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Symmetries of Spacetimes and Riemannian Manifolds

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Symmetries of Spacetimes and Riemannian Manifolds

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Dedication

TO

SHREE SATYA SAI BABA

AND

OUR TEACHERS

May our teachers be auspicious

May the bestower of wisdom be auspicious

May the Lord who manifested the Universe be auspicious

Contents

Preface

This book provides an upto date information on metric, connection and curvature symmetries used in geometry and physics. More specifically, we present the characterizations and classifications of Riemannian and Lorentzian manifolds (in particular, the spacetimes of general relativity) admitting metric (i.e., Killing, homothetic and conformal), connection (i.e., affine conformal and projective) and curvature symmetries. Our approach, in this book, has the following outstanding features:

(a) It is the first-ever attempt of a comprehensive collection of the works of a very large number of researchers on all the above mentioned symmetries.

(b) We have aimed at bringing together the researchers interested in differential geometry and the mathematical physics of general relativity by giving an invariant as well as the index form of the main formulas and results.

(c) Attempt has been made to support several main mathematical results by citing physical example(s) as applied to general relativity.

(d) Overall the presentation is self contained, fairly accessible and in some special cases supported by an extensive list of cited references.

(e) The material covered should stimulate future research on symmetries.

Chapters 1 and 2 contain most of the prerequisites for reading the rest of the book. We present the language of semi-Euclidean spaces, manifolds, their tensor calculus; geometry of null curves, non-degenerate and degenerate (lightlike) hypersurfaces. All this is described in invariant as well as the index form.

Chapter 3 deals with a brief coverage of the theory of Lie-derivatives, Lie group and their Lie algebras. We show that, whereas, the set of Killing, conformal, affine and projective vector fields have a finite dimensional Lie algebra structure, unfortunately, homothetic and curvature collineations do not share this important property.

Chapter 4 focuses on the kinematics of fluid spacetimes, Einstein field equations, energy conditions, globally hyperbolic spacetimes and exact solutions of field equations discussed in the rest of the book.

Chapter 5 deals with divergence theorems (including latest information on its restricted validity for semi-Riemannian manifolds) and integral formulas as related to the existence of Killing and affine symmetries on Riemannian manifolds. We show that, in relativity, the existence of aforementioned symmetries is closely related to their kinematic and dynamical properties. Precisely, while the Killing symmetry is characterized by expansion-free and shear-free spacetimes, the affine symmetry is relevant to a restricted class of spacetimes with non-zero constant expansion and non-zero constant shear.

In chapter 6 we present homothetic and conformal symmetries of Riemannian manifolds with respect to compact hypersurfaces and general relativistic spacetimes with respect to their kinematic / dynamic properties. We highlight the fact that conformal symmetries are characterized by shear-free spacetimes. Since a very large number of papers deal with conformal motions, we close this chapter with a summary on most of their main results.

Chapter 7 includes an upto date coverage on the existence of conformal collineations, their restricted use in relativity, very limited known work on projective collineations and extensive description of curvature collineations. We conclude that the general (global and local) study of conformal collineations in semi - Riemannian manifolds is a very potential open problem. Also, in relativity, this symmetry has an important role in the study of fluids with non-constant shear, a counter part of the theory of shear-free cosmology.

Chapter 8 deals with latest work on the fundamental question of determining when the symmetry of the geometry (defined by a given metric and / or curvature symmetry vector field) is inherited by all the source terms of a matter tensor of Einstein field equations. Physically, there is a close connection of inheriting conformal Killing vectors (**CKV's**) with the relativistic thermodynamics of fluids since for a distribution of massless particles in equilibrium the inverse temperature function is inheriting **CKV**. Also we have presented general results on semi - Riemannian manifolds with curvature and Ricci inheritance symmetries.

Finally, in chapter 9, we have presented a basic introduction to Kaehlerian, contact and globally framed geometric structures on smooth manifolds; followed by a review of work done on various symmetries of these structured manifolds. We have included what is known on their semi - Riemannian structures. In particular, a totally geodesic lightlike hypersurface of a class of Lorentz framed manifold is a Killing horizon. Also, a globally hyperbolic spacetime can carry an almost contact or a framed metric structure. Complex manifolds have two interesting classes of Kaehler manifolds, namely, (i) Calabi-Yau manifolds which have their application in superstring theory (see Candelas [26]) and (ii) Teichmuller spaces applicable to general relativity (see Tromba [191]).

This book could be recommended as a nice resource for graduate students taking courses or pursuing research in differential geometry and / or general relativity.

Both authors are grateful to all authors of books and articles whose work has been used in preparing this book. We acknowledge both the University of Windsor (Canada) and the University of New Haven (USA) for providing facilities and research support. Finally, it is a pleasure to thank Kluwer Academic Publishers for their support in the preparation of the camera - ready manuscript and excellent care in publishing this volume.

Krishan L. Duggal
Ramesh Sharma
March 15, 1999

Chapter 1

Preliminaries

In this chapter we review the algebraic preliminaries on semi-Euclidean vector spaces, subspaces of Minkowski spaces (including degenerate subspaces) and the algebraic classification of the electromagnetic tensor field in R^4_1. Finally, we review some algebraic structures (needed in this book), namely, groups, Lie groups and Lie algebras.

1.1 Semi-Euclidean Vector Spaces

Let V be a real n-dimensional vector space with a symmetric bilinear mapping $g : V \times V \to \mathbf{R}$. We say that g is **positive (negative) definite** on V if $g(v,v) > 0\,(g(v,v) < 0)$ for any non-zero $v \in V$. On the other hand, if $g(v,v) \geq 0\,(g(v,v) \leq 0)$ for any $v \in V$ and there exists a non-zero $u \in V$ with $g(u,u) = 0$, we say that g is **positive (negative) semi-definite** on V.

Let $B = \{u_1, \dots, u_n\}$ be an arbitrary basis of V. Then, g can be expressed by an $n \times n$ symmetric matrix $G = (g_{ij})$, where

$$g_{ij} = g(u_i, u_j), \qquad (1 \leq i, j \leq n).$$

G is called the **associated matrix** of g with respect to the basis B. We assume that $rank\ G = n \iff g$ is **non-degenerate** on V. The non-degenerate g on V is called a **semi-Euclidean metric (scalar product)** and then (V, g) is said to be a **semi-Euclidean vector space**, for which $g(u,v) = u \cdot v$ where $\cdot$ is the usual dot product. It is well known that for a semi-Euclidean $V \neq 0$, there exists an **orthonormal basis** $E = \{e_1, \dots, e_n\}$ such that

$$g(v,v) = -\sum_{i=1}^{p} (v^i)^2 + \sum_{a=p+1}^{p+q} (v^a)^2, \tag{1.1}$$

where $p + q = n$ and (v^i) are the coordinate components of v with respect to E. Thus, with respect to (1.1) G is a diagonal matrix of canonical form:

$$diag(- \dots - + \dots +) \tag{1.2}$$

The sum of these diagonal elements (also called the **trace** of the canonical form) is called the **signature** of g and the number of negative signs in (1.2) is called the **index** of V. Two special cases are important both for geometry and physics. First, g is positive (negative) definite for which V is Euclidean with zero (n) index. Secondly, if the index of g is 1, then g is called a **Minkowski metric** and V is called a **Minkowski space**, used in the study for special relativity.

Define a mapping (called **norm**), of a semi-Euclidean V, by

$$\| \cdot \| : V \to \mathbf{R}; \qquad \| v \| = | \, g(v,v) \, |^{\frac{1}{2}}, \qquad \forall v \in V.$$

$\| v \|$ is called the length of v. A vector v is said to be
spacelike, if $g(v,v) > 0$ or $v = 0$
timelike, if $g(v,v) < 0$
lightlike (null, isotropic), if $g(v,v) = 0$ and $v \neq 0$.
The set of all null vectors in V, denoted by Λ , is called the **null cone** of V, *i.e.*,

$$\Lambda = \{v \in (V - \{0\}), g(v,v) = 0\}.$$

For a semi-Euclidean V, a unit vector u is defined by $g(u,u) = \pm 1$ and, as in the case of Euclidean spaces, we say that u is of length 1. Also, it is important to mention that, in case g is semi-definite, orthogonal vectors (i.e., $u \perp v$ if $g(u,v) = 0$) are not necessarily at right angles to each other. For example, a null vector is a non-zero vector that is orthogonal to itself.

Throughout this book, the signature of g will be of the form as given by (1.2) with respect to the orthonormal basis of V.

Since, in general, semi-Euclidean V has three types of vectors (spacelike, timelike, null), it is some times desirable to transform a given orthonormal basis $E = \{e_1, \ldots, e_n\}$ into another basis which contains some null vectors. To construct such a basis we let $\{e_1, \ldots, e_p\}$ and $\{e_{p+1}, \ldots, e_{p+q}\}$, $p + q = n$, be unit timelike and spacelike vectors, respectively. In general, following three cases arise:

Case 1 ($p < q$). Construct the following vectors

$$N_i = \frac{1}{\sqrt{2}}(e_{p+i} + e_i) \quad ; \quad N_i^\star = \frac{1}{\sqrt{2}}(e_{p+i} - e_i). \tag{1.3}$$

Here each N_i and $N_i^\star$ are null vectors which satisfy

$$g(N_i, N_j) = g(N_i^\star, N_j^\star) = 0, \qquad g(N_i, N_j^\star) = \delta_{ij}, \qquad i,j \in \{1, \ldots, p\}. \tag{1.4}$$

Thus $\{N_1, \ldots, N_p, N_1^\star, \ldots, N_p^\star, e_{2p+1}, \ldots, e_{p+q} = e_n\}$ is a basis of V which contains $2p$ null vectors and $q - p$ spacelike vectors.

Case 2 ($q < p$). For this case we set

$$N_a = \frac{1}{\sqrt{2}}(e_{p+a} + e_a), \qquad N_a^\star = \frac{1}{\sqrt{2}}(e_{p+a} - e_a), \tag{1.5}$$

to obtain the relations (1.4) but with i, j replaced by $a, b \in \{1, \ldots, q\}$. Then, the basis of V contains $2q$ null vectors $\{N_1, \ldots, N_q, N_1^\star, \ldots, N_q^\star\}$ and $p - q$ timelike vectors $\{e_{q+1}, \ldots, e_p\}$.

Case 3 $p = q$. This is a special case for which $n = 2p = 2q$ is even and the null transformed basis of V is $\{N_1, \dots, N_p, N_1^\star, \dots, N_p^\star\}$.

We say that V is a **proper semi-Euclidean space** if $p \cdot q \neq 0$. It follows from the above three cases that, in general, there exists a basis

$$B = \{N_1, \dots, N_r, N_1^\star, \dots, N_r^\star, u_1, \dots, u_s\} \tag{1.6}$$

for a proper semi-Riemannian space (V, g). In the sequel we call such a basis, given by (1.6), a **quasi-orthonormal basis** which satisfies the following conditions:

$$g(N_i, N_j) = g(N_i^\star, N_j^\star) = 0 \quad ; \quad g(N_i, N_j^\star) = \delta_{ij},$$

$$g(u_a, N_i) = g(u_a, N_i^\star) = 0 \quad ; \quad g(u_a, u_b) = \epsilon_a \delta_{ab}, \tag{1.7}$$

for any $i, j \in \{1, \dots, r\}$, $a, b \in \{1, \dots, s\}$, $2r + s = n$ and $\epsilon_a = \pm 1$.

EXAMPLE 1. A quasi-orthonormal basis of a 4-dimensional Minkowski space, denoted by R_1^4, is given by

$$B = \{N, N^\star, u, v\} \tag{1.8}$$

where N, $N^\star$ and u,v are real null and spacelike vectors respectively and they satisfy the relations (1.4). If $B^\star = \{\theta_1, \theta_2, \theta_3, \theta_4\}$ is the dual basis of B, then

$$g(X, X) = 2\theta_1(X)\theta_2(X) + (\theta_3(X))^2 + (\theta_4(X))^2, \qquad \forall X \in R_1^4. \tag{1.9}$$

It is important to note that, in above example, the associated matrix G of g will not be diagonalizable.

EXAMPLE 2. Consider the complexified vector space $(R_1^4)^c$ consisting of vectors $u + iv$, $u, v \in R_1^4$, $i = \sqrt{-1}$, with the operations

$$(u + iv) + (u' + iv') = (u + u') + i(v + v')$$

$$(a + ib)(u + iv) = (au - bv) + i(av + bu),$$

for any $a, b \in R$ and $u, v, u', v' \in R_1^4$. The scalar product g on $(R_1^4)^c$ is given by

$$g(u + iv, u' + iv') = g(u, u') - g(v, v') + i(g(u, v') + g(v, u')).$$

Thus, the quasi-orthonormal basis given by (1.6) can be transformed into the so called **null tetrad** of $(R_1^4)^c$ (see Kramer et al. [118, page 44])

$$T = \{\ell, k, m, \bar{m}\}, \qquad \ell = N,\ k = -N^\star,$$

$$m = \frac{1}{\sqrt{2}}(u + iv), \qquad \bar{m} = \frac{1}{\sqrt{2}}(u - iv), \tag{1.10}$$

where m and $\bar{m}$ are complex conjugate null vectors and

$$g(\ell, k) = -1 \quad ; \quad g(m, \bar{m}) = 1 \tag{1.11}$$

are the only non-zero scalar products. If $T^\star = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ is the dual basis of T, then

$$g(X,X) = 2[\omega_3(X)\omega_4(X) - \omega_1(X)\omega_2(X)], \text{ for any vector } X. \tag{1.12}$$

1.2 Subspaces of Minkowski Spaces

Let W be a subspace of an n-dimensional Minkowski space (V, g). Then, although $g|W$ on $W \times W$ is also a symmetric bilinear form on W, it is not always non-degenerate on W. For example, any null vector of V generates a one dimensional subspace W of V for which $g|W$ is degenerate. In fact, we have the following three mutually exclusive possibilities of W:

1. $g|W$ is positive definite $\iff$ W is Euclidean,
2. $g|W$ is Lorentz $\iff$ W is Minkowskian,
3. $g|W$ is degenerate $\iff$ W is lightlike.

The category into which a subspace W falls is called its **causal character**. Consider the subspace $W^\perp$, called W **perp** defined by

$$W^\perp = \{v \in V\,; g(v,w) = 0\,, \forall w \in W\}. \tag{1.13}$$

Then, it is well known that for all the three categories of W

$$dim(W) + dim(W^\perp) = n, \qquad (W^\perp)^\perp = W.$$

However, in general, $W \cap W^\perp \neq \{0\}$. For example, consider $W = \{(x, y, x) \in R^3_1\,; x, y \in R\}$. Then, $W^\perp = \{(x, 0, x) \in R^3_1\,; x \in R\}$ and $W \cap W^\perp = W^\perp$. In fact, one can show (see O'Neill [157, page 49]) that $W \cap W^\perp = \{0\} \iff W$ is a non-degenerate subspace of V.

PROPOSITION 1 (Duggal-Bejancu [57, page 7]). *Let g be a proper semi-Euclidean metric on an n-dimensional vector space V of index p. Then, there exists a subspace $\bar{W}$ of V of dimension $min\{p, n-p\}$ and no larger, such that $g|\bar{W} = 0$.*

Proof. Let $E = \{e_1, ..., e_n\}$ be an orthonormal basis of V. Define g by

$$g(x,y) = -\sum_{i=1}^{p} x^i y^i + \sum_{a=p+1}^{n} x^a y^a, \qquad \forall x, y \in R^n_p,$$

where (x^i) and (y^i) are the coordinates of x and y with respect to E. Suppose $2p < n$. Now define a p-dimensional subspace

$$\bar{W} = Span\{u_1 = e_1 + e_{p+1}, \ldots, u_p = e_p + e_{2p}\}.$$

It follows from above that $g|\bar{W} = 0$. Choose a null vector $N = \sum_{i=1}^n N^i e_i$ such that $g(N, u_a) = 0, \forall a \in \{1, \dots, p\}$. Thus we have $N^1 = N^{p+1}, \dots, N^p = N^{2p}$. Since $\| N \| = 0$ and $\{e_1, \dots, e_{2p}\}$ and $\{e_{p+1}, \dots, e_n\}$ are timelike and spacelike respectively, we conclude that $N^{2p+1} = \dots = N^n = 0$. Hence, $N = \sum_{a=1}^{p} N^a u_a$. Thus, there is no subspace larger than $\bar{W}$ on which g vanishes. Similar arguments apply for $2p \geq n$.

EXAMPLE. Consider an orthonormal basis $E = \{e_1, e_2, e_3, e_4\}$ of R_1^4 with e_1 and $\{e_2, e_3, e_4\}$ a unit timelike and three spacelike vectors respectively. Let $W = Span\{e_2, e_3, e_4\}$ be a spacelike subspace of R_1^4. Then, $W^\perp = Span\{e_1\}$ is timelike, $W \oplus W^\perp = R_1^4$ and $W \cap W^\perp = \{0\}$. Following exactly as in the proof of Proposition 1, by setting $p = 1$ and $\bar{W} = Span\{u = e_1 + e_2\}$, it is easy to see that $g|\bar{W} = 0$ and there does not exist a larger subspace of R_1^4 than $\bar{W}$ on which g vanishes.

PROPOSITION 2. *Suppose $(W, \bar{g})$ is a real m-dimensional lightlike subspace of an n-dimensional semi-Euclidean vector space (V, g), where $\bar{g}$ is the induced metric from g on W. Let $\bar{W} \subset W$ be the largest subspace of V on which g vanishes. Then, the complement subspace to $\bar{W}$ in W is always non-degenerate.*

Proof. Let S denote the complementary subspace to $\bar{W} \subset W$, *i.e.* ,

$$W = S \perp \bar{W}. \tag{1.14}$$

Choose a non-zero vector $u \in S$ such that $g(u, x) = 0$, $\forall x \in S$. Now, (1.14) implies that $g(u, y) = 0$, $\forall y \in \bar{W}$. This means that $u \in \bar{W}$, which contradicts the assumption that S and $\bar{W}$ are complementary. Thus, S is non-degenerate.

Following the terminology used in Duggal-Bejancu [57, page 5], a complementary subspace to $\bar{W}$ in W is called a **screen subspace** of W.

COROLLARY. *Under the hypothesis of proposition 2 if $V = R_1^n$, then the screen space S is always a spacelike subspace of R_1^n.*

Let $dim(\bar{W}) = d < m$. Since S is non-degenerate with respect to $\bar{g}$, it is itself a semi-Euclidean space. Thus, there exists an orthonormal basis $\{e_{d+1}, ..., e_m\}$ of S which can be extended to construct a basis $\bar{B} = \{N_1, ..., N_d, e_{d+1}, \dots, e_m\}$ of W with $\bar{W} = Span\{N_1, \dots, N_d\}$. The matrix of $\bar{g}$ with respect to $\bar{B}$ is of the form:

$$(\bar{g}) = \begin{pmatrix} O_{d,d} & O_{d,m-d} \\ O_{m-d,d} & \epsilon_a \delta_{ab} \end{pmatrix}$$

where $a, b \in \{d+1, \dots, m\}$, $\qquad \epsilon_a = \bar{g}(e_a, e_a)$.

EXAMPLE. Consider a quasi-orthonormal basis of R_1^4 given by $B = \{N, N^\star, u, v\}$ as per equation (1.8). Then, $dim(\bar{W}) = 1$ with $g|\bar{W} = 0$. Let $\bar{W} = Span\{N\}$. Now construct a lightlike hypersurface $W = Span\{N, u, v\}$ of R_1^4 so that $\bar{W} \subset W$. Then $S = Span\{u, v\}$ is spacelike, $W^\perp = \bar{W}$, $W + W^\perp \subset R_1^4$ and $W \cap W^\perp = W^\perp$. The matrix of the induced metric $\bar{g}$ on W is of the form:

$$(\bar{g}) = \begin{pmatrix} O & O & O \\ O & 1 & O \\ O & O & 1 \end{pmatrix}, \qquad det(\bar{g}) = 0.$$

Similarly another lightlike hypersurface $W^\star = Span\{N^\star, u, v\}$ of R_1^4 can be constructed such that $\bar{W} = Span\{N^\star\}$.

Next, consider an m-dimensional lightlike subspace W of an n-dimensional proper semi-Euclidean space V. Then a quasi-orthonormal basis

$$B = \{N_1, \dots, N_r, N_1^\star, \dots, N_r^\star, u_1, \dots, u_t\}$$

such that $W = Span\{N_1, \dots, N_r\}$, if $m = r + s$ and $1 \leq s \leq t$, or $W = Span\{N_1, \dots, N_m\}$, if $m \leq r$, is called a quasi-orthonormal basis of V along W. For example, consider the quasi-orthonormal basis (1.8) for which $W = Span\{N\}$, $W = Span\{N, u\}$ and $W = Span\{N, u, v\}$ are one, two and three dimensional lightlike subspaces of R_1^4, respectively.

REMARK. It is clear from the discussion so far that the primary difference between the lightlike subspace W and the non-degenerate subspaces of V is that in the first case $W^\perp$ intersects W. The reader will see in chapters 2 and 4 how to deal with this special lightlike case while discussing null curves and lightlike hypersurfaces of spacetimes.

1.3 Electromagnetism in R_1^4

Let R_1^4 be a 4-dimensional Minkowski space with coordinate system (x^0, x^1, x^2, x^3), where $x^0 = ct$ is the time coordinate and c is the velocity of light. Let g be the **Minkowski metric** defined by

$$g(x, y) = -x^0y^0 + x^1y^1 + x^2y^2 + x^3y^3, \qquad \forall x, y \in R_1^4.$$

The associated matrix G, of g, with respect to an orthonormal basis is given by

$$G = (g) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad a, b, \dots \in \{0, 1, 2, 3\}.$$

In the sequel, we set $\frac{\partial}{\partial x^a} = \partial_a$. Suppose $X = X^a \partial_a$ (summation in a) is a space like vector field, X^a has partial derivatives with respect to the coordinates (x^a). Then, the **curl** and the **divergence** of X are given by

$$curl\, X = (\partial_2 X^3 - \partial_3 X^2)\, \partial_1 + (\partial_3 X^1 - \partial_1 X^3)\, \partial_2 + (\partial_1 X^2 - \partial_2 X^1)\, \partial_3,$$

$$div\, X = \partial_a X^a \text{ (summation in a)} \qquad ; \qquad \partial_a X = (\partial_a X^b)\partial_b.$$

The **Maxwell equations** are

$$curl\, H - \frac{1}{c}\frac{\partial E}{\partial t} = 4\pi\rho\frac{u}{c}, \qquad div\, E = 4\pi\rho,$$

$$curl\, E + \frac{1}{c}\frac{\partial H}{\partial t} = 0, \qquad div\, H = 0,$$

where H and E are the **magnetic and electric fields** respectively, ρ is the **charge density** and u is the local velocity of the charge. The current density $j = \rho u$ and ρ are the sources of H and E. The Maxwell equations are consistent only if the following (called **continuity equations**) holds:

$$\frac{\partial \rho}{\partial t} + div(\rho u) = 0.$$

In vacuum ($\rho = 0$) and for $u = c$, the Maxwell equations represent **electromagnetic waves** in R_1^4. Furthermore, there exists a 3-dimensional spacelike vector field A and a **scalar potential** ϕ such that

$$H = curl\, A \quad ; \quad E = -grad\, \phi - \frac{1}{c}\frac{\partial A}{\partial t}.$$

Introduce a 4-vector Φ with components

$$\Phi = (\Phi^0, \Phi^1, \Phi^2, \Phi^3) = (\phi, A^1, A^2, A^3)$$

whose covariant components are $\Phi_0 = -\phi$, $\Phi_i = A^i$ for $(i = 1, 2, 3)$. Thus, we have the following components of H and E.

$$\begin{aligned} H^1 &= \partial_2\Phi_3 - \partial_3\Phi_2 \quad , \quad & E^1 &= \partial_1\Phi_0 - \partial_0\Phi_1, \\ H^2 &= \partial_3\Phi_1 - \partial_1\Phi_3 \quad , \quad & E^2 &= \partial_2\Phi_0 - \partial_0\Phi_2, \\ H^3 &= \partial_1\Phi_2 - \partial_2\Phi_1 \quad , \quad & E^3 &= \partial_3\Phi_0 - \partial_0\Phi_3. \end{aligned}$$

The above six components of H and E can be represented by a single equation

$$F_{ab} = \partial_a\Phi_b - \partial_b\Phi_a, \qquad a, b = (0, 1, 2, 3)$$

for which F_{ab} has the following matrix form:

$$(F_{ab}) = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & H^3 & -H^2 \\ E^2 & -H^3 & 0 & H^1 \\ E^3 & H^2 & -H^1 & 0 \end{pmatrix} \tag{1.15}$$

where we call F_{ab} the **Minkowski electromagnetic tensor field.**

Using F_{ab} and g, we define a linear operator F_a^b by

$$F_a^b = g^{bc}F_{ca}.$$

The eigenvalues of the matrix (F_a^b) are given by

$$F_a^b X^a = \alpha X^b$$

where α is a solution of the equation

$$det \mid F_a^b - \lambda \delta_a^b \mid = 0.$$

By direct calculations, we get the following minimum recurrent relation for the above equation:

$$\lambda^4 - (\mid E \mid^2 - \mid H \mid^2)\lambda^2 - (H \cdot E)^2 = 0. \tag{1.16}$$

Now recall from linear algebra the well-known **Cayley-Hamilton Theorem** which states " *Any square matrix satisfies the algebraic equation of its characteristic polynomial* ." Using this theorem, we conclude that the associated linear operator F_a^b of the electromagnetic field F_{ab} satisfies the following characteristic polynomial equation (see Adler et al. [1, page 508] for details):

$$F^4 - (\mid E \mid^2 - \mid H \mid^2)F^2 - (H \cdot E)^2 I_4 = O_4, \tag{1.17}$$

where $K = (\mid E \mid^2 - \mid H \mid^2)^2 + 4(H \cdot E)^2 \geq 0$. According to Ruse-Synge [40] classification, we say that F is **non-singular (non-null) or singular (null)** $\Longleftrightarrow$ $K \neq 0$ or $K = 0$. Following are possible cases of F with their respective eigenvalues and their characteristic polynomial equations.

General non-singular F. $\mid E \mid^2 - \mid H \mid^2 \neq 0$ and $H \cdot E \neq 0$

Eigenvalues. $(a, -a, b, -b)$, a real and b complex numbers.

Subcase 1. $\mid E \mid^2 - \mid H \mid^2 = 0$ and $H \cdot E \neq 0$

Eigenvalues. $(a, -a, ia, -ia)$, $a = (H \cdot E)^{\frac{1}{2}}$, $i = \sqrt{-1}$.

Subcase 2. $\mid E \mid^2 - \mid H \mid^2 \neq 0$ and $H \cdot E = 0$

Eigenvalues. $(a, a, 0, 0)$ or $(0, 0, ia, ia)$, $\Longleftrightarrow$ $\mid E \mid^2 > \mid H \mid^2$ or $\mid H \mid^2 > \mid E \mid^2$.

Singular F. $\mid E \mid^2 - \mid H \mid^2 = 0$ and $H \cdot E = 0$ with eigenvalues $(0, 0, 0, 0)$.

Finally, the characteristic polynomial equations for the two subcases of non-singular and the singular F are:

Subcase 1. $F^4 - (H \cdot E)^2 I_4 = O_4$, $rank(F) = 4$

Subcase 2. $F^3 - (\mid E \mid^2 - \mid H \mid^2)F = 0_4$, $rank(F) = 2$

Singular F. $F^3 = 0_4$, $rank(F) = 0$

In this book we concentrate on the study of non-singular subcase 2 and the singular F which are physically important cases of the electromagnetic theory, in particular reference to general relativity.

1.4 Algebraic Structures

A given set $(G, \circ)$, where $\circ$ denotes a **binary operation**, is called a **group** if it satisfies the following conditions:

1. $a, b \in G$ implies that $a \circ b \in G$, that is, G is closed.
2. $a, b, c \in G$ implies that $a \circ (b \circ c) = (a \circ b) \circ c$, that is, G is associative.
3. There is an identity element $e \in G$ such that $a \circ e = e \circ a = a \;\forall a \in G$.
4. For every $a \in G$ there is its inverse element $a^{-1} \in G$ such that $a \circ a^{-1} = a^{-1} \circ a = e$.

A group is said to be **Abelian (commutative)** if $a \circ b = b \circ a \;\forall a, b \in G$. A simple example is the set of all integers, with addition as binary operation, which is an Abelian group. For non-Abelian groups we mention a set S_n of all the permutations of n symbols ($n \geq 3$) with the binary operation as composition of mappings. S_n is a symmetric group of degree n.

A subgroup H of a group G is a subset of G which itself is a group with respect to the same binary operation as that of G. For example, the symmetric group of S_n has a subgroup S_{n-1} which does not change the position of any one (say the first one) of the objects. Also, the set of all even permutations is a subgroup of S_n. However, the set of all odd permutations do not form a subgroup since the identity of S_n (being even permutation) is not its element.

A mapping f from a group $(G, \circ)$ to a group $(G', \star)$ is called a **homomorphism** if $f(a \circ b) = f(a) \star f(b)$, where $\circ$ and $\star$ denote the respective binary operations of G and G', $\forall a, b \in G$. In particular, G and G' are said to **isomorphic** if f is one-to-one and onto.

A particular type of group, called **Lie group** (has many physical uses) whose formal definition is given in section 3.1 of chapter 3. Here we say that a Lie group is a continuous group whose elements have neighborhoods mapping 1-to-1 open sets of R^n for some n. A well-known example is the **translation group** of R^n for which $x \to x + a$, $a =$ constant. The group operation is simple addition.

A **Lie algebra** is a vector space $\mathcal{L}$ with a binary operation $\mathcal{L} \times \mathcal{L} \to \mathcal{L}$ such that $(\xi_1, \xi_2) \to [\xi_1, \xi_2]$ and satisfies the following two axioms:

$$[\xi_1, \xi_2] + [\xi_2, \xi_1] = 0 \qquad \text{(skew-symmetric)} \tag{1.18}$$

$$[\xi_1, [\xi_2, \xi_3]] + [\xi_2, [\xi_3, \xi_1]] + [\xi_3, [\xi_1, \xi_2]] = 0 \qquad \text{(Jacobi identity)} \tag{1.19}$$

for every $\xi_1, \xi_2, \xi_3 \in \mathcal{L}$. The product $[\,,]$ is called the **Lie bracket** or the **commutator** of any two elements of $\mathcal{L}$. The dimension of $\mathcal{L}$ is , by definition, the dimension of its underlying vector space. Any vector space can be made into a Lie algebra. A simple example is by setting $[\xi_1, \xi_2] = 0$ for all ξ_1, ξ_2 , we obtain an Abelian Lie algebra. A detailed study on Lie algebra structure, its relation with Lie group and several examples has been given in chapter 3.

Chapter 2

Semi-Riemannian Manifolds and Hypersurfaces

In this chapter we present a review of differentiable manifolds, tensor fields, covariant and exterior derivatives and linear connection. Using the Levi-Civita connection we brief on the geometry of semi-Riemannian manifolds and their non-degenerate hypersurfaces. In the last two sections we deal with the basic results on null curves and lightlike hypersurfaces of 4 dimensional Lorentz manifolds. On null curves we show the existence of an affine parameter and its Frenet frame, consisting of two real null and two spacelike vectors. We prove the existence of Levi-Civita induced connection on totally geodesic lightlike hypersurfaces. The main formulas and results are expressed by using both the invariant form and the index form.

2.1 Differentiable Manifolds

Given a set M, a **topology** on M is a family T of open subsets of M such that

1. the empty set and M are in T,

2. the intersection of any two members of T is in T,

3. the union of an arbitrary collection of members of T is in T.

In the above case, (M,T) is called a **topological space** whose elements are the open sets of T. This means that M can have many topologies depending on the choice of T. In the sequel, we assume that M is a topological space with a given T. M is a **Hausdorff topological space** if for every $p, q \in M$, $p \neq q$, there exist non intersecting neighborhoods $\mathcal{U}_1$ and $\mathcal{U}_2$ respectively. A neighborhood of p in M is an open set that contains p. A system of open sets of T is called a **basis** if its every open set is a union of the set of the system.

DEFINITION 1. *An n-dimensional manifold M is a topological Hausdorff space whose each point has a neighborhood homeomorphic to an open set in R^n.*

The Hausdorff condition is not absolutely necessary, although is assumed most often. The open neighborhood of each point admits a **coordinate system** which determines the position of points and the topology of that neighborhood. For a smooth transformation of two such coordinate systems and also taking care of the intersecting neighborhoods, we need the concept of **differentiable manifolds** as follows:

A homeomorphism $\varphi : M \to R^n$, mapping an open set $\mathcal{U}$ of M onto an open set $\varphi(\mathcal{U})$ of R^n, is called a chart (by mathematicians), or a local coordinate system (by physicists). In this way we assign to each point p in $\mathcal{U}$, the n local coordinates $x^1, \ldots, x^n$ and then $\mathcal{U}$ is called a local coordinate neighborhood. If p is in the intersection $\mathcal{U}_1 \cap \mathcal{U}_2$ of two local coordinate neighborhoods $\mathcal{U}_1$ and $\mathcal{U}_2$ with respect to charts φ_1 and φ_2, then we say that φ_1 and φ_2 are C^r- compatible if $\mathcal{U}_1 \cap \mathcal{U}_2$ is non-empty and $\varphi_2 o \varphi_1^{-1} : \varphi_1(\mathcal{U}_1 \cap \mathcal{U}_2) \to \varphi_2(\mathcal{U}_1 \cap \mathcal{U}_2)$ and its inverse are C^r.

DEFINITION 2. *An n-dimensional differentiable manifold M is a topological space together with a family $\mathcal{A} = \{\mathcal{U}_\alpha, \varphi_\alpha\}$ of local coordinate neighborhoods such that*

1. *The union of $\mathcal{U}_\alpha$'s is M,*
2. *Any two charts of M are C^r - compatible,*
3. *Any two members of $\mathcal{A}$ are C^r - compatible.*

The family $\mathcal{A}$ is called a maximal atlas on M. M is called a C^∞ manifold (also called **smooth manifold**) or **analytic manifold** according as r is ∞ or $r = \omega$. A C^r manifold with boundary is defined in the same manner as above, but by replacing R^n with the lower half $\frac{1}{2}R^n$ which is the set of all n-tuples $(x^1, \ldots, x^n)$ with the usual topology of R^n, $x^1 \le 0$. The boundary of M (denoted by ∂M) is the set of all points whose images, under a chart, lie on the boundary of $\frac{1}{2}R^n$, that is, on the set of all n-tuples $(0, x^2, \ldots, x^n)$.

A trivial example of a manifold is R^n. Other examples are 2-sphere, cylinder, tori, projective spaces and Minkowski spacetime. Details on these and much more may be seen in standard texts on differential geometry, for example Kobayashi and Nomizu [111, 1965].

An atlas $\mathcal{A} = \{\mathcal{U}_\alpha, \phi_\alpha\}$ of M is said to be locally finite if for each p in M, there is a local coordinate neighborhood $\mathcal{U}$ which intersects with only finitely many out of $\mathcal{U}_\alpha$'s. Another atlas $\mathcal{B} = \{\mathcal{V}_\beta, \psi_\beta\}$ of M is called a refinement of the atlas $\mathcal{A}$, if each $\mathcal{V}_\beta$ is contained in some $\mathcal{U}_\alpha$. A manifold M is said to be **paracompact** if for every atlas $\mathcal{A}$ there is a locally finite refined atlas $\mathcal{B}$ of $\mathcal{A}$. All manifolds considered in this book will be assumed paracompact.

2.2 Tensor Fields

We brief the essential properties of tensors on a smooth manifold M. For details the reader is referred to standard books on this subject such as Bishop-Goldberg

[15, 1986]. Let $F(p)$ denote the set of all real-valued smooth functions defined on some neighborhood of a point p of M. A **tangent vector** of M at p is a mapping $X_p : F(p) \to R$, such that

$$X_p(af + bg) = aX_pf + bX_pg\,, \qquad X_p(fg) = (X_pf)g(p) + f(p)X_pg \tag{2.1}$$

for any $f, g \in F(p)$ and any real numbers a, b. The set of all tangent vectors, at p, forms a vector space, called the **tangent space** under the following addition and scalar multiplication:

$$(X_p + Y_p)f = X_pf + Y_pf \quad , \quad (aX_p)f = aX_pf$$

where X_p and Y_p are tangent vectors at p, $f \in F(p)$ and a is a real number. The tangent space of M at p is denoted by $T_p(M)$. It is easy to see that $dim T_p(M) = dim M = n$, such that n vectors $\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\}$ form the basis (called a coordinate basis) of $T_p(M)$ with respect to a local coordinate system (x^i), $i = 1, \ldots, n$, on a neighborhood $\mathcal{U}$ of p. Let $f \in F(p)$ and q any other point in the intersection of $\mathcal{U}$ and domain of f such that x^i and $(x^i)(p)$ are the coordinates of q and p respectively. Then expanding f about p we get

$$f(q) = f(p) + \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(p)(x^i - x^i(p)),$$

where $f(p)$, $\frac{\partial f}{\partial x^i}(p)$ and $x^i(p)$ are real constants. Operating above equation by X_p we find

$$X_pf = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(p)X_p(x^i) = X_p(x^i)(\frac{\partial}{\partial x^i})_pf.$$

Therefore it follows that

$$X_p = X^i(p)(\frac{\partial}{\partial x^i})_p,$$

where $X^i(p) = X_p(x^i)$ are the components of X_p.

A smooth curve on M is a smooth map $C : (a, b) \to M$ where (a, b) is an open interval of R. Let $p = C(t_\circ)$ be a point on the curve C in M and $f \in F(p)$. Then, a tangent vector X_p, to C at p, of M is given by $X_pf = \frac{d}{dt}(f \circ C)|_{t_\circ}$. With respect a local coordinate system (x^i) around p, $X_pf = \frac{\partial}{\partial x^i}(f \circ \phi^{-1})\frac{dx^i}{dt}$, where ϕ is the chart that determines the local coordinates. Thus, a tangent vector X_p of M can be viewed as a tangent vector to a curve C with components $\frac{dx^i}{dt}$. Let

$$T(M) = U_{p \in M} T_p(M)$$

be the union of all tangent spaces over M. A **vector field** X on M is a mapping $X : M \to T(M)$ such that X_p is a vector in $T_p(M)$. Let $x^i(p)$ be a coordinate system of neighborhood $\mathcal{U}$ of p. Let the associated coordinate basis be denoted by ∂_i. Then,

$$X_p = X^i_p \partial_i \quad , \quad X^i_p = X_p(x^i).$$

At this point we drop the suffix p for discussing the properties of vector fields which will be denoted by X, Y, Z etc. Define the Lie bracket $[X, Y]$, of any two vector fields X and Y, by

$$[X, Y]f = X(Yf) - Y(Xf), \qquad \forall f \in F(M). \tag{2.2}$$

Using the local expressions $X = X^i \partial_i$ and $Y = Y^j \partial_j$, it is easy to show that

$$[X, Y] = (X^i \partial_i Y^j - Y^i \partial_i X^j) \partial_j$$

is also a vector field on M for any X, Y on M. The Lie bracket has the following properties:

1. $[aX + bY, Z] = a[X, Z] + b[Y, Z]$: $a, b \in R$ (bilinear)
2. $[X, Y] = -[Y, X]$ (skew-symmetry)
3. $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$: (Jacobi identity)
4. $[f\,X, g\,Y] = f\,g\,[X, Y] + f(X\,g)Y - g(Y\,f)X, \quad \forall f, g \in F(M)$.

It follows from above properties that the set of all tangent vector fields on M, denoted by $\mathcal{X}(M)$, has a **Lie-algebra** structure with respect to the Lie-bracket operation.

Every $T_p(M)$ has a **dual vector space** $T^\star_p(M)$(also called **cotangent space**) of the same dimension and its elements are linear maps

$$\omega : T_p(M) \to R. \tag{2.3}$$

Let $f \in F(M)$ and $X \in \mathcal{X}(M)$. It follows from (2.1) that $X(f)$ is a function called a derivative on $F(M)$ and conversely, every derivative on $F(M)$ comes from some vector field on M. In other words, $X(f) = df(X)$ where d is the symbol for ordinary derivative. Relating this with (2.3) we conclude that

$$\omega = df : T_p(M) \to R \quad , \quad \omega = \omega_i dx^i$$

where the coordinate differentials dx^i form a **dual basis** of $T^\star_p(M)$ having the property

$$(dx^i)(\frac{\partial}{\partial x^j}) = \delta^i_j$$

with respect to a coordinate system (x^i). The elements of $T^\star_p(M)$ are called **differential 1-forms or covectors**. For any $p \in M$ consider the vector space

$T^r_s(M)_p$ of all $(r+s)$-linear mappings

$$T_p : \underbrace{T^\star_p(M) \times \ldots \times T^\star_p(M)}_{r-times} \times \underbrace{T_p(M) \times \ldots \times T_p(M)}_{s-times} \to R$$

An element of $T^r_s(M)_p$ is a tensor of type (r,s) at p. Let

$$T^r_s(M) = \cup_{p \in M} T^r_s(M)_p$$

be the union of all tensor spaces of type (r,s) with respect to all smooth curves through a point p. A **tensor field** on M is a mapping

$$T : M \to T^r_s(M)$$

such that T_p is a tensor at that point p. Then, locally we have

$$T = T^{i_1 \ldots i_r}_{j_1 \ldots j_s}(p) \partial_{i_1} \otimes \ldots \otimes \partial_{i_r} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_s},$$

where $T^{i_1 \ldots i_r}_{j_1 \ldots j_s}(p)$ are components of T for the following local basis of $T^r_s(M)_p$

$$\{\partial_{i_1} \otimes \ldots \otimes \partial_{i_r} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_s}\}.$$

The tensor fields of types $(r,0)$ and $(0,s)$, with $r, s \geq 1$, are called **contravariant** and **covariant** with orders r and s respectively. For example, a vector field and a differential 1-form are contravariant and covariant tensor fields of type $(1,0)$ and (0,1) respectively.

A contravariant or a covariant tensor field T of order ≥ 2 is **symmetric** if T remains invariant with respect to transposing any two of the arguments of its components. In case each such transposition produces a change in sign of the original tensor, then T is called **skew-symmetric**. A skew-symmetric tensor of type $(0,s)$ is called a **differential** s**-form**, locally expressed by (see Vaisman [194, page 157]):

$$\omega = \frac{1}{s!} \omega_{i_1 \ldots i_s} dx^{i_1} \wedge \ldots \wedge dx^{i_s},$$

where $\omega_{i_1 \ldots i_s}$ are skew-symmetric with respect to any pair of arguments and $\wedge$ denotes the wedge product operator. A differential form of order 2 is also known as a **bivector**. The wedge product has the following properties:

1. $(\theta \wedge \omega) \wedge \eta = \theta \wedge (\omega \wedge \eta)$
2. If θ is a p-form and ω a q-form, then $\theta \wedge \omega = (-1)^{pq} \omega \wedge \theta$
3. $(\theta_1 + \theta_2) \wedge \omega = \theta_1 \wedge \omega + \theta_2 \wedge \omega$ for p-forms θ_1 , θ_2 and ω any form.

Thus the wedge product is a generalization of cross product of vectors in R^3. The space of all p-forms $(0 \leq p \leq n)$ on M, with the wedge product $\wedge$, becomes an algebra called **exterior or Grassmannian algebra**.

2.3 Covariant and Exterior Derivatives

A **linear affine connection** on M is a function $\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ such that

$$\begin{aligned} \nabla_{fX+gY} Z &= f(\nabla_X Z) + g(\nabla_Y Z), \qquad \nabla_X f = Xf, \\ \nabla_X (fY + gZ) &= f\nabla_X Y + g\nabla_X Z + (Xf)Y + (Xg)Z, \end{aligned}$$

for arbitrary vector fields X, Y, Z and smooth functions f, g on M. ∇_X is called **covariant derivative operator** and $\nabla_X Y$ is called **covariant derivative** of Y with respect to X. Define a tensor field ∇Y, of type $(1,1)$, and given by $(\nabla Y)(X) = \nabla_X Y$, for any Y. Also, $\nabla_X f = Xf$ is the covariant derivative of f long X. The covariant derivative of a 1-form ω is given by

$$(\nabla_X \omega)(Y) = X(\omega(Y)) - \omega(\nabla_X Y). \tag{2.4}$$

In order to find local expressions for the geometric quantities induced by a linear connection, we consider the natural basis $\{\partial_i\}$, $i \in (1, \ldots, n)$, on a coordinate neighborhood $\mathcal{U}$ and set

$$\nabla_{\partial_j} \partial_i = \Gamma^k_{ji} \partial_k, \tag{2.5}$$

where Γ^k_{ji} are n^3 local components of ∇ on M. In the sequel we use ; a symbol for the covariant derivative.
If $X = X^i \partial_i \, , Y = Y^j \partial_j$ and $\omega = \omega_i dx^i$ then the following are local expressions:

$$\nabla_X f = X^i \partial_i f \qquad , \qquad \nabla_X Y = Y^i_{;k} X^k \partial_i, \tag{2.6}$$

$$Y^i_{;k} = \partial_k Y^i + \Gamma^i_{kj} Y^j \tag{2.7}$$

$$\omega_{i;j} = \partial_j \omega_i - \Gamma^k_{ji} \omega_k \tag{2.8}$$

Finally, the covariant derivative of a tensor T of type (r,s) along a vector field X is a tensor field $\nabla_X T$, of type (r,s), given by

$$\begin{aligned} (\nabla_X T)(\omega^1, \ldots, \omega^r, Y_1, \ldots, Y_s) &= X(T(\omega^1, \ldots, \omega^r, Y_1, \ldots, Y_r)) \\ &- \sum_{\alpha=1}^{r} T(\omega^1, \ldots, \nabla_X \omega^\alpha, \ldots, \omega^r, Y_1, \ldots, Y_s) \\ &- \sum_{t=1}^{s} T(\omega^1, \ldots, Y_1, \ldots, \nabla_X Y_t, \ldots, Y_s) \end{aligned}$$

for any vector field X, r covariant vectors $\omega^1, \ldots, \omega^r$ and s contravariant vectors $Y_1, \ldots, Y_s$. Note that ∇T of T is a tensor of type $(r, s+1)$. In local coordinates, we have

$$\begin{aligned} T^{i_1 \ldots i_r}_{j_1 \ldots j_s;k} &= \partial_k T^{i_1 \ldots i_r}_{j_1 \ldots j_s} + \sum_{h=1}^{r} T^{i_1 \ldots i_{h-1} q i_{h+1} \ldots i_r}_{j_1 \ldots j_s} \Gamma^{i_h}_{qk} \\ &- \sum_{t=1}^{s} T^{i_1 \ldots i_r}_{j_1 \ldots j_{t-1} q j_{t+1} \ldots j_s} \Gamma^q_{j_t k} \end{aligned}$$

In particular, for a tensor of type $(1,2)$, we have

$$T^h_{ij;k} = \partial_k T^h_{ij} + \Gamma^h_{kt} T^t_{ij} - \Gamma^t_{kj} T^h_{ti} - \Gamma^t_{ki} T^h_{jt}, \tag{2.9}$$

where T^h_{ij} are the components of T. The **curvature tensor** of M with linear connection ∇ is a $(1,3)$ tensor field R defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{2.10}$$

where X, Y, Z are arbitrary smooth vector fields on M. The **torsion tensor,** denoted by T, of ∇ is a $(1,2)$ tensor defined by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y].$$

Obviously, R is skew-symmetric in the first two slots. In case T (resp. R) vanishes on M we say that ∇ is **torsion-free or symmetric** (resp. **flat**) **linear connection** on M. In this book we always assume that ∇ is a symmetric linear connection. Then it satisfies the following two **Bianchi's identities**:

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0, \tag{2.11}$$

$$(\nabla_X R)(Y,Z,W) + (\nabla_Y R)(Z,X,W) + (\nabla_Z R)(X,Y,W) = 0. \tag{2.12}$$

In local coordinates, a symmetric connection satisfies

$$\Gamma^i_{jk} = \Gamma^i_{kj} \tag{2.13}$$

and the Bianchi's identities (2.11) and (2.12) are expressed by

$$R^i_{jkl} + R^i_{klj} + R^i_{ljk} = 0, \tag{2.14}$$

$$R^i_{jkl;m} + R^i_{jlm;k} + R^i_{jmk;l} = 0. \tag{2.15}$$

A vector field Y on M is said to be parallel with respect to a linear connection ∇ if for any vector field X on M it is covariant constant, i.e., $\nabla_X Y = 0$. It follows from (2.7) that Y is parallel on M if and only if its local components Y^i, with respect to a natural basis $\{\partial_i\}$, satisfy the following differential equation:

$$\partial_j Y^i + \Gamma^i_{kj} Y^k = 0. \tag{2.16}$$

In general a tensor field T on M is parallel with respect to ∇ if it is covariant constant with respect to any vector field X on M. Let C be a smooth curve on M given by the equations

$$x^i = x^i(t), \qquad t \in I \subset R, \qquad i = 1, \dots, n.$$

Then a tangent vector field V to C is given by

$$V = \frac{dx^i}{dt} \partial_i.$$

Thus, a vector field Y is said to be parallel along C, with respect to ∇, if $\nabla_V Y = 0$. Using this and (2.7) we conclude that Y is parallel along C if and only if

$$\frac{dY^k}{dt} + \Gamma^k_{ij} Y^i \frac{dx^j}{dt} = 0. \tag{2.17}$$

The curve C is called a **geodesic** if V is parallel along C, i.e., if $\nabla_V V = fV$ for some smooth function f along C. It is possible to find a new parameter s along C such that f is zero along C and then the geodesic equation $\nabla_V V = 0$ can be expressed, in local coordinate system (x^i), as

$$\frac{d^2x^k}{ds^2} + \Gamma^k_{ji} \frac{dx^j}{ds} \frac{dx^i}{ds} = 0. \tag{2.18}$$

The parameter s is called an **affine parameter**. Two affine parameters s_1 and s_2 are related by $s_2 = as_1 + b$, where a and b are constants. If the connection ∇ is smooth (or C^r), then the theory of differential equations certifies that , given a point p of M and a tangent vector X_p, there is a **maximal geodesic** $C(s)$ such that $C(0) = p$ and $\frac{dx^i}{ds}|_{s=0} = X^i_p$. If C is defined for all values of s, then it is said to be **complete**, otherwise incomplete.

The **exterior derivation** is a differential operator, denoted by d, which assigns to each p-form ω, a $(p+1)$-form $d\omega$ defined by

$$(d\omega)(X_1, \ldots, X_{p+1}) = \frac{1}{p+1}\{\sum_{i=1}^{p+1}(-1)^{i+1} X_i \omega(X_1, \ldots, \widehat{X}_i, \ldots, X_{p+1})$$

$$+ \sum_{1 \le i < j \le 1+p} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_{p+1})\}$$

where the hood means the term in that particular slot is omitted. In particular, for a 1-form ω and a 2-form Ω ,

$$\begin{aligned}
(d\omega)(X, Y) &= \frac{1}{2}\{X(\omega(Y)) - Y(\omega(X) - \omega([X, Y])\} \\
(d\Omega)(X, Y, Z) &= \frac{1}{3}\{X(\Omega(Y, Z)) - Y(\Omega(X, Z)) + Z(\Omega(X, Y)) \\
&- \Omega([X, Y], Z) + \Omega([X, Z], Y) - \Omega([Y, Z], X)\}.
\end{aligned}$$

For example, if $\omega = \omega_i dx^i$ and $\Omega = \frac{1}{2}\Omega_{ij} dx^i \wedge dx^j$, then

$$\begin{aligned}
d\omega &= \frac{1}{2!}(d\omega)_{ij} dx^i \wedge dx^j, \\
d\Omega &= \frac{1}{3!}(d\Omega)_{ijk} dx^i \wedge dx^j \wedge dx^k, \\
(d\omega)_{ij} &= \partial_i \omega_j - \partial_j \omega_i, \\
(d\Omega)_{ijk} &= \partial_i \Omega_{jk} + \partial_j \Omega_{ki} + \partial_k \Omega ij.
\end{aligned}$$

The exterior derivation d has the following properties:

(1) For a smooth function f on M, df is a 1-form (also called the gradient-form of f) such that $(df)X = Xf$, for any vector field X.

(2) For a p-form ω and a q-form θ

$$d(\omega \wedge \theta) = (d\omega) \wedge \theta + (-1)^p \omega \wedge d\theta.$$

(3) $d(d\omega) = 0$ for any p-form ω, (Poincare Lemma).

(4) d is linear with respect to the addition of any two p-forms.

2.4 Semi-Riemannian Geometry

Let M be a real n-dimensional smooth manifold and g a symmetric tensor field of type $(0,2)$ on M. Thus g assigns, to each point $p \in M$, a symmetric bilinear form g_p on the tangent space $T_p(M)$. Suppose g_p is non-degenerate and of constant index for all $p \in M$. This condition means (see section 1 of chapter 1) that each $T_p(M)$ is an n-dimensional semi-Euclidean space. Let $X_p = X^i \partial_i$ and $g_{ij} = g(\partial_i, \partial_j)$, $i, j \in (1, \ldots, n)$ and $\{\partial_i\}$ the natural basis of $T_p(M)$. Then X_p is called

Spacelike	if $g_{ij}X^iX^j > 0$ or $X_p = 0$,	
Timelike	if $g_{ij}X^iX^j < 0$,	
Lightlike	if $g_{ij}X^iX^j = 0$	and $X_p \neq 0$.

Set of all null vectors of $T_p(M)$ is called the **null cone** at p, defined by

$$\Lambda_p = \{X_p \in (T_p(M) - \{0\}), \qquad g_{ij}X^iX^j = 0\}.$$

Based on above, g is called a **semi-Riemannian metric** (**metric tensor field**) and (M, g) is called a **semi-Riemannian manifold** (see O'Neill [157]). For example, M is **Riemannian** or **Lorentzian manifold** according as g is of index 0 or 1 respectively. In case $0 < index < n$, then we say that M is a **proper semi-Riemannian manifold**.

The metric g splits each tangent space at each point $p \in M$ into three categories; namely (i) spacelike (ii) timelike (iii) null vectors. The category into which a tangent vector falls is called its **causal character**. A curve C in M also belongs to one of the three categories. For a vector field X on M we say that X is spacelike (resp. timelike or null) according as $g(X,X) > (resp. < or\, 0)$.

It is well-known that Riemannian metrics always exist on a paracompact manifold. Unfortunately, in general, the existence of a non-degenerate metric on M can not be guaranteed. However, it is possible to construct a Lorentz metric, on a smooth M, which has a Riemannian metric. Indeed, let $(M, \bar{g})$ be a Riemannian manifold with a unit vector field V, i.e., $\bar{g}(V,V) = 1$, whose associated 1-form v is defined by $v(X) = \bar{g}(X, V)$. Define another metric g on M by

$$g(X,Y) = \bar{g}(X,Y) - 2v(X)v(Y), \qquad \forall X, Y \in \mathcal{X}(M).$$

It is easy to check that g is a Lorentz metric such that V is a unit timelike vector field with respect to g, i.e., $g(V,V) = -1$.

In general, a Lorentz manifold (M, g) may not have a globally defined timelike vector field. If (M, g) admits a global timelike vector field, then, it is called a **time orientable Lorentz manifold**, physically known as a **spacetime manifold**. The foundations of general relativity are based on a 4-dimensional spacetime manifold.

A linear connection ∇ on a semi-Riemannian manifold (M, g) is called a **metric connection** if g is parallel with respect to ∇, i.e.,

$$(\nabla_X g)(Y, Z) = X(g(Y, Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) = 0. \tag{2.19}$$

In terms of local coordinates system, we have

$$g_{ij;k} = \partial_k g_{ij} - g_{ih}\Gamma^h_{jk} - g_{jh}\Gamma^h_{ik} = 0 \tag{2.20}$$

where

$$\Gamma^h_{ij} = \frac{1}{2} g^{hk}\{\partial_j g_{ki} + \partial_i g_{kj} - \partial_k g_{ij}\} \tag{2.21}$$

$$g^{ik} g_{kj} = \delta^i_j \qquad , \qquad \Gamma^h_{ij} = \Gamma^h_{ji}.$$

Furthermore, if we set

$$\Gamma_{k,ij} = g_{kh}\Gamma^h_{ij}$$

then the equation (2.20) becomes

$$g_{ij;k} = \partial_k g_{ij} - \Gamma_{i,jk} - \Gamma_{j,ik} = 0.$$

The connection coefficients $\Gamma_{k,ij}$ and Γ^h_{ij} are called the **Christoffel symbols of first and second type** respectively. A fundamental result in semi-Riemannian geometry states (see O'Neill [157]) that there exists a torsion-free metric connection ∇ ($\nabla g = 0$), called **Levi-Civita connection**, which satisfies the following identity:

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X(g(Y, Z)) + Y(g(X, Z)) - Z(g(X, Y)) \\ &+ g([X, Y], Z) + g([Z, X], Y) - g([Y, Z], X) \end{aligned}$$

for any vector fields X, Y and Z on M. Throughout this book, we shall assume that ∇ is the Levi-Civita connection, unless otherwise stated and the semi-Riemannian curvature tensor R satisfies the two Bianchi's identities (2.11) and (2.12). The **Riemannian curvature tensor** of type $(0, 4)$ is defined by

$$\begin{aligned} R(X, Y, Z, U) &= g(R(X, Y)Z, U), \qquad \forall X, Y, Z, U \text{ on } M, \ i.e., \\ R_{ijhk} &= R(\partial_h, \partial_k, \partial_j, \partial_i) = g_{it} R^t{}_{jhk}. \end{aligned} \tag{2.22}$$

Then by direct calculations we get

$$\begin{aligned} R(X, Y, Z, U) + R(Y, X, Z, U) &= 0, \\ R(X, Y, Z, U) + R(X, Y, U, Z) &= 0, \\ R(X, Y, Z, U) - R(Z, U, X, Y) &= 0. \end{aligned} \tag{2.23}$$

The local expressions of the curvature tensor and its identities (2.23) are:

$$R^t jhk = \partial_h \Gamma^t_{jk} - \partial_k \Gamma^t jh + \Gamma^m jk \Gamma^t mh - \Gamma^m_{jh} \Gamma^t_{mk}$$

$$R_{ijkh} + R_{jikh} = 0,$$

$$R_{ijkh} + R_{ijhk} = 0,$$

$$R_{ijhk} - R_{hkij} = 0.$$

An orthonormal basis of $T_p(M)$ is called an **orthonormal frame** on M at p. Then, a set of n vector fields $\{E_1, \ldots, E_n\}$ defined on a coordinate neighborhood $\mathcal{U} \subset M$ and preserving their causal character along $\mathcal{U}$ such that $\{E_1, \ldots, E_n\}_p$ is a basis for each $p \in \mathcal{U}$, is called a **local orthonormal frame field** on M. Based on above definition we have

$$g(E_i, E_j) = \epsilon_i \, \delta_{ij} \text{ (no summation in } i) \qquad X = \sum_{i=1}^{n} \epsilon_i \, g(X, E_i) \, E_i,$$

where $\{\epsilon_i\}$ is the signature of $\{E_i\}$. Thus we obtain

$$g(X, Y) = \sum_{i=1}^{n} \epsilon_i \, g(X, E_i) \, g(Y, E_i). \tag{2.24}$$

The **Ricci tensor** *Ric* and the **Ricci operator** Q of types $(0, 2)$ and $(1, 1)$ are defined by

$$Ric(X, Y) = \sum_{i=1}^{n} \epsilon_i \, g\,(R(E_i, X)Y, E_i), \qquad g(QX, Y) = Ric(X, Y), \ i.e., \tag{2.25}$$

$$R_{ij} = R^m{}_{imj}\,, \qquad S^i_j = R_{kj} g^{ki}.$$

If Ricci tensor vanishes on M we say that M is **Ricci flat**. If $n > 2$ and Ricci tensor is given by

$$Ric = kg, \qquad k \text{ is a constant,} \tag{2.26}$$

then M is called an **Einstein manifold**. For $n = 2$, any M is Einstein but k in (2.26) is not necessarily constant. The **scalar curvature**, denoted by r , is defined by

$$r = \sum_{i=1}^{n} \epsilon_i \, Ric(E_i, E_i) = g^{ij} R_{ij} \tag{2.27}$$

By using (2.27) in (2.26) we deduce that M is an Einstein manifold if and only if r is constant and

$$Ric = \frac{r}{n} g. \tag{2.28}$$

The **gradient** of a smooth function f is defined as a vector field, denoted by $grad\, f$, and given by

$$\begin{aligned} g(grad\, f, X) &= X(f), \qquad i.e., \\ grad\, f &= \sum_{i,j=1}^{n} g^{ij} \partial_i f \partial_j. \end{aligned} \tag{2.29}$$

The **divergence** and the **curl** of a vector field X is a smooth function and a 2-form, respectively, denoted by $div\, X$ and $curl\, X$, and given by

$$\begin{aligned} div\, X &= X^m;m = \frac{\partial X^m}{\partial x^m} + \Gamma^m_{km} X^k \\ curl\, X &= \frac{1}{2}(\partial_j X_i - \partial_i X_j) dx^i \wedge dx^j. \end{aligned} \tag{2.30}$$

The **Laplacian** of f, denoted by $\triangle f$, is given by

$$\begin{aligned} \triangle f &= div(grad\, f) \\ &= \sum_{i,j=1}^{n} g^{ij} \{ \frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial f}{\partial x^k} \}. \end{aligned} \tag{2.31}$$

The **Weyl conformal curvature tensor** C of type (1,3) is defined by

$$\begin{aligned} C(X,Y)Z &= R(X,Y)Z + \frac{1}{n-2}\{Ric(X,Z)Y - Ric(Y,Z)X + g(X,Z)QY \\ &- g(Y,Z)QX\} - r\{(n-1)(n-2)\}^{-1}\{g(X,Z)Y - g(Y,Z)X\}, \; i.e., \\ C^h{}_{kij} &= R^h{}_{kij} + \frac{1}{n-2}\{\delta^h_j R_{ki} - \delta^h_i R_{kj} + g_{ki} R^h_j - g_{kj} R^h_i\} \\ &+ r\{(n-1)(n-2)\}^{-1}\{\delta^h_i g_{kj} - \delta^h_i g_{ki}\}. \end{aligned} \tag{2.32}$$

The tensor C vanishes for $n = 3$ (see Weyl [199]). Let $\bar{g} = \Omega^2 g$ be a conformal transformation of g where Ω is smooth positive real function on M. It is well-known that C is invariant under any such conformal transformation of the metric. If (M, g) is conformally related with a semi-Euclidean flat metric $\bar{g}$ then we say that (M, g) is **conformally flat manifold** with conformally flat metric g. It follows from a theorem of Weyl [199] that (M, g) is conformally flat $\Longleftrightarrow$ $C \equiv 0$ for $n > 3$. A 3-dimensional (M, g) is conformally flat if and only if

$$(R_{ij} - \frac{r}{4} g_{ij})_{;k} = (R_{ik} - \frac{r}{4} g_{ik})_{;j}.$$

Suppose P is a non-degenerate plane of $T_p(M)$. Then, according to section 1.1 of chapter 1, the associated matrix G_p of g_p, with respect to an arbitrary basis $B = \{u, v\}$, is of *rank* 2 and given by

$$G_p = \begin{pmatrix} g_{uu} & g_{uv} \\ g_{uv} & g_{vv} \end{pmatrix}, \qquad det(G_p) \neq 0. \tag{2.33}$$

Define a real number

$$K(P) = K_p(u,v) = \frac{R(u,v,v,u)}{det(G_p)}, \tag{2.34}$$

where $R(u,v,v,u)$ is the 4- linear mapping on $T_p(M)$ by the curvature tensor as given in (2.22). The smooth function K which assigns to each non-degenerate tangent plane P the real number $K(P)$ is called the **sectional curvature** of M, which is independent of the basis $B = \{u, v\}$. M is a **semi-Riemannian manifold of constant curvature** c, denoted by $M(c)$, if K is a constant c for which its curvature tensor field R is given by (see O'Neill [157, page 80])

$$\begin{aligned} R(X,Y)Z &= c\{g(Y,Z)X - g(X,Y)Z\}, \qquad i.e., \\ R^h{}_{kij} &= c\{\delta_i^h g_{jk} - \delta_j^h g_{ki}\}. \end{aligned} \tag{2.35}$$

2.5 Semi-Riemannian Hypersurfaces

Suppose Σ and M are smooth n and m $(n < m)$ dimensional manifolds respectively and $i : \Sigma \to M$ a smooth mapping such that each point $p \in \Sigma$ has an open neighborhood $\mathcal{U}$ for which i restricted to $\mathcal{U}$ is one to one and $i^{-1} : i(\mathcal{U}) \to \Sigma$ are smooth. Then we say that $i(\Sigma)$ is an **immersed submanifold** of M. If this condition globally holds then $i(\Sigma)$ is called an **embedded submanifold** of M. The embedded submanifold has a natural manifold structure inherited from the manifold structure on Σ via the embedding mapping i. At each point $i(p)$ of $i(\Sigma)$, the tangent space is naturally identified with an n-dimensional subspace of the tangent space of M at $i(p)$. If $n = m - 1$, then Σ is called a hypersurface of M. An example of a hypersurface is the boundary ∂M of a smooth manifold with boundary (see section 2.1). If M is a **compact orientable manifold** with boundary ∂M, then ∂M is also orientable. Recall that an m-dimensional manifold M is orientable if M has a continuous nowhere vanishing m-form.

In this book we need the following results on the differential geometry of non-degenerate (non-null) hypersurfaces. Let g be a semi-Riemannian metric, of index r, on M. Then the embedding i induces a metric γ on Σ such that

$$\gamma(X,Y)|_p = g(i_\star X, i_\star Y)|_{i(p)}$$

for every $X, Y \in T_p(\Sigma)$. Here $i_\star$ is the differential map of i defined by $i_\star : T_p \to T_{i(p)}$ and $(i_\star X)(f) = X(f \circ i)$ for an arbitrary smooth function f in a neighborhood of $i(p)$ of $i(\Sigma)$. If g is positive definite, so is γ. However, if g is indefinite then γ is either non-degenerate or degenerate. In this section we assume that γ is non-degenerate and, therefore $(i(\Sigma), \gamma)$ is a **semi-Riemannian hypersurface** of (M, g) for which $m = n + 1$. The case of **degenerate (null, lightlike) hypersurfaces** has been discussed in section 2.7. Henceforth, we write p and Σ instead of $i(p)$ and $i(\Sigma)$ respectively. Since for any $p \in \Sigma$, $T_p(\Sigma)$ is a hyperplane of the semi-Euclidean space $T_p(M)$, as per equation (1.13) of chapter 1, we define

$$T_p(\Sigma)^\perp = \{V \in T_p(M) : \; g(V,W) = 0, \; \forall W \in T_p(\Sigma)\},$$

the normal space of Σ in M. Since Σ is a hypersurface of M, $dim(T_p(\Sigma)^{\perp}) = 1$. Following is the direct sum decomposition:

$$T_p(M) = T_p(\Sigma) \oplus T_p(\Sigma)^{\perp}, \qquad T_p(\Sigma) \cap T_p(\Sigma)^{\perp} = \{0\}, \tag{2.36}$$

at each point $p \in \Sigma$. Here, both the tangent and the normal subspaces are non-degenerate. At this point we drop the suffix p for discussing the general properties of the induced geometric objects on Σ. Let $\bar{\nabla}$ and ∇ be the Levi-Civita connections on M and Σ respectively. Then, using (2.36), we obtain

$$\begin{aligned} \bar{\nabla}_X Y &= \nabla_X Y + B(X,Y)N, && \textbf{Gauss formula} \\ \bar{\nabla}_X N &= -\epsilon\, A X, && \textbf{Weingarten formula} \end{aligned} \tag{2.37}$$

for any tangent vectors X and Y of Σ. Here $g(N,N) = \epsilon = \pm 1$ such that N belongs to $\Sigma^{\perp}$ and $\nabla_X Y$, $A X$ belong to the tangent space. Also B is the **second fundamental form** and A denotes the **shape operator** defined by

$$g(B(X,Y), N) = \gamma(A X, Y).$$

We say that Σ is **totally geodesic hypersurface** in M if

$$B = 0 \Leftrightarrow A = 0. \tag{2.38}$$

A point p of Σ is said to be **umbilical** if

$$B(X,Y)_p = k\,\gamma(X,Y)_p, \qquad \forall X, Y \in T_p(\Sigma), \tag{2.39}$$

where $k \in R$ and depends on p. Above definition is independent of any coordinate neighborhood around p. Σ is **totally umbilical** in M if every point of Σ is umbilical, i.e., if $B = \rho\,\gamma$ where ρ is a smooth function.

With respect to an orthonormal basis $\{E_1, \ldots, E_n\}$ of $T_p(\Sigma)$, the **mean curvature vector** μ of Σ is defined by

$$\mu = \frac{Trace(B)}{n} = \frac{1}{n}\sum_{i=1}^{n} \epsilon_i B(E_i, E_i), \;\; \gamma(E_i, E_i) = \epsilon_i.$$

It is easy to show that μ is independent of any coordinate neighborhood around p. The characteristic equations of **Gauss** and **Codazzi** are respectively

$$\begin{aligned} < \bar{R}(X,Y)Z, W > &= < R(X,Y)Z, W > + \epsilon < B(X,Z), B(Y,W) > \\ &- \epsilon < B(Y,Z), B(X,W) > \\ \epsilon < \bar{R}(X,Y)Z, N > &= (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) \end{aligned} \tag{2.40}$$

where $\bar{R}$ and R denote the curvature tensors of M and Σ respectively, $X, Y, Z, W \in \mathcal{X}(\Sigma)$ and $<,>$ is the inner product with respect to the metrics g and γ as and where applicable.

2.6 Null Curves of Lorentz Manifolds

Let (M, g) be a real 4-dimensional Lorentz manifold and C a smooth curve of M. Then, with respect to a local coordinate neighborhood $\mathcal{U}$ on C and a corresponding local parameter t, C is locally given by

$$x^i = x^i(t),\ i \in \{1, 2, 3, 4\},\ \forall t \in I, \tag{2.41}$$

where I is an open interval of R. The non-zero tangent vector field on $\mathcal{U}$ is

$$\frac{d}{dt} \equiv (\frac{dx^1}{dt}, \frac{dx^2}{dt}, \frac{dx^3}{dt}, \frac{dx^4}{dt}).$$

Assume that C is a **null (lightlike) curve**. This means that any tangent vector to C is a null vector. Thus C is a null curve iff locally at each point of $\mathcal{U}$ we have

$$g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} = 0, \qquad g_{ij} = g(\partial_i, \partial_j), \qquad i, j \in \{1, 2, 3, 4\}.$$

Denote $T(C) = \cup_{p \in C} T_p(C)$ and $T(C)^\perp = \cup_{p \in C} T_p(C)^\perp$ where

$$T(C)^\perp = \{X \in T(M) :\ g(X, \frac{d}{dt}) = 0\},$$

and each $T_p(C)$ is also null subspace of $T_p(M)$ with

$$T(C) \cap T(C)^\perp = T(C), \qquad dim(T(C)^\perp)_p = 3.$$

Using the proposition 2 of chapter 1 (page 5), let S be the 2-dimensional complementary and spacelike **screen distribution** to $T(C)$ in $T(C)^\perp$, for every $p \in C$. Since S is non-degenerate, we have the following decomposition:

$$T(M) = S \oplus S^\perp, \qquad S \cap S^\perp = \{0\}, \tag{2.42}$$

where $T(M) = \cup_{p \in M} T_p(M)$ and $S^\perp$ is a 2-dimensional complementary orthogonal and non-degenerate distribution in M.

THEOREM 2.1. *Let C be a null curve of a 4-dimensional Lorentz manifold (M, g), with a screen distribution S to $T(C)$ in $T(C)^\perp$. Then, with respect to each coordinate neighborhood $\mathcal{U}$ of C, there exists a unique one dimensional null distribution $E = \cup_{p \in C} E_p$, where $E_p = span\{N_p\}$ such that N is unique and satisfies*

$$g(\frac{d}{dt}, N) = 1, \qquad g(N, N) = g(N, X) = 0, \qquad \forall X \in \mathcal{X}(S|_{\mathcal{U}}). \tag{2.43}$$

PROOF. As $T(C)$ is in $S^\perp$, consider a complementary one dimensional distribution E to $T(C)$ in $S^\perp$. Since $rank(S^\perp) = 2$, there exists a non-zero vector field W in E such that $g(\frac{d}{dt}, W) \neq 0$ on $\mathcal{U}$, otherwise (2.42) implies that g is degenerate on $T(M)$ at least at one point of $\mathcal{U}$. Define $N \in \mathcal{X}(M|_{\mathcal{U}})$ by

$$N = \frac{1}{g(\frac{d}{dt}, W)} \{W - \frac{g(W, W)}{2g(\frac{d}{dt}, W)} \frac{d}{dt}\}. \tag{2.44}$$

A direct calculation implies that N is a null vector field and satisfies (2.43). N is unique since it does not depend on E and W. Consider another tangent vector field $\frac{d}{dt^\star}$ of C with respect to another coordinate neighborhood $\mathcal{U}^\star$, such that $\mathcal{U} \cap \mathcal{U}^\star \neq \phi$. Then, on $\mathcal{U} \cap \mathcal{U}^\star$, $\frac{d}{dt^\star} = \frac{dt}{dt^\star} \cdot \frac{d}{dt}$. Let $N^\star$ be the unique vector field given by (2.44) on $\mathcal{U}^\star$ with respect to $\frac{d}{dt^\star}$. Then, $N^\star = \frac{dt^\star}{dt} N$. Thus there exists a unique 1-dimensional null distribution E over C with N given by (2.44).

Using above theorem and (2.42) we have the following decomposition:

$$T(M) = (T(C) + E) \oplus S, \tag{2.45}$$

where $+$ denotes non-orthogonal complementary sum and

$$T(C) + E = span\{\frac{d}{dt}, N\}.$$

Let $\bar{\nabla}$ denote the Levi-Civita connection on M. Using (2.19), (2.43) and (2.45) and S spacelike, we obtain the following equations (here we set $\frac{d}{dt} \equiv \lambda$):

$$\begin{aligned}
\bar{\nabla}_\lambda \lambda &= h\lambda + \sigma U, \\
\bar{\nabla}_\lambda N &= -hN + \tau U + \beta V, \\
\bar{\nabla}_\lambda U &= -\tau\lambda - \sigma N + \gamma U, \\
\bar{\nabla}_\lambda V &= -\beta\lambda - \gamma U,
\end{aligned} \tag{2.46}$$

where h and $\{\sigma, \tau, \beta, \gamma\}$ are smooth functions on $\mathcal{U}$ and $\{U, V\}$ is an orthonormal basis of S with respect to $\mathcal{U}$. Using the standard terminology from geometry we call

$$F = \{\frac{d}{dt}, N, U, V\}, \tag{2.47}$$

a **Frenet frame** on M along the null curve C with respect to a screen distribution S. The functions σ, τ, β and γ are called **curvature, torsion, bitorsion, and bicurvature functions** respectively and the four equations (2.46) are called the **Frenet equations** of C with respect to F. In order to express the Frenet equations in the simple possible form we show the following:

(a) *It is possible to find a parameter on C such that $h = 0$ in the Frenet equations (2.46), using the same screen space S.*

(b) *If $\sigma \neq 0$ on $\mathcal{U}$, then there exists another screen distribution $\bar{S}$ which induces another Frenet frame $\bar{F}$ on $\mathcal{U}$ such that its corresponding bicurvature function $\bar{\gamma} = 0$ on $\mathcal{U}$.*

To deal with the possibility **(a)**, consider another coordinate neighborhood $\mathcal{U}^\star$, and its corresponding Frenet frame $F = \{\frac{d}{dt^\star}, N^\star, U^\star, V^\star\}$ such that $\mathcal{U} \cap \mathcal{U}^\star \neq \phi$. Since we are using same screen distribution for this case, we have the following transformation equations:

$$\frac{d}{dt^\star} = \frac{dt}{dt^\star} \cdot \frac{d}{dt}, \qquad N^\star = \frac{dt^\star}{dt} N,$$

$$U^\star = \alpha_1 U + \alpha_2 V, \qquad V^\star = \alpha_3 U + \alpha_4 V, \tag{2.48}$$

where the 2×2 matrix $(\alpha_a(p))$ is an element of the orthogonal group $O(2)$ for any p and the four smooth functions α_a are on $\mathcal{U} \cap \mathcal{U}^\star$. Using $F^\star$ in the Frenet equations and (2.48), we obtain

$$\frac{d^2 t}{dt^{\star 2}} + h(\frac{dt}{dt^\star})^2 = h^\star \frac{dt}{dt^\star},$$

$$\sigma^\star \alpha_1 = \sigma(\frac{dt}{dt^\star})^2, \qquad \sigma^\star \alpha_2 = 0. \tag{2.49}$$

Now consider a differential equation $\frac{d^2 t}{dt^{\star 2}} - h^\star \frac{dt}{dt^\star} = 0$ whose general solution comes from

$$t = a \int_{t_0}^{t^\star} exp(\int_{s_0}^{s} h^\star(t^\star) dt^\star) ds + b, \qquad a, b \in R. \tag{2.50}$$

It follows from (2.48) that any solution of (2.50), with $a \neq 0$, can be taken as a particular parameter on C such that $h = 0$. Denote such a particular parameter by $P = \frac{t-b}{a}$ where t is the general parameter as given in equation (2.50). We call P a **distinguished parameter** of C, in terms of which if we set $h = 0$ and $\frac{d}{dP} \equiv \mu$, then the Frenet equations (2.46) become

$$\bar{\nabla}_\mu \mu = \sigma U, \qquad \bar{\nabla}_\mu N = \tau U + \beta V,$$

$$\bar{\nabla}_\mu U = -\tau \mu - \sigma N + \gamma V, \qquad \bar{\nabla}_\mu V = -\beta \mu - \gamma U. \tag{2.51}$$

Suppose $\sigma \neq 0$ on $\mathcal{U} \cap \mathcal{U}^\star$. It follows from (2.49) that $\sigma^\star \neq 0$ on $\mathcal{U} \cap \mathcal{U}^\star$ and thus $\alpha_2 = 0$. Since (α_a) is an orthogonal matrix, we infer that $\alpha_1 = \pm 1 = \alpha_4$, $\alpha_3 = 0$. Thus, for $\sigma \neq 0$ on $\mathcal{U} \cap \mathcal{U}^\star$ we have the following relations:

$$\sigma^\star = \pm\, \sigma(\frac{dt}{dt^\star})^2, \; \tau^\star = \pm\, \tau, \; \beta^\star = \pm\, \beta, \; \gamma^\star = \pm\, \gamma \frac{dt}{dt^\star}.$$

Now we deal with the case (b). Let $F' = \{\frac{d}{dt'}, N', U', V'\}$ be another Frenet frame of C with respect to another screen distribution S', a parameter t' and a coordinate neighborhood $\mathcal{U}'$ such that $\mathcal{U} \cap \mathcal{U}' \neq \phi$. We set $\sigma \neq 0$ on $\mathcal{U}$. Then the general transformations relating elements of F and F' are given by

$$\frac{d}{dt'} = \frac{dt}{dt'} \frac{d}{dt}, \qquad N' = -\frac{1}{2} \frac{dt}{dt'} (A^2 + B^2) \frac{d}{dt} + N \frac{dt'}{dt} + AU + BV, \tag{2.52}$$

$$\begin{aligned} U' &= \alpha_1 U + \alpha_2 V - (\alpha_1 A + \alpha_2 B) \frac{d}{dt'}, \\ V' &= \alpha_3 U + \alpha_4 V - (\alpha_3 A + \alpha_4 B) \frac{d}{dt'}, \end{aligned} \tag{2.53}$$

where A, B and the four α's (same as defined in the case (b)) are smooth functions on $\mathcal{U} \cap \mathcal{U}'$. Then, by using (2.53) and first Frenet equation for F and F', we get

$$\frac{d^2t}{dt'^2} + h(\frac{dt}{dt'})^2 + \sigma A(\frac{dt}{dt'})^3 = h'\frac{dt}{dt'},$$

$$\sigma'\alpha_1 = \sigma(\frac{dt}{dt'})^2, \qquad \sigma'\alpha_2 = 0.$$

Since by hypothesis, $\sigma \neq 0$ on $\mathcal{U}$, it follows from a procedure similar to the case (a) that $\alpha_1 = \pm 1 = \alpha_4$ and $\alpha_2 = \alpha_3 = 0$. Thus, (2.52) remains same but (2.53) and the corresponding relations are as follows:

$$\begin{aligned}
U' &= \pm(U - A\frac{d}{dt'}), \qquad V' = \pm(V - B\frac{d}{dt'}),\\
\sigma' &= \pm(\frac{dt}{dt'})^2,\\
\tau' &= \pm(\tau + Ah' + \frac{\sigma}{2}(A^2 + B^2)\frac{dt}{dt'} + \frac{dA}{dt'} - B\gamma\frac{dt}{dt'}),\\
\beta' &= \pm(\beta + Bh' + \frac{dB}{dt'} + A\gamma\frac{dt}{dt'}),\\
\gamma' &= \pm(\gamma + B\sigma\frac{dt}{dt'})\frac{dt}{dt'}. \qquad (2.54)
\end{aligned}$$

To show that $\gamma' = 0$ on $\mathcal{U}$ we define N', U' and V' in terms of F on $\mathcal{U}$ by the following relations:

$$\begin{aligned}
N' &= -\frac{1}{2}(\frac{\gamma}{\sigma})\frac{d}{dt} + N - \frac{\gamma}{\sigma}V,\\
U' &= U, \qquad V' = V + \frac{\gamma}{\sigma}\frac{d}{dt}.
\end{aligned}$$

It is easy to see that with respect to another intersecting coordinate neighborhood there exists another screen distribution satisfying above relations. By setting $t' = t$ and $B = -\frac{\gamma}{\sigma}$ in (2.54) we conclude that $\gamma' = 0$ on $\mathcal{U}$.

In conclusion, we have shown the existence of a Frenet frame $F = \{\frac{d}{dP}, N, U, V\}$, with respect to a distinguished parameter P, for which $h = 0$ and t is replaced by P in the Frenet equations (2.46). Also for $\sigma \neq 0$, γ vanishes.

To be consistent with the notations used in relativity for a null tetrad, we set

$$\frac{d}{dP} = \ell, \qquad -N = k, \qquad \nabla_\ell X = \dot{X}.$$

Then, the Frenet equations, with respect to $F = \{\ell, k, U, V\}$ are

$$\dot{\ell} = \sigma U, \qquad \sigma \neq 0,$$

$$\begin{aligned}
\dot{k} &= -\tau U - \beta V, \\
\dot{U} &= -\tau \ell + \sigma k, \\
\dot{V} &= -\beta \ell.
\end{aligned} \tag{2.55}$$

In case σ vanishes on $\mathcal{U}$, then with respect to the distinguished parameter P, the first Frenet equation takes the following familiar form:

$$\frac{d^2 x^i}{dP^2} + \Gamma^i_{jk} \frac{dx^j}{dP} \frac{dx^k}{dP} = 0, \qquad i, j, k \in \{1, 2, 3, 4\},$$

where Γ^i_{jk} are the Christoffel symbols of the second type (see page (19)). Therefore, C is a **null geodesic** of M. Also, the converse of above assertion holds. Thus, we have proved the following theorem (also valid for an n-dimensional Lorentz manifold, details of which may be seen in Duggal-Bejancu [57, page 59]).

THEOREM 2.2. *Let C be a null curve of a Lorentz manifold M. Then there exists a distinguished parameter P on C with respect to which C is a null geodesic of M iff its first curvature function vanishes identically on C.*

At this point we let M be time orientable, that is , M is a physical spacetime of general relativity. We relate the distinguished Frenet frame $F = \{\ell, k, U, V\}$ with the Newman-Penrose [NP] formalism (see Kramer et al. [118]), in the following way. At each point of $\mathcal{U} \subset C$ consider two complex vectors m and $\bar{m}$ and define the following complex Frenet frame:

$$F^c = \{\ell, k, m, \bar{m}\}, \qquad m = \frac{1}{\sqrt{2}}(U + iV), \qquad \bar{m} = \frac{1}{\sqrt{2}}(U - iV)$$

called the **null frame** on $\mathcal{U}$. Recall from (1.11) of chapter 1 that all scalar products of the base vectors of F^c vanish, excepting $< \ell, k > = -1$ and $< m, \bar{m} > = 1$. From (2.55) we derive the following Frenet equations with respect to F^c:

$$\begin{aligned}
\dot{\ell} &= K_1(m + \bar{m}), \\
\dot{k} &= \bar{K}_2 m + K_2 \bar{m} \\
\dot{m} &= K_2 \ell + K_1 k, \\
\dot{\bar{m}} &= \bar{K}_2 \ell + K_1 k,
\end{aligned}$$

where $\sqrt{2}K_1 = \sigma$ and $\sqrt{2}K_2 = -\tau + i\beta$. Hence , C has a real curvature K_1 and a complex curvature K_2 with respect to the null frame F^c.

EXAMPLE. Consider a null curve in R^4_1 given by

$$C : \frac{1}{\sqrt{2}}(\sinh t, cosh\, t, sin\, t, cos\, t), \qquad t \in R.$$

Then

$$\frac{d}{dt} \equiv \lambda = \frac{1}{\sqrt{2}}(\cosh t, sinh\, t, cos\, t, -sin\, t),$$

$$\bar{\nabla}_\lambda \lambda = \frac{1}{\sqrt{2}}(sinh\, t, cosh\, t, -sin\, t, -cos\, t) = U.$$

Choose $V = \frac{1}{\sqrt{2}}(sinh\, t, cosh\, t, sin\, t, -cos\, t)$ to obtain

$$N = \frac{1}{\sqrt{2}}(-cosh\, t, -sinh\, t, cos\, t, -sin\, t).$$

Then, the Frenet equations (2.46) with respect to $F = \{\frac{d}{dt}, N, U, V\}$ give

$$h = 0, \qquad \sigma = 1, \qquad \tau = 0, \qquad \beta = 1, \qquad \gamma = 0.$$

REMARK. It is important to mention that the curve C of above example lies on the pseudo sphere S_1^3 of R_1^4. As is well-known, in theory of curves of R^3, given a regular curve, there exists a Frenet frame of unit tangent , normal and binormal vector fields such that the normal and the binormal vectors are expressed in terms of the tangent vector and its directional covariant derivatives. Following the same idea, it is possible to express $\{N, U, V\}$ in terms of ℓ and its covariant derivatives up to the order 3. Details on this and the study on null curves in n-dimensional Lorentz manifolds (including the fundamental existence and uniqueness theorem and some results on 3-dimensional Lorentz manifolds) is available in Duggal-Bejancu [57, chapter 3].

2.7 Lightlike Hypersurfaces of Lorentz Manifolds

Let (Σ, γ) be a hypersurface of a real 4-dimensional Lorentz manifold (M, g). Suppose Σ is lightlike, i.e., γ is degenerate on Σ. Then, both the tangent space $T_p(\Sigma)$ and the normal space $T_p(\Sigma)^\perp$ are degenerate. Moreover, as per section 1.2 of chapter 1, we have

$$T_p(\Sigma) \cap T_p(\Sigma)^\perp = T_p(\Sigma)^\perp, \qquad dim(T_p(\Sigma)^\perp) = 1.$$

In this section we follow the notations of sections 2.5 and 2.6 of this chapter and also drop the suffix p. Let S be the 2-dimensional complementary and spacelike distribution to $T(\Sigma)^\perp$ in $T(\Sigma)$. Since S is non-degenerate, we have the following decomposition:

$$T(M) = S \oplus S^\perp, \qquad S \cap S^\perp = \{0\}, \tag{2.56}$$

where $S^\perp$ is a 2-dimensional complementary orthogonal non-degenerate distribution of $T(M)$. Let $T(\Sigma)^\perp$ be spanned by $\{\ell\}$, where ℓ is a real null vector.

THEOREM 2.3. *Let (Σ, γ, S) be a lightlike hypersuface of a 4-dimensional Lorentz manifold (M, g). Then, with respect to each coordinate neighborhood $\mathcal{U}$ of Σ, there exists a unique null distribution $E = \cup_{p\in\Sigma} E_p$, where E is spanned by a unique null vector field N such that*

$$g(\ell, N) = 1, \qquad g(N, N) = g(N, X) = 0, \qquad \forall X \in \mathcal{X}(S|_{\mathcal{U}}). \tag{2.57}$$

PROOF. Since $T(\Sigma)^{\perp}$ is contained in $S^{\perp}$, consider a complementary distribution E to $T(\Sigma)^{\perp}$ in $S^{\perp}$. Now $rank(S^{\perp}) = 2$ implies that their exists a non-zero vector field W in E such that $g(\ell, W)$ is non-zero on $\mathcal{U}$, otherwise it follows from (2.56) that $S^{\perp}$ is degenerate at least at one point of $\mathcal{U}$. As in the case of null curves, define on $\mathcal{U}$ a null vector field

$$N = \frac{1}{g(\ell, W)}\{W - \frac{g(W,W)}{2g(\ell, W)}\ell\}.$$

The rest of the proof is common with the proof of Theorem 2.1.

Based on above theorem and (2.56) we have the following decomposition:

$$T(M) = S \oplus (T(\Sigma)^{\perp} + E) = T(\Sigma) + E, \tag{2.58}$$

where $+$ denotes non-orthogonal complementary sum and

$$T(\Sigma) = S \oplus T(\Sigma)^{\perp}. \tag{2.59}$$

To find a local field of basis on Σ adapted to the decomposition (2.59), we observe that $T(\Sigma)^{\perp}$ is integrable since it is a 1-dimensional distribution on Σ. Thus, there exists a local coordinate system $\{x^0, x^1, x^2\}$ such that $\partial_0 \in T(\Sigma)^{\perp}|_{\mathcal{U}}$. Therefore, one can take $\{\partial_0, \partial_1, \partial_2\}$ as the natural basis with respect to which γ is expressed by the following matrix form:

$$(\gamma) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \gamma_{11} & \gamma_{12} \\ 0 & \gamma_{21} & \gamma_{22} \end{pmatrix}, \qquad det(\gamma) = 0, \tag{2.60}$$

where the coefficients $\gamma_{ij} = \gamma(\partial_i, \partial_j)$, $(i, j = 1, 2)$, are functions of (x^0, x^1, x^2) and $det(\gamma_{ij}) \neq 0$. Thus, according to the general law of transformations (for details see Duggal-Bejancu [57, page 90]), there exists a local field of basis

$$\{\partial_0, \delta_1, \delta_2\}, \qquad S = span\{\delta_1, \delta_2\} \tag{2.61}$$

adapted to the decomposition (2.59), such that

$$\delta_i = \partial_i - A_i(x)\, \partial_0, \qquad i = 1, 2,$$

and $A_i(x^0, x^1, x^2)$ are 2 differentiable functions on Σ. If we take N as defined by (2.57) and replace ℓ by ∂_0, then

$$\mathcal{B} = \{\partial_0, N, \delta_1, \delta_2\} \tag{2.62}$$

is the corresponding local basis on M, and then the matrix form of g is given by

$$(g) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \gamma_{11} & \gamma_{12} \\ 0 & 0 & \gamma_{21} & \gamma_{22} \end{pmatrix} \tag{2.63}$$

where the coefficients γ_{ij} are the four functions as in (2.60). Let $\{\omega^0, \omega^1, \omega^2, \omega^3\}$ be the set of dual basis of $\mathcal{B}$. Then, the line elements ds and ds' of g and γ are given by

$$\begin{aligned} ds^2 &= 2d\omega^0 d\omega^2 + ds'^2, \\ ds'^2 &= \gamma_{ij}\, d\omega^i d\omega^j. \end{aligned} \tag{2.64}$$

EXAMPLE. Consider the unit pseudo sphere S_1^3 of R_1^4 given by the equation

$$-(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 = 1.$$

Cut S_1^3 by the hyperplane $x^0 - x^1 = 0$, to obtain a lightlike surface Σ of S_1^3 with

$$T(\Sigma)^{\perp} = span\{\ell = \partial_0 + \partial_1\}.$$

Consider a screen distribution $S = span\{U = x^3\partial_2 - x^2\partial_3\}$. Then, by using theorem 2.3 we get

$$N = -\frac{1}{2}\{(1 + (x^0)^2)\partial_0 + ((x^0)^2 - 1)\partial_1 + 2x^0x^2\partial_2 + 2x^0x^3\partial_3\}.$$

Another well-known example is the null cone Λ of R_1^4. Reader is invited to study this example as an exercise by setting $M = R_1^4$ and $\Sigma = \Lambda$ and show that the theorem 2.3 and its related results hold. In this book (as appropriate) we have discussed several other examples of lightlike hypersurfaces.

Let $\bar{\nabla}$ and ∇ be the Levi-civita connection and an affine torsion-free connection on M and Σ respectively. Then using (2.58) we get

$$\begin{aligned} \bar{\nabla}_X Y &= \nabla_X Y + B(X,Y)N, \\ \bar{\nabla}_X N &= -A_N X + \tau(X)N, \end{aligned} \tag{2.65}$$

for any tangent vectors X,Y , $\nabla_X Y$ and $A_N X$ belonging to Σ while $B(X,Y)$ a function, τ a 1-form on Σ and N belong to E. It is easy to check that B, A_N and τ are a symmetric E-valued bilinear form, a linear operator and a E-valued 1-form respectively, locally defined on Σ. Also,

$$g\,(B(X,Y), N) = g\,(A_N X, Y), \qquad \forall X, Y \in \mathcal{X}(\Sigma). \tag{2.66}$$

Following the terminology used for the structure equations (2.37), we call the two equations of (2.65) the **Gauss** and the **Weingarten** formulas, respectively, with the understanding that the decompositions of these equations are complementary but not orthogonal. Thus it is reasonable to call B and A_N the **second fundamental form** and the **shape operator** of the lightlike hypersurface Σ.

Clearly, the above structure equations have been obtained by using a screen distribution S which allows [through the decomposition (2.58)] to define above mentioned induced geometric objects on Σ. Since the geometry of hypersurfaces should be independent of the choice of S, we consider another screen distribution S' on Σ with corresponding B' and E'. Then, it follows from the Gauss equation that $B(X,Y) = g(\bar{\nabla}_X Y, \ell) = B'(X,Y), \forall X, Y \in \mathcal{X}(\Sigma)$. Thus, $B = B'$ on $\mathcal{U}$, therefore, the local second fundamental form of Σ is independent of the choice of screen distribution. It follows from (2.66) that the same is true for A_N. However, unfortunately, the invariance of the induced affine connection and τ imposes some geometric conditions on the embedding of Σ in M, as stated in the following two propositions (for details see Duggal-Bejancu [57, pages 84-87]):

PROPOSITION 1. *Let (Σ, γ, S) be a lightlike hypersurface of (M, g). Then, the induced affine connection ∇ is invariant with respect to S iff the second fundamental form of Σ vanishes identically on Σ.*

PROPOSITION 2. *Let (Σ, γ, S) be a lightlike hypersurface of (M, g). Suppose τ and τ' are 1-forms on $\mathcal{U}$ in Σ with respect to two null vectors ℓ and ℓ'. Then, $d\tau = d\tau'$ on $\mathcal{U}$.*

Based on the Gauss and Weingarten equations (2.65), with respect to an affine connection $\bar{\nabla}$ on Σ, the **Gauss-Codazzi equations** will be the same as given by (2.40).

Now, the question is whether there exists a unique Levi-Civita (metric) induced connection on Σ. Unfortunately, contrary to the non-degenerate case, in general, this is not true. Indeed, if we set $\gamma\,(X, N) = \omega(X)$ for every $X \in \Sigma$ and then using the Gauss equation of (2.65), we obtain

$$\begin{aligned}
(\bar{\nabla}_X g)(Y, Z) &= X(g(Y, Z) - g(\bar{\nabla}_X Y, Z) - g(Y, \bar{\nabla}_X Z) \\
&= X(\gamma(Y, Z)) - \gamma(\nabla_X Y, Z) - \gamma(Y, \nabla_X Z) \\
&- B(X, Y)g(Z, N) - B(X, Z)g(Y, N) \\
&= (\nabla_X \gamma)(Y, Z) - B(X, Y)\omega(Z) - B(X, Z)\omega(Y) \\
&= 0.
\end{aligned}$$

Thus, ∇ is a metric connection (i.e., $\nabla\gamma = 0$) iff B vanishes identically on Σ, which means iff M is **totally geodesic lightlike hypersurface** of M. Finally, it follows from proposition 1 that the existence of unique Levi-Civita connection on Σ does not depend on the screen distribution. Thus, we have the following result:

THEOREM 2.4. *Let (Σ, γ) be a lightlike hypersurface of a 4-dimensional Lorentz manifold (M, g). Then the following statements are equivalent:*

(1) *Σ is totally geodesic.*

(2) *The second fundamental form B vanishes on Σ.*

(3) *There exists a unique metric (Levi-Civita) induced connection on Σ.*

(4) *$T(\Sigma)^{\perp}$ is a Killing distribution on Σ.*

In this book, for physical applications, we are only interested in the study of totally geodesic lightlike hypersurfaces. Based on this assumption, we state (proof is straightforward) the following theorem:

THEOREM 2.5. *Let (Σ, γ, S) be a lightlike hypersurface of (M, g) such that $rank(A_N) = 2$ at any point of Σ. Then, Σ is totally geodesic iff the induced connection on Σ has the same curvature tensor as the metric connection on M*, i.e.,

$$\bar{R}(X, Y)Z = R(X, Y)Z, \qquad \forall X, Y, Z \in \mathcal{X}(\Sigma|_{\mathcal{U}}). \tag{2.67}$$

Thus for a totally geodesic Σ the Gauss-Codazzi equations (2.40) imply that

$$\begin{aligned} g(\bar{R}(X, Y)Z, PW) &= \gamma(R(X, Y)Z, PW), \\ g(\bar{R}(X, Y)Z, N) &= \gamma(R(X, Y)Z, N), \\ g(\bar{R}(X, Y)Z, \ell) &= 0, \end{aligned} \tag{2.68}$$

for all $X, Y, Z, W \in \mathcal{X}(\Sigma)$ where $P : T(\Sigma) \to S$ is the projection morphism with respect to the decomposition (2.59).

Using the local field of basis (2.61) on Σ and (2.25), we obtain the following local expression of the induced Ricci tensor:

$$Ric(X, Y) = \gamma^{ij}\gamma(R(X, \delta_i)Y, \delta_j) + \gamma(R(X, \partial_0)Y, N). \tag{2.69}$$

By using first Bianchi identity (2.11), (2.23) and (2.68) we obtain

$$Ric(X, Y) - Ric(Y, X) = \gamma(Ric(X, Y)\,\partial_0, N).$$

Hence, in general, Ric is not symmetric and, therefore, can not be associated with the induces Ricci curvature field on Σ, without imposing some geometric condition(s). We now deal with this problem. Replace X and Y by δ_i and δ_j and using the Gauss-Codazzi equations we get (here $i, j = 1, 2$):

$$\begin{aligned} R_{ij} - R_{ji} &= 2d\tau(\delta_i, \delta_j), \\ R_{0i} - R_{i0} &= 2d\tau(\partial_0, \delta_i). \end{aligned}$$

Thus, above equations and the proposition 2 imply the following result.

THEOREM 2.6 (Katsuno [104]). *Let (Σ, γ, S) be a lightlike hypersurface of a 4-dimensional Lorentz manifold (M, g). Then the induced Ricci tensor Σ is symmetric iff each 1-form τ induced by S is closed , i.e., $d\tau = 0$ on any $\mathcal{U}$ of Σ.*

At this point, for simplicity, we use the local pseudo-orthonormal frame $\{\ell, N, U, V\}$ for $T_p(M)$ for which $\gamma_{ij} = \delta_{ij}$ Let $T_p(\Sigma) = span\{\ell, U, V\}$. Then,

$$\gamma(X, Y) = u(X)u(Y) + v(X)v(Y), \qquad i.e., \qquad \gamma_{ij} = u_i u_j + v_i v_j, \tag{2.70}$$

$$Ric(X, Y) = \gamma(R(U, X)U, Y) + \gamma(R(V, X)V, Y) - \gamma(R(Y, N)\ell, X), \tag{2.71}$$

where $\{u, v\}$ is the dual set of $\{U, V\}$.

THEOREM 2.7. *Let (Σ, γ, S) be a lightlike hypersurface of a 4-dimensional Lorentz manifold (M, g) with the induced metric connection ∇ on Σ. Then, the local expression of the induced Ricci curvature field Ric and the scalar curvature r,* *with respect to a local pseudo-orthonormal basis* $\{\ell, U, V\}$ *is given by*

$$\begin{aligned} Ric(X,Y) &= 2d\theta\,(U,V)\,\gamma(X,Y) - 2d\phi\,(\ell,N)\omega\,(X)\omega\,(Y), \\ r &= 4d\theta\,(U,V) \qquad \forall X,Y \in \mathcal{X}(\Sigma|_{\mathcal{U}}) \end{aligned} \tag{2.72}$$

where θ , ϕ *and* ω *are* 1*-forms with* $\omega\,(N) = 1$.

PROOF. Since ∇ is metric connection, $(\nabla_X \gamma)(X,Y) = 0$ for every $X, Y, Z \in \mathcal{X}(\Sigma)$. Setting Y and Z the base vectors ℓ, U, V successively in this covariant constant equation for γ, we obtain the following

$$\begin{aligned} \nabla_X U &= \theta\,(X)V + \psi\,(X)\ell, \\ \nabla_X V &= -\theta\,(X)U + \lambda\,(X)\ell, \\ \nabla_X \ell &= \phi\,(X)\ell, \end{aligned} \tag{2.73}$$

where θ, ψ, λ and ϕ are some 1-forms on Σ. Replacing Z by the three base vectors U, V, ℓ successively in the curvature tensor equation(2.10), a lengthy but straightforward computation provides the following equations:

$$\begin{aligned} R(X,Y)U &= 2\,d\theta\,(X,Y)V + 2\,d\tau\,(X,Y)\ell, \\ R(X,Y)V &= -2\,d\theta\,(X,Y)U + 2\,d\tau'(X,Y)\ell, \\ R(X,Y)\ell &= 2\,d\phi\,(X,Y)\ell, \end{aligned} \tag{2.74}$$

where we set

$$d\tau = \theta \wedge \lambda + \psi \wedge \phi + d\psi,$$

$$d\tau' = \psi \wedge \theta + \lambda \wedge \phi + d\lambda.$$

Using above equations in the metric and the Ricci tensors (2.70) and (2.71) respectively, we obtain

$$\begin{aligned} Ric(X,Y) &= 2\,d\theta\,(U,V)\gamma\,(X,Y) - 2\,d\phi\,(\ell,N)\omega\,(X)\omega\,(Y) \\ &+ 2\,d\tau\,(U,V)v(X)\omega\,(Y) + 2\,d\tau'(U,V)u(X)\omega\,(Y), \end{aligned}$$

where $u(X) = \gamma\,(U,X)$, $v(X) = \gamma\,(V,X)$ and $\omega\,(N) = 1$. Since $d\tau\,(U,V)$ and $d\tau'\,(U,V)$ are induced by the screen distribution S, it follows from theorem 2.6 that they both vanish iff Ric is the induced symmetric Ricci curvature tensor field on Σ. Thus, (2.72) holds where r is obtained by contraction.

COROLLARY 1. *Under the hypothesis of theorem 2.6,* Σ *is empty (Ricci-flat) iff both the* 1*-forms* θ *and* ω *are closed (i.e.,* $d\theta = 0 = d\omega$*) on* Σ.

COROLLARY 2. *Under the hypothesis of theorem 2.6,* Σ *is an Einstein space (*$Ric = constant.\gamma$*) iff* $d\theta$ *is constant and* ω *is a closed* 1*-form on* Σ.

REMARKS. We close this section with the following remarks. All the results presented in this section also hold for a general case of lightlike hypersurfaces of n-dimensional semi-Riemannian manifolds. Detailed study of those general results is available in Duggal-Bejancu [57, chapter 4], including the physically important case of lightlike Monge hypersurfaces of R_q^{m+2}.

Lightlike hypersurfaces of semi-Euclidean spaces have been studied by Bonnor [23], Cagnac [25], Dallmer [38] and many more.

In particular, Kupeli [122] has done considerable work on degenerate submanifolds of semi-Riemannian manifolds.

In case the ambient space is a Lorentz manifold, we recognize the work of Dautcourt [39], Katsuno [104], Lemmer [126], Rosca [164], Carter [29], Israel [102], Larsen [125], Barrabes-Israel [6] and Racz-Wald [163] etc.

Also see Julch (1974), Kammerer (1967), Pinl (1940), Taub (1957) and Swift (1992) all cited in Duggal-Bejancu [57].

Chapter 3

Lie Derivatives and Symmetry Groups

In this chapter we present basic results on Lie derivatives, Lie groups and their Lie algebras. This is then followed by preliminaries on isometric, conformal, affine, projective and curvature symmetries on a general semi-Riemannian manifold.

3.1 Integral Curves and Lie Derivatives

Let V be a vector field on a real n-dimensional smooth manifold. The **integral curves (orbits)** of V are given by the following system of ordinary differential equations:

$$\frac{dx^i}{dt} = V^i(x(t)), \qquad i \in \{1, \ldots, n\}, \tag{3.1}$$

where (x^i) is a local coordinate system on M and $t \in I \subset R$. It follows from the well-known theorem on the existence and uniqueness of the solution of (3.1) that for any given point, with a local coordinate system, there is a unique integral curve defined over a part of the real line.

Consider a mapping ϕ from $[-\delta, \delta] \times \mathcal{U}$ ($\delta > 0$ and $\mathcal{U}$ an open set of M) into M defined by

$$\phi : (t, x) \to \phi(t, x) = \phi_t(x) \in M,$$

satisfying the following conditions:

(1) $\phi_t : x \in \mathcal{U} \to \phi_t(x) \in M$ is a diffeomorphism of $\mathcal{U}$ onto the open set $\phi_t(\mathcal{U})$ of M, for every $t \in [-\delta, \delta]$,

(2) $\phi_{t+s}(x) = \phi_t(\phi_s(x))$, $\forall t, s, t + s \in [-\delta, \delta]$ and $\phi_s(x) \in \mathcal{U}$.

In the above case, we say that the family ϕ_t is a **1-parameter group of local transformations** on M. The mapping ϕ is then called a **local flow** on M. Using the equation (3.1) it has been proved (see Kobayashi-Nomizu [111, page 13] that the vector field V generates a local flow on M. If each integral curve of V is defined on the entire real line, we say that V is a **complete vector field** and it generates a **global flow** on M. A set of local (resp. complete) integral curves is called a **local congruence (resp. congruence) of curves** or **stream lines** of V.

In the following we show how the flow ϕ is used to transform any object, say Ω, on M into another one of the same type as Ω, with respect to a point transformation $\phi_t : x^i \to x^i + tV^i$ along an integral curve through x^i. We denote by $\bar{\Omega}(x^i)$ the pull-back of $\Omega(x^i + tV^i)$ to the point x^i through the inverse mapping of ϕ_t. This defines a differentiable operator, denoted by L_V, which assigns to an arbitrary Ω another object $L_V\Omega$ of the same type as Ω and given by

$$(L_V\Omega)(x^i) = \lim_{t\to 0} \frac{1}{t}[\bar{\Omega}(x^i) - \Omega(x^i)]. \tag{3.2}$$

The operator L_V is called the **Lie derivative** with respect to V. It is important to mention that above definition (3.2) holds for local as well as global flows. Following are basic properties of Lie derivatives:

(1) $L_V(aX + bY) = aL_VX + bL_VY$, for all $a, b \in R$ and $X, Y \in \mathcal{X}(M)$: (linearity)

(2) $L_V(T \otimes S) = (L_VT) \otimes S + T \otimes L_VS$, where $\otimes$ is the tensor product of any two objects T and S: (Leibnitz rule)

(3) L_V commutes with the contraction operator.

It follows from above properties that in order to compute the Lie derivative of any arbitrary Ω, it is sufficient to know the Lie derivatives of a function, a vector field and a 1-form. Indeed, the Lie derivatives of all other objects of higher order can be obtained by using the operations of tensor analysis and above three properties. Thus, we now compute these three basic Lie derivatives.

Functions. Let f be a scalar function on M. Then,

$$\begin{aligned} L_Vf &= \lim_{t\to 0} \frac{1}{t}(f(x^i + tV^i) - f(x^i)) \\ &= \lim_{t\to 0} \frac{1}{t}(f(x^i) + tV^j\partial_j f - f(x^i)) \\ &= V^j\partial_j f, \end{aligned}$$

where we neglect t^2 and higher powers. Index-free expression is

$$L_Vf = V(f). \tag{3.3}$$

Vector Fields. Let $X = X^i\partial_i$ be a vector field on M. Then, the pull-back of X, from $x^i + tV^i$ to x^i, is given by

$$\begin{aligned} \bar{X}^j(x) &= X^k(x + tV)\partial_k(x^j - tV^j) \\ &= \{X^k(x) + tV^i\partial_i X^k + O(t^2)\}\{\delta^j_k - t\,\partial_k V^j\} \\ &= X^j(x) + t\,(V^i\partial_i X^j - X^k(x)\partial_k V^j) + O(t^2). \end{aligned}$$

Hence, using above in (3.2) with $\Omega = X$, we get

$$L_V X^j = V^i \partial_i X^j - X^i \partial_i V^j. \tag{3.4}$$

As per the Lie-bracket (2.2), in terms of index-free notation, we have

$$L_V X = [V, X]. \tag{3.5}$$

<u>**1-forms**</u>. Let $\omega = \omega_i dx^i$ be a 1-form (linear differential form) on M. We could proceed by using the pull-back of ω and the definition (3.2), but, we prefer as follows: Let $X = X^i \partial_i$ be an arbitrary vector field on M. Then, the contraction $\omega(X) = \omega_i X^i$ is a real scalar function, say f. Now, from (3.3) we obtain

$$L_V \omega(X) = L_V f = V(f) = V(\omega(X)).$$

Then, using the linear property of L_V, we get

$$L_V \omega(X) = (L_V \omega)(X) + \omega(L_V X) = V(\omega(X)).$$

Therefore, substituting the value of $L_V X$ from (3.5), we obtain

$$\begin{aligned} (L_V \omega)(X) &= V(\omega(X)) - \omega[V, X] \\ (L_V \omega)_i &= V^j \partial_j(\omega_i) + \omega_j \partial_i (V^j). \end{aligned} \tag{3.6}$$

It follows from above three Lie derivatives that if V is a vector field of class, say C^m, then the Lie derivative of a function, a vector field and a 1-form is of the same type but of class C^{m-1}. This is also true for higher geometric/tensor quantities. Let T be a tensor (or just a geometric object) of type (r, s). Then using above results and the theory of tensor analysis, we obtain the following general formulae for its Lie derivative with respect to a vector field V:

$$\begin{aligned} (L_V T)(\omega^1, \dots, \omega^r, X_1, \dots, X_s) &= V(T(\omega^1, \dots, \omega^r, X_1, \dots, X_s)) \\ &- \sum_{a=1}^{r} T(\omega^1, \dots, L_V \omega^a, \dots, \omega^r, X_1, \dots, X_s) \\ &- \sum_{A=1}^{s} T(\omega^1, \dots, \omega^r, X_1, \dots, [V, X_A], \dots, X_s), \end{aligned}$$

where $\omega^1, \dots, \omega^r$ and $X_1, \dots, X_s$ are r 1-forms and s vector fields respectively.

In particular, if T is a tensor of type $(1, 1)$ then

$$(L_V T)(X) = V(T(X)) - T([V, X]),$$

for an arbitrary vector field X on M. The Lie derivative of the metric tensor g (covariant tensor of type $(0, 2)$) and its Levi-Civita connection ∇ are given by

$$\begin{aligned} (L_V g)(X, Y) &= V(g(X, Y)) - g([V, X], Y) - g(X, [V, Y]) \\ &= g(\nabla_X V, Y) + g(\nabla_Y V, X) \end{aligned} \tag{3.7}$$

for arbitrary vector fields X and Y on M and Levi-Civita (metric) connection ∇ of g as defined in chapter 2. Locally, we have

$$\begin{aligned} L_V g_{ij} &= \nabla_i v_j + \nabla_j v_i \\ &= v_{j\,;\,i} + v_{i\,;\,j} \quad , \qquad v_i = g_{ij} V^j. \end{aligned} \tag{3.8}$$

$$\begin{aligned} (L_V \nabla)(X,Y) &= L_V \nabla_X Y - \nabla_{[V,X]} Y - \nabla_X [V,Y] \\ &= [V, \nabla_X Y] - \nabla_{[V,X]} Y - \nabla_X (\nabla_V Y - \nabla_Y V) \\ &= \nabla_V \nabla_X Y - \nabla_X \nabla_V Y - \nabla_{[V\,X]} Y + \nabla_X \nabla_Y V - \nabla_{\nabla_X\,Y} V \\ &= \nabla_X \nabla_Y V - \nabla_{\nabla_X\,Y} + R(V,X)Y. \end{aligned}$$

In terms of local coordinates, we have

$$\begin{aligned} L_V \Gamma^i_{jk} &= \nabla_j \nabla_k V^i + R^i{}_{kmj} V^m \\ &= \frac{1}{2} g^{im} (\nabla_j (L_V g_{km}) + \nabla_k (L_V g_{jm}) - \nabla_m (L_V g_{jk})), \end{aligned} \tag{3.9}$$

where Γ^i_{jk} are the Christoffel symbols of the second kind (see page 19), with respect to the metric tensor g_{ij}. Setting $\nabla_X Y = \nabla(X,Y)$, we can prove the following commutative formulas:

$$\begin{aligned} L_V(\nabla_X Y) - \nabla_X(L_V Y) - \nabla_{[V,X]} Y &= (L_V \nabla)(X,Y) \\ (L_V(\nabla_X \omega) - \nabla_X(L_V \omega) - \nabla_{[V,X]} \omega)Y &= -\omega((L_V \nabla)(X,Y)) \\ \{(L_V(\nabla_X T) - \nabla_X(L_V T) - \nabla_{[V,X]} T\}Y &= (L_V \nabla)(X,TY) \\ &- \; T((L_V \nabla)(X,Y)) \\ \nabla_X (L_V \nabla)(Y,Z) - \nabla_Y (L_V \nabla)(X,Z) &= (L_V R)(X,Y,Z) \end{aligned}$$

where ω and T are a 1-form and a $(1,1)$ tensor field respectively and X, Y, Z are arbitrary vector fields on M, or, in local coordinates,

$$\begin{aligned} L_V(\nabla_i Y^k) - \nabla_i(L_V Y^k) &= (L_V \Gamma^k{}_{ij}) Y^j \\ L_V(\nabla_i \omega_j) - \nabla_i(L_V \omega_j) &= -(L_V \Gamma^k{}_{ij}) \omega_k \\ L_V(\nabla_i T^j_k) - \nabla_i(L_V T^j_k) &= (L_V \Gamma^j{}_{im}) T^m_k - T^j_m L_V(\Gamma^m{}_{ik}) \\ \nabla_i L_V \Gamma^j{}_{km} - \nabla_k L_V \Gamma^j{}_{im} &= L_V R^j{}_{mik}. \end{aligned}$$

The Lie derivative of a p-form $\omega = a_{i_1 \ldots i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p}$, with respect to V, is given by

$$L_V\, \omega = (L_V\, a_{i_1 \ldots i_p})\, dx^{i_1} \wedge \ldots \wedge dx^{i_p}$$

where $\wedge$ is the wedge product (see page 14) operator. Following identity holds

$$L_V\, \omega = d i_V\, \omega + i_V\, d\, \omega \tag{3.10}$$

where d denotes the exterior derivative operator and i_V is the inner product such that $(i_V \omega)(X_2, \ldots, X_p) = \omega(V, X_2, \ldots, X_p)$. It, therefore, follows that L_V and d

commute, that is, $L_V(d\omega) = d(L_V\omega)$.

The reader may find proofs of above formulas and further information on Lie derivatives in standard books such as Kobayashi-Nomizu [111].

3.2 Lie Groups and Lie Algebras

A **Lie group** G is a set which is both a group and a differentiable manifold such that the following group operations

$$G \times G \to G \qquad \text{given by} \qquad (a,b) \to ab$$

$$G \to G \qquad \text{given by} \qquad a \to a^{-1}$$

are differentiable. The identity element of the group is denoted by e. The dimension of Lie group G is the dimension of G as a manifold. A one dimensional Lie group is usually called a 1-parameter group. A **local Lie group** is a neighborhood of the identity element e of the group. Thus, a **1-parameter local group** is isomorphic to an interval $I \subset R$ including the origin. A simple example is the n-dimensional vector space R^n which is both a group (with additive operation + of its vectors) and a manifold. Another example is the unit circle $S^1 = \{e^{i\theta} : \theta \in R(mod 2\pi)\}$ in the complex plane. The group operations are defined by $e^{i\theta}.e^{i\phi} = e^{i(\theta+\phi)}$ and $(e^{i\theta})^{-1} = e^{-i\theta}$. This Lie group is denoted by $U(1)$.

It can be shown that G has a unique analytic structure with which the group operations can be written as convergent power series. A **Lie subgroup** of G is a subgroup of G which is also a Lie group and a submanifold. In the following we list some examples of Lie groups and their Lie subgroups which are frequently used in geometry and physics.

$GL(n,C)$: **(General linear group)** $n \times n$ non-singular matrices of complex elements and real dim. $2n^2$.

$GL(n,R)$: **(General linear group)** $n \times n$ non-singular matrices of dim. n^2.

$SL(n,C)$: **(Special linear group)** $\{A \in GL(n,C) : det(A) = 1\}$ with real dim. $2n^2 - 2$.

$SL(n,R)$: **(Special linear group)** $\{A \in GL(n,R) : det(A) = 1\}$ with dim. n^2-1.

$O(n)$: **(Orthogonal group)** $\{A \in GL(n,R) : AA^t = I\}$ with dim. $\frac{n(n-1)}{2}$.

$SO(n)$: **(Special orthogonal group)** $\{A \in O(n) : det(A) = 1\}$ with dim. $\frac{n(n-1)}{2} - 1$

$U(n)$: **(Unitary group)** $\{A \in GL(n,C) : A^{-1} = \bar{A}^t\}$ with real dim. n^2.

$SU(n)$: **(Special unitary group)** $\{A \in U(n) : det(A) = 1\}$ with real dim. n^2-1.

One of the principal uses of the group structure in physics is to study the invariance properties of a manifold M under investigation. For this purpose, we need the concept of a **Lie algebra structure** as follows. Let $\mathcal{L}$ be a set of vector fields on a region $\mathcal{U}$ of M. If $\mathcal{L}$ is a vector space under addition and is closed under the Lie-bracket operation (which means that $[A, B] \in \mathcal{L}$ for every $A, B \in \mathcal{L}$) then $\mathcal{L}$ is a Lie algebra of all those vector fields on $\mathcal{U}$. Obviously, the set of all smooth vector fields on $\mathcal{U}$ is a Lie algebra. However, for physical reasons it has been useful to single out some Lie sub algebras of vector fields on the Lie group G, which we now study.

Let a and b be in a Lie group G. We do not assume that G is Abelian. The **right translation** $R_a : G \to G$ and the **left translation** $L_a : G \to G$ of b by a are defined as $R_a b = ba$ and $L_a b = ab$ respectively. Since the translations are diffeomorphism, their differentials are $R_{a*} : T_b(G) \to T_{ba}(G)$ and $L_{a*} : T_b(G) \to T_{ab}(G)$ respectively. A vector field X of G is said to be **right invariant** (resp. **left invariant**) if $R_{a*}X_b = X_{ba}$ (respectively $L_{a*}X_b = X_{ab}$) for any $a, b \in G$. Following properties can be easily verified:

$$R_a R_b = R_{ba},\ L_a L_b = L_{ab},\ L_a R_b = R_b L_a,\ L_{a^{-1}} = (L_a)^{-1},\ R_{a^{-1}} = (R_a)^{-1}.$$

Denote by $\mathcal{G}$ the set of all right invariant vector fields on G. Then $\mathcal{G}$ is a Lie sub algebra of the Lie algebra of all vector fields on $\mathcal{G}$. Indeed,

$$\begin{aligned} R_{a*}[X, Y]_b &= [R_{a*}X_b, R_{a*}Y_b] \\ &= [X_{ba}, Y_{ba}] = [X, Y]_{ba}. \end{aligned}$$

$\mathcal{G}$ is called the Lie algebra of the Lie group G, denoted by $\mathcal{L}(\mathcal{G})$. There is an isomorphism between $\mathcal{L}(\mathcal{G})$ and the tangent space $T_e(G)$ given by the following rule: If V is in $T_e(G)$ then the vector field X_V, defined by $(X_V)_a = R_{a*}V\,\forall a \in G$, is a right invariant vector field. Conversely, a right invariant vector field X defines a unique vector X_e in $T_e(G)$. Consequently, $dim(\mathcal{L}(\mathcal{G})) = dim\,(T_e(G)) = dim\,(G)$. Note that an analogous definition would follow if we had formed the Lie algebra of G by using all left invariant vector fields.

At this point we assume that G is an n-dimensional Lie group with Lie algebra $\mathcal{L}(\mathcal{G})$ of right invariant vector fields, with a basis $B = \{X_1, \ldots, X_n\}$. Since the Lie-bracket operator is closed in $\mathcal{L}(\mathcal{G})$, we can write

$$[X_i, X_j] = C^k_{ij} X_k, \tag{3.11}$$

where C^k_{ij} are n^3 **structure constants** and called the **structure equations**. From the skew symmetry and cyclic property of the Lie-brackets and (3.11) it follows that

$$C^k_{ij} = -C^k_{ji},$$

$$C^t_{is} C^s_{jk} + C^t_{js} C^s_{ki} + C^t_{ks} C^s_{ij} = 0.$$

Since the basis B is not unique, under a change of basis C^k_{ij} transform as components of a $(1, 2)$ tensor, called the **structure tensor** which is unique for every Lie group and its Lie algebra.

It is easy to see that every Lie group G has its Lie algebra $\mathcal{L}(\mathcal{G})$. Indeed, every a of G is the image of the identity element e under right translation by a and every vector in $T_e(G)$ has one to one correspondence to a vector in $\mathcal{L}(\mathcal{G})$. Conversely, it has been proved (see Warner [198]) that for every Lie algebra, with structure equations (3.11), there exists one and only one (up to isomorphism) **simply connected** (this means that every closed curve can be continuously contracted to a point) Lie group. Thus, the Lie algebra $\mathcal{L}(\mathcal{G})$ is completely characterized by its structure constants. However, it is important to note that above characterization holds only locally since the structure constants do not, in general, recover a Lie group globally. Finally, if all structure constants vanish then the Lie algebra and its associated Lie group is said to be Abelian.

EXAMPLE 1. R^n is a group and a manifold with respect to vector addition. Its 1-parameter subgroups are the straight lines through the origin. Since this Lie group is Abelian, its Lie algebra is the tangent vector space T_e, at the identity e, endowed with the trivial Lie-bracket $[X, Y] = 0$ for all $X, Y \in T_e$.

EXAMPLE 2. Consider the general linear group $GL(n, R)$ of dimension n^2. The identity element e is the unit $n \times n$ matrix. The tangent space T_e can be identified with R^{n^2} by representing any of its element as a matrix at e. Then, its Lie algebra, denoted by $g\ell(n, R)$, is isomorphic to T_e of R^{n^2} at e. Right translation is defined by

$$a = (a_{ij}) \to ab = (a_{ij})(b_{jk}) = (ab)_{ik}.$$

Let $A \in g\ell(n, R)$ and x^{ij} be the standard coordinates on R^{n^2}. Then,

$$A_e = \sum_{i,j} A_{ij}(\partial_{ij})_e,$$

where $\{\partial_{ij}\}_e$ are n^2 base vectors of T_e. Therefore,

$$L_{ji} = f_L^{-1}(\partial_{ij})_e,$$

where $f_L : g\ell(n, R) \to T_e$, $A \to A_e$ provides the basis $\{L_{ji}\}$ of $g\ell(n, R)$ with any member

$$A = A_{ij}L_{ji},$$

and for every $A, B \in g\ell(n, R)$, we have

$$\begin{aligned} [A, B] &= AB - BA \\ &= (A_{ik}B_{kj} - B_{ik}A_{kj})L_{ji}. \end{aligned}$$

Thus, the Lie-bracket is closed. Note that the treatment is similar for the Lie algebra of $GL(n, C)$, using example 2. The reader is invited to find the Lie algebra structures of $O(n)$, $SO(n)$, $U(n)$ and $SU(n)$.

3.3 Transformation Groups

Let G and M be finite dimensional connected Lie group and smooth connected manifold respectively. A **local G-transformation group** acting on M is a smooth mapping

$$\phi : D \to M, \qquad D \qquad \text{an open subset of} \qquad G \times M, \qquad \text{such that}$$

1. $D_p = \{a \in G : (a,p) \in D\}, \forall p \in M$, is a connected open neighborhood of the identity e of G.

2. $\phi(e,p) = p$, $\forall p \in M$.

3. If $(b,p), (a, \phi(a,p))$ and (ab,p) are all in D then $\phi(ab,p) = \phi(a, \phi(b,p))$.

ϕ is called a **global G-transformation group** (or **Lie transformation group**) acting on M if $D = G \times M$. Let $\mathcal{G}$ denote the Lie algebra of left invariant vector fields (treatment is identical for the right invariant case) on G and $\mathcal{X}(M)$, the Lie algebra of smooth vector fields on M with respect to the usual Lie-bracket operation. Define the following mappings:

$$\phi^p : D_p \to M \qquad \text{by} \qquad \phi^p(a) = \phi(a,p) \tag{3.12}$$

and

$$\phi^+ : \mathcal{G} \to \mathcal{X}(\mathcal{M}) \tag{3.13}$$

such that $(\phi^+(X))_p = \phi^p_\star(X)_e$ for all $p \in M$ and $X \in \mathcal{G}$. Here the map $\phi_\star$ denotes the differential (tangent) map of ϕ. The map ϕ^+ is called the **infinitesimal generator** of ϕ. Based on above we state the following fundamental theorem that connects the local and infinitesimal G-transformation groups on M.

THEOREM 3.1 (Palais [159]). *The infinitesimal generator ϕ^+ of a local G-transformation group ϕ on M is an infinitesimal G-transformation group on M. Conversely, any infinitesimal G-transformation group on M is the infinitesimal generator of some local G-transformation group on M.*

Thus, loosely speaking, finite dimensional sub algebras of the Lie algebra of vector fields on M arise as a result of local G-transformation group actions on M and conversely. For example, the local R-transformation groups on M are just the local 1-parameter groups of local transformations.

In the sequel, we set $\phi(a,p) = \phi_a(p) = ap$ such that G acts differentially on M to the left, that is, $(ab)p = a(bp)$ for all $a, b \in G$. A Lie transformation group G acts **transitively to the left** (treatment is identical for the right transitivity) if, for every $p, q \in M$, there exists an element $a \in G$ such that $\phi_a(p) = q$. A manifold M is said to be a **homogeneous space** of the Lie group G if there is a transitive smooth action of G on M. There are several examples (see the end of this section) of homogeneous spaces. The map ϕ_a gives rise to a transformation $\phi_a : G \to G$ under the bijection $a \to \phi_a$, for every $a \in G$. Thus, the **infinitesimal transformation group** of G is generated by the associated map ϕ^+ as defined by (3.13). In particular, if $\phi_a(p) = p$ for every $p \in M$ implies that $a = e$, then we say that G acts **effectively** on M and for this case ϕ^+ is a Lie algebra isomorphism (whereas, in general, it is a Lie algebra homeomorphism). Thus, we state the following theorem:

THEOREM 3.2 (Palais [159]) *The infinitesimal group of a Lie transformation group G acting on M is a finite dimensional Lie algebra of complete vector fields on M (and is isomorphic to $\mathcal{G}$ if the action is effective). Conversely, a non-trivial finite dimensional Lie algebra of complete vector fields on M is the infinitesimal group of a unique connected Lie transformation group G on M.*

Let G be a group of transformations of M. Then, G is called a Lie group of transformations of M if G is a Lie group such that $\phi : G \times M \to M$ is smooth and if whenever X is a complete vector field on M such that the corresponding global R-transformation group ϕ has the property that the set of transformations $\{\phi_t : t \in R\}$ of M is a subset of G (that is X tangent to G), then this set $\{\phi_t\}$ is actually a 1-parameter subgroup of G.

THEOREM 3.3 (Palais [159]) *Let G be a group of transformations of M and A the set of complete vector fields on M which are tangent to G. Then, G is a Lie group of transformations of M iff A is a non-trivial finite dimensional Lie algebra.*

Based on above theorem, the Lie algebra A coincides with the infinitesimal group of G when G acts effectively on M as a Lie transformation group. For a Lie transformation group G on M an equivalence relation, $\sim$, can be defined by $p \sim q$ ($p, q \in M$) if there is an $a \in G$ such that $\phi_a(p) = q$, that is, G acts transitively to the left (also holds for right) on M. The equivalence class containing a point p of M is called the **orbit** of p. For a fixed p, the set $I_p = \{a \in G : \phi_a(p) = p\}$ is a closed subgroup of G, called the **isotropic group** at p. I_p is either **discrete** (which means that its each element has a neighborhood which contains no other element of I_p) or a Lie subgroup of G and if $p \sim q$ then either I_p and I_q are discrete or they are isomorphic. The map G/I_p (the right coset of I_p) to M, defined by $I_p a = pa$ is smooth and one-to-one onto. Moreover, for G compact this map is a homeomorphism. G acts freely on M if ϕ_e is the only transformation that leaves any point p of M fixed, that is, if $\phi_a(p) = p$ implies that $a = e$. For this particular case, the elements of the Lie algebra homomorphism ϕ^+ (see equation (3.13)) are non vanishing vector fields on M.

EXAMPLE 1. A simple but physically important example is the action of $GL(n, R)$ on R^n defined by the map $\phi : G \times R^n \to R^n$ such that $\phi(A, x) = Ax$ (multiplication of the $n \times n$ matrix A by the $n \times 1$ column vector $x \in R^n$). G acts differentially to the right by $\phi(AB, x) = \phi(A, \phi(B, x))$ since $AB(x) = A(Bx)$ holds due to associative property of matrix products.

EXAMPLE 2. Consider the action of G on $R^n - \{0\}$ with the map ϕ as defined in example 1. This action is also transitive. Let $p = (1, 0, \ldots, 0)$ be a fixed point of R^n. Then, this isotropic group of p is the set of matrices of the form

$$\begin{pmatrix} I & A \\ O & B \end{pmatrix}, \qquad B \in GL(n-1, R), \qquad A \in R^{n-1}$$

where O is a column of $n-1$ zeroes. The multiplication is given by

$$(B, A)(B', A') = (BB', AB' + A').$$

Here (B', A') belongs to the **semi-direct product** of $GL(n-1, R)$ and R^{n-1}. Observe that $R^n - \{0\}$ is homogeneous space since the action of G on it is transitive.

EXAMPLE 3. Let $H \subset GL(2, R)$ be the subgroup of all matrices of the form $A = \begin{pmatrix} a & b \\ o & 1 \end{pmatrix}$, where $a > o$ and b are real numbers. It is left as an exercise to show that H is a Lie group and acts on R by

$$\phi(A, x) = ax + b.$$

EXAMPLE 4. Consider $O(n) \subset GL(n, R)$. This is left as an exercise to show that the action of $O(n)$ on R^n is smooth and the orbits correspond to concentric spheres having one-to-one correspondence with real numbers $r \geq 0$ defined by the mapping of each sphere to its radius.

EXAMPLE 5. Let $M = S^2$, the unit sphere in R^3 defined by the set $\{x \in R^3 : |x| = 1\}$ and $G = Z_2 = \{a, a^2 = e\}$, the discrete cyclic group of order 2. Then, $a(x) = -x$ and $e(x) = x$ defines an action of Z_2 on S^2. It is left as an exercise to show that the action $\phi : Z_2 \times S^2 \to S^2$ is free and that the quotient space S^2/Z_2 is the real projective 2-space.

EXAMPLE 6. The reader is invited to show that $O(n)$ acts transitively on S^{n-1} and, therefore, S^{n-1} is homogeneous space of the Lie group $O(n)$.

EXAMPLE 7. Consider the set $I_p = \{a \in G : \phi_a(p) = p\}$, the closed subgroup (called isotropy group of a fixed point p of M) and the map: G/I_p (the left coset of I_p) to M defined by $aI_p \to ap$. It is left as an exercise to show that G acts transitively on G/I_p by $a(bI_p) = (ab)I_p$ and, therefore, G/I_p is a homogeneous space of G, with $dim(G/I_p) = dim(G) - dim(I_p)$.

3.4 Isometric and Conformal Symmetries

Given a smooth manifold M, the group of all smooth transformations of M is a very large group. This leads to the study of those transformations of M which leave a certain physical/geometric quantity invariant. To study one such class of invariant transformations, we let (M, g) be a real n-dimensional smooth semi-Riemannian manifold. Recall that a diffeomorphism $\phi : M \to M$ is an **isometry** of M if it leaves **invariant** the metric tensor g. This means that

$$g(\phi_\star X, \phi_\star Y) = g(X, Y), \qquad \forall X, Y \in \mathcal{X}(M), \tag{3.14}$$

where $\phi_\star$ denotes the differential (tangent) map of ϕ. Since each tangent mapping $(\phi_\star)_p$, at $p \in M$, is a linear isomorphism of $T_p(M)$ on $T_{\phi(p)}(M)$, it follows that ϕ is an isometry iff $(\phi_\star)_p$ is a linear isometry for any $p \in M$. The set of all isometries of M forms a group under composition of mappings. Myers and Steenrod [147] proved that the group of all isometries of a Riemannian manifold (with positive definite g) is a Lie group. For analogous results on semi-Riemannian manifolds see O'Neill [157, chapter 9]. In relativity, groups of isometries are known as **isometric symmetry groups**. In general, **symmetry groups** arise as groups of transformations of M or local G-transformation groups acting on M which have some special property respect to a geometric object on M. To see how isometric symmetry is related to a local G-transformation group we proceed as follows:

Let V be a smooth vector field on M and $\mathcal{U}$ a neighborhood of each $p \in M$ with local coordinate system (x^i). Let the integral curves of V, through any point q in $\mathcal{U}$, be defined on an open interval $(-\epsilon, \epsilon)$ for $\epsilon > 0$. For each $t \in (-\epsilon, \epsilon)$ we define a map ϕ_t on $\mathcal{U}$ such that for q in $\mathcal{U}$, $\phi_t(q)$ is that point with parameter value t on the integral curve of V through q. If ϕ_t is an isometry then V is called a **Killing vector field**. Since $\phi_t(x^i) = x^i + tV^i$ and ϕ_t being isometric map we have

$$\partial_k(x^i + tV^i)\,\partial_m(x^j + tV^j)\,g_{ij}(x + tV) = g_{km}$$

which, after a simple calculation by expanding $g_{ij}\,(x + tV)$ up to first order in t, yields

$$V^i\,\partial_i\,g_{jk} + \partial_j\,(V^i)\,g_{ik} + \partial_k(V^i)g_{ji} = 0$$

Using the Lie derivative operator L_V, the above equation can be rewritten as

$$L_V g_{ij} = 0, \qquad i.e., \qquad L_V g = 0. \tag{3.15}$$

In terms of the Levi-Civita connection ∇ on M, (3.15) reduces to

$$\nabla_i V_j + \nabla_j V_i = 0 \tag{3.16}$$

where $V_i = g_{ij}V^j$ is the associated 1-form of V. The above partial differential equations (3.16) are well-known as **Killing equations** (named after the geometer W. Killing). A physical example is the isometry of a spacetime such that g is invariant in time, that is, $L_T g = 0$, where we take a time coordinate t for which $T = \partial_t$. Such spaces are called **static spacetimes**. The Killing equations show that if V is a Killing vector field then the local geometry remains invariant while moving along the local 1-parameter group of local transformations ϕ_t generated by V. It follows easily that the set of all Killing vector fields of M forms a sub algebra of the Lie algebra of all vector fields on M. Indeed, a linear combination $aV + bW$ (a, b real constants) of any two Killing vector fields V and W is obviously Killing and the Lie-bracket $[V, W]$ is also Killing since $L_{[V,W]}g = L_V L_W g - L_W L_V g = 0$. A set of Killing vector fields is said to be dependent if one of them is a linear combination of others with constant coefficients. Thus, there may be more independent Killing vector fields than the dimension of the manifold M. To illustrate this fact we consider the following examples.

EXAMPLE 1. Let $V = (V_1, V_2)$ be a Killing vector field on a 2-dimensional Euclidean space R^2. Since R^2 is flat, the Killing equations (3.16) reduce to $\partial_1 V_1 = 0$, $\partial_2 V_2 = 0$ and $\partial_1 V_2 + \partial_2 V_1 = 0$ and, therefore, $V_1 = -c_1 y + c_2$ and $V_2 = c_1 x + c_1$. Consider the transformation

$$x \to x - tV_1, \qquad y \to y + tV_2$$

which obviously has 2 independent translations and one rotation (cf. $x \to x\cos\theta - y\sin\theta$ and $y \to x\sin\theta + y\cos\theta$, $\theta = tc_1$ for every small t). Thus, R^2 admits 3 independent Killing vectors.

EXAMPLE 2. Let $V = (V_0, V_1, V_2, V_3)$ be a Killing vector on a Minkowski space R^4_1. Since R^4_1 is flat, the Killing equations (3.16) reduce to

$$\partial_i V_j + \partial_j V_i = 0 \qquad (i, j = 0, 1, 2, 3).$$

It is easy to see that each V_i is at most of the first order in terms of some local coordinates, say x^i. The constant solutions $V_{(i)j} = \delta_{ij}$ correspond to spacetime translations. Let $V_i = a_{ij}\, x^j$ (a_{ij} being constants), then it follows from the above Killing equations that $a_{ij} = a_{ji}$. Hence there are $C^4_2 = 6$ independent solutions (Killing vectors) of this form, 3 of which are solutions corresponding to 3 spatial rotations about the x^a-axis given by

$$V_{(a)0} = 0 \qquad \textit{and} \qquad V_{(a)b} = \epsilon_{abc}\, x^c \qquad (a, b, c = 1, 2, 3)$$

and the other 3 correspond to 3 Lorentz boosts along x^a-axis, given by

$$V_{(a)0} = x^a \qquad \textit{and} \qquad V_{(a)b} = -\delta_{ab}\, x^0.$$

In general, an n-dimensional Minkowski spacetime has $\frac{n(n+1)}{2}$ independent Killing vectors, n of which generate translations, $(n-1)$ generate boosts and $\frac{(n-1)(n-2)}{2}$ generate space rotations. Any semi-Riemannian manifold which admits maximal $(\frac{n(n+1)}{2})$ Killing vector fields is called a **maximally symmetric manifold** (also called **manifold of constant curvature**).

Let G denote some abstract group isomorphic to the group of isometries of M, having r independent Killing vectors V_a $(a = 1, \ldots, r)$. Then, the Lie algebra structure equations (3.11) can be replaced by

$$[V_a,\, V_b] = C^d_{ab}\, V_d. \tag{3.17}$$

We say that a manifold M is **invariant** under the group G if there are r $(dim(G) = r)$ independent Killing vectors V_a satisfying (3.17). This is all we need to relate the isometric symmetry group with the transformation groups as presented in previous section 3.3. To help understand such a relation we present the following simple example:

EXAMPLE 3. Consider the 2-sphere S^2 with the standard metric $ds^2 = d\theta \otimes d\theta + \sin^2\theta\, d\phi \otimes d\phi$ for a local coordinate system $(\theta, \phi) = (x^1, x^2)$. Then, the only surviving Christoffel symbols are

$$\Gamma^1_{22} = -\sin\theta\cos\theta \qquad \textit{and} \qquad \Gamma^2_{12} = \cot\theta.$$

Thus, the Killing equations (3.16) expand into the following equations:

$$
\begin{aligned}
\partial_1 V_1 &= 0, && (3.18)\\
\partial_2 V_2 + \sin\theta\cos\theta V_1 &= 0, && (3.19)\\
\partial_1 V_2 + \partial_2 V_1 - 2\cot\theta V_2 &= 0, && (3.20)
\end{aligned}
$$

where the indices 1 and 2 stand for θ and ϕ respectively. Equation (3.18) gives $V_1 = f(\phi)$ which substituted in (3.19) provides $V_2 = -F(\phi)\sin\theta\cos\theta + g(\theta)$, where $F(\phi) = \int f(\phi)d\phi$. Now substituting these in (3.20) and separating the variables we obtain

$$
\begin{aligned}
\frac{dg}{d\theta} - 2\cot\theta g(\theta) &= c,\\
\frac{df}{d\phi} + F(\phi) &= c,
\end{aligned}
$$

where c is a real constant. Separating the variables while solving above two equations, we get $f = A\sin\phi + B\cos\phi$. Thus,

$$
\begin{aligned}
V_1 &= A\sin\phi + B\cos\phi,\\
V_2 &= (A\cos\phi - B\sin\phi)\sin\theta\cos\theta + C\sin^2\theta
\end{aligned}
$$

for some constants A,B, and C. Hence,

$$
\begin{aligned}
V &= V^1\partial_1 + V^2\partial_2\\
&= (A\sin\phi + B\cos\phi)\partial_1 + (C + A\cos\phi\cot\theta - B\sin\phi\cot\theta)\partial_2,
\end{aligned}
$$

whence it follows from the standard metric of S^2 that $V_1 = V^1$ and $V_2 = \sin^2\theta\, V^2$. Thus the basis vectors of the Lie algebra consists of three Killing vectors, namely

$$
\begin{aligned}
K_1 &= -\cos\phi\partial_1 + \cot\theta\sin\phi\partial_2,\\
K_2 &= \sin\phi\partial_1 + \cot\theta\cos\phi\partial_2,\\
K_3 &= \partial_2.
\end{aligned}
$$

Since these three vectors K_1, K_2 and K_3 generate rotations about the three axis x, y, and z of R^3, it follows that they generate the Lie algebra of the special orthogonal group $SO(3)$. This shows that S^2 is the homogeneous space $SO(3)/SO(2)$ and has 3 independent Killing vectors. In general, it is left as an exercise to prove that $S^n = SO(n+1)/SO(n)$ has $\frac{n(n+1)}{2}$ independent Killing vector fields and it is a homogeneous space.

Physically, following three classes of spacetimes have been extensively studied in general relativity, with respect to symmetry groups.

(A) A 4-dimensional spacetime manifold M with transitive symmetry group G such that $dim(G) = 4$. M is then called **homogeneous in space and time (ST-homogeneous)**. The metric of this class is the same at all points of M. Such a model is known as a **cosmological model**.

(B) Same as model (A) with $dim(G) = 3$. Thus, G generates 3-dimensional **invariant hypersurfaces**. M is then called **spacially homogeneous**. Note that some hypersurfaces may not be spacelike , but their 1-parameter family do form the submanifold structure. Thus, the metric depends on only one variable, which is independent of each invariant hypersurface.

(C) A particular case of the model (B) for which although G generates 3-dimensional invariant hypersurfaces but $dim(G) > 3$. This model is also called spacially homogeneous model.

Details on above models (with some examples) may be seen , for example, in Ryan-Shepley [166, chapter 6].

As a generalization of isometric symmetry, we now present a brief account on **conformal symmetry** which leaves the metric tensor g conformally invariant, under a given point transformation of a smooth semi-Riemannian manifold (M, g). This means that a map $\phi : M \to M$ is a conformal symmetry if

$$g(\phi_\star X, \phi_\star Y) = e^{2f} g(X, Y), \qquad \forall X, Y \in \mathcal{X}(M) \tag{3.21}$$

where $\phi_\star$ is the differential (tangent) map of ϕ and f is a scalar function on M. The set of all conformal maps, satisfying (3.21), form a group of conformal motions under composition of mappings. Proceeding as in the case of isometries, let V be a smooth vector field on M and $\mathcal{U}$ denote a neighborhood of each $p \in M$ with local coordinate system. Let the integral curve of V, through any point q in $\mathcal{U}$, be defined on an open interval $(-\epsilon, \epsilon)$ for $\epsilon > 0$. For each t in this interval we define a map ϕ_t on $\mathcal{U}$ such that for q in $\mathcal{U}$, $\phi_t(q)$ is that point with parameter value t on the integral curve of V through q. Then, V generates a local 1-parameter group of local transformations $\phi_t : x^i \to x^i + tV^i$. If ϕ_t satisfies the conformal equation (3.21) then we say that V is a **conformal vector field** (briefly denoted by **CKV**). In local coordinates, V conformal implies that

$$\partial_k \left(x^i + tV^i\right) \partial_m \left(x^j + tV^j\right) g_{ij}(x + tV) = e^{2f} g_{km}(x).$$

After expanding $g_{ij}(x + tV)$ up to first order in t, and then using the Lie derivative operator L_V, we obtain

$$t(L_V g_{ij}) = (e^{2f} - 1) g_{ij}.$$

As t is small, so is f. Contracting above equation with g^{ij} gives $2t\, div\, V = 2\, f\, n$. Set $f = t\, \sigma$ and then expanding $e^{2t\sigma}$ up to first order in t, we get

$$L_V g_{ij} = \nabla_i V_j + \nabla_j V_i = 2\sigma g_{ij}. \tag{3.22}$$

The above partial differential equations (3.22) are well-known as **conformal Killing equations**, which reduce to the Killing equations (discussed earlier) whenever $\sigma = 0$. In the sequel, we call V a **proper CKV** if σ is non-constant. The conformal Killing equations show that if V is a **CKV** field then the local geometry is conformally invariant while moving along the local 1-parameter group of local transformations ϕ_t generated by V. In particular V is **homothetic** or Killing according as σ is a no-zero constant or zero. As seen before we get

$$\sigma = \frac{1}{n} \nabla_i V^i = \frac{1}{n} (\partial_i V^i + V^j \partial_j \ln \sqrt{|g|}), \qquad g = det(g_{ij}).$$

Suppose V and W are **CKV** fields on M such that $L_V g = 2\,\sigma_1\, g$ and $L_W g = 2\,\sigma_2\, g$. Then, obviously $aV + bW$ is also a **CKV** for any constants a and b. Consider

$$\begin{aligned} L_{[V,W]}g &= L_V L_W g - L_W L_V g = L_V(2\sigma_2 g) - L_W(2\sigma_1 g) \\ &= 2(V\sigma_2)g + 4\sigma_1\sigma_2\, g - 2(W\sigma_1)g - 4\sigma_1\sigma_2 g \\ &= 2(V\sigma_2 - W\sigma_1)g. \end{aligned}$$

Thus it follows from above that the set of all proper **CKV** fields form a finite dimensional sub algebra of the Lie algebra of all the vector fields on M. We have restricted the case to proper **CKV** since it is noteworthy that if V and W are homothetic, then $V\sigma_1 = 0 = W\sigma_2$ and, therefore, [V,W] is a Killing vector field.

CONCLUSION. *The sets of all proper* **CKV** *fields and all Killing vector fields on M form a finite dimensional Lie algebra, known as* **conformal algebra** *and* **Killing algebra** *respectively.* **Homothetic vector fields** *do form a group but not the* **Lie algebra structure**.

A simple example is the 2-sphere with metric $g = d\theta \otimes d\theta + \sin^2\theta d\phi \otimes d\phi$ for a local coordinate system (θ, ϕ). It is easy to see that $V = \sin\theta\partial_\theta$ is a **CKV** since $L_V g = 2(\cos\theta)g$. Another example is R^n with the dilation vector $V = X^i\partial_i$ with respect to a natural coordinate basis $\{\partial_i\}$. However, for this case, $L_V\delta_{ij} = \partial_i X_j + \partial_j X_i = 2\delta_{ij}$ and V is homothetic.

The following decomposition of the tensor $\nabla_i V_j$ in the equation (3.16) has been very useful in the study of **CKV**'s. Set

$$\nabla_i V_j = \sigma g_{ij} + F_{ij} \tag{3.23}$$

$$\nabla_k F_{ij} = R_{ijkm}V^m - 2g_{k[j}\nabla_{i]}\sigma \tag{3.24}$$

$$\nabla_i\nabla_j\sigma = -(n-2)^{-1}\{V^k\nabla_k L_{ij} + 2\sigma L_{ij} + 2R_{k(j}F^k_{i)}\} \tag{3.25}$$

$$L_{ij} = R_{ij} - \frac{r}{2(n-1)}g_{ij}$$

where F_{ij} is called the **conformal Killing bivector** of V. On the differentiability of V, observe that the above three equations show that the third and higher derivatives of V are all determined by V and its first and second order derivatives. In other words, they may be regarded as a system of first order differential equations for V, F, σ and $\nabla_i\sigma$. Hence, the smoothness of V is guaranteed. Also, any two global **CKV** fields are equal everywhere on M if their corresponding quantities V, F, σ and $\nabla_i\sigma$ agree at any point of M. Thus, the maximum dimension of the conformal algebra is equal to

$$n(V) + \frac{1}{2}n(n-1)(F) + 1(\sigma) + n(\nabla_i\sigma) = \frac{1}{2}(n+1)(n+2),$$

where $n(V)$ etc. means dimension n due to V etc. It is well-known (see Yano [211]) that this maximum dimension occurs if the manifold (M, g) is conformally flat. As an example, we refer to Maartens-Maharaj [131] who found the conformal algebra of the full G_{15} (15-parameter group) of conformal motions in **Robertson-Walker**

spacetime which is conformally flat. Reader will find more information on the Robertson-Walker spacetime models in chapter 4.

Related to above are semi-Riemannian manifolds (M, g) of constant curvature (see equation(2.35)). Eisenhart [66] has proved that in any n-dimensional M of constant curvature, there exists a coordinate system $\{x^i\}$ such that the distance element of g becomes

$$ds^2 = (a_1 + \ldots + a_n)^{-2} \sum_{i=1}^{n} \epsilon_i (dx^i)^2, \tag{3.26}$$

where each $\epsilon_i = \pm 1$, according to the signature of M, and

$$a_i = \epsilon_i (a(x^i)^2 + 2b_i x^i + c_i)$$

with no summation over i and the constants a_i, b_i and c_i satisfy

$$4 \sum_i (ac_i - b_i^2) = c(\text{constant curvature}).$$

It is clear from (3.26) that every M, of constant curvature, is conformal to a flat space, and, hence, belongs to a class of semi-Riemannian manifolds admitting a group of conformal motions of the maximal order $\frac{1}{2}(n+1)(n+2)$.

A particular case of a **CKV** is called a **special CKV** (denoted by **SCKV**) if $\nabla_i \nabla_j \sigma = 0$. The set of all **SCKV**'s also form a sub algebra of the conformal algebra. Since for this case $\nabla_j \sigma$ is covariant constant, this quantity will not effect the dimension of its sub algebra. Therefore, the maximum dimension of **SCKV** sub algebra is

$$\frac{1}{2}(n+1)(n+2) - n = \frac{n(n+1)}{2} + 1.$$

3.5 Affine, Projective and Curvature Collineations

In this section we assume that M is smooth, simply connected and its curvature is non-zero over any non-empty open set of M, that is M is non-flat. Let V and ∇ be a global vector field and a symmetric affine connection on M respectively. V is said to define a symmetry called **affine collineation** iff V leaves ∇ invariant, i.e.,

$$L_V \nabla = 0, \qquad i.e., \qquad L_V \Gamma^i_{jk} = 0. \tag{3.27}$$

This is equivalent to

$$\nabla_j \nabla_k V^i + R^i_{kmj} V^m = 0. \tag{3.28}$$

The vector V is then called an **affine Killing vector** (or an **affine vector**) field. Following the previous sections we say that V gives rise to a 1-parameter group of local transformations ϕ_t of M. It follows from (3.27) that for an affine collineation

ϕ_t preserves the geodesics (and also affine parameters on the geodesics) of M.

In general one can always split the tensor $\nabla_i V_j$ into its symmetric and anti-symmetric parts as follows:

$$\nabla_i V_j = K_{ij} + F_{ij} \quad , \quad (K_{ij} = K_{ji}, F_{ij} = -F_{ji}). \tag{3.29}$$

Then, it follows that

$$\nabla_k K_{ji} = 0, \tag{3.30}$$
$$\nabla_k F_{ji} = R_{ijkm} V^m. \tag{3.31}$$

Suppose (M, g) is a semi-Riemannian manifold and ∇ is the Levi-Civita (metric) connection of g. Then, for an affine vector field V, it is easy to see that

$$(L_V R)(X,Y,Z) = 0, \qquad i.e., \qquad L_V R^m{}_{ijk} = 0 \Rightarrow L_V R_{ij} = 0. \tag{3.32}$$

Also, we deduce from (3.29) and (3.30) that V is affine iff

$$L_V g_{ij} = 2\,K_{ij}, \tag{3.33}$$

where K_{ij} is a **covariant constant symmetric tensor field** on M. In particular, if $K_{ij} = a g_{ij}$, for some constant a, then it follows from (3.22) that V is homothetic or Killing according as $a \neq 0$ or $a = 0$. Note that due to the condition (3.30) a can not be non-constant. Thus, an affine vector field can never be a proper **CKV**.

Following exactly as presented in section 3.4, one can show that the set of all proper affine vector fields is a finite dimensional sub algebra of the Lie algebra of all the vector fields on an n-dimensional (M, g). To find the maximum dimension of this sub algebra we first notice that the equations (3.29), (3.30) and (3.31) demand that V is C^2 and since g is smooth, it implies that the third and higher derivatives of V all exist. Therfore, smoothness of V is guaranteed. Now regard above referred three equations as a closed system of first order differential equations for V, F_{ij} and K_{ij}. Thus, any two global affine vector fields are equal everywhere on M if their corresponding quantities V, F_{ij} and K_{ij} agree at any point of M. Since K_{ij} is covariant constant, this quantity will not affect the dimension of affine algebra. Thus, the maximum dimension of the affine algebra of a non-flat (M, g) is

$$n(V) + \frac{1}{2}n(n-1)(F) = \frac{n(n+1)}{2}.$$

This maximum dimension occurs when M is of non-zero constant curvature. It is important to mention that if we include the trivial case of flat M into consideration then as stated by Yano [211, page 24], the possible dimension of affine Lie algebra is n^2+n. In chapter 5 the reader will find some examples of affine vector fields.

A generalization of the affine symmetry is called **projective collineation** defined by a vector field V satisfying

$$(L_V \nabla)(X,Y) = p(X)Y + p(Y)X, \qquad \forall X, Y \in \mathcal{X}(M) \tag{3.34}$$

and for some 1-form. In local coordinates, this condition is

$$L_V \Gamma^i_{jk} = \delta^i_j p_k + \delta^i_k p_j, \qquad p_j = \nabla_j p. \tag{3.35}$$

V is then called a **projective collineation (briefly PC) vector field**. Let us write, in any coordinate system, the following general equations

$$\nabla_i V_j = \frac{1}{2} h_{ij} + F_{ij} \qquad (h_{ij} = h_{ji}, F_{ij} = -F_{ji}) \tag{3.36}$$

$$L_V g_{ij} = h_{ij} \tag{3.37}$$

Then, it is easy to see that V is projective iff

$$\nabla_k h_{ij} = 2 g_{ij} p_k + g_{ik} p_j + g_{jk} p_i \tag{3.38}$$

Also the following holds

$$\nabla_k F_{ji} = R_{ijkm} V^m + g_{ik} p_j - g_{jk} p_i. \tag{3.39}$$

As in the case of **SCKV**'s we say that if V is proper **PC** ($p_i \neq 0$) and $\nabla_j \nabla_i p = 0$ on M, then V is called a **special projective collineation [SPC] vector field**. Thus, $\nabla_i p$ is nowhere zero covariant constant vector field on M.

PC is also called a **symmetry of geodesics** since the local 1-parameter group of transformations, generated by V, maps geodesics (auto parallel curves) into geodesics. Also, V is affine iff $p_i = 0$ and for which $h_{ij} = K_{ij}$, the covariant constant symmetric tensor field.

Following exactly as presented in section 3.4, one can show that the set of all **PC** vector fields is a finite dimensional sub algebra of the Lie algebra of all the vector fields on M. To find the maximum dimension of this sub algebra we first notice that the equations (3.38) and (3.39) demand that V is C^3 and since g is smooth, it implies that V is smooth. Now, regard above referred equations and (3.36) as a closed system of first order differential equations for V, h_{ij} and F_{ij} and the fact that h_{ij} is not covariant constant for a proper V, we conclude that the maximum dimension of its **projective algebra** is equal to

$$n(V) + \frac{1}{2} n(n+1)(K_{ij}) + \frac{1}{2} n(n-1)(F_{ij}) + n(p_i) = n^2 + 2n$$

for which M is projectively flat and, therefore, its Weyl projective curvature tensor vanishes.

If V is **SPC** then p_i being covariant constant will not affect the dimension of its special sub algebra. Thus, the maximum dimension of this special projective sub algebra is $n^2 + 2n - n = n^2 + n$.

Finally, we discuss the group structure of another type of symmetry defined by a vector field V that leaves the Riemann curvature tensor invariant, that is,

$$L_V R^i{}_{jkm} = 0. \tag{3.40}$$

Such vector fields are called **curvature collineations**, briefly denoted by **CC**. Obviously, it follows from (3.32) that every affine collineation is a **CC**. However, the converse is not necessarily true. Moreover, every **CC** also satisfies

$$L_V R_{ij} = 0. \tag{3.41}$$

and, then V is called **Ricci collineation vector field**, briefly denoted by **RC**. In this case also, in general, the converse is not true. Using the general equations (3.36) and (3.37), one can prove (see Katzin et al [106]) that h_{ij} satisfies the following identity:

$$(h_{ij;k} + h_{jk;i} - h_{ik;j})_{;m} - (h_{mj;k} + h_{jk;m} - h_{mk;j})_{;i} = 0. \tag{3.42}$$

A **CC** is said to be a **special curvature collineation**, briefly denoted by **SCC**, if

$$h_{im;jk} = 0. \tag{3.43}$$

It is easy to see that if M admits a **SCC**, say V, then it also admits a nowhere covariant constant vector field which is $U_j = V^i{}_{;ij}$.

Relations between **CC** and other symmetries (discussed so far) depend on pre-assigned value of the tensor h_{ij} as follows (see Katzin at el. [106]).

(1) Every isometric and every homothetic symmetry is a **CC**.

(2) Every affine collineation is a **CC**.

(3) If a vector field V is both a **CKV** and a **PC**, then it is necessarily homothetic (see Yano [211, page 167]). Using this result and statement (2), we conclude that V is necessarily a **CC**.

(4) A **PC** is a **CC** iff it is an **SPC**.

(5) Every **CC** and every **RC** in an Einstein space ($r \neq 0, n > 2$) is an isometric symmetry.

(6) In a Ricci flat manifold every **PC** is a **CC**.

Although **CC**'s have been extensively studied since their introduction by Katzin et al. [106] in 1969, a detailed analysis of their group structure is due to a recent work of Hall and da Costa [83, 85], in particular reference to their differences from affine and conformal symmetries. It is important to briefly mention those differences as follows:

We have seen that the sets of Killing, conformal, affine and projective collineation vector fields on M have all of them a finite dimensional Lie algebra structure since they all satisfy a closed system of three partial differential equations and also such a closed system guarantees that the corresponding symmetry vector field is smooth. Unfortunately, (3.42) does not provide such a closed system and, therefore, neither the smoothness nor the finite dimensional Lie group structure of **CC**'s can be guaranteed. If we impose the condition of smoothness, then of course, the set of all smooth **CC**'s on M do form an infinite dimensional Lie group structure. However, for this case one loses those **CC**'s which are not smooth. On the other hand, if we assume that the set of **CC**'s is C^m ($m < \infty$), then one loses the Lie algebra structure. Moreover, contrary to the cases of other symmetries, due to the lack of closed system, a **CC** vector field V may not be determined uniquely even if we specify V and all its covariant derivatives at some point $p \in M$. This also may happen even if we assume that V is smooth.

These properties of **CC**'s and much more detailed information on their existence, within the scope of general relativity, and on semi-Riemannian manifolds has been discussed in chapter 7.

REMARKS. First, the **CC**'s are quite different than the affine and the conformal symmetries. Second, the set of homothetic vector fields do not form a Lie group structure. Third, as the subject matter develops, the reader will find some more higher symmetries which do not have Lie group structure but their properties have been useful in dealing with specific problems in mathematics and physics.

Finally, we quote the following comment of Daniel Rockmore[1] on a recent book, **The Universe and the Teacup: The Mathematics of Truth and Beauty** by K. C. Cole, 1998, ISBN 0-151-00323-8.

"The connections between symmetry and beauty are a well-trodden area, with Hermann Weyl's Symmetry the classical reference. Ms. Cole sees invariance and symmetry as a way to get from truth to beauty, adding that deep truths can be defined as invariants - things that do not change no matter what; how invariants are defined by symmetries, which in turn define which properties of nature are conserved, no matter what. These are the selfsame symmetries that appeal to the senses in art and music and natural forms like snowflakes and galaxies. The fundamental truths are based on symmetry, and there's a deep kind of beauty in that."

[1]Book Review, Notices of the AMS, March 1999, Vol. 40, No. 3, page 351.

Chapter 4

Spacetimes of General Relativity

In this chapter we discuss some of the physically important spacetimes of general relativity. We start with basic information on the kinematic quantities with respect to timelike, spacelike and null congruence. This is followed by Einstein's field equations, energy conditions for physically meaningful exact solutions, spacetimes of constant curvature, spatially homogeneous cosmological models, asymptotically flat spaces and plane wave solutions. We also provide latest information on a large class of globally hyperbolic spacetimes.

4.1 Spacetimes and Kinematic Quantities

Let (M, g) denote a 4-dimensional spacetime manifold of **general relativity**. This means that M is a smooth (C^∞) connected Hausdorff 4-dimensional manifold and g is a time orientable Lorentz metric of normal hyperbolic signature $(-+++)$. The continuity of M has been observed experimentally for distances down to 10^{-15} cms and, therefore, should be sufficient for the general theory of relativity unless the density becomes to about $10^{58} gms/cm^3$. In this book we assume that the density is sufficient to maintain the continuity of the spacetime under investigation.

The set of all **integral curves** given by a unit timelike (spacelike or null) vector field u is called the **congruence of timelike (spacelike or null) curves**. We first consider timelike curves, also called **flow lines**. The **acceleration of the flow lines** along u is given by $\nabla_u u$ or $u^a{}_{;b} u^b$ where ∇ is the Levi-Civita connection on M and $(0 \leq a, b \leq 3)$. The **projective tensor**, defined by

$$h_{ab} = g_{ab} + u_a u_b \tag{4.1}$$

is used to project a tangent vector at a point p in the spacetime into a spacelike vector orthogonal to u at p. The rate of change of the separation of flow lines from a timelike curve, say C, tangent to u is given by the **expansion tensor**

$$\theta_{ab} = h^c_a h^d_b u_{(c\,;\,d)}. \tag{4.2}$$

The **volume expansion** θ, the **shear tensor** σ_{ab} , the **vorticity tensor** ω_{ab} and the **vorticity vector** ω^a are defines as follows:

$$\begin{aligned}
\theta &= div\, u = \theta_{ab}h^{ab} && (4.3)\\
\sigma_{ab} &= \theta_{ab} - \frac{\theta}{3}h_{ab} && (4.4)\\
\omega_{ab} &= h_a^c h_b^d u_{[c;d]} && (4.5)\\
\omega^a &= \frac{1}{2}\eta^{abcd}u_b\omega_{cd} && (4.6)\\
\eta^{abcd} &= g^{ae}g^{bf}g^{cg}g^{dh}\,\eta_{efgh}\\
\eta_{efgh} &= (4!)\sqrt{-g}\,\delta^0_{[e}\delta^1_f\delta^2_g\delta^3_{h]},
\end{aligned}$$

where η^{abcd} is the Levi-Civita volume-form. The equation (4.5) measures the rate at which the timelike curves rotate about an integral curve of u. The covariant derivative of u can be decomposed as follows:

$$u_{a;b} = \omega_{ab} + \sigma_{ab} + \frac{\theta}{3}h_{ab} - u_b(u^c_{;a}u_c). \qquad (4.7)$$

The rate at which the expansion θ changes along the flow lines of u is given by the following **Raychaudhuri equation**:

$$u\theta = \frac{d\theta}{ds} = -R_{ab}u^a u^b + 2\omega^2 - 2\sigma^2 - \frac{1}{3}\theta^2 + div(\nabla_u u), \qquad (4.8)$$

where $\omega^2 = \frac{1}{2}\omega_{ab}\omega^{ab}$ and $\sigma^2 = \frac{1}{2}\sigma_{ab}\sigma^{ab}$ are both non-negative and s is a parameter of an integral curve of u.

Next we consider the **congruence of spacelike curves**, which was established by Greenberg [80]. A set of spacelike curves defined by

$$x^a = x^a(\eta^\alpha, \tau)$$

forms a congruence of spacelike curves, where the three parameters η^α $(\alpha = 1, 2, 3)$ identify the particular curve of spacelike congruence and τ is some arc length parameter along each curve measured from some initial cross section of the congruence. The unit tangent vector to a curve C, at any given point p of C, is defined by

$$n^a = (\frac{dx^a}{d\tau})_p, \qquad n_a n^a = +1.$$

Since we assume that n forms a vector field, we may write

$$n_a n^a_{;b} = 0$$

with respect to the Levi-Civita connection ∇ on M. In order to observe the given congruence of spacelike curves in the neighborhood of a point p, we introduce an observer at that point p with a unit timelike velocity vector w^a $(w_a w^a = -1)$ orthogonal to n^a at p (i.e., $w_a n^a = 0$). It is easy to see that w^a orthogonal to n^a is not a unique timelike unit vector. Indeed, the vector $u_a = w_a + v_a$ is another unit vector orthogonal to n^a at p if $v_a n^a = 0$ and $v_a v^a = -2w_a v^a$. The projection tensor is defined by

$$p_{ab} = g_{ab} + w_a w_b - n_a n_b \tag{4.9}$$

such that $p_{ab} = p_{ba}$, $p^a_b\, p^b_c = p^a_c$, $p^a_a = 2$, $p_{ab}\, w^b = p_{ab}\, n^b = 0$. Consider another curve $C^\star$ (with the same value of the parameter τ) defined by

$$\delta x^a = \frac{\delta x^a}{\delta \eta^\alpha}\, \delta \eta^\alpha.$$

It is easy to show that $\int_n \delta x^a = 0$ (see Greenberg [80]). Now we can obtain a connecting vector which is orthogonal to w^a and n^a at p, by using the projection tensor (4.9), in the following way. Define

$$\delta_\star x^a = p^a_b\, \delta x^b$$

another connecting vector. Then, using the properties of p_{ab} we get

$$\delta_\star x^a\, w_a = \delta_\star x^a\, n_a = 0.$$

Thus, one can decompose any connecting vector δx^a into a linear combination of a spacelike vector orthogonal to n^a, a timelike vector parallel to w^a and a spacelike vector parallel to n^a as follows:

$$\delta x^a = \delta_\star x^a - (\delta x^b w_b) w^a + (\delta x^b n_b) n^a. \tag{4.10}$$

Let $\delta_\star v^a$ be a quantity which expresses the rate of change of separation of any two spacelike curves, as seen by an observer moving with a 4-velocity vector w^a. Then

$$\delta_\star v^a = p^a_b \frac{D}{d\tau}(\delta_\star x^b)$$

such that $\delta_\star v^a$ is perpendicular to both w^a and n^a. Here D stands for the operator of **intrinsic derivative** (see Willmore [200]). Using the fact that $\int_n \delta x^a = 0$, the projection tensor, $\frac{Dn^b}{d\tau} = n^b{}_{;c}\, n^c$ and (4.10), we obtain

$$\delta_\star v^a = p^a_c\, p^c_d\, n^d{}_{;c} + B^{ad}\, \delta x_d,$$

where

$$B^{ad} = -p^a_b (n^b{}_{;c}\, w^c - \frac{Dw^b}{d\tau})\, w^d.$$

Greenberg [80] defined the following transport law given by

$$\frac{Dw^a}{d\tau} = n^a_{;c}\, w^c + w^a n_{b;c} w^b\, w^c - n^a \frac{Dn_b}{d\tau}\, w^b \tag{4.11}$$

according to which B^{ab} vanishes, which is necessary (as we will soon see) if the rotation and the shear tensors of the curves of the congruence lie in 2-space of the screen erected orthogonal to the spacelike curves at p. Define the **expansion tensor** by

$$A_{ab} = p^d_a\, p^c_b\, n_{d;c}. \tag{4.12}$$

Then using the transport law (4.11) we conclude that

$$\delta_\star v_a = A_{ab}\,\delta x^b = A_{ab}\,\delta_\star x^b$$

where $A_{ab}\,w^b = A_{ab}\,n^b = 0$. Decomposing A_{ab} into its irreducible parts we get

$$A_{ab} = \mathcal{R}_{ab} + \frac{1}{2}\,\bar{\theta}\,p_{ab} + \bar{\sigma}_{ab}, \tag{4.13}$$

with

$$\begin{aligned}
\mathcal{R}_{ab} &= A_{[ab]} = p^c_a\,p^d_b\,n_{[c;d]}, &(4.14)\\
\bar{\theta} &= A^a_a = p^{ab}\,n_{a\,;\,b}, &(4.15)\\
\bar{\sigma}_{ab} &= A_{ab} - \frac{1}{2}A^c_c\,p_{ab}. &(4.16)
\end{aligned}$$

It is easy to see that

$$\mathcal{R}_{ab}\,w^b = 0,\ \mathcal{R}_{ab}\,n^b = 0,\ \bar{\sigma}_{ab}\,w^b = 0,\ \bar{\sigma}_{ab}\,n^b = 0,\ \bar{\sigma}^a_a = 0.$$

Here $\mathcal{R}_{ab}$, $\bar{\theta}$, and $\bar{\sigma}_{ab}$ are called , respectively, the **rotation tensor**, the **expansion** and the **shear tensor** of the curves of the congruence, as measured by the observer w^a. Finally, the covariant derivative of n can be decomposed as follows:

$$\begin{aligned}
n_{a;b} &= \mathcal{R}_{ab} + \frac{1}{2}\,\bar{\theta}\,p_{ab} + \bar{\sigma}_{ab} + \overset{\bullet}{n}_a\,n_b - \overset{\circ}{n}_a\,w_b\\
&+ w_a(n^c\,w_{c\,;\,b}) + w_a w_b(n^c\,\overset{\circ}{w}_c) - w_a n_b(n^c\,\overset{\bullet}{w}_c), &(4.17)
\end{aligned}$$

where we set $\overset{\bullet}{X}{}^a = \frac{DX^a}{d\tau}$ and $\overset{\circ}{X}{}^a = X^a_{;b}w^b$ for any vector field X on M. It is possible to introduce an orthonormal basis $\{w, q, r, n\}$ at each point p of the tangent space $T_p(M)$ such that w is the unit timelike velocity vector of the observer and the 2-dimensional spacelike screen space is generated by $\{q, r\}$. Then, we get

$$\begin{aligned}
g_{ab} &= -w_a w_b + q_a q_b + r_a r_b + n_a n_b, &(4.18)\\
p_{ab} &= q_a q_b + r_a r_b.
\end{aligned}$$

REMARK. Vorticity vector being orthogonal to the 4-velocity vector of the fluid at all points of the vortex line, the theory of spacelike congruence has been used in the study of relativistic fluid dynamics. Also, the kinematic properties of the spacelike electric and magnetic fields play an important role in relativistic electrodynamics of continuous media (for references see [80] and [141]). In general relativity, Greenberg [80] introduced the theory of spacelike congruence and applied its kinematic properties to a congruence of vortex lines. Later on, Mason and Tsamparlis[141] investigated congruence of electric and magnetic field lines. In this book, the reader will find (wherever appropriate) an up-to-date information on the kinematics and the dynamics of spacelike (also timelike) congruence with respect to a given symmetry vector of a spacetime manifold.

Finally, we consider a **congruence of null geodesics** given by a null vector field ℓ. As explained in the section 2.6 of chapter 2, since the arc-length parameterization is not possible for a null curve C generated by $\{\ell\}$, we use a Frenet frame

$$F = \{\frac{d}{dP} = \ell, k, U, V\}$$

with respect to a **distinguished parameter** P whose Frenet equations are given by (2.55). Recall that

$$g(\ell,\ell) = g(k,k) = 0, g(\ell,k) = -1, g(U,U) = g(V,V) = 1.$$

Furthermore, for the congruence of null geodesic

$$\dot{\ell} = \sigma U = 0 \Rightarrow \sigma = 0,$$

where σ is the curvature function of the null curve C. We also use the 2-dimensional screen distribution S which is complementary to the tangent space $T(C)$ in $(T(C))^{\perp}$ at every point of C. Thus, it is possible to define a projection operator $\hat{h}_{ab}$ by

$$\hat{h}_{ab} = u_a u_b + v_a v_b, \tag{4.19}$$

where

$$u_a = g_{ab}U^b \qquad v_a = g_{ab}V^b. \tag{4.20}$$

Here $\hat{h}_{ab}$ is a positive definite metric induced by g_{ab} on the screen distribution S. The inverse metric is given by $\hat{h}^{ab} = \hat{h}_{cd}g^{ca}g^{db}$. Then, any tensor (or just geometric) quantity on M can be projected onto its hatted component in S by using the projection operator (4.19). Then, the covariant derivative of the projection null vector $\hat{\ell}$ can be decomposed as follows:

$$\hat{\ell}_{a\,;\,b} = \frac{\hat{\theta}}{2}\,\hat{h}_{ab} + \hat{\sigma}_{ab} + \hat{\omega}_{ab}, \tag{4.21}$$

where $\hat{\theta}$, $\hat{\sigma}_{ab}$ and $\hat{\omega}_{ab}$ are respectively called **expansion**, **shear** and **twist** of the null congruence and given by

$$\begin{aligned}
\hat{\theta} &= \hat{h}^{ab}\,\hat{\ell}_{a\,;\,b},\\
\hat{\sigma}_{ab} &= \hat{\ell}_{(a\,;\,b)} - \frac{\hat{\theta}}{2}\,\hat{h}_{ab},\\
\hat{\omega}_{ab} &= \hat{\ell}_{[a\,;\,b]}
\end{aligned}$$

and they satisfy (see Hawking-Ellis [94, page 88]

$$\ell\,\hat{\theta} = \frac{d\hat{\theta}}{dP} = -\frac{\hat{\theta}^2}{2} + 2\,\hat{\omega}^2 + 2\,\hat{\sigma}^2 - R_{ab}\,\ell^a\,\ell^b, \tag{4.22}$$

where $2\,\hat{\omega}^2 = \hat{\omega}_{ab}\,\hat{\omega}^{ab}$ and $2\,\hat{\sigma}^2 = \hat{\sigma}_{ab}\,\hat{\sigma}^{ab}$ are both non-negative. Observe that the equation (4.22) is the analogous of the **Raychaudhuri equation** (4.8) of timelike geodesics.

4.2 Matter Tensor and Einstein's Field Equations

The matter distribution on the 4-dimensional spacetime manifold (M, g) can be expressed by tensorial equations with regard to the Levi-Civita connection of the metric tensor g. The matter field is given by a symmetric tensor field, denoted by T_{ab}, called the **energy momentum tensor** such that

1. T_{ab} vanishes on an open set $\mathcal{U}$ of M iff the matter fields vanish on $\mathcal{U}$.

2. T_{ab} is divergence-free, that is,

$$T^a_{b\,;\,a} = 0. \tag{4.23}$$

Let V be a Killing vector field (*i.e.*, $V_{a\,;\,b} + V_{b\,;\,a} = 0$) on M. Then, (4.23) can be integrated along V as follows. Denoting the vector field $T^a_b\, V^b$ by W^a and computing its divergence we get

$$W^a_{\;;\,a} = T^a_{b\,;\,a}\, V^b + T^{ab}\, V_{b\,;\,a} = 0$$

since T_{ab} is symmetric and divergence-free and V is Killing. Now integrating over a compact orientable region $\mathcal{D}$ with boundary $\partial\mathcal{D}$ and using the Gauss divergence theorem (see Schultz [169, page 148]) gives

$$\int_{\partial\mathcal{D}} W^a\, d\sigma_a = \int_{\mathcal{D}} W^a_{\;;\,a} dV = 0, \tag{4.24}$$

where $d\sigma_a = N_a ds$ (N_a is a unit vector field normal to $\partial\mathcal{D}$ and ds is the elementary surface element of $\partial\mathcal{D}$) and dV is the volume element of $\mathcal{D}$. The equation (4.24) shows that the total flux of the component of T_{ab} along a Killing vector V taken over a closed surface is zero. Consequently, this result provides a conservation law. We now present a brief description of two energy momentum tensor fields.

(a) Electromagnetic fields. The electric and magnetic fields can be combined into a single tensor field $F = (F_{ab})$ on M, that is, F is a skew-symmetric (2-form) tensor of type $(0, 2)$. The complex self-dual electromagnetic tensor field $F^\star$ is defined by

$$\begin{aligned} F^\star_{ab} &= F_{ab} + i\,\tilde{F}_{ab}, \qquad i = \sqrt{-1} \\ \tilde{F}_{ab} &= \frac{1}{2}\,\epsilon_{abcd}\, F^{cd}. \end{aligned}$$

Here, ϵ_{abcd} is the Levi-Civita tensor field. The **Maxwell equations** are:

$$F_{[ab\,;\,c]} = 0, \tag{4.25}$$

$$F^{ab}_{\;\;;\,b} = \frac{4\pi}{c} J^a, \tag{4.26}$$

where c is the velocity of light and J^a is the 4-current vector which is conserved. This means that it satisfies the **continuity equation** defined by

$$J^a_{\;;\,a} = 0. \tag{4.27}$$

Alternatively, following are the invariant forms of above three equations

$$\begin{aligned} dF &= 0, \\ div\, f &= \frac{4\pi}{c} J, \\ div\, J &= 0, \end{aligned}$$

where $g(X, fY) = F(X, Y), \forall X, Y \in \mathcal{X}(M)$ and J is current 1-form. The energy momentum tensor is:

$$T_{ab} = \frac{1}{4} F_{cd} F^{cd} g_{ab} - F_{ac} F^c_b. \tag{4.28}$$

By virtue of Poincare' lemma and (4.25) we obtain $F = dA$ for some 1-form A, called the **electromagnetic potential**. Obviously, A is defined upto the addition of the gradient form of a scalar function (gauge). Also, for a source-free ($J = 0$) electromagnetic field, the Maxwell equations are conformally invariant.

For the local study of this field, we let $E = \{e_0, e_1, e_2, e_3\}$ be a local orthonormal basis of the tangent space $T_p(M)$ at a point $p \in M$. Define the **Newman-Penrose null tetrad** $T = \{\ell, k, m, \bar{m}\}$ where

$$\ell = \frac{1}{\sqrt{2}}(e_0 + e_1) \quad , \quad k = \frac{1}{\sqrt{2}}(e_0 - e_1),$$

$$m = \frac{1}{\sqrt{2}}(e_2 + ie_3) \quad , \quad \bar{m} = \frac{1}{\sqrt{2}}(e_2 - ie_3).$$

Here the only surviving components are $g(\ell, k) = -1$ and $g(m, \bar{m}) = 1$. Following Debney-Zund [40], we consider the following three complex functions, called **Maxwell scalar fields**:

$$\begin{aligned} \phi_0 &= 2F_{ab}\ell^a m^b, & (4.29) \\ \phi_1 &= F_{ab}(\ell^a k^b + \bar{m}^a m^b), & (4.30) \\ \phi_2 &= 2F_{ab}\bar{m}^a m^b. & (4.31) \end{aligned}$$

Then, the general form of F_{ab} is given by

$$\begin{aligned} F_{ab} &= -2Re\phi_1 \ell_{[a}k_{b]} + 2iIm\phi_1 m_{[a}\bar{m}_{b]} + \phi_2 \ell_{[a}m_{b]} \\ &+ \bar{\phi}_2 \ell_{[a}\bar{m}_{b]} - \phi_0 k_{[a}\bar{m}_{b]} - \bar{\phi}_0 k_{[a}m_{b]}. \end{aligned} \tag{4.32}$$

Define a complex invariant

$$K = \frac{1}{2}(F_{ab}F^{ab} + iF_{ab}F^{\star ab}), \tag{4.33}$$

which, in terms of the Maxwell scalars, is expressed by

$$K = 2({\phi_1}^2 - \phi_0\phi_2). \tag{4.34}$$

Since at any point p of M, $T_p(M)$ is a Minkowski space, it follows that

$$Re(K) = |E|^2 - |H|^2 \quad , \quad Im(K) = -2E \cdot H,$$

where E and H are the electric and the magnetic fields as described in section 3 of chapter 1. F is **non-singular (non-null)** or **singular (null)** according as $K \neq 0$ or $K = 0$. In either case, by a Lorentz transformation, one can set two of the three Maxwell scalars zero (see Debney-Zund [40]) which we do so. Thus, we have the following two classes of F:

$$\begin{array}{lccc} \textbf{Non-singular F} & \phi_0 = \phi_2 = 0 & , & \phi_1 \neq 0, \qquad (4.35) \\ \textbf{Singular F} & \phi_0 = \phi_1 = 0 & , & \phi_2 \neq 0. \qquad (4.36) \end{array}$$

According to above classification, using (4.32) in (4.28), we get the following canonical forms of the energy momentum tensor field:

$$\textbf{Non-singular} \quad T_{ab} = \frac{1}{2}|\phi_1|^2(m_{(a}\bar{m}_{b)} + k_{(a}\ell_{b)}), \tag{4.37}$$

$$\textbf{Singular} \quad T_{ab} = \frac{1}{2}|\phi_2|^2\ell_a\ell_b. \tag{4.38}$$

From (4.37) and (4.38) it follows that the non-singular or the singular T_{ab} admits two or one principal null directions respectively. Physically, non-singular and singular matter fields correspond to **non-radiating** and **radiating** fields respectively. For the singular case, the real principal null direction ℓ corresponds to the propagation of the **plane waves** (see Ruse [165]). For both the cases of F, the energy momentum tensor fields is trace-free ($T \equiv T^a_a = 0$).

(b) Perfect fluids. The fluid matter is described by its **mass density** ρ and a congruence of timelike curves, called the flow lines. Suppose u is the unit timelike vector field of a congruence. Then, the fluid current vector is defined by $j = \rho u$, such that j is conserved, that is $j^a_{;a} = 0$. The flow of the current vector provides **internal energy**, denoted by ϵ, as a function of ρ. For a perfect fluid, $\mu = \rho(1+\epsilon)$ is called the **energy density** and $p = \rho^2 \frac{d\epsilon}{d\rho}$ is the **pressure** of the fluid. The energy momentum tensor of a perfect fluid is given by (see Hawking-Ellis [94, page 69]

$$T_{ab} = (\mu + p)u_a u_b + p g_{ab}, \; \mu + p \neq 0, \; \mu > 0. \tag{4.39}$$

A perfect fluid is said to be **isentropic** if it is subject to a **barotropic equation of state** $\mu = \mu(p)$. Cases which are of particular physical interest are the **dust matter field** ($p = 0$), the **radiation field** ($p = \frac{\mu}{3}$) and the **stiff matter** ($p = \mu$). The conservation equation (4.23) entails the following **energy equation**:

$$(\mu + p)u^a_{\,;\,b}\, u^b = -p_{\,;\,b}\,(g^{ab} + u^a u^b). \tag{4.40}$$

One may also have a fluid with a conserved electric charge, that is, $div\, J = 0$, where $J = eu$ is the electric current. The energy momentum tensor of the charged fluid is then given by

$$T_{ab} = (\mu + p)u_a u_b + p g_{ab} + \frac{1}{4}(F_{cd}F^{cd})g_{ab} - F_{ac}F^c_b. \tag{4.41}$$

The energy equation (4.40) then becomes

$$(\mu + p)\, u^a_{\,;\,b} u^b = -p_{\,;\,b}\,(g^{ba} + u^b u^a) + F^a_b J^b. \tag{4.42}$$

Some other combinations (such as involving shear tensor and heat conduction vector etc) have also been discussed in this book as and when appropriate.

Energy Conditions. For finding energy conditions we need to determine how a prescribed T_{ab} determines the geometry of spacetime by considering the following **Einstein-Hilbert action**:

$$I(g) = \int_{\mathcal{D}} (A(r + 2\Lambda) + L) dV$$

and vary g over a closed region $\mathcal{D}$, where A, r, L and Λ are a suitable coupling constant, scalar curvature, the matter Lagrangian and the cosmological constant respectively. Varying g over $\mathcal{D}$ implies that the action $I(g)$ is stationary if the **Einstein field equations** (see Hawking-Ellis [94])

$$R_{ab} - (\frac{r}{2} - \Lambda) g_{ab} = 8\pi T_{ab} \tag{4.43}$$

are satisfied, where 8π is due to the fact that (4.43) reduces to the **Newtonian Poisson equation** $\nabla^2 \phi = 4\pi \rho$ for a weak field. Contracting (4.43), we get

$$r = 4\Lambda - 8\pi T \quad , \quad T = T^a_a. \tag{4.44}$$

The left and the right hand sides of the Einstein field equations describe the geometry and the physics respectively of the spacetime under investigation. Considerable work has been done on **exact solutions** of (4.43) by prescribing physically meaningful choices of T_{ab} (see Kramer et al. [118]). For a physically reasonable exact solution, T_{ab} must satisfy the following energy conditions.

(A) Weak Energy Condition. For any non-spacelike vector V in $T_p(M)$, at each point p of M,

$$T_{ab} V^a V^b \geq 0.$$

In particular, with respect to a perfect fluid (4.39) if $V = u$, then it is easy to see that $T_{ab} u^a u^b = \mu$ (which being non-negative local energy density), the weak energy condition is satisfied. Moreover, if the Einstein equations hold with $\Lambda = 0$, then $R_{ab} u^a u^b = 4\pi(\mu + 3p) > 0$ since $\mu + 3p > 0$. Also, for any null vector ℓ in $T_p(M)$, weak energy condition implies from (4.43) that $R_{ab} \ell^a \ell^b \geq 0$.

(B) Dominant Energy Condition. Besides satisfying **(A)** for a timelike vector field V, $T^a_b V^b$ must be a non-spacelike vector. Physically this means that the local energy flow vector is non-spacelike along with the non-negative local energy density. In terms of any orthonormal basis $\{E_a\}$, the dominant energy condition is equivalent to $T_{00} \geq |T_{ab}|$, for $a, b = 0, 1, 2, 3$. Thus, the dominant energy condition is the weak energy condition plus the condition that the pressure should not exceed the energy density. This also implies that the matter can not travel faster than the speed of light.

(C) Strong Energy Condition. A spacetime M satisfies the **strong energy condition** if for any non-spacelike vector V of M

$$R_{ab} V^a V^b \geq 0 \tag{4.45}$$

holds. Moreover, if the Einstein equations hold then, for all timelike V, this condition implies that

$$T_{ab}V^aV^b \geq (\frac{T}{2} - \frac{\Lambda}{8\pi})V^aV_a, \tag{4.46}$$

which reduces to $T_{ab}V^aV^b = 0$ for all null vectors. It follows from (4.45) that when $\Lambda = 0$, the strong energy condition , for all non-spacelike vectors, is equivalent to the condition

$$T_{ab}V^aV^b \geq (\frac{T}{2})V^aV_a,$$

as stated in Hawking-Ellis [94, page 95]. Observe that the condition (4.45) is also called the **timelike (resp. null) congruence condition**, at any point p of M, according as V is timelike (resp. null). It follows from **(A)** that the weak energy condition implies prevention of a congruence of null geodesic by the matter to expand. Assuming that the vorticity is zero, we get from (4.22) that

$$\frac{d\hat{\theta}}{dP} = -\frac{\hat{\theta}^2}{2} - 2\,\hat{\sigma}^2 - R_{ab}\,\ell^a\,\ell^b.$$

Hence $\hat{\theta}$ would decrease along ℓ if $R_{ab}\,\ell^a\,\ell^b \geq 0$ which is the null convergence condition. Consequently, the weak energy condition implies null convergence condition. In particular, consider T_{ab} representing singular (null) electromagnetic field. Then, it follows from the Einstein field equations that $R_{ab}\ell^a\ell^b = 0$ and therefore, the strong energy condition is satisfied.

The reader will find in this book (wherever appropriate) physical interpretation of the energy conditions with respect to some more specified energy momentum tensors.

4.3 Spacetimes of Constant Curvature

The simplest spacetime of this category is the flat Minkowski space (R^4_1, η_{ab}) where η_{ab} denotes the flat normal hyperbolic metric with the distance element given by

$$ds^2 = \eta_{ab}dx^adx^b = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2. \tag{4.47}$$

Here we set $x^0 = t$, the time coordinate with $c = 1$, x^1, x^2 and x^3 are the three space coordinates. It is well-known that, for this metric, the spacelike hypersurfaces (t constant) are a family of **Cauchy hypersurfaces** which cover the whole of R^4_1. Thus, R^4_1 is a product space

$$(R^4_1 = R \times N,\, \eta = -dt^2 \oplus G),$$

where (N, G) is a 3-dimensional Euclidean space. It is important to mention that not every spacelike hypersurface of R^4_1 is a Cauchy hypersurface (details on above may be seen in Hawking-Ellis [94, page 119]).

Besides Cauchy hypersurfaces we are also interested in the existence of a pair of **lightlike (null) hypersurfaces** as follows. Consider a spherical coordinate system

(t, r, θ, ϕ) such that $x^1 = r \sin\theta \sin\phi$, $x^2 = r \sin\theta \cos\phi$, $x^3 = r\cos\theta$. Then, (4.47) transforms into

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{4.48}$$

Above metric is singular at $r = 0$ and $\sin\theta = 0$. We, therefore, choose the ranges $0 < r < \infty$, $0 < \theta < \pi$ and $0 < \phi < 2\pi$ for which it is a regular metric. Actually two such charts are needed to cover the full R^4_1. Now we take two null coordinates v and w, with respect to a pseudo-orthonormal basis, such that $v = t + r$ and $w = t - r$ $(v \geq w)$. Thus, we can transform (4.48) as

$$ds^2 = -dvdw + \frac{1}{4}(v - w)^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{4.49}$$

where $-\infty < v, w < \infty$. The absence of the terms dv^2 and dw^2 in (4.49) imply that the hypersurfaces $\{w =$ constant $\}$ and $\{v =$ constant $\}$ are **lightlike hypersurfaces** since $w_{;a}w_{;b}\eta^{ab} = 0 = v_{;a}v_{;b}\eta^{ab}$. Thus, there exists a pair of lightlike hypersurfaces of R^4_1. Relating this example with the discussion on lightlike hypersurfaces (see section 2.7 of chapter 2), we say that a leaf of the 2-dimensional screen distribution S is topologically a 2-sphere, with coordinate system $\{\theta, \phi\}$, and can be seen as the intersection of the two hypersurfaces $w =$ constant and $v =$ constant.

In relativity, the null coordinates $v(w)$ are called **advanced (retarded) time coordinates** and are physically related to **incoming (outgoing) spherical waves** traveling at the speed of light.

Penrose transformed these null coordinates to a new set of null coordinates p and q defined by $\tan p = v$ and $\tan q = w$ $(-\frac{\pi}{2} < p, q < \frac{\pi}{2})$ to a new metric

$$ds^2 = \sec^2 p \sec^2 q(d\bar{s}^2)$$

which is conformal to the metric $d\bar{s}^2$ given by

$$d\bar{s}^2 = -4dpdq + \sin^2(p - q)(d\theta^2 + \sin^2\theta d\phi^2).$$

Furthermore, let $t' = p - q$ and $r' = p - q$. Then

$$d\bar{s}^2 = -(dt')^2 + (dr')^2 + \sin^2 r'(d\theta^2 + \sin^2\theta d\phi^2), \tag{4.50}$$

where $-\pi < t' + r', t' - r' < \pi$ and $r' \geq 0$. It turns out that the complete Minkowski space is given by

$$ds^2 = \frac{1}{4}\sec^2(\frac{t' + r'}{2})\sec^2(\frac{t' - r'}{2})d\bar{s}^2 \tag{4.51}$$

where $d\bar{s}^2$ is as given in (4.50) which is locally the **Einstein static universe** (see Hawking-Ellis [94, page 139]).

Thus, using (4.51) we can extend (4.50) analytically to the complete Einstein static universe which is topologically $R^1 \times S^3$ with $-\infty < t' < \infty$ and (r', θ, ϕ) as standard coordinates on S^3. Hence, we conclude from (4.51) that the entire Minkowski space has been conformally compactified to the region as given in (4.50).

Recall that two Lorentzian manifolds of the same dimension and of constant sectional curvature are locally isometric (see Wolf [201, page 69]). Thus, in particular, any Lorentzian manifold of zero sectional curvature is locally isometric to Minkowski spacetime.

Before discussing any other spacetime manifold, at this point we briefly provide the basic information on a physically significant class of Lorentzian manifolds which are **globally hyperbolic**. It turns out that several of the spacetimes, discussed in this book (including the Minkowski space) belong to this class or an extension of this class.

A spacetime (M, g) is said to be globally hyperbolic if there exists a spacelike hypersurface S such that every endless causal curve intersects S once and only once. Such a hypersurface (if it exists) is called a **Cauchy surface**. If M is globally hyperbolic, then (a) M is homeomorphic to $R \times S$, where S is a hypersurface of M, and for each t, $\{t\} \times S$ is a Cauchy surface, (b) if S' is any compact hypersurface without boundary, of M, then S' must be a Cauchy surface (see Eardley et al. [64], also Beem et al. [8, page 65]). It is obvious from above discussion that Minkowski spacetime is globally hyperbolic. Now we highlight as to why globally hyperbolic spacetimes are physically important and also present a mathematical technique to construct an extension of this class. Recall the following fundamental theorem of Hopf-Rinow [98] on compact and complete smooth Riemannian manifolds.

HOPF-RINOW THEOREM. *For any connected Riemannian manifold N, the following are equivalent:*

(a) *N is metric complete, i.e., every* **Cauchy sequence** *converges.*

(b) *N is geodesic complete, i.e., the exponential map is defined on the entire tangent space $T_p(N)$ at each point $p \in N$.*

(c) *Every closed bounded subset of N is compact.*

Thus the Hopf-Rinow theorem maintains the equivalence of metric and geodesic completeness and, therefore, guarantees the completeness of all Riemannian metrics, for a compact smooth manifold, with the existence of minimal geodesics. Also, if any one of the **(a)** through **(c)** holds then the Riemannian function is obviously finite-valued and continuous. In the non-compact case, it is known through the work of Nomizu-Ozeki [152] that every non-compact Riemannian manifold admits a complete metric. Unfortunately, there is no analogue to the Hopf-Rinow theorem for a general Lorentzian manifold. In fact, we know now that the metric completeness and the geodesic completeness are unrelated for arbitrary Lorentz manifolds and their **causal structure** requires that a complete manifold must independently be spacelike, timelike and null complete. The singularity theorems (see Hawking-Ellis [94]) confirm that not all Lorentz manifolds are metric and/ geodesic complete. Also, the Lorentz distance function fails to be finite and/ or continuous for an arbitrary spacetime (see Beem et al. [8]).

Based on above, it is natural to ask if there exists a class of spacetimes which shares some of the conditions of the Hopf-Rinow theorem. It has been shown in the work of Beem et al. [8] that the globally hyperbolic spacetimes turn out to be the most closely related physical model sharing some properties of Hopf-Rinow theorem. Indeed, **timelike Cauchy completeness** and **finite compactness** are equivalent and the Lorentz distance function is finite and continuous for this class (see Beem et al. [8]). Thus, the globally hyperbolic spacetimes are physically important. Although the Minkowski spacetime and the Einstein static universe are globally hyperbolic, to include some more physically important models (such as Robertson-Walker and Schwarzschild spaces, discussed in next two sections), we need a more extended case of the product spaces, called **warped product** which we now explain.

The **warped product** of two Riemannian manifolds (M_1, g_1) and (M_2, g_2) was first introduced by Bishop-O'Neill [16] and defined as

$$(M = M_1 \times_h M_2,\, g = g_1 \oplus_h g_2)$$

where h is a smooth function on M_1. Following this, in 1981, O'Neill [157] generalized this concept for semi-Riemannian manifolds. In particular, Beem et al. used O'Neill's Warped product to construct a large extended class of globally hyperbolic Lorentz manifolds as follows:

Let $(M_1,\, g_1)$ and $(M_2,\, g_2)$ be Lorentz and Riemannian manifolds respectively. Let $h : M_1 \to (0, \infty)$ be a C^∞ function and $\Pi : M_1 \times M_2 \to M_1$, $\bar{\Pi} : M_1 \times M_2 \to M_2$ the projection maps given by $\Pi(p, q) = p$ and $\bar{\Pi}(p, q) = q$ for every $(p, q) \in M_1 \times M_2$. Then, define the metric g given by

$$g(X, Y) = g_1(\Pi_\star X,\, \Pi_\star Y) + h(\Pi(p, q))\, g_2(\bar{\Pi}_\star X,\, \bar{\Pi}_\star Y), \tag{4.52}$$

where $\Pi_\star$ and $\bar{\Pi}_\star$ are respectively tangent maps. Based on above we state the following theorem:

THEOREM 4.1 (Beem et al. [8]). *Let $(M_1,\, g_1)$ and $(M_2,\, g_2)$ be Lorentzian and Riemannian manifolds respectively. Then, the Lorentzian warp product manifold $M = (M_1 \times_h M_2, g = g_1 \oplus_h g_2)$ is globally hyperbolic iff both the following conditions hold:*

(1) *$(M_1,\, g_1)$ is globally hyperbolic.*

(2) *$(M_2,\, g_2)$ is a complete Riemannian manifold.*

To use this theorem for another physically important example, we discuss the spacetimes $(M,\, g)$ of non-zero constant curvature c for which the curvature tensor is given by (see also (2.35) of chapter 2):

$$R_{abcd} = c\,(g_{bd} g_{ac} - g_{bc} g_{ad}).$$

Contracting above with g^{ac} and g^{bd} successively provides

$$R_{bd} = 3\, c\, g_{bd} \quad , \quad r = 12\, c. \tag{4.53}$$

Thus M is Einstein. Using above two equations in the Weyl curvature tensor C (see (2.32) of chapter 2), we deduce that M is conformally flat, that is,

$$C_{abcd} \equiv 0.$$

Consequently, the spacetime (and, in general, a semi-Riemannian manifold) has non-zero constant curvature iff it is Einstein and conformally flat. Now it follows from the information in section 3.4 of chapter 3 that the constant curvature 4-dimensional spacetime admits a maximal 10-parameter group of isometries and hence is a **homogeneous spacetime**. In particular, the spacetime of constant positive curvature is called the **de-Sitter spacetime** which is topologically $R^1 \times S^3$ with metric

$$ds^2 = -dt^2 + a^2 \cosh^2(\frac{t}{a})\{dr^2 + \sin^2 r(d\theta^2 + \sin^2 \theta d\phi^2)\}, \tag{4.54}$$

where a is non-zero constant. The spacial slices ($t =$ constant) are 3-spheres and they belong to a family of Cauchy surfaces. Introducing a new timelike coordinate $t' = 2\arctan(exp(\frac{t}{2})) - \frac{\pi}{2}$, we get

$$ds^2 = a^2 \cosh^2(\frac{t'}{a})\, d\bar{s}^2, \qquad (-\frac{\pi}{2} < t' < \frac{\pi}{2})$$

where the metric $d\bar{s}^2$ is given by (4.50). This shows that the de-Sitter spacetime is locally conformal to the Einstein static universe. The spacetime of constant negative curvature is known as **anti-de-Sitter spacetime** which is topologically $S^1 \times R^3$. Its universal covering space of constant curvature $c = -1$ (constructed by unwrapping the circle S^1) has the metric

$$ds^2 = -\cosh^2 r\, dt'^2 + dr^2 + \sinh^2 r\,(d\theta^2 + \sin^2 \theta\, d\phi^2),$$

where the coordinates $(t', r, \theta, \phi\ )$ cover the entire spacetime. Introducing a new coordinate $r' = 2\arctan(e^r) - \frac{\pi}{2}$, $0 \le r' \le \frac{\pi}{2}$, we observe that

$$ds^2 = (\cosh^2 r')\, d\bar{s}^2,$$

where $d\bar{s}^2$ is given by (4.50). Thus, we conclude that both the de-Sitter and the anti-de-Sitter spacetimes are locally conformal to the Einstein static universe which, in turn, is locally conformal to the Minkowski spacetime. Also both these spaces are conformally flat and, hence, are of Petrov type O (see Appendix A).

Observe that $C_{abcd} = 0$ and constant curvature condition is equivalent to $R_{ab} = \frac{r}{4}\, g_{ab}$. Substituting the values of R_{ab} and r from (4.53) in the Einstein field equations (4.43) we conclude that these two spaces may be regarded as solutions of the field equations with the cosmological constant $\Lambda = \frac{1}{4} r$. Finally, it is important to mention that based on (4.54) and theorem 4.1, de-Sitter spacetime is globally hyperbolic.

4.4 Spacially Homogeneous Cosmological Models

Since the Einstein field equations are a complicated set of nonlinear partial differential equations, we often assume certain relevant symmetry conditions for a satisfactory representation of our universe. Through extragalactic observations we know that the universe is approximately spherically symmetric about an observer. In fact, it would be more reasonable to assume that the universe is isotropic, that is, approximately spherical symmetric about each point in spacetime. This means (see Walker [197]) the universe is **spacially homogeneous**, that is, admits a 6-parameter group G_6 of isometries whose surfaces of transitivity are spacelike hypersurfaces of constant curvature. This further means that any point on one of these hypersurfaces is equivalent to any other point on the same hypersurface. Such a spacetime is called **Robertson-Walker spacetime** with metric

$$ds^2 = -dt^2 + S^2(t)d\Sigma^2, \tag{4.55}$$

where $d\Sigma^2$ is the metric of a spacelike hypersurface Σ with spherical symmetry and constant curvature $c = 1, -1$ or 0. With respect to a local spherical coordinate system (r, θ, ϕ), this metric is given by

$$d\Sigma^2 = dr^2 + f^2(r)(d\theta^2 + \sin^2\theta d\phi^2), \tag{4.56}$$

where $f(r) = \sin r$, $\sinh r$ or r according as $c = 1, -1$ or 0. The range of the coordinates is restricted from 0 to 2π or from 0 to ∞ for $c = 1$ or -1 respectively. For $c = 1$, Σ is diffeomorphic to S^3 and hence is compact. For $c = -1$, 0 it is possible to compactify Σ by identifying suitable points (see Löbell [124]). However, it follows from a result of Yano-Bochner (see theorem 5.3 of chapter 5) that for $c = -1$, the compact Σ would not admit any global Killing vector field even though it can have Killing vectors at each of its points. Also, for $c = 0$, a compact Σ can have only 3-parameter group G_3 of isometries. Thus, in both cases of $c = -1$ and 0, the compactified Σ can not be isotropic (which requires a G_6). Since the isotropy of Robertson-Walker spacetime is necessary, we conclude that Σ is non-compact for $c = -1$ or 0. Isotropy suggests that the energy momentum tensor of the Robertson-Walker spacetime is the form of a perfect fluid (see equation (4.39)). The metric (4.55) shows that the energy density and the pressure depend only on the time coordinate. Thus, the space coordinates are comoving and the fluid is made up of galaxies as particles. The function $S(t)$ can, therefore, be taken as the distance between two nearby galaxies at time t. The Raychaudhuri equation (4.8), the energy equation (4.40) and the Einstein field equations (4.43) provide

$$\begin{aligned} \dot{\mu} &= -3(\mu + p)\frac{\dot{S}}{S}, && (4.57)\\ 4\pi(\mu + 3p) - \Lambda &= -3\frac{\ddot{S}}{S}, && (4.58)\\ 3\left(\frac{\dot{S}}{S}\right)^2 &= 8\pi\mu + \Lambda - \frac{3c}{S^2}, && (4.59) \end{aligned}$$

with the reasonable assumption that $\mu > 0$ and $p \geq 0$. Present days experiments confirm that the universe is now going through the stage of negligible pressure $p \to 0$

(known as dust case). Thus with $\Lambda = 0 = p$, we obtain

$$\begin{aligned} \frac{4\pi}{3}\mu &= \frac{M}{S^3}, \\ 3\dot{S}^2 - \frac{6M}{S} &= -3c = \frac{E}{M}, \end{aligned} \tag{4.60}$$

where E is the sum of kinetic and potential energies and M = constant. The equation (4.60) is known as **Friedmann equation** and the corresponding solution is called the **Friedmann-Robertson-Walker (FRW) universe**. Rescaling t by T such that $dT = \frac{dt}{S(t)}$ and then integrating the Friedmann equation provides

$$\begin{aligned} S &= \frac{E}{3}(\cosh T - 1), \qquad t = \frac{E}{3}(\sin T - T), \qquad c = -1, \\ S &= T^2, \qquad t = \frac{T^3}{3}, \qquad c = 0, \\ S &= -(\frac{E}{3})(1 - \cos T), \qquad t = -(\frac{E}{3})(T - \sin T), \qquad c = 1. \end{aligned}$$

Robertson-Walker universe is not singularity free model. In fact, the most striking feature of this model is that its singularity is not based on a particular choice of coordinates explained below. Firstly we observe from (4.48), with $\Lambda = 0$, that S can not be constant. This means that the universe is either expanding or contracting. Based on this and (4.57) we conclude that the density decreases or increases according as the universe expands or contracts respectively. Thus, as $S \to 0$, the density approaches ∞ and the spacetime becomes singular at $S = 0$. This also shows that the universe had a beginning a finite time ago (the so-called big-bang singularity when $\mu \to \infty$). In order that the present day physical laws remain valid, we must therefore exclude the case $S = 0$ from this model.

Relating above model with the previous cases we studied, using the rescaled coordinate T in the metric (4.55), we obtain

$$ds^2 = S^2(t)d\bar{s}^2, \qquad d\bar{s}^2 = -dT^2 + d\Sigma^2. \tag{4.61}$$

The second part of above equation is the Einstein static universe which further is conformally flat. We, therefore, conclude that the Robertson-Walker spacetimes are conformally flat.

Now using the framework of Lorentzian warped products, we show that all Robertson-Walker spacetimes are globally hyperbolic. We know that $d\Sigma^2$ is the Riemannian metric of the spacelike hypersurface Σ. Set $M_0 = (a, b)$ for $(-\infty \leq a, b \leq \infty$) as 1-dimensional space with negative definite metric $-dt^2$. Define $S^2(t) = h(t)$ where $h : (a, b) \to (0, \infty)$. Then, it follows from the metric (4.55) and the discussion on warped product (see page 68) that a Robertson-Walker spacetime $(M,\, g)$ can be written as a Lorentzian warped product

$$(M_0 \times_h \Sigma,\, g = -dt^2 \oplus_h d\Sigma^2).$$

Since we know from the discussion so far that $(\Sigma,\, d\Sigma^2)$ is an isotropic Riemannian manifold, it follows from a work of Wolf [201, page 289] that this class of hypersurfaces coincides with the class of two point homogeneous Riemannian manifolds and, therefore, belongs to a class of complete manifolds. Also, the map $\Pi : M_0 \times_h \Sigma \to R$, given by $\Pi(t,p) = t$, is a smooth timelike function on M whose each level surface $\Pi^{-1}(t_0) = \{t_0\} \times \Sigma$ is a Cauchy surface. Consequently, it follows from the Theorem 4.1 that all Robertson-Walker spacetimes are globally hyperbolic.

Finally, we present a simple and less restricted class of solutions by dropping the isotropy and retaining the spacial homogeneity. This model is based on the assumption that at some point in the past there might have been anisotropies whose effect could not be neglected. Thus we assume the existence of an r-parameter Abelian group of isometries, whose orbits are locally spacelike hypersurfaces. Also set $\Lambda = 0 = p$. Then there exists a comoving coordinate system (t,x,y,z) with the metric

$$ds^2 = -dt^2 + X^2(t)dx^2 + Y^2(t)dy^2 + Z^2(t)dz^2. \tag{4.62}$$

Define a function $S(t)$ by $S^3 = XYZ$. Then it follows from the conservation law that $\frac{4}{3}\pi\mu = \frac{M}{S^3}$, where M is a suitable constant. This provides the following general solution

$$\begin{aligned}
X &= S\left(\frac{t^{2/3}}{S}\right)^{2\sin\alpha}, \\
Y &= S\left(\frac{t^{2/3}}{S}\right)^{2\sin(\alpha+\frac{2\pi}{3})}, \\
Z &= S\left(\frac{t^{2/3}}{S}\right)^{2\sin(\alpha+\frac{4\pi}{3})}, \\
S^3 &= \frac{9}{2}Mt(t+A),
\end{aligned}$$

where $A(>0)$ is constant which determines the magnitude of the anisotropy and $(-\frac{\pi}{6} < \alpha \leq \frac{\pi}{2})$ determines the direction in which the most rapid expansion takes place. Thus, it is clear from the above solution that at the beginning $(t = 0)$ our universe was at a highly anisotropic singular state, and then reaching a nearly isotropic phase for large t which is almost the same as the **Einstein de-Sitter universe**.

As a closing remark, it is important to mention that although we have considered above stated solution for $\Lambda = 0 = p$ (a belief that our universe at present is at this state), one can also find the properties of such models for a general equation of state. In particular, one may use a perfect fluid with the general equation of state $\mu = \mu(p)$ or a combination of two or more kinds of matter. However, it is now known that, within the framework of general relativity, the behavior near the singularity $(t = 0)$ is the same as discussed in the pressure free (dust) case.

4.5 Asymptotically Flat Spacetimes

One of the important areas of research in general relativity is the study of isolated systems, such as the sun and a host of stars in our universe. It is now well known that such isolated systems can best be understood by examining the local geometry of the spacetimes which are **asymptotically flat**, that is, their metric is flat at a large distance from a centrally located observer. It is the purpose of this section to discuss such physical spacetimes.

First we define **stationary and static spacetimes**. A spacetime is stationary if it has a 1-parameter group of isometries with timelike orbits. Equivalently, a spacetime is stationary if it has a timelike Killing vector field, say V. A static spacetime is stationary with the additional condition that V is hypersurface orthogonal, that is, there exists a spacelike hypersurface Σ orthogonal to V. The general form of the metric of a static spacetime can be written as

$$ds^2 = -A^2(x^1, x^2, x^3)dt^2 + B_{\alpha\beta}(x^1, x^2, x^3)dx^\alpha dx^\beta \,,$$

where $A^2 = -V_a V^a$ and $\alpha, \beta = 1, 2, 3$. Static spacetimes have both the time translation symmetry ($t \to t+$ constant) and the time reflection symmetry ($t \to -t$). A spacetime is said to be **spherically symmetric** if its isometry group has a subgroup isometric to $SO(3)$ and its orbits are 2-spheres.

Let (M, g) be a 4-dimensional isolated system with a Lorentz metric g and 3-dimensional spherical symmetry. Choose a coordinate system (t, r, θ, ϕ) for which the most general form of g is given by

$$ds^2 = -e^\lambda dt^2 + e^\nu dr^2 + A\,dr\,dt + B\,r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

where λ, ν, A, and B are functions of t and r only due to the 3-dimensional symmetry. The inherent freedom in choosing some of the coefficients allows to consider a Lorentz transformation such that $A = 0$ and $B = 1$. Using this in above general metric we get

$$ds^2 = -e^\lambda dt^2 + e^\nu dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{4.63}$$

for which

$$\begin{aligned} &g_{00} = -e^\lambda, \qquad g_{11} = e^\nu, \qquad g_{22} = r^2, \qquad g_{33} = r^2\sin^2\theta, \\ &g_{ab} = 0, \,\forall a \neq b, \qquad |g| = -r^4\sin^2\theta e^{\lambda+\nu}. \end{aligned}$$

Assume that M is Ricci-flat, that is, $R_{ab} = 0$. Finding the Christoffel symbols of the second type, we calculate the four non-zero components of the Ricci tensor (for their formulas see pages 19 and 20) and then equating them to zero entails the following three independent equations:

$$\begin{aligned} \partial_r\lambda &= \frac{e^\lambda - 1}{r}, \\ \partial_r\nu &= \frac{1 - e^\lambda}{r}, \\ \partial_t\nu &= 0. \end{aligned}$$

Adding first and second equations provide $\partial_r(\lambda + \nu) = 0$. Thus, $\lambda + \nu = f(t)$. Now integrating first equation and then using $\partial_r\lambda = -\partial_r\nu$, we get

$$e^\lambda = e^{-\nu} = (1 - \frac{2m}{r}),$$

where m is a positive constant. Thus, (4.63) takes the form

$$ds^2 = -(1 - \frac{2m}{r})\, dt^2 + (1 - \frac{2m}{r})^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{4.64}$$

This solution is due to Schwarzschild for which M is the **exterior Schwarzschild spacetime** $(r > 2m)$ with m and r as the mass and the radius of a spherical body. If we consider all values of r, then the metric (4.64) is singular at $r = 0$ and $r = 2m$. Hawking-Penrose's (see [94, page 150]) work showed that $r = 0$ is an **essential singularity** and the singularity $r = 2m$ can be removed by extending (M, g) to another larger manifold say (M', g') as follows. Let

$$r' = \int (1 - \frac{2m}{r})^{-1} dr = r + 2m \, log\,(r - 2m),$$

be a transformation with a new coordinate system (u, r, θ, ϕ), where $u = t + r'$ is an advanced null coordinate. Then, (4.64) takes the following new form:

$$ds^2 = -(1 - \frac{2m}{r})\, du^2 + 2du\, dr + r^2(d\theta^2 + \sin^2 d\phi^2), \tag{4.65}$$

which is known as **Eddington-Finkelstein metric** and is non-singular for all values of r. Similarly, if we use a retarded null coordinate $v = t - r'$, then (4.64) takes the form

$$ds^2 = -(1 - \frac{2m}{r})\, dv^2 - 2dv\, dr + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{4.66}$$

Observe that above two extended spacetimes have a pair of lightlike hypersurfaces defined by $r = 2m$. In each case, the corresponding hypersurface is a section $(\theta, \phi =$ constant) of the spacetime such that each point represents a 2-sphere of area $4\pi r^2$. In terms of the theory of lightlike hypersurfaces (see section 2.7 of chapter 2) we say that the corresponding 2-sphere is a leaf of the screen distribution of the lightlike hypersurface.

Birkhoff's theorem (see [94, Appendix B]) states that a spherically symmetric vacuum solution is necessarily static. Thus, the Schwarzschild solution is the only solution for any spherical symmetric vacuum field equations. The vector field ∂_t is timelike Killing and is also a gradient. Note that Schwarzschild solution is asymptotically flat since $1 - \frac{2m}{r} \to 1$ as $r \to \infty$ and then the metric (4.64) is flat.

Finally, consider a transformation of t and r to new coordinates T and R related by

$$R^2 - T^2 = (\frac{r}{2m} - 1)\, e^{r/2} \qquad , \qquad t = 2m\, log\,(\frac{R+T}{R-T}).$$

Then, (4.64) takes the form

$$ds^2 = \frac{32m^3 e^{-r/2m}}{r}(-dT^2 + dR^2) + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{4.67}$$

The metric (4.67) is well known as **Kruskal metric** and $\{T, R\}$ are called **Kruskal coordinates**. This metric is regular everywhere except $r = 0$ and is the unique analytic inextendible local solution of the Schwarzschild metric (4.64). The singularity at $r = 0$ can not be removed since at that moment the curvature scalar

$R_{abcd}R^{abcd}$ blows up.

Another solution of this category is due to Reissner-Nordström, which represents the spacetime $(M,\, g)$ outside a spherically symmetric body having an electric charge e but no spin or magnetic dipole. Following a procedure similar to the case of Schwarzschild solution, the metric of this spacetime can be expressed by

$$\begin{aligned} ds^2 &= -(1-\frac{2m}{r}+\frac{e^2}{r^2})\,dt^2+(1-\frac{2m}{r}+\frac{e^2}{r^2})^{-1}\,dr^2 \\ &+ r^2(d\theta^2+\sin^2\theta d\phi^2) \end{aligned} \tag{4.68}$$

for a local coordinate system $(t, r, \theta, \phi\ )$. This metric is also asymptotically flat as for $r \to \infty$, it approaches the Minkowski metric and in particular if $e = 0$ then this is Schwarzschild metric. It is singular at $r = 0$ and $r = m \pm (m^2 - e^2)^{\frac{1}{2}}$ if $e^2 \leq m^2$. While $r = 0$ is an essential singularity, the other two can be removed as follows. Consider a transformation

$$r' = \int (1-\frac{2m}{r}+\frac{e^2}{r^2})^{-1}\,dr.$$

Let $(u, r, \theta, \phi\ )$ be a new coordinate system, with respect to (4.68), such that $u = t + r'$ the advanced null coordinate. Then, this metric transforms into

$$ds^2 = -(1-\frac{2m}{r}+\frac{e^2}{r^2})\,du^2+2\,du\,dr+r^2(d\theta^2+\sin^2\theta\,d\phi^2)$$

which is regular for values of r and represents an extended spacetime $(M',\, g')$ such that M is embedded in M' and g' is g on the image of M on M'. In this extended space there exists a local region, say $\mathcal{U}$, of M such that $r = m$ and $e^2 = m^2$, for which $g'|\mathcal{U}$ is degenerate and its metric is given by

$$ds^2 = 0\cdot dv^2+r^2(d\theta^2+\sin^2\theta\,d\phi^2). \tag{4.69}$$

Thus, there exists an **asymptotically flat region** $\mathcal{U}$ in M which is isometric to some open set of M' including a **lightlike hypersurface** say $(\Sigma,\, g'\,|\,\mathcal{U},\, r = m,\, e^2 = m^2)$ with metric (4.69). Similar is the case if we choose a retarded coordinate transformation $v = t - r'$. Also, as explained previously, the two lightlike hypersurfaces can be embedded in a maximally extended manifold (details may be seen in Hawking-Ellis [94, p.157]). It is noteworthy that the **Reissner-Nordström solution** (4.68) is the unique spherically symmetric asymptotically flat solution of the Einstein Maxwell field equations.

If $e^2 > m^2$ then the metric (4.68) is singular only at $r = 0$ and then it is called the **exterior Reissner-Nordström solution** which is globally hyperbolic. To prove this we let

$$M_1 = \{(t, r) \in R^2 : e^2 > m^2 \qquad \text{or} \qquad e^2 = 0 \qquad \text{and} \qquad r > 2m\}$$

be endowed with a metric

$$g_1 = -(1-\frac{2m}{r}+\frac{e^2}{r^2})\,dt^2+(1-\frac{2m}{r}+\frac{e^2}{r^2})^{-1}\,dr^2.$$

Set $h = r^2$ and let M_2 the unit 2-sphere with the usual spherical Riemannian metric g_2 of constant sectional curvature 1 induced by the inclusion mapping $S^2 \to R^3$. Then, $(M = M_1 \times_h M_2 \,,\, g = g_1 \oplus_h g_2)$ is the exterior **Reissner-Nordström spacetime** which is also globally hyperbolic (since it satisfies the two conditions of the theorem 4.1).

It is noteworthy that all the three solutions discussed so far are of Petrov type D (see Appendix A).

Since, in general, the astronomical bodies are rotating, their exterior solution is not exactly spherically symmetric. The properties of such rotating massive and isolated objects can best be understood through a solution known as **Kerr solution** of the stationary axisymmetric asymptotically flat spaces. These solutions are given by

$$\begin{aligned} ds^2 &= -dt^2 + \rho^2\left(\frac{dr^2}{\Delta} + d\theta^2\right) + (r^2 + a^2)\sin^2\theta\, d\phi^2 \\ &+ \frac{2mr}{\rho^2}(a\sin^2\theta d\phi - dt)^2, \end{aligned} \tag{4.70}$$

where $\rho^2 = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2\,r\,m + a^2$, m and a are constants representing, respectively, the mass and the angular momentum per unit mass (measured from infinity). The metric is not time symmetric, that is, it is not invariant under $t \to -t$. The stationary and axisymmetric property means that there exists a 2-parameter Abelian group of isometries. This metric has only essential singularity at $r = 0$ if $a^2 > m^2$ and $\Delta > 0$. If this condition is ignored then there are other removable singularities. Just as the Reissner-Nordström solution is a charged version of Schwarzschild solution, so there is a family of charged Kerr solutions (details may be seen in Hawking-Ellis [94]).

Above four examples of isolated objects belong to a class of **unbounded asymptotically flat objects**. In order to study a general class (which includes **bounded asymptotically flat spacetimes**) we recall the following :

DEFINITION (Penrose [161]). *A spacetime (M, g) is said to be asymptotically simple if there exists another spacetime (M', g') and an embedding $\theta : M \to M'$, with smooth boundary ∂M in M' and a scalar Ω (say C^3 at least) on M such that*

1. *$M' - \partial M$ is conformal to M with $g' = \Omega^2 g$.*

2. *$\Omega > 0$ in $M' - \partial M$ and $\Omega = 0$ on ∂M with $d\Omega \neq 0$.*

3. *Every null geodesic in M' contains two end points of ∂M.*

PROPOSITION (Penrose [161]).

(a) *If M satisfies the Einstein empty equations near ∂M then ∂M is lightlike. This also follows if matter is present near ∂M provided the stress energy tensor is trace free, that is, Maxwell field is present near ∂M.*

(b) *∂M consists of two disjoint pieces ∂M_+ and ∂M_- each topologically $R \times S^2$. ∂M_+ (future null infinity) respectively ∂M_- (past null infinity) bounds M to the future (respectively past).*

(c) *The Weyl tensor vanishes at ∂M. Thus, M is asymptotically flat.*

Thus, according to above proposition, there exists a Lightlike hypersurface ∂M, of M, such that the property of boundedness allows one to attach the two end points of ∂M_+ and ∂M_- to M. Observe that one must exclude the three previously discussed solutions from this case as those represent unbounded objects, and, therefore, their null geodesics do not have end points. To accommodate those three solutions in a larger class of asymptotically flat spacetimes, Penrose introduced the following concept.

A spacetime (M, g) is called **weakly asymptotically simple** if there exists an asymptotically simple M_0 with corresponding M_0' such that for some open set $\mathcal{U}$ of M_0' including ∂M, the region $M \cap \mathcal{U}$ is isometric with an open subset of M. This definition covers the three earlier solutions since it is easy to see that these objects contain an infinite sequence of asymptotically flat regions isometric to open sets of weakly asymptotically simple spacetimes. If we choose only one such flat region, then there exists a simple pair of lightlike hypersurfaces of M. Consequently, based on Penrose's work, we now have a large class of physically significant asymptotically flat spacetimes, some of which are globally hyperbolic.

4.6 Plane Wave Solutions

There are many exact local solutions of Einstein field equations. However, only few are known global solutions. One of them is the **plane wave solution** of the vacuum field equations which is homeomorphic to R^4 with metric

$$ds^2 = 2\,du\,dv + dy^2 + dz^2 + H(y, z, u)\,du^2, \tag{4.71}$$

where $H = (y^2 - z^2)\,f(u) - 2\,y\,z\,g(u)$ and $f(u)$, $g(u)$ are arbitrary C^3 functions determining the amplitude and polarization of the waves. This solution is invariant under G_5 acting multiply transitively on the lightlike hypersurfaces $u =$ constant. An extra Killing vector field arises if $f(u) = \cos 2u$ and $g(u) = \sin 2u$. As a special cases of plane wave solutions are those which admit a covariant constant null vector say ℓ , that is, $\nabla \ell = 0$. Such special non-flat vacuum solutions are called **plane-fronted waves with parallel rays (denoted by *pp*-waves)** which were first discovered by Brinkman (see Kramer et al. [118, page 234]) and since have been widely studied. Such a null field is non-expanding, non-rotating and its null congruence curves are orthogonal trajectories of a family of hypersurfaces. Precisely, a non-flat vacuum field is a *pp*-wave iff any one of the following conditions hold.

1. The complex curvature tensor $P_{abcd} = R_{abcd} + i\,\overset{\star}{R}_{abcd}$ (where $\overset{\star}{R}$ is dual of R) is recurrent, that is, $P_{abcd;e} = q_e P_{abcd}$.

2. The null field admits a 1-parameter group of isometries with null trajectories, that is, a null Killing vector field has vanishing eigenvalues of the curvature tensor.

3. It admits a covariant constant null bivector.

4. The rays (null vectors along which the wave propagates) are trajectories of an affine collineation.

THEOREM 4.2 (Brinkman cited in [118]). *A vacuum solution of Einstein field equations can be mapped on to another vacuum solution iff both solutions admit a covariant constant null vector field, that is, both are pp-waves.*

Finally, we mention that the plane-fronted gravitational waves are of type N (see Appendix A) vacuum solutions with a null vector field quadruply repeated.

EXAMPLE. Let (Σ, γ) be a totally geodesic (vanishing second fundamental form) lightlike hypersurface of a 4-dimensional spacetime manifold (M, g). It follow from the theorem 2.6 of chapter 2 that the induced Ricci tensor on Σ is given by

$$Ric(X, Y) = 2\, d\theta(U, V)\gamma(X, Y) - 2\, d\phi(\ell, N)\, \omega(X)\, \omega(Y), \tag{4.72}$$

with respect to an induced pseudo-orthonormal basis $\{\ell, U, V\}$. Here scalar curvature is $r = 4\, d\theta(U, V)$ and $X, Y \in \mathcal{X}(\Sigma)$. Suppose $r = 0$. Then, in terms of local coordinates, (4.72) reduces to

$$R_{ij} = -2\, d\phi(\ell, N)\, \ell_i\, \ell_j \quad , \quad (1 \leq i, j \leq 3). \tag{4.73}$$

Now suppose (M, g) represents plane wave solutions given by the metric (4.71). A lengthy computation of all the Ricci components indicates that $(M,\, g)$ admits a covariant constant null bivector iff its Ricci tensor is given by

$$R_{ab} = \frac{1}{2}(H_{yy} + H_{zz})\ell_a\, \ell_b \quad , \quad (a, b = 0, 1, 2, 3). \tag{4.74}$$

Since (Σ, γ) is totally geodesic in (M, g), the induced Ricci tensor on Σ will have the same form as (4.74) but $(a, b = 1, 2, 3)$. Comparing (4.74) and (4.73), we conclude that (Σ, γ) belongs to a class of lightlike hypersurfaces whose ambient spacetime $(M,\, g)$ represents plane-fronted gravitational waves with parallel rays, such that

$$d\phi(\ell, N) = -\frac{1}{4}(H_{yy} + H_{zz}).$$

Chapter 5

Killing and Affine Killing Vector Fields

We start this chapter with upto date information on divergence theorems and integral formulas. In particular, we provide new information on the validity of divergence theorem for semi-Riemannian manifolds with boundary. Then we review on the existence of Killing and affine Killing vectors and their kinematic and dynamic properties. We show that while Killing symmetry is characterized by a class of expansion-free and shear-free spacetimes, proper affine Killing symmetry is relevant to non-zero expansion and non-zero shear.

5.1 Divergence Theorems

In this section, we present important results from the theory of integration on manifolds.

THEOREM 5.1 (Stokes). *Let M be a compact orientable m-dimensional manifold with boundary ∂M and ω an $(m-1)$- form on M. Then*

$$\int_{\partial M} \omega = \int_M d\,\omega. \tag{5.1}$$

If M has a Riemannian metric g, then, with respect to a unit outward-pointing vector field N normal to ∂M, the **Stokes theorem** has the following form (due to Gauss):

THEOREM 5.2 (Divergence Theorem). *Let M be a compact orientable Riemannian manifold with boundary ∂M. For a smooth vector field V on M, we have*

$$\int_M div\, V\, d\,v = \int_{\partial M} g(N, V)\, d\,S, \tag{5.2}$$

where N is the unit normal to ∂M, $d\,v$ is the volume element of M and $d\,S$ is the surface element of ∂M. For semi-Riemannian M, the validity of divergence theorem is not obvious due to the possible existence of degenerate metric coefficient

$g_{ii} = 0$ for some index i. Thus the boundary ∂M may become degenerate at some of its points or it may be a lightlike hypersurface of M. In both these cases, there is no well defined outward normal. This question was first raised by Duggal [52] and since then recently Ünal [193] (also see Garcia-Rio and Kupeli [70]) have overcome a part of this difficulty by assuming some conditions on the degenerate portion of ∂M. To the best of our knowledge, the problem of dealing with the divergence theorem for a general case of semi-Riemannian M is still open. In the following we now briefly present the work of Ünal.

Let ν be the index of the metric tensor g of a semi-Riemannian manifold M with boundary ∂M (possibly $\partial M = \phi$). Then the induced tensor $g_{\partial M}$ on ∂M is also symmetric but not necessary a metric tensor on ∂M because it may be degenerate at some or all points of ∂M. Let ∂M_+ , ∂M_- and ∂M_0 be the subsets of points where the non-zero vectors orthogonal to ∂M are spacelike, timelike and lightlike respectively. Thus,

$$\partial M = \partial M_+ \cup \partial M_- \cup \partial M_0$$

where the three subsets are pairwise disjoint. It is straightforward to show that ∂M_+ and ∂M_- are open submanifolds of ∂M. Let $\partial M' = \partial M_+ \cup \partial M_-$ which obviously is also an open submanifold of ∂M. Recall that the Stokes theorem embodies the following, called, **Gauss theorem** (which is valid for any semi-Riemannian manifold):

THEOREM 5.3 (Gauss). *Let M be a compact orientable semi-Riemannian manifold with boundary ∂M. For a smooth vector field V on M, we have*

$$\int_M (div\, V)\epsilon = \int_{\partial M} i_V \epsilon, \tag{5.3}$$

where $\epsilon = \sqrt{|g|}dx^1 \wedge \ldots \wedge dx^m$ is the volume element on M and $g = det(g_{ij})$ with respect to a suitable local coordinate system $(x^1, \ldots, x^m)$. Here i denotes the operator of inner product.

We use the same symbol N for the unit normal vector field to ∂M induced on $\partial M'$. Also, we denote by $\epsilon_{\partial M'} = i_N \epsilon$ the induced volume element on ∂M_+ and ∂M_0 when restricted to these submanifolds respectively. Assume that ∂M_0 has measure zero (see Bröcker-Jänich [24]) in ∂M.

THEOREM 5.4 (Ünal [193]). *Let (M, g) be an oriented semi-Riemannian manifold with boundary ∂M (possibly $\partial M = \phi$) and semi-Riemannian volume element ϵ. Let V be a vector field on M with compact support. If ∂M_0 has measure zero in ∂M then,*

$$\int_M (div\, V)\epsilon = \int_{\partial M_+} g(N, V)\epsilon_{\partial M'} - \int_{\partial M_-} g(N, V)\epsilon_{\partial M'}. \tag{5.4}$$

PROOF. Since ∂M_0 has measure zero in ∂M, it follows from the Gauss theorem 5.3 that

$$\int_M (div)\epsilon = \int_{\partial M'} i_V\, \epsilon = \int_{\partial M_+} i_V\, \epsilon + \int_{\partial M_-} i_V\, \epsilon. \tag{5.5}$$

Choose an orthonormal basis $\{e_1 = N(p), e_2, \ldots, e_m\}$ for $T_p(M)$ at a point $p \in M$. Then, the vector field V at p is given by

$$V = g(N, N)g(N, V)N + \sum_{a=2}^{m} g(e_a, e_a)g(V, e_a)\, e_a.$$

Therefore,

$$\begin{aligned}
(i_V\,\epsilon)(e_2, \ldots, e_m) &= \epsilon\,(V, e_2, \ldots, e_m) \\
&= \epsilon\,[g(N, N)g(N, V)N \\
&+ \sum_{a=2}^{m} g(e_a, e_a)g(V, e_a)e_a, e_2, \ldots, e_m] \\
&= \epsilon\,[g(N, N)g(N, V)N, e_2, \ldots, e_m] \\
&= g(N, N)g(N, V)(i_N\,\epsilon)(e_2, \ldots, e_m) \\
&= g(N, N)g(N, V)\,\epsilon_{\partial M'}(e_2, \ldots, e_m).
\end{aligned}$$

Hence,

$$i_V\,\epsilon = \begin{cases} g(N, V)\,\epsilon_{\partial M'} & \text{on } \partial M_+ \\ -g(N, V)\,\epsilon_{\partial M'} & \text{on } \partial M_- . \end{cases} \tag{5.6}$$

Using (5.6) in (5.5) we get (5.4) which proves the theorem.

THEOREM 5.5 (Ünal [193]). *Let (M, g) be an oriented semi-Riemannian manifold with boundary ∂M (possibly $\partial M = \phi$) and semi-Riemannian volume element ϵ. If V is a vector field on M, with compact support, such that V is tangent to ∂M at the points of ∂M_0, then (5.4) holds.*

PROOF. V tangent to ∂M at the points of ∂M_0 implies that $i_V\,\epsilon = 0$ on ∂M_0. Hence, following as in the proof of theorem 5.4, we claim that (5.4) holds.

REMARK. The reader is invited to study divergence theorem in semi-Riemannian manifolds, with boundary, without any geometric conditions such as stated in previous two theorems. Also, the growing importance of semi-Riemannian manifolds with null boundary ∂M (seen as a lightlike hypersurface of M) in mathematical physics (see, for example, recent book by Duggal-Bejancu [57]) demands further research on this open problem of "*divergence theorem for null boundary* ".

5.2 Killing Vector Fields on Riemannian Manifolds

In this section, following Yano [211], we study the existence of Killing vector fields on a **compact and orientable n-dimensional Riemannian manifold** M with positive definite metric tensor g. We assume that M is without boundary until otherwise stated. This means that the divergence theorem equation (5.2) reduces to

$$\int_M div\, V = 0, \tag{5.7}$$

for a smooth vector field V on M. In particular, we are interested in the case when $V = grad\, f$, for some smooth function f. Then,

$$\int_M \triangle f = 0, \tag{5.8}$$

where, as per equation (2.31) of chapter 2, $\triangle f = div(grad\, f) = \nabla_i \nabla^i f$. First we compute $grad(f^2)$ as $\nabla^i(f^2) = 2\, f \nabla^i f$ and then taking the divergence on both sides of this result provides

$$\triangle f^2 = 2(|grad\, f|^2 + f \triangle f). \tag{5.9}$$

Integrating (5.9) over M and then using (5.8) provides

$$\int_M (|\nabla f|^2 + f \triangle f) = 0. \tag{5.10}$$

From above we conclude the following results.

Hopf's Lemma. *If $\triangle f = 0$ for a smooth function f on a compact orientable Riemannian manifold M without boundary, then f is constant on M.*

Furthermore, if we assume $\triangle f \geq 0$ (or $\triangle f \leq 0$) on M, then $\triangle f = 0$ and by Hopf's lemma we have the following general result:

Bochner's Lemma. *If $\triangle f \geq 0$ (or $\triangle f \leq 0$) on M, then f is constant on M.*

Now suppose that f is non-constant and $\triangle f = -k\, f$, for a constant k on M. Then, it follows from (5.10) that k is positive. In particular, let $f = |V|^2$, where V is a smooth vector field on M. Then,

$$\begin{aligned} \triangle f &= (\nabla^i \nabla_i)(V^j V_j) = \nabla^i[(\nabla_i V^j)\, V_j + V^j(\nabla_i V_j)] \\ &= 2\,[(\nabla^i \nabla_i V^j) V_j + |\nabla V|^2]. \end{aligned}$$

Integrating above expression over M and then using (5.2) we obtain

$$\int_M < \triangle V, V> + |\nabla V|^2 = 0, \tag{5.11}$$

where $\triangle V = (\nabla^i \nabla_j V^i)\partial_i$, $V = V^i \partial_i$, $|\nabla V|^2 = (\nabla_i V_j)(\nabla^i V^j)$ and $<,>$ is the inner product with respect to the Riemannian metric g on M. From (5.11) we obtain the following result.

PROPOSITION 1 (Bochner [22]). *If the second covariant derivative of a vector field V vanishes on M, then the first covariant derivative of V also vanishes on M.*

Using V we consider another vector field $\nabla_V V - (div\, V)\, V$ and calculate its divergence as follows:

$$
\begin{aligned}
div\,(\nabla_V V - (div\,V)V) &= \nabla_i (V^j \nabla_j V^i - (\nabla_j V^j)\, V^i) \\
&= (\nabla_i V^j)(\nabla_j V^i) + V^j \nabla_i \nabla_j V^i \\
&- (\nabla_i \nabla_j V^j)\, V^i - (\nabla_i V^i)^2 \\
&= (\nabla_i V_j)(\nabla^j V^i) - (\nabla_i V^i)^2 \\
&+ V^i (\nabla_j \nabla_i V^j - \nabla_i \nabla_j V^j) \\
&= (\nabla_i V_j)(\nabla^j V^i) - (\nabla_i V^i)^2 + V^i R^j_{kji} V^k \\
&= (\nabla_i V_j)(\nabla^j V^i) - (\nabla_i V^i)^2 + R_{ki} V^k V^i.
\end{aligned}
$$

Integrating above equation over M and then using (5.2) we obtain the following integral formula.

$$
\int_M R_{ij} V^i V^j + (\nabla_i V_j)(\nabla^j V^i) - (div\,V)^2 = 0. \tag{5.12}
$$

If $\omega = V_i\, dx^i$ denotes the associated 1-form of V, then

$$
\begin{aligned}
|d\omega|^2 &= (\nabla_i V_j - \nabla_j V_i)(\nabla^i V^j - \nabla^j V^i) \\
&= 2\,(|\nabla V|^2 - (\nabla_i V_j)(\nabla^j V^i)).
\end{aligned}
$$

Using above result in (5.12) we get

$$
\int_M Ric\,(V, V) + |\nabla V|^2 - \frac{1}{2}\,|d\,\omega|^2 - (div\,V)^2 = 0. \tag{5.13}
$$

Furthermore, since

$$
\begin{aligned}
|\,L_V g\,|^2 &= (\nabla_i V_j + \nabla_j V_i)(\nabla^i V^j + \nabla^j V^i) \\
&= 2\,(|\nabla V|^2 + (\nabla_i V_j)(\nabla^j V^i)),
\end{aligned}
$$

using this and the one previous equation to eliminate $|d\omega|^2$ from (5.13) we obtain

$$
\int_M Ric\,(V, V) + \frac{1}{2}\,|L_V\, g|^2 - |\nabla V|^2 - (div\,V)^2 = 0. \tag{5.14}
$$

Denote by $\mathcal{D}$ the **de-Rham Laplacian** defined by

$$
\mathcal{D}\, V = Q\, V + \triangle\, V,
$$

where we set Q the $(1,1)$ associated tensor field of the Ricci tensor whose local components are R^j_i. Then, it follows from (5.11) and (5.14) that

$$
\int_M < Q\, V + \triangle\, V, V > + \frac{1}{2} |L_V\, g|^2 - (div\,V)^2 = 0. \tag{5.15}
$$

Now consider a smooth function f, then

$$
div\,(f\,V) = \nabla_i\,(f\,V^i) = (\nabla_i\, f) V^i + f\,(\nabla_i\, V^i).
$$

Integrating above result over M and then again using the divergence theorem 5.2, we obtain

$$\int_M <V, \nabla f> + f(div\, V) = 0, \tag{5.16}$$

which is a generalization of the divergence theorem. In particular, if $f = div\, V$, then using (5.16) we can transform (5.15) as

$$\int_M <Q\,V + \triangle\, V + \frac{n-2}{2} grad(div\, V), V> + \frac{1}{2}|L_V g - \frac{2}{n}(div\, V)g|^2 = 0.$$

At this point we let V be a **Killing vector field**, that is, $\nabla_i V_j + \nabla_j V_i = 0$. This implies that $div\, V = 0$ and if we assume that V is the gradient of some function f, then $\triangle f = 0$. Thus, by Hopf's lemma f is constant on M. For a Killing V, the integral formula (5.14)reduces to

$$\int_M (\,Ric(V,\, V) - |\nabla\, V|^2) = 0. \tag{5.17}$$

Thus, we conclude the following result from (5.17).

THEOREM 5.6 (Bochner [22]) *If the Ricci curvature of M is negative semi-definite, then a Killing vector field V on M is covariant constant (parallel). On the other hand, if the Ricci curvature on M is negative definite, then a Killing vector field other than zero does not exist on M.*

Using the identity $\nabla_i \nabla_j V_k - \nabla_j \nabla_i V_k = -R^t{}_{kij} V_t$ for a Killing vector field V, it is easy to see that $\mathcal{D}\, V = 0$. Conversely, if $\mathcal{D}\, V = 0$ and $div\, V = 0$, then V is Killing. Thus,

THEOREM 5.7. *A vector field V on M is Killing iff*

$$\mathcal{D}\, V = 0 \quad , \quad div\, V = 0. \tag{5.18}$$

Now let M be a compact orientable Riemannian manifold with compact orientable boundary ∂M. For this purpose we need to calculate the following divergence, with respect to a vector field V:

$$\begin{aligned} div[(\nabla_i V^j + \nabla^j V_i) V^i - (div\, V) V^j] &= (\nabla_j \nabla_i V^j + \nabla_j \nabla^j V_i) V^i - \nabla_j(\nabla_i V^i) V^j \\ &+ (\nabla_i V^j + \nabla^j V_i)\nabla_j V^i - (\nabla_i V^i)\nabla_j V^j \\ &= Ric(V,\, V) + < \triangle\, V, V> \\ &+ \frac{1}{2}|L_V g|^2 - (div\, V)^2. \end{aligned}$$

Integrating above equation over M and then using theorem 5.2 provides

$$\int_M <QV + \Delta V, V> + \frac{1}{2}|L_V g|^2 - (div\, V)^2$$

$$= \int_{\partial M} (L_V g)(V, N) - (div\, V) <V, N>.$$

Using $\mathcal{D}V = QV + \Delta V$ and assuming that V is Killing we obtain the following:

THEOREM 5.8 (Hsiung [99], Yano-Ako [212]). *A vector field V on a compact orientable manifold M, with compact orientable boundary ∂M, is Killing iff*

$$\Delta V + QV = 0 \quad , \quad div\, V = 0 \quad \text{on} \quad M \quad \text{and}$$

$$(L_V g)(V, N) = 0 \quad \text{on} \quad \partial M. \tag{5.19}$$

For a **Killing vector field** V, we have the following integrability conditions (see Yano [211, page 24]):

$$L_V \nabla = 0, \qquad L_V R = 0, \qquad L_V C = 0. \tag{5.20}$$

In terms of a local coordinate system, above conditions are expressed by

$$L_V \Gamma^i_{jk} = 0 \qquad L_V R^i{}_{jkm} = 0, \qquad L_V C^i{}_{jkm} = 0.$$

Therefore, the Ricci tensor R_{ij} and the scalar curvature r satisfy

$$L_V R_{ij} = 0 \quad , \quad L_V r = 0. \tag{5.21}$$

Using the Killing equation (3.16), of chapter 3, we see that $L_V \nabla = 0$ is equivalent to

$$\nabla_j \nabla_k V^i + R^i{}_{kmj} V^m = 0. \tag{5.22}$$

The integrability condition $L_V \nabla = 0$ also implies that a Killing vector is obviously an affine Killing, but, the converse is not necessarily true (as per two equations (3.27) and (3.33) of chapter 3). In the following we show that the converse also holds if M is compact and without boundary. Let V be **affine Killing** on compact M without boundary. Contracting (5.22) with g^{jk} yields

$$\Delta V^i + R^i_m V^m = 0. \tag{5.23}$$

Now if we contract (5.22) with $i = k$ then $\nabla_j (div\, V) = 0$, which means that $div\, V$ is constant on M. On the other hand, since $\int_M div\, V = 0$, for M without boundary, we conclude that $div\, V = 0$. Substituting this and (5.23) in the integral formula (5.15) gives $\int_M |L_V g|^2 = 0$ which implies that $L_V g = 0$ and V is Killing. Thus we have the following:

THEOREM 5.9 (Yano [211]) *An affine Killing vector field on a compact orientable Riemannian manifold , without boundary, is Killing.*

For example, it is easy to see that an affine Killing vector field in a non-flat Riemannian manifold of constant curvature (and also non-Ricci-flat Einstein manifold) is Killing. Later on we have presented some examples of **affine Killing vector** fields which are not Killing.

5.3 Killing and Affine Symmetries on Semi-Riemannian Manifolds

Let (M, g) be a semi-Riemannian manifold with an indefinite metric g. Suppose M is compact orientable and without boundary. To the best of our knowledge, it is an open problem to verify some (or all) results discussed so far which may or may not hold for a class of proper semi-Riemannian manifolds. On the other hand, if M has the boundary ∂M, then, as mentioned before in section 5.1, the divergence theorem is not obviously valid for a general case. However, based on a recent work of $\ddot{U}$nal (see theorems 5.4 and 5.5) and proceeding exactly as given on pages 80 and 81, one can prove the following two results:

THEOREM 5.10. *Let $(M,\, g)$ be a compact semi-Riemannian manifold with boundary $\partial\, M$ such that its lightlike part $\partial\, M_0$ has measure zero in $\partial\, M$. A vector field V on M is Killing iff*

$$\triangle\, V + Q\, V = 0 \quad , \quad div\, V = 0 \quad \text{on} \quad M$$

$$(L_V g)(V, N) = 0 \quad \text{on} \quad \partial M' = \partial M_+ \cup \partial M_- \tag{5.24}$$

where N is the unit normal vector field to ∂M induced on $\partial M'$ and all eigenvalues of $L_V\, g$ are real.

THEOREM 5.11. *Let (M, g) be a compact orientable semi-Riemannian manifold with boundary $\partial\, M$. Suppose M admits a vector field V, with compact support, such that V is tangent to $\partial\, M$ at the points of $\partial\, M$ and all eigenvalues of $L_V\, g$ are real. Then, V is Killing iff (5.23) holds.*

Recall that M is connected iff any two points can be joined by a piecewise smooth curve segment. Let M be just a semi-Riemannian manifold. Suppose γ and γ' denote a geodesic and its tangent vector field on M respectively. A vector field V on γ is called a **Jacobi vector field** if it satisfies the following, called the **Jacobi differential equation**:

$$\nabla_{\gamma'}\, \nabla_{\gamma'}\, V = R(\gamma', V)\gamma'. \tag{5.25}$$

Recall the following formula (see page 40)

$$(L_V \nabla)(X, Y) = \nabla_X\, \nabla_Y\, V - \nabla_{\nabla_X Y} V + R(V, X)Y, \tag{5.26}$$

for arbitrary vector fields X, Y on M. Setting $X = Y = \gamma'$ in above formula , we get

$$(L_V \nabla)(\gamma', \gamma') = \nabla_{\gamma'}\, \nabla_{\gamma'}\, V + R(V, \gamma')\, \gamma'. \tag{5.27}$$

Thus it follows from (5.27) and (5.25) that V along γ is a Jacobi vector field iff $(L_V\nabla)(\gamma', \gamma') = 0$. As a immediate consequence of this (where we use the definition of affine Killing symmetry) we can state the following:

PROPOSITION 2. *If V is an affine Killing vector field on a semi-Riemannian manifold and γ is a geodesic, then V is a Jacobi vector field along γ.*

Contracting (2.26) with respect to X, Y and then expressing the result in local coordinates, we get

$$g^{kj} L_V \Gamma^i_{jk} = \triangle V^i + R^i_j V^j.$$

It is well-known that a vector field V, on a semi-Riemannian manifold, is a geodesic vector field if the right hand side of above equation vanishes. It, therefore, follows that V is a geodesic iff $g^{jk} L_V \Gamma^i_{jk} = 0$. In particular, as an immediate consequence of this we can state the following result:

PROPOSITION 3. *An affine Killing vector field in a connected semi-Riemannian manifold is a geodesic vector field, but, the converse need not hold.*

Finally, we close this section by stating the following two results on Killing symmetry (details may be seen in O'Neill [157, pages 252, 254]).

PROPOSITION 4. *If a Killing vector field V is covariant constant at any one point of a connected semi-Riemannian manifold M, then $V = 0$ at every point of M.*

PROPOSITION 5. *On a connected complete semi-Riemannian manifold every Killing vector field is complete.*

5.4 Killing Symmetries in General relativity

In relativity, since the Einstein field equations are a complicated set of non-linear differential equations, most explicit solutions (see Kramer et al. [118]) have been found by the use of Killing symmetries. This is due to the fact that the Killing symmetries leave the Levi-Civita connection and all the curvature quantities invariant (see equations (5.20) and (5.21)). Also, by operating Lie-derivative on both sides of the Einstein equation (4.43), with respect to a Killing vector field V on M, it is easy to see that any prescribed matter tensor T_{ab} is invariant, that is,

$$L_V T_{ab} = 0, \qquad (1 \leq a, b \leq 4). \tag{5.28}$$

In the following we discuss the kinematic and the dynamic properties of Killing symmetries using timelike , spacelike and null congruence as presented in section 4.1 of chapter 4. Let (M, g) be a 4-dimensional spacetime of general relativity with a Killing vector field V. In general, for a unit vector X, we let $L_V X^a = f\, X^a + Y^a$, for some function f and a vector Y such that $Y^a X_a = 0$. Contracting with X_a, using $L_V(X_a X^a) = 0$ and $L_V X_a = L_V(g_{ab} X^b)$, we obtain $f = 0$. Thus, the effect of a Killing vector V on any unit vector X is given by

$$L_V X^a = Y^a \quad , \quad L_V X_a = Y_a \quad , \quad Y^a X_a = 0. \tag{5.29}$$

Given two orthogonal unit vectors X and Z timelike and spacelike respectively, it is always possible to generate a null vector $N = \frac{X+Z}{\sqrt{2}}$ or $\frac{X-Z}{\sqrt{2}}$. Consider first choice. Then, using (5.29) for $L_V X = Y$ and $L_V Z = W$, we obtain

$$L_V N = \frac{Y+W}{\sqrt{2}} \quad , \qquad X \cdot W + Z \cdot Y = 0.$$

In particular, let X be the fluid flow velocity vector u ($u^a u_a = -1$) or a unit spacelike vector n ($n^a n_a = 1$, $n^a u_a = 0$). Also, let N be ℓ or k (the two principal null directions of TM) such that $\ell = \frac{u+n}{\sqrt{2}}$, $k = \frac{u-n}{\sqrt{2}}$ and $g(\ell, k) = -1$. Then, we have

$$L_V u^a = Y^a , \quad L_V u_a = Y_a , \quad Y \cdot u = 0, \tag{5.30}$$
$$L_V n^a = W^a , L_V n_a = W_a , \quad W \cdot n = 0, \tag{5.31}$$
$$L_V \ell^a = \frac{Y^a + W^a}{\sqrt{2}}, \quad L_V \ell_a = \frac{Y_a + W_a}{\sqrt{2}}, \tag{5.32}$$
$$L_V k^a = \frac{Y^a - W^a}{\sqrt{2}}, \quad L_V k_a = \frac{Y_a - W_a}{\sqrt{2}}, \tag{5.33}$$

such that

$$W \cdot u + Y \cdot n = 0 , \quad \ell \cdot (Y + W) = 0 , \quad k \cdot (Y - W) = 0.$$

In general, let $V = \lambda u + \xi$ such that $\lambda = -u_a V^a$ and $\xi^a u_a = 0$. Then, using (see equation (4.7) of chapter 4)

$$u_{a;b} = \sigma_{ab} + \frac{1}{3} \theta \, h_{ab} + \omega_{ab} - \dot{u}_a \, u_b,$$

we obtain

$$L_V u_a = \dot{\lambda} u_a + \lambda \, (\dot{u}_a - (log \, \lambda)_{,a}) + \xi^b \, \dot{u}_b \, u_a + 2 \, \omega_{ab} \, \xi^b, \tag{5.34}$$

where the over dot indicates the covariant derivative in the direction of the fluid flow. Now equating the second equation of (5.30) with (5.34) followed by successive contractions by u^a and h^{ab} implies

$$\dot{\lambda} = -\dot{u}_a \, V^a , \quad Y_a = 2 \, \omega_{ab} \, V^b + \lambda \, (\dot{u}_a - (log \, \lambda)_{,a}). \tag{5.35}$$

There are three special cases of interest which we discuss separately.

Case 1. $V = \lambda \, u$, then, $\xi \equiv 0$, $\omega_{ab} V^b = 0$ and $\dot{u}_a \, V^a = 0$. Also $Y \equiv 0$ (since $L_V u$ is clearly parallel to u). Thus, (5.35) reduces to

$$\dot{\lambda} = 0 , \quad \dot{u}_a = (log \, \lambda)_{,a}.$$

Contracting successively the Killing equation $L_V g_{ab} = 0$ with the **projection tensor** $h^{ab} = g^{ab} + u^a \, u^b$ (of the **congruence of timelike curves**) and $h^a_c \, h^b_d - \frac{1}{3} \, h^{ab} \, h_{cd}$ we obtain $\theta \equiv 0 \equiv \sigma_{ab}$. Thus, we have proved the following theorem (the converse is straightforward):

THEOREM 5.12. *A 4-dimensional spacetime (M, g) admits a timelike Killing vector field V parallel to the fluid flow velocity vector u ($V = \lambda u$, $\lambda > 0$) iff*

(1) *M is expansion-free , that is, $\theta \equiv 0$,*

(2) *M is shear-free, that is, $\sigma_{ab} \equiv 0$,*

(3) $L_V u = 0\,, \qquad \dot{\lambda} = 0\,, \qquad \dot{u}_a = (log\,\lambda)_{,a}\,,$

where σ_{ab}, θ and $\dot{u}$ are the shear, expansion and acceleration of the timelike congruence generated by u.

It is important to mention that above results are purely kinematic and, therefore, valid for any fluid matter tensor. To further examine the dynamic properties of this case we let (M, g) satisfy the **Einstein field equations** of perfect fluid for which

$$R_{ab} - \frac{1}{2}\,(r - 2\Lambda)g_{ab} = T_{ab} = \mu\, u_a\, u_b + p\, h_{ab}. \tag{5.36}$$

Lie derivative of (5.36) with respect to V, using (5.28) and $L_V u_a = 0$ provides

$$L_V \mu = 0\,, \qquad L_V p = 0.$$

In the above case, we say that the three physical quantities (μ, p, u) inherit symmetry (in particular these three quantities are invariant for Killing symmetry) with respect to a Killing vector field. The reader will find in chapter 8 of this book formal definition of **symmetry inheritance** with respect to a given symmetry vector and its important role in the study of fluid spacetimes.

Case 2. $V = \xi$, then $\lambda = 0$ and $V^a u_a = 0$. Therefore, (5.35) reduces to

$$\dot{u}_a\, V^a = 0\,, \qquad Y_a = 2\,\omega_{ab}\, V^b.$$

In particular we are interested in the following result:

THEOREM 5.13. *A 4-dimensional spacetime (M, g) admits a spacelike Killing vector field $V = \alpha n$ $(n \cdot n = -1\,, u \cdot n = 0\,, \alpha > 0)$ iff*

(1) *M is expansion free, that is, $\bar{\theta} \equiv 0$,*

(2) *M is shear free, that is, $\bar{\sigma}_{ab} \equiv 0$,*

(3) $\overset{\bullet}{n}_a = -(log\alpha)_{,a} \qquad \overset{\bullet}{\alpha} = 0,$

(4) $w_a \overset{\circ}{n}{}^a = 0\,, \qquad p^b_a(\overset{\circ}{n}_b + w^c\, n_{c;b}) = 0,$

where $\bar{\sigma}_{ab}$ and $\bar{\theta}$ are, respectively the shear and the expansion of the spacelike congruence generated by n and as measured by an observer with four velocity w $(w_a\, w^a = -1, w_a\, n^a = 0)$.

PROOF. Suppose M admits a spacelike Killing vector $V = \alpha\, n$. Contracting the Killing equation $V_{a;b} + V_{b;a} = 0$ in turn with the tensors $w^a\, w^b$, $w^a\, p^{bc}$, $n^a\, n^b$, $n^a\, p^{bc}$ and $p^{ac}\, p^{bd}$, with the aid of (4.15) and (4.16) on page 59, we obtain

(a) $w^a\, w^b$: $\quad w_a \overset{\circ}{n}{}^a = 0,$

(b) $w^a\, n^b$: $\quad w^a\,(\overset{\bullet}{n}_a + (log\,\alpha)_{,a}) = 0,$

(c) $w^a\, p^{bc}$: $\quad p^{ab}\,(\overset{\circ}{n}_b + w^c\, n_{c;b}) = 0,$

(d) $n^a\, n^b$: $\quad \overset{\bullet}{\alpha} = 0,$

(e) $n^a\, p^{bc}$: $\quad p^{ab}\,(\overset{\bullet}{n}_b + (log\,\alpha)_{,b}) = 0,$

(f) $p^{ac}\, p^{bd}$: $\quad \bar{\sigma}_{ab} + \frac{1}{2}\,(\bar{\theta})\, p_{ab} = 0.$

Taking the trace of (f) and then using $\bar{\sigma}^a_a \equiv 0$, we obtain $\theta \equiv 0$ and, therefore $\bar{\sigma}_{ab} \equiv 0$. So, (1) and (2) holds. Now (b) and (e) implies that $\overset{\bullet}{n}_a + (log\,\alpha)_{,a} = f\, n_a$, for some function f. Contracting this with n^a provides $f = (log\,\alpha)^\bullet$. Then, using (d) implies that $f = 0$. This together with (d) means that (3) holds. Finally, (4) follows directly from (a) and (c). The converse follows by the reverse argument.

Also, it is easy to see that , for $V = \alpha\, n$, we have

$$L_V u_a = 2\,\alpha\,\omega_{ab}\, n^b\,, \qquad L_V n_a = 0. \tag{5.37}$$

Again we mention that above results are purely kinematic, and, therefore, valid for any fluid matter field. To examine the dynamic properties of this case, we let M satisfy the **Einstein field equations** (5.36). Taking the Lie derivative of (5.36), with respect to $V = \alpha\, n$, using (5.28) and (5.37) we obtain

$$L_V \mu = 0\,, \qquad L_V p = 0.$$

Thus, it follows that, for $V = \alpha\, n$, the three physical quantities (μ, p, u) will inherit Killing symmetry if either V (and, therefore, n) is parallel to the **vorticity vector** ω^a (since $\omega_{ab}\,\omega^b = 0$) or the vorticity vanishes.

Case 3. $V = \ell = \frac{u+n}{\sqrt{2}}$ (results are similar for $V = k$), one of the principal null directions. Using the equation (4.21) which is

$$\ell_{a;b} = \frac{\hat{\theta}}{2}\,\hat{h}_{ab} + \hat{\sigma}_{ab} + \hat{\omega}_{ab},$$

and the Killing equation $\ell_{a;b} + \ell_{b;a} = 0$ and then contracting with $\hat{h}_{ab}$ provides $\hat{\theta} = 0$ and, therefore, $\hat{\sigma}_{ab} = 0$. Also general $V = \lambda\, u + \xi$ provides $V = \ell$ or k, $\lambda = \frac{1}{\sqrt{2}}$ and $\xi = (\,\frac{1}{\sqrt{2}}$ or $-\frac{1}{\sqrt{2}}\,)n$. Thus we can state the following theorem:

THEOREM 5.14. *A* 4*-dimensional spacetime* (M, g) *admits a null Killing vector field* V *(aligned with any of the two principal directions* ℓ *or* k *) iff*

(1) *M is expansion free, that is,* $\hat{\theta} \equiv 0,$

(2) *M is shear-free, that is,* $\hat{\sigma}_{ab} \equiv 0,$

(3) $L_\ell\, u_a = \sqrt{2}\,\omega_{ab}\, n^b = -L_k\, u_a, \quad L_k\, n_a = \sqrt{2}\,\omega_{ab}\, n^b = -\, L_\ell\, n_a.$

where $\hat{\sigma}_{ab}$ and $\hat{\theta}$ are respectively the shear and the expansion of the null congruence generated by V.

Again above results are purely kinematic and for dynamic results we let M satisfy the Einstein field equations (5.36). Taking the Lie derivative of field equations, with respect to ℓ or k, using (5.28) and **(3)** of theorem 5.14 we get

$$L_V\,\mu = 0\,, \qquad L_V\,p = 0\,, \qquad V = \ell \text{ or } k.$$

Thus, if V is a null vector field (aligned with ℓ or k) then the three quantities (μ, p, u) will inherit symmetry with respect to V if either the spacelike congruence vector n is parallel to the vorticity vector ω or the vorticity vanishes.

We now illustrate above kinematic and dynamic results by the following example.

Gödel Spacetimes. The line element is given by

$$ds^2 = -(dx^0)^2 - 2\,e^{Ax^1}\,dx^0\,dx^2 + (dx^1)^2 + \frac{1}{2}\,e^{2\,Ax^1}\,(dx^2)^2 + (dx^3)^2,$$

where A is a non-zero constant and $(t = x^0, x^1, x^2, x^3)$ are local coordinates. The non-zero Christoffel symbols are

$$\Gamma^0_{01} = \Gamma^0_{10} = A\,,\ \Gamma^2_{01} = \Gamma^2_{10} = -Ae^{-Ax^1}\,,$$

$$\Gamma^0_{12} = \Gamma^0_{21} = (A/2)\,e^{Ax^1} = \Gamma^1_{02} = \Gamma^1_{20} = \Gamma^1_{22}\,,$$

and

$$u^a = \delta^a_0\,,\ \dot{u}^a = 0\,,\ \omega^a = (A/\sqrt{2})\,\delta^a_3\,,$$

$$\sigma_{ab} = 0 = \theta\,,\quad \omega_{a;b} = 0.$$

This metric admits the following five Killing vectors

$$\begin{aligned}
V^a_1 &= u^a\,,\quad V^a_2 = \delta^a_2\,,\\
V^a_3 &= -2e^{-Ax^1}\,u^a + Ax^2\,\delta^a_1 + (e^{-2Ax^1} - \frac{1}{2}(Ax^2))\delta^a_2\,,\\
V^a_4 &= \delta^a_3\,,\quad V^a_5 = \delta^a_1 - A\,x^2\,\delta^a_2.
\end{aligned}$$

Here V_1, V_2 and V_3 are timelike , V_4 is spacelike and V_5 is timelike , spacelike or null depending on the values of A, x^1 and x^2. The reader is invited to verify other kinematic and dynamic properties of Gödel spacetimes.

For the case of null Killing vector fields, we consider the Einstein field equations such that the matter tensor belongs to a class of **pure radiation fields**, that is,

$$T_{ab} = \phi^2\,\ell_a\,\ell_b, \tag{5.38}$$

where ϕ is some function on M. Let $V = \lambda\,u + \xi$ be the general expression of a Killing vector field on M. Taking L_V of (5.38) and using (5.28), we obtain

$$(L_V \ell_a)\,\ell_b + \ell_a\,(L_V \ell_b) = -2\,V^c\,(log\,\phi)_{;c}\,\ell_a\,\ell_b.$$

Contracting both sides of above equation with k^b, we obtain

$$L_V \ell_a = V^c (log\,\phi)_{;c}\, \ell_a. \tag{5.39}$$

Above result is same if we replace ℓ by the other principal null direction k. Thus, we also have

$$L_V\, k_a = V^c (log\,\phi)_{;c}\, k_a. \tag{5.40}$$

The field equations, with matter field (5.38), have a relation to the null Einstein Maxwell spacetimes for which ϕ is one of the three Maxwell scalars of the electromagnetic field F (see equation (4.38) of chapter 4) and F is singular. If this is the case, then related to our theorem 5.14 , we state the following (without proof) which is a reformulation of the Goldberg-Sachs theorems [77] on shear-free geodesic null congruence.

THEOREM 5.15 (Mariot and Robinson cited in [118, page 90]). *An arbitrary spacetime (M, g) admits a geodesic shear free null congruence iff M admits an electromagnetic null field (test field) satisfying the Maxwell equations.*

Furthermore, if ϕ is constant, then we say that $(M,\ g)$ is homogeneous null Einstein Maxwell spacetime. Note that for ϕ constant, the equations (5.38) and (5.39) indicate that the two principal directions are invariant with respect to the Killing vector field V. For this class we have the following main result (see Kramer et al. [118, page 119]).

THEOREM 5.16. *The plane waves*

$$ds^2 = 2\,dz\,d\bar{z} - 2\,e^{\epsilon\,u}\,du\,dv - 2\,du^2\,(a^2\,z\,\bar{z} + b\,Re\,(z^2\,e^{-ic\,u}))$$

represents all homogeneous null Einstein Maxwell fields and all vacuum homogeneous solutions with a multiple-transitive group.

Note that the spacetimes, with plane waves solutions, admit a covariant constant null vector field which may be aligned with one of the principal null vector fields ℓ or k. To help reader understand the physical significance of this condition, we recall that such spacetimes are called **plane-fronted gravitational wave** with parallel rays, briefly denoted by *pp*-**waves** and has been widely studied. In general, plane waves can be interpreted as gravitational fields at a large distance from finite bodies.

Now let (M, g) belong to a class of **Einstein Maxwell spacetimes** for which the electromagnetic field F_{ab} is non-singular. As explained in section 4.2 of chapter 4, for this class ϕ_1 is the only surviving Maxwell scalar. We are interested in a simple F_{ab} and, therefore, ϕ_1 is either real or pure imaginary. For this sub case, it follows from (4.32) that its canonical form is given by

$$F_{ab} = -2\,Re(\phi_1)\,\ell_{[a}n_{b]} \;\text{ or }\; 2\,i\,Im(\phi_1)\,m_{[a}\bar{m}_{b]}\,, \quad det(F_{ab}) = 0. \tag{5.41}$$

Consider a homogeneous spacetimes for which ϕ_1 is constant (see Kramer et al [118, page 120]). Set $|{\phi_1}^2| = 1$ for both the real or pure imaginary cases.

Based on above, we prove that under a reasonable geometric condition, M admits a 2-parameter Abelian group of Killing vector fields. For this purpose, define a $(1,1)$ tensor field $f \equiv (f^a_b)$, on the tangent space $T_p(M)$, at each point $p \in M$, such that

$$f^a_b = g^{ac}\, F_{cb}\,,\ i.e.,\quad F(X,Y) = g(X, fY), \tag{5.42}$$

for every $X, Y \in \mathcal{X}(M)$. Since most of the results also hold globally, we prefer using index free notations (reader may use information from chapter 2 for its local coordinates version). It follows from the well-known **Cayley-Hamilton theorem**, that f satisfies its own minimum characteristic polynomial equation: $f^3 \pm f = 0$, where the sign $\pm$ depends on the choice of $Im(\phi)$ or $Re(\phi)$. We choose (other case is similar)

$$f^3 + f = 0\,, \qquad rank(f) = 2. \tag{5.43}$$

Since F is skew-symmetric, (5.42) implies that f is skew-symmetric with respect to g, that is,

$$g(X, f\,Y) + g(f\,X, Y) = 0\,, \qquad \forall X, Y \in \mathcal{X}(M). \tag{5.44}$$

In the above case, we say that M admits a differential geometric structure (g, f) called **metric f-structure**, first introduced by Yano [207] in 1963. It is important to mention that a homogeneous spacetime (M, g), with simple F, inherits a metric f-structure without imposing any geometric condition. Using $rank\,(f) = 2$, we consider two complementary orthogonal projection operators P and Q defined by

$$P = -f^2 \quad , \quad Q = f^2 + I\,, \tag{5.45}$$

where I is the identity operator. Then, there exist two complementary orthogonal distributions D and $D^\perp$, with $dim(D) = dim(D^\perp) = 2$ and

$$f\,P = P\,f = f\,,\ f\,Q = Q\,f = 0\,,\ f^2\,P = -P\,,\ f^2\,Q = 0. \tag{5.46}$$

Based on above, we have the following decomposition of TM:

$$TM = D \oplus D^\perp\,,\ D \cap D^\perp = \{0\}\,,\ f\,D = D\,,\ f(D^\perp) = 0. \tag{5.47}$$

Consider a real orthonormal frame $\{\ T, U, V, W\ \}$ of the tangent space $T_p(M)$ at each point p of M and denote its dual set by $\{t, u, v, w\ \}$, such that T is timelike and others are spacelike. Then, $\forall X, Y \in \mathcal{X}(M)$,

$$\begin{aligned} g(T,X) &= -t(X) \quad g(U,X) = u(X), \\ g(V,X) &= v(X) \quad\ g(W,X) = w(X), \\ g(X,Y) &= -t(X)t(Y) + u(X)u(Y) + v(X)v(Y) + w(X)(Y). \end{aligned} \tag{5.48}$$

The equation (5.47) allows to set $D = span\,\{U, V\}$ and $D^\perp = span\,\{T, W\}$ which we use. Thus, it follows from (5.45) and (5.46) that

$$fU = V, \quad fV = -U, \quad fT = 0 = fW,$$

$$u \circ f = v, \quad v \circ f = -u, \quad t \circ f = 0 = w \circ f,$$

$$fX = u(X)V - v(X)U,$$

$$f^2X = -X + t(X)T + w(X)W.$$

PROPOSITION 6. *Let (M, g) be a homogeneous spacetime with a simple non-singular electromagnetic field F and a metric f-structure (g, f). Then*

$$L_T F = 0, \qquad L_W F = 0\,,$$

where $\{T, W\}$ spans the distribution $D^{\perp}$.

PROOF. Using $L_X = d\,i(X) + (i\,X)d$ (see equation (3.10) of chapter 3), where $i\,(X)$ is the inner product by X, we get $L_T F = d\,i(T)\,F + i\,(T)\,d\,F = d\,i(T)\,F)$ since $d\,F = 0$. Now $(iT\,F)Y = F(T, Y) = w \circ f\,Y = 0$, $\forall Y \in \mathcal{X}(M)$. Thus, $L_T = 0$. Similarly, it follows that $L_W F = 0$.

Using differential geometric terminology (see Goldberg-Yano [78]), we say that M has a **normal metric f-structure** if the torsion tensor S_f of f vanishes, that is, if

$$S_f \equiv N_f(X, Y) + dt(X, Y)T + dw(X, Y)W = 0\,, \tag{5.49}$$

where

$$N_f(X, Y) \equiv [f, f](X, Y) = [fX, f\,Y] + f^2[X, Y] - f([X, f\,Y] + [fX, Y])$$

is the Nijenhuis tensor field of f. Details on Nijenhuis tensor and its properties may be seen in Kobayashi-Nomizu [111, page 141]. Furthermore, it follows from the work of Newlander-Nirenberg [150] that M has an **integrable f-structure** iff its Nijenhuis tensor vanishes. Physically it is desirable that the f-structure is integrable. However, there are considerable amount of geometric results free from the condition of normality and integrability. Details on this may be seen in Goldberg-Yano [78] and many other papers referred there in.

PROPOSITION 7. *Let (M, g) be a homogeneous spacetime with a simple non-singular electromagnetic tensor field F and a normal f-structure (g, f, N_f). Then*

$$L_T f = 0\,, \quad L_W f = 0\,, \quad [T, W] = 0. \tag{5.50}$$

PROOF. Setting $Y = T$ in (5.49) and then taking the inner product of the resulting equation with T and W successively, we obtain

$$dt\,(X, T) = 0\,, \quad dw\,(X, T) = 0.$$

Now applying f to (5.49), with $Y = T$, and then using above and f-structure equations (5.48) we get $L_T f = 0$. Similarly, by setting $Y = T$ in (5.49) we obtain $L_W f = 0$. Finally, setting $X = T$ and $Y = W$ in the Nijenhuis tensor it follows that $[T, W] = 0$.

THEOREM 5.17 (Duggal [43]). *Let (M, g) be a homogeneous spacetime with a simple electromagnetic field F and a normal f-structure (g, f, N_f). Then, M admits a 2-parameter group of Killing vector fields.*

PROOF. Using (5.42) and proposition 6, we obtain

$$\begin{aligned}(L_T F)(X,Y) &= T\,F(X,Y) - F([T,X],Y) - F(X,[T,Y]) \\ &= T\,g(X,fY) - g([T,X],fY) - g(X,[T,fY]) \\ &= (L_T g)(X,fY) = 0.\end{aligned}$$

Thus, $(L_T g)(X,Z) = 0$ for all $X \in \mathcal{X}(\mathcal{M})$ and all $Z \in D$. On the other hand, $(L_T g)(T,W) = -g([T,T],W) - g(T,[T,W]) = 0$ due to (5.50). Thus, $(L_T g)(X,Y) = 0$ for all $X, Y \in \mathcal{X}(M)$. The proof is similar for $L_W g = 0$. Since $[T,W] = 0$, we conclude that M admits a 2-parameter group, generated by $\{T, W\}$, of Killing vector fields.

Observe that T and W are timelike and spacelike Killing vectors. It is possible to get a 2-parameter group, generated by $\{U, V\}$, of two spacelike Killing vector fields by reversing the role of D and $D^{\perp}$. As an example, we state the following known solution for which the details may be seen in Kramer et al. [118, page 120].

THEOREM 5.18. *The only Einstein Maxwell field which is homogeneous and has a homogeneous non-singular Maxwell field is the Bertotti-Robinson solution*

$$ds^2 = A^2(d\theta^2 + \sin^2\theta\, d\phi^2 + dx^2 + sinh^2 x\, dt^2)$$

where (t, x, θ, ϕ) and A are local coordinates and an arbitrary constant respectively.

This solution has two families of orthogonal 2-surfaces having equal and opposite curvatures. Relating these 2-surfaces with D and $D^{\perp}$, we observe that they are integrable. Thus **Bertotti-Robinson spacetime** has an integrable f-structure.

Also, see Duggal [42, 43, 44, 45] and Duggal-Moskal [58], on applications of f-structures in Einstein Maxwell spacetimes with Killing symmetry, where general case of non-singular (not necessarily simple) and singular F is studied.

REMARK. The subject matter on Killing vectors is indeed very wide and can not be completely covered in one book. We have included as much possible material which is consistent with some of the following chapters, of this book, on higher symmetries. The reader is invited to see Kramer et al. [118] and several references there in for more information on Killing vectors. Finally, we state the following important result of Moncrief and Isenberg [146]:

Analytic vacuum and electro vacuum spacetimes, containing a compact lightlike hypersurface ruled by closed generators, do have a non-trivial Killing symmetry.

5.5 Affine Collineations in General Relativity

Let (M, g) be a 4-dimensional spacetime manifold of general relativity. Recall from chapter 3 that a vector field V is said to define a symmetry called **affine collineation** iff V leaves the Levi-Civita connection of g invariant, which is further equivalent to

$$L_V g_{ab} = 2\,K_{ab}\,, \qquad K_{ab;c} = 0, \tag{5.51}$$

where K_{ab} is a covariant constant symmetric tensor of order 2. The vector V is then called an **affine vector field**. V is proper affine if K_{ab} is other than g_{ab}.

In general, for an n-dimensional manifold (M, g), the existence of proper K_{ab} has its roots back in 1923, when Eisenhart [65] proved that a Riemannian M admits a proper K_{ab} iff M is reducible. This means that M is a **product manifold** of the form $(M = M_1 \times M_2\,,\, g = g_1 \oplus g_2\,)$ and there exists a local coordinate system in terms of which the distance element of g is given by

$$ds^2 = g_{ab}(x^c)\,dx^a\,dx^b + g_{AB}(x^C)\,dx^A\,dx^B,$$

where $a, b, c = 1, \dots, r\,, A, B, C = r+1, \dots, n$ and $1 \le r \le n$. Based on above, it follows that *an irreducible Riemannian manifold does not admit any proper affine vector field.*

Eisenhart's result was generalized by Patterson [160], in 1951, showing that a semi-Riemannian M admitting a proper K_{ab} is reducible if the matrix of K_{ab} has at least two distinct characteristic roots at any point of M. Since then, a general characterization of affine collineations remains open. However, for a 4-dimensional spacetime M, this problem has been completely resolved (see Hall and da Costa [83]). Global study requires the spacetime to be simply connected (which means that any closed loop through any point can be shrunk continuously to that point), and for local considerations one may restrict to a simply connected region. We now know, from the works of above mentioned researchers, that

If a simply connected spacetime M admits a global, nowhere zero, covariant constant proper K_{ab}, then one of the following three possibilities exist:

(a) *There exists locally a timelike or spacelike, nowhere zero covariant constant vector field ξ such that*

$$K_{ab} = \eta_a\,\eta_b \qquad , \qquad \eta_a = g_{ab}\,\xi^b \tag{5.52}$$

and M is then locally decomposable into $(1+3)$ *spacetime.*

(b) *There exists locally a null, nowhere zero, covariant constant vector field ξ such that K_{ab} is as given in the case* **(a)** *but M, in general, is not reducible.*

(c) *M is locally reducible into a* $(2+2)$ *spacetime and no covariant constant vector exists unless M decomposes further into* $(1+1+2)$ *spacetime (a special case of* **(b)***). For the latter case, there exist two covariant constant vector fields for two such proper covariant constant tensors of order* 2.

The metrics of locally decomposable spacetimes can be written as one of the following:

$$
\begin{aligned}
ds^2 &= \epsilon\,(dt)^2 + g_{ab}\,dx^a\,dx^b \qquad (a,b=1,2,3), &(5.53)\\
ds^2 &= \epsilon_1\,(dt)^2 + \epsilon_2\,(dx^1)^2 + G_{pq}\,dx^p\,dx^q \qquad (p,q=2,3),\\
ds^2 &= g_{AB}\,dx^a\,dx^b + G_{pq}\,dx^p\,dx^q \qquad (A,B=0,1), &(5.54)
\end{aligned}
$$

where ϵ, ϵ_1 and ϵ_2 are ± 1 and the signatures of the lower dimensional metrics are compatible with over all Lorentz signature. The lower dimensional matrics are further indecomposable.

The set S of all second order covariant constant symmetric tensors, on M, is a vector space which, if M is non-flat, satisfies $1 \leq dim(S) = n \leq 4$ (see Hall and da Costa [83]. If T is the subspace of members of S that arise from affine collineations, then $dim(T) = m \leq n$. Furthermore, if M admits an r-dimensional algebra of affine collineations, then the Killing sub algebra has dimension $r - m$. Obviously, M admits a homothetic vector field iff g is in T. For simply connected non-flat M, or simply connected region of M, $dim(S) = 1, 2$ or 4 and for proper affine collineations $dim(S) = 2$ or 4.

Define an **affine bivector** (skew-symmetric tensor) $F_{ab} = \frac{1}{2}(V_{a;b} - V_{b;a})$ with respect to an affine vector field. Suppose $dim(S) = 2$ so that M admits two proper affine vectors, say V and W. In general, it is possible to write

$$
\begin{aligned}
V_{a;b} &= A\,K_{ab} + B\,g_{ab} + F_{ab},\\
W_{a;b} &= C\,K'_{ab} + D\,g_{ab} + F'_{ab},
\end{aligned}
$$

where F and F', K and K' are affine bivectors and covariant constant tensors of V and W respectively. Also, A, B, C are reals such that $A, C \neq 0$. It is easy to see that $C\,V - A\,W$ is either homothetic or Killing. Thus, there is only one proper affine vector and any other is a linear combination of this one and homothetic or Killing. This case corresponds to metrics (5.53) or (5.54). When, $dim(S) = 4$, there are 3 independent affine vectors in the above sense.

Using Einstein field equations, with zero cosmological constant, one can show that (see Hall et al [92]) the existence of a proper affine collineation eliminates all vacuum spacetimes except the plane waves, all perfect fluids for which $p \neq \mu$ and all non-null Einstein Maxwell fields except the $(2+2)$ locally decomposable case. Hence, affine collineations have a very restrictive use, particularly, in the study of exact solutions. However, the reader will find in this book that such symmetries are useful in several other aspects such as field equations with viscous fluids.

Consistent with the previous section, we now discuss kinematic and dynamic properties of affine collineations. Let (M, g) admit an affine vector field V. In the sequel, we set $f_X = K_{ab}X^a\,X^b$ for any vector field X of M. Following as in the case of Killing symmetry, one can show the effect of V on any non-null unit vector X is given by

$$
\begin{aligned}
L_V X^a &= -\epsilon\,f_X\,X^a + Y^a,\\
L_V X_a &= -\epsilon\,f_X\,X_a + Y_a,
\end{aligned}
$$

where Y is some vector orthogonal to X and $\epsilon = +1$ or -1 according as X is spacelike or timelike. Also, choose a null vector $N = \frac{X+Z}{\sqrt{2}}$, where X and Z are spacelike and timelike unit vectors respectively.

Now consider a fluid flow velocity vector u ($u^a u_a = -1$), a unit spacelike vector n orthogonal to u and the two principal null directions $\ell = \frac{u+n}{\sqrt{2}}$ and $k = \frac{u-n}{\sqrt{2}}$. Then, using above general equations, we obtain

$$L_V u^a = f_u u^a + Y^a, \qquad L_V u_a = -f_u u_a + 2 K_{ab} u^b + Y_a, \tag{5.55}$$

$$L_V n^a = -f_n n^a + W^a, \qquad L_V n_a = -f_n n_a + 2 K_{ab} n^b + W_a, \tag{5.56}$$

$$L_V \ell^a = S^a + \frac{Y^a + W^a}{\sqrt{2}}, \qquad \sqrt{2} S^a = f_u u^a - f_n n^a, \tag{5.57}$$

$$L_V k^a = Q^a + \frac{Y^a - W^a}{\sqrt{2}}, \qquad \sqrt{2} Q^a = f_u u^a + f_n n^a, \tag{5.58}$$

$$W^a u_a + Y^a n_a = -2 K_{ab} u^a n^b.$$

In general, let $V = \lambda u + \xi$ such that $\lambda = -u_a V^a$ and $\xi^a u_a = 0$. Then using the covariant derivative of u_a we obtain (5.34) which together with (5.55), gives upon contraction with u and h^{ac} ,

$$\dot{\lambda} = -\dot{u}_a \xi^a - f_u \,, \; Y_a = 2 \omega_{ab} \xi^b + \lambda(\dot{u}_a - (log\lambda)_{,a}) - 2K_{ab} u^b - f_u u_a. \tag{5.59}$$

There are three special cases of interest which we discuss separately.

Case 1. $V = \lambda u$, then , $\xi \equiv 0$.

THEOREM 5.19. *A 4-dimensional spacetime (M, g) admits a timelike collineation vector field V parallel to the fluid flow velocity vector u ($V = \lambda u, \lambda > 0$) iff*

(1) $L_V u^a = f_u u^a$, $\quad L_V u_a = -f_u u_a + 2 K_{ab} u^b$,

(2) $\dot{u}_a = (log\, \lambda)_{,a} + \frac{1}{\lambda}(f_u u_a + 2 K_{ab} u^b)$,

(3) $\theta = \frac{1}{\lambda} h^{ab} K_{ab}$,

(4) $\sigma_{cd} = \frac{1}{\lambda}(h^a_c h^b_d - \frac{1}{3} h^{ab} h_{cd}) K_{ab}$,

where σ_{ab}, θ , $\dot{u}$ are the shear, expansion and acceleration of the timelike congruence generated by u.

PROOF. Since V is parallel to u, $Y \equiv 0$ which implies from (5.55) and (5.59) that **(1)** and **(2)** hold. Now contracting the affine Killing equation $V_{a;b} + V_{b;a} = 2 K_{ab}$ with the projection tensor h^{ab} and also with $h^a_c h^b_d - \frac{1}{3} h^{ab} h_{cd}$ we obtain **(3)** and **(4)**. The converse also holds.

Case 2. $V = \xi$, then $\lambda = 0$ and $V \cdot u = o$. Therefore, (5.59) reduces to

$$\dot{u} \cdot V = -f_u \,, \qquad Y_a = 2 \omega_{ab} V^b - f_u u_a - 2 K_{ab} u^b.$$

It, therefore, follows from above equation that if V is parallel to the vorticity vector ω^a or if the vorticity is zero, then $Y_a = 0$ iff u is an eigenvector of K_{ab}. In particular, if V is parallel to n, then we have the following result.

THEOREM 5.20. *A 4-dimensional spacetime (M, g) admits a spacelike affine vector field $V = \alpha\, n$ $(\, n \cdot n = 1, u \cdot n = 0, \alpha > 0)$ iff*

(1) $\quad L_V\, n^a = -f_n\, n^a\,, \qquad L_V\, n_a = -f_n\, n_a + 2\, K_{ab}\, n^b\,,$

(2) $\quad L_V u^a = -f_u\, u^a - 2\, K^a_b\, u^b + 2\, \alpha\, \omega^a_b\, n^b\,,$

(3) $\quad L_V u_a = -f_u\, u_a + 2\, \alpha\, \omega_{ab}\, n^b\,,$

(4) $\quad \overset{\bullet}{n}_a = -(log\, \alpha)_{,a} + \frac{2}{\alpha}\, K_{ab}\, n^b\,, \qquad \overset{\bullet}{\alpha} = f_n\,,$

(5) $\quad \bar{\theta} = \frac{1}{\alpha}\, p^{ab}\, K_{ab}\,,$

(6) $\quad \bar{\sigma}_{ab} = \frac{1}{\alpha}\, (p^c_a\, p^d_b - \frac{1}{2}\, p^{cd}\, p_{ab}) K_{cd},$

where $\bar{\theta}$ and $\bar{\sigma}_{ab}$ are respectively expansion and shear of the spacelike congruence generated by n.

PROOF. Proceeding exactly as in the case of theorem 5.13, one can show that all the six results hold.

Case 3. $V = \ell = \frac{u+n}{\sqrt{2}}$ (the results are similar if $V = k$) is one of the principal null directions. It follows from the general expression of $V = \lambda\, u + \xi$, that for $V = \ell$, $\lambda = \frac{1}{\sqrt{2}}$ and $\xi = \frac{n}{\sqrt{2}}$. Using this, the equation (4.21), of chapter 4, and the affine Killing equation $V_{a;b} + V_{b;a} = 2\, K_{ab}$, we can state the following theorem.

THEOREM 5.21. *A 4-dimensional spacetime (M, g) admits a null affine vector field V (aligned with one of the principal null directions ℓ or k) iff*

(1) $\quad \hat{\theta} = \hat{h}^{ab}\, K_{ab}\,,$

(2) $\quad \hat{\sigma}_{ab} = (\hat{h}^c_a\, \hat{h}^d_b - \frac{1}{2}\, \hat{h}_{ab}\, \hat{h}^{cd}) K_{cd},$

(3) $\quad L_V\, u_a = \sqrt{2}\, \omega_{ab}\, n^b,$

(4) $\quad L_V n_a = -\sqrt{2}\, \omega_{ab}\, n^b,$

where $\hat{\theta}$ and $\hat{\sigma}_{ab}$ are respectively the expansion and the shear of the null congruence generated by ℓ or k.

Note that all the above three theorems are purely kinematic results and their dynamic properties depend on the existence of a covariant constant K_{ab} (based on recent work of Hall and da Costa [83]) and an allowable matter tensor. In general let $(M,\, g)$ satisfy the Einstein field equations (4.43) with zero cosmological constant. Taking its Lie derivative with respect to an affine vector V, we get

$$L_V T_{ab} = (R_{cd}\, K^{cd})\, g_{ab} - r\, K_{ab}, \tag{5.60}$$

where we have used $L_V R_{ab} = 0$ and $L_V r = -2 R_{ab} K^{ab}$. Thus, contrary to the case of Killing symmetry, T_{ab} is not invariant with respect to a proper affine collineation symmetry. Thus, not only that many physical cases (such as perfect fluids) must be excluded for this symmetry, it is not very effective in generating exact solutions due to the non-invariant property of matter tensor. However, in the area of anisotropic fluids this symmetry has some interesting properties which we discuss as follows. Consider a fluid with a preferred direction of pressure anisotropy and no energy flux, for which we have

$$T_{ab} = \mu\, u_a\, u_b + p_{||}\, n_a\, n_b + p_\perp\, S_{ab}, \tag{5.61}$$

where μ is the total energy density, n is the unit spacelike vector along the dynamically preferred direction, $S_{ab} = h_{ab} - n_a\, n_b$ is the projection tensor into the local 2-planes of pressure isotropy (for which $S_{ab}\, u^b = 0 = S_{ab}\, n^b$), and $p_{||}$ and $p_\perp$ are the pressure along and orthogonal to n respectively. We assume that $p_{||} \neq p_\perp$, since otherwise T_{ab} reduces to a perfect fluid which is not allowed for this symmetry. Taking the Lie derivative of both sides of (5.61), with respect to V, and then using (5.51), (5.55), (5.56) and (5.61), we obtain

$$\begin{aligned}
\{L_V\, \mu + 2\,(\mu + p_\perp) f_u\} u_a\, u_b \;&+\; (L_V\, p_\perp)\, S_{ab} + \{L_V\, p_{||} - 2\,(p_{||} - p_\perp) f_n\} n_a\, n_b \\
&+\; 2(\mu + P_\perp) u_{(a} Y_{b)} + 2(p_{||} - p_\perp) n_{(a} W_{b)} + 2 p_\perp K_{ab} \\
&+\; 4(\mu + p_\perp) u_{(a} K_{b)c}\, u^c + 4(p_{||} - p_\perp)\, n_{(a} K_{b)c}\, n^c \\
&=\; \{\frac{1}{2}(\mu - p_{||}) K_{cd}\, S^{cd} + \frac{1}{2}(\mu + 2p - p_{||}) f_u \\
&+\; \frac{1}{2}(\mu - 2p_\perp + p_{||}) f_n\}(S_{ab} - u_a u_b + n_a n_b) \\
&-\; (\mu - 2p_\perp - p_{||}) K_{ab}.
\end{aligned} \tag{5.62}$$

Contracting (5.62) in turn with $u^a\, u^b, u^a\, n^b, u^a\, S^{bc}, n^a\, n^b, n^a\, S^{ab}$ and $S^{ac}\, S^{bd} - \frac{1}{2} S^{ab}\, S^{cd}$, we obtain

(a) $L_V\, \mu = -\frac{1}{2}(\mu - p_{||}) K_{ab} S^{ab} + \frac{1}{2}(\mu + 2p_\perp + p_{||}) f_u - \frac{1}{2}(\mu - 2p_\perp + p_{||}) f_n$,

(b) $L_V\, p_{||} = L_V\, \mu$,

(c) $2\, L_V\, p_\perp = (\mu + 2p_\perp + p_{||}) f_u + (\mu - 2p_\perp + p_{||}) f_n$,

(d) $(\mu + p_{||})\, n^a\, Y_a = -(\mu + 2p_\perp + p_{||})\, K_{ab}\, u^a\, n^b$,

(e) $(\mu + p_\perp)\, S^{ab}\, Y_b = -(\mu + 2p_\perp + p_{||})\, S^{ab}\, K_{bc}\, u^c$,

(f) $(p_{||} - p_\perp)\, S^{ab}\, W_b = -(\mu - 2p_\perp + p_{||})\, S^{ab}\, K_{bc}\, n^c$,

(g) $(\mu - p_{||})(S^c_a\, S^d_b - \frac{1}{2} S_{ab}\, S^{cd})\, K_{cd} = 0.$

PROPOSITION 1. *Let (M, g) be a 4-dimensional spacetime with a timelike affine vector field V parallel to the fluid velocity vector u ($V = \lambda u$, $\lambda > 0$). If M satisfies the Einstein field equations with zero cosmological constant and matter tensor prescribed by (5.61), then*

(1) $\mu + 2p_\perp + p_{||} = 0$ *or* u *is an eigenvector of* K_{ab} ,

(2) $\mu = p_{||}$ *or* $S^c_a S^d_b \sigma_{cd} = -(\sigma_{cd} n^c n^d) S_{ab}$.

Proof follows immediately by using theorem 5.19 , items (d), (e) and (g).

Thus, it is clear from above proposition that if $\mu \neq p_{||}$, then the projection of the shear tensor into the local 2-planes of pressure isotropy is isotropic, that is, the preferred direction of the pressure anisotropy is also a preferred shear direction.

PROPOSITION 2. *Let (M, g) be a 4-dimensional spacetime with a spacelike affine vector field V parallel to n ($V = \alpha n , n \cdot n = 1 , n \cdot u = 0 , \alpha > 0$). If M satisfies the Einstein field equations with zero cosmological constant and matter tensor prescribed by (5.60), then*

(1) *If $\mu + p_{||} \neq 0 \neq \mu + p_\perp$ and u is an eigenvector of K_{ab}, then either the vorticity vector ω is parallel to V or the vorticity vanishes,*

(2) $\mu = 2p_\perp - p_{||}$ *or* n *is an eigenvector of* K_{ab},

(3) $\mu = p_{||}$ *or* $(S^c_a S^d_b - \frac{1}{2} S_{ab} S^{cd})K_{cd} = 0$.

Proof follows immediately by using the items (d) to (g). Note that by using the projection tensor p_{ab}, of the spacelike congruence, it is possible to relate the direction of shear with the direction of pressure anisotropy. This is left as an exercise.

EXAMPLE 1. Consider the Robertson-Walker metric in spherical coordinates (t, r, θ, ϕ) given by

$$ds^2 = dt^2 - S^2(t)\,((1 - K r^2)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2),$$

where $K = 0, \pm 1$. Let $V^a = \lambda(t)\, \delta^a_t$ be a timelike vector parallel to the fluid flow vector $u^a = \delta^a_t$. Calculating for an affine Killing equation $V_{a;b} + V_{b;a} = 2 K_{ab}$, we obtain

$$\begin{aligned} V_{a;b} &= K_{ab} = \delta^t_a\, \delta^t_b\, \dot\lambda \\ &- \lambda S \dot S [\delta^r_a \delta^r_b (1 - K r^2)^{-1} + \delta^\theta_a \delta^\theta_b r^2 + \delta^\phi_a \delta^\phi_b r^2 \sin^2\theta]. \end{aligned} \tag{5.63}$$

Since K_{ab} is covariant constant, $V_{(a;b);c} = 0$. Calculating this later equation, we get $\lambda \dot S - S \dot\lambda = 0$ and $\ddot\lambda = 0$. Thus, we obtain

$$\lambda = a\, S(t) \qquad S = b\, t + c, \tag{5.64}$$

for some constants a, b and c. Thus, we have shown that V is a time like vector field parallel to u such that K_{ab} is given by (5.63) and λ and S are related by (5.64). Furthermore, Collinson [36] has proved the following:

(1) *Non-empty Robertson-Walker spacetimes admitting a proper affine vector field are necessarily static.*

(2) *Non-empty static Robertson-Walker spacetimes admit one and only one independent proper affine vector field in the sense that any other proper affine vector can be obtained by adding a homothetic vector field to the generator.*

EXAMPLE 2. The Einstein static universe, which is simply connected and complete manifold $M = R^1 \times S^3$, with the metric

$$ds^2 = -dt^2 + dr^2 + \sin^2 r(d\theta^2 + \sin^2\theta\, d\phi^2)$$

admits [83] an 8-dimensional transitive Lie group of affine transformations generated by the global proper affine vector field $V = t\partial_t$.

EXAMPLE 3. Consider the Gödel spacetime model as given on page 91. It has been mentioned there that $V_4^a = \delta_3^a$ is a spacelike Killing vector. If we set $V^a = f(x^2)V_4^a$, such that the function $f(x^2)$ satisfies the condition $f_{,22} = 0$, then V is an affine vector field (details may be seen in [83]).

In general, it was shown in [128] that a non-Einstein conformally flat Riemannian manifold can admit an affine vector field for which K_{ab} is a linear combination of the metric tensor and the Ricci tensor. This result also holds for any non-recurrent, non-conformally flat and non-Einstein manifold which is conformally recurrent with a locally gradient recurrent vector [128]. Thus, affine vector fields in such spaces are proper since they are neither Killing nor homothetic.

EXAMPLE 4. Consider the *pp*-waves metric (see Kramer et al [118]):

$$ds^2 = -2\, f(u,x,y)\, du^2 - 2\, du\, dv + dx^2 + dy^2,$$

where u, v are retarded/advanced timelike coordinates. The null rays are given by $\{u, x, y$ constants $\}$. Lightlike hypersurfaces are generated by $\{u =$ constant $\}$ and the spacelike 2-surfaces are called wave surfaces. Since they are flat, the waves are plane fronted. This metric admits a covariant constant null Killing vector field ξ such that

$$\xi = \partial_v\,,\ \eta_a = -u_{;a}\,,\ \eta_{a;b} = 0\,,\ \eta_a = g_{ab}\xi^b.$$

Thus the rays are non-twisting, non-expanding and non-shearing and hence parallel. Also, it has been shown by Maartens [130] that above relations imply the existence of a proper affine vector field $V = u\,\partial_v$.

REMARK. In conclusion we say that although the Killing symmetry is characterized by expansion-free and shear-free spacetimes, the affine collineation symmetry is relevant to a class of spacetimes with non-zero expansion and non-zero shear.

Chapter 6

Homothetic and Conformal Symmetries

We present upto date information on the kinematic and dynamic properties of homothetic and conformal motions in general relativity followed by their use in fluid spacetimes and Riemannian manifolds.

6.1 Homothetic Symmetry in General Relativity

Let (M, g) be a 4-dimensional spacetime manifold. A vector field V is said to be **homothetic** if

$$L_V\, g_{ab} = 2\, c\, g_{ab}, \qquad c = \text{constant}, \tag{6.1}$$

or

$$V_{a;\, b} + V_{b;\, a} = 2\, c\, g_{ab}, \qquad (1 \leq a, b \leq 4).$$

V is proper homothetic if $c \neq 0$ otherwise it reduces to a Killing vector field. Since c is constant, for a homothetic V, the following integrability conditions (identical to the case of Killing symmetry) hold:

$$L_V\, \Gamma^a{}_{bc} = 0, \qquad L_V\, R^a{}_{bcd} = 0, \qquad L_V\, C^a{}_{bcd} = 0.$$

Therefore,

$$L_V\, R_{ab} = 0, \qquad L_V\, r = -2\, c\, r.$$

Also, by operating Lie-derivative, with respect to V, on both sides of the Einstein field equations (4.43) we see that any matter tensor T_{ab} is invariant, that is,

$$L_V\, T_{ab} = 0.$$

Thus, as in the case of Killing symmetry, the Levi-Civita connection, all the curvature quantities (except the scalar curvature which is preserved up to a constant factor) are invariant under homothetic vector field.

Following exactly as in case of section 5.4 of chapter 5, one can show that the effect of a homothetic V on the fluid flow velocity vector u $(u \cdot u = -1)$ or a unit spacelike vector $(n \cdot n = 1,\, n \cdot u = 0)$ or the two principal null directions ℓ and k, is given by

$$
\begin{aligned}
L_V\, u^a &= -cu^a + Y^a\,, \quad L_V\, u_a = c\, u_a + Y_a\,, \quad Y \cdot u = 0, && (6.2)\\
L_V\, n^a &= -c\, n^a + W^a\,, \quad L_V\, n_a = c\, n_a + W_a\,, \quad W \cdot n = 0, && (6.3)\\
L_V\, \ell &= -c\, k + \frac{Y+W}{\sqrt{2}}\,, \quad L_V\, k = -c\, \ell + \frac{Y-W}{\sqrt{2}}\,, && (6.4)
\end{aligned}
$$

such that

$$
W \cdot u + Y \cdot n = 0\,, \quad \ell \cdot (Y+W) = 0\,, \quad k \cdot (Y-W) = 0.
$$

One could proceed by using the technique as explained in chapter 5, but we prefer using the work of McIntosh [142] in which he has discussed the kinematic properties of a general homothetic vector field V. Let P denote the projection tensor for a non-null V, with components

$$
P_{ab} = g_{ab} - \frac{V_a\, V_b}{V \cdot V}, \qquad P_{ab} V^b = 0. \tag{6.5}
$$

Decompose $V_{a;b}$ into its symmetric and skew-symmetric parts such that

$$
V_{a;b} = c\, g_{ab} + F_{ab}, \qquad F_{ab} = V_{[a;b]}, \tag{6.6}
$$

where, according to the terminology used my McIntosh [142], F_{ab} is called the **homothetic bivector**. Also, V satisfies

$$
V_{a;bc} - V_{a;cb} = R^d_{abc}\, V_d. \tag{6.7}
$$

THEOREM 6.1 (McIntosh [142]). *If V is a non-null homothetic vector field, it is shear free and has constant expansion given by $\theta = 3\,c$.*

PROOF. Let the relative velocities of neighboring particles be represented by a tensor A_{ab} and given as

$$
A_{ab} = P_{ac}\, P_{bc}\, V^{c;d} = \theta_{ab} + \omega_{ab},
$$

where θ_{ab} and ω_{ab} are the **expansion** and the **vorticity tensors**. Using $V^{c;d} = c\, g^{cd} + F^{cd}$, we obtain

$$
\theta_{ab} = \theta_{(ab)} = \sigma_{ab} + \frac{1}{3}\, \theta\, P_{ab} = c\, P_{ab}.
$$

Contracting above equation with g^{ab} and using $g^{ab}\, \sigma_{ab} = 0$ entails $\theta = 3\,c$, and , therefore, $\sigma_{ab} \equiv 0$.

Now the **vorticity tensor** can be written as follows:

$$
\omega_{ab} = \omega_{[ab]} = F_{ab} + \frac{(F_{ca}\, V_b - F_{cb}\, V_a)\, V^c}{V \cdot V}.
$$

Thus, the **vorticity vector** ω is given by

$$\omega^a = \frac{1}{2}\eta^{abcd} V_b\, \omega_{c;d} = \frac{1}{2}\eta^{abcd} V_b\, V_{c;d}. \tag{6.8}$$

Following two corollaries can be proved (left as an exercise) using the theory of non-null congruence, (6.2) and (6.3).

COROLLARY 1. *If V is timelike homothetic vector field parallel to the fluid flow vector u ($V = \lambda u$, $\lambda > 0$), then the following holds:*

$$\begin{aligned} L_V\, u^a &= -c\, u^a\,, \qquad \dot{\lambda} = c\,, \\ \dot{u} &= (log\, \lambda)_{;a} + (\frac{c}{\lambda})\, u_a. \end{aligned}$$

COROLLARY 2. *If V is a spacelike homothetic vector field parallel to a unit spacelike vector n ($V = \alpha\, n$, $\alpha > 0$) with $n \cdot u = 0$, then the following holds:*

$$\begin{aligned} L_V\, n^a &= -c\, n^a\,, \qquad L_V\, u^a = -c\, u^a + 2\, \alpha\, \omega^a_b\, n^b\,, \\ \overset{\bullet}{n}_a &= -(log\, \alpha)_{;a} + (\frac{c}{\alpha})\, n_a\,, \qquad \overset{\bullet}{\alpha} = c. \end{aligned}$$

Now let V be a **null homothetic vector field**, that is, $V^a\, V_a = 0$ which when differentiated gives $V^a\, V_{a;b} = 0 = V^a(2\, c\, g_{ab} - V_{b;a})$ or

$$V_{a;b}\, V^b = 2\, c\, V_a. \tag{6.9}$$

This means that V is always tangent to a geodesic. Thus,

$$V_a = \lambda\, \ell_a\,, \qquad \ell_{a;b}\, \ell^b = 0,$$

where ℓ is a null geodesic congruence (treatment is same if we replace ℓ by the other principal direction k). It, therefore, follows from (6.9) that λ satisfies

$$2\, c = \lambda_{,a}\, \ell^a.$$

THEOREM 6.2. *If V is a null homothetic vector field, then it is parallel to one of the principal null vectors, say ℓ ($V = \lambda\, \ell$, $\lambda > 0$), which is geodesic, shear free and has constant expansion given by $\theta = \frac{2\, c}{\lambda}$.*

Proof follows by using (6.9) and contracting the homothetic equation (6.1) with the projection tensor $\hat{h}^{ab}$ (see (4.19) page 60) and $\hat{h}^a_c\, \hat{h}^b_d - \frac{1}{2}\hat{h}^{ab}\, \hat{h}_{cd}$.

According to McIntosh [142], we take F_{ab} as a test electromagnetic field of any homothetic vector field. Thus, using the section 4.2(a) of chapter 4 on electromagnetic fields and (6.6), we get

$$F^{ab}{}_{;b} = -V^{b;a}{}_{;b} = R^{ab}\, V_b = 4\, \pi\, J^a\,, \qquad c = 1, \tag{6.10}$$

where J is the 4-current vector of the **test electromagnetic field** generated by F. In a vacuum or any spacetime with $R^{ab}\,V_a = 0$, F is source-free, that is, $J \equiv 0$. Based on (6.10), the equation (6.7) can be rewritten as $F_{ab;c} - F_{ac;b} = R^d_{abc}\,V_d$. Using this last equation and the Maxwell's equation $F_{[ab;c]} = 0$, provides

$$V_{a;bc} = F_{ab;c} = R_{abcd}\,V^d.$$

Using the equation $L_V\,r = -2\,c\,r$, it is easy to see that J satisfies the continuity equation $J^a_{;a} = 0$. Also, since $R_{abcd}\,V^c\,V^d = 0$, it implies that $\nabla_V\,F = 0$.

THEOREM 6.3 (McIntosh [142]). *If a non-null homothetic vector field V is parallel to its source vector J, then V is Killing.*

PROOF. Using the vorticity vector ω as given by (6.8), we write

$$\begin{aligned} \omega_{[a;b]} &= c\,F^\star_{ab} + \frac{1}{2}\,\eta_{abcd}\,V^c\,R^{de}\,V_e \\ &= c\,F^\star_{ab} + 2\,\pi\,\eta_{abcd}\,V^c\,J^d. \end{aligned}$$

In the index-free notation, above are the components of the following differential equation:

$$d\,\omega = c\,F^\star + 4\,\pi\,(V \wedge J)^\star. \tag{6.11}$$

Now $V \parallel J$ implies $V \wedge J = 0$ and $dd\omega = 0$ implies $c\,F^\star = 4c\pi\,J^\star = 0$. Thus, $c = 0$ and, therefore, V is Killing.

EXAMPLE. Let (M, g) be an Einstein spacetime with $R_{ab} = \lambda\,g_{ab}$. Then, (6.10) provides $\lambda\,V = 4\,\pi\,J$ and, therefore, based on theorem 6.3, V is Killing.

Furthermore, if $J \equiv 0$ then (6.11) reduces to $d\,\omega = c\,F^\star$. This means either $c = 0$ and ω is locally the gradient of some function or ω is proportional to a 4-potential from which F can be generated.

THEOREM 6.4. *An electovac spacetime does not admit any proper null homothetic vector.*

PROOF. Covariant derivative $V^a\,V_{a;b} = 0$ and use of (6.6) and (6.10) gives

$$F_{ab}\,F^{ab} + 4\,c^2 + 4\,\pi\,J \cdot V = 0. \tag{6.12}$$

Recall that for a general form of F (see page 62), we defined a complex invariant $K = \frac{1}{2}(F_{ab}\,F^{ab} + i\,F_{ab}\,F^{\star ab}) = 2\,({\phi_1}^2 - \phi_0\,\phi_2)$, where ϕ_0, ϕ_1, ϕ_2 are three Maxwell scalars. Using from this the value of $F_{ab}\,F^{ab}$ in (6.12), we obtain

$${\phi_1}^2 + c^2 + \pi\,J \cdot V = 0. \tag{6.13}$$

Now for a null homothetic V the corresponding bivector F belongs to the singular electromagnetic fields which implies that $\phi_1 = 0$. Thus, (6.13) further reduces to

$$c^2 + \pi\,J \cdot V = 0. \tag{6.14}$$

Since J vanishes in an electovac spacetime, (6.14) implies that $c = 0$ and, therefore, V is Killing which proves the theorem.

Now we discuss the effect of Einstein equations (4.43), with zero cosmological constant, and a perfect fluid matter tensor $T_{ab} = (\mu + p)u_a\, u_b + p\, g_{ab}$. Then

$$R_{ab} = 8\,\pi\{(\mu + p)\, u_a\, u_b + \frac{1}{2}(\mu - p)\, g_{ab}\},$$

and, it follows from (6.10) that

$$J = (\mu - p)\, V + 2\,(\mu + p)(u \cdot V)\, u. \tag{6.15}$$

Now using theorem 6.3 and (6.15), we can prove the following theorem.

THEOREM 6.5 (McIntosh [142]). *If a spacetime (M, g) which contains a perfect fluid with 4-velocity vector u, density μ and pressure p admits a non-null homothetic vector field V with corresponding source vector J, then one of the following cases hold:*

(a) $\mu \neq p$, $J \neq 0$: *Any of the conditions* $u \cdot V = 0, u \parallel V$, $u \cdot J = 0$ *or* $u \parallel J$ *gives* $V \parallel J$ *and thus* $c = 0$ *and then* V *is Killing ; except in the special case* $u \cdot J = 0,\ 3\,p + \mu = 0,\ u \cdot V \neq 0$.

(b) $J = 0$: $V = (u \cdot V)\, u$ *and* $3\,p + \mu = 0$ *or* $u \cdot V = 0$ *and* $\mu = p$.

(c) $\mu = p$: $J = 4\,\mu(u \cdot V)\, u$

For a perfect fluid with V null, it follows from (6.15) that

$$J \cdot V = 2\,(\mu + p)(u \cdot V)^2,$$

so that using (6.14) in above equation we get

$$2\,\pi(\mu + p)(u \cdot V)^2 + c^2 = 0.$$

THEOREM 6.6. *If a spacetime (M, g) which contains a perfect fluid with 4-velocity vector u, density μ and pressure p admits a proper null homothetic vector field V with corresponding source vector J, then V can not be orthogonal to u.*

EXAMPLE 1. Minkowski spacetime with metric: $ds^2 = -dt^2 + \delta_{ij}\, dx^i\, dx^j$ $(i, j = 1, 2, 3)$. $V = t\,\partial_t + x^i\,\partial_i$ is homothetic which is unique only upto spacial translations $V \to V + a^i\,\partial_i$ and time translation $V \to V + a\,\partial_t$, where each a^i and a are constants.

EXAMPLE 2. Robertson-Walker spacetimes ($c = 0$) with metric $ds^2 = -dt^2 + S^2(t)\, dx^i\, dx^i$, and $t\,\frac{\dot{S}}{S} =$ constant. $V = t\,\partial_t + (1 - t\,\frac{\dot{S}}{S})\, x^i\,\partial_i$ is homothetic The restriction $t\,\frac{\dot{S}}{S} =$ constant means S is proportional to some constant power, say b, of t (see Eardley [63]). If $b \geq \frac{2}{3}$, then these are solutions for a fluid matter with the equation of state $p = (\frac{2}{3b} - 1)\mu$. Other equations of state generally do not permit a homothetic vector field. For example, the **hot big-bang model of universe** is asymptotically self-similar which means that it admits homothetic vector only before and after onset of the matter dominance.

EXAMPLE 3. Robertson-Walker spacetimes ($c \neq 0$) with metric as given by (4.55). None of these models, except the unphysical case $\mu + 3p = 0$, admit a proper homothetic vector (see Eardley [63]). These models approximate to $c = 0$ models very close to the big-bang model and at very early times when the intrinsic scale was dynamically unimportant and then these models are asymptotically self-similar. At very large times, $c = -1$ models approach Minkowski spacetime, and hence admit asymptotically homothetic symmetry.

EXAMPLE 4. Kasner vacuum spacetimes. The metric is given by

$$ds^2 = -dt^2 + \sum_a t^{2p_i} \left(dx^i\right)^2, \qquad \sum p_i = \sum {p_i}^2 = 1 \; (p_i = \text{constant}).$$

Any homothetic vector field is expressed by $V = t\,\partial_t + \sum_i (1 - p_i)\, x^i\, \partial_i$. This solution was first studied by Schucking and Heckmann [168], in 1958, with metric $ds^2 = -dt^2 + \sum_i t^{2p_i}\,(t + t_0)^{(4/3 - 2p_i)}\left(dx^i\right)^2$, $t_0 =$ constant, which represents anisotropic dust universes. For $t << t_0$, these are approximated to Kasner spacetimes and for $t \geq t_0$, these correspond to dust solutions of **Friedmann Robertson-Walker spacetimes**. In each of these regions of t, these models are asymptotically self-similar.

For details on above example and some more examples we refer to Hsu and Wainwright [100] and other references given in this paper. It is also notable that the maximum dimension of the homothetic algebra in 4-dimensional spacetime M is $\frac{4(4+1)}{2} + 1 = 11$ and this occurs when M is flat. For non-flat conformally flat (resp. non-conformally flat) M, the maximum is 8 (resp. 7) and both belong to homogeneous (resp. Petrov type N homogeneous) plane waves solution. To understand how the plane waves solution occurs, let V be a proper homothetic vector field in M. Then, the set of points where V vanishes is called the set of isolated points, or a part of a null geodesic of M. It is the latter condition which characterizes plane waves that are conformally flat or Petrov type N. Also, if a proper homothetic V vanishes at a point p, then the Weyl tensor is of Petrov type **N**, **III** or **O**. For details on Petrov classification see Appendix A.

EXAMPLE 5. A vacuum or non-vacuum plane-wave (M, g) with metric

$$ds^2 = 2\, f_{ab}(u) x^A\, x^B\, du^2 + 2\, du\, dv + \delta_{AB}\, dx^A\, dx^B, \qquad (A, B = 2, 3),$$

admits a proper homothetic symmetry group H_6 and a Killing subgroup $G_5 \subset H_6$. Moreover, G_5 is transitive on the **lightlike hypersurfaces** $u =$ constant. The generator of H_6 is $V = 2\, v\, \partial_v + x^A\, \partial_A$. Physically, such a symmetry means that for each observer, V generates a transformation which is a composition of a boost in the direction of propagation of a wave, with a scalar transformation which exactly cancels the **Doppler shift** induced by the boost, leaving that wave invariant.

For detailed study on homothetic transformations with fixed points in spacetimes and a characterization of generalized plane waves solutions, we refer Hall [82]. Also, for study on homothety groups see Hall and Steele [91].

6.2 Homothetic Symmetry and Cauchy Surfaces

A **Cauchy surface** is a spacelike hypersurface of a spacetime manifold (M, g) such that each non-spacelike curve intersects it once and only once. For example, $\{t = \text{constant}\}$ are examples of Cauchy surfaces in Minkowski spactime with local coordinates (t, x, y, z). It is important to note that the Cauchy property is not only of the surface itself but also of the whole M in which the surface is embedded. This means that if a single point is removed out from M, then the resulting spacetime does not admit any Cauchy surface. Among well-known spacetimes not admitting any Cauchy surface are: anti-de Sitter, plane waves and the Reissner-Nordström solutions.

In 1970, Geroch [73] proved an important theorem which guarantees that a **globally hyperbolic spacetime** may be written as a topological (not necessarily metric) product $R^1 \times S$ where S is a Cauchy surface, and, therefore, for each t, $\{t\} \times S$ is a Cauchy surface. In this section we assume that (M, g) is globally hyperbolic with a Cauchy surface denoted by Σ. Global hyperbolicity implies the following:

(1) M is homotopic to $R \times \Sigma$,

(2) If Σ' is any compact spacelike hypersurface without boundary and embedded in M, then it must be a Cauchy surface,

(3) If (M, g) satisfies the vacuum Einstein equations, then g may be completely determined from a set of Cauchy data specified on Σ or if M satisfies the Einstein equations coupled to a well-posed hyperbolic system of matter equations, then the coupled system has the same property.

The last implication **(3)** allows one to do essentially all of the analysis on a Cauchy surface, which is Riemannian with the pull back induced metric γ given by

$$\gamma_{ab} = i^\star \, g_{ab} \tag{6.16}$$

where $i^\star$ is the differential map induced by the embedding $i : \Sigma \to M$. If N denotes the unit timelike normal and ∂_a, ∂_b vector fields tangent to $i(\Sigma)$, then the components of the second fundamental form B are given by

$$B_{ab} = -g(\nabla_{\partial_a} N, \partial_b), \tag{6.17}$$

where we let $(1 \leq a, b, c, d \leq 3)$ and $(0 \leq i, j, k, m \leq 3)$. Then we say that the pair (γ_{ab}, B_{ab}) is a complete set of data (called **Cauchy data**) for the induced gravitational field on $i(\Sigma)$. Furthermore, if the matter fields are present, then those fields also need their own Cauchy data. The mean curvature is defined by

$$\mu = \frac{1}{3} \, Trace(B) = \frac{1}{3} \, B_{ab} \, \gamma^{ab}.$$

Define the following traceless tensor

$$A^a_b = B^a_b - \frac{1}{3}\,\tau\,\gamma^a_b, \qquad \tau = trace(B). \tag{6.18}$$

Since M is globally hyperbolic, for some function, say f, it is possible to write any vector V as

$$V = f\,N + X, \tag{6.19}$$

where X is tangent to $i(\Sigma)$. The Einstein field equations (4.43), with zero cosmological constant, yield the following constraint equations:

$$\begin{aligned} r + \frac{2}{3}\tau^2 - |A|^2 &= 16\,\pi\,T_{00}, \qquad T_{00} = T(N,N), \\ \nabla_a\,A^a_b &= \nabla_b\,\tau - 8\,\pi\,T_{0b}, \qquad T_{ob} = T(N,\partial_b), \end{aligned}$$

where ∇_a is the surface(Σ)-intrinsic covariant derivative compatible with γ. Suppose V is homothetic (see equation (6.1)). Then,

$$\begin{aligned} \hat{L}_V\,\gamma_{ab} &= 2\,c\,\gamma_{ab}, \qquad \hat{L}_V\,A^a_b = -c\,A^a_b, \\ \hat{L}_V\,\tau &= -(c\,\tau + 3\,\nabla_N\,c) = -\,c\,\tau, \end{aligned}$$

where $\hat{L}_V$ is the surface-projected Lie derivative along V. Let $V = \partial_t$ (the time evolution vector field: see Berger [13]). Then, we obtain the following homothetic evolution equations from Einstein field equations;

$$2\,c\,\gamma_{ab} = \hat{L}_X\,\gamma_{ab} - 2\,f(A_{ab} + \frac{1}{3}\,\tau\,\gamma_{ab}), \tag{6.20}$$

$$\begin{aligned} -c\,A^a_b &= f[(R^a_b - \frac{1}{3}\,\gamma^a_b\,r) + \tau\,A^a_b - 8\,\pi(T^a_b - \frac{1}{3}\,T\,\gamma^a_b)] \\ &- (\nabla^a\,\nabla_b\,f - \frac{1}{3}\,\gamma^a_b\,\nabla^2\,f) + \hat{L}_X\,A^a_b, \end{aligned} \tag{6.21}$$

$$\begin{aligned} -(c\,\tau + 3\,N(f)) &= f[|A|^a + 4\,\pi(T_{00} + T) + \frac{1}{3}\,\tau^2] \\ &+ \hat{L}_X\,\tau - \nabla^2\,f. \end{aligned} \tag{6.22}$$

Following Eardley et al [64] we say that a spacetime obeys the **mixed energy condition** if at any point p on Σ

(a) *the strong energy condition* ($T_{00} + T^a_a|_p \geq 0$) *holds , and*

(b) *equality in* **(a)** *implies that all components of* T *are zero, that is,* $T_{ij}|_p = 0$.

Thus, mixed energy condition is slightly stronger than strong energy condition. For example, a perfect fluid will satisfy the mixed energy condition $3p + \mu > 0$ everywhere.

THEOREM 6.6 (Eardley et al [64]). *Let (M, g) be a globally hyperbolic space time which*

(a) *satisfies the Einstein field equations for a stress tensor T obeying the mixed energy condition and the dominant energy condition ;*

(b) *admits a homothetic vector field V of g; and*

(c) *admits a compact hypersurface of constant mean curvature.*

Then either M is an expanding hyperbolic model with metric

$$ds^2 = e^{\lambda t}(-dt^2 + h_{ab}\,dx^a\,dx^b), \tag{6.23}$$

with $h_{ab}\,dx^a\,dx^b$ a 3-dimensional Riemannian metric of constant negative curvature on a compact manifold and T vanishing, or V is Killing.

PROOF. By hypothesis, τ and c are constant, If $c = 0$, then, V is Killing and so the theorem is obvious. Thus, we assume $c \neq 0$. The trace of (6.20) is

$$3\,c = -\,f\,\tau + div\,X, \tag{6.24}$$

where divergence is taken with respect to the 3-metric γ. Integrating (6.24) over the hypersurface $i(\Sigma)$ we get

$$3\,c\,\Omega = -\,\tau \int_{i(\Sigma)} f, \tag{6.25}$$

Ω denotes the volume of $i(\Sigma)$ and $\int\, div\,X = 0$ by divergence theorem. Since c and Ω are non-zero, it follows from (6.25) that $\tau \neq 0$. Putting $\tau =$ constant in (6.22) yields the following elliptic equation in f:

$$\begin{aligned}(U - \nabla^2)\,f &= -3\,c\,\tau \\ U &= |A|^2 + \frac{1}{3}\,\tau^2 + 4\,\pi(T_{00} + T^a_a).\end{aligned} \tag{6.26}$$

Now $\tau \neq 0$ implies by the mixed energy condition that $U > 0$ on $i(\Sigma)$. Since the elliptic operator $(U - \nabla^2)$ is positive and the right hand side of (6.26) is constant, we expect that f is strictly bounded away from zero on $i(\Sigma)$ and is opposite in sign to $c\,\tau$. Indeed consider the case $c\,\tau < 0$. At a point p in $i(\Sigma)$ where f attains its global minimum, $\nabla^2 f(p) \geq 0$. Therefore, (6.26) implies that $U\,f(p) > 0$. Since $U(p) > 0$, $f(p) > 0$ at its global minimum and hence $f < 0$ everywhere.
Next, integrating (6.26) over $i(\Sigma)$ and using divergence theorem provides

$$\int_{i\Sigma} U\,f = -c\,\tau\,\Omega. \tag{6.27}$$

Multiplying (6.25) by $\tau/6$ and adding to (6.27) we get

$$\int_{i(\Sigma)} [|A|^2 + 4\,\pi(T_{00} + T^a_a)] = 0. \tag{6.28}$$

Since the integrand in (6.28) is everywhere ≥ 0 or ≤ 0 according as $c\,\tau\ < 0$ or > 0 it must vanish everywhere. Also, since c is everywhere non-zero, $A^a_b = 0$ and $T_{00} + T^a_a = 0$ everywhere. Hence, by mixed energy condition, $T_{ij} = 0$ on $i(\Sigma)$. Consequently, $U = \tau^2/3 =$ constant and the unique solution f of (6.26) is given by

$$f = -\frac{3\,c}{\tau} = constant. \tag{6.29}$$

Also, as $A^a_b = 0$, the second fundamental form B is expressed as

$$B_{ab} = \frac{1}{3}\,\tau\,\gamma_{ab}. \tag{6.30}$$

Substituting all these consequences in (6.21) we obtain

$$R_{ab} = -\frac{2}{9}\,\tau^2\,\gamma_{ab}\ ,\ \ r = -\frac{2}{3}\tau^2. \tag{6.31}$$

Observe that the equation (6.30) shows that the hypersurface $i(\Sigma)$ is **totally umbilical** (see (2.39) of chapter 2) in M and the equation (6.31) implies that the metric γ has negative constant curvature. Only the standard hyperbolic metric satisfies the condition (6.31), and they are specified by the choice of global topology and the choice of a single scale factor. Substituting $A^a_a = 0$ and (6.31) in (6.20), we find that X is a Killing vector of the 3-geometry. However, it follows from a theorem of Bochner [22] that the standard hyperbolic metrics admit no non-zero global Killing vector field. Hence, $X \equiv 0$ which means that V is orthogonal to $i(\Sigma)$. Thus, the allowable initial data, for the hypothesis of this theorem, turns out to be

$$\gamma_{ab} = \frac{9\,h_{ab}}{\tau^2}\ ,\ B_{ab} = \frac{\tau}{3}\,\gamma_{ab}$$

for a non-zero constant τ and $T_{ij} \equiv 0$. Here h_{ab} is a metric of constant negative curvature with scalar curvature -6. From the dominant energy condition it follows that the matter tensor T vanishes throughout the spacetime. Finally, one can check, by a straightforward calculation, that the vacuum spacetime development of the initial data is of the form (6.23), which completes the proof.

As an application of the above theorem, consider the Einstein-Yang-Mills equations [203], with the gauge group chosen to be a compact Lie group. The Lie-algebra-valued Yang-Mills field F has the components

$$F_{ij} = D_i\,A_j - D_j\,A_i + [A_i, A_j],$$

where A_i and D_i are the gauge potential and the spacetime covariant derivative operator (with respect to g) respectively. The **Einstein-Yang-Mills equations** are

$$\begin{aligned} R_{ij} - \frac{1}{2}\,r\,g_{ij} &= 8\,\pi\,T_{ij}, \\ T_{ij} &= \frac{1}{4}F_{km}F^{km}g_{ij} - F_{ik}F^k_j, \\ D^i\,F_{ij} + [A^i, F_{ij}] &= 0. \end{aligned}$$

Above system of equations satisfy mixed and dominant energy conditions. It is easy to show that if the condition **(a)** of theorem 6 is replaced by [**(a)** *satisfies the Einstein-Yang-Mills equations*], then following the proof of this theorem one can show that either M is expanding hyperbolic model with metric (6.23) and field $F \equiv 0$ everywhere or V is Killing.

Another application comes from a massless scalar field ψ coupled to gravity for which the massless **Einstein-Klein-Gordon equations** are Einstein field equations with

$$\begin{aligned} T_{ij} &= (D_i\,\psi)(D_j\,\psi) - \frac{1}{2}\,g_{ij}(D^k\,\psi)(D_k\,\psi), \\ D^i\,D_i\,\psi &= 0. \end{aligned} \tag{6.32}$$

In this case, since T_{ab} does not satisfy the mixed energy condition, we state the following theorem:

THEOREM 6.7 (Eardley et al. [64]). *Let (M,g) be a globally hyperbolic spacetime which*

(a) *satisfies the Einstein-Klein-Gordon equations,*

(b) *admits a homothetic vector field V of g,*

(c) *admits a compact hypersurface of constant mean curvature.*

Then, either M is an expanding hyperbolic model with metric (6.23) and ψ is constant everywhere, or V is Killing.

PROOF. Since for this case $T_{00} + T^a_a = 2\,(N(\psi))^2$, if the left side vanishes, ψ being not necessarily constant, T_{ij} need not vanish on $i(\Sigma)$. If we follow the proof of the theorem 6.6 upto (6.30), then we have

$$N(\psi) = 0, \qquad T_{00} = \frac{1}{2}\,(\nabla^a\,\psi)(\nabla_a\,\psi), \qquad T^0_a = 0,$$

$$T_{ab} = (\nabla_a\,\psi)(\nabla_b\,\psi) - \frac{1}{2}\,\gamma_{ab}\,(\nabla^c\,\psi)(\nabla_c\,\psi).$$

Therefore, the equation (6.21) provides

$$R_{ab} = 8\,\pi\,(\nabla_a\,\psi)(\nabla_b\,\psi) - \frac{2\,\tau^2}{9}\,\gamma_{ab}. \tag{6.33}$$

The use of contracted Bianchi identity $\nabla_a\,R^a_b = \frac{1}{2}\,\nabla_b\,r$ in (6.33) implies that $(\nabla_a\,\psi)(\nabla^2\,\psi) = 0$. Hence ψ is constant on $i(\Sigma)$. Note that since $\nabla^2\,\psi = 0$ outside the support of $\nabla_a\,\psi$, therefore, $\nabla^2\,\psi = 0$ on $i(\Sigma)$ and hence, by Hopf's lemma (see page 82) $\psi =$ constant on $i(\Sigma)$. Hence, the massless wave equation (6.32) has trivial Cauchy data ($\psi =$ constant , $N(\psi) = 0$) on $i(\Sigma)$. Thus, $\psi =$ constant on M. This completes the proof.

Note that, in general, X may induce a homothetic, Killing or higher symmetry on a hypersurface of a semi-Riemannian manifold and may not vanish. This general result is due to Sharma-Duggal [180], as follows.

Let (M,g) be an n-dimensional semi-Riemannian manifold which admits a homothetic vector field V, defined by

$$(L_V\,g)(Y,Z) = 2\,c\,g(Y,Z), \qquad \forall Y,Z \in \mathcal{X}(M).$$

Consider (Σ, γ) a semi-Riemannian hypersurface of (M, g) such that γ is the induced metric of g on Σ. From equation (2.36) of chapter 2, it is possible to express V as

$$V = U + f\,N, \tag{6.34}$$

where U is tangent to Σ, f is a smooth function on Σ and N is unit normal ($g(N,N) = \epsilon = \pm 1$, according as N is spacelike or timelike respectively) to Σ. Thus we have

$$\begin{aligned}(L_V\, g)(Y,Z) &= (L_{U+f\,N} g)(Y,Z) \\ &= (L_U \gamma)(Y,Z) + f\,\{g(\nabla_Y N, Z) + g(\nabla_Z N, Y)\},\end{aligned}$$

where ∇ is the Levi-Civita connection on M and Y, Z arbitrary tangent vectors to Σ. Using the Gauss and Weingarten formulas (see equation (2.37) of chapter 2), we obtain

$$(L_U\gamma)(Y,Z) = 2\,c\,\gamma(Y,Z) + 2\,\epsilon\,f\,B(Y,Z), \tag{6.35}$$

where B is the second fundamental form as defined on page 23. In local coordinates (6.35) is written as

$$L_U \gamma_{ab} = 2\,c\,\gamma_{ab} + 2\,\epsilon\,f\,B_{ab}. \tag{6.36}$$

We now discuss the following important cases:

(i) Σ is **totally geodesic**, that is, $B \equiv 0$. Then it follows from (6.36) that U is also homothetic on Σ.

(ii) Σ is **totally umbilical**, that is, $B = \rho\gamma$, for some smooth function ρ on Σ. Then, (6.36) becomes

$$L_U \gamma_{ab} = 2\,(c + \epsilon\,f\,\rho)\gamma_{ab}. \tag{6.37}$$

The following sub cases, of (6.37), arise

(a) $c + \epsilon\,f\,\rho = 0 \Rightarrow U$ is Killing.

(b) $c + \epsilon\,f\,\rho \neq 0$ and $f\,\rho =$ constant $\Rightarrow U$ is homothetic on Σ.

(c) $c + \epsilon\,f\,\rho \neq 0$ and $f\,\rho$ is non-constant $\Rightarrow U$ is conformal on Σ.

Note that cases other that **(i)** and **(ii)** are also geometrically important, which have been discussed in later chapters. Finally, observe that the existence of proper homothetic vector field on a semi-Riemannian manifold is restricted by the following result.

PROPOSITION. *A homothetic vector field on a compact semi-Riemannian manifold without boundary is Killing.*

PROOF. Let V be a homothetic vector field. Then,

$$g(\nabla_X V, Y) + g(\nabla_Y V, X) = 2\,c\,g(X,Y)$$

for some constant c. Contracting this equation provides $div\,V = c\,n$ (here $n = dim(M)$), and then integrating it gives (with the use of divergence theorem) $0 = c\,n\,vol\,(M)$. Hence, $c = 0 \Rightarrow V$ is Killing.

6.3 Homothetic Symmetry and Compact Hypersurfaces

In this section we assume that (M, g) is an n-dimensional semi-Riemannian orientable manifold other than compact but without boundary. First we remark that for indefinite hypersurfaces, the eigenvalues and eigenvectors of the second fundamental form need not be all real (for positive definite case, they all are real). some of them may be complex and null. Thus, for an indefinite hypersurface, the real eigenvectors may not span its tangent space at every point. According to the terminology used by Fialkow [67], an indefinite hypersurface is called proper if the eigenvalues of the second fundamental form are real and no eigenvectors are null at every point. In this section we consider this class of proper hypersurfaces. Based on this we state and prove the following theorem, which was first proved by Yano [211] for Riemannian manifolds.

THEOREM 6.8. *Let (M, g) be an n-dimensional orientable semi-Riemannian manifold which*

(a) *is other than compact and without boundary,*

(b) *admits a homothetic vector field V of g,*

(c) *admits a compact proper hypersurface (Σ, γ) without boundary and with constant mean curvature.*

If the inner product of V and the unit normal vector N has constant sign on Σ and the Ricci curvature of M along N is non-negative on Σ, then Σ is totally umbilical and Ricci curvature of M vanishes on Σ.

PROOF. Let (e_a) be a pseudo-orthonormal base of the tangent space $T_p(\Sigma)$ at each point p of Σ such that $g(e_a, e_b) = \epsilon_a \, \delta^a_b$, for $(a, b = 1, ..., n-1)$ and $\epsilon_a = \pm 1$ according as e_a is spacelike or timelike. Then, setting a vector field $X = e_a$ in the Codazzi equation $\bar{R}(X, Y)N = \epsilon((\nabla_Y B))X - (\nabla_X B)Y)$, and then taking inner product with e_a and summing over a gives

$$\bar{Ric}\,(Y, N) = \epsilon(Y(tr.B) - (div\, B)Y). \tag{6.38}$$

Let V be a homothetic vector field on M. Then, equations (6.34) and (6.35) hold. Substituting $Y = U = V - f\,N$ in (6.38) we get

$$\bar{Ric}\,(V,\, N) - f\,\bar{Ric}\,(N, N) = \epsilon\,(U(tr.B) - (div\, B)U). \tag{6.39}$$

On the other hand the following holds

$$\begin{aligned} (div\, B)U &= div\, BU - \sum_{a=1}^{n-1} \epsilon_a \, \gamma(\nabla_{e_a} U, B\, e_a), \\ \gamma\,(\nabla_{e_a} U, B\, e_a) &= (\nabla_a \, U_b)B^{ab}. \end{aligned}$$

Therefore, (6.35) implies that

$$\sum_{a=1}^{n-1} \epsilon_a \, \gamma\,(\nabla_{e_a} U, B\, e_a) = c\, tr.\, B + \epsilon\, f\, tr.\, B^2.$$

Hence, (6.39) becomes

$$\begin{aligned} \bar{Ric}\,(V,N) - f\,\bar{Ric}\,(N,N) &= \epsilon\{U(tr.\,B) - div\,BU) \\ &+ c\,tr.\,B + \epsilon\, f\, tr.B^2\}. \end{aligned} \tag{6.40}$$

Differentiating $g(V,X) = g(U,X)$ along Σ provides

$$g(\bar{\nabla}_Y\, V, X) + \epsilon\, f\, B\,(Y,X) = \gamma\,(\nabla_Y U, X).$$

Contracting above result with respect to X, Y, through the basis (e_a) gives

$$(n-1)\, c + \epsilon\, f\, tr.\, B = div\, U.$$

Now integrating it over Σ and using the divergence theorem, we get

$$c \int_\Sigma dv + \epsilon \int_\Sigma (f\, M_1)\, dv = \int_\Sigma (div\, U)\, dv = 0, \tag{6.41}$$

where dv is the volume element and $M_1 = \frac{tr.\,B}{n-1}$ = the mean curvature of Σ. Next, integrating (6.40) over Σ, and using the fact that Σ is proper (hence B is diagonalizable) we obtain

$$\begin{aligned} \int_\Sigma \bar{Ric}\,(U,N)\, dv &= (n-1)\,\epsilon\,\{U(M_1) + c\, M_1 \\ &+ \epsilon\, f\,((n-1)\, M_1^2 - (n-2) M_2)\}dv, \end{aligned} \tag{6.42}$$

where

$$M_2 = \frac{2\sum_{a<b} k_a\, k_b}{(n-1)(n-2)}$$

is the second mean curvature. Here each k_a denotes the principal curvature of Σ. M_1 and M_2 are related by (see Yano [211, page 97])

$$Tr.\, B^2 = (n-1)((n-1)\, {M_1}^2 - (n-2)\, M_2).$$

Now differentiating $\epsilon\, f = g(V,N)$ along Σ gives

$$\epsilon\,(\bar{\nabla}\, f)(Y) = g((\bar{\nabla}\, V)\, Y, N) - \epsilon\, g\,(BU, Y).$$

A second derivative along Σ provides

$$\begin{aligned} \epsilon\,(\bar{\nabla}\bar{\nabla}\, f)(X,Y) &= g\,((\bar{\nabla}\bar{\nabla}\, V)(X,Y) + B(X,Y)(\bar{\nabla}\, V)N, N) \\ &- \epsilon\,\{g(\bar{\nabla}\, V)Y, BX) - g\,(\nabla_X BU, Y)\}. \end{aligned} \tag{6.43}$$

Now V homothetic on M implies $g\,((\bar{\nabla}\, V)N, N) = \epsilon\, c$ and $L_V\, \bar{\nabla} = 0$, that is, $(\bar{\nabla}\bar{\nabla}\, V)(X,Y) + \bar{Ric}\,(V,X)Y = 0$ for any vector fields X, Y on M. Using these results, putting $X = Y = e_a$ in (6.43), multiplying by ϵ_a and summing over a we obtain

$$\epsilon\, \nabla^2\, f = -\,\bar{Ric}\,(V,N) - \epsilon\, div\,(BU).$$

Thus integrating above equation over Σ yields

$$\int_\Sigma \bar{Ric}\,(V,N)\, dv = 0. \tag{6.44}$$

By hypothesis, $M_1 =$ constant. Using this in (6.41) gives

$$c\,vol\,(\Sigma) + \epsilon\, M_1 \int_\Sigma f\, dv = 0. \tag{6.45}$$

Also, using (6.44) in (6.42) provides

$$\int_\Sigma [\frac{f}{n-1}\, \bar{R}ic\,(N, N) + \epsilon\, c\, M_1 + f\,\{(n-1)\, {M_1}^2 - (n-2)\, M_2\}]dv = 0.$$

Eliminating c from (6.45) and the above equation gives

$$\int_\Sigma f\,\{\bar{R}ic\,(N, N) + (n-1)(n-2)({M_1}^2 - M_2)\}dv = 0. \tag{6.46}$$

By hypothesis, f has constant sign (because ϵ is either 1 or -1 on Σ), and $\bar{R}ic\,(N, N) \geq 0$. Hence, (6.46) implies that ${M_1}^2 - M_2 = 0$. This means that (see Yano [211, page 105])

$$\frac{1}{(n-1)^2(n-2)} \sum_{a<b} (k_a - k_b)^2 = 0.$$

Therefore, $k_1 = k_2 = - - - = k_{n-1}$ and hence Σ is totally umbilical in M. In addition, it follows that $\bar{R}ic\,(N, N) = 0$ on Σ. This completes the proof.

REMARK 1. The theorem 6.8 is applicable to two important cases, first when Σ is a hypersurface of a Riemannian manifold, and second when Σ is a spacelike hypersurface of a Lorentz manifold (note that a spacelike hypersurface is obviously proper). In particular, the Eardley et al.'s theorem 6.7 is a special case where M is a 4-dimensional globally hyperbolic spacetime satisfying conditions as stated therein. Another special case of the theorem 6.8 is the following result:

THEOREM 6.9 (Alias et al. [3]). *Let M be a spacetime admitting a timelike homothetic vector field V and satisfying the timelike convergence condition (TCC), that is, the Ricci curvature is non-negative along all timelike vectors tangent to M. Then, every compact spacelike hypersurface of constant mean curvature in M is totally umbilical.*

REMARK 2. The hypothesis "$g(V, N)$ *has constant sign on* Σ" of theorem 6.8 is automatically satisfied by the hypothesis "V *is timelike*" of theorem 6.9, since N is timelike. Furthermore, the TCC mentioned in the hypothesis of theorem 6.9 is also called the strong energy condition (see Beem et al. [8, page 45]).

6.4 Homothetic Symmetry and Lightlike Hypersurfaces

In this section, let $(M,\, g)$ be a 4-dimensional spacetime manifold of general relativity and $(\Sigma,\, \gamma)$ a **lightlike (null) hypersurface** of M. We use the mathematical theory presented in section 2 of chapter 2. Recall that γ is the induced degenerate metric, of g, on Σ. Let V be a homothetic vector field satisfying $L_V\, g = 2\, c\, g$, $c =$ constant. Then from the equation (2.58) of chapter 2, it is possible to express V as

$$V = U + f\, N,$$

where N is the unique null vector spanning the null distribution E (see theorem 2.3). U is tangent to Σ and f is a smooth function on Σ. Thus we have

$$\begin{aligned}(L_V\, g)(X,\, Y) &= (L_{U + f\, N}\, N)(X,\, Y)\\ &= (L_U\, \gamma)(X,\, Y) + f\{g(\bar{\nabla}_X\, N,\, Y) + g(\bar{\nabla}_Y\, N,\, X)\}\end{aligned}$$

for arbitrary X, Y tangent to Σ. Using the Gauss and Weingarten formulas: $\bar{\nabla}_X\, Y = \nabla_X\, Y + B(X,\, Y)N$, $\bar{\nabla}_X\, N = -\, A_N\, X + \tau\,(X)\, N$ (see equation (2.65) of chapter 2), we obtain

$$(L_U\, \gamma)(X,\, Y) = 2\, c\, \gamma(X,\, Y) + f\, \{\gamma(A_N\, X,\, Y) + \gamma(X,\, A_N\, Y)\}, \tag{6.47}$$

where we have used the equation (2.66) of chapter 2. It is important to note that, contrary to the non-degenerate case, as per the Weingarten formula, for the lightlike hypersurface, $\gamma(A_N\, X,\, Y) \neq \gamma(X,\, A_N,\, Y)$. Also, in general, the induced connection ∇ on Σ is just a linear connection, and, therefore, γ is not a metric connection. Thus, in the absence of any unique Levi-Civita connection, the authors do not know any possible geometric meaning of the general equation (6.47). In particular, Duggal-Bejancu [57] have proved (see theorem 2.4 of chapter 2) that "*there exists a unique Levi-Civita (metric) connection ∇, induced by $\bar{\nabla}$, on Σ iff Σ is totally geodesic ($B \equiv 0$)*". In the sequel, we assume that Σ is totally geodesic in M. Based on this, we can state the following result:

PROPOSITION. *Let $(\Sigma,\, \gamma)$ be a lightlike hypersurface of a 4-dimensional spacetime $(M,\, g)$. Then the tangential component of a homothetic vector field on M induces a homothetic vector field on Σ iff Σ is totally geodesic in M.*

EXAMPLE. Recall from the theorem 2.7, of chapter 2, that the local expression of the induced Ricci tensor R_{ab} and its scalar curvature r, of a totally geodesic Σ in M, is given by

$$\begin{aligned}R_{ab} &= 2\, d\,\theta\,(E_1,\, E_2)\, \gamma_{ab} - 2\, d\,\phi\,(\ell,\, N)\, \ell_a\, \ell_b\,,\\ r &= 4\, d\,\theta\,(E_1,\, E_2),\end{aligned} \tag{6.48}$$

where θ, ϕ are 1-forms on Σ and $\{\ell, E_1, E_2\}$ is a local basis such that $\ell \cdot N = 1$, $\ell \cdot \ell = 0$ and E_1, E_2 are unit spacelike vectors. Let M satisfy the Einstein's field equations with the energy momentum tensor of the **pure radiation fields**. This means that $r = 0$. Since Σ is totally geodesic in M, using the theorem 2.5, of chapter 2, and (6.48), we conclude that the induced Einstein's field equations, on Σ, are of the form

$$R_{ab} = -\, 2\, d\,\phi\,(\ell,\, N)\, \ell_a\, \ell_b,$$

which in fact represents pure radiation fields where ℓ is one of the principal null directions (see Kramer et al. [118, page 69]). A special case of this class is the example 5, of section 6.1, for which the spacetime M admits a proper homothetic symmetry group H_6 whose metric is given by

$$ds^2 = 2\, f_{AB}\,(u) x^A\, x^B\, du^2 + 2\, du\, dv + \delta_{AB}\, dx^A\, dx^B, \qquad (A,\, B = 2, 3).$$

The lightlike hypersurface Σ of M is defined by $u =$ constant.

6.5 Conformal Motions in General Relativity

Let $(M,\, g)$ be a 4-dimensional spacetime of general relativity which admits a **conformal Killing vector (CKV) field** V with conformal function σ satisfying

$$L_V\, g_{ab} = 2\, \sigma\, g_{ab}\,, \quad \text{or} \quad V_{a;b} + V_{b;a} = 2\, \sigma\, g_{ab}, \tag{6.49}$$

which reduces to homothetic or Killing vector field whenever σ is non-zero constant or zero respectively. V is called proper **CKV** if σ is non-constant. The second set of equations of (6.49) are called **Conformal Killing equations**. V satisfies the following integrability conditions (also valid for any Riemannian or semi-Riemannian) manifold (see Yano [211]):

$$\begin{aligned} (L_V\, \nabla)(X, Y) &= (X\, \sigma)\, Y + (Y\, \sigma)\, X - g\,(X, Y) D\, \sigma, \\ (L_V\, R)(X, Y, Z) &= (\nabla\, \nabla\, \sigma)(X, Z) Y - (\nabla\, \nabla\, \sigma)(Y, Z) X \\ &+ g\,(X, Z)\, \nabla_Y\, D\, \sigma - g\,(Y, Z)\, \nabla_X\, D\, \sigma, \end{aligned}$$

where $D\, \sigma$ is the gradient vector field of σ. In local coordinates, above equations are

$$L_V\, \Gamma^a_{bc} = \delta^a_b\, \sigma_c + \delta^a_c\, \sigma_b - g_{bc}\, \sigma^a, \tag{6.50}$$

$$L_V\, R^a{}_{bcd} = -\, \delta^a_c\, \sigma_{b\,;\,d} + \delta^a_d\, \sigma_{b\,;\,c} - \sigma^a{}_{;\,c}\, g_{bd} + \sigma^a{}_{;\,d}\, g_{cb}. \tag{6.51}$$

Following results can be easily deduced (6.51)

$$L_V\, R_{ab} = -\, 2\, \sigma_{b;a} - \triangle\, \sigma\, g_{ab}, \tag{6.52}$$

$$L_V\, r = -\, 2\, \sigma\, r - 6\, \triangle\, \sigma, \tag{6.53}$$

$$L_V\, G_{ab} = 2\, [\triangle\, \sigma\, g_{ab} - \sigma_{b\,;\,a}], \tag{6.54}$$

$$L_V\, C^a{}_{bcd} = 0. \tag{6.55}$$

where $G_{ab} \equiv R_{ab} - \frac{1}{2}\, r\, g_{ab}$ is the Einstein's tensor and $\triangle\, \sigma = div\, (grad\, \sigma)$. Recall from chapter 3, that a particular case of **CKV** is called a **special CKV** if $\sigma_{a\,;\,b} = 0$ (which also implies that $\triangle\, \sigma = 0$). Thus, for a special **CKV**, the Einstein's tensor remains invariant as seen from (6.54). We therefore, conclude that although the Einstein's tensor is invariant, for any matter tensor, under Killing and homothetic symmetries, unfortunately, this does not hold for a proper **CKV**. For this very reason, Killing and homothetic symmetries are preferred in finding exact solutions of the highly nonlinear Einstein's equations. On the other hand, a conformal symmetry preserves the causal character of the spacetime manifold. Also we do have cosmological spacetimes, which admit proper **CKV's**.

For example, **Robertson-Walker spacetimes** have a 9-parameter group of conformal motions (see Maartens-Maharaj [131]). There are many exact solutions known with conformal symmetry, but in most of those cases, the symmetry is actually homothetic and in most spatially compact or asymptotically flat cases, conformal symmetries are full isometries (see Eardley [63]). It is also clear from the integrability conditions (6.50) and (6.51) that the use of conformal symmetry places severe restrictions on the connection and the curvature of M. We now discuss some key results indicating specific restrictions on spacetimes. First we need the following lemma:

LEMMA 1. *If V is a non-null vector field satisfying $V_i C^i{}_{jkl} = 0$ in a 4-dimensional semi-Riemannian manifold M, then M is conformally flat.*

PROOF. Consider a local orthonormal basis $(e_i)_{i=1,2,3,4}$ of the tangent space, where e_4 is the normalized V and $(e_a)_{a=1,2,3}$ are orthonormal vectors orthogonal to e_4. Recall that the Weyl conformal tensor has following properties

1. $C_{ijkm} = C_{kmij} = -C_{jikm}$, $C_{[ijkl]} = 0$.
2. $C^i_{jim} = 0$ (complete tracelessness).

Using above two properties we can easily see through combinatorial computations that $C_{abcd} = 0$. By hypothesis, $(e_4)_i C^i{}_{jkm} = 0$. Thus, we conclude that $C \equiv 0$, which proves the lemma.

THEOREM 6.10 (Collinson-French [37]). *A non-flat vacuum spacetime with a proper conformal vector field is of type N, and represents a plane-fronted gravitational wave with parallel rays.*

PROOF. The proof in [37] uses the Newman-Penrose null tetrad formalism, however, we prefer to follow Eardley at al. [64] which needs lemma 1. As M is vacuum $\Rightarrow R_{ij} = 0$. Thus it follows from (6.52) that

$$\nabla_i \sigma_j = 0, \tag{6.56}$$

where $\sigma_j = \nabla_j \sigma$. Differentiating (6.56) and anti-symmetrization we get $\sigma_i R^i{}_{jkm} = 0$. But, as M is vacuum, this implies

$$\sigma_i C^i{}_{jkm} = 0. \tag{6.57}$$

Now from (6.56) we see that $\sigma^i \sigma_i$ is constant, and hence the causal character of σ^i at one point is the same as at every point of M. If σ^i is non-null, then lemma 1 shows that M is conformally flat, and being vacuum, is flat. This contradicts the hypothesis. Therefore σ^i is lightlike which further implies from (6.56) that σ^i generates a non rotating, shear-free, divergence free, null geodesic congruence. Also, the equation (6.57) shows that the Weyl tensor is of Petrov type N (see Appendix A), with principal congruence σ^i. Thus, M is a plane-fronted gravitational wave with parallel rays whose metric is of the form:

$$ds^2 = -2\, H(u, x^C)\, du^2 - 2\, du\, dr + \delta_{AB}\, dx^A\, dx^B, \tag{6.58}$$

where A, $B = 2, 3$ and the vacuum Einstein equations require that H satisfies the 2-dimensional Laplace equation

$$\delta^{AB}\, \partial_A\, \partial_B\, H(u, x^C) = 0.$$

REMARK. Eardley et al. [64] solved the conformal field equations for (6.58) in vacuum and non-vacuum spacetimes. Also following result resolve the problem of existence of conformal motions in **asymptotically flat spacetimes**.

THEOREM 6.11 (Eardley et al. [64]). *Let (M, g) be a globally hyperbolic spacetime which*

(a) *is spatially asymptotically flat,*

(b) *satisfies the Einstein's field equations for a stress tensor T obeying the dominant energy condition, with T asymptotically of order $O(r^{-4})$,*

(c) *admits a* **CKV** *field V which asymptotically approaches the dilation vector $x^a\, \partial_a$.*

Then (M, g) is a Minkowski spacetime.

PROOF. Let M admit a coordinate system $(x^i)_{i=0,1,2,3}$ such that for a Minkowski metric η_{ij} we have

$$g_{ij} = \eta_{ij} + O(r^{-1}) \tag{6.59}$$

and in particular, for $a, b = 1, 2, 3$,

$$g_{ab} = (1 + \frac{2\,m}{r})\, \delta_{ab} + O(r^{-1}) \tag{6.60}$$

with the condition that

$$\partial_c\, g_{ab} = -\frac{2\,m\,x^c}{r^3}\, \delta_{ab} + O(r^{-3}) \tag{6.61}$$

$$\partial_0\, g_{ab} = O(r^{-2}). \tag{6.62}$$

In this proof we use the positive energy theorem, that is, $m \geq 0$, since by hypothesis (b) it is applicable. For this purpose we let the second derivative of g_{ab} hold as $r \to \infty$. Now we work in a spacelike hypersurface defined by $x^0 = 0$. It follows from (6.62) that the second fundamental form $B_{ab} = O(r^{-2})$. Setting the conformal vector field $V = f\, N + X$ (see (6.19)), we specify

$$f = c + O(r^{-1}) \tag{6.63}$$

$$X^a = x^a + f^a\,(x^b) + O(r^{-1}) \tag{6.64}$$

as $r \to \infty$, where c is a constant and f^a are homogeneous functions of degree 0. Assuming that the first derivatives of (6.63) and (6.64) are well-behaved, we conclude from (6.61) $\sim$ (6.64) that the conformal function σ is given by

$$\sigma = 1 + O(r^{-1}). \tag{6.65}$$

Using equations (6.59) $\sim$ (6.64) in the following evolution equation (see (6.20), section 6.2 where we replace c by σ)

$$2\,\sigma\, g_{ab} = -2(f\, A_{ab} + \frac{f}{3}\,\tau\, g_{ab}) + L_X\, g_{ab}$$

we obtain

$$\begin{aligned} 2\,\sigma(1+\frac{2m}{r})\,\delta_{ab} &= -\frac{2m}{r}\,\delta_{ab} + 2\,(1+\frac{2m}{r})\,\delta_{ab} \\ &+ \partial_a\, f_b + \partial_b\, f_a + O(r^{-2}). \end{aligned}$$

Contracting above equation with $\frac{x^a\,x^b}{r^2}$ and using the homogeneity $(x^b\,\partial_b\, f^a = 0)$, provides

$$\sigma = 1 - \frac{m}{r} + O(r^{-2}).$$

By hypothesis, we have

$$T_{ij} = O(r^{-4}), \qquad \partial_k\, T_{ij} = O(r^{-5}).$$

Taking the Lie-derivative of the Einstein equations $(R_{ij} = T_{ij} - \frac{1}{2}\, g_{ij}\, T)$, using one of the integrability condition (6.52), and the above fall-off rates, we get

$$\nabla_i\, \nabla_j\, \sigma = O(r^{-4}).$$

In particular, the spacial components provide

$$\frac{(6\,x^a\,x^b\,/r^2) - 2\,\delta_{ab}}{r^3}\, m = O(r^{-4}).$$

Hence $m = 0$, and positive energy theorem implies that the spacetime is Minkowski. This completes the proof.

THEOREM 6.12 (Garfinkle-Tian [72]). *Let (M, g) be a solution of the vacuum Einstein equations with non-zero cosmological constant Λ. Let V be a conformal vector field which is not Killing field. Then M is locally isometric to de Sitter spacetime if $\Lambda > 0$ or anti-de Sitter spacetime if $\Lambda < 0$.*

PROOF. Contracting the vacuum $(T_{ab} \equiv 0)$ field equations (4.43), of chapter 4, with g^{ab} provides

$$R_{ab} = \Lambda\, g_{ab}. \tag{6.66}$$

Lie-differentiating it along the conformal vector field V and using (6.52) we get

$$\nabla_a\, \sigma_b = -\frac{\Lambda\,\sigma}{3}\, g_{ab}. \tag{6.67}$$

Differentiating (6.67) and anti-symmetrizing provides

$$\sigma^b\, R^a{}_{bcd} = \frac{\Lambda}{3}(\sigma\,\delta^a_c - \sigma\,\delta^a_d).$$

Using (6.66) in the above equation and (2.32) of chapter 2 we get

$$\sigma_a\, C^a{}_{bcd} = 0. \tag{6.68}$$

Since **Lemma 1** applies here we have either (i) M is conformally flat, or (ii) σ^a is null. We will rule out the possibility (ii). Assume $C \neq 0$ at some point p. Since C is smooth, there must be some open neighborhood $\mathcal{U}$ of p such that C is non-zero at its each point. Again, the use of **Lemma 1** shows that σ^a is null or 0. Now taking the inner product of (6.68) with σ^a we get, in $\mathcal{U}$, $\sigma_a \sigma^a = 0$, whence, by differentiating we have

$$\sigma_a \sigma_b = \frac{\Lambda}{3} \sigma^2 g_{ab}.$$

Consequently, $\sigma_a = 0$ and $\sigma = 0$ on $\mathcal{U}$. But σ satisfies (6.67) everywhere on M and hence σ vanishes on M. Thus, V is Killing which contradicts the hypothesis. Therefore, $C \equiv 0$ on M. Using this and (6.66) in (2.32) of chapter 2, we get

$$R_{abcd} = \frac{\Lambda}{3}(g_{ac} g_{bd} - g_{ad} g_{bc}).$$

Hence M is of constant curvature, and locally isometric to de Sitter spacetime for $\Lambda > 0$, and anti-de Sitter spacetime for $\Lambda < 0$. This completes the proof.

THEOREM 6.13 (Sharma [171]). *A spacetime (M, g), with parallel Weyl conformal tensor and a non-homothetic conformal vector field V, is of Petrov type N or O. In case M is of type N and the Einstein tensor is invariant under V, then M represents plane-fronted gravitational waves with parallel rays.*

PROOF. Consider the commutative formula (see [211])

$$\begin{aligned} L_V \nabla_a C^b{}_{cde} \quad - \quad & \nabla_a L_V C^b{}_{cde} = (L_V \Gamma^b_{am}) C^m{}_{cde} - (L_V \Gamma^m_{ac}) C^b{}_{mde} \\ - \quad & (L_V \Gamma^m_{ad}) C^b{}_{cme} - (L_V \Gamma^m_{ae}) C^b{}_{cdm}. \end{aligned} \tag{6.69}$$

Left hand side of above equation vanishes since by hypothesis $\nabla C = 0$ and V conformal $\Rightarrow L_V C = 0$ as per equation (6.55). Now using these results and (6.50) in (6.69), then contracting at a and b followed by inner product with σ^a provides

$$\sigma^a \sigma^a C^b{}_{cde} = 0.$$

It shows that, either the spacetime is conformally flat ($C \equiv 0$), or σ^a is null (since V is non-homothetic). In the latter case, M is of type N and the quadruply repeated principal null direction of the Weyl tensor is given by σ^a. Since by hypothesis, the Einstein tensor is invariant under V, it follows after little computation that $\nabla_a \sigma_b = 0$. Hence, σ_a generates a non-rotating, shear-free, divergence-free, null geodesic congruence. Thus M represents a plane-fronted gravitational wave with parallel rays, This completes the proof.

At this point we state the following results (without proof) which also demonstrate restrictions placed by conformal symmetry.

Garfinkle [71]: A conformal vector field in an asymptotically Minkowskian, vacuum spacetime with positive Bondy energy, is Killing.

Sharma [172]: A spacetime (M, g), with divergence-free Weyl conformal tensor C and a non-homothetic conformal vector field, is locally either of type O or N.

In the later case, the quadruply repeated principal null direction of C is given by $D\sigma$, the gradient vector field of conformal scalar function σ. Moreover, the null sectional curvature with respect to $D\sigma$ is non-negative if M satisfies the null convergence condition.

REMARK 1. Theorem 6.12 also holds for any 4-dimensional Einstein space of arbitrary signature, admitting a conformal vector field. In higher dimensions we have the following result.

Kerckhove [109]: Let $(M,\, g)$ be an n-dimensional Einstein manifold of arbitrary signature with $R_{ab} = k(n-1)\, g_{ab}$ and $k \neq 0$. Let $C(p)$ be the set of closed conformal vector fields at $p \in M$. If each subspace $C(p)$ is a non-degenerate subspace of $T_p(M)$, whose dimension d is independent of p, then M is locally isometric to a warped product $B \times_f F$. The base $(B,\, g_B)$ is a d-dimensional space of constant sectional curvature k ; the fiber $(F,\, g_F)$ is an $(n-d)$-dimensional Einstein manifold with $Ric_F = c(n-d-1)\, g_F$ for some constant c ; and the gradient of the warped function f is a closed conformal vector field on B satisfying $g(grad\, f,\, grad\, f) = c - k\, f^2$.

REMARK 2. Theorem 6.13 also holds for any 4-dimensional semi-Riemannian manifold. Thus, if a 4-dimensional semi-Riemannian manifold M, with parallel Weyl conformal tensor, admits a non-homothetic conformal vector field, then either M is conformally flat, or the gradient of the conformal scalar is null.

REMARK 3. In spite of above evidence of severe restrictions placed by conformal symmetry, extensive work has been done which shows that **CKV's** play a key role in investigating deep physical insight on some astrophysical and cosmological questions, in particular reference to fluid spacetimes as presented in next section.

6.6 Relativistic Fluids and Conformal Symmetry

In this section we review the kinematic and dynamic properties of a fluid spacetime $(M,\, g)$ with a **CKV** field V and their use in the study of relativistic fluids. Following exactly as in the case of section 5.4 of chapter 5, one can show that the effect of a **CKV** on the fluid 4-vector u $(u\cdot u = -1)$ or a unit spacelike vector n $(n\cdot n = 1,$ $n\cdot u = 0)$ or the two principal null directions ℓ and k is given by

$$\begin{aligned}
L_V\, u^a &= -\sigma\, u^a + Y^a, \;\; L_V\, u_a = \sigma\, u_a + Y_a, \;\; Y\cdot u = 0, && (6.70)\\
L_V\, n^a &= -\sigma\, n^a + W^a, \;\; L_V\, n_a = \sigma\, n_a + W_a, \;\; W\cdot n = 0, && (6.71)\\
L_V\, \ell &= -\sigma\, k + \frac{Y+W}{\sqrt{2}}, \;\; L_V\, k = -\sigma\, \ell + \frac{Y-W}{\sqrt{2}}, && (6.72)
\end{aligned}$$

such that

$$W\cdot u + Y\cdot n = 0, \;\; \ell\cdot(Y+W) = 0, \;\; k\cdot(Y-W) = 0.$$

It is important to mention that the equations (6.70) and (6.71) were first proved by Maartens et al. [134] to show that the statement in the literature (see [95, 59]) that Y or W vanishes is not valid in general. They provided the following counter example. Consider a perfect fluid Friedmann Robertson Walker (FRW) spacetime with metric

$$ds^2 = r^2(t)(-\,dt^2 + dx^2 + dy^2 + dz^2)$$

for which fluid flow velocity vector $u = r^{-1}\,\partial_t$. The vector $V = t\,\partial_x + x\,\partial_t$ is a **CKV** (see [134]), and

$$L_V\, u = [V,\, u] = x(r^{-1})_{,t}\,\partial_t - r^{-1}\,\partial_x \neq -\sigma\, u.$$

Let $V = \lambda\, u + \xi$ such that $\lambda = -\,u \cdot V$ and $\xi \cdot u = 0$. Then, using the covariant derivative of u_a (see equation (4.7) of chapter 4) we obtain

$$L_V\, u_a = (\dot{\lambda} + \dot{u})\, u_a + \lambda\, \dot{u}_a - \lambda_{,b}\, h^b_a + 2\,\omega_{ab}\, V^b, \tag{6.73}$$

where $h_{ab} = g_{ab} + u_a\, u_b$ is the projection tensor. Now equating the second equation of (6.70) with (6.73) followed with successive contractions by u^a and h^{ab} provides

$$\sigma = \dot{\lambda} + \dot{u}_a\, V^a, \quad Y_a = 2\,\omega_{ab}\, V^b + \lambda\{\dot{u}_a + (log\,(\lambda^{-1}))_{,b}\, h^b_a\}. \tag{6.74}$$

There are three special cases which we discuss separately.

Case 1. $V = \lambda\, u$, then, $\xi \equiv 0$, $\omega_{ab}\, V^b = 0$ and $\dot{u}_a\, V^a = 0$. Also $Y \equiv 0$ (since $L_V\, u$ is clearly parallel to u).

THEOREM 6.14 (Oliver-Davis [156]). *A spacetime $(M,\, g)$ admits a timelike* **CKV** *field V parallel to the fluid flow vector u ($V = \lambda\, u$, $\lambda > 0$),iff*

(1) *M is shear-free, that is, $\sigma_{ab} \equiv 0$,*

(2) $\dot{u}_a = (log\,\lambda)_{,a} + (\frac{\theta}{3})\, u_a$,

(3) $\theta = \frac{3\sigma}{\lambda}$,

where σ_{ab}, θ and $\dot{u}_a$ are the shear, expansion and acceleration of the timelike congruence generated by u.

Proof follows by contracting successively the conformal Killing equation (6.49) with $h^a_c\, h^b{}_d - \frac{1}{3}\, h^{ab}\, h_{cd}$ and h^{ab} and then using (6.74).

Case 2. $V = \xi$, then $\lambda = 0$ and $V \cdot u = 0$. Therefore, (6.74) reduces to

$$\sigma = \dot{u}_a\, V^a, \quad Y_a = 2\,\omega_{ab}\, V^b. \tag{6.75}$$

It follows from the first equation of (6.75) that a **CKV** field V orthogonal to u is necessarily a Killing vector if V is also orthogonal to $\dot{u}$ or if the flow is geodesic. Also, by the second equation of (6.75) if V is parallel to the vorticity vector ω^a ($\omega_{ab}\,\omega^b = 0$), or if the vorticity is zero, then the flow V maps flow lines into flow lines, that is , $Y \equiv 0$.

THEOREM 6.15 (Mason-Tsamparlis [141]). *A spacetime (M, g) admits a spacelike* **CKV** *field $V = \alpha\, n$ $(\, n \cdot n = 1,\, n \cdot u = 0,\, \alpha > 0)$ iff*

(1) *M is shear-free, that is, $\bar{\sigma}_{ab} = 0$,*

(2) $L_V\, n = -\,\sigma\, n, \quad L_V\, u_a = -\,\sigma\, u_a + 2\,\alpha\,\omega_{ab}\, n^b$,

(3) $\sigma = \frac{\alpha}{2}\,\bar{\theta}, \quad \overset{\bullet}{n}_a = -\,(log\, \alpha)_{,a} + \frac{\bar{\theta}}{2}\, n_a.$

Proof follows the pattern of theorem 5.13 of chapter 5.

Now let V be a null **CKV**, that is, $V^a\, V_a = 0$ which when differentiated gives $V^a\, V_{a;b} = 0 = V^a\,(2\,\sigma\, g_{ab} - V_{b;a})$ or

$$V_{a;b}\, V^b = 2\,\sigma\, V_a, \tag{6.76}$$

so that V is always tangent to geodesic. Thus, we can set

$$V = \lambda\,\ell, \quad \lambda > 0 \quad (\ell \text{ is a null geodesic congruence}).$$

Note that the treatment is same if we replace ℓ by the other principal direction k.

THEOREM 6.16. *If V is a null* **CKV** *field on a spacetime (M, g), then it is parallel to one of the two principal null vectors, say ℓ $(\, V = \lambda\,\ell,\, \lambda > 0)$, which is geodesic, shear-free and has expansion given by $\hat{\theta} = 2\,\frac{\hat{\sigma}}{\lambda}$.*

Proof is common with the theorem 6.2 on homothetic vector fields.

An important physical result of the theorems 6.14 and 6.15 is a connection between conformal motions and **material curves** in the fluid spacetime. A **material curve** in a fluid is a curve which consists at all times of the same fluid particles and, therefore, it moves with the fluid as it evolves. This means, for example, the integral curves of a vector field V are material curves iff

$$L_V\, u = \tau\, u, \qquad \tau = \text{ function.} \tag{6.77}$$

It follows from theorem 6.14 that the integral curves of a timelike **CKV** field V, parallel to u, are material curves for which $\tau = -\,\sigma$. In particular, (6.77) holds for a homothetic $(\tau = -\,c)$ or Killing $(\tau = 0)$ or affine $(\tau = f_u)$ vector field. However, it follows from theorem 6.15 that in general (6.77) does not hold for any **CKV**. We now brief the following two results on material curves (proofs are given in [156] and [141] respectively).

PROPOSITION 1. *If a rotational fluid spacetime admits a timelike* **CKV** *parallel to u, then the vortex lines are material lines in the fluid.*

PROPOSITION 2. *Suppose a fluid spacetime admits a spacelike conformal motion with symmetry vector parallel to n, where $n \cdot n = 1$ and $n \cdot u = 0$.*

(1) *If the fluid is irrotational $(\omega = 0)$, then the integral curves of n must be material curves in the fluid.*

(2) *If the vorticity of the fluid is nonzero ($\omega \neq 0$), then the integral curves of n are material curves in the fluid iff they are vortex lines.*

Observe that all the above stated results are purely kinematic and are valid for any fluid matter field. To further examine the dynamic properties of **CKV's**, we let $(M,\, g)$ satisfy the Einstein's field equations

$$R_{ab} - \frac{1}{2}\, r\, g_{ab} = T_{ab}.$$

Using (6.49), (6.52) and (6.53), the Lie derivative of the field equations provides

$$L_V\, T_{ab} = 2\,((\triangle\,\sigma)\, g_{ab} - \sigma_{;\,ab}). \tag{6.78}$$

We now consider a perfect fluid for which T_{ab} is prescribed by

$$T_{ab} = (\mu + p)\, u_a\, u_b + p\, g_{ab}.$$

Expanding the left-hand side of (6.78) using T_{ab} for a perfect fluid, and then contracting, in turn, with $u^a\, u^b$, h^{ab}, $u^a\, h^b_c$ and $h^a_c\, h^b_d - \frac{1}{3}\, h^{ab}\, h_{cd}$, we obtain (here we set $\sigma_{;\,ab} u^a\, u^b \equiv \sigma_u$) the following:

$$\begin{aligned} L_V\, \mu &= -2\,\sigma\,\mu - 2\,\triangle\,\sigma - 2\,\sigma_u, \\ L_V\, p &= -2\,\sigma\, p - \frac{4}{3}\,\triangle\,\sigma - \frac{2}{3}\,\sigma_u, \\ (\mu + p)\, Y_a &= 2\,(\sigma_{;\,ab}\, u^b + \sigma_u\, u_a), \\ \sigma_{;\,ab} &= \frac{1}{3}\{\, g_{ab}\,(\triangle\,\sigma + \sigma_u) + u_a\, u_b(\triangle\,\sigma - 2\,\sigma_u)\} \\ &- \sigma_{;\,bc}\, u^c\, u_a - \sigma_{;\,ac}\, u^c\, u_b. \end{aligned} \tag{6.79}$$

To find physical equations of state, we note that the following holds for any **CKV**.

$$[(T^{ab} + \frac{1}{2}\, r\, g^{ab})V_b]_{;a} = -\,3\,\triangle\,\sigma. \tag{6.80}$$

Now consider the following two special cases:

Case 1. V is parallel to u. Then

$$(T^{ab} + \frac{1}{2}\, r\, g^{ab})V_b = -\,(\frac{\mu + 3p}{2})\, V^a$$

which when substituted in (6.80) and then using (6.79) and $V^a_{;a} = 4\,\sigma$ provides

$$\sigma\,(\mu + 3p) = 2\,(\sigma_u + 3\,\triangle\,\sigma). \tag{6.81}$$

Case 2. V is perpendicular to u. This includes V is parallel to spacelike unit vector n or principal null vector ℓ. Proceeding as above we get

$$\sigma\,(\mu - p) = \frac{2}{3}\,(\sigma_u - 4\,\triangle\,\sigma). \tag{6.82}$$

In particular, let V be a **Special Conformal vector field**. This means that $\sigma_{;ab} = 0$. Then, above two equations reduce to

$$\sigma\,(\mu + 3p) = 0, \qquad V \parallel u, \tag{6.83}$$

$$\sigma\,(\mu - p) = 0, \qquad V \perp u. \tag{6.84}$$

Thus, for a special **CKV** field V, we have

(1) If $V \parallel u$ then either $\sigma = 0 \Rightarrow V$ is Killing, or $\mu + 3p = 0$.

(2) If $V \perp u$ then either $\sigma = 0 \Rightarrow V$ is Killing, or $p = \mu$.

DISCUSSION. If V is a special **CKV** then, by definition, $\sigma_{;ab} = 0 \Rightarrow$ that the spacetime M admits a covariant constant vector field σ_a. Thus, according to the work of Hall and da Costa [83], a proper ($\sigma \neq 0$) special **CKV** cannot be a perfect fluid except $p \neq \mu$ (also see page 97 of this book). However, unfortunately, for $p \neq \mu$, the only solution **(1)** is not desirable since $\mu + 3p = 0$ is non physical condition. Also, the stiff equation of state ($p = \mu$), in **(2)**, is not possible. Therefore, the repeated statement in the literature (for example, see [95, 134]) " **A special CKV in general relativity can be a perfect fluid with stiff equation of state ($p = \mu$) is not true**". On the contrary, we now know the following:

PROPOSITION 3. *A special* **CKV** *in general relativity cannot be a perfect fluid, with zero cosmological constant.*

In general, a large number of papers deal with conformal motions in perfect fluid spacetimes. In the following we present a partial list with summary.

Maartens-Maharaj [131]. Recall that a spacetime (M, g) admitting a maximal conformal group is conformally flat. Maartens-Maharaj studied Robertson-Walker spacetimes, which are conformally flat having maximal conformal group G_{15} with G_6 of Killing vectors and a timelike conformal vector normal to the homogeneous hypersurfaces t = constant. They proved that M admits a 9-parameter group of conformal motions.

Kramer [115, 116]. These papers deal with a general case of perfect fluid stationary and axisymmetric exact solutions with rigid rotation. The first paper [115] deals with conformal vector commuting with two Killing vectors and proved that the only solution, under these assumptions, is the Schwarzschild interior solution which is static and conformally flat. Second paper [116] deals with a conformal vector whose commutation with each Killing vector is an arbitrary linear combination of the Killings. It is proved that there does not exist such solution of Einstein equations.

Kramer-Carot [117]. Based on the assumptions in [116], apart from the static solutions, there are no stationary axisymmetric, rigidly rotating perfect fluid spacetimes admitting a proper **CKV**.

Sharma [177]. The Weyl conformal tensor of a general perfect fluid spacetime M is divergence-free iff M is shear-free, irrotational and its energy density is constant over the spacelike hypersurface orthogonal to the 4-velocity vector. If M admits a proper conformal vector field, then it is conformally flat and locally either of type O or N. In the later case, the quadruply repeated principal null direction of C is given by $D\sigma$, the gradient of conformal function σ. The null sectional curvature, with respect to $D\sigma$, is non-negative if M satisfies the null convergence condition.

Mars-Senovilla [138, 139]. Let M be an axially symmetric spacetime with a **CKV** field V and the axial Killing vector K. They proved the following:

(1) A one and only one V commutes with K. Thus, if they do not commute, there is at least one 3-dimensional conformal group.

(2) If V is timelike and non-commuting with K, then, either V and K are the only symmetries, or else there is at least one 4-dimensional conformal group.

(3) Given one V and only one more Killing vector, K commutes with both.

(4) If M is also stationary with one V, then K commutes with both.

REMARKS. Thus, the 3-dimensional conformal group of stationary and axisymmetric M, with one V can not be arbitrary but, rather it can only take one of the few forms in which K commutes with the other two symmetries. Similar results hold for cylindrically symmetric spacetimes. Mars-Senovilla [139] also generalized the results of Kramer-Carot [117] to differentially rotating perfect fluids by finding solutions arising when a V is added to the stationary and axial symmetry.

All conformally flat perfect fluid solutions are known: they are either the generalized interior Schwarzschild solutions with zero expansion or the generalized Friedmann solutions for non-zero expansion (see Kramer et al. [118]). The only solutions admitting an equation of state $p = p(\mu)$ are the FRW models.

Also see Van der Bergh [195], Dyer et al. [62], Sussman [183], Castejon-Amenedo-Coley [30], Sussman-Lake [184], Maharaj-Maharaj [136, 137], Kitamura [110], Maartens et al. [132, 133], Capocci-Hall [27] and many more referred there in.

Herrera et al. [95, 96, 97]. They solved the Einstein equations for spherical symmetric distributions of anisotropic matter (see equation (5.61) of chapter 5), with a special **CKV**, and matched them with the Schwarzschild exterior metric on the boundary of the matter. Two families of solutions represent, respectively, expanding and contracting spheres which asymptotically tend to a static sphere. A third family of solutions describes oscillating black holes. However, due to the work of Hall and da Costa [83, 85], any discussion on the stiff equation of state is no longer valid.

Maartens et al. [134]. They studied kinematic and dynamics of special **CKV's** in anisotropic fluids. It is shown that both the integral curves of the fluid velocity vector u and the unit spacelike vector n of the anisotropy are material curves. Here also any of their remark on perfect fluid, with stiff equation of state, is not valid.

Now we let (M, g) be a class of Einstein Maxwell spacetimes with the electromagnetic tensor field F_{ab} and a **CKV** field V. If we take F_{ab} as a test electromagnetic field of any V and decompose $V_{a;\,b}$ into its symmetric and skew-symmetric parts such that

$$V_{a;\,b} = \sigma\, g_{ab} + F_{ab}, \qquad F_{ab} = V_{[a;\,b]}, \tag{6.85}$$

then F_{ab} is called **conformal Killing bivector** which satisfies

$$F_{ab;\,c} = R_{abcd}\,V^d - 2\,g_{c[b}\,\sigma_{a]}, \tag{6.86}$$

$$\sigma_{b;\,a} = -\frac{1}{2}\,\{V^c\,L_{ab;\,c} + 2\,\sigma\,L_{ab} + 2\,R_{c(b}\,F^c_{a)}\}, \tag{6.87}$$

$$L_{ab} \equiv R_{ab} - \frac{r}{6}\,g_{ab}.$$

Observe that the unit vector tangent to several important spacelike vectors satisfy the property $n \cdot u = 0$. Examples are the unit vector tangent to vortex lines in a rotational fluid, for which the local vorticity vector is defined by

$$\omega^a = \frac{1}{2}\,\eta^{abcd}\,u_b\,u_{c;\,d},$$

and the unit vector tangent to electric and magnetic field lines in an electrically conducting fluid, where the electric (E) and magnetic (H) 4-vectors are defined by

$$E^a = F^{ab}\,u_b, \qquad H^a = \frac{1}{2}\,\eta^{abcd}\,u_b\,F_{cd}.$$

If the current vector $J^a = F^{ab}_{\;;\,b}$ and $V \cdot u = 0$ in a perfect fluid, then it follows that

$$\sigma_{,\,a} = \frac{1}{6}\,(p - \mu)\,V_a - \frac{1}{3}\,J_a$$

and, therefore, $\sigma_a = 0$ (for which V is homothetic) iff

$$J = \frac{1}{2}(p - \mu)\,V.$$

Thus, either $J \parallel V$ or J vanishes iff V is homothetic. But, McIntosh (see theorems 6.3 and 6.6) has shown that for such proper homothetic V, J necessarily vanishes. Thus, we state the following:

PROPOSITION 4. *If a* **CKV** *field V is orthogonal to the fluid velocity vector u and also parallel to the source vector J, then V is Killing.*

Now we let the electric field E vanish. Physically, the almost vanishing of E can occur in a field for the idealized limit of infinite electric conductivity even when the anisotropy of the electric conductivity due to the magnetic field H is taken into account. Also, let vorticity be nonzero and $V \parallel H$. Then, it is a direct consequence of Maxwell's equations that the magnetic field lines are material curves. Hence, it follows from proposition 2(2) that the magnetic lines must coincide with the vortex lines and $n = H\,/|H| = \omega\,/|\omega|$.

Although the spherical symmetric static solutions have been extensively studied, very limited work has been done on non-static models. Therefore, in the following we review a recent work non-static models for anisotropic magneto fluids with conformal symmetry, which has an extensive list of related references.

Duggal et al. [55]. They have obtained a class of spherically symmetric non-static models with conformal symmetry and relativistic anisotropic magneto fluid matter. These models have a special feature that the flow lines are shear-free but expanding and non-geodesic. In addition, they derived some conformally flat solutions leading to generalized version of Robertson-Walker models which are of Petrov type I, D or O (see Appendix A).

A null Einstein Maxwell field is defined by $R_{ab} = \frac{1}{2} |\phi_2|^2 \ell_a \ell_b$ (see equation (4.37) of chapter 4), where ϕ_2 is the Maxwell scalar of singular F.

THEOREM 6.17 (Hall-Carot [86]). *Let (M, g) be a null Einstein-Maxwell field. If the Weyl tensor does not vanish on M and if M admits a proper* **CKV** *field, then $M = W \cup (M \setminus W)$, where W is open and the Petrov type is III at each point of W and N at each point of $M \setminus W$. The null ray vector ℓ is properly recurrent ($\nabla \ell = p \otimes \ell$, for a non-zero 1-form p) on W and scalable to a covariant constant null vector field on int $(M \setminus W)$ which is everywhere locally isometric to a generalized pp-wave.*

REMARK. The subject matter on conformal motions is indeed very wide and can not be covered in one book. There are many papers referred in those papers we discussed so far which may interest the readers.

6.7 Conformal Motions in Riemannian Manifolds

Let $(M,\, g)$ be a connected Riemannian manifold. The fundamental theorem of Hopf-Rinow (see page 47) says that a compact Riemannian manifold is complete. In order to include as much possible results we state main theorems with references where the proofs are available.

Yano [205]. If the Ricci curvature of a compact orientable Riemannian manifold, without boundary, is non-positive, a conformal vector field has a vanishing covariant derivative (hence Killing), and if the Ricci curvature is negative-definite, a conformal vector field other than the zero vector field does not exist on M.

Ishihara-Tashiro [101]. Let $(M,\, g)$ be a complete Riemannian manifold of dimension $n \geq 2$. In order for M to admit a nontrivial solution σ for a system of partial differential equations $\nabla \nabla \sigma = -\frac{1}{n} (\nabla^2 \sigma)\, g$, it is necessary and sufficient that M be conformal to a sphere in the $(n+1)$-dimensional Euclidean space.

Yano-Nagano [213]. If a complete connected Einstein manifold $(M,\, g)$, of dimension $n \geq 2$, admits a non-Killing conformal vector field, then M is isometric to a sphere in an $(n+1)$-dimensional Euclidean space.

Lichnerowicz [127], Yano-Obata [215], Bishop-Goldberg [14]. Let a compact Riemannian manifold M admit a non-Killing conformal vector field V, with conformal function σ such that one of the following conditions hold:

(1) The 1-form associated with V is exact,

(2) $grad\,\sigma$ is an eigenvalue of the Ricci tensor with constant eigenvalues,

(3) $L_V\,Ric = f\,g$, for some smooth function f.

Then M is isometric to a sphere.

Goldberg [75], Yano [210]. In order that a compact Riemannian manifold M, of dimension $n > 2$, with scalar curvature $r =$ constant and admitting non-Killing conformal vector field (with conformal function σ) be isometric to a sphere, it is necessary and sufficient that

$$\int_M (Ric - \frac{r}{n}\,g\,)(grad\,\sigma\,,\,grad\,\sigma)\,dv = 0.$$

Goldberg [75], Yano [210]. If M is complete, of dimension $n > 2$, with r =constant > 0, and if it admits a non-Killing conformal vector field, with conformal function σ, then

$$\sigma^2\,r^2 \leq n\,(n-1)^2\,|\nabla\,\nabla\,\sigma|^2,$$

and equality holds iff M is isometric to a sphere.

Yano [211]. If a complete Riemannian manifold M, of dimension $n > 2$, with scalar curvature r admits a non-Killing conformal vector field V that leaves the length of the Ricci tensor Ric invariant, that is, $V(|Ric|) = 0$, then M is isometric to a sphere.

A corollary of Yano's [211] result for $r =$ constant and $|Ric| =$ constant, was obtained by Lichnerowicz [127] and another corollary, for M homogeneous, was obtained by Goldberg-Kobayashi [76]. Note that Yano's [211] theorem also holds if Ricci tensor is replaced by the curvature tensor.

Katsurada [105]. Let Σ be a closed orientable hypersurface, of constant mean curvature, of an orientable Einstein manifold M. If M admits a conformal vector field V that is nowhere tangential to Σ, then σ is totally geodesic.

REMARK 1. The hypothesis "M is Einstein" in Katsurada's result can be replaced by the following weaker assumption.

"*The normal vector field to Σ is an eigenvector of the Ricci tensor of M*".

At this point we recall that the group $C(M\,,g)$ of all conformal diffeomorphism of a Riemannian manifold M is a Lie group with respect to the compact-open topology. Let $C_0(M\,,g)$ be the connected component of the identity of $C(M\,,g)$. If g and g' are conformal, then $C(M\,,g) = C(M\,,g')$. The group $I(M\,,g)$ of all isometries of M is a closed subgroup of $C(M\,,g)$. A subgroup G of $C(M,\,g)$ is said to be essential if G is not contained in $I(M\,,e^{2f}\,g)$ for any smooth function f on M, and is inessential otherwise. Assuming $dim(M) > 2$, M connected compact and smooth, **Obata [155]** has proved the following:

(1) $C_0(M, g)$ is essential iff M is conformal to a Euclidean n-sphere S^n.

(2) If M has constant scalar curvature k, then $C_0(M, g)$ is essential iff k is positive and M is isometric to a Euclidean n-sphere $S^n(k)$ of radius $\frac{1}{\sqrt{k}}$.

REMARK 2. Recall that all known examples of compact Riemannian manifolds, with positive sectional curvature carry a positively curved metric with a continuous Lie group as its group of isometries and thus they carry a Killing vector field. A theorem of M. Berger [12] says that such a Killing vector field must be singular at least at one point if the manifold is even dimensional. Berger's result remains true for conformal vector fields. On the other hand, the Euler characteristic of a closed odd dimensional manifold is zero. There are many examples of odd dimensional closed positively curved Riemannian manifolds carrying non-singular Killing vector fields. A simple example is the round 3-sphere S^3 which admits 3 pointwise linearly independent Killing vector fields while no two of them commute. This is obvious if one considers S^3 from the Lie group theoretic point of view. Generalizing Berger's result to odd dimensional manifold, we state the following result (Proof is available in [203]).

Yang [204]. On a closed odd dimensional Riemannian manifold of positive sectional curvature, each pair of commutative conformal vector fields are dependent at least at one point.

REMARK 3. Since there is no analogue to the Hopf-Rinow theorem (and also the divergence theorem is not valid) for a general semi-Riemannian manifold, it remains an open problem to verify some (or all) results discussed when the metric is of an arbitrary signature. Recently, there seems to be some interest in the study of conformal fields in semi-Riemannian manifolds. For example, we refer three papers of Kühnel-Rademacher [119, 120, 121]. Also, we mentioned earlier that Garfinkle-Tian's theorem 6.12 has been generalized by Kerckhove [109] for any Einstein space of arbitrary signature and Sharma's theorem 6.13 also holds for any 4-dimensional semi-Riemannian manifold.

Chapter 7

Connection and Curvature Symmetries

In this chapter we study properties of semi-Riemannian (in particular Lorentzian) manifolds subject to three higher symmetries. We present up-to-date information on the existence of conformal collineations (whose roots go back to H.Levy's [129] work in 1926), their restricted use in relativity, very limited known work on projective collineations and a large number of papers on curvature collineations, in particular their kinematic and dynamic properties with respect to perfect fluid spacetimes.

7.1 Conformal Collineations

Recall that a **conformal vector field** V on an n-dimensional semi-Riemannian manifold $(M\,,g)$ is defined by

$$L_V\, g = 2\,\sigma\, g \tag{7.1}$$

for a smooth function σ on M. Equation (7.1) used in the equation (3.9), of chapter 3, yields

$$L_V\, \Gamma^k_{ij} = \delta^k_i\, \sigma_j + \delta^k_j\, \sigma_i - g_{ij}\, \sigma^k, \tag{7.2}$$

where $\sigma_j = \partial_j\,(\sigma)$. Above relation in index-free notation is

$$(L_V\, \nabla)(X,Y) = (X\,\sigma)\,Y + (Y\,\sigma)\,X - g(X,Y)\,D\,\sigma, \tag{7.3}$$

where $\nabla(X,Y) = \nabla_X\, Y$ for any vector fields X, Y on M and $D\,\sigma$ is the gradient vector field of σ. Thus (7.1) implies (7.3), however the converse is not necessarily true. This means that a vector field V satisfying (7.3) need not be a **CKV** and, therefore, defines another symmetry, called **conformal collineation** (see Tashiro [189]) and V is then called an **affine conformal vector field** (see Sharma-Duggal [180]), briefly denoted by **ACV**.

PROPOSITION 1 (Sharma-Duggal [180]). *A vector field V on a semi-Riemannian manifold $(M\,,g)$ is an* **ACV** *iff*

$$L_V\, g = 2\,\sigma\, g + K, \tag{7.4}$$

where K is a **second order covariant constant** ($\nabla K = 0$) **symmetric tensor field**.

PROOF. Recall the following commutative formula (see page 39 of chapter 3)

$$(L_V \nabla_X g - \nabla_X L_V g - \nabla_{[V,X]} g)(Y,Z) = -g((L_V \nabla)(X,Y),Z) - g((L_V \nabla)(X,Z),Y)$$

Using $\nabla g = 0$ and (7.3) in above formula we get

$$\nabla_X (L_V g - 2\sigma g) = 0,$$

from which the equation (7.4) follows immediately. This completes the proof.

REMARK 1. Proposition 1 reveals that an **ACV** deviates from a **CKV** field by a **second order covariant constant symmetric tensor** K, unless K is proportional to g. If K is different than g, then we say that V is a proper **ACV**. Moreover, V is called a **special ACV** if it is proper and $\nabla\nabla\sigma = 0$. On the existence of a proper **ACV** and a **special ACV** and specific restrictions on the ambient manifold M, we recall that, in 1932, Eisenhart [65] proved "*If a Riemannian M admits such a tensor K, independent of g, then M is reducible.*" Thus, we have

PROPOSITION 2 (Tashiro [189]). *An irreducible Riemannian manifold admits no* **ACV** *which is not a* **CKV**.

In 1950, Patterson [160] proved "*If a semi-Riemannian M admitting such a tensor K, independent of g, is reducible then the matrix (K_{ij}) has at least two distinct characteristic roots at any point of M.*" Thus, for a semi-Riemannian M, a general characterization of an **ACV** still remains open, and, an irreducible M may admit this symmetry.

Based on above stated Patterson's condition, we can state the following main results on **ACV** on reducible semi-Riemannian manifolds (first proved by Tashiro [189] for Riemannian manifolds):

THEOREM 7.1. *If a locally reducible semi-Riemannian manifold M has at least three parts, one of which is flat, then an* **ACV** *on M is the sum of an affine vector and a conformal vector. If, in addition, M is complete, then the* **ACV** *is affine vector.*

EXAMPLE. Mason and Maartens [140] constructed the following example which supports first part of above theorem. Let M be a 4-dimensional **Einstein static fluid spacetime** with metric

$$ds^2 = -dt^2 + (1-r^2)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2)$$

and the fluid velocity vector $u^a = \delta^a_0$ ($a = 0,1,2,3$). This spacetime admits a **CKV**

$$V^a_1 = (1-r^2)^{1/2} \{\cos t \, u^a - r \sin t \, \delta^a_1\}$$

and a proper **affine vector** $V_2^a = t\,u^a$. Since this spacetime metric is reducible, it can be easily verified that a combination $V = V_1 + V_2$ is a proper **ACV** such that

$$\begin{aligned} V^a &= [t + (1-r^2)^{1/2} \cos t]\, u^a - r(1-r^2)^{1/2} \sin t\, \delta_1^a, \\ \sigma &= -(1-r^2)^{1/2} \sin t, \qquad K_{ab} = -2\, t_{,a}\, t_{,b}. \end{aligned}$$

THEOREM 7.2. *If a semi-Riemannian manifold M has constant scalar curvature and have a flat part, then an* **ACV** *on M is the sum of an affine and a conformal vector fields.*

THEOREM 7.3. *If a semi-Riemannian manifold M has at least three parts and no part is locally flat, then an* **ACV** *on M is an affine vector. If, in addition, M is complete, then the* **ACV** *is Killing.*

THEOREM 7.4. *Let a semi-Riemannian manifold M be of constant scalar curvature and have no flat part. If M itself is irreducible or M is the product of two irreducible parts whose scalar curvatures are signed opposite to each other, then an* **ACV** *on M is a* **CKV**. *Otherwise, it is an affine vector.*

THEOREM 7.5. *A globally defined* **ACV** *on a semi-Euclidean space is necessarily an affine vector field.*

In 1926, H. Levy [129] proved that a second order covariant constant non-singular (recently Sharma [173] removed "non-singular" condition) symmetric tensor in a space of constant curvature is proportional to the metric tensor. Thus, we now can state:

THEOREM 7.6. *A semi-Riemannian manifold of constant curvature admits no* **ACV** *which is not a* **CKV**.

REMARK 2. It follows from above theorem that a large class of manifolds, with constant curvature, is excluded from the study on **ACV's**. Physically, for example, **de-Sitter or anti-de-Sitter spacetimes** cannot carry a proper **ACV** or a **special ACV**.

In general, for an **ACV** field V, the following identities hold.

$$\begin{aligned} (L_V\, R)(X,Y,Z) &= (\nabla\nabla\sigma)(X,Z)Y - (\nabla\nabla\sigma)(Y,Z)X \\ &+ g(X,Z)\,\nabla_Y\, D\,\sigma - g(Y,Z)\,\nabla_X\, D\,\sigma, \\ (L_V\, Ric)(X,Y) &= -(\Delta\,\sigma)\, g(X,Y) - (n-2)(\nabla_X\,\nabla\,\sigma)Y, \\ (L_V\, Q)X &= -\{(n-2)\,\nabla_X\, D\,\sigma + (\Delta\,\sigma)X + 2\,\sigma\, Q\,X + H\,Q\,X\}, \\ L_V\, r &= -\{2(n-1)\,\Delta\,\sigma + 2\,\sigma\, r + Tr.\, H\,Q\}, \\ (L_V\, C)(X,Y,Z) &= (n-2)^{-1}\{g(Y,Z)H\,Q\,X - g(X,Z)H\,Q\,Y \\ &- K(Y,Z)Q\,X + K(X,Z)\,Q\,Y\} \\ &+ \frac{1}{(n-1)(n-2)}\{r\,(K(Y,Z)X - K(X,Z)Y) \\ &- Tr.\,H\,Q\,(\,g(Y,Z)X - g(X,Z)Y)\}, \end{aligned}$$

where $\triangle\sigma = div\,(grad\,\sigma)$, $g(QX,Y) = Ric(X,Y)$ and $g(HX,Y) = K(X,Y)$. Observe that, for a general **ACV** or **special ACV**, contrary to the case of **CKV**, $L_V\,C \neq 0$. This raises the following open question:

Find condition(s) on $(K,\,\sigma)$ of a proper **ACV** *or a* **special ACV** *field V such that $L_V\,C = 0$.*

To understand the geometric description of an **ACV** field V on M, we let $\{f_t :$ $t =$ a real parameter $\}$ be a 1-parameter group of local infinitesimal transformations generated by V. Then, (7.3) and (7.4) can be expressed as

$$\begin{aligned} \bar{\nabla}_X Y &= \nabla_X Y + t\,\{X(\sigma)Y + Y(\sigma)X - g(X,\,Y)\,grad\,\sigma\}, & (7.5)\\ \bar{g} &= (1+2\,t\,\sigma)g + t\,K, & (7.6) \end{aligned}$$

where $\bar{g}(X\,,Y) = g\,(d\,f_t\,X,\,d\,f_t\,Y)$, $\bar{\nabla}_X Y = \nabla_{(df_t)X}\,(d\,f_t)Y$ and $d\,f_t$ denotes the **Jacobian differential** of f_t. The proof of the following proposition is straightforward, where we use (7.4), (7.5) and (7.6).

PROPOSITION 3 (Sharma-Duggal [180]). *If a semi-Riemannian manifold M admits a proper* **ACV**, *then*

(a) *A non null vector retains its causal character.*

(b) *A null vector U retains its causal character iff $K(U,\,U) = 0$.*

(c) *A geodesic does not, in general, remain a geodesic unless it is null (like a conformal motion), but the transformed geodesic will not be null unless $K(U,\,U) = 0$ (unlike a conformal motion).*

(d) *In general, the set of all* **ACV's** *do not form a Lie group structure (unlike a conformal motion).*

Let M be a compact orientable semi-Riemannian manifold with boundary $\partial\,M$. As stated in chapter 5, the **Stokes theorem** (see pages 79 to 81) is not obviously valid due to the possible existence of degenerate metric coefficient $g_{ii} = 0$. For this reason, in 1991, Duggal [52] considered only those M for which the Stokes theorem is valid (excluding the degenerate case), called **regular semi-Riemannian manifolds**. The term "**regular**" has its roots in the study of separability structures for M (see Benenti-Francaviglia [11]), where the regular separability structures were restricted by excluding the degenerate case.

THEOREM 7.7 (Duggal [52]). *A vector field V in a compact orientable regular semi-Riemannian manifold $(M,\,g)$, with boundary $\partial\,M$, is a proper* **ACV** *iff*

(a) $\int_{\partial\,M} (K - \frac{tr.H}{n}\,g)\,(V,\,N)\,ds \neq 0$

(b) $\mathcal{D}\,V = -\,(n-2)\,grad\,\sigma \in M$

where $\mathcal{D}\,V = Q\,V + \triangle\,V$, σ, K and H are the **de-Rham Laplacian**, *affine conformal function, covariant constant tensor of type $(0,2)$ and its associated $(1,\,1)$ tensor respectively.*

PROOF. Recall the following identity of Yano [211, page 121]

$$\begin{aligned} g(\mathcal{D}X - \frac{n-2}{2}\, grad\,(div\, X),\, X) \; - \; \frac{1}{2}\,|L_X\, g - \frac{2}{n}\,(div\, X)\, g|^2 \\ = \; \delta\,[i(X)(L_X\, g) - \frac{2}{n}\,(div\, X)\eta\,], \end{aligned} \tag{7.7}$$

where η is a 1-form associated with X and $\delta(\omega) = -\nabla_i\, w^i$ for a 1-form ω. Let $X = V$ an **ACV** on M. Contracting (7.4) provides

$$div\, V = n\,\sigma + \frac{1}{2}\, Tr.H, \qquad \mathcal{D}\, V = -\,(n-2)\, grad\,\sigma. \tag{7.8}$$

Integrating (7.7) over a class of regular M, using Stokes theorem and then (7.8) we obtain

$$\int_M |K - \frac{Tr.H}{n}\, g\,|^2\, d\,V = 2\int_{\partial\, M} (K - \frac{Tr.H}{n}\, g\,)\,(V,\, N)\, ds, \tag{7.9}$$

where, for regular M, we assume that the integrand in the left term of (7.9) is non-negative. V reduces to a **CKV** iff $K = (\frac{Tr.H}{n})g$. Since g is nondegenerate, it implies that the right hand side of (7.9) is non-zero iff V is a proper **ACV**. Thus, **(a)** and **(b)** hold, which completes the proof.

COROLLARY 1. *An* **ACV** *field V in a compact orientable regular semi-Riemannian manifold with boundary, for which V vanishes on boundary, is a* **CKV**.

COROLLARY 2. *A compact orientable regular semi-Riemannian manifold without boundary admits no* **ACV** *other than a* **CKV**.

NOTE. In 1995, Ünal [193] strengthened Duggal's above results by proving the validity of Stokes theorem under the hypothesis of theorems 5.4 and 5.6 (see pages 80 and 81). However, the general case, with no geometric conditions on M, still remains open.

The problem of finding an allowable prescription for the covariant constant tensor K, on a spacetime of general relativity, has been completely solved by Hall and da Costa (see page 96 of this book). For Riemannian and semi-Riemannian M, we have the following upto date results.

Levine-Katzin [128]. A second order covariant constant symmetric tensor K in a **conformally flat Riemannian space** is a linear combination of the metric tensor and the Ricci tensor.

Grycak [79]. He showed that above result also holds for a non-recurrent conformally recurrent Riemannian manifold with a locally exact recurrence form.

Duggal [52]. The following approach has been used to find a general mathematical equation for generating such a K-tensor for an **ACV** vector field V.

The equation (7.4) can be rewritten as

$$\nabla_{(i} V_{j)} = 2\,\sigma\, g_{ij} + K_{ij}, \qquad \nabla_k K_{ab} = 0.$$

Define the following tensor fields

$$\begin{aligned} F_{ij} &= \nabla_{[i} V_{j]}, \\ v_i &= (n-1)\,\nabla_i\,\sigma, \\ D_{ij} &= R_{ij} - \frac{r}{2(n-1)}\, g_{ij}. \end{aligned} \tag{7.10}$$

The following identities hold:

$$\begin{aligned} \nabla_i V_j &= \sigma_{ij} + F_{ij} + \frac{1}{2} K_{ij}, \\ \nabla_i F_{jk} &= \frac{2}{(n-2)}\, g_{i[k} v_{j]} + R_{kjim}\, V^m, \\ C_{ijkm} &= R_{ijkm} - (n-2)^{-1}\{ g_{i[k}\, D_{m]j} - g_{j[} D_{m]i}\}, \\ \nabla_j v_i &= \frac{\mu}{n-1}\, g_{ij} - L_{V_k}\, D_{ij} - \frac{r}{2(n-1)}\, K_{ij}, \end{aligned} \tag{7.11}$$

where we set $2\,\mu = K^i_j\, R^j_i$. We observe that the identity (7.11) provides an allowable prescription for a K-tensor of an **ACV** if $r \neq 0$ and $L_V\, D_{ij}$ is prescribed or known for a given Riemannian or semi-Riemannian M. We illustrate this point by considering following two cases.

(A). Consider all M with an **ACV** V subject to invariant $D(L_V\, D = 0)$. Motivation comes from the fact that the **invariant quantities** have substantial use in mathematical physics. For example, one knows that angular momentum will be important in a physical problem only if the problem is invariant under the rotations associated with at least one of the vector fields. In relativity, if a spacetime admits a symmetry vector (examples are Killing and homothetic) which leaves the **Einstein's tensor equations** invariant, then it has been very effective in finding exact solutions. Based on this we have the following result.

THEOREM 7.8 (Duggal [52]). *Let (M, g) be a semi-Riemannian manifold, with non-zero scalar curvature r and an* **ACV** *field V. Let the tensor D_{ij}, defined by (7.10), be invariant with respect to V. Then, K-tensor of V is of the form*

$$K = (\frac{2\,\mu}{r})\, g - \frac{2(n-1)(n-2)}{r}\, \nabla\nabla\,\sigma.$$

Moreover, under this hypothesis, there exists no **special ACV** *other than a* **special CKV**.

(B) Einstein Manifolds $(n > 2)$. For this case it is easy to see that $D = \frac{r(n-2)}{2n(n-1)}\, g$ for which $L_V\, D = \frac{r(n-2)}{2n(n-1)}\,[2\,\sigma\, g + K]$ is known. We, thus, state the following:

THEOREM 7.9 (Duggal [52]). *A second order covariant constant symmetric tensor K in an Einstein manifold is of the form*

$$K = \phi\, g - \frac{n(n-2)}{r} \nabla\nabla\sigma, \quad \phi = \frac{Tr.H - 2\,(n-2)\,\sigma}{2\,(n-1)}.$$

Also M belongs to Petrov type **N**. *Moreover, an Einstein manifold admits no* **special ACV** *(which means that $\nabla\nabla\sigma = 0$) other than a* **Killing vector field**.

7.2 Conformal collineations in Relativity

Let (M, g) be a 4-dimensional spacetime manifold with an **ACV** field V. The equation (7.4) can be rewritten as

$$V_{(b;\,a)} = \sigma\, g_{ab} + \frac{1}{2}\, K_{ab}.$$

A decomposition of $V_{a;\,b}$ into its symmetric and skew-symmetric parts and the use of above equation provides

$$V_{a;\,b} = \sigma\, g_{ab} + F_{ab} + \frac{1}{2}\, K_{ab}, \tag{7.12}$$

where, using the standard terminology (see McIntosh [142] and Duggal [49]), $F_{ab} = V_{[a;\,b]}$ is called a **conformal collineation bivector**. Let V be non-null. Define the projection tensor h, with the components

$$h_{ab} = g_{ab} - \epsilon\,\alpha^2\, V_a\, V_b \tag{7.13}$$

such that $h_{ab}\, V^b = 0$ and $V \cdot V = \epsilon\,\alpha^2\,(\epsilon = +1$ or -1 for spacelike or timelike V respectively) and $\alpha > 0$ is a real function.

THEOREM 7.10 (Duggal [49]). *If a spacetime admits a non-null* **ACV** *field V $(\,V \cdot V = \epsilon\,\alpha^2, \epsilon = \pm 1, \alpha > 0)$ then*

$$\sigma_{ab} = (2\,\alpha)^{-1}\,[h_a^c\, h_b^d - \frac{1}{3}\, h^{cd}\, h_{ab}]\, K_{cd}, \tag{7.14}$$

$$\theta = (\alpha)^{-1}\,[3\,\sigma + \frac{1}{2}\, h^{ab} K_{ab}], \tag{7.15}$$

where σ_{ab} and θ are the **shear tensor** *and the* **expansion** *of V.*

PROOF. Consider the tensor V_{ab} (which represents the relative velocities of neighboring particles) expressed by

$$V_{ab} = h_{ac}\, h_{bd}\, V^{c;\,d} = \theta_{ab} + \omega_{ab},$$

where θ_{ab} and ω_{ab} are the **expansion and vorticity tensors** respectively. Substituting the value of $V^{c;\,d}$ from (7.12) in above, we obtain

$$\begin{aligned} \theta_{ab} = \theta_{(ab)} &= \sigma_{ab} + \frac{1}{3}\,\theta\, h_{ab} \\ &= \frac{1}{2}\, h_{ac}\, h_{bd}\, K^{cd} + \sigma\, h_{ab}. \end{aligned} \tag{7.16}$$

Contracting (7.16) with g^{ab} and using $g^{ab}\sigma_{ab} = 0$, we obtain (7.14), which when substituted in (7.16) provides (7.15). This completes the proof.

CRITICAL REMARKS. In 1984, Herrera et al. [95] proved that for **special conformal motions**, the stiff equation of state ($p = \mu$), for a perfect fluid, is singled out in a unique way, if the generating **CKV** is orthogonal to the fluid velocity vector. In 1986, Duggal and Sharma [59] studied the same problem by using **special conformal collineation** and proved that the stiff equation of state is not singled out and cited some other solutions. In 1987, Mason and Maartens [140] further studied kinematics and dynamics of **conformal collineations** with respect to anisotropic fluids and also cited that **Einstein static spacetime** can carry a proper **ACV**. After the publication of above papers, Hall and da Costa [83] proved that the existence of a covariant constant second order tensor, other than the metric tensor, must exclude some spacetimes, in particular, perfect fluids (see page 96 of this book). Hence, the stiff equation of state in Herrera et al. [95] paper is no longer a valid solution for a **special CKV**. In the mean time, Coley and Tupper [31] used Hall and da Costa's work to show that the spacetimes admitting a **special ACV** are precisely the spacetimes that admit a **special CKV**. Therefore, Duggal and Sharma's [59] claim of solutions other than the stiff equation of state is also no longer valid. Thus, use of special **ACV** has not been effective in producing any new exact solutions.

On the other hand, in the case of proper **ACV**, there are strong indications that this symmetry has an important role in the study of fluids with shear, a counter part of the theory of shear-free cosmology (see theorem 7.10).

Also, on the existence of a physically meaningful K-tensor for a proper **ACV**, Duggal [52] proposed the following. Consider the **Einstein field equations**

$$G_{ab} \equiv R_{ab} - \frac{1}{2}\, r\, g_{ab} = T_{ab},$$

where G_{ab} is the **Einstein tensor**. Suppose M admits an **ACV** field V. Then, taking L_V of both sides of above field equations and using (7.4) and the integrability conditions (see page 136), we obtain

$$L_V\, G_{ab} = (\mu - 2 \triangle \sigma)\, g_{ab} - 2\nabla_a \nabla_b \sigma - \frac{r}{2} K_{ab}.$$

It is known that the invariance of G is physically desirable since that amounts to invariance of matter tensor, useful in finding exact solutions. Thus, assuming $L_V\, G = 0$ provides

$$K = \frac{2}{r}(\mu - 2 \triangle \sigma)\, g - \frac{4}{r}\nabla\nabla\sigma, \qquad r \neq 0.$$

Observe that it is immediate from above value of K that a **special ACV** is necessarily a **special CKV** for which G is obviously invariant.

Finally, on exact solutions we mention the work of Tello-Llanos [190] who used the necessary condition

$$h_{ab}\, R^b{}_{cde} + h_{cb}\, R^b{}_{ade} = 0$$

for the existence of an **ACV** field V, where $L_V g = h$, and also used Hall's classification (see [81]) of h. Suppose S is the subspace of independent 2-forms spanning the curvature 2-form say Ω_{ab}. Then, we have the following three cases:

Case 1: Dim(S) is 2 or 3 and the members of S have a common eigenvector with zero eigenvalue. Then,

$$L_V\, g_{ab} = h_{ab} = 2\,\sigma g_{ab} + f\, W_a\, W_b,$$

where W is the unique solution of $\Omega_{ab} W^b = 0$ and σ, f are smooth functions. In particular, it has been claimed that the spacetime with the interior of a **spherically symmetric matter distribution** can carry a proper conformal collineation vector field if W is non-null. For null W, conformal collineation reduces to a **CKV**.

Case 2: Dim(S) $= 1$. If Ω_{ab} is a spacelike bivector, then there exists a proper conformal collineation vector field with the spherically symmetric matter distribution.

Case 3: S is spanned by a simple bivector. This case does not admit any proper conformal collineation.

CONCLUSION. So far we know that **proper collineations** do exist in the **Einstein static universe** and in the interior of **spherical symmetric matter distributions**.
Also, the general study of **conformal collineations** in semi-Riemannian manifolds is still a very potential open problem. In particular, there is a need to have extensive study on global properties of proper **ACV's**.

7.3 Projective Collineations

Let M be a differentiable manifold with a symmetric affine (not necessarily metric) connection ∇. A continuous group of local diffeomorphism of M is said to be a **projective collineation** (briefly, **PC**), if it maps geodesics (auto parallel curves) into geodesics, and the generator of this group is called a **projective vector field**. If a **PC** also preserves the affine parameters of the geodesics, then it is called an **affine collineation**. A vector field V is a **PC** iff

$$(L_V\, \nabla)(X, Y) = p(X) Y + p(Y) X \tag{7.17}$$

for arbitrary vector fields X, Y and some 1-form p on M. In particular, if ∇ is the metric (Levi-Civita) connection induced from a semi-Riemannian metric g on M, then, (7.17) can be written (in a local coordinate system (x^i)) as

$$V^i_{;jk} + R^i_{kmj}\, V^m = \delta^i_j\, p_k + \delta^i_k\, p_j,$$

where $p_j = \partial_j\, p$ and the scalar function $p = (n+1)^{-1}\,(div\, V)$. Thus p_j is locally an exact form. Recall from section 3.5 of chapter 3 that the maximum dimension of the projective algebra (algebra of all **PC** vector fields) of M is $n^2 + n$ for which M is projectively flat and, therefore, its **Weyl projective tensor**

$$W^i{}_{jkm} = R^i{}_{jkm} - (n-1)^{-1}\,(\delta^i_k\, R_{jm} - \delta^i_m\, R_{jk}) \tag{7.18}$$

vanishes on $M \Leftrightarrow M$ is of constant curvature. Also recall that a **PC** vector field V satisfies (3.36) through (3.39) as stated on page 53. In addition, V satisfies the following integrability conditions:

$$L_V R^i{}_{jkm} = \delta^i_m p_{j\,;\,k} - \delta^i_k p_{j\,;\,m}, \tag{7.19}$$

$$L_V R_{ij} = (1-n)\, p_{i\,;j}, \tag{7.20}$$

$$L_V W^i{}_{jkm} = 0. \tag{7.21}$$

Since it is not possible to provide details on a large number of results for **PC** in one book, in the following we only state them with sufficient references and or hints for their proofs.

THEOREM 7.11. *If a Riemannian manifold M (dim. > 2) admits a non-affine* **PC** *for which its Weyl projective tensor is covariant constant, then M is of constant curvature.*

For proof we refer Yano and Nagano's work available in Yano [206]. Also note that if M is locally symmetric then, by definition $\nabla R = 0$ which further implies that $\nabla W = 0$. Thus, the hypothesis of theorem 7.11 is satisfied for a locally symmetric M.

THEOREM 7.12. *If a compact symmetric manifold M admits a* **PC***, then either M has constant curvature or the* **PC** *is an isometry.*

Hint. See Kostant [114] which says "*An affine collineation in a compact Riemannian (not necessarily orientable) manifold is necessarily Killing*".

THEOREM 7.13. *In a compact Einstein space with negative scalar curvature r, there exists no non-zero* **PC**.

The proof follows by using the following commutative formula

$$(L_V R_{ij})_{;\,k} - L_V (R_{ij\,;\,k}) = R_{mj}\, L_V \Gamma^m{}_{ik} + R_{im}\, L_V \Gamma^m{}_{jk}$$

and (7.20) along with the theorem 5.6 (see [22] of chapter 5).

THEOREM 7.14 (Nagano [148]). *Let M be a complete Einstein space of positive constant scalar curvature r. If M admits a non-affine* **PC***, then M is of constant curvature $\frac{r}{n(n-1)}$.*

The proof is based on the following theorem:

THEOREM 7.15 (Obata [154]). *In order for a complete and simply connected Riemannian manifold M to admit a non-constant function f satisfying*

$$f_{;\,ijk} + c\,(2\,g_{ij}\, f_{,k} + g_{ik}\, f_{,\,j} + g_{kj}\, f_{,\,i}) = 0$$

for some positive constant c, it is necessary and sufficient that M is isometric to a Euclidean sphere of constant curvature c.

THEOREM 7.16 (Tanno [186]). *Let M be an n-dimensional complete Riemannian manifold with positive constant scalar curvature $r = n(n-1)k$. If M admits a non-affine* **PC** *that leaves the tensor $G_{ij} = R_{ij} - (r/n)\, g_{ij}$ invariant, then M has constant curvature k.*

THEOREM 7.17 (Sumitomo [182]). *If a Ricci-symmetric ($R_{ij\,;\,k} = 0$) Riemannian manifold M has a non-isometric* **PC**, *then M is Einstein.*

THEOREM 7.18. *Let M be a complete Riemannian manifold with harmonic curvature ($R^i{}_{jkl\,;\,i} = 0$) and positive scalar curvature. Then M with a non-affine* **PC** *is of positive constant curvature* **(see Yamauchi (1979) cited in [202]).**

THEOREM 7.19 (Yamauchi [202]). *If a compact Riemannian manifold M with constant scalar curvature admits a non-affine* **PC**, *then M is of positive constant curvature.*

Recall (see Tanno [188]) that any eigenfunction corresponding to the k-th eigenvalue $k(n+k)$ of the Laplacian on the standard n-sphere S^n is given by the restriction of a homogeneous polynomial of degree k which is harmonic function in E^{n+1}. Based on this we state the following (proof is available in [188]):

THEOREM 7.20. *Any eigenfunction f corresponding to the second eigenvalue $2(n+2)$ of the Laplacian on the standard sphere (S^n, g) satisfies the following differential equations (in particular, grad f defines a non-affine* **PC**).

$$f_{;\,ijk} + 2\, g_{ij}\, f_{,k} + g_{jk}\, f_{,i} + g_{ik}\, f_{,j} = 0.$$

THEOREM 7.21 (Hall-Lonie [87]). *Let M be a smooth manifold of dimension $n \geq 3$ with positive definite metric g and a* **PC** *vector field V. Then*

(1) *If for a point x in M, there is a non-zero decomposable 2-form at x with components $G^{ij} = u^i\, v^j - v^i\, u^j$, where u and v are vectors at x, satisfying $R^i{}_{jkm}\, u^k\, v^m = 0$, then u and v are eigenvectors of the symmetric tensor $p_{i\,;\,j}$ at x with the same eigenvalue.*

(2) *If for each x in M, there is a coordinate neighborhood $\mathcal{U}$ of x and a smooth vector field K on $\mathcal{U}$ satisfying $R^i_{jkm}\, K^m = 0$ on $\mathcal{U}$, then $p_{i\,;j} = 0$ and, therefore, V is a* **special PC** *on M.*

(3) *If for each x in M, there is a coordinate neighborhood $\mathcal{U}$ of x and a smooth decomposable 2-form G on $\mathcal{U}$ satisfying the hypothesis of* **(1)**, *along with the prescription $p_{i\,;\,j} = f\, g_{ij}$, for some function f, then $f = 0$ and, therefore, V is a* **special PC** *on M.*

THEOREM 7.22 (Hall-Lonie [87]). *Let M be a smooth connected manifold of dimension $n \geq 3$ with smooth positive definite metric g and associated Levi-Civita connection Γ. Then*

(1) *A non-affine* **PC** *on M is special iff the holonomy group of M with respect to Γ is reducible.*

(2) *If Γ is not flat and if M admits a special projective vector field, then M is not geodesic complete with respect to Γ.*

7.4 Projective collineations in Relativity

Compared to other symmetries (discussed so far), there is very little available on **PC's** in the context of general relativity. This is surprising since **PC's** are, in fact, the most general transformations preserving geodesics that represent the trajectories of test particles. Also, we have seen in previous section quite an extensive work done on semi-Riemannian manifolds. One possible reason may be that the **causal structure** is not preserved by **PC**, but the same objection could be raised against affine and curvature collineations. At the end of this section we will present some open problems on **PC**, in particular reference to their study in fluid spacetimes.

Barnes [5] has characterized vacuum spacetimes ($R_{ab} = 0$), with a **PC** in the following way. Recall that $(a, b, c = 0, 1, 2, 3)$. $R_{ab} = 0 \Rightarrow p_{a;b} = 0$. If $p_a \neq 0$, then it is a covariant constant vector field and hence the spacetime is either flat, or a *pp*-**wave**. If $p_a = 0$, then **PC** reduces to an **affine collineation** for which the spacetime is decomposable or a *pp*-wave or it further reduces to a **homothetic** vector field. Barnes also proved the following:

THEOREM 7.23 (Barnes [5]). *The only 4-dimensional proper (non-vacuum) Einstein spacetimes admitting a non-affine* **PC** *are of constant curvature.*

THEOREM 7.24 ((Hall and Lonie [88]). *If M is a non-conformally flat Einstein spacetime (the Weyl conformal tensor does not vanish over a non-empty open subset of M) and admits a* **PC**, *then this* **PC** *is affine.*

NOTE. If the Petrov type is same all over the spacetime, above theorem shows that the only Einstein spaces admitting a non-affine **PC** are conformally flat. Their existence (also true for an arbitrary semi-Riemannian proper Einstein manifold) has been shown in [5].

Hall and Lonie [88] have also studied **PC** for the following four classes of spacetimes M, based on the algebraic properties of the curvature tensor.

Class A. There exists a unique vector X (which is locally smooth) at each point of M such that

$$R_{abcd} X^d = 0. \tag{7.22}$$

If we view the components of this type of curvature tensor as a 6×6 matrix R_{AB}, then $rank(R_{AB}) = 2$ or 3.

Class B. $Rank(R_{abcd}) = 2$ everywhere and spanned by an orthogonal timelike - spacelike pair of simple bivectors. In this class, there is no solution of (7.22) for X.

It is possible to choose a local null tetrad $\{\ell, n, u, v\}$ such that $\ell \cdot n = u \cdot u = v \cdot v = 1$, ℓ and n are real null vectors satisfying

$$\begin{aligned} \ell_{a\,;\,b} &= \ell_a\,\theta_b, \qquad n_{a\,;\,b} = -\,n_a\,\theta_b, \\ R_{abcd} &= a\,F_{ab}\,F_{cd} + b\,F^{\star}_{ab}\,F^{\star}_{cd}, \end{aligned}$$

where a and b are non-zero functions, θ_b is a 1-form on M and

$$F_{ab} = 2\,\ell_{(a}n_{b)}, \qquad F^{\star}_{ab} = 2\,u_{[a}v_{b]}.$$

Class C. $Rank(R_{abcd}) = 1$ and there are exactly 2 independent solutions of (7.22) for X which may also be chosen locally smooth.

Class D. This class is other than the above three classes with non-zero curvature tensor of *rank* $2, 3, 4, 5$ or 6. Also, there is no solution of (7.22) for X.

Observe that if M admits a global parallel vector field X, then obviously X satisfies (7.22) at any point and, therefore, M (if not flat) is of class **A** or **C**. If two such independent vector fields occur, then non-flat M is of class **C**.

To relate above classification with the existence of **PC**, we consider the following equation for a symmetric tensor, say X_{ab}:

$$X_{ab}\,R^b\,cde + X_{cb}\,R^b\,ade = 0. \tag{7.23}$$

A trivial solution of (7.23) is $X_{ab} = \lambda\,g_{ab}$, for some function λ. Based on above stated classification, following are the possible prescriptions for X_{ab}:

Class A. $X_{ab} = \alpha\,g_{ab} + \beta\,X_a\,X_b$

Class B. $X_{ab} = \alpha\,g_{ab} + \frac{\beta}{2}F_{ab}$

Class C. $X_{ab} = \alpha\,g_{ab} + \beta\,X_a\,X_b + \gamma\,X'_a\,X'_b + \delta\,(X_a\,X'_b + X'_a\,X_b)$

Class D. $X_{ab} = \lambda\,g_{ab}$

where $\alpha, \beta, \gamma, \delta$ and λ are some scalars and in case of class **C** X and X' are independent solutions of (7.22). Based on above, we state the following four theorems of **Hall and Lonie** [88], where their proofs are available.

THEOREM 7.25. *Let M be a spacetime which admits a* **PC** *vector field V and $R_{abcd}\,G^{cd} = 0$ for some bivector G at a point x of M. Then if G is simple, its blade consists entirely of eigenvectors of $p_{a\,;\,b}$ each with the same eigenvalue at x. If g is non-simple, then each of the two principal complementary blades of G consists of eigenvectors of $p_{a\,;\,b}$ with the same eigenvalue at x. If four or more independent bivectors G exist at x with this property, then $P_{a\,;\,b}$ is proportional to g_{ab}.*

THEOREM 7.26. *Let M be a spacetime admitting a* **PC** *vector V and each point x of M have a neighborhood $\mathcal{U}$ such that $R_{abcd}\,X^d = 0$ for some smooth vector field X on $\mathcal{U}$. Then $p_{a\,;\,b} = 0$ on M, and, therefore, V reduces to a curvature collineation.*

THEOREM 7.27. *There exist no proper* **PC** *in spacetimes of class* **A** *and* **B**.

THEOREM 7.28. *If a Class* **C** *spacetime M admits two nowhere zero independent global covariant vector fields, or if it admits no such vector fields, then M admits no proper* **PC**.

REMARKS. Since **PC's** preserve geodesics, they are closely related to the spacetime connection Γ. Thus it may be useful to describe **PC's** with respect to a spacetime classification based on Γ, that is, the classification of M by its **holonomy group**. For some work done on this, we refer Hall and Lonie [88] and Hall [84].

In comparison with other symmetries (Killing, homothetic, conformal, affine etc.), no work has been done on the kinematic and the dynamic properties of **PC's** for a given fluid spacetime. There is a timely need for further research on this open problem.

7.5 Curvature Collineations

According to the early observations of Noether [151], the existence of certain geometric symmetry properties described by continuous groups of motions or collineations lead to **conservation laws** in the form of **first integrals** (i.e., constants of motions) of a dynamical system. The relation between the existence of affine or projective collineations in the spacetimes of general relativity was first developed by Davis and Moss [41] and further studied by Katzin et al. [106, 107, 108] with respect to symmetry called **curvature collineation (CC)** defined by the existence of a vector field V, on an n-dimensional semi-Riemannian manifold (M, g), which leaves the curvature tensor invariant. This means

$$L_V \, R^i{}_{jkl} = 0. \tag{7.24}$$

Obviously, (7.24) implies, by contraction, that V is also a **Ricci collineation (RC)** vector field, that is

$$L_V \, R_{ij} = 0. \tag{7.25}$$

However, it is easy to see that **RC** does not imply **CC**. Set

$$L_V \, g_{ij} = V_{i\,;\,j} + V_{j\,;\,i} = h_{ij}. \tag{7.26}$$

PROPOSITION 1. *A necessary condition for a vector field V to define a* **CC** *is*

$$h_{ij\,;kl} - h_{ij\,;lk} = 0. \tag{7.27}$$

PROOF. Taking the Lie derivative of the curvature tensor identity

$$g_{ij} \, R^i{}_{klm} + g_{ik} \, R^i{}_{jlm} = 0$$

with respect to V and using (7.26) implies (7.27).

In particular, multiplying (7.27) by $\sqrt{g}\, g^{ij}\, g^{kl}$, we obtain the well-known Komar's identity [112]:

$$\{\sqrt{g}(V^{i\,;j} - V^{j\,;i})\}_{;\,ji} = \{[\sqrt{g}(V^{i\,;j} - V^{j\,;i})]_{;\,j}\}_{;\,i} = 0,$$

where $g = |det\, g_{ij}|$. Physically, Komar's identity plays the role of a **conservation law generator** in general relativity, when M admits symmetry properties. As Komar's identity holds for all vector fields on M, it follows that the necessary condition (7.27), for a **CC**, places no restriction on this symmetry vector V. Hence, **CC's** are necessary symmetry properties of M that are embraced by the group of general curvilinear coordinate transformations in M. There is one exception, in case of pure radiation fields of gravitational radiation when M is a Ricci flat spacetime of general relativity. For this case Komar [113] has proved that $(V^{i\,;j} - V^{j\,;i})_{;\,j}$ vanishes for a Killing V if $R_{ij} = 0$ everywhere.

Katzin et al [106] have shown that V is a **CC** iff

$$(h_{ij\,;\,k} + h_{jk\,;\,i} - h_{ik\,;\,j})_{;m} - (h_{mj\,;\,k} + h_{jk\,;\,m} - h_{mk\,;\,j})_{;i} = 0.$$

A **CC** vector field is said to be special, denoted by **SCC**, if

$$(L_V\, \Gamma^i{}_{jk})_{;l} = 0 \tag{7.28}$$

which implies that

$$(h_{im\,;\,j} + h_{mj\,;\,i} - h_{ij\,;\,m})_{;k} = 0.$$

Symmetrizing above result with respect to i and m gives

$$h_{im\,;\,jk} = 0. \tag{7.29}$$

It is easy to see that (7.29) implies (7.28). Hence we have

PROPOSITION 2. *A vector field V in a semi-Riemannian manifold M defines an* **SCC** *iff (7.29) holds.*

Contracting (7.29) with g^{im} one can show that if M admits an **SCC** then it admits a nowhere **covariant constant vector field**, say $U_j = V^i{}_{;\,ij}$. Recall that relations between **CC** and affine, **CKV** and **PC** have been discussed on page 54. For some more relations we state the following (proofs are given in [106]):

PROPOSITION 3. *A conformal collineation (in particular, conformal motion) is a* **CC** *iff $\sigma_{i\,;\,j} = 0$, where $\sigma_i = \partial_i\, \sigma$ and $h = 2\,\sigma\, g + K$ defines the conformal collineation.*

PROPOSITION 4. *In a Ricci-flat manifold every conformal collineation is a* **CC** *and hence a special conformal collineation.*

We know that every 2-dimensional $(M,\, g)$ is Einstein for which $R_{ij} = \frac{r}{2}\, g_{ij}$ but r is not necessarily constant. Lie differentiating it along an **RC** vector field V gives

$$0 = (L_V\, r) g_{ij} + r\, L_V\, g_{ij}.$$

Since $r \neq 0$, it follows that V is a **CKV**. Hence we have

PROPOSITION 5. *Every* **RC** *vector field in a 2-dimensional semi-Riemannian manifold, with non-zero scalar curvature, is a* **CKV**.

We know (see Eisenhart [66]) that the geodesic equation

$$\frac{dt^i}{ds} + \Gamma^i{}_{jk}\, t^j\, t^k = 0\, (t^i = \frac{dx^i}{ds})$$

admits an m-th order **first integral** of the form

$$A_{i_1\, i_2 \ldots i_m}\, t^{i_1}\, t^{i_2} \ldots t^{i_m} = const.,$$

where the tensor A is symmetric in all indices, iff

$$P\{A_{i_1\, i_2 \ldots i_m\, ;\, i_{m+1}}\} = 0.$$

Here $P\{\ \}$ indicates the sum of the terms obtained by cyclic permutation of all free indices within the braces. In particular, for $m = 1$, the geodesics have linear **first integrals** $A_{i_1}\, t^{i_1} = const.$ if $A_{i_1\,;\,i_2} + A_{i_2\,;\,i_1} = 0$ which are the Killing equations. Hence from proposition 2, we have

THEOREM 7.29. *If a semi-Riemannian manifold M admits an* **SCC**, *defined by a vector field V, then it also admits a Killing vector defined by the parallel vector field $V^i{}_{;i}{}^j$. The geodesics admit a linear first integral $(V^i{}_{;\,ij})\, t^j = constant$.*

We now state and prove how a **CC** gives rise to a **conservation law generator**. The results also hold for a general n-dimensional spacetime M.

THEOREM 7.30. *If a 4-dimensional spacetime M, with $r = 0$ and $R_{ij} \neq 0$ admits an* **RC** *vector field V (in particular, a* **CC**)*, then there exists a covariant conservation law generator of the form*

$$(\sqrt{g}T^i{}_j\, V^j)_{;\, i} \equiv (\sqrt{g}T^i{}_j\, V^j)_{,\, i} = 0, \quad g = |det\, g_{ij}|, \tag{7.30}$$

where T is the energy momentum tensor.

PROOF. $0 = L_V\, r = L_V\, (g^{ij}\, R_{ij}) = R_{ij}\, L_V\, g^{ij}$. Using $L_V\, g^{ij} = -\, g^{ik}\, g^{jl}\, L_V\, g_{kl}$ in this equation we get

$$R^{kl}\, L_V\, g_{kl} = 0. \tag{7.31}$$

The symmetry of R^{kl} and $L_V\, g_{kl} = V_{k\,;\,l} + V_{l\,;\,k}$ reduces (7.31) to

$$R^i{}_j\, V^j{}_{;\, i} = 0. \tag{7.32}$$

From the twice-contracted Bianchi identity and $r = 0$, we get $R^i{}_{j\,;\,i} = 0$. Hence (7.32) implies $(R^i{}_j\, V^j)_{;\, i} = 0$. On the other hand, the Einstein field equations, for $r = 0$, are $R^i{}_j = 8\,\pi\, T^i{}_j$. Hence, (7.30) holds, proving the theorem.

For null-electromagnetic and pure null-gravitational fields admitting **CC**, the conservation laws of the type first constructed by Sachs [167] follow from a symmetry argument. Indeed, the null-electromagnetic field has $T_{ij} = R_{ij} = \mu\, l_i\, l_j$, where μ is a scalar and l one of the principal null vector fields. Without any loss of generality, assume $l^i{}_{;j}\, l^j = 0$. Let this spacetime admit a **CC** vector field V. Then, it follows from theorem 7.30 that

$$(\sqrt{g}\, T^{ij}\, V_j)_{;\,i} = (\sqrt{g}\, \mu\, l^i)_{;\,i}\, l^j\, V_j = 0.$$

This equation provides Sach's conservation law: $(\sqrt{g}\, \mu\, l^i)_{,i} = 0$ for $l^j\, V_j \neq 0$. The close analogy between null electromagnetic and null gravitational fields imply that null spacetimes can be defined by the **Bel-Robinson tensor** (see [167])

$$T^{ijkm} = 2\, R^{irjs}\, R^k{}_r{}^m{}_s = \nu\, l^i\, l^j\, l^k\, l^m, \tag{7.33}$$

where ν is a scalar. For vacuum case ($R_{ij} = 0$), we have

$$T^{ijkm}{}_{;\,m} = 0.$$

The following theorem is immediate from above results:

THEOREM 7.31. *If a null spacetime, defined by (7.33), admits a* **CC** *vector field V, then it admits a conservation law of the form*

$$(\,\sqrt{g}\, T^{ijkm}\, V_k)_{;\,m} = 0,$$

or a proper conservation law generator of the form

$$(\,\sqrt{g}\, T^{ijkm}\, V_i\, V_j\, V_k)_{;\,m} = 0.$$

NOTE. It can be easily seen from theorem 7.31 that the existence of a **CC** in a null spacetime implies Sach's conservation law.

EXAMPLE 1. Consider the following line element

$$ds^2 = (dx)^2 + (dy)^2 + C(u)(dz)^2 - (dt)^2, \qquad u = x - t, \tag{7.34}$$

which represents a null-electromagnetic (plane-wave) radiation spacetime. After a lengthy computation (details are in [106]), one can show that this spacetime admits a proper **CC** vector V with components

$$\begin{aligned}
V^1 &= -\, C(u)\,\{\,\frac{1}{2}\, D(u)\,_{,u}\, z^2 + E(u)\,_{,u}\, z\} + P(\,u,\, y\,) \\
&+ \frac{1}{2}\,(x+t)\,\{\,2\, D(u) + M(u)\}, \\
V^2 &= V^2\,(u,\, y), \\
V^3 &= D(u)\, z + E(u), \\
V^4 &= V^1 - A\,(\,\frac{R_{1313}}{C(u)}\,)^{-\frac{1}{2}},
\end{aligned}$$

where A is an arbitrary constant; $D(u)$, $E(u)$ and $P(u,\, y)$ are arbitrary functions and $M(u) = A\,\{\,B\, log\,(R_{\,1313})\,_{,u} + B\,_{,u}\}$, with $B = (\,c(u)\,/\,R_{1313}\,)^{1/2}$.

Based on above material, we state (with references for their proofs) the following results on semi-Riemannian manifolds M with a **CC**:

Katzin et al. [107, 108].

(a) *M admitting a parallel vector field $U_i = S_{,i}$ (S a non-constant function), admits* **(i)** *a proper* **SCC** $V^{(2)i} = S^2 U^i$, **(ii)** *a proper affine collineation* $V^{(1)i} = S\,U^i$, *and* **(iii)** *a Killing vector* $V^{(0)i} = U^i$. *The generators* $V^{(\alpha)i}$ ($\alpha = 0, 1, 2$) *define a 3-parameter group of proper* **SCC's** *which is Abelian if U is null.*

(b) *Under the hypothesis of the above result* **(a)**, *M also admits a proper* **CC** $V = F'(S)\,U$ *where F is a function of S such that* $F'''' \neq 0$. *If U is null, then M admits an r-parameter Abelian group of proper* **CC's** $V^{(\alpha)i} = F_{(\alpha)}\,U^i$, *where each $F_{(\alpha)}$ is a function of S and* $F''''_{(\alpha)} \neq 0$ ($\alpha = 1, 2, \ldots, r$ *is any given set of r linearly independent but otherwise arbitrary scalar functions).*

Collinson [36]. *The only* **CC's** *admitted by an empty spacetime, not of Petrov type* **N**, *are conformal motions.*

EXAMPLE 2. Consider the vacuum *pp*-waves with metric

$$ds^2 = -\,2\,du\,dv + 2\,H\,dv^2 + 2\,dz\,d\bar{z},$$

where u, v are real coordinates, z is a complex coordinate and $\frac{\partial H}{\partial u} = \frac{\partial^2 H}{\partial z\,\partial \bar{z}} = 0$. This spacetime admits the most general proper **CC** vector V with components

$$\begin{aligned} V^1 &= 0, \;\; V^2 = -\,2\,a\,u - \dot{a}\,z\,\bar{z} - \dot{\bar{b}}\,z - \dot{b}\,\bar{z} + c(v), \\ V^3 &= a(v)\,z + b(v), \qquad \text{if} \end{aligned}$$

$$\dot{c} - 2\,\dot{a}\,u - \ddot{a}\,z\,\bar{z} - \bar{z}(\ddot{b} - a\,\bar{\alpha}\,b) - z(\ddot{\bar{b}} - 2\,\alpha\,b) = \frac{\alpha}{2}\;(\alpha \neq 0).$$

Aichelburg [2]. *The vacuum pp-waves admit lightlike proper* **CC's** *in the direction of the covariant constant null vector.*

Vaz and Collinson [196]. *They showed that Type* **N** *empty Robinson-Trautman spacetimes with non-twisting geodesic rays admit a proper* **CC**.

Tello-Llanos [190]. *He obtained a Robertson-Walker cosmological model (in the presence of spherical symmetric distribution of matter) with a zero deceleration parameter, that admits a* **CC**.

REMARK. We have already seen that the study of conformal symmetry in spacetimes is more complicated than that of affine (including isometric and homothetic) symmetries due to the lack of a certain type of linearity (which is present in the affine case). From this point of view, **CC's** are worse, because, not only do they lack this linearity, they can fail to possess a "*finite dimensionality*". Also, **CC** vector fields do form vector space, but, may not form a Lie algebra. These intriguing features of **CC's** have been described at the end of chapter 3.

Inspite of these negative aspects of **CC's**, recently, in [85], Hall and da Costa presented a mathematical basis for **CC's** by using their algebraic classification as described on page 147 of this book. We now state their results.

(1) *Every* **CC** *in a spacetime of type* **A** *is necessarily homothetic, which is not true for other types.*

(2) *Let $(M,\, g)$ be a 2-dimensional connected Riemannian or Lorentzian manifold. Then,* **(i)** *a vector field in M is a* **CC** *iff it is an* **RC**, **(ii)** **CC's** *are smooth and form a Lie algebra $K(M)$ of finite dimension $0, 1, 2$ or 3 (dimension 2 can not occur if M is simply connected), and a non-trivial member of $K(M)$ cannot vanish on a non-empty open subset of M, and* **(iii)** *a vector field is a* **CC** *iff it is a* **CKV** *with harmonic conformal factor.*

7.6 Ricci Collineations in Relativity

Let (M, g) be a 4-dimensional spacetime of general relativity with an **RC** vector field V. We use the range of indices $a, b, c, \ldots, \in \{0, 1, 2, 3\}$. By explicitly writing equation (7.24) followed by one contraction and the contracted Bianchi identity gives

$$(R^a{}_b V^b)_{;a} = 0. \tag{7.35}$$

Physically, (7.35) is important in the study of fluid spacetimes. To illustrate this point, recall that the Einstein equations for a perfect fluid are given by

$$R_{ab} - \frac{1}{2} g_{ab} = T_{ab} = \mu\, u_a\, u_b + p\, h_{ab}, \tag{7.36}$$

where μ, p and h_{ab} are the energy density, the isotropic pressure relative to the fluid flow velocity vector u and the projection tensor respectively. Using (7.35) in (7.36) for any T_{ab} provides

$$[(T^{ab} - \frac{1}{2} T\, g^{ab})\, V_b\,]_{;a} = 0, \qquad T = -r. \tag{7.37}$$

Equation (7.37), known as **dynamic equation**, serves as the basis for generating solutions for a variety of fluid spacetimes, subject to a given symmetry discussed so far. Here we consider perfect fluids. Observe that Tsamparlis and Mason [192] studied **RC's** subject to **imperfect fluids**.

THEOREM 7.32 (Oliver-Davis [156]). *A perfect fluid spacetime M, with Einstein field equations (7.36), admits a timelike* **RC** *vector field V parallel to the fluid flow velocity vector u ($V = \lambda\, u$, $\lambda > 0$) iff*

$$\begin{aligned} (\mu + 3p)\{\dot{u}_a - (log\, \lambda)_{,a} - \theta\, u_a\} &= 0, & (7.38)\\ \{(\mu + 3p)\, \lambda\, u^a\}_{;\, a} &= 0, & (7.39)\\ (\mu - p)\, \sigma_{ab} &= 0, & (7.40) \end{aligned}$$

where θ and σ_{ab} are the expansion and the shear tensor respectively.

PROOF. The definition of Lie derivative provides

$$L_{\lambda u} R_{ab} = \lambda \, [\dot{R}_{ab} + 2 \, u^c \, R_{c(a} \, (log \, \lambda)_{,b)} + 2 \, R_{c(a} \, u^c_{;b)}]. \tag{7.41}$$

Let $V = \lambda u$ be an **RC** vector field in M. Using $L_V R_{ab} = 0$ and (7.36) in (7.41) we obtain

$$\begin{aligned}(\dot{\mu} + 3 \dot{p}) \, u_a \, u_b + (\dot{\mu} - \dot{p}) h_{ab} \; &+ \; 4 \, (\mu + p) \, \dot{u}_{(a} \, u_{b)} \\ &- \; 2 \, (\mu + 3 p) \, u_{(a} \, (log \, \lambda)_{,b)} + (\mu - p) \, u_{(a \, ; \, b)} \\ &= \; 0. \end{aligned} \tag{7.42}$$

By contracting (7.42) in turn, with $u^a \, u^b$, $u^a \, h^b_c$, $h^a_c \, h^b_d - \frac{1}{3} \, h^{ab} \, h_{cd}$ and then using (see equation (4.7), page 57)

$$u_{a \, ; \, b} = \sigma_{ab} + (\theta \, / \, 3) \, h_{ab} + \omega_{ab} - \dot{u}_a \, u_b$$

we obtain, respectively

$$\dot{\mu} + 3 \dot{p} + 2 \, (\mu + 3 p) \, (log \, \lambda)^{\cdot} = 0, \tag{7.43}$$

$$(\mu + 3 p) \, [\dot{u}_a - (log \, \lambda)_{,a} - (log \, \lambda)^{\cdot} \, u_a \,] = 0, \tag{7.44}$$

$$(\dot{\mu} - \dot{p}) + \frac{2}{3} \, (\mu - p) \, \theta = 0 \tag{7.45}$$

and (7.40). Now we need the **energy conservation equation** along a fluid particle world line, which follows from Einstein's field equations:

$$\dot{\mu} = - \, (\mu + p) \, \theta. \tag{7.46}$$

Eliminating $\dot{\mu}$ and $\dot{p}$, by using (7.46) and (7.45) in (7.43), we obtain

$$(\mu + 3 p) \, (log \, \lambda)^{\cdot} = (\mu + 3 p) \, \theta. \tag{7.47}$$

Using (7.47) in (7.44), we recover (7.38). Now, we rewrite (7.43) in the form

$$(\mu + 3 p)^{\cdot} + 2 \, (\mu + 3 p) \, (log \, \lambda)^{\cdot} = 0. \tag{7.48}$$

If (7.47) is used to replace the term $(\mu + 3 p) \, (log \, \lambda)^{\cdot}$ in (7.48), then (7.39) follows directly. The converse is straightforward. This completes the proof.

DISCUSSION. In order to describe a physically reasonable perfect fluid, under the hypothesis of theorem 7.31, we assume the state of equation $p = p(\mu)$ satisfies energy condition:

$$\mu > 0, \qquad p \geq 0, \qquad 0 \leq (dp \, / \, d\mu) \leq 1. \tag{7.49}$$

The covariant derivative of $p = p(\mu)$ along a fluid particle world line provides $\dot{p} = (dp \, / \, d\mu) \, \dot{\mu}$. Using this, the energy equation (7.46) and $(log \, \lambda)^{\cdot}$ from (7.47) in (7.43), we obtain

$$\{(\mu + p) \, \frac{dp}{d\mu} - \frac{1}{3} \, (\mu + 5 p)\} \, \theta = 0.$$

If $\theta \neq 0$, then above equation reduces to the following homogeneous differential equation of degree zero

$$\frac{dp}{d\mu} = \frac{\mu + 5\,p}{3\,(\mu + p)}. \tag{7.50}$$

Tsamparlis and Mason [192] have shown that, under the hypothesis of theorem 7.31, stiff equation of state ($p = \mu$) is the only solution of (7.50), satisfying the energy condition (7.49). Thus, it follows from (7.40) that a spacetime with **RC** may have nonzero shear. If $\theta = 0$, $\dot{u} = 0$ and $p \neq \mu$ then it is left as an exercise that V reduces to a Killing vector field.

Recall that a **material curve** in a fluid is a curve that always consists of the same fluid particles and therefore it moves with the fluid as the fluid evolves. The following result is straightforward:

COROLLARY 1. *Under the hypothesis of theorem 7.32, vortex lines are material lines if the fluid satisfies the energy condition (7.50).*

Now we state the following results using the spacelike congruence (see pages 58 and 59). For proofs see Tsamparlis-Mason [192], where they have also studied **imperfect fluids** with cosmological constant.

THEOREM 7.33. *A perfect fluid spacetime M, with Einstein field equations (7.36) and T_{ab}, admits a spacelike* **RC** *vector field $V = \alpha n$ $(n \cdot n = 1, u \cdot n = 0, \alpha > 0\ )$ iff*

(i) $(\mu + 3\,p)\,\omega_{ab}\,n^b = \frac{1}{2}\,(\mu - p)N_a$,

(ii) $(\mu - p)\,\bar{\sigma}_{ab} = 0$,

(iii) $(\mu - p)\,\{\overset{\bullet}{n}_a + (log\,\alpha)_{,a} - \frac{1}{2}\,\bar{\theta}\,n_a\} = 0$,

(iv) $(\mu - p)\,(\,\frac{1}{2}\,\bar{\theta} + n_a\,\dot{u}^a) = 0$,

(v) $\{(\mu - p)\,V^a\,\}_{;\,a} = 0$,

where, by definition, $N^a = h^a{}_b\,\dot{n}^b - \overset{\bullet}{u}{}^a + (n_b\ \overset{\bullet}{u}{}^b\,)\,n^a$ and $\bar{\theta}$ and $\bar{\sigma}_{ab}$ are the expansion and shear of the spacelike congruence generated by n.

COROLLARY 2. *Under the hypothesis of theorem 7.33, following holds*

(a) $\omega = 0$. *Then either $p = \mu$ or the integral curves of n are material curves.*

(b) $\omega \neq 0$. *If $p = \mu \neq 0$ then $n^a = \pm\frac{\omega^a}{\omega}$. The converse is true even if $\mu \neq p$. Moreover, if the integral curves of n are material curves and $\mu + 3\,p \neq 0$, then also $n^a = \pm\frac{\omega^a}{\omega}$.*

Tsamparlis-Mason [192] have studied theorem 7.33 and found some exact solutions. Also, in this paper they considered a variety of imperfect fluids with cosmological constant.

If a given symmetry vector V of M leaves the matter tensor invariant ($L_V T_{ab} = 0$), then we say that M also has symmetry called **matter collineation** (denoted by **MC**). Well-known examples are Killing and homothetic symmetries. A counter example is proper conformal symmetry.

THEOREM 7.34 (Hall-Rendall [89]). *Let M be a spacetime manifold. Then, generically, any vector field V on M which simultaneously satisfies* **CC** *and* **MC** *is a homothetic vector field.*

Carot et al. [28]. They obtained examples of **MC** in dust fluids, including Szekeres spacetime: $ds^2 = -dt^2 + e^\lambda dr^2 + e^\sigma (dx^2 + dy^2)$ for smooth functions λ and σ and null fluid spacetimes.

Melbo et al [144]. They studied **RC** symmetry in Gödel-type spacetimes.

Hall, Roy and Vaz [90]. They studied **RC** for various decomposable spacetimes.

For $(1+3)$ timelike case: $ds^2 = -dt^2 + h_{ab}\, dx^a\, dx^b$, where h_{ab} depends on x^a, it has been shown that any **RC** takes the form $f\, \partial_t + W$, where f is an arbitrary function of the coordinates x^a in some coordinate neighborhood, and W induces an **RC** on each of the isometric hypersurfaces $t =$ constant. If the Ricci tensor is of *rank* 3, then the hypersurface **RC's** form a finite dimensional Lie algebra of vector fields in the hypersurface. The $(1+3)$ spacelike case is similar.

In the $(2+2)$ case there exists a local null tetrad of smooth vector fields $\{l, n, u, v\}$ $(l \cdot n = u \cdot u = v \cdot v = 1$ and all other inner products vanish), where l, n are null, with $R_{ab} = a\,(u_a\, u_b + v_a\, v_b) + b\,(l_a\, n_b + n_a\, l_b)$. Here $a = a(u^1, u^2)$ and $b = b(u^3, t)$. If an **RC** vector field is $V = X_T + X_S$, where X_T and X_S are tangent to 2-manifolds, then **RC's** from a finite (resp. infinite) dimensional vector space according as a and b are nowhere nonzero or one of them is zero.

In the $(1+1+2)$ case, an **RC** vector is $V = V_T + c\, W_1 + d\, W_2$, where V_T is tangent to the 2-dimensional submanifold, W_1, W_2 are tangent to the two 1-dimensional submanifolds and c, d are arbitrary functions. The vector space of all such **RC's** is infinite dimensional.

Chapter 8

Symmetry Inheritance

This chapter deals with the fundamental question of determining when the symmetries of the geometry (defined by a given metric and / or curvature symmetry vector field) is inherited by all the source terms of a prescribed matter tensor of Einstein field equations. We start discussing symmetry inheritance with respect to a conformal vector field, and the source terms of imperfect fluid matter tensor. This is then followed by recent work done on semi-Riemannian manifolds, subject to a prescribed symmetry, called "**curvature inheritance**" vector field (a generalization of curvature collineations). Finally, the geometric results so obtained are applied to spacetimes having curvature and / or **"Ricci inheritance"** vector fields. Several physical results are presented.

8.1 Inheriting CKV Fields

Let (M, g) be a spacetime of general relativity with a **CKV** field V defined by

$$L_V\, g_{ab} = 2\,\sigma\, g_{ab}. \tag{8.1}$$

If a perfect fluid M, with field equations defined by equation (7.36) of chapter 7, is self-similar, i.e., admits a homothetic vector field V ($\sigma =$ const.), then it is easy to see that the density, pressure and fluid velocity satisfy

$$L_V\, \mu = -\,2\,\sigma\, \mu, \qquad L_V\, p = -\,2\,\sigma\, p, \qquad L_V\, u_a = \sigma\, u_a. \tag{8.2}$$

In the above case, we say that these three source terms inherit the spacetime symmetry defined by V. This concept of symmetry inheritance was first introduced, in 1989, by Coley and Tupper [31], and, they did extensive work on special conformal Killing vector **SCKV** fields. For proper **CKV**, they concluded that, even in the case of a perfect fluid source, a proper conformal motion will not, in general, map fluid flow conformally (i.e., $L_V\, u \neq -\,\sigma\, u$), and, therefore, the symmetries can not be inherited in the sense of their definition. Also, since there are very few spacetimes admitting **SCKV's**, and proper **CKV's** have greater physical significance, Coley and Tupper [31] suggested that the concept of symmetry inheritance be modified for proper **CKV's**. They, therefore, proposed future research on "*What actually is meant by symmetry inheritance in spacetimes admitting* **CKV's**". In, 1990, Coley-Tupper [33, 34] studied inheriting proper **CKV's** with the condition that the flow lines are mapped conformally by V. In this section, we present Coley-Tupper's work on **inheriting CKV fields**.

Suppose M satisfies Einstein field equations with **imperfect matter tensor** (i.e., viscous, heat-conducting fluids), defined by

$$T_{ab} = \mu\, u_a\, u_b + p\, h_{ab} - 2\eta\, \sigma_{ab} + q_a\, u_b + q_b\, u_a, \tag{8.3}$$

where μ, p, q^a, $\eta\,(\geq 0)$, h_{ab}, σ_{ab} are the energy density, the isotropic pressure, the **heat flux vector** relative to the fluid velocity vector u, the **shear viscosity coefficient**, the projection tensor and the shear tensor respectively. Let V be a homothetic vector field. Then, in general, the symmetry is not inherited by all the six source terms appearing in T_{ab}. However, it has been shown by Eardley [63] that if self-similarity is imposed on the complete solution (using dimensional considerations) then this symmetry is inherited and the following holds:

$$\begin{aligned} L_V\, \mu &= -2\sigma\, \mu, \quad L_V\, p = -2\sigma\, p, \quad L_V\, u_a = \sigma\, u_a, \\ L_V\, q_a &= -\sigma\, q_a, \quad L_V\, \sigma_{ab} = \sigma\, \sigma_{ab}, \quad L_V\, \eta = -\sigma\, \eta. \end{aligned} \tag{8.4}$$

According to Coley-Tupper, we say that the symmetries of V are inherited if (8.4) holds. For a special conformal Killing vector **SCKV** field ($\sigma_{a\,;\,b} = 0$), there is no self-similarity unless the **SCKV** is homothetic. Coley-Tupper [31] have found various conditions under which (8.4) holds for a **SCKV**. We need the following kinematic and dynamic results for the six source terms of (8.3). Recall from equation (6.70) of chapter 6, that for the fluid velocity vector u, we have

$$L_V\, u^a = -\sigma\, u^a + v^a\,, \qquad L_V\, u_a = \sigma\, u_a + v_a\,, \qquad v \cdot u = 0. \tag{8.5}$$

Note that $u_a\, L_V\, u^a = -\, u^a\, L_V\, u_a = \sigma$. The heat flux vector q^a is not a unit vector, so let $q^a\, q_a = Q^2$. By an argument similar to that used in establishing (8.5) it can be shown that,

$$L_V\, q^a = (Q^{-1}\, L_V\, Q - \sigma)\, q^a + w^a, \qquad w \cdot q = 0. \tag{8.6}$$

Since $u \cdot q = 0$, it follows that

$$-\, u_a\, L_V\, q^a = -\, u^a\, L_V\, q_a = q_a\, L_V\, u^a = q^a\, L_V\, u_a \equiv P, \tag{8.7}$$

a scalar quantity. This implies that $P = v \cdot q = -\, u \cdot w$.

Using $L_V\, \Gamma^a{}_{bc} = \delta^a_b\, \sigma_c + \delta^a_c\, \sigma_b - g_{bc}\, \sigma^a$ and (8.5) we get

$$L_V\, u_{a\,;\,b} = \sigma\, u_{a\,;\,b} + g_{ab}\, \sigma_{,\,c}\, u^c - \sigma_{,\,a}\, u_b + v_{a\,;\,b}\,, \tag{8.8}$$

$$L_V\, \theta = -\sigma\, \theta + 3\, \sigma_{,\,a}\, u^a + v^a{}_{;\,a}\,, \tag{8.9}$$

$$L_V\, h_{ab} = 2\, \sigma\, h_{ab} + 2\, u_{(a}\, v_{b)}\,, \tag{8.10}$$

where $\theta = u^a{}_{;a}$ and h_{ab} are the expansion scalar for the fluid velocity congruence and projection tensor respectively. Using the expression of shear tensor σ_{ab} (see equation (4.4), chapter 4) expressed as follows

$$\sigma_{ab} = \frac{1}{2}\, (\, u_{a\,;\,c}\, h^c{}_b + u_{b\,;\,c}\, h^c{}_a\,) - \frac{1}{3}\, \theta\, h_{ab}$$

and (8.8) - (8.10), after a lengthy calculation we obtain

$$\begin{aligned} L_V\,\sigma_{ab} &= \sigma\,\sigma_{ab} - \frac{1}{3}\,h_{ab}\,v^c{}_{;c} - \frac{2}{3}\,\theta\,u_{(a}\,v_{b)} + \dot{v}_{(a}\,u_{b)} \\ &+ v_{(a\,;\,b)} + \dot{u}_{(a}\,v_{b)} + u_{(a}\,u_{b)\,;\,c}\,v^c. \end{aligned} \tag{8.11}$$

Recall the following dynamic results (see page 119) for a **CKV**:

$$L_V\,R_{ab} = -2\,\sigma_{b\,;\,a} - (\triangle\,\sigma)\,g_{ab}, \tag{8.12}$$

$$L_V\,r = -2\,\sigma\,r - 6\,\triangle\,\sigma, \tag{8.13}$$

$$L_V\,G_{ab} = 2\,\{\,(\triangle\,\sigma)\,g_{ab} - \sigma_{b\,;\,a}\,\}, \tag{8.14}$$

where $\triangle\,\sigma = g^{ab}\,\sigma_{a\,;\,b}$ and Einstein's equations are of the form

$$G_{ab} \equiv R_{ab} - \frac{1}{2}\,r\,g_{ab} = T_{ab}$$

and, therefore, we obtain

$$L_V\,T_{ab} = 2\,(\triangle\,\sigma)\,g_{ab} - 2\,\sigma_{a\,;\,b}. \tag{8.15}$$

At this point we let V be a **SCKV** for which $\sigma_{a\,;\,b} = 0$. Thus, (8.15) becomes

$$L_V\,T_{ab} = 0. \tag{8.16}$$

By taking the Lie-derivative of both sides of (8.3) and then using (8.16), we obtain

$$\begin{aligned} L_V\,T_{ab} &= (L_V\,\mu)\,(\,u_a\,u_b\,) + (L_V\,p)\,(\,h_{ab}\,) + 2\,\sigma\,(\,\mu\,u_a\,u_b + p\,h_{ab}\,) \\ &+ 2\,(\,\mu + p\,)\,v_{(a}\,u_{b)} - 2\,\sigma_{ab}\,L_V\,\eta \\ &- 2\,\eta\,L_V\,\sigma_{ab} + 2\,(\,Q^{-1}\,L_v\,Q + 2\,\sigma\,)\,u_{(a}\,q_{b)} \\ &+ 2\,q_{(a}\,v_{b)} + 2\,u_{(a}\,w_{b)} = 0. \end{aligned} \tag{8.17}$$

Contracting (8.17) in turn with $u^a\,u^b$, h^{ab}, $u^a\,h^b{}_c$, $h^{ac}\,h^{bd} - \frac{1}{3}\,h^{ab}\,h^{cd}$, q^b, $q^a\,u^b$ and $q^a\,q^b$, and simplifying we get

$$L_V\,\mu = -2\,\sigma\,\mu - 2\,P, \tag{8.18}$$

$$L_V\,p = -2\,\sigma\,p - \frac{2}{3}\,P, \tag{8.19}$$

$$\begin{aligned} 2\,\eta\,\sigma_{ab}\,v^b &= w_a - P\,u_a + (\mu + p)\,v_a \\ &+ (\,Q^{-1}\,L_V\,Q + 2\,\sigma\,)\,q_a, \end{aligned} \tag{8.20}$$

$$L_V\,(\eta\,\sigma_{ab}) = 2\,\eta\,\sigma_{c(a}\,u_{b)}\,v^c + q_{(a}\,v_{b)} - \frac{1}{3}\,P\,h_{ab}, \tag{8.21}$$

$$\begin{aligned} 2\,q^b\,L_V\,(\eta\,\sigma_{ab}) &= \frac{1}{3}\,P\,q_a + Q^2\,v_a + \{\,(\mu + p)\,P \\ &+ Q\,L_V\,Q + 2\,\sigma\,Q^2\,\}\,u_a, \end{aligned} \tag{8.22}$$

$$2\,\eta\,\sigma_{ab}\,q^a\,v^b = Q\,(\,L_V\,Q + 2\,\sigma\,Q\,) + (\mu + p)\,P, \tag{8.23}$$

$$2\,q^a\,q^b\,L_V\,(\,\eta\,\sigma_{ab}\,) = \frac{4}{3}\,P\,Q^2. \tag{8.24}$$

In the following we state main results of paper [31] (proofs follow from the equations (8.18) - (8.24)):

THEOREM 8.1. *Let V be a* **SCKV** *of a spacetime M satisfying Einstein's field equations for an imperfect fluid with the matter tensor (8.3). Then*

(1) *If either $L_V\, u = \sigma\, u$ or $L_V\, (\eta\, \sigma_{ab}) = 0$, the symmetries of V are inherited.*

(2) *If q_a is an eigenvector of σ_{ab} and $L_V\, q_a = -\,\sigma\, q_a$, then the symmetries of V are inherited.*

THEOREM 8.2. *If V is a* **Killing vector field** *satisfying Einstein's field equations for an imperfect fluid with energy momentum tensor (8.3), then the necessary and sufficient condition for the symmetries of V to be inherited is that $L_V\, (\eta\, \sigma_{ab}) = 0$ or, equivalently, $L_V\, u = 0$.*

CONCLUSIONS. In [31], Coley-Tupper found all spacetimes which admit a **SCKV** satisfying the dominant energy condition. Their conclusions are summarized as follows:

(a) Perfect fluid spacetimes (in particular, **FRW** models) do not admit a proper **SCKV**. Note that, this result invalidates the repeated statement in the literature (see for example, [96, 134]) "**An SCKV can admit a perfect fluid**".

(b) All the **SCKV** spacetimes can be interpreted as representing either viscous or anisotropic fluids, or both of these, except for some vacuum plane-wave solutions. These **SCKV** spacetimes may also admit null electromagnetic field and a perfect fluid with an electromagnetic field.

(c) Based on the item **(b)**, they presented some examples compatible with the theorems 8.1 and 8.2.

(d) They also studied the following sub cases of the general imperfect fluid. **(i)** A viscous fluid with no heat conduction ($q^a = 0$). **(ii)** A heat conducting perfect fluid ($\eta\, \sigma_{ab} = 0$) which will always inherit the symmetry.

(e) By studying the equation (8.21) for a proper **CKV**, it can be shown that the right hand side of this equation will have a non-zero term $\sigma_{;\,ab}$, and, therefore, $L_V\, (\eta\, \sigma_{ab})$ can not vanish. Thus, the symmetries of a proper **CKV** can not be inherited in the sense of the definition given in [31].

In 1990, Coley-Tupper [33] modified the definition of inheriting **CKV** as follows:

DEFINITION. The spacetime symmetries of a **CKV** field V are said to be inherited if fluid flow lines are mapped conformally by V.

Based on above definition, they studied a special class of spacetimes M, called "**synchronous spacetimes**" in which (locally) the metric can be written as

$$ds^2 = -\, dt^2 + H_{\alpha\,\beta}\, dx^\alpha\, dx^\beta, \qquad (\alpha, \beta = 1, 2, 3) \tag{8.25}$$

where $H_{\alpha\beta}$ depend on all four coordinates t, x, y, z. If, in this coordinate system, the four velocity vector u is comoving, that is $u^a = \delta^a_0$, then M is an **orthogonal synchronous (OS) spacetime**. Example: All spatially homogeneous, including orthogonal spatially homogeneous (**OSH**), models.

THEOREM 8.3 (Coley-Tupper [33]). **OS** *perfect fluid spacetimes, other than FRW spacetimes, do not admit any inheriting proper* **CKV**. *General orthogonal spatially homogeneous spacetimes, other than FRW models and spacetimes with viscosity for which* $[\, L_V\, \eta + \eta\, \dot{V} + \ddot{V}\,] = 0$, *admit no proper inheriting* **CKV**.

PROOFS of the above stated "no-go results" and the following summarized below are available in the cited references.

THEOREM 8.4 (Coley-Tupper [34]). *If a perfect fluid, spherically symmetric spacetime M (with comoving 4-velocity u), in which the perfect fluid satisfies a barotropic equation of state and the energy conditions are satisfied, admits a proper inheriting global* **CKV** *field V and anyone of the following conditions is satisfied,*

(a) *the metric is a generalized static metric,*

(b) *the metric is a generalized Kantowski-Sachs metric,*

(c) *V is parallel or orthogonal to u,*

(d) *the spacetime is shear-free,*

(f) *the spacetime is conformally flat,*

then the spacetime is necessarily a **FRW** *spacetime or a special* **GGBW** *spacetime.*

NOTE. Here, **Kantowski-Sachs** spacetimes have a metric of the form

$$ds^2 = -\,dt^2 + A^2\,(t)\,dr^2 + B^2\,(t)\,(\,d\theta^2 + \sin^2\,\theta\,d\phi^2\,),$$

which can be generalized to A and B functions of all the four coordinates. For details on this and many more **spherically symmetric spacetimes** (such as **GGBW** spacetimes) admitting a proper **CKV** see [33, 34].

REMARK. There are several physical reasons for studying inheriting **CKV**. In the following we briefly mention three papers (cited in Coley-Tupper [33]) using Inheriting **CKV** in physically significant spacetimes.

Israel (1972). He has shown that for a distribution of massless particles in equilibrium, the inverse temperature function vector $V^a_T = \frac{1}{T}\,u^a$ is a **CKV** (where T is the temperature). Since V_T is parallel to u, it maps the flow lines conformally. Thus, as per Coley-Tupper's definition, the inverse temperature function is an inheriting **CKV**. Consequently, there is a close connection of inheriting **CKV's** with the **relativistic thermodynamics of fluids**.

Stephani (1982). This paper uses the following form of the **entropy production density** (denoted by $S^a_{;a}$):

$$S^a_{;a} = -(T^{ab} - \mu u^a u^b - p h^{ab}) L_{V_T} g_{ab},$$

which by the generalized second law of thermodynamics, is always non-negative. For the imperfect fluid (8.3), above equation reduces to

$$S^a_{;a} = \frac{1}{2}(2\eta \sigma^{ab} - u^a q^b - u^b q^a) L_{V_T} g_{ab}.$$

Therefore, complete, exact reversible thermodynamics (i.e., $S^a_{;a} = 0$) is only possible if the inverse temperature function V^a_T is an inheriting **CKV**.

Tauber and Weinberg (1961). They have argued that the isotropy of the **cosmic microwave background** can be used to imply the existence of an inheriting **CKV**, since if the cosmic microwave background is isotropic about each observer moving with the (timelike) velocity congruence, then there exists (locally) a **CKV** parallel to u and hence inheriting.

8.2 Curvature Inheritance

Let (M, g) be an n-dimensional semi-Riemannian manifold of an arbitrary signature. In 1992, Duggal [53] generalized the concept of **Curvature collineations (CC)** by introducing a new symmetry called **curvature inheritance (CI)** defined by a vector field V, of M, satisfying

$$L_V R^i{}_{jkm} = 2\alpha R^i{}_{jkm}, \tag{8.26}$$

where $\alpha = \alpha(x^i)$ is a smooth function on M. A **CI** reduces to **curvature collineation (CC)** when $\alpha = 0$, otherwise, we say that V is a proper **curvature inheritance vector (CIV) field**. Contracting (8.26) implies

$$L_V R_{ij} = 2\alpha R_{ij}. \tag{8.27}$$

A vector field V satisfying (8.27) is called **Ricci inheritance vector (RIV) field**. Thus, in general, any **CIV** is **RIV**, but the converse may not hold (see proposition 3, page 170). In particular, **RIV** reduces to **Ricci collineation (RC)** when $\alpha = 0$. Otherwise, for $\alpha \neq 0$, it is called a proper **RIV**.

Following is the motivation for above defined new symmetry. Firstly, a critical review of **CC** (including **RC**) indicates that their relation with conformal symmetry is severely restricted to a rare case of special **CKV** which implies the existence of a covariant constant hypersurface orthogonal and geodesic vector. Thus, according to Hall and da Costa [83], such spaces must admit either two null eigenvectors or a repeated null vector of the energy momentum tensor. In particular, **FRW** and perfect fluid spacetimes are excluded. This information was not available to Katzin et al. [106] in 1969, when they introduced **CC**. The reader will see that a proper **CIV** (including **RIV**) can be related with a proper **CKV**. This is certainly an improvement over the use of **CC's** since proper **CKV's** have greater physical significance. Indeed, (a) there do exist proper **CKV** (nine in general) in **FRW** spacetimes and perfect fluids [131], (b) they preserve the causal character of curves and (c) they are useful in astrophysics and cosmology. Secondly, the use of a **CIV** or **RIV** with a **CKV** completely solves the **symmetry inheritance** problem with respect to a perfect fluid and an anisotropic fluid (see theorems 8.14 and 8.15). We set

$$L_V\, g_{ij} = V_{i\,;\,j} + V_{j\,;\,i} \equiv h_{i\,j}, \tag{8.28}$$

and use the following form of the Lie-derivative of curvature tensor:

$$L_V\, R^i{}_{jkm} = (\,L_V\, \Gamma^i{}_{jm}\,)_{\,;\,k} - (\,L_V\, \Gamma^i{}_{jk}\,)_{\,;\,m}.$$

In general, for a **CIV** V, using (8.26) - (8.28) we derive the following identities:

$$\begin{aligned} L_V\, R^i_j &= 2\,\alpha\, R^i_j - R^k_j\, h^i_k, \\ L_V\, r &= 2\,\alpha\, r - r', \qquad r' \equiv 2\, R^i_j\, V^j{}_{;\,i}, && (8.29) \\ L_V\, C^i{}_{jkm} &= 2\,\alpha\, C^i{}_{jkm} + D^i{}_{jkm}, && (8.30) \\ L_V\, W^i{}_{jkm} &= 2\,\alpha\, W^i{}_{jkm}, && (8.31) \end{aligned}$$

where $C^i{}_{jkm}$, $W^i{}_{jkm}$ are **conformal curvature tensor, Weyl projective tensor** (see pages 21 and 142) respectively and

$$\begin{aligned} D^i{}_{jkm} \;&\equiv\; \frac{1}{(n\,-\,2)}\,[\,h_{jm}\, R^i_k - h_{jk}\, R^i_m + g_{jk}\, R^e_m\, h^i_e - g_{jm}\, R^e_k\, h^i_e\,] \\ &+\; \frac{1}{(n-1)(n-2)}\,[\,\delta^i_m\,(r\, h_{jk} - r'\, g_{jk}) - \delta^i_k\,(r\, h_{jm} - r'\, g_{jm})]. \end{aligned} \tag{8.32}$$

PROPOSITION 1. *A necessary condition for a vector field V to be* **CIV** *is*

$$h_{ij\,;\,km} - h_{ij\,;\,mk} = 0. \tag{8.33}$$

PROOF. The curvature tensor satisfies the identity: $g_{ij}\, R^i{}_{kme} + g_{ik}\, R^i_{jme} = 0$. Taking its Lie-derivative, with respect to V, and using (8.26) and (8.28) provides the following general identity

$$h_{ij}\, R^i{}_{kme} + h_{ik}\, R^i{}_{jme} = 0, \tag{8.34}$$

which, by means of the Ricci identity, reduces to (8.33).

Multiplying (8.33) by $\sqrt{g}\, g^{im}\, g^{jk}$, we obtain **Komar's identity** [112], i.e.,

$$(\,\sqrt{g}\,(\,V^{i\,;\,j} - V^{j\,;\,i}\,)\,)_{\,;\,ji} = 0\,, \qquad g = det\,|(g_{ij})|.$$

PROPOSITION 2 (Duggal [54]). *If an n-dimensional semi-Riemannian manifold $(M,\, g)$ admits a* **CIV** *field V, then*

$$(R^i_m\, V^m)_{\,;\,i} = \alpha\, r. \tag{8.35}$$

PROOF. By explicitly writing (8.26) and then contracting j with i and finally using the contracted Bianchi identity we obtain (8.34).

REMARKS. (a) As the necessary condition (8.33) for a **CIV** is independent of the function α of (8.26) and same as that of **CC** (see equation (7.27), chapter 7), we highlight that **CIV's** retain the fundamental general transformation properties of **CC's**.

(b) Observe that the proposition 2 is a generalization of an earlier result by Collinson [35] for which $\alpha = 0$. Later on reader will see that the relation (8.35) serves as the basis for generating some new solutions for a variety of fluid spacetimes due to the appearance of non-zero term $\alpha\, r$ on its righthand side.

(c) It follows from equations (8.30) and (8.31) that while the Weyl projective tensor W inherits the symmetry defined by **CIV** V, the conformal curvature tensor is non-inheriting due to the extra tensor D which is not necessarily zero. This raises the following problem:

"*Find condition(s) on M, with a proper* **CI** *symmetry, such that D vanishes.*"

In particular, it immediately follows from (8.30) that if M is conformally flat, that is, if C vanishes, then D also vanishes. However, the converse does not hold, so, in general, above problem still remains open.

Since D involves h and g, the answer of above open problem may lie in finding possible prescriptions for the symmetric tensor h of equation (8.28), representing the change in g, with respect to a **CIV** V. For this purpose, we let the general solution of the identity (8.34) be of the form:

$$L_V\, g \equiv h = 2\,\sigma\, g + T, \tag{8.36}$$

where T is a second order symmetric tensor and σ is a function on M. This provides the following two Types (labeled **I** and **II**) of M.

TYPE I. $T \equiv 0$, therefore, V is also a **CKV**.

TYPE II. $T \neq 0$ and $T \neq$ (function) g, therefore, V is non-conformal.

REMARK 1. At this point we mention that as the curvature components arise on a manifold admitting a connection which may not be a metric connection, the natural question of "why relate" curvature with metric symmetries needs to be answered. We cite two reasons arising from geometry and physics.

Geometrically, it is well known that the set of **CKV's** forms a finite-dimensional Lie-algebra structure, even if these **CKV's** are at least C^3. Contrary to this, the Lie algebra structure of smooth curvature symmetry vectors is not necessarily finite-dimensional. If the differentiability condition is less than smooth, then we may lose the Lie algebra structure of **CIV's** or **CC's** (see Remark on page 151). Thus, a relation of metric and curvature symmetries is desirable to precisely distinguish the types of spaces with a finite dimensional Lie algebra.

Physically, the answer lies in the study of geodesic deviation to a congruence of timelike geodesics, given by $\nabla_t^2 v^i = R^i{}_{jkm}\, u^j\, u^k\, v^m$, where u, v are the tangent vectors to the observer's timelike geodesic world line, the "connecting vector" joining the observer to a neighboring particles, and t is the proper time of a 4-dimensional spacetime M.

Recall that, based on the algebraic classification of Hall [81], there are 4 possible prescriptions for the equation (8.36), h in terms of g and T, for any curvature symmetry (see page 146) where **Type I** corresponds to Hall's **class D** and the **Type II** includes any one of his **classes A, B or C**.

On the **Type I** case, in particular, we have the following result:

THEOREM 8.5 (Duggal [53]). *Every proper* **CIV** *V in an Einstein manifold $(r \neq 0)$ is a proper* **CKV**. *Moreover, if $n > 2$ then $\alpha = \sigma$.*

PROOF. For an n-dimensional Einstein manifold, $R_{ij} = \frac{r}{n}\, g_{ij}$, whose Lie-derivative with respect to V and the use of (8.27) and (8.29) completes the proof.

In general, following results have been proved in [53, 54].

THEOREM 8.6. *Let a semi-Riemannian manifold (M, g) which admits a proper* **CIV** *V that is also a proper* **CKV**. *Then*

(a) *M is necessarily conformally flat.*

(b) $\sigma_{;\, ij} = \frac{\alpha}{n-2}\left[\frac{r}{n-1}\, g_{ij} - 2\, R_{ij}\right]$.

(c) $\triangle\, \sigma + \frac{r}{n-1} = 0$.

(d) $\alpha = \sigma + V^i\, \partial_i\, (log\, \sqrt{r})$.

(e) *If V is special* **CKV** *or homothetic or Killing, then, there exists no* **CIV** *other than a* **CC** *(i. e., $\alpha = 0$).*

REMARK 2. Firstly, it is immediate from the item **(e)** that, contrary to the restricted case of **CC's** with **SCKV**, proper **CIV's** are relevant to greater physically significant proper **CKV's**. Secondly, item **(a)** raises the question of finding all conformally flat manifolds which admit **CIV**, including those which also admit **CKV**. Physically, possible candidates are nine **CKV's** of conformally flat spacetimes [131].

THEOREM 8.7 (Duggal [53]). *A semi-Riemannian manifold (M, g) with a* **PC** *admits no* **CIV** *other than a* **CC**.

PROOF. Suppose M admits a **PC** vector V. Then, we know that $L_V\, W^i{}_{jkm} = 0$ (see equation 7.21, page 143). If V is also a **CIV**, it follows from (8.31) that either $\alpha = 0 \Rightarrow V$ is **CC** or $W = 0 \Rightarrow M$ has constant curvature and so M is Einstein. For the latter case, theorem 8.5 says that **PC** is also a proper **CKV**. However, we know from a result of Yano [206, page 167] that if V is both **PC** and **CKV**, then it is homothetic. Using this we conclude that $\alpha = 0$ is the only possibility which proves the theorem.

EXAMPLE 1. Let M be an n-dimensional Einstein manifold ($R_{ij} = \frac{r}{n}\, g_{ij}$, $n > 2$), with a **CIV** V satisfying (8.26). Then it follows from theorems 8.5 and 8.6 (item **(b)**) that $\alpha = \sigma$ and

$$\alpha_{;ij} = \rho\, g_{ij}, \qquad \rho \equiv \frac{-\,r\,\alpha}{n(n-1)}. \tag{8.37}$$

Petrov [162] has quoted a result of Sinyukov (1957), namely that if an M admits a vector field α_i satisfying (8.37) for a nonzero scalar function ρ, then a system of coordinates exists in which the metric takes the form

$$ds^2 = g_{11}\,(dx^1)^2 + (g_{11})^{-1}\, A_{pq}\,(x^2, \ldots, x^n)\, dx^p\, dx^q,$$

where $p,\, q \neq 1$, $g_{11} = [2 \int \rho\, dx^1 + c]^{-1}$ and $p = p(x^1)$. Thus, this example of M, with above metric is compatible with the theorems 8.5 and 8.6. Reader is invited to study this example for a 4-dimensional Einstein spacetime M to precisely find the form of its metric and a **CIV**. Earlier, Coley-Tupper [31] used this example and found exactly one inheriting **SCKV**.

The Type **II** case is much more complicated since it involves an unknown symmetric tensor T. A systematic study of this case is wide open for further research. In particular, Duggal [54] and Sharma-Duggal [181] have done some work by assuming that T is a covariant constant symmetric tensor K so that (8.36) is of the form

$$L_V\, g_{ij} = 2\,\sigma\, g_{ij} + K_{ij}, \qquad K_{ij\,;\,k} = 0. \tag{8.38}$$

This means that V is also an **affine conformal vector(ACV)** (see equation (7.4), chapter 7). Recall that for an **ACV**, the following identities hold:

$$L_V\, R_{ij} \;=\; -\,(\triangle\,\sigma)\, g_{ij} - (n-2)\,\sigma_{\,;\,ij}, \tag{8.39}$$

$$L_V\, r \;=\; -\,2\,(n-1)\,\triangle\,\sigma - 2\,r\,\sigma - 2\,\mu, \quad 2\,\mu = R^i_j\, K^j_i\,, \tag{8.40}$$

$$L_V\, C^i{}_{jkm} \;=\; D^i{}_{jkm}. \tag{8.41}$$

THEOREM 8.8 (Duggal [54]). *Suppose M belongs to Type* **II**, *with respect to a* **CIV** *V, which is also an* **ACV**, *satisfying (8.38). Then, either V is a* **CC** *or M is necessarily a non-Einstein conformally flat manifold.*

PROOF. By hypothesis, V satisfies (8.30) and (8.41). This implies that either $\alpha = 0$ (i.e., V is **CC**) or $C = 0$ (i.e., M is conformally flat), since D is never proportional to C. On the other hand, it follows from theorem 8.5 that any M of Type **II** is non-Einstein. This completes the proof.

COROLLARY 1. *Let M admit a* **CIV** *or an* **RIV** *V which is also an* **ACV**. *Then the following holds:*

$$\begin{aligned} L_V\, R^i_j &= 2\,(\alpha - \sigma)\, R^i_j - K^{im}\, R_{mj}, \\ L_V\, r &= 2\,(\alpha - \sigma) r - 2\,\mu, \\ 2\,\alpha\, R_{ij} &= -\,(n-2)\,\sigma_{\,;\,ji} - (\triangle\,\sigma)\, g_{ij}, \\ \alpha\, r &= (1-n)\,\triangle\,\sigma. \end{aligned}$$

Proceeding exactly as discussed in Katzin et al. [108], one can show that, under the hypothesis of theorem 8.8, there are essentially two types of non-Einstein conformally flat manifolds, viz., locally reducible with a nonzero r and irreducible for which r is necessarily zero. In the reducible case, $M = M_1 \times M_{n-1}$ with metric

$$ds^2 = e_1\,(dx^1)^2 + \{\,1 - \frac{M_0}{4}\sum_{i=2}^{n} e_i\,(x^i)^2\,\}^{-2}\,\{\sum_{j=2}^{n}(dx^j)^2\,\},$$

where $e's$ are ± 1, M_0 is the constant curvature of M_{n-1}, and $2 \le i, j \le n$. There exists an **ACV** field $V = (v^1(x^1)\,;\, V^i(x^2, \ldots, x^n))$, where V^1 is arbitrary and V^i is a **CKV** on M_{n-1}. Moreover r is constant such that

$$\begin{aligned} R^i{}_{jkm} &= M_0\,(\delta^i_k\,g_{jm} - \delta^i_m\,g_{jk}), \\ r &= (n-1)(n-2)\,M_0, \\ R_{abcd\,;\,e} &= 0, \qquad R_{ab\,;\,c} = 0, \qquad R^a_{bcd} = 0, \end{aligned}$$

for a, b, c or $d = 1$.

EXAMPLE 2. A physical example of above metric is the following **Einstein static cosmological spacetime**

$$ds^2 = -\,dt^2 + \{\,1 - \frac{M_0}{4}\,S^2\,\}^{-2}\,(dx^2 + dy^2 + dz^2),$$

where $S^2 = x^2 + y^2 + z^2$ and $r = 6\,M_0$. Based on Tashiro's theorem 7.1 (stated on page 135), this metric admits an **ACV**, say V. By calculating the curvature components and computing their Lie-derivatives with respect to V, it is left as an exercise to show that V is also a **CIV**.

It is known that the invariance of the matter tensor (i.e., $L_V T_{ab} = 0$), with respect to a symmetry vector V, has been very effectively useful in relativity. V is then called a **Matter Collineation vector (MCV)**. For example, any Killing or homothetic vector is a **MCV** but a proper **CKV** is not. Following result on **MCV** is immediate from the corollary 1.

COROLLARY 2. *Let M admit a* **CIV** *or an* **RIV** *V which is also an* **ACV**, *defined by (8.38). Then V is an* **MCV** *if*

$$K_{ij} = 2\,r^{-1}\,\{(\mu - r\,\alpha)\,g_{ij} + 2\,\alpha\,R_{ij}\}, \qquad r \ne 0. \tag{8.42}$$

In the following we state two recent results using the Ricci inheriting vectors which are also **(i)** affine conformal **(ii)** projective collineation vectors respectively. For proofs see Sharma-Duggal [181].

THEOREM 8.9 (Sharma - Duggal [181]) . *Let V be an* **RIV** *as well as an* **ACV** *on a semi-Riemannian manifold M with divergence-free curvature tensor. Then*

(a) *the scalar curvature $r = 0$, or*

(b) *$r \neq 0$ and V is an affine collineation, or*

(c) *M is a non-Ricci flat Einstein manifold with V as a* **CKV**. *If the dimension of M is 4, then M has constant curvature and for higher dimensions it is a warped product $B \times_f F$ where B is a constant curvature submanifold totally geodesic in M and F is Einstein totally umbilical in M, provided the span of closed* **CKV's** *on M has constant dimension. Moreover, if, in particular, M is locally symmetric then it has constant curvature.*

THEOREM 8.10 (Sharma - Duggal [181]). *Let V be an* **RIV** *as well as a* **PC** *on a semi-Riemannian manifold with divergence-free curvature tensor. Then*

(a) *the scalar curvature $r = 0$ and $\nabla_i p^i = 0$, or*

(b) *$r \neq 0$ and $p = \alpha +$ constant, or*

(c) *$r \neq 0$ and V is an affine collineation.*

In all cases, the gradient of the scalar field p is an eigenvector of the Ricci tensor with eigenvalue r/n.

REMARK 3. Discussion so far indicates that proper **CIV's** of Type **I** have direct interplay with proper **CKV's**. Although we have presented upto date limited work done on Type **II**, by two prescriptions of T, in general, this case is wide open for further research. Finally, observe that a relation of any of **SCKV**, homothetic, Killing and **PC** reduces a **CI** to a **CC**. However, a proper **RIV** (which is not a **CIV**) has an interplay with a **PC** (see theorem 8.10).

8.3 Ricci Inheritance in Relativity

In this section we primarily discuss two papers of Duggal [53, 54], unless otherwise cited. Let $(M,\, g)$ be a 4-dimensional perfect fluid spacetime manifold with Einstein field equations and the energy momentum tensor given by

$$R_{ab} - \frac{1}{2} r\, g_{ab} = T_{ab}, \tag{8.43}$$

$$T_{ab} = \mu\, u_a\, u_b + p\, h_{ab}, \tag{8.44}$$

where recall that μ, p, and $h_{ab}\,(= g_{ab} + u_a\, u_b\,)$ are the energy density, the isotropic pressure relative to the fluid velocity vector u and the projection tensor, respectively. Here we set $0 \leq a, b, c \leq 3$. Since Ricci curvature plays a direct role in kinematics and dynamics of general relativistic fluid spacetimes, we let M admit an **RIV** V defined by the equation (8.27). As a **CIV** is also an **RIV**, using (8.35) in (8.43) we obtain

$$[\,(T^{ab} - \frac{1}{2} T\, g^{ab})\, V_b\,]_{;a} = \alpha\, r\,, \qquad T = -\, r, \tag{8.45}$$

which serves as the basis for generating exact solutions for a variety of fluid spacetimes.

THEOREM 8.11. *A perfect fluid spacetime M, with Einstein field equations (8.43) and T_{ab}, defined by (8.44), admits a timelike* **RIV** *V parallel to the fluid flow vector $u\,(V = \lambda u,\ \lambda > 0\,)$ iff*

$$
\begin{aligned}
(\mu + 3p)\,[\,\dot{u} - (\log\lambda)_a - \theta\,u_a\,] &= 4\,\alpha\,\lambda^{-1}\,\mu\,u_a\,, && (8.46)\\
\{\lambda\,(\mu + 3p\,)\,u^a\,\}_{;a} &= -\,2\,\alpha\,r, && (8.47)\\
(\,\mu - p\,)\,\sigma_{ab} &= 0, && (8.48)
\end{aligned}
$$

where recall that θ and σ_{ab} are the expansion and the shear tensor respectively.

PROOF. The definition of Lie derivative provides

$$L_{\lambda u}\,R_{ab} = \lambda\,\{\dot{R}_{ab} + 2\,u^c\,R_{c(a}(\,\log\lambda\,)_{,\,b} + 2\,R_{c(a}\,u^c_{\;;\,b}\,\}.$$

Let $V = \lambda\,u$ be an **RIV** field in M. Using (8.27) and (8.43) with (8.44) in above equation, we obtain

$$
\begin{aligned}
(\dot{\mu} + 3\,\dot{p})\,u_a\,u_b + (\dot{\mu} - \dot{p})\,h_{ab} \;&+\; 4\,(\mu + p)\,\dot{u}_{(a}\,u_{b)}\\
&-\; 2\,(\mu + 3p)u_{(a}\,(\log\lambda)_{,b)} + (\mu - p)\,u_{(a\,;\,b)}\\
&=\; 4\,\alpha\,\lambda^{-1}\,R_{ab}. && (8.49)
\end{aligned}
$$

By contracting (8.49) in turn, with $u^a\,u^b$, $u^a\,h^b_c$, $h^a_c\,h^b_d - \frac{1}{3}\,h^{ab}\,h_{cd}$ and then using (see equation (4.7), page 57)

$$u_{a;\,b} = \sigma_{ab} + (\theta/3)\,h_{ab} + \omega_{ab} - \dot{u}_a\,u_b$$

we obtain, respectively

$$
\begin{aligned}
\dot{\mu} + 3\,\dot{p} + 2\,(\mu + 3\,p)[\,(\log\lambda)^{\cdot} - \alpha\,\lambda^{-1}\,] &= 0\,, && (8.50)\\
(\mu + 3\,p)\,[\,\dot{u} - (\log\lambda)_{,\,a} - (\log\lambda)^{\cdot}\,u_a\,] &= 0\,, && (8.51)\\
(\dot{\mu} - \dot{p}) + \frac{2}{3}\,(\mu - p)(\theta - 3\,\alpha\,\lambda^{-1}\,) &= 0\,, && (8.52)
\end{aligned}
$$

and, the equation (8.48). Eliminating $\dot{\mu}$ and $\dot{p}$ (by using the energy conservation equation (7.46), chapter 7 and (8.52) in (8.50)), we obtain

$$(\mu + 3\,p)(\,\log\lambda\,)^{\cdot} = (\,\mu + 3\,p\,)\,\theta + 4\,\mu\,\alpha\,\lambda^{-1}. \qquad (8.53)$$

Using (8.53) in (8.51) we recover (8.46). Now rewriting (8.50) in the form

$$(\,\mu + 3\,p\,)^{\cdot} + 2\,(\,\mu + 3\,p\,)\{\,(\log\lambda)^{\cdot} - \alpha\,\lambda^{-1}\} = 0$$

and using (8.53) to replace the term $(\mu + 3\,p)(\log\lambda)^{\cdot}$ in above equation, we recover (8.47). The converse is straightforward. This completes the proof.

COROLLARY 1. *Under the hypothesis of theorem 8.11, if M satisfies the energy conditions: $\mu > 0,\ p \geq 0,\ 0 \leq \frac{dp}{d\mu} \leq 1$, then*

$$\{\,(\mu + p)\,\frac{dp}{d\mu} - \frac{1}{3}\,(\,\mu + 5\,p)\,\}\,\theta = 2\,\alpha\,\lambda^{-1}\,(\mu - p). \qquad (8.54)$$

In order to find a physically meaningful equation of state $p = p(\mu)$ for the perfect fluid spacetime M, we need further relations to eliminate the variables θ, λ and α from the equation (8.54). One way of getting the required relations is to relate the Ricci curvature symmetry **RIV** with the metric symmetries either of Type **I** or Type **II** as a solution of the general equation (8.36). We choose the Type **I** so that V is also a **CKV**. A comparison of theorem 6.14, page 125, with theorem 8.11, provides

COROLLARY 2. *Under the hypothesis of theorem 8.11, if the* **RIV** *V is also a* **CKV** *then*

$$\sigma_{ab} \equiv 0\,, \quad \sigma = \dot{\lambda}\,, \quad 2\,\mu\,\alpha = -\,\sigma\,(\mu + 3\,p).$$

Case 1 : $\alpha = 0$. V reduces to an **RC** vector field and it is known (see Tsamparlis-Mason [192]) that the stiff equation of state ($p = \mu$) is the only solution of (8.54) if $\theta \neq 0$. If $\theta = 0$ then V further reduces to a Killing vector field. Moreover, if **RC** is also a **CKV**, then it follows from corollary 2 that either $\sigma = 0$ (i.e., V is Killing) or $\mu + 3\,p = 0$ which is unphysical solution. Furthermore, , it is known [106] that "*A* **CKV** *is an* **RC** *iff it is an* **SCKV**". Thus, not only that we have an unphysical solution if $\alpha = 0$, this special case is also very rare, severely restricted and according to Hall and da Costa [83], its stiff equation of state must be excluded.

Case 2: $\alpha \neq 0$ and $\theta = 0$. V is a proper **RIV** field and the stiff equation of state $p = \mu$ is a physically valid solution of (8.54). Moreover, contrary to the case of **RC**, vanishing of θ does not reduce V to any lower symmetry such as Killing.

Case 3: $\alpha \neq 0$ and $\theta \neq 0$. Since further relations are needed to find an equation of state $p = p(\mu)$ for this case, we assume that V is also a **CKV**. From the results of theorem 6.14 and corollary 2 in (8.54) we get $6\,\alpha\,\mu = -\,\lambda\,\theta\,(\mu + 3\,p)$. Using this in (8.54) we obtain the following separable variables differential equation:

$$\frac{dp}{p} = \frac{d\mu}{\mu}$$

whose general solution is

$$p = c\,\mu\,, \qquad 0 < c \leq 1 \quad \text{is a constant.}$$

CONCLUSIONS.

(a) The general equation of state $p = c\,\mu$ provides a variety of physical solutions for a proper **RIV** which is a proper **CKV**, including $p = \mu$.

(b) The unphysical case ($\mu + 3\,p = 0$) does not appear at all if $\alpha \neq 0$.

(c) It follows from theorem 8.6 that a proper **RIV** which is also a proper **CKV** is necessarily conformally flat. Thus, in particular, an **FRW** spacetime (which admits, in general, nine proper **CKV's** [131]) can admit a proper **RIV**. On the other hand, if $\alpha = 0$, then **FRW** spacetimes must be excluded for those **RC's** which are also **SCKV**.

EXAMPLE. Consider the following metric of a de-Sitter spacetime

$$ds^2 = -\,dt^2 + e^{2\,m\,t}\,(dx^2 + dy^2 + dz^2),$$

where m is a constant. This metric admits a proper **CKV**, $V = (e^{m\,t}, 0, 0, 0)$, for which $\sigma = m\,e^{m\,t}$. A straightforward computation of the components $R^a{}_{bcd}$ and then taking Lie derivative, with respect to V, indicates that V is a **CIV** and, therefore, an **RIV** with $\alpha = \sigma$.

PROPOSITION 3 (Sharma-Duggal [181]). *If V is* **RIV** *and a* **CKV** *then*

$$L_V\,R^a{}_{bcd} = 2\,\alpha\,(R^a{}_{bcd} - C^a{}_{bcd}).$$

REMARK. In general, above equation implies that $L_V\,R_{bc} = 2\,\alpha\,R_{bc}$, but not necessarily conversely. Also, proposition 3 tells us that the converse holds if V is a **CKV**. Based on this we say that all the results presented in previous section 2 for those **CIV** which are **CKV's** also hold for **RIV's**.

APPLICATION. Denote by $\mathcal{G}_p$ the 4-dimensional Grassman manifold of 2-dimensional subspaces of the tangent spaces $T_p(M)$ at each $p \in M$. Let $\bar{\mathcal{G}}_p$ be the 4-dimensional open submanifold of $\mathcal{G}_p$. Then, the **sectional curvature** is defined by

$$K_p\,(E) = \frac{R_{abcd}\,E^{ab}\,E^{cd}}{2\,g_{a[c}\,g_{d]b}\,E^{ab}\,E^{cd}}\,,$$

where $E \in \bar{\mathcal{G}}_p$ and E_{ab} is any non-null simple bivector whose blade is E. Let V be a smooth and nowhere zero vector field on M with f_t the corresponding smooth and local diffeomorphism generated by V. Suppose E and $f_{t^\star}\,(E)$ are spanned by U and $W \in T_p(M)$ so that $f_{t^\star}\,(U)$ and $f_{t^\star}\,(W) \in T_q(M)$, respectively (here $f_{t^\star}$ is the differential of f_t and q is another point of M). Then the vector field V (or the mapping f_t) is called **sectional curvature - preserving** if

$$K_p\,(E) = K_q\,(f_{t^\star}\,(E)).$$

THEOREM 8.12 (Hall et al. [92]). *Let $(M,\,g)$ be a conformally flat generalized plane-wave spacetime. Then, a global smooth vector field V on M is sectional-preserving iff, for some function α on M,*

$$L_V\,R_{ab} = 2\,\alpha\,R_{ab}\,, \qquad L_V\,g_{ab} = 2\,\alpha\,g_{ab}.$$

NOTE. Thus, above result of Hall et al. is a physical application on the existence of a sectional-preserving global vector field V on a conformally flat generalized plane-wave spacetime, which is an **RIV** as well as a **CKV** with $\sigma = \alpha$.

It is important to mention that, in the same paper [92], Hall et al. studied a Lie algebra structure of the set of sectional-preserving vector fields on M, satisfying the hypothesis of theorem 8.12. Since the existence of a finite-dimensional Lie algebra structure of the set of symmetry vectors is very important property both mathematically and physically, we state (only main part) of their following theorem:

THEOREM 8.13 (Hall et al. [92]). *The set of global sectional curvature-preserving vector fields on a conformally flat generalized plane wave spacetime, is a finite-dimensional subalgebra of the Lie algebra of* **CKV's**.

At this point we let T_{ab} be an imperfect fluid of the form (8.3) of a spacetime $(M,\, g)$, with a proper **CKV** V. Taking the Lie derivative of both sides of the field equations, with respect to V, and proceeding exactly as presented on pages 158 and 159 by performing the same contractions, we obtain (set: $\sigma_{\,;\,ab}u^a\, u^b = \sigma_u$)

$$
\begin{aligned}
L_V\,\mu &= -2\sigma\,\mu - 2\,P - 2\,\triangle\,\sigma - 2\sigma_u\,,\\
L_V\,p &= -2\sigma\,p - \frac{2}{3}\,P + \frac{4}{3}\,\triangle\,\sigma - \frac{2}{3}\,\sigma_u\,,\\
2\,\eta\,u^b\,L_V\,\sigma_{ab} &= 2\,\sigma_{\,;\,cb}\,u^b\,h^c_a - w_a + (P + 2\,\sigma_u\,)\,u_a\\
&- (\mu + p)\,v_a - (Q^{-1}\,L_V\,Q + 2\,\sigma)\,q_a\,,\\
L_V\,(\eta\,\sigma_{ab}) &= 2\,\eta\,\sigma_{c(a}\,u_{b)}\,v^c + q_{(a}\,v_{b)} + \sigma_{\,;\,ab}\\
&+ \sigma_{\,;\,cb}\,u^c\,u_a + \sigma_{\,;\,ac}\,u^c\,u_b\\
&- \frac{1}{3}\,(P + \triangle\,\sigma\,)\,h_{ab} + (\,u_a\,u_b - \frac{1}{3}\,h_{ab}\,)\,\sigma_u.
\end{aligned}
$$

For a perfect fluid, above equations reduce to the following:

$$
\begin{aligned}
L_V\,\mu &= -2\,\sigma\,\mu - 2\,\triangle\,\sigma - 2\sigma_u\,,\\
L_V\,p &= -2\,p + \frac{4}{3}\,\triangle\,\sigma - \frac{2}{3}\,\sigma_u\,,\\
(\mu + p)\,v_a &= 2\,\sigma_{\,;\,bc}\,u^b\,h^c_a + 2\,\sigma_u\,u_a\,,\\
\sigma_{\,;\,ab} &= \frac{1}{3}\,(\triangle\,\sigma + \sigma_u\,)\,h_{ab} - u_a\,u_b\,\sigma_u\\
&- \sigma_{\,;\,cb}\,u^c\,u_a - \sigma_{\,;\,ac}\,u^c\,u_b.
\end{aligned}
$$

Now consider M with anisotropic fluid whose energy momentum tensor is

$$T_{ab} = \mu\,u_a\,u_b + p_{\|}\,n_a\,n_b + p_{\perp}\,p_{ab},$$

where $p_{\|}$, $p_{\perp}$ are the pressures parallel and perpendicular to a unit spacelike vector n orthogonal to u and $p_{ab} = h_{ab} - n_a\,n_b$ is the projection tensor onto the two planes orthogonal to u and n, respectively.

Following exactly as in the case of imperfect fluids, taking the Lie-derivative of the field equations, with anisotropic fluids, and contracting in turn with the tensors $u^a\,u^b$, $n^a\,n^b$, p^{ab}, $u^a\,n^b$, $u^a\,p^{bc}$, we get (here we set: $\sigma_{\,;\,ab}\,n^a\,n^b = \sigma_n$)

$$
\begin{aligned}
L_V\,\mu &= -2\,\sigma\,\mu - 2\,\triangle\,\sigma - 2\sigma_u\,,\\
L_V\,p_{\|} &= -2\,\sigma\,p_{\|} + 2\,\triangle\,\sigma - 2\,\sigma_n\,,\\
L_V\,p_{\perp} &= 2\,\sigma\,p_{\perp} + 2\,\triangle\,\sigma - \sigma_{\,;\,ab}\,p^{ab}\,,\\
(\mu + p_{\|})\,n^b\,v_b &= 2\,\sigma_{\,;\,ab}\,u^a\,n^b\,,\\
(\mu + p_{\perp})\,p^{bc}\,v_b &= 2\,\sigma_{\,;\,ab}\,u^a\,p^{bc}.
\end{aligned}
$$

Recall from section 1 that Coley and Tupper studied spacetime **symmetry inheritance** subject to a **CKV** which is a metric symmetry. To extend the concept of symmetry inheritance for spacetimes subject to a **CKV** which is also an **RIV** (relating metric and curvature symmetries) we make the following definition.

DEFINITION. Given a spacetime $(M,\, g,\, \Omega_i\, :\, i = 1, \ldots, p)$, where Ω_i are source terms of a matter tensor field T_{ab} of the Einstein field equations of M. Suppose M admits a **CKV** V which is also an **RIV** or a **CIV**. Then M will be said to inherit the symmetry corresponding to V if

$$L_V\, \Omega_i = f_i\, \Omega_i, \tag{8.55}$$

where each f_i is a function corresponding to each Ω_i.

This definition includes Coley-Tupper's definition used in [31] if V is only a **SCKV** for which their six symmetry equations (labeled by (8.4)) satisfy (8.55) such that $f_i \in \{-2\,\sigma, -\sigma, \sigma\}$. We now state the following three theorems using above definition (for proofs see Duggal [53]).

THEOREM 8.14. *Let a perfect fluid spacetime M, with $\mu + p \neq 0$, admit a proper* **CKV** *V which is also an* **RIV**. *Then the symmetries of V are inherited and the following set of equations holds*:

$$L_V\, \mu = 2\,(\alpha - \sigma)\,\mu\,, \qquad L_V\, u = -\,\sigma\, u\,, \qquad L_V\, p = 2\,(\alpha - \sigma)\, p.$$

THEOREM 8.15. *Let an anisotropic fluid spacetime M, with $\mu + p_{\|} \neq 0$ and $\mu + p_{\perp} \neq 0$, admit a proper* **CKV** *V which is also an* **RIV**. *Then the symmetries of V are inherited and the following set of equations holds*:

$$\begin{aligned} L_V\, \mu &= 2\,(\alpha - \sigma)\,\mu, \qquad L_V\, p_{\|} = 2\,(\alpha - \sigma)\, p_{\|}\,, \\ L_V\, p\perp &= 2\,(\alpha - \sigma)\, p_{\perp}, \qquad L_V\, u = -\,\sigma\, u. \end{aligned}$$

THEOREM 8.16. *Let an imperfect fluid spacetime M, with $q^a \neq 0$ and $\mu + p \neq 0$, admit a proper* **CKV** *V which is also an* **RIV**. *Suppose the fluid flow lines are mapped conformally by V (i.e., $L_V\, u = -\,\sigma\, u$). Then the symmetries of V are inherited and the following set of equations holds*:

$$\begin{aligned} L_V\, \mu &= 2\,(\alpha - \sigma)\,\mu\,, \quad L_V\, p = 2\,(\alpha - \sigma)\, p\,, \quad L_V\, \sigma_{ab} = \sigma\, \sigma_{ab}, \\ L_V\, \eta &= -\,(2\,\alpha + \sigma)\,\eta\,, \quad L_V\, q_a = (2\,\alpha - \sigma)\, q_a. \end{aligned}$$

REMARKS.

(a) Observe that Coley-Tupper's condition (see their modified definition on page 159) is not required for perfect as well as anisotropic fluids, since for these two cases fluid flows lines are automatically mapped conformally by a proper **CKV** which is also an **RIV**.

(b) There are potential open problems in the study of symmetry inheritance of metric and or curvature symmetries. In particular, no work has been done on the symmetry inheritance of those **CIV's** or **RIV's** which are non - conformal and, therefore, their spacetimes are of Type **II** (see page 163).

Chapter 9

Symmetries of Some Geometric Structures

In this chapter we first provide basic background information on Kaehlerian, contact and globally framed structures on smooth manifolds, needed for the rest of sections. This is then followed by a review of up-to date work done on metric and curvature symmetries of manifolds endowed with each one of the these three geometric structures. Since most of the results also hold globally, we prefer using index-free notations. For readers who may be unfamiliar with these geometric structures, we have provided extensive references on their mathematical background in bibliography.

9.1 Kaehler Manifolds

Let M be a C^∞ real $2n$-dimensional manifold, covered by coordinate neighborhoods with coordinates (x^i), where i runs over $1, 2, \ldots, n, \bar{1}, \bar{2}, \ldots, \bar{n}$ and the index a runs over $1, 2, \ldots, n$. M can be seen as a **complex manifold** of dimension n if we define complex coordinates $(z^a = x^a + \sqrt{-1}\, x^{\bar{a}})$ on a neighborhood of $z \in M$ such that the intersection of any two such coordinate neighborhoods is regular. There exists an endomorphism J (a tensor field J of type (1, 1)) of the tangent space $T_p(M)$, at each point p of M, such that

$$J(\partial / \partial x^a) = \partial / \partial x^{\bar{a}}, \qquad J(\partial / \partial x^{\bar{a}}) = -(\partial / \partial x^a),$$

and hence $J^2 = -I$, where I is the identity morphism of $T(M)$.

Now the question is whether the above property is sufficient for the existence of a complex structure on M. Fukami-Ishihara [69], in 1955, answered this question in negative by proving that 6-dimensional sphere S^6 has no complex structure but its tangent bundle admits such an endomorphism. We call a real $2n$-dimensional M, endowed with J satisfying $J^2 = -I$, an **almost complex manifold** and J its **almost complex structure** tensor. It is known that an almost complex manifold is even-dimensional. The tensor field N of type (1, 2) defined by

$$N_J(X, Y) = [JX, JY] - J^2[X, Y] - J([X, JY] + [JX, Y]), \qquad (9.1)$$

for any $X, Y \in \mathcal{X}(M)$, is called **Nijenhuis tensor field** of J. Newlander-Nirenberg [150] proved that J defines a complex structure on M iff N vanishes on M.

Consider a Riemannian metric g on (M, J). We say that the pair (J, g) is an **almost Hermitian structure** on M, and M is an **almost Hermitian manifold** if

$$g(JX, JY) = g(X, Y), \qquad \forall X, Y \in \mathcal{X}(M). \qquad (9.2)$$

Moreover, if J defines a complex structure on M, then (J, g) and M are called **Hermitian structure** and **Hermitian manifold**, respectively. The fundamental 2-form Ω of an almost Hermitian manifold is defined by

$$\Omega(X, Y) = g(X, JY), \qquad \forall X, Y \in \mathcal{X}(M). \qquad (9.3)$$

A Hermitian metric on an almost complex M is called a **Kaehler metric** and then M is called a **Kaehler manifold** if Ω is closed, i.e.,

$$3\, d\Omega(X, Y, Z) = (\nabla_X \Omega)(Y, Z) + (\nabla_Y \Omega)(Z, X) + (\nabla_Z \Omega)(X, Y) = 0, \qquad (9.4)$$

where ∇ is the Riemannian connection of g. Kaehler geometry has its roots back in 1933, introduced by Kaehler [103]. It is known (see Kobayashi-Nomizu [111]) that the Kaehlerian condition (9.4) is equivalent to $\nabla J = 0$. For a Kaehler manifold, the following identities hold:

$$\begin{aligned} R(X, Y)\, JZ &= J\, R(X,Y)Z\,, \quad Q\, J = J\, Q, \\ Ric(X, Y) &= \frac{1}{2}\{Tr.J \circ R(X, JY)\}, \end{aligned}$$

where Q is the Ricci operator. A conformally flat (in particular, constant curvature) Kaehler manifold is necessarily flat. Since the 2-form ρ, defined by $\rho(X, Y) = Ric(X, JY)$, on Kaehler manifold is closed, it therefore represents the first Chern class C_1. Complex manifolds with a Ricci-flat Kaehler metric are called **Calabi-Yau manifolds**. In 1977, Yau [216] proved the existence of a Ricci-flat Kaehler metric on any complex manifold, with $C_1 = 0$, and which admits a Kaehler metric. The Calabi-Yau manifolds have their application in physics in **super string theory** which is based on a 10-dimensional manifold $\bar{M} = V_4 \times M$, where V_4 is ordinary spacetime and M is a 6-dimensional manifold which is at least approximately Ricci-flat. Candelas et al. [26] have shown that for dimensional supersymmetry to be unbroken, M must be Kaehler. Also, Lebrun [123] has proved that a 2-dimensional complex manifold C^2 admits a complete non-flat Ricci-flat Kaehler metric, isometric to the Euclidean Taub-Nut metric discovered by Hawking [93].

Another interesting example of Kaehler manifolds comes from **Teichmuller theory** which we now explain. Let M be a smooth 2-dimensional manifold of genus $g \geq 2$. Let C be the set of all complex structures on M and $\mathcal{D}_0$ the group of all orientation preserving diffeomorphism of M, that are homotopic to the identity.

Then the moduli space $\mathcal{T}(M) = C\,/\mathcal{D}_0$ is called the **Teichmuller moduli space** of M and has dimension $= 6\,g - 6$. It is known (see Tromba [191]) that the Teichmuller space becomes a Kaehler space of negative curvature if it is assigned a Weil-Petersson metric (an incomplete Riemannian metric). For an application of Teichmuller space to $2+1$ dimensional spacetimes we refer Moncrief [145].

Now consider a vector U at a point p of a Kaehler manifold M. Then the pair $(U,\, JU)$ determines a plane (since JU is obviously orthogonal to U) element called a **holomorphic section**, whose curvature K is given by

$$K = \frac{g(R(U,\, JU)JU,\, U)}{(g(U,\, U))^2},$$

and is called the **holomorphic sectional curvature** with respect to U. If K is independent of the choice of U at each point, then $K = c$, an absolute constant. A simply connected complete Kaehler manifold of constant sectional curvature c is called a **complex space-form**, denoted by $M(c)$, which can be identified with the complex projective space $P_n\,(c)$, the open ball D_n in C^n or C^n according as $c > 0$, $c < 0$ or $c = 0$. The curvature tensor of $M(c)$ is given by

$$\begin{aligned} R(X,\, Y)Z &= \frac{c}{4}\,[g\,(Y,\, Z)X - g\,(X,\, Z)Y + g\,(JY,\, Z)JX \\ &- g\,(JX,\, Z)JY + 2\,g\,(X,\, JY)JZ]. \end{aligned}$$

A contraction of the above equation shows that $M(c)$ is Einstein, that is

$$Ric(X,\, Y) = \frac{n+1}{2}\,c\,g\,(X,\, Y).$$

Above stated results also hold for a class of semi-Riemannian manifolds (see Barros-Romero [7]), subject to the following restrictions on the signature of its indefinite metric g. For the endomorphism J, satisfying $J^2 = -I$, the eigenvalues of J are $i = \sqrt{-1}$ and $-i$ each one of multiplicity n. As J is real, and satisfies (9.2), the only possible signatures of g are $(2p,\, 2q)$ with $p + q = n$. For example, indefinite g can not be a Lorentz metric and therefore, for real J, a spacetime can not be Hermitian and or Kaehlerian. Subject to this restriction, according to Barros and Romero [7], $(J,\, g)$ and M are called **indefinite Hermitian (or Kaehler) structures and manifolds** respectively.

EXAMPLES. Barros and Romero [7] constructed C^n_q, $CP^n_q\,(c)$ and $CH^n_q\,(c)$, as representative examples of simply connected **indefinite complex space-forms** according as $c = 0$, $c > 0$ and $c < 0$, respectively. Here, $2\,q$ is the index of g. For instance, C^n_q can be identified with $(R^{2n}_{2q}\,,\, J,\, g)$ where J and g are defined by

$$J(x^1, \bar{x}^1, \ldots, x^n, \bar{x}^n) = (-\bar{x}^1, x^1, \ldots, -\bar{x}^n, x^n),$$

$$\begin{aligned} g((x^1, \bar{x}^1, \ldots, x^n, \bar{x}^n), (u^1, \bar{u}^1, \ldots, u^n, \bar{u}^n)) = & - \sum_{i=1}^{q} (x^i\, u^i + \bar{x}^i\, \bar{u}^i) \\ & + \sum_{a=q+1}^{q} (x^a\, u^a + \bar{x}^a\, \bar{u}^a). \end{aligned}$$

Finally we mention that for an application of Hermitian and Kaehlerian structures in general relativity, Flaherty [68] modified these structures by using a complex valued endomorphism J, defined by

$$J = \sqrt{-1}(\omega^1 \otimes e_1 - \omega^2 \otimes e_2 + \omega^3 \otimes e_3 - \omega^4 \otimes e_4),$$

where $\{e_a\} = \{l, k, m, \bar{m}\}$ the Newman-Penrose (NP) null tetrad such that l, k are real null vectors and m is a complex null vector and $\{\omega^1, \omega^2, \omega^3, \omega^4\}$ is its dual basis (see pages 3 and 4 for details). It is easy to see that this complex J satisfies the condition (9.2) of an almost Hermitian structure with respect to a Lorentz metric g of a spacetime. If the integrability condition ($\mathbf{N} = 0$) is satisfied (see equation (9.1)), then the Lorentz metric g can be locally expressed as

$$g = A\, d\, z^0\, d\, \bar{z}^0 + B\, d\, z^1\, d\, \bar{z}^1 + C\, d\, z^0\, d\, \bar{z}^1 + D\, d\, z^1\, d\, \bar{z}^0,$$

for some functions A, B, C, and D and for a complex coordinates system (z^0, z^1, $\bar{z}^0$, $\bar{z}^1$). Well-known physical examples are type-D (see Appendix **A**) vacuum spacetimes, which include Schwarzschild and Kerr solutions (see Hawking-Ellis [94, pages 149, 161].

9.2 Symmetries of Kaehler Manifolds

In this section we state results of the following geometers (for details on some proofs we refer Yano [209]):

M. Ako, M. Apte, J. Hano, S. Kobayashi, A. Lichnerowicz, Y. Matsushima, T. Nagano, K. Nomizu, S. Sasaki, J. A. Schouten, S. Tachibana.

DEFINITION. A vector field V on a complex (resp. almost complex) manifold is said to be **analytic (resp. almost analytic)** if its structure tensor J is invariant with respect to an infinitesimal transformation of V, that is, $L_V\, J = 0$.

One can verify that if V is analytic (also called holomophic) on a Kaehler manifold, then so is JV. Moreover, for any two analytic vector fields U and V, their Lie-bracket $[U, V]$ is also analytic.

THEOREM 9.1. *In a compact Kaehler manifold an analytic divergence free vector field is Killing.*

PROOF. $(L_V\, J)X = 0$ implies $L_V\, JX = J\, L_V\, X$. In terms of covariant derivative operator, this means that $\nabla_{JX}\, V = J\, \nabla_X\, V$ which expressed in local coordinates gives

$$J^i{}_j\, \nabla_i\, V_k + J^i{}_k\, \nabla_j\, V_i = 0.$$

Operating above equation by $J^{mj}\, \nabla_m$ gives

$$\triangle\, V^k + R^k{}_j\, V^j = 0, \tag{9.5}$$

where $\triangle = g^{ij} \nabla_i \nabla_j$. In view of hypothesis that $div\, V = 0$ and the following integral formula (see page 84) for a compact Riemannian manifold

$$\int_M \{\triangle V^k + R^k{}_j V^j + \frac{n-1}{n} \nabla^k (div\, V)\} V_k + \frac{1}{2} |L_V g_{ij} - \frac{1}{n} (div\, V) g_{ij}|^2 = 0$$

we obtain $L_V g = 0$, which completes the proof.

THEOREM 9.2. *A vector field V on a compact Kaehler manifold M is analytic iff $\triangle V + Q V = 0$. Thus, a Killing V on M is analytic.*

PROOF. For a Kaehler M, by a straightforward calculation, one can obtain

$$\begin{aligned} \nabla_j [(\nabla^j V^i V_i - J^{jk} J^{li} (\nabla_k V_i) V_l] &= (\triangle V^m + R^m_i V^i) V_m \\ &+ \frac{1}{2} |L_V J|^2. \end{aligned}$$

Since left side of above is a divergence, using divergence theorem we get

$$\int_M (\triangle V^m + R^m{}_i V^i) V_m + \frac{1}{2} |L_V J|^2 = 0.$$

Thus, the proof is immediate. This result also holds for an almost Kaehler M.

PROPOSITION 1. *If V is a Killing vector on a compact Kaehler manifold, then JV is an analytic gradient vector.*

PROPOSITION 2. *A conformal vector field V on a compact Kaehler manifold M is Killing and hence analytic. If M is almost Kaehler, then V is Killing if $dim(M) > 2$ and almost analytic if $dim(M) \geq 2$.*

PROPOSITION 3. *An analytic vector field V on a compact Kaehler manifold, which admits a Ricci collineation ($L_V R_{ij} = 0$), is Killing.*

PROPOSITION 4. *If an almost analytic vector field on a compact Kaehler manifold is a projective collineation (**PC**) vector, then it is Killing.*

PROPOSITION 5. *An analytic vector field V on a compact Kaehler Einstein manifold M with non-zero curvature decomposes uniquely as $V = U + JW$, where U and W are Killing.*

NOTE. Proposition 5 is due to Y. Matsushima, which was further generalized by A. Lichnerowicz for M with constant scalar curvature. Details on the proofs of above propositions, and many more side results, are available in Yano [208]. Finally, we state the following three results for which M may not be compact.

THEOREM 9.3 (Goldberg [75]). *A conformal vector field, whose metrically associated 1-form is closed, on a Kaehler manifold is analytic and homothetic vector field.*

Recall from section 5 (of chapter 5) and section 2 (of chapter 7) that the existence of a second order parallel tensor (other than a constant multiple of the metric tensor) is necessary for the existence of affine or conformal collineation symmetries. Related to this we have the following results.

THEOREM 9.4 (Sharma [173]). *A second order parallel tensor on a Kaehler manifold of constant curvature is a linear combination (with constant coefficients) of the Kaehlerian metric and its fundamental 2-form.*

THEOREM 9.5 (Sharma [173]). *An affine Killing vector field in a non-flat Kaehler manifold of constant curvature is Killing and analytic.*

REMARK. Since the publication of Barros-Romero's [7] work on indefinite Kaehler manifolds, there has been potential scope for further research on **indefinite Kaehler manifolds** and their symmetries.

9.3 Contact Manifolds

The theory of **contact structures** has its roots in differential equations, optics and phase space of a **dynamical system** (for details see Arnold [4], Maclane [135], Nazaikinskii et al. [149] and many more references therein).

A $(2n+1)$-dimensional differentiable manifold M is called a **contact manifold** if it has a global differential 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. For a given **contact form** η, there exists a unique vector field ξ, called the **characteristic vector field**, satisfying

$$\eta(\xi) = 1, \qquad (d\eta)(\xi, X) = 0, \qquad \forall X \in \mathcal{X}(M).$$

The definition of the contact structure implies that the **contact distribution** (subbundle of $T(M)$) D , given by $\eta = 0$, is far from being integrable. It is known (see Sasaki in Blair [17]) that the maximum dimension of an integral submanifold of D is n. A Riemannian metric g is said to be an associated metric, of the contact structure, if there exists a tensor field ϕ, of type (1, 1) such that

$$\begin{aligned} d\eta(X, Y) &= g(X, \phi Y), \qquad g(X, \xi) = \eta(X), \\ \phi^2(X) &= -X + \eta(X)\xi, \qquad \forall X, Y \in \mathcal{X}(M). \end{aligned}$$

These metrics can be constructed by the polarization of $d\eta$ evaluated on a local orthonormal basis of the tangent space with respect to an arbitrary metric, on the contact subbundle D. The structure (ϕ, η, ξ, g) on M is called a **contact metric structure** and its associated manifold is called a **contact metric manifold** which is orientable and odd dimensional with $n \geq 3$. Thus, a contact metric manifold is an analogue of an almost Kaehler manifold, in odd dimensions. For details we mention the reader Sasaki's work cited in Blair [17].

For a contact metric structure, the tensor field h defined by $h = \frac{1}{2} L_\xi \phi$, is self-adjoint, that is, $g(hX, Y) = g(X, hY)$ and also satisfies

$$h\xi = 0, \quad Tr.h = 0, \quad Tr.h\phi = 0, \quad h\phi = -\phi h.$$

Thus, if E is an eigenvector of h with eigenvalue λ then ϕE is also an eigenvector with eigenvalue $-\lambda$. Following formulas hold for a contact manifold (see Blair [17]):

$$\begin{aligned} \nabla_\xi \phi = 0, \qquad \nabla_X \xi = -\phi X - \phi h X,& \\ l X - \phi l \phi X = -2(h^2 + \phi^2)X,& \\ (\nabla_\xi h)X = \phi(X - h^2 X - lX),& \\ Ric(\xi, \xi) = 2n - Tr.h^2,& \end{aligned} \tag{9.6}$$

where $lX = R(X, \xi)\xi$, which measures the sectional curvature of a plane section containing ξ, called ξ-sectional curvature. Furthermore, a contact metric manifold satisfies the following identities:

$$(L_\xi g)(X, Y) = 2g(h\phi X, Y), \qquad (div\,\phi)X = -2n\eta, \tag{9.7}$$

$$Q\xi = \nabla^2 \xi + 4n\xi, \qquad Q\xi = 2n\xi + T, \tag{9.8}$$

where Q and T are Ricci operator and a vector field metrically equivalent to the 1-form $div(h\phi)$ respectively. The contact metric structure is called a **K-contact structure** if ξ is Killing, which further $\Leftrightarrow h = 0$ (follows from first equation of (9.7)) $\Leftrightarrow Ric(\xi, \xi) = 2n$ (which follows from (9.6)) $\Leftrightarrow div(h\phi) = 0$ (follows from second equation of (9.8)). We say that M has a **normal contact structure** if

$$N_\phi + 2d\eta \otimes \xi = 0,$$

where N_ϕ is the Nijenhuis tensor field of ϕ (see Blair [17]). A normal contact metric manifold is called a **Sasakian manifold**. It is known that Sasakian manifolds are K-contact but the converse holds only if $dim(M) = 3$. In the following we brief some key results on contact manifolds.

Blair [17] proved that there are no flat Riemannian metrics associated to a contact structure on a contact manifold of dimension > 3; in contrast the 3-dimensional torus admits a contact structure for which there is a flat associated metric. This result was further generalized by

Z. Olszak (see in Blair-Sharma [20]) who proved that there are no contact metric structures of constant curvature unless the constant is 1 and in which case the structure is Sasakian. This raised an open question as to whether or not there are 3-dimensional contact metric manifolds of constant curvature other than 0 or 1. That these are the only possibilities was answered by

Blair-Sharma [20] who proved that a 3-dimensional locally symmetric contact manifold is of constant curvature 0 or 1. Note that this result generalized in dimension 3 an earlier result of **Tanno (cited in Sharma [178])** who proved that a locally symmetric K-contact manifold is of constant curvature.

Blair-Sierra [21] proved that a 5-dimensional locally symmetric contact manifold is locally isometric to either $S^5\,(1)$ or $E^3 \times S^2\,(4)$ or the product of a line and two hyperbolic planes. The last case is ruled out by an unpublished result of A. M. Pastore. In dimensions ($>$ 5) the problem of characterizing a locally symmetric contact manifold is still open.

Tanno [187] studied a special family of contact metric manifolds by requiring that ξ belongs to a distribution, called k-nullity distribution $N(k) : p \longrightarrow N_p\,(k)$, for every $p \in M$, defined by

$$N_p\,(k) = \{Z \in T_p\,(M) : R(X,\,Y)Z = k(g(Y,\,Z)X - g(X,\,Z)Y)\}, \tag{9.9}$$

for any $X, Y \in T_p\,(M)$ and a real number k. It is known that for $k = 1$, condition (9.9) means M is Sasakian and for $k = 0$, M is flat in dimension 3 and otherwise it is locally the Riemannian product $E^{n+1} \times S^n\,(4)$. Related to this we have

Blair et al. [19] proved that for a 3-dimensional contact metric manifold M, the following three conditions are equivalent:

(a) $Q\,\phi = \phi\,Q$, where Q is the Ricci operator.

(b) ξ belongs to the k-nullity distribution $N(k)$.

(c) $Q = (\frac{r}{2} - k)I + (3\,k - \frac{r}{2})\,\eta \otimes \xi$.

For $k = 1$, any one of the conditions imply that M is Sasakian. For $k\, <\, 1$, any one of the conditions imply $Q = 2\,k\,\eta \otimes \xi$. Examples of the last case are left-invariant metrics on $SU(2)$ for $k > 0$, and $SL(2,\,R)$ for $k < 0$.

EXAMPLES. Standard examples of contact manifolds are **(i)** the odd dimensional spheres, **(ii)** tangent or cotangent sphere bundles, **(iii)** the 3-dimensional Lie-groups $SU(2)$ and $SL(2,\,R)$.

In Thermodynamics, we have an example due to Gibbs which is given by the contact form $du - T\,ds + p\,dv$ (u is the energy, T is the temperature, s is the entropy, p is the pressure and v is the volume) whose zeros define the laws of thermodynamics. Details may be seen in Arnold [4].

In 1990, Duggal [51] introduced a larger class of contact manifolds as follows. We use the same notations of geometric objects as above. A (2n + 1)-dimensional smooth manifold $(M,\,g)$ is called an **(ϵ)-almost contact metric manifold** if

$$\phi^2 = -\,I + \eta \otimes \xi, \quad \eta\,(\xi) = 1,$$

$$g(\xi,\,\xi) = \epsilon, \quad g(\phi\,X,\,\phi\,Y) = g(X,\,Y) - \epsilon\,\eta(X)\,\eta(Y), \tag{9.10}$$

where $\epsilon = 1$ or -1 according as ξ is spacelike or timelike and $rank(\phi) = 2\,n$. It is important to mention that in above definition ξ is never a lightlike vector field. If $(d\eta)\,(X,\,Y) = g(X,\,Y)$, for every $X, Y \in \mathcal{X}(M)$, then we say that M is an

(ϵ)-**contact metric manifold**. If $\epsilon = -1$ and the contact distribution $D(\eta = 0)$ is positive definite, then g is Lorentzian and we say that the underlying almost contact manifold M is an **almost contact spacetime**.

THEOREM 9.6 (Duggal [51]). *An odd dimensional globally hyperbolic spacetime is an almost contact spacetime.*

PROOF. Consider a $2n$-dimensional almost Hermitian manifold (S, G, J) defined by

$$J^2 = -I, \qquad G(J\bar{X}, J\bar{Y}) = G(\bar{X}, \bar{Y}), \qquad \forall \bar{X}, \bar{Y} \in \mathcal{X}(S),$$

where (S, G) is a compact Riemannian manifold. Construct a globally hyperbolic spacetime $M = \{R \otimes S\,,\, g = -\,dt^2 + G\}$ (see page 67 for conditions on S). Denote a vector field on M by $X = (\eta(X)\,\frac{d}{dt}, \bar{X})$ where $\bar{X}$ is tangent to S, t and η are the time coordinate of R and a 1-form on M respectively. Set $\eta = dt$ so that $\xi = (\frac{d}{dt}, 0)$ is a global timelike vector field on M. Then, with

$$\phi(\eta(X)\,\frac{d}{dt}, \bar{X}) = (0, J\bar{X}),$$

$$g\,\{(\eta(X)\,\frac{d}{dt}, \bar{X}), (\eta(Y)\,\frac{d}{dt}, \bar{Y})\} = G(\bar{X}, \bar{Y}) - \eta(X)\,\eta(Y),$$

one can verify that both the structure equations of (9.10) hold for $\epsilon = -1$. Thus M is an almost contact spacetime, which completes the proof.

An (ϵ)-contact metric structure which is normal is called an (ϵ)-**Sasakian structure**. As in the Riemannian case, we have the following existence theorem:

THEOREM 9.7 (Bejancu-Duggal [9]). *An (ϵ)-almost contact metric structure is (ϵ)-Sasakian iff*

$$(\nabla_X\,\phi)\,Y = g\,(X, Y)\,\xi - \epsilon\,\eta\,(Y)\,X, \qquad \forall X, Y \in \mathcal{X}(M).$$

EXAMPLE. Consider a $(2n+1)$-dimensional Minkowski spacetime $(M,\, g)$ with local coordinates $(x^i\,,\, y^i,\, t)$ and $i = 1, \ldots, n$. M being time oriented admits a global timelike vector field, say ξ. Define a 1-form $\eta = \frac{1}{2}\,(dt - \sum_1^n y^i\, dx^i\,)$ so that $\xi = 2\,\partial_t$. With respect to the natural field of frames $\{\partial_{x^i}\,;\, \partial_{y^i},\, \partial_t\,\}$, define a tensor field ϕ of type (1, 1) by its matrix

$$(\phi) = \begin{pmatrix} 0_{n,\,n} & I_n & 0_{n,\,1} \\ -I_n & 0_{n,\,n} & 0_{n,\,1} \\ 0_{1,\,n} & -y^i & 0 \end{pmatrix}$$

Define a Lorentzian metric $\bar{g}$ with line element given by

$$ds^2 = \frac{1}{4}\,\{\sum_1^n (\,(dx^i)^2 + (dy^i)^2\,) - \eta \otimes \eta\}$$

Then, with respect to an orthonormal basis $\{E_i\,,\, E_{n+i}\,,\, \xi\}$ such that

$$\begin{aligned} E_i &= 2\,\partial_i, \quad E_{n+i} = 2\,\partial_{n+i}, \\ \phi\,E_i &= 2\,(\partial_i - y^i\,\partial_t), \\ \phi\,E_{n+i} &= 2\,(\partial_i + y^i\,\partial_t), \end{aligned}$$

it is easy to verify that the spacetime (M, g) has a Sasakian structure. For Riemannian case, we refer to Blair [17, page 99] in which it is shown that R^{2n+1} is a Sasakian space form with $c = -3$.

9.4 Symmetries of Contact Manifolds

DEFINITION 1. A vector field V on a contact manifold M is said to define an **infinitesimal contact transformation** if $L_V\,\eta = f\,\eta$, for some differentiable function f on M. In particular, for $f = 0$, V is called an **infinitesimal strict contact transformation**.

In 1962, Tanno (see in [179]) proved that if V is an infinitesimal strict contact transformation then $L_V\,\xi = 0$.

DEFINITION 2. A vector field V on a contact metric manifold M is said to be an **infinitesimal automorphism** if V leaves the structure tensors η, ξ, ϕ and g invariant.

Proposition 1. *If a vector field V on a contact metric manifold satisfies any two of the following conditions, then it satisfies the remaining third one:*

$$(i)\ L_V\,\eta = 0,\ \ (ii)\ L_V\,\phi = 0,\ \ (iii)\ L_V\,g = 0.$$

PROOF. (i) and (ii) $\Rightarrow$ (iii). Indeed, taking Lie-derivative of $g(\xi, X) = \eta(X)$ along V and then using (i) and $L_V\,\xi = 0$, we get $(L_V\,g)(\xi, X) = 0$. Next, Lie-derivative of $g(X, \phi Y) = (d\eta)(X, Y)$ and then using (i) and (ii) implies $(L_V\,g)(X, \phi Y) = 0$. Thus, (iii) holds. By Lie-differentiating $\eta(\phi X) = 0$ and $g(\xi, \xi) = 1$ one can show that (ii) and (iii) $\Rightarrow$ (i). Finally, (iii) and (i) $\Rightarrow$ (ii) follows by Lie-differentiating $(d\eta)(X, Y) = g(X, \phi Y)$. This completes the proof.

THEOREM 9.8. *If a* **PC** *vector field V on a contact metric manifold M leaves ϕ invariant, then V is an infinitesimal automorphism.*

Proof. Since V is **PC**, we have

$$L_V\,\Gamma^i{}_{jk} = \delta^i{}_j\,p_k + \delta^i{}_k\,p_j,$$

where $p_i = (2n+2)^{-1}\,\partial_i\,(div\,V)$ and $dim(M) = 2n+1$. Recall the following commutative formula

$$\begin{aligned} L_V\,\nabla_j\,\phi^i{}_h - \nabla_j\,L_V\,\phi^i{}_h &= (L_V\,\Gamma^i{}_{jk})\phi^k{}_h - \phi^i{}_k\,(L_V\,\Gamma^k{}_{jh}) \\ &= \delta^i{}_j\,p_k\,\phi^k{}_h - p_h\,\phi^i{}_j. \end{aligned} \tag{9.11}$$

Contracting (9.11) at j and i and using $L_V\,\phi = 0$, we get

$$L_V\,\nabla_i\,\phi^i{}_h = (2\,n+1)\,p_k\,\phi^k{}_h.$$

Using the second identity of (9.7), above can be rewritten as

$$L_V\,\eta_h = -\,\frac{2n+1}{2n}\,p_k\,\phi^k{}_h. \tag{9.12}$$

The inner product of (9.11) with ξ^i and the use of $\nabla_\xi \phi = 0$ (see page 179) gives

$$(L_V \xi^j) \nabla_j \phi^i{}_h = - \eta^i p_k \phi^k{}_h. \tag{9.13}$$

Also, Lie-differentiating the identity $\phi^2 = -I + \eta \otimes \xi$ gives

$$L_V \xi^i = -(L_V \eta_k) \eta^k \eta^i = 0$$

since (9.12) implies $(L_V \eta_h) \eta^h = 0$. Thus, (9.12) and (9.13) yields $L_V \eta_h = 0$. Consequently, it follows from proposition 1 that V is Killing, which completes the proof.

PROPOSITION 2. *If an infinitesimal contact transformation V preserves the volume element of the associated metric, then V is an infinitesimal strict contact transformation.*

Proof is straightforward by using the Lie-derivative of the volume element $\eta \wedge (d\eta)^n$.

THEOREM 9.9. *A conformal vector field V on a contact metric manifold is Killing if V leaves any one of the tensors η, ξ, ϕ, $d\eta$ invariant.*

Proof. If $L_V \eta = 0$, then $\eta(\xi) = 1 = g(\xi, \xi)$ implies that $\eta(L_V \xi) = 0$ and $(L_V g)(\xi, \xi) + g(L_V \xi, \xi) = 0$. Using this in the conformal Killing equation $(L_V g = 2\sigma g)$ we get $\sigma = 0 \Rightarrow V$ is Killing. Next, if $L_V \xi = 0$, then Lie-derivative of $g(\xi, \xi) = 1 \Rightarrow \sigma = 0$. If $L_V \phi = 0$, then Lie-derivative of $\eta(\phi X) = 0$, $\phi \xi = 0$ and $(d\eta)(X, Y) = g(X, \phi Y)$ implies $\sigma = 0$. Finally, if $L_V d\eta = 0$, then $L_V \phi = 0$ and hence, as discussed before, $\sigma = 0$. This completes the proof.

Proofs of the following results on automorphism are left as an exercise:

(a) *If a transformation f of a K-contact manifold M leaves ϕ invariant and the scalar curvature of M is a constant $\neq -2n$, then f is an automorphism.*

(b) *If the scalar curvature of a K-contact manifold M is non-constant but bounded, then a transformation f of M leaving ϕ invariant, is an automorphism.*

(c) *If a transformation f of an Einstein contact metric manifold leaves ϕ invariant, then f is an automorphism.*

(d) *A vector field V on a compact contact metric manifold leaves ϕ invariant iff V is an infinitesimal automorphism.* Hint: see in Tanno [185].

(e) *If a transformation f of a compact contact metric manifold leaves ϕ invariant, then f is an automorphism of this structure. The set of all such transformations forms a compact Lie-group.* Hint: see Tanno [185].

(f) *If a conformal transformation f of a contact metric manifold leaves η invariant upto a positive function, then f is an automorphism of the contact metric structure.*

Proofs of the following results are available in the cited references:

In [175], Sharma proved that a second order parallel symmetric tensor field on a K-contact manifold is a constant multiple of the associated metric. He generalized this result in [176] for a contact metric manifold with ξ-sectional curvature $K(\xi, X)$ non-vanishing and independent of the choice of X. This was further generalized in [178] for a contact metric manifold with non-vanishing $K(\xi, X) = K(\xi, \phi X)$ everywhere and for all X orthogonal to ξ. Recently, Ghosh-Sharma have done further work on this as stated in item (1) below:

Ghosh and Sharma [74].

(1) *A second order parallel tensor field on a contact metric manifold M with non-vanishing ξ-sectional curvature $K(\xi, X)$ is a constant multiple of the associated contact metric. Hence, any affine Killing or conformal collineation vector field on M is homothetic or conformal respectively.*

(2) *If V is a non-Killing* **PC** *vector field on a 3-dimensional Sasakian manifold M, then M is locally isometric to the unit 3-sphere.*

Okumura [158]. *If a Sasakian manifold M of dimension > 3, admits a non-isometric infinitesimal conformal transformation V, then V is special concircular and hence, if, in addition, M is complete and connected, then it is isometric to a unit sphere.*

Sharma and Blair [179].

(1) *Let V be a non-isometric infinitesimal conformal transformation on a 3-dimensional Sasakian manifold M which leaves the scalar curvature of M invariant. Then, M is of constant curvature 1 and V is special concircular.*

(2) *Let M be a contact metric manifold with ξ in the k-nullity distribution $N(k)$ and V a conformal vector field on M. For dim$(M) > 3$, either M is Sasakian or V is Killing. In the second case, V is an infinitesimal automorphism of the contact metric structure except when $k = 0$. Moreover, for $k = 0$, a Killing vector field orthogonal to ξ cannot be an infinitesimal automorphism of the contact metric structure. For dim$(M) = 3$, M is either flat or Sasakian or V is an infinitesimal automorphism of the contact metric structure.*

Based on above result **(1)** and Obata's [153] 1962 result (see page 133, chapter 6) we have the following characterization of a unit 3-sphere.

"*Among all complete and simply connected 3-dimensional Sasakian manifolds, only the unit 3-sphere admits a non-isometric infinitesimal conformal transformation that leaves the scalar curvature invariant.*"

NOTE 1. The set of all conformal vector fields leaving the scalar curvature invariant is a subalgebra of the conformal algebra. Also recall that a conformal vector field V ($L_V\, g = 2\,\sigma\, g$) is called an **infinitesimal special concircular transformation** if $\nabla\,\nabla\,\sigma = (-\,c_1\,\sigma + c_2)$, for some constants c_1 and c_2. The proof of above result **(1)** uses Lie-derivative of the following identity (see page 180):

$$Q = (\frac{r}{2} - 1)\,g + (3 - \frac{r}{2})\,\eta \otimes \xi$$

for a 3-dimensional Sasakian manifold M, along the conformal vector field, the fact that M has constant curvature iff $r = 6$ and the following lemma (proof is straightforward):

LEMMA. *A homothetic vector field on a K-contact manifold is Killing.*

NOTE 2. Based on the above result **(2)** following holds:

"*The existence of a non-Killing conformal vector field on contact manifolds, with ξ in $N(k)$, singles out the Sasakian manifolds for which $k = 1$.*"

Sharma [174].

(1) *If a conformal vector field V on a contact metric manifold is either collinear with, or orthogonal to ξ at each point, then V is Killing.*

(2) *A curvature collineation on a K-contact manifold is Killing.*

(3) *Let the characteristic vector field ξ of a contact metric (but not K-contact) manifold M of dimension ,$n+1$, define a curvature collineation. For $n = 1$, M is flat. For $n > 1$, the curvature operator Q annihilates h and the contact distribution is orthogonally decomposed into the eigenspaces $[\,0\,]$, $[\,1\,]$ and $[\,-1\,]$ of h corresponding to the eigenvalues 0, 1 and -1. Moreover, the sectional curvature of M satisfies*

$$\begin{aligned} K(\xi,\,[\,0\,]) = 0, \quad K(\xi,\,[\,\pm 1\,]) &= K([\,0\,],\,[\,\pm 1\,]) \\ &= K([\,+1\,],\,[\,-1\,]) = 0. \end{aligned}$$

Also if Ric$(\xi,\,\xi) < 2$ for $n > 1$, then M is locally isometric to $E^{n+1} \times S^n$ (4).

Sharma's result **(2)** has been extended by the following two results:

Sharma [178].

(1) *Let M be a contact metric manifold with non-vanishing $K(\xi,\,X) = K(\xi,\,\phi\,X)$ everywhere and for all $X \perp \xi$. Then a curvature collineation is homothetic.*

(2) *Let M be a contact metric manifold with non-vanishing ξ-sectional curvature. Then, a curvature collineation on M is homothetic.*

REMARK. Recent interest in the geometry of semi-Riemannian manifolds, and, in particular, publications by Duggal [51], Bejancu-Duggal [9] (also a few isolated papers by others) has opened a potential scope of further research on **indefinite contact and Sasakian manifolds** and their symmetries.

9.5 Globally Framed Manifolds

In 1963, Yano [207] introduced **f-structure**, on a real $(2n+r)$-dimensional smooth manifold M, defined by a tensor field f of type (1, 1) such that

$$f^3 + f = 0, \qquad rank(f) = 2n. \tag{9.14}$$

Corresponding to two complementary projection operators P and Q applied to $T(M)$, defined by

$$(i)\quad P = -f^2, \qquad (ii)\quad Q = f^2 + I, \tag{9.15}$$

where I is the identity operator, there exist two complementary distributions D and $D^\perp$ such that $dim(D) = 2n$ and $dim(D^\perp) = r$. The following relations hold

$$fP = Pf = f, \qquad fQ = Qf = 0, \qquad f^2 P = -P, \qquad f^2 Q = 0. \tag{9.16}$$

Thus, we have an **almost complex distribution**, defined by $(D,\, J = f|D,\, J^2 = -I)$, and f acts on $D^\perp$ as a **null operator**. It follows that

$$T(M) = D \oplus D^\perp, \qquad D \cap D^\perp = \{0\}.$$

It is clear from (9.16) that an f-structure is a generalization of **almost complex** and **almost contact** structures according as $r = 0$ and $r = 1$ respectively. Yano [207] has proved that the existence of an f-structure is equivalent to the reducibility of the structure group of $T(M)$ to $U(n) \times O(r)$. Assume that $D_x^\perp$ is spanned by r globally defined orthonormal vectors $\{\xi_a\}$ at each point $x \in M$, $(1 \le a, b, \ldots, \le r)$, with its dual set $\{\eta^a\}$. Then, (9.15) implies

$$f^2 = -I + \eta^a \otimes \xi_a. \tag{9.17}$$

In the above case, M is called a **globally framed (or simply a framed) manifold** (see Blair [18]) and we denote its **framed structure** by $M(f, \xi_a, \eta^a)$. The equation (9.17), together with (9.14) implies

$$f\,\xi_a = 0, \qquad \eta^a \circ f = 0, \qquad \eta^a(\xi_b) = \delta^a_b. \tag{9.18}$$

PROPOSITION 1 (Goldberg-Yano [78]). *For a globally framed manifold, the following identities hold:*

$$\begin{aligned} X(\eta^a(Y)) &= (L_X \eta^a)(Y) + \eta^a([X, Y]), \\ d\eta^a(\xi_b, X) &= (L_{\xi_b} \eta^a)(X), \\ d\eta^a(fX, Y) &= (L_{fX} \eta^a)(Y). \end{aligned}$$

Now we consider a metric g on a framed manifold M. We say that M has an associated metric structure if f is skew symmetric with respect to g. This means that

$$g(fX, Y) + g(X, fY) = 0, \qquad \forall X, Y \in \mathcal{X}(M). \tag{9.19}$$

For a framed manifold M, (9.17), (9.18) and (9.19) imply

$$g(f X, f Y) = g(X, Y) - \epsilon_a \eta^a(X)(\eta^a(Y), \tag{9.20}$$

where we set $g(X, \xi_a) = \epsilon_a \eta^a(X)$ and $\epsilon_a = +1$ or -1 according as the corresponding ξ_a is spacelike or timelike. In the above case, we say that M is a **metric framed manifold**. If g is a Riemannian metric (each $\epsilon_a = 1$) then it is always possible to associate M with a metric structure (see Blair [17]) satisfying (9.20). However, unfortunately, it is not possible to associate M with an arbitrary metric g, as explained below.

For a metric framed M, its $2n$-dimensional distribution $(D, g|D, J = f|D, J^2 = -1)$ has an **indefinite almost Hermitian structure**. Thus as stated on page 175, the only possible signatures of $g|D$ are $(2p, 2q)$ with $p + q = n$. In general, at best we have the following result.

THEOREM 9.10 (Duggal-Bejancu [57]). *Let $M(f, \xi_a, \eta^a)$ be a framed structure on (M, h_0) where h_0 is a semi-Riemannian metric on M. Suppose $\{\xi_a\}$ is orthonormal with respect to h_0 and $h_0(\xi_a, \xi_a) = -\epsilon_a$ for every $(1 \leq a \leq n)$. Then, there exists a symmetric tensor field g of type $(0, 2)$ on M, satisfying (9.20).*

PROOF. Define a metric h by

$$h(X, Y) = h_0(f^2 X, f^2 Y) + \epsilon_a \eta^a(X)\eta^a(Y).$$

To prove that h is also a semi-Riemannian metric, we first note that $h(X, \xi_a) = \epsilon_a \eta^a(X)$ and, therefore, $h(\xi_a, \xi_b) = \epsilon_a$. Let D_0 and D_1 be respective distributions spanned by $\{\xi_a\}$ and complementary orthogonal to D_0 with respect to h_0. Thus, for any X belonging to D_1,

$$\begin{aligned} h(X, X) &= h_0(-X + \eta^a(X)\xi_a, -X + \eta^b(x)\xi_b) + \epsilon_c \eta^c(X)\eta^c(X) \\ &= h_0(X, X), \end{aligned}$$

since $h_0(X, \xi_a) = 0$ and $h_0(\xi_a, \xi_a) = -\epsilon_a$. Thus, h is also a semi-Riemannian metric on M of the same index as h_0 is on D_1. Finally, let g be a symmetric tensor of type $(0, 2)$, such that

$$g(X, Y) = \frac{1}{2}\{h(X, X) + h(f X, f Y) + \epsilon_a \eta^a(X)\eta^a(Y)\},$$

which satisfies (9.20) as desired. This completes the proof.

NOTE. Above theorem was first proved in [9] for an **almost contact manifold**, in which case also, in general, there does not exist an **(ϵ)- almost contact metric structure**. On the brighter side, we now show that there always exists a Lorentz metric g associated to a framed manifold (same as the case of Riemannian manifolds).

PROPOSITION 2 (Duggal-Bejancu [57]). *Let M be a $(2n + r)$- dimensional framed manifold. Then, M admits a Lorentz metric g satisfying (9.20).*

PROOF. Since M has a framed structure, by (9.17) there exists a global orthonormal basis $\{\xi_a\}$ for the distribution $D^\perp$. Also, for a Lorentz M the distribution (D, J) must be Riemannian and, therefore, $D^\perp$ is either Lorentzian for $n \geq 2$ or is generated by a single global timelike vector when $n = 1$. Thus, a framed M, with Lorentz metric, admits a **line element field** which is an assignment of a pair of equal and opposite vectors $(X, -X)$ at each point x of M (see Hawking-Ellis [94, page 39]). Based on this and the fact that M is paracompact it follows that M admits a Lorentz metric, say h_0, such that $h_0(\xi_a, \xi_a) = -\epsilon_a$, for every $(1 \leq a \leq n)$. Assume that $\{\xi_a\}$ is orthonormal to h_0. Now define h and g as in theorem 9.10, with $\epsilon_1 = -1$ and all others $+1$. Then, it follows that g is an associated metric to M satisfying (9.20), which completes the proof.

Following the terminology introduced by Duggal [50], we say that $M(f, g, \xi_a, \eta^a)$ is a **Lorentz framed structure** and M is a **Lorentz framed manifold** if g is Lorentzian and M satisfies (9.14) - (9.20).

EXAMPLE 1. Let $\bar{M}(G, J)$ be a $2n$-dimensional almost Hermitian manifold where g is its associated Riemannian metric, $J^2 = -I$ and $G(J\bar{X}, \bar{Y}) = G(\bar{X}, \bar{Y})$ for every $\bar{X}, \bar{Y} \in \mathcal{X}(\bar{M})$. Construct a product manifold:

$$M = \{R_1^r \times \bar{M},\ g = -(dx^1)^2 + \sum_{a=2}^{r} (dx^a)^2 \oplus G\}, \tag{9.21}$$

where R_1^r is an r-dimensional Minkowski spacetime, with local coordinates $(x^1, \ldots, x^r)$ and g is a Lorentz metric on M. Denote a vector field on M by $X = (\eta^a(X)\,\partial_{x^a}, \bar{X})$, where $\eta^a(X)$ are r smooth functions on M and $\bar{X} \in \mathcal{X}(\bar{M})$. Thus, there exist r vector fields $\xi_a = (\partial_{x^a}, 0)$ and their r duals $\eta^a = dx^a$ such that ξ_1 and $\xi_2, \ldots, \xi_r$ are timelike and spacelike respectively. Define a (1, 1) tensor field f on M by

$$f(\eta^a(X)\,\partial_{x^a}, \bar{X}) = (0, J\bar{X}). \tag{9.22}$$

It follows from (9.21) that

$$g(X, Y) = -\eta^1(X)\eta^1(Y) + \sum_{a=2}^{r} \eta^a(X)\eta^a(Y) + G(\bar{X}, \bar{Y}). \tag{9.23}$$

Thus, using (9.22) and (9.23) we recover a **Lorentz framed structure** on M. Indeed, we have

$$\begin{aligned} f^2 &= -I + \eta^a \otimes \xi_a, \qquad f(\xi_a) = 0 = \eta^a \circ f, \\ f^3 + f &= 0, \qquad rank(f) = 2n, \\ g(fX, fY) &= g(X, Y) + \eta^1(X)\eta^1(Y) - \sum_{a=2}^{r} \eta^a(X)\eta^a(Y). \end{aligned}$$

Based on above example, if $\bar{M}$ is compact and R_1^r is replaced by an r-dimensional spacetime manifold, then we have the following result:

THEOREM 9.11 (Duggal [50]). *A globally hyperbolic (in particular, de-Sitter) spacetime can carry a framed metric structure.*

EXAMPLE 2. Recall from pages 92 and 93 that a homogeneous spacetime, with simple **electromagnetic field** F, inherits a Lorentz framed structure.

Another interesting class of framed manifolds comes from the geometry of **Cauchy Riemann (CR) manifolds** as follows. A real $(2n + r)$ - dimensional smooth manifold M is said to have a **Cauchy Riemann (CR) structure** if it has a $2n$-dimensional **holomorphic subspace** H_x, of the tangent space $T_x(M)$, at each point x of M such that the subbundle H is endowed with a complex structure J satisfied by $J^2 = -I_H$. To complete a basis for $T_x(M)$, one needs a complementary set of r vectors which exists subject to certain integrability conditions (see Newlander-Nirenberg [150]). In differential geometry, we let M be a submanifold of an **almost Hermitian manifold** $(\bar{M}, \bar{g}, \bar{J} : \bar{J}^2 = -I)$, $\bar{g}$ and $\bar{J}$ are the Hermitian associated metric and almost complex operator respectively of $\bar{M}$. Then, (M, g) is called a **CR submanifold** of $\bar{M}$ if there exists a real invariant distribution D (i.e., $\bar{J}|D = J : J\,D = D$) such that $D = Re\,(H \oplus \bar{H})$, its orthogonal complementary distribution $D^{\perp}$ is anti-invariant (i.e., $J\,D^{\perp} \subset T(M)^{\perp}$ and $T(\bar{M}) = T(M) \oplus T(M)^{\perp}$). We have the following result as a characterization of **CR** submanifolds in terms of f - structures:

THEOREM 9.12 (Yano-Kon [214, page 87]). *In order for a submanifold M of a Kaehlerian manifold $\bar{M}$ to be a* **CR** *submanifold, it is necessary and sufficient that M and the normal bundle of M, both, have an f - structure.*

Since the subject matter on **CR** submanifolds is out of scope of this book, we refer Yano-Kon [214], Sharma [170], Duggal [48] and Duggal - Bejancu [57] for details on the geometry of Riemannian, semi-Riemannian, Lorentzian and lightlike **CR** submanifolds respectively and many more cited in these references such as Bejancu (1978), Blair-Chen (1979) and Chen (1973, 1981).

9.6 Symmetries of Framed Manifolds

A framed structure is said to be **Normal** (see Goldberg-Yano [78]) if the **torsion tensor** S_f of f is zero, i.e., if

$$S_f \equiv N_f + 2\,d\eta^a \otimes \xi_a = 0, \tag{9.24}$$

where N_f is the **Nijenhuis tensor field** of f, as defined by the equation (5.49). Define a 2-form Ω on M by

$$\Omega\,(X, Y) = g\,(f\,X, Y), \qquad \forall X, Y \in \mathcal{X}(M). \tag{9.25}$$

PROPOSITION 1 (Goldberg-Yano [78]). *The following identities hold on a normal framed manifold*

$$L_{\xi_a}\,\eta^b = 0, \quad [\,\xi_a, \xi_b\,] = 0, \quad L_{\xi_a}\,f = 0,$$

$$d\eta^a(fX, Y) + d\eta^a(X, fY) = 0.$$

Let $M(f, \xi_a, \eta^a)$ and $M'(f', U_a, u^a)$ be two framed structures of the same rank on two smooth manifolds M and M' respectively. A diffeomorphism μ of M onto M' is called an **isomorphism** of M onto M' if

$$\mu_\star \circ f = f' \circ \mu_\star, \qquad \mu_\star \xi_a = U_a,$$

where $\mu_\star$ denotes the induced map on the tangent spaces. If $M = M'$ and their two structures are identical, then μ is said to be an **automorphism** of M. The set of all automorphisms of M clearly forms a group.

PROPOSITION 2 (Goldberg-Yano [78]). *The group of automorphisms of a compact normal framed structure is a Lie group.*

A normal framed manifold, with closed Ω (i.e., $d\Omega = 0$), is called a K**-manifold**. Since $\eta^1 \wedge \ldots \eta^r \wedge \Omega^n \neq 0$, a K-manifold is orientable. Furthermore, we say that a K-manifold is a C**-manifold** if each $d\eta^a = 0$.

PROPOSITION 3. *On a K-manifold, its fundamental 2-form Ω is invariant with respect to each ξ_a, that is*

$$L_{\xi_a}\Omega = 0.$$

PROOF. Using the operator $L_X = di(X) + (iX)d$, where $i(X)$ is the inner product by X, we get $L_{\xi_a}\Omega = di(\xi_a)\Omega + i(\xi_a)d\Omega$. Now, $(i\xi_a\Omega)Y = \Omega(\xi_a, Y) = \eta^a \circ f(Y) = 0$ for all $Y \in \mathcal{X}(M)$. Also, $d\Omega = 0$. Thus, $L_{\xi_a}\Omega = 0$ which completes the proof.

THEOREM 9.13. *On a K-manifold, there exists an r-parameter group of isometries generated by the set of Killing vector fields $\{\xi_a\}$.*

THEOREM 9.14. *A C-manifold M is locally a decomposable manifold of the form $M = N^{2n} \times L^r$, where N^{2n} is a Kaehler manifold and L^r is an* **Abelian group manifold**, *that is, r one dimensional manifolds.*

NOTE. For the proofs of above two theorems, we refer Blair [18] if g is Riemannian and Duggal [43, 50] if g is Lorentzian.

REMARK. Observe that for $r = 1$, a framed manifold is an almost contact manifold, and, for this case all the symmetry properties (discussed in section 9.4) will hold. Also, in 1972, Goldberg (see cited in [50]) proved "*A covariant constant ($\nabla f = 0$) even dimensional framed Riemannian manifold carries a Kaehlerian structure*". Based on this result of Goldberg, it follows that all the symmetry properties (discussed in section 9.2) will also hold for a class of even dimensional covariant constant framed Riemannian manifolds. Finally, it is important to mention that based on his 1972 result, Goldberg studied a generalization of Kaehlerian geometry which includes the generalizations due to Lichnerowicz (1951) and Chen (1957) (both cited in [50]), in particular, reference to the topology of compact even dimensional framed manifolds.

9.7 Killing Horizon

In this section we first brief on the concept of **Killing horizon** in a general semi-Riemannian manifold and then show an interplay between a class of **lightlike hypersurfaces** (see section 2.7, chapter 2) of Lorentz framed manifolds and Killing horizons. Due to highly technical nature of this section, some details (not discussed here) are left for the readers to see in Carter [29] and Israel [102] and other references therein.

Consider an open region $\mathcal{U}$ on an m-dimensional semi-Riemannian manifold $(M,\, g)$ such that there exists a continuous r-parameter group of Killing vector fields generating an r-dimensional sub-tangent space V_x at every $x \in \mathcal{U}$, where $(1 \leq r \leq m-1)$. Then, the group is said to be **orthogonally transitive** if the r-dimensional orbits of the group are orthogonal to a family of $(m-r)$-dimensional surfaces. Let the respective decomposition be

$$T_x\,(M) = W_x + V_x\,, \qquad dim(W_x) = m - r. \tag{9.26}$$

The group is said to be **invertible** at a point $x \in \mathcal{U}$ if there is an **isometry** leaving x fixed , at x, but leaves unaltered the sense of the direction orthogonal to V_x at the same point $x \in \mathcal{U}$. The existence of such an isometry obviously implies that it is an involution and uniquely determined. It follows from the decomposition (9.26) that a group is invertible at a point x only if both V_x and W_x are non-singular. Indeed, if any one is lightlike (null), then there exists a self orthogonal null direction in that one, and, therefore, the group can not be invertible. Carter [29] has shown that for an **Abelian group**, the non-singular orthogonal transitive condition is also sufficient for the group to be invertible. A trivial example is the case of 1-parameter group which is Abelian, and, therefore the orthogonal transitivity and the invertibility are equivalent for a non-null Killing vector. Before defining Killing horizon, it is appropriate to relate above material with a class of Lorentz framed structure in the following way.

Suppose M has a Lorentz framed structure $M\,(f,\, g,\, \xi_a,\, \eta^a)$ as defined in section 9.5, such that M is a **C**-manifold, satisfying theorem 9.14. Then, based on this theorem and the orthogonal complementary decomposition $T_x\,(M) = D_x \oplus {D_x}^{\perp}$, for every $x \in \mathcal{U}$, it follows that the Abelian isometry group, generated by $\{\,\xi_1, \ldots, \xi_r\,\}$ acts orthogonal transitively on M with non-singular orthogonally complementary tangent sub-spaces D_x and ${D_x}^{\perp}$ for any $x \in \mathcal{U}$. Since, each Killing vector ξ_a is non-null, the invertibility of the Abelian group also holds.

Now coming back to the concept of Killing horizon, let $(\Sigma,\, \gamma)$ be a **lightlike hypersurface** of $(M,\, g)$ as presented in section 2.7, chapter 2. Then, Σ is said to be a **local isometry horizon (LIH)** with respect to a group of isometry if

(a) Σ is invariant under the group.

(b) Each null geodesic generator is a **trajectory** of the group.

In particular, a lightlike hypersurface $(\Sigma,\, \gamma)$ which is an **LIH** with respect to a 1-parameter group (or sub-group) is said to be a **Killing horizon**. This means (in terms of our case) that a Killing horizon is a lightlike hypersurface Σ of M whose generating null vector can be normalized so as to coincide with one of the Killing vectors ξ_a. Physically, an **LIH**, with respect to a 4-dimensional spacetime manifold M, has the following significant role. A particle on an **LIH**, of M, may immediately be traveling at the speed of light along the single null generator but standing still relative to its surroundings. This happens because, by definition of an **LIH**, the variable affine parameter along null geodesic leaves invariant both the intrinsic structure of M and the position of the lightlike hypersurface as an **LIH**. On the existence of **LIH** and Killing horizon, we have the following key result:

THEOREM 9.15 (Carter [29]). *Let $\mathcal{U}$ be an open subregion of an m-dimensional C^2 manifold with a C^1 semi-Riemannian metric, such that there is a continuous group of isometries whose surfaces of transitivity has constant dimension r in $\mathcal{U}$. Let N be the subset (closed in $\mathcal{U}$) where the surfaces of transitivity become null and suppose that they are never more than single null (i.e., the rank of the induced metric on the surface of transitivity drops from r to $r-1$ on N, but never lower). Then if the group is orthogonally transitive in $\mathcal{U}$, it follows that N is the union of non-intersecting hypersurfaces which are* **LIH's** *with respect to the group, and consequently (since N is closed in $\mathcal{U}$) that the boundaries of N are members of the family.*

COROLLARY. *Let the postulates of theorem 9.15 be satisfied. Then, if the group is Abelian, each of the resulting* **LIH's** *is a Killing horizon.*

Recall from section 2.7 (chapter 2) that, any lightlike hypersurface Σ has a quasi-orthonormal basis of its $T_x(\Sigma)$ consisting of exactly one real null vector and all others spacelike vectors. Thus we do have a generating null vector, say ℓ, of Σ. Assume that (Σ, γ) is totally geodesic in a C-manifold (M, g) which is equivalent to the induced γ a metric connection. This is necessary to talk about induced Killing symmetry in Σ. Finally, we let ξ_1 be timelike and, therefore, all $\{\xi_2, \dots, \xi_r\}$ are spacelike Killing vectors in C-manifold M. Set

$$\ell = \frac{\xi_1 + \xi_2}{\sqrt{2}}, \qquad \ell \in T_x(\Sigma).$$

Thus, based on above, the theorem 9.14 and the fact that ℓ is Killing (it follows from the statement (4) of theorem 2.4 in chapter 2) we conclude that a totally geodesic lightlike hypersurface (Σ, γ), of a C-manifold (M, g), satisfies the postulates of theorem 9.15, such that Σ is one of the boundaries of N. Since the group manifold L^r is Abelian, the following holds:

THEOREM 9.16. *A totally geodesic lightlike hypersurface of a Lorentz C-manifold is a Killing horizon.*

Appendix A

The Petrov Classification

The Weyl tensor C_{abcd}, due to its symmetries, can be represented as a 6×6 matrix, denoted by C_{AB}. One can then classify the Weyl tensor by using the following simple approach. The Weyl symmetry $C^c_{abc} = 0$ allows us to write C_{AB} as

$$C_{AB} = \begin{pmatrix} A & B \\ B^T & -A \end{pmatrix}$$

where A and B are 3×3 matrices, such that A is symmetric and both A and B are trace free. Above matrix can also be described by the complex 3×3 matrix $D = A + i\,B$ for which there are the following three possible Jordan forms:

Petrov Type I

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}$$

Petrov Type II

$$\begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & -2\,\alpha \end{pmatrix}$$

Petrov Type III

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

where the trace free condition on A and B (and hence on D) is used. Following are subcases of Petrov Type **I** and **II**:

Petrov Type D	Subcase of type **I** where $\alpha = \beta$.
Petrov Type O	Subcase of type **I** where $\alpha = \beta = \gamma = 0 \iff C \equiv 0$.
Petrov Type N	Subcase of type **II** where $\alpha = 0$.

An alternative and very useful version of the Petrov classification is due to the work of L. Bel [10], known as the **Bel criteria**. In this criteria, null eigen bivectors and their principal null directions play an important role. Define the **complex self dual Weyl tensor** $\overset{+}{C}_{abcd}$ by

$$\overset{+}{C}_{abcd} = C_{abcd} + i \overset{\star}{C}_{abcd},$$

where $\overset{\star}{C}$ is the dual of C. Let a null direction k satisfy the following

$$k_{[e}C_{a]bc[d}k_{f]}\, k^b\, k^c = 0. \tag{A.1}$$

Then the Bel criteria are as follows:

(1) C is Petrov type **I** if there are exactly 4 distinct null vectors (called its principal null directions) k satisfying (A.1).

(2) C is Petrov type **II** if there are two coincident null directions k satisfying (A.1).

(3) C is Petrov type **III** if there are three coincident null directions k satisfying (A.1). Also C is of type **III** $\iff \overset{+}{C}_{abcd}\, k^a\, k^c = 0$.

(4) C is Petrov type **N** if there are all four coincident null directions k satisfying (A.1). Also C is of type **N** $\iff \overset{+}{C}_{abcd}\, k^d = 0$.

(5) C is Petrov type **D** if its principal null directions are coincident pairs.

(6) The Petrov type **O** is characterized by zero Weyl tensor and it does not single out any null direction.

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