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APPLICATIONS OF RIEMANNIAN AND EINSTEIN–WEYL GEOMETRY IN THE THEORY OF SECOND-ORDER ORDINARY DIFFERENTIAL EQUATIONS

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We consider some properties of four-dimensional Riemannian spaces whose metric coefficients are associated with the coefficients of second-order nonlinear differential equations, and we study the properties of three-dimensional Einstein–Weyl spaces related to the dual equations $b'' = g(a, b, b')$, where the function $g(a, b, b')$ satisfies a special partial differential equation.

1. Introduction

The second-order ordinary differential equations (ODEs) of the type

$$y'' + a_1(x, y){y'}^3 + 3a_2(x, y){y'}^2 + 3a_3(x, y)y' + a_4(x, y) = 0 \tag{1}$$

are connected with nonlinear dynamic systems of the form

$$\frac{dx}{dt} = P(x, y, z, \alpha_i), \qquad \frac{dy}{dt} = Q(x, y, z, \alpha_i), \qquad \frac{dz}{dt} = R(x, y, z, \alpha_i),$$

where α_i are parameters. For example, the Lorentz system

$$\dot{X} = \sigma(Y - X), \qquad \dot{Y} = rX - Y - ZX, \qquad \dot{Z} = XY - bZ,$$

which manifests chaotic properties for some parameter values, is equivalent to the equation

$$y'' - \frac{3}{y}{y'}^2 + \left(\alpha y - \frac{1}{x}\right)y' + \epsilon x y^4 - \beta x^3 y^4 - \beta x^2 y^3 - \gamma y^3 + \delta\frac{y^2}{x} = 0, \tag{2}$$

where

$$\alpha = \frac{b + \sigma + 1}{\sigma}, \qquad \beta = \frac{1}{\sigma^2}, \qquad \gamma = \frac{b(\sigma + 1)}{\sigma^2}, \qquad \delta = \frac{(\sigma + 1)}{\sigma}, \qquad \epsilon = \frac{b(r - 1)}{\sigma^2}.$$

The properties of this equation were investigated in [1–5] using the theory of invariants. According to this theory [6–10], all equations of type (1) can be segregated into two classes depending on whether $\nu_5 = 0$ or $\nu_5 \neq 0$; the expression for ν_5 is

$$\nu_5 = L_2(L_1 L_{2x} - L_2 L_{1x}) + L_1(L_2 L_{1y} - L_1 L_{2y}) - a_1 L_1^3 + 3a_2 L_1^2 L_2 - 3a_3 L_1 L_2^2 + a_4 L_2^3,$$

where L_1 and L_2 are

$$L_1 = \frac{\partial}{\partial y}(a_{4y} + 3a_2 a_4) - \frac{\partial}{\partial x}(2a_{3y} - a_{2x} + a_1 a_4) - 3a_3(2a_{3y} - a_{2x}) - a_4 a_{1x},$$

$$L_2 = \frac{\partial}{\partial x}(a_{1x} - 3a_1 a_3) + \frac{\partial}{\partial x}(a_{3y} - 2a_{2x} + a_1 a_4) - 3a_2(a_{3y} - 2a_{2x}) + a_1 a_{4y}.$$

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For the equations with $\nu_5 = 0$, Liouville discovered the series of semi-invariants starting from

$$w_1 = \frac{1}{L_1^4}\left[L_1^3(\alpha' L_1 - \alpha'' L_2) + R_1(L_1^2)_x - L_1^2 R_{1x} + L_1 R_1(a_3 L_1 - a_4 L_2)\right],$$

where $R_1 = L_1 L_{2x} - L_2 L_{1x} + a_2 L_1^2 - 2a_3 L_1 L_2 + a_4 L_2^2$ and

$$\alpha = a_{2y} - a_{1x} + 2(a_1 a_3 - a_2^2),$$

$$\alpha' = a_{3y} - a_{2x} + a_1 a_4 - a_2 a_3,$$

$$\alpha'' = a_{4y} - a_{3x} + 2(a_2 a_4 - a_3^2).$$

In the case where $\nu_5 \neq 0$, the semi-invariants are

$$\nu_{m+5} = L_1 \frac{\partial \nu_m}{\partial y} - L_2 \frac{\partial \nu_m}{\partial x} + m\nu_m \left(\frac{\partial L_2}{\partial x} - \frac{\partial L_1}{\partial y}\right),$$

and the corresponding series of absolute invariants is

$$[5t_m - (m-2)t_7 t_{m-2}]\nu_5^{2/5} = 5\left(L_1 \frac{\partial t_{m-2}}{\partial y} - L_2 \frac{\partial t_{m-2}}{\partial x}\right),$$

where $t_m = \nu_m \nu_5^{-m/5}$.

2. Riemannian spaces and the ODE theory

To construct the Riemannian spaces related to Eq. (1), we begin with the geodesic equations of the two-dimensional space A_2 endowed with an affine (or Riemannian) connection,

$$\ddot{x} + \Gamma^1_{11}\dot{x}^2 + 2\Gamma^1_{12}\dot{x}\dot{y} + \Gamma^1_{22}\dot{y}^2 = 0,$$

$$\ddot{y} + \Gamma^2_{11}\dot{x}^2 + 2\Gamma^2_{12}\dot{x}\dot{y} + \Gamma^2_{22}\dot{y}^2 = 0.$$

This system of equations is equivalent to the single equation

$$y'' - \Gamma^1_{22}{y'}^3 + (\Gamma^2_{22} - 2\Gamma^1_{12}){y'}^2 + (2\Gamma^2_{12} - \Gamma^1_{11})y' + \Gamma^2_{11} = 0,$$

which is of type (1) with a special choice of the coefficients $a_i(x, y)$.

Equation (1) with arbitrary coefficients $a_i(x, y)$ can be considered as the geodesic equation in the two-dimensional space A_2

$$\ddot{x} - a_3\dot{x}^2 - 2a_2\dot{x}\dot{y} - a_1\dot{y}^2 = 0,$$

$$\ddot{y} + a_4\dot{x}^2 + 2a_3\dot{x}\dot{y} + a_2\dot{y}^2 = 0$$

with the projective connection with the components

$$\Pi_1 = \begin{vmatrix} -a_3 & -a_2 \\ a_4 & a_3 \end{vmatrix}, \qquad \Pi_2 = \begin{vmatrix} -a_2 & -a_1 \\ a_3 & a_2 \end{vmatrix}.$$

The curvature tensor of this type of connection is

$$R_{12} = \frac{\partial \Pi_2}{\partial x} - \frac{\partial \Pi_1}{\partial y} + [\Pi_1, \Pi_2]\,,$$

and its components are

$$R^1_{112} = a_{3y} - a_{2x} + a_1 a_4 - a_2 a_3 = \alpha', \qquad R^1_{212} = a_{2y} - a_{1x} + 2(a_1 a_3 - a_2^2) = \alpha,$$

$$R^2_{112} = a_{3x} - a_{4y} + 2(a_3^2 - a_2 a_4) = -\alpha'', \qquad R^2_{212} = a_{2x} - a_{3y} + a_3 a_2 - a_1 a_4 = -\alpha'.$$

To construct the Riemannian space related to the equation of type (1), we use the Riemannian extension W^4 of the space A_2 with the connection Π^k_{ij} [11]. The corresponding metric is

$$ds^2 = -2\Pi^k_{ij}\xi_k\, dx^i\, dx^j + 2\, d\xi_i\, dx^i.$$

In our case (where $\xi_1 = z$ and $\xi_2 = \tau$), it is

$$ds^2 = 2(za_3 - \tau a_4)\, dx^2 + 4(za_2 - \tau a_3)\, dx\, dy + 2(za_1 - \tau a_2)\, dy^2 + 2\, dx\, dz + 2\, dy\, d\tau. \tag{3}$$

We therefore formulate the following statement.

Proposition 1. *For a given equation of type* (1), *we have the Riemannian space with metric* (3), *a part of whose geodesic curves are integral curves of this equation.*

The calculation of geodesics of the space W^4 with metric (3) results in the system of equations

$$\frac{d^2x}{ds^2} - a_3\left(\frac{dx}{ds}\right)^2 - 2a_2\frac{dx}{ds}\frac{dy}{ds} - a_1\left(\frac{dy}{ds}\right)^2 = 0,$$

$$\frac{d^2y}{ds^2} + a_4\left(\frac{dx}{ds}\right)^2 + 2a_3\frac{dx}{ds}\frac{dy}{ds} + a_2\left(\frac{dy}{ds}\right)^2 = 0,$$

$$\begin{aligned}\frac{d^2z}{ds^2} &+ [z(a_{4y} - \alpha'') - \tau a_{4x}]\left(\frac{dx}{ds}\right)^2 + 2[za_{3y} - \tau(a_{3x} + \alpha'')]\frac{dx}{ds}\frac{dy}{ds} + \\ &+ [z(a_{2y} + \alpha) - \tau(a_{2x} + 2\alpha')]\left(\frac{dy}{ds}\right)^2 + 2a_3\frac{dx}{ds}\frac{dz}{ds} - 2a_4\frac{dx}{ds}\frac{d\tau}{ds} + 2a_2\frac{dy}{ds}\frac{dz}{ds} - 2a_3\frac{dy}{ds}\frac{d\tau}{ds} = 0,\end{aligned}$$

$$\begin{aligned}\frac{d^2\tau}{ds^2} &+ [z(a_{3y} - 2\alpha') - \tau(a_{3x} - \alpha'')]\left(\frac{dx}{ds}\right)^2 + 2[z(a_{2y} - \alpha) - \tau a_{2x}]\frac{dx}{ds}\frac{dy}{ds} + \\ &+ [za_{1y} - \tau(a_{1x} + \alpha)]\left(\frac{dy}{ds}\right)^2 + 2a_2\frac{dx}{ds}\frac{dz}{ds} - 2a_3\frac{dx}{ds}\frac{d\tau}{ds} + 2a_1\frac{dy}{ds}\frac{dz}{ds} - 2a_2\frac{dy}{ds}\frac{d\tau}{ds} = 0.\end{aligned}$$

This system of equations admits the integral of motion

$$2(za_3 - \tau a_4)\dot{x}^2 + 4(za_2 - \tau a_3)\dot{x}\dot{y} + 2(za_1 - \tau a_2)\dot{y}^2 + 2\dot{x}\dot{z} + 2\dot{y}\dot{\tau} = 1.$$

We note that first two equations of this system are equivalent to Eq. (1).

We have thus constructed the four-dimensional Riemannian space with metric (3) and the connection

$$\Gamma_1 = \begin{vmatrix} -a_3 & -a_2 & 0 & 0 \\ a_4 & a_3 & 0 & 0 \\ z(a_{4y} - \alpha'') - \tau a_{4x} & za_{3y} - \tau(a_{3x} + \alpha'') & a_3 & -a_4 \\ z(a_{3y} - 2\alpha') - \tau(a_{3x} - \alpha'') & z(a_{2y} - \alpha) - \tau a_{2x} & a_2 & -a_3 \end{vmatrix},$$

$$\Gamma_2 = \begin{vmatrix} -a_2 & -a_1 & 0 & 0 \\ a_3 & a_2 & 0 & 0 \\ za_{3y} - \tau(a_{3x} + \alpha'') & z(a_{2y} + \alpha) - \tau(a_{2x} + 2\alpha') & a_2 & -a_3 \\ z(a_{2y} - \alpha) - \tau a_{2x} & za_{1y} - \tau(a_{1x} + \alpha) & a_1 & -a_2 \end{vmatrix},$$

$$\Gamma_3 = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_3 & a_2 & 0 & 0 \\ a_2 & a_1 & 0 & 0 \end{vmatrix}, \qquad \Gamma_4 = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -a_4 & -a_3 & 0 & 0 \\ -a_3 & -a_2 & 0 & 0 \end{vmatrix}.$$

The curvature tensor of this metric is

$$R^1_{112} = -R^3_{312} = -R^2_{212} = R^4_{412} = \alpha',$$

$$R^1_{212} = -R^4_{312} = \alpha, \qquad R^2_{112} = -R^3_{412} = -\alpha'',$$

$$R^1_{312} = R^1_{412} = R^2_{312} = R^2_{412} = 0,$$

$$R^3_{112} = 2z(a_2\alpha'' - a_3\alpha') + 2\tau(a_4\alpha' - a_3\alpha''),$$

$$R^4_{212} = 2z(a_3\alpha' - a_2\alpha) + 2\tau(a_3\alpha - a_2\alpha'),$$

$$R^3_{212} = z(\alpha_x - \alpha'_y + a_1\alpha'' - a_3\alpha) + \tau(\alpha''_y - \alpha'_x + a_4\alpha - a_2\alpha''),$$

$$R^4_{112} = z(\alpha'_y - \alpha_x + a_1\alpha'' - a_3\alpha) + \tau(\alpha'_x - \alpha''_y + a_4\alpha - a_2\alpha'').$$

Using the expessions for components of the projective curvature of the space A_2,

$$L_1 = \alpha''_y - \alpha'_x + a_2\alpha'' + a_4\alpha - 2a_3\alpha',$$

$$L_2 = \alpha'_y - \alpha_x + a_1\alpha'' + a_3\alpha - 2a_2\alpha',$$

we can write the components of the curvature tensor as

$$R^4_{112} = z(L_2 + 2a_2\alpha' - 2a_3\alpha) - \tau(L_1 + 2a_3\alpha' - 2a_4\alpha),$$

$$R^3_{212} = z(-L_2 + 2a_1\alpha'' - 2a_2\alpha') + \tau(L_1 + 2a_3\alpha' - -2a_2\alpha''),$$

$$R_{13} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\alpha' & 0 & 0 \\ \alpha' & 0 & 0 & 0 \end{vmatrix}, \qquad R_{14} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \alpha'' & 0 & 0 \\ -\alpha'' & 0 & 0 & 0 \end{vmatrix},$$

$$R_{23} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\alpha & 0 & 0 \\ \alpha & 0 & 0 & 0 \end{vmatrix}, \qquad R_{24} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \alpha' & 0 & 0 \\ -\alpha' & 0 & 0 & 0 \end{vmatrix}, \qquad R^i_{j34} = 0.$$

The Ricci tensor $R_{ik} = R^l_{ilk}$ of the space M^4 has the components

$$R_{11} = 2\alpha'', \qquad R_{12} = 2\alpha', \qquad R_{22} = 2\alpha,$$

and the scalar curvature $R = g^{in} g^{km} R_{nm}$ of the space M^4 vanishes: $R = 0$.

We now introduce the tensor

$$L_{ijk} = \nabla_k R_{ij} - \nabla_j R_{ik} = R_{ij;k} - R_{ik;j}$$

with the components

$$L_{112} = -L_{121} = 2L_1, \qquad L_{221} = L_{212} = -2L_2.$$

Invariants of Eq. (1) can be expressed through the covariant derivatives of the curvature tensor and the values L_1 and L_2.

The Weyl tensor of the space M^4

$$C_{lijk} = R_{lijk} + \frac{1}{2}(g_{jl}R_{ik} + g_{ik}R_{jl} - g_{jk}R_{il} - g_{il}R_{jk}) + \frac{R}{6}(g_{jk}g_{il} - g_{jl}g_{ik})$$

has the single component $C_{1212} = tL_1 - zL_2$. Investigating the equation

$$|\, R_{ab} - \lambda g_{ab} \,| = 0,$$

where the matrix R_{ab} is constructed from the components of the Riemannian tensor,

$$R_{1412} = \alpha'', \qquad R_{2412} = \alpha', \qquad R_{2312} = -\alpha, \qquad R_{3112} = \alpha',$$

$$R_{1212} = z(\alpha_x - \alpha'_y + a_1\alpha'' - 2a_2\alpha' + a_3\alpha) + t(\alpha''_y - \alpha'_x - a_4\alpha + 2a_3\alpha' - a_3\alpha''),$$

we find that the space M^4 is of type N, and all second-order invariants of the space M^4 vanish.

Remark 1. The spaces with metric (3) are flat for Eq. (1) with the conditions $\alpha = 0$, $\alpha' = 0$, and $\alpha'' = 0$ imposed on the coefficients $a_i(x, y)$. For such equations, the components of the projective curvature vanish ($L_1 = 0$ and $L_2 = 0$), and these conditions acquire the form $y'' = 0$ after a pointlike transformation.

On the other hand, there are examples of Eq. (1) with the conditions $L_1 = 0$ and $L_2 = 0$ but with $\alpha \neq 0$, $\alpha' \neq 0$, and $\alpha'' \neq 0$. For such equations, the curvature of the corresponding Riemannian spaces does not vanish. In fact, the equation

$$y'' + 2e^{\varphi}{y'}^3 - \varphi_y {y'}^2 + \varphi_x y' - 2e^{\varphi} = 0,$$

where the function $\varphi(x, y)$ is a solution of the Wilczynski–Tzitzeika nonlinear equation

$$\varphi_{xy} = 4e^{2\varphi} - e^{-\varphi},$$

which is integrable by the inverse-transform method, satisfies the conditions $L_1 = 0$ and $L_2 = 0$, but $\alpha \neq 0$, $\alpha' \neq 0$, and $\alpha'' \neq 0$.

Remark 2. The properties of the spaces with metrics (3) for Eq. (2) with the coefficients leading to chaotic behavior of the solutions ($\sigma = 10$, $b = 8/3$, and $r > 24$) are especially interesting. In particular, studying the equations of geodesic deviation

$$\frac{d^2\eta^i}{ds^2} + 2\Gamma^i_{lm}\frac{dx^m}{ds}\frac{d\eta^l}{ds} + \frac{\partial\Gamma^i_{kl}}{\partial x^j}\frac{dx^k}{ds}\frac{dx^l}{ds}\eta^j = 0$$

can be useful.

We now consider some applications of soliton theory for studying the properties of equations of type (1). These applications are based on representing metric (3) in the form

$$ds^2 = 2z(a_3\,dx^2 + 2a_2\,dx\,dy + a_1\,dy^2) - 2\tau(a_4\,dx^2 + 2a_3\,dx\,dy + a_2\,dy^2) + 2\,dx\,dz + 2\,dy\,d\tau$$

or

$$ds^2 = 2z\,ds_1^2 - 2\tau\,ds_2^2 + 2\,dx\,dz + 2\,dy\,d\tau.$$

For example, the metric corresponding to the equation

$$y'' + H^2(x,y){y'}^3 + 3y' = 0$$

is

$$ds^2 = 2z(dx^2 + H^2\,dy^2) - 4\tau\,dx\,dy + 2\,dx\,dz + 2\,dy\,d\tau,$$

which contains the two-dimensional part

$${ds_1}^2 = dx^2 + H^2\,dy^2 \tag{4}$$

connected with the KdV equation

$$K_y + KK_x + K_{xxx} = 0,$$

which is satisfied by the curvature $K(x,y)$ of metric (4).

The metric corresponding to the equation

$$y'' + {y'}^3 + 3\cos H(x,y){y'}^2 + y' = 0$$

is related to the integrable equation

$$H_{xy} = \sin H.$$

3. Relation to surface theory

The metric for Eq. (1) can also be used to construct surfaces. One possibility is to embed two-dimensional surfaces, which are generalizations of the surfaces of translation, in a given four-dimensional space. The equations for the coordinates $Z^i(x,y)$ of such surfaces are

$$\frac{\partial^2 Z^i}{\partial x \partial y} + \Gamma^i_{jk}\frac{\partial Z^j}{\partial x}\frac{\partial Z^k}{\partial y} = 0.$$

From the compatibility condition for this system, we can obtain the coefficients $a_i(x,y)$ and the corresponding second-order ODEs.

Another possibility for studying two-dimensional surfaces in a space with metric (3) is to consider the section

$$x = x, \qquad y = y, \qquad z = z(x,y), \qquad \tau = \tau(x,y)$$

in such spaces. From the expressions

$$dz = z_x\,dx + z_y\,dy, \qquad d\tau = \tau_x\,dx + \tau_y\,dy,$$

we obtain the metric

$$ds^2 = 2(z_x + za_3 - \tau a_4)\,dx^2 + 2(\tau_x + z_y + 2za_2 - 2\tau a_3)\,dx\,dy + 2(\tau_y + za_1 - \tau a_2)\,dy^2.$$

We use this representation below to investigate particular cases of Eq. (1).

1. The functions z and τ chosen in the forms

$$z_x + za_3 - \tau a_4 = 0, \qquad \tau_x + z_y + 2za_2 - 2\tau a_3 = 0, \qquad \tau_y + za_1 - \tau a_2 = 0$$

determine flat surfaces, and the substitution $z = \Phi_x$, $\tau = \Phi_y$ allows reducing them to the system

$$\Phi_{xx} = a_4\Phi_y - a_3\Phi_x, \qquad \Phi_{xy} = a_3\Phi_y - a_2\Phi_x, \qquad \Phi_{yy} = a_2\Phi_y - a_1\Phi_x,$$

which is compatible with the conditions $\alpha = 0$, $\alpha' = 0$, and $\alpha'' = 0$.

2. We choose the functions $z = \Phi_x$ and $\tau = \Phi_y$ satisfying the system of equations

$$\Phi_{xx} = a_4\Phi_y - a_3\Phi_x, \qquad \Phi_{yy} = a_2\Phi_y - a_1\Phi_x,$$

where the coefficients $a_i(x, y)$ have the forms

$$a_4 = R_{xxx}, \qquad a_3 = -R_{xyy}, \qquad a_2 = R_{xyy}, \qquad a_1 = R_{yyy},$$

and the function $R(x, y)$ is the solution of the WDVV equation

$$R_{xxx}R_{yyy} - R_{xxy}R_{xyy} = 1.$$

These functions correspond to Eq. (1) in the form

$$y'' - R_{yyy}{y'}^3 + 3R_{xyy}{y'}^2 - 3R_{xxy}y' + R_{xxx} = 0.$$

Choosing the coefficients a_i to be

$$a_4 = -2\omega, \qquad a_1 = 2\omega, \qquad a_3 = \frac{\omega_x}{\omega}, \qquad a_2 = -\frac{\omega_y}{\omega},$$

we obtain the system

$$\Phi_{xx} + \frac{\omega_x}{\omega}\Phi_x + 2\omega\Phi_y = 0, \quad \Phi_{yy} + 2\omega\Phi_x + \frac{\omega_y}{\omega}\Phi_y = 0,$$

whose compatibility condition

$$\frac{\partial^2 \log\omega}{\partial x \partial y} = 4\omega^2 + \frac{\kappa}{\omega}$$

is the Wilczynski–Tzitzeika equation.

Remark 3. The linear system of equations for the WDVV equation determines some surfaces in the three-dimensional projective space. The canonical form of this system is [12]

$$\Phi_{xx} - R_{xxx}\Phi_y + \left(\frac{R_{xxxy}}{2} - \frac{R_{xxy}^2}{4} - \frac{R_{xxx}R_{xxy}}{2}\right)\Phi = 0,$$

$$\Phi_{yy} - R_{yyy}\Phi_x + \left(\frac{R_{yyyx}}{2} - \frac{R_{xyy}^2}{4} - \frac{R_{yyy}R_{xxy}}{2}\right)\Phi = 0.$$

The relations between Wilczynski invariants for the linear system pertain to various types of surfaces. Some of them are connected with solutions of the WDVV equation.

Remark 4. We consider the system of equations $\xi_{i,j}+\xi_{j,i} = 2\Gamma^k_{ij}\xi_k$ on the Killing vectors of metric (3). In components, this system has the form

$$\xi_{1x} = -a_3\xi_1 + a_4\xi_2 + (zA - ta_{4x})\xi_3 + (zE + tF)\xi_4,$$

$$\xi_{2y} = -a_1\xi_1 + a_2\xi_2 + (zC + tD)\xi_3 + (za_{1y} - tH)\xi_4,$$

$$\xi_{1y} + \xi_{2x} = 2[-a_2\xi_1 + a_3\xi_2 + (za_{3y} - tB)\xi_3 + (zG - ta_{2x})\xi_4,$$

$$\xi_{1z} + \xi_{3x} = 2[a_3\xi_3 + a_2\xi_4], \qquad \xi_{1t} + \xi_{4x} = 2[-a_4\xi_3 - a_3\xi_4],$$

$$\xi_{2z} + \xi_{3y} = 2[a_2\xi_3 + a_1\xi_4], \qquad \xi_{2t} + \xi_{4y} = -2[a_3\xi_3 - a_2\xi_4],$$

$$\xi_{3z} = 0, \qquad \xi_{4t} = 0.$$

In the particular case where $\xi_3 = \xi_4 = 0$ and $\xi_i = \xi_i(x, y)$, we obtain the system of equations

$$\xi_{1x} = -a_3\xi_1 + a_4\xi_2, \qquad \xi_{2y} = -a_1\xi_1 + a_2\xi_2,$$

$$\xi_{1y} + \xi_{2x} = 2[-a_2\xi_1 + a_3\xi_2],$$

which is equivalent to the system of equations on $z = z(x, y)$ and $\tau = \tau(x, y)$.

Remark 5. The Beltrami–Laplace operator

$$\Delta = g^{ij}\left(\frac{\partial^2}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial}{\partial x^k}\right)$$

is useful for investigating the properties of metric (3). For example, the equation $\Delta\Psi = 0$ becomes

$$(ta_4 - za_3)\Psi_{zz} + 2(ta_3 - za_2)\Psi_{zt} + (ta_2 - za_1)\Psi_{tt} + \Psi_{xz} + \Psi_{yt} = 0,$$

and solutions of this equation are connected with the geometry of metric (3). For example, setting the expression $\Psi = \exp[zA + tB]$ in the equation $\Delta\Psi = 0$, we obtain the conditions $A = \Phi_y$, $B = -\Phi_x$, and

$$a_4\Phi_y^2 - 2a_3\Phi_x\Phi_y + a_2\Phi_x^2 - \Phi_y\Phi_{xx} + \Phi_x\Phi_{xy} = 0.$$

Using solutions of the eikonal equation

$$g^{ij}\frac{\partial F}{\partial x^i}\frac{\partial F}{\partial x^j} = 0$$

or

$$F_xF_z + F_yF_t - (ta_4 - za_3)F_zF_z - 2(ta_3 - za_2)F_zF_t - (ta_2 - za_1)F_tF_t = 0,$$

we can investigate the properties of isotropic surfaces in a space with metric (3). The solution in the form $F = A(x, y)z^2 + B(x, y)t^2$ corresponds to the following choice of the coefficients for Eq. (1):

$$a_1 = \frac{AB_x}{2B^2} + \frac{A_x}{B}, \qquad a_4 = -\frac{BA_y}{2A^2} - \frac{B_y}{A}, \qquad a_2 = -\frac{B_y}{2B}, \qquad a_3 = \frac{A_x}{2A}.$$

Remark 6. Metric (3) admits a tetradic representation $g_{ij} = \omega^a_i \omega^b_j \eta_{ab}$, where

$$\eta_{ab} = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix}.$$

We then obtain $ds^2 = 2\omega^1\omega^3 + 2\omega^2\omega^4$, where

$$\begin{aligned}
&\omega^1 = dx + dy, && \omega^2 = dx + dy + \frac{1}{t(a_2 - a_4)}(dz - dt), \\
&\omega^4 = -t(a_4 dx + a_2 dy), && \omega^3 = z(a_3 dx + a_1 dy) + \frac{1}{(a_2 - a_4)}(a_2 dz - a_4 dt), \\
&a_1 + a_3 = 2a_2, && a_2 + a_4 = 2a_3.
\end{aligned}$$

Remark 7. Some equations on curvature tensors in the space M^4 are related to ODEs. For example, the equation

$$R_{ij;k} + R_{jk;i} + R_{ki;j} = 0 \tag{5}$$

results in the conditions on the coefficients $a_i(x, y)$

$$\begin{aligned}
&\alpha''_x + 2a_3\alpha'' - 2a_4\alpha' = 0, \\
&\alpha_y + 2a_1\alpha' - 2a_2\alpha = 0, \\
&\alpha''_y + 2\alpha'_x + 4a_2\alpha'' - 2a_4\alpha - 2a_3\alpha' = 0, \\
&\alpha_x + 2\alpha'_y - 4a_3\alpha + 2a_2\alpha' + 2a_1\alpha'' = 0.
\end{aligned}$$

Solutions of this system provide examples of second-order equations related to the space M^4 with condition (5) imposed on the Ricci tensor. The simplest examples are

$$y'' - \frac{3}{y}y'^2 + y^3 = 0, \qquad y'' - \frac{3}{y}y'^2 + y^4 = 0.$$

4. Einstein–Weyl geometry in the theory of second-order ODEs

The relation between equations expressed in form (1) and the equation $b'' = g(a, b, b')$, where the function $g(a, b, b')$ satisfies the PDE

$$\begin{aligned}
&g_{aacc} + 2cg_{abcc} + 2gg_{accc} + c^2 g_{bbcc} + 2cgg_{bccc} + \\
&\quad + g^2 g_{cccc} + (g_a + cg_b)g_{ccc} - 4g_{abc} - 4cg_{bbc} - \\
&\quad - cg_c g_{bcc} - 3gg_{bcc} - g_c g_{acc} + 4g_c g_{bc} - 3g_b g_{cc} + 6g_{bb} = 0,
\end{aligned}$$

was studied by Cartan [13] from the geometric standpoint. He showed that the Einstein–Weyl triples parameterize the families of curves of Eq. (5), which is dual to Eq. (1). Some examples of solutions of Eq. (5) were obtained in [2].

We now consider examples of the Einsten–Weyl spaces. The basic facts about the Einstein–Weyl spaces are the following [14].

A Weyl space is a smooth manifold endowed with a conformal metric $g_{ij}(x)$, with a symmetric connection

$$G^k_{ij} = \Gamma^k_{ij} - \frac{1}{2}(\omega_i \delta^k_j + \omega_j \delta^k_i - \omega_l g^{kl} g_{ij})$$

and with the condition of covariant derivation $D_i g_{kj} = \omega_i g_{kj}$, where $\omega_i(x)$ are the components of the vector field.

The Weyl connection G^k_{ij} has a curvature tensor W^i_{jkl}, and the Ricci tensor W^i_{jil} is not symmetric in general, $W^i_{jil} \neq W^i_{lij}$.

A Weyl space satisfying the Einstein condition

$$\frac{1}{2}(W_{jl} + W_{lj}) = \lambda(x) g_{jl}(x)$$

for some function $\lambda(x)$ is called an *Einstein–Weyl space*.

The components of the Weyl connection in the three-dimensional space with the metric $ds^2 = dx^2 + dy^2 + dz^2$ are

$$2G_1 = \begin{vmatrix} -\omega_1 & -\omega_2 & -\omega_3 \\ \omega_2 & -\omega_1 & 0 \\ \omega_3 & 0 & -\omega_1 \end{vmatrix}, \qquad 2G_2 = \begin{vmatrix} -\omega_2 & \omega_1 & 0 \\ -\omega_1 & -\omega_2 & -\omega_3 \\ 0 & \omega_3 & -\omega_2 \end{vmatrix}, \qquad 2G_3 = \begin{vmatrix} -\omega_3 & 0 & \omega_1 \\ 0 & -\omega_3 & \omega_2 \\ -\omega_1 & -\omega_2 & -\omega_3 \end{vmatrix}.$$

From the equations of the Einstein–Weyl space

$$W_{[ij]} = \frac{W_{ij} + W_{ji}}{2} = \lambda g_{ij},$$

we obtain the system of equations

$$\omega_{3x} + \omega_{1z} + \omega_1\omega_3 = 0, \qquad \omega_{3y} + \omega_{2z} + \omega_2\omega_3 = 0, \qquad \omega_{2x} + \omega_{1y} + \omega_1\omega_2 = 0,$$
$$2\omega_{1x} + \omega_{2y} + \omega_{3z} - \frac{\omega_2^2 + \omega_3^2}{2} = 2\lambda, \qquad 2\omega_{2y} + \omega_{1x} + \omega_{3z} - \frac{\omega_1^2 + \omega_3^2}{2} = 2\lambda,$$
$$2\omega_{3z} + \omega_{2y} + \omega_{1x} - \frac{\omega_1^2 + \omega_2^2}{2} = 2\lambda.$$

The first three equations result in the Chazy equation [15] $R''' + 2RR'' - 3R'^2 = 0$ on the function

$$R = R(x + y + z) = \omega_1 + \omega_2 + \omega_3,$$

where $\omega_i = \omega_i(x + y + z)$, and these functions generalize the classical Chazy equation.

The Einstein–Weyl geometry of the metric $g_{ij} = \mathrm{diag}(1, -e^U, -e^U)$ and the vector $\omega_i = (2U_z, 0, 0)$ are determined by solutions of the equation [16]

$$U_{xx} + U_{yy} = (e^U)_{zz},$$

which, upon the substitution $U = U(x + y = \tau, z)$, is equivalent to the equation

$$U_\tau = (e^{U/2})_z,$$

which has many-valued solutions.

Considering the Einstein–Weyl structure for the metric $ds^2 = dy^2 - 4\,dx\,dz - 4u\,dt^2$, we obtain the dispersionless KP equation [17] $(U_t - UU_x)_x = U_{yy}$.

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