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Quasi-minimal rotational surfaces in pseudo-Euclidean four-dimensional space

Research Article

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Abstract: In the four-dimensional pseudo-Euclidean space with neutral metric there are three types of rotational surfaces with two-dimensional axis – rotational surfaces of elliptic, hyperbolic or parabolic type. A surface whose mean curvature vector field is lightlike is said to be quasi-minimal. In this paper we classify all rotational quasi-minimal surfaces of elliptic, hyperbolic and parabolic type, respectively.

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1. Introduction

A spacelike surface in the Minkowski 4–space $\mathbb{E}_1^4$ whose mean curvature vector H is lightlike at each point is called *marginally trapped*. The concept of trapped surfaces, introduced in 1965 by Roger Penrose [19], plays an important role in general relativity and the theory of cosmic black holes. Recently, classification results on marginally trapped surfaces have been obtained imposing some extra conditions on the mean curvature vector, the Gauss curvature or the second fundamental form. In particular, marginally trapped surfaces with positive relative nullity were classified by B.-Y. Chen and J. Van der Veken in [8]. They also proved the non–existence of marginally trapped surfaces in Robertson–Walker spaces with positive relative nullity [9] and classified marginally trapped surfaces with parallel mean curvature vector in Lorentz space forms [10]. For a recent survey on marginally trapped surfaces, see also [4].

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In the four-dimensional Minkowski space there are three types of rotational surfaces with two-dimensional axis – rotational surfaces of elliptic, hyperbolic or parabolic type, known also as surfaces invariant under spacelike rotations, hyperbolic rotations or screw rotations, respectively. A *rotational surface of elliptic type* is an orbit of a regular curve under the action of the orthogonal transformations of $\mathbb{E}_1^4$ which leave a timelike plane point-wise fixed. Similarly, a *rotational surface of hyperbolic type* is an orbit of a regular curve under the action of the orthogonal transformations of $\mathbb{E}_1^4$ which leave a spacelike plane point-wise fixed. A *rotational surface of parabolic type* is an an orbit of a regular curve under the action of the orthogonal transformations of $\mathbb{E}_1^4$ which leave a degenerate plane point-wise fixed.

The marginally trapped surfaces in Minkowski 4-space which are invariant under spacelike rotations (rotational surfaces of elliptic type) were classified by S. Haesen and M. Ortega in [15]. The classification of marginally trapped surfaces in $\mathbb{R}_1^4$ which are invariant under boost transformations (rotational surfaces of hyperbolic type) was obtained in [14] and the classification of marginally trapped surfaces which are invariant under screw rotations (rotational surfaces of parabolic type) is given in [16].

The local theory of marginally trapped surfaces in the Minkowski 4-space $\mathbb{E}_1^4$ was developed by the present authors in [13]. There we proved that any marginally trapped surface is uniquely determined up to a motion in $\mathbb{E}_1^4$ by seven invariant functions. As an application we found all marginally trapped meridian surfaces in $\mathbb{E}_1^4$.

Some classification results for rotational surfaces in three-dimensional space forms satisfying some classical extra conditions have also been obtained. For example, a classification of all timelike and spacelike hyperbolic rotational surfaces with non-zero constant mean curvature in the three-dimensional de Sitter space $\mathbb{S}_1^3$ is given in [17] and a classification of the spacelike and timelike Weingarten rotational surfaces of the three types in $\mathbb{S}_1^3$ is found in [18]. In [12] we described all Chen spacelike rotational surfaces of hyperbolic or elliptic type.

Pseudo-Riemannian geometry has many important applications in physics. According to the words of Bang-Yen Chen in his new book *Pseudo-Riemannian Geometry, δ-Invariants and Applications*, 2011: "Spacetimes are the arenas in which all physical events take place" [4]. In recent times, physics and astrophysics have played a central role in shaping the understanding of the universe through scientific observation and experiment. The use of higher dimensional pseudo-Riemannian manifolds in physics has led to many new developments in string theory.

In the pseudo-Riemannian geometry there is an important subject closely related with marginally trapped surfaces, namely quasi-minimal surfaces. A Lorentz surface in a pseudo-Riemannian manifold is called *quasi-minimal*, if its mean curvature vector is lightlike at each point of the surface. Borrowed from general relativity, some authors call the quasi-minimal Lorentz surfaces in a pseudo-Riemannian manifold also *marginally trapped*. We shall use the notion of a quasi-minimal surface.

The classification of quasi-minimal surfaces with parallel mean curvature vector in the pseudo-Euclidean space $\mathbb{E}_2^4$ is obtained in [6]. In [1] B.-Y. Chen classified quasi-minimal Lorentz flat surfaces in $\mathbb{E}_2^4$. As an application, he gave the complete classification of biharmonic Lorentz surfaces in $\mathbb{E}_2^4$ with lightlike mean curvature vector. Several other families of quasi-minimal surfaces have also been classified. For example, quasi-minimal surfaces with constant Gauss curvature in $\mathbb{E}_2^4$ were classified in [2, 11]. Quasi-minimal Lagrangian surfaces and quasi-minimal slant surfaces in complex space forms were classified, respectively, in [5] and [7]. For an up-to-date survey on quasi-minimal surfaces, see also [3].

In the present paper we consider three types of rotational surfaces in the four-dimensional pseudo-Euclidean space $\mathbb{E}_2^4$, namely rotational surfaces of elliptic, hyperbolic, and parabolic type, which are analogous to the three types of rotational surfaces in the Minkowski space $\mathbb{E}_1^4$.

In Theorem 3.3 we find all quasi-minimal rotational surfaces of elliptic type. In Theorem 3.6 we describe all quasi-minimal rotational surfaces of hyperbolic type and in Theorem 3.9 we describe the construction of all quasi-minimal rotational surfaces of parabolic type.

Our idea to study quasi-minimal rotational surfaces in the pseudo-Euclidean space $\mathbb{E}_2^4$ was motivated by the results of S. Haesen and M. Ortega for marginally trapped rotational surfaces in the Minkowski space $\mathbb{E}_1^4$ [14–16].

2.   Preliminaries

Let $\mathbb{E}_2^4$ be the pseudo–Euclidean 4-space endowed with the canonical pseudo–Euclidean metric of index 2 given by

$$g_0 = dx_1^2 + dx_2^2 - dx_3^2 - dx_4^2,$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system of $\mathbb{E}_2^4$. As usual, we denote by $\langle \, , \rangle$ the indefinite inner scalar product with respect to g_0.

A vector v is called *spacelike* (respectively, *timelike*) if $\langle v, v \rangle > 0$ (respectively, $\langle v, v \rangle < 0$). A vector v is called *lightlike* if it is nonzero and satisfies $\langle v, v \rangle = 0$.

A surface M^2 in $\mathbb{E}_2^4$ is called *Lorentz* if the induced metric g on M^2 is Lorentzian. Thus at each point $p \in M^2$ we have the following decomposition

$$\mathbb{E}_2^4 = T_p M^2 \oplus N_p M^2$$

with the property that the restriction of the metric onto the tangent space $T_p M^2$ is of signature $(1, 1)$, and the restriction of the metric onto the normal space $N_p M^2$ is of signature $(1, 1)$.

Denote by ∇ and ∇' the Levi Civita connections of M^2 and $\mathbb{E}_2^4$, respectively. Let x and y denote vector fields tangent to M^2 and let ξ be a normal vector field. The formulas of Gauss and Weingarten give a decomposition of the vector fields $\nabla'_x y$ and $\nabla'_x \xi$ into a tangent and a normal component:

$$\nabla'_x y = \nabla_x y + \sigma(x, y);$$
$$\nabla'_x \xi = -A_\xi x + D_x \xi,$$

which define the second fundamental form σ, the normal connection D, and the shape operator A_ξ with respect to ξ. In general, A_ξ is not diagonalizable.

It is well known that the shape operator and the second fundamental form are related by the formula

$$\langle \sigma(x, y), \xi \rangle = \langle A_\xi x, y \rangle.$$

The mean curvature vector field H of the surface M^2 is defined as $H = \dfrac{1}{2} \operatorname{tr} \sigma$.

A surface M^2 is called *minimal* if its mean curvature vector vanishes identically, i.e. $H = 0$. A natural extension of minimal surfaces are quasi–minimal surfaces. The surface M^2 is *quasi–minimal* if its mean curvature vector is lightlike at each point, i.e. $H \neq 0$ and $\langle H, H \rangle = 0$. Obviously, quasi–minimal surfaces are always non–minimal.

3.   Lorentz rotational surfaces in pseudo-Euclidean 4-space

Let $Oe_1 e_2 e_3 e_4$ be a fixed orthonormal coordinate system in the pseudo-Euclidean space $\mathbb{E}_2^4$, i.e. $\langle e_1, e_1 \rangle = \langle e_2, e_2 \rangle = 1$, $\langle e_3, e_3 \rangle = \langle e_4, e_4 \rangle = -1$.

First we consider rotational surfaces of elliptic type. Let $c : \widetilde{z} = \widetilde{z}(u), \ u \in J$ be a smooth spacelike curve, parameterized by

$$\widetilde{z}(u) = (x_1(u), x_2(u), r(u), 0); \quad u \in J.$$

The curve c lies in the three-dimensional subspace $\mathbb{E}_1^3 = \operatorname{span}\{e_1, e_2, e_3\}$ of $\mathbb{E}_2^4$. Without loss of generality we assume that c is parameterized by the arc-length, i.e. $(x_1')^2 + (x_2')^2 - (r')^2 = 1$. We assume also that $r(u) > 0, \ u \in J$.

Let us consider the surface $\mathcal{M}'$ in $\mathbb{E}_2^4$ defined by

$$\mathcal{M}' : z(u, v) = (x_1(u), x_2(u), r(u) \cos v, r(u) \sin v); \quad u \in J, \ v \in [0; 2\pi). \tag{1}$$

The tangent space of $\mathcal{M}'$ is spanned by the vector fields

$$z_u = (x_1', x_2', r'\cos v, r'\sin v);$$
$$z_v = (0, 0, -r\sin v, r\cos v).$$

Hence, the coefficients of the first fundamental form of $\mathcal{M}'$ are

$$E = \langle z_u, z_u \rangle = 1; \quad F = \langle z_u, z_v \rangle = 0; \quad G = \langle z_v, z_v \rangle = -r^2(u).$$

The surface $\mathcal{M}'$, defined by (1), is a Lorentz surface in $\mathbb{E}_2^4$, obtained by the rotation of the spacelike curve c about the two-dimensional Euclidean plane Oe_1e_2. We call $\mathcal{M}'$ a *rotational surface of elliptic type*.

We can also obtain a rotational surface of elliptic type in $\mathbb{E}_2^4$ using rotation of a timelike curve about the two-dimensional plane Oe_3e_4. Indeed, if c is a timelike curve lying in the three-dimensional subspace $\text{span}\{e_1, e_3, e_4\}$ of $\mathbb{E}_2^4$ and parameterized by

$$\widetilde{z}(u) = (r(u), 0, x_3(u), x_4(u)); \quad u \in J,$$

then the surface, defined by

$$z(u, v) = (r(u)\cos v, r(u)\sin v, x_3(u), x_4(u)); \quad u \in J, \ v \in [0; 2\pi)$$

is a Lorentz rotational surface of elliptic type. It can be obtained by a surface of the form (1) using an orthogonal transformation.

Next, we consider rotational surfaces of hyperbolic type. Let $c : \widetilde{z} = \widetilde{z}(u), \ u \in J$ be a smooth spacelike curve, lying in the three-dimensional subspace $\mathbb{E}_1^3 = \text{span}\{e_1, e_2, e_4\}$ of $\mathbb{E}_2^4$ and parameterized by

$$\widetilde{z}(u) = (r(u), x_2(u), 0, x_4(u)); \quad u \in J.$$

Without loss of generality we assume that c is parameterized by the arc-length, i.e. $(r')^2 + (x_2')^2 - (x_4')^2 = 1$. We assume also that $r(u) > 0, \ u \in J$.

Now we consider the surface $\mathcal{M}''$ in $\mathbb{E}_2^4$ defined by

$$\mathcal{M}'' : z(u, v) = (r(u)\cosh v, x_2(u), r(u)\sinh v, x_4(u)); \quad u \in J, \ v \in \mathbb{R}. \tag{2}$$

The tangent space of $\mathcal{M}''$ is spanned by the vector fields

$$z_u = (r'\cosh v, x_2', r'\sinh v, x_4');$$
$$z_v = (r\sinh v, 0, r\cosh v, 0),$$

and the coefficients of the first fundamental form of $\mathcal{M}''$ are

$$E = \langle z_u, z_u \rangle = 1; \quad F = \langle z_u, z_v \rangle = 0; \quad G = \langle z_v, z_v \rangle = -r^2(u).$$

The surface $\mathcal{M}''$, defined by (2), is a Lorentz surface in $\mathbb{E}_2^4$, obtained by hyperbolic rotation of the spacelike curve c about the two-dimensional Lorentz plane Oe_2e_4. We call $\mathcal{M}''$ a *rotational surface of hyperbolic type*.

Similarly, we can obtain a rotational surface of hyperbolic type using hyperbolic rotation of a timelike curve lying in $\text{span}\{e_2, e_3, e_4\}$ about the two-dimensional Lorentz plane Oe_2e_4. Indeed, if c is a timelike curve parameterized by

$$\widetilde{z}(u) = (0, x_2(u), r(u), x_4(u)); \quad u \in J,$$

then the surface, defined by

$$z(u,v) = (r(u)\sinh v, x_2(u), r(u)\cosh v, x_4(u)); \quad u \in J, \ v \in \mathbb{R}$$

is a Lorentz rotational surface of hyperbolic type.

Rotational surfaces of hyperbolic type can also be obtained by hyperbolic rotations of spacelike or timelike curves about the two–dimensional Lorentz planes Oe_1e_3, Oe_1e_4 and Oe_2e_3. We are not going to consider all of them, since they are constructed in a similar way and can be obtained from (2) by orthogonal transformations.

Now, let us consider rotational surfaces of parabolic type in $\mathbb{E}_2^4$. For convenience we shall use the pseudo–orthonormal base $\{e_1, e_4, \xi_1, \xi_2\}$ of $\mathbb{E}_2^4$, such that $\xi_1 = \frac{e_2+e_3}{\sqrt{2}}$, $\xi_2 = \frac{-e_2+e_3}{\sqrt{2}}$. Note that

$$\langle \xi_1, \xi_1 \rangle = 0; \quad \langle \xi_2, \xi_2 \rangle = 0; \quad \langle \xi_1, \xi_2 \rangle = -1.$$

Let c be a spacelike curve lying in the subspace $\mathbb{E}_1^3 = \text{span}\{e_1, e_2, e_3\}$ of $\mathbb{E}_2^4$ and parameterized by

$$\widetilde{z}(u) = x_1(u)\, e_1 + x_2(u)\, e_2 + x_3(u)\, e_3; \quad u \in J,$$

or equivalently,

$$\widetilde{z}(u) = x_1(u)\, e_1 + \frac{x_2(u) + x_3(u)}{\sqrt{2}}\, \xi_1 + \frac{-x_2(u) + x_3(u)}{\sqrt{2}}\, \xi_2; \quad u \in J.$$

Denote $f(u) = \frac{x_2(u)+x_3(u)}{\sqrt{2}}$, $g(u) = \frac{-x_2(u)+x_3(u)}{\sqrt{2}}$. Then

$$\widetilde{z}(u) = x_1(u)\, e_1 + f(u)\, \xi_1 + g(u)\, \xi_2.$$

Without loss of generality we assume that c is parameterized by the arc–length, i.e. $(x_1')^2 + (x_2')^2 - (x_3')^2 = 1$, or equivalently $(x_1')^2 - 2f'g' = 1$.

We define a rotational surface of parabolic type in the following way:

$$\mathcal{M}''' : z(u,v) = x_1(u)\, e_1 + f(u)\, \xi_1 + (-v^2 f(u) + g(u))\, \xi_2 + \sqrt{2}\, v f(u)\, e_4; \quad u \in J, \ v \in \mathbb{R}. \tag{3}$$

The tangent vector fields of $\mathcal{M}'''$ are

$$z_u = x_1'\, e_1 + \sqrt{2}\, v f'\, e_4 + f'\, \xi_1 + (-v^2 f' + g')\, \xi_2;$$
$$z_v = \sqrt{2}\, f\, e_4 - 2vf\, \xi_2.$$

Hence, the coefficients of the first fundamental form of $\mathcal{M}'$ are

$$E = \langle z_u, z_u \rangle = 1; \quad F = \langle z_u, z_v \rangle = 0; \quad G = \langle z_v, z_v \rangle = -2f^2(u).$$

The surface $\mathcal{M}'''$, defined by (3), is a Lorentz surface in $\mathbb{E}_2^4$, which we call *rotational surface of parabolic type.* The rotational axis is the two–dimensional plane spanned by e_1 (a spacelike vector field) and ξ_1 (a lightlike vector field).

Similarly, we can obtain a rotational surface of parabolic type using a timelike curve lying in the subspace $\text{span}\{e_2, e_3, e_4\}$ as follows. Let c be a timelike curve given by

$$\widetilde{z}(u) = x_2(u)\, e_2 + x_3(u)\, e_3 + x_4(u)\, e_4; \quad u \in J.$$

We consider the lightlike vector fields $\overline{\xi}_1 = \frac{e_2 + e_4}{\sqrt{2}}$, $\overline{\xi}_2 = \frac{-e_2 + e_4}{\sqrt{2}}$. Then the parametrization of c is expressed as

$$\widetilde{z}(u) = x_3(u)\, e_3 + \overline{f}(u)\, \overline{\xi}_1 + \overline{g}(u)\, \overline{\xi}_2,$$

where $\overline{f}(u) = \frac{x_2(u) + x_4(u)}{\sqrt{2}}$, $\quad \overline{g}(u) = \frac{-x_2(u) + x_4(u)}{\sqrt{2}}$.

Now, let us consider the surface defined as follows.

$$z(u, v) = \sqrt{2}\, v\overline{f}(u)\, e_1 + x_3(u)\, e_3 + \overline{f}(u)\, \overline{\xi}_1 + (v^2\overline{f}(u) + \overline{g}(u))\, \overline{\xi}_2; \quad u \in J, \ v \in \mathbb{R}. \tag{4}$$

The surface, given by (4), is a Lorentz surface in $\mathbb{E}_2^4$ whose coefficients of the first fundamental form are

$$E = \langle z_u, z_u \rangle = -1; \quad F = \langle z_u, z_v \rangle = 0; \quad G = \langle z_v, z_v \rangle = 2\overline{f}^2(u).$$

This surface is also a *rotational surface of parabolic type*, where the rotational axis is the two-dimensional plane spanned by e_3 (a timelike vector field) and $\overline{\xi}_1$ (a lightlike vector field). It can be transformed in a surface of the form (3) by an orthogonal transformation.

In what follows, we find all quasi-minimal surfaces in the three classes of rotational surfaces: elliptic type, hyperbolic type, and parabolic type.

3.1. Quasi-minimal rotational surfaces of elliptic type

Let us consider the surface $\mathcal{M}'$ in $\mathbb{E}_2^4$ defined by (1). Since the generating curve c is a spacelike curve parameterized by the arc-length, i.e. $(x_1')^2 + (x_2')^2 - (r')^2 = 1$, then $(x_1')^2 + (x_2')^2 = 1 + (r')^2$ and $x_1' x_1'' + x_2' x_2'' = r' r''$. We shall use the following orthonormal tangent frame field:

$$X = z_u; \qquad Y = \frac{z_v}{r},$$

and the normal frame field $\{n_1, n_2\}$, defined by

$$
\begin{aligned}
n_1 &= \frac{1}{\sqrt{1 + (r')^2}}\, (-x_2', x_1', 0, 0); \\
n_2 &= \frac{1}{\sqrt{1 + (r')^2}}\, (r' x_1', r' x_2', (1 + (r')^2) \cos v, (1 + (r')^2) \sin v).
\end{aligned}
\tag{5}
$$

Note that

$$\langle X, X \rangle = 1; \quad \langle X, Y \rangle = 0; \quad \langle Y, Y \rangle = -1;$$

$$\langle n_1, n_1 \rangle = 1; \quad \langle n_1, n_2 \rangle = 0; \quad \langle n_2, n_2 \rangle = -1.$$

The second partial derivatives of $z(u, v)$ are expressed as follows

$$
\begin{aligned}
z_{uu} &= (x_1'', x_2'', r'' \cos v, r'' \sin v); \\
z_{uv} &= (0, 0, -r' \sin v, r' \cos v); \\
z_{vv} &= (0, 0, -r \cos v, -r \sin v).
\end{aligned}
\tag{6}
$$

By a straightforward computation, using (5) and (6), we obtain the components of the second fundamental form:

$$
\begin{array}{ll}
\langle z_{uu}, n_1 \rangle = \dfrac{1}{\sqrt{1 + (r')^2}}(x_1' x_2'' - x_1'' x_2'); & \langle z_{uu}, n_2 \rangle = -\dfrac{r''}{\sqrt{1 + (r')^2}}; \\[2mm]
\langle z_{uv}, n_1 \rangle = 0; & \langle z_{uv}, n_2 \rangle = 0; \\[2mm]
\langle z_{vv}, n_1 \rangle = 0; & \langle z_{vv}, n_2 \rangle = r\sqrt{1 + (r')^2}.
\end{array}
$$

Hence,

$$\sigma(z_u, z_u) = \frac{x_1' x_2'' - x_1'' x_2'}{\sqrt{1 + (r')^2}} \, n_1 + \frac{r''}{\sqrt{1 + (r')^2}} \, n_2,$$
$$\sigma(z_u, z_v) = 0,$$
$$\sigma(z_v, z_v) = \qquad\qquad -r\sqrt{1 + (r')^2} \, n_2.$$

With respect to the orthonormal frame field $\{X, Y\}$ we get the formulas:

$$\sigma(X, X) = \frac{x_1' x_2'' - x_1'' x_2'}{\sqrt{1 + (r')^2}} \, n_1 + \frac{r''}{\sqrt{1 + (r')^2}} \, n_2,$$
$$\sigma(X, Y) = 0, \tag{7}$$
$$\sigma(Y, Y) = \qquad\qquad -\frac{\sqrt{1 + (r')^2}}{r} \, n_2.$$

Formulas (7) imply that the Gauss curvature K of the rotational surface of elliptic type $\mathcal{M}'$ is

$$K = -\frac{r''}{r} \tag{8}$$

and the normal mean curvature vector field H is

$$H = \frac{1}{2r\sqrt{1 + (r')^2}} \left(r(x_1' x_2'' - x_1'' x_2') \, n_1 + (rr'' + (r')^2 + 1) \, n_2 \right). \tag{9}$$

Equalities (8) and (9) imply the following two statements.

Proposition 3.1.
The rotational surface of elliptic type $\mathcal{M}'$, defined by (1), is flat if and only if $r'' = 0$.

Proposition 3.2.
The rotational surface of elliptic type $\mathcal{M}'$, defined by (1), is minimal if and only if $x_1' x_2'' - x_1'' x_2' = 0$ and $rr'' + (r')^2 + 1 = 0$.

In the present paper we are interested in quasi–minimal rotational surfaces, so we assume that $(x_1' x_2'' - x_1'' x_2')^2 + (rr'' + (r')^2 + 1)^2 \neq 0$.

It follows from (5) that

$$\nabla_X' n_1 = -\frac{x_1' x_2'' - x_1'' x_2'}{\sqrt{1 + (r')^2}} \, X + \frac{r'}{1 + (r')^2} (x_1' x_2'' - x_1'' x_2') \, n_2,$$
$$\nabla_Y' n_1 = 0,$$
$$\nabla_X' n_2 = \frac{r''}{\sqrt{1 + (r')^2}} \, X + \frac{r'}{1 + (r')^2} (x_1' x_2'' - x_1'' x_2') \, n_1, \tag{10}$$
$$\nabla_Y' n_2 = \frac{\sqrt{1 + (r')^2}}{r} \, Y.$$

We can distinguish two special classes of rotational surfaces of elliptic type.

I. Let $x_1' x_2'' - x_1'' x_2' = 0$, $rr'' + (r')^2 + 1 \neq 0$. In this case from the first two equalities of (10) we get

$$\nabla_X' n_1 = 0; \quad \nabla_Y' n_1 = 0,$$

which imply that the normal vector field n_1 is constant. Hence, the rotational surface of elliptic type $\mathcal{M}'$ lies in the hyperplane $\mathbb{E}_2^3$ of $\mathbb{E}_2^4$ orthogonal to n_1, i.e. $\mathcal{M}'$ lies in the hyperplane $\mathbb{E}_2^3 = \mathrm{span}\{X, Y, n_2\}$.

Moreover, the mean curvature vector field of $\mathcal{M}'$ is:

$$H = \frac{rr'' + (r')^2 + 1}{2r\sqrt{1 + (r')^2}}\, n_2.$$

Hence, $\langle H, H \rangle = 0$ if and only if $H = 0$ (i.e. $\mathcal{M}'$ is minimal). Consequently, there are no quasi-minimal rotational surfaces of elliptic type in the class $x_1'x_2'' - x_1''x_2' = 0$.

II. Let $rr'' + (r')^2 + 1 = 0$, $x_1'x_2'' - x_1''x_2' \neq 0$. In this case, it can be proved that $\mathcal{M}'$ does not lie in any hyperplane of $\mathbb{E}_2^4$. But, since the mean curvature vector field is

$$H = \frac{x_1'x_2'' - x_1''x_2'}{2\sqrt{1 + (r')^2}}\, n_1,$$

we have again that $\langle H, H \rangle = 0$ if and only if $H = 0$. Consequently, there are no quasi-minimal rotational surfaces of elliptic type in the class $rr'' + (r')^2 + 1 = 0$.

Further we shall consider general rotational surfaces of elliptic type, i.e. we assume that $x_1'x_2'' - x_1''x_2' \neq 0$ and $rr'' + (r')^2 + 1 \neq 0$ in an open interval $I \subset J$. In the next theorem we give a local description of all quasi-minimal rotational surfaces of elliptic type.

Theorem 3.3.
Given a smooth positive function $r(u) : I \subset \mathbb{R} \to \mathbb{R}$, define the functions

$$\varphi(u) = \eta \int \frac{rr'' + (r')^2 + 1}{r(1 + (r')^2)}\, du, \quad \eta = \pm 1,$$

and

$$x_1(u) = \int \sqrt{1 + (r')^2}\, \cos\varphi(u)\, du,$$
$$x_2(u) = \int \sqrt{1 + (r')^2}\, \sin\varphi(u)\, du.$$

Then the spacelike curve $c : \widetilde{z}(u) = (x_1(u), x_2(u), r(u), 0)$ is a generating curve of a quasi-minimal rotational surface of elliptic type.

Conversely, any quasi-minimal rotational surface of elliptic type is locally constructed as above.

Proof. Let $\mathcal{M}'$ be a general rotational surface of elliptic type generated by a spacelike curve $c : \widetilde{z}(u) = (x_1(u), x_2(u), r(u), 0)$; $u \in J$. We assume that c is parameterized by the arc-length and $x_1'x_2'' - x_1''x_2' \neq 0$, $rr'' + (r')^2 + 1 \neq 0$ for $u \in I \subset J$.

It follows from (9) that $\mathcal{M}'$ is quasi-minimal if and only if

$$r^2(x_1'x_2'' - x_1''x_2')^2 - (rr'' + (r')^2 + 1)^2 = 0,$$

or equivalently

$$r(x_1'x_2'' - x_1''x_2') = \eta(rr'' + (r')^2 + 1), \qquad \eta = \pm 1. \tag{11}$$

Since the curve c is parameterized by the arc-length, we have $(x_1')^2 + (x_2')^2 = 1 + (r')^2$, which implies that there exists a smooth function $\varphi = \varphi(u)$ such that

$$x_1'(u) = \sqrt{1 + (r')^2}\, \cos\varphi(u),$$
$$x_2'(u) = \sqrt{1 + (r')^2}\, \sin\varphi(u). \tag{12}$$

Using (12) we get $x_1'x_2'' - x_1''x_2' = (1 + (r')^2)\varphi'$. Hence, condition (11) for quasi-minimality of $\mathcal{M}'$ is written in terms of $r(u)$ and $\varphi(u)$ as follows:

$$\varphi' = \eta \frac{rr'' + (r')^2 + 1}{r(1 + (r')^2)}. \tag{13}$$

Consequently, the mean curvature vector field of a quasi-minimal rotational surface of elliptic type is given by the formula

$$H = \frac{rr'' + (r')^2 + 1}{2r\sqrt{1 + (r')^2}}(\eta\, n_1 + n_2).$$

Formula (13) allows us to recover $\varphi(u)$ from $r(u)$, up to integration constant. Using formulas (12), we can recover $x_1(u)$ and $x_2(u)$ from the functions $\varphi(u)$ and $r(u)$, up to integration constants. Consequently, the quasi-minimal rotational surface of elliptic type $\mathcal{M}'$ is constructed as described in the theorem.

Conversely, given a smooth function $r(u) > 0$, we can define the function

$$\varphi(u) = \eta \int \frac{rr'' + (r')^2 + 1}{r(1 + (r')^2)}\, du,$$

where $\eta = \pm 1$, and consider the functions

$$x_1(u) = \int \sqrt{1 + (r')^2}\, \cos\varphi(u)\, du,$$
$$x_2(u) = \int \sqrt{1 + (r')^2}\, \sin\varphi(u)\, du.$$

A straightforward computation shows that the curve $c : \tilde{z}(u) = (x_1(u), x_2(u), r(u), 0)$ is a spacelike curve generating a quasi-minimal rotational surface of elliptic type according to formula (1). $\qquad\square$

3.2. Quasi-minimal rotational surfaces of hyperbolic type

Now, we shall consider the rotational surface of hyperbolic type $\mathcal{M}''$ defined by (2). The generating curve c is a spacelike curve parameterized by the arc-length, i.e. $(r')^2 + (x_2')^2 - (x_4')^2 = 1$, and hence $(x_4')^2 - (x_2')^2 = (r')^2 - 1$. We assume that $(r')^2 \neq 1$, otherwise the surface lies in a 2-dimensional plane. Denote by ε the sign of $(r')^2 - 1$.

As in the elliptic case, we use the following orthonormal tangent frame field:

$$X = z_u; \qquad Y = \frac{z_v}{r},$$

and the normal frame field $\{n_1, n_2\}$, defined by

$$n_1 = \frac{1}{\sqrt{\varepsilon((r')^2 - 1)}}\left(0, x_4', 0, x_2'\right);$$
$$n_2 = \frac{1}{\sqrt{\varepsilon((r')^2 - 1)}}\left((1 - (r')^2)\cosh v, -r'x_2', (1 - (r')^2)\sinh v, -r'x_4'\right). \tag{14}$$

The orthonormal frame field $\{X, Y, n_1, n_2\}$ satisfies

$$\langle X, X \rangle = 1; \quad \langle X, Y \rangle = 0; \quad \langle Y, Y \rangle = -1;$$
$$\langle n_1, n_1 \rangle = \varepsilon; \quad \langle n_1, n_2 \rangle = 0; \quad \langle n_2, n_2 \rangle = -\varepsilon.$$

Calculating the second partial derivatives of $z(u, v)$ we obtain

$$z_{uu} = (r''\cosh v, x_2'', r''\sinh v, x_4'');$$
$$z_{uv} = (r'\sinh v, 0, r'\cosh v, 0); \tag{15}$$
$$z_{vv} = (r\cosh v, 0, r\sinh v, 0).$$

Formulas (14) and (15) imply that the components of the second fundamental form of $\mathcal{M}''$ are:

$$\langle z_{uu}, n_1 \rangle = \frac{1}{\sqrt{\varepsilon((r')^2 - 1)}}(x_4'x_2'' - x_4''x_2'); \qquad \langle z_{uu}, n_2 \rangle = \frac{r''}{\sqrt{\varepsilon((r')^2 - 1)}};$$
$$\langle z_{uv}, n_1 \rangle = 0; \qquad \langle z_{uv}, n_2 \rangle = 0;$$
$$\langle z_{vv}, n_1 \rangle = 0; \qquad \langle z_{vv}, n_2 \rangle = \frac{r(1 - (r')^2)}{\sqrt{\varepsilon((r')^2 - 1)}}.$$

Hence, we obtain the following formulas for the second fundamental form σ:

$$\sigma(X, X) = \frac{\varepsilon(x_4'x_2'' - x_4''x_2')}{\sqrt{\varepsilon((r')^2 - 1)}} n_1 - \frac{\varepsilon r''}{\sqrt{\varepsilon((r')^2 - 1)}} n_2,$$
$$\sigma(X, Y) = 0, \tag{16}$$
$$\sigma(Y, Y) = \qquad + \frac{\varepsilon((r')^2 - 1)}{r\sqrt{\varepsilon((r')^2 - 1)}} n_2.$$

Formulas (16) imply that the Gauss curvature K of the rotational surface of hyperbolic type $\mathcal{M}''$ is

$$K = -\frac{r''}{r} \tag{17}$$

and the normal mean curvature vector field H is

$$H = \frac{\varepsilon}{2r\sqrt{\varepsilon((r')^2 - 1)}} \left(r(x_4'x_2'' - x_4''x_2') n_1 - (rr'' + (r')^2 - 1) n_2 \right). \tag{18}$$

The next two statements follow directly from equalities (17) and (18).

Proposition 3.4.
The rotational surface of hyperbolic type $\mathcal{M}''$, defined by (2), is flat if and only if $r'' = 0$.

Proposition 3.5.
The rotational surface of hyperbolic type $\mathcal{M}''$, defined by (2), is minimal if and only if $x_4'x_2'' - x_4''x_2' = 0$ and $rr'' + (r')^2 - 1 = 0$.

We assume that $(x_4'x_2'' - x_4''x_2')^2 + (rr'' + (r')^2 - 1)^2 \neq 0$, since we are interested in quasi-minimal rotational surfaces.

Similarly to the elliptic case it follows from (14) that

$$\nabla_X' n_1 = \frac{x_2'x_4'' - x_2''x_4'}{\sqrt{\varepsilon((r')^2 - 1)}} X + \frac{r'}{\varepsilon((r')^2 - 1)}(x_2'x_4'' - x_2''x_4') n_2,$$
$$\nabla_Y' n_1 = 0. \tag{19}$$

We distinguish the following two special classes of rotational surfaces of hyperbolic type.

I. Let $x_2'x_4'' - x_2''x_4' = 0$, $rr'' + (r')^2 - 1 \neq 0$. Using (19) we get that in this case

$$\nabla_X' n_1 = 0; \quad \nabla_Y' n_1 = 0,$$

which imply that the rotational surface of hyperbolic type $\mathcal{M}''$ lies in the hyperplane span$\{X, Y, n_2\}$.

The mean curvature vector field of $\mathcal{M}''$ is:

$$H = \frac{\varepsilon(1-(r')^2-rr'')}{2r\sqrt{\varepsilon((r')^2-1)}}\, n_2.$$

Hence, $\langle H, H \rangle = 0$ if and only if $H = 0$. Consequently, there are no quasi-minimal rotational surfaces of hyperbolic type in the class $x_2' x_4'' - x_2'' x_4' = 0$.

II. Let $rr'' + (r')^2 - 1 = 0$, $\ x_2' x_4'' - x_2'' x_4' \neq 0$. In this case $\mathcal{M}''$ does not lie in any hyperplane of $\mathbb{E}_2^4$ and the mean curvature vector field is

$$H = \frac{\varepsilon(x_4' x_2'' - x_4'' x_2')}{2\sqrt{\varepsilon((r')^2-1)}}\, n_1.$$

Hence, we have again that $\langle H, H \rangle = 0$ if and only if $H = 0$. Consequently, there are no quasi-minimal rotational surfaces of hyperbolic type in the class $rr'' + (r')^2 - 1 = 0$.

Further we consider general rotational surfaces of hyperbolic type, i.e. we assume that $x_2' x_4'' - x_2'' x_4' \neq 0$ and $rr'' + (r')^2 - 1 \neq 0$ in an open interval $I \subset J$. The following theorem gives a local description of all quasi-minimal rotational surfaces of hyperbolic type.

Theorem 3.6.

Case (A). *Given a smooth positive function $r(u) : I \subset \mathbb{R} \to \mathbb{R}$, such that $(r')^2 > 1$, define the functions*

$$\varphi(u) = \eta \int \frac{rr'' + (r')^2 - 1}{r(1-(r')^2)}\, du, \quad \eta = \pm 1,$$

and

$$\begin{aligned} x_2(u) &= \int \sqrt{(r')^2 - 1}\, \sinh \varphi(u)\, du, \\ x_4(u) &= \int \sqrt{(r')^2 - 1}\, \cosh \varphi(u)\, du. \end{aligned}$$

Then the spacelike curve $c : \tilde{z}(u) = (r(u), x_2(u), 0, x_4(u))$ is a generating curve of a quasi-minimal rotational surface of hyperbolic type.

Case (B). *Given a smooth positive function $r(u) : I \subset \mathbb{R} \to \mathbb{R}$, such that $(r')^2 < 1$, define the functions*

$$\varphi(u) = \eta \int \frac{rr'' + (r')^2 - 1}{r(1-(r')^2)}\, du, \quad \eta = \pm 1,$$

and

$$\begin{aligned} x_2(u) &= \int \sqrt{1 - (r')^2}\, \cosh \varphi(u)\, du, \\ x_4(u) &= \int \sqrt{1 - (r')^2}\, \sinh \varphi(u)\, du. \end{aligned}$$

Then the spacelike curve $c : \tilde{z}(u) = (r(u), x_2(u), 0, x_4(u))$ is a generating curve of a quasi-minimal rotational surface of hyperbolic type.

Conversely, any quasi-minimal rotational surface of hyperbolic type is locally described by one of the cases given above.

Proof.

Let $\mathcal{M}''$ be a general rotational surface of hyperbolic type generated by a spacelike curve $c : \tilde{z}(u) = (r(u), x_2(u), 0, x_4(u))$; $u \in J$. We assume that c is parameterized by the arc-length and $x_2' x_4'' - x_2'' x_4' \neq 0$, $\ rr'' + (r')^2 - 1 \neq 0$ in an interval $I \subset J$.

Formula (18) implies that $\mathcal{M}''$ is quasi-minimal if and only if

$$r(x_2' x_4'' - x_2'' x_4') = \eta(rr'' + (r')^2 - 1), \qquad \eta = \pm 1. \tag{20}$$

Since our considerations are local, we can assume that either $\varepsilon = 1$ in some open interval $I_0 \subset I$ or $\varepsilon = -1$ in an open interval $I_1 \subset I$. We study the restriction of $\mathcal{M}''$ on I_0, respectively I_1.

If $\varepsilon = 1$, then using that $(x_4')^2 - (x_2')^2 = (r')^2 - 1$ and $(r')^2 - 1 > 0$ we obtain that there exists a smooth function $\varphi = \varphi(u)$ such that

$$
\begin{aligned}
x_2'(u) &= \sqrt{(r')^2 - 1}\ \sinh \varphi(u), \\
x_4'(u) &= \sqrt{(r')^2 - 1}\ \cosh \varphi(u).
\end{aligned}
\tag{21}
$$

The last equalities imply $x_4' x_2'' - x_4'' x_2' = ((r')^2 - 1)\varphi'$. Hence, condition (20) for quasi-minimality of $\mathcal{M}''$ is written in terms of $r(u)$ and $\varphi(u)$ as follows:

$$
\varphi' = \eta\,\frac{1 - (r')^2 - rr''}{r((r')^2 - 1)}.
\tag{22}
$$

Then, the mean curvature vector field is given by the formula

$$
H = \frac{1 - (r')^2 - rr''}{2r\sqrt{(r')^2 - 1}}(\eta\, n_1 + n_2).
$$

Using (22) we can recover $\varphi(u)$ from $r(u)$, up to integration constant, and using (21), we can recover $x_2(u)$ and $x_4(u)$ from the functions $\varphi(u)$ and $r(u)$, up to integration constants. Consequently, if $\varepsilon = 1$ the restriction of the quasi-minimal rotational surface of hyperbolic type $\mathcal{M}''$ on I_0 is constructed as described in case (A) of the theorem.

If $\varepsilon = -1$, then there exists a smooth function $\varphi = \varphi(u)$ such that

$$
\begin{aligned}
x_2'(u) &= \sqrt{1 - (r')^2}\ \cosh \varphi(u), \\
x_4'(u) &= \sqrt{1 - (r')^2}\ \sinh \varphi(u).
\end{aligned}
$$

As in the previous case we get that condition (20) for quasi-minimality of $\mathcal{M}''$ is:

$$
\varphi' = \eta\,\frac{rr'' + (r')^2 - 1}{r(1 - (r')^2)},
$$

and the mean curvature vector field is given by the formula

$$
H = \frac{rr'' + (r')^2 - 1}{2r\sqrt{1 - (r')^2}}(\eta\, n_1 + n_2).
$$

Hence, we can recover $\varphi(u)$ from $r(u)$, and $x_2(u)$, $x_4(u)$ from $\varphi(u)$ and $r(u)$, up to integration constants. Consequently, if $\varepsilon = -1$ the restriction of the quasi-minimal rotational surface of hyperbolic type $\mathcal{M}''$ on I_1 is constructed as described in case (B) of the theorem.

Conversely, given a smooth function $r(u) > 0$, we can define the function

$$
\varphi(u) = \eta \int \frac{rr'' + (r')^2 - 1}{r(1 - (r')^2)}\, du, \quad \eta = \pm 1,
$$

and consider the functions

$$
\begin{aligned}
x_2(u) &= \int \sqrt{(r')^2 - 1}\ \sinh \varphi(u)\, du, \\
x_4(u) &= \int \sqrt{(r')^2 - 1}\ \cosh \varphi(u)\, du,
\end{aligned}
\qquad \text{case (A)}
$$

or

$$
\begin{aligned}
x_2(u) &= \int \sqrt{1 - (r')^2}\ \cosh \varphi(u)\, du, \\
x_4(u) &= \int \sqrt{1 - (r')^2}\ \sinh \varphi(u)\, du.
\end{aligned}
\qquad \text{case (B)}
$$

A straightforward computation shows that the curve $c : \widetilde{z}(u) = (r(u), x_2(u), 0, x_4(u))$ is a spacelike curve generating a quasi-minimal rotational surface of hyperbolic type according to formula (2). $\qquad\square$

3.3. Quasi-minimal rotational surfaces of parabolic type

Now we shall consider the rotational surface of parabolic type $\mathcal{M}'''$ in $\mathbb{E}_2^4$ defined by formula (3) with respect to $\{e_1, e_4, \xi_1, \xi_2\}$, where $\xi_1 = \frac{e_2+e_3}{\sqrt{2}}$, $\xi_2 = \frac{-e_2+e_3}{\sqrt{2}}$. Recall that

$$\langle \xi_1, \xi_1 \rangle = 0; \quad \langle \xi_2, \xi_2 \rangle = 0; \quad \langle \xi_1, \xi_2 \rangle = -1.$$

The generating curve c is a spacelike curve parameterized by the arc-length, i.e. $(x_1')^2 + (x_2')^2 - (x_3')^2 = 1$, and hence $(x_1')^2 = 1 + 2f'g'$; $x_1'x_1'' = g'f'' + f'g''$.

We use the following orthonormal tangent frame field:

$$
\begin{aligned}
X &= z_u = x_1'\, e_1 + \sqrt{2}\,vf'\, e_4 + f'\, \xi_1 + (-v^2f' + g')\, \xi_2; \\
Y &= \frac{z_v}{\sqrt{2}f} = e_4 - \sqrt{2}v\, \xi_2;
\end{aligned}
$$

and the normal frame field $\{n_1, n_2\}$, defined by

$$
\begin{aligned}
n_1 &= e_1 + \frac{x_1'}{f'}\, \xi_2; \\
n_2 &= x_1'\, e_1 + \sqrt{2}\,vf'\, e_4 + f'\, \xi_1 + \frac{1 + f'g' - v^2(f')^2}{f'}\, \xi_2.
\end{aligned}
\tag{23}
$$

The second partial derivatives of $z(u,v)$ are expressed as follows

$$
\begin{aligned}
z_{uu} &= x_1''\, e_1 + \sqrt{2}\,vf''\, e_4 + f''\, \xi_1 + (-v^2f'' + g'')\, \xi_2; \\
z_{uv} &= \sqrt{2}f'\, e_4 - 2vf'\, \xi_2; \\
z_{vv} &= -2f\, \xi_2.
\end{aligned}
\tag{24}
$$

By a straightforward computation from (23) and (24) we obtain the components of the second fundamental form:

$$
\begin{array}{ll}
\langle z_{uu}, n_1 \rangle = \dfrac{x_1''f' - x_1'f''}{f'}; & \langle z_{uu}, n_2 \rangle = -\dfrac{f''}{f'}; \\[2mm]
\langle z_{uv}, n_1 \rangle = 0; & \langle z_{uv}, n_2 \rangle = 0; \\[2mm]
\langle z_{vv}, n_1 \rangle = 0; & \langle z_{vv}, n_2 \rangle = 2ff'.
\end{array}
$$

Hence, we obtain the following formulas for the second fundamental form σ:

$$
\begin{aligned}
\sigma(X, X) &= \frac{x_1''f' - x_1'f''}{f'}\, n_1 + \frac{f''}{f'}\, n_2, \\
\sigma(X, Y) &= 0, \\
\sigma(Y, Y) &= \qquad\qquad\quad -\frac{f'}{f}\, n_2.
\end{aligned}
\tag{25}
$$

Formulas (25) imply that the Gauss curvature K of the rotational surface of parabolic type $\mathcal{M}'''$ is expressed as

$$K = -\frac{f''}{f} \tag{26}$$

and the mean curvature vector field H is

$$H = \frac{1}{2ff'}\left(f(x_1''f' - x_1'f'')\, n_1 + (ff'' + (f')^2)\, n_2\right). \tag{27}$$

Using equalities (26) and (27) we get the following two statements.

Proposition 3.7.
The rotational surface of parabolic type $\mathcal{M}'''$, defined by (3), is flat if and only if $f'' = 0$.

Proposition 3.8.
The rotational surface of parabolic type $\mathcal{M}'''$, defined by (3), is minimal if and only if $x_1''f' - x_1'f'' = 0$ and $ff'' + (f')^2 = 0$.

We assume that $(x_1''f' - x_1'f'')^2 + (ff'' + (f')^2)^2 \neq 0$, since we study quasi-minimal rotational surfaces.

It follows from (23) that

$$
\begin{aligned}
\nabla'_X n_1 &= -\frac{x_1''f' - x_1'f''}{f'} X + \frac{x_1''f' - x_1'f''}{f'} n_2, \\
\nabla'_Y n_1 &= 0, \\
\nabla'_X n_2 &= \frac{f''}{f'} X + \frac{x_1''f' - x_1'f''}{f'} n_1, \\
\nabla'_Y n_2 &= \frac{f'}{f} Y.
\end{aligned}
\tag{28}
$$

As in the elliptic and hyperbolic cases we distinguish two special classes of rotational surfaces of parabolic type.

I. Let $x_1''f' - x_1'f'' = 0$, $ff'' + (f')^2 \neq 0$. In this case from the first two equalities of (28) we get

$$
\nabla'_X n_1 = 0; \quad \nabla'_Y n_1 = 0,
$$

which imply that the normal vector field n_1 is constant and hence, the rotational surface of parabolic type $\mathcal{M}'''$ lies in the hyperplane $\mathbb{E}_2^3 = \mathrm{span}\{X, Y, n_2\}$ of $\mathbb{E}_2^4$.

In this case the mean curvature vector field of $\mathcal{M}'''$ is:

$$
H = \frac{ff'' + (f')^2}{2ff'} n_2,
$$

which implies that $\langle H, H \rangle = 0$ if and only if $H = 0$. Consequently, there are no quasi-minimal rotational surfaces of parabolic type in this class.

II. Let $ff'' + (f')^2 = 0$, $x_1''f' - x_1'f'' \neq 0$. In this case the mean curvature vector field is

$$
H = \frac{x_1''f' - x_1'f''}{2f'} n_1,
$$

which implies again that $\langle H, H \rangle = 0$ if and only if $H = 0$. Consequently, there are no quasi-minimal rotational surfaces of parabolic type in this special class.

Further we consider general rotational surfaces of parabolic type, i.e. we assume that $x_1''f' - x_1'f'' \neq 0$ and $ff'' + (f')^2 \neq 0$ in an open interval $I \subset J$. In the following theorem we give a local description of all quasi-minimal rotational surfaces of parabolic type.

Theorem 3.9.
Given a smooth function $f(u) : I \subset \mathbb{R} \to \mathbb{R}$, define the functions

$$
\varphi(u) = f'(u) \left(C + \eta \left(-\frac{1}{f'(u)} + \int \frac{du}{f(u)} \right) \right), \quad \eta = \pm 1, \ C = const,
$$

and

$$
x_1(u) = \int \varphi(u)\,du; \qquad g(u) = \int \frac{\varphi^2(u) - 1}{2f'(u)}\,du.
$$

Then the curve $c : \widetilde{z}(u) = x_1(u)\,e_1 + f(u)\,\xi_1 + g(u)\,\xi_2$ is a spacelike curve generating a quasi-minimal rotational surface of parabolic type.

Conversely, any quasi-minimal rotational surface of parabolic type is locally constructed as described above.

Proof. Let $\mathcal{M}'''$ be a general rotational surface of parabolic type generated by a spacelike curve $c : \widetilde{z}(u) = x_1(u)\, e_1 + f(u)\, \xi_1 + g(u)\, \xi_2;\ u \in J$. We assume that c is parameterized by the arc-length and $x_1''f' - x_1'f'' \neq 0,\ \ ff'' + (f')^2 \neq 0$ for $u \in I \subset J$.

Equality (27) implies that $\mathcal{M}'''$ is quasi-minimal if and only if

$$f(x_1''f' - x_1'f'') = \eta(ff'' + (f')^2), \qquad \eta = \pm 1. \tag{29}$$

Hence, the mean curvature vector field of a quasi-minimal rotational surface of parabolic type is given by the formula

$$H = \frac{1}{2}\left(\ln|ff'|\right)'(\eta\, n_1 + n_2).$$

We denote $\varphi(u) = x_1'(u)$. Since c is parameterized by the arc-length, we have $(x_1')^2 = 1 + 2f'g'$, which implies that $g'(u) = \frac{\varphi^2(u)-1}{2f'(u)}$. The last equality allows us to recover $g(u)$ from the functions $\varphi(u)$ and $f(u)$, up to integration constant. Condition (29) for quasi-minimality of $\mathcal{M}'''$ is written in terms of $f(u)$ and $\varphi(u)$ as follows:

$$\varphi' - \frac{f''}{f'}\varphi = \eta\left(\frac{f''}{f'} + \frac{f'}{f}\right). \tag{30}$$

We consider (30) as a differential equation with respect to $\varphi(u)$. Then the general solution of (30) is given by the formula

$$\varphi(u) = e^{-\int p(u)du}\left(C + \int q(u)\, e^{\int p(u)du}\, du\right), \tag{31}$$

where $p(u) = -\frac{f''}{f'},\ \ q(u) = \eta\left(\frac{f''}{f'} + \frac{f'}{f}\right)$. Calculating the integrals in formula (31) we obtain

$$\varphi(u) = f'(u)\left(C + \eta\left(-\frac{1}{f'(u)} + \int \frac{du}{f(u)}\right)\right), \quad \eta = \pm 1,\ C = const, \tag{32}$$

which allows us to recover $\varphi(u)$ from $f(u)$.

Hence, the quasi-minimal rotational surface of parabolic type $\mathcal{M}'''$ is locally constructed as described in the theorem.

Conversely, given a smooth function $f(u)$, we can define the function $\varphi(u)$ by formula (32) and consider the functions

$$x_1(u) = \int \varphi(u)\, du; \qquad g(u) = \int \frac{\varphi^2(u)-1}{2f'(u)}\, du.$$

A straightforward computation shows that the curve $c : \widetilde{z}(u) = x_1(u)\, e_1 + f(u)\, \xi_1 + g(u)\, \xi_2$ is a spacelike curve generating a quasi-minimal rotational surface of parabolic type according to formula (3). $\qquad \square$

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