# CGU Math 381 (Image Processing), Spring '18 – Homework #3

Corner deteection, and an introduction to Fourier Analysis.

*Due on Friday 3/30, in the box in front of Prof. Micheli's office.*
*Song, Zhengming, Claremont Graduate University*

**Reading:** Fourier Analysis is treated in Chapters 7, 8 and 9 of *Principles of Digital Image Processing: Core Algorithms* (i.e. Volume 2) by Burger and Burge. The Harris corner detector is treated Chapter 4 of the same volume, and (in more detail) in the instructor's class notes, that are now available online.

Write, on top of the first page of your assignment: Name (<u>LAST</u>, First), your University or College, the HW#, and acknowledge other students with whom you may have worked (just write "*Worked with* ..."). For the computational problems where you are asked to write computer code you may choose the programming language that you prefer, such as *Python* (I am learning it myself!) or *Matlab*.

**Problem 3.1** (time shifting). Consider the *shifted* delta function $\delta_{t_0}(t) = \delta(t - t_0)$. **(a)** Show that for any continuous-time signal $s(t)$, $t \in \mathbb{R}$, it is the case that $(s * \delta_{t_0})(t) = s(t - t_0)$ (that is, the convolution with $\delta_{t_0}$ hs the effect of time-shifting by $t_0$). **(b)** find the Fourier Transform $\mathcal{F}[\delta_{t_0}](f)$ by direct computation. **(c)** Finally, combine the above results with the convolution theorem (it states that $\mathcal{F}[g*h](f) = G(f)H(f)$) to find the Fourier Transform of the shifted signal $s(t - t_0)$ in terms of the Fourier Transform $S(f)$ of $s(t)$.

**Solution 3.1.**     a

$$
\begin{aligned}
(s * \delta_{t_0})(t) &= \int_{-\infty}^{+\infty} s(\tau)\delta_{t_0}(t - \tau)d\tau \\
&= \int_{-\infty}^{+\infty} s(\tau)\delta(t - t_0 - \tau)d\tau \\
&= \int_{-\infty}^{+\infty} s(t - t_0)\delta(t - t_0 - \tau)d\tau \\
&= s(t - t_0) \int_{-\infty}^{+\infty} \delta(t - t_0 - \tau)d\tau \\
&= s(t - t_0) \int_{-\infty}^{+\infty} \delta(\tau - (t - t_0))d\tau \\
&= s(t - t_0) \cdot 1 \\
&= s(t - t_0)
\end{aligned}
$$

b

$$
\begin{aligned}
\mathcal{F}[\delta t_0](f) &= \int_{-\infty}^{+\infty} \delta_{t_0}(\tau)e^{-i2\pi f\tau}d\tau \\
&= \int_{-\infty}^{+\infty} \delta(\tau - t_0)e^{-i2\pi f\tau}d\tau \\
&\stackrel{t=\tau-t_0}{=} \int_{-\infty}^{+\infty} \delta(t)e^{-i2\pi f(t+t_0)}dt \\
&= e^{-i2\pi t_0} \int_{-\infty}^{+\infty} \delta(t)e^{-i2\pi ft}dt \\
&= e^{-i2\pi t_0} \cdot e^{-i2\pi f0} \cdot \int_{-\infty}^{+\infty} \delta(t)dt \\
&= e^{-i2\pi t_0} \cdot 1 \cdot 1 \\
&= e^{-i2\pi t_0}
\end{aligned}
$$

c

$$
\begin{aligned}
\mathcal{F}[s(t - t_0)](f) &= \mathcal{F}[s * \delta_{t_0}(t)](f) \\
&= \mathcal{S}(f)\mathcal{D}(f) \\
&= \mathcal{S}(f) \cdot \mathcal{F}[\delta_{t_0}](f) \\
&= e^{-i2\pi ft_0}\mathcal{S}(f)
\end{aligned}
$$

**Problem 3.2** (the causal exponential). Fix a constant $\alpha > 0$, and consider the continuous-time signal $s(t) = e^{-\alpha t} H(t)$, $t \in \mathbb{R}$, known as the *causal exponential* , where $H(t)$ is the Heaviside function (that is, $H(t) = 1$ for $t \geq 0$ and $H(t) = 0$ for $t < 0$). **(a)** Compute its Fourier transform $S(f)$, $f \in \mathbb{R}$ (this is example #26 in the Fourier pairs table that was distributed in class). **(b)** Compute the convolution $r(t) = (s * s)(t)$, $t \in \mathbb{R}$ (this may be the first and last time that you actually compute a convolution 'by hand', but it iss the kind of thing that you should do at least once in your life!). **(c)** Finally, using the result from part (a), what is the Fourier transform $R(f)$ of the continuous-time signal $r(t)$ from part (b)?

**Solution 3.2.**  a

$$
\begin{aligned}
S(f) &= \mathcal{F}[s](f) \\
&= \int_{-\infty}^{+\infty} e^{-\alpha t} H(t) e^{-i2\pi f t} dt \\
&= \int_{0}^{+\infty} e^{-\alpha t} e^{-i2\pi f t} dt \\
&= \int_{0}^{+\infty} e^{-\alpha + i2\pi f\, t} dt \\
&= \frac{e^{-(\alpha + i2\pi f)t}}{-(\alpha + i2\pi f)} \Big|_{0}^{+\infty} \\
&= 0 - \frac{-1}{\alpha + i2\pi f} \\
&= \frac{1}{\alpha + i2\pi f}
\end{aligned}
$$

b

$$
\begin{aligned}
r(t) &= (s * s)(t) \\
&= \int_{-\infty}^{+\infty} s(\tau) s(t - \tau) d\tau \\
&= \int_{-\infty}^{+\infty} e^{-\alpha \tau} H(\tau) e^{-\alpha(t-\tau)} H(t - \tau) d\tau \\
&= e^{-\alpha t} \int_{-\infty}^{+\infty} H(\tau) H(t - \tau) d\tau \\
&= e^{-\alpha t} t H(t)
\end{aligned}
$$

c

$$
R(f) = \mathcal{F}[r](f) = S \cdot S(f) = \frac{1}{(\alpha + i2\pi f)^2}
$$

**Problem 3.3** (FT of separable 2D signals). Suppose that a signal $s(x, y)$, with $(x, y) \in \mathbb{R}^2$, may be written as: $s(x, y) = f(x)g(y)$, where $f$ and $g$ are (continuous-space) signals. Such 2D signals are called *separable*. Show that the Fourier transform of $s$ is $S(u, v) = F(u)G(v)$, where $F(u) = \mathcal{F}[f](u)$ and $G(v) = \mathcal{F}[g](v)$.

**Solution 3.3.**

$$
\begin{aligned}
S(u, v) &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} s(x, y) e^{-i2\pi(ux+vy)} dx dy \\
&= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x)g(y) e^{-i2\pi(ux+vy)} dx dy \\
&= \int_{-\infty}^{+\infty} g(y) e^{-i2\pi vy} \int_{-\infty}^{+\infty} f(x) e^{-i2\pi ux} dx dy \\
&= \int_{-\infty}^{+\infty} g(y) e^{-i2\pi vy} \mathcal{F}(u) dy \\
&= \mathcal{F}(u) \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(y) e^{-i2\pi vy} dy \\
&= \mathcal{F}(u)\mathcal{G}(v)
\end{aligned}
$$

**Problem 3.4** (differential operators and convolution). Consider two continuous-space signals $f(x, y)$ and $g(x, y)$, with $(x, y) \in \mathbb{R}^2$. **(a)** Show that $\nabla^2(f * g) = f * (\nabla^2 g)$, where $\nabla^2$ donotes the Laplace operator in the variables $(x, y)$. **(b)** Express the Fourier transform of $\frac{\partial^2 f}{\partial x \partial y}$ in terms of $F(u, v) = \mathcal{F}[f](u, v)$.

**Solution 3.4.**    a

$$
\begin{aligned}
\nabla^2(f * g)(x, y) &= \nabla^2 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(u, v) g(x - u, y - v) du dv \\
&= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(u, v) \cdot \nabla^2 g(x - u, y - v) du dv \\
&= (f * \nabla^2 g)(x, y)
\end{aligned}
$$

b

$$
\begin{aligned}
\frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}(f)\right] \\
&= \frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}\left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(u,v) e^{i2\pi(ux+vy)} du dv\right)\right] \\
&= \frac{\partial}{\partial y}\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(u,v) \frac{\partial}{\partial x} e^{i2\pi(ux+vy)} du dv\right] \\
&= \frac{\partial}{\partial y}\left[\int_{-\infty}^{+\infty} i2\pi u \int_{-\infty}^{+\infty} F(u,v) e^{i2\pi(ux+vy)} du dv\right] \\
&= \left[\int_{-\infty}^{+\infty} i2\pi u \int_{-\infty}^{+\infty} F(u,v) \frac{\partial}{\partial y} e^{i2\pi(ux+vy)} du dv\right] \\
&= \left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (i2\pi)^2 uv F(u,v) e^{i2\pi(ux+vy)} du dv\right]
\end{aligned}
$$

Hence,

$$
\mathcal{F}\left[\frac{\partial^2 f}{\partial x \partial y}\right](u, v) = -4\pi^2 uv F(u, v)
$$

**Problem 3.5** (symmetries). Consider a continous-space signal $s(x, y)$, $(x, y) \in \mathbb{R}^2$. Show that if the signal $s$ is real and has even symmetry, i.e. $s(x, y) = \overline{s(x, y)} = s(-x, -y) = s(-x, -y)$, (where the bar indicates complex conjugation) then its Fourier transform $S(u, v)$ is also real, with even symmetry.

**Solution 3.5.**

$$
\begin{aligned}
S(u, v) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} s(x, y) e^{-i2\pi(ux+vy)} dx dy \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} s(x, y)[\cos(2\pi(ux+vy)) - i\sin(2\pi(ux+vy)]dx dy \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} s(x, y)\cos(2\pi(ux+vy))dx dy + -i\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} s(x, y)\sin(2\pi(ux+vy))dx dy \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} s(x, y)\cos(2\pi(ux+vy))dx dy + 0
\end{aligned}
$$

The last equality hold because $\sin(\cdot)$ is a odd function, while $s(x, y)$ is a event function, hence, the product of those two functions are odd, and the integral of a odd function is 0. In addition, since both $s(x, y)$ and $\cos(\cdot)$ are real, $S(u, v)$ is real.

$$
\begin{aligned}
S(-u, -v) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} s(x, y) e^{i2\pi(-ux-vy)} dx dy \\
&\overset{x=-\alpha, y=-\beta}{=} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} s(x, y) e^{-i2\pi(u\alpha+v\beta)} d\alpha d\beta \\
&= S(u, v)
\end{aligned}
$$

**Problem 3.6** (UMF in the frequency domain). Remember that the sharpening technique known as *unmask filtering* is defined as follows, for continuous-space images $f(x, y)$. First of all, a smoothing (low-pass) fiter, such as a Gaussian filter $h(x, y) = \frac{1}{2\pi\sigma^2}\exp\left(-\frac{1}{2}\frac{x^2+y^2}{\sigma^2}\right)$ is chosen (note that since $h$ has volume 1, we have $H(0, 0) = 1$). Then the smooth compunent of $f$ is computed via convolution: $m = h * f$. The sharp component of the image $f(x, y)$ (also known as the *unsharp mask*) is $s(x, y) = f(x, y) - m(x, y)$.

3

The sharpened image is then defined as $g(x, y) = f(x, y) + a \cdot s(x, y)$, where the constant $a > 0$ is the *sharpening strentgh*. For a generic smoothing convolutional kernel $h(x, y)$ with Fourier transform $H(u, v)$, show (as we did in class) that we can write, in frequency, $G(u, v) = L(u, v) \cdot F(u, v)$; express the frequency response $L(u, v)$ in terms of the function $H(u, v)$ and the constant $a$. Also, write an explicit expression for $L(u, v)$ when $h(x, y)$ is the Gaussian function given above, and plot the graph of $z = L(u, v)$ for one or two particular choices of $\sigma$ and $a$ (since $L(u, v)$ turns out to be a radial fucntion, i.e. such that its graph is symmetric with respect to rotations about the $z$-axis, you may just plot the cross section for $v = 0$).

**Solution 3.6.**

$$
\begin{aligned}
g(x, y) &= f(x, y) + a \cdot s(x, y) \\
&= f(x, y) + a \cdot f(x, y) - a \cdot h(x, y) * f(x, y)
\end{aligned}
$$

$$
\mathcal{G}(u, v) = F(u, v) + aF(u, v) + aH(u, v)F(u, v) = (1 + a + aH(u, v))F(u, v)
$$

Hence,

$$
\mathcal{L}(u, v) = (1 + a + aH(u, v)) = (1 + a - ae^{-2\pi a^2(u^2 + v^2)})
$$

**Problem 3.7** (DFT and IDFT). In class we defined the Discrete Fourier Transform (DFT) of a discrete- and finite-time signal $s_n$, $n = 0, 1, \ldots, M - 1$ as follows:

$$
S_k = \sum_{n=0}^{M-1} s_n e^{-i2\pi kn/M} \qquad k = 0, 1, \ldots, M - 1.
$$

Show that $s_n$ can be recovered via the Inverse Discrete Fourier Transform: $s_n = \dfrac{1}{M} \displaystyle\sum_{k=0}^{M-1} S_k e^{+i2\pi kn/M}$.

*Hint:* You should first prove and then use the following *orthogonality* of discrete complex exponentials:

$$
\sum_{k=0}^{M-1} e^{i2\pi km/M} e^{-i2\pi kn/M} = \begin{cases} M & \text{if } m = n \\ 0 & \text{otherwise.} \end{cases}
$$

**Solution 3.7.** First let's prove the orthogonality of discrete complex exponential:

*Proof.*

$$
\sum_{k=0}^{M-1} e^{i2\pi km/M} e^{-i2\pi kn/M} = \sum e^{2\pi ki(m-n)/M}
$$

if $m = n$, we have

$$
\sum e^{2\pi ki(m-n)/M} = \sum 1 = M
$$

if $m \neq n$, we have

$$
\begin{aligned}
\sum e^{2\pi ki(m-n)/M} &= \sum [e^{\frac{2\pi i(m-n)}{M}}]^k \\
&= \frac{1 - (e^{\frac{2\pi i(m-n)}{M}})^M}{1 - e^{\frac{2\pi i(m-n)}{M}}}
\end{aligned}
$$

Given that $m, n \in [0, 1, \cdots, M - 1]$ hence, the denominator cannot be 0. however, in the numerator,

$$
e^{2\pi i(m-n)} = \cos(2\pi(m - n)) + i\sin(2\pi(m - n)) = 1 + 0
$$

, which makes the numerator 0.

Hence,

$$\sum_{k=0}^{M-1} e^{i2\pi km/M} e^{-i2\pi kn/M} = \begin{cases} M & \text{if } m = n \\ 0 & \text{otherwise.} \end{cases}$$

$\square$

Next, we proof the main statement of this problem

*Proof.*

$$
\begin{aligned}
s_n &= \frac{1}{M} \sum_{k=0}^{M-1} S_k e^{i2\pi kn/M} \\
&= \frac{1}{M} \sum_{k=0}^{M-1} \sum_{m=0}^{M-1} s_m e^{-i2\pi km/M} e^{i2\pi kn/M} \\
&= \frac{1}{M} \sum_{m=0}^{M-1} s_m \sum_{k=0}^{M-1} e^{-i2\pi km/M} e^{i2\pi kn/M} \\
&= \frac{1}{M} \cdot s_n \cdot M \\
&= s_n
\end{aligned}
$$

$\square$

**Problem 3.8** (2D DFT of sine function). Consider the 2D signal: $s_{m,n} = \sin(2\pi k_0 m + 2\pi \ell_0 m)$, with $m = 0, 1, 2, \ldots, M-1$ and $n = 0, 1, 2, \ldots, N-1$; also, $k_0 \in \left\{0, \frac{1}{M}, \frac{2}{M}, \ldots, \frac{M-1}{M}\right\}$ and $\ell_0 \in \left\{0, \frac{1}{N}, \frac{2}{N}, \ldots, \frac{N-1}{N}\right\}$ are constants. Show that its DFT is $S_{k\ell} = \frac{iMN}{2}\left[\delta(k + Mk_0, \ell + N\ell_0) - \delta(k - Mk_0, \ell - N\ell_0)\right]$, where $\delta$ is the 2D discrete delta function: $\delta(k, \ell) = \begin{cases} 1 & \text{for } k = \ell = 0, \\ 0 & \text{for } 1 \leq k \leq M-1 \text{ or } 1 \leq k \leq N-1. \end{cases}$

**Solution 3.8.**

$$
\begin{aligned}
S_{k,l} &= \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} s_{m,n} e^{-i2\pi(\frac{mk}{M} + \frac{nl}{N})} \\
&= \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \sin(2\pi k_0 m + 2\pi l_0 n) e^{-i2\pi(\frac{mk}{M} + \frac{nl}{N})} \\
&= \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \left[-\frac{i}{2}\left(e^{i(2\pi k_0 m + 2\pi l_0 n)} - e^{-i(2\pi k_0 m + 2\pi l_0 n)}\right)\right] e^{-i2\pi(\frac{mk}{M} + \frac{nl}{N})} \\
&= \frac{i}{2} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \left[e^{-i2\pi(k_0 m + l_0 n) - i2\pi(\frac{mk}{M} + \frac{nl}{N})} - e^{i2\pi(k_0 n + l_0 n) - i2\pi(\frac{mk}{M} + \frac{nl}{N})}\right] \\
&= \frac{i}{2}\left\{\sum_{m=0}^{M-1} e^{-i2\pi m(k_0 + \frac{k}{M})} \sum_{n=0}^{N-1} e^{-i2\pi n(l_0 + \frac{l}{N})} - \sum_{m=0}^{M-1} e^{i2\pi m(k_0 - \frac{k}{M})} \sum_{n=0}^{N-1} e^{i2\pi n(l_0 - \frac{l}{N})}\right\}
\end{aligned}
$$

if $Mk_0 + k = 0$, $\sum_{m=0}^{M-1} e^{-i2\pi m(k_0 + \frac{k}{M})} = \sum_{m=0}^{M-1} 1 = M$, else,

$$
\begin{aligned}
\sum_{m=0}^{M-1} e^{-i2\pi m(k_0 + \frac{k}{M})} &= \sum_{m=0}^{M-1}\left[e^{-i2\pi(k_0 + \frac{k}{M})}\right]^m \\
&= \frac{1 - \left(e^{-i2\pi(k_0 + \frac{k}{M})}\right)^M}{1 - e^{e^{-i2\pi(k_0 + \frac{k}{M})}}} \\
&= \frac{1 - e^{-i2\pi(Mk_0 + k)}}{1 - e^{e^{-i2\pi(k_0 + \frac{k}{M})}}} \\
&= \frac{1 - (\cos(2\pi(Mk_0 + k)) - i\sin(2\pi(Mk_0 + k)))}{1 - e^{e^{-i2\pi(k_0 + \frac{k}{M})}}} \\
&= 0
\end{aligned}
$$

Hence, we can write $\sum_{m=0}^{M-1} e^{-i2\pi m(k_0 + \frac{k}{M})} = M\delta(k + Mk_0)$, $\sum_{n=0}^{N-1} e^{-i2\pi n(l_0 + \frac{l}{N})} = N\delta(l + Nl_0)$, $\sum_{m=0}^{M-1} e^{i2\pi m(k_0 - \frac{k}{M})} = M\delta(k - Mk_0)$, and $\sum_{n=0}^{N-1} e^{i2\pi n(l_0 - \frac{l}{N})} = N\delta(l - Nl_0)$. Then, we have:

$$S_{k\ell} = \frac{iMN}{2}\left[\delta(k + Mk_0, \ell + N\ell_0) - \delta(k - Mk_0, \ell - N\ell_0)\right]$$

**Problem 3.9** (Harris corner detector[1]). In class we introduced the Harris corner detector:

$$E_{(i,j)}(u, v) = \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} w(k - i, \ell - j)\big[I(k, \ell) - I(k - u, \ell - v)\big]^2,$$

which measures the change of the image $I$ around the location $(i, j)$ due to a small displacement $(u, v)$. Typically, $w$ is a $(2K + 1) \times (2K + 1)$ filter of ones centered around the origin, but is could also be a rotationally invariant Gaussian mask. In class we showed[2] that we can approximate $(\star)$ with

$$E_{(i,j)}(u, v) = \begin{bmatrix} u & v \end{bmatrix} \cdot \overline{M}(i, j) \cdot \begin{bmatrix} u \\ v \end{bmatrix},$$

where
$$\overline{M}(i, j) = \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} w(k - i, \ell - j) \begin{bmatrix} I_x^2(k, \ell) & I_x(k, \ell)I_y(k, \ell) \\ I_x(k, \ell)I_y(k, \ell) & I_y^2(k, \ell) \end{bmatrix}$$

is called the *local structure matrix*. We saw that we have evidence of the existence of a corner at $(i, j)$ when both eigenvalues $\lambda_1(i, j)$ and $\lambda_2(i, j)$ of $\overline{M}(i, j)$ are large, and of comparable magnitude. This happens when the so-called *corner response function* $\boxed{Q_\alpha(i, j) = \det\big(\overline{M}(i, j)\big) - \alpha\big[\mathrm{Tr}\big(\overline{M}(i, j)\big)\big]^2}$

is largest (typically the parameter $\alpha$ is chosen somewhere in the range between 0.04 and 0.15; this is discussed at the very end of the class notes).

**(a)** Choose $\alpha = 0.10$, and produce a contour plot of the corner response function with respect the eigenvalues (i.e., of $Q = \lambda_1\lambda_2 - \alpha(\lambda_1 + \lambda_2)^2$). You should get a plot similar to Fig. 5 of Harris' paper[1]. (Feel free to use *Mathematica* , as it has one-line commands that allow you to achieve this.)

**(b)** Now download the images `image-polygons.gif`, `image-house.gif`, *and* other images of your own choice. Write code that computes the function $Q_\alpha(i, j)$, and plot, on top the image, markers (e.g. red crosses or bullets) that correspond to points within the image where $Q_\alpha$ is above a threshold $t$ of your choice. *Remark:* There are several paramenters that you can play with. For my own code, I have used a $9 \times 9$ filter $w$ of ones, $\alpha = 0.05$, and a threshold for $Q_\alpha$ of $t = 10^7$.

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[1]C. Harris and M. Stephens: A Combined Corner and Edge Detector. In *Proceedings of the Fourth Alvey Vision Conference*, pages 147–151. University of Manchester, August 31—September 2, 1988.

[2]The fact that $(u, v)$ is small allowed us to use a first-order Taylor expansion for the term in square brackets in $(\star)$.