Formula Derivation for Engineering & Physics Interview

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Simple Harmonic Motion

Displacement

$$x = x_0 \sin(\omega t)$$

Velocity

$$v = \frac{dx}{dt} = \omega x_0 \cos(\omega t)$$

Acceleration

$$a = \frac{dv}{dt} = \omega^2 x_0(-\sin(\omega t)) = -\omega^2 x$$

Maximum velocity

$$v = \omega x_0 \cos(\omega t)$$
$$\text{when } \cos(\omega t) = 1$$
$$v_{\max} = \omega x_0$$

Another equation for velocity

$$x = x_0 \sin(\omega t) \text{ and } v = \omega x_0 \cos(\omega t)$$
$$\frac{x}{x_0} = \sin(\omega t) \text{ and } \frac{v}{\omega x_0} = \cos(\omega t)$$
$$(\frac{x}{x_0})^2 + (\frac{v}{\omega x_0})^2 = 1$$
$$\omega^2 x^2 + v^2 = \omega^2 x_0{}^2$$
$$v^2 = \omega^2 (x_0{}^2 - x^2)$$
$$v = \omega\sqrt{x_0{}^2 - x^2}$$

Period of mass-spring system (horizontal)

$$F = kx$$
$$a = \frac{kx}{m}$$
$$\frac{k}{m} = \omega^2$$
$$\omega = \sqrt{\frac{k}{m}}$$
$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

Period of mass-spring system (vertical)

$$\text{when at rest, } mg = kx_i$$
$$F = k(x_i + x) - mg = kx$$
$$a = \frac{kx}{m}$$
$$\omega = \sqrt{\frac{k}{m}}$$
$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

Period of pendulum motion

$$F = mg\sin\theta$$
$$a = \frac{mg\sin\theta}{m} = g\sin\theta$$
$$\because \theta \text{ is very small,}$$
$$\therefore a = g\theta$$
$$a = r\alpha = l\alpha$$
$$l\alpha = g\theta$$
$$\alpha = \frac{g}{l}\theta$$
$$\therefore \omega = \sqrt{\frac{g}{l}}$$
$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}$$

Period of water in U-pipe

$$F = \frac{m}{L} \bullet 2hg$$

$$a = \frac{F}{m} = \frac{2g}{L} h$$

$$\omega = \sqrt{\frac{2g}{L}}$$

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{L}{2g}}$$

Period of a floating block

$$B = W = mg$$

$$mg = \rho_l g V_i = \rho_l g A h_i$$

$$B' = \rho_l g V_i' = \rho_l g A \left(h_i + dh \right)$$

$$F_{net} = B' - mg = \rho_l g A \left(h_i + dh \right) - \rho_l g A h_i$$

$$F_{net} = \rho_l g A dh$$

$$a = \frac{\rho_l g A dh}{m}$$

$$\omega = \sqrt{\frac{\rho_l g A}{m}}$$

$$T = 2\pi \sqrt{\frac{m}{\rho_l g A}}$$

Discharging a capacitor

$$C = \frac{Q}{V}$$

$$\because V_R = V_C \text{ after a long time,}$$

$$\therefore IR = \frac{Q}{C}$$

$$I = -\frac{dQ}{dt}$$

$$-\frac{dQ}{dt}R = \frac{Q}{C}$$

$$-\frac{RC}{Q}dQ = dt$$

$$\int -\frac{RC}{Q}dQ = \int dt + c$$

$$-RC\ln Q = t + c$$

$$\text{when } t = 0, c = -RC\ln Q_0$$

$$\ln\frac{Q}{Q_0} = \frac{-t}{RC}$$

$$e^{-\frac{t}{RC}} = \frac{Q}{Q_0}$$

$$Q = Q_0 e^{-\frac{t}{RC}}$$

Charging a capacitor

$$V = V_C + V_R = \frac{Q}{C} + IR$$

$$I = \frac{dQ}{dt}$$

$$V = \frac{Q}{C} + \frac{dQ}{dt}R$$

$$\frac{CV - Q}{RC} = \frac{dQ}{dt}$$

$$\frac{1}{RC}dt = \frac{1}{CV - Q}dQ$$

$$\int \frac{1}{RC}dt = \int \frac{1}{CV - Q}dQ + c$$

$$\frac{t}{RC} = -\ln(CV - Q) + c$$

$$\text{when } t = 0, c = \ln(CV) = \ln Q_f$$

$$-\frac{t}{RC} = \ln(Q_f - Q) - \ln Q_f = \ln \frac{Q_f - Q}{Q_f}$$

$$e^{-\frac{t}{RC}} = 1 - \frac{Q}{Q_f}$$

$$\therefore Q = Q_f(1 - e^{-\frac{t}{RC}})$$

Pressure of an ideal gas

$$\Delta p = mv - mv = -2mv$$

$$F = \frac{\Delta p}{\Delta t}$$

$$\Delta t = \frac{2l}{v}$$

$$\therefore F = \frac{2mv}{2l / v} = \frac{mv^2}{l}$$

$$P = \frac{F}{A} = \frac{mv^2 / l}{l^2} = \frac{mv^2}{l^3} \text{(for a cubic container)}$$

This is the pressure exerted by one particle

$$P = \frac{Nm\left\langle c^2 \right\rangle}{l^3}$$

This is the pressure exerted by all particles

$$P = \frac{1}{3}\frac{Nm\left\langle c^2 \right\rangle}{l^3}$$

This is because the molecules are moving in all three dimisions equally instead of travelling in the same direction

$$Nm = M_{gas}$$

$$\because P = \frac{M_{gas}\left\langle c^2 \right\rangle}{3l^3}$$

$$\rho = \frac{M_{gas}}{V} = \frac{M_{gas}}{l^3}$$

$$\therefore P = \frac{1}{3}\rho\left\langle c^2 \right\rangle$$

Kepler's 2nd law

$$dA = \pi r^2 \frac{d\theta}{2\pi}$$

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{1}{2}r^2\omega = \frac{1}{2}rv$$

$$p_{ang} = I\omega = mr^2\omega = mrv$$

$$\frac{dA}{dt} = \frac{L}{2m}$$

Average kinetic energy of an ideal gas

$$\because P = \frac{1}{3}\frac{Nm\left\langle c^2 \right\rangle}{l^3}$$

$$PV = nRT$$

$$\therefore \frac{1}{3}\frac{Nm\left\langle c^2 \right\rangle}{l^3}V = nRT$$

$$\frac{1}{3}Nm\left\langle c^2 \right\rangle = nRT$$

$$N_A = \frac{N}{n}$$

$$\frac{N_A}{R}m\left\langle c^2 \right\rangle = 3T$$

$$\frac{1}{2}m\left\langle c^2 \right\rangle = \frac{3}{2}\frac{R}{N_A}T$$

$$\frac{R}{N_A} = k\left(\text{Boltzmann constant}\right)$$

$$\therefore \left\langle E_k \right\rangle = \frac{3}{2}kT$$

Gravitational potential energy

$$F = \frac{GMm}{r^2}$$

$$W = \int F dr = \int \frac{GMm}{r^2} dr$$

$$W = \int_{\infty}^{r} \frac{GMm}{r^2} dr$$

$$= -\frac{GMm}{r} + \frac{GMm}{\infty}$$

$$= -\frac{GMm}{r}$$

Electrical potential energy

$$F = \frac{Q_1 Q_2}{4\pi\varepsilon_0 r^2}$$

$$W = \int -F dr = \int -\frac{Q_1 Q_2}{4\pi\varepsilon_0 r^2} dr$$

$$= \int_{\infty}^{r} -\frac{Q_1 Q_2}{4\pi\varepsilon_0 r^2} dr$$

$$= \frac{Q_1 Q_2}{4\pi\varepsilon_0 r} - \frac{Q_1 Q_2}{4\pi\varepsilon_0 \infty}$$

$$= \frac{Q_1 Q_2}{4\pi\varepsilon_0 r}$$

Escape velocity

$$E_{k_0} + E_{p_0} = E_k' + E_p'$$

$$\frac{1}{2} m v_{esc}{}^2 - \frac{GMm}{R_E} = 0 + 0$$

$$v_{esc}{}^2 = \frac{2GM}{R_E}$$

$$v_{esc} = \sqrt{\frac{2GM}{R_E}}$$

Kepler's 3rd law

$$F_c = F_g$$
$$mr\omega^2 = \frac{GMm}{r^2}$$
$$\omega = \frac{2\pi}{T}$$
$$m\frac{4\pi^2}{T^2} = \frac{GMm}{r^3}$$
$$\therefore T^2 = \frac{4\pi^2}{GM}r^3$$

RMS current for AC circuit

$$I = I_{\max}\sin\omega t$$
$$I^2 = \mathrm{I}_{\max}{}^2\sin^2(\omega t)$$
$$\langle I^2\rangle = \frac{\int_0^T \mathrm{I}_{\max}{}^2\sin^2(\omega t)\,dt}{T}$$
$$= \frac{I_{\max}{}^2\int_0^T 1-\cos(2\omega t)dt}{2T}$$
$$= \frac{I_{\max}{}^2\left[t-\frac{1}{2\omega}\sin(2\omega t)\right]_0^T}{2T}$$
$$= \frac{I_{\max}{}^2 T}{2T}$$
$$= \frac{I_{\max}{}^2}{2}$$
$$\therefore I_{rms} = \frac{\sqrt{2}}{2}I_{\max}$$

Equation of a capacitor

$$\Phi = EA$$
$$\Phi = \frac{Q}{\varepsilon_0}$$
$$\therefore EA = \frac{Q}{\varepsilon_0}$$
$$\frac{V}{d}A = \frac{Q}{\varepsilon_0}$$
$$Q = \frac{\varepsilon_0 A}{d}V$$
$$Q = CV$$

Radioactive decay

$$\lambda = -\frac{dN}{Ndt}$$
$$-\lambda dt = \frac{1}{N}dN$$
$$\int -\lambda dt = \int \frac{1}{N}dN + c$$
$$\ln N = -t + c$$
$$\text{when } t = 0,$$
$$N = N_0$$
$$\therefore c = \ln N_0$$
$$\ln N = -\lambda t + \ln N_0$$
$$\ln \frac{N}{N_0} = -\lambda t$$
$$N = N_0 e^{-\lambda t}$$

Doppler effect

$$\Delta\lambda\text{: distnace the source moved in a period}$$
$$\Delta\lambda = v_s T = \frac{v_s}{f}$$
$$f' = \frac{v}{\lambda'} = \frac{v}{\lambda - \Delta\lambda} = \frac{v}{\frac{v}{f} - \frac{v_s}{f}} = \frac{v}{v - v_s} f$$

Inductor (when switch is closed)

$$\varepsilon - IR - L\frac{dI}{dt} = 0$$

$$\frac{\varepsilon}{R} - I = \frac{L}{R}\frac{dI}{dt}$$

$$\text{suppose } \frac{\varepsilon}{R} - I = \frac{L}{R}\frac{dI}{dt} = x$$

$$\frac{dx}{dI} = -1$$

$$dx = -dI$$

$$\therefore x + \frac{L}{R}\frac{dx}{dt} = 0$$

$$\frac{1}{x}dx = -\frac{R}{L}dt$$

$$\int_{x_0}^{x}\frac{1}{x}dx = -\frac{R}{L}\int_0^t dt + c$$

$$\ln\frac{x}{x_0} = -\frac{Rt}{L} + c$$

$$\text{when } t = 0,$$

$$x = x_0, c = 0$$

$$\therefore x = x_0 e^{-\frac{Rt}{L}}$$

$$\frac{\varepsilon}{R} - I = \left(\frac{\varepsilon}{R} - I_0\right)e^{-\frac{Rt}{L}}$$

$$\because I_0 = 0,$$

$$\therefore I = \frac{\varepsilon}{R}\left(1 - e^{-\frac{Rt}{L}}\right)$$

Inductor (when switch is opened)

$$IR = -L\frac{dI}{dt}$$

$$IRdt = -LdI$$

$$-\frac{R}{L}dt = \frac{1}{I}dI$$

$$\int -\frac{R}{L}dt = \int \frac{1}{I}dI + c$$

$$-\frac{R}{L}t = \ln\frac{I}{I_0} + c$$

when $t = 0$,

$$I = I_0,$$

$$\therefore c = 0$$

$$\frac{I}{I_0} = e^{-\frac{R}{L}t}$$

$$I = I_0 e^{-\frac{R}{L}t}$$