International Journal of Geometric Methods in Modern Physics
Vol. 13, No. 9 (2016) 1650107 (21 pages)
© World Scientific Publishing Company
DOI: 10.1142/S0219887816501073

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A note on para-holomorphic Riemannian–Einstein manifolds

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Received 30 March 2015
Accepted 9 May 2016
Published 29 June 2016

The aim of this note is the study of Einstein condition for para-holomorphic Riemannian metrics in the para-complex geometry framework. First, we make some general considerations about para-complex Riemannian manifolds (not necessarily para-holomorphic). Next, using a one-to-one correspondence between para-holomorphic Riemannian metrics and para-Kähler–Norden metrics, we study the Einstein condition for a para-holomorphic Riemannian metric and the associated real para-Kähler–Norden metric on a para-complex manifold. Finally, it is shown that every semi-simple para-complex Lie group inherits a natural para-Kählerian–Norden Einstein metric.

Keywords: Para-complex manifold; para-Norden metric; para-holomorphic Riemannian metric; Einstein metric.

Mathematics Subject Classification 2010: 53C15, 53C25, 53C56

1. Introduction

A (holomorphic) complex Riemannian manifold is a complex manifold M, together with a (holomorphic) complex tensor field G that is a complex scalar product (i.e. nondegenerate, symmetric, $\mathbb{C}$-bilinear form) on each holomorphic tangent space of M. Geometrical aspects of the complex Riemannian manifolds with analytic (holomorphic) metrics and their applications to mathematical physics have been investigated by many authors, see for instance [8, 10–12, 16, 21, 23, 27, 32]. The holomorphic Riemannian geometry possesses an underlying real geometry consisting of a pseudo-Riemannian metric of neutral signature for which the (integrable) almost complex structure tensor is anti-orthogonal. This leads to the notion of an anti-Kählerian manifold (also known as Kähler–Norden manifold [15, 24, 25, 31] or

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B-manifold [10]) that is a complex manifold with an anti-Hermitian metric and a parallel almost complex structure. In [3], it is proved that a metric on such a manifold must be the real part of a holomorphic metric. There is studied the Einstein condition for anti-Kählerian metrics and it is shown that the complexification of a given Einstein metric leads to a method of generating new solutions of Einstein equations from a given one. Some generalized Einstein conditions on holomorphic Riemannian manifolds are studied in [25].

Although the almost product Einstein manifolds are studied in [2], our aim in this note is to formulate some analogous results as in [3, 25] concerning the Einstein condition for para-holomorphic Riemannian metrics in terms of para-complex geometry.

The notion of almost para-complex structure (or almost product structure) on a smooth manifold was introduced in [22] and a survey of further results on para-complex geometry (including para-Hermitian and para-Kähler geometry) can be found for instance in [6, 7]. Also, other further significant developments are due in some recent surveys [1, 4, 5], where some aspects concerning the geometry of para-complex manifolds are presented sistematically by analogy with the geometry of complex manifolds using some para-holomorphic coordinate systems. See also [9, 17, 18, 20].

The paper is organized as follows. In Sec. 2, following [1, 4, 17], we briefly recall some basic notions used in the para-complex geometry. In Sec. 3, we define para-complex Riemannian metrics on para-complex manifolds, we prove that the real part of such a metric is a para-Norden (or almost product Riemannian) metric and following the construction from the complex case [11, 12], we make some general considerations about the Levi–Civita and characteristic connections on para-complex Riemannian manifolds. Also, a Schur type theorem concerning the para-holomorphic curvature is presented and we write Einsten equations in our setting. In Sec. 4, starting from a one-to-one coorespondence between para-holomorphic Riemannian metrics and real para-Kähler–Norden metrics on a para-complex manifold, we prove an equivalence between Einstein condition with real constant for para-holomorphic Riemannian metrics and for the associated para-Kähler metric, giving an analogous result from the case of anti-Kählerian–Einstein manifolds [2, 3]. Also, the case when the para-holomorphic Riemannian metric is Einstein with a para-complex constant is also analyzed in a similar manner to the case of Kähler–Norden metrics [25]. In the last section, as an example of our study, it is shown that every semi-simple para-complex Lie group inherits a natural para-Kählerian–Norden–Einstein metric.

We notice that other problems related to generalized Einstein condition as in [25] can be addressed in the context of para-holomorphic Riemannian manifolds. Also, an important example of anti-Kählerian–Einstein metric on the tangent bundle of a space form is given in [26] and a similar study can be also analyzed in the context of para-complex geometry.

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The main methods used here are similar and closely related to those used in the study of complex Riemannian manifolds [11, 12, 16] and anti-Kählerian manifolds [2, 3, 25, 31].

2. Preliminaries and Settings in Para-Complex Geometry

The algebra of para-complex numbers is defined as the vector space $C = \mathbb{R} \times \mathbb{R}$ with multiplication given by

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1x_2 + y_1y_2, x_1y_2 + y_1x_2), \quad \forall (x_1, y_1), (x_2, y_2) \in C.$$

Setting $e = (0, 1)$, then $e^2 = (1, 0) = 1$ and we can write $C = \mathbb{R} + e\mathbb{R} = \{z = x + ey \,|\, x, y \in \mathbb{R}\}$.

The conjugation of an element $z = x + ey \in C$ is defined as usual by $\overline{z} = x - ey$ and $\operatorname{Re} z = x$ and $\operatorname{Im} z = y$ are called the *real part* and *imaginary part* of the para-complex number z.

A *para-complex* structure on a real finite dimensional vector space V is defined as an endomorphism $I \in \operatorname{End}(V)$ which satisfies $I^2 = \mathrm{Id}$, $I \neq \pm\mathrm{Id}$ and the following two eigenspaces $V^{\pm} := \ker(\mathrm{Id} \pm I)$ corresponding to the eigenvalues ± 1 of I have the same dimension. Such a pair (V, I) is called a *para-complex vector space*. Consequently, an *almost para-complex structure* on a smooth manifold M is defined as an endomorphism $I \in \operatorname{End}(TM)$ with the property that (T_xM, I_x) is a para-complex vector space, for every $x \in M$. Moreover, an almost para-complex structure I on M is said to be *integrable* if the distributions $T^{\pm}M = \ker(\mathrm{Id} \mp I)$ are both integrable, and in this case I is called a *para-complex structure* on M. A manifold M endowed with a para-complex structure is called a *para-complex manifold*. The para-complex dimension of a para-complex manifold M is the integer $n = \dim_C M := (\dim_{\mathbb{R}} M)/2$.

Given two almost para-complex manifolds (M, I_M) and (N, I_N), a smooth map $f : (M, I_M) \to (N, I_N)$ is called *para-holomorphic* (*respectively*: *anti-para-holomorphic*) if

$$df \circ I_M = I_N \circ df \quad (\text{respectively: } df \circ I_M = -I_N \circ df). \tag{2.1}$$

Moreover, an (anti-)para-holomorphic map $f : (M, I_M) \to C$ is called (*anti*)-*para-holomorphic function*.

As usual, the Nijenhuis tensor N_I associated to an almost para-complex structure I is defined by

$$N_I(X, Y) := [IX, IY] - I[IX, Y] - I[X, IY] + [X, Y], \tag{2.2}$$

for every $X, Y \in \Gamma(TM)$, and according to [4] we have that I is integrable iff $N_I = 0$. The Frobenius theorem implies, see [5], the existence of local coordinates (z^a_+, z^a_-), $a = 1, \dots, n = \dim_C M$ on para-complex manifold (M, I), such that $T^+M = \operatorname{span}\{\partial/\partial z^a_+\}$, $T^-M = \operatorname{span}\{\partial/\partial z^a_-\}$, $a \in \{1, \dots, n\}$. Such (real) coordinates are called *adapted coordinates* for the para-complex structure I.

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As in the complex case, on every para-complex manifold (M, I_M) we can define an atlas of para-holomorphic local charts $(U_\alpha, \varphi_\alpha)$, such that the transition functions $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \subset C^n \to \varphi_\beta(U_\alpha \cap U_\beta) \subset C^n$ are para-holomorphic functions in the sense of (2.1). Moreover, to every (real) adapted coordinate system (z_+^a, z_-^a), $a \in \{1, \ldots, n\}$ on U_α we can associate a para-holomorphic system (z^a), $a = 1, \ldots, n$ by setting

$$z^a = \frac{z_+^a + z_-^a}{2} + e\frac{z_+^a - z_-^a}{2} := x^a + ey^a, \quad a \in \{1, \ldots, n\}. \tag{2.3}$$

According to [4], z^a are para-holomorphic functions in the sense of (2.1) and the transition functions between two para-holomorphic coordinate systems are also para-holomorphic. Equivalently, if $(\widetilde{z}^b)$, $b = 1, \ldots, n$ is a para-holomorphic coordinate system on U_β, with $\widetilde{z}^b = \widetilde{x}^b + e\widetilde{y}^b$, then the following para-Cauchy–Riemann equations hold (see for instance [17]):

$$\frac{\partial \widetilde{x}^b}{\partial x^a} = \frac{\partial \widetilde{y}^b}{\partial y^a}, \quad \frac{\partial \widetilde{x}^b}{\partial y^a} = \frac{\partial \widetilde{y}^b}{\partial x^a}, \quad a, b \in \{1, \ldots, n\}. \tag{2.4}$$

In this case, on each U_α, I is given by

$$I\left(\frac{\partial}{\partial x^a}\right) = \frac{\partial}{\partial y^a}, \quad I\left(\frac{\partial}{\partial y^a}\right) = \frac{\partial}{\partial x^a}. \tag{2.5}$$

Now, we consider the para-complexification of the tangent bundle TM as the $\mathbb{R}$-tensor product $T_C M = TM \otimes_{\mathbb{R}} C$ and its decomposition $T_C M = T^{1,0}M \oplus T^{0,1}M$ produced by C-linear extension of I to $T_C M$, where

$$T_x^{1,0}M = \{Z = T_{x,C}M \mid IZ = eZ\} = \{X + eIX \mid X \in T_x M\},$$
$$T_x^{0,1}M = \{Z = T_{x,C}M \mid IZ = -eZ\} = \{X - eIX \mid X \in T_x M\},$$

are the eigenspaces of I with eigenvalues $\pm e$. Also, if I is integrable, that is (M, I) is a para-complex manifold, the para-complex vectors

$$\frac{\partial}{\partial z^a} = \frac{1}{2}\left(\frac{\partial}{\partial x^a} + e\frac{\partial}{\partial y^a}\right),$$
$$\frac{\partial}{\partial z^{\overline{a}}} = \frac{1}{2}\left(\frac{\partial}{\partial x^a} - e\frac{\partial}{\partial y^a}\right)$$

form a basis of the spaces $T_x^{1,0}M$ and $T_x^{0,1}M$.

Remark 1 ([4]). A C-valued function $f : M \to C$ on a para-complex manifold (M, I) is para-holomorphic iff it satisfies

$$\frac{\partial f}{\partial z^{\overline{a}}} = 0, \quad \forall\, a \in \{1, \ldots, n\}, \tag{2.6}$$

where (z^a) are local para-holomorphic coordinates on (M, I) and $z^{\overline{a}} = \overline{z}^a$.

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3. Para-Complex Riemannian Manifolds

Let M be a para-complex manifold of para-complex dimension n and denote by (M, I) the manifold considered as a real $2n$-dimensional manifold with the induced almost para-complex structure I.

Definition 2. A *para-complex Riemannian metric* on M is a covariant symmetric 2-tensor field $G : \Gamma(T_C M) \times \Gamma(T_C M) \to C$, which is non-degenerate at each point of M and satisfies

$$G(\overline{Z}_1, \overline{Z}_2) = \overline{G(Z_1, Z_2)}, \quad \text{for every } Z_1, Z_2 \in \Gamma(T_C M), \tag{3.1}$$

$$G(Z_1, Z_2) = 0, \quad \text{for every } Z_1 \in \Gamma(T^{1,0}M) \quad \text{and} \quad Z_2 \in \Gamma(T^{0,1}M). \tag{3.2}$$

It is easy to see that the relation (3.2) is equivalent to

$$G(IZ_1, IZ_2) = G(Z_1, Z_2), \quad \text{for every } Z_1, Z_2 \in \Gamma(T_C M), \tag{3.3}$$

where we have denoted again by I the C-linear extension of I to $T_C M$. Thus a para-complex Riemannian metric on M is completely determined by its values on $\Gamma(T^{1,0}M)$.

Definition 3. The pair (M, G) consisting in a para-complex manifold M and a para-complex Riemannian metric G on M, will be called a *para-complex Riemannian manifold*.

If (z^a), $a = 1, \ldots, n$ is a para-holomorphic coordinate system on M, such that $\Gamma(T_C M) = \operatorname{span}\{\partial/\partial z^a, \partial/\partial z^{\overline{a}}\}$, we put

$$G_{AB} = G\left(\frac{\partial}{\partial z^A}, \frac{\partial}{\partial z^B}\right), \quad A, B \in \{1, \ldots, n, \overline{1}, \ldots, \overline{n}\}. \tag{3.4}$$

Then, for a para-complex Riemannian metric G, the defining conditions (3.1) and (3.2) can be expressed locally as

$$G_{\overline{A}\,\overline{B}} = \overline{G_{AB}}, \quad G_{a\overline{b}} = G_{\overline{a}b} = 0. \tag{3.5}$$

Definition 4. A para-complex Riemannian metric G on a para-complex manifold M is called *para-holomorphic Riemannian metric* if the local components G_{ab} are para-holomorphic functions, i.e.

$$\frac{\partial G_{ab}}{\partial z^{\overline{c}}} = 0, \quad \text{for every } c \in \{1, \ldots, n\}. \tag{3.6}$$

In this case, the pair (M, G) is called a *para-holomorphic Riemannian* manifold.

As in the case of complex Riemannian manifolds (see [11]) for a given para-complex metric G on M, we define the tensor field $\widetilde{G}$ on M by setting

$$\widetilde{G}(Z_1, Z_2) = (G \circ I)(Z_1, Z_2) := G(IZ_1, Z_2), \quad \text{for every } Z_1, Z_2 \in \Gamma(T_C M). \tag{3.7}$$

This metric is said to be associated with G (also called *twin metric*), and locally, it satisfies

$$\widetilde{G}_{ab} = eG_{ab} \quad \text{and} \quad \widetilde{G}_{\overline{a}\,\overline{b}} = -eG_{\overline{a}\,\overline{b}}. \tag{3.8}$$

Also, we notice that given a para-complex Riemannian manifold (M, G), the para-complex Riemannian metric G induces a real Riemannian metric g on the real manifold (M, I) by setting

$$g(X, Y) = 2\,\mathrm{Re}\,G(\widehat{X}, \widehat{Y}), \quad X, Y \in \Gamma(TM), \tag{3.9}$$

where $\widehat{X} = (1/2)(X + eIX), \widehat{Y} = (1/2)(Y + eIY) \in \Gamma(T^{1,0}M)$, which satisfies

$$\begin{gathered} g(IX, IY) = g(X, Y) \quad \text{or, equivalently,} \\ g(IX, Y) = g(X, IY) \quad \text{for every } X, Y \in \Gamma(TM). \end{gathered} \tag{3.10}$$

Such a real metric is also known as an almost product Riemannain metric or para-Norden metric, and the para-Norden manifold (M, I, g) will be called the *realization* of (M, G). The para-Norden manifolds are studied for instance in [29, 30].

Conversely, every para-Norden metric on the real manifold (M, I) induces a para-complex Riemannian metric on the para-complex manifold M by setting

$$G(\widehat{X}, \widehat{Y}) = \frac{1}{2}(g(X, Y) + eg(X, IY)), \quad X, Y \in \Gamma(TM) \tag{3.11}$$

and $\widehat{X} = (1/2)(X + eIX)$, $\widehat{Y} = (1/2)(Y + eIY) \in \Gamma(T^{1,0}M)$ as above, and next we extend G to have the conditions (3.1) and (3.2), which is possible because of (3.10).

Given any linear connection D on a para-complex manifold (M, I), with respect to a para-holomorphic coordinate system, we put

$$D_{\frac{\partial}{\partial z^A}} \frac{\partial}{\partial z^B} = L^C_{AB} \frac{\partial}{\partial z^C}.$$

We notice that the covariant differentiation, which is defined for real vector fields in $\Gamma(TM)$, can be extended by para-complex linearity on para-complex vector fields from $\Gamma(T_C M)$. Then $L^{\overline{C}}_{\overline{A}\,\overline{B}} = \overline{L^C_{AB}}$, where $\overline{\overline{A}} = A$.

Definition 5. A (real) linear connection D on (M, I) is called *almost para-complex* if $DI = 0$.

Definition 6. A para-Norden manifold (M, I, g) is called *para-Kähler–Norden* manifold if the Levi–Civita connection ∇ of g is almost para-complex.

Similary to the complex case [24, 25, 31] (see also [2, 3]), we have the following one-to-one corresponce between the para-Kähler–Norden metrics and para-holomorphic Riemannian metrics on a para-complex manifold (M, I).

Proposition 7. *Let (M, I) be a para-complex manifold. If G is a para-holomorphic Riemannian metric (M, I) then g defined in (3.9) is a para-Kähler–Norden metric on (M, I), and conversely if g is a para-Kähler metric on the (real) manifold (M, I) then G defined in (3.11) is a para-holomorphic Riemannian metric on (M, I).*

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By direct calculus, we easily obtain

Proposition 8. *A linear connection D on M is almost para-complex iff* $L^{\overline{c}}_{ab} = L^{c}_{a\overline{b}} = 0$.

Now, let us denote by ∇ and $\widetilde{\nabla}$ the Levi–Civita connections of G and $\widetilde{G}$, respectively. Then, as usual, the Christoffel symbols of G are given by

$$\Gamma^{C}_{AB} = \frac{1}{2}G^{CD}\left(\frac{\partial G_{BD}}{\partial z^{A}} + \frac{\partial G_{AD}}{\partial z^{B}} - \frac{\partial G_{AB}}{\partial z^{D}}\right), \tag{3.12}$$

where $(G^{AB})_{n\times n}$ denotes the inverse matrix of $(G_{AB})_{n\times n}$, and similarly for the Christoffel symbols $\widetilde{\Gamma}^{C}_{AB}$ of $\widetilde{G}$.

Taking into account (3.5) and (3.8), we have the following relations which relate the Christoffel symbols of G and $\widetilde{G}$, respectively

$$\widetilde{\Gamma}^{c}_{ab} = \Gamma^{c}_{ab} = \frac{1}{2}G^{cd}\left(\frac{\partial G_{bd}}{\partial z^{a}} + \frac{\partial G_{ad}}{\partial z^{b}} - \frac{\partial G_{ab}}{\partial z^{d}}\right) \tag{3.13}$$

$$\widetilde{\Gamma}^{\overline{c}}_{ab} = -\Gamma^{\overline{c}}_{ab} = \frac{1}{2}G^{\overline{c}\,\overline{d}}\frac{\partial G_{ab}}{\partial z^{\overline{d}}}, \quad \widetilde{\Gamma}^{c}_{\overline{a}b} = \Gamma^{c}_{\overline{a}b} = \frac{1}{2}G^{cd}\frac{\partial G_{bd}}{\partial z^{\overline{a}}}. \tag{3.14}$$

By analogy with the complex case [11], we define the fundamental tensor Φ on a para-complex Riemannian manifold by setting

$$\Phi(Z_1, Z_2) = \widetilde{\nabla}_{Z_1}Z_2 - \nabla_{Z_1}Z_2, \quad \text{for every } Z_1, Z_2 \in \Gamma(T_C M). \tag{3.15}$$

By this definition, we deduce

$$\Phi(\overline{Z}_1, \overline{Z}_2) = \overline{\Phi(Z_1, Z_2)}, \quad \text{for every } Z_1, Z_2 \in \Gamma(T_C M). \tag{3.16}$$

Using (3.15), (3.13), (3.14) and (3.16) the nonvanishing components of the fundamental tensor Φ are given by

$$\Phi^{\overline{c}}_{ab} = G^{\overline{c}\,\overline{d}}\frac{\partial G_{ab}}{\partial z^{\overline{d}}} \quad \text{and} \quad \Phi^{c}_{\overline{a}\,\overline{b}} = \overline{\Phi^{\overline{c}}_{ab}}. \tag{3.17}$$

Also, from (3.15) and (3.17) we have

Proposition 9. *The fundamental tensor of a para-complex Riemannian manifold (M, G) satisfies*

$$\Phi(Z_1, Z_2) = \Phi(Z_2, Z_1), \quad \Phi(IZ_1, Z_2) = -I\Phi(Z_1, Z_2), \quad \forall\, Z_1, Z_2 \in \Gamma(T_C M). \tag{3.18}$$

Remark 10. If (M, I, g) is the realization of a para-complex Riemannian manifold (M, G) we can define as in (3.15) the fundamental tensor for real vector fields, and the property (3.16) of Φ implies that Φ is the para-complex extension of the real fundamental tensor on (M, I, g).

In the following, we extend the study from [11] to the para-complex case, and we shall construct a *characteristic* linear connection on a para-complex Riemannian manifold.

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We consider the fundamental tensor of type $(0,3)$ defined by

$$\Psi(Z_1,Z_2,Z_3)=G(\Phi(Z_1,Z_2),Z_3),\quad \text{for every } Z_1,Z_2,Z_3\in\Gamma(T_CM). \tag{3.19}$$

In a para-holomorphic coordinate system on M, we have locally

$$\Psi_{AB,C}=\Phi^D_{AB}G_{DC}, \tag{3.20}$$

and the nonvanishing componets of $\Psi_{AB,C}$ are

$$\Psi_{ab,\overline{c}}=\frac{\partial G_{ab}}{\partial z^{\overline{c}}}\quad\text{and}\quad \Psi_{\overline{a}\,\overline{b},c}=\overline{\Psi_{ab,\overline{c}}}. \tag{3.21}$$

We have

Theorem 11. *On every para-complex Riemannian manifold (M,G) there exists a unique linear connection D with local coefficients L^C_{AB} such that*

(i) *D is symmetric, that is $L^C_{AB}=L^C_{BA}$;*
(ii) *D is almost para-complex, that is $L^{\overline{c}}_{ab}=L^c_{a\overline{b}}=0$;*
(iii) *The covariant derivatives $D_aG_{bc}=\partial G_{bc}/\partial z^a-L^d_{ab}G_{dc}-L^d_{ac}G_{bd}$ vanish.*

Proof. If we define the local coefficients of D by

$$L^C_{AB}=\Gamma^C_{AB}+\frac{1}{2}\Phi^C_{AB}-\frac{1}{2}G^{CD}(\Psi_{DA,B}+\Psi_{DB,A}), \tag{3.22}$$

where Γ^C_{AB} are the para-complex Christoffel symbols of G, then by direct calculus we obtain that D satisfies the conditions of theorem.

Also, if D' is another connection which satisfies all the conditions of theorem, with local coefficients L'^C_{AB}, we denote by $D^C_{AB}=L^C_{AB}-L'^C_{AB}$ the difference tensor. Then, we easily obtain

$$D^C_{AB}=D^C_{BA},\quad D^{\overline{c}}_{ab}=D^c_{a\overline{b}}=0,\quad D^d_{ab}G_{dc}+D^d_{ac}G_{ab}=0, \tag{3.23}$$

which implies $D^C_{AB}=0$, that is $D=D'$, and the uniqueness then follows. □

The linear connection from the above theorem, will be called the *characteristic connection* of the para-complex Riemannian manifold (M,G).

The defining equality (3.22) of the characteristic connection and the properties of the fundamental tensor imply.

Corollary 12. *On every para-complex Riemannian manifold (M,G) there exists a unique linear connection D such that*

(i) *D is symmetric;*
(ii) *D is almost para-complex;*
(iii) *$D_AG_{BC}=\Psi_{BC,A}$, i.e. the covariant derivative of the metric G is the fundamental tensor Ψ.*

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Remark 13. The third condition of Theorem 11 says that the nonvanishing components of the tensor $D_A G_{BC}$ are

$$D_{\overline{a}} G_{bc} = \Psi_{bc,\overline{a}} \quad \text{and} \quad D_a G_{\overline{b}\,\overline{c}} = \overline{D_{\overline{a}} G_{bc}}. \tag{3.24}$$

On the realization of a para-complex Riemannian manifold we have

Corollary 14. *If (M, I, g) is the realization of a para-complex Riemannian manifold (M, G), then the characteristic connection D on (M, I, g) is the unique connection which satisfies the conditions*

(i) *D is symmetric*;
(ii) *D is almost para-complex*;
(iii) *$(D_X g)(Y, Z) = (D_{IX} g)(IY, Z)$, for every $X, Y, Z \in \Gamma(TM)$.*

The defining equality (3.22) and (3.21) imply that the nonvanishing coefficients of the caracteristic connection D are

$$L_{ab}^{c} = \Gamma_{ab}^{c} \quad \text{and} \quad L_{\overline{a}\,\overline{b}}^{\overline{c}} = \overline{L_{ab}^{c}}, \tag{3.25}$$

that is, D is completely determined on $\Gamma(T^{1,0}M)$.

We notice that a vector field $Z = Z^a(\partial/\partial z^a)$ on a para-complex manifold is para-holomorphic if Z^a are para-holomorphic functions. Also, according to [18, Lemma 2.1.6], a vector field $\widehat{X} = (1/2)(X + eIX)$ is para-holomorphic iff

$$(\mathcal{L}_X I)Y = [X, IY] - I[X, Y] = 0, \quad \forall\, Y \in \Gamma(TM). \tag{3.26}$$

In what follows we denote the set of para-holomorphic vector fields on (M, I) by $\Gamma_{\text{ph}}(T^{1,0}M)$.

Definition 15. A linear connection D on M is called *para-holomorphic* if $D_{Z_1} Z_2 \in \Gamma_{\text{ph}}(T^{1,0}M)$ for arbitrary para-holomorphic vector fields Z_1, Z_2.

We have

Proposition 16. *The characteristic connection D of a para-complex Riemannian manifold (M, G) is para-holomorphic iff the para-complex Christoffel symbols $L_{ab}^{c} = \Gamma_{ab}^{c}$ are para-holomorphic functions.*

As a direct consequence of (3.21), (3.13), (3.14), Corollary 12 and (3.22), we get

Theorem 17. *For every para-complex Riemannian manifold (M, G), the following assertions are equivalent*:

(i) *The fundamental tensor Φ (or Ψ) vanishes*;
(ii) *The local components G_{ab} of the metric G are para-holomorphic functions*;
(iii) *The Levi–Civita connection ∇ of G is almost para-complex, that is $\nabla I = 0$*;
(iv) *The characteristic connection D is metrical with respect to G, that is $DG = 0$*;
(v) *The Levi–Civita connection ∇ coincides with the characteristic connection D.*

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Let R be the characteristic curvature tensor of the characteristic connection D, defined as usual by

$$R(X,Y)Z = [D_X, D_Y]Z - D_{[X,Y]}Z, \quad \text{for every } X,Y,Z \in \Gamma(T_C M).$$

The local components of R are given by

$$R\left(\frac{\partial}{\partial z^A}, \frac{\partial}{\partial z^B}\right)\frac{\partial}{\partial z^C} = R^D_{C,AB}\frac{\partial}{\partial z^D}, \tag{3.27}$$

and the nonvanishing components of R are

$$R^d_{c,ab} = \frac{\partial L^d_{cb}}{\partial z^a} - \frac{\partial L^d_{ca}}{\partial z^b} + L^f_{cb}L^d_{fa} - L^f_{ca}L^d_{fb}, \quad R^{\overline{d}}_{\overline{c},\overline{a}\,\overline{b}} = \overline{R^d_{c,ab}}, \tag{3.28}$$

$$R^d_{c,\overline{a}b} = \frac{\partial L^d_{bc}}{\partial z^{\overline{a}}}, \quad R^{\overline{d}}_{\overline{c},a\overline{b}} = \overline{R^d_{c,\overline{a}b}}. \tag{3.29}$$

It is easy to see that $R^d_{c,\overline{a}b} = 0$ if and only if D is a para-holomorphic connection. Also, the characteristic Riemann curvature tensor of D is defined as usual by $\mathcal{R}(Z_1,Z_2,Z_3,Z_4) = G(R(Z_1,Z_2)Z_3,Z_4)$ and its local components are $R_{ABCD} = G_{DF}R^F_{C,AB}$. Its nonvanishing components are

$$R_{abcd} = G_{df}R^f_{c,ab} \quad \text{and} \quad R_{\overline{a}bcd} = G_{df}R^f_{c,\overline{a}b}, \tag{3.30}$$

and their para-complex conjugates.

Moreover, every nondegenerate 2-plane in $T^{1,0}_z M$ is called a *para-holomorphic* 2-plane, and the *para-holomorphic* characteristic sectional curvature for a given 2-plan $P = \operatorname{span}\{Z_1, Z_2\}$, where $Z_1, Z_2 \in \Gamma(T^{1,0}_z M)$, $z \in M$ is defined by

$$K_z(P) = \frac{\mathcal{R}(Z_1,Z_2,Z_1,Z_2)}{G(Z_1,Z_1)G(Z_2,Z_2) - (G(Z_1,Z_2))^2}. \tag{3.31}$$

The following Schur type theorem holds.

Theorem 18. *Let (M,G) be a connected para-holomorphic Riemannian manifold of para-complex dimension $n \geq 3$. If the para-holomorphic sectional curvature does not depend on the 2-plane P, then (M,G) is of constant para-holomorphic sectional curvature.*

In the end of this section we describe the Einstein equations for para-complex Riemannian manifolds. The associated characteristic Ricci tensor Ric is locally given by

$$\operatorname{Ric}\left(\frac{\partial}{\partial z^C}, \frac{\partial}{\partial z^A}\right) = \operatorname{Ric}_{CA} = R^B_{C,AB}, \tag{3.32}$$

and its nonvanishing components are

$$\operatorname{Ric}_{ca} = R^b_{c,ab}, \quad \operatorname{Ric}_{c\overline{a}} = R^b_{c,\overline{a}b}, \quad \operatorname{Ric}_{\overline{c}\,\overline{a}} = \overline{\operatorname{Ric}_{ca}}, \quad \operatorname{Ric}_{\overline{c}a} = \overline{\operatorname{Ric}_{c\overline{a}}}. \tag{3.33}$$

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The function ρ defined by

$$\rho = G^{CA}\mathrm{Ric}_{CA} = G^{ca}\mathrm{Ric}_{ca} + G^{\overline{c}\,\overline{a}}\mathrm{Ric}_{\overline{c}\,\overline{a}} \tag{3.34}$$

is called the *scalar curvature* of D and it is a real function.

The equation

$$\mathrm{Ric} - \frac{\rho}{2}G = 8\pi c T \tag{3.35}$$

is called the *Einstein equation* of the para-complex Riemannian manifold (M, G). In Eq. (3.35), the left-hand side is called the *Einstein curvature* which is constructed using the para-complex Riemannian metric G, while in the right hand side we have a tensor T called the *stress–energy–momentum tensor*, which represents the matter and energy that generate the gravitational field of potentials (G_{AB}). The constant c is the gravitational constant. Locally, the Einstein equation is expressed as

$$\mathrm{Ric}_{AB} - \frac{\rho}{2}G_{AB} = 8\pi c T_{AB}. \tag{3.36}$$

If the Einstein equation holds, then taking into account (3.5) it follows that $\mathrm{Ric}_{a\overline{b}} = 8\pi c T_{a\overline{b}}$. In the empty leave space (no matter, no energy) we have $T_{AB} = 0$, and contracting (3.36) with G^{AB} one gets $\rho = 0$ and so it reduces to

$$\mathrm{Ric}_{AB} = 0. \tag{3.37}$$

Consequently, $\mathrm{Ric}_{ab} = \mathrm{Ric}_{\overline{a}\,\overline{b}} = 0$.

Letting $E_{AB} = \mathrm{Ric}_{AB} - (\rho/2)G_{AB}$ and $E^A_B = G^{AC}E_{CB}$, the divergence of E is defined by

$$\operatorname{div} E = E^A_{B|A}, \tag{3.38}$$

where "|" denotes the covariant derivative with respect to ∇ and we have $\operatorname{div} E = 0$. The proof is based on the second Bianchi identity $\sum_{\mathrm{cycl}}(\nabla_X R)(Y, Z) = 0$ written in a local basis $\{\partial/\partial z^A\}$ of $\Gamma(T_C M)$. Assuming the Einstein equation holds, by using $\operatorname{div} E = 0$, we must have

$$\operatorname{div} T = 0, \tag{3.39}$$

which is called the *continuity condition* for para-complex Riemannian manifold (M, G).

Finally, by analogy with the complex case, see [16], the following result concerning the Einstein condition for para-complex Riemannian manifolds holds.

Definition 19. The para-complex Riemannian manifold (M, G) is said to be *characteristic Einstein* if $\mathrm{Ric}_{c\overline{a}} = 0$ and $\mathrm{Ric}_{ca} = fG_{ca}$, where $f = f_1 + ef_2$ is a para-complex valued function on M.

Theorem 20. *Let (M, G) be a m-dimensional para-complex Riemannian characteristic Einstein manifold with $m \geq 3$. Then the characteristic scalar curvature $\rho_0 = G^{ca}\mathrm{Ric}_{ca}$ is an anti-para-holomorphic function on M and* $\mathrm{Ric}_{ca} = (\rho_0/m)G_{ca}$.

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4. Para-Holomorphic Riemannain–Einstein Manifolds

We recall that a (real) metric g on the (real) manifold M is said to be an *Einstein metric* if

$$\mathrm{Ric}(g) = \lambda g, \tag{4.1}$$

where λ is a real constant and $\mathrm{Ric}(g)$ denotes the Ricci tensor of the metric g.

In this section, we prove that by taking the real part of a para-holomorphic Einstein metric on a para-complex manifold (M, I) of para-complex dimension n one gets a real Einstein manifold of real dimension $2n$ obtaining a result similar to Theorem 5.1 from [3] from the anti-Kählerian manifolds case.

Let (M, G) be a para-holomorphic Riemannian manifold. Then, as we already noticed in the previous section the relations (3.9) and (3.11) establish a one-to-one correspondence between the para-Kähler–Norden metrics on the (real) manifold (M, I) and the para-holomorphic metrics on the para-complex manifold M.

Although we can follow an argument similar from [2, 3], for a better presentation of the notions that we use, in this section we denote the para-holomorphic Riemannain metric G by $\widehat{g}$ and we follow an argument similar to [25, 31] for Kähler–Norden manifolds.

Without loss of generality, we consider the (real) vector fields $X, Y, \ldots, \in \Gamma(TM)$ such that $\widehat{X}, \widehat{Y}, \ldots, \in \Gamma_{\mathrm{ph}}(T^{1,0}M)$, are para-holomorphic vector fields on the para-complex manifold (M, I), that is the relation (3.26) holds. Then, we easily obtain

$$[IX, Y] = [X, IY] = I[X, Y], \quad [IX, IY] = [X, Y], \quad [\widehat{X}, \widehat{Y}] = \widehat{[X, Y]} =: [X, Y]\widehat{}. \tag{4.2}$$

Also, by a direct calculation, we have that for every para-complex function $f = \mathrm{Re}\, f + e\, \mathrm{Im}\, f$ on M, and every vector field $X \in \Gamma(TM)$, the following relation holds

$$f\widehat{X} = ((\mathrm{Re}\, f)X + (\mathrm{Im}\, f)X)\widehat{}, \tag{4.3}$$

and, moreover, if f is para-holomorphic, then the para-Cauchy–Riemann equations imply

$$X(\mathrm{Re}\, f) = (IX)(\mathrm{Im}\, f), \quad (IX)(\mathrm{Re}\, f) = X(\mathrm{Im}\, f), \quad \widehat{X}f = X(\mathrm{Re}\, f) + eX(\mathrm{Im}\, f). \tag{4.4}$$

Now, for every real tangent space $T_{z,\mathbb{R}}M$, $z \in M$, we can choose an adapted orthonormal (real) frame $\{e_a, Ie_a\}$, $a \in \{1, \ldots, n\}$, such that

$$g(e_a, e_b) = g(Ie_a, Ie_b) = \delta_{ab}, \quad g(e_a, Ie_b) = 0, \quad a, b \in \{1, \ldots, n\}. \tag{4.5}$$

Then, we obtain an adapted para-complex frame $\{\widehat{e_a}\}$, $a \in \{1, \ldots, n\}$, for $\Gamma(T_z^{1,0}M)$, where $\widehat{e}_a = (1/2)(e_a + eIe_a)$ for which $\widehat{g}(\widehat{e}_a, \widehat{e}_b) = (1/2)\delta_{ab}$.

Let ∇ and $\widehat{\nabla}$ be the Levi–Civita connections of the para-Kähler–Norden metric g and of the para-holomorphic Riemannian metric $\widehat{g}$, respectively. According to the

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discussion from the previous section, $\widehat{\nabla}$ is a para-holomorphic connection, and also, by the symmetry of ∇ and using (4.2), we obtain

$$\nabla_{IX} Y = I \nabla_X Y, \quad \forall X, Y \in \Gamma(TM). \tag{4.6}$$

Let us consider now the Koszul formula which gives the Levi–Civita connection $\widehat{\nabla}$ of $\widehat{g}$

$$\begin{aligned} 2\widehat{g}(\widehat{\nabla}_{\widehat{X}} \widehat{Y}, \widehat{Z}) &= \widehat{X}(\widehat{g}(\widehat{Y}, \widehat{Z})) + \widehat{Y}(\widehat{g}(\widehat{X}, \widehat{Z})) - \widehat{Z}(\widehat{g}(\widehat{X}, \widehat{Y})) \\ &\quad - \widehat{g}([\widehat{X}, \widehat{Z}], \widehat{Y}) - \widehat{g}([\widehat{Y}, \widehat{Z}], \widehat{X}) + \widehat{g}([\widehat{X}, \widehat{Y}], \widehat{Z}). \end{aligned} \tag{4.7}$$

and similar, we can write this formula for the real metric g, with ∇, g, X, Y and Z, respectively.

Using (3.11), (4.2) and (4.4), it follows

$$\begin{aligned} \widehat{X}\widehat{g}(\widehat{Y}, \widehat{Z}) &= \frac{1}{2}(Xg(Y, Z) + eXg(Y, IZ)), \\ \widehat{g}([\widehat{X}, \widehat{Y}], \widehat{Z}) &= \frac{1}{2}(g([X, Y], Z) + eg([X, Y], IZ)), \\ Zg(X, IY) &= (IZ)g(X, Y). \end{aligned}$$

Now, by the above formulas, (4.2) and the Koszul formula (4.7) for the real metric g, the relation (4.7) becomes

$$2\widehat{g}(\widehat{\nabla}_{\widehat{X}} \widehat{Y}, \widehat{Z}) = g(\nabla_X Y, Z) + eg(\nabla_X Y, IZ) = 2\widehat{g}(\widehat{\nabla_X Y}, \widehat{Z}), \tag{4.8}$$

which implies the following important relation

$$\widehat{\nabla}_{\widehat{X}} \widehat{Y} = \widehat{\nabla_X Y}. \tag{4.9}$$

In the sequel we consider the Riemann curvature tensors R and $\widehat{R}$ of ∇ and $\widehat{\nabla}$, respectively. Taking into account that ∇ is almost para-complex, i.e. $\nabla I = 0$, and also using (4.2) and (4.6), we obtain that R is totally pure (or I-symmetric), that is (see also [30])

$$R(X, Y)I = R(IX, Y) = R(X, IY) = IR(X, Y). \tag{4.10}$$

By direct calculus, using (4.2) and (4.9), it follows that the Riemann curvature tensors R and $\widehat{R}$ are related by

$$\widehat{R}(\widehat{X}, \widehat{Y})\widehat{Z} = (R(X, Y)Z)\widehat{\ }. \tag{4.11}$$

Now, let us consider the Ricci tensor fields associated to the metrics g and $\widehat{g}$, respectively, given by

$$\begin{aligned} &\mathrm{Ric}(g)(X, Y) = \mathrm{Tr}\{Z \mapsto R(Z, X)Y\} \quad \text{and} \\ &\mathrm{Ric}(\widehat{g})(\widehat{X}, \widehat{Y}) = \mathrm{Tr}\{\widehat{Z} \mapsto \widehat{R}(\widehat{Z}, \widehat{X})\widehat{Y}\}, \end{aligned} \tag{4.12}$$

and let us denote by Q and $\widehat{Q}$ the associated Ricci operators, given by

$$g(QX, Y) = \mathrm{Ric}(g)(X, Y) \quad \text{and} \quad \widehat{g}(\widehat{Q}\widehat{X}, \widehat{Y}) = \mathrm{Ric}(\widehat{g})(\widehat{X}, \widehat{Y}). \tag{4.13}$$

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We have

Proposition 21. *The Ricci tensors* $\mathrm{Ric}(g)$, $\mathrm{Ric}(\widehat{g})$ *and the Ricci operators* Q, $\widehat{Q}$ *satisfy the following relations*

$$\mathrm{Ric}(g)(IX,Y)=\mathrm{Ric}(g)(X,IY),\quad \mathrm{Ric}(g)(IX,IY)=\mathrm{Ric}(g)(X,Y),\quad QI=IQ \tag{4.14}$$

and

$$\mathrm{Ric}(\widehat{g})(\widehat{X},\widehat{Y})=\frac{1}{2}(\mathrm{Ric}(g)(X,Y)+e\,\mathrm{Ric}(g)(X,IY)),\quad \widehat{Q}\widehat{X}=\widehat{QX}. \tag{4.15}$$

Proof. The relations (4.14) follow directly from the defining relations (4.12) and (4.13) and using (4.10).

For the first relation of (4.15), using the orthonormal frame $\{e_a, Ie_a\}$, $a\in\{1,\dots,n\}$, we have

$$\begin{aligned}\mathrm{Ric}(g)(X,Y)&=\sum_a(g(R(e_a,X)Y,e_a)+g(R(Ie_a,X)Y,Ie_a))\\&=2\sum_a(g(R(e_a,X)Y,e_a)),\end{aligned}$$

where we have also used (4.10) and (3.10). Next, using the adapted para-complex frame $\{\widehat{e}_a\}$, $a\in\{1,\dots,n\}$ and the formulas (3.11), (4.10) and (4.11) we obtain

$$\begin{aligned}\mathrm{Ric}(\widehat{g})(\widehat{X},\widehat{Y})&=2\sum_a\widehat{g}(\widehat{R}(\widehat{e}_a,\widehat{X})\widehat{Y},\widehat{e}_a)=2\sum_a\widehat{g}((R(e_a,X)Y)\widehat{\ },\widehat{e}_a)\\&=\sum_a(g(R(e_a,X)Y,e_a)+eg(R(e_a,X)Y,Ie_a))\\&=\sum_a(g(R(e_a,X)Y,e_a)+eg(R(e_a,X)IY,e_a)),\end{aligned}$$

which together with the previous equality implies the first relation of (4.15). This together with (3.11) gives the following relation for the Ricci operators Q and $\widehat{Q}$

$$\begin{aligned}\widehat{g}(\widehat{Q}\widehat{X},\widehat{Y})&=\mathrm{Ric}(\widehat{g})(\widehat{X},\widehat{Y})=\frac{1}{2}(\mathrm{Ric}(g)(X,Y)+e\mathrm{Ric}(g)(X,IY))\\&=\frac{1}{2}(g(QX,Y)+eg(QX,IY))=\widehat{g}(\widehat{QX},\widehat{Y}),\end{aligned}$$

which proves the second relation of (4.15). $\square$

The first relation of (4.15) leads to the announced result, that is

Theorem 22. *Let us suppose that* (M,I,g) *is a para-Kählerian–Norden manifold, that is a para-complex manifold of para-complex dimension* n *endowed with a para-holomorphic Riemannian metric* $\widehat{g}\equiv(\widehat{g}_{ab}(z))$, $a,b\in\{1,\dots,n\}$ *and with a real metric* $g\equiv(g_{jk}(x))$, $j,k\in\{1,\dots,2n\}$ *given by* $g=2\,\mathrm{Re}\,\widehat{g}$. *Then the para-holomorphic*

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metric $\widehat{g}$ is Einstein with the real constant λ if and only if the real metric g is Einstein metric with the same constant.

Remark 23. We notice that starting from the original para-Kählerian–Norden metric g on a para-complex manifold (M, I), the real twin metric can be considered, that is $h(X,Y) := (g \circ I)(X,Y) = g(IX,Y)$, for every $X, Y \in \Gamma(TM)$. We find

$$h(X,Y) = 2\,\mathrm{Im}\,\widehat{g}(\widehat{X},\widehat{Y}), \quad \forall\, X, Y \in \Gamma(TM). \tag{4.16}$$

Moreover, if we denote by ∇ the covariant differentiation of the Levi–Civita connection associated to the para-Kählerian–Norden metric g, then we have (see [30])

$$\nabla h = \nabla g \circ I + g \circ \nabla I = 0. \tag{4.17}$$

The above relation says that, the Levi–Civita connection of g coincides with the Levi–Civita connection of h, thus they have the same real and para-complex Riemann and Ricci tensors (see also the discussion from the previous section). In the real case only one of two twin metrics can be Einsteinian. In para-complex case the Einstein condition $\mathrm{Ric}(\widehat{g}) = \lambda\widehat{g}$ implies $\mathrm{Ric}(\widehat{h}) = e\lambda\widehat{h}$, i.e. both para-holomorphic metrics $\widehat{g}$ and $\widehat{h}$ are Einstein metrics at the same time. Also, we can conclude that the metric h is an Einstein metric with an imaginary cosmological constant.

If the para-holomorphic metric $\widehat{g}$ is Einstein with para-complex constant $\widehat{\lambda}$, that is

$$\mathrm{Ric}(\widehat{g}) = \widehat{\lambda}\widehat{g}, \quad \widehat{\lambda} \in C, \tag{4.18}$$

then, similarly to the Kähler–Norden manifolds from the complex case, see [25], we can describe the following generalization of Theorem 22.

We consider the real scalar curvatures K, K^* of g, and the para-holomorphic scalar curvature $\widehat{K}$ of $\widehat{g}$, that is

$$K = \mathrm{Tr}\,Q, \quad K^* = \mathrm{Tr}(IQ), \quad \widehat{K} = \mathrm{Tr}(\widehat{Q}).$$

We have

Proposition 24. *The real scalar curvatures K, K^* and the para-holomorphic scalar curvature $\widehat{K}$ are related by*

$$\widehat{K} = \frac{1}{2}(K + eK^*). \tag{4.19}$$

Proof. Using (4.14) and (3.10), we obtain the following expressions for K and K^*:

$$K = \sum_a (g(Qe_a, e_a) + g(QIe_a, Ie_a)) = 2\sum_a (g(Qe_a, e_a)),$$

and

$$K^* = \sum_a (g(IQe_a, e_a) + g(IQIe_a, Ie_a)) = 2\sum_a (g(Qe_a, Ie_a)).$$

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Now, using (4.15) and (3.11), we obtain

$$\widehat{K} = 2\sum_a \widehat{g}(\widehat{Q}\widehat{e}_a, \widehat{e}_a) = 2\sum_a \widehat{g}(\widehat{Qe_a}, \widehat{e}_a) = \sum_a (g(Qe_a, e_a) + eg(Qe_a, Ie_a))$$

which proves (4.19). □

Now, by applying the para-Cauchy–Riemann equations to the para-holomorphic function $\widehat{K}$ and taking into account that $\operatorname{Re}\widehat{K} = K/2$ and $\operatorname{Im}\widehat{K} = K^*/2$, we get

$$dK(X) = XK = (IX)K^* = dK^*(IX) \quad \text{and}$$
$$dK(IX) = (IX)K = XK^* = dK^*(X),$$

which implies

$$d\widehat{K}(\widehat{X}) = \frac{1}{2}(dK(X) + edK(IX)). \tag{4.20}$$

Then, the following theorem, which is an analogue of Theorem 1 from [25] for Kähler–Norden manifolds, holds.

Theorem 25. *The para-holomorphic Riemannian manifold $(M, \widehat{g})$ is para-holomorphic Einstein with para-complex constant $\widehat{\lambda} = \lambda_1 + e\lambda_2$ iff*

$$\operatorname{Ric}(g)(X, Y) = \lambda_1 g(X, Y) + \lambda_2 g(X, IY). \tag{4.21}$$

Moreover, in the formula (4.21), we have $\lambda_1 = K/2n$ and $\lambda_2 = K^*/2n$.

Proof. Taking into account the formula (3.11) and the first relation of (4.15), we see that (4.18) holds iff

$$\begin{aligned}\operatorname{Ric}(g)(X, Y) + e\operatorname{Ric}(g)(X, IY) &= \lambda_1 g(X, Y) + \lambda_2 g(X, IY)\\ &\quad + e(\lambda_2 g(X, Y) + \lambda_1 g(X, IY)),\end{aligned}$$

which is equivalent to (4.21). Moreover, using (4.21), it follows that the Ricci operator Q satisfies

$$QX = \lambda_1 X + \lambda_2 IX \quad \text{and} \quad IQX = \lambda_1 IX + \lambda_2 X.$$

Now, the shape constants λ_1, λ_2 can be obtained from these two relations using the definitions of K and K^*, respectively. Indeed, using the orthonormal frame $\{e_a, Ie_a\}$, $a \in \{1, \ldots, n\}$, we have

$$\begin{aligned}K = \operatorname{Tr} Q &= \sum_a (g(Qe_a, e_a) + g(QIe_a, Ie_a)\\ &= \sum_a (g(\lambda_1 e_a + \lambda_2 Ie_a, e_a) + g(\lambda_1 Ie_a + \lambda_2 e_a, Ie_a))\\ &= \sum_a (\lambda_1 g(e_a, e_a) + \lambda_1 g(Ie_a, Ie_a)) = 2n\lambda_1\end{aligned}$$

and similarly, we obtain $K^* = 2n\lambda_2$. □

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5. Para-Complex Lie Groups as Para-Holomorphic Riemannian Einstein Manifolds

Definition 26. A para-complex Lie group, is a group G, which is also a para-complex manifold, such that the group multiplication $\phi : G \times G \to G$, $\phi(z, v) = z \cdot v$ and the inverse map $z \in G \mapsto z^{-1} \in G$ are para-holomorphic.

Let U be a coordinate neighborhood of the identity u_G of an m-parameter para-complex Lie group G. The coordinates of u_G are identified with $\{0, \ldots, 0\} \in C^m$, while the coordinates of elements of z, v, w of U will be denoted by $\{z^a\}$, $\{v^a\}$, $\{w^a\}$, respectively, $a, b, c, \ldots, \in \{1, \ldots, m\}$. The map $\phi : G \times G \to G$ given by $w = \phi(z, v)$ is represented para-holomorphically by m equations $w^\alpha = \phi^a(z, v)$, in which $\{\phi^a\}$ denotes a set of r para-complex-valued para-holomorphic functions on $G \times G$, where $\phi^a(z, v)$ is an abbreviated notation for $\phi^a(z^1, \ldots, z^m, v^1, \ldots, v^m)$. Since $z = u_G \cdot z = z \cdot u_G$ for all $z \in G$, it follows that up to and including second order terms

$$w^a = \phi^a(z, v) = z^a + v^a + A^a_{bc} z^b v^c + \cdots, \tag{5.1}$$

where the 3-index symbols A^a_{bc} are para-complex constants (in a given para-holomorphic coordinate system) in terms of which the structure constants of G are defined as $C^a_{bc} = A^a_{bc} - A^a_{cb}$.

Let us denote

$$\Phi^a_b(z, v) = \frac{\partial \phi^a(z, v)}{\partial z^b}, \quad \Psi^a_b(z, v) = \frac{\partial \phi^a(z, v)}{\partial v^b}, \tag{5.2}$$

such that by (5.1)

$$\Phi^a_b(z, 0) = \Psi^a_b(0, v) = \delta^a_b. \tag{5.3}$$

The derivatives (5.2) give rise to the definitions of the following para-holomorphic functions on G:

$$\begin{aligned} \overset{*}{\chi}{}^a_b(z) &= \Phi^a_b(0, z), \quad \chi^a_b(z) = \Psi^a_b(z, 0), \\ \lambda^a_b(z) &= \Phi^a_b(z, z^{-1}), \quad \overset{*}{\lambda}{}^a_b(z) = \Psi^a_b(z, z^{-1}), \end{aligned} \tag{5.4}$$

it being noted as a direct consequence of (5.3)

$$\overset{*}{\chi}{}^a_b(0) = \chi^a_b(0) = \lambda^a_b(0) = \overset{*}{\lambda}{}^a_b(0) = \delta^a_b. \tag{5.5}$$

Using the same technique as in the real (complex) case, [13, 28, 14], we obtain that $\overset{*}{\chi}{}^a_b(z) = \widetilde{\lambda}^a_b(z)$, where $\widetilde{\lambda}^a_b(z)$ denotes the elements of the para-holomorphic matrix that is inverse to $(\lambda^a_b(z))$ and $\overset{*}{\lambda}{}^a_b(z) = \widetilde{\chi}^a_b(z^{-1})$, where $\widetilde{\chi}^a_b(z)$ denotes the elements of the para-holomorphic matrix that is inverse to $(\chi^a_b(z))$. Also, we can consider the left and right invariant para-holomorphic 1-forms on the para-complex

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Lie group G defined by $\widetilde{\chi}^a = \widetilde{\chi}^a_b(z)dz^b$ and $\lambda^a = \lambda^a_b(z)dz^b$, respectively. Then

$$\Gamma^a_{bc}(z) = \widetilde{\lambda}^a_d(z)\left(\frac{\partial\lambda^d_b(z)}{\partial z^c} + \frac{1}{2}C^d_{pq}\lambda^p_b(z)\lambda^q_c(z)\right) = \frac{1}{2}\widetilde{\lambda}^a_d(z)\left(\frac{\partial\lambda^d_b(z)}{\partial z^c} + \frac{\partial\lambda^d_c(z)}{\partial z^b}\right), \tag{5.6}$$

defines the local coefficients of an unique torsion-free para-holomorphic connection on G.

The torsion-free para-holomorphic connection from (5.6) is always metric with respect to the para-holomorphic tensor field $g \in (T^{1,0}G)^* \otimes (T^{1,0}G)^*$ whose local components are given by

$$g_{ab}(z) = C_{pq}\lambda^p_a(z)\lambda^q_b(z), \tag{5.7}$$

where $C_{ab} = C^c_{ad}C^d_{bc}$ are the para-complex Cartan–Killing elements of the para-complex Lie group G. Moreover, if the para-complex Lie group G is semi-simple, that is $\det g_{ab} \neq 0$ (or equivalently $\det C_{ab} \neq 0$), it is the only symmetric para-holomorphic connection for which this is the case.

Remark 27. For the case of para-holomorphic metric tensor g from (5.7) its symmetry is guaranted from the expression of para-complex Cartan–Killing elements C_{ab}. If G is semi-simple then the para-holomorphic connection coefficients from (5.6) admit a representation in terms of the para-complex Christoffel symbols of (5.7).

Remark 28. The para-holomorphic metric tensor g_{ab} from (5.7) is not in general unique such that the torsion-free para-holomorphic connection from (5.6) is metric with respect to it (see the construction from the real case [28]).

As usual, the para-holomorphic curvature tensor of the torsion-free para-holomorphic connection from (5.6) must be specified as

$$R^d_{c,ab} = -\frac{1}{4}\widetilde{\lambda}^d_f C^f_{pq}C^q_{rs}\lambda^p_c\lambda^r_a\lambda^s_b. \tag{5.8}$$

Then, the para-holomorphic Ricci tensor is

$$R_{ca} = R^b_{c,ab} = -\frac{1}{4}C^f_{pq}C^q_{rf}\lambda^p_c\lambda^r_a, \tag{5.9}$$

or, in terms of para-complex Cartan–Killing elements

$$R_{ca} = -\frac{1}{4}C_{pr}\lambda^p_c\lambda^r_a.$$

By comparing this para-holmorphic tensor with the para-holomorphic metric tensor from (5.7) it is seen that the para-holomorphic Ricci tensor satisfies

$$R_{ab} = -\frac{1}{4}g_{ab}, \tag{5.10}$$

which implies that every para-complex Lie group is locally *para-holomorphic Einsteinian*.

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Now, as well as we noticed, if G is a semi-simple para-complex Lie group the para-holomorphic metric tensor from (5.7) is symmetric and nondegenerated. Thus, according to discussion from Sec. 4

$$ds^2 = 2\,\mathrm{Re}[g_{ab}(z)dz^a \otimes dz^b] \tag{5.11}$$

defines a para-Kählerian–Norden metric on G. Consequently, we have

Theorem 29. *Every semi-simple para-complex Lie groups is a para-Kählerian–Norden Einstein space with respect to the para-Kählerian–Norden metric defined by* (5.11).

Also, the following proposition holds.

Proposition 30. *If the para-complex Lie group is semi-simple then its para-holomorphic curvature scalar is constant and it is given by*

$$g^{ab}R_{ab} = -\frac{1}{4}(\dim_C G). \tag{5.12}$$

Moreover, it is natural to consider the type $(0,4)$ para-holomorphic curvature tensor associated with (5.7) and (5.8) as

$$R_{abcd} = g_{df}R^f_{c,ab}, \tag{5.13}$$

and, the explicit expression of this para-holomorphic tensor is given by

$$R_{abcd} = -\frac{1}{4}C_{tf}C^f_{pq}C^q_{rs}\lambda^p_a\lambda^t_b\lambda^r_c\lambda^s_d. \tag{5.14}$$

The *para-holomorphic sectional curvature* $k(Z,W)$ of G with respect a pair of para-holomorphic vector fields $Z, W \in \Gamma_{ph}(T^{1,0}(G))$ can be written in accordance with the standard formula

$$k(Z,W)(g_{ac}g_{bd} - g_{ad}g_{bc})Z^aZ^cW^bW^d = R_{abcd}Z^aZ^cW^bW^d. \tag{5.15}$$

Finally, we notice that similarly to the real case [28], the following two theorems hold.

Theorem 31. *The para-holomorphic sectional curvature of a para-complex Lie group G with respect to every pair of right-invariant para-holomorphic vector fields is constant.*

Theorem 32. *The covariant derivatives of the components of $R^d_{c,ab}$ with respect to the torsion-free para-holomorphic connection from* (5.6) *vanish identically.*

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