Assignment 02

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1 Thermodynamics relations

1.1 (a)

$$\begin{aligned} K_S &= -V(\frac{\partial P}{\partial V})_S \\ &= -V\Big[(\frac{\partial P}{\partial V})_T + (\frac{\partial P}{\partial T})_V(\frac{\partial T}{\partial V})_S\Big] \\ &= -V\Big[(\frac{\partial P}{\partial V})_T + (\frac{\partial S}{\partial T})_V(\frac{\partial P}{\partial S})_V(\frac{\partial T}{\partial V})_S\Big] \\ &= -V\Big[(\frac{\partial P}{\partial V})_T - (\frac{\partial S}{\partial T})_V(\frac{\partial T}{\partial V})_S^2\Big] \\ &= K_T + C_V\frac{\alpha^2 K_S^2 T}{C_P^2}V \end{aligned} \tag{1}$$

$$\frac{C_p}{C_V} = \frac{(\frac{\partial S}{\partial T})_P}{(\frac{\partial S}{\partial T})_V} \quad = \frac{(\frac{\partial S}{\partial P})_T(\frac{\partial P}{\partial T})_S}{(\frac{\partial S}{\partial V})_T(\frac{\partial V}{\partial T})_S} = \frac{(\frac{\partial P}{\partial V})_S}{(\frac{\partial P}{\partial V})_T} \quad = \frac{K_S}{K_T} \tag{2}$$

From (1) and (2),

$$K_S = K_T + C_V\frac{\alpha^2 K_S^2 T}{C_P^2}V = K_T(1 + C_V\frac{\alpha^2 K_T T}{C_V^2}) = K_T(1+\alpha\gamma T) \tag{3}$$

1.2 (b)

For pressure derivative of bulk modulus:

$$K_T^{'} = (\frac{\partial K_T}{\partial P})_T = (\frac{\partial K_T}{\partial V})_T(\frac{\partial V}{\partial P})_T = -\frac{V}{K_T}(\frac{\partial K_T}{\partial V})_T = -\Big(\frac{\partial ln(K_T)}{\partial ln(V)}\Big)_T \tag{4}$$

For Anderson-Grüneisen parameter:

$$\Big(\frac{\partial ln(\alpha)}{\partial ln(V)}\Big)_T = \frac{V}{\alpha}(\frac{\partial \alpha}{\partial V})_T = -\frac{1}{\alpha V}(\frac{\partial V}{\partial T})_P + \frac{1}{\alpha}\Big[\frac{\partial(\frac{\partial V}{\partial T})_P}{\partial V}\Big]_T = -1 + V(\frac{\partial T}{\partial V})_P\Big[\frac{\partial(\frac{\partial V}{\partial T})_P}{\partial V}\Big]_T \tag{5}$$

$$
\begin{aligned}
-\Big(\frac{\partial ln(K_T)}{\partial ln(V)}\Big)_P &= -\frac{V}{K_T}\Big(\frac{\partial K_T}{\partial V}\Big)_P \\
&= \frac{V}{K_T} \quad \Bigg\{(\frac{\partial P}{\partial V})_T + V\Big[\frac{\partial(\frac{\partial P}{\partial V})_T}{\partial V}\Big]_P\Bigg\} \\
&= -\frac{V}{K_T}(\frac{\partial P}{\partial V})_T - \frac{V^2}{K_T}(\frac{\partial P}{\partial V})_T^2\Big[\frac{\partial(\frac{\partial V}{\partial P})_T}{\partial V}\Big]_P \\
&= (\frac{\partial V}{\partial T})_P(\frac{\partial T}{\partial P})_V(\frac{\partial P}{\partial V})_T + V(\frac{\partial P}{\partial V})_T(\frac{\partial T}{\partial V})_P(\frac{\partial V}{\partial P})_T\Big[\frac{\partial(\frac{\partial V}{\partial T})_P}{\partial V}\Big]_T \\
&= -1 + V(\frac{\partial T}{\partial V})_P\Big[\frac{\partial(\frac{\partial V}{\partial T})_P}{\partial V}\Big]_T
\end{aligned}
\tag{6}
$$

From (5) and (6) we note that

$$
\delta_T = \Big(\frac{\partial ln(\alpha)}{\partial ln(V)}\Big)_T = -\Big(\frac{\partial ln(K_T)}{\partial ln(V)}\Big)_P \tag{7}
$$

2 Adiabatic temperature gradient of Earth's mantle

2.1 (a)

$$
(\frac{dT}{dz})_S = (\frac{\partial T}{\partial P})_S(\frac{\partial P}{\partial z})_S = (\frac{\partial V}{\partial S})_P(\frac{\partial P}{\partial z})_S = (\frac{\partial T}{\partial S})_P(\frac{\partial V}{\partial T})_P(\frac{\partial P}{\partial z})_S = \frac{T}{c_p \rho V}\cdot V\alpha\cdot\rho g = \frac{g\alpha T}{c_p} \tag{8}
$$

2.2 (b)

$$
\frac{dT}{dz} = \frac{g\alpha T}{c_p} = \frac{10\times 3\times 10^{-5}\times 1600}{1} = 0.48\ Km^{-1} \tag{9}
$$

3 Composition of the Earth's mantle

3.1 (1)

The temperature of intersting depth 355km and 450km can be derived from reference temperature at 660km consider a stable adiabatic temperature gradient.

$$
T(z) = (z - z_0)(\frac{dT}{dz})_S + T_{z_0} \tag{10}
$$

The 3^{rd} Birch-Murnaghan equation of state is related to desity at certain temperature and is:

$$
P = \frac{3}{2}K_{T0}\Big[(\frac{V_0(T)}{M}\rho)^{\frac{7}{3}} - (\frac{V_0(T)}{M}\rho)^{\frac{5}{3}}\Big]\Bigg\{1 + \frac{3}{4}(K_T' - 4)\Big[(\frac{V_0(T)}{M}\rho)^{\frac{2}{3}} - 1\Big]\Bigg\} \tag{11}
$$

where,

$$
V_0(T) = V_0(T_0)e^{\int_{T_0}^{T}\alpha(T')dT'} \tag{12}
$$

$$K_{T0}(T) = K_{T0}(T_0) + \left(\frac{\partial K_T}{\partial T}\right)_p (T - T_0) \tag{13}$$

$$\alpha(T^{'}) \approx \alpha_0 + \alpha_1 T + \alpha_2 \frac{1}{T^2} \tag{14}$$

$$M = X_{Mg}M_{Mg} + X_{Fe}M_{Fe} \tag{15}$$

$$V_0(T_0) = X_{Mg}V_{Mg} + X_{Fe}V_{Fe} \tag{16}$$

All these parameters of right hand side of (12)-(15) can be obtain from table 3 of the homework set 2.

There are three possible phases of $(Mg, Fe)_2SiO_4$ —— Olivine, $\beta - Phase$ and Spinel phase. By numerically solving (11) for each depth with certain pressure predicted by PREM in table II for each phases, the corresponding theoretical densities are derived and recorded in the first three columns in the Table 1 while the PREM density is recorded in the last column:

	$\rho_{Olivine}$	$\rho_{\beta-phase}$	ρ_{Spinel}	ρ_{PREM}
335km	3.6119	3.8281	3.9095	3.5164
450km	3.6700	3.8815	3.9617	3.7868

Table 1: The theoretical density of each phase in each depth with PREM pressure (unit: g/cm^3)

In comparison the three possible phases' theoretical density to corresponding PREM density, the mantle mineral phase at depth 355km is likely to be Olivine while wadsleyite at 450km.

3.2 (2)

Based on the conclusion in Qustion (1), the density of Olivine and Wadsleyite at 400km is:

$$\begin{cases} \rho_{Olive} = 3.6389\ g/cm^3 \\ \rho_{Wadsleyite} = 3.8527\ g/cm^3 \end{cases} \tag{17}$$

The density contrast is then

$$\phi = \frac{\rho_{Wadsleyite} - \rho_{Olive}}{\rho_{Olive}} = 0.0588 \tag{18}$$