# Algorithms. A Collection of Problems

November 1, 2014

★ Trivial.

★★ Easy. Directly application of the textbook result.

★★★ Medium. One observation or one easy trick should do the work.

★★★★ Difficult. May require one clever trick or even more than one tricks.

★★★★★ Difficult. May require more than one clever tricks.

1. ★ (Longest path in a DAG) We are given a directed acyclic graph $G$ and two specific vertices $s$ and $t$ in $G$. Each edge in the graph has a length. Design a polynomial time algorithm that finds the longest path from $s$ to $t$. (if there is no path from $s$ to $t$, your algorithm should be able to detect the fact.)

2. ★★ (Finding the maximum area polygon) We are given a unit circle and $n$ points on the circle. Design a polynomial time algorithm that, given a number $m < n$, finds $m$ points (out of $n$ points) such that the area of the polygon formed by the $m$ points is maximized.

3. ★ (Longest palindrome subsequence) A palindrome is a nonempty string over some alphabet that reads the same forward and backward. For example, $aaaabaaaa$, 00000, $abcddcba$ are all palindrome. Give an efficient algorithm to find the longest palindrome that is a subsequence of a given input string. Your algorithm should run in time better than $O(n^3)$.

4. ★★ (Matrix-chain multiplication) We state the matrix-chain multiplication problem as follows: given a chain of matrices $A_1A_2 \dots A_n$ where $A_i$ has dimension $p_{i-1} \times p_i$. (Note that the number of columns of $A_i$ must be the same as the number of rows of $A_{i+1}$.) Assume multiplying a $x \times y$ matrix with a $y \times z$ matrix takes $xyz$ scalar multiplications. Design a polynomial time to find a fully parenthesization of the product $A_1 \dots A_n$ in a way that minimizes the number of scalar multiplications.

   For example, consider $A_1A_2A_3$ of three matrices with the dimensions of $10 \times 100, 100 \times 5, and 5 \times 50$, respectively. If we multiply according to the parenthesization $((A_1A_2)A_3)$, we perform $5000 + 2500 = 7500$ scalar multiplications. However, if instead we multiply according to the $A_1(A_2A_3)$, we perform $25,000 + 50,000 = 75,000$ scalar multiplications. Clearly, the first one is much better.

5. ★★ (Viterbi algorithm) We can use dynamic programming on a directed graph $G(V; E)$ for speech recognition. Each edge $(u, v) \in E$ is labeled with a sound from a finite set $\Sigma$ of sounds. The labeled graph is a formal model of a person speaking a restricted language. Each path in the graph starting from a distinguished vertex $v_0 \in V$ corresponds to a possible sequence of sounds produced by the model. We define the label of a directed path to be the concatenation of the labels of the edges on that path.

(a) Describe a polynomial time algorithm that, given an edge-labeled graph $G$ with distinguished vertex $v_0$ and a sequence $s = \langle \sigma_1, \ldots, \sigma_k \rangle$ of sounds from $\Sigma$, returns a path in G that begins at $v_0$ and has s as its label $s$, if any such path exists. Otherwise, the algorithm should return NO-SUCH-PATH.

(b) Now, suppose that every edge $(u, v)$ has an associated nonnegative probability $p(u, v)$ of traversing the edge. and thus producing the corresponding sound. The sum of the probabilities of the edges leaving any vertex equals 1. The probability of a path is defined to be the product of the probabilities of its edges. We can view the probability of a path beginning at $v_0$ as the probability that a random walk beginning at $v_0$ will follow the specified path, where we randomly choose which edge to take leaving a vertex $u$ according to the probabilities of the available edges leaving $u$. Extend your answer to part (a) so that if a path is returned, it is a most probable path starting at $v_0$ and having label $s$. Analyze the running time of your algorithm.

6. ★★ (Edit distance) In order to transform one string $x$ to a target string $y$, we can perform various edit operations. Our goal is, given $x$ and $y$, to produce a series of edits that change $x$ to $y$. We may choose from among edit operations:

(a) Insert a letter, (e.g., changing 100 to 1001 takes one insertion)

(b) Delete a letter, (e.g., changing 100 to 10 takes one deletion)

(c) Replace a letter by another (e.g., you need do one replacement to change 100 to 000).

Design an efficient algorithm that finds a series of edit operations that change $x$ to $y$ and the total number of edits is minimized.

7. ★★★★ (Four Russians Speedup) In this problem, we explore an interesting trick to speed up a dynamic program. We use the edit distance problem as an example (See the last problem). We assume the size of the alphabet is constant. Suppose the two strings are $S_1$ and $S_2$, both of length $n$. To start with, your solution for the last problem must run in $O(n^2)$ time. If it is so, in the dynamic program for the problem, you need to fill a two dimensional table $M$. In fact, we can fill out this table in a more clever way such that the running time can be improved to $O(n^2/\log n)$ (I know, it is a small improvement and of little practical interests for this particular problem. But the same trick has been used somewhere else to make a huge difference). In this trick, we need to a bit preprocessing to make many small tables. Then we fill only a subset of entries of the dynamic program table. The value of a table entry we are filling depends on the values of some entries we have already filled and the list we made in the beginning.

To make is a little bit more formal. We define a $t$-block to be a $t \times t$ squares in the dynamic programming table. Let $t = \frac{\log n}{4}$. We first observe that the distance values in a $t$-block starting in position $(i, j)$ are a function of the values of its first row and first colum and substrings $S_1[i, \ldots, i+t-1]$ and $S_2[i, \ldots, i+t-1]$.

Now, let us observe another interesting fact: In any row or column, the values of two adjacent cells differ by at most 1.

- Please prove this fact.

We say two $t$-blocks $B_1, B_2$ are offset-equivalent if there is a number $c$, such that $B_1[i, j] = B_2[i, j]+c$ for all $i, j$.

- There are $C$ types of $t$-blocks (up to offset-equivalence). Show how large $C$ is.

It is obvious we can fill a $t$-block in $O(t^2)$ times. Therefore, filling all $C$ types of blocks takes $O(Ct^2)$ times. These are the small tables we produce for preprocessing. How, we start to fill up the dynamic table $M$. We are going to fill up only $O(n^2/\log n)$ entries.

- We have given you enough hints. Now, it is up to you to develop the entire algorithm which should run in $O(n^2/\log n)$ time. In particular, you need to describe which entries of $M$ that need to be filled and how to compute their values.

8. ★★ Solve the following two recurrences:
   (a) $T(n) = 2T(n/2) + n \log n$.
   (b) $T(n) = 2T(\sqrt{n}) + \log n$.

9. ★★★ (Maximal Common Subsequence) String $C$ is a subsequence of string $A$ if $C$ can be obtained by deleting some letters from $A$. For example, $ade$ is a substring of $abcde$. We are given two string $A = a_1a_2\dots a_m$ and $B = b_1b_2\dots b_n$. Design an algorithm that finds a common subsequence of $A$ and $B$ using $O(mn)$ time and $O(m+n)$ space. (You will get half of the points if you can find a polynomial time algorithm, but the running time (and/or) the space are worse that the stated.)

10. ★ (Stick Game) There is a pile of $n$ sticks. Two players $A$ and $B$ take turns removing 1 or 4 sticks. $A$ starts first. The players who removes the last stick wins. Design a polynomial time algorithm (polynomial in $n$) that decides which player has a winning strategy (i.e., no matter how the opponent plays, the player can take certain moves to win the game.)

11. ★★★ (Monge Matrix) An $m \times n$ array $A$ of real number is a *Monge matrix* if

$$A[i,j] + A[k,l] \le A[i,l] + A[k,j], \forall i < k \text{ and } j < l.$$

Answer the following questions. Let $f(i)$ the column index of the leftmost minimum element of row $i$. For example:

$$\begin{pmatrix} 10 & 9 & 12 & 10 \\ 11 & 10 & 13 & 10 \\ 9 & 8 & 10 & 7 \\ 11 & 10 & 11 & 8 \end{pmatrix}$$

is a Monge Matrix. $f(1) = 2, f(2) = 10, f(3) = 4, f(4) = 4$.

   (a) Show that $f(1) \le f(2) \le \dots \le f(m)$.
   (b) Given an $m \times n$ Monge matrix, Design an algorithm that computes $f(1), \dots, f(m)$ in $O(m + n \log m)$ time.

Just a side note: In fact, there is an $O(m+n)$ time algorithm to compute the row minima for Monge Matrix. It is called the SMAWK algorithm (The initials of five authors).

12. ★★ (Unit tasks scheduling) We are given a set of unit-time tasks. Each task $i$ is supposed to finish by time $d_i$ ($d_i$ is an integer). Each task $i$ is also associated with a penalty $w_i$ if $i$ is not finished by time $d_i$. There is no penalty if we finish task $i$ by its deadline. Design a polynomial time that find a schedule that minimizes the total penalty.

13. ★★ (Coin changing) Suppose that the coins are in the denominations that are powers of $c$, i.e., $c^0, c^1, c^2, \ldots, c^k$, for some integers $c > 1, k \geq 1$. You are asked to change for $n$ cents using the fewest number of coins. Show that the greedy algorithm yields an optimal solution. (The greedy algorithm first tries the coin with the largest denomination , then the coin with the second largest denomination, and so on).

14. ★★ (Schedule to minimize completion time) You are given a set $S = \{a_1, a_2, \ldots, a_n\}$ tasks. Task $a_i$ is released at time $r_i$ (you can not start $a_i$ before $r_i$) and requires $p_i$ units of time to process. You machine can process one task at each time. Assume *preemption* is allowed, so that you can suspend a task and resume it at a later time. For a particular schedule $S$, we denote the completion time of task $a_i$ (the time when $a_i$ is finished) to be $c_S(a_i)$ Give a polynomial time algorithm that finds a schedule $S$ of all tasks and minimizes the total completion time $\sum_{i=1}^{n} c_S(a_i)$.

15. ★★★ We are given an unweighted undirected graph $G$. Let $M$ be a matching in $G$ that has no augmenting path of length smaller than $2t + 1$. Let $M^*$ be the maximum matching in $G$. Show that $|M| \geq \frac{t}{t+1}|M^*|$.

16. ★★★ We are given $n$ jobs and one server. Each job $j$ is associated with a profit $p_j$, a release time $r_j$ and completion time $c_j$. If we decide to schedule job $j$, the server has to process it continuously from time $r_j$ to $c_j$ and we can get a profit $p_j$. No partial profit can be obtained if the job is not finished. The server can process at most $k$ jobs at any time. Design a polynomial time algorithm that finds a feasible schedule such that the total profit we can get is maximized.

17. ★★★ Given an undirected graph $G(V, E)$, a feedback set is a set $X \subseteq V$ with the property that $G - X$ has no cycles. The undirected feedback set problem asks whether $G$ contains a feedback set of size at most $k$. Show that the problem is NP-complete.

18. ★★ (Rearrangeable Matrix) Let $M = \{m_{ij}\}_{1 \leq i,j \leq n}$ be an $n \times n$ $0/1$ matrix. We say $M$ is rearrangeable if there exist a permutation $\pi$ of rows and a permutation $\tau$ of columns such that the matrix $M' = \{m'_{ij}\}_{1 \leq i,j \leq n}$ obtained by permuting the rows of $M$ according to $\pi$ and permuting the columns of $M$ according to $\tau$ has all its diagonal entries (i.e., $m'_{ii}$ for $1 \leq i \leq n$) equal to 1.

    (a) Give an example of a matrix $M$ that is not rearrangeable, but for which at least one entry in each row and each column is equal to 1.(3 pts)

    (b) Give a polynomial time algorithm that determines whether a matrix is rearrangeable. (7 pts)

19. ★★★ (Turán's bound) We are given a graph $G(V, E)$. The degree of a vertex $v$, denoted by $deg(v)$, is defined to be the number of edges incident on $v$. A subset $S$ of vertices is called *an independent set* if there is no edge connecting any two vertices in $S$. Let $\alpha(G)$ be the size (i.e., number of vertices) of the maximum independent set in graph $G$. Prove the following inequality:

$$\alpha(G) \geq \sum_{v \in V} \frac{1}{deg(v) + 1}.$$

    (Hint: Consider the greedy algorithm which repeatedly adds the vertex with lowest degree and delete all its neighbors.)

20. ★★★ (Multiple Interval Scheduling) In this problem, we have a single processor which can only work on one job at any single point in time. We are also given a set of jobs, each of them requiring a

set of disjoint intervals of time during which it needs to be processed on the processor. For example, a job could require the processor form 10am to 11am and 1pm to 2pm. We would like to answer the following question: For a set of jobs and a given number $k$, is it possible to process at least $k$ jobs in the processor. Show that the problem is NP-complete. (Hint: Maximum Independent Set is a possible candidate for the reduction.)

21. ★★★ You have $m$ slow machines and $k$ fast machines. You are given a set of $n$ jobs. Job $i$ takes time $t_i$ to process on a slow machine and time $t_i/2$ to process on a fast machine. Assign each job to a machine such that the makespan is minimized. (The makespan is the maximum load of any machine.) Design a polynomial time approximation algorithm that produces an assignment with a makespan at most 3 times the optimum.

22. ★★ (Densest $k$-Subgraph) We are given an undirected graph $G(V, E)$. The densest $k$-subgraph problem asks for a subset $S$ of $k$ vertices such that the number of edges in the induced graph $G[S]$ is maximized. Formulate the decision version of the problem and show it is NP-Complete.

23. ★★★ (maximum coverage) We are given a ground set $U$ of $n$ element and a collection of subsets $S_1, \ldots, S_m$. The goal is to choose $k$ subsets and maximize the cardinality of their union.

    (a) Formulate the decision version of the problem and show it is NP-Complete.

    (b) Design a polynomial time approximation algorithm for the problem. Your algorithm should have an approximation ratio of $1 - \frac{1}{e}$ in order to get full credit.

24. ★★★ (Polynomial Multiplication) You are given $2n$ numbers $a_1, b_1, \ldots, a_n, b_n$. Consider the polynomial $P(x) = \prod_{i=1}^{n}(a_i x + b_i)$. Design an algorithm that computes the expansion of $P(x)$. In order to get full credit, your algorithm should run in time $o(n^2)$ (Note: small $o$).

25. ★★ (Longest increasing subsequence) You are given a sequence $S$ of $n$ numbers $a_1, a_2, \ldots, a_n$. A subsequence $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ is increasing if $i_1 < i_2 < \ldots < i_k$ and $a_{i_1} < a_{i_2}, \ldots < a_{i_k}$. For example, $S = \{2, 10, 7, 11, 8, 6, 12\}$ is the given sequence and $\{2, 10, 11, 12\}$ is an increasing subsequence. Design a polynomial time algorithm that finds an longest increasing subsequence in the given sequence $S$.

26. There is a round table divided into 4 equal quadrants, with one cup in each quadrant. The quadrants are labeled with letters (A, B, C, D) that do not move. Initially, each cup is randomly face-up or face-down. You are blindfolded and put in front of the table. On each turn of the game, you instruct a genie to flip the cups in whichever positions you choose (e.g., you may say ”flip the cups in A and B”), possibly choosing no cups or possibly all four. The genie complies. At this point, if all the cups on the table are face-up, the genie will tell you that you have won the game and are free to go. If not, he rotates the cups randomly (possibly not rotating them) and you play another turn.

    (a) ★★ Give a strategy to win this game in a finite number of moves (the solution is not unique).

        Remark: the outcome of ”rotation” the four cups is one of the four possible positions: the cups originally at (A,B,C,D) can be at (A,B,C,D), (B,C,D,A), (C,D,A,B), or (D,A,B,C). It is not an arbitrary permutation.

    (b) ★★★★ Show that there is a strategy that guarantees a win if the number of cups $N$ is a power of 2.

(c) ★★★★ Show that there is no strategy that guarantees a win if the number of cups $N$ is not a power of 2. (You can get partial credits if you can give a proof for the case where $N$ is odd and larger than 3.)

27. ★★★★★ This problem is a generalization of the above problem. There are fixed positive integers $N$ and $K$, each at least 2. We have an unknown vector $V$ of length $N$ of integers between 0 and $K-1$. During each move, we propose another vector $W$ of length $N$, of integers between 0 and $K-1$. A genie selects a secret integer $J$ between 0 and $N-1$ and rotates $W$ by $J$ positions: $\mathsf{Rot}(W, J)$ is the last $N-J$ entries of $W$, followed by the first $J$ entries of $W$. The genie then replaces $V$ by $(V + \mathsf{Rot}(W, J))(\text{mod } K)$. If the new value of $V$ is all 0, we win; otherwise we continue to play. For what pairs of integers $(N, K)$ is there a fixed finite sequence of moves (a sequence of vectors $W$) such that if we apply this sequence in order, we are guaranteed to win by the time the sequence is finished? Prove that these pairs admit such a sequence, and that no other pairs do.

28. ★★★ In this problem, we are going to work with a very popular class of objects in computer science - graphs (图). A graph consists a set $V$ of vertices (点) and a set $E$ of edges (边). An edge connects two vertices. The degree of a vertex $v$, denoted by $deg(v)$, is defined to be the number of edges incident on $v$. Now, we consider a very classic problem in computer science - the traveling salesman problem. A path is a set of edges of the form $\{(v_1, v_2), (v_2, v_3), (v_3, v_4), \ldots, (v_{k-1}, v_k)\}$ and a cycle is a path whose head and tail are the same vertex. A Hamitonian cycle is a cycle that goes through all vertices exactly once. See Figure 2 for an example. Prove the following statement: Every graph with $n \geq 3$ vertices and minimum degree at least $\lceil n/2 \rceil$ has a Hamitonian cycle. (Hint: consider the longest path.)

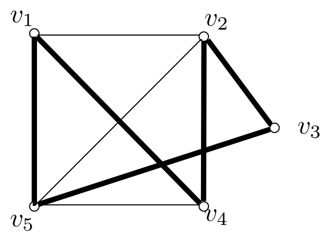
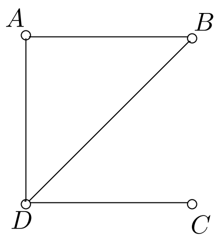


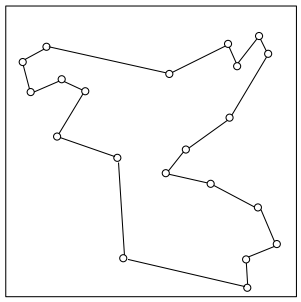
Figure 1: (i) A graph $G$. The vertices are $v_1, v_2, v_3, v_4, v_5$. There are 8 edges. The degree of $v_1$ is 3 and the degree of $v_2$ is 4. $\{(v_1, v_4), (v_4, v_2), (v_2, v_3), (v_3, v_5), (v_5, v_1)\}$ is Hamitonian cycle (indicated by the thicker edges).

29. ★★ Catch a car without knowing anything. We have an infinite road (i.e., $(-\infty, +\infty)$) and a car whose initial position $p$ (at time 0) is some unknown integer. The car is running at a fixed velocity $v$. $v$ is some unknown but fixed integer. Note that $v$ can be negative which means the car is heading towards $-\infty$. At each time step, you can make at most one query of the following form: Is the car at position $x$ now? (you can choose the integer $x$). Design a query strategy such that you can guarantee you will get a "yes" answer in finite time. (HINT: It is easy to see that the position of the car at time $t$ is $p + vt$.)

30. ★★★ Turán's bound. In this problem, we are going to work with a very popular class of objects in computer science - graphs (图). A graph consists a set $V$ of vertices (点) and a set $E$ of edges (边). An edge connects two vertices. The degree of a vertex $v$, denoted by $deg(v)$, is defined to be



(i)



(ii)

Figure 2: (i) A graph $G$. The vertices are $A, B, C, D$. The edges are $AB, AD, BD, DC$. The degree of $A$ is 2 and the degree of $D$ is 3. The set {A,C} is an independent set of size two, which also happens to be the maximum one. So, $\alpha(G) = 2$. But $\{A, B, C\}$ is not an independent set, due to the existence of edge $AB$. (ii) A traveling salesman tour.

the number of edges incident on $v$. See Figure 2(i) for an example. A subset $S$ of vertices is called *an independent set (*独立集*)* if there is no edge connecting any two vertices in $S$. Let $\alpha(G)$ be the size (i.e., number of vertices) of the maximum independent set in graph $G$. Prove the following inequality:

$$\alpha(G) \geq \sum_{v \in V} \frac{1}{deg(v) + 1}.$$

31. ★★★★ Traveling salesman in the unit square. In this problem, we are considering a very classic problem in computer science - the traveling salesman problem. We are given a set of $n$ points in the unit square ($[0, 1] \times [0, 1]$). For two points $v, u$, we use $|uv|$ to denote the Euclidean distance between $u$ and $v$. A traveling salesman tour is a tour that starts at some starting point, goes through all points exactly once and ends at the starting point. See Figure 2(ii) for an example.

    a. Prove that there is a tour $\{v_1, v_2, \ldots, v_n, v_1\}$ (i.e., the tour visits $v_1$ first, then $v_2$ and so on) such that

    $$|v_1v_2|^2 + |v_2v_3|^2 + \ldots + |v_{n-1}v_n|^2 + |v_nv_1|^2 \leq 4$$

    .

    b. Prove that there is a tour $\{v_1, v_2, \ldots, v_n, v_1\}$ such that

    $$|v_1v_2| + |v_2v_3| + \ldots + |v_{n-1}v_n| + |v_nv_1| \leq 2\sqrt{n}$$

32. (a) ★ Show that a graph with maximum degree $\Delta$ can be colored in $\Delta + 1$ colors.

    (b) ★ We are given a 3-colorable graph $G(V, E)$. In other words, it is possible to color the vertices of $G$ using 3 colors such that no edge is incident on two vertices with the same color. Show that 2-coloring the neighbors of any vertex (if possible) can be done in polynomial time.

    (c) ★★★★ We are given a 3-colorable graph $G(V, E)$. Design a polynomial time algorithm that finds an $O(\sqrt{|V|})$-coloring of $G$.

33. ★★★★★ You have a graph $G$. Each edge $e$ has two positive costs $A_e$ and $B_e$. Find a spanning tree $T$ that minimizes $(\sum_{e \in T} A_e) \times (\sum_{e \in T} B_e)$. You algorithm should run in polynomial time.

34. ★★★ Initially, we have $n$ empty bins. In each round, we throw a ball into a uniformly random chosen bin. Let $T$ be number of rounds needed such that no bin is empty. Show that $\mathbb{E}[T] = nH_n$ where $H_n = \sum_{i=1}^{n} \frac{1}{n}$.

35. ★★★ Show that finding a min-cost matching with exactly $k$ edges in a bipartite graph can be solved in polynomial time.

36. ★★★ A vertex cover is a set of vertices such that each edge is incident on at least one vertex in the set. An independent set is a set of vertices such that no edge joins two vertices in the set (i.e., no two vertices in the set are adjacent.) Prove the following statements:

    (a) In any bipartite graph, the number of edges in a maximum matching equals the number of vertices in a minimum vertex cover.

    (b) In any bipartite graph, the number of vertices in a maximum independent set equals the total number of vertices minus the number of edges in a maximum matching.

37. ★★★★ Given an undirected edge weighted graph $G(V, E)$, design a polynomial time algorithm that finds a minimum weight subset $E'$ of edges such that each vertex is incident on at least one edges in $E'$.

38. ★★★★ A cycle cover of an undirected graph is a collection of vertex-disjoint cycles such that each vertex belongs to exactly one cycle. We require each cycle contains at least three vertices. In an edge weighted graph, the weight of a cycle cover is the sum of the weights of all its edges. Given an undirected edge weighted graph $G(V, E)$, design a polynomial time algorithm that finds a cycle cover with the minimum total weight.

39. ★★★A Wolf, a Goat, and a Cabbage

    (a) A man finds himself on a riverbank with a wolf, a goat, and a head of cabbage. He needs to transport all three to the other side of the river in his boat. However, the boat has room for only the man himself and one other item (either the wolf, the goat, or the cabbage). In his absence, the wolf would eat the goat, and the goat would eat the cabbage. Show how the man can get all these "passengers" to the other side.

    (b) We consider a generalization of the above problem. We are given $n$ objects on a riverbank. We are also given a set of pairs of objects. Each pair of objects means that the two objects cannot stay on either side of the river together without the man's supervision. For example, in the above problem, the set of pairs is (Wolf, Goat), (Goat, Cabbage). Now, your boat can hold the man and at most $k$ other objects where the input $k$ is smaller than $n$. Design an algorithm to decide whether it is possible for the man to get all objects to the other side of the river. What is the running time of your algorithm?

40. ★★★Without the help of a computer or calculator, find the total sum of the digits in all integers from 1 to a million, inclusive. Write down the computation details.

41. ★★★(Rumor Spreading) There are $n$ people, each in possession of a different rumor. They want to share all the rumors through a series of bilateral conversations (e.g., via a telephone). Devise an efficient (in terms of the total number of conversations) algorithm for this task. Assume that in every conversation both parties exchange all the rumors they know at the time. (You can assume $n$ is a power of 2 first.)

42. ★★★We are given a sequence of distinct numbers $A_1, A_2, A_3, ,, A_n$. For each number, its position in the sequence and its position in the sorted sequence (in increasing order) differ by at most $k$, where $k$ is much smaller than $n$. Design an algorithm that sort the sequence in time less than $O(n \log n)$. Of course, the running time of your algorithm should depend on $k$.

43. ★★★★(One Coin for Freedom) A jailer offers to free two imprisoned programmers-we call them A and B-if they manage to win the following guessing game. The jailer sets up an 8 by 8 board with one coin on each cell, some tails up and the others tails down. While B is absent, the jailer points out to A the board's cell the jailer has selected to be guessed by B. Prisoner A is required to turn over exactly one coin on the board before leaving the room. Then B enters and guesses the selected cell. A and B are allowed to plan their strategy beforehand, but there should be no communication between them after the game begins. Of course, B is allowed to see the board after he enters the room and even to perform any calculations he may wish to do. Can the prisoners win their freedom or is the game impossible to win?

44. ★★★(Discrepancy) We are given a set $U$ of $n$ elements and a family $F$ of $n$ subsets of $U$. Each subset in $F$ consist of $n/2$ elements. We need an polynomial time algorithm to color each element in $U$ into two colors, red and blue, so that for any subset $S$ in $F$,

$$|R(S) - B(S)| = o(n).$$

Here $R(S)$ is the number of red elements in $S$ and $B(S)$ is the number of blue elements in $S$. Note we have small $o()$ notation here. If your algorithm is randomized, your algorithm should succeed with probability at least 0.999.

45. ★★★★You are given an array of $n$ distinct integers. Output the length of the longest interval of consecutive integers (if there are more than one longest interval, output any of them). For example, the array is $\{0, 1, 5, 3, 2, 6, 8, 11, 7\}$, the output should be $4$ (the longest interval is $\{5, 6, 7, 8\}$). You algorithm should run in $O(n)$ time. (Hint: You can assume there is a hash table and each insertion and lookup operation take $O(1)$ time. We give 0 point if your answer takes $O(n \log n)$ time or more).

46. ★★★An undirected Eularian graph is a connected graph in which all nodes have even degree. You job is to design an efficient algorithm that traverse an undirected Eularian graph so that each edge is visited exactly once.

47. ★★★A directed Eularian graph is a strongly connected graph in which the indegree of each node is equal to its outdegree. You job is to design an efficient algorithm that traverse a directed Eularian graph so that each edge is visited exactly once.

48. ★★★★(Chinese postman problem) We are given an edge-weighted undirected graph $G(V, E)$. Find a shortest tour that visit each edge at least once. (You can assume the graph is connected. Hint: For Eularian graphs (all nodes have even degree), there is a tour visiting each edge exactly once.)

49. ★★★★We are given a directed graph $G(V, E)$ and a positive integer $k$. You are allowed to delete at most $k$ edges from the graph and the goal is to minimize

$$\max_v |\text{out-degree}(v) - \text{in-degree}(v)|.$$

You algorithm should run in polynomial time. (Hint: Solve the Chinese postman problem first and think about flow.)

50. We have $5$ questions in this problem.

    (a) ★Your friend claims that she can distinguish most of the time (i.e. with some reasonable probability) between Coke and Pepsi by just taking a sip from a can that contains either Coke (可口可乐) or Pepsi （百事可乐）. You want to organize an experiment with which you can actually test (with high probability) whether she is honest. You have at your disposal enough Coke and Pepsi, and you also have (unbiased) coins in your pocket. Design an experiment that within a reasonable amount of time you can test whether her assertion is true. （Important remark: you tell your friend all the details of the experiment before the test starts.）

    (b) ★The Graph Isomorphism problem is given two graphs G1 and G2 decide whether G1 and G2 are isomorphic (同构) (we say G1 and G2 are isomorphic if we can obtain G2 by relabeling the vertices of G1 and vice versa). What is the best deterministic or randomized algorithm you can get for this problem?

    (c) ★★Suppose that in addition to your computational power you have access to an all-powerful prover to which (i) you can ask any question you want but (ii) you cannot trust whether he is telling the truth (like what happens in the first part of this question). How can you use such a prover in order to design an efficient algorithm (queries/answers to the Prover count as one time step) for testing isomorphism? (The prover's goal is to convince you that two graphs are isomorphic. However, if the prover lies, the algorithm should be able to find that out.)

    (d) ★★★★Suppose that in addition to your computational power you have access to an all-powerful prover to which (i) you can ask any question you want but (ii) you cannot trust whether he is telling the truth (like what happens in the first part of this question). How can you use such a prover in order to design an efficient randomized algorithm (queries/answers to the Prover count as one time step) for testing non-isomorphism? (The prover's goal is to convince you that two graphs are non-isomorphic. However, if the prover lies, the algorithm should be able to find that out.)

    (e) ★★★★★, In the above the prover cannot see the coins the randomized algorithm is flipping. Can you still use such a prover if the prover sees all the random choices made by your algorithm?

51. Consider the following interval packing problem we discussed in the class: given a list of intervals $[a_i, b_i]$, $i = 1, \dots, n$ with weights $c_1, \dots, c_n$ and an integer $k$, find a maximum weight subset of intervals such that no points is contained in more than $k$ of them.

    (a) find the LP formulation for the problem and argue why the LP captures the problem.

    (b) Write down the dual.

    (c) Try to solve the dual using simple combinatorial algorithm (i.e., without using general LP algorithm like simplex or Ellipsoid algorithm).

    (d) Give a simple combinatorial algorithm for the un-weigthted interval packing problem. (Hint: the algorithm is a simple greedy algorithm)

    Ref: Note on scheduling intervals on-line, Ulrich Faigle, Willem Nawijn.

52. ★★★★Let $X \subset \mathbb{R}^3$ be a compact polyhedral body (i.e., a finite union of convex polytope). Suppose for every plane $P$, the intersection $X \cap P$ is contractible (meaning it is simple polygon without holes). Show $X$ is convex.

53. ★★★★We are given a set $\mathcal{H}$ of halfplanes and set of points $P$ in $\mathbb{R}^2$. Each halfplane $h$ in $\mathcal{H}$ is associated with a positive weight $w(h) \in \mathbb{R}^+$. Our goal is to find a minimum weight subset $S \subseteq \mathcal{H}$ that covers all points in $P$, i.e., $P \subset \cup_{h \in S} h$ and $\sum_{h \in S} w(h)$ is minimized. Try to make your algorithm as fast as possible. (hint: use dynamic programming).

    REF: Weighted Geometric Set Cover Problem Revisited.

54. ★★★★We are given a set $\mathcal{H}$ of halfplanes and set of points $P$ in $\mathbb{R}^2$. Each point $p$ in $P$ is associated with a positive weight $w(p) \in \mathbb{R}^+$. Our goal is to find a minimum weight subset $S \subseteq P$ that all halfplanes in $\mathcal{H}$ are hitted, i.e., $S \cap h \neq \emptyset$ for all $h \in \mathcal{H}$ and $\sum_{p \in S} w(p)$ is minimized. Try to make your algorithm as fast as possible. (hint: use dynamic programming).

    REF: Weighted Geometric Set Cover Problem Revisited.

55. ★★★There is a set $P$ of points in $\mathbb{R}^2$ which lie in the region $B = [-\infty, +\infty] \times [0, 1]$. There are also a set $\mathcal{D} = \{D_1, \ldots, D_n\}$ of disks, all with radius 1. The center of each disk are not in $B$. The union of all disks covers all points in $P$, i.e., $P \subset \cup_{i=1}^{n} D_i$. The disk $D_i$ has a positive weight $w_i \in \mathbb{R}^+$. Design an efficient algorithm that find a subset of disks that covers all points and has minimum total weight. If all weights are the same, say 1, can you find a faster algorithm?

    REF: Constant-factor approximation for minimum-weight dominating sets in unit disk graphs.

56. ★★★★★Given a set of disks, find an $\epsilon$-net of size $O(1/\epsilon)$. (Sampling+ resampling as Clarkson, to be detailed).

57. ★★★Given a set $\mathcal{H}$ of halfspaces (in general positions) in $\mathbb{R}^d$, design an efficient algorithm that decides whether $\mathcal{H}$ can cover the whole space $\mathbb{R}^d$. If $\mathcal{H}$ can cover $\mathbb{R}^d$, show there is always a subset of $\mathcal{H}$ with at most $d+1$ halfspaces that also covers $\mathbb{R}^d$.

58. Let us consider the following shortest path game $\mathcal{G}$ between two players, red and blue. There is an directed edge weighted graph with a source node $s$ and a destination $t$. Suppose for now all edge weights are positive. Each node of the graph is controlled by either the red or blue player. Each directed edge is either from a red node to a blue node, or from a blue node to a red node. Initially, there is token at the source node $s$, which is controlled by the red player. If the token is at a red node (say $v$), the red player can choose an outgoing edge $(v, w)$, which sends the token to node $w$ (note $w$ should be a blue node). Similarly, if the token is at a blue node (say $b$), the blue player can choose an outgoing edge $(b, c)$, which sends the token to node $c$ (note $c$ should be a red node). The red player's goal is to minimize the total weight of path (the sum of the weights of all edges traversed) of reaching $t$ and the blue player's goal is to maximize the total weight. We use $\mathsf{val}(\mathcal{G})$ to denote the above total weight under the optimal strategies of both players. Note there are two possible outcomes of the game: (1) under the optimal strategies of both players, the game terminates in finite steps (reaching $t$), in which case the value of game $\mathsf{val}(\mathcal{G})$ is finite; (2) under the optimal strategies of both player, the game can go on forever, and the value of game $\mathsf{val}(\mathcal{G})$ is $+\infty$; Show the following:

    (a) ★★★Show that there is an optimal *positional strategy* for both the red and blue players. A positional strategy is simply to choose a specific outgoing edge for each node.

    (b) ★★★Show how to find the optimal strategy for this game in polynomial time.

    (c) ★★★★Suppose there are some edges with negative weights, but there is no negative cycles. Show how to find the optimal strategy for this game in polynomial time.

(d) ★★★★Suppose there are some edges with negative weights, and there may be some negative cycles. All edges weights are in $[-M, M]$. Try to find the optimal strategy in time $M\mathsf{poly}(n)$.

(e) ⧫ (a good research problem) Suppose there are some edges with negative weights, and there may be some negative cycles. All edges weights are in $[-M, M]$. Try to find the optimal strategy in time $o(M)\mathsf{poly}(n)$.

(f) ⧫ (for truly adventurous people) Suppose there are some edges with negative weights, and there may be some negative cycles. Show how to find the optimal strategy for this game in polynomial time.

59. There are $n$ players. Each player holds a private 0/1 bit. Player 1 is the coordinator. The coordinator wants to compute the parity (i.e., determine there are even number of 1s or odd number of 1s).

    (a) ★★★In each round, your protocol can choose a player and let the player to broadcast a bit (depending on your protocol, the bit can be either the player's own bit, or other encoded information). Upon the broadcast, each other player can receive the bit with probability $1 - \epsilon$ for some small constant $0 < \epsilon < 1/3$, and the flipped bit with probability $\epsilon$ (all flips are independent of each other). Show if every player broadcasts her bits for $k = O(\log n)$ times, the coordinator can compute the parity correctly with probability $0.9$.

    (b) ★★★★Show the number of broadcasts can be reduced to $O(n \log\log n)$ by designing a cleverer protocol (hint: divide the players into groups of size $O(\log n)$ each and try to learn the parity for each group). (sidenote: surprisingly, $\Theta(n \log\log n)$ is the optimal bound for the problem, i.e., we can not do better than that.)

    (c) ★★★★★Show that with $O(n \log\log n)$ broadcasts, the coordinator can even learn all bits correctly with probability 0.9. (hint: again first learn the parity of groups (we need different grouping method though), then use the parity of the groups to correct the errors.)

60. There are $n$ straight lines $\ell_1, \dots, \ell_n$ in $\mathbb{R}^2$ (assuming general positions).

    (a) ★★★Design an $O(n \log n)$ time algorithm for the following decision problem: Given two vertical slab $W$ bounded by two vertical line $x = a$ and $x = b$, compute how many intersection points of $\{\ell_i\}_{i\in[n]}$ are there in $W$.

    (b) ★★★Given a vertical slab, show how to uniformly sample $q$ intersection points in $W$ in $O(n \log n + q)$ time.

    (c) ★★★★Use the above result, show how to find the intersection point with the $k$th largest $x$-coordinate in $o(n^2)$ time (you can design a randomized algorithm that succeeds with high probability).

61. ★★★★We are given $n$ mutually disjoint disks in $\mathbb{R}^2$. Show that there is a rectangle $R$ such that

    (a) The number of disks inside $R$ is at least $\Omega(n)$;

    (b) The number of disks outside $R$ is at least $\Omega(n)$;

    (c) The number of disks intersecting the boundary of $R$ is at most $O(\sqrt{n})$.

    (Hint: First find a smallest square $S$ (with center $a$ and side length $\ell$) containing $n/10$ points. For $0 \le t \le 1$, let $S_t$ be the square with center $a$ and size length $(1+t)\ell$. Show there is a $S_t$ for some $t$

which meets our need. First consider two cases: (1) all disks has diameter $\geq \ell/\sqrt{n}$; (2) all disks has diameter $\leq \ell/\sqrt{n}$. )

(The above is a special case of the geometric separator theorem. The above result, combined with the Koebe's circle packing theorem, implies the well known Lipton-Tarjan planar separator theorem (see http://en.wikipedia.org/wiki/Planar_separator_theorem). For the circle packing theorem, see http://www.cs.yale.edu/homes/spielman/course/lect3.ps )

62. ★★★★In this problem, we aim to solve linear programming in $\mathbb{R}^2$ in linear time using the so-called prune-and-search technique. We are given a set $H = \{h_1, \dots, h_n\}$ of halfplanes in general positions. and a vector $c \in \mathbb{R}^2$. We wish to minimize $cx$ over the feasible region $F = \cup_{i=1}^{n} h_i$. Without loss of generality, we assume $c = (0, 1)$, i.e., we would like to find the lowest point in $F$. Let $\ell_i$ be the bounding line of $h_i$. We further assume for simplicity that each $h_i$ lies above its bounding line $\ell_i$. Please complete the details for the following algorithm.

    The prune-and-search technique works as follows: We first pair up all $\ell_i$s into disjoint pairs. Then we compute all intersection points $x_1, \dots, x_{n/2}$ of all pairs. Now, find the median $x_m$ of all $x_i$s in linear time. We can then decide in linear time whether $x^* = x_m$, or $x^* > x_m$, or $x^* < x_m$. Then, we can safely remove $n/4$ halfplanes that have no effect on the optimal solution (show we can remove 1 halfplane from $n/4$ pairs). Then solve the problem for the remaining halfplanes.

63. (★★★★$-$) In this problem, we aim to develop a PTAS for the maximum independent set problem in unit disk graph. Suppose we are given $n$ unit disks in $\mathbb{R}^2$. Our goal is to select a subset $S$ of mutually disjoint disks such that $|S|$ is maximized. The problem is NP-hard (we are not going to prove it in this problem). A PTAS (i.e, $1 - \epsilon$ approximation for any given $\epsilon > 0$) for this problem can be obtained as follows. Let $L$ be a positive integer (which may depend only on $\epsilon$). Suppose $0 \leq a, b \leq L$ are two integers. Consider the horizontal lines $h_i : y = a + iL$ for $i \in \mathbb{Z}$ the vertical lines $v_j : x = b + jL$ for $i \in \mathbb{Z}$. Delete all disks that intersect any $h_i$ or $v_j$. Then, solve the problem optimally for the resulting instance in polynomial time (hint: time $n^{f(L)}$ for some function $f$; if $L$ only depends on $\epsilon$, $n^{f(L)}$ would be a polynomial running time).

    A key to the analysis is to show that there is a choice of $a$ and $b$ (for appropriate value $L$) such that

    $$\text{optimal value for the resulting instance} \geq (1 - \epsilon)\mathsf{OPT}.$$

    Please complete the details of the above algorithm. (The above technique is called the *shifting* technique, which is a standard technique for many geometric politicization problems)

    ★★★★Modify the above technique and obtain a PTAS for the minimum dominating set problem in unit disk graphs.

64. ★★★We are given a finite set $S$ of rectangles and a rectangle $R$ in the plane. Is there a way of placing the rectangles of $S$ inside $R$, so that no pair of the rectangles intersect, and all the rectangles have their edges parallel of the edges of $R$? Show the problem is NPC.

65. ★★★★Prove that the disk cover problem is NP-Hard. In this problem, given a set $P$ of points in $\mathbb{R}^2$ and a radius $r$, one should find the smallest number of disks of radius $r$ that cover $P$.