2023 阿里巴巴全球数学竞赛

球状闪电

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$$v = ar + r^3 - r^5.$$

这里$r(t)$ 表示球状闪电的半径，而t 是时间变量。初始时刻，没有球状闪电，即$r(0) = 0$。相应地，我们也有$v(0) = 0$。而$a \in \mathbb{R}$ 可以被人为控制，您可以通过拉动一个控制杆来迅速的改变a 的值。我们给它的预设值是$a = -1$。"

你："做的漂亮，博士！a 是我们的唯一控制方式吗？这似乎并不能把球状闪电启动起来。"

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科学家："是的，长官。我们为此设计了四个控制方案。

请问长官您觉得这些方案如何？"

你看了一下这些选项，发现其中可行的方案有（）。

- A 设置$a = 2$, 踢一下仪器，等球状闪电半径严格超过$\sqrt{2}$，再设置$a = -\frac{1}{2}$；
- B 设置$a = 3$, 踢一下仪器，等球状闪电半径严格超过$\sqrt{2}$，再设置$a = -\frac{1}{3}$；
- C 设置$a = 4$, 踢一下仪器，等球状闪电半径严格超过$\sqrt{2}$，再设置$a = -\frac{1}{4}$；
- D 设置$a = 5$, 踢一下仪器，等球状闪电半径严格超过$\sqrt{2}$，再设置$a = -\frac{1}{5}$。

设两个凸八面体O_1, O_2的每个面都是三角形, 且O_1在O_2的**内部**. 记$O_1(O_2)$的棱长之**和**为$\ell_1(\ell_2)$. 当我们计算ℓ_1/ℓ_2时, 可能得到以下哪个(些)值?（多选题）

0.64

1

1.44

1.96

4

A 与 B 二人进行“抽鬼牌”游戏。游戏开始时，A 手中有n张两两不同的牌。B 手上有$n+1$张牌，其中n张牌与 A 手中的牌相同，另一张为“鬼牌”，与其他所有牌都不同。游戏规则为：

i) 双方交替从对方手中抽取一张牌，A 先从 B 手中抽取。

ii) 若某位玩家抽到对方的牌与自己手中的某张牌一致，则将两张牌丢弃。

iii) 最后剩一张牌（鬼牌）时，持有鬼牌的玩家为输家。

假设每一次抽牌从对方手上抽到任一张牌的概率都相同，请问下列n中哪个n使 A 的胜率最大？

$n = 31$

$n = 32$

$n = 999$

$n = 1000$

对所有的n，A 的胜率都一样

某个城市有10条东西向的公路和10条南北向的公路，共交于100个路口. 小明从某个路口驾车出发，经过每个路口恰一次，最后回到出发点. 在经过每个路口时，向右转不需要等待，直行需要等待1分钟，向左转需要等待2分钟. 设小明在路口等待总时间的最小可能值是S分钟，则

$S < 50$；

$50 \leq S < 90$；

$90 \leq S < 100$；

$100 \leq S < 150$；

$S \geq 150$.

设$n \geq 2$ 是给定正整数. 考虑$n \times n$ 矩阵$X = (a_{i,j})_{1 \leq i,j \leq n}$ ($a_{i,j} = 0$ 或者1) 的集合.

(1) 证明: 存在这样的X 满足$\det X = n-1$.

(2) 若$2 \leq n \leq 4$, 证明$\det X \leq n-1$.

(3) 若$n \geq 2023$, 证明存在X 使得$\det X > n^{\frac{n}{4}}$.

对实数r, 用$||r||$表示r和最近的整数的距离：$||r|| = \min\{|r-n| : n \in \mathbb{Z}\}$.

1. 试问是否存在非零实数s, 满足$\lim_{n\to\infty} ||(\sqrt{2}+1)^n s|| = 0$?

2. 试问是否存在非零实数s, 满足$\lim_{n\to\infty} ||(\sqrt{2}+3)^n s|| = 0$?

某公司要招聘一名员工，有N 人报名面试。假设N 位报名者所具有该职位相关的能力值两两不同，且招聘委员会能观察到的能力值排名与其真实能力值排名吻合。委员会决定采取如下招聘程序：

1. 招聘委员会按随机顺序逐个面试候选人，且他们能观察到当时所见候选人的相对排名。比如委员会面试到第m 位候选人时，他们拥有的信息是前m 位面试者的相对排名，但不知后$N-m$ 位候选人的能力情况。
2. 每面试完一位候选人，委员会需当即决定是否给他/她发工作offer。
3. 如果委员会决定给某位候选者发offer，那么这位候选者以概率p 接受，以概率$1-p$ 拒绝，且独立于(之前) 所有其他面试者的决定。如果该候选人接受offer，那么委员会将不再继续面试接下去的候选人。如果该候选人拒绝offer，那么委员会将继续面试下一位。
4. 如果委员会决定不给某位面试者发offer，那么他们将继续面试下一位候选人，且不能再回头去找前面已经面试过的人。
5. 反复该面试程序，直到有候选者接受offer。如果没有候选者接收该工作，那么委员会面试完所有的N 位候选者。

由于N 位面试者的顺序是完全随机的，因此他们能力的排名在$N!$ 的可能性中是均匀分布。且委员会所具有的全部信息是当前面试过的候选人的相对排名。委员会的任务是，在遵守如上程序的前提下，找到一个策略，使得招到N 位候选者中能力最优者的概率最大化。

问题如下：

(a) 考虑如下策略。委员会先面试前$m-1$ 位候选者，不管其能力排名如何，都不发工作offer。从第m 位开始，一旦看到能力在所面试过候选人中的最优者，即发工作offer。如对方拒绝，则继续面试直到下一位当前最优者[1]出现。试证明：对于任意的N，都存在一个$m=m_N$，使得依靠上述策略找到(所有N 位候选人中) 最优者的概率值，在所有可能的策略所给出的概率值中是最大的。

(b) 假设$p=1$。当$N\to+\infty$，求$\frac{m_N}{N}$ 的极限。

(c) 对一般的$p\in(0,1)$，当$N\to+\infty$，求$\frac{m_N}{N}$ 的极限。

[1] “当前最优者”指当前被面试者在所有被面试过的人(包括被发offer 并婉拒的人)中的最优者。

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(A). 设置$a = 2$, 踢一下仪器，等球状闪电半径严格超过$\sqrt{2}$，再设置$a = -\frac{1}{2}$；

(B). 设置$a = 3$, 踢一下仪器，等球状闪电半径严格超过$\sqrt{2}$，再设置$a = -\frac{1}{3}$；

(C). 设置$a = 4$, 踢一下仪器，等球状闪电半径严格超过$\sqrt{2}$，再设置$a = -\frac{1}{4}$；

(D). 设置$a = 5$, 踢一下仪器，等球状闪电半径严格超过$\sqrt{2}$，再设置$a = -\frac{1}{5}$。

1 答案　选**(B)**。

我们记变化率方程为

$$v = f(r; a).$$

如果$v > 0$ 则r 随时间增长；如果$v < 0$ 则r 随时间下降；如果$v = 0$ 则r 保持不变。

我们首先注意到$f(0,a)=0$，即$r=0$ 永远是一个根。但是变化率函数的非负实根数量受a 的取值影响。事实上，我们可以算出来$f(r,a)=0$ 的所有的根：

$$r_1=0,\quad r_2=-\frac{\sqrt{1-\sqrt{4a+1}}}{\sqrt{2}},\quad r_3=\frac{\sqrt{1-\sqrt{4a+1}}}{\sqrt{2}},$$

$$r_4=-\frac{\sqrt{1+\sqrt{4a+1}}}{\sqrt{2}},\quad r_5=\frac{\sqrt{1+\sqrt{4a+1}}}{\sqrt{2}}.$$

下面我们分类讨论，当$a>0$ 的时候，我们有两个非负实根$r_1=0$ 和$r_5>0$。我们容易验证，当$r\in(0,r_5)$ 时$v>0$，但$r\in(r_5,+\infty)$ 时$v<0$。于是当$a>0$ 的时候，如果我们踢一下机器，就能启动球状闪电，闪电的半径逐渐增大到r_5，但不会超过r_5。

为了使得半径严格超过$\sqrt{2}$，我们需要令$r_5>\sqrt{2}$。所以启动时，我们需要令$a>2$。这样排除了选项(A) 。

当$-\frac{1}{4}<a<0$ 的时候，我们有三个非负实根，从小到大依次是$r_1=0,\ r_3>0$ 和$r_5>0$。特别地，$r_5<1$ 且当$r\in(r_5,+\infty)$ 时，$v<0$ 半径缩小。如果此刻$r=\sqrt{2}$，半径会逐步缩小直到$r=r_5$，但不会小于r_5。所以此时，球状闪电不能完全消失。这样，排除了选项(D)。

当$a=-\frac{1}{4}$ 时，我们有两个非负实根$r_1=0$ 和$r_5=\frac{1}{\sqrt{2}}$。类似上述情况，如果此刻$r=\sqrt{2}$，半径会逐步缩小直到$r=r_5$，但不会小于r_5。所以此时，球状闪电不能完全消失。这样，排除了选项(C)。

当$a<-\frac{1}{4}$ 时，我们只有一个非负实根$r_1=0$，且当$r>0$ 时，$v<0$。所以球状闪电会逐渐完全消失。选项(B) 的确是合理的选项。

第2题 设两个凸八面体O_1, O_2的每个面都是三角形, 且O_1在O_2的**内部**. 记$O_1(O_2)$的棱长之和为$\ell_1(\ell_2)$. 当我们计算ℓ_1/ℓ_2时, 可能得到以下哪个(些)值?（多选题）

(A). 0.64

(B). 1

(C). 1.44

(D). 1.96

(E). 4

2 答案 选 (A) (B) (C) (D)。

说明：在60-70年代全苏中学生数学奥林匹克中，有过这样一个题：“四面体V_1位于四面体V_2内部, 证明V_1的棱长之和小于V_2的棱长之和的$\frac{4}{3}$倍”. 这里反直觉的地方在于，如果是二维平面上一个三角形位于另一个三角形内部, 那么小三角形不仅面积是严格小于大三角形的, 周长也是如此. 而在三维情形, 虽然体积和表面积的大小关系是保持的，但棱长之和的大小关系会被破坏.

这道题的“出处”应该是两个波兰数学家于1962年发表的：

Holsztyński, W. and Kuperberg, W., *O pewnej wlasnósci czworościanów*, Wiadomości Matematyczne 6 (1962), 14-16. 这文章用波兰语写的，自然没有什么人知道，然后1977年他们出了一个英文版

Holsztyński, W. and Kuperberg, W., *On a Property of Tetrahedra*, Alabama J. Math. 1(1977), 40-42.

到了1986年, Alabama大学的Carl Linderholm把这个结果推广到了高维欧氏空间中的单形:

定理: 对于$\mathbb{R}^n$中的两个m维单形S和T(前者完全位于后者的内部), 和任意正整数$1 \leqslant r \leqslant m$. 存在常数$B_{m,r}$, 使得$S$的所有$r$维面的面积之和不超过$T$的所有$r$维面的面积之和的$B_{m,r}$倍. 这里$B_{m,r}$的具体数值计算如下: 设$m+1=(r+1)q+s$(带余除法), 则

$$B_{m,r} = \frac{q^{r+1-s}(q+1)^s}{m+1-r}.$$

(CARL LINDERHOLM, AN INEQUALITY FOR SIMPLICES, Geometriae Dedicata (1986) 21, 67-73.)

回到本题, 这里的选项(A)是平凡的, 关键是要说明:

- 为什么(B)、(C)和(D)可以实现?
- 为什么(E)不能实现?

这里需要的数学知识大致有:

(A) 一些几何拓扑: 每个面都是三角形的凸八面体, 共有$3 \times 8/2 = 12$条棱, 于是由Euler公式, 顶点数为6.

(B) 一点点图论: 如果有一个顶点引出5条棱, 那么简单讨论可知必有另一个顶点也引出5条棱, 这个八面体的各顶点度数为$(5,5,4,4,3,3)$. 除此之外, 唯一可能的情形就是每个顶点都引出4条棱(如正八面体).

(C) 一点点凸几何知识: 因为我们考虑的都是凸八面体, 所以八面体的任意两点之间距离的最大值必定在某两个顶点之间实现.

- 如果大八面体的每个顶点都引出4条棱, 且最大距离ℓ在两个不相邻顶点A和B之间实现, 那么因为另四个顶点与这两个顶点均相邻, 所以大八面体的棱长之和至少是4ℓ(且在另四个顶点到直线AB的距离充分小的时候可以充分接近), 而对于小八面体来说, 假设也是每个顶点引出4条棱, 让三个顶点趋近于A, 另三个趋近于B, 其棱长之和会趋近于6ℓ. 这样所有小于1.5的比例均可实现. (所以有选手会选(A),(B),(C))

- 如果大八面体的两点间最大距离是在两个度数为3的顶点之间实现的, 那么大八面体的棱长之和至少是3ℓ(且在另四个顶点到直线AB的距离充分小的时候可以充分接近), 而对于小八面体来说, 仍假设每个顶点引出4条棱, 让三个顶点趋近于A, 另三个趋近于B, 其棱长之和会趋近于6ℓ.这样所有小于2的比例均可实现. 而如果此时小八面体与大八面体的拓扑结构相同, 且两个度数为5的顶点非常接近, 与此同时另4个顶点非常接近, 那么比例上限可提高到8/3.

- 作简单的分类讨论可知, 如果大八面体的两顶点间最大距离ℓ_2是在一个度数为a的顶点和一个度数为b的顶点之间实现的(不管它们是否相邻), 那么大八面体的各棱长度之和大于$\min(a,b)\ell$, 而小八面体的棱长之和显然不超过12ℓ, 所以(E)是不可能实现的.

第3题 A 与 B 二人进行“抽鬼牌”游戏。游戏开始时，A 手中有n张两两不同的牌。B 手上有$n+1$张牌，其中n张牌与 A 手中的牌相同，另一张为“鬼牌”，与其他所有牌都不同。游戏规则为：

i) 双方交替从对方手中抽取一张牌，A 先从 B 手中抽取。

ii) 若某位玩家抽到对方的牌与自己手中的某张牌一致，则将两张牌丢弃。

iii) 最后剩一张牌（鬼牌）时，持有鬼牌的玩家为输家。

假设每一次抽牌从对方手上抽到任一张牌的概率都相同，请问下列n中哪个n使 A 的胜率最大？

(A). $n=31$

(B). $n=32$

(C). $n=999$

(D). $n=1000$

(E). 对所有的n，A 的胜率都一样

3 答案 选 (B)。

记初始A手上n张牌时A的胜率为a_n，则

$$a_1=\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2}a_1, \tag{1}$$

故有$a_1=\frac{2}{3}$。而

$$a_2=\frac{2}{3}+\frac{1}{3}\cdot\frac{1}{3}a_2, \tag{2}$$

故有$a_2=\frac{3}{4}$。

我们可以得到递推公式

$$a_n=\frac{n}{n+1}a_{n-2}+\frac{1}{n+1}\frac{1}{n+1}a_n+\frac{1}{n+1}\frac{n}{n+1}p_{n,n-1}, \tag{3}$$

其中右端第一项为A未抽中鬼牌的情况，这时B无论抽中什么都能成功配对（鬼牌在B手上），这时A手上有$n-2$张牌，B手上有$n-1$张牌且A先手。右端第二项为A，B均抽中对方手上的鬼牌的情况，右端第三项为A抽中B手上的鬼牌而B没抽中A手上的鬼牌的情况，而$p_{n,n-1}$为A先手，手上有包含鬼牌的n张牌，B手上有不包含鬼牌的$n-1$张牌时A的胜率。我们有

$$p_{n,n-1}=1-a_{n-2}, \tag{4}$$

这是因为A无论抽到哪一张均能配对，此时变为A手上有包含鬼牌的$n-1$张牌，B手上有不包含鬼牌的$n-2$张牌且为B先手，故此时B的胜率为a_{n-2}而A的胜率为$1-a_{n-2}$。

因此

$$a_n = \frac{n}{n+1}a_{n-2} + \frac{1}{n+1}\frac{1}{n+1}a_n + \frac{n}{(n+1)^2} - \frac{n}{(n+1)^2}a_{n-2} \tag{5}$$

进而可得

$$\begin{aligned} a_n &= \frac{n}{n+2}a_{n-2} + \frac{1}{n+2} \\ &= \frac{n}{n+2}(\frac{n-2}{n}a_{n-4} + \frac{1}{n}) + \frac{1}{n+2} \\ &= ... \end{aligned} \tag{6}$$

若n为奇数，由递推可得

$$a_n = \frac{n+3}{2(n+2)}, \tag{7}$$

若n为偶数，由递推可得

$$a_n = \frac{n+4}{2(n+2)}. \tag{8}$$

因此

- $a_{31} = \frac{17}{33}$
- $a_{32} = \frac{9}{17}$
- $a_{999} = \frac{501}{1001}$
- $a_{1000} = \frac{251}{501}$

答案为(B)，即A初始手上有32张牌时A的胜率最大。

第4题 某个城市有10条东西向的公路和10条南北向的公路，共交于100个路口. 小明从某个路口驾车出发，经过每个路口恰一次，最后回到出发点. 在经过每个路口时，向右转不需要等待，直行需要等待1分钟，向左转需要等待2分钟. 设小明在路口等待总时间的最小可能值是S分钟，则

(A). $S < 50$；

(B). $50 \leq S < 90$；

(C). $90 \leq S < 100$；

(D). $100 \leq S < 150$；

(E). $S \geq 150$.

4 答案 选 (C)。

由题意知小明行驶的路线是一条不自交的闭折线. 将每个路口看作一个顶点，那么他行驶的路线可以看成是一个100边形（有的内角可能是平角，也有大于平角的内角）. 由多边形内角和公式知这个100边形的所有内角之和为$98 \times 180°$. 注意内角只能是$90°$，$180°$和$270°$，设$90°$有a个，$270°$有b个，那么$90a + 270b + 180(100 - a - b) = 98 \times 180$，整理得$a - b = 4$. 如果小明在这条路上是顺时针行驶的，那么$90°$内角对应右转，$180°$内角对应直行，$270°$内角对应左转，他在路口等待的总时间是$(100 - a - b) + 2b = 100 - (a - b) = 96(min)$；如果小明在这条路上是逆时针行驶的，那么$90°$内角对应左转，$180°$内角对应直行，$270°$内角对应右转，他在路口等待的总时间是$(100 - a - b) + 2a = 100 + (a - b) = 104(min)$. 因此，$S = 96$，选项(C)正确.

注：如果小明的起点/终点处的转弯时间不计，那么等待的总时间还可以减少2分钟（选择一个左转的位置作为起点），这样$S = 94$，但不影响选择的选项.

第5题 设$n \geq 2$ 是给定正整数. 考虑$n \times n$ 矩阵$X = (a_{i,j})_{1 \leq i,j \leq n}$ ($a_{i,j} = 0$ 或者1) 的集合.

(1) 证明: 存在这样的X 满足$\det X = n-1$.

(2) 若$2 \leq n \leq 4$, 证明$\det X \leq n-1$.

(3) 若$n \geq 2023$, 证明存在X 使得$\det X > n^{\frac{n}{4}}$.

5 答案

(1)若X有一行全为0或者有两行相等, 则$\det X = 0$; 若X有一行只有一个1, 则可约化到$(n-1)$阶矩阵的情形; 若X有一行全为1, 还有一行有$n-1$个1, 则可约化到有一行只有一个1的情形, 进一步约化到$(n-1)$阶矩阵的情形. 若以上都不发生, 则X的各行有很少的可能性, 我们可以逐个讨论.

(2)取$X' = (a_{i,j})_{1 \leq i,j \leq n}$, 其中

$$a_{i,j} = 1 - \delta_{i,j}, \quad 1 \leq i, j \leq n.$$

则$\det X' = (-1)^{n-1}(n-1)$. 若$n$ 是奇数, 令$X = X'$. 若n是偶数, 令X为调换X'的最后两行所得矩阵. 则$\det X = n-1$.

(3)当$n = 2^k - 1$时, 令

$$Y = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{\otimes k}.$$

则Y是元素为± 1的$(n+1) \times (n+1)$矩阵, 且

$$\det Y = (\sqrt{2^k})^{2^k} = 2^{k 2^{k-1}}.$$

注意Y的最后一行为$\alpha_{n+1} = (\underbrace{1, \ldots, 1}_{n+1})$. 记$t_i = \pm 1$为$Y$的第$i$行的最后一个元素$(1 \leq i \leq n)$. 令

$$\beta_i' = \frac{1}{2}(t_i \alpha_i - \alpha_{n+1}).$$

去掉β_i'的最后一个元素(其等于0), 得到一个n行向量β_i. 令

$$X' = (\beta_1, \ldots, \beta_n)^t.$$

记

$$t = \prod_{1 \leq i \leq n} t_i = \pm 1.$$

则X' 是元素为0, 1的$n \times n$矩阵, 且

$$\det X' = t 2^{(k-2)2^{k-1}+1}.$$

若有必要, 则调换X'的最后两行, 可以得到一个元素为0, 1 的$n\times n$矩阵, 满足$\det X=2^{(k-2)2^{k-1}+1}$.

不妨设$2^k-1\le n<2^{k+1}-1$. 当$2^k-1\le n<3\cdot 2^{k-1}$且$n\ge 2023$时, 则$k\ge 11$. 存在元素为0, 1的$n\times n$矩阵$X$, 满足

$$\det X\ge 2^{(k-2)2^{k-1}+1}>2^{(k-2)2^{k-1}}.$$

另一方面,

$$n^{\frac{n}{4}}<(2^{k+1})^{3\cdot 2^{k-3}}=2^{3(k+1)\cdot 2^{k-3}}.$$

由于$k\ge 11$, 得

$$(k-2)2^{k-1}\ge 3(k+1)\cdot 2^{k-3}.$$

这样, $\det X>n^{\frac{n}{4}}$.

当$3\cdot 2^{k-1}\le n<2^{k+1}-1$且$n\ge 2023$时, 则$k\ge 10$. 存在元素为0, 1的$n\times n$矩阵$X$, 满足

$$\det X\ge 2^{(k-2)2^{k-1}+1}2^{(k-3)\cdot 2^{k-2}+1}>2^{(3k-7)2^{k-2}}.$$

另一方面,

$$n^{\frac{n}{4}}<(2^{k+1})^{2^{k-1}}=2^{(k+1)\cdot 2^{k-1}}.$$

由于$k\ge 10$, 得

$$(3k-7)2^{k-2}>(k+1)\cdot 2^{k-1}.$$

这样, $\det X>n^{\frac{n}{4}}$.

第6题 对实数r, 用$||r||$表示r和最近的整数的距离：$||r|| = \min\{|r-n| : n \in \mathbb{Z}\}$.

1. 试问是否存在非零实数s, 满足$\lim_{n\to\infty} ||(\sqrt{2}+1)^n s|| = 0$?

2. 试问是否存在非零实数s, 满足$\lim_{n\to\infty} ||(\sqrt{2}+3)^n s|| = 0$?

6 答案

1. 存在，取$s=1$即可。设$(\sqrt{2}+1)^n = x_n + \sqrt{2}y_n$, 则$(-\sqrt{2}+1)^n = x_n - \sqrt{2}y_n$. 从而$x_n^2 - 2y_n^2 = (-1)^n$. 由此$|x_n + \sqrt{2}y_n - 2x_n| = |\sqrt{2}y_n - x_n| = \frac{|2y_n^2 - x_n^2|}{\sqrt{2}y_n + x_n} \to 0$.

2. 不存在。反证法，假设s满足$(\sqrt{2}+3)^n s = m_n + \epsilon_n$, 其中$\lim_{n\to\infty} \epsilon_n = 0$. 记$\alpha = \sqrt{2}+3, \bar{\alpha} = -\sqrt{2}+3$. 考虑幂级数

$$\frac{s}{1-\alpha x} = \sum_{n=0}^{\infty} m_n x^n + \sum_{n=0}^{\infty} \epsilon_n x^n.$$

$(1-\alpha x)(1-\bar{\alpha}x) = 1 - 6x + 7x^2$, 上式两边乘以$1-6x+7x^2$可得

$$s(1-\bar{\alpha}x) = (1-6x+7x^2)\sum_{n=0}^{\infty} m_n x^n + (1-6x+7x^2)\sum_{n=0}^{\infty} \epsilon_n x^n. \tag{9}$$

设$(1-6x+7x^2)\sum_{n=0}^{\infty} m_n x^n = \sum_{n=0}^{\infty} p_n x^n, (1-6x+7x^2)\sum_{n=0}^{\infty} \epsilon_n x^n = \sum_{n=0}^{\infty} \eta_n x^n$, 则$p_n \in \mathbb{Z}, \lim_{n\to\infty} \eta_n = 0$. 因为(9)左边是一次式，从而右边满足$p_n + \eta_n = 0, n \geq 2$. n充分大时η_n很小，所以必有$p_n = \eta_n = 0$, 即(9)右边两项均为多项式。因此

$$\sum_{n=0}^{\infty} \epsilon_n x^n = \frac{G(x)}{1-6x+7x^2}.$$

右边写成部分分式形如$H(x) + \frac{A}{1-\bar{\alpha}x} + \frac{B}{1-\alpha x}$. 因为$\lim_{n\to\infty} \epsilon_n = 0$, 所以左边的收敛半径至少为1，而$\alpha, \bar{\alpha}$均大于1，所以必须$A = B = 0$. 这样当$n$充分大时，$\epsilon_n = 0$, 从而$(\sqrt{2}+3)^n s = m_n \in \mathbb{Z}$, 矛盾！

第7题 某公司要招聘一名员工，有N 人报名面试。假设N 位报名者所具有该职位相关的能力值两两不同，且招聘委员会能观察到的能力值排名与其真实能力值排名吻合。委员会决定采取如下招聘程序：

1. 招聘委员会按随机顺序逐个面试候选人，且他们能观察到当时所见候选人的相对排名。比如委员会面试到第m 位候选人时，他们拥有的信息是前m 位面试者的相对排名，但不知后$N-m$ 位候选人的能力情况。
2. 每面试完一位候选人，委员会需当即决定是否给他/她发工作offer。
3. 如果委员会决定给某位候选者发offer，那么这位候选者以概率p 接受，以概率$1-p$ 拒绝，且独立于(之前) 所有其他面试者的决定。如果该候选人接受offer，那么委员会将不再继续面试接下去的候选人。如果该候选人拒绝offer，那么委员会将继续面试下一位。
4. 如果委员会决定不给某位面试者发offer，那么他们将继续面试下一位候选人，且不能再回头去找前面已经面试过的人。
5. 反复该面试程序，直到有候选者接受offer。如果没有候选者接收该工作，那么委员会面试完所有的N 位候选者。

由于N 位面试者的顺序是完全随机的，因此他们能力的排名在$N!$ 的可能性中是均匀分布。且委员会所具有的全部信息是当前面试过的候选人的相对排名。委员会的任务是，在遵守如上程序的前提下，找到一个策略，使得招到N 位候选者中能力最优者的概率最大化。

问题如下：

(a) 考虑如下策略。委员会先面试前$m-1$ 位候选者，不管其能力排名如何，都不发工作offer。从第m 位开始，一旦看到能力在所面试过候选人中的最优者，即发工作offer。如对方拒绝，则继续面试直到下一位当前最优者[1]出现。试证明：对于任意的N，都存在一个$m=m_N$，使得依靠上述策略找到(所有N 位候选人中) 最优者的概率值，在所有可能的策略所给出的概率值中是最大的。

(b) 假设$p=1$。当$N\to+\infty$，求$\frac{m_N}{N}$ 的极限。

(c) 对一般的$p\in(0,1)$，当$N\to+\infty$，求$\frac{m_N}{N}$ 的极限。

7 答案 对于任意的$1\le k\le N$，我们令Z_k 为委员会完全略过前$k-1$ 位面试者，而从第k位开始采取最优策略的最终所得，即N位候选者中能力最高者接受该工作的概率。则我们有

$$Z_k \ge Z_{k+1}\ .$$

(a) 如果委员会面试了第k 位候选人，且其能力在前k 位被面试者中居首，那么委员会发出offer。在这个事件下，委员会最终找到能力值最高者的条件概率为

$$Y_k = \frac{pk}{N} + (1-p)Z_{k+1}\ .$$

[1]“当前最优者”指当前被面试者在所有被面试过的人(包括被发offer 并婉拒的人)中的最优者。

由此可知，委员会向第k 位面试者发出offer，当且仅当其在前k 位中的能力值最高，且

$$\frac{pk}{N} + (1-p)Z_{k+1} \geq Z_{k+1} \, , \tag{10}$$

即$\frac{k}{N} \geq Z_{k+1}$[2]. 由于$\{\frac{k}{N}\}_k$ 递增，而$\{Z_k\}_k$ 递减，且$Z_k \leq \frac{N-k+1}{N}$，易见不等式(10) 必定对某一个$k \geq N-1$ 成立。由此可知，最优策略可以通过选择某个m 来达到，也即满足不等式(10) 的k 中的最小值。此外，如果$k = m$ 满足不等式(10)，则任意的$k \geq m$ 也满足。因此，对于第m 位之后的“当前最优者”，也应当发放offer。

(b) 令p_m 为委员会采取(a) 中的策略，且找到能力最高者的概率。当$p = 1$ 时，被发offer 的候选者一定会接受，所以这时选中能力最高者对应的是事件

$$\bigcup_{k=m}^{N} A_k$$

的不相交并集，其中A_k 对应的事件为第k 位候选人是N 位中的能力最高者，且被面试了。这样的话，事件A_k 的概率为

$$\mathbf{P}(A_k) = \frac{1}{N} \cdot \frac{m-1}{k-1} \, ,$$

其中$\frac{1}{N}$ 对应的是这位候选者是能力最高者的概率，$\frac{m-1}{k-1}$ 是他/她被面试到的概率，即前$k-1$ 位中的相对能力最高者在前$m-1$ 位中。这样，我们就有

$$p_m = \frac{m-1}{N} \sum_{k=m}^{N} \frac{1}{k-1} \, .$$

易见p_m 先曾后减，因此其最优值$m^* = m_N$ 应为满足

$$p_m \geq p_{m+1}$$

的最小的m，即满足

$$\sum_{k=m+1}^{N} \frac{1}{k-1} \leq 1$$

的最小m。当N 很大时，由左端的近似逼近为$\log(N/m)$ 可知，$\frac{m_N}{N} \to \frac{1}{e}$.

(c) 对一般的于$p \in (0,1)$，同样的，p_m 的值为下列不相交事件并的概率：

$$\bigcup_{k=m}^{N} A_k \, ,$$

其中A_k 对应的事件为第k 位候选人是N 人中的能力最高者、被面试了、并且接受了offer. 我们有

$$p_m = \frac{p}{N} \sum_{k=m}^{N} q_k \, ,$$

[2]如果这个不等式不满足，则略过第k 个人，且从第$k+1$ 位开始采取最优策略，则得到最佳求职者的概率将更大。

其中q_k 为假定第k 位候选人为N 人中能力最高者后，他/她被面试的条件概率。则我们有

$$q_k = \Big(\frac{m-1}{m} + \frac{1-p}{m}\Big)\Big(\frac{m}{m+1} + \frac{1-p}{m+1}\Big)\cdots\Big(\frac{k-2}{k-1} + \frac{1-p}{k-1}\Big) = \frac{\Gamma(m)\Gamma(k-p)}{\Gamma(k)\Gamma(m-p)} ,$$

其中Γ 为经典的Γ 函数。据此，我们得到如果从第m 位开始，委员会找到(所有N 位候选人中)能力值最高者的概率为

$$p_m = \frac{p}{N} \cdot \frac{\Gamma(m)}{\Gamma(m-p)} \sum_{k=m}^{N} \frac{\Gamma(k-p)}{\Gamma(k)} ,$$

p_m 对于m 先增后减。由Γ 函数的近似及积分对求和的逼近，我们计算得，当N 非常大时，让p_m 最大化的m_N 的值满足

$$\frac{m_N}{N} \to p^{\frac{1}{1-p}} .$$

当$p = 1$ 时，极限为$1/e$.

2023 Alibaba Global Mathematics Competition

Ball lightening

As a chief officer of a secret mission, you had the following conversation with the leading scientist.

Scientist: "Chief, we have mastered the control law of ball lightning. We found that the rate of change of the radius of ball lightning in the laboratory $v(t)$ satisfies the following equation.

$$v = ar + r^3 - r^5.$$

Here $r(t)$ represents the radius of ball lightning, and t is the time variable. At the initial moment, there is no ball lightning, that is, $r(0) = 0$. Accordingly, we also have $v(0) = 0$. And $a \in \mathbb{R}$ can be artificially controlled. You can quickly change the value of a by pulling a control lever. We set its preset value to $a = -1$."

You: "Well done, Doctor! Is a our only way of control? It doesnt seem to be able to start the ball lightning."

Scientist: "You're right, Chief. We do have another way of control, which is to kick the instrument."

You: "Doctor, are you kidding me? Kick it?"

Scientist: "Yes, if you kick it, the value of $r(t)$ will instantly increase by ε (ε is much smaller than 1)."

You: "I see. That's helpful indeed. Our test goal today is to start the ball lightning, make its radius strictly exceed $\sqrt{2}$, and then let it gradually disappear completely."

Scientist: "Yes, Chief. We have designed four control schemes for this.

What do you think of these schemes, Chief?"

You looked at these options and found that the feasible schemes are ( ).

Set $a = 2$, kick the instrument, wait for the ball lightning radius to strictly exceed $\sqrt{2}$, then set $a = -\frac{1}{2}$;

Set $a = 3$, kick the instrument, wait for the ball lightning radius to strictly exceed $\sqrt{2}$, then set $a = -\frac{1}{3}$;

Set $a = 4$, kick the instrument, wait for the ball lightning radius to strictly exceed $\sqrt{2}$, then set $a = -\frac{1}{4}$;

Set $a = 5$, kick the instrument, wait for the ball lightning radius to strictly exceed $\sqrt{2}$, then set $a = -\frac{1}{5}$.

Let O_1, O_2 be two convex octahedron whose faces are all triangles, and O_1 is *inside* O_2. Let the *sum* of edge kengths of O_1(resp. O_2) be ℓ_1(resp. ℓ_2). When we calculate ℓ_1/ℓ_2, which value(s) among the following can be obtained? (Multiple Choice)

0.64

1

1.44

1.96

4

Two players, A and B, play a game called "draw the joker card". In the beginning, Player A has n different cards Player B has $n + 1$ cards, n of which are the same with the n cards in Player A's hand, and the rest one is a Joker (different from all other n cards). The rules are

i) Player A first draws a card from Player B, and then Player B draws a card from Player A, and then the two players take turns to draw a card from the other player.

ii) if the card that one player drew from the other one coincides with one of the cards on his/her own hand, then this player will need to take out these two identical cards and discard them.

iii) when there is only one card left (necessarily the Joker), the player who holds that card loses the game.

Assume for each draw, the probability of drawing any of the cards from the other player is the same. Which n in the following maximises Player A's chance of winning the game?

$n = 31$

$n = 32$

$n = 999$

$n = 1000$

For all choices of n, A has the same chance of winning

There are 10 horizontal roads and 10 vertical roads in a city, and they intersect at 100 crossings. Bob drives from one crossing, passes every crossing exactly once, and return to the original crossing. At every crossing, there is no wait to turn right, 1 minute wait to go straight, and 2 minutes wait to turn left. Let S be the minimum number of total minutes on waiting at the crossings, then

$S < 50$;

$50 \leq S < 90$;

$90 \leq S < 100$;

$100 \leq S < 150$;

$S \geq 150$.

Let $n \geq 2$ be a given positive integer. Consider the set of $n \times n$ matrices $X = (a_{i,j})_{1 \leq i,j \leq n}$ with entries 0 and 1.

(1) show that: there exists such an X with $\det X = n - 1$.

(2) when $2 \leq n \leq 4$, show that $\det X \leq n - 1$.

(3) When $n \geq 2023$, show that there exists an X with $\det X > n^{\frac{n}{4}}$.

For a real number r, set $||r|| = \min\{|r-n| : n \in \mathbb{Z}\}$, where $|\cdot|$ means the absolute value of a real number.

1. Is there a nonzero real number s, such that $\lim_{n\to\infty} ||(\sqrt{2}+1)^n s|| = 0$?
2. Is there a nonzero real number s, such that $\lim_{n\to\infty} ||(\sqrt{2}+3)^n s|| = 0$?

A company has one open position available, and N candidates applied (N is known). Assume the N candidates' abilities for this position are all different from each other (in other words, there is a non-ambiguous ranking among the N candidates), and the hiring committee can observe the full relative ranking of all the candidates they have interviewed, and their observed rankings are faithful with respect to the candidates' true abilities. The hiring committee decides the following rule to select one candidate from N:

1. The committee interviews the candidates one by one, at a completely random order. They observe information on candidates' relative ranking regarding their abilities for the position. The only information available to them after interviewing m candidates is the relative ranking among these m people.
2. After each interview, the committee decides whether to offer the candidate the position or not.
3. If they decide to offer the position to the candidate just interviewed, then the candidate will accept the job with probability p, and decline the offer with probability $1-p$, independently with all other candidates. If the selected candidate accepts the offer, then he/she gets the job, and the committee stops interviewing the remaining candidates. If he/she declines the offer, then the committee proceed to interviewing the next candidate.
4. If they decide not to offer the position to the candidate just interviewed, then they proceed to interviewing the next candidate, and they can not turn back to previously interviewed candidates any more.
5. The committee continues this process until a candidate is selected and accepts the job, or until they finish interviewing all N candidates if the position has not been filled before, whichever comes first.

Since the interview order of the candidates are completely random, each ranking has equal probability among the $N!$ possibilities. The committee's mission is to maximise the probability of getting the candidate with the highest ranking (among N candidates) for the job constrained to the above selection process.

Here are the questions

(a) Fix $1 \leq m \leq N$, and consider the following strategy. The committee interviews the first $m-1$ candidates, and do not give offer to any of them regardless of their relative rankings. Starting from the m-th candidate, the committee offers him/her the position whenever the candidate's relative ranking is the highest among all previously interviewed candidates. If he/she declines the offer, then the committee continues the interview until the next relatively best candidate[1], and then repeat the process when applicable.

Show that for every N, there exists $m = m_N$ such that the above strategy maximises the probability of getting the best candidate among all possible strategies.

[1] "Relatively best candidate" refers to the candidate with the highest ability among all candidates who have been interviewed (including those who are offered the position and declined).

(b) Suppose $p = 1$. What is the limit of $\frac{m_N}{N}$ as $N \to +\infty$?

(c) For $p \in (0, 1)$, what is the limit of $\frac{m_N}{N}$ as $N \to +\infty$?

2023 Alibaba Global Mathematics Competition

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Here $r(t)$ represents the radius of ball lightning, and t is the time variable. At the initial moment, there is no ball lightning, that is, $r(0) = 0$. Accordingly, we also have $v(0) = 0$. And $a \in \mathbb{R}$ can be artificially controlled. You can quickly change the value of a by pulling a control lever. We set its preset value to $a = -1$."

You: "Well done, Doctor! Is a our only way of control? It doesnt seem to be able to start the ball lightning."

Scientist: "You're right, Chief. We do have another way of control, which is to kick the instrument."

You: "Doctor, are you kidding me? Kick it?"

Scientist: "Yes, if you kick it, the value of $r(t)$ will instantly increase by ε (ε is much smaller than 1)."

You: "I see. That's helpful indeed. Our test goal today is to start the ball lightning, make its radius strictly exceed $\sqrt{2}$, and then let it gradually disappear completely."

Scientist: "Yes, Chief. We have designed four control schemes for this.

What do you think of these schemes, Chief?"

You looked at these options and found that the feasible schemes are ( ).

(A). Set $a = 2$, kick the instrument, wait for the ball lightning radius to strictly exceed $\sqrt{2}$, then set $a = -\frac{1}{2}$;

(B). Set $a = 3$, kick the instrument, wait for the ball lightning radius to strictly exceed $\sqrt{2}$, then set $a = -\frac{1}{3}$;

(C). Set $a = 4$, kick the instrument, wait for the ball lightning radius to strictly exceed $\sqrt{2}$, then set $a = -\frac{1}{4}$;

(D). Set $a = 5$, kick the instrument, wait for the ball lightning radius to strictly exceed $\sqrt{2}$, then set $a = -\frac{1}{5}$.

1 Answer The answer is **(B)**.

We introduce the following notation for the rate function

$$v = f(r; a).$$

When $v > 0$, r is increasing in time. When $v < 0$, r is decreasing in time. When $v = 0$, r remains unchanged.

We can find all the roots of $f(r, a) = 0$, which we list in the following:

$$r_1 = 0, \quad r_2 = -\frac{\sqrt{1-\sqrt{4a+1}}}{\sqrt{2}}, \quad r_3 = \frac{\sqrt{1-\sqrt{4a+1}}}{\sqrt{2}},$$

$$r_4 = -\frac{\sqrt{1+\sqrt{4a+1}}}{\sqrt{2}}, \quad r_5 = \frac{\sqrt{1+\sqrt{4a+1}}}{\sqrt{2}}.$$

When $a > 0$, we have two nonnegative roots: $r_1 = 0$ and $r_5 > 0$. Clearly, when $r \in (0, r_5)$, $v > 0$; and when $r \in (r_5, +\infty)$, $v < 0$. Thus, when $a > 0$ and if we kick the instrument, we can start the ball lightening, and its radius will grow to r_5 (but it will not exceed r_5).

To make the radius exceed $\sqrt{2}$, we need $r_5 > \sqrt{2}$. This means in the starting phase, we need $a > 2$, and thus Scheme (A) fails.

When $-\frac{1}{4} < a < 0$, we have three nonnegative roots, which satisfy $0 = r_1 < r_3 < r_5$. In particular, we have $r_5 < 1$ and when $r \in (r_5, +\infty)$, $v < 0$. This means, if we start with $r = \sqrt{2}$, the radius is getting smaller, but it will not become smaller than r_5. Therefore, the ball lightening will not vanish completely. Hence, Scheme (D) fails.

When $a = -\frac{1}{4}$, similar to the previous case, the radius will not be smaller than $r_5 = \frac{1}{\sqrt{2}}$, and the ball lightening will not vanish completely. Hence, Scheme (C) fails.

When $a < -\frac{1}{4}$, we have only one nonnegative root $r_1 = 0$. When $r > 0$, we always have $v < 0$, and thus the ball lightening will vanish completely. This means Scheme (B) works.

2 Let O_1, O_2 be two convex octahedron whose faces are all triangles, and O_1 is *inside* O_2. Let the *sum* of edge kengths of O_1(resp. O_2) be ℓ_1(resp. ℓ_2). When we calculate ℓ_1/ℓ_2, which value(s) among the following can be obtained? (Multiple Choice)

(A). 0.64

(B). 1

(C). 1.44

(D). 1.96

(E). 4

2 Answer

The answer is **(A) (B) (C) (D)**.

Comments In the 60's - 70's, the following question appeared in All-Union Math Olympiad of USSR: A tetradehron V_1 sits inside another tetrahedron V_2, prove that the sum of edge lengths of V_1 does not exceed $\frac{4}{3}$ times that of V_2. What is anti-intuitive is that, on a plane, if a triangle sits inside another triangle, then not only the area of the first triangle is strictly smaller than that of the second one, but the perimeter also is. Now in a three dimensional situation, though the "order" of volume and surface is still kept, it is not the case for the sum of edge lengths.

The "origine" of the problem is likely the following paper in Polish:

Holsztyński, W. and Kuperberg, W., *O pewnej wlasnósci czworościanów*, Wiadomości Matematyczne 6 (1962), 14-16.

They published an English version some 15 years later:

Holsztyński, W. and Kuperberg, W., *On a Property of Tetrahedra*, Alabama J. Math. 1(1977), 40 42.

Then in 1986, Carl Linderholm of the University of Alabama generalized the above result to higher dimensional Euclidean spaces:
Theorem. *Let S and T be two m-dimensional simplexes in $\mathbb{R}^n$, the first being inside the second, and $1 \leqslant r \leqslant m$. The there exists constants $B_{m,r}$, such that the sum of all r-dimensional faces of S does not exceed $B_{m,r}$ times that of T. Here $B_{m,r}$is calculated as follows: Let $m+1 = (r+1)q + s$(Euclidean division), then*

$$B_{m,r} = \frac{q^{r+1-s}(q+1)^s}{m+1-r}.$$

(CARL LINDERHOLM, AN INEQUALITY FOR SIMPLICES, Geometriae Dedicata (1986) 21, 67-73.)

Now back to the current problem, the Choice (A) is trivial, so we focus on:

- why (B),(C) and (D) can be realized?
- why (E) cannot?

The mathematics that we need here is:

(A) a little geometric topology: an octahedron with all faces being triangles has $3 \times 8/2 = 12$ edges, so by Euler's Formula, the number of vertices is 6.

(B) a bit of graph theory: if one vertex has degree 5, then by a very easy argument one has another vertex with degree 5 also, and the degrees of the vertices are $(5, 5, 4, 4, 3, 3)$. The only other possibility is that every vertex has degree 4 (like that of a regular ocrahedron).

(C) a little bit of convex geometry: as we consider convex octahedron, so the maximum distance of two points on it must be attained between two vertices.

- If every vertex of the big octahedron is of degree 4,and the maximum distance $\ell$is realized between two vertices A and B that are NOT adjacent, then as the other four vertices are all adjacent to them, so ℓ_2 is at least $4\ell_2$ (and can be arbitrarily close to that valur when the other four vertices are close enough to line AB), and for the small octahedron, if every vertex is of degree 4, we can make three vertices very close to A, while the other three very close to B, so ℓ_1 would be very close to $6\ell_2$. Hence any ratio less than 1.5 is realizable. (so the Choices (A),(B) and (C))

- If the maximum distance ℓ is realized between two vertices of degree 3 in the big octahedron, then ℓ_2 is at least $3\ell_2$(and can be arbitrarily close to that valur when the other four vertices are close enough to line AB), while for the small octahedron, we can still take each vertex to be of degree 4, and three of them very close to A, while the other three very close to B, so ℓ_1 would be very close to $6\ell_2$. Hence any ratio less than 2 is realizable. (so the Choice (D))

  Actually, if the small octahedron has the some topological configuration as that of the big one, and the two vertices of degree 5 are very close to each other, while the other four vertices are very close together, then the ratio can actually approach 8/3.

- After some easy case by case discussion, we conclude that, if the maximum distance ℓ is realized between a vertex of degree a and a vertex of degree b (whether they are adjacent or not), one has always ℓ_2 is at last $\min(a, b)\ell$, while obviously ℓ_1 cannot exceed $12\ell_2$, So (E)is impossible.

3 Two players, A and B, play a game called "draw the joker card". In the beginning, Player A has n different cards. Player B has $n+1$ cards, n of which are the same with the n cards in Player A's hand, and the rest one is a Joker (different from all other n cards). The rules are

i) Player A first draws a card from Player B, and then Player B draws a card from Player A, and then the two players take turns to draw a card from the other player.

ii) if the card that one player drew from the other one coincides with one of the cards on his/her own hand, then this player will need to take out these two identical cards and discard them.

iii) when there is only one card left (necessarily the Joker), the player who holds that card loses the game.

Assume for each draw, the probability of drawing any of the cards from the other player is the same. Which n in the following maximises Player A's chance of winning the game?

(A). $n = 31$

(B). $n = 32$

(C). $n = 999$

(D). $n = 1000$

(E). For all choices of n, A has the same chance of winning

3 Answer

The answer is **(B)**.

We denote a_n to be the probability that A wins the game when A has n cards in the beginning. So we have

$$a_1 = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} a_1. \tag{1}$$

Therefore, $a_1 = \frac{2}{3}$. In addition, we have

$$a_2 = \frac{2}{3} + \frac{1}{3} \cdot \frac{1}{3} a_2, \tag{2}$$

so we conclude that $a_2 = \frac{3}{4}$.

Actually, we can obtain the following induction formula

$$a_n = \frac{n}{n+1} a_{n-2} + \frac{1}{n+1} \frac{1}{n+1} a_n + \frac{1}{n+1} \frac{n}{n+1} p_{n,n-1} \tag{3}$$

where the first term on the RHS is the scenario when A does not draw the joker card from B. In this case, no matter which card B draws from A, this card would match one of the cards that B has in his hand (because B holds the joker card). Then A will have $n-2$ cards and B has $n-1$ cards, with A drawing from B first and B holding the joker card. The second term on the RHS is the scenario when A first draws the joker card from B, and then B draws the joker card from A. The third term on the RHS is the scenario when A draws the joker card from B but B does not draw the joker card from A, and $p_{n,n-1}$ is the probability for A to win the game when A draws first with n cards including a joker card, B draws next with $n-1$ cards that do not include the joker card. We have

$$p_{n,n-1} = 1 - a_{n-2}, \tag{4}$$

because no matter which card A draws from B, A would have one card in hand that match this drawn card from B (because the joker card is in A's hand). Therefore, after A's drawing, A will have $n-1$ cards including the joker card, B will have $n-2$ cards without the joker card, and B draws first. In this case, the probability for B to win will be a_{n-2}, so we have $p_{n,n-1} = 1 - a_{n-2}$.

Therefore,

$$a_n = \frac{n}{n+1}a_{n-2} + \frac{1}{n+1}\frac{1}{n+1}a_n + \frac{n}{(n+1)^2} - \frac{n}{(n+1)^2}a_{n-2}, \tag{5}$$

and we can simplify the above equation to

$$\begin{aligned} a_n &= \frac{n}{n+2}a_{n-2} + \frac{1}{n+2} \\ &= \frac{n}{n+2}(\frac{n-2}{n}a_{n-4} + \frac{1}{n}) + \frac{1}{n+2} \\ &= ... \end{aligned} \tag{6}$$

By induction, if n is an odd number, then

$$a_n = \frac{n+3}{2(n+2)}. \tag{7}$$

On the other hand, if n is an even number, then by induction we have

$$a_n = \frac{n+4}{2(n+2)}. \tag{8}$$

Therefore, we conclude that

- $a_{31} = \frac{17}{33}$
- $a_{32} = \frac{9}{17}$
- $a_{999} = \frac{501}{1001}$
- $a_{1000} = \frac{251}{501}$.

So the correct answer is (B), and $n = 32$ initial cards will give A the biggest chance of winning.

4 There are 10 horizontal roads and 10 vertical roads in a city, and they intersect at 100 crossings. Bob drives from one crossing, passes every crossing exactly once, and return to the original crossing. At every crossing, there is no wait to turn right, 1 minute wait to go straight, and 2 minutes wait to turn left. Let S be the minimum number of total minutes on waiting at the crossings, then

(A). $S < 50$

(B). $50 \leq S < 90$

(C). $90 \leq S < 100$

(D). $100 \leq S < 150$

(E). $S \geq 150$.

4 Answer

The answer is **(C)**.

Obviously, the route of driving is a non-self-intersecting closed polyline. Regard each crossing as a vertex, then the route is regarded as a 100-gon.An interior angle may be greater than or equal to a straight angle.. By the formula of the sum of the angles of the polygon, the sum of all interior angles is $98 \times 180°$. Note that the interior angle can only be $90°$, $180°$ or $270°$, if there are a angles of $90°$, b angles of $270°$, then $90a + 270b + 180(100 - a - b) = 98 \times 180$, so $a - b = 4$. If Bob drives clockwise, then $90°$, $180°$ and $270°$ corrsponds to turn right, go straight and turn left, respectively. The total time on waiting at the crossings is $(100 - a - b) + 2b = 100 - (a - b) = 96(min)$; If Bob drives clockwise, then $90°$, $180°$ and $270°$ corrsponds to turn left, go straight and turn right, respectively. The total time on waiting at the crossings is $(100 - a - b) + 2a = 100 + (a - b) = 104(min)$. Therefore, $S = 96$, and (C) is correct.

Note: If we ignore the waiting time on the beginning/ending crossing, the total time on waiting can be decreased by 2 minutes (Bob can choose a left-turn crossing as the beginning), we have that $S = 94$, but do not affect the correct choice.

5 Let $n \geq 2$ be a given positive integer. Consider the set of $n \times n$ matrices $X = (a_{i,j})_{1 \leq i,j \leq n}$ with entries 0 and 1.

(1) show that: there exists such an X with $\det X = n - 1$.

(2) when $2 \leq n \leq 4$, show that $\det X \leq n - 1$.

(3) When $n \geq 2023$, show that there exists an X with $\det X > n^{\frac{n}{4}}$.

5 Answer

(1)If X has a zero row or two equal rows, then $\det X = 0$; if X has a row with only one nonzero entry, it reduces to $(n-1)$ matrix case; if X has a row with n nonzero entries and a row with $(n-1)$ nonzero entries, it reduces to the case that X has a row with only one nonzero entry and further reduces to $(n-1)$ matrix case. When the above all not happen, then rows of X have few possibilities and one could take a case by case verification.

(2)take $X' = (a_{i,j})_{1 \leq i,j \leq n}$ where

$$a_{i,j} = 1 - \delta_{i,j}, \quad 1 \leq i, j \leq n.$$

Then, $\det X' = (-1)^{n-1}(n-1)$. If n is odd, let $X = X'$. If n is even, get X by switching the first two rows of X'. Then, $\det X = n - 1$.

(3)when $n = 2^k - 1$, set

$$Y = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{\otimes k}.$$

Then, Y is an $(n+1) \times (n+1)$ matrix with entries ± 1 and having

$$\det Y = (\sqrt{2^k})^{2^k} = 2^{k 2^{k-1}}.$$

The last row of Y is equal to $\alpha_{n+1} = (\underbrace{1, \ldots, 1}_{n+1})$. Write $t_i = \pm 1$ for the last entry of the i-th row α_i of Y $(1 \leq i \leq n)$. Put

$$\beta_i' = \frac{1}{2}(t_i \alpha_i - \alpha_{n+1}).$$

Removing the last entry of β_i' (which is 0), we get an n row vector β_i. Put

$$X' = (\beta_1, \ldots, \beta_n)^t$$

and write

$$t = \prod_{1 \leq i \leq n} t_i = \pm 1.$$

Then, X' is an $n \times n$ matrix with entries 0 and 1 and we have

$$\det X' = t 2^{(k-2)2^{k-1}+1}.$$

Switching the first two rows of X' if necessary, we get an $n \times n$ matrix X with entries $0, 1$ such that $\det X = 2^{(k-2)2^{k-1}+1}$.

Assume that $2^k - 1 \le n < 2^{k+1} - 1$. When $2^k - 1 \le n < 3 \cdot 2^{k-1}$ and $n \ge 2023$, we have $k \ge 11$. There exists an $n \times n$ matrix X with entries $0, 1$ such that

$$\det X \ge 2^{(k-2)2^{k-1}+1} > 2^{(k-2)2^{k-1}}.$$

We have

$$n^{\frac{n}{4}} < (2^{k+1})^{3 \cdot 2^{k-3}} = 2^{3(k+1) \cdot 2^{k-3}}.$$

Due to $k \ge 11$, we get

$$(k-2)2^{k-1} \ge 3(k+1) \cdot 2^{k-3}.$$

Then, $\det X > n^{\frac{n}{4}}$.

When $3 \cdot 2^{k-1} \le n < 2^{k+1} - 1$ and $n \ge 2023$, we have $k \ge 10$. There exists an $n \times n$ matrix X with entries $0, 1$ such that

$$\det X \ge 2^{(k-2)2^{k-1}+1} 2^{(k-3) \cdot 2^{k-2}+1} > 2^{(3k-7)2^{k-2}}.$$

We have

$$n^{\frac{n}{4}} < (2^{k+1})^{2^{k-1}} = 2^{(k+1) \cdot 2^{k-1}}.$$

Due to $k \ge 10$, we get

$$(3k-7)2^{k-2} > (k+1) \cdot 2^{k-1}.$$

Then, $\det X > n^{\frac{n}{4}}$.

6 For a real number r, set $||r|| = \min\{|r-n| : n \in \mathbb{Z}\}$, where $|\cdot|$ means the absolute value of a real number.

1. Is there a nonzero real number s, such that $\lim_{n\to\infty} ||(\sqrt{2}+1)^n s|| = 0$?
2. Is there a nonzero real number s, such that $\lim_{n\to\infty} ||(\sqrt{2}+3)^n s|| = 0$?

6 Answer

1. Yes. We prove that $s = 1$ has the property. Denote $(\sqrt{2}+1)^n = x_n + \sqrt{2}y_n$, where $x_n, y_n \in \mathbb{Z}$. Then $(-\sqrt{2}+1)^n = x_n - \sqrt{2}y_n$ and $x_n^2 - 2y_n^2 = (-1)^n$. It follows that $|x_n + \sqrt{2}y_n - 2x_n| = |\sqrt{2}y_n - x_n| = \frac{|2y_n^2 - x_n^2|}{\sqrt{2}y_n + x_n} \to 0$.

2. No. We prove this by contradiction. Assume that there is some real number $s \neq 0$ such that $(\sqrt{2}+3)^n s = m_n + \epsilon_n$, where $\lim_{n\to\infty} \epsilon_n = 0$. Denote $\alpha = \sqrt{2}+3, \bar{\alpha} = -\sqrt{2}+3$. Consider the power series:

$$\frac{s}{1-\alpha x} = \sum_{n=0}^{\infty} m_n x^n + \sum_{n=0}^{\infty} \epsilon_n x^n.$$

Since $(1-\alpha x)(1-\bar{\alpha}x) = 1 - 6x + 7x^2$, multiplying both sides of the above equation by $1 - 6x + 7x^2$ we get

$$s(1-\bar{\alpha}x) = (1-6x+7x^2)\sum_{n=0}^{\infty} m_n x^n + (1-6x+7x^2)\sum_{n=0}^{\infty} \epsilon_n x^n. \tag{9}$$

Denote $(1-6x+7x^2)\sum_{n=0}^{\infty} m_n x^n = \sum_{n=0}^{\infty} p_n x^n, (1-6x+7x^2)\sum_{n=0}^{\infty} \epsilon_n x^n = \sum_{n=0}^{\infty} \eta_n x^n$, where $p_n \in \mathbb{Z}, \lim_{n\to\infty} \eta_n = 0$. Because the left hand side of (9) is a polynomial of degree 1it holds that $p_n + \eta_n = 0, n \geq 2$. Since $\lim_{n\to\infty} \eta_n = 0$, we get $p_n = \eta_n = 0$ when n is large enough. As a consequnce, $(1-6x+7x^2)\sum_{n=0}^{\infty} m_n x^n$ and $(1-6x+7x^2)\sum_{n=0}^{\infty} \epsilon_n x^n$ are polynomials. So we have

$$\sum_{n=0}^{\infty} \epsilon_n x^n = \frac{G(x)}{1-6x+7x^2}.$$

Write the right hand side as $H(x) + \frac{A}{1-\bar{\alpha}x} + \frac{B}{1-\alpha x}$, where $H(x)$ is a polynomial and A, B are constants. Since $\lim_{n\to\infty} \epsilon_n = 0$, the radius of convergence of the power series in the left hand side is at least 1. While α and $\bar{\alpha}$ are larger than 1 A and B must be zero. Hence $\epsilon_n = 0$ for large n. It follows that $(\sqrt{2}+3)^n s = m_n \in \mathbb{Z}$ for large n. It's a contradiction!

7 A company has one open position available, and N candidates applied (N is known). Assume the N candidates' abilities for this position are all different from each other (in other words, there is a non-ambiguous ranking among the N candidates), and the hiring committee can observe the full relative ranking of all the candidates they have interviewed, and their observed rankings are faithful with respect to the candidates' true abilities. The hiring committee decides the following rule to select one candidate from N:

1. The committee interviews the candidates one by one, at a completely random order. They observe information on candidates' relative ranking regarding their abilities for the position. The only information available to them after interviewing m candidates is the relative ranking among these m people.
2. After each interview, the committee decides whether to offer the candidate the position or not.
3. If they decide to offer the position to the candidate just interviewed, then the candidate will accept the job with probability p, and decline the offer with probability $1-p$, independently with all other candidates. If the selected candidate accepts the offer, then he/she gets the job, and the committee stops interviewing the remaining candidates. If he/she declines the offer, then the committee proceed to interviewing the next candidate.
4. If they decide not to offer the position to the candidate just interviewed, then they proceed to interviewing the next candidate, and they can not turn back to previously interviewed candidates any more.
5. The committee continues this process until a candidate is selected and accepts the job, or until they finish interviewing all N candidates if the position has not been filled before, whichever comes first.

Since the interview order of the candidates are completely random, each ranking has equal probability among the $N!$ possibilities. The committee's mission is to maximise the probability of getting the candidate with the highest ranking (among N candidates) for the job constrained to the above selection process.

Here are the questions

(a) Fix $1 \leq m \leq N$, and consider the following strategy. The committee interviews the first $m-1$ candidates, and do not give offer to any of them regardless of their relative rankings. Starting from the m-th candidate, the committee offers him/her the position whenever the candidate's relative ranking is the highest among all previously interviewed candidates. If he/she declines the offer, then the committee continues the interview until the next relatively best candidate[1], and then repeat the process when applicable.

Show that for every N, there exists $m = m_N$ such that the above strategy maximises the probability of getting the best candidate among all possible strategies.

[1] "Relatively best candidate" refers to the candidate with the highest ability among all candidates who have been interviewed (including those who are offered the position and declined).

(b) Suppose $p = 1$. What is the limit of $\frac{m_N}{N}$ as $N \to +\infty$?

(c) For $p \in (0, 1)$, what is the limit of $\frac{m_N}{N}$ as $N \to +\infty$?

7 Answer For any $1 \le k \le N$, let Z_k be the expected probability of winning (getting the best candidate) when skipping the first $m-1$ candidates interviewed (not giving offers to them no matter their relative rankings), and starting optimal strategy from the m-th candidate. Then, we have

$$Z_k \ge Z_{k+1} \ .$$

(a) If the committee interviews the k-th candidate, and he/she is the relatively best one among the k people, then the committee will make him/her an offer. Under this event, the conditional probability that the committee gets the best person (among the N people) for the job is

$$Y_k = \frac{pk}{N} + (1-p)Z_{k+1} \ .$$

Hence, the committee offers the k-th candidate the position if and only if he/she is the best among the first k candidates, and

$$\frac{pk}{N} + (1-p)Z_{k+1} \ge Z_{k+1} \ , \tag{10}$$

which is equivalent to $\frac{k}{N} \ge Z_{k+1}$. Since $\{\frac{k}{N}\}_k$ is increasing in k while $\{Z_k\}_k$ is decreasing, and that $Z_k \le \frac{N-k+1}{N}$, there exists $k \le N-1$ such that (10) holds. We then conclude that there exists $m = m_N$ (which is the smallest k such that (10) holds) such that the above strategy maximises the probability of getting the best person.

(b) Let p_m be the probability of adopting the strategy in part (a) (offering the relatively best candidate starting from the m-th person) and getting the best person. We want to find $m = m_N$ that maximises p_m. When $p = 1$, p_m is the probability of the following disjoint union of events:

$$\bigcup_{k=m}^{N} A_k \ ,$$

where A_k is the event that the k-th candidate is the best among the N people and he/she is interviewed. Then, we have

$$\mathbf{P}(A_k) = \frac{1}{N} \cdot \frac{m-1}{k-1} \ ,$$

where $\frac{1}{N}$ is the probability that the k-th person is the best among the N, and $\frac{m-1}{k-1}$ is the probability that he/she is interviewed conditioned on him being the best. Thus, we have

$$p_m = \frac{m-1}{N} \sum_{k=m}^{N} \frac{1}{k-1} \ .$$

Since p_m first increases in m and then decreases, the optimal $m = m_N$ that maximises p_m should be the smallest m that satisfies

$$p_m \geq p_{m+1} \ ,$$

or equivalently,

$$\sum_{k=m+1}^{N} \frac{1}{k-1} \leq 1 \ .$$

The left hand side is approximately $\log(N/m)$ when N is large. Hence, $m_N \to \frac{1}{e}$ as $N \to +\infty$.

(c) For general $p \in (0,1) p_m$ is the probability of the following disjoint union of events:

$$\bigcup_{k=m}^{N} A_k \ ,$$

where A_k is the event that the k-th candidate is the best among N, he/she is interviewed, and accepts the offer. Hence, we have

$$p_m = \frac{p}{N} \sum_{k=m}^{N} q_k \ ,$$

where q_k is the probability that the k-th person is interviewed conditioned on him/her being the best. Thus, we have

$$q_k = \Big(\frac{m-1}{m} + \frac{1-p}{m}\Big)\Big(\frac{m}{m+1} + \frac{1-p}{m+1}\Big) \cdots \Big(\frac{k-2}{k-1} + \frac{1-p}{k-1}\Big) = \frac{\Gamma(m)\Gamma(k-p)}{\Gamma(k)\Gamma(m-p)} \ ,$$

where Γ is the standard Γ function. Hence, we get

$$p_m = \frac{p}{N} \cdot \frac{\Gamma(m)}{\Gamma(m-p)} \sum_{k=m}^{N} \frac{\Gamma(k-p)}{\Gamma(k)} \ .$$

Again, since p_m first increases and then decreases in m, by asymptotics of Γ functions and integral approximations to summation, we get the optimal $m = m_N$ satisfies the asymptotics

$$\frac{m_N}{N} \to p^{\frac{1}{1-p}} \ .$$

When $p = 1$, the limit is $\frac{1}{e}$, which agrees with part (b).