Examination Syllabus

The 9th S. T. Yau College Student Mathematics Contest

Syllabuses on Algebra and Number Theory

Linear Algebra

Abstract vector spaces; subspaces; dimension; matrices and linear transformations; matrix algebras and groups; determinants and traces; eigenvectors and eigenvalues, characteristic and minimal polynomials; diagonalization and triangularization of operators; invariant subspaces and canonical forms; inner products and orthogonal bases; reduction of quadratic forms; hermitian and unitary operators, bilinear forms; dual spaces; adjoints. tensor products and tensor algebras;

Integers and polynomials

Integers, Euclidean algorithm, unique decomposition; congruence and the Chinese Remainder theorem; Quadratic reciprocity ; Indeterminate Equations. Polynomials, Euclidean algorithm, uniqueness decomposition, zeros; The fundamental theorem of algebra; Polynomials of integer coefficients, the Gauss lemma and the Eisenstein criterion; Polynomials of several variables, homogenous and symmetric polynomials, the fundamental theorem of symmetric polynomials.

Group

Groups and homomorphisms, Sylow theorem, finitely generated abelian groups. Examples: permutation groups, cyclic groups, dihedral groups, matrix groups, simple groups, Jordan-Holder theorem, linear groups (GL(n, F) and its subgroups), p-groups, solvable and nilpotent groups, group extensions, semi-direct products, free groups, amalgamated products and group presentations.

Ring

Basic properties of rings, units, ideals, homomorphisms, quotient rings, prime and maximal ideals, fields of fractions, Euclidean domains, principal ideal domains and unique factorization domains, polynomial and power series rings, Chinese Remainder Theorem, local rings and localization, Nakayama's lemma, chain conditions and Noetherian rings, Hilbert basis theorem, Artin rings, integral ring extensions, Nullstellensatz, Dedekind domains,algebraic sets, Spec(A).

Module

Modules and algebra   Free and projective; tensor products; irreducible modules and Schur’s lemma; semisimple, simple and primitive rings; density and Wederburn theorems; the structure of finitely generated modules over principal ideal domains, with application to abelian groups and canonical forms; categories and functors; complexes, injective modues, cohomology; Tor and Ext.

Field

Field extensions, algebraic extensions, transcendence bases; cyclic and cyclotomic extensions; solvability of polynomial equations; finite fields; separable and inseparable extensions; Galois theory, norms and traces, cyclic extensions, Galois theory of number fields, transcendence degree, function fields.

Group representation

Irreducible representations, Schur's lemma, characters, Schur orthogonality, character tables, semisimple group rings, induced representations, Frobenius reciprocity, tensor products, symmetric and exterior powers, complex, real, and rational representations.

Lie Algebra

Basic concepts, semisimple Lie algebras, root systems, isomorphism and conjugacy theorems, representation theory.

Combinatorics (TBA)

References:

Strang, Linear algebra, Academic Press.

I.M. Gelfand, Linear Algebra

《整数与多项式》冯克勤 余红兵著 高等教育出版社

Jacobson, Nathan Basic algebra. I. Second edition. W. H. Freeman and Company, New York, 1985. xviii+499 pp.

Jacobson, Nathan Basic algebra. II. Second edition. W. H. Freeman and Company, New York, 1989. xviii+686 pp.

S. Lang, Algebra, Addison-Wesley

冯克勤，李尚志，查建国，章璞，《近世代数引论》

刘绍学，《近世代数基础》

J. P. Serre, Linear representations of finite groups

J. P. Serre: Complex semisimple Lie algebra and their representations

J. Humphreys: Introduction to Lie algebra and representation theory, GTM 009.

W. Fulton, Representation theory, a First Course, GTM 129.

Syllabuses on Analysis and Differential Equations

Calculus and mathematical analysis

Derivatives, chain rule; maxima and minima, Lagrange multipliers; line and surface integrals of scalar and vector functions; Gauss', Green's and Stokes' theorems. Sequences and series, Cauchy sequences, uniform convergence and its relation to derivatives and integrals; power series, radius of convergence, convergence of improper integrals. Inverse and implicit function theorems and applications; the derivative as a linear map; existence and uniqueness theorems for solutions of ordinary differential equations, explicit solutions of simple equations.; elementary Fourier series.

Complex analysis

Analytic function, Cauchy's Integral Formula and Residues, Power Series Expansions, Entire Function, Normal Families, The Riemann Mapping Theorem, Harmonic Function, The Dirichlet Problem Simply Periodic Function and Elliptic Functions, The Weierstrass Theory Analytic Continuation, Algebraic Functions, Picard's Theorem

Point set topology of Rn

Countable and uncountable sets, the axiom of choice, Zorn's lemma.
Metric spaces. Completeness; separability; compactness; Baire category; uniform continuity; connectedness; continuous mappings of compact spaces.
Functions on topological spaces. Equicontinuity and Ascoli's theorem; the Stone-Weierstrass theorem; topologies on function spaces; compactness in function spaces.

Measure and integration

Measures; Borel sets and contor sets; Lebesgue measures; distributions; product measures.
Measurable functions. approximation by simple functions; convergence in measure;

Construction and properties of the integral; convergence theorems;
Radon-Nykodym theorem; Fubini's theorem; mean convergence.
Monotone functions; functions of bounded variation and Borel measures;
Absolute continuity, convex functions; semicontinuity.

Banach and Hilbert spaces

Lp spaces; C(X); completeness and the Riesz-Fischer theorem; orthonormal bases; linear functionals; Riesz representation theorem; linear transformations and dual spaces; interpolation of linear operators; Hahn-Banach theorem; open mapping theorem; uniform boundedness (or Banach-Steinhaus) theorem; closed graph theorem. Basic properties of compact operators, Riesz-Fredholm theory, spectrum of compact operators. Basic properties of Fourier series and the Fourier transform; Poission summation formula; convolution.

Basic partial differential equations

First order partial differential equations, linear and quasi-linear PDE, Wave equations: initial condition and boundary condition, well-poseness, Sturn-Liouville eigen-value problem, energy functional method, uniqueness and stability of solutions Heat equations: initial conditions, maximal principle and uniqueness and stability Potential equations: Green functions and existence of solutions of Dirichlet problem, harmonic functions, Hopf’s maximal principle and existence of solutions of Neumann’s problem, weak solutions, eigen-value problem of the Laplace operator Generalized functions and fundamental solutions of PDE

References:

Rudin, Principles of mathematical analysis, McGraw-Hill.

Courant, Richard; John, Fritz Introduction to calculus and analysis. Vol. I. Reprint of the 1989 edition.Classics in Mathematics. Springer-Verlag, Berlin, 1999.

Courant, Richard; John, Fritz Introduction to calculus and analysis. Vol. II. With the assistance of Albert A. Blank and Alan Solomon. Reprint of the 1974 edition. Springer-Verlag, New York, 1989.

V. I. Arnold, Ordinary Differential Equations, Springer-Verlag, Berlin, 2006.

Valerian Ahlfors, An Introduction to the Theory of Analytic Functions of One Complex Variable

K. Kodaira, Complex Analysis

Rudin, Real and complex analysis

龚升，简明复分析

Royden, Real Analysis, except chapters 8, 13, 15.

E.M. Stein and R. Shakarchi; Real Analysis: Measure Theory, Integration, and Hilbert Spaces, Princeton University Press, 2005

周民强, 实变函数论, 北京大学出版社, 2001

夏道行等，《实变函数论与泛函分析》，人民教育出版社.

Peter D. Lax, Functional Analysis, Wiley-Interscience, 2002.

《Basic Partial Differential Equations》, D. Bleecker, G. Csordas 著, 李俊杰 译，高等教育出版社，2008.

《数学物理方法》，柯朗、希尔伯特著。

Syllabuses on Computational Mathematics and Applied Mathematics

Interpolation and approximation

Trigonometric interpolation and approximation, fast Fourier transform; approximations by rational functions; polynomial and spline interpolations and approximation; least-squares approximation.

Nonlinear equation solvers

Convergence of iterative methods (bisection, Newton's method, quasi-Newton’s methods and fixed-point methods)for both scalar equations and systems, finding roots of polynomials.

Linear systems and eigenvalue problems

Classical and modern iterative method for linear systems and eigenvalue problems, condition number and singular value decomposition.

Numerical solutions of ordinary differential equations

Single step methods and multi-step methods, stability, accuracy and convergence; absolute stability, long time behavior; numerical methods for
stiff ODE’s.

Numerical solutions of partial differential equations

Finite difference method, finite element method and spectral method: stability, accuracy and convergence, Lax equivalence theorem.

References:

[1] C. de Boor and S.D. Conte, *Elementary Numerical Analysis, an algorithmic approach*, McGraw-Hill, 2000.

[2] G.H. Golub and C.F. van Loan, *Matrix Computations, third edition*, Johns Hopkins University Press, 1996.

[3] E. Hairer, P. Syvert and G. Wanner, *Solving Ordinary Differential Equations,* Springer, 1993.

[4] B. Gustafsson, H.-O. Kreiss and J. Oliger, *Time Dependent Problems and Difference Methods,* John Wiley Sons, 1995.

[5] Lloyd N. Trefethen and David Bau, *Numerical linear algebra*, SIAM, 1997.

[6] Susanne Brenner and Ridgway Scott, *The Mathematical Theory of Finite Element Methods*, Springer, 2010.

Ordinary differential equations and dynamical systems

ODE and dynamical systems, critical points, phase space & stability analysis; Hamiltonian dynamical systems; ODE system with gradient structure, conservative ODE's.

Partial differential equations and applications

Basic theory for elliptic, parabolic, and hyperbolic PDEs; calculus of variations: Euler-Lagrange equations; shock waves and rarefaction waves in scale conservation laws; method of characterization, weak formulation, energy estimates, maximum principle; Hamilton-Jacobi equations, Lax-Milgram, Fredholm operator.

Mathematical modeling, simulation, and applied analysis

Scaling behavior and asymptotics analysis, stationary phase analysis, boundary layer analysis, qualitative and quantitative analysis of mathematical models, Monte-Carlo method.

Linear and nonlinear programming

Simplex method, interior method, penalty method, Newton's method, homotopy method and fixed point method, dynamic programming.

References:

[1] W. D. Boyce and R. C. DiPrima, *Elementary Differential Equations*, Wiley, 2009.

[2] V. I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer.

[3] F.Y.M. Wan, *Introduction to Calculus of Variations and Its Applications*, Chapman & Hall, 1995

[4] J. Keener, "*Principles of Applied Mathematics*", Addison-Wesley, 1988.

[5] C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers*, 1999.

[6] A. J. Chorin and J. E. Marsden, *A Mathematical Introduction to Fluid Mechanics*, 2000.

Graph theory and Discrete Mathematics

Hamiltonian, coloring, network flow, network algorithm, connectivity, spanning tree, connectivity testing, bipartite graphs, trees, breadth/depth first search.

Computational Number Theory

Primality, integer factorization; greatest common divisor; Chinese Remainder Theorem; modular arithmetic.

Computational geometry and discrete geometry

Convex hull, Delaunay triangulation, Voronoi diagram, arrangement, discrete curvature, discrete Ricci flow.

References:

[1] A. Bondy and U. S. R. Murty: "*Graph theory*", GTM, Springer, 1976.

[2] T. H. Cormen, C. E. Leiserson, R. L. Rivest, & C. Stein, "*Introduction to Algorithms*", MIT Press, 2009.

[3] S. L. Devaloss and Joseph O'Rourke, "*Discrete and Computational Geometry*", Princeton University Press, 2011.

[4] Mark De Berg, "*Computational Geometry: Algorithms and Applications*", Springer, 2008.

[5] Xianfeng Gu and S. T. Yau, "*Computational conformal geometry*", International Press, 2003.

Syllabuses on Geometry and Topology

Space curves and surfaces

Curves and Parametrization, Regular Surfaces; Inverse Images of Regular Values.
Gauss Map and Fundamental Properties; Isometries; Conformal Maps; Rigidity of the Sphere.

Topological space

Space, maps, compactness and connectedness, quotients; Paths and Homotopy. The Fundamental Group of the Circle. Induced Homomorphisms. Free Products of Groups. The van Kampen Theorem. Covering Spaces and Lifting Properties; Simplex and complexes. Triangulations. Surfaces and its classification.

Differential Manifolds

Differentiable Manifolds and Submanifolds, Differentiable Functions and Mappings; The Tangent Space, Vector Field and Covector Fields. Tensors and Tensor Fields and differential forms. The Riemannian Metrics as examples, Orientation and Volume Element; Exterior Differentiation and Frobenius's Theorem; Integration on manifolds, Manifolds with Boundary and Stokes' Theorem.

Homology and cohomology

Simplicial and Singular Homology. Homotopy Invariance. Exact Sequences and Excision. Degree. Cellular Homology. Mayer-Vietoris Sequences. Homology with Coefficients. The Universal Coefficient Theorem. Cohomology of Spaces. The Cohomology Ring. A Kunneth Formula. Spaces with Polynomial Cohomology. Orientations and Homology. Cup Product and Duality.

Riemannian Manifolds

Differentiation and connection, Constant Vector Fields and Parallel Displacement

Riemann Curvatures and the Equations of Structure   Manifolds of Constant Curvature,
Spaces of Positive Curvature, Spaces of Zero Curvature, Spaces of Constant Negative Curvature

References:

M. do Carmo , Differentia geometry of curves and surfaces.
Prentice- Hall, 1976 (25th printing)

Chen Qing and Chia Kuai Peng, Differential Geometry

M. Armstrong,   Basic Topology   Undergraduate texts in mathematics

W.M. Boothby, An Introduction to Differentiable Manifolds and Riemannian Geometry
Academic Press, Inc., Orlando, FL, 1986

M. Spivak, A comprehensive introduction to differential geometry

N. Hicks, Notes on differential geometry, Van Nostrand.

T. Frenkel, Geometry of Physics

J. Milnor, Morse Theory

A Hatcher, Algebraic Topology (http://www.math.cornell.edu/~hatcher/AT/ATpage.html)

J. Milnor, Topology from the differentiable viewpoint

R. Bott and L. Tu, Differential forms in algebraic topology

V. Guillemin, A. Pollack, Differential topology

Syllabus on Probability Theory and Statistics

Probability Theory

Random variable, Expectation, Independence

Variance and covariance, correlation, moment

Various distribution functions

Multivariate distribution

Characteristic function, Generating function

Various modes of convergence of random variables

Law of large numbers

Random series

Central limit theorem

Bayes formula, Conditional probability

Conditional expectation given a sigma-field

Markov chains

References:

Rick Durrett, Probability: Theory and Examples, Cambridge University Press, 2010

Kai-Lai Chung ,　A Course in Probability Theory, New York, 1968, 有中译本(钟开莱: 概率论教程，　机械工业出版社，　2010）

Statistics

Distribution Theory and Basic Statistics

Families of continuous distributions: normal, chi-sq, t, F, gamma, beta; Families of discrete distributions: multinomial, Poisson, negative binomial; Basic statistics: sample mean, variance, median and quantiles.

Testing

Neyman-Pearson paradigm, null and alternative hypotheses, simple and composite hypotheses, type I and type II errors, power, most powerful test, likelihood ratio test, Neyman-Pearson Theorem, generalized likelihood ratio test.

Estimation

Parameter estimation, method of moments, maximum likelihood estimation, criteria for evaluation of estimators, Fisher information and its use, confidence interval.

Bayesian Statistics

Prior, posterior, conjugate priors, Bayesian estimator.

Large sample properties

Consistency, asymptotic normality, chi-sq approximation to likelihood ratio statistic.

References:

Casella, G. and Berger, R.L. (2002). Statistical Inference (2nd Ed.) Duxbury Press.

茆诗松，程依明，濮晓龙，概率论与数理统计教程（第二版），高等教育出版社，2008.

陈家鼎，孙山泽，李东风，刘力平，数理统计学讲义，高等教育出版社，2006.

郑明，陈子毅，汪嘉冈，数理统计讲义，复旦大学出版社，2006.

陈希孺，倪国熙，数理统计学教程，中国科学技术大学出版社，2009.