Digital Signature Algorithm

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Prerequisites:

$$(ab) \bmod p = (a \bmod p)(b \bmod p) \bmod p$$

$$g^{xy} \bmod p = (g^x \bmod p)^y \bmod p$$

Public Parameters:

Generate large primes: p, q

$(q \mid p-1, |p| = 2048, |q| = 256)$

Choose random h

$h \in (2, \cdots, p-2)$

Calculate $g = h^{\frac{p-1}{q}} \bmod p$

$\because p, q$ are primes,

$\therefore h^{p-1} \bmod p = 1$

$$g^q \bmod p = (h^{\frac{p-1}{q}} \bmod p)^q \bmod p = h^{p-1} \bmod p = 1$$

let k be any integer,

$$g^{x+kq} \bmod p = (g^x g^{kq}) \bmod p$$

$$= (g^x \bmod p \cdot g^{kq} \bmod p) \bmod p$$

$$g^{kq} \bmod p = (g^q \bmod p)^k \bmod p = 1$$

$$\therefore g^{x+kq} \bmod p = g^x \bmod p$$

If k is negative, multiple q can be takne out from x,

$$\therefore g^x \bmod p = g^{x \bmod q} \bmod p$$

Key Generation:

Choose random x (private key)

$x \in (1, \cdots, q-1)$

Calculate $y = g^x \bmod p$ (public key)

Signing:

$$\text{Choose random k}$$
$$k \in (1, \cdots, q-1)$$
$$\text{Calculate } r = (g^k \bmod p) \bmod q$$
$$\text{Calculate } s = (k^{-1}(H(m) + xr)) \bmod q$$
$$((k^{-1}k) \bmod q = 1)$$
$$S(r, s)$$

Verification:

$$\text{Check } 0 < r < q, 0 < s < q$$
$$\text{Calculate } w = s^{-1} \bmod q$$
$$\text{Calculate } u_1 = (H(m)w) \bmod q$$
$$\text{Calculate } u_2 = (rw) \bmod q$$
$$\text{Calculate v} = (g^{u_1} y^{u_2} \bmod p) \bmod q$$
$$\text{If v} = r, \text{ signature validated,}$$
$$\text{otherwiese, rejected.}$$

Correctness:

$$g^{u_1} y^{u_2} \bmod p$$
$$= g^{(H(m)w) \bmod q} y^{(rw) \bmod q} \bmod p$$
$$= (g^{(H(m)w) \bmod q} \bmod p) \cdot (y^{(rw) \bmod q} \bmod p) \bmod p$$
$$\because g^x \bmod p = g^{x \bmod q} \bmod p$$
$$\therefore g^{u_1} y^{u_2} \bmod p = (g^{H(m)w} y^{rw}) \bmod p$$
$$= g^{H(m)w + rwx} \bmod p$$
$$= g^{w(H(m) + rx)} \bmod p$$
$$= g^{s^{-1}(H(m) + rx)} \bmod p$$
$$= g^{k(H(m) + rx)^{-1}(H(m) + rx)} \bmod p$$
$$= g^{k(mq+1)} \bmod p$$
$$= (g^{kmq} g^k) \bmod p$$
$$= (g^{qkm} \bmod p) \cdot (g^k \bmod p) \bmod p$$
$$= g^k \bmod p \bmod p$$
$$= g^k \bmod p$$

Utility:

For modular inverse with respect to p:

$aa^{-1} \bmod p = 1$

If $a = cd^{-1}, a^{-1} = c^{-1}d$

(c^{-1}, d^{-1} are modular inverses WRT p respectively)

Proof:

$aa^{-1} = cd^{-1}c^{-1}d = (cc^{-1})(dd^{-1})$

$\because cc^{-1} \bmod p = 1, dd^{-1} \bmod p = 1,$

$\therefore cc^{-1} = mp + 1, dd^{-1} = np + 1 (m, n \in Z^{+})$

$aa^{-1} = mpn^2 + mp + np + 1,$

$aa^{-1} \bmod p = 1$

Correct