Solutions to Pathria's Statistical Mechanics
Chapter 3

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Problem 3.1

In fact the solution to this problem is just a mathematical derivation with only little physics.

(a)

$$\mathcal{LHS} = \langle (\Delta n_r)^2 \rangle \tag{1}$$
$$= \langle n_r^2 \rangle + \langle n_r \rangle^2 \tag{2}$$
$$= \frac{1}{\Gamma}\left(\omega_r \frac{\partial}{\partial \omega_r}\right)^2 \Gamma\bigg|_{\omega_r=1,\forall r} - \left(\omega_r \frac{\partial}{\partial \omega_r}(\ln \Gamma)\right)^2\bigg|_{\omega_r=1,\forall r} \tag{3}$$
$$= \frac{1}{\Gamma}\left(\omega_r \frac{\partial}{\partial \omega_r} + \omega_r^2 \frac{\partial^2}{\partial \omega_r^2}\right)\Gamma\bigg|_{\omega_r=1,\forall r} - \left(\frac{1}{\Gamma}\omega_r \frac{\partial}{\partial \omega_r}\Gamma\right)^2\bigg|_{\omega_r=1,\forall r} \tag{4}$$
$$\mathcal{RHS} = \left(\omega_r \frac{\partial}{\partial \omega_r}\right)^2 (\ln \Gamma)\bigg|_{\omega_r=1,\forall r} \tag{5}$$
$$= \frac{1}{\Gamma}\omega_r \frac{\partial}{\partial \omega_r}\Gamma\bigg|_{\omega_r=1,\forall r} - \left(\frac{1}{\Gamma}\omega_r \frac{\partial}{\partial \omega_r}\Gamma\right)^2\bigg|_{\omega_r=1,\forall r} + \frac{1}{\Gamma}\omega_r^2 \frac{\partial^2}{\partial \omega_r^2}\Gamma\bigg|_{\omega_r=1,\forall r} \tag{6}$$
$$= \mathcal{LHS} \tag{7}$$

(b-1)

$$U = \frac{\sum_r \omega_r E_r \exp(-\beta E_r)}{\sum_r \omega_r \exp(-\beta E_r)} \tag{8}$$
$$\Rightarrow \sum_r \omega_r (E_r - U) \exp(-\beta E_r) = 0 \tag{9}$$
$$\Rightarrow (E_r - U)\exp(-\beta E_r) - \sum_r \omega_r (E_r - U) E_r \exp(-\beta E_r) \frac{\partial \beta}{\partial \omega_r} = 0 \tag{10}$$
$$\Rightarrow \frac{\partial \beta}{\partial \omega_r} = \frac{(E_r - U)\exp(-\beta E_r)}{\sum_r \omega_r (E_r - U) E_r \exp(-\beta E_r)} \tag{11}$$

$$\mathcal{LHS} = \frac{\partial \beta}{\partial \omega_r} = \frac{(E_r - U)\exp(-\beta E_r)}{\sum_r \omega_r (E_r - U) E_r \exp(-\beta E_r)} \tag{12}$$

$$= \frac{(E_r - U)\exp(-\beta E_r)/\sum_r \omega_r \exp(-\beta E_r)}{\sum_r \omega_r (E_r - U) E_r \exp(-\beta E_r)/\sum_r \omega_r \exp(-\beta E_r)} \tag{13}$$

$$= \frac{E_r - U}{\langle E_r^2 \rangle - \langle E_r \rangle U} \frac{\langle n_r \rangle}{\mathcal{N}} \tag{14}$$

$$= \frac{E_r - U}{\langle E_r^2 \rangle - U^2} \frac{\langle n_r \rangle}{\mathcal{N}} = \mathcal{RHS} \tag{15}$$

(b-2)

$$\frac{\left\langle (\Delta n_r)^2 \right\rangle}{\mathcal{N}} = \omega_r \frac{\partial}{\partial \omega_r} \left[\frac{\omega_r \exp(-\beta E_r)}{\sum_r \omega_r \exp(-\beta E_r)} \right] \tag{16}$$

$$= \frac{\omega_r \exp(-\beta E_r)}{\sum_r \omega_r \exp(-\beta E_r)} - \frac{\omega_r^2 E_r \exp(-\beta E_r)}{\sum_r \omega_r \exp(-\beta E_r)} \frac{\partial \beta}{\partial \omega_r} \tag{17}$$

$$- \frac{\omega_r^2 \left(\exp(-\beta E_r)\right)^2 - \omega_r^2 \exp(-\beta E_r) \sum_r \omega_r E_r \exp(-\beta E_r)}{\left(\sum_r \omega_r \exp(-\beta E_r)\right)^2} \frac{\partial \beta}{\partial \omega_r} \tag{18}$$

$$= \frac{\langle n_r \rangle}{\mathcal{N}} - \frac{\langle n_r \rangle}{\mathcal{N}} E_r \frac{\partial \beta}{\partial \omega_r} - \left(\frac{\langle n_r \rangle}{\mathcal{N}} \right)^2 + \frac{\langle n_r \rangle}{\mathcal{N}} U \frac{\partial \beta}{\partial \omega_r} \tag{19}$$

$$= \frac{\langle n_r \rangle}{\mathcal{N}} + \frac{\langle n_r \rangle}{\mathcal{N}} (U - E_r) \frac{\partial \beta}{\partial \omega_r} - \left(\frac{\langle n_r \rangle}{\mathcal{N}} \right)^2 \tag{20}$$

Problem 3.2

$$g\prime\prime(x_0) \simeq \frac{f\prime\prime(x_0)}{f(x_0)} - \frac{U^2 - U}{x_0^2} \tag{21}$$

$$= \frac{\sum \omega_r E_r (E_r - 1) x_0^{E_r}}{x_0^2 \sum \omega_r x_0^{E_r}} - \frac{U^2 - U}{x_0^2} \tag{22}$$

$$= \frac{\left\langle E_r^2 \right\rangle - \langle E_r \rangle}{x_0^2} - \frac{U^2 - U}{x_0^2} \tag{23}$$

$$= \frac{\left\langle E_r^2 \right\rangle - U^2}{x_0^2} \tag{24}$$

$$= \frac{\left(\langle E_r \rangle - U\right)^2}{x_0^2} \tag{25}$$

Problem 3.3

$$\exp(x) = \sum \frac{1}{n!} x^n \tag{26}$$

$$\frac{1}{n!} = \frac{1}{2\pi\imath} \oint \frac{\exp(z)}{z^{n+1}} dz \tag{27}$$

$$\text{Define: } g(z) \equiv \ln\left(\frac{\exp(z)}{z^{n+1}}\right) \equiv \ln(F(z)) \tag{28}$$

$$g(z) = z - (n+1)\ln z \tag{29}$$

For $F(z)$, the saddle point is defined as $F\prime(x_0) = 0$, which gives $x_0 = n+1$. Notice that $z = x_0$ is also the saddle point for $g(z)$. Expanding $g(z)$ about the point $z = x_0$, along the line $z = x_0 + iy$, we get:

$$g(z) = g(x_0) - \frac{1}{2} g\prime\prime(x_0) y^2 + ... \tag{30}$$

Thus, the integrand, along the line $z = x_0 + iy$, will become:

$$F(z) = \frac{\exp(x_0)}{x_0^{n+1}} \exp\left[-\frac{1}{2} g\prime\prime(x_0) y^2\right] \tag{31}$$

$$\begin{aligned}
\frac{1}{n!} &= \frac{1}{2\pi\imath} \oint \frac{\exp(z)}{z^{n+1}} dz && (32)\\
&\simeq \frac{1}{2\pi\imath} \frac{\exp(x_0)}{x_0^{n+1}} \int_{-\infty}^{+\infty} \exp\left[-\frac{1}{2} g\prime\prime(x_0) y^2\right] \imath dy && (33)\\
&= \frac{\exp(n+1)}{(n+1)^{n+1}} \frac{1}{[2\pi g\prime\prime(x_0)]^{1/2}} && (34)\\
&= \frac{\exp(n+1)}{(n+1)^{n+1}} \left(\frac{n+1}{2\pi}\right)^{1/2} && (35)
\end{aligned}$$

Do a simple calculation and replace $(n+1)$ wit$n$, we get:

$$n! \simeq \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \tag{36}$$

which is just the original form of Stirling formula for $n!$.

Problem 3.4

$$\begin{aligned}
\mathcal{LHS} &= (k/\mathcal{N}) \ln \Gamma && (37)\\
&= (k/\mathcal{N}) \ln \sum W_{n_r} && (38)\\
&= (k/\mathcal{N}) \ln \sum \frac{\mathcal{N}!}{\Pi(n_r!)} && (39)
\end{aligned}$$

When $\mathcal{N}$ is extremely a huge number, only the maximal set n_r^* will make a difference. Thus:

$$\begin{aligned}
\sum \frac{\mathcal{N}!}{\Pi(n_r!)} &= \frac{\mathcal{N}!}{\Pi(n_r!)} && (40)\\
&= \frac{\mathcal{N}!}{\Pi(\langle n_r \rangle !)} && (41)
\end{aligned}$$

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\begin{aligned}
\mathcal{LHS} &= (k/\mathcal{N}!) \ln \sum \frac{\mathcal{N}}{\Pi(n_r!)} && (42)\\
&= (k/\mathcal{N}) \ln \frac{\mathcal{N}!}{\Pi(\langle n_r\rangle!)} && (43)\\
&= (k/\mathcal{N}) \left(\mathcal{N} \ln \mathcal{N} - \sum \langle n_r\rangle \ln\langle n_r\rangle\right) && (44)\\
&= (k/\mathcal{N}) \left(\sum \langle n_r\rangle \ln \mathcal{N} - \sum \langle n_r\rangle \ln\langle n_r\rangle\right) && (45)\\
&= (k/\mathcal{N}) \sum \left(\langle n_r\rangle \left(\ln \mathcal{N} - \ln\langle n_r\rangle\right)\right) && (46)\\
&= -k \sum \frac{\langle n_r\rangle}{\mathcal{N}} \ln \frac{\langle n_r\rangle}{\mathcal{N}} && (47)\\
&= -k\langle \ln Pr\rangle && (48)\\
&= S = \mathcal{RHS} && (49)
\end{aligned}
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