S-T Yau College Math Contests 2012 Oral Exam
Geometry and Topology

1. (a) Show that for any $k \in \mathbb{Z}$, there exists a continuous map $f : S^1 \times S^1 \to S^2$ of degree k.

(b) Let Σ_2 be the closed surface of genus 2. Show that any continuous map $f : S^1 \times S^1 \to \Sigma_2$ has degree 0.

2. Let Σ_2 be the closed orientable surface of genus 2.
  (a) What is $G = \pi_1(\Sigma_2)$?
  (b) Why is G non-abelian?
  (c) Why does G contain a subgroup of index 7?
  (d) Show that Σ_2 is not a non-trivial cover of any orientable surface.
  (e) Show that Σ_2 is a non-trivial cover of a space.

3. State and prove the Crofton formula.

4. State and sketch the proof of Bonnet-Meyer's theorem.

5. What is a Jacobi field? Compute it in the space form M_k^n. What is the relationship between Jacobi fields and the exponential map.

6. What is the Cartan-Hadamard theorem for non-positively curved spaces? Why is the space a $K(\pi, 1)$ space?

7. Let S be a compact surface in $\mathbb{R}^3$ with smooth boundary and S has negative curvature. Show that S is contained in convex hull of ∂S.