T \hat{\Gamma}_\ell(\hat{C} - \hat{D}\hat{\Theta}^\Delta) + (\hat{\Theta}^\Delta)^T \hat{R}_\ell \hat{\Theta}^\Delta = 0, \quad s \in [t_{\ell+1}, t_N], \\ \hat{\Gamma}_\ell(t_N) = \hat{G}_\ell. \end{cases}$$

6. CONVERGENCE

In the previous section, for any given partition Δ of $[0, T]$, a closed-loop Δ -equilibrium strategy $(\Theta^\Delta(\cdot), \hat{\Theta}^\Delta(\cdot))$ is constructed. The goal of this section is to establish the convergence of $(\Theta^\Delta(\cdot), \hat{\Theta}^\Delta(\cdot))$ as $\|\Delta\| \rightarrow 0$. To achieve this, we first recall the equations (5.25)–(5.28), and (5.33). Practically, we solve the equations in the following order:

$$\begin{aligned} (P_{N-1}, \hat{P}_{N-1}) \text{ on } [t_{N-1}, t_N] &\Rightarrow (\tilde{\Gamma}_{N-2}, \Gamma_{N-2}, \hat{\Gamma}_{N-2}) \text{ on } [t_{N-1}, t_N] \\ &\Rightarrow (P_{N-2}, \hat{P}_{N-2}) \text{ on } [t_{N-2}, t_{N-1}] \Rightarrow (\tilde{\Gamma}_{N-3}, \Gamma_{N-3}, \hat{\Gamma}_{N-3}) \text{ on } [t_{N-2}, t_N] \\ &\Rightarrow (P_{N-3}, \hat{P}_{N-3}) \text{ on } [t_{N-3}, t_{N-2}] \Rightarrow (\tilde{\Gamma}_{N-4}, \Gamma_{N-4}, \hat{\Gamma}_{N-4}) \text{ on } [t_{N-3}, t_N] \\ &\Rightarrow \dots \end{aligned}$$

Denote

$$\begin{cases} Q^\Delta(s) = \sum_{k=0}^{N-1} Q(s, t_k) I_{(t_k, t_{k+1}]}(s), & \hat{Q}^\Delta(s) = \sum_{k=0}^{N-1} \bar{Q}(s, t_k) I_{(t_k, t_{k+1}]}(s), \\ R^\Delta(s) = \sum_{k=0}^{N-1} R(s, t_k) I_{(t_k, t_{k+1}]}(s), & \hat{R}^\Delta(s) = \sum_{k=0}^{N-1} \hat{R}(s, t_k) I_{(t_k, t_{k+1}]}(s). \end{cases}$$

The dependence on the partition Δ is indicated. Denote

$$\begin{cases} P^\Delta(s) = \sum_{k=0}^{N-1} P_k(s) I_{(t_k, t_{k+1}]}(s), \\ \hat{P}^\Delta(s) = \sum_{k=0}^{N-1} \hat{P}_k(s) I_{(t_k, t_{k+1}]}(s), \end{cases} \quad s \in [0, t_N].$$

Then

$$(6.1) \quad \begin{cases} \dot{P}^\Delta + P^\Delta A + A^T P^\Delta + C^T P^\Delta C + Q^\Delta \\ \quad - (P^\Delta B + C^T P^\Delta D)(R^\Delta + D^T P^\Delta D)^{-1}(B^T P^\Delta + D^T P^\Delta C) = 0, \\ \quad s \in \bigcup_{k=0}^{N-1} (t_k, t_{k+1}), \\ P^\Delta(t_N) = G_{N-1}, \quad P^\Delta(t_{N-1}) = \Gamma_{N-2}(t_{N-1}), \quad \dots, \quad P^\Delta(t_1) = \Gamma_0(t_1), \end{cases}$$

and

$$\begin{cases} \dot{\hat{P}}^\Delta + \hat{P}^\Delta \hat{A} + \hat{A}^T \hat{P}^\Delta + \hat{C}^T P^\Delta \hat{C} + \hat{Q}^\Delta \\ \quad - (\hat{P}^\Delta \hat{B} + \hat{C}^T P^\Delta \hat{D})(\hat{R}^\Delta + \hat{D}^T P^\Delta \hat{D})^{-1}(\hat{B}^T \hat{P}^\Delta + \hat{D}^T P^\Delta \hat{C}) = 0, \\ \quad s \in \bigcup_{k=0}^{N-1} (t_k, t_{k+1}), \\ \hat{P}^\Delta(t_N) = \hat{G}_{N-1}, \quad \hat{P}^\Delta(t_{N-1}) = \hat{\Gamma}_{N-2}(t_{N-1}), \quad \dots, \quad \hat{P}^\Delta(t_1) = \hat{\Gamma}_0(t_1). \end{cases}$$

A straightforward computation shows that the above are equivalent to the following:

$$\begin{cases} \dot{P}^\Delta + P^\Delta(A - B\Theta^\Delta) + (A - B\Theta^\Delta)^T P^\Delta + Q^\Delta \\ \quad + (C - D\Theta^\Delta)^T P^\Delta (C - D\Theta^\Delta) + (\Theta^\Delta)^T R^\Delta \Theta^\Delta = 0, \\ \quad s \in \bigcup_{k=0}^{N-1} (t_k, t_{k+1}), \\ P^\Delta(t_N) = G_{N-1}, \quad P^\Delta(t_{N-1}) = \Gamma_{N-2}(t_{N-1}), \quad \dots, \quad P^\Delta(t_1) = \Gamma_0(t_1), \end{cases}$$

and

$$\begin{cases} \dot{\hat{P}}^\Delta + \hat{P}^\Delta(\hat{A} - \hat{B}\hat{\Theta}^\Delta) + (\hat{A} - \hat{B}\hat{\Theta}^\Delta)^T \hat{P}^\Delta + \hat{Q}^\Delta \\ \quad + (\hat{C} - \hat{D}\hat{\Theta}^\Delta)^T P^\Delta (\hat{C} - \hat{D}\hat{\Theta}^\Delta) + (\hat{\Theta}^\Delta)^T \hat{R}^\Delta \hat{\Theta}^\Delta = 0, \quad s \in [t_k, t_{k+1}), \\ \quad s \in \bigcup_{k=0}^{N-1} (t_k, t_{k+1}), \\ \hat{P}^\Delta(t_N) = \hat{G}_{N-1}, \quad \hat{P}^\Delta(t_{N-1}) = \hat{\Gamma}_{N-2}(t_{N-1}), \quad \dots, \quad \hat{P}^\Delta(t_1) = \hat{\Gamma}_0(t_1). \end{cases}$$

Now, we may establish some comparisons.

First, on $(t_{N-1}, t_N]$, we compare

$$\begin{cases} \dot{P}_{N-1} + P_{N-1}(A - B\Theta^\Delta) + (A - B\Theta^\Delta)^T P_{N-1} + Q_{N-1} \\ \quad + (C - D\Theta^\Delta)^T P_{N-1} (C - D\Theta^\Delta) + (\Theta^\Delta)^T R_{N-1} \Theta^\Delta = 0, \\ P_{N-1}(t_N) = G_{N-1} \end{cases}$$

with

$$\begin{cases} \dot{\tilde{\Gamma}}_\ell + \tilde{\Gamma}_\ell(A - B\Theta^\Delta) + (A - B\Theta^\Delta)^T \tilde{\Gamma}_\ell + Q_\ell \\ \quad + (C - D\Theta^\Delta)^T \tilde{\Gamma}_\ell (C - D\Theta^\Delta) + (\Theta^\Delta)^T R_\ell \Theta^\Delta = 0, \quad s \in (t_{N-1}, t_N), \\ \tilde{\Gamma}_\ell(t_N) = G_\ell. \end{cases}$$

Since $\ell \mapsto Q_\ell$, $\ell \mapsto R_\ell$ and $\ell \mapsto G_\ell$ are non-decreasing, we have

$$(6.2) \quad 0 \leq \tilde{\Gamma}_\ell(s) \leq \tilde{\Gamma}_{\ell+1}(s) \leq \cdots \leq \tilde{\Gamma}_{N-2}(s) \leq P_{N-1}(s), \\ s \in (t_{N-1}, t_N], \quad 0 \leq \ell \leq N-2.$$

Then, from (5.27), one has

$$(6.3) \quad 0 \leq \Gamma_\ell(s) \leq \Gamma_{\ell+1}(s) \leq \cdots \leq \Gamma_{N-2}(s), \\ s \in (t_{N-1}, t_N], \quad 0 \leq \ell \leq N-2.$$

Next, on $(t_{N-2}, t_{N-1}]$, comparing

$$\begin{cases} \dot{P}_{N-2} + P_{N-2}(A - B\Theta^\Delta) + (A - B\Theta^\Delta)^T P_{N-2} + Q_{N-2} \\ \quad + (C - D\Theta^\Delta)^T P_{N-2} (C - D\Theta^\Delta) + (\Theta^\Delta)^T R_{N-2} \Theta^\Delta = 0, \\ P_{N-2}(t_N) = \Gamma_{N-2}(t_{N-1}) \end{cases}$$

with

$$\begin{cases} \dot{\tilde{\Gamma}}_\ell + \tilde{\Gamma}_\ell(A - B\Theta^\Delta) + (A - B\Theta^\Delta)^T \tilde{\Gamma}_\ell + Q_\ell \\ \quad + (C - D\Theta^\Delta)^T \tilde{\Gamma}_\ell (C - D\Theta^\Delta) + (\Theta^\Delta)^T R_\ell \Theta^\Delta = 0, \quad s \in (t_{N-2}, t_{N-1}), \\ \tilde{\Gamma}_\ell(t_{N-1}) = \Gamma_\ell(t_{N-1}) \leq \Gamma_{N-2}(t_{N-1}), \end{cases}$$

making use of the monotonicity of $\ell \mapsto Q_\ell$, $\ell \mapsto R_\ell$ and $\ell \mapsto G_\ell$, we have

$$(6.4) \quad 0 \leq \tilde{\Gamma}_\ell(s) \leq \tilde{\Gamma}_{\ell+1}(s) \leq \cdots \leq \tilde{\Gamma}_{N-3}(s) \leq P_{N-2}(s), \\ s \in (t_{N-2}, t_{N-1}], \quad 0 \leq \ell \leq N-3.$$

Again, from (5.27), one has

$$(6.5) \quad 0 \leq \Gamma_\ell(s) \leq \Gamma_{\ell+1}(s) \leq \cdots \leq \Gamma_{N-3}(s), \\ s \in (t_{N-2}, t_{N-1}], \quad 0 \leq \ell \leq N-3.$$

By induction, one has the following:

$$(6.6) \quad 0 \leq \tilde{\Gamma}_\ell(s) \leq \tilde{\Gamma}_{\ell+1}(s) \leq \cdots \leq \tilde{\Gamma}_k(s) \leq P_{k+1}(s) = P^\Delta(s), \\ s \in (t_{k+1}, t_{k+2}], \quad 0 \leq \ell \leq k \leq N-2.$$

Thus,

$$(6.7) \quad 0 \leq \tilde{\Gamma}_\ell(s) \leq P^\Delta(s), \quad s \in (t_{\ell+1}, t_N], \quad 0 \leq \ell \leq N-2.$$

Next, on $[t_{N-1}, t_N]$, comparing

$$\begin{cases} \dot{\hat{P}}_{N-1} + \hat{P}_{N-1}(\hat{A} - \hat{B}\hat{\Theta}^\Delta) + (\hat{A} - \hat{B}\hat{\Theta}^\Delta)^T \hat{P}_{N-1} + \hat{Q}_{N-1} \\ \quad + (\hat{C} - \hat{D}\hat{\Theta}^\Delta)^T \hat{P}_{N-1} (\hat{C} - \hat{D}\hat{\Theta}^\Delta) + (\hat{\Theta}^\Delta)^T \hat{R}_{N-1} \hat{\Theta}^\Delta = 0, \quad s \in (t_{N-1}, t_N], \\ \hat{P}_{N-1}(t_N) = \hat{G}_{N-1} \end{cases}$$

with

$$\begin{cases} \dot{\hat{\Gamma}}_\ell + \hat{\Gamma}_\ell(\hat{A} - \hat{B}\hat{\Theta}^\Delta) + (\hat{A} - \hat{B}\hat{\Theta}^\Delta)^T \hat{\Gamma}_\ell + \hat{Q}_\ell \\ \quad + (\hat{C} - \hat{D}\hat{\Theta}^\Delta)^T \hat{\Gamma}_\ell(\hat{C} - \hat{D}\hat{\Theta}^\Delta) + (\hat{\Theta}^\Delta)^T \hat{R}_\ell \hat{\Theta}^\Delta = 0, & s \in (t_{N-1}, t_N), \\ \hat{\Gamma}_\ell(t_N) = \hat{G}_\ell, \end{cases}$$

since $\ell \mapsto Q_\ell$, $\ell \mapsto R_\ell$ and $\ell \mapsto G_\ell$ are non-decreasing, making use of (6.7), we have

$$0 \leq \hat{\Gamma}_\ell(s) \leq \hat{\Gamma}_{\ell+1}(s) \leq \cdots \leq \hat{\Gamma}_{N-2}(s) \leq \hat{P}_{N-1}(s), \\ s \in (t_{N-1}, t_N], \quad 0 \leq \ell \leq N-2.$$

On $[t_{N-2}, t_{N-1}]$, we compare

$$\begin{cases} \dot{\hat{P}}_{N-2} + \hat{P}_{N-2}(\hat{A} - \hat{B}\hat{\Theta}^\Delta) + (\hat{A} - \hat{B}\hat{\Theta}^\Delta)^T \hat{P}_{N-2} + \hat{Q}_{N-2} \\ \quad + (\hat{C} - \hat{D}\hat{\Theta}^\Delta)^T \hat{P}_{N-2}(\hat{C} - \hat{D}\hat{\Theta}^\Delta) + (\hat{\Theta}^\Delta)^T \hat{R}_{N-2} \hat{\Theta}^\Delta = 0, & s \in (t_{N-2}, t_{N-1}], \\ \hat{P}_{N-2}(t_{N-1}) = \hat{\Gamma}_{N-2}(t_{N-1}) \end{cases}$$

with

$$\begin{cases} \dot{\hat{\Gamma}}_\ell + \hat{\Gamma}_\ell(\hat{A} - \hat{B}\hat{\Theta}^\Delta) + (\hat{A} - \hat{B}\hat{\Theta}^\Delta)^T \hat{\Gamma}_\ell + \hat{Q}_\ell \\ \quad + (\hat{C} - \hat{D}\hat{\Theta}^\Delta)^T \hat{\Gamma}_\ell(\hat{C} - \hat{D}\hat{\Theta}^\Delta) + (\hat{\Theta}^\Delta)^T \hat{R}_\ell \hat{\Theta}^\Delta = 0, & s \in (t_{N-1}, t_N), \\ \hat{\Gamma}_\ell(t_N) = \hat{G}_\ell. \end{cases}$$

Again, by the monotonicity of $\ell \mapsto Q_\ell$, $\ell \mapsto R_\ell$ and $\ell \mapsto G_\ell$, together with (6.7), we have

$$0 \leq \hat{\Gamma}_\ell(s) \leq \hat{\Gamma}_{\ell+1}(s) \leq \cdots \leq \hat{\Gamma}_{N-3}(s) \leq \hat{P}_{N-2}(s), \\ s \in (t_{N-2}, t_{N-1}], \quad 0 \leq \ell \leq N-3.$$

Inductively,

$$0 \leq \hat{\Gamma}_\ell(s) \leq \hat{\Gamma}_{\ell+1}(s) \leq \cdots \leq \hat{\Gamma}_k(s) \leq \hat{P}_{k+1}(s) = \hat{P}^\Delta(s), \\ s \in (t_{k+1}, t_{k+2}], \quad 0 \leq \ell \leq k \leq N-2.$$

Therefore,

$$(6.8) \quad 0 \leq \hat{\Gamma}_\ell(s) \leq \hat{P}^\Delta(s), \quad s \in (t_\ell, t_N], \quad 0 \leq \ell \leq N-2.$$

We now introduce

$$(6.9) \quad \begin{cases} \dot{\hat{\Pi}}^\Delta + \hat{\Pi}^\Delta \hat{A} + \hat{A}^T \hat{\Pi}^\Delta + \hat{C}^T \hat{\Pi}^\Delta \hat{C} + \hat{Q}^\Delta = 0, & s \in [t_0, t_N], \\ \hat{\Pi}^\Delta(t_N) = \hat{G}_N. \end{cases}$$

Comparing the above with

$$\begin{cases} \dot{\hat{P}}^\Delta + \hat{P}^\Delta(\hat{A} - \hat{B}\hat{\Theta}^\Delta) + (\hat{A} - \hat{B}\hat{\Theta}^\Delta)^T \hat{P}^\Delta + \hat{Q}^\Delta \\ \quad + (\hat{C} - \hat{D}\hat{\Theta}^\Delta)^T \hat{P}^\Delta(\hat{C} - \hat{D}\hat{\Theta}^\Delta) + (\hat{\Theta}^\Delta)^T \hat{R}^\Delta \hat{\Theta}^\Delta = 0, & s \in \bigcup_{\ell=0}^{N-1} (t_\ell, t_{\ell+1}), \\ \hat{P}^\Delta(t_N) = \hat{G}_{N-1}, \quad \hat{P}^\Delta(t_{\ell+1}) = \hat{\Gamma}_\ell(t_{\ell+1}), & 0 \leq \ell \leq N-2, \end{cases}$$

on $(t_{N-1}, t_N]$, we have

$$\hat{P}^\Delta(s) \leq \hat{\Pi}^\Delta(s).$$

Now,

$$\hat{P}^\Delta(t_{N-1}) = \hat{\Gamma}_{N-2}(t_{N-1}) \leq \hat{P}^\Delta(t_{N-1} + 0) \leq \hat{\Pi}^\Delta(t_{N-1}).$$

Thus,

$$\hat{P}^\Delta(s) \leq \hat{\Pi}^\Delta(s), \quad s \in (t_{N-2}, t_{N-1}].$$

Inductively, we obtain

$$\hat{P}^\Delta(s) \leq \hat{\Pi}^\Delta(s), \quad s \in [0, T].$$

Hence,

$$(6.10) \quad 0 \leq \hat{\Gamma}_k(s) \leq \hat{P}^\Delta(s) \leq \hat{\Pi}^\Delta(s), \quad s \in (t_{k+1}, t_N], \quad 0 \leq k \leq N-2.$$

Since

$$\begin{aligned} |\hat{\Pi}^\Delta(s)| &\leq \|\hat{G}\| + \int_s^T \left[(2\|\hat{A}\| + \|\hat{C}\|^2) |\hat{\Pi}^\Delta(r)| + \|\hat{Q}\| \right] dr \\ &\leq \|\hat{G}\| + \|\hat{Q}\|T + (2\|\hat{A}\| + \|\hat{C}\|^2) \int_s^T |\hat{\Pi}^\Delta(r)| dr, \end{aligned}$$

by Gronwall's inequality, we obtain

$$\|\hat{\Pi}^\Delta\| \leq (\|\hat{G}\| + \|\hat{Q}\|T) e^{(2\|\hat{A}\| + \|\hat{C}\|^2)T}.$$

Hence, $\hat{\Gamma}_k(\cdot)$ and $\hat{P}^\Delta(\cdot)$ are uniformly bounded (independent of partition Δ).

Next, we want to show the uniform boundedness of $\tilde{\Gamma}_k(\cdot)$, $\Gamma_k(\cdot)$, $P_\ell(\cdot)$. To this end, we introduce

$$\begin{cases} \dot{\Pi}^\Delta + \Pi^\Delta A + A^T \Pi^\Delta + C^T \Pi^\Delta C + Q^\Delta = 0, & s \in \bigcup_{\ell=0}^{N-1} (t_\ell, t_{\ell+1}), \\ \Pi^\Delta(t_N) = G_N, \quad \Pi^\Delta(t_{N-1}) = \Gamma_{N-2}(t_{N-1}), \quad \dots, \quad \Pi^\Delta(t_1) = \Gamma_0(t_1), \end{cases}$$

and comparing it with the equation (6.1), we see that

$$(6.11) \quad 0 \leq P^\Delta(s) \leq \Pi^\Delta(s), \quad s \in (t_0, t_N].$$

Now, on $(t_{N-1}, t_N]$,

$$|\Pi^\Delta(s)| \leq \|G\| + \|Q\|(t_N - s) + (2\|A\| + \|C\|^2) \int_s^{t_N} |\Pi^\Delta(r)| dr,$$

and (noting (5.27))

$$|\Gamma_\ell(s)| \leq |G_{N-1}| + \|Q\|(t_N - s) + \int_s^{t_N} \left(\|\hat{A} - \hat{B}\hat{\Theta}^\Delta\| |\Gamma_\ell(r)| + \|\hat{C} - \hat{D}\hat{\Theta}^\Delta\|^2 |\tilde{\Gamma}_\ell(r)| \right) dr.$$

On the other hand, under (H4),

$$\hat{\Theta}^\Delta = (\hat{R} + \hat{D}^T P^\Delta \hat{D})^{-1} (\hat{B}^T \hat{P}^\Delta + \hat{D}^T P^\Delta \hat{D} \tilde{C}).$$

Thus,

$$\begin{aligned} |\Theta^\Delta| &= |(\hat{R} + \hat{D}^T P^\Delta \hat{D})^{-1} (\hat{B}^T \hat{P}^\Delta + \hat{D}^T P^\Delta \hat{D} \tilde{C})| \\ &\leq \frac{\|\hat{B}\|}{\delta} \|\hat{P}^\Delta\| + |(\hat{R} + \hat{D}^T P^\Delta \hat{D})^{-1} \hat{D}^T P^\Delta \hat{D}| \|\tilde{C}\| \\ &\leq \frac{\|\hat{B}\|}{\delta} \left(\|\hat{G}\| + \|\hat{Q}\|T \right) e^{(2\|\hat{A}\| + \|\hat{C}\|^2)T} + \|\tilde{C}\|. \end{aligned}$$

Consequently,

$$|\Gamma_\ell(s)| \leq \|G\| + \|Q\|(t_N - s) + \int_s^{t_N} (K_\Gamma |\Gamma_\ell(r)| + K_\Pi |\Pi^\Delta(r)|) dr,$$

with

$$K_\Gamma = \|\hat{A} - \hat{B}\Theta^\Delta\|, \quad K_\Pi = \|\hat{C} - \hat{D}\hat{\Theta}^\Delta\|^2.$$

Now, on $(t_{N-2}, t_{N-1}]$,

$$\begin{aligned} |\Pi^\Delta(s)| &\leq |\Gamma_\ell(t_{N-1})| + \|Q\|(t_{N-1} - s) + (2\|A\| + \|C\|^2) \int_s^{t_{N-1}} |\Pi^\Delta(r)| dr \\ &\leq \|G\| + \|Q\|(t_N - s) + \int_{t_{N-1}}^{t_N} (K_\Gamma |\Gamma_\ell(r)| + K_\Pi |\Pi^\Delta(r)|) dr \\ &\quad + (2\|A\| + \|C\|^2) \int_s^{t_{N-1}} |\Pi^\Delta(r)| dr. \end{aligned}$$

Making use of (6.7) and (6.11), on $(t_{N-2}, t_{N-1}]$, one has

$$\begin{aligned} |\Gamma_\ell(s)| &\leq |\Gamma_\ell(t_{N-1})| + \|Q\|(t_{N-1} - s) \\ &\quad + \int_s^{t_{N-1}} \left(\|\hat{A} - \hat{B}\hat{\Theta}^\Delta\| |\Gamma_\ell(r)| + \|\hat{C} - \hat{D}\hat{\Theta}^\Delta\|^2 |\tilde{\Gamma}_\ell(r)| \right) dr \\ &\leq \|G\| + \|Q\|(t_N - s) + \int_s^{t_N} (K_\Gamma |\Gamma_\ell(r)| + K_\Pi |\Pi^\Delta(r)|) dr. \end{aligned}$$

By induction, we see that for any $s \in (t_{\ell+1}, t_N]$,

$$\begin{aligned} |\Pi^\Delta(s)| &\leq \|G\| + \|Q\|(t_N - s) + \int_{t_{N-1}}^{t_N} (K_\Gamma |\Gamma_\ell(r)| + K_\Pi |\Pi^\Delta(r)|) dr \\ &\quad + (2\|A\| + \|C\|^2) \int_s^{t_{N-1}} |\Pi^\Delta(r)| dr, \end{aligned}$$

and

$$|\Gamma_\ell(s)| \leq \|G\| + \|Q\|(t_N - s) + \int_s^{t_N} (K_\Gamma |\Gamma_\ell(r)| + K_\Pi |\Pi^\Delta(r)|) dr.$$

Hence,

$$\begin{aligned} |\Pi^\Delta(s)| + |\Gamma_k(s)| &\leq 2\|G\| + 2\|Q\|T + K \int_s^T (|\Pi^\Delta(r)| + |\Gamma_k(r)|) dr, \\ &\quad s \in (t_{k+1}, T], \end{aligned}$$

with $K > 0$ being an absolute constant. Hence, by Gronwall's inequality, one obtains

$$|\Pi^\Delta(s)| + |\Gamma_\ell(s)| \leq K, \quad s \in (t_{\ell+1}, t_N],$$

uniformly in Δ . Then we obtain the uniform boundedness of $P^\Delta(\cdot)$, $\tilde{\Gamma}_\ell(\cdot)$ and $\bar{\Gamma}_\ell(\cdot)$. We may state the above as the following proposition.

Proposition 6.1. *Let (H1)–(H4) hold. Then $(P^\Delta(\cdot), \hat{P}^\Delta(\cdot))$ and $(\tilde{\Gamma}_\ell(\cdot), \Gamma_\ell(\cdot), \bar{\Gamma}_\ell(\cdot))$ are bounded uniformly in Δ .*

The following result essentially gives some estimates on the jumps.

Proposition 6.2. *Let (H1)–(H4) hold. Then*

$$(6.12) \quad \begin{aligned} & |\Gamma_\ell(s) - \Gamma_{\ell-1}(s)| + |\tilde{\Gamma}_\ell(s) - \tilde{\Gamma}_{\ell-1}(s)| \\ & \leq K \left[|G_\ell - G_{\ell-1}| + \int_s^{t_N} (|Q_\ell(r) - Q_{\ell-1}(r)| + |R_\ell(r) - R_{\ell-1}(r)|) dr \right], \\ & \hspace{25em} s \in [t_{\ell+1}, t_N], \end{aligned}$$

and

$$(6.13) \quad \begin{aligned} & |\hat{\Gamma}_\ell(s) - \hat{\Gamma}_{\ell-1}(s)| \\ & \leq K[|G_\ell - G_{\ell-1}| + |\hat{G}_\ell - \hat{G}_{\ell-1}| + \int_s^{t_N} (|Q_\ell(r) - Q_{\ell-1}(r)| \\ & \quad + |R_\ell(r) - R_{\ell-1}(r)| + |\hat{Q}_\ell(r) - \hat{Q}_{\ell-1}(r)| + |\hat{R}_\ell(r) - \hat{R}_{\ell-1}(r)|) dr], \\ & \hspace{25em} s \in [t_{\ell+1}, t_N]. \end{aligned}$$

Proof. Under (H1)–(H4), we know that $P_k(\cdot), \hat{P}_k(\cdot)$ are uniformly bounded. Thus, all the coefficients of the Lyapunov equations (5.26)–(5.28) and (5.33) are bounded. Now, comparing the equations for $\Gamma_\ell(\cdot)$ and $\Gamma_{\ell-1}(\cdot)$, we have

$$(6.14) \quad \begin{aligned} & |\Gamma_\ell(s) - \Gamma_{\ell-1}(s)| \\ & \leq |G_\ell - G_{\ell-1}| + K \int_s^{t_N} \left(|\Gamma_\ell(r) - \Gamma_{\ell-1}(r)| + |\tilde{\Gamma}_\ell(r) - \tilde{\Gamma}_{\ell-1}(r)| \right) dr \\ & \quad + K \int_s^{t_N} (|Q_\ell(r) - Q_{\ell-1}(r)| + |R_\ell(r) - R_{\ell-1}(r)|) dr, \quad s \in [t_{\ell+1}, t_N]. \end{aligned}$$

Next, comparing the equations for $\tilde{\Gamma}_\ell(\cdot)$ and $\tilde{\Gamma}_{\ell-1}(\cdot)$, we see that for $s \in (t_{N-1}, t_N]$,

$$\begin{aligned} |\tilde{\Gamma}_\ell(s) - \tilde{\Gamma}_{\ell-1}(s)| & \leq |G_\ell - G_{\ell-1}| + K \int_s^{t_N} |\tilde{\Gamma}_\ell(r) - \tilde{\Gamma}_{\ell-1}(r)| dr \\ & \quad + K \int_s^{t_N} (|Q_\ell(r) - Q_{\ell-1}(r)| + |R_\ell(r) - R_{\ell-1}(r)|) dr. \end{aligned}$$

For $s \in (t_{N-2}, t_{N-1}]$,

$$\begin{aligned} |\tilde{\Gamma}_\ell(s) - \tilde{\Gamma}_{\ell-1}(s)| & \leq |\Gamma_\ell(t_{N-1}) - \Gamma_{\ell-1}(t_{N-1})| + K \int_s^{t_{N-1}} |\tilde{\Gamma}_\ell(s) - \tilde{\Gamma}_{\ell-1}(r)| dr \\ & \quad + K \int_s^{t_{N-1}} (|Q_\ell(r) - Q_{\ell-1}(r)| + |R_\ell(r) - R_{\ell-1}(r)|) dr \\ & \leq |G_\ell - G_{\ell-1}| + K \int_{t_{N-1}}^{t_N} \left(|\Gamma_\ell(r) - \Gamma_{\ell-1}(r)| + |\tilde{\Gamma}_\ell(r) - \tilde{\Gamma}_{\ell-1}(r)| \right) dr \\ & \quad + K \int_{t_{N-1}}^{t_N} (|Q_\ell(r) - Q_{\ell-1}(r)| + |R_\ell(r) - R_{\ell-1}(r)|) dr \\ & \quad + K \int_s^{t_{N-1}} \left(|\tilde{\Gamma}_\ell(s) - \tilde{\Gamma}_{\ell-1}(r)| + |Q_\ell(r) - Q_{\ell-1}(r)| + |R_\ell(r) - R_{\ell-1}(r)| \right) dr \\ & \leq |G_\ell - G_{\ell-1}| + K \int_s^{t_N} \left(|\Gamma_\ell(r) - \Gamma_{\ell-1}(r)| + |\tilde{\Gamma}_\ell(r) - \tilde{\Gamma}_{\ell-1}(r)| \right) dr \\ & \quad + K \int_s^{t_N} (|Q_\ell(r) - Q_{\ell-1}(r)| + |R_\ell(r) - R_{\ell-1}(r)|) dr. \end{aligned}$$

For $s \in (t_{N-3}, t_{N-2}]$,

$$\begin{aligned}
 |\tilde{\Gamma}_\ell(s) - \tilde{\Gamma}_{\ell-1}(s)| &\leq |\Gamma_\ell(t_{N-2}) - \Gamma_{\ell-1}(t_{N-2})| + K \int_s^{t_{N-2}} |\tilde{\Gamma}_\ell^\Delta(s) - \tilde{\Gamma}_{\ell-1}^\Delta(r)| dr \\
 &\quad + K \int_s^{t_{N-2}} [|Q_\ell(r) - Q_{\ell-1}(r)| + |R_\ell(r) - R_{\ell-1}(r)|] dr \\
 &\leq |G_\ell - G_{\ell-1}| + K \int_{t_{N-2}}^{t_N} \left(|\Gamma_\ell(r) - \Gamma_{\ell-1}(r)| + |\tilde{\Gamma}_\ell(r) - \tilde{\Gamma}_{\ell-1}(r)| \right) dr \\
 &\quad + K \int_{t_{N-2}}^{t_N} (|Q_\ell(r) - Q_{\ell-1}(r)| + |R_\ell(r) - R_{\ell-1}(r)|) dr \\
 &\quad + K \int_s^{t_{N-2}} \left(|\tilde{\Gamma}_\ell(s) - \tilde{\Gamma}_{\ell-1}(r)| + |Q_\ell(r) - Q_{\ell-1}(r)| + |R_\ell(r) - R_{\ell-1}(r)| \right) dr \\
 &\leq |G_\ell - G_{\ell-1}| + K \int_s^{t_N} \left(|\Gamma_\ell(r) - \Gamma_{\ell-1}(r)| + |\tilde{\Gamma}_\ell(r) - \tilde{\Gamma}_{\ell-1}(r)| \right) dr \\
 &\quad + K \int_s^{t_N} (|Q_\ell(r) - Q_{\ell-1}(r)| + |R_\ell(r) - R_{\ell-1}(r)|) dr.
 \end{aligned}$$

By induction, we see that

$$\begin{aligned}
 &|\tilde{\Gamma}_\ell(s) - \tilde{\Gamma}_{\ell-1}(s)| \\
 &\leq |G_\ell - G_{\ell-1}| + K \int_s^{t_N} \left(|\Gamma_\ell(r) - \Gamma_{\ell-1}(r)| + |\tilde{\Gamma}_\ell(r) - \tilde{\Gamma}_{\ell-1}(r)| \right) dr \\
 &\quad + K \int_s^{t_N} (|Q_\ell(r) - Q_{\ell-1}(r)| + |R_\ell(r) - R_{\ell-1}(r)|) dr, \quad s \in [t_{\ell+1}, t_N].
 \end{aligned}$$

Combining (6.14), we obtain

$$\begin{aligned}
 &|\Gamma_\ell(s) - \Gamma_{\ell-1}(s)| + |\tilde{\Gamma}_\ell(s) - \tilde{\Gamma}_{\ell-1}(s)| \\
 &\leq 2|G_\ell - G_{\ell-1}| + K \int_s^{t_N} \left(|\Gamma_\ell(r) - \Gamma_{\ell-1}(r)| + |\tilde{\Gamma}_\ell(r) - \tilde{\Gamma}_{\ell-1}(r)| \right) dr \\
 &\quad + K \int_s^{t_N} (|Q_\ell(r) - Q_{\ell-1}(r)| + |R_\ell(r) - R_{\ell-1}(r)|) dr, \quad s \in [t_{\ell+1}, t_N].
 \end{aligned}$$

By Gronwall's inequality, we obtain (6.12). Similarly, comparing equations for $\hat{\Gamma}_\ell(\cdot)$ and $\hat{\Gamma}_{\ell-1}(\cdot)$, we obtain

$$\begin{aligned}
 |\hat{\Gamma}_\ell(s) - \hat{\Gamma}_{\ell-1}(s)| &\leq |\hat{G}_\ell - \hat{G}_{\ell-1}| + K \int_s^{t_N} |\hat{\Gamma}_\ell(r) - \hat{\Gamma}_{\ell-1}(r)| dr \\
 &\quad + \int_s^{t_N} \left(|\hat{Q}_\ell(r) - \hat{Q}_{\ell-1}(r)| + |\hat{R}_\ell(r) - \hat{R}_{\ell-1}(r)| + |\tilde{\Gamma}_\ell(r) - \tilde{\Gamma}_{\ell-1}(r)| \right) dr.
 \end{aligned}$$

By Gronwall's inequality, we obtain (6.13). □

Now, for given partition Δ of $[0, T]$ and any $s \in [0, t_N]$, let us denote

$$(6.15) \quad \left\{ \begin{array}{l} \tilde{\Gamma}^\Delta(s, \tau) = \sum_{\ell=0}^{N-1} \tilde{\Gamma}_\ell^\Delta(s) I_{(t_{\ell+1}, t_{\ell+2}]}(\tau), \\ \Gamma^\Delta(s, \tau) = \sum_{\ell=0}^{N-1} \Gamma_\ell^\Delta(s) I_{(t_{\ell+1}, t_{\ell+2}]}(\tau), \\ \bar{\Gamma}^\Delta(s, \tau) = \sum_{\ell=0}^{N-1} \bar{\Gamma}_\ell^\Delta(s) I_{(t_{\ell+1}, t_{\ell+2}]}(\tau), \end{array} \right. \quad 0 \leq \tau \leq s \leq t_N,$$

with the extension on $[0, t_N] \times [0, t_N]$ as follows:

$$\tilde{\Gamma}^\Delta(s, \tau) = \tilde{\Gamma}^\Delta(\tau, \tau), \quad \Gamma^\Delta(s, \tau) = \Gamma^\Delta(\tau, \tau), \quad \bar{\Gamma}^\Delta(s, \tau) = \bar{\Gamma}^\Delta(\tau, \tau), \quad s \in [0, \tau].$$

Note that $s \mapsto (\Gamma^\Delta(s, \tau), \bar{\Gamma}^\Delta(s, \tau))$ is continuous on $[0, T]$, whereas $s \mapsto \tilde{\Gamma}^\Delta(s, \tau)$ might have jumps at $s = t_\ell$, with $t_\ell \in [\tau, T]$.

From Proposition 6.1, we have the uniform boundedness of $P^\Delta(\cdot)$, $\hat{P}^\Delta(\cdot)$, $\tilde{\Gamma}^\Delta(\cdot, \cdot)$, $\Gamma^\Delta(\cdot, \cdot)$, and $\hat{\Gamma}^\Delta(\cdot, \cdot)$. This implies the equicontinuity of $s \mapsto (P^\Delta(s), \hat{P}^\Delta(s), \tilde{\Gamma}^\Delta(s, \tau), \Gamma^\Delta(s, \tau), \hat{\Gamma}^\Delta(s, \tau))$ on each (t_k, t_{k+1}) , for fixed $\tau \in [0, T]$. Also, from Proposition 6.1, we see the jump sizes of the above functions are controlled by the mesh size $\|\Delta\|$ of the partition Δ , thanks to the continuity of functions $(s, t) \mapsto (Q(s, t), \hat{Q}(s, t), R(s, t), \hat{R}(s, t), G(t), \hat{G}(t))$.

Finally, we will need the following result, which is a kind of extension of the well-known Arzela-Ascoli Theorem.

Lemma 6.3. *Let $f_k : [0, T] \rightarrow \mathbb{R}^n$, $k \geq 1$, be a sequence of uniformly bounded piecewise continuous functions. Suppose that for any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ and a $k_0(\varepsilon) \geq 1$ such that*

$$(6.16) \quad |f_k(t) - f_k(s)| < \varepsilon, \quad \forall |t - s| < \delta(\varepsilon), \quad k > k_0(\varepsilon).$$

Then there exists a subsequence $f_{k_j}(\cdot)$ and a continuous function $f : [0, T] \rightarrow \mathbb{R}^n$ such that

$$\lim_{j \rightarrow \infty} \|f_{k_j}(\cdot) - f(\cdot)\|_\infty = 0.$$

Sketch of the proof. Let $\mathbb{Q}[0, T] \equiv \{r_1, r_2, \dots\}$ be the set of all rational numbers in $[0, T]$. Without loss of generality, let us assume that all the functions $f_k(\cdot)$ are right-continuous. First consider the sequence $\{f_k(r_1)\}_{k \geq 1}$ which is bounded. Thus, we may have a convergent subsequence, denoted by $\{f_{\sigma_1(j)}(r_1)\}_{j \geq 1} \subseteq \{f_k\}_{k \geq 1}$. Next, consider sequence $\{f_{\sigma_1(j)}(r_2)\}_{j \geq 1}$. By the boundedness, we again have a convergent subsequence $\{f_{\sigma_2(j)}(r_2)\}_{j \geq 1} \subseteq \{f_{\sigma_1(j)}(r_2)\}_{j \geq 1}$. Continuing the procedure and using a usual diagonal argument, we obtain a subsequence $\{f_{k_j}(\cdot)\}_{j \geq 1}$ and a function $f(r)$ defined for all $r \in \mathbb{Q}[0, T]$ such that $f_{k_j}(r) \rightarrow f(r)$ for all $r \in \mathbb{Q}[0, T]$. By (6.16), we see that $f(\cdot)$ is bounded and continuous on $\mathbb{Q}[0, T]$ which is dense in $[0, T]$. Then one can extend it naturally to a continuous function on $[0, T]$. Further, we see that our convergence holds. \square

With the same argument, we see that the above result holds if $[0, T]$ is replaced by $[0, T] \times [0, T]$ (actually, $[0, T]$ can be replaced by any compact metric space). The point here is that the functions $f_k(\cdot)$ are not necessarily continuous, which will be the case in our problem.

Having the above preparation, we are now at the position of stating and proving the following convergence theorem.

Theorem 6.4. *Let (H1)–(H4) hold. Then*

$$(6.17) \quad \lim_{\|\Delta\| \rightarrow 0} (|\tilde{\Gamma}^\Delta(s, \tau) - \Gamma(s, \tau)| + |\Gamma^\Delta(s, \tau) - \Gamma(s, \tau)| + |\hat{\Gamma}^\Delta(s, \tau) - \hat{\Gamma}(s, t)| \\ + |P^\Delta(s) - \Gamma(s, s)| + |\hat{P}^\Delta(s) - \hat{\Gamma}(s, s)|) = 0,$$

uniformly in $(s, \tau) \in [0, T] \times [0, T]$ with $\Gamma(\cdot, \cdot)$ and $\hat{\Gamma}(\cdot, \cdot)$ satisfying (4.26).

Proof. Let a partition $\Delta : 0 = t_0 < \cdots < t_N = T$ of $[0, T]$ be given. By our condition and Proposition 6.1, we have the boundedness of $\tilde{\Gamma}^\Delta(s, \tau)$, $\Gamma^\Delta(s, \tau)$ and $\hat{\Gamma}^\Delta(s, \tau)$. Then noting that $\Gamma^\Delta(\cdot, \cdot)$, etc., satisfy proper differential equations, we have

$$(6.18) \quad \begin{cases} |\Gamma_s^\Delta(s, \tau)| + |\hat{\Gamma}_s^\Delta(s, \tau)| \leq K, & s \neq \tau, \quad \tau \neq t_k, \quad 0 \leq k \leq N, \\ |\tilde{\Gamma}_s^\Delta(s, \tau)| \leq K, & s \neq \tau, t_k, \quad 0 \leq k \leq N, \\ |\dot{P}^\Delta(s)| + |\dot{\hat{P}}^\Delta(s)| \leq K, & s \neq t_k, \quad 0 \leq k \leq N, \end{cases}$$

with some uniform constant $K > 0$ (independent of partition Δ). The above implies that

$$|\Gamma^\Delta(s, \tau) - \Gamma^\Delta(\bar{s}, \tau)| + |\hat{\Gamma}^\Delta(s, \tau) - \hat{\Gamma}^\Delta(\bar{s}, \tau)| \leq K|s - \bar{s}|, \\ s, \bar{s} \in [0, T], \quad \tau \in [0, T].$$

On the other hand, by Proposition 6.2, for any $\tau, \bar{\tau} \in [0, T]$ with

$$|\tau - \bar{\tau}| < \min_{0 \leq k \leq N-1} |t_{k+1} - t_k|,$$

we have

$$|\tilde{\Gamma}^\Delta(s, \tau) - \tilde{\Gamma}^\Delta(s, \bar{\tau})| + |\Gamma^\Delta(s, \tau) - \Gamma^\Delta(s, \bar{\tau})| + |\hat{\Gamma}^\Delta(s, \tau) - \hat{\Gamma}^\Delta(s, \bar{\tau})| \leq \bar{\omega}(\|\Delta\|),$$

with $\bar{\omega}(\cdot)$ being a modulus of continuity. Then by Lemma 6.3, we have

$$\lim_{\|\Delta\| \rightarrow 0} \left(|\tilde{\Gamma}^\Delta(s, \tau) - \tilde{\Gamma}(s, \tau)| + |\Gamma^\Delta(s, \tau) - \Gamma(s, \tau)| + |\hat{\Gamma}^\Delta(s, \tau) - \hat{\Gamma}(s, \tau)| \right) = 0,$$

for some continuous functions $\tilde{\Gamma}(\cdot, \cdot)$, $\Gamma(\cdot, \cdot)$, and $\hat{\Gamma}(\cdot, \cdot)$.

Next, let $s \in [t_{k+1}, t_{k+2})$. Then

$$|P^\Delta(s) - \Gamma^\Delta(s, s)| \leq |P_k(s) - P_k(t_{k+1})| + |\Gamma_k(s, s) - \Gamma_k(s, t_{k+1})| \leq K\|\Delta\|,$$

$$|\hat{P}^\Delta(s) - \hat{\Gamma}^\Delta(s, s)| \leq |\hat{P}_k(s) - \hat{P}_k(t_{k+1})| + |\hat{\Gamma}_k(s) - \hat{\Gamma}_k(t_{k+1})| \leq K\|\Delta\|,$$

and for $0 \leq \tau \leq s \leq T$, let $\tau \in [t_{k+1}, t_{k+2})$, and $s \in [t_{\ell+1}, t_{\ell+2})$, with $\ell \geq k$. Then

$$|\tilde{\Gamma}^\Delta(s, \tau) - \Gamma^\Delta(s, \tau)| = |\tilde{\Gamma}_k^\Delta(s) - \Gamma_k^\Delta(s)| \\ \leq |\tilde{\Gamma}_k^\Delta(s) - \tilde{\Gamma}_k^\Delta(t_{\ell+2})| + |\Gamma_k^\Delta(t_{\ell+2}) - \Gamma_k^\Delta(s)| \leq K\|\Delta\|.$$

Hence, we have

$$\lim_{\|\Delta\| \rightarrow 0} \left(|\tilde{\Gamma}^\Delta(s, \tau) - \Gamma(s, \tau)| + |P^\Delta(s) - \Gamma(s, s)| + |\hat{P}^\Delta(s) - \hat{\Gamma}(s, s)| \right) = 0.$$

Consequently,

$$\lim_{\|\Delta\| \rightarrow 0} \left(|\Theta^\Delta(s) - \Theta(s)| + |\hat{\Theta}^\Delta(s) - \hat{\Theta}(s)| \right) = 0,$$

with

$$\begin{cases} \Theta(s) = [R(s, s) + D(s)\Gamma(s, s)D(s)]^{-1} [B(s)^T\Gamma(s, s) + D(s)^T\Gamma(s, s)C(s)], \\ \hat{\Theta}(s) = [\hat{R}(s, s) + \hat{D}(s)\Gamma(s, s)\hat{D}(s)]^{-1} [\hat{B}(s)^T\hat{\Gamma}(s, s) + \hat{D}(s)^T\Gamma(s, s)\hat{C}(s)]. \end{cases}$$

Then we see that $\Gamma(\cdot, \cdot)$ and $\hat{\Gamma}(\cdot, \cdot)$ satisfy (4.26). \square

A careful observation of the equations (5.26)–(5.28) for $\tilde{\Gamma}_k, \Gamma_k, \bar{\Gamma}_k$ shows that for given k , $\Gamma_k(\cdot)$ and $\bar{\Gamma}_k(\cdot)$ are continuous on $[t_{k+1}, t_N]$, whereas $\tilde{\Gamma}_k(\cdot)$ has possible jumps at $t_{k+1}, t_{k+2}, \dots, t_{N-1}$. Therefore, it is difficult to pass to the limit in the equation for $\tilde{\Gamma}_k(\cdot)$. On the other hand, since at each point t_{k+1}, \dots, t_N , $\tilde{\Gamma}_k(\cdot)$ coincides with $\Gamma_k(\cdot)$, we are able to show that $\tilde{\Gamma}_k(\cdot)$ and $\Gamma_k(\cdot)$ actually have the same limit, and hence the limit process for (5.26) can be avoided.

Note that the above convergence result gives an existence of a solution $(\Gamma(\cdot, \cdot), \hat{\Gamma}(\cdot, \cdot))$ to the Riccati equation system (4.26). To be complete, we have the following result.

Theorem 6.5. *Let (H1)–(H2) hold. Then (4.26) admits a unique solution.*

Proof. We need only to prove the uniqueness. We denote

$$\Lambda(s, t) = \begin{pmatrix} \Gamma(s, t) \\ \hat{\Gamma}(s, t) \end{pmatrix}, \quad 0 \leq t \leq s \leq T, \quad \Lambda_0(\tau) = \begin{pmatrix} G(\tau) \\ \hat{G}(\tau) \end{pmatrix},$$

and rewrite (4.26) as follows:

$$\begin{cases} \Lambda_s(s, \tau) = F(s, \tau, \Lambda(s, \tau), \Lambda(s, s)), & 0 \leq \tau \leq s \leq T, \\ \Lambda(T, \tau) = \Lambda_0(\tau), \end{cases}$$

where $F(s, \tau, \gamma, \bar{\gamma})$ is some continuous map which is differentiable in $(\gamma, \bar{\gamma})$. Now, suppose there are two solutions $\Lambda^1(\cdot, \cdot)$ and $\Lambda^2(\cdot, \cdot)$. Then

$$\hat{\Lambda}(\cdot, \cdot) = \Lambda^1(\cdot, \cdot) - \Lambda^2(\cdot, \cdot)$$

satisfies

$$\begin{cases} \hat{\Lambda}_s(s, \tau) = F_\gamma(s, \tau)\hat{L}(s, \tau) + F_{\bar{\gamma}}(s, \tau)\hat{\Lambda}(s, s), & 0 \leq \tau \leq s \leq T, \\ \hat{\Lambda}(T, \tau) = 0, \end{cases}$$

for some continuous (matrix-valued) functions $F_\gamma(s, \tau)$ and $F_{\bar{\gamma}}(s, \tau)$. If we let $\hat{\Phi}(\cdot, \cdot)$ be the fundamental matrix of $F_\gamma(\cdot, \cdot)$, then

$$\hat{\Lambda}(s, \tau) = \int_s^T \hat{\Phi}(r, s)\Phi_{\bar{\gamma}}(r, \tau)\hat{\Lambda}(r, r)dr, \quad 0 \leq \tau \leq s \leq T.$$

Setting $s = \tau$ in the above, we end up with the following:

$$\hat{\Lambda}(\tau, \tau) = \int_\tau^T \hat{\Phi}(r, \tau)\Phi_{\bar{\gamma}}(r, \tau)\hat{\Lambda}(r, r)dr, \quad 0 \leq \tau \leq T.$$

This is a linear homogeneous Volterra integral equation with a continuous kernel.

Hence, it is necessary that

$$\widehat{\Lambda}(s, s) = 0, \quad s \in [0, T].$$

Then,

$$\widehat{\Lambda}(s, \tau) = 0, \quad 0 \leq \tau \leq s \leq T,$$

which leads to the uniqueness of the solution of (4.26). \square

Combining the above, we can easily obtain a proof of Theorem 4.5.

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